Contact Anosov actions with smooth invariant bundles

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Tese de Doutorado do Programa de Pós-Graduação em
Matemática (PPG-Mat)

SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

Data de Depósito:

Assinatura: _____

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Contact Anosov actions with smooth invariant bundles

Doctoral dissertation submitted to the Institute of Mathematics and Computer Sciences – ICMC-USP, in partial fulfillment of the requirements for the degree of the Doctorate Program in Mathematics.
EXAMINATION BOARD PRESENTATION COPY

Concentration Area: Mathematics

Advisor: Prof. Dr. Carlos Alberto Maquera Apaza

USP – São Carlos
February 2018

Ficha catalográfica elaborada pela Biblioteca Prof. Achille Bassi
e Seção Técnica de Informática, ICMC/USP,
com os dados inseridos pelo(a) autor(a)

d447c de Almeida, Uirá Norberto Matos
Contact Anosov actions with smooth invariant
bundles / Uirá Norberto Matos de Almeida;
orientador Carlos Alberto Maquera Apaza. -- São
Carlos, 2018.
163 p.

Tese (Doutorado - Programa de Pós-Graduação em
Matemática) -- Instituto de Ciências Matemáticas e
de Computação, Universidade de São Paulo, 2018.

1. Anosov Actions. 2. Contact Structures. 3.
Geometric Structures. 4. Algebraic Action. I.
Maquera Apaza, Carlos Alberto, orient. II. Título.

Uirá Norberto Matos de Almeida

Ações Anosov de contato com fibrados invariantes suaves

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *EXEMPLAR DE DEFESA*

Área de Concentração: Matemática

Orientador: Prof. Dr. Carlos Alberto Maquera Apaza

USP – São Carlos
Fevereiro de 2018

*I dedicate this thesis to everyone who realised
that self-knowledge is a person's truest possession and a most valued treasure.*

ACKNOWLEDGEMENTS

Foremost, I thank CAPES and CNPq for financial support and fellowship.

I would like to express my gratitude to my advisor Prof. Carlos Maquera for the continuous support of my Ph.D study and research, for his patience, guidance, motivation and insightful comments.

Besides my advisor, I would like to thank Prof. Thierry Barbot, who supervised me during my stay in France, for his encouragement, hard questions and immense knowledge.

My sincere thanks to all my professors, who taught, guided and helped me during this part of the never ending journey of learning. In particular I thank Prof. Carlos Grossi and Prof. Sasha Ananin who helped me to broaden my horizons with respect to mathematics and look at areas beyond my own.

I thank my family, for their continuous, unwavering and unconditional support, for their love and patience through the whole process of my PhD.

Last but not the least, I thank (in alphabetical order) Paulo Borges, Philip Chiovetto, Felipe Dreilick, Rafael Esquines, Ciro Ferreira, Iago Israel, Christian Lemos, Ana Luchesi, Murilo Luiz, Guilherme Pereira, João dos Reis, and many others, dearest friends who supported me, both visibly in the most invisible ways, who made the darkest moments brighter and the bright ones shine like the sun, who made my weights lighter and my joyful moments more plentiful, who cheered me up even when they didn't know I needed cheering. I did not mention everybody and every list will be undoubtedly incomplete and I apologise to all those people whose names were not included for one reason or another.

*“As above, so below,
as within, so without,
as the universe, so the soul”
(Hermes Trismegistus)*

*“Acessamos o universo sob a lente de nossa percepção,
portanto as coisas que observamos no universo,
não podem ser mais que reflexos das estruturas internas de nossa mente”
(Carlos Grossi)*

RESUMO

DE ALMEIDA, U. N. M. **Ações Anosov de contato com fibrados invariantes suaves**. 2018. 163 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

O problema da classificação dos sistemas Anosov são de grande interesse dentro da teoria dos sistemas dinâmicos. Os principais exemplos conhecidos são de natureza algébrica e foi levantada na década de 1960 a conjectura de que estes são os únicos exemplos (SMALE, 1967). Esta conjectura se mostrou falsa para fluxos Anosov (ações de $\mathbb{R}$), onde foram construídos contraexemplos em variedades de dimensões ímpares ((HANDEL; THURSTON, 1980) e (BARTHELMÉ *et al.*, 1980)). Estes contra exemplos no entanto são de natureza patológica, e sob hipóteses um pouco mais fortes, por exemplo, suavidade dos fibrados invariantes, a conjectura permanece em aberto.

Em 1992, foi publicado um artigo (BENOIST; FOULON; LABOURIE, 1992) provando que fluxos de contato Anosov com fibrados invariantes suaves são de fato algébricos. Neste trabalho procuramos generalizar o resultado obtido em (BENOIST; FOULON; LABOURIE, 1992). Para isso criamos uma definição adequada para ações de $\mathbb{R}^k$ contato Anosov, que generalizam a noção de fluxo de contato Anosov, e seguindo a estratégia de prova utilizada em (BENOIST; FOULON; LABOURIE, 1992), obtivemos uma generalização parcial deste resultado.

Palavras-chave: Ações Anosov, Estruturas de Contato, Estrutura Geométrica, Ações algébricas.

ABSTRACT

DE ALMEIDA, U. N. M. **Contact Anosov actions with smooth invariant bundles**. 2018. 163 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

The problem of classifying the Anosov systems is of great interest in the theory of dynamical systems. The most important known examples are of algebraic nature and it has been conjectured on 1960's by S. Smale ([SMALE, 1967](#)) that these are in fact the only examples. This conjecture has been proved false for Anosov flows, where counter examples had been constructed for odd dimensional manifolds (([HANDEL; THURSTON, 1980](#)) and ([BARTHELMÉ *et al.*,](#))). This non algebraic examples however are very pathological, and with some stronger hypothesis, for example, smoothness of the invariant bundles, the conjecture remains open.

In 1992, it was published a paper ([BENOIST; FOULON; LABOURIE, 1992](#)) which proved that contact Anosov flows with smooth invariant bundles are in fact algebraic. In this monograph we seek to generalize the result obtained in ([BENOIST; FOULON; LABOURIE, 1992](#)). For this end, we create an adequate definition for contact Anosov $\mathbb{R}^k$ -actions, and following the proof strategy used in ([BENOIST; FOULON; LABOURIE, 1992](#)) we obtained a partial generalization of this result.

Keywords: Anosov Actions, Contact Structures, Geometric Structures, Algebraic Actions.

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INTRODUCTION

The problem of classification is ever present in the world of mathematics, it has old roots, perhaps as old as mathematics itself, however, only after the turn of the 20th century, did it gain such a central role. A naive person may think it is just another expression of humanity's propensity for labelling things. However, in the fields of mathematics, the problem of classification has such a robust and natural setting, that to call it merely a human propensity does not make it justice. In few words, the problem of classification can be stated in the following way: Given an abstract objet, and some of its properties, is there some "nice" list of families of examples such that this object belongs to one of those families?. ? In the field of Dynamical Systems, the problem of classification gave its first steps with the work of Henri Poincaré who realized that, given the difficulty of solving, explicitly, a differential equation, the best one could hope was to give some information about the qualitative behaviour of the solutions and that this qualitative behaviour can, sometimes, be enough to classify some dynamical systems.

It was some decades after Poincaré that the importance of hyperbolic dynamics was recognized. The seminal work of D. Anosov ([ANOSOV, 1967](#)), and latter on the works of S. Smale ([SMALE, 1967](#)) shed some light on the importance of the hyperbolic behaviour for the study of general dynamical systems. Since their introduction, the most important examples of globally hyperbolic dynamics (Anosov Systems) have been algebraic in nature: toral automorphisms, their suspensions, and geodesic flows on (strictly) negatively curved manifolds, it was, therefore, only natural to conjecture ([SMALE, 1967](#)) that Anosov systems were, in fact, algebraic, up to some (continuous) conjugation. This conjecture has been proved false for Anosov Flows, counter examples having been constructed by M. Handel and W. Thurston (([HANDEL; THURSTON, 1980](#))) on dimension 3, and by T. Barthelme, C. Bonatti, A. Gogolev and F. R. Hertz (([BARTHELMÉ *et al.*,](#))) on arbitrary odd dimensions. However the conjecture remains open for actions of higher rank abelian groups, but it can be shown that under some conditions Anosov flows and diffeomorphisms are in fact algebraic. For example, it was shown by J. Franks and S. Newhouse ([FRANKS, 1970](#)), ([NEWHOUSE, 1970](#)) that $\mathbb{Z}^k$ Anosov actions

of co-dimension one, are topologically conjugated to hyperbolic toral automorphisms.

A related conjecture on the same direction was given by A. Verjovsky ([VERJOVSKY, 1974](#)) that states that Anosov flows with co-dimension one on manifolds of dimension ≥ 4 are topologically conjugate to a suspension of toral hyperbolic automorphism. This conjecture was latter generalized to Anosov $\mathbb{R}^k$ actions on manifolds of dimension $\geq 3 + k$. Some results in this direction were given by J. Plante ([PLANTE, 1981](#)), E. Ghys ([GHYS., 1988](#)) and T. Barbot and C. Maquera ([BARBOT; MAQUERA, 2011](#)).

Suspension of algebraic Anosov diffeomorphisms have the remarkable property that the invariant bundles are jointly integrable, therefore, they exclude the rather important examples of Anosov flows given by geodesic flows on negatively curved manifolds, for these flows are actually Reeb flows of a contact structure defined by the sub bundle $E^+ \oplus E^-$ which is, therefore, nowhere integrable.

Taking into account this example, we cite another paper by E. Ghys who showed that contact Anosov flows on manifolds of dimension 3, with smooth invariant bundles, are in fact geodesic flows over surfaces of constant negative curvature. Latter on, in 1992, this result was generalized by Y. Benoist, P. Foulon and F. Labourie (BFL - ([BENOIST; FOULON; LABOURIE, 1992](#))), who showed that smoothness of the invariant Anosov bundles implies that Anosov contact flows are in fact algebraic (up to smooth conjugation, flow re-parametrization and finite coverings). It is important to notice that the smoothness condition is in fact necessary, P. Foulon and B. Hasselblatt have shown (([FOULON; HASSELBLATT, 2013](#))) an example of contact Anosov flow which is not algebraic. It was BFL's paper that motivated this work. We hoped to obtain a similar result for Anosov actions of $\mathbb{R}^k$, confirming the standing conjecture about the algebricity of higher rank abelian Anosov actions for some particular cases. We obtained partial success:

Theorem 1.0.1. If M is a manifold which supports an Anosov action associated with a "generalized k -contact structure" with smooth invariant bundles, then it is smoothly conjugated to a "quasi-algebraic" action. If, moreover, the Anosov action is faithful and $H^1(\pi_1(M), \mathbb{R}) = 0$ then, our "quasi-algebraic" model (G, K, Γ, α) can be taken with G semisimple.

We also prove a kind of converse result

Theorem 1.0.2. Let G be a semisimple Lie group $\mathfrak{a}$ a Cartan subspace with centralizer $\mathfrak{a} \oplus \mathfrak{k}$, and let K be the compact group associated with $\mathfrak{k}$. Let Γ be a uniform lattice on G acting freely on G/K . Then there exists a generalized k -contact structure such that its associated action is precisely the Weyl chamber action (that is, the right translation on $\Gamma \backslash G/K$ by the exponentials of $\mathfrak{a}$)

We also consider some cases with stronger hypothesis and prove the following two results:

Theorem 1.0.3. Under the hypothesis of Theorem 1.0.1, let $(G, K, \Gamma, \mathfrak{a})$ be our quasi-algebraic model. Suppose that G is an algebraic group. Then $\mathfrak{a}$ contained in the Cartan subspace of G

Theorem 1.0.4. Under the hypothesis of Theorem 1.0.1, let $(G, H, \Gamma, \mathfrak{a})$ be our quasi-algebraic model. Suppose that H is a compact subgroup, then the action is the Weyl chamber action.

This work is organized in the following way:

Chapter 2: After we fix some notations on section 2.1, the sections 2.2 and 2.3 are devoted to review some notions and results about Gromov's geometric structures that will be of use later on.

Chapter 3: The section 3.2 is devoted to defining our main object of study, Anosov actions associated with a generalized k -contact structure. On section 3.3 we construct a family of examples prove Theorem 1.0.2. We also give a precise definition of algebraic actions (Definition 3.3.1) and quasi-algebraic actions (Definition 3.3.2).

Chapter 4: This chapter is devoted to the proof of first part of Theorem 1.0.1 which follows the following structure

- Choose an adequate open subset Ω of M where a Lie pseudo-group acts transitively (section 4.1).
- Show that this open set is locally modelled after some homogeneous space G'/H' (section 4.2).
- Show that this local model can be extended to the whole manifold M and is complete (sections 4.3 and 4.4).

Chapter 5: This chapter is devoted to the proof of second part of Theorem 1.0.1, that is, to the study the Lie algebras $\mathfrak{g}'$ and $\mathfrak{h}'$ to obtain a reduction from $G' = G \times \mathbb{R}^k$ to G of the action (section 5). We also prove the Theorems 1.0.3 and 1.0.4

A large part of the strategy of the proof was outlined on BFL's paper ((BENOIST; FOULON; LABOURIE, 1992)).

PRELIMINARIES

The main goals of this chapter are to fix our notations and to present some aspects of the Gromov theory of geometric structures (Sections 2.2 and 2.3) which are crucial for the proof of our main theorem.

2.1 Notations

In this chapter we establish some notations.

- Plus or Minus ($\pm$) convention:

It is usual in mathematics to use the symbols $\pm$ or $\mp$ to denote the possibility to choose either $+$ or $-$ on some expression. This may cause some ambiguity if the symbols $\{\pm, \mp\}$ appear more than once in a single expression. For example, what could mean the expression: $a \pm b \pm c \mp d$? Do we make a choice for each symbol $\{\pm, \mp\}$?

On this monograph, we adopt the convention that each expression allows for a single choice (as far as signs are concerned): top sign or bottom sign. For example the expression $a \pm b \pm c \mp d$ can only have two possible meanings:

$$a + b + c - d \quad (\text{top sign})$$

or

$$a - b - c + d \quad (\text{bottom sign})$$

- Differential Manifolds

- We denote by M^m a smooth manifold of dimension m . If the dimension is either unimportant or already fixed, we omit the index m , and denote simply by M .

- If ω is any p -form $p \geq 2$, and $k \geq 1$ then we denote

$$(\omega)^k = \omega^k = \underbrace{\omega \wedge \cdots \wedge \omega}_{k\text{-times}}$$

- If ω is a $p+1$ -differential form on M and X is a vector field on M , the interior product $i_X \omega$ is the p -form defined as

$$i_X \omega(Y_1, \dots, Y_p) = \omega(X, Y_1, \dots, Y_p)$$

- The Lie derivative of a tensor T along a vector field X is denoted by $\mathcal{L}_X T$.

- Vector bundles Consider $\pi : E \rightarrow M$ a smooth vector bundle over a smooth manifold M .

- If $U \subset M$ is an open set, we denote by $\Gamma(U, E)$ the space of local sections over U .
- If E is a subbundle of the tangent bundle, the elements of $\Gamma(U, E)$ are called vector fields. If we want to emphasize the fact that $U \neq M$, we call them local vector fields.
- If $p \in M$, we denote by E_p the fiber $\pi^{-1}(p)$.
- Let $\alpha \in \Gamma(U, E)$, then we denote $\alpha_p = \alpha(p)$.

- Lie groups and algebras

- Lie groups will be denoted by capital Latin letters: G, H, K , etc.
- Lie algebras will be denoted by lower-case Gothic Latin letters: $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$, etc.
- Let G be a Lie group, it's Lie algebra will be denoted by $\text{Lie}(G)$.
- Let $(\mathfrak{g}, [\cdot \cdot])$ be a real Lie algebra, then $\mathfrak{g}_{\mathbb{C}}$ will denote it's complexification $\mathfrak{g} \oplus i\mathfrak{g}$ with induced Lie bracket $[\cdot \cdot]_{\mathbb{C}}$:

$$[u + iv, x + iy]_{\mathbb{C}} := ([u, x] - [v, y]) + i([v, x] + [u, y])$$

2.2 Gromov's Geometric Structures

In this section, we recall the basic facts about the theory of Gromov's geometric structures. The theory is developed in a most general way, and follows closely the exposition by Benoist, Y. (BENOIST, 1997) and Feres, R. (FERES, 2002)

2.2.0.1 Definitions and notations

First, let fix some notations and conventions that shall be used on this chapter. Let M, N be manifolds of dimensions m and n respectively. We shall denote by $J_p^k(M, N)$ the space of

k -jets¹ from M to N on p . The elements of $J_p^k(M, N)$ will be denoted by either $j_p^k(f)$ or $j_f^k(p)$ or $j^k(f)(p)$. We shall put

$$J^k(M, N) = \bigsqcup_{p \in M} J_p^k(M, N).$$

Both $J^k(M, N)$ (and $J_p^k(M, N)$) have a natural structure of smooth manifold. Let (U, x) and (V, y) be local charts of M and N

$$\begin{aligned} J^k(U, V) &\longrightarrow x(U) \times y(V) \times \text{Lin}(\mathbb{R}^m, \mathbb{R}^n) \times \bigoplus_{j=2}^k \text{Sym}^j(m, n) \\ j_p^k(f) &\longmapsto (u, \varphi(u), \varphi'(u), \dots, \varphi^{(k)}(u)) \end{aligned}$$

where $u = x(p)$, $\varphi = y \circ f \circ x^{-1}$, $\text{Lin}(\mathbb{R}^m, \mathbb{R}^n)$ denote the space of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ and $\text{Sym}^j(m, n)$ denotes the space of j -multi linear symmetric maps from $\mathbb{R}^m$ to $\mathbb{R}^n$.

We shall also denote by $Gl^k(m) \subset J_0^k(\mathbb{R}^m, \mathbb{R}^m)$ the space of k -jets of local diffeomorphisms from $\mathbb{R}^m$ to $\mathbb{R}^m$ that fixes the origin. A local map from M to N is a map $f : U \rightarrow V \subset M$ defined on some open subset U of M . And for local diffeomorphisms we shall mean local maps that are diffeomorphisms. The composition law

$$j_0^k(\varphi) \cdot j_0^k(\psi) := j_0^k(\varphi \circ \psi)$$

makes $Gl^k(m)$ a group. It is, in fact, a real, algebraic linear Lie group. We shall often identify $j_0^k(\varphi) \in Gl^k(m)$ with its Taylor polynomial's coefficients: $j_0^k(\varphi) \leftrightarrow (\varphi'(0), \dots, \varphi^{(k)}(0))$.

Lets also consider $F^k(M)$ the space of k linear frames on M , that is the space of k -jets of local diffeomorphisms from a neighbourhood of the origin in $\mathbb{R}^m$ to M . More precisely

$$\begin{aligned} F^k(M) &:= \{(p, j_0^k(f)) \in M \times J_0^k(\mathbb{R}^m, M); f : U_0 \rightarrow V \subset M, U_0 \text{ open subset of } \mathbb{R}^m \\ &\quad f \text{ is a diffeomorphism such that } f(0) = p\} \end{aligned}$$

The canonical projection $\pi : F^k(M) \rightarrow M$ defines a $Gl^k(m)$ -principal bundle, where the action of $Gl^k(m)$ on $F^k(M)$ is given by:

$$j_0^k(f) \cdot j_0^k(\varphi) := j_0^k(f \circ \varphi) \quad \forall j_0^k(f) \in F^k(M) \quad j_0^k(\varphi) \in Gl^k(m)$$

Definition 2.2.1. Let Σ be a smooth manifold, and let $\lambda : Gl^k(m) \times \Sigma \rightarrow \Sigma$ be a smooth action. A geometric structure of order k and type λ (sometimes we say, of type Σ , if the action is understood), is a $Gl^k(m)$ -equivariant map $\sigma : F^k(M) \rightarrow \Sigma$, that is

$$(x \cdot g) = \lambda(g^{-1}, \sigma(x)) = g^{-1} \cdot \sigma(x)$$

¹ A k -jet at p is a class of equivalent functions whose derivatives of order less then or equal to k coincides on some neighbourhood of p .

If Σ is a real smooth algebraic manifold and the action λ is algebraic, then σ is said to be an A -structure.

Remark 2.2.2. A geometric structure can also be thought as a section of the bundle $\Sigma \times_{Gl^k(m)} F^k(M)$

Definition 2.2.3. An automorphism of σ is a diffeomorphism F of M , such that

$$\sigma \circ F_{(k)} = \sigma$$

where

$$\begin{aligned} F_{(k)} : F^k(M) &\rightarrow F^k(M) \\ j_0^k(\varphi) &\mapsto j_0^k(F \circ \varphi). \end{aligned}$$

If $H : U \rightarrow V$ is a local diffeomorphism, then the local automorphism is defined in the same way with some obvious modifications:

$$\begin{aligned} H_{(k)} : F^k(U) &\rightarrow F^k(V) \\ j_0^k(\varphi) &\mapsto j_0^k(H \circ \varphi) \end{aligned}$$

and we require that

$$\sigma|_{F^k(V)} \circ H_{(k)} = \sigma|_{F^k(U)}$$

The (pseudo) group of (local) automorphisms is denoted by $\text{Aut}(\sigma)(\text{Aut}^{loc}(\sigma))$.

Example 2.2.4. An example of especial interest for us is given by the pseudo-riemannian metrics. Let g be a pseudo-riemannian metric of signature s on M . We shall see how g can be understood as an A -structure of order 1 on M . Let Σ denote the space of all symmetric bilinear forms of signature s on $\mathbb{R}^m$. Clearly, Σ can be identified with the space $GL(m, \mathbb{R})/O(s, m-s)$, where $O(p, q)$ denotes the isotropy group of the bilinear form

$$Q(x, x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

$GL(m, \mathbb{R})$ acts on Σ by conjugation, that is, for a bilinear form B of signature s , and $g \in GL(m, \mathbb{R})$ we have,

$$(g \cdot B)(u, v) := B(g^{-1}u, g^{-1}v).$$

Now, if we identify $Gl^1(m)$ with $GL(m, \mathbb{R})$ ($(j_0^1(\varphi)) \leftrightarrow \varphi'(0)$), and we identify

$$F^1(M) = \{(p, \xi) ; p \in M, \xi : \mathbb{R}^m \rightarrow T_p M \text{ is a linear isomorphism}\}$$

then, we define the A-structure $\tilde{g}$ of type Σ by

$$\begin{aligned}\tilde{g} : F^1(M) &\rightarrow \Sigma \\ (p, \xi) &\mapsto \tilde{g}(p, \xi)(u, v) := g_p(\xi u, \xi v)\end{aligned}$$

It is clear that the (local) automorphisms of $\tilde{g}$ are in fact the (local) isometries of g . This motivates Gromov to call the automorphisms of a geometric structure, an isometry for that geometric structure, but we shall reserve the name isometry for the automorphisms of the geometric structure given by a Riemannian metric.

Like many things in differential geometry, geometric structures can be glued together from compatible local data. Let us make this notion more precise.

Definition 2.2.5. Let σ be a geometric structure of order k on M and let $x : U' \rightarrow U \subset \mathbb{R}^m$ be a local chart of M . We define the local form $\sigma_x : U \rightarrow \Sigma$ of σ with respect to x as

$$\sigma_x(u) := \sigma(j_0^k(x^{-1} \circ \tau_u)) \quad \tau_u : h \mapsto h + u$$

Remark 2.2.6. If $y : V' \rightarrow V$ is another local chart, and $\varphi := y \circ x^{-1}$ has non empty domain, then

$$\begin{aligned}j_0^k(x^{-1} \tau_u) &= j_0^k(y^{-1} \circ \varphi \circ \tau_u) \\ &= j_0^k(y^{-1} \circ \tau_{\varphi(u)}) \cdot g_\varphi^k(u)\end{aligned}$$

where $g_\varphi^k(u) = j_0^k(\tau_{-\varphi(u)} \circ \varphi \circ \tau_u)$ can be seen as the Taylor polynomial of φ at u .²

Thus,

$$g_\varphi^k(u) \sigma_x(u) = \sigma_y(\varphi(u)) \quad (2.1)$$

Proposition 2.2.7. Consider a manifold Σ with a $Gl^k(m)$ action. Suppose that, for a given atlas of M , for any coordinate system $x : U \rightarrow U' \subset \mathbb{R}^m$ of this atlas we have a map $\sigma_x : U \rightarrow \Sigma$ such that, for any two coordinate systems x, y , with non zero intersection, we have,

$$g_\varphi^k(u) \sigma_x(u) = \sigma_y(\varphi(u))$$

where $\varphi := y \circ x^{-1}$ and $g_\varphi^k(u) = j_0^k(\tau_{-\varphi(u)} \circ \varphi \circ \tau_u)$. Then, there exists a unique geometric structure $\sigma : F^k(M) \rightarrow \Sigma$ such that its local form is given by $\{\sigma_x\}$

Proof. Notice that if $U' \subset \mathbb{R}^m$, is an open set then we have the identification $F^k(U') = U' \times Gl^k(m)$. To see this, we consider, without loss of generality, U' to be an open ball around the

² From this interpretation, it's easy to see that $g_\varphi^k(u)^{-1} = g_{\varphi^{-1}}^k(\varphi(u))$ and $g_{\varphi \circ \psi}^k(u) = g_\varphi^k(\psi(u)) g_\psi^k(u)$

origin and notice that any local diffeomorphism from U' to itself can be uniquely written as a composition of a translation with a diffeomorphism that preserves the origin.

This identification allow us to identify the bundle $F^k(U') \times_{Gl^k(m)} \Sigma$ with $U' \times \Sigma$. The maps σ_x , therefore, allow us to build a section ξ_x of this bundle, and moreover, the coordinates x allow us to view this as a subsection of the bundle $F^k(U) \times_{Gl^k(m)} \Sigma$.

It is a classical result that local, compatible smooth sections of a smooth bundle glue together on a global section. And the condition $g_\varphi^k(u)\sigma_x(u) = \sigma_y(\varphi(u))$ does means, in fact, that the sections ξ_x are compatible. $\square$

2.2.0.2 Rigidity

Intuitively, rigidity (of order r) means that the $r + 1$ -jets of automorphisms of some structure, are determined by the r -jets of automorphisms. To make this notion precise, we shall define something called the prolongation of a geometric structure.

Definition 2.2.8. Consider Σ with a $Gl^k(m)$ action and $r = l + k$, $l \geq 0$. Define $\Sigma^l := J_0^l(\mathbb{R}^m, \Sigma)$. There exists a natural action λ^l of $Gl^r(m)$ on Σ^l given by:

$$\begin{aligned} \lambda^l : \quad Gl^r(m) \times \Sigma^l &\longrightarrow \Sigma^l \\ (j_0^r(\varphi, j_0^l((s))) &\longmapsto j_0^l((g_\varphi^k \cdot s) \circ \varphi^{-1}) = j_0^l(u \mapsto (g_\varphi^k(\varphi^{-1}(u))s(\varphi^{-1}(u)))) \end{aligned}$$

Moreover, the natural map $\Sigma^l \rightarrow \Sigma$ is $Gl^r(m)$ equivariant.

A prolongation of degree l of a geometric structure σ is a geometric structure σ^l which makes the following diagram commutative and equivariant

$$\begin{array}{ccc} F^{k+l}(M) & \xrightarrow{\sigma^l} & \Sigma^l \\ \downarrow p_k^{k+l} & & \downarrow \pi \\ F^k(M) & \xrightarrow{\sigma} & \Sigma \\ \downarrow p_k & & \\ M & & \end{array}$$

Remark 2.2.9. We must verify that λ_l actually defines an action:

For $j_0^r(\varphi), j_0^r(\psi) \in Gl^r(m)$ and $j_0^l(s) \in \Sigma^l$ we have:

$$\begin{aligned}
 (j_0^r(\varphi)j_0^r(\psi)) \cdot j_0^l(s) &= j_0^r(\varphi \circ \psi) \cdot j_0^l(s) = j_0^l((g_{(\varphi \circ \psi)}^k \cdot s) \circ (\varphi \circ \psi)^{-1}) \\
 &= j_0^l(((g_\varphi^k \circ \psi) \cdot g_\psi^k) \cdot s) \circ (\varphi \circ \psi)^{-1}) \\
 &= j_0^l(u \mapsto (g_\varphi^k \circ \psi)(\varphi \circ \psi)^{-1}(u) \cdot (g_\psi^k)(\varphi \circ \psi)^{-1}(u) \cdot s((\varphi \circ \psi)^{-1}(u))) \\
 &= j_0^l(u \mapsto g_\varphi^k(\varphi^{-1}(u)) \cdot (g_\psi^k)(\psi^{-1} \circ \varphi^{-1}(u)) \cdot s(\psi^{-1} \circ \varphi^{-1}(u))) \\
 &= j_0^l((g_\varphi^k \cdot ((g_\psi^k \circ \psi^{-1}) \cdot s \circ \psi^{-1})) \circ \varphi^{-1}(u)) \\
 &= j_0^r(\varphi) \cdot (j_0^l((g_\psi^k \circ \psi^{-1}) \cdot s \circ \psi^{-1})) = j_0^r(\varphi) \cdot (j_0^r(\psi) \cdot j_0^l(s))
 \end{aligned}$$

Proposition 2.2.10. For any $l \geq 0$, there exists a natural l -prolongation of σ given by:

$$\begin{aligned}
 \sigma^l : F^r(M) &\longrightarrow \Sigma^l \\
 j_0^r(f) &\longmapsto j_0^l(u \mapsto \sigma(j_0^k(f \circ \tau_u)))
 \end{aligned}$$

Proof. We must verify that σ^l is an $Gl^r(m)$ -equivariant map, but before that, assuming that σ^l have been correctly defined. Then it is clear that the Diagram 2.2.8 is commutative and equivariant, that is both π and p_k^{k+l} are $Gl^r(m)$ -equivariant³.

Now, to prove that σ^l is $Gl^r(m)$ -equivariant, let $j_0^r(\varphi) \in Gl^r(m)$ and $j_0^r(f) \in F^r(M)$. From the definition of g_φ^k , it follows that

$$j_0^k(f \circ \varphi \circ \tau_u) = j_0^k(f \circ \tau_{\varphi(u)}) \cdot g_\varphi^k(u)$$

Also note that $g_\varphi^k(u)$ can be understood as an element of $Gl^k(m)$. Thus, as σ is $Gl^k(m)$ -invariant

$$\begin{aligned}
 \sigma^l(j_0^r(f) \cdot j_0^r(\varphi)) &= \sigma^l(j_0^r(f \circ \varphi)) = j_0^l(u \mapsto \sigma(j_0^k(f \circ \varphi \circ \tau_u))) \\
 &= j_0^l(u \mapsto \sigma(j_0^k(f \circ \tau_{\varphi(u)}) \cdot g_\varphi^k(u))) \\
 &= j_0^l(u \mapsto g_\varphi^k(u)^{-1} \cdot \sigma(j_0^k(f \circ \tau_{\varphi(u)}))) \\
 &= j_0^l(u \mapsto g_{\varphi^{-1}}^k(\varphi(u)) \cdot \sigma(j_0^k(f \circ \tau_{\varphi(u)}))) \\
 &= j_0^l((g_{\varphi^{-1}}^k \cdot \sigma(j_0^k(f \circ \tau_\bullet))) \circ \varphi) \\
 &= \lambda^l(j_0^r(\varphi^{-1}), \sigma(j_0^k(f \circ \tau_\bullet))) \\
 &= \lambda^l(j_0^r(\varphi)^{-1}, \sigma(v \mapsto j_0^k(f \circ \tau_v))) = j_0^r(\varphi)^{-1} \cdot \sigma^l(j_0^r(f))
 \end{aligned}$$

□

³ We understand that $Gl^r(m)$ acts on $F^k(M)$ and Σ via the natural map

$$Gl^r(m) \ni j_0^{k+l}(\varphi) \mapsto j_0^k(\varphi) \in Gl^k(m)$$

Definition 2.2.11. Let $D^k(M) \subset J^k(M, M)$ be the open set of all k -jets of local diffeomorphisms of M . We define

$$\text{Aut}_{pq}^k(\sigma) := \{j_p^k(F) \in D^k(M); F(p) = q; \sigma(j_0^k(F \circ f)) = \sigma(j_0^k(f)); \forall j_0^k(f) \in F_p^k(M)\}$$

From the $GL^k(m)$ -equivariance of σ it follows that we can substitute the "for all" on the above definition with "for some". For $r = k + l$, $l \geq 0$ we also define

$$\text{Aut}_{pq}^r(\sigma) := \text{Aut}_{pq}^r(\sigma^l)$$

For convenience, we also put $\text{Aut}^r(\sigma) = \bigsqcup_{p,q \in M} \text{Aut}_{pq}^r(\sigma)$. The elements of $\text{Aut}^r(\sigma)$ are called infinitesimal automorphisms of order r .

Definition 2.2.12. There is an obvious projection $\pi_{pq}^r : \text{Aut}_{pq}^{r+1}(\sigma) \rightarrow \text{Aut}_{pq}^r(\sigma)$. A Gromov r -rigid geometric structure (or simply r -rigid geometric structure) is a geometric structure such that π_{pp}^r is injective for every $p \in M$.

Lemma 2.2.13. Pseudo-riemannian metrics are 1-rigid.

Proof. Let's first remember the proof of a well known result:

If $\varphi, \psi : U \rightarrow M$ are local isometries (in the traditional sense) then $\varphi = \psi$ if, and only if, there exists a point x in each connected component of U such that $d\varphi_x = d\psi_x$.

The proof is rather easy: we suppose that U is connected, and let $V = \{y \in U; \varphi(y) = \psi(y) \text{ and } d\varphi_y = d\psi_y\}$. It is clear that V is closed and the isometries φ and ψ coincide on V . It remains to show that V is open. Let $y \in V$ and let W be a normal neighbourhood of y . Then, for every $z \in W$ there exists $v = v(z) \in T_y M$ such that $z = \exp_y(v)$. This implies that

$$\begin{aligned} \varphi(z) &= \varphi(\exp_y(v(z))) = \exp_{\varphi(y)} d\varphi_y v(z) \\ &= \exp_{\psi(y)} d\psi_y v(z) = \psi(\exp_y(v(z))) = \psi(z) \end{aligned}$$

and thus $z \in V$.

Consider $g : GL(m, \mathbb{R}) \rightarrow \Sigma \equiv GL(m, \mathbb{R})/O(s, m-s)$ the geometric structure associated with a pseudo Riemannian metric, and put $\tilde{g} = g^1$ be the 1-prolongation of g . We want to show that $\pi_{pp}^2 : \text{Aut}_{pp}^2(\tilde{g}) \rightarrow \text{Aut}_{pp}^1(g)$ is injective. But $j_0^1(\varphi) \in \text{Aut}_{pp}^1(g)$ if, and only if φ is a local isometry at p , and, as we have seen, the local isometry φ is completely determined by its first derivative, which means that π_{pp}^2 is injective. $\square$

Two important results about geometric structure are the following:

Theorem 2.2.14 (Gromov, (GROMOV, 1988), Theorem 1.6.F). If σ is a rigid geometric structure of order r on M , there exists k and an open dense, $\text{Aut}^{r+k}(\sigma)$ -invariant, subset $M_0 \subset M$ such that every infinitesimal automorphisms in $\text{Aut}_{v_1, v_2}^{r+k}(\sigma)$, $v_1, v_2 \in M_0$, extends to a unique local automorphism. In other words, the map

$$\begin{aligned} \text{Aut}_{v_1, v_2}^{loc}(\sigma) &\rightarrow \text{Aut}_{v_1, v_2}^{r+k}(\sigma) \\ f &\mapsto j_{v_1}^{r+k}(f) \end{aligned}$$

is injective for every $v_1, v_2 \in M_0$.

Corollary 2.2.15. If σ is rigid, then there exists an open dense subset M_0 such that, for sufficiently large r , the orbits (on M_0) of $\text{Aut}^{loc}(\sigma)$ and $\text{Aut}^r(\sigma)$ coincide.

Theorem 2.2.16 (Gromov, (GROMOV, 1988), Theorem 3.1). If σ is an algebraic (possibly not rigid) geometric structure of order r on M , then, for every k there exists a finite partition of M in locally closed subsets

$$M = M_0^k \cup M_1^k \cup \dots \cup M_{s_k}^k$$

such that

1. M_0^k is an open dense subset in M and, for every $1 \leq l \leq s_k$, the union $M_l^k \cup M_{l+1}^k \cup \dots \cup M_{s_k}^k$ is closed in M .
2. This partition is $\text{Aut}^{r+k}(\sigma)$ -invariant.
3. The partition M^{k+1} is a refinement of the partition M^k .
4. The Aut^{r+k-1} -orbits in M_0^k are closed smooth submanifolds.

As a consequence of those two theorems, we obtain

Theorem 2.2.17 (Open-Dense orbit theorem). [Gromov, (GROMOV, 1988), Corollary 3.3.A] Let $\sigma : F^k(M) \rightarrow \Sigma$ be an A -structure that is rigid (for some r), then a dense orbit of $\text{Aut}^{loc}(\sigma)$ is open.

2.3 Killing fields

The goal of this chapter is to study the infinitesimal symmetries of a geometric structure, that is, the vector fields that integrates to automorphisms of the geometric structure. After a short technical preparation in the first section, we prove, in the second section, that such vector fields forms a Lie algebra. In the third section, we explore an interesting consequence of the existence of an affine connection which is "compatible" with a geometric structure.

2.3.1 Canonical form on $F^k(M)$

Let $P^k = F^k(\mathbb{R}^m)$. For $k \neq 1$ this bundle has no natural group structure, however, it does have a distinguished element e , that is, the k -jet of the identity at the origin $o \in \mathbb{R}^m$. We shall denote by $\mathfrak{P}^k$ the tangent space $T_e P^k$. Note that for $k = 1$ this is precisely the Lie algebra of the affine Lie group and for $k = 0$ this is in fact $\mathbb{R}^n$ itself. The canonical form on $\mathcal{F}^k$ shall be a differential 1-form which takes values on $\mathfrak{P}^{k-1}$.

Remark 2.3.1. For $r > k$ the projection $P^r \rightarrow P^k$ induces a projection $\mathfrak{P}^r \rightarrow \mathfrak{P}^k$. The kernel of the projection $\mathfrak{P}^k \rightarrow \mathfrak{P}^0$ is in fact the Lie algebra of the group $Gl^k(m)$.

Let $u = j_0^k(f) \in F^k(M)$ and u' the image of u on the projection $F^k(M)(M) \rightarrow F^{k-1}(M)$. The local diffeomorphism f induces a map around the distinguished element e

$$\begin{aligned} \tilde{f} : P^{k-1} &\longrightarrow F^{k-1}(M) \\ j_0^{k-1}(g) &\longmapsto \tilde{f}(j_0^{k-1}(g)) = j_0^{k-1}(f \circ g) \end{aligned}$$

Notice that the map $\tilde{f}$ is actually a diffeomorphism.

The differential of $\tilde{f}$ at the distinguished element (which can be identified with $j_e^1(\tilde{f})$) maps $\mathfrak{P}^{k-1}$ onto $T_{u'} F^{k-1}(M)$. This linear map depends only on u and not on the choice of f , and thus, we shall denote it $\tilde{u}$.

Definition 2.3.2. Let Z be a tangent vector of $F^k(M)$ at u , that is, $Z \in T_u F^k(M)$. Let $Z' \in T_{u'} F^{k-1}(M)$ be the image of Z under the projection $F^k(M) \rightarrow F^{k-1}(M)$. The canonical form θ_k on $F^k(M)$ is a $\mathfrak{P}^{k-1}$ valued 1-form defined by

$$\theta_k(Z) = \tilde{u}^{-1}(Z')$$

It is immediate from the definitions, that if $r > k$ and if v is the image of u under the projection $F^r(M) \rightarrow F^k(M)$, the following diagram:

$$\begin{array}{ccc} \mathfrak{P}^r & \xrightarrow{\tilde{u}} & T_{u'} F^r(M) \\ \downarrow & & \downarrow \\ \mathfrak{P}^k & \xrightarrow{\tilde{v}} & T_{v'} F^k(M) \end{array}$$

is commutative, where v' is the image of v under the projection $F^k(M) \rightarrow F^{k-1}(M)$. Thus we also have the following commutative diagram

$$\begin{array}{ccc}
T(F^r(M)) & \xrightarrow{\theta_r} & \mathfrak{P}^{r-1} \\
\downarrow & & \downarrow \\
T(F^k(M)) & \xrightarrow{\theta_k} & \mathfrak{P}^{k-1}
\end{array}$$

Lemma 2.3.3. Let M and M' be two m dimensional manifolds, and $f : M \rightarrow M'$ be a local diffeomorphism. Then, f induces a map $f_{(k)} : F^k(M) \rightarrow F^k(M')$. Let θ_k and θ'_k be the canonical forms on $F^k(M)$ and $F^k(M')$ respectively. Then

$$f_{(k)}^*(\theta'_k) = \theta_k$$

Proof. Notice that we have the following commutative diagram

$$F^k(M) \xrightarrow{f_{(k)}} F^k(M')$$

$$F^{k-1}(M) \xrightarrow{f_{(k-1)}} F^{k-1}(M')$$

Thus, if Z is a tangent vector of $F^k(M)$ at u , then $((f_{(k)})_* Z)' = (f_{(k-1)})_*(Z')$. Let $w = f_{(k)}(u)$. Thus,

$$f_{(k)}^*(\theta'_k)(Z) = \theta'_k((f_{(k)})_* Z) = \tilde{w}^{-1}((f_{(k-1)})_* Z')$$

on the other hand, if $u = j_0^k(h)$, then $v = j_0^k(f \circ h)$, also, for $j_0^{k-1}(g) \in P^{k-1}$,

$$\widetilde{f \circ h}(j_0^{k-1}(g)) = j_0^{k-1}(f \circ h \circ g) = f_{(k-1)}(j_0^{k-1}(h \circ g)) = f_{(k-1)} \circ \tilde{h}(j_0^{k-1}(g))$$

thus

$$\tilde{w} = j_0^1(\widetilde{f \circ h}) = j_0^1(f_{(k-1)} \circ \tilde{h}) = (f_{(k-1)})_*(j_0^1(\tilde{h})) = (f_{(k-1)})_* \circ \tilde{u}$$

and the result follows. $\square$

Corollary 2.3.4. Every local diffeomorphism of M lifted to $F^k(M)$, leaves the canonical form invariant.

There is a kind of converse of this lemma

Lemma 2.3.5. Let M and M' be two m -dimensional manifolds, and let $\tilde{f} : F^k(M) \rightarrow F^k(M')$ be a fiber preserving transformation such that

$$\tilde{f}^*(\theta'_k) = \theta_k$$

Then, $\tilde{f} = f_{(k)}$ for some local diffeomorphism f .

Proof. Because $\tilde{f}$ is fiber preserving, it induces a transformation f from M to M' . We shall prove that $\tilde{f} = f_{(k)}$. Let $\Phi = f_{(k)}^{-1} \circ \tilde{f}$. Then, Φ is a fiber preserving transformation that induces the identity transformation on M . Moreover, it leaves θ_k invariant, that is, if $Z \in T_u F^k(M)$ then $\theta_k(Z) = \theta_k(J_* Z)$. But, we notice, that the diagram 2.3.1 can be generalized for diffeomorphisms that preserves the fiber, that is, if Φ is a local diffeomorphism on $F^k(M)$ that preserves the fiber, then, it induces a local diffeomorphism ϕ on $F^{k-1}(M)$, and we have

$$\begin{array}{ccc} F^k(M) & \xrightarrow{\Phi} & F^k(M) \\ \downarrow & & \downarrow \\ F^{k-1}(M) & \xrightarrow{\phi} & F^{k-1}(M) \end{array}$$

Repeating the same considerations of the previous lemma, we have

$$\tilde{u}^{-1}(Z') = \theta_k(Z) = \theta_k(\Phi_* Z) = (\phi_* \tilde{u})^{-1}(Z')$$

which means that $\tilde{u} = \phi_* \tilde{u}$. As $\tilde{u}$ is an isomorphism, this means that $\phi_* = Id$. Which means that $\Phi = Id$, as we desired. □

2.3.2 Lie Algebra of Killing Vector fields

Definition 2.3.6. Let $\sigma : F^k(M) \rightarrow \Sigma$ be a geometric structure on M . A Killing field for σ is a vector field X such that the diffeomorphism defined by the flow $\varphi_t : M \rightarrow M$ are automorphism, that is, $\varphi_t \in \text{Aut}(\sigma)$.

Notice that if φ_t is the flow of a vector field X on M , then, it induces a one parameter family of diffeomorphisms $(\varphi_t)_{(k)}$ on $F^k(M)$. The infinitesimal generator of this family will be denoted by $X_{(k)}$ and is called the Natural Lift of X to $F^k(M)$.

Lemma 2.3.7. A vector field X is a Killing field for σ if, and only if, $d\sigma(X_{(k)}) = 0$

Proof. If X is a Killing field, then the flow $\varphi_t \in \text{Aut}^{loc}(\sigma)$, that means that

$$\sigma((\varphi_t)_{(k)}) = \sigma$$

But this means that $(\varphi_t)_{(k)} \in \text{Aut}^k(\sigma)$. Thus, $d\sigma$ will collapse the vectors tangent to $(\varphi_t)_{(k)}$. That is $d\sigma(X_{(k)}) = 0$.

Reversing the implications above, we obtain the other implication. □

There is a good characterization of the vector fields on $F^k(M)$ which are natural lifts, it uses the canonical form on $F^k(M)$.

Theorem 2.3.8. For each vector field X of M , there exists a unique vector field $\tilde{X}$ on $F^k(M)$ such that

1. $\tilde{X}$ is invariant by the action of $Gl^k(m)$.
2. $\mathcal{L}_{\tilde{X}}\theta_k = 0$, where θ_k is the canonical form on $F^k(M)$.
3. $\pi\tilde{X}_u = X_{\pi(u)}$ for every $u \in F^k(M)$

And conversely, if $\tilde{X}$ satisfies 1 – 2, then there exists unique vector field X on M satisfying 3.

Proof. Let $\tilde{X} = X_{(k)}$ be the natural lift of X to $F^k(M)$. By definition, $X_{(k)}$ satisfies the third property. Also, as the flow $(\varphi_t)_{(k)}$ commutes with the right action of $Gl^k(m)$, it follows that $\tilde{X}$ is $Gl^k(m)$ -invariant. The second property follows from Corollary 2.3.4.

To prove the uniqueness, let Y be another lift of X to $F^k(M)$. Let $\tilde{\psi}_t$ be the flow generated by Y . The first property means that $\tilde{\psi}_t$ commutes with the action of $Gl^k(m)$. This means that we can project it to ψ_t a one parameter group of diffeomorphisms on M . From Lemma 2.3.5, this means that $\tilde{\psi}_t = (\psi_t)_{(k)}$, and from the third property, this means that $\psi_t = \varphi_t$, and thus, $\tilde{\psi}_t = (\varphi_t)_k$ and therefore $Y = X_{(k)}$.

Conversely, if $\tilde{X}$ is a vector field satisfying 1 – 2, then, for each $x \in M$ let $u \in F^k(M)$ such that $x = \pi(u)$ and set $X_x = \pi(\tilde{X}_u)$. From 1, X_x is independent of the choice of u and thus X satisfies 3. The uniqueness is trivial. $\square$

Corollary 2.3.9. The space of Killing vector fields of σ on M is a Lie algebra.

Proof. From Theorem 2.3.8, we have

$$[X, \tilde{Y}] = [\tilde{X}, \tilde{Y}]$$

for any vector fields X, Y on M . And thus,

$$d\sigma([X, Y]_{(k)}) = d\sigma([X_{(k)}, Y_{(k)}])$$

From Lemma 2.3.7, it follows that if X and Y are Killing vector fields, then so is $[X, Y]$. $\square$

Corollary 2.3.10. Let $J^k(TM)$ be the space of k -jets of sections of TM . For every $u \in F^k(M)$, $\pi(u) = x \in M$, the spaces $T_u F^k(M)$ and $J_x^k(TM)$ can be naturally identified.

Proof. Let X be a local vector field on M around x . Taking $X_{(k)}$ the natural lift of X , it is easy to see that $X_{(k)}(u)$ depends only on $j_x^k(X)$. The uniqueness of the lift makes the map

$$J_x^k(TM) \rightarrow T_u F^k(M)$$

injective. Analysing the dimensions, we conclude that this is a isomorphism.

In particular, this means, that every tangent vector of $T_u F^k(M)$ can be extended to a projectable vector field. $\square$

Lemma 2.3.11. Fix a point $o \in M$. Let $\mathfrak{h}'$ denote the space of germs of killing fields that vanish at o . Then, $\mathfrak{h}'$ is the Lie algebra of $H' = \text{Aut}_{oo}^{loc}(\sigma)$.

Proof. We have not actually defined the space $\text{Aut}_{xy}^{loc}(\sigma)$. It is defined as the space of germs of local automorphisms sending x to y .

From definition, a local vector field X defined on a neighbourhood U of o is a (local) Killing vector field if, and only if, its flow is a local automorphism: $\sigma \circ (\varphi_t)_{(k)} = \sigma$. More precisely:

$$j_x^k(\varphi_t) \in \text{Aut}_{x\varphi_t(x)}^k(\sigma) \quad ; \quad \forall x \in U \quad ; \quad \forall t \in (-\varepsilon, \varepsilon)$$

If X vanishes at o , we have

$$j_o^k(\varphi_t) \in \text{Aut}_{oo}^k(\sigma) \quad ; \quad \forall t \in \mathbb{R}$$

This means that $\mathfrak{h}' \subset \text{Lie}(H')$. To prove the opposite inclusion, we must show, that every one parameter subgroup of H' is of the form $j_o^k(\varphi_t)$ for φ_t the flow of a Killing vector field that vanishes at o .

Let $\gamma(t) \subset H'$ be a smooth one parameter subgroup. For each t , $\gamma(t)$ is the germ of a local automorphism at o , that fixes o . We take a representative $\gamma(t) = \psi_t$ such that

$$\psi_t \text{ is a local diffeomorphism near } o \in M \text{ and } j_o^k(\psi_t) \in \text{Aut}_{oo}^k(\sigma) \quad (2.2)$$

But ψ_t is a smooth one parameter family of local diffeomorphisms, which has infinitesimal generator Y , and (2.2) means precisely that Y is a Killing field that vanish at o , which finishes the proof. $\square$

2.3.3 Killing Lie algebra and compatible connections

Now, let us consider a smooth connection ∇ on M . We denote by T and R the torsion and curvature tensors. We shall suppose that $\nabla T = 0$. We define the derivation $A_X = \mathcal{L}_X - \nabla_X$.

As this derivation is zero on the function algebra $C^\infty(M)$, it follows that it is induced by a $(1, 1)$ -tensor, and thus, it makes sense to calculate $A_X v$ for any tangent vector v of M .

Remember that this is also true for the torsion and curvature tensors, that is, for tangent vectors $v_1, v_2 \in T_p M$, $T(v_1, v_2)$ is a well defined element of $T_p M$ ⁴, and likewise, $R(v_1, v_2)$ is well defined endomorphism of $T_p M$.

We have:

$$\begin{aligned} R(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} = [\mathcal{L}_X - A_X, \nabla_Y] - \nabla_{[X, Y]} \\ &= [\mathcal{L}_X, \nabla_Y] - \nabla_{[X, Y]} - [A_X, \nabla_Y] \end{aligned}$$

Definition 2.3.12. Let M be a manifold with a connection. A vector field X on M is called an infinitesimal affine transformation if the flow ϕ_t of X is affine, that is it preserves the connection.

The following theorem is classic:

Lemma 2.3.13 (Kobayashi, S and Nomizu, K. ([KOBAYASHI; NOMIZU, 1963](#)), Section VI.2 Proposition 2.2). Let M be a manifold with a connection, then, X is an infinitesimal affine transformation if, and only if,

$$[\mathcal{L}_X, \nabla_Y] = \nabla_{[X, Y]} \quad \text{for every vector field } Y \text{ on } M$$

Corollary 2.3.14. If X is an infinitesimal affine transformation then,

$$R(X, Y)Z = [A_X, \nabla_Y]Z = A_X(\nabla_Y(Z)) - \nabla_Y(A_X Z) = (\nabla_Y(A_X))Z$$

Let's fix a point $o \in M$ and let $V_o = T_o(M)$. Consider the space $\text{End}(V_o) \times V_o$ and endow it with a bracket operation

$$[(A, a), (B, b)] = ([A, B] - R(a, b), Ab - Ba + T(a, b))$$

This is not, in general, a Lie bracket, for the Jacobi Identity is not usually satisfied. However nothing prevents a subspace of $\text{End}(V_o) \times V_o$ to be a Lie algebra.

⁴ Let $Y = \sum a_i X_i$, $Z = \sum b_j X_j$ be vector fields of M . We have:

$$\begin{aligned} T(Y, Z) &= \nabla_Y Z - \nabla_Z Y - [Y, Z] \\ &= \sum_{i,j} a_i \nabla_{X_i} b_j X_j - b_j \nabla_{X_j} a_i X_i - [a_i X_i, b_j X_j] \\ &= \sum_{i,j} a_i b_j (\nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j]) \end{aligned}$$

thus, $T(Y, Z)_p$ depends only on Y_p and Z_p

Lemma 2.3.15. Suppose that M has a smooth linear connection ∇ such that ∇ is invariant by $\text{Aut}^{loc}(\sigma)$. Consider $\mathfrak{g}'$ the space of germs of Killing vector fields at $o \in M$ (it is of course, a Lie algebra) and consider the map

$$\begin{aligned}\theta_o : \mathfrak{g}' &\rightarrow \text{End}(V_o) \times V_o \\ X &\mapsto ((A_X)_o, X_o)\end{aligned}$$

Then, θ_o is injective and preserves the bracket operation. In particular, its image is a Lie algebra.

Proof. First notice that the hypothesis on the connection ∇ implies that the Killing fields are infinitesimal affine transformations.

To prove that θ_o is injective we shall show that the germ of a Killing vector field X at o depends only on $(A_X)_o$ and X_o . To see this, consider a smooth curve $\gamma(t)$ on M passing through o . We denote by $a(t) = A_{X(\gamma(t))}$, $v(t) = \dot{\gamma}(t)$ and $x(t) = X(\gamma(t))$. From the Corollary 2.3.14, and $A_X Y = -\nabla_Y X - T(X, Y)$, it follows that

$$\begin{aligned}\nabla_{v(t)} a(t) &= R(x(t), v(t)) \\ \nabla_{v(t)} x(t) &= -T(x(t), v(t)) - a(t)v(t)\end{aligned}$$

We have therefore a Cauchy problem and thus, the germ of X at o depends only on the initial data: $(A_X)_o$ and X_o .

Now, we must show that θ_o preserves the bracket operation. First, observe that

$$\begin{aligned}A_{[X,Y]} + \nabla_{[X,Y]} &= \mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y] = [A_X + \nabla_X, A_Y + \nabla_Y] \\ &= [A_X, A_Y] + [A_X, \nabla_Y] + [\nabla_X, A_Y] + [\nabla_X, \nabla_Y] \\ &= [A_X, A_Y] + [A_X, \nabla_Y] + [\nabla_X, A_Y] + R(X, Y) + \nabla_{[X,Y]}\end{aligned}$$

On the other hand $[\nabla_X, A_Y] = \nabla_X(A_Y)$ and thus:

$$A_{[X,Y]} = R(X, Y) + [A_X, A_Y] + \nabla_X(A_Y) - \nabla_Y(A_X)$$

Now suppose that X and Y are Killing vector fields. In particular, X and Y are infinitesimal affine transformations and from Corollary 2.3.14, we have:

$$R(X, Y) = (\nabla_Y(A_X))$$

and thus

$$A_{[X,Y]} = [A_X, A_Y] + R(Y, X)$$

Now, as $[X, Y] = \nabla_X Y - \nabla_Y X - T(X, Y)$ and

$$A_X Y - A_Y X = \nabla_X Y + T(Y, X) - \nabla_Y X - T(X, Y) = [X, Y] + T(Y, X)$$

we have

$$\begin{aligned} \theta_o([X, Y]) &= ((A_{[X, Y]})_o, [X, Y]_o) \\ &= ([A_X, A_Y]_o + R(Y, X)_o, (A_X Y)_o - (A_Y X)_o - T(X, Y)_o) \\ &= [((A_X)_o, X_o), ((A_Y)_o, Y_o)] = [\theta_o(X), \theta_o(Y)] \end{aligned}$$

Which finishes the proof. □

Remark 2.3.16. Notice that this lemma gives an upper bound on the dimension of the Lie algebra of Killing vector fields if there exists a compatible connection.

CONTACT ANOSOV ACTIONS

In this section we define Anosov actions of abelian groups and present a few already known results. Our goal is to study Anosov action associated with a specific geometric structure, the generalized k -contact structure (Subsection 3.2.1). With this goal in mind, we present some related definitions¹ already present in the literature: Contact Pairs (Subsection 3.2.2.1), r -Contact Structures (Subsection 3.2.2.2), Multicontact Structures (Subsection 3.2.2.3), Polycontact Structures (Subsection 3.2.2.4) and Pluricontact structures (Subsection 3.2.2.4). The following diagram clarifies which condition implies which:

We also give sufficient conditions (Proposition 3.2.56) for a Pluricontact structure to also be a generalized k -contact structure.

3.1 Anosov actions

Definition 3.1.1. Consider a compact smooth manifold M and a smooth action $\phi : \mathbb{R}^k \times M \rightarrow M$. This action is said to be Anosov if there exists an element $a \in \mathbb{R}^k$, called an Anosov element,

¹ For the sake of clarity, we took the liberty of renaming some of these definitions.

such that ϕ_a acts on M normally hyperbolically, that is, there exists a, $d\phi^a$ invariant, continuous, splitting $TM = E^+ \oplus T\phi \oplus E^-$, where $T\phi$ is the distribution tangent to the orbits, such that, there exists positive constants C, λ for which

$$\|d\phi^{ta}(u^+)\| \leq Ce^{-t\lambda} \|u^+\| \quad \forall t > 0 \quad \forall u^+ \in E^+ \quad (3.1)$$

$$\|d\phi^{ta}(u^-)\| \leq Ce^{t\lambda} \|u^-\| \quad \forall t < 0 \quad \forall u^- \in E^- \quad (3.2)$$

Remark 3.1.2. Notice that as M is compact, the estimates (3.1) do not depend on the choice of metric (that is, the constants may depend on the metric but the subbundles still contract and expand uniformly)((HIRSCH; PUGH; SHUB, 1970)).

Remark 3.1.3. The set $\mathcal{A}(\phi)$ of Anosov elements forms an open set of $\mathbb{R}^k$, and each connected component of $\mathcal{A}(\phi)$, is an open cone. Moreover, Anosov elements on the same open cone will have the same invariant distributions ((BARBOT; MAQUERA, 2013)).

Theorem 3.1.4 ((HIRSCH; PUGH; SHUB, 1970)). The bundles $E^+, E^-, E^+ \oplus T\phi$, and $T\phi \oplus E^-$ are integrable, with smooth leaves. The resulting foliations $\mathcal{F}^{ss}, \mathcal{F}^{uu}, \mathcal{F}^{ws}, \mathcal{F}^{wu}$, are called strong stable, strong unstable, weak stable and weak unstable foliations. In general, those foliations are not smooth, they are simply Hölder continuous.

Definition 3.1.5. A group action $G \curvearrowright X$ is said to be topologically transitive if it admits a dense orbit. If $Y \subset X$, we say that the action is topologically transitive on Y if Y is G -invariant and there exists an orbit which is dense in Y .

Theorem 3.1.6 (Spectral decomposition for Anosov actions (ARBIETO; MORALES, 2009)). Let $A : \mathbb{R}^k \rightarrow \text{Diff}(M)$ be an Anosov action. Then, the non-wandering² set $NW(A)$ of A is a finite union $NW(A) = \Omega_1 \cup \dots \cup \Omega_l$ of disjoint compact and invariant sets Ω_i . Moreover, each Ω_i cannot be further subdivided in two compact, non-empty, disjoint and invariant subsets and the action A on each Ω_i is topologically transitive on Ω_i .

Corollary 3.1.7. An Anosov action $\phi : \mathbb{R}^k \times M \rightarrow M$ that preserves a volume is topologically transitive.

Proof. Let $x \in M$ and let $K \subset \mathbb{R}^k$ be a compact set, suppose that $K \subset B_R(0) \subset \mathbb{R}^k$, and let $g \in B_R(0)^c$. Then, $\phi(g, \cdot)$ defines a volume preserving diffeomorphism of M . Fix an arbitrary neighbourhood U of x . From the Poincaré recurrence theorem, for almost every $y \in U$, there exists $n \in \mathbb{N}$ such that $\phi(n \cdot g, y) \in U$. In particular, $(n \cdot g) \cdot U \cap U \neq \emptyset$. As $(n \cdot g) \in B_{nR}(0)^c \subset B_R(0)^c \subset K^c$ we conclude that x is non-wandering. Thus, $NW(\phi) = M$. From the connectedness of M , it follows from Theorem 3.1.6, that ϕ is topologically transitive. $\square$

² Let $G \curvearrowright M$ be an action of a topological group G on a topological manifold M . We shall say that a point $x \in M$ is non-wandering, if, for every neighbourhood U of x and for every compact subset $S \subset G$, there exists $g \in S^c$ such that $gU \cap U \neq \emptyset$. The set of non-wandering points is clearly invariant by the action of G .

3.2 Generalized k -contact Structure

In this subsection we present the definition and basic properties of a generalized k -contact structure, and what means a generalized k -contact Anosov action. For the sake of completeness, we also present the definition and some relevant properties of two stronger (and previously known, (BANDE; HADJAR, 2005), (MONTANO, 2017), (BOLLE, 1996)) structures: Contact pair structures (introduced and studied by Gianluca Bande and Amine Hadjar (BANDE; HADJAR, 2005)) and k -contact structures (studied by Philippe Bolle (BOLLE, 1996), Beniamino Cappelletti (MONTANO, 2017), Adriano Tomassini and Luigi Vezzoni (TOMASSINI; VEZZONI, 2008), among others).

Afterwards, we also explore two (weaker) possible ways to generalize the definition of contact structures (Multicontact structures (VITAGLIANO, 2015) and Polycontact structures (APOSTOLOV *et al.*, 2017) and (ERP, 2010)), and how they relate to the generalized k -contact structure.

We point out some differences between those definitions and briefly expose the context in which they arose.

3.2.1 Generalized k -contact structures

Definition 3.2.1. A generalized k -contact structure on a manifold M of dimension $2n + k$ is a collection of, k non zero, linearly independent, 1-forms $\{\alpha_1, \dots, \alpha_k\}$ and a splitting $TM = I \oplus F$, $\dim I = k$, such that, for every $1 \leq j \leq k$ we have,

1. $F = \bigcap_{i=1}^k \ker \alpha_i$
2. $d\alpha_j|_F$ is non degenerate;
3. $\ker(d\alpha_j) = I$

We denote this structure by $(M, \alpha, TM = I \oplus F)$.

Remark 3.2.2. It follows from the definition that, for every $j = 1, \dots, k$ $\alpha_1 \wedge \dots \wedge \alpha_k \wedge d\alpha_j^n$ is a volume form.

Remark 3.2.3. As $\dim(F) = 2n$, the condition

$$d\alpha_j|_F \text{ is non degenerate,}$$

is equivalent to

$$d\alpha_j^n|_F \neq 0$$

Lemma 3.2.4. The 1-forms that define a generalized k -contact structure have constant rank³ $2n + 1$

Proof. As $\dim I = k$ we have $\dim(F) = 2n$, and the second item implies that $d\alpha_j(p)^n \neq 0$ for each point $p \in M$. As $\ker(d\alpha_j) = I$ and $TM = I \oplus F$ then, $d\alpha_j^{n+1}(p) = 0$ for every point $p \in M$. Thus, $\text{rank}(\alpha_j)$ on each point is either $2n$ or $2n + 1$. Finally, as $(\alpha_j)|_{\ker d\alpha_j} \neq 0$ then α_j has odd rank everywhere, that is, $\text{rank}(\alpha_j) = 2n + 1$ at every point. $\square$

Lemma 3.2.5. For each j , there is a unique vector field $X_j \in \Gamma(M, I)$ such that $\alpha_i(X_j) = \delta_{ij}$. These vector fields are called Reeb vector fields.

Moreover, the Reeb vector fields commute one with each other:

$$[X_i, X_j] = 0$$

Proof. By definition, the characteristic space of each α_i is contained in $\mathcal{I}$. As $(\alpha_1 \wedge \cdots \wedge \alpha_k)|_{\mathcal{I}} \neq 0$, then, for each $i = 1, \dots, k$,

$$I \cap \bigcap_{j \neq i} \ker \alpha_j \neq \{0\}$$

A dimension analysis shows that in fact $\dim(I \cap \bigcap_{j \neq i} \ker \alpha_j) = 1$ and thus there exists for each point $p \in M$ a unique vector $X_j(p) \neq 0$ such that $i_{X_j} d\alpha_i = 0$, and $i_{X_j} \alpha_i = \delta_{ij}$.

To see that they commute, we shall show that

$$i_{[X_i, X_j]} \alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_l^n = 0$$

As $\alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_l^n$ is a volume form, this means that $[X_i, X_j] = 0$.

First we notice that for any i, l we have $d\alpha_i \wedge d\alpha_l^n = 0$, and thus, for any $1 \leq j \leq k$ we have

$$d(\alpha_j \wedge d\alpha_l^n) = 0$$

and more generally, for any $1 \leq j_1 < \dots < j_s \leq k$ we have

$$d(\alpha_{j_1} \wedge \cdots \wedge \alpha_{j_s} \wedge d\alpha_l^n) = 0$$

Thus, for any $1 \leq j \leq k$

$$(d \circ i_{X_j}) \alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_l^n = d(\alpha_1 \wedge \cdots \wedge \alpha_{j-1} \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k \wedge d\alpha_l^n) = 0$$

and similarly

$$(d \circ i_{X_j} \circ i_{X_i}) \alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_l^n = 0$$

³ Remember that the rank of a differential form ω of rank p at a point y is the co-dimension of the characteristic space

$$\mathcal{C}(\omega)(y) = \{X \in T_p M ; i_X \omega(y) = 0 \text{ and } i_X d\omega(y) = 0\}$$

Moreover, as $\alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_l^n$ is a volume form, we have

$$d(\alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_l^n) = 0$$

Thus, using Cartan's formula⁴:

$$\begin{aligned} i_{[X_i, X_j]} \alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_l^n &= [i_A, \mathcal{L}_B] \alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_l^n \\ &= [i_A, i_B \circ d + d \circ i_B] \alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_l^n = 0 \end{aligned}$$

□

Remark 3.2.6. Thus, a generalized k -contact structure comes with a canonical $\mathbb{R}^k$ action ϕ , called a generalized k -contact action. It is also clear that I is precisely $T\phi$, the distribution tangent to the action. Moreover, if the splitting $TM = I \oplus F$ is smooth, then so are the vector fields X_j and therefore so is the action ϕ .

It is sometimes convenient to denote a generalized k -contact structure on M as the 4-tuple (M, α, ϕ, F) , where $\alpha = (\alpha_1, \dots, \alpha_k)$, ϕ denotes the canonical generalized k -contact action and $F \leq TM$ is the ϕ -invariant subbundle where $d\alpha_j$ is non degenerate.

Example 3.2.7. There exist an easy way to build examples of generalized k -contact manifolds from an existing one. Let $(B, \alpha, TB = I \oplus F)$ be a generalized r contact manifold and $\pi : M \rightarrow B$ a principal torus bundle (with standard fiber $\mathbb{T}^l$). Let $Y_i, i = 1, \dots, l$ be the vector fields that generates the $\mathbb{T}^l$ -action. Consider on each torus, fibers of the bundle, the canonical 1-forms $\xi_1, \dots, \xi_l$, $\xi_i(Y_j) = \delta_{ij}$. Consider a flat connection on M , and let $TM = H \oplus V$ the associated decomposition into horizontal (H) and vertical (V) bundles. Using this connection we can extend the forms $\xi_1, \dots, \xi_l$ to global forms on M .

In fact, the as the bundle $\pi : M \rightarrow B$ is a principal torus bundle, the vertical part V of the splitting $TM = H \oplus V$ has a canonical splitting $V = \mathbb{R}Y_1 \oplus \cdots \oplus \mathbb{R}Y_l$ given by the $\mathbb{T}^l$ action on M . We define the forms $\xi_1, \dots, \xi_l$ by

$$\xi_i(Y_j) = \delta_{ij} ; \quad \xi_i(H) = 0$$

As the vertical distribution is involutive, it follows that $d\xi_i(Y_l, Y_k) = 0$ for any l, k . It is known (Nesterov, A. (NESTEROV, 2000) Lemma 3.12) that for any horizontal vector field Y ,

⁴ The Cartan's formula is $\mathcal{L}_X = i_X \circ d + d \circ i_X$ where $\mathcal{L}_X$ denotes the Lie derivative. A classical consequence (Kobayashi, S and Nomizu, K. (KOBAYASHI; NOMIZU, 1963), Section I.3 Proposition 3.10) of Cartan's formula is the identity

$$i_{[A, B]} = [i_A, \mathcal{L}_B]$$

the commutator $[Y_i, Y]$ is horizontal. Therefore, the 1-forms ξ_i satisfies, for any horizontal vector field W ,

$$d\xi_i(Y_l, W) = Y_l(\underbrace{\xi_i(W)}_{=0}) - W(\underbrace{\xi_i(Y_l)}_{=1}) - \underbrace{\xi_i([Y_l, W])}_{=0} = 0 \quad (3.3)$$

and thus $i_{Y_l} d\xi = 0$. Moreover, as the connection is flat, the horizontal distribution is involutive and therefore, for any two horizontal vector fields W_1, W_2 we have

$$d\xi_i(W_1, W_2) = W_1(\xi_i(W_2)) - W_2(\xi_i(W_1)) - \xi_i([W_1, W_2]) = 0 \quad (3.4)$$

As we have the splitting $TB = I \oplus F$, the fibres H_p of the horizontal bundle H can be identified with the fibres $T_{\pi(p)}B$ of the tangent bundle TB , we have an induced splitting $H = \hat{I} \oplus \hat{F}$. We denote $\hat{V} = V \oplus \hat{I} \subset TM$. From equations 3.4 and 3.3 we obtain that $\hat{I} \subset \ker d\xi_i$ and thus, it is clear that, for any $(j_1, \dots, j_l) \in \{1, \dots, r\}^l$, we have the following generalized $(l+r)$ -contact structure on M :

$$(M, \xi_1 + \pi^* \alpha_{j_1}, \dots, \xi_l + \pi^* \alpha_{j_l}, \pi^* \alpha_1, \dots, \pi^* \alpha_r, TM = \hat{V} \oplus \hat{F})$$

In particular, if B is a contact manifold (equivalently, 1-contact or generalized 1-contact manifold), and $M \rightarrow B$ is a principal torus bundle with a flat connection, then, the induced structure we have constructed above is a $(l+1)$ -contact manifold.

Lemma 3.2.8. Let $(M, \alpha, TM = I \oplus F)$ be a generalized k -contact structure, and let $B = (\beta_{ij}) \in M_{k \times k}(\mathbb{R})$. We define $\eta_i = \alpha_i - \sum_{j=1}^k \beta_{ij} \alpha_j$. Then, if B is sufficiently small, $(M, \eta, TM = I \oplus F)$ is a generalized k -contact structure.

Proof. As B is small, then $Id - B$ is invertible, then

$$\bigcap \ker(\eta_i) = \bigcap \ker \alpha_i = F$$

Also, we can write

$$d\eta_i = (1 - \beta_{ii})d\alpha_i + \sum_{j \neq i} \beta_{ij} d\alpha_j$$

where β_{ij} are small. And thus

$$d\eta_i^n = d\alpha_i^n + \omega$$

where ω is small. As non degeneracy is an open condition, if ω is small enough, then $d\eta_i$ is non degenerate. Similarly, $d\eta_i^n = d\alpha_i^n + \omega_0$ where ω_0 is small, and therefore

$$\eta_1 \wedge \dots \wedge \eta_k \wedge d\eta_j^n = \det(Id - B) \alpha_1 \wedge \dots \wedge \alpha_k (d\alpha_i^n + \omega_0)$$

which is non zero for small ω_0 .

Finally, $I \subset \ker(d\eta_j)$, but as $TM = I \oplus F$ and $d\eta_j$ is non degenerate on F (for small perturbation of the identity B), then $\ker(d\eta_j) \subset I$ $\square$

A similar argument proves the following Lemma

Lemma 3.2.9. Consider a manifold M of dimension $2n + k$, a splitting, $TM = I \oplus F$, $\dim I = k$ and, linear independent, non vanishing 1-forms $\alpha_1, \dots, \alpha_k$ such that

- $F = \bigcap_i \ker(\alpha_i)$
- $d\alpha_k$ is non degenerate on F and $I = \ker(d\alpha_k)$
- $I \subset \ker d\alpha_j$ for $j = 1, \dots, k-1$

Then, there exists $B = (\beta_{ij}) \in GL(\mathbb{R}^{n-1})$ and a change of coordinates $\eta_j = \sum_i \beta_{ij} \alpha_j$, $j = 1, \dots, k-1$ such that $(M, \eta_1, \dots, \eta_{k-1}, \alpha_k, TM = I \oplus F)$ is a generalized k -contact manifold

Remark 3.2.10. Notice that under the conditions of the Lemma 3.2.9, we have well defined vector fields $X_1, \dots, X_k$ tangent to I such that $\alpha_j(X_i) = 1$. The same arguments of Lemma 3.2.5 shows that these vector fields commute with each other, and therefore defines an action of $\mathbb{R}^k$ on M .

With some additional conditions, we can improve Lemma 3.2.8

Lemma 3.2.11. Under the conditions of the previous Lemma 3.2.8, suppose that the action is topologically transitive. For $B \in GL(\mathbb{R}^k)$, denote by $\eta = B\alpha$ the change of coordinates $\eta_i = \sum_j \beta_{ij} \alpha_j$. Then, there exists a Zariski open subset of $\Lambda \subset GL(\mathbb{R}^k)$ such that $(M, B\alpha, TM = I \oplus F)$ is generalized k -contact.

Proof. Like in the previous lemma, we have

$$E = \bigcap_{i=1}^k \alpha_i = \bigcap_{i=1}^k \eta_i$$

and

$$I \subset \ker d\eta_j \quad \forall j = 1, \dots, k$$

we must show that for a Zariski open set Λ , the conditions

- $d\eta_j|_F$ is non degenerate for every $1 \leq j \leq k$;
- $\eta_1 \wedge \dots \wedge \eta_k \wedge d\eta_j^n$ is a volume form.

are satisfied.

Now, denote

$$d\alpha^J = d\alpha_1^{J_1} \wedge \dots \wedge d\alpha_k^{J_n}$$

for any multi-index $J = (J_1, \dots, J_k) \in \mathbb{N}^k$. Then, we can write

$$d\eta_j^n = \sum_{|J|=n} Q_J d\alpha^J$$

where Q_J is some polynomial in the variables β_{ij} and $|J| = J_1 + \dots + J_k$

Notice that $d\alpha^J$ is clearly a top form over F , and thus $d\alpha^J = f_J d\alpha_k^n$ for some function f_J . As the action is topologically transitive, and the forms $d\alpha^J$ and $d\alpha_1$ are invariant by this action, this function is constant over M . Thus, we write

$$d\eta_j^n = P_j(B) d\alpha_k^n$$

where $P_j(B)$ is a polynomial on the coefficients β_{ij} of B . We also write

$$\eta_1 \wedge \dots \wedge \eta_k \wedge d\eta_j^k = P_j(B) \det(B) \alpha_1 \wedge \dots \wedge \alpha_k \wedge d\alpha_k^n$$

And thus, conditions 3.2.1 are satisfied when $P_j(B) \neq 0$. This polynomial is non zero, for $P_j(Id) = 1$, and thus, the conditions 3.2.1 are met for B in a Zariski open set. $\square$

Remark 3.2.12. We can understand the re-parametrization $\eta = B\alpha$ as a re-parametrization of the action, that is, let $(M, \alpha, TM = I \oplus F)$ be a generalized k -contact structure, $X_1, \dots, X_k$ the associated vector fields. For every j , we denote by ϕ^j the flow associated with the vector field X_j . Let $e_1, \dots, e_k$ be the standard basis on $\mathbb{R}^k$. The associated action $\phi \mathbb{R}^k \curvearrowright M$ is given by

$$\phi(e_j) = \phi^j$$

Let $B = (\beta_{ij}) \in GL(\mathbb{R}^k)$ and $f_j = \sum_{i=1}^k \beta_{ij} e_i$ be another basis on $\mathbb{R}^k$ and $Y_j = \phi(f_j)$. Let us define 1-forms on M by

$$\eta_j(Y_i) = \delta_{ij} \ ; \ \eta_j(E) = 0$$

then

$$\eta = B^{-1} \alpha$$

Definition 3.2.13. A generalized k -contact action ϕ will be called generalized k -contact Anosov, if the action is Anosov.

Lemma 3.2.14. If $\phi : \mathbb{R}^k \rightarrow M$ is a generalized k -contact action, then there exists $B \in GL(\mathbb{R}^k)$ such that every Reeb vector field is Anosov and they have the same invariant splitting $TM = T\phi \oplus E^+ \oplus E^-$

Proof. As the action preserves the volume form $\alpha_1 \wedge \dots \wedge \alpha_k \wedge d\alpha_j^n$, the action is topologically transitive (Corollary 3.1.7). Thus, the previous lemma (Lemma 3.2.11) implies that for almost any linear re-parametrization of the action, the corresponding structure is still generalized k -contact. It is known (Remark 3.1.3) that the set of Anosov elements is a disjoint union of connected open cones, and moreover, the invariant splitting $TM = T\phi \oplus E^+ \oplus E^-$ depends only on the choice of open cone, thus just choose a re-parametrization that puts every Reeb vector field in the same open cone. $\square$

Definition 3.2.15. Let $(M, \alpha, TM = I \oplus F, \phi)$ be a generalized k -contact Anosov action. The parametrization α will be called adapted if every Reeb vector field is Anosov.

Remark 3.2.16. The Lemma 3.2.14 proves that every generalized k -contact Anosov action admits an adapted parametrization.

Remark 3.2.17. Another way to see this definition is to start with an Anosov $\mathbb{R}^k$ -action ϕ , consider the action as given by a family of commuting vector fields X_j , where each X_j is the vector field associated with an Anosov element. Using the splitting $TM = T\phi \oplus E^+ \oplus E^-$ we define the dual 1-forms $\alpha_1, \dots, \alpha_k$, and we suppose that each of those forms have constant rank $2n + 1$ and satisfies:

$$\alpha_1 \wedge \dots \wedge \alpha_k \wedge d\alpha_j^n \text{ is a volume form.}$$

Remark 3.2.18. For the rest of this text, $\alpha_1, \dots, \alpha_k$ will always denote the 1-forms associated with an adapted parametrization of a generalized k -contact Anosov action, and $X_1, \dots, X_k$ the associated dual Anosov vector fields. We shall also denote each volume form $dM_j = \alpha_1 \wedge \dots \wedge \alpha_k \wedge d\alpha_j^n$.

Remark 3.2.19. Notice that $\dim E^+ = \dim E^- = n$. This follows from the fact that, for some fixed j , $d\alpha_j$ restricted to $E^+ \oplus E^-$ is a ϕ -invariant symplectic form, the hyperbolic dynamics will ensure that $E^\pm$ are Lagrangian subspaces.

Lemma 3.2.20. The distribution F in the generalized k -contact structure is non integrable.

Proof. Suppose that F is integrable. From Frobenius Theorem, this is equivalent to $[F, F] \subset F$. Take two vector fields $Z, W \in \Gamma(M, F)$, in particular, we have

$$\alpha_j(Z) = \alpha_j(W) = \alpha_j([Z, W]) = 0$$

and thus

$$d\alpha_j(Z, W) = Z(\alpha_j(W)) - W(\alpha_j(Z)) - \alpha_j([Z, W]) = 0$$

which contradicts the fact that $d\alpha_j$ is non degenerate on F . $\square$

Remark 3.2.21. The distribution F is maximally non integrable in the following sense: For every vector field Z tangent o F there exists a vector field W also tangent to F such that $[Z, W]$ is not tangent to F .

Remark 3.2.22. Consider, on a manifold M of dimension $2n + k$, a distribution $F \subset TM$, $\dim F = 2n$, and let $\mathcal{J}$ be the $C^\infty(M, \mathbb{R})$ -submodule of 1-forms on M that vanishes on F .

Let $\mathcal{J}^k$ denote the space of k -forms of the type $\xi_1 \wedge \dots \wedge \xi_k$ with $\xi_j \in \mathcal{J}$, $(d\mathcal{J})^{(n)}$ the space of $2n$ -forms of the type $d\eta_1 \wedge \dots \wedge d\eta_n$, $\eta_j \in \mathcal{J}$, $(d\mathcal{J})^n$ the space of $2n$ -forms of the type $(d\eta)^n$, $\eta \in \mathcal{J}$.

Let $\mathcal{J}^k \wedge (d\mathcal{J})^n$ (resp. $\mathcal{J}^k \wedge (d\mathcal{J})^{(n)}$) denote the space of $2n+k$ -forms of the type $\omega \wedge \Omega$, where $\omega \in \mathcal{J}^k$ and $\Omega \in (d\mathcal{J})^n$ (resp. $\Omega \in (d\mathcal{J})^{(n)}$)

We can compare the notions of k -contact structure and generalized k -contact structure in the following way:

- If F is the $2n$ -plane distribution of a k -contact structure, then, every non-zero element of $J^k \wedge (dJ)^{(n)}$ is a volume form.
- If F is the $2n$ -plane distribution of a generalized k -contact structure, then, there exists non-zero element of $J^k \wedge (dJ)^n$ which is a volume form.

The first item follows from the fact that $J = \text{Span}_{C^\infty(M, \mathbb{R})} \{\alpha_1, \dots, \alpha_k\}$, therefore $J^k = \{f\alpha_1 \wedge \dots \wedge \alpha_k ; f \in C^\infty(M, \mathbb{R} \setminus \{0\})\}$ and $J^{k+1} = 0$.

Moreover $dJ = \{fd\alpha_1 + df \wedge \eta ; f \in C^\infty(M, \mathbb{R}), \eta \in J\}$, and thus, if $d\eta \in dJ$, then $\alpha_1 \wedge \dots \wedge \alpha_k \wedge d\eta = f\alpha_1 \wedge \dots \wedge \alpha_k \wedge d\alpha_1$.

By induction we see that $\alpha_1 \wedge \dots \wedge \alpha_k \wedge d\eta_1 \wedge \dots \wedge d\eta_n = f_1 \dots f_n \alpha_1 \wedge \dots \wedge \alpha_k \wedge d\alpha_1^n$.

The second item is clear.

3.2.2 Other contact like structures

3.2.2.1 Contact Pairs

The notion of contact pair was introduced by Bande, G. And Hadjar, A. to study certain principal torus bundles on surfaces.

Definition 3.2.23. A contact pair (C.P.) on a smooth manifold M of dimension $2h+2k+2$ is a couple of 1-forms (α, η) satisfying:

- $d\alpha^{h+1} = d\eta^{k+1} = 0$ for some positive integers h, k .
- $\alpha \wedge d\alpha^h$ and $\eta \wedge d\eta^k$ vanish nowhere.
- $\Omega_{\alpha, \eta} = \alpha \wedge d\alpha^h \wedge \eta \wedge d\eta^k$ is a volume form

This contact pair is said to be of type (h, k) . We shall usually denote $\Omega = \Omega_{\alpha, \eta}$ when there is no confusion.

Example 3.2.24. The easiest way to construct a manifold with a contact pair is to multiplying two contact manifolds: If (M_1^{2h+1}, α) and (M_2^{2k+1}, η) are contact manifolds (α and η their

respective contact forms), then we can consider α and η as 1-forms on the product $M_1 \times M_2$ ⁵. It is clear that (α, η) is a contact pair on $M_1 \times M_2$ of type (h, k) .

Another construction given by Bande. G and Hadjar, A. (BANDE; HADJAR, 2005), is by taking a contact manifold (M^{2h+1}, α) with a contactomorphism f . The suspension manifold $M \times [0, 1]/(x, 1) \sim (f(x), 0)$ is a contact manifold of type $(h, 0)$. Bande. G and Hadjar, A. also construct examples on nilpotent Lie groups of dimension 4 and 6.

Remark 3.2.25. Notice that, if M is a manifold with a Contact Pair (α, η) of type (h, k) then

$$(M, \alpha + \eta, \alpha - \eta, TM = (\ker(d\alpha) \cap \ker(d\eta)) \oplus (\ker \alpha \cap \ker \eta))$$

is a generalized 2-contact manifold. In this sense, generalized k -contact structures generalize the notion of Contact Pairs, in particular, we have the following lemma.

Lemma 3.2.26 (Bande, G. Hadjar, A. (BANDE; HADJAR, 2005), Theorem 2.2). We can associate to a contact pair (α, η) on M vector fields, called Reeb vector fields, X_α and X_η characterized by:

- $i_{X_\alpha} \Omega_\alpha = i_{X_\eta} \Omega_\eta = 0$
- $\alpha(X_\alpha) = \eta(X_\eta) = 1$

Those vector fields are unique. Moreover, $[X_\alpha, X_\eta] = 0$, and both α and η are X_α and X_η invariant.

Which means that associated with a contact pair, there is an action of $\mathbb{R}^2$, and the contact pair is invariant by this action.

3.2.2.2 r -Contact structures

The notion of r -contact manifolds was used by Philippe Bolle in (BOLLE, 1996) to study certain class of co-isotropic submanifolds⁶ of a symplectic manifold. The r -contact condition was used to generalize the well known contact condition^{7,8}. Like in the usual contact case, we

⁵ We consider the projections $\pi_j : M_1 \times M_2 \rightarrow M_j$ and consider $\tilde{\alpha} = \pi_1^*(\alpha)$ and $\tilde{\eta} = \pi_2^*(\eta)$. We abuse notation and set $\tilde{\alpha} = \alpha$ and $\tilde{\eta} = \eta$

⁶ Given a symplectic manifold (M, ω) , a submanifold S of co-dimension p is called co-isotropic if $\dim(\ker \omega|_S) = p$ everywhere on S

⁷ For a given symplectic manifold (M, ω) , a co-dimension 1 submanifold S is of contact type if S is a contact manifold (S, α) , co-isotropic and $d\alpha = \omega|_S$

⁸ Philippe Bolle's motivation is that such co-isotropic manifolds admits a canonical $\mathbb{R}^k$ -action, and his goal was to prove that under certain conditions, there is a non trivial element X of $\mathbb{R}^k$, whose associated flow admits a non trivial, periodic, non contractible orbit. This is result towards a generalization of Weinstein Conjecture, which propose the existence of a such orbits for a Reeb flow on an arbitrary, compact, contact manifold.

can forget the ambient manifold and consider the k -contact condition as a geometric structure on its own. In this case, the r -contact is an extension of the usual contact structure in the sense that for $r = 1$ we obtain the traditional contact structure.

Definition 3.2.27. A manifold M of dimension $2n + r$ is said to be r -contact, if there exists nowhere zero, linearly independent, 1-forms $\alpha_1, \dots, \alpha_r$ such that

1. $d\alpha_i = d\alpha_j$ for every $1 \leq i, j \leq r$.
2. $d\alpha_i^{n+1} = 0$
3. $\alpha_1 \wedge \dots \wedge \alpha_r \wedge d\alpha_i^n$ is a volume form on M .

We denote it by $(M, \alpha_1, \dots, \alpha_r)$, or simply (M, α) .

Remark 3.2.28. Philippe Bolle was in a situation with an ambient symplectic manifold, and he studied co-isotropic submanifolds. The first condition is rewritten as $d\alpha_i = \omega|_M$ where ω is the symplectic form on the ambient manifold. Together with the fact that he is interested in co-isotropic submanifolds, this modification means that the second condition was superfluous (and in fact it was not included in Bolle's definition).⁹

Remark 3.2.29. It is easy to see that generalized k -contact structures are in fact generalizations of r -contact structures. Moreover, if M is a manifold with a Contact Pair (α, η) of type (h, k) and, either k or h is even, then

$$(M, \alpha + \eta, \alpha - \eta)$$

is a 2-contact manifold.

Lemma 3.2.30 (Philippe Bolle, (BOLLE, 1996)). For a given r -contact manifold M , there exist unique, linearly independent, vector fields $X_1, \dots, X_r$ such that

- $\alpha_j(X_i) = \delta_{ij}$, for every i, j .
- $i_{X_i}d\alpha_j = 0$ for every i, j .

These vector fields are called Reeb vector fields and they satisfy:

$$[X_i, X_j] = 0 \quad \forall i, j$$

⁹ To be precise, Bolle considers a symplectic manifold (N, ω) of dimension $2(n + r)$, and is interested in co-isotropic study submanifolds M of co-dimension r , that is, $\dim(\ker \omega|_M) = r$ everywhere. He defined a submanifold M to be of r -contact type if it is co-isotropic, and has r , linearly independent 1-forms $\alpha_1, \dots, \alpha_r$ such that $d\alpha_i = \omega|_M$ and $\alpha_1 \wedge \dots \wedge \alpha_r \wedge \omega|_M^n$ is a volume form. The co-isotropy condition implies that $\omega|_M^{n+1} \equiv 0$.

Remark 3.2.31. Other authors who studied the r -contact manifolds ((TOMASSINI; VEZZONI, 2008),(MONTANO, 2017)) do not ask explicitly for the second condition ($d\alpha_i^{n+1} = 0$), but it is a necessary condition to obtain the Reeb vector fields above¹⁰ not explicitly stated by the authors. For example, consider $\mathbb{R}^5$ with coordinates (x_1, y_1, x_2, y_2, z) and consider

$$\Sigma = y_1 dx_1 + y_2 dx_2$$

$$\alpha_1 = dz + dx_1 + \Sigma$$

$$\alpha_1 = dy_1 + dx_1 + \Sigma$$

$$\alpha_1 = dy_1 + \Sigma$$

It is clear that $d\alpha_j = d\Sigma$ for every j , and a quick computation shows that

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge d\Sigma = dz \wedge dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$$

However $\ker(d\Sigma) = \text{Span}\{\partial_z\}$. This means that it is impossible to have 3 linearly independent vector fields X_1, X_2, X_3 such that $\alpha_i(X_j) = \delta_{ij}$ and $X_i \in \ker(d\alpha_j)$.

3.2.2.3 Multicontact

Luca Vitagliano introduced the notion of multicontact structure as the higher co-dimension counterpart of multisymplectic¹¹ structure. It was previously known (Rogers (ROGERS, 2012)) that there is an L_∞ -algebra¹² $\mathcal{P}$ which is the multisymplectic structures analogue of the Poisson algebra¹³. Vitagliano constructed, for the multicontact structure an L_∞ -algebra $\mathcal{J}$ which is the higher dimensional analogue of the Jacobi bundle¹⁴, that is, this $\mathcal{J}$ and $\mathcal{M}$ interacts in a similar

¹⁰ For example, Cappelletti (MONTANO, 2017) on page 2. clearly states the existence of r linearly independents vector fields $\xi_1, \dots, \xi_r$ such that $\alpha_j(\xi_i) = \delta_{ij}$ and $\xi_i \in \ker d\alpha_j$

¹¹ A closed $(n+1)$ -form ω on M is called pre- n -plectic if $\ker(\omega)$ has constant rank. If $\ker(\omega) = 0$ everywhere, ω is called a n -plectic form, and defines a multiplectic structure on M . (Cantrijn et al. (CANTRIJN; IBROT; LÉON, 1999))

¹² Recall that a L_∞ algebra is a $\mathbb{Z}$ -graded vector space V and multi-linear maps $f_n : V^{\otimes n} \rightarrow V$ of degree $n - 2$ called brackets that satisfies a (graded) skew symmetry property and a (strong homotopy) Jacobi property. For more details see (STASHEFF,)

¹³ A Poisson algebra is a Lie algebra $(V, [\cdot, \cdot])$ where for each $a \in V$, $[a, \cdot] : V \rightarrow V$ is a derivation of V . The algebra of smooth function on a symplectic manifold, endowed with the Poisson bracket is an example of a Poisson algebra

¹⁴ A Jacobi bundle on M is a line bundle $E \rightarrow M$ equipped with a Lie bracket on its local sections. The Reeb vector field X associated with a contact manifold (M, α) gives a trivial line bundle and for any two sections fX and gX of this bundle, the map bellow defines a Lie bracket on the space of sections

$$\{f, g\} = dg(\Lambda(df)) + fdg(X) - gdf(X)$$

where $\Lambda : T^*M \rightarrow TM$ is defined as

$$\Lambda(\xi)\alpha = 0 \text{ and } \Lambda(\xi)(d\alpha) = \xi(X)\alpha - \xi$$

way as the Jacobi bundle and Poisson algebra.

Consider a manifold M and $E \subset TM$ a linear subbundle. Let us denote by N the quotient bundle TM/E . We have, for any open set $U \subset M$ the following exact sequence:

$$0 \rightarrow \Gamma(U, C) \rightarrow \Gamma(U, TM) \rightarrow \Gamma(U, N) \xrightarrow{\theta} 0$$

Where θ denotes the natural projection map. We also have a well defined bilinear map called the curvature form of E ,

$$\begin{aligned} B : \Gamma(M, E) \times \Gamma(M, E) &\rightarrow \Gamma(M, N) \\ (Z, W) &\mapsto \theta([Z, W]) \end{aligned}$$

Lemma 3.2.32. The curvature form B is tensorial, that is, it induces a bundle map

$$B : \Lambda^2 E \rightarrow N = TM/E$$

Proof. Let us show that for any $p \in M$ and local vector fields Z, W , in neighbourhood U of p , tangents to E , the value $B(Z, W)(p)$ depends only on the values $Z(p)$ and $W(p)$.

In fact, consider Z', W' vector fields on U such that $Z'(p) = Z(p)$ and $W'(p) = W(p)$. Then, we write $Z' = fZ$ and $W' = gW$ for (smooth) functions f, g such that $f(p) = g(p) = 1$. As Z and W are tangent to E , it follows that $\theta(Z) = \theta(W) = 0$ and, thus, we have

$$\begin{aligned} B(Z', W') &= \theta([fZ, gW]) \\ &= \theta(Z(g)W - W(f)Z + fg[Z, W]) \\ &= Z(g)\theta(W) - W(f)\theta(Z) + fg\theta([Z, W]) \\ &= fg\theta([Z, W]) \end{aligned}$$

in particular,

$$B(Z', W')(p) = \underbrace{f(p)g(p)}_{=1} \theta([Z, W])(p) = \theta([Z, W])(p) = B(Z, W)(p)$$

□

Lemma 3.2.33. Let $M \rightarrow Q$ be a G -principal bundle and $E \subset TM$ is the horizontal space of a connection on M . Then $-B$ coincides with the usual $\mathfrak{g}$ -valued curvature form, where $\mathfrak{g}$ denotes the Lie algebra of G , restricted to horizontal vector fields.

Proof. Let $A \in \Gamma(M, T^*M \otimes \mathfrak{g})$ denote a connection form on the G -principal bundle $\pi : M \rightarrow Q$. The horizontal distribution E associated with this connection is defined (pointwise) by $E_p = \ker(A_p)$ where A_p is the induced map

$$A_p = A(p) : T_p M \rightarrow \mathfrak{g}$$

This allow us to identify $T_p M / E_p = \mathfrak{g}$.

The usual curvature form is defined as $\Omega_A = dA + A \wedge A$. Thus, for horizontal vector fields Z, W we have

$$\begin{aligned}\Omega_A(Z, W) &= dA(Z, W) + [A(Z), A(W)] \\ &= Z(A(W)) - W(A(Z)) - A([Z, W]) = -A([Z, W])\end{aligned}$$

and thus

$$\Omega_A(Z, W)(p) = -A([Z, W])(p) = -[Z, W](p) \text{ Mod } E_p = -\theta([Z, W])(p) = -B(Z, W)(p)$$

as we desired. $\square$

The characteristic distribution associated with E is the distribution

$$\mathcal{C}(E) = \{Y \in \Gamma(M, E) ; B(Y, \cdot) = 0\}$$

Definition 3.2.34 (Multicontact structure: Luca Vitagliano ([VITAGLIANO, 2015](#))). A distribution E is called pre-multicontact if the characteristic distribution $\mathcal{C}(E)$ has constant rank. If $\mathcal{C}(E) = 0$, E is called a multicontact distribution.

Remark 3.2.35. Notice that the characteristic distribution can be defined locally, that is, we can consider, for any open set $U \subset M$, the local distribution $\mathcal{C}_U(E)$ of local vector fields $Y \in \Gamma(U, E)$ such that $B_U(Y, Z) = 0$ for any $Z \in \Gamma(U, E)$. Then, a vector field $Y \in \mathcal{C}(E)$ if, and only if, for any open set U , the restriction of Y_U of Y to U belongs to $\mathcal{C}_U(E)$.

Lemma 3.2.36. If $(M, \alpha, TM = I \oplus F)$ is a generalized k -contact structure, then (M, F) is a multicontact structure.

Proof. As $d\alpha_j$ is non degenerate on F , for any non vanishing vector field Z tangent to F , there exists another non vanishing vector field W tangent to F , such that $d\alpha_j(Z, W)$ is nowhere zero.

On the other hand,

$$d\alpha_j(Z, W) = \underbrace{Z(\alpha_j(W))}_{=0} - \underbrace{W(\alpha_j(Z))}_{=0} - \alpha_j([Z, W]) = -\alpha_j([Z, W])$$

Moreover, the splitting $TM = I \oplus F$ allow us to identify $TM/F = I$, and thus,

$$B(Z, W) = \sum_j \alpha_j([Z, W])X_j$$

Therefore a vector field Z belongs to the characteristic distribution $\mathcal{C}(F)$ if, and only if, $\alpha_j([Z, W]) = 0$ for every vector field W tangent to E and every $j \in \{1, \dots, k\}$.

As we already noticed, there is no such vector field and thus, the characteristic distribution is the zero section of TM . $\square$

3.2.2.4 Polycontact and Pluricontact

Van Erp introduced the notion of Polycontact to study a certain algebra of pseudo-differential operators. More precisely, he used the curvature form to associate with each tangent space the structure of a 2-step nilpotent Lie algebra and understood that polycontact manifolds are (in some sense) micro locally modelled after groups of Heisenberg type. Van Erp points out that the usual Heisenberg (pseudo differential) calculus on contact manifolds (developed by Taylor (TAYLOR, 1984)) is generalized to the polycontact context, defining the algebra $\Psi_H^*(M) = \bigoplus_d \Psi_H^d(M)$ of Heisenberg pseudo differential operators¹⁵. Van Erp showed that the existence of a projection with infinite dimensional kernel on $\Psi_H^*(M)$ is equivalent to the non degeneracy condition on the Levi form.

Definition 3.2.37 (Polycontact structure: Erik Van Erp (ERP, 2010)). A distribution E defines a polycontact structure, if for every open set U , and for any 1-form $\alpha \in \Gamma(U, N^*)$, that doesn't vanishes on U , the following bilinear map is non degenerate

$$\Gamma(U, E) \times \Gamma(U, E) \xrightarrow{\alpha \circ B_U} \mathbb{R}$$

where B_U denotes the restriction of B to sections defined on U , and non degeneracy means that for any vector field $Z \in \Gamma(U, E)$ and any point $p \in U$, there exists $W \in \Gamma(U, E)$ such that $\alpha_p(B_U(Z, W)(p)) \neq 0$.

Apostolov et al ((APOSTOLOV *et al.*, 2017)) relaxed the contact condition further still, defining what we will call a pluricontact structure (they did not use this terminology, calling (M, E) a contact structure of co-dimension l and dimension $n = \text{rank}(E)/2$. We will use the terminology Pluricontact in accordance to Choi, W. and Ponge, R. (CHOI; PONGE, 2017)¹⁶. Their goal is to extend the theory of toric contact manifolds to the higher co-dimension. After defining the appropriate higher co-dimension analogue of the moment polytope of a compact toric symplectic manifold, which they called the Grassmanian momentum map, they give a classification of the contact toric manifolds in terms of the cohomology of a sheaf on the image of the Grassmanian momentum map.

Definition 3.2.38 (Pluricontact structure: Apostolov et al (APOSTOLOV *et al.*, 2017)). A distribution E defines a pluricontact structure, if for every $p \in M$, there exists $\alpha \in (TM/E)^*$

¹⁵ We notice here that on the usual contact case, the Darboux theorem says that there exists no local invariants for contact manifolds, and thus, local (and microlocal) analysis on contact manifolds are equivalent to local (and microlocal) analysis on Heisenberg groups, which carries a natural contact structure. In the case of polycontact, there exists no unique local model, but the Heisenberg type Lie algebra structure on each tangent space of the manifold M allows to extend the Heisenberg calculus to the polycontact context.

¹⁶ On their work Choi, W. and Ponge, R study Carnot manifolds, that is, manifolds whose tangent spaces have a graded Lie algebra structure. Polycontact and pluricontact structures naturally induces such structures, and were given as examples on (CHOI; PONGE, 2017).

such that

$$\alpha_p \circ B_p : E_p \times E_p \rightarrow \mathbb{R}$$

is non degenerate. Where B_p denotes the restriction of the curvature form $B : \Lambda^2 E \rightarrow TM/E$ to $E_p \times E_p$.

Lemma 3.2.39 (Van Erp, (ERP, 2010), Lemma 3). The following are equivalent

- The distribution E defines a polycontact structure
- For every (local) 1-form α which vanishes on E , the restriction of $d\alpha$ to E is non degenerate.

This lemma allow us to prove

Proposition 3.2.40. If a distribution E defines a multicontact structure, then it defines a polycontact structure

Proof. In fact, let $\alpha \in \Gamma(U, N^*)$ be a local 1-form that vanishes on E , and let Z be a vector field on U tangent to E which is non vanishing. As $d\alpha$ is non degenerate on E , it follows that there exists W , another vector field on U tangent to E and non-vanishing, such that $d\alpha(Z, W) = -\alpha([Z, W]) \neq 0$. And thus, $\theta([Z, W]) \neq 0$, that is, $\mathcal{C}_U(E) = 0$ for every U , and thus $\mathcal{C}(E) = 0$. $\square$

Corollary 3.2.41. If $(M, \alpha, TM = I \oplus F)$ is a generalized k -contact structure, then (M, F) is a polycontact structure and a pluricontact structure.

Proposition 3.2.42. Let (M, E) be a polycontact structure and suppose that,

[Condition A:] There exists a unidimensional distribution $I \subset TM$ such that the image of B_U Lies on $\Gamma(U, I)$

. Let X be a non zero vector fields which generates this distribution.

Then, E is associated with a l -contact structure (in the sense of (BOLLE, 1996)).

Proof. Notice that the curvature form B was normalized so, for every section α of E^0 , the restriction of $d\alpha$ to $\Lambda^2 E$ is precisely $\alpha \circ B$.

As the image of B is tangent to I , we can write, for any vector fields Z, W tangents to E , $B(Z, W) = f_{Z, W} X$.

Let $\alpha_1, \dots, \alpha_l \in \Gamma(M, E^0)$ such that $\alpha_j(X) = 1$ for every j and $\alpha_1(p), \dots, \alpha_l(p)$ are linearly independent for every $p \in M$, then, for any vector fields Z, W tangents to E , and any i we have

$$d\alpha_i(Z, W) = \alpha_j \circ B(Z, W) = \alpha_j(f_{Z, W} X) = f_{Z, W}$$

and thus, $d\alpha_i = d\alpha_j$ for any $i, j \in \{1, \dots, s\}$.

As $d\alpha_1 = \dots = d\alpha_l$ are non degenerate on E , it follows that $\alpha_1 \wedge \dots \wedge \alpha_l \wedge d\alpha_j$ is a volume form on M as desired. $\square$

Remark 3.2.43. If we use the fact that the curvature form B is tensorial, we can rewrite the Condition A as

[Condition A':] The map $B : \Lambda^2 E \rightarrow TM/E$ has constant rank 1.

In what follows (M, E) will always be a pluricontact structure. Our goal is to follow the exposition of Apostolov et al. ([APOSTOLOV et al., 2017](#)) and prove a sufficient and necessary condition for the reciprocal of the Corolary 3.2.41, that is, when does a pluricontact structure also defines a generalized k -contact structure.

Definition 3.2.44. Let and let E^0 denote the annihilator of E , that is,

$$E^0 = \{\alpha \in T^*M ; \alpha(E) = 0\}$$

and let, for every open set $U \subset M$

$$\mathcal{U}_E(U) = \{\alpha \in \Gamma(U, E^0) \mid \alpha \circ B \text{ is non degenerate}\}$$

The set

$$\mathcal{U}_E = \bigcup_{\substack{U \subset M \\ U \text{ open}}} \{\alpha_p \in E^0 ; \forall \alpha \in \mathcal{U}_E(U) \quad \forall p \in U\}$$

is called the non-degeneracy locus of E and is an open submanifold of the total space of the bundle $E^0 \rightarrow M$.

A section $\alpha : M \rightarrow \mathcal{U}_E$ is called a contact form of E .

Remark 3.2.45. Using the notations and definitions of this section and Lemma 3.2.9, we can rewrite the definition of generalized k -contact structure as follows:

A generalized k -contact structure on M is a pluricontact structure (M, E) such that $I = TM/E$ is a trivial bundle and there exists $\alpha \in E^0 = I^*$ a contact form such that $I = \ker(d\alpha)$.

Definition 3.2.46. A (local) vector field X is called an infinitesimal (local) pluricontactomorphism for (M, E) if $\mathcal{L}_X \Gamma(U, E) \subset \Gamma(U, E)$. We denote the Lie algebra of infinitesimal pluricontactomorphisms by $\text{con}(M, E)$

Lemma 3.2.47 (Apostolov et al. ([APOSTOLOV et al., 2017](#)), Proposition 2). For any contact form $\alpha : M \rightarrow \mathcal{U}_E$, we have $TM = E \oplus \mathcal{K}^\alpha$, where $\mathcal{K}^\alpha$ is the ortogonal complement of E via $d\alpha$.

Definition 3.2.48. The distribution $\mathcal{K}^\alpha$ defined above is called the Reeb distribution of α .

Definition 3.2.49. A pluricontact action of a Lie algebra $\mathfrak{g}$ on (M, E) is a Lie algebra homomorphism $K : \mathfrak{g} \rightarrow \mathfrak{con}(M, E)$. We define $\kappa : \mathfrak{g} \times M \rightarrow TM$ by $\kappa(g, p) = K(g)(p)$, and we set $\mathcal{K}^\mathfrak{g}$ the distribution given by

$$\mathcal{K}_p^\mathfrak{g} = \text{Span}_{\mathbb{R}}\{K(g)(p) ; g \in \mathfrak{g}\}$$

Definition 3.2.50. A pluricontact action $K : \mathfrak{g} \rightarrow \mathfrak{con}(M, E)$ is said to be transversal if

- $\text{rank } \mathcal{K}^\mathfrak{g} = l$ everywhere on M
- $\mathcal{K}^\mathfrak{g} \cap E$ is the zero section of TM .

Or, equivalently, if $TM = E \oplus \mathcal{K}^\mathfrak{g}$.

Lemma 3.2.51 (Apostolov et al. ([APOSTOLOV et al., 2017](#))). For a transverse pluricontact action $K : \mathfrak{g} \rightarrow \mathfrak{con}(M, E)$, there exists a unique $\mathfrak{g}$ -valued 1-form $\eta = \eta^\mathfrak{g}$ on M satisfying

- $\ker(\eta^g) = E$
- $\eta^g(K(g)) = g ; \forall g \in \mathfrak{g}$

This form is called the canonical $\mathfrak{g}$ -form associated with K

Lemma 3.2.52 (Apostolov et al. ([APOSTOLOV et al., 2017](#)), Lemma 1). Let $K : \mathfrak{g} \rightarrow \mathfrak{con}(M, E)$ be a transverse pluricontact action and $\eta = \eta^\mathfrak{g}$ be the canonical $\mathfrak{g}$ -form associated with K . Let $\tilde{B}$ be the extension of the levi form B by zero to $\Gamma(M, TM) = \Gamma(M, E) \oplus \Gamma(M, \mathcal{K}^\mathfrak{g})$.

Then, for any $g \in \mathfrak{g}$ we have

$$\begin{aligned} (\mathcal{L}_{K(g)}\eta)(X) + [g, \eta(X)]_\mathfrak{g} &= 0 \forall X \in \Gamma(M, TM) \\ d\eta &= \eta \circ \tilde{B} - \frac{1}{2}\eta \wedge \eta \end{aligned}$$

Definition 3.2.53. Let $K : \mathfrak{g} \rightarrow \mathfrak{con}(M, E)$ be a transverse pluricontact action, and let η be the canonical $\mathfrak{g}$ -form associated with K . Let $\lambda \in \mathfrak{g}^*$ and define $\eta^\lambda : M/E \rightarrow E^0$ by

$$\eta^\lambda(p)(W) := \lambda(\eta_p(W))$$

so $\eta^\lambda(K(g)) = \lambda(g)$, and thus

$$d\eta|_E^\lambda = \lambda \circ d\eta|_E = \eta^\lambda \circ B$$

For $\lambda \in [\mathfrak{g}, \mathfrak{g}]^0 \subset \mathfrak{g}^*$, we call the pair $(E, d\eta|_E^\lambda)$ the induced Levi structure. It is said to be non degenerate if η^λ is a contact form.

Remark 3.2.54 (Apostolov et al. ([APOSTOLOV et al., 2017](#)), page 7). If α is a contact form of (M, E) such that $\alpha(K(v))$ is constant for every v , then the Reeb distribution $\mathcal{K}^\alpha$ coincides with $\mathcal{K}^\mathfrak{g}$. In particular,

$$\mathcal{K}^{\eta^\lambda} = \mathcal{K}^\mathfrak{g}$$

Definition 3.2.55. Let $K : \mathfrak{g} \rightarrow \text{con}(M, E)$ be an injective, transverse, pluricontact abelian (That is, $\mathfrak{g} = \mathbb{R}^k$) action. Let $\lambda \in \mathfrak{g}^* \setminus \{0\}$. Then $(\mathfrak{g}, \lambda)$ is called a Levi pair if $(E, d\eta|_E^\lambda)$ is non degenerate.

Proposition 3.2.56. Suppose that $(M, E, K : \mathbb{R}^k \rightarrow \text{con})$ is an abelian, injective, transverse, pluricontact action that admits a Levi pair $(\mathfrak{g}, \lambda)$. Then, it defines a generalized k -contact structure on (M, E) .

Proof. We have $\lambda : \mathbb{R}^k \rightarrow \mathbb{R}$ non zero. We choose a basis $e_1, \dots, e_k$ of $\mathbb{R}^k$, such that $\lambda(e_1) = 1$ and $\lambda(e_2) = \dots = \lambda(e_k) = 0$.

Now, consider the canonical Lie algebra valued 1-form η can be written as $\eta = (\eta_1, \dots, \eta_k)$, where each η_j is an usual $\mathbb{R}$ -valued 1-form. For our choice of basis $e_1, \dots, e_k$ of $\mathbb{R}^k$, we denote $X_j = K(e_j) \in \text{con}(M, E)$. The condition

$$\eta^{\mathfrak{g}}(K(g)) = g \ ; \ \forall g \in \mathfrak{g}$$

means

$$\eta_j(X_i) = \delta_{ij}$$

As K is injective, $\mathcal{K}^{\mathfrak{g}} = \text{Span}\{X_1, \dots, X_k\}$ is k -dimensional.

In particular, this means that $\eta_1, \dots, \eta_k$ are linearly independent.

The condition $\ker(\eta^{\mathfrak{g}}) = E$ means

$$E = \bigcap_j \ker(\eta_j)$$

The condition

$$(\mathcal{L}_{K(g)}\eta)(X) + [g, \eta(X)]_{\mathfrak{g}} = 0 \forall X \in \Gamma(M, TM)$$

means

$$\mathcal{L}_{X_j}\eta_i = d\eta_i(X_j, \cdot) = 0$$

We also have, for vector fields Z, W tangents to E , and $p \in M$

$$\begin{aligned} d\eta|_E^\lambda(Z, W)(p) &= \eta^\lambda \circ B(Z, W)(p) \\ &= \lambda(\eta_p(B(Z, W)(p))) \\ &= \lambda(\eta_{1p}(B(Z, W)(p)), \dots, \eta_{kp}(B(Z, W)(p))) = \eta_{1p}(B(Z, W)(p)). \end{aligned}$$

As $d\eta|_E^\lambda$ is non degenerate, this means that $\eta_1 \circ B$ is also non degenerate. But $d\eta_j = \eta_j \circ B$ and thus $d\eta_1$ is non degenerate on E .

The condition $\mathcal{K}^{\mathfrak{g}} = \text{Span}\{X_1, \dots, X_k\}$ means that $\mathcal{K}^{\mathfrak{g}} \subset \ker(d\eta_j)$ for every j .

Finally, the decomposition $TM = \mathcal{K}^{\mathfrak{g}} \oplus E$ proves that

$$\mathcal{K}^{\mathfrak{g}} = \ker(d\eta_1)$$

Thus, from Lemma 3.2.9, there exists a change of coordinates $\alpha_j = \sum_{i=1}^k \beta_{ij} \eta_i$ such that $(M, \alpha_1, \dots, \alpha_k, TM = \mathcal{K}^{\mathfrak{g}} \oplus E)$ is indeed a generalized k -contact structure on M .

□

3.3 Examples and Counter-examples

3.3.1 An example: $SO(k, k+n)/SO(n)$

Our goal in this section is to give an explicit construction of a generalized k -contact Anosov action on $SO(k, k+n)/SO(n)$ which is $SO(k, k+n)$ -invariant, and therefore descends to a generalized k -contact Anosov action on $\Gamma \backslash SO(k, k+n)/SO(n)$ for any uniform lattice $\Gamma \subset SO(k, k+n)$, acting freely on $SO(k, k+n)/SO(n)$. Latter on, we give a more geometric view of this objects.

The algebraic nature of our example will allow us to generalize this construction on Theorem 3.3.26.

This examples are actually the first examples of generalized k -contact Anosov actions we found before realizing that the general Theorem 3.3.26 is true. Moreover, using the machinery and notations used in the first part, we shall also prove (on sections 3.3.1.3 and 3.3.1.4) that the associated action does not come from a Contact Pair (in the case of $k = 2$) or a k -contact structure (in the case of general $k \geq 2$).

Before we construct the examples, we define the notion of algebraic action and give a criterion for an algebraic action be Anosov.

Definition 3.3.1. An algebraic (nilpotent) action is given by a quadruple $(G, K, \Gamma, \mathfrak{h})$, where

- G is a connected Lie group with Lie algebra $\mathfrak{g}$
- K is a compact subgroup of G with Lie algebra $\mathfrak{k}$
- $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$, contained in the normalizer $N_{\mathfrak{g}}\mathfrak{k}$ of $\mathfrak{k}$ and such that $\mathfrak{k} \cap \mathfrak{h} = \{0\}$.
- Γ is a uniform lattice in G acting freely on G/K .

The Lie group H associated with $\mathfrak{h}$ acts (locally free) on the quotient $\Gamma \backslash G/K$ by right multiplication.

A related, but weaker, notion is the notion of quasi-algebraic action.

Definition 3.3.2. A quasi-algebraic (nilpotent) action is given by a quadruple $(G, K, \Gamma, \mathfrak{a})$, where

- G is a connected Lie group with Lie algebra $\mathfrak{g}$
- K is a closed subgroup of G with Lie algebra $\mathfrak{k}$
- $\mathfrak{a}$ is a nilpotent subalgebra of $\mathfrak{g}$, contained in the normalizer $N_{\mathfrak{g}}\mathfrak{k}$ of $\mathfrak{k}$ and such that $\mathfrak{k} \cap \mathfrak{a} = \{0\}$.
- Γ is a discrete subgroup G acting freely on G/K such that $\Gamma \backslash G/K$ is compact.

The Lie group H associated with $\mathfrak{a}$ acts (locally free) on the quotient $\Gamma \backslash G/K$ by right multiplication.

Lemma 3.3.3. (BARBOT; MAQUERA, 2013) An algebraic action $(G, K, \Gamma, \mathfrak{h})$ is Anosov, if, and only if, there exists an element $h \in \mathfrak{h}$ and an $ad(h)$ invariant splitting $\mathfrak{g} = \mathcal{U} \oplus \mathcal{S} \oplus \mathfrak{h} \oplus \mathfrak{k}$ such that the eigenvalues of $ad(h)|_{\mathcal{U}}$ (resp. $ad(h)|_{\mathcal{S}}$) have strictly positive (resp. negative) real part.

We shall consider $SO(k, k+n)$ as the matrix subgroup of $GL(\mathbb{R}^{2k+n})$ of elements that preserves the bilinear form

$$\langle u, v \rangle = \sum_{i=1}^k u_i v_i - \sum_{i=k+1}^{2k+n} u_i v_i.$$

Using this identification, we can consider the Lie algebra $\mathfrak{so}(k, k+n)$ also as a matrix Lie algebra given by

$$\mathfrak{so}(k, k+n) = \left\{ \begin{pmatrix} \mathfrak{so}(k) & C & X^t \\ C^t & \mathfrak{so}(k) & Z^t \\ X & Z & D \end{pmatrix} ; C \in M_{k \times k}; X, Z \in M_{n \times k}; D \in \mathfrak{so}(n) \right\} \quad (3.5)$$

We consider $\mathcal{J} \subset M_{k \times k}$ the set of diagonal matrices and embed $\mathcal{J}$ on $\mathfrak{so}(k, k+n)$ in a natural way, that is

$$\mathcal{J} = \left\{ \begin{pmatrix} 0 & J & 0 \\ J & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(k, k+n) ; J \in M_{k \times k} \text{ is diagonal} \right\} \quad (3.6)$$

It is clear that $\mathcal{J}$ induces an action of $\mathbb{R}^k$ on $SO(k, k+n)$. Moreover, as this action leaves $SO(n)$ invariant, it descends to an action on $SO(k, k+n)/SO(k)$.

3.3.1.1 The action is Anosov

Proposition 3.3.4. Let Γ be an uniform lattice in $SO(k, k+n)$, acting freely in $SO(k, k+n)/SO(k)$. If $J = \text{diag}(\lambda_1, \dots, \lambda_k) \in \mathcal{J}$ is such that $0 \neq \lambda_i \neq \pm \lambda_j$ for all $i \neq j$, then J is an Anosov element of the action (by right multiplication) $\mathcal{J} \curvearrowright \Gamma \backslash SO(k, k+n)/SO(k)$

Proof. The main tool of the proof is Lemma 3.3.3. It is quite clear that the action is algebraic. It remains to study the adjoint action of $\mathcal{J}$ on $\mathfrak{g} = \mathfrak{so}(k, k+n)$. We will show that there exists a $\mathcal{J}$ invariant splitting, $\mathfrak{g} = \mathcal{J} \oplus F \oplus \mathfrak{so}(n)$ such that for a $J \in \mathcal{J}$ under the conditions of the proposition, it's eigenvalues are real and non zero on F . By grouping together the eigenspaces corresponding to the positive and negative eigenvalues we obtain the desired $F = \mathcal{U} \oplus \mathcal{S}$ splitting.

Let $J \in \mathcal{J}$ and

$$M = \begin{pmatrix} R_1 & C & X^t \\ C^t & R_2 & Z^t \\ X & -Z & D \end{pmatrix} \in \mathfrak{so}(k, k+n)$$

and we set

$$F = \left\{ M \in \mathfrak{so}(k, k+n) ; D = 0 ; C = \begin{pmatrix} 0 & * & * & \dots & * \\ * & 0 & * & \dots & * \\ \vdots & & \ddots & & \vdots \\ * & \dots & * & * & 0 \end{pmatrix} \right\}$$

then:

$$[J, M] = \begin{pmatrix} JC^t - CJ & JR_2 - R_1J & JX^t \\ JR_1 - R_2J & JC - C^tJ & JZ^t \\ XJ & -ZJ & 0 \end{pmatrix}$$

We want to consider the eigenvalues of this action, more specifically, we want to show that for any $J = \text{diag}(\lambda_1, \dots, \lambda_k)$ such that $0 \neq \lambda_i \neq \pm \lambda_j$ for all $i \neq j$ there exists a J invariant splitting $\mathfrak{g} = \mathcal{J} \oplus \mathcal{S} \oplus \mathcal{U} \oplus \mathfrak{so}(n)$ where the action of J on $\mathcal{S}$ (resp $\mathcal{U}$) has positive (resp negative) eigenvalues. Now consider $J = \text{diag}(\lambda_1, \dots, \lambda_k)$. First observe that J acts on the X and Z part by multiplication, and the eigenvalue problem reduces to $JX = \alpha X$ (similarly for Z). Thus, each element λ_j corresponds to an eigenvalue with n dimensional eigenspace. Now suppose that $R_2 = 0, X = 0, Z = 0$, we obtain

$$[J, M] = \begin{pmatrix} JC^t - CJ & -R_1J & 0 \\ JR_1 & JC - C^tJ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the eigenvalue problem $[J, M] = \alpha M$ we obtain

$$0 = JC - C^t J \quad (3.7)$$

$$R_1 = \alpha(JC^t - CJ) \quad (3.8)$$

$$C = -\alpha R_1 J \quad (3.9)$$

After a little manipulation we obtain

$$R_1 = \alpha^2(J^2 R^1 + R^1 J^2)$$

Now write E_{ij} , $1 \leq i < j \leq k$ for the generators of $\mathfrak{so}(k)$ that contains a 1 in the i, j position and a -1 on the j, i position. Taking $R_1 = E_{ij}$, then

$$\alpha^2(J^2 R^1 + R^1 J^2) = (\lambda_i^2 - \lambda_j^2) E_{ij}$$

Thus, $\alpha_{ij}^\pm = \frac{\pm 1}{\sqrt{\lambda_i^2 - \lambda_j^2}}$ are eigenvalues with corresponding eigenvectors given by solving 3.7 above. Moreover, from the hypothesis $\lambda_i \neq \pm \lambda_j \neq 0$, these eigenvalues are real and non zero. Doing a similar calculation for $R_1 = 0$, $X = 0$, $Z = 0$ we obtain the other eigenvectors. Dimensional analysis show that this is in fact a complete set of eigenvectors¹⁷. Now it is just a matter of grouping our eigenvectors in positive and negative associated eigenvalues. The corresponding spanned spaces $\mathcal{U}$ and $\mathcal{S}$ satisfies the hypothesis of Lemma 3.3.3. $\square$

3.3.1.2 The action is generalized k -contact

Our goal in this section is to show, that the action we considers comes from a generalized k -contact action.

The general strategy for the proof used in this section will be used latter (on section 3.3.4) to generalize this example. Roughly speaking, the idea is to choose 1-forms on $\mathcal{S}$ and extend those to $\mathfrak{so}(k, k+n)$ by using the splitting $\mathfrak{so}(k, k+n) = \mathcal{S} \oplus \mathcal{S} \oplus \mathcal{U} \oplus \mathfrak{so}(n)$ obtained in the previous section. Afterwards we show that if we avoid certain hyperplanes (finitely many), the chosen 1-forms will define the generalized k -contact structure.

Remember that a k -contact structure on M is a collection of 1-forms satisfying certain properties (Definition 3.2.1). To compensate our lack of information about Γ we shall define left invariant one forms on $SO(k, k+n)/SO(n)$. The left invariance means that those forms will descend to 1-forms on $\Gamma \backslash SO(k, k+n)/SO(n)$.

Now, our goal is to define some specific left invariant one forms on $SO(k, k+n)/SO(n)$ and make some computations. If we denote by $\Omega^q(G/K, \mathbb{R})^G$ the space of G -invariant q -forms

¹⁷ By a complete set of eigenvector, we mean that they generate F , that is if we sum the corresponding eigenspaces with $\mathcal{S}$ and $\mathfrak{so}(n)$ we recuperate the whole $\mathfrak{g}$.

on G/K and $\text{Hom}(\Lambda^q \mathfrak{g}/\mathfrak{k}, \mathbb{R})^K$ the space of K -invariant, q -multi-linear, alternating maps on $\mathfrak{g}/\mathfrak{k}$, then, it is known that

$$\Omega^q(G/K, \mathbb{R})^G = \text{Hom}(\Lambda^q \mathfrak{g}/\mathfrak{k}, \mathbb{R})^K$$

We shall work first on $\Lambda^q \mathfrak{g}^*$.

First, we establish some notations and do some preliminary computations. We denote by $C_{ij} \in M_{k \times k}$ the matrix whose only non zero coordinate is the i, j coordinate, whose value is 1. As before, we denote $E_{ij} = C_{ij} - C_{ji}$.

From $C_{ij}C_{kl} = \delta_{jk}C_{il}$ we obtain

$$E_{ij}C_{kl} = \delta_{jk}C_{il} - \delta_{ik}C_{jl} \quad (3.10)$$

$$C_{kl}E_{ij} = \delta_{il}C_{kj} - \delta_{jl}C_{ki} \quad (3.11)$$

Now, consider $\mathcal{J} \subset M_{k \times k}$ the subspace of diagonal matrices, and embed it into $\mathfrak{so}(k, k+n)$:

$$\mathcal{J} = \left\{ \begin{pmatrix} 0 & J & 0 \\ J & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(k, k+n) ; J \text{ is diagonal} \right\}$$

We take $\alpha \in \mathcal{J}^*$ and we consider α as a linear form on $\mathfrak{so}(k, k+n)$ by extending it to zero according to the obvious basis. By definition, this linear form is zero on $\mathfrak{so}(n)$ and also $SO(n)$ -invariant¹⁸, thus, it defines a left invariant 1-form on $SO(k, k+n)$ which descends to a 1-form on $SO(k, k+n)/SO(n)$.

We shall make our calculations on $SO(k, k+n)$. For left invariant vector fields A, B , we have that $\alpha(A)$ and $\alpha(B)$ are constants, and thus the usual formula

$$d\alpha(A, B) = A(\alpha(B)) - B(\alpha(A)) - \alpha([A, B])$$

reduces to $d\alpha(A, B) = -\alpha([A, B])$. Thus, for the purpose of our calculations, we are interested only in the diagonal portion of the $M_{k \times k}$ on $\mathfrak{so}(k, k+n)$, that is, for two given matrices $M_1, M_2 \in \mathfrak{so}(k, k+n)$, in the expression of

$$[M_1, M_2] = \begin{pmatrix} R_1 & C & X^t \\ C^t & R_2 & Z^t \\ X & Z & D \end{pmatrix}, R_1, R_2 \in \mathfrak{so}(k), D \in \mathfrak{so}(n), X, Z \in M_{n \times k}, C \in M_{k \times k}$$

¹⁸ Just notice that $\mathcal{J}$ is in the normalizer of $\mathfrak{so}(n)$ and $\mathcal{J} \cap \mathfrak{so}(n) = \{0\}$

we are only interested in the matrix C . With this notation, we shall write

$$\llbracket M_1, M_2 \rrbracket = C$$

Moreover, as we are actually interested only in the diagonal elements of C , we shall denote

$$\text{Diag} \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix} = \begin{pmatrix} c_{11} & 0 & \dots & 0 \\ 0 & c_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{nn} \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{22} \\ \vdots \\ c_{nn} \end{pmatrix}$$

and define

$$\llbracket M_1, M_2 \rrbracket = \text{Diag} \llbracket M_1, M_2 \rrbracket$$

If we write

$$M_1 = \begin{pmatrix} R_1 & A & X^t \\ A^t & R_2 & Z^t \\ X & Z & D \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} \tilde{R}_1 & \tilde{A} & \tilde{X}^t \\ \tilde{A}^t & \tilde{R}_2 & \tilde{Z}^t \\ \tilde{X} & \tilde{Z} & \tilde{D} \end{pmatrix}$$

then

$$\llbracket M_1, M_2 \rrbracket = R_1 \tilde{A} + A \tilde{R}_2 + X^t \tilde{Z} - \tilde{R}_1 A - \tilde{A} R_2 - \tilde{X}^t Z$$

In what follows, we will make frequent use of the different injections $M_{k \times k}, \mathfrak{so}(k), M_{n \times k} \hookrightarrow \mathfrak{so}(k, k+n)$, and thus, we fix the following notation:

For any $1 \leq i \leq j \leq k$ we will denote

$$F_{ij} = \begin{pmatrix} E_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad G_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad H_{ij} = \begin{pmatrix} 0 & C_{ij} & 0 \\ C_{ji} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We will also make no distinction between $X, Z \in M_{n \times k}$ and their images:

$$\begin{pmatrix} 0 & 0 & X^t \\ 0 & 0 & 0 \\ X & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Z^t \\ 0 & -Z & 0 \end{pmatrix}$$

We shall denote by $X_{ij}, Z_{ij} \in M_{n \times k}$ the matrix with 1 in the i, j coordinate and zero in all others.

It is clear that for $\{s, t\} \neq \{i, j\}$ we have

$$\llbracket F_{ij}, H_{st} \rrbracket = \llbracket G_{ij}, H_{st} \rrbracket = 0$$

and therefore

$$\begin{aligned} d\alpha^{2k(k-1)}(F_{12}, \dots, F_{k-1,k}, G_{12}, \dots, G_{k-1,k}, H_{12}, H_{21}, \dots, H_{k-1,k}, H_{k,k-1}) = \\ = \pm \Pi_{i < j} d\alpha^2(F_{ij}, G_{ij}, H_{ij}, H_{ji}) \end{aligned}$$

Now, from $\llbracket F_{ij}, G_{ij} \rrbracket = 0$ and $\llbracket H_{ij}, H_{ji} \rrbracket = 0$, we obtain

$$d\alpha^2(F_{ij}, G_{ij}, H_{ij}, H_{ji}) = \quad (3.12)$$

$$= 2d\alpha(F_{ij}, H_{ij})d\alpha(G_{ij}, H_{ji}) - 2d\alpha(F_{ij}, H_{ji})d\alpha(G_{ij}, H_{ij}) \quad (3.13)$$

$$= 2\alpha(\llbracket F_{ij}, H_{ij} \rrbracket)\alpha(\llbracket G_{ij}, H_{ji} \rrbracket) - 2\alpha(\llbracket F_{ij}, H_{ji} \rrbracket)\alpha(\llbracket G_{ij}, H_{ij} \rrbracket) \quad (3.14)$$

Now,

$$\llbracket F_{ij}, H_{ij} \rrbracket = E_{ij}C_{ij} = -C_{jj}.$$

$$\llbracket G_{ij}, H_{ji} \rrbracket = -C_{ji}E_{ij} = -C_{jj}.$$

$$\llbracket F_{ij}, H_{ji} \rrbracket = E_{ij}C_{ji} = C_{ii}.$$

$$\llbracket G_{ij}, H_{ij} \rrbracket = -C_{ij}E_{ij} = C_{ii}.$$

Thus,

$$d\alpha^2(F_{ij}, G_{ij}, H_{ij}, H_{ji}) = 2\alpha(H_{jj})^2 - 2\alpha(H_{ii})^2 \quad (3.15)$$

Moreover,

$$\llbracket X_{ij}, Z_{st} \rrbracket = X_{ji}Z_{st} = \delta_{is}C_{jt}$$

and thus,

$$\llbracket X_{ij}, Z_{st} \rrbracket = 0 \quad \text{if } j \neq t \text{ and } i \neq s$$

So,

$$d\alpha^{kn}(X_{11}, Z_{11}, \dots, X_{nk}, Z_{nk}) = \quad (3.16)$$

$$= \prod_{\substack{1 \leq s \leq n \\ 1 \leq t \leq k}} d\alpha(X_{st}, Z_{st}) \quad (3.17)$$

$$= (-1)^{kn} \prod_{\substack{1 \leq s \leq n \\ 1 \leq t \leq k}} \alpha(\llbracket X_{st}, Z_{st} \rrbracket) \quad (3.18)$$

$$= (-1)^{kn} \prod_{\substack{1 \leq s \leq n \\ 1 \leq t \leq k}} \alpha(H_{tt}) \quad (3.19)$$

If we choose an appropriate $\alpha \in \mathcal{J}^*$ that satisfies

$$0 \neq \alpha(H_{jj}) \neq \pm \alpha(H_{ii}) \quad \forall i \neq j, \quad (3.20)$$

then, from 3.12, 3.15 and 3.16 it follows that $d\alpha^{kn+2k(k-1)} \neq 0$ and moreover, $\ker(d\alpha) = \mathfrak{so}(n) \oplus \mathcal{I}$.

Finally, as α and its differential $d\alpha$ vanish everywhere on $\mathfrak{so}(n)$ and are $SO(n)$ invariant, they correspond to forms on the quotient $SO(k, k+n)/SO(n)$ and our calculations remain valid, that is, $d\alpha$ has constant rank $2kn + 2k(k-1) + 1$. Choosing a basis $\alpha_1, \dots, \alpha_k$ of the dual space $\mathcal{I}^*$ satisfying the condition 3.22, we obtain that for every $j \in \{1, \dots, k\}$

$$\alpha_1 \wedge \dots \wedge \alpha_k \wedge d\alpha_j^{kn+k(k-1)} \text{ is a volume form on } SO(k, k+n)/SO(n)$$

Therefore, such 1-forms $\alpha_1, \dots, \alpha_k$ defines a generalized k -contact structure on $SO(k, k+n)/SO(n)$. Moreover, by definition, the generalized k contact action is the action of the Cartan subspace $\mathcal{I}$.

3.3.1.3 It is not k -contact

The goal of this section is to prove that the action of the Cartan subspace we have been considering in the previous two sections is not associated with a k -contact structure (Definition 3.2.27). More precisely, we will prove the following proposition

Proposition 3.3.5. Consider $M = SO(k, k+n)/SO(n)$, $k > 1$, let $\mathcal{I} \subset \mathfrak{so}(k, k+n)$ be the abelian subalgebra given by 3.6 and $\mathcal{E} = \mathcal{U} \oplus \mathcal{I} \subset \mathfrak{so}(k, k+n)$ (Proof of Proposition 3.7). Let I and E be the corresponding left $SO(k, k+n)$ -invariant distributions on M . Then, there does not exist left invariant 1-forms $\alpha_1, \dots, \alpha_k$ on M such that these forms define a k -contact structure on M and such that

$$\bigcap_{j=1}^k \ker \alpha_j = E \text{ and } I = \ker d\alpha_j \text{ } 1 \leq j \leq k$$

Proof. As before (3.5) we consider $\mathfrak{so}(k, k+n)$ as matrix Lie algebra

$$\mathfrak{so}(k, k+n) = \left\{ \begin{pmatrix} \mathfrak{so}(k) & C & X^t \\ C^t & \mathfrak{so}(k) & Z^t \\ X & Z & D \end{pmatrix} ; C \in M_{k \times k}; X, Z \in M_{n \times k}; D \in \mathfrak{so}(n) \right\} \quad (3.21)$$

and

$$\mathcal{I} = \left\{ \begin{pmatrix} R_1 & C & X^t \\ C^t & R_2 & Z^t \\ X & Z & D \end{pmatrix} \in \mathfrak{so}(k, k+n) ; R_1 = R_2 = 0 ; X = Z = 0 ; D = 0 ; C \text{ is diagonal} \right\}$$

We also have $H_{ij} \in \mathfrak{so}(k, k+n)$ defined by

$$H_{ij} = \begin{pmatrix} 0 & C_{ij} & 0 \\ C_{ji} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $C_{ij} \in C \in M_{k \times k}$ is the matrix with value 1 on the coordinate (i, j) and value 0 otherwise.

The condition

- $\alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_i^n$ is a volume form on M

on the definition of k -contact, means that $d\alpha_i$ must be non degenerate on $E = \cap \ker \alpha_j$. From previous considerations, we know that left invariant 1-forms on M can be identified with $SO(n)$ -invariant linear forms on $\mathfrak{so}(k, k+n)/\mathfrak{so}(n)$. The additional condition $\alpha|_E = 0$ implies that α can in fact, be identified with an element of $\mathcal{S}^*$. In particular, the values (and certain properties) of $d\alpha$ depends only on the values of α on $\mathcal{S}$ ¹⁹, more precisely, it depends on the values of α on the following subspace of $\mathcal{S}$

$$[\mathfrak{so}(k, k+n), \mathfrak{so}(k, k+n)] \subset \mathcal{S} = \text{Span}_{\mathbb{R}}\{H_{11}, \dots, H_{kk}\}$$

However, the computations on the previous chapter show us that

$$[\mathfrak{so}(k, k+n), \mathfrak{so}(k, k+n)] = \text{Span}_{\mathbb{R}}\{H_{11}, \dots, H_{kk}\}$$

Explicitly,

$$[F_{ij}, H_{ij}] = -H_{jj}$$

Thus, suppose that α_1, α_2 are two of the 1-forms that define the k -contact structure. We have $d\alpha_1 = d\alpha_2$ if, and only if $\alpha_1(H_{jj}) = \alpha_2(H_{jj})$, $\forall j = 1, \dots, k$. However, $H_{11}, \dots, H_{kk}$ is in fact a basis of $\mathcal{S}$, and therefore,

$$d\alpha_1 = d\alpha_2 \Leftrightarrow \alpha_1 = \alpha_2$$

which contradicts the hypothesis that α_1, α_2 are linearly independent. $\square$

Remark 3.3.6. For the case $k = 1$ the Cartan action is in fact the geodesic flow on a negatively curved symmetric space (Katok, A. and Hasselblatt, B (KATOK; HASSELBLATT, 1997)), it is therefore a contact flow²⁰. Moreover, if $k = 1$, both the k -contact structure and the generalized k -contact structure reduces to the usual contact structure. This explain why the condition $k > 1$ on the proposition is necessary.

¹⁹ In fact, most of the work in the previous section was to prove that $d\alpha$ is non degenerate on E if, and only if

$$0 \neq \alpha(H_{jj}) \neq \pm \alpha(H_{ii}) \quad \forall i \neq j, \quad (3.22)$$

²⁰ It is a well established (if somewhat folkloric) fact of the contact geometry that the unit tangent bundle has canonical contact structure and it's Reeb flow gives the geodesic flow (For a complete proof in the slightly more general context of Finsler geodesic see Dorner, M. and Geiges, H. and Zehmisch, K.. (DÖRNER; GEIGES; ZEHMISCH, 2016)).

3.3.1.4 For $k = 2$ it is not a contact pair

The goal of this section is to prove that, for $k = 2$ the action of the Cartan subspace we have been considering is not associated with a contact pair (Definition 3.2.23). More precisely, we will prove the following proposition

Proposition 3.3.7. Consider $M = SO(2, 2 + n)/SO(n)$, let $\mathcal{J} \subset \mathfrak{so}(2, 2 + n)$ be the abelian subalgebra given by 3.6 and $\mathcal{E} = \mathcal{U} \oplus \mathcal{J} \subset \mathfrak{so}(k, k + n)$ (Proof of Proposition 3.7). Let I and E be the corresponding left $SO(2, 2 + n)$ -invariant distributions on M . Then, there does not exist left invariant 1-forms α, η on M such that these forms define a contact pair on M where at least one Reeb vector field is Anosov, and such that

$$\ker \alpha \cap \ker \eta = E \quad \text{and} \quad I = \ker d\alpha \cap \ker d\eta$$

Proof. As before, we consider $\mathfrak{so}(2, 2 + n)$ as matrix Lie algebra

$$\mathfrak{so}(2, 2 + n) = \left\{ \begin{pmatrix} 0 & r_1 & a & b & X^t \\ -r_1 & 0 & c & d & \\ a & c & 0 & r_2 & Z^t \\ b & d & -r_2 & 0 & \\ X & & Z & & D \end{pmatrix} ; a, b, c, d, r_1, r_2 \in \mathbb{R}; \right. \quad (3.23)$$

$$\left. X, Z \in M_{n \times 2}; D \in \mathfrak{so}(n) \right\}$$

and

$$\mathcal{J} = \left\{ \begin{pmatrix} 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & d & \\ a & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & \\ 0 & & 0 & 0 & \end{pmatrix} \in \mathfrak{so}(2, 2 + n) ; a, d \in \mathbb{R} \right\}$$

As we have previously seen, left invariant 1-forms on M that satisfies the desired conditions correspond to elements in $\mathcal{J}^*$, therefore, the condition α, η are linearly independent²¹ means that the choice of α and η corresponds (by duality) to a choice of basis for $\mathcal{J}$.

We shall take a basis $\{X_\alpha, X_\eta\}$ defined as

$$X_\alpha = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a & \\ 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \\ 0 & & 0 & 0 & \end{pmatrix} \quad X_\eta = \begin{pmatrix} 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 1 & \\ d & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \\ 0 & & 0 & 0 & \end{pmatrix}$$

²¹ The hypothesis that $\alpha \wedge \eta \wedge d\alpha^h \wedge d\eta^k$ is a volume form implies that α and η are linearly independent (otherwise $\alpha \wedge \eta = 0$.)

where $ad \neq 1$. And let α, η be the corresponding dual basis.

From the Section 3.3.1.1, the Anosov condition means that either $a \notin \{0, \pm 1\}$ or $d \notin \{0, \pm 1\}$.

Our goal is to prove that for any choice of constants $a, d \in \mathbb{R} \setminus \{\pm 1\}$ such that $ad \neq 1$, the pair (α, η) is not a contact pair.

As $\{X_\alpha, X_\eta\}$ is not the canonical basis of $\mathcal{J}$, it is convenient to see how the canonical basis is written in terms of X_α, X_η .

Let $J \in \mathcal{J}$ written as

$$J = \begin{pmatrix} 0 & 0 & S & 0 & 0 \\ 0 & 0 & 0 & T & 0 \\ S & 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad S, T \in \mathbb{R}$$

then

$$J = \lambda_\alpha X_\alpha + \lambda_\eta X_\eta$$

where

$$\lambda_\alpha = \lambda_\alpha(S, T) = \frac{S - dT}{1 - ad} \quad ; \quad \lambda_\eta = \lambda_\eta(S, T) = \frac{T - aS}{1 - ad}$$

If we use the notations of Section 3.3.1.2, we can write

$$J = SH_{11} + TH_{22}$$

and

$$\lambda_\alpha(S, T) = S\alpha(H_{11}) + T\alpha(H_{22})$$

$$\lambda_\eta(S, T) = S\eta(H_{11}) + T\eta(H_{22})$$

In particular,

$$\alpha(H_{22}) = \lambda_\alpha(0, 1) = \frac{-d}{1 - ad}$$

$$\eta(H_{11}) = \lambda_\eta(1, 0) = \frac{-a}{1 - ad}$$

and

$$\alpha(H_{11}) = \frac{1}{1 - ad} = \eta(H_{22})$$

From the computations of Section 3.3.1.2 we have

$$\begin{aligned} d\alpha^{2n}(X_{11}, Z_{11}, \dots, X_{2n}, Z_{2n}) \\ = \Pi_{1 \leq s \leq n} \alpha(H_{11}) \alpha(H_{22}) = \frac{(-d^n)}{(1-ad)^{2n}} \neq 0 \end{aligned}$$

and similarly

$$d\eta^{2n}(X_{11}, Z_{11}, \dots, X_{2n}, Z_{2n}) = \frac{(-a^n)}{(1-ad)^{2n}} \neq 0$$

This shows that the rank of α and η must be at least $2n$, and if (α, η) is a contact pair of order (h, k) , then, by dimensional analysis, we have three distinct possibilities:

- $(h, k) = (n+2, n)$
- $(h, k) = (n+1, n+1)$
- $(h, k) = (n+2, n+2)$

Now, still using the notation from Section 3.3.1.2, $\mathfrak{so}(k) \oplus \mathfrak{so}(k)$ is generated by the matrices $\{F_{ij}, G_{ij}\}_{1 \leq i < j \leq k}$, as $k = 2$ we set $F = F_{12}$ and $G = G_{12}$.

From previous computations

$$\begin{aligned} d\alpha(F, H_{21}) &= \alpha(H_{11}) = \frac{1}{1-ad} \neq 0 \\ d\eta(G, H_{21}) &= -\eta(H_{11}) = -\frac{1}{1-ad} \neq 0 \end{aligned}$$

Which implies that $h \geq n+1$ and $k \geq n+1$. However we also have

$$d\alpha^2(F, G, H_{12}, H_{21}) = 2(\alpha(H_{11})^2 - \alpha(H_{22})^2) = \frac{2(1+d)(1-d)}{(1-ad)^2}$$

and

$$d\eta^2(F, G, H_{12}, H_{21}) = \frac{2(1+a)(1-a)}{(1-ad)^2}$$

If $d \neq \pm 1$ we conclude that $d\alpha$ has rank $n+2$, and if $a \neq \pm 1$ we conclude that $d\eta$ has rank $n+2$. The dimensional analysis implies that we must have $d, a \in \{+1, -1\}$. But this contradicts the Anosov condition, which finishes the proof.

□

3.3.2 Geometric interpretation of $SO(k, k+n)/SO(n)$

In this subsection, we shall revisit the example in the previous section with a more geometric flavour. After setting some definitions and basic results we define an action which we prove to be Anosov. To do so we define an left invariant metric and make some explicit calculations. Afterwards, we define a family of 1-forms and prove the necessary conditions to show that they define a generalized k -contact Anosov structure.

Remark 3.3.8. In what follows, small Latin letters x, y, z, a, b, c will denote vectors (with dimensions to be specified), with coordinates $x_i, y_i, z_i, a_i, b_i, c_i$, respectively.

3.3.3 Definitions

Consider on $\mathbb{R}^{k,k+n}$ the $2k+n$ -dimensional real vector space endowed with a non degenerate bilinear form of signature $(k, k+n)$, explicitly, let $x, y, a, b \in \mathbb{R}^k$ and $c, z \in \mathbb{R}^n$, then,

$$(a, b, c) \cdot (x, y, z) = \sum_{i=1}^k a_i y_i + b_i x_i + \sum_{j=1}^n c_j z_j$$

Lemma 3.3.9. We can identify $SO(k, k+n)/SO(n)$ with the set of $2k$ - vectors (u^p, v^p) , $u^p = (x^p, y^p, z^p)$, $v^p = (a^p, b^p, c^p)$, $p = 1, \dots, k$, such that

1. $u^p \cdot u^q = 0 = v^p \cdot v^q$
2. $u^p \cdot v^q = \delta_{pq}$
3. $\dim(\text{Span}\langle u^1, v^1, \dots, u^k, v^k \rangle) = 2k$

With this identification, we will denote an element of $SO(k, k+n)/SO(n)$ by the pair $(\bar{u}, \bar{v})$, where $\bar{u} = (u^1, \dots, u^k)$ and $\bar{v} = (v^1, \dots, v^k)$

Proof. If we denote by M the set defined above, then, $SO(k, k+n)$ acts on M transitively, and the isotropy subgroup is precisely $SO(n)$. $\square$

Lemma 3.3.10. Using the above identification, we can also identify $T_{(\bar{u}, \bar{v})}M$ with the set of $2k$ -vectors $(\dot{u}^p, \dot{v}^p)$, $1 \leq p \leq k$, such that

1. $u^p \cdot \dot{u}^q + u^q \cdot \dot{u}^p = 0$
2. $v^p \cdot \dot{v}^q + v^q \cdot \dot{v}^p = 0$
3. $u^p \cdot \dot{v}^q + v^q \cdot \dot{u}^p = 0$

As before, we denote an element of $T_{(\bar{u}, \bar{v})}M$ by the pair $(\dot{u}, \dot{v})$, where $\dot{u} = (\dot{u}^1, \dots, \dot{u}^k)$ and $\dot{v} = (\dot{v}^1, \dots, \dot{v}^k)$.

Proof. We just differentiate the defining equations of M . The third condition that defines M is not relevant for it is an open condition. $\square$

3.3.3.1 Anosovness

For each $(\bar{u}, \bar{v}) \in M$ we consider the decomposition $\mathbb{R}^{k,k+n} = V_{(\bar{u}, \bar{v})} \oplus W_{(\bar{u}, \bar{v})} \oplus Z_{(\bar{u}, \bar{v})}$ where

$$V_{(\bar{u}, \bar{v})} = \text{Span}\langle u^1, \dots, u^k \rangle$$

$$W_{(\bar{u}, \bar{v})} = \text{Span}\langle v^1, \dots, v^k \rangle$$

$$Z_{(\bar{u}, \bar{v})} = (V_{(\bar{u}, \bar{v})} \oplus W_{(\bar{u}, \bar{v})})^\perp$$

Now, we write for, $(\dot{u}, \dot{v}) \in T_{(\bar{u}, \bar{v})}M$ the vectors $\dot{u}^p$ and $\dot{v}^p$ in this basis:

$$\dot{u}^p = \sum_{i=1}^k \mu_{pi} u^i + v_{pi} v^i + w^p \quad (3.24)$$

$$\dot{v}^p = \sum_{i=1}^k \mu'_{pi} u^i + v'_{pi} v^i + w^{p'} \quad (3.25)$$

$$(3.26)$$

$w^p, w^{p'} \in Z$.

From conditions (3.3.9) we have

$$\dot{u}^p \cdot v^q = \sum_i \mu_{pi} u^i \cdot v^q = \sum_i \mu_{pi} \delta_{iq} = \mu_{pq}$$

$$\dot{u}^p \cdot u^q = v_{pq}$$

$$\dot{v}^p \cdot u^q = v'_{pq}$$

$$\dot{v}^p \cdot v^q = \mu'_{pq}$$

Lemma 3.3.11. For $(\dot{u}, \dot{v}) \in T_{(\bar{u}, \bar{v})}M$, the formula

$$\|(\dot{u}, \dot{v})\|^2 = \sum_{p \geq q} |\mu_{pq}|^2 + |v_{pq}|^2 + |\mu_{pq'}|^2 + |v'_{pq}|^2 + \sum_q \|w^q\|^2 + \|w^{q'}\|^2$$

defines an $SO(k, k+n)$ invariant metric on M .

Proof. Let us first check that this is in fact a norm, that is,

$$\|(\dot{u}, \dot{v})\| = 0 \Rightarrow (\dot{u}, \dot{v}) = 0$$

But $\|(\dot{u}, \dot{v})\|^2 = 0$ implies that for every $p \geq q$

$$|\mu_{pq}| = |v_{pq}| = |\mu_{pq'}| = |v'_{pq}| = \|w^q\| = \|w^{q'}\| = 0$$

From 3.3.10, for every p, q

$$0 = u^p \cdot \dot{v}^q + v^q \cdot \dot{u}^p = v'_{qp} + \mu_{pq}$$

$$0 = u^p \cdot \dot{u}^q + u^q \cdot \dot{u}^p = v_{qp} + v_{pq}$$

$$0 = v^p \cdot \dot{v}^q + v^q \cdot \dot{v}^p = \mu'_{qp} + \mu'_{pq}$$

And thus, $\mu_{pq}, \mu'_{pq}, v_{pq}, v'_{pq}$ are zero for any p, q , as we wanted.

Let us check that this is $SO(k, k+n)$ invariant. First, notice that $SO(k, k+n)$ preserves the splitting $Z \oplus Z^\perp$, and moreover, it acts on Z by orthogonal transformations. Finally, notice that it also preserves all the products²², and thus, it preserves the quantities $\mu_{pq}, \mu'_{pq}, v_{pq}$ and v'_{pq} for they are given by products.

It is straightforward to show that this norm actually satisfies the parallelogram law, and thus, comes from an inner product, and thus define a metric on M .

□

Remark 3.3.12. Notice that we have also prove that the coordinates $\mu_{pq}, \mu'_{pq}, v_{pq}$ and v'_{pq} are completely determined if we know just their values for $p \geq q$.

Theorem 3.3.13. For a given $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$, and a collection of k vectors $\bar{u} = (u^1, \dots, u^k)$ we denote $e^\lambda \bar{u} = (e^{\lambda_1} u^1, \dots, e^{\lambda_k} u^k)$. Then, the action of $\mathbb{R}^k$ on M given by

$$\begin{aligned} \phi : \mathbb{R}^k \times M &\rightarrow M \\ (\lambda, (\bar{u}, \bar{v})) &\mapsto \phi_\lambda(\bar{u}, \bar{v}) = (e^\lambda \bar{u}, e^{-\lambda} \bar{v}) \end{aligned}$$

is Anosov.

Lemma 3.3.14. If ϕ is the action above, then $d\phi_\lambda$ acts on the tangent space by

$$(d\phi_\lambda)_{(\bar{u}, \bar{v})}(\dot{u}, \dot{v}) = (e^\lambda \dot{u}, e^{-\lambda} \dot{v})$$

In particular, if we write

$$\dot{u}^p = \sum_{i=1}^k \mu_{pi} u^i + v_{pi} v^i + w^p$$

and

$$e^\lambda \dot{u}^p = \sum_{i=1}^k \mu_{pi}^\lambda u^i + v_{pi}^\lambda v^i + w^{\lambda p}$$

then $w^{\lambda p} = e^\lambda w^p$, and similarly for $\dot{v}^p$

²² By definition, $SO(k, k+n)$ is the set of transformations that preserves the products.

Proof. Notice that we can understand our manifold M as embedded in the linear space $(\mathbb{R}^{k,k+n})^{2k}$, and in this case, the action is actually linear, therefore, its differential is itself. $\square$

Proof of Theorem 3.3.13. For $\lambda = (\lambda_1, \dots, \lambda_k)$ and $(\dot{u}, \dot{v}) \in T_{(\bar{u}, \bar{v})}M$ we have

$$\begin{aligned} \|d\phi_{t\lambda}(\dot{u}, \dot{v})\|^2 &= \sum_{p \geq q} e^{(\lambda_p - \lambda_q)t} |\mu_{pq}|^2 + e^{(\lambda_p + \lambda_q)t} |\nu_{pq}|^2 + e^{-(\lambda_p - \lambda_q)t} |\mu'_{pq}|^2 + e^{-(\lambda_p + \lambda_q)t} |\nu'_{pq}|^2 \\ &\quad + \sum_q e^{2\lambda_q t} \|w^q\|^2 + e^{-2\lambda_q t} \|w^{q'}\|^2 \end{aligned}$$

We shall suppose that $\lambda_p > \lambda_q > 0$ if $p > q$.

We define

$$\begin{aligned} \mathcal{S} &:= \{(\dot{u}, \dot{v}) ; \mu_{pq} = \nu_{pq} = 0 \ \forall p \geq q ; \mu'_{pp} = w^p = 0 \ \forall p\} \\ \mathcal{U} &:= \{(\dot{u}, \dot{v}) ; \mu'_{pq} = \nu'_{pq} = 0 \ \forall p \geq q ; \mu_{pp} = w^{p'} = 0 \ \forall p\} \end{aligned}$$

and

$$\begin{aligned} \lambda_0^+ &= \max_{p \geq q} \{(\lambda_p - \lambda_q), (\lambda_p + \lambda_q), (2\lambda_p)\} \\ \lambda_0^- &= \min_{p \geq q} \{(\lambda_p - \lambda_q), (\lambda_p + \lambda_q), (2\lambda_p)\} \end{aligned}$$

Then, for $t > 0$

$$\begin{aligned} \|(d\phi_{t\lambda})|_{\mathcal{S}}(\dot{u}, \dot{v})\|^2 &= \sum_{p \geq q} e^{-(\lambda_p - \lambda_q)t} (1 - \delta_{pq}) |\mu'_{pq}|^2 + e^{-(\lambda_p + \lambda_q)t} |\nu'_{pq}|^2 + \sum_q e^{-2\lambda_q t} \|w^{q'}\|^2 \\ &\leq e^{\lambda_0^+ t} \sum_{p \geq q} |\mu'_{pq}|^2 + |\nu'_{pq}|^2 + \sum_q \|w^{q'}\|^2 \leq e^{\lambda_0^+ t} \|(\dot{u}, \dot{v})\|^2 \end{aligned}$$

and analogously,

$$\|(d\phi_{-t\lambda})|_{\mathcal{U}}(\dot{u}, \dot{v})\|^2 \leq e^{-\lambda_0^- t} \|(\dot{u}, \dot{v})\|^2$$

That is, the distribution $\mathcal{U}$ (resp. $\mathcal{S}$) is expanded (resp. contracted) uniformly.

Define also

$$\begin{aligned} \mathcal{I} &:= \{(\dot{u}, \dot{v}) ; \mu_{pq} = \nu'_{pq} = 0 \ \forall p > q \\ &\quad \nu_{pq} = \mu_{pq'} = 0 \ \forall p \geq q \\ &\quad w^{p'} = w^p = 0 \ \forall p\} \end{aligned}$$

that is, $\mathcal{I}$ is the set of $(\dot{u}, \dot{v}) \in T_{(\bar{u}, \bar{v})}M$ such that, in the basis (3.24), we have

$$\begin{aligned} \dot{u}^p &= \mu_{pp} u^p \\ \dot{v}^p &= \nu'_{pp} v^p \end{aligned}$$

The condition $u^p \cdot \dot{v}^p + v^p \cdot \dot{u}^p = 0$ implies $\mu_{pp} = -\nu'_{pp}$, and thus

$$\mathcal{I} = \{(\dot{u}, \dot{v}) \in T_{(\bar{u}, \bar{v})}M ; (\dot{u}^p, \dot{v}^p) = (\lambda_p u^p, -\lambda_p v^p) \text{ for some } \lambda_1, \dots, \lambda_k \in \mathbb{R}\}$$

that is

$$\mathcal{J} = \{(\dot{u}, \dot{v}) \in T_{(\bar{u}, \bar{v})}M; (\dot{u}, \dot{v}) = \lambda(\bar{u}, \bar{v}) \text{ for some } \lambda \in \mathbb{R}^k\}$$

To see that this distribution is tangent to the action, take a curve $\gamma = (\bar{u}(t), \bar{v}(t)) = (e^{\lambda t} \bar{u}_0, e^{-\lambda t} \bar{v}_0)$. It is clear that $(\dot{u}(t), \dot{v}(t)) = (\lambda \bar{u}(t), -\lambda \bar{v}(t))$ is the field of tangent vectors along γ .

It remains to show that $TM = \mathcal{U} \oplus \mathcal{J} \oplus \mathcal{U}$. This is the content of the following lemma □

Lemma 3.3.15. Let $\mathcal{S}$, $\mathcal{U}$ and $\mathcal{J}$ defined as in the previous proof. Then

$$TM = \mathcal{U} \oplus \mathcal{J} \oplus \mathcal{U}$$

Proof. Let $(\dot{u}, \dot{v}) \in T_{(\bar{u}, \bar{v})}M$, and let us write

$$\begin{aligned} \dot{u}^p &= \sum_j \mu_{pj} u^j + v_{pj} v^j + w^j \\ &= \underbrace{\mu_{pp} u^p}_{\dot{u}_{\mathcal{S}}^p} + \underbrace{\sum_{j>p} \mu_{pj} u^j}_{\dot{u}_{\mathcal{S}}^p} + \underbrace{\sum_{j<p} \mu_{pj} u^j + \sum_j v_{pj} v^j + v_{pp} v^p + w^p}_{\dot{u}_{\mathcal{U}}^p} \\ \dot{v}^p &= \sum_j \mu'_{pj} u^j + v'_{pj} v^j + w^p \\ &= \underbrace{v'_{pp} v^p}_{\dot{v}_{\mathcal{S}}^p} + \underbrace{\sum_j \mu'_{pj} u^j + \sum_{j<p} v'_{pj} v^j + w^p}_{\dot{v}_{\mathcal{S}}^p} + \underbrace{\sum_{j>p} v_{pj} v^j}_{\dot{v}_{\mathcal{U}}^p} \end{aligned}$$

It is a straightforward calculation to show that $(\dot{u}_{\mathcal{S}}, \dot{v}_{\mathcal{S}})$, $(\dot{u}_{\mathcal{J}}, \dot{v}_{\mathcal{J}})$ and $(\dot{u}_{\mathcal{U}}, \dot{v}_{\mathcal{U}})$ are tangent to M . Moreover,

$$(\dot{u}, \dot{v}) = (\dot{u}_{\mathcal{S}}, \dot{v}_{\mathcal{S}}) + (\dot{u}_{\mathcal{J}}, \dot{v}_{\mathcal{J}}) + (\dot{u}_{\mathcal{U}}, \dot{v}_{\mathcal{U}})$$

and

$$\begin{aligned} (\dot{u}_{\mathcal{S}}, \dot{v}_{\mathcal{S}}) &\in \mathcal{U} \\ (\dot{u}_{\mathcal{J}}, \dot{v}_{\mathcal{J}}) &\in \mathcal{J} \\ (\dot{u}_{\mathcal{U}}, \dot{v}_{\mathcal{U}}) &\in \mathcal{U} \end{aligned}$$

□

Remark 3.3.16. In the proof of Theorem 3.3.13 above, we have assumed $\lambda_p > \lambda_q > 0$ if $p > q$ and we have shown that such $\lambda = (\lambda_1, \dots, \lambda_k)$ is an Anosov element. We do not need this

condition. We only must suppose that $\lambda_p \neq \lambda_q \neq 0$ for $p \neq q$, but the study of the signs in this general case becomes cumbersome. That is, for the different choices of λ , we must define different $\mathcal{U} = \mathcal{U}_\lambda$ and $\mathcal{S} = \mathcal{S}_\lambda$. However, whatever the case may be, the direct sum distribution $\mathcal{S}_\lambda \oplus \mathcal{U}_\lambda$ is the same.

3.3.3.2 Generalized k -contact

Consider, for $j = 1, \dots, k$ the 1-forms η_j defined by

$$(\eta_j)_{(\bar{u}, \bar{v})}(\dot{u}, \dot{v}) = u^j \cdot \dot{v}^j$$

Clearly η_j is $SO(k, k+n)$ -invariant.

Lemma 3.3.17. $\mathcal{S} \oplus \mathcal{U} = \bigcap_{j=1}^k \ker \eta_j$

Proof. First, we observe that

$$\bigcap_{j=1}^k \ker \eta_j = \{(\dot{u}, \dot{v}) ; u^p \cdot \dot{v}^p = \mu_{pp} = v'_{pp} = 0 \ \forall p\}$$

and thus

$$\mathcal{S} \oplus \mathcal{U} \subset \bigcap_{j=1}^k \ker \eta_j$$

Now, we take $(\dot{u}, \dot{v}) \in \bigcap_{j=1}^k \ker \eta_j$ and write its decomposition $TM = \mathcal{S} \oplus \mathcal{U} \oplus \mathcal{V}$ according to Lemma 3.3.15.

$$(\dot{u}, \dot{v}) = (\dot{u}_{\mathcal{S}}, \dot{v}_{\mathcal{S}}) + (\dot{u}_{\mathcal{U}}, \dot{v}_{\mathcal{U}}) + (\dot{u}_{\mathcal{V}}, \dot{v}_{\mathcal{V}})$$

By definition,

$$(\dot{u}_{\mathcal{S}}^p, \dot{v}_{\mathcal{S}}^p) = (\mu_{pp} u^p, v'_{pp} v^p),$$

and thus, $(\dot{u}_{\mathcal{S}}, \dot{v}_{\mathcal{S}}) = 0$, that is $(\dot{u}, \dot{v}) \in \mathcal{U} \oplus \mathcal{V}$

□

Let us write $u^p = (x^p, y^p, z^p)$, $v^p = (a^p, b^p, c^p)$, $p = 1, \dots, k$, then, in this coordinates we have

$$\eta_p = u^p \cdot \dot{v}^p = \sum_{i=1}^k x_i^p db_i^p + y_i^p da_i^p + \sum_{l=1}^n z_l^p dc_l^p$$

and

$$d\eta_p = \sum_{i=1}^k dx_i^p \wedge db_i^p + dy_i^p \wedge da_i^p + \sum_{l=1}^n dz_l^p \wedge dc_l^p$$

Lemma 3.3.18. For any $(\dot{u}, \dot{v}) \in \mathcal{S}$, we have $i_{(\dot{u}, \dot{v})} d\eta_p = 0$

Proof. First, we observe that we are seeing the form $d\eta_p$ as a 2-form on the linear space $(\mathbb{R}^{k,k+n})^{2k}$ and $(\mathbb{R}^{k,k+n})^{2k}$ with itself. Thus, we shall write

$$(\dot{u}, \dot{v}) = \sum_{j=1}^p \dot{u}^j \partial_{u^j} + \dot{v}^j \partial_{v^j}$$

where

$$\dot{u}^p \partial_{u^p} = \sum_{l=1}^k \dot{x}_l^p \partial_{x_l^p} + \dot{y}_l^p \partial_{y_l^p} + \sum_{l=1}^n \dot{z}_l^p \partial_{z_l^p}$$

and analogous for $\dot{v}^p$.

We have seen that

$$(\dot{u}, \dot{v}) \in \mathcal{J} \Leftrightarrow \exists (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k ; (\dot{u}^p, \dot{v}^p) = (\lambda_p u^p, -\lambda_p v^p) \quad \forall p$$

And thus,

$$(\dot{u}, \dot{v}) = \sum_{j=1}^k \lambda_p u^p \partial_{u^p} - \lambda_p v^p \partial_{v^p}$$

It is clear that

$$i_{\partial_{u^q}} d\eta_p = i_{\partial_{v^q}} d\eta_p = 0 \quad \text{for } q \neq p$$

therefore

$$\begin{aligned} i_{(\dot{u}, \dot{v})} d\eta_p &= i_{\lambda_p u^p \partial_{u^p}} d\eta_p - i_{\lambda_p v^p \partial_{v^p}} d\eta_p \\ \lambda_p &= \lambda_p \left(\sum_{i=1}^k x_i^p \wedge db_i^p + y_i^p \wedge da_i^p + \sum_{l=1}^n z_l^p \wedge dc_l^p \right) \\ &\quad - \lambda_p \left(\sum_{i=1}^k -dx_i^p \wedge b_i^p - dy_i^p \wedge a_i^p + \sum_{l=1}^n -dz_l^p \wedge c_l^p \right) \\ &= \lambda_p \left(\sum_{i=1}^k x_i^p db_i^p + y_i^p da_i^p + b_i^p dx_i^p + a_i^p dy_i^p + \sum_{l=1}^n z_l^p dc_l^p + c_l^p dz_l^p \right) \end{aligned}$$

That is,

$$\begin{aligned} i_{(\dot{u}, \dot{v})} d\eta_p(\partial_{x_l^q}) &= \lambda b_l^q \\ i_{(\dot{u}, \dot{v})} d\eta_p(\partial_{y_l^q}) &= \lambda a_l^q \\ i_{(\dot{u}, \dot{v})} d\eta_p(\partial_{z_l^q}) &= \lambda c_l^q \end{aligned}$$

and analogous for the tangent vectors $\partial_{a_l^q}$, $\partial_{b_l^q}$ and $\partial_{c_l^q}$.

Thus, for a general tangent vector $(\hat{u}, \hat{v}) = \sum_{j=1}^k \hat{u}^j \partial_{u^j} + \hat{v}^j \partial_{v^j}$ we have

$$\begin{aligned} i_{(\hat{u}, \hat{v})} d\eta_p((\hat{u}, \hat{v})) &= \lambda_p \left(\sum_{i=1}^k x_i^p \hat{b}_i^p + y_i^p \hat{a}_i^p + b_i^p \hat{x}_i^p + a_i^p \hat{y}_i^p + \sum_{l=1}^n z_l^p \hat{c}_l^p + c_l^p \hat{z}_l^p \right) \\ &= \lambda_p (u^p \cdot \hat{v}^p + v^p \cdot \hat{u}^p) \end{aligned}$$

But, if $(\hat{u}, \hat{v}) \in T_{(\bar{u}, \bar{v})}M$ then $u^p \cdot \hat{v}^p + v^p \cdot \hat{u}^p = 0$ and thus $i_{(\dot{u}, \dot{v})}d\eta_p((\hat{u}, \hat{v}))$ as we wanted. $\square$

Remark 3.3.19. From the previous computations, it is clear that for two vectors $(\dot{u}, \dot{v}), (\hat{u}, \hat{v}) \in T_{(\bar{u}, \bar{v})}M$ we have

$$d\eta_j((\dot{u}, \dot{v}), (\hat{u}, \hat{v})) = \dot{u}^j \cdot \hat{v}^j - \dot{v}^j \cdot \hat{u}^j$$

Remark 3.3.20. Consider an invertible matrix $B = (\beta_{ij}) \in M_{k \times k}$. Then, the 1-forms

$$\alpha_i := \sum_j \beta_{ij} \eta_j$$

satisfies

$$\bigcap_{j=1}^k \ker \alpha_j = \bigcap_{j=1}^k \ker \eta_j$$

and moreover,

$$i_{(\dot{u}, \dot{v})}d\alpha_j = 0$$

Theorem 3.3.21. For almost every coordinate change B , the 1-forms $\alpha_i = \sum_j \beta_{ij} \eta_j$ define a generalized k -structure on M .

Proof. We consider a change off coordinates $B = (\beta_{ij})$ such that $\beta_{is} \neq \pm \beta_{it} \neq 0$ for every s, t .

We will show that for every i , $d\alpha_i$ is non degenerate on $\mathcal{U} \oplus \mathcal{S}$. To avoid cluttering the notation, we suppress the index i , that is we consider a linear combination $\alpha = \sum_j \beta_j \eta_j$ such that $\beta_s \neq \pm \beta_t \neq 0$ for every j . We want to show that $d\alpha$ is non degenerate on $\mathcal{U} \oplus \mathcal{S}$. As each η_j is $SO(k, k+n)$ -invariant, then so is $d\alpha$ and as $SO(k, k+n)$ acts transitively on M , then we only have to study $d\alpha$ on a single point.

Take the point $(\bar{u}, \bar{v})$ given by

$$u^p = (e_p, \vec{0}_k, \vec{0}_n) \in \mathbb{R}^{k, k+n} \quad \text{and} \quad v^p = (\vec{0}_k, e_p, \vec{0}_n) \in \mathbb{R}^{k, k+n}$$

where e_p denotes the p 'th element of the canonical basis $\{e_1, \dots, e_k\}$ of $\mathbb{R}^k$, and $\vec{0}_d$ denotes the zero vector of $\mathbb{R}^d$. Clearly $(\bar{u}, \bar{v}) \in M$.

On this point, if $(\dot{u}, \dot{v})$ has coordinates

$$\dot{u}^p = (\dot{x}^p, \dot{y}^p, \dot{z}^p)$$

$$\dot{v}^p = (\dot{a}^p, \dot{b}^p, \dot{c}^p)$$

then we can write the conditions 3.3.10 as

$$1. \quad \dot{y}_p^i + \dot{y}_i^p = 0$$

$$2. \quad \dot{a}_p^i + \dot{a}_i^p = 0$$

$$3. \dot{x}_p^i + \dot{b}_i^p = 0$$

In particular

$$y_p^p = a_p^p = 0 \quad \text{and} \quad x_p^p + b_p^p = 0 \quad (3.27)$$

That, is, if we restrict $d\alpha = \sum_{p,i=1}^k \beta_p dx_i^p \wedge db_i^p + dy_i^p \wedge da_i^p + \sum_{l=1}^n dz_l^p \wedge dc_l^p$ to $T_{(\bar{u}, \bar{v})}M$ then we can use the identities

$$1. dy_p^i + dy_i^p = 0$$

$$2. da_p^i + da_i^p = 0$$

$$3. dx_p^i + db_i^p = 0$$

in particular, $dy_p^p = 0$, $da_p^p = 0$ and $dx_p^p \wedge db_p^p = 0$ and we obtain

$$\begin{aligned} d\alpha &= \sum_{p,i=1}^k \beta_p (dx_i^p \wedge db_i^p + dy_i^p \wedge da_i^p + \sum_{l=1}^n dz_l^p \wedge dc_l^p) \\ &= \sum_{l,p} \beta_p dz_l^p \wedge dc_l^p + \left(\sum_{p>i} + \sum_{p<i} + \sum_{p=i} \right) \beta_p dx_i^p \wedge db_i^p + \beta_p dy_i^p \wedge da_i^p \\ &= \sum_{l,p} dz_l^p \wedge dc_l^p + \sum_{p>i} (\beta_p - \beta_i) dx_i^p \wedge db_i^p + (\beta_p + \beta_i) dy_i^p \wedge da_i^p \end{aligned}$$

It is clear that for $\beta_p \neq \pm \beta_i \neq 0$ we have

$$\ker(d\alpha|_{T_{(\bar{u}, \bar{v})}M}) = \{(\dot{u}, \dot{v}); \dot{u}^p = (t_1 e_p, t_2 e_p, \vec{0}_n); \dot{v}^p = (t_3 e_p, t_4 e_p, \vec{0}_n)\}$$

where t_1, t_2, t_3 and t_4 are arbitrary values constrained by the equations $(\dot{u}, \dot{v}) \in T_{(\bar{u}, \bar{v})}M$. That is, $t_2 = t_3 = 0$ and $t_1 + t_4 = 0$.

This means that, over $T_{(\bar{u}, \bar{v})}M$, $\ker(d\alpha)$ is precisely the points of the form $(\lambda \bar{u}, -\lambda \bar{v})$, which we have seen is the subspace $\mathcal{S}$ tangent to the orbit of the action ϕ .

We conclude that $d\alpha$ is non degenerate on $TM/\mathcal{S} = \mathcal{S} \oplus \mathcal{U}$

□

3.3.4 Algebraic actions

The goal of this section is to generalize the construction in the previous section and prove Theorem 3.3.26 below. Before we state the theorem, let us recall some definitions and results about Lie algebras.

Definition 3.3.22. Consider a Lie algebra $\mathfrak{g}$. A Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ is an abelian subalgebra, such that $ad(x)$ is semisimple²³ for every $x \in \mathfrak{c}$, and maximal for these properties.

²³ Recall that an endomorphism T of a (finite dimensional) vector space V is called simple if any invariant subspace is trivial, and it is called semisimple if it is a sum $T_1 \oplus \cdots \oplus T_l \in \text{End}(V_1 \oplus \cdots \oplus V_l)$ of simple endomorphisms $T_j \in \text{End}(V_l)$

Definition 3.3.23. Consider a Lie algebra $\mathfrak{g}$. A Cartan subspace $\mathfrak{a}$ of $\mathfrak{g}$ is an abelian subalgebra, such that, for every $x \in \mathfrak{a}$, the linear map $ad(x)$ is hyperbolic (its eigenvalues are all real and positive), and maximal for these properties.

Lemma 3.3.24. If $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{g}$, then it's centralizer $Z_{\mathfrak{g}}(\mathfrak{a})$ can be written as

$$Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{k} \oplus \mathfrak{a}$$

where $\mathfrak{k}$ is the Lie algebra of a compact subgroup.

Lemma 3.3.25. If $\mathfrak{g}^0$ is a real Lie algebra and $\mathfrak{h}^0$ is a Cartan subalgebra of $\mathfrak{g}^0$, then, the complexification $\mathfrak{h} = \mathfrak{h}_{\mathbb{C}}^0 = \mathfrak{h} \oplus i\mathfrak{h}$ of $\mathfrak{h}^0$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}^0$ (the complexification of $\mathfrak{g}^0$).

We are ready to enunciate the main theorem of this section.

Theorem 3.3.26. Consider a real, connected, semisimple, non compact, Lie group G , with finite center and with Lie algebra $\mathfrak{g}$. Let $\mathfrak{a}$ be the Cartan subspace and $K \subset G$ the compact group associated with the compact part of the center of $\mathfrak{a}$. Consider also a uniform lattice Γ in G acting freely on G/K . The action of the Cartan subspace $\mathfrak{a}$ on $\Gamma \backslash G/K$, is an generalized k -contact Anosov action.

Before we prove this theorem, we give some definitions and previous known results that will be used in our proof.

Proposition 3.3.27. (HOF, 1985) Consider G a non compact semisimple real Lie group with finite center and Lie algebra $\mathfrak{g}$. Consider a torsion free, uniform lattice, Γ in G and a Cartan subspace $\mathfrak{a}$ of G . Let $\mathfrak{a} \oplus \mathfrak{k}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, and $K \subset G$ the compact subgroup with Lie algebra $\mathfrak{k}$ (see Lemma 3.3.24). The algebraic action $(G, K, \Gamma, \mathfrak{a})$ is always Anosov.

Remark 3.3.28. By the Proposition 3.3.27, the action in Theorem 3.3.26 is Anosov. It remains to show that this action is associated with a generalized k -contact structure. The main tool is a structure theorem of real semisimple Lie algebras as exposed by Holger Kammeyer ((KAMMEYER, 2014)). In what follows we develop the necessary notations to enunciate Kammeyer's theorem.

Definition 3.3.29. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The Killing form B on $\mathfrak{g}^0$ is the bilinear form

$$\begin{aligned} B : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathbb{F} \\ (x, y) &\mapsto B(x, y) := \text{Tr}(ad(x) \circ ad(y)) \end{aligned}$$

A Cartan involution of $\mathfrak{g}$, is a involution map $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ such that the bilinear map

$$\begin{aligned} B_{\theta} : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathbb{F} \\ (x, y) &\mapsto B_{\theta}(x, y) := B(x, \theta y) \end{aligned}$$

is positive definite.

Remark 3.3.30. If $\mathfrak{g}$ is a semisimple Lie algebra and $\mathbb{F}$ has characteristic 0, then the Killing form is non degenerate and a Cartan involution always exists. The (generalized) eigenspaces $\mathfrak{k}, \mathfrak{p}$ associated with the eigenvalues $+1$ and -1 of θ , gives a θ -invariant decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

called the Cartan decomposition of $\mathfrak{g}$.

Remark 3.3.31. If $\mathfrak{g}^0$ is a real Lie algebra and θ is a Cartan involution on $\mathfrak{g}^0$, there exists a unique $\mathbb{C}$ -linear extension $\theta_{\mathbb{C}}$ of θ to the complexification $\mathfrak{g} = \mathfrak{g}^0 \oplus i\mathfrak{g}^0$ of $\mathfrak{g}^0$. We shall abuse notation and denote $\theta_{\mathbb{C}} = \theta$.

Lemma 3.3.32 (([HELGASON, 1979](#)),([KAMMEYER, 2014](#))). Consider $\mathfrak{g}^0$ a real semisimple Lie algebra, a Cartan involution θ on $\mathfrak{g}^0$, and the corresponding Cartan decomposition $\mathfrak{g}^0 = \mathfrak{k} \oplus \mathfrak{p}$.

There exists a maximal abelian, θ -stable subalgebra $\mathfrak{h}^0 \subseteq \mathfrak{g}^0$ such that $\mathfrak{a}^0 = \mathfrak{h}^0 \cap \mathfrak{p}$ is a maximal abelian subalgebra in $\mathfrak{p}$.

Lemma 3.3.33 (Helgason, ([HELGASON, 1979](#)), Chapter XI). Under the above notations, $\mathfrak{h}^0$ is a Cartan subalgebra of $\mathfrak{g}^0$ and $\mathfrak{a}$ is a Cartan subspace²⁴ of $\mathfrak{g}^0$.

Definition 3.3.34. Let $\mathfrak{g}$ be a reductive Lie algebra over a field $\mathbb{F}$ of characteristic zero. A Cartan subalgebra $\mathfrak{h}$ is called a splitting for $\mathfrak{g}$ if $ad_{\mathfrak{g}}(x)$ is triangularizable over $\mathbb{F}$ for every $x \in \mathfrak{h}$. We call the pair $(\mathfrak{h}, \mathfrak{g})$ a $\mathbb{F}$ -splitting Lie algebra, or simply a splitting Lie algebra.

Lemma 3.3.35. If $\mathfrak{g}$ is a Lie algebra over an algebraically closed field $\mathbb{F}$, then every Cartan subalgebra $\mathfrak{c}$ is a splitting for $\mathfrak{g}$.

Definition 3.3.36. Consider a splitting Lie algebra $(\mathfrak{h}, \mathfrak{g})$, and denote $\mathfrak{h}^*$ the dual vector space of $\mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$ and $\mathfrak{h} \subset E \subset \mathfrak{g}$ an $ad(\mathfrak{h})$ -invariant vector space, we denote

$$E^{\lambda} = E^{\lambda}(\mathfrak{h}) := \{Y \in E ; (ad(Z) - \lambda(Z)Id)^n Y = 0 ; \text{ for some } n \in \mathbb{N} \ \forall Z \in \mathfrak{h}\}$$

and we define

$$\Delta(\mathfrak{h}, E) = \{\lambda \in \mathfrak{h}^* \setminus \{0\} ; E^{\lambda} \neq \{0\}\}$$

The elements of $\Delta(\mathfrak{h}, \mathfrak{g})$ are called roots of $(\mathfrak{h}, \mathfrak{g})$ and the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Delta(\mathfrak{h}, \mathfrak{g})} \mathfrak{g}^{\lambda}$$

is called the root space decomposition.

²⁴ Helgason does not call this space a Cartan subspace. However, he shows that for any $x \in \mathfrak{a}$, $ad(x)$ is hyperbolic. Moreover, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ denotes the Iwasawa decomposition, and $g = k + a + n$ is the decomposition of $g \in \mathfrak{g}$, then we can write $ad(g) = KAN$, where K is elliptic, A is hyperbolic and N is unipotent. Moreover, $K = Id$ (resp $N = Id$) if and only if $k = 0$ (resp $n = 0$). Therefore, $\mathfrak{a}$ is maximal and thus a Cartan subspace of $\mathfrak{g}^0$.

Remark 3.3.37. If $(\mathfrak{h}, \mathfrak{g})$ is a $\mathbb{F}$ -splitting Lie algebra, where $\mathbb{F}$ is algebraically closed, then E^λ is at most one dimensional and

$$E^\lambda \{Y \in E ; ad(Z)Y = \lambda(Z)Y ; \forall Z \in \mathfrak{h}\}$$

Definition 3.3.38. The set $\{Y \in \mathfrak{h} ; \lambda(Y) \neq 0 \ \forall \lambda \in \Delta(\mathfrak{h}, \mathfrak{g})\}$ has a finite number of connected components called Weyl chambers. For a fixed choice of Weyl chamber $\mathscr{W}$, we can define

$$\Delta^+(\mathfrak{h}, \mathfrak{g}) = \Delta^+(\mathfrak{h}, \mathfrak{g}, \mathscr{W}) = \{\lambda \in \Delta(\mathfrak{h}, \mathfrak{g}) ; \lambda(x) > 0 \ \forall x \in \mathscr{W}\}$$

The roots on $\Delta^+(\mathfrak{h}, \mathfrak{g})$ are called positive roots, with respect to $\mathscr{W}$, and we speak about "choice of positivity", meaning that a Weyl chamber was chosen and the positive roots are those that are positive with respect to this Weyl chamber.

Definition 3.3.39. A positive root is called simple if it is not a positive linear combination of other positive roots. The set of simple positive roots will be denoted by $\Delta_S^+(\mathfrak{h}, \mathfrak{g})$.

Consider a real semisimple Lie algebra $\mathfrak{g}^0$ with a Cartan involution θ . Consider the Cartan subalgebra $\mathfrak{h}^0$ obtained in Lemma 3.3.32. Let $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}^0$ the complexification of $\mathfrak{g}^0$, then, $\mathfrak{h} = \mathfrak{h}_{\mathbb{C}}^0$ (the complexification of $\mathfrak{h}^0$) is a Cartan subalgebra of $\mathfrak{g}$ and therefore $(\mathfrak{h}, \mathfrak{g})$ is a splitting Lie algebra.

We let $\Delta_S^+(\mathfrak{h}, \mathfrak{g}) = \{\alpha_1, \dots, \alpha_l\}$ be a choice of simple positive roots.

Denote by σ the anti-linear complex automorphism of $\mathfrak{g}$ given by the conjugation with respect to $\mathfrak{g}^0$, that is,

$$\begin{aligned} \sigma : \mathfrak{g} &= \mathfrak{g}^0 \oplus i\mathfrak{g}^0 \rightarrow \mathfrak{g}^0 \oplus i\mathfrak{g}^0 \\ g &= g_1 + ig_2 \mapsto \sigma(g) = g_1 - ig_2 \end{aligned}$$

For each $\alpha \in \Delta(\mathfrak{h}, \mathfrak{g})$ we define

$$\alpha^\sigma(h) = \overline{\alpha(\sigma(h))}.$$

Notice that if $x_\alpha \in g^\alpha$, then

$$[h, \sigma(x_\alpha)] = \sigma([\sigma(h), x_\alpha]) = \sigma(\alpha(\sigma(h))x_\alpha) = \overline{\alpha(\sigma(h))}\sigma(x_\alpha)$$

and therefore $(\sigma(x_\alpha)) \in g^{\alpha^\sigma}$, that is α^σ is also a root.

We also denote by $\Delta_{\mathbb{R}}$ the space of real roots (roots fixed by σ), $\Delta_{i\mathbb{R}}$ the space of imaginary roots (roots fixed by the Cartan involution θ , and therefore vanishes on $\mathfrak{a}$) and $\Delta_{\mathbb{C}}$ the complex roots (neither real or imaginary). We set $\Delta_{\mathbb{C}}^{+*}$ be $\Delta_{\mathbb{C}}^+$ with one element removed from each pair $\{\alpha, \alpha^\sigma\}$. We also denote $\Delta_{\mathbb{C}}^* = \Delta_{\mathbb{C}}^{+*} \cup -\Delta_{\mathbb{C}}^{+*}$.

Theorem 3.3.40 (Kammeyer, (KAMMEYER, 2014), Theorem 4.1). There exists a partition $\Delta(\mathfrak{h}, \mathfrak{g}) = \Delta^1 \cup \Delta^0$ such that $\Delta(\mathfrak{h}, \mathfrak{g}) \cap \Delta_{i\mathbb{R}} \subset \Delta^0$ and $\mathcal{H}^1 = \{H_\alpha^1; \alpha \in \Delta^1\}$ and $\mathcal{H}^0 = \{H_\alpha^0; \alpha \in \Delta^0\}$ are basis of $\mathfrak{a}$ and $\mathfrak{h}^0 \cap \mathfrak{k}$ and a basis $\mathcal{B}$ of $\mathfrak{g}^0$ given by

$$\mathcal{B} := \{X_\alpha^0, X_\alpha^1; \alpha \in \Delta_{i\mathbb{R}}^+ \cup \Delta_{\mathbb{C}}^*\} \cup \{Z_\alpha; \alpha \in \Delta_{\mathbb{R}}\} \cup \mathcal{H}^1 \cup \mathcal{H}^2$$

such that,

1. $[H_\alpha^i, H_\beta^j] = 0$ for every α, β .
2. $[H_\alpha^i, X_\beta^j] = c_{\alpha\beta}^{ij} X_\beta^{i+j+1}$.
3. $[H_\alpha^i, Z_\beta] = [H_\alpha^i, X_\beta^{\frac{\text{sgn}(\beta)-1}{2}}]$.
4. For $\beta \notin \{-\alpha, -\alpha^\sigma\}$; $[X_\alpha^i, X_\beta^j] = (-1)^{ij} \gamma_{\alpha\beta} X_{\alpha+\beta}^{i+j} + \text{sgn}(\alpha) \gamma_{\alpha^\sigma\beta} X_{\alpha^\sigma+\beta}^{i+j}$.
5. For $\alpha \in \Delta_{\mathbb{C}}^*$; $[X_\alpha^i, X_{-\alpha}^j] = (-1)^{ij} H_\alpha^{1+i+j}$ where $2H_\alpha^0$ and H_α^1 are non zero $\mathbb{Z}$ -linear combinations in $\mathcal{H}^0$ and $\mathcal{H}^1$ respectively.
6. For $\alpha \in \Delta_{i\mathbb{R}}^+$; $[X_\alpha^0, X_\alpha^1] = \tilde{H}_\alpha^0$, where $\tilde{H}_\alpha^0$ is a nonzero $\mathbb{Z}$ linear combination of elements H_β^0 , where $\beta \in \Delta(\mathfrak{h}, \mathfrak{g}) \cap \Delta_{i\mathbb{R}} \subset \Delta^0$.
7. $[Z_\alpha, Z_{-\alpha}] = -\text{sgn}(\alpha) \hat{H}_\alpha^1$ where $2\hat{H}_\alpha^1$ is a non zero $\mathbb{Z}$ -linear combination in $\mathcal{H}^1$.
8. $[Z_\alpha, X_\beta^i] = [X_\alpha^{\frac{\text{sgn}(\alpha)-1}{2}}, X_\beta^i]$.

and the constants $c_{\alpha\beta}^{ij}$ and $\gamma_{\alpha\beta}$ are half integers and satisfy.

1. $\gamma_{\alpha,\beta} = \gamma_{-\alpha,-\beta}$
2. $\gamma_{\alpha\beta} = \pm(r+1)$ where r is the largest integer such that $\beta - r\alpha \in \Delta(\mathfrak{h}, \mathfrak{g})$
3. $c_{\alpha\beta}^{ij} = -c_{\alpha\beta}^{i,j+1}$
4. $c_{\alpha,\beta}^{ij} = (-1)^{i+1} c_{\alpha,\beta^\sigma}^{ij}$
5. $c_{\alpha,\beta}^{1j} = 0$ if $\beta \in \Delta_{i\mathbb{R}}$.

Remark 3.3.41. On items 5 and 6 on Theorem 3.3.40 above, we have imposed that $\mathcal{H}^0$, $\mathcal{H}^1$, $\mathcal{H}_\alpha^0$ and $\mathcal{H}_\alpha^1$ are non zero. These conditions were not stated explicitly on Kammeyer's paper ((KAMMEYER, 2014)). They are, however, a clear consequence of the proof. Those vectors are obtained by constructive methods, and a careful analysis of the construction show them to be non zero.

We are now ready to prove Theorem 3.3.26

Proof of Theorem 3.3.26. The idea of the proof is to imitate the construction in section 3.3. More precisely, we shall take a splitting $\mathfrak{g}_0 = \mathfrak{a} \oplus (\mathfrak{k} \cap \mathfrak{h}_0) \oplus \mathcal{N}^+ \oplus \mathcal{N}^-$ where $\mathcal{N}^\pm$ are the nilpotent parts of opposing parabolic subalgebras.

Using the idea of section 3.3, we shall take 1-forms on $\mathfrak{a}^*$ and using this splitting, extend it to 1-forms on $\mathfrak{g}_0$ and, like in section 3.3, we shall prove that if we chose 1-forms $\eta_1, \dots, \eta_k$ that are non zero on certain vectors, then, they define the desired generalized k -contact structure.

As we have already remarked (Remark 3.3.28), under the conditions of our theorem, the action is Anosov, moreover, almost²⁵ every element of $\mathfrak{a}$ is Anosov.

Thus it remains to choose the 1-forms

Let us denote

$$\mathcal{N}^\pm = \text{Span}_{\mathbb{R}} \{X_\alpha^i, Z_\beta, \alpha \in \Delta_{\mathbb{C}}^{*\pm}, \beta \in \Delta_{\mathbb{R}}^\pm\}$$

,

$$\mathcal{K} = \text{Span}_{\mathbb{R}} \{X_\alpha^0, X_\alpha^1; \alpha \in \Delta_{\mathbb{R}}^+\} \oplus \text{Span}_{\mathbb{R}} \{\mathcal{H}^0\}$$

and

$$\mathcal{H} = \text{Span}_{\mathbb{R}} \{\mathcal{H}^1\}$$

We have the decomposition

$$\mathfrak{g}_0 = \mathcal{K} \oplus \mathcal{H} \oplus \mathcal{N}^+ \oplus \mathcal{N}^- \quad (3.28)$$

where $\mathcal{H}$ is a Cartan subspace and $\mathcal{K}$ is a compact subalgebra. Take $\lambda \in \mathcal{H}^*$. We can use the decomposition (3.28) to extend λ to a linear functional on $\mathfrak{g}_0$, and therefore we can understand it as a left invariant 1-form on G . Moreover, as $\mathcal{K}$ is in the centre of $\mathcal{H}$, than λ is K -invariant and, therefore, can be seen as a left invariant 1-form on G/K . Because it is left invariant, it pass on to the quotient $\Gamma \backslash G/K$.

We want conditions to show that its differential $d\lambda$ is non degenerate on $\mathcal{N}^+ \oplus \mathcal{N}^-$. We just have to choose λ that does not vanishes on $H_\alpha^1, \hat{H}_\alpha^1$ ²⁶. For such λ , we obtain that for each element $Y \in \{X_\alpha^i, Z_\beta, \alpha \in \Delta_{\mathbb{C}}^{*\pm}, \beta \in \Delta_{\mathbb{R}}^\pm\}$ basis of $\mathcal{N}^+ \oplus \mathcal{N}^-$, there exists Y^* in the same basis such that $d\lambda(Y, Y^*) = -\lambda([Y, Y^*]) \neq 0$ as we wanted. Moreover, the conditions on λ are open, and we can choose a basis of such λ satisfying the condition. It is clear that this basis will define the generalized r -contact structure on G/K .

□

²⁵ Here, almost every element means that the complement is a finite union of hyperplanes.

²⁶ Such λ does exists, for $H_\alpha^1, \hat{H}_\alpha^1$ are always non zero and are finite in number.

MAIN THEOREM

The goal of this section is to prove the following theorem

Theorem 4.0.1. Let (M^{2n+k}, α, ϕ) be a generalized k -contact Anosov action. Then, it is smoothly conjugated to a quasi algebraic action.

The strategy of our proof is to endow our manifold with a $(G', G'/H')$ -structure that is compatible with the Anosov action. This compatibility means that it give us the desired smooth conjugacy, and moreover, the induced action on G'/H' is of algebraic nature, that is, it is the right action of an abelian subalgebra. The proof can be outlined in the following steps.

Step 1: Define a rigid Gromov geometric structure compatible with our given structure and dynamics. In particular, $\mathbb{R}^k$ acts on M by automorphisms of this structure. As the Anosov action is transitive, the open-dense orbit theorem will ensure the existence of an open dense subset $\Omega \subset M$ which is locally homogeneous. This is done in Subsection 4.1.

Step 2: Fix a point with compact orbit in Ω . Let H' be the group of local automorphisms that stabilizes this point. Denote its Lie algebra by $\mathfrak{h}'$. Let $\mathfrak{g}'$ be the Lie algebra of Killing vector fields of the geometric structure around this point. Let $\hat{G}'$ be the associated connected simply connected Lie group, and $\hat{H}'$ the connected subgroup with Lie algebra $\mathfrak{h}'$. Our goal is to show that Ω can be locally modelled after $\hat{G}'/\hat{H}'$. This is done in the Subsection 4.2.4. The second (and most technical) step of the proof is to show that $\hat{H}'$ is closed in $\hat{G}'$ (Proposition 4.2.36). This requires some technical preparation which is done over the Subsections 4.2.1, 4.2.2 and 4.2.3.

Step 3: $\hat{G}'/\hat{H}'$ is a model space for Ω . The third step (Section 4.3) is to show that we can extend this structure, that is $\Omega = M$.

Step 4: The last step of the Part 1 is to show that the geometric structure we defined (now in the sense of Ehreshman-Thurston) is complete, that is, the developing map is a covering map (Proposition 4.4.1).

4.1 The geometric structure

We shall now, define a rigid geometric structure associated with our generalized k contact Anosov action. Let $m = 2n + k$ and $V = \mathbb{R}^m$. Denote by $Gr^n(V)$ the Grassmanian of n -planes in V and define Σ as

$$\Sigma := \left\{ (A_1, \dots, A_k, e^+, e^-, \omega_1, \dots, \omega_k), \text{ with} \right. \\
\begin{aligned}
& A_j \in V \ \forall j \quad e^\pm \in Gr^n(V) \text{ such that} \\
& V = e^+ \oplus \mathbb{R}A_1 \oplus \dots \oplus \mathbb{R}A_k \oplus e^- \\
& \omega_j \in \Lambda^2(V^*) \text{ such that, for each } j \in \{1, \dots, k\} \\
& (\omega_j)|_{e^+ \oplus e^-} \text{ is non degenerated} \\
& \ker \omega_j = \mathbb{R}A_1 \oplus \dots \oplus \mathbb{R}A_k
\end{aligned}
\left. \right\}$$

It is clear that Σ is algebraic and have a natural algebraic action of $GL(m, \mathbb{R})$. Now we define an A -structure of order 1 and type Σ on M by:

$$\begin{aligned} \sigma : F^1(M) &\rightarrow \Sigma \\ (p, \xi) &\mapsto \xi^{-1}((X_1)_p, \dots, (X_k)_p, (E^+)_p, (E^-)_p, (d\alpha_1)_p, \dots, (d\alpha_k)_p) \end{aligned}$$

Remark 4.1.1. Notice that our geometric structure also encodes the data of the 1-forms defining the geometric structure, for given a point in Σ , there exists unique linear functionals $\tilde{\alpha}_1, \dots, \tilde{\alpha}_k$ on V such that $\tilde{\alpha}_i(A_j) = \delta_{ij}$ and $e^+ \oplus e^- \subset \ker(\tilde{\alpha}_i)$. However, the geometric structure deal with infinitesimal data, and thus cannot encode how such linear functionals should relate with the bilinear forms $\omega_1, \dots, \omega_k$.

Notice that the Killing vector fields of this geometric structure, are precisely the vector fields Y such that for every $j = 1, \dots, k$

$$\begin{aligned} [Y, X_j] &= 0 \\ [Y, E^\pm] &\subset E^\pm \\ \mathcal{L}_Y \alpha_j &= 0 \end{aligned}$$

Moreover, notice that for each j , we can define a pseudo-Riemannian metric g_j on M by

$$\begin{aligned} g_j(X_i, X_l) &= \delta_{il} \\ g_j(X_i, Z^+) &= g_j(X_i, Z^-) = 0 \\ g_j(W^+, Z^+) &= g_j(W^-, Z^-) = 0 \\ g_j(Z^+, W^-) &= d\alpha_j(Z^+, W^-) \end{aligned}$$

for all $1 \leq i, l \leq k$, $Z^+, W^+ \in \Gamma(M, E^+)$, $Z^-, W^- \in \Gamma(M, E^-)$.

Proposition 4.1.2. The geometric structure σ is rigid.

Remark 4.1.3. Essentially, the rigidity of the geometric structure σ follows from the fact that its automorphisms actually preserves a pseudo-Riemannian metric, which are rigid. In what follows we formalize this intuition.

Proof. Define Σ_0 as the following

$$\begin{aligned} \Sigma_0 := \left\{ (A_1, \dots, A_k, e^+, e^-, \omega), \text{ with} \right. \\ A_j \in V \forall j \quad e^\pm \in Gr^n(V) \text{ such that} \\ V = e^+ \oplus \mathbb{R}A_1 \oplus \dots \oplus \mathbb{R}A_k \oplus e^- \\ \omega \in \Lambda^2(V^*) \text{ such that} \\ \omega|_{e^+ \oplus e^-} \text{ is non degenerate} \\ \left. \ker \omega = \mathbb{R}A_1 \oplus \dots \oplus \mathbb{R}A_k \right\} \end{aligned}$$

and for each j , define the geometric A -structure σ_j of order 1 and type Σ_0

$$\begin{aligned} \sigma_j : F^1(M) &\rightarrow \Sigma_0 \\ (p, \xi) &\mapsto \xi^{-1}((X_1)_p, \dots, (X_k)_p, (E^+)_p, (E^-)_p, (d\alpha_j)_p) \end{aligned}$$

Suppose that each σ_j is rigid. This means that form some $l \geq 1$ we have

$$Aut_{pp}^{l+1}(\sigma_j) \xrightarrow{\pi_j^l} Aut_{pp}^l(\sigma_j)$$

is injective for all j . It is clear that

$$Aut_{pp}^l(\sigma) = \bigcap_j Aut_{pp}^l(\sigma_j).$$

Thus the map

$$\bigcap_j Aut_{pp}^{l+1}(\sigma_j) \rightarrow \bigcap_j Aut_{pp}^l(\sigma_j)$$

is injective and thus σ is rigid.

The following lemma proves that σ_j is rigid. □

Lemma 4.1.4. For $j = 1, 2$, we have the inclusion $\text{Aut}_{pp}^j(\sigma_j) \hookrightarrow \text{Aut}_{pp}^j(g_j)$, for every $p \in M$, where g_j is understood as an A -structure and σ_j is 1-rigid.

Proof. The inclusion is quite clear. To avoid cluttering the notation, we shall denote $\sigma = \sigma_j$ and $g = g_j$. To understand that σ is rigid, we consider Σ_σ and Σ_g , the Σ 's of the geometric structures σ and g respectively. Suppose that there exists a map $\rho : \Sigma_\sigma \rightarrow \Sigma_g$ ($GL(\mathbb{R}^m)$ -invariant) such that $\rho \circ \sigma = g$. And suppose that g is r -rigid. Then, consider $\phi \in \text{Aut}^k(\sigma)$. We have the following diagram

$$\begin{array}{ccc} F^k(M) & \xrightarrow{\sigma} & \Sigma_\sigma \\ \downarrow \phi & \searrow g & \downarrow \rho \\ F^k(M) & \xrightarrow{g} & \Sigma_g \end{array}$$

It is clear that ϕ is also an automorphism of g , moreover, if ψ is an automorphism of g^l , the diagram above induces a diagram

$$\begin{array}{ccc} F^{k+l}(M) & \xrightarrow{\sigma^l} & \Sigma_\sigma^l \\ \downarrow \psi & \searrow g^l & \downarrow \rho \\ F^{k+l}(M) & \xrightarrow{g^l} & \Sigma_g^l \end{array}$$

We obtained the following situation

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \text{Aut}^{r+1}(\sigma) & \xrightarrow{p_k^{r+1}(\sigma)} & \text{Aut}^r(\sigma) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \text{Aut}^{r+1}(g) & \xrightarrow{p_k^{r+1}(g)} & \text{Aut}^r(g) & \longrightarrow & \cdots \end{array}$$

As both the vertical arrows and $p_k^{r+1}(g)$ are injective, then, so are $p_k^{r+1}(\sigma)$, as we wanted.

It remains to construct the map ρ :

$$(A_1, \dots, A_k, e^+, e^-, \lambda) \mapsto \begin{bmatrix} \bar{\lambda} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where we understand λ as a bilinear form with matrix $\bar{\lambda}$ and use the identification $\mathbb{R}^m = e^+ \oplus e^- \oplus \mathbb{R}A_1 \oplus \cdots \oplus \mathbb{R}A_k$. $\square$

Remark 4.1.5. The previous lemma generalizes in the following way. Consider two geometric structures on M^m σ_1 and σ_2 of types Σ_1 and Σ_2 and orders r_1 and r_2 respectively. Suppose that for some $r \geq 0$ and $s_1, s_2 \geq 0$ such that $r_1 + s_1 = r_2 + s_2 = r$ there exists a $Gl^r(m)$ -equivariant map

$$\rho : \Sigma_1^{s_1} \rightarrow \Sigma_2^{s_2}$$

such that $\rho \circ \sigma_1^{s_1} = \sigma_2^{s_2}$. Then, if σ_2 is rigid, so is σ_1 .

Corollary 4.1.6. The pseudo-group of local diffeomorphisms of M that preserves the distributions $E^\pm$ and X_j and the 1-forms α_j have a dense orbit Ω .

Proof. This follows from the topological transitivity of the action and Gromov's open dense orbit theorem.. $\square$

Definition 4.1.7. The open and dense set $\Omega \subset M$ where the pseudo-group $\text{Aut}^{loc}(\sigma)$ acts transitively will be called the domain of the geometric structure.

Now, for a given Anosov element X with flow ϕ_t , we consider the strong stable and unstable leaves through p , $F_X^\pm(p)$.

$$F^\pm(p) = F_X^\pm(p) = \{q \in M \mid \lim_{t \rightarrow \mp\infty} d(\phi_t(q), \phi_t(p)) = 0\}$$

where d is the distance function associated with the riemmanian metric.

Lemma 4.1.8. For a given Anosov element X , consider the set

$$\Delta = \Delta_X = \{p \in M \mid F_X^+(p) \subset \Omega \text{ and } F^-(p) \subset \Omega\}$$

Then, Δ is dense in M .

Proof. First, we notice that compact orbits are dense in M (they are dense in the non wandering set, which we have proved to be all M). As Ω is open and dense, it follows that compact orbits are dense in the entire Ω .

We shall prove that Δ contains every compact orbit in Ω . It is known that the Anosov foliations depends only on the open connected cone of Anosov elements we chose. Let $p \in \Omega$ with a compact orbit. We can take an Anosov element X' with flow ϕ'_t in the same cone as X such that $\phi'_{t_0}(p) = p$.

Now, let $q \in F^+(p)$, then,

$$\begin{aligned} 0 &= \lim_{t \rightarrow +\infty} d(\phi'_t(q), \phi'_t(p)) = \lim_{n \rightarrow \mp\infty} d(\phi'_{nt_0}(q), \phi'_{nt_0}(p)) \\ &= \lim_{n \rightarrow \mp\infty} d(\phi'_{nt_0}(q), p) \end{aligned}$$

thus $\varphi'_{n_0}(q) \rightarrow p$. As Ω is open, and invariant by the action, it follows that $q \in \Omega$. We have proved that $F^+(p) \subset \Omega$. A similar argument proves the case $F^-(p)$.

□

4.2 The model space

Fix a point $v_0 \in \Omega$ with compact orbit, and let $\mathfrak{g}'$ be the set of germs of Killing vector fields at v_0 . Let $\mathfrak{h}'$ be the Lie algebra of the isotropy group H' of v_0 . Let $\hat{G}'$ be the connected, simply connected Lie group associated to $\mathfrak{g}'$ and $\hat{H}'$ be the connected subgroup of $\hat{G}'$ associated with $\mathfrak{h}'$. Our goal in this section is to prove that $\hat{H}'$ is closed in $\hat{G}'$. To do this, we must study the Lie algebra $\mathfrak{g}'$ in great detail. Our main tool for this study will come from a collection of connections ∇^j on TM which are "compatible" with the geometric structure.

On Subsection 4.2.1, we build this connections and give some of their properties. These connections will induce connections the determinant bundle $\Lambda E^\pm$ (bundle of volume forms on $E^\pm$) (Subsection 4.2.2). This connections will turn out to be flat (Theorem 4.2.8, Definition 4.2.9 Theorem 4.2.10, Theorem 4.2.23)

On Subsection 4.2.3, we consider lie algebra $\mathfrak{g}'$ and using parallel sections of the determinant bundle $\Lambda E^\pm$, we construct $\mathfrak{g}$ the semisimple part of the Levy decomposition of $\mathfrak{g}'$ (Lemma 5.1.1).

On subsection 4.2.4 we finally show that $\hat{H}'$ is closed in $\hat{G}'$ (Proposition 4.2.36) and construct a model for the open dense subset Ω (Lemma 4.2.44).

4.2.1 An adapted connection

In this section we define a collection of affine connections on M which are compatible with our geometric structure, that is, for which the Killing vector fields are infinitesimal affine transformations. This will allow us to use Lemma 2.3.15 to better understand our geometric structure. We also obtain an explicit formula for the torsion, and obtain some properties for the curvature, which we shall use later to prove that the Lie algebra of Killing vector fields is reductive. Our definition of the connection is inspired by the definition given at (BENOIST; FOULON; LABOURIE, 1992).

Consider a k -contact Anosov action on M .

Lemma 4.2.1. For every $j \in \{1, \dots, k\}$, and real valued linear functionals $C_j^\pm : \mathbb{R}^k \rightarrow \mathbb{R}$ there exists unique smooth connection ∇^j on M that satisfies

$$\nabla^j d\alpha_j = 0 ; \nabla^j \alpha_i = 0 \quad \forall i ; \nabla^j(E^\pm) \subset E^\pm ; \nabla^j T\phi \subset T\phi \quad (4.1)$$

and for $X \in \Gamma(M, T\phi)$, $Z^\pm \in \Gamma(M, E^\pm)$

$$\nabla_{Z^+}^j Z^- = p^-([Z^+, Z^-]) \quad (4.2)$$

$$\nabla_{Z^-}^j Z^+ = p^+([Z^-, Z^+]) \quad (4.3)$$

$$\nabla_X^j Z^\pm = [X, Z^\pm] + C_j^\pm(X)Z^\pm \quad (4.4)$$

Where $p^\pm$ denotes the projection $TM \rightarrow E^\pm$ and we use the natural identification $\mathbb{R}^k = T_p\phi = \mathbb{R}X_1(p) \oplus \cdots \oplus \mathbb{R}X_k(p)$ to compute $C^\pm(X)$.

Proof. Notice that a connection can be seen as a bilinear map

$$\nabla : \Gamma(M, TM) \times \Gamma(M, TM) \rightarrow \Gamma(M, TM)$$

which satisfies a Leibniz condition

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y ; \quad \forall f \in C^\infty(M, \mathbb{R})$$

and is $C^\infty(M, \mathbb{R})$ -linear in the first coordinate. The smooth splitting $TM = E^+ \oplus RX_1 \oplus \cdots \oplus \mathbb{R}X_k \oplus E^-$ induces a splitting

$$\Gamma(M, TM) = \Gamma(M, E^+ \oplus) \Gamma(M, RX_1) \oplus \cdots \oplus \Gamma(M, \mathbb{R}X_k) \oplus \Gamma(M, E^-)$$

and bi-linearity means that we can define our connection on each component of this splitting.

Notice that the conditions (4.2), (4.3), (4.4) defines a bilinear map on appropriate subspaces of $\Gamma(M, TM) \times \Gamma(M, TM)$, which satisfies the Leibniz condition and is $C^\infty(M, \mathbb{R})$ -linear in the first coordinate, it remains to define ∇ on $\Gamma(M, E^+) \times \Gamma(M, E^+)$, $\Gamma(M, E^-) \times \Gamma(M, E^-)$ and on $\Gamma(M, TM) \times \Gamma(M, \mathbb{R}X_i)$, that is, we must specify, for $Z^\pm, G^\pm \in \Gamma(M, E^\pm)$ and $G \in \Gamma(M, TM)$, the vector fields

$$\nabla_{G^+}^j Z^+ ; \quad \nabla_{G^-}^j Z^- \quad \text{and} \quad \nabla_G^j X_j$$

We actually will see that the conditions (4.1) - (4.4) forces what must be these vector fields, and therefore the obtained connection is unique.

First, we shall prove that a connection ∇^j which satisfies the hypothesis of the Lemma 4.2.1, also satisfies $\nabla^j X_i = 0$ for every $1 \leq i \leq k$. This is of course bilinear and satisfies the Leibniz condition. For the sake of clearer notation, we shall avoid using the j superscript on the connection, and simply denote ∇ instead of ∇^j .

From $\nabla \alpha_i = 0$ we have, for any vector fields Y, Z ,

$$0 = (\nabla_Y \alpha_i)(X_s) = Y(\alpha_i(X_s)) - \alpha_i(\nabla_Y X_s) = -\alpha_i(\nabla_Y X_s)$$

and from $\nabla d\alpha_j = 0$ and the fact that $X_s \in \ker(d\alpha_j)$ we obtain

$$\begin{aligned} 0 &= ((\nabla_Y d\alpha_j)(X_s, Z)) = Y(d\alpha_j(X_s, Z)) - d\alpha_j(\nabla_Y X_s, Z) - d\alpha_j(X_s, \nabla_Y Z) \\ &= -d\alpha_j(\nabla_Y X_s, Z) = -(i_{\nabla_Y X_s} d\alpha_j)(Z) \end{aligned}$$

Thus, $i_{\nabla_Y X_s} d\alpha_j = 0$ and $i_{\nabla_Y X_s} \alpha_i = 0$ therefore $i_{\nabla_Y X_s} dM_j = 0$. But dM_j is a volume form and, thus, $\nabla_Y X_s = 0$.

Now, we just have to define $\nabla_{G^\pm} Z^\pm$. For this, we notice that $\nabla d\alpha_j = 0$ means that for every vector field A, B, C we have

$$0 = (\nabla_C d\alpha_j)(A, B) = \mathcal{L}_C(d\alpha_j(A, B)) + d\alpha_j(\nabla_C A, B) - d\alpha_j(A, \nabla_C B)$$

But $d\alpha_j$ restricted to $E^+ \oplus E^-$ is non degenerate, thus we define $\nabla_{G^\pm} Z^\pm$ to be the unique vector field tangent to $E^\pm$ which satisfies.

$$d\alpha_j(\nabla_{G^\pm} Z^\pm, Y^\mp) = \mathcal{L}_{G^\pm}(d\alpha_j(Z^\pm, Y^\mp)) - d\alpha_j(Z^\pm, \nabla_{G^\pm} Y^\mp)$$

This is of course bilinear and $C^\infty(M, \mathbb{R})$ -linear on the $G^\pm$ coordinate. It remains to see that it satisfies the Leibniz condition: Let f be a smooth function on M , then by definition

$$\begin{aligned} d\alpha_j(\nabla_{G^\pm} fZ^\pm, Y^\mp) &= \mathcal{L}_{G^\pm}(d\alpha_j(fZ^\pm, Y^\mp)) - d\alpha_j(fZ^\pm, \nabla_{G^\pm} Y^\mp) \\ &= \mathcal{L}_{G^\pm}(fd\alpha_j(Z^\pm, Y^\mp)) - fd\alpha_j(Z^\pm, \nabla_{G^\pm} Y^\mp) \\ &= \mathcal{L}_{G^\pm}(f)(d\alpha_j(Z^\pm, Y^\mp)) + f\mathcal{L}_{G^\pm}(d\alpha_j(Z^\pm, Y^\mp)) - fd\alpha_j(Z^\pm, \nabla_{G^\pm} Y^\mp) \\ &= (d\alpha_j(\mathcal{L}_{G^\pm}(f)Z^\pm, Y^\mp)) + fd\alpha_j(\nabla_{G^\pm} Z^\pm, Y^\mp) \\ &= (\nabla_{fG^\pm} Z^\pm + \mathcal{L}_{G^\pm}(f)Z^\pm, Y^\mp) \end{aligned}$$

and thus $\nabla_{G^\pm} fZ^\pm = \nabla_{fG^\pm} Z^\pm + \mathcal{L}_{G^\pm}(f)Z^\pm$ as wanted.

□

Remark 4.2.2. Notice that, by construction, for all j , the connection ∇^j defined above is compatible with our geometric structure, that is, ∇^j is invariant by $\text{Aut}^{loc}(\sigma)$, moreover, the smoothness of the invariant bundles means that this connection is smooth.

Lemma 4.2.3. Let T^j be the torsion of the connection ∇^j above, denote by $C_{ij}^\pm = C_j^\pm(X_i)$ and $S_{ij} = C_{ij}^+ p^+ + C_{ij}^- p^-$. Then,

$$\begin{aligned} T^j(A, B) &= \sum_i (\alpha_i([p^+ B, p^- A] + [p^- B, p^+ A])X_i + \alpha_i(A)S_{ij}B - \alpha_i(B)S_{ij}A) \\ &= \sum_i d\alpha_i(A, B)X_i + \alpha_i(A)S_{ij}B - \alpha_i(B)S_{ij}A \end{aligned}$$

and $\nabla^j T^j = 0$. Moreover, if K^j denotes the curvature of ∇^j , then for any $A, Z \in \Gamma(M, E^+)$ and $B \in \Gamma(M, E^-)$ we have:

$$K^j(A, B)Z = 0$$

Proof. The first step of the proof is to compute the torsion, by the bi-linearity of the torsion tensor, we can decompose the tangent space and compute the torsion on each pair of components.

To show that $\nabla^j T^j = 0$ we show, using some straightforward computations, that $\nabla_Z^j T^j(A, B) = \sum c_i X_i$ for some functions $c_i = c_i(A, B, Z)$ and then, we use the Bianchi identity to conclude that $\nabla_Z^j T^j(A, B)$ must be tangent to $E^+ \oplus E^-$. The Bianchi Identity will also be used to compute $K^j(A, B)Z$ for $A, Z \in \Gamma(M, E^+)$ and $B \in \Gamma(M, E^-)$.

Let's compute the torsion. In what follows, we shall omit the index j and write $\nabla = \nabla^j$
 $T = T^j$ and $C^\pm = C_j^\pm$

First, note that for X tangent to $T\phi$ and $Z = Z^\pm \in \Gamma(M, E^\pm)$

$$\begin{aligned} T(X, Z) &= \nabla_X Z - \nabla_Z X - [X, Z] \\ &= [X, Z] + C^\pm(X)Z - [X, Z] = C^\pm(X)Z \end{aligned}$$

Moreover, for given 2-form ω and vector fields G, Z, Y we have

$$\begin{aligned} i_G \circ i_Z \circ i_Y d\omega &= \omega([Z, Y], G) + \omega(Z, [G, Y]) + \mathcal{L}_Y(\omega(Z, G)) - \mathcal{L}_Z(\omega(Y, G)) + \\ &\quad + \omega(Y, [Z, G]) + d(\omega(Y, Z))(G) \end{aligned}$$

So, if we take $\omega = d\alpha_j$ and $Y = Y^\pm$ and $Z = Z^\pm$, we know that $d\alpha_j(Y, Z) = 0$ and $dd\alpha_j = 0$, that is

$$\begin{aligned} 0 &= d\alpha_j([Z^\pm, Y^\pm], G) + d\alpha_j(Z^\pm, [G, Y^\pm]) + \mathcal{L}_{Y^\pm}(d\alpha_j(Z^\pm, G)) - \mathcal{L}_{Z^\pm}(d\alpha_j(Y^\pm, G)) + \\ &\quad + d\alpha_j(Y^\pm, [Z^\pm, G]) \end{aligned}$$

If we take $G = G^\mp$ we obtain

$$d\alpha_j(T(Y^\pm, Z^\pm), G^\mp) = 0$$

On the other hand,

$$T(Y^\pm, Z^\pm) = \nabla_{Y^\pm} Z^\pm - \nabla_{Z^\pm} Y^\pm + [Y^\pm, Z^\pm]$$

and thus, $T(Y^\pm, Z^\pm) \subset E^\pm$ and therefore $d\alpha_j(T(Y^\pm, Z^\pm), G^\mp) = 0$ implies

$$T(Y^\pm, Z^\pm) = 0.$$

Now, if we write $Id = p^+ + p^- + \sum_i \alpha_i(\cdot) X_i$ then

$$\begin{aligned}
T(Y^\pm, Z^\mp) &= p^\mp([Y^\pm, Z^\mp]) - p^\pm([Z^\mp, Y^\pm]) - [Y^\pm, Z^\mp] \\
&= (p^\pm + p^\mp - Id)([Y^\pm, Z^\mp]) = -\sum_i \alpha_i([Y^\pm, Z^\mp])X_j \\
&= \sum_j d\alpha_i(Y^\pm, Z^\mp)X_j
\end{aligned}$$

If we write $A = Z^+ + X + Z^-$ and $B = W^+ + Y + W^-$, then $d\alpha_i(A, B) = d\alpha_i(Z^+, W^-) + d\alpha_i(Z^-, W^+)$ and we obtain from the bi-linearity of T

$$\begin{aligned}
T(A, B) &= T(Z^+, W^-) + T(Z^-, W^+) + T(X, W^+ + W^-) - T(Y, Z^+ + Z^-) \\
&= \sum_i (d\alpha_i(Z^+, W^-) + d\alpha_i(Z^-, W^+))X_i + C^+(X)W^+ + C^-(X)W^- - C^+(Y)Z^+ - C^-(Y)Z^-
\end{aligned}$$

If we write $C_{ij}^\pm = C_j^\pm(X_i)$ then $C(X) = C_j^\pm(X) = \sum_i \alpha_i(X)C_{ij}^\pm$

If we write $S_{ij} = C_{ij}^+p^+ + C_{ij}^-p^-$

$$T^j(A, B) = \sum_i d\alpha_i(A, B)X_i + \alpha_i(A)S_{ij}B - \alpha_i(B)S_{ij}A$$

Now, let's prove that $\nabla T^j = 0$.

As before, we continue to omit the index j . In particular, we write S_i for S_{ij}

From $\nabla X_i = 0$ we have $\nabla_Z(fX_i) = \mathcal{L}_Z(f)X_i$. Moreover, the projections $p^\pm$ commutes with ∇ and thus:

$$\begin{aligned}
\nabla_Z(T(A, B)) &= \sum_i \left(\mathcal{L}_Z(d\alpha_i(A, B))X_j + \mathcal{L}_Z(\alpha_i(A))S_iB + \alpha_i(A)S_i\nabla_ZB - \right. \\
&\quad \left. - \mathcal{L}_Z(\alpha_i(B))S_iA + \alpha_i(B)S_i\nabla_ZA \right)
\end{aligned}$$

On the other hand, we have

$$0 = (\nabla_Z\alpha_i)(A) = \mathcal{L}_Z(\alpha_i(A)) - \alpha_i(\nabla_ZA)$$

so

$$\begin{aligned}
\nabla_Z(T(A, B)) &= \sum_i \left(\mathcal{L}_Z(d\alpha_i(A, B))X_j + \alpha_i(\nabla_ZA)S_iB + \alpha_i(A)S_i\nabla_ZB - \right. \\
&\quad \left. - \alpha_i(\nabla_ZB)S_iA + \alpha_i(B)S_i\nabla_ZA \right)
\end{aligned}$$

Thus,

$$\nabla_Z T(A, B) - T(\nabla_Z A, B) - T(A, \nabla_Z B) = \quad (4.5)$$

$$= \sum_i \left(\mathcal{L}_Z(d\alpha_i(A, B)) - d\alpha_i(\nabla_Z A, B) - d\alpha_i(A, \nabla_Z B) \right) X_i \quad (4.6)$$

$$= \sum_i ((\nabla_Z d\alpha_i)(A, B)) X_i \quad (4.7)$$

Notice that if either A or B belongs to the central direction, $T\phi$ the above formula is zero as we wanted. The formula also vanishes if both A and B belong to the (un)stable distribution. From symmetry, we can consider $A \in \Gamma(M, E^+)$ and $B \in \Gamma(M, E^-)$.

First we shall consider $Z = X_s$ for some $s \in \{1, \dots, k\}$.

In this case $d\alpha_i(\nabla_Z A, B) = -\alpha_i([Z, A] + C^+(Z)A, B)$ and $d\alpha_i(A, \nabla_Z B) = -\alpha_i(A, [Z, B] + C^-(Z)B)$ and thus, as $C^+ + C^- = 0$ we obtain

$$\begin{aligned} \nabla_Z T(A, B) - T(\nabla_Z A, B) - T(A, \nabla_Z B) &= \\ &= -\sum_i \left(\mathcal{L}_Z(\alpha_i([A, B])) - \alpha_i([Z, A], B) + [A, [Z, B]] \right) X_i \\ &= -\sum_i \left(\mathcal{L}_Z(\alpha_i([A, B])) - \alpha_i([Z, [A, B]]) \right) X_i \\ &= -\sum_i \left(\mathcal{L}_Z(\alpha_i([A, B])) - [A, B](\alpha_i(Z)) - \alpha_i([A, B], Z) \right) X_i \\ &= -\sum_i d\alpha_i(Z, [A, B]) X_i = 0 \end{aligned}$$

Now we consider $Z = Z^+ \in \Gamma(M, E^+)$. Let U be a set and E a vector space. For any map $\Xi : U \times U \times U \rightarrow E$, we denote the cyclic sum over $a, b, c \in U$ as:

$$\mathfrak{S}(\Xi(a, b, c)) := \Xi(a, b, c) + \Xi(b, c, a) + \Xi(c, a, b)$$

using such notation, the well known Bianchi formula can be written as

$$\mathfrak{S}(K(A, B)Z) = \mathfrak{S}(T(T(A, B), Z)) + \nabla_Z T(A, B)$$

where K denotes the curvature of the connection, that is

$$K(A, B) = \nabla_A \nabla_B - \nabla_B \nabla_A - \nabla_{[A, B]}$$

Now, for Z, A tangent to E^+ we have $T(A, Z) = 0$, and for $B \in \Gamma(M, E^-)$ we obtain

$$\begin{aligned} \mathfrak{S}(T(T(A, B), Z)) &= T(T(A, B), Z) + T(T(B, Z), A) \\ &= \sum_i d\alpha_i(A, B) T(X_i, Z) + \sum_i d\alpha_i(B, Z) T(X_i, A) \\ &= \sum_i d\alpha_i(A, B) C_i^+ Z + d\alpha_i(B, Z) C_i^+ A \end{aligned}$$

and thus $\mathfrak{S}(T(T(A, B), Z))$ is tangent to E^+ .

As $\nabla E^\pm \subset E^\pm$ it follows that

$$\mathfrak{S}(K(A, B)Z) \in \Gamma(M, E^+ \oplus E^-)$$

As $\nabla_Z T(A, B) = \sum_i ((\nabla_Z d\alpha_i)(A, B))X_i$ it follows that $\nabla_Z T(A, B) = 0$.

Moreover, from

$$\mathfrak{S}(K(A, B)Z) = \mathfrak{S}(T(T(A, B), Z)) \in \Gamma(M, E^+)$$

and

$$\mathfrak{S}(K(A, B)Z) = \underbrace{K(A, B)Z}_{\in \Gamma(M, E^+)} + \underbrace{K(B, Z)A}_{\in \Gamma(M, E^+)} + \underbrace{K(Z, A)B}_{\in \Gamma(M, E^-)}$$

it follows that $K(Z, A)B = 0$ □

Corollary 4.2.4. For any $i \in \{1, \dots, k\}$ we have $\nabla^j d\alpha_i = 0$

Proof. It follows from the previous proof that

$$(\nabla_Z T)(A, B) = \sum_i ((\nabla_Z d\alpha_i)(A, B))X_i.$$

As $\nabla T = 0$ it follows that $(\nabla_Z d\alpha_i)(A, B) = 0$ □

Lemma 4.2.5. If K is the curvature tensor of ∇ then $K(E^\pm, E^\pm) = 0$

Proof. Let ω be any 2-form. Denote by $H(A, D) = (\nabla_B \omega)(C, D)$. Then

$$\begin{aligned} (\nabla_A \nabla_B \omega)(C, D) &= (\nabla_A H)(C, D) = AH(C, D) - K(\nabla_A C, D) - H(C, \nabla_A D) \\ &= A(B\omega(C, D) - \omega(\nabla_B C, D) - \omega(C, \nabla_B D)) \\ &\quad - (B\omega(\nabla_A C, D) - \omega(\nabla_B \nabla_A C, D) - \omega(\nabla_A C, \nabla_B D)) \\ &\quad - (B\omega(C, \nabla_A D) - \omega(\nabla_B C, \nabla_A D) - \omega(C, \nabla_B \nabla_A D)) \end{aligned}$$

and

$$(\nabla_{[A, B]}\omega)(C, D) = \mathcal{L}_{[A, B]}(\omega(C, D)) - \omega(\nabla_{[A, B]}C, D) - \omega(C, \nabla_{[A, B]}D)$$

thus

$$(K(A, B)\omega)(C, D) = \omega(K(B, A)C, D) + \omega(C, K(B, A)D)$$

in particular, for $\omega = d\alpha_i$, as $\nabla d\alpha_i = 0$ (and therefore $K(A, B)d\alpha_i = 0$), we have:

$$0 = d\alpha_i(K(B, A)C, D) + d\alpha_i(C, K(B, A)D)$$

From the previous lemma, $K(Y^\pm, Z^\pm)G^\mp = 0$ for any $Y^\pm, Z^\pm$ tangent to $E^\pm$ and $G^\mp$ tangent to $E^\mp$, and therefore, for any $W^\pm$ tangent to $E^\pm$,

$$d\lambda(K(Y^\pm, Z^\pm)W^\pm, G^\mp) = d\lambda(K(Y^\pm, Z^\pm)G^\mp, W^\pm) = 0$$

and thus, $K(Y^\pm, Z^\pm)W^\pm = 0$. Evidently $K(Y^\pm, Z^\pm)X_j = 0$ and we can conclude that $K(Y^\pm, Z^\pm) = 0$.

□

Lemma 4.2.6. Consider the curvature K as a 2-form taking values on $\text{End}(\mathbb{R}^{2n+k})$. As the connection preserves the splitting $TM = E^+ \oplus T\phi \oplus E^-$, we can consider the restriction of this 2-form to the sub-bundles $E^\pm$. In this case we have

$$K(X_l, Y)|_{E^\pm} = 0 \quad \forall l \in \{1, \dots, k\}, \quad Y \in TM$$

Proof. Once again we use the linearity of the curvature tensor on each coordinate to facilitate our computations. This allow us to break the proof in in three cases:

Case 1: $Y = X_i$

Case 1: $Y = X_i$

Case 2: $Y = Y^\mp$

Case 3: $Y = Y^\pm$

We consider a section $Z^\pm$ of $E^\pm$. Then,

$$\begin{aligned} \nabla_{X_l} \nabla_{X_i} Z^\pm &= \nabla_{X_l} ([X_i, Z^\pm] + C_i^\pm Z^\pm) \\ &= [X_i, [X_l, Z^\pm]] + C_l^\pm [X_i, Z^\pm] + C_i^\pm [X_l, Z^\pm] + C_i^\pm C_l^\pm Z^\pm \end{aligned}$$

and thus, from Jacobi Identity, it follows that

$$K(X_l, X_i)Z^\pm = [X_i, [X_l, Z^\pm]] - [X_l, [X_i, Z^\pm]] = [Z^\pm, [X_l, X_i]] = 0$$

Case 2: $Y = Y^\mp$

We have

$$\begin{aligned} \nabla_{X_l} \nabla_{Y^\mp} Z^\pm &= \nabla_{X_l} (p^\pm [Y^\mp, Z^\pm]) \\ &= [X_l, p^\pm [Y^\mp, Z^\pm]] + C_l^\pm p^\pm [Y^\mp, Z^\pm] \end{aligned}$$

and

$$\begin{aligned}\nabla_{Y^\mp} \nabla_{X_l} Z^\pm &= \nabla_{Y^\mp} ([X_l, Z^\pm] + C_l^\pm Z^\pm) \\ &= p^\pm [Y^\mp, [X_l, Z^\pm]] + C_l^\pm p^\pm [Y^\mp, Z^\pm]\end{aligned}$$

Thus

$$\begin{aligned}K(X_l, Y^\mp) Z^\pm &= \nabla_{X_l} \nabla_{Y^\mp} Z^\pm - \nabla_{Y^\mp} \nabla_{X_l} Z^\pm - \nabla_{[X_l, Y^\mp]} Z^\pm \\ &= [X_l, p^\pm [Y^\mp, Z^\pm]] - p^\pm [Y^\mp, [X_l, Z^\pm]] - p^\pm [[X_l, Y^\mp], Z^\pm] \\ &\quad + C_l^\pm p^\pm [Y^\mp, Z^\pm] - C_l^\pm p^\pm [Y^\mp, Z^\pm] \\ &= [X_l, p^\pm [Y^\mp, Z^\pm]] - p^\pm [Y^\mp, [X_l, Z^\pm]] - p^\pm [[X_l, Y^\mp], Z^\pm]\end{aligned}$$

Now, from the integrability, $[X_l, p^\pm A] = p^\pm [X_l, A]$, and therefore, we write

$$K(X_l, Y^\mp) Z^\pm = p^\pm ([X_l, [Y^\mp, Z^\pm]] - [Y^\mp, [X_l, Z^\pm]] - [[X_l, Y^\mp], Z^\pm])$$

From Jacobi's identity, it follows that $K(X_l, Y^\mp) Z^\pm = 0$

Case 3: $Y = Y^\pm$

Lets denote $\star := d\alpha_j(K(X_l, Y^\pm) Z^\pm, Z^\mp)$. It is clear that $K(X_l, Y^\pm) Z^\pm$ is completely determined by $\star$. To simplify, in what follows, we denote $X = X_l$, $Y = Y^\pm$, $Z^+ = Z^\pm$ and $Z^- = Z^\mp$.

From $\nabla d\alpha_j = 0$ we have

$$\begin{aligned}d\alpha_j(\nabla_X \nabla_Y Z^+, Z^-) &= X(d\alpha_j(\nabla_Y Z^+, Z^-)) - d\alpha_j(\nabla_Y Z^+, \nabla_X Z^-) \\ &= X(d\alpha_j(\nabla_Y Z^+, Z^-)) - (Y(d\alpha_j(Z^+, \nabla_X Z^-)) - d\alpha_j(Z^+, \nabla_Y \nabla_X Z^-))\end{aligned}$$

and thus

$$\begin{aligned}\star &= d\alpha_j(\nabla_X \nabla_Y Z^+, Z^-) - d\alpha_j(\nabla_Y \nabla_X Z^+, Z^-) - d\alpha_j(\nabla_{[X, Y]} Z^+, Z^-) \\ &= X(d\alpha_j(\nabla_Y Z^+, Z^-)) - Y(d\alpha_j(Z^+, \nabla_X Z^-)) + d\alpha_j(Z^+, \nabla_Y \nabla_X Z^-) - \\ &\quad (Y(d\alpha_j(\nabla_X Z^+, Z^-)) - X(d\alpha_j(Z^+, \nabla_Y Z^-)) - d\alpha_j(Z^+, \nabla_X \nabla_Y Z^-)) - \\ &\quad ([X, Y](d\alpha_j(Z^+, Z^-)) - d\alpha_j(Z^+, \nabla_{[X, Y]} Z^-)) \\ &= X(d\alpha_j(\nabla_Y Z^+, Z^-) + d\alpha_j(Z^+, \nabla_Y Z^-)) - \\ &\quad Y(d\alpha_j(\nabla_X Z^+, Z^-) + d\alpha_j(Z^+, \nabla_X Z^-)) + \\ &\quad [X, Y](d\alpha_j(Z^+, Z^-)) + d\alpha_j(Z^-, \nabla_Y \nabla_X Z^- - \nabla_X \nabla_Y Z^- - \nabla_{[Y, X]} Z^-)\end{aligned}$$

From the previous case, we know that the last term is zero, and from

$$Y(d\alpha_j(Z^+, Z^-)) = d\alpha_j(\nabla_Y Z^+, Z^-) + d\alpha_j(Z^+, \nabla_Y Z^-)$$

we obtain

$$\star = X(Y(d\alpha_j(Z^+, Z^-))) - Y(X(d\alpha_j(Z^+, Z^-))) - [X, Y](d\alpha_j(Z^+, Z^-)) = 0$$

which finishes the proof. $\square$

Lemma 4.2.7. The geodesics of ∇ which are tangent to the unstable (stable) distribution E^{uu} (E^{ss}) are complete.

Proof. Associated with the connection ∇ we have the concepts of exponential map and normal neighbourhoods, which are analogous to the Riemannian case. On such a neighbourhood, the exponential map is give us the geodesic. Since M is compact, there exists a constant $c > 0$ such that for every tangent vector Y of M with $\|Y\| < c$ we can integrate the geodesic with initial condition Y to a time equal or greater the one.

Now, suppose that we have $Y \in E^{uu}$, this means that there exists t_0 such that for $t < t_0$ we have

$$\|d\phi(ta, \cdot)Y\| < c$$

and through $d\phi(ta, \cdot)Y$ we can integrate the geodesic to a time equal or greater then one. But the action ϕ is affine (with respect to ∇), which means it transport geodesics, and therefore, we can integrate the geodesic through Y to a time equal or greater then one. As Y is arbitrary, the geodesics tangent to E^{uu} are complete. $\square$

4.2.2 Bundle of volume forms

We now study the determinant bundle of $E^\pm$, i.e the bundle $\Lambda^\pm$ of volume forms on $E^\pm$, that is, we consider the vector bundle $E^\pm$ and take the exterior n -power $\Lambda^n(E^\pm)^*$, where $n = \dim E^\pm$. It is easy to see that this bundle is of rank 1.

This bundle can be endowed with a connection $(\nabla^j)'$, induced from the connection ∇^j on $E^\pm$. We shall abuse notation and write $(\nabla^j)' = \nabla^j$.

Our goal is to prove

Theorem 4.2.8. The bundle $\Lambda^\pm$ endowed with the connection ∇^j has curvature tensor $\Omega^{j,\pm}$ identically zero.

A consequence of this theorem is that the associated bundle, $\tilde{\Lambda}^\pm$, on the universal cover $\tilde{M}$ of M , also has zero curvature and, therefore, there exists global parallel sections, unique up to

a constant. Those parallel sections will be of great importance for our study of the Lie algebra $\mathfrak{g}'$.

In what follows, we shall (usually) drop the superscript $\pm$, and whenever it is needed we shall choose the stable bundle E^+ , but know that all the constructions done can be done for both the stable and the unstable bundle.

The curvature $\Omega^j = \Omega^{j,\pm}$ on this bundle is given¹ by

$$\Omega^{j,\pm}(X, Y) = \text{Tr}(K(X, Y)|_{E^\pm})$$

Definition 4.2.9. Observe that Θ defines a bilinear, non degenerate, skew symmetric form on $E^+ \oplus E^-$. Therefore, we can define an operator $B^j = B^{j,\pm}$ on $E^+ \oplus E^-$ by

$$\Omega^j(Y, Z) = d\alpha_j(B^j Y, Z) \quad (4.8)$$

We can further extend B^j by setting

$$\alpha_i(B^j Z) = 0 \quad \forall i$$

Our strategy to prove that (Λ^+, ∇^j) is flat, is to show that B^j is zero. To see this, we will show that B^j is both nilpotent and semisimple.

Our goal for this section is to prove

Theorem 4.2.10. The operator B^j is nilpotent

Before proving this, let's build a primitive of Ω^j . Let $\zeta = \zeta^j$ be a section of Λ^+ . If Λ^+ is trivial, we can assume this section to be never vanishing, otherwise, we take a section ζ modulo sign (that is, a section of $|\Lambda^+|^2$).

Now, remember that the connection form β of a vector bundle E (with respect to a local frame s) is given by

$$\nabla^E s = s \cdot \beta$$

where β can be understood as a $\text{End}(E)$ valued form. On the case at hand, our bundle is unidimensional, and our local frame is just a single section ζ . We define thus the (real valued) one form $\beta = \beta^j$ by

$$\nabla_Z^j \zeta^j = \beta^j(Z) \zeta^j$$

¹ Check the Appendix B, equation (B.4)

² The fiber of $|\Lambda^+|$ is actually $\mathbb{R}/\{\pm Id\} = [0, +\infty)$

Finally, observe that the curvature form Ω of E (with respect to the local frame s) is given (in terms of the connection form β), by:

$$\Omega = d\beta + \beta \wedge \beta$$

On the case at hand, β is actually a "normal" real valued one form³, and therefore, $\beta \wedge \beta = 0$, thus,

$$\Omega^j = d\beta^j$$

Lemma 4.2.11. Consider the connection form β^j and curvature form Ω^j on Λ^+ . Consider also the constants $C_i^\pm = C_{ij}^\pm$ (equivalently, the linear map $C^\pm = C_j^\pm : \mathbb{R}^k \rightarrow \mathbb{R}$) involved in the definition of the connection (Lemma 4.2.1). We write dM_j for the volume form $\alpha_1 \wedge \cdots \wedge \alpha_k \wedge d\alpha_j^n$. Then, for appropriate choice of constants $C_{ij}^\pm$ we have

$$\int_M \beta^j(X_j) dM_j = 0$$

Before we prove this lemma, it will be necessary to give a new definition. This definition will be used to choose the appropriate constants $C_{ij}^\pm$

Definition 4.2.12. [Entropy of a invariant subbundle] Let F be a sub-bundle of TM that is invariant by a flow ψ_t . Choose a volume form dx of M and a never vanishing section ζ of the determinant bundle of F , that is of the volume forms of F (if necessary, quotient the bundle modulo sign). We define, for $t \in \mathbb{R}$

$$a_t = a_t(F) = \int_M \log |\det(d\psi_t)_F| dx$$

where the determinant is taken with the help of the section ζ . Now, observe that for $t, s \in \mathbb{R}$, we have:

$$\begin{aligned} a_{t+s} &= \int_M \log |\det(d\psi_{t+s})_F| dx \\ &= \int_M \log |\det(d(\psi_t \circ \psi_s))_F| dx = \int_M \log |\det(d\psi_t d\psi_s)_F| dx \\ &= \int_M \log |\det(d\psi_t)_F \det(d\psi_s)_F| dx = \int_M \log |\det(d\psi_t)_F| + \log |\det(d\psi_s)_F| dx = a_t + a_s \end{aligned}$$

This means that a_t is a continuous one parameter subgroup of $\mathbb{R}$. Therefore, there exists $\mathfrak{s} = \mathfrak{s}(F) \in \mathbb{R}$ such that

$$a_t = \mathfrak{s} \cdot \text{Vol}(M) \cdot t$$

where $\text{Vol}(M) = \int_M dx$. Notice that $\mathfrak{s}$ does not depend on the choice of the section. The quantity $\mathfrak{s}$ is called the entropy of the sub-bundle F .

³ More technically, as our bundle is one dimensional, then our connection form has values on a one dimensional Lie algebra, which is therefore, abelian, and thus, $\beta \wedge \beta(A, B) = [\beta(A), \beta(B)] = 0$

Now, we consider the entropy $\mathfrak{s}_j^\pm$ of the fiber bundles $E^\pm$ invariant by the flow ϕ_t^j of X_j (the volume form is dM_j). As $E^+ \oplus E^-$ admits a non degenerate bilinear antisymmetric form that is invariant by ϕ (the form Θ), we obtain that the entropy of $E^+ \oplus E^-$ is zero⁴, that is

$$\mathfrak{s}_j^+ + \mathfrak{s}_j^- = 0$$

We choose our constants $C_{ij}^\pm$ such that

$$\frac{\delta_{ij}\mathfrak{s}_j^\pm}{n} = C_{ij}^\pm \quad (4.9)$$

Proof of Lemma 4.2.11. In what follows we avoid using the index j that indicates which connection we are using. But whenever the index j is used, it indicates the very same index of the connection.

Let $\tau(t)$ be the parallel transport of ∇ along the orbits of the flow of X_{j_j} starting at the point p_0 . Let ϕ_t be the flow along X_j , and let $p = \phi_t(p_0)$. From the definition of ∇ , a vector field $Y \in E^+$ is parallel with respect to ϕ if

$$0 = \nabla_{\phi_t} Y = \nabla_{X_j} Y = [X_j, Y] + C_{jj}^+ Y$$

If we denote

$$s = C_{jj}^+ = \frac{\mathfrak{s}_j^+}{n}$$

Then Y is parallel with respect to ϕ if $[X_j, Y] + sY = 0$. If we write $Y_p = \tau(t)Y_0$ then,

$$\tau(t)|_{E^\pm} = e^{-st} d\phi_t$$

In fact:

$$\begin{aligned} \mathcal{L}_{X_j} Y &= \mathcal{L}_{X_j} (e^{-st} d\phi_t Y_0) \\ &= \frac{d}{dt} (e^{-st}) d\phi_t Y_0 + e^{-st} \mathcal{L}_{X_j} (d\phi_t Y_0) \end{aligned}$$

But $\mathcal{L}_{X_j} (d\phi_t Y_0) = 0$ because $(d\phi_t Y_0)_{\phi_s} = d\phi_{t+s} Y_0$ and thus

$$\lim_{s \rightarrow 0} \frac{d\phi_{-s} d\phi_t (Y_0)_{\phi_s(p_0)} - d\phi_t (Y_0)}{s} = \lim_{s \rightarrow 0} \frac{d\phi_{-s} d\phi_{t+s} (Y_0) - d\phi_t (Y_0)}{s} = \lim_{s \rightarrow 0} \frac{d\phi_t (Y_0) - d\phi_t (Y_0)}{s} = 0$$

⁴ Remember that a symplectic matrix S has determinant 1, and from

$$(d\alpha_j)_p(u, v) = (d\alpha_j)_{\phi_t(p)}(d\phi_t u, d\phi_t v)$$

we obtain

$$(\det(\phi_t)|_{E^+ \oplus E^-})^2 = 1$$

On the other hand $\frac{d}{dt}(e^{-st}) = -se^{-st}$, and thus, $[X_j, Y] = -sY$ and therefore Y is parallel as we wanted.

Now, let $\Delta = \Delta^j(t)$ be the determinant of the parallel transport restricted to E^+ (the section ζ is used to compute the determinant). We have

$$\Delta(t) = e^{-nst} \det(d\phi_t^v|_{E^+})$$

From $\nabla X_j = 0$ we obtain

$$\beta(X_j)\zeta(X_j) = (\nabla_{X_j}\zeta)(X_j) = X_j(\zeta(X_j))$$

Also, as ζ was chosen to be never vanishing, we have:

$$\begin{aligned} \beta^j(X_j) &= \frac{X_j(\zeta^j(X_j))}{\zeta^j(X_j)} \\ &= \frac{d}{dt} \log(\Delta(t)) \\ &= \frac{d}{dt} (\log(\det(d\phi_t)|_{E^+})) - ns \\ &= \frac{d}{dt} (\log(\det(d\phi_t)|_{E^+})) - \mathfrak{s}^\pm \end{aligned}$$

But, $\mathfrak{s}^+$ is defined by

$$\mathfrak{s}^+ \text{Vol}(M)t = \int_M \log(\det(d\phi_t)|_{E^+}) dM_j$$

Taking the derivative with respect to t on both sides we obtain

$$\int_M \beta^j(X_j) dM_j = 0$$

□

Lemma 4.2.13. For every $p \in \{1, \dots, n\}$, we have $(\Omega^j)^p \wedge d\alpha_j^{n-p} = 0$.

Proof. We only prove in the case $j = k$. The other cases are similar. It is clear that the form $\Omega = \Omega^k$ is invariant by the action, but by the topological transitivity of the action, there is a unique volume form invariant by it, and thus, there exists constants c_p such that

$$\alpha_1 \wedge \dots \wedge \alpha_k \wedge \Omega^p \wedge d\alpha_k^{n-p} = c_p dM_k$$

If these constants c_p are zero, the proof is finished. So we shall suppose that they are not zero and $\alpha_1 \wedge \dots \wedge \alpha_k \wedge \Omega^p \wedge d\alpha_k^{n-p}$ is a true volume form on M .

Thus, using $\Omega = d\beta$ and integrating by parts, we have:

$$\begin{aligned} c_p \int dM_k &= \int \alpha_1 \wedge \cdots \wedge \alpha_k \wedge \Omega^p \wedge d\alpha_k^{n-p} \\ &= \int \beta \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \Omega^{p-1} \wedge d\alpha_k^{n-p+1} \end{aligned}$$

Now, we write $\beta = \sum_j f_j \alpha_j + \eta$ where η vanishes on $T\phi$. Clearly⁵

$$\beta \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \Omega^{p-1} \wedge d\alpha_k^{n-p+1} = f_k \alpha_1 \wedge \cdots \wedge \alpha_k \wedge \Omega^{p-1} \wedge d\alpha_k^{n-p+1}$$

and from $\eta|_{T\phi} = 0$ it follows $\beta(X_k) = f_k$. And thus

$$\begin{aligned} c_p \int dM_k &= \int \beta^k(X_k) \alpha_1 \wedge \cdots \wedge \alpha_k \Omega^{p-1} \wedge d\alpha_k^{n-p+1} \\ &= c_{p-1} \int \beta(X^k) dM_k = 0 \end{aligned}$$

as wanted. □

We are finally ready to prove Theorem 4.2.10:

Proof of Theorem (4.2.10). First we notice that B^j leaves E^+ invariant (this follows from Lema 4.2.5). Consider now the eigenvalues $\xi_1, \dots, \xi_n$ of $B^j|_{E^+}$. It follows from the Lemma 4.2.13 that, for every $p \in \{1, \dots, n\}$ we have

$$\sum_{i_1 < \dots < i_p} \xi_{i_1} \cdots \xi_{i_p} = 0 \quad (4.10)$$

And thus, $\xi_i = 0$ for every i , and thus B^j is nilpotent. □

4.2.3 The Lie algebra $\mathfrak{g}'$

We have already studied the Lie algebra of Killing fields of a general rigid geometric structure in the sense of Gromov. Now, we devote this section to the study of the particular algebra that arises from our particular geometric structure σ associated with a contact pair and

⁵ We have seen that $K(X_s, \cdot) = 0$ for every $s \in \{1, \dots, k\}$, and thus, $i_{X_s} \Omega = 0$. Thus if we write $\beta = \sum_j f_j \alpha_j + \eta$, then

$$\begin{aligned} \beta \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \Omega^{p-1} \wedge d\alpha_k^{n-p+1} &= \\ &= f_k \alpha_1 \wedge \cdots \wedge \alpha_k \wedge \Omega^{p-1} \wedge d\alpha_k^{n-p+1} + \eta \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \Omega^{p-1} \wedge d\alpha_k^{n-p+1} \end{aligned}$$

but $\eta \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \Omega^{p-1} \wedge d\alpha_k^{n-p+1}$ is also an action invariant volume form, and thus differs from dM_k by a constant. As $i_{X_k} \eta \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \Omega^{p-1} \wedge d\alpha_k^{n-p+1} = 0$ it follows that this constant is null.

an Anosov action. In the first section we remember and fix some notations, we also show that the Killing vector fields are infinitesimal affine transformations. In the second section we show that the Lie algebra of Killing vector fields is reductive and in the third section, after some technical preparation we give a explicit construction of the semisimple part of the Levi decomposition.

4.2.3.1 Compatibility and the Isotropy group

From Gromov's open-dense orbit theorem (Theorem 2.2.17), we obtain an open dense set $\mathcal{U}$, where the pseudo-group $\text{Aut}^{loc}(\sigma)$ acts transitively. It is well known that compact orbits of topologically transitive Anosov actions are dense. Moreover, compact orbits are homeomorphic to a torus. We shall take a point $v_0 \in \mathcal{U}$ that has a compact orbit (do not confuse the orbit of the Anosov action with the orbit of local automorphism of the geometric structure). The tangent space $T_{v_0}M$ will be denoted by V_0 .

As before, we take $\mathfrak{g}'$ the Lie algebra of germs of killing vector fields of σ at v_0 . As previously observed, this is the Lie algebra of germs at v_0 of vector fields Y such that

$$\begin{aligned} [Y, X_j] &= 0 \\ \mathcal{L}_Y d\alpha_i &= 0 \end{aligned}$$

Notice that the first condition also implies $[Y, E^\pm] \subset E^\pm$;

We take $\mathfrak{h}' \subset \mathfrak{g}'$ the Lie subalgebra of (germs of) Killing vector fields that fixes v_0 . As we have seen, this is precisely the Lie algebra of the Lie group $H' = \text{Aut}_{v_0 v_0}^{loc}(\sigma)$.

We also take $\mathfrak{q}'^\pm$ to be the Lie subalgebras of the vector fields Y such that $Y_{v_0} \in (E^\pm)_{v_0}$. And finally, we take $\mathfrak{p}'^\pm$ to be the Lie subalgebra $\mathfrak{q}'^\pm \oplus \mathbb{R}X_1 \oplus \cdots \oplus \mathbb{R}X_k$.

Lemma 4.2.14. Every Killing vector field is an infinitesimal affine transformation for the connections $\nabla = \nabla^j$ defined on Lemma 4.2.1.

Proof. From Lemma 2.3.13, we must check that every $Y \in \mathfrak{g}'$ satisfies

$$[\mathcal{L}_Y, \nabla_Z] = \nabla_{[Y, Z]} \quad \text{for every vector field } Z \text{ on } M$$

that is, for every vector field W we have

$$0 = [\mathcal{L}_Y, \nabla_Z]W - \nabla_{[Y, Z]}W \tag{4.11}$$

$$= [Y, \nabla_Z W] - \nabla_Z[Y, W] - \nabla_{[Y, Z]}W \tag{4.12}$$

As the above expressions are linear in W , we just have to prove that 4.11 is valid for W tangent to each of the distributions $T\phi$, E^+ and E^- . The linearity on Z give us the same freedom.

The case $W = X_s$ for some s this is trivial, for $\nabla X_s = 0$ and as $Y \in \mathfrak{g}'$, then $[Y, X_s] = 0$.

For $W \in E^\pm$ and $Z = X_s$ for some s , we obtain

$$\begin{aligned} [Y, \nabla_Z W] - \nabla_Z [Y, W] - \nabla_{[Y, Z]} W &= \\ &= [Y, [X_s, W] + C_s^\pm W] - ([X_s, [Y, W]] + C_s^\pm [Y, W]) \\ &= [Y, [X_s, W]] + [X_s, [W, Y]] = [W, [X_s, Y]] = 0 \end{aligned}$$

Now, for $W \in E^\pm$ and $Z \in E^\mp$ we obtain

$$\begin{aligned} [Y, \nabla_Z W] - \nabla_Z [Y, W] - \nabla_{[Y, Z]} W &= \\ &= [Y, p^\pm [Z, W]] - p^\pm [Z, [Y, W]] - p^\pm [[Y, Z], W] \\ &= [Y, p^\pm [Z, W]] - p^\pm ([Z, [Y, W]] + [[Y, Z], W]) \\ &= [Y, p^\pm [Z, W]] - p^\pm ([Y, [Z, W]]) \end{aligned}$$

But, as $[Y, E^\pm] \subset E^\pm$, we have, that for any vector field G , $[Y, p^\pm G] = p^\pm [Y, G]$, and the above equations equals to zero.

Finally, it remains to see the case $Y, Z \in E^\pm$. For this we use the fact that $d\alpha = d\alpha_j$ is non degenerated. We take $G \in E^\mp$ and we shall prove that

$$d\alpha([Y, \nabla_Z W] - \nabla_Z [Y, W] - \nabla_{[Y, Z]} W, G) = 0$$

First we remember, that for $A \in E^\pm$ and $B \in E^\mp$ we have $d\alpha(A, B) = -\alpha([A, B])$. Moreover, for any vector field $A = A^+ + \sum_i a_i X_i + A^-$ where $A^\pm \in E^\mp$ we have

$$\alpha_j([Y, A]) = \alpha_j([Y, a_j X_j]) = Y(a_j) = Y(\alpha_j(A)) \quad (4.13)$$

Now we compute

$$\begin{aligned}
d\alpha([Y, \nabla_Z W], G) &= -\alpha([Y, \nabla_Z W], G) \\
&= -\alpha([Y, G], \nabla_Z W) + [G, \nabla_Z W], Y) \\
&= d\alpha([Y, G], \nabla_Z W) - Y(d\alpha(G, \nabla_Z W)) \\
&= Z(d\alpha([Y, G], W)) - d\alpha(\nabla_Z [Y, G], W) \\
&\quad - Y(Z(d\alpha(G, W))) + Y(d\alpha(\nabla_Z G, W)) \\
&= Z(d\alpha([Y, G], W)) - Y(Z(d\alpha(G, W))) \\
&\quad - d\alpha([Z, [Y, G]], W) + Y(d\alpha([Z, G], W)) \\
&= Y(Z(d\alpha(W, G))) - Z(\alpha([Y, G], W)) \\
&\quad + \alpha([Z, [Y, G]], W) - Y(\alpha([Z, G], W))
\end{aligned}$$

We also compute

$$\begin{aligned}
d\alpha(\nabla_Z [Y, W], G) &= Z(d\alpha([Y, W], G)) - d\alpha([Y, W], \nabla_Z G) \\
&= \alpha([Y, W], [Z, G]) - Z(\alpha([Y, W], G)) \\
&= \alpha([Y, [W, [Z, G]]]) + [W, [[Z, G], Y]] - Z(\alpha([Y, W], G)) \\
&= Y(\alpha([W, [Z, G]])) + \alpha([W, [Z, G], Y]) - Z(\alpha([Y, W], G))
\end{aligned}$$

and using the Jacobi identity for 4 terms

$$[x, [y, [z, w]]] + [y, [x, [w, z]]] + [z, [w, [x, y]]] + [w, [z, [y, x]]] = 0 \quad (4.14)$$

we obtain

$$\begin{aligned}
d\alpha(\nabla_{[Y, Z]} W, G) &= [Y, Z](d\alpha(W, G)) - d\alpha(W, \nabla_{[Y, Z]} G) \\
&= [Y, Z](d\alpha(W, G)) + \alpha([W, [[Y, Z], G]]) \\
&= [Y, Z](d\alpha(W, G)) + \alpha([G, [W, [Z, Y]]] + [Z, [Y, [G, W]]] + [Y, [Z, [W, G]]]) \\
&= [Y, Z](d\alpha(W, G)) + \alpha([G, [W, [Z, Y]]] + [Z, [Y, [G, W]]]) + Y(\alpha([Z, [W, G]]))
\end{aligned}$$

Summing every term we obtain

$$Z(Y(d\alpha(W, G))) + Z(\alpha([Y, W], G) + [[G, Y], W]) \quad (4.15)$$

$$+ Y(\alpha([W, G], Z)) + \alpha([G, [W, [Z, Y]]] + [Z, [Y, [G, W]]]) \quad (4.16)$$

$$+ \alpha([W, [Y, [Z, G]]]) + \alpha([W, [Z, [G, Y]]]) \quad (4.17)$$

Using the Jacobi identity and $\alpha_j([Y, A]) = Y(\alpha_j(A))$ the first line is zero and the last line simplifies to $\alpha([W, [G, [Z, Y]]])$. Now we just have to use the 4 terms Jacobi identity (4.14) to conclude that

$$[G, [W, [Z, Y]]] + [Z, [Y, [G, W]]] + [W, [G, [Z, Y]]] = [Y, [Z, [W, G]]]$$

Which reduces the above expression 4.15 to $Y(\alpha([W, G], Z)) + \alpha([Y, [Z, [W, G]]])$. From 4.13, this is zero. Which finishes the proof. $\square$

It follows, from Lemma 2.3.15 that we can identify $\mathfrak{g}'$ with a subset of $\text{End}(V_0) \oplus V_0$ via the map

$$\begin{aligned}\theta_o : \mathfrak{g}' &\rightarrow \text{End}(V_0) \times V_0 \\ Y &\mapsto ((A_Y)_{v_0}, Y_{v_0})\end{aligned}$$

where $A_Y = L_Y - \nabla_Y$, and with the following Lie algebra structure:

$$[(A, a), (B, b)] = ([A, B] - R(a, b), Ab - Ba + T(a, b))$$

where T denotes the torsion tensor and R denotes the curvature tensor. The following lemma allow us to better understand the map θ_o .

Lemma 4.2.15. Consider the map $i : H' \rightarrow GL(V_0)$ given by $i(\phi) = d\phi_{v_0}$. Then, this map is injective and di coincides with the restriction of θ_o to $\mathfrak{h}'$.

Proof. Injectiveness follows from rigidity. Now to prove the second assertion, we consider $Y \in \mathfrak{h}'$ and $\psi^t := \exp(tY)$. Let Z be a vector field around v_0 . As ψ^t fixes v_0 , the parallel transport in this orbit reduces to the identity, and therefore, $(\nabla_Y Z)_{v_0} = 0$. We have

$$di_e(Y) = \left. \frac{d}{dt} \right|_{t=0} i(\exp(tY)) = \left. \frac{d}{dt} \right|_{t=0} (\psi_*^t)$$

but, for a vector field Z around v_0

$$\lim_{t \rightarrow 0} \frac{1}{t} (\psi_*^t Z - Z) = \lim_{t \rightarrow 0} \frac{1}{-t} (\psi_*^{-t} Z - Z) = -(\mathcal{L}_Y Z)_{v_0} = (\nabla_Y Z)_{v_0} - (\mathcal{L}_Y Z)_{v_0} = (A_Y)_{v_0} Z_{v_0}$$

and thus,

$$di_e(Y) = (A_Y)_{v_0}$$

as we wanted. $\square$

This lemma allow us to identify $\mathfrak{g}'/\mathfrak{h}'$ with V_0 (and similarly, $\mathfrak{q}'^\pm/\mathfrak{h}'$ with $(E^\pm)_{v_0}$). We shall also identify $\mathfrak{g}'$ and $\mathfrak{h}'$ with their images $\theta_o(\mathfrak{g}')$ and $\theta_o(\mathfrak{h}')$. And also H' with $j(H')$.

Moreover, notice that while the map θ_o depends on the connection (so it would be more correct to denote θ_o^j), the restriction of the map to $\mathfrak{h}'$ does not.

Let φ_t be the flow of X given by an Anosov element. As the Anosov elements form an open cone on $\mathbb{R}^k$, we have some freedom to choose this element. We shall chose an element, such that v_0 is a periodic point of φ_t of period t_0 . Consider the map $(d\varphi_{t_0})_{v_0} \in H' \subset GL(V_0)$. As H' is algebraic, it has a finite number of connected components, and thus, there exists $n_0 \in \mathbb{N}$

such that $(d\varphi_{nt_0})_{v_0}$ is in the connected component (of H') of the identity. We set $l_0 = (d\varphi_{nt_0})_{v_0}$, and L_0 the logarithm of the hyperbolic part of the Iwasawa decomposition of l_0 ⁶. In particular $\exp L_0$ have positive eigenvalues.

Lemma 4.2.16. The following assertions are valid:

1. H' is an algebraic subgroup of $GL(V_0)$
2. $L_0 \in \mathfrak{h}'$
3. We can choose an iterate l_0^r , $r \geq 0$ of l_0 such that the eigenvalues of the corresponding L_0 (that is, the logarithm of the hyperbolic part of l_0^r) on $E_{v_0}^+$ (resp. $E_{v_0}^-$) are strictly negative (resp. positive).

Proof. 1. From the demonstration of the open-dense orbit theorem, it follows that

$$H' = \text{Aut}_{v_0 v_0}^{\text{loc}}(\sigma) = \text{Aut}_{v_0 v_0}^r(\sigma)$$

for some r . As $\text{Aut}_{v_0 v_0}^r(\sigma)$ is algebraic by construction, it coincides with its image on $GL(V_0)$.

2. As H' is algebraic, it contains both the hyperbolic and the elliptic part of its elements (Helgason ([HELGASON, 1979](#)), IX.§7 Lemma 7.1, or Seco et al. ([PATRÃO; SANTOS; SECO, 2008](#))).
3. This will follow from the Anosov property of the action. In fact, as $L_0 \in \mathfrak{h}'$, we use the identification θ_o and write $L_0 = (\tilde{L}_0, 0)$, where $\tilde{L}_0 = (\nabla_{L_0} - \mathcal{L}_{L_0})_{v_0}$. However, as we have seen (Lemma [4.2.15](#)), the restriction of θ_o to $\mathfrak{h}'$ coincides with the differential of $H' \ni h \mapsto dh_{v_0} \in GL(V_0)$.

Thus, $\tilde{L}_0$ acts on $Z_0 \in E_{v_0}$ precisely as the (hyperbolic part of the) differential of an Anosov element. Let us call this differential T .

While it may not be true that

$$\|T|_{E_{v_0}^+}\| \leq 1$$

It is true that

$$\|T^r|_{E_{v_0}^+}\| \leq 1$$

for some r . Thus, T^r restricted to $E_{v_0}^+$ have eigenvalues of module less than one, and therefore its hyperbolic part restricted to $E_{v_0}^+$ have eigenvalues on the open interval $]0, 1[$, and thus its logarithm have strictly negative eigenvalues.

A similar argument works for the $E_{v_0}^-$ case.

□

⁶ While it is not true, in general, that the logarithm of an element of a Lie group exist, the logarithm map is well defined on the hyperbolic part of the Iwasawa decomposition

4.2.3.2 $\mathfrak{g}'$ is reductive

Theorem 4.2.17. The Lie algebra $\mathfrak{g}'$ is reductive and its centre is

$$\mathcal{J} = \bigoplus_j \mathbb{R}X_j$$

This theorem will follow from the following lemma:

Lemma 4.2.18. The nil-radical of $\mathfrak{g}'$ is $\mathcal{J}$.

Proof of Theorem 4.2.17. First observe that the lemma implies that $\mathcal{J}$ is in fact the centre of $\mathfrak{g}'$, just notice that the center is in fact a nilpotent ideal of $\mathfrak{g}'$. Consider a solvable ideal $\mathfrak{a}$ of $\mathfrak{g}'$. Then there exists ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_l$ such that $[\mathfrak{a}_j, \mathfrak{a}_j] \subset \mathfrak{a}_{j+1}$ and

$$\mathfrak{a} = \mathfrak{a}_0 \geq \mathfrak{a}_1 \geq \dots \geq \mathfrak{a}_l \geq \mathfrak{a}_{l+1} = 0$$

In this setting, $\mathfrak{a}_l$ is an abelian ideal, and therefore is nilpotent. From Lemma 4.2.18. It follows $\mathfrak{a}_l \subset \mathcal{J}$.

Now, $[\mathfrak{a}_{l-1}, \mathfrak{a}_{l-1}] \subset \mathfrak{a}_l \subset \mathcal{J}$ and thus

$$[[\mathfrak{a}_{l-1}, \mathfrak{a}_{l-1}], \mathfrak{a}_{l-1}] \subset [\mathcal{J}, \mathfrak{a}_{l-1}] = 0$$

that is, $\mathfrak{a}_{l-1}$ is a nilpotent ideal of $\mathfrak{g}'$ and therefore $\mathfrak{a}_{l-1} \subset \mathcal{J}$. Proceeding by induction, we obtain $\mathfrak{a} \subset \mathcal{J}$ and thus $\mathcal{J}$ is in fact the radical of $\mathfrak{g}'$, which is therefore reductive. $\square$

Proof of Lemma 4.2.18. Notice that $\mathcal{J}$ is inside the centre $Z(\mathfrak{g}')$ by definition. Take a nilpotent ideal $\mathfrak{j}'$. As $\exp L_0$ is hyperbolic, this means that L_0 has real eigenvalues and we can write

$$\mathfrak{g}' = \bigoplus_{i \in \mathbb{R}} \mathfrak{g}'_i ; \quad \mathfrak{j} = \bigoplus_{i \in \mathbb{R}} \mathfrak{j}_i ; \quad T_{v_0}M = V_0 = \bigoplus_{i \in \mathbb{R}} V_{0,i}$$

where $\mathfrak{g}'_i$ (respectively $\mathfrak{j}_i$ and $V_{0,i}$) is the (generalized) eigenspace associated with the eigenvalue i of the action of L_0 on $\mathfrak{g}'$ (respectively $\mathfrak{j}$ and V_0).

Remember that we are identifying $\mathfrak{g}'$ with its image under the map $\theta_o : \mathfrak{g}' \rightarrow \text{End}(V_0) \oplus V_0$. Also remember that $\mathfrak{h}$ is identified with its image (via θ_o) on $\text{End}(V_0)$, and thus, we identify $L_0 = (L_0, 0)$. Now, for $i \neq 0$ let $(A, y) \in \mathfrak{j}_i$, then, for every $(A', y') \in \mathfrak{g}'_{-i}$ we have:

$$(A'', y'') =: [(A, y), (A', y')] \in \mathfrak{j}_0$$

Notice that $\mathfrak{j}_0 \subset \mathfrak{g}'_0$, and

$$\begin{aligned} \mathfrak{g}'_0 &= \{(Q, q) \in \mathfrak{g}'; [L_0, (Q, q)] = 0\} \\ &= \{(Q, q) \in \mathfrak{j}; ([L_0, Q], L_0 q) = 0\} \end{aligned}$$

but, the eigenvalues of L_0 restricted to $E^+ \oplus E^-$ are strictly non zero, and thus, $\mathfrak{g}'_0 \subset \mathfrak{h}' \oplus \mathcal{J}$. Now, as $(A, y) \in \mathfrak{j}_i$ for $i \neq 0$, we have that $L_0 y = iy$, and thus $y \in E^+ \oplus E^-$, it follows that, $\alpha_j(y) = 0$. And analogous for y' , that is, $(A', y') \in \mathfrak{g}'_{-i}$, and thus, $L_0 y' = -iy'$ and thus, $y' \in E^+ \oplus E^-$, therefore $\alpha_j(y') = 0$

On the other hand, $V_{0,0} \cap V_0^+ = 0$ ⁷ and, as $(A, y) \in \mathfrak{j}_i$ and $(A', y') \in \mathfrak{g}'_{-i}$, this means, that $Ay' = 0 = A'y$. Moreover, either $y = p^+ y$ and $y' = p^- y'$ or $y = p^- y$ and $y' = p^+ y'$.

Therefore,

$$\begin{aligned} (A'', y'') &= [(A, y), (A', y')] = ([A, A'] - R(y, y'), T(y, y')) \\ &= ([A, A'] - R(y, y'), \sum_j \lambda_j([y'^+, y^-] + [y'^-, y^+])X_j + \lambda_j(y)S_j(y') - \lambda_j(y')S_j(y)) \\ &= ([A, A'] - R(y, y'), \sum_j \lambda_j([y', y])X_j) \end{aligned}$$

Now, consider $\theta_o(X_j) = ((A_{X_j})_{v_0}, X_{j,v_0})$. As

$$\begin{aligned} A_{X_j} Y &= \mathcal{L}_{X_j} Y - \nabla_{X_j} Y \\ &= \mathcal{L}_{X_j} Y - [X_j, Y] - S_j(Y) = -S_j(Y) \end{aligned}$$

we have, that $\theta_o(X_j) = (-S_{j,v_0}, X_{j,v_0}) = (-S_j, X_j)$. Thus,

$$(A'', y'') - \sum_j \lambda_j([y', y])\theta_o(X_j) = \sum_j ([A, A'] - R(y, y') + \lambda_j([y', y])S_j, 0)$$

That is,

$$(A'', y'') \text{Mod } \mathfrak{a} = \sum_j ([A, A'] - R(y, y') + \lambda_j([y', y])S_j, 0)$$

Now, $\mathfrak{j}$ is a nilpotent ideal, and therefore the adjoint of its bracket has null trace for every invariant subspace stable (Foulon, P. and Labourie, F. (FOULON; LABOURIE, 1989). Proof of Proposition 3.7) by $\mathfrak{j}$, that is

$$\text{Tr}|_{E_{v_0}^\pm}(A'', y'') = 0$$

Notice that $\mathcal{J}$ is contained in the centre, and therefore it doesn't contribute to the trace, and thus

$$\text{Tr}|_{E_{v_0}^+}(A'', y'') \text{Mod } \mathcal{J} = 0$$

From the linearity of the trace, we have

$$0 = \text{Tr}|_{E_{v_0}^+}[A, A'] + \text{Tr}|_{E_{v_0}^+}R(y, y') + \sum_j \text{Tr}|_{E_{v_0}^+}\alpha_j([y', y])S_j$$

⁷ We just observe that $V_0^+ = \bigoplus_{i>0} V_{0,i}$.

The first term is clearly zero, while the trace of a projection is given by the dimension of the subspace it projects into, and thus $Tr_{E_{v_0}^\pm} S_j = nC_j^\pm$. We obtain

$$\begin{aligned} 0 &= d\lambda(B^j y, y') + \sum_i \alpha_i([y', y] nC_i^+) \\ &= d\alpha_j(B^j y, y') + n \sum_i C_i^\pm d\alpha_i(nC_i^+ y, y') \end{aligned}$$

But, we have chosen the constants $C_i^\pm = C_{ij}^\pm = \frac{\delta_{ij} s_j^\pm}{n}$ and thus

$$0 = d\alpha_j((B^j + \mathfrak{s}_j^+)y, y')$$

As this is true for every y' , it follows $(B^j + \mathfrak{s}_j^+)y = 0$. But $\mathfrak{s}_j^+ \neq 0$ and B^j is nilpotent (and thus doesn't have a non zero eigenvalue), therefore $y = 0$.

We conclude that $\mathfrak{j}_i \subset \mathfrak{h}'^8$. Thus,

$$\mathfrak{j} \subset \mathfrak{h}' \oplus \mathcal{I}$$

Now, $\mathfrak{h}'$ can't contain an ideal of $\mathfrak{g}'^9$, so $\mathfrak{h}' \cap (\mathfrak{j} \oplus \mathcal{I}) = 0$ and we obtain $\mathfrak{j} \subset \mathcal{I}$ as wanted. $\square$

4.2.3.3 The semisimple part of the Levi decomposition

4.2.3.3.1 Flatness of the determinant Bundle

Before we construct the semisimple part of the Levi decomposition we must first return to a technical aspect. We finish the proof of the Theorem 4.2.8

Definition 4.2.19. Let $\mathfrak{g}$ be a Lie algebra, a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ is a nilpotent subalgebra that is equal to its own normalizer, that is

$$[X, Y] \in \mathfrak{c} \quad \forall X \in \mathfrak{c} \implies Y \in \mathfrak{c}$$

Lemma 4.2.20. There exists a Cartan subalgebra $\mathfrak{c}'$ of $\mathfrak{g}'$ such that $\mathfrak{c}' \subset \mathfrak{h}' \oplus \mathcal{I}$

The proof will use the following classical Lie algebra theory, those results will be stated without proof.

⁸ We proved that for $(A, y) \in \mathfrak{j}_i$, $y = 0$

⁹ Let $(A, 0) \in \mathfrak{h}'$, and chose $q \in V_0$ such that $Aq \neq 0$, we have

$$[(A, 0), (0, q)] = (0, Aq) \notin \mathfrak{h}'$$

Lemma 4.2.21 (Bourbaki, (BOURBAKI, 2005), VII.§2.3 Proposition 10). Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a}$ an abelian Lie subalgebra and $\mathfrak{b}$ the commutant of $\mathfrak{a}$ in $\mathfrak{g}$. Then, the Cartan subalgebras of $\mathfrak{b}$ are the Cartan subalgebras of $\mathfrak{g}$ containing $\mathfrak{a}$.

(Proof of lemma (4.2.20)). We take $\mathfrak{a} = \text{Span}_{\mathbb{R}}\{L_0\}$, and $\mathfrak{b} := \{y \in \mathfrak{g}; [y, L_0] = 0\} = \mathfrak{g}'_0$. We take $\mathfrak{c}$ a Cartan subalgebra of $\mathfrak{g}'_0$. From the previous lemma, this is a Cartan subalgebra of $\mathfrak{g}'$ containing L_0 . As $\mathfrak{c} \subset \mathfrak{g}'_0 \subset \mathfrak{h}' \oplus \mathcal{J}$ the proof is finished. $\square$

Lemma 4.2.22. The linear operator B^j defined on 4.8 is semisimple.

Proof. As we have seen, there exists a Cartan subalgebra $\mathfrak{c}'$ of $\mathfrak{g}'$ contained in $\mathfrak{h}' \oplus \mathcal{J}$. From Lemma 3.3.25, $\mathfrak{c}'_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}'_{\mathbb{C}}$, and from Lemma 3.3.35, $(\mathfrak{c}'_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ is splitting.

Consider the corresponding root space decomposition:

$$\mathfrak{g}'_{\mathbb{C}} = \mathfrak{c}'_{\mathbb{C}} \oplus \bigoplus_{\lambda \in \Delta(\mathfrak{c}'_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})} (\mathfrak{g}'_{\mathbb{C}})^{\lambda}$$

As $\mathbb{C}$ is algebraically closed, $(\mathfrak{g}'_{\mathbb{C}})^{\lambda}$ is one-dimensional.

Now we identify $T_{v_0}M = V_0 = \mathfrak{g}'/\mathfrak{h}'$, and consider the action of B^j on V_0 , it commutes with the action of H' , for H' is the space of isometries that fixes v_0 . Therefore, it commutes with the action of $\mathfrak{c}'$ (It commutes with the action of X_j trivially, and thus with the action of $\mathfrak{h}' \oplus \mathcal{J}$).

We can consider the adjoint action of $\mathfrak{c}'_{\mathbb{C}}$ on $\mathfrak{g}'_{\mathbb{C}}$ and, thus, on $\mathfrak{g}'_{\mathbb{C}}/\mathfrak{h}'_{\mathbb{C}}$. As we observed, the root spaces on $\mathfrak{g}'_{\mathbb{C}}/\mathfrak{h}'_{\mathbb{C}}$ are unidimensional. As the action of B^j commutes the adjoint action of $\mathfrak{c}'_{\mathbb{C}}$, they share the same invariant subspaces and, therefore, the eigenspaces of B^j are also unidimensional. Thus, B^j is semisimple. $\square$

We are now ready to prove

Theorem 4.2.23 (Theorem 4.2.8). The bundle $\Lambda^{\pm}$ endowed with the connection ∇^j has zero curvature.

Proof. We defined the operator B^j by $\Omega^j(Y, Z) = d\alpha_j(B^j Y, Z)$. From Theorem 4.2.10 B^j is nilpotent, and from Lemma 4.2.22, it is also semisimple. Therefore $B^j = 0$ and $\Lambda^{\pm}$ has curvature zero. $\square$

Corollary 4.2.24. As the bundle $\tilde{\Lambda}^+$ is flat, it must have a unique (up to a constant) parallel section ζ_j .

4.2.3.3.2 A semisimple Lie algebra

In the previous paragraph, we showed the existence of an unique (up to a constant) parallel sections ζ_j of the determinant bundle $\tilde{\Lambda}^+$.

Definition 4.2.25. We define the maps $d\chi_j : \mathfrak{g}' \rightarrow \mathbb{R}$ by:

$$\forall Y \in \mathfrak{g}' \quad \mathcal{L}_Y(\zeta_j) = d\chi_j(Y)\zeta_j$$

The uniqueness of ζ_j means that this is a well defined map

Lemma 4.2.26. For $X \in \mathcal{J}$ and ζ a section of Λ^+ we have

$$\nabla_X^j \zeta - \nabla_X^i \zeta = (\alpha_i(X)\mathfrak{s}_i^\pm - \alpha_j(X)\mathfrak{s}_j^\pm)\zeta$$

Proof. First, we observe that $\nabla_X^j Y^\pm = [X, Y^\pm] + \alpha_j(X)C_{jj}^\pm Y^\pm$, and thus,

$$\nabla_X^j Y^\pm - \nabla_X^i Y^\pm = (\alpha_j(X)C_{jj}^\pm - \alpha_i(X)C_{ii}^\pm)Y^\pm$$

Now we take $Y_1, \dots, Y_n$ local basis for $E^\pm$, from the formula

$$\nabla_X \zeta(Y_1, \dots, Y_n) = X(\zeta(Y_1, \dots, Y_n)) - \sum_l \zeta(Y_1, \dots, \nabla_X Y_l, \dots, Y_n)$$

we obtain

$$\begin{aligned} (\nabla_X^j \zeta - \nabla_X^i \zeta)(Y_1, \dots, Y_n) &= \sum_l \zeta(Y_1, \dots, \nabla_X^i Y_l - \nabla_X^j Y_l, \dots, Y_n) \\ &= \sum_l (\alpha_i(X)C_{ii}^\pm - \alpha_j(X)C_{jj}^\pm) \zeta(Y_1, \dots, Y_l, \dots, Y_n) \\ &= (\alpha_i(X)\mathfrak{s}_i^\pm - \alpha_j(X)\mathfrak{s}_j^\pm) \zeta(Y_1, \dots, Y_l, \dots, Y_n) \end{aligned}$$

□

Proposition 4.2.27. Let us write $\zeta_i = f_{ij}\zeta_j$ then $X(f_{ij}) = 0$ if, and only if $\alpha_i(X)\mathfrak{s}_i = \alpha_j(X)\mathfrak{s}_j$.

Proof. From the previous lemma,

$$\begin{aligned} (\alpha_i(X)\mathfrak{s}_i^\pm - \alpha_j(X)\mathfrak{s}_j^\pm)\zeta_i &= \nabla_X^j \zeta_i - \nabla_X^i \zeta_i \\ &= \nabla_X^j \zeta_i = \nabla_X^j f_{ij} \zeta_j \\ &= X(f_{ij})\zeta_j + f_{ij} \nabla_X^j \zeta_j \\ &= \frac{X(f_{ij})}{f_{ij}} \zeta_i \end{aligned}$$

that is $X(f_{ij}) = 0$ if and only if

$$\alpha_i i(X) \mathfrak{s}_i - \alpha_j (X) \mathfrak{s}_j = 0$$

.

□

Remark 4.2.28. From the previous lemma, we obtained that, for $i \neq j$,

$$\frac{X^i(f_{ij})}{f_{ij}} = \mathfrak{s}_i$$

Lemma 4.2.29. For every $1 \leq j \leq k$, let $d\chi_j : \mathfrak{g}' \rightarrow \mathbb{R}$ be the character¹⁰, given by Definition 4.2.25. We have:

$$\mathcal{L}_{X^j} \zeta_j = \mathfrak{s}_j \zeta_j$$

Moreover $d\chi_j(L_0) = \text{Tr}_{|E^\pm}(L_0) > 0$.

Proof. First of all, notice that $d\chi_j$ is in fact a Lie algebra morphism. In fact,

$$\begin{aligned} d\chi_j([Y, Z]) \zeta_j &= \mathcal{L}_{[Y, Z]} \zeta_j = [\mathcal{L}_Y, \mathcal{L}_Z] \zeta_j \\ &= \mathcal{L}_Y \mathcal{L}_Z \zeta_j - \mathcal{L}_Z \mathcal{L}_Y \zeta_j \\ &= \mathcal{L}_Y (d\chi_j(Z) \zeta_j) - \mathcal{L}_Z (d\chi_j(Y) \zeta_j) \\ &= d\chi_j(Y) d\chi_j(Z) \zeta_j - d\chi_j(Z) d\chi_j(Y) \zeta_j = [d\chi_j(Y), d\chi_j(Z)] \zeta_j \end{aligned}$$

As wanted. Now, let's prove $d\chi_j(X^j) = \mathfrak{s}_j$.

We shall write

$$Y = (Y_0, \dots, Y_n),$$

$$Y^{(k)} = (Y_0, \dots, Y_{k-1}, Y_{k+1}, \dots, Y_n)$$

and for indexes $\alpha_1 < \dots < \alpha_l, l < n$

$$Y^{(\alpha_1 \dots \alpha_l)} = (Y_0, \dots, Y_{\alpha_1-1}, Y_{\alpha_1+1}, \dots, Y_{\alpha_l-1}, Y_{\alpha_l+1}, \dots, Y_n)$$

Let ∇ be a connection on $\Lambda^n E^\pm$ and ζ be a (closed) flat section of $\Lambda^n E^\pm$.

$$0 = \nabla_{Y_0} \zeta(Y^{(0)}) = Y_0(\zeta(Y^{(0)})) - \sum_k \zeta(Y_1, \dots, \nabla_{Y_0} Y_k, \dots, Y_n)$$

$$\begin{aligned} 0 &= d\zeta(Y) = \sum_{k=0}^n (-1)^k Y_k (\zeta(Y^{(k)})) + \sum_{i < k} (-1)^{i+k} \zeta([Y_i, Y_k], Y^{ik}) \\ &= \sum_{k=0}^n (-1)^k Y_k (\zeta(Y^{(k)})) + \sum_{0 < i < k} (-1)^{i+k} \zeta([Y_i, Y_k], Y^{ik}) + \sum_{k=1}^n (-1)^k \zeta([Y_0, Y_k], Y^{0k}) \end{aligned}$$

¹⁰ Once again we avoid the usage of the superscript " $\pm$ ". The more correct notation is $d\chi_j^\pm$

Also

$$\begin{aligned}
d(i_{Y_0}\zeta)(Y^{(0)}) &= \sum_{k=1}^n (-1)^{k-1} Y_k(\zeta(Y_0, Y^{(0k)})) + \sum_{0 < i < k} (-1)^{i+k} \zeta(Y_0, [Y_i, Y_k], Y^{(0ik)}) \\
&= Y_0(\zeta(Y)) - \sum_{k=0}^n (-1)^k Y_k(\zeta(Y^{(k)})) - \sum_{0 < i < k} (-1)^{i+k} \zeta([Y_i, Y_k], Y^{(ik)}) \\
&= Y_0(\zeta(Y)) + \sum_{k=1}^n (-1)^k \zeta([Y_0, Y_k], Y^{0k}) \\
&= \sum_{k=1}^n \zeta(Y_1, \dots, \nabla_{Y_0} Y_k - [Y_0, Y_k], \dots, Y_n)
\end{aligned}$$

Now, we notice that for $Y_0 = X_j$, $\nabla = \nabla^j$ and $\zeta = \zeta_j$ we have $\nabla_{X_j}^j Y = [X^j, Y] + C_{jj}^+ p^+ Y + C_{jj}^- p^- Y$, thus, if we take $Y_1, \dots, Y_n$ a basis of E^+ we obtain $\nabla_{X_j}^j Y_l - [X_j, Y_l] = C_{jj}^+ Y_l$ for every $l = 1, \dots, n$, that is

$$\mathcal{L}_{X_j} \zeta(Y^{(0)}) = d(i_{X_j} \zeta)(Y^{(0)}) = \sum_{k=1}^n \zeta(Y_1, \dots, C_{jj}^+ Y_k, \dots, Y_n) = n C_{jj}^+ \zeta(Y^{(0)})$$

but $n C_{jj}^+ = \mathfrak{s}_j (= \mathfrak{s}_j^+)$ and thus $\mathcal{L}_{X_j} \zeta_j = \mathfrak{s}_j \zeta_j$ as we wanted.

Now, for any $Y \in \mathfrak{h}'$ we have $d\chi(Y) = Tr_{|E^\pm}(Y)$. Here we understand Y as an element on $End(V_0)$. As this immersion actually depends on the connection¹¹, the trace may depend on the choice of Anosov element. However, as we have seen, when restricted to $\mathfrak{h}'$ the map θ_0 is actually the differential of the map $j : H' \rightarrow GL(V_0)$, $j(h) = dh_{v_0}$ which does not depend on the choice of the connection. Thus, for $Y \in \mathfrak{h}'$ we have

$$d\chi(Y) = Tr_{|E^\pm}(Y) \quad (4.18)$$

and the trace formula doesn't depend on the choice of Anosov element and associated connection, etc. $\square$

Corollary 4.2.30. We have:

$$X^i - \frac{\mathfrak{s}_i L_0}{d\chi_i(L_0)} \in \ker d\chi_i \cap (\ker d\chi_j)^c \quad i \neq j$$

Proof. First, we notice that $d\chi_1(L_0) = d\chi_2(L_0)$ and write $r_i = \frac{\mathfrak{s}_i}{d\chi_i(L_0)}$. We also write $f = f_{ij}$ (that is: $\zeta_i = f \zeta_j$).

That $X^i - r_i L_0 \in \ker d\chi_i$ is clear. Now notice that

$$\begin{aligned}
\mathcal{L}_{X_j - r_j L_0} \zeta_i &= \mathcal{L}_{X_j} f \zeta_j - r_j d\chi_i(L_0) \zeta_i \\
&= X_j(f) \zeta_j + f \mathcal{L}_{X_j} \zeta_j - r_j d\chi_i(L_0) \zeta_i \\
&= \left(\frac{X_j(f)}{f} + s_j - \frac{s_j d\chi_i(L_0)}{d\chi_j(L_0)} \right) \zeta_i = \frac{X_j(f)}{f} \zeta_i \neq 0.
\end{aligned}$$

$\square$

¹¹ Remember, the map $\theta_0 : \mathfrak{g}' \rightarrow End(V_0) \times V_0$ is defined as $\theta_0(Y) := (\mathcal{L}_Y - \nabla_Y, Y_{v_0})$

Corollary 4.2.31. For any $i, j \in \{1, \dots, k\}$, $i \neq j$ we have $d\chi_j(X_i) = 0$.

Proof. From previous lemmas, we have

$$\begin{aligned} \mathcal{L}_{X_j}\zeta_i &= \mathcal{L}_{X_j}(f_{ij}\zeta_j) = X_j(f_{ij})\zeta_j + f_{ij}\mathcal{L}_{X_j}\zeta_j \\ &= \left(\frac{X_j(f_{ij})}{f_{ij}} + \mathfrak{s}_j \right) \zeta_i \\ &= \left((\alpha_i(X_j)\mathfrak{s}_i - \alpha_j(X_j)\mathfrak{s}_j)\lambda^i(X_j) + \mathfrak{s}_j \right) \zeta_i = (0 - \mathfrak{s}_j + \mathfrak{s}_j)\zeta_i = 0 \end{aligned}$$

□

Remark 4.2.32. Notice that the Corollary 4.2.30 implies that the hyperplanes $\ker d\chi_j$, $j = 1, \dots, k$, are in general position, that is $\cap_j \ker d\chi_j$ have co-dimension k . To see this, we notice that for every j ,

$$\left(X^j - \frac{\mathfrak{s}_j L_0}{d\chi_j(L_0)} \right) \notin \cap_j \ker d\chi_j$$

It is clear that $\{X^1 - \frac{\mathfrak{s}_1 L_0}{d\chi_1(L_0)}, \dots, X^k - \frac{\mathfrak{s}_k L_0}{d\chi_k(L_0)}\}$ are linearly independent.

Remark 4.2.33. Consider the Lie algebra $\cap_j \ker d\chi_j$. Then, the map $\cap_j \ker d\chi_j \rightarrow \mathfrak{g}'/\mathcal{I}$ is an isomorphism. In fact, from the previous remark this map is indeed between spaces of the same dimension, thus, we just have to prove that it is an injection. Now, let $X = \sum_l b_l X_l$ be an element of $\mathcal{I}$. Then

$$d\chi_j(X) = \sum_l b_l d\chi_j(X_l) = b_j \mathfrak{s}_j$$

That is, $X \in \cap_j \ker d\chi_j \Leftrightarrow b_j = 0 \forall j$, that is the map is injective.

Corollary 4.2.34. The Lie algebra $\mathfrak{g}$ is semisimple

Proof. From the previous remark $\mathfrak{g} = \mathfrak{g}'/\mathcal{I}$ and thus is semisimple. □

We can also build a Lie group H , by taking characters $\chi_j : H' \rightarrow (\mathbb{R}, \cdot)$ defined by

$$h_* \zeta_j = \chi_j(h) \zeta_j$$

and taking $H = \cap_j \ker \chi_j$. It is clear that $\text{Lie}(H) = \mathfrak{g} \cap \mathfrak{h}' = \mathfrak{h}$. It is clear that H is a closed subgroup of H' . Moreover, as H' is algebraic, then so is H .

4.2.4 Building the Model Space

Our goal is to build a model space of our manifold M , that is, we shall build a homogeneous space $\hat{V} = G'/H'$ such that M is locally modelled on $\hat{V}$ by G' maps and the $(G', \hat{V})$ structure is complete. First let's remember the necessary ingredients.

- We have the Killing Lie algebra of our geometric structure, that is, the germs at v_0 (a point with a compact orbit) of vector fields Y such that $[Y, X_j] = 0$ and $\mathcal{L}_Y d\alpha_j = 0$. We remember that the first condition also implies, $[Y, E^\pm] \subset E^\pm$, and together with the second condition they implies $\mathcal{L}_Y \alpha_j = 0$. The connected simply connected associated Lie group is $\hat{G}'$.
- We have the subalgebra $\mathfrak{h}' \subset \mathfrak{g}'$ of vector fields Y such that $Y_0 = 0$. This is precisely the Lie algebra of the isotropy group of v_0 : $H' = \text{Aut}^{loc}(v_0)$.
- We have the subalgebras $\mathfrak{q}'^\pm$ of vector fields $Y \in \mathfrak{g}'$ such that $Y_{v_0} \in E^\pm$ and the Lie subalgebras $\mathfrak{p}'^\pm = \mathfrak{q}' \pm \mathcal{I}$
- We have the Lie algebra $\mathfrak{g}$ which is isomorphic to $\mathfrak{g}'/\mathcal{I}$, and a Lie group H such that

$$\text{Lie}(H) = \mathfrak{h} = \mathfrak{g} \cap \mathfrak{h}'$$

- We have the associated connected Lie subgroups $\hat{H}', \hat{G}, \hat{H}, \hat{Q}'^\pm \subset \hat{G}'$

Remark 4.2.35. Notice that if $\hat{G}_0$ is the connected, simply connected, Lie group associated with $\mathfrak{g}$, then $\hat{G}_0 \times \mathbb{R}^k$ is a connected, simply connected, Lie group with Lie algebra $\mathfrak{g} \oplus \mathbb{R}^k = \mathfrak{g}'$, and thus, $\hat{G}' = \hat{G}_0 \times \mathbb{R}^k$, and $\hat{G}_0 = \hat{G}$, that is $\hat{G}$ is also simply connected.

Proposition 4.2.36. The subgroup $\hat{H}'$ is closed in $\hat{G}'$

Before proving this proposition, let's see a technical lemma

Lemma 4.2.37. Consider $G' = G \times \mathbb{R}^k$ a Lie group and $H' \subset G'$ be a Lie subgroup. Then, H' is closed in G' if, and only if, $H' \cap G$ is closed in G .

Proof. Notice that we only need to prove the lemma for the case $k = 1$. In fact, if the lemma is true for $k = 1$, by induction, suppose that for $1 \geq l$ we have $H_0 \subset G_0 \times \mathbb{R}^l$ is closed if and only if $H_0 \cap G_0$ is closed. Then, let $H' \subset G \times \mathbb{R}^{l+1}$ such that $H' \cap H$ is closed, then, by induction, $H_0 = H' \cap G \times \mathbb{R}^l$ is closed, and thus, applying the induction hypothesis again we obtain H' is closed in $(G \times \mathbb{R}^l) \times \mathbb{R} = G \times \mathbb{R}^{l+1}$.

Now we prove the case $k = 1$.

Consider $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{R}$ the Lie algebra of G' , and $\mathfrak{h}'$ the Lie algebra of H' . If $\mathfrak{h}' \subset \mathfrak{g}$ then $H' \subset G$ and there is nothing to prove. Now, suppose that $\mathfrak{h}' \not\subset \mathfrak{g}$, then, there exists an element $v \in \mathfrak{h}$ with non zero $\mathbb{R}$ component. Take $R = \exp(\mathbb{R}v)$. The non zero $\mathbb{R}$ component of v implies that R is closed in H' and in G' . We have the inclusion $H'/R \subset G'/R$, and the identifications

$$H'/R = G \cap H \quad \text{and} \quad G'/R = G$$

As the quotient map $\pi : G' \rightarrow G'/R$ is continuous, the inverse image of closed sets are closed, and thus, $\pi^{-1}(G \cap H) = H'$ is closed.

□

Proof of Proposition 4.2.36. First, let's recall that H' is an algebraic subgroup of $GL(E_{v_0})$. We also have that $\mathfrak{g}'/\mathfrak{h}' = E_{v_0} = V_0$ therefore the map

$$\begin{aligned} j : H' &\rightarrow \text{Aut}(\mathfrak{g}') \\ h' &\mapsto (Y \mapsto h'_*(Y)), \end{aligned}$$

is injective¹².

That is, we can identify both H and H' with algebraic subgroups of $\text{Aut}(\mathfrak{g}')$ and therefore closed.

As $\mathfrak{g}$ is semisimple, the map $\text{Ad} : \hat{G} \rightarrow \text{Aut}(\mathfrak{g}')$ is a covering onto its image. This implies that $\text{Lie}(\hat{G}) = \text{Lie}(\text{Ad}(\hat{G}))$ and therefore

$$\text{Lie}(j(H)) = \text{Lie}(H) = \mathfrak{h} = \text{Lie}(\hat{H}) = \text{Lie}(\text{Ad}(\hat{H}))$$

And thus $(\text{Ad}(\hat{H}))_0 = j(H)_0$. As $\hat{H}$ is connected, that is $\hat{H}$ is closed in $\hat{G}$. By our previous lemma $\hat{H}'$ is also closed in $\hat{G}'$.

□

Remark 4.2.38. As $\hat{H}'$ is closed we can take the homogeneous space $\hat{V} = \hat{G}'/\hat{H}'$ we consider the base point $\hat{v}_0 \in \hat{V}$ the induced flows on $\hat{V}$ given by

$$\hat{\phi}_t^j(\hat{v}) = \exp(tX_j) \cdot \hat{v}$$

Clearly those flows commute and they define an action of $\mathbb{R}^k$ on $\hat{V}$. We shall call it the Anosov action on $\hat{V}$.

Remark 4.2.39. We consider the associated vector fields $\hat{X}_j$. We also consider $\hat{E}^\pm$ the tangent distribution to the foliation $\hat{\mathcal{F}}^\pm := \{g\hat{Q}^\pm/\hat{H}' | g\hat{Q}^\pm \in \hat{G}'/\hat{Q}^\pm\}$. We will call this the strong stable (unstable) foliations on $\hat{V}$.

It is clear that $T(\hat{V}) = \hat{E}^+ \oplus \mathbb{R}\hat{X}_1 \oplus \cdots \oplus \mathbb{R}\hat{X}_k \oplus \hat{E}^-$.

¹² In fact, consider $Y = (A, Y_0) \in \mathfrak{g}' \subset \text{End}(V_0) \times V_0$. Then, $h'_*Y = (dh'_{v_0} \circ A, dh'_{v_0}Y_0)$. But $\mathfrak{g}'/\mathfrak{h}' = V_0$ means that the composite map $\mathfrak{g}' \rightarrow \text{End}(V_0) \times V_0 \rightarrow V_0$ is surjective, and thus $h'_*Y = Y$ for all $Y \in \mathfrak{g}'$ implies $dh'_{v_0}Y_0 = Y_0$ for all $Y_0 \in V_0$ and thus $dh'_{v_0} = \text{Id}$. But we have already seen (Lemma 4.2.15) that the map $H' \ni h' \rightarrow dh'_{v_0} \in GL(V_0)$ is injective, and thus, h' must be the identity.

Finally, for $Y, Z \in \mathfrak{g}'$ the alternating bilinear maps

$$B_j(Y, Z) := (d\alpha_j)_{v_0}(Y_{v_0}, Z_{v_0})$$

is well defined and $\hat{H}'$ -invariant, which allow us to extend them to 2-forms ω_j on $\hat{V}$.

The map

$$\theta(\exp(Y) \cdot v_0) = \exp(Y) \cdot \hat{v}_0$$

is a local diffeomorphism between a neighbourhood of v_0 and a neighbourhood of $\hat{v}_0$ such that $\theta_* X_j = \hat{X}_j$, $\theta_* E^\pm = \hat{E}^\pm$ and $\theta_* \omega_i = d\alpha_i$.

The transitivity of $\text{Aut}^{loc}(\sigma)$ on Ω gives us local charts from Ω to $\hat{G}'/\hat{H}'$ which preserves sends $(E^\pm, X_j, d\alpha_j)$ to $(\hat{E}^\pm, \hat{X}_j, B_j)$.

Remark 4.2.40. The local charts we defined above do not yet define a $(\hat{G}', \hat{V})$ -structure on Ω , because we do not have any control of the transition maps, that is, we obtained an atlas

$$\{f_i : U_i \rightarrow V_i \subset \hat{V}\} \quad (4.19)$$

of Ω , and the change of coordinates of this atlas

$$f_i \circ f_j^{-1} : f_j(U_i \cap U_j) \rightarrow f_i(U_i \cap U_j)$$

preserves the $(\hat{E}^\pm, \hat{X}_j, B_j)$ -structure. However, we do not know if the transition maps $f_i \circ f_j^{-1}$ are the restriction of an element of $\hat{G}'$.

In what follows, we shall construct a (slightly) larger group G' , such that the atlas 4.19 will in fact be an atlas of a $(G', \hat{V})$ -structure on Ω .

Define the group G' of diffeomorphisms of $\hat{V}$ that preserves $\hat{E}^\pm$, $\hat{X}_j$, and ω_j for every j . This is a Lie group and its identity connected component is precisely the image of $\hat{G}'$ in $\text{Diff}(\hat{V})$. Define H' the isotropy group of $\hat{v}_0$ in G' . The following lemma proves that there is no confusion¹³.

Lemma 4.2.41. We have the following identifications

$$H' \equiv \text{Aut}^{loc}(\hat{v}_0) \equiv \{A \in \text{Aut}(\mathfrak{g}') | A(\mathfrak{q}^\pm) \subset \mathfrak{q}^\pm; A(\mathfrak{h}') \subset \mathfrak{h}'; A(X_j) = X_j \ \omega_j \circ (A \times A) = \omega_j \}$$

¹³ Remember that we have previously defined H' as the isotropy group of our chosen point $v_0 \in M$, that is $H' = \text{Aut}_{v_0}^{loc}(\sigma)$. As the map θ above is a local diffeomorphism from a neighbourhood of v_0 to a neighbourhood of $\hat{v}_0$ that preserves the structure, we identify $\text{Aut}^{loc}(v_0) = \text{Aut}^{loc}(\hat{v}_0)$

Proof. First we observe that $H' \subset \text{Aut}^{loc}(\hat{v}_0)$. This is clear, as global diffeomorphisms can be restricted to local ones.

Lets call the right side of the above equivalence H_1 . Remember that $\text{Aut}^{loc}(\hat{v}_0)$ is the group of germs at $\hat{v}_0$ of local diffeomorphisms of $\hat{V}$ that preserves $\hat{X}_j, \hat{E}^\pm$ and $d\alpha_j$, and thus, we can identify

$$\text{Aut}^{loc}(\hat{v}_0) = \text{Aut}_{v_0}^{loc}(\sigma)$$

As we have already seen (proof of Proposition 4.2.36), the map

$$\begin{aligned} j : \text{Aut}_{v_0}^{loc}(\sigma) &\rightarrow \text{Aut}(\mathfrak{g}') \\ h' &\mapsto (Y \mapsto h'_*(Y)), \end{aligned}$$

is injective. Thus we can understand $\text{Aut}^{loc}(\hat{v}_0)$ as a subgroup of $\text{Aut}(\mathfrak{g}')$. It is clear that the image of j is contained in H_1 . Thus we have the following inclusions

$$H' \subset \text{Aut}^{loc}(\hat{v}_0) \subset H_1$$

Now, as $\hat{G}'$ is simply connected,

$$\text{Aut}(\mathfrak{g}') = \text{Aut}(\hat{G}')$$

and thus, an element A of H_1 is also an automorphism of $\hat{G}'$. The condition $A(\mathfrak{h}') \subset \mathfrak{h}'$ implies that it preserves $\hat{H}'$, and thus induces an diffeomorphism of $\hat{V}$. The other conditions on A means that it preserves the A -structure, which shows that $H_1 \subset H'$. $\square$

Corollary 4.2.42. Every diffeomorphism from a connected open set of $\hat{V}$ to another that preserves $\hat{X}_j, \hat{E}^\pm$ and ω is a restriction of an element of G'

Proof. It suffices to prove in the case of a diffeomorphism that fixes $\hat{v}_0$. The corollary follows from the equality $H' = \text{Aut}^{loc}(\hat{v}_0)$. $\square$

Remark 4.2.43. Notice that we can identify $\hat{V} = \hat{G}'/\hat{H}' = G'/H'$. As H' is algebraic, it follows that H' and G' have a finite number of connected components. This means that up to finite covering of M , we can suppose that $G' = \hat{G}'$.

Lemma 4.2.44. The diffeomorphisms θ from an open set of Ω to an open set of $\hat{V}$ such that $\theta_*X_j = \hat{X}_j$, $\theta_*E^\pm = \hat{E}^\pm$ and $\theta^*\omega_j = d\alpha_j$ forms a maximal atlas of a $(G', \hat{V})$ - structure over Ω

Proof. By construction the domains of definition of those diffeomorphisms covers Ω , and from the previous lemma, it follows that the change of local coordinates are given by the elements of G' . $\square$

Corollary 4.2.45. For every simple connected open set $O \subset \Omega$ there exists an "developing map" $\theta : O \rightarrow \hat{V}$. That is, a local diffeomorphism θ such that $\theta_* X_j = \hat{X}_j$, $\theta_* E^\pm = \hat{E}^\pm$ and $\theta^* \omega = d\lambda$.

Proof. classical □

Remark 4.2.46. As we have already observed in the chapter introduction, this developing map is compatible with the Anosov action. It is clear that the induced action on the local model G'/H' is given by

$$\begin{aligned} \mathbb{R}^k \times G'/H' &\rightarrow G'/H' \\ (t_1, \dots, t_k, gH') &\mapsto gH' \exp(t_1 \hat{X}_1 + \dots + t_k \hat{X}_k) \end{aligned}$$

In particular, should Ω be equal to M , and the developing map $\tilde{M} \rightarrow G'/H'$ be a diffeomorphism, then, the canonical representation $\rho : \pi_1(M) \rightarrow G'$ give a diffeomorphism

$$M = \pi_1(M) \backslash \tilde{M} \rightarrow \rho(\pi_1(M)) \backslash G'/H'$$

which smoothly conjugates our original action with an quasi-algebraic one.

We can build a connection $\hat{\nabla}$ on $\hat{V}$ in a similar to the one on M . It is clear that G' is a group of affine transformation with respect to $\hat{\nabla}$. This connection is also compatible with the A -structures on M and $\hat{V}$, which allow us to prove the following lemma:

Lemma 4.2.47. The geodesics of $\hat{\nabla}$ tangents to the distributions $\hat{E}^+$ are complete

Proof. First we recall, from Lemma 4.1.8, that, for a given connected cone of Anosov elements on $\mathbb{R}^k$, the set

$$\Delta = \{p \in M \mid F^+(p) \subset \Omega \text{ and } F^-(p) \subset \Omega\}$$

where $F^\pm(p) = \{q \in M \mid \lim_{t \rightarrow \mp\infty} d(\varphi_t(q), \varphi_t(p)) = 0\}$ ¹⁴, is dense on M .

Consider $\hat{Y} \in \hat{E}^+$ with base-point $\hat{p} \in \hat{V}$. Lets prove that a geodesic with initial condition $\hat{Y}$, integrates to up to time greater then 1. There exists a connected, simple connected open set $O \subset \Omega$, and a developing map $\theta : O \rightarrow \hat{V}$ and $Y \in E_v^+$ such that, $\theta_* Y = \hat{Y}$. From the density of Δ and the transitivity of the pseudo-group $Aut^{loc}(\sigma)$, we can move the base point v of Y a little bit and suppose that $v \in \Delta$. Now, we have already proved that the geodesics in Ω tangents to E^+ are complete, thus, they integrate up to a time greater the one. In particular, the geodesic $t \rightarrow \gamma(t)$, starting from v with direction Y integrate up to time greater the one. As $\nabla E^+ \subset E^+$, we have $\dot{\gamma} \in E^+$, thus, γ never leaves the leaf $F^+(v)$. As $v \in \Delta$, $\gamma(t) \in \Omega$, and we can suppose that $\gamma([0, 1]) \subset O$ ¹⁵. The curve $t \mapsto \theta \circ \gamma$ is our desired geodesic. □

¹⁴ Notice that $F^\pm(p)$ only depends on the choice of open Anosov cone.

¹⁵ The developing map is defined for any simple connected open set, We just take O' to be a tubular neighbourhood of γ , and θ' the developing map of O' that coincides with θ on the point v

Remark 4.2.48. At first we may be tempted to just use the completeness of the geodesics in M . But we build our $(G', \bar{V})$ -structure on Ω , which means that the developing maps are defined on the universal cover $\tilde{\Omega}$, and not on $\tilde{M}$. That is, maybe our geodesic passes through a point in $M \setminus \Omega$, and in this case, we won't be able to develop our geodesic on $\bar{V}$.

Our next section will solve this problem by extending the $(G', \bar{V})$ -structure to all M .

4.3 Extending the Structure

First, we shall define appropriated local coordinates for M using the completeness of the connection along the strong leaves.

Let $\varphi_t^j, j = 1, \dots, k$ be the flows of the vector fields X_j . Define, for every $v \in M$ the map

$$\begin{aligned} \Psi_v : T_v M = E_v^+ \oplus \bigoplus_j \mathbb{R}X_j \oplus E_v^- &\rightarrow M \\ Y = Y^+ + t_1 X_1 + \dots + t_k X_k + Y^- &\mapsto \Psi_v(Y) = \varphi_{t_1}^1 \circ \dots \circ \varphi_{t_k}^k(\exp^\nabla(\tau_{Y^+}(Y^-))) \end{aligned}$$

where $\exp^\nabla$ is the exponential with respect to the connection ∇ and $\tau_{Y^+}(Y^-)$ is the parallel transport for time $t = 1$ along the geodesic $t \mapsto \exp^\nabla(tY^+)$ of the vector Y^- .

We define, in an analogous way, for every $\tilde{v} \in \tilde{M}$ and $\hat{v} \in \hat{V}$ the maps $\tilde{\Psi} : T_{\tilde{v}}\tilde{M} \rightarrow \tilde{M}$ and $\hat{\Psi} : T_{\hat{v}}\hat{V} \rightarrow \hat{V}$.

From the previous section we see that those maps can be defined even for arbitrary large $Y^+, \tilde{Y}^+$ and $\hat{Y}^+$.

Definition 4.3.1. An open set $O \subset M$ will be called A -star-shaped with respect to $v \in O$ if there exists an open set $U \subset T_v M$ such that

- Ψ_v is an diffeomorphism from U to O .
- If $Y = Y^+ + t_1 X_1 + \dots + t_k X_k + Y^-$ is in U , then, for every $s \in [0, 1]$, so are
 - $Y^+ + t_1 X_1 + \dots + t_k X_k + sY^-$
 - $sY^+ + t_1 X_1 + \dots + t_k X_k$
 - $t_1 X_1 + \dots + st_j X_j + \dots + t_k X_k$ for every $j \in \{1, \dots, k\}$

In particular, O is contractible and therefore simple connected.

We have defined the A -star-shaped open sets in such way that they are well behaved with respect to the affine transformations, that is, if O is an A -star-shaped open set with respect to v and θ is a developing map from O to $\hat{V}$, then

$$\theta \circ \Psi_v = \hat{\Psi}_{\theta(v)} \circ T_v \theta \quad (4.20)$$

Lemma 4.3.2. Let $v \in \Delta$ and O be an A -star-shaped open set with respect to v . Then, there exists an open dense subset $O' \subset O \cap \Omega$ which is also A -star-shaped with respect to v .

Proof. We consider the open set $U \subset T_v M$ associated with the A -star-shaped set O and define

$$U' := \{Y = Y^+ + t_1 X_1 + \cdots + t_k X_k + Y^- \in U \mid \Psi_v(Y^+ + t_1 X_1 + \cdots + t_k X_k + s Y^-) \in \Omega \forall s \in [0, 1]\}$$

We take $O' = \Psi_v(U')$. It is clearly an open set. Let us see that it is also dense.

For any open set $W \subset \Omega \cap O$ take $p \in W \cap \Delta$. We write $p = \Psi_v(Y_0^+ + t_{01} X_1 + \cdots + t_{0k} X_k + Y_0^-)$. We must show that $\Psi_v(Y_0^+ + t_{01} X_1 + \cdots + t_{0k} X_k + s Y_0^-) \in \Omega$ for every $s \in [0, 1]$.

Remember that, by definition, Ψ_v is defined by travelling a certain distance along the unstable foliation, then travelling a certain distance along the stable foliation, and acting a certain vector on the result.

As Δ is invariant by the action, we have that $\Psi_v(Y_0^+ + Y_0^-) \in \Delta$.

By definition, the stable foliation of $\Psi_v(Y_0^+ + Y_0^-)$ is contained in Ω , in particular, $\Psi_v(Y_0^+ + s Y_0^-) \in \Omega$. But Ω is invariant by the action, and thus

$$\Psi_v(Y_0^+ + t_{01} X_1 + \cdots + t_{0k} X_k + s Y_0^-) \in \Omega$$

It remains to show that O' is A -star-shaped.

As Ω is invariant by the action and $v \in \Delta \subset \Omega$, we have

$$t_1 X_1 + \cdots + t_k X_k \in U' \Leftrightarrow \Psi_v(+t_1 X_1 + \cdots + s t_j X_j + \cdots + t_k X_k) \in \Omega$$

that is

$$t_1 X_1 + \cdots + t_k X_k \in U' \Leftrightarrow +t_1 X_1 + \cdots + s t_j X_j + \cdots + t_k X_k \in U'$$

We must now show, that for any $s \in [0, 1]$ we have

$$Y^+ + t_1 X_1 + \cdots + t_k X_k \in U' \Leftrightarrow s Y^+ + t_1 X_1 + \cdots + t_k X_k \in U'$$

that is

$$Y^+ + t_1 X_1 + \cdots + t_k X_k \in U' \Leftrightarrow \Psi_v(s Y^+ + t_1 X_1 + \cdots + t_k X_k) \in \Omega$$

But $v \in \Delta$ and thus, its unstable foliation is in Ω . In particular, $\Psi_v(s Y^+) \in \Omega$ for every s . As Ω is invariant by the action, $\Psi_v(s Y^+ + t_1 X_1 + \cdots + t_k X_k) \in \Omega$

□

Lemma 4.3.3. Fix a background metric g on M and let us denote, for $v \in M, r \in \mathbb{R}_{>0}$ and $Y = Y^+ + t_1 X_1 + \cdots + t_k X_k + Y^- \in T_v M$:

$$B^r(v) = \{Y \in T_v M; \|Y\|^2 \stackrel{\text{def}}{=} g(Y^+, Y^+) + |t_1|^2 + \cdots + |t_k|^2 + g(Y^-, Y^-) < r^2\}$$

and

$$B_r(v) = \{p \in M; \text{dist}_g(c, p) < r\}$$

Then, there exists $\delta, \varepsilon > 0$ such that for every $v \in M$,

$$\Psi_v : B^\delta(v) \rightarrow M$$

is a diffeomorphism onto it's image and

$$B_\varepsilon(v) \subset \Psi_v(B^\delta(v))$$

Proof. The proof is a variation of the usual argument for the exponential map and relies on the compactness of M .

For each point $p \in M$ consider $\varepsilon_p < 1$ such that $(\Psi_p)_{|_{B^p(\varepsilon_p)}}$ is injective¹⁶. As injectiveness is an open condition, there exists a neighbourhood $p \in U_p \subset M$ such that

$$(\Psi_q)_{|_{B^q(\varepsilon_p)}} \text{ is injective}$$

As M is a manifold, it is always possible to take a locally finite subcover $\{U_{p_i}\}_{i \in I}$. We define $h : M \rightarrow \mathbb{R}$ by

$$h(p) = \min\{\varepsilon_{p_i}; p \in U_{p_i}\}.$$

This function is a locally constant positive function on M . Take $g \in C(M, \mathbb{R}_{>0})$ such that $0 < g(p) \leq h(p)$ for all $p \in M$.

¹⁶ We must check that Ψ_v is in fact a local diffeomorphism at the origin. If $\|Y\|$ is sufficiently small, then the exponential map is a diffeomorphism onto it's image and it's differential at the origin is the identity, and we have, for $Y = Y^+ + t_1 X_1 + \cdots + t_k X_k + Y^-$:

$$\begin{aligned} (d\Psi_v)_0 Y &= \frac{d}{ds} \Big|_{s=0} (\Psi_v(sY)) \\ &= \frac{d}{ds} \Big|_{s=0} (\varphi_{st_1}^1 \circ \cdots \circ \varphi_{st_k}^k (\exp^\nabla(\tau_{sY^+}(sY^-)))) \\ &= t_1 X_1 + \cdots + t_k X_k + t_1 \cdot \tilde{\cdot} \cdot t_k \frac{d}{ds} \Big|_{s=0} (\tau_{sY^+}(sY^-)) \end{aligned}$$

where $t_1 \cdot \tilde{\cdot} \cdot t_k = \Pi_{t_j \neq 0} t_j$. Now, $\gamma(s) := \tau_{sY^+}(sY^-)$ is a curve on TM which we write as $\gamma(s) = (\alpha(s), \beta(s))$ where $\alpha(s) \in M$ and $\beta(s) \in T_{\alpha(s)} M$. By definition, $\alpha(s)$ is actually the geodesic on M with starting vector Y^+ , and $\beta(s)$ is the parallel transport of sY^- to the desired point. It is easy to see that $\frac{d}{ds} \Big|_{s=0} (\tau_{sY^+}(sY^-)) = Y^+ + Y^-$

Define for any $g \in C(M, \mathbb{R}_{>0})$ the submanifold

$$A_g = \{(p, Y) \in TM ; \|Y\|_p < g(p)\}$$

and define

$$\mathcal{C} := \{g \in C(M, \mathbb{R}_{>0}) ; \Psi|_{A_g} \text{ is injective}\}$$

Where $\Psi : TM \rightarrow M \times M$ is defined as

$$\Psi(p, Y) = (p, \Psi_p(p, Y))$$

We consider the map $f : M \rightarrow (0, 1]$ defined as

$$f(p) = \max\{1, r \in \mathbb{R} ; \Psi_p|_{B^p(r)} \text{ is injective}\}$$

As $\mathcal{C}$ is not empty, we can also write

$$f(p) = \max_{g \in \mathcal{C}} \{g(p)\}$$

As $\mathcal{C}$ is a family of continuous positive limited function, then so is f .

As M is compact, there exists $0 < \delta \leq 1$ such that

$$f(p) \geq \delta \quad \forall p \in M$$

Now, we consider the map

$$\begin{aligned} M &\rightarrow \mathbb{R} \\ p &\mapsto \max\{r ; B_p(r) \subset \Psi_p(B^p(\delta))\} \end{aligned}$$

By similar arguments this map is also continuous and positive and from the compacity of M it admits a non zero minimum ε . $\square$

Corollary 4.3.4. For any dense subset $U \subset M$, we can cover M by A -star shaped open sets with base points in U .

Proposition 4.3.5. $\Omega = M$

Proof. We just need to prove that the $(G', \hat{V})$ -structure on Ω can be extended to M .

Using the Corollary 4.3.4, we take a cover $\{O_{v_i}\}$ of M by A -star shaped open sets with basepoints $v_i \in \Delta$, and using Lemma 4.3.2 we consider open, subsets $O'_{v_i} \subset \Omega \cap O_{v_i}(\varepsilon)$ dense in O_{v_i} and also A -star-shaped with respect to v_i .

In particular, O'_{v_i} is contractible, and therefore, simple connected. there exists, thus, a developing map $\theta : O'_{v_i} \rightarrow \hat{V}$. We define $\hat{\theta} : O_{v_i} \rightarrow \hat{V}$ by

$$\hat{\theta} = \Psi_{\hat{\theta}(v_i)} \circ (T_{v_i} \theta) \circ \Psi_{v_i}^{-1}$$

It follows from 4.20 that $\hat{\theta}$ is an extension of θ . From the density of $\mathcal{O}'_{v_i}$ in $\mathcal{O}_{v_i}$, it follows that $\hat{\theta}_*(X_j) = \hat{X}_j$, $\hat{\theta}_*(E^\pm) = \hat{E}^\pm$ and $\hat{\theta}^*(\omega) = d\lambda$. We conclude that $\hat{\theta}$ is a developing map¹⁷. That is, we managed to extend the $(G', \hat{V})$ -structure to M . $\square$

4.4 Completeness of the structure

Finally we shall prove that the extended structure build in the previous section is complete, that is

Proposition 4.4.1. Let $\tilde{M}$ be the universal cover of M and let $\theta : \tilde{M} \rightarrow \hat{V}$ be a developing map of the $(G', \hat{V})$ -structure. Then, θ is a covering map..

Proof. To prove our proposition, we must find, for each $w \in \hat{V}$ a neighbourhood O such that $\theta^{-1}(O)$ is the disjoint union of open sets $\{O_i\}_{i \in I}$ such that the restriction of θ to each O_i is a diffeomorphism onto O .

Let $w \in \hat{V}$ and $v_{ii \in I} = \theta^{-1}(w)$. Consider O an A -star-shaped open set with respect to w and $U \subset T_w \hat{V}$ the associated open set.

We define, for $i \in I$

$$O_i = \tilde{\Psi}_{v_i}((T_{v_i} \theta)^{-1}(U))$$

It follows from 4.20 that θ induces an diffeomorphism θ_i from O_i to O .

Lets show that $O_i \cap O_j = \emptyset$ if $i \neq j$. For this, we consider the set

$$O' = \{w' \in O \mid \theta_i^{-1}(w') = \theta_j^{-1}(w')\}$$

The set O' is obviously closed¹⁸.

On the other hand, O' is also open, for θ_i and θ_j are the restriction of a θ to a certain open sets. If there is a intersection (which is the case if $\theta_i^{-1}(w') = \theta_j^{-1}(w')$ for some w'), then $\theta_i = \theta_j$ on this intersection V' , and thus $\theta(V') \subset O'$.

As $w \notin O'$ if $i \neq j$ and O is connected (actually contractible), it follows that $O' = \emptyset$.

Finally, we shall show that $\theta^{-1}(O) = \cup_{i \in I} O_i$

First we define the map

$$\tilde{\Phi} : T\tilde{M} \rightarrow T\tilde{M}\tilde{Y} = \tilde{Y}^+ + t_1\tilde{X}_1 + \dots + t_k\tilde{X}_k + \tilde{Y}^- \mapsto (\tilde{\varphi}_{t_1}^1)_* \dots (\tilde{\varphi}_{t_k}^k)_* (\tilde{\tau}_{(\tilde{\tau}_{\tilde{Y}^+} \tilde{Y}^-)}(\tilde{\tau}_{\tilde{Y}^+} \tilde{Y}^-))$$

and in a similar way the map $\hat{\Phi} : T\hat{V} \rightarrow T\hat{V}$. It is clear that

¹⁷ A developing map is a local diffeomorphism ($\hat{\theta}$ is the composite of local diffeomorphisms) that preserves the invariant bundles and the 2-form.

¹⁸ O' is the intersection of two closed sets, the inverse images by continuous functions of a point

$$T\theta \circ \tilde{\Phi} = \hat{\Phi} \circ T\theta$$

Moreover, $\tilde{\Phi}$ is a diffeomorphism¹⁹

Now, consider $v' \in \hat{V}$ such that $\theta(v') \in O$. We can write $\theta(v') = \tilde{\Psi}_w(\hat{Y})$ for $\hat{Y} \in U$. Consider

$$\tilde{Y} := (\tilde{\Phi})^{-1}((T_{v'}\theta)^{-1}(\hat{\Phi}(\hat{Y})))$$

and let v be the basepoint of $\tilde{Y}$. We have

$$\begin{aligned} T\theta(\tilde{Y}) &= \hat{\Phi}^{-1} \circ T\theta \circ \tilde{\Phi}(\tilde{Y}) \\ &= \hat{\Phi}^{-1} \circ T\theta(T_{v'}\theta)^{-1}(\hat{\Phi}(\hat{Y})) = \hat{\Phi}^{-1}(\hat{\Phi}(\hat{Y})) = \hat{Y} \end{aligned}$$

Thus the base point v must be v_i for some $i \in I$. It remains to prove that

$$\tilde{\Psi}_{v_i}(\tilde{Y}) = v'$$

This will conclude the proof, for we have shown that $v' \in O_i$. □

Lemma 4.4.2. If $\tilde{Z}$ has basepoint a , and $\Phi^{-1}(\tilde{Z})$ has basepoint b , then, $\tilde{\Psi}_b \circ \tilde{\Phi}(\tilde{Z}) = a$.

Proof. Notice that if we take an element $T_b\tilde{M} \ni \tilde{Y} = \tilde{Y}^+ + t_1\tilde{X}_1 + \cdots + t_k\tilde{X}_k + \tilde{Y}^-$, then, we can understand the map Ψ_b in the following way.

We take the geodesic with starting vector $\tilde{Y}^+$ and transport along it, to time one, the vector $\tilde{Y}^-$. Now we take a geodesic with this starting vector and take it's time one and apply $\varphi_{t_1}^1 \circ \cdots \circ \varphi_{t_k}^k$.

¹⁹ in fact, we shall write explicit formulas for its inverse. Consider the smooth maps $\tilde{f}^\pm, \tilde{g}^\pm : T\tilde{M} \rightarrow T\tilde{M}$ defined by

$$\begin{aligned} \tilde{f}^\pm(\tilde{Y}) &= \tilde{\tau}_{\tilde{Y}^\pm}\tilde{Y} \\ \tilde{g}^\pm(\tilde{Y}) &= \tilde{\tau}_{-\tilde{Y}^\pm}\tilde{Y} \end{aligned}$$

It is clear that $\tilde{f}^\pm \circ \tilde{g}^\pm = \tilde{g}^\pm \circ \tilde{f}^\pm = Id$. which makes $\tilde{f}^\pm$ diffeomorphisms. We can write

$$\tilde{\Phi}(\tilde{Y}) = (\tilde{\varphi}_{t_1}^1)_* \cdots (\tilde{\varphi}_{t_k}^k)_*(\tilde{f}^- \circ \tilde{f}^+(\tilde{Y}))$$

. Let $\tilde{Z} = \tilde{Z}^+ + t_1\tilde{X}_1 + \cdots + t_k\tilde{X}_k + \tilde{Z}^-$ and let

$$(\tilde{\varphi}_{-t_1}^1)_* \cdots (\tilde{\varphi}_{-t_k}^k)_*(\tilde{Z}) := \tilde{W}$$

The inverse $\tilde{\Phi}^{-1}$ is given by

$$\tilde{\Phi}^{-1}(\tilde{Z}) = g^+ \circ g^-(\tilde{W})$$

The key element here is the fact that if we write

$$\tilde{\Phi}(\tilde{Y}^+ + t_1\tilde{X}_1 + \cdots + t_k\tilde{X}_k + \tilde{Y}^-) = \tilde{Z}^+ + s_1\tilde{X}_1 + \cdots + s_k\tilde{X}_k + \tilde{Z}^-$$

, then $s_j = t_j$ for every j .

We can understand the map Φ in the following way. We take the geodesic with starting vector $\tilde{Y}^+$ and transport along it, to time one, both the vector $\tilde{Y}^-$. We now take a geodesic with this starting vector and take it's time one. We transport the original vector $\tilde{Y}$ along both of this segments, and apply $D(\varphi_{t_1}^1 \circ \dots \circ \varphi_{t_k}^k)$ to the result. It is clear that $\Phi(\tilde{Y})$ will have basepoint $\Psi(Y)$. We take $\tilde{Y} = \Phi^{-1}(\tilde{Z})$ and the result follows.

□

Remark 4.4.3. It is a classical theorem (Mostow, ([MOSTOW, 1950](#)), page 617, Corollary 1) in the theory of homogeneous spaces that if G is a simply connected Lie group and H is a connected closed subgroup of G , then G/H is simply connected. Thus, the developing map of Theorem [4.4.1](#) is in fact a diffeomorphism.

GROUP REDUCTION

The goal of this chapter is to study the quasi-algebraic action obtained in the last chapter. In particular, under some additional hypothesis, we obtain a reduction of the group G' , to a semisimple one of rank greater or equal to k . To be precise, our goal is to prove the following theorem

Theorem 5.0.1. Let (M^{2n+k}, α, ϕ) be a generalized k -contact Anosov action. Suppose, moreover, that the action is faithful and $H^1(\pi_1(M), \mathbb{R}) = 0$. Then it is smoothly conjugated to a quasi algebraic action (G, K, Γ, α) where G is semisimple of rank greater or equal to k and α is contained in the Cartan subspace of $Lie(G)$

Step 1: Construct a semisimple lie group that acts transitively on M (Lemma 5.1.3)

Step 2: Show that the distributions tangent to the weak stable and weak unstable foliations corresponds to opposite parabolic subalgebras $\mathfrak{p}^\pm$ of $\mathfrak{g}$ (Theorem 5.1.9).

Step 3: Use the Langlands decomposition of the parabolic subalgebras $\mathfrak{p}^\pm$ to obtain a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus L \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

where $\mathfrak{h}$ is the centralizer of L . This requires some technical preparation and is done in

Step 4: Show that the composition map $\pi_1(M) \rightarrow G \times \mathbb{R}^k \rightarrow G$ defines a discrete representation of $\pi_1(M)$. This is in fact the only step of the proof which uses the hypothesis that the Anosov action is faithful and $H^1(\pi_1(M), \mathbb{R}) = 0$.

It is an easy consequence from the previous chapter that $\tilde{M} = G'/H' = (G \times \mathbb{R}^k)/H'$, where G is semisimple.

In the Section 5.1 we show that subgroup G acts transitively on G'/H' , which allows us to identify $\tilde{M} = G/H$, $H = G \cap H'$. We will also show that $\mathfrak{p}^\pm = \mathfrak{g} \cap \mathfrak{p}'^\pm$ are opposite parabolic

subalgebras (Theorem 5.1.9) and their intersection correspond precisely to the algebra of the subgroup which stabilizes an orbit (Corollary 5.1.7).

In the Section 5.2 we do some technical preparation, which we will use in Section 5.3 to obtain a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus L \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

where $\mathfrak{h}$ is the centralizer of L . And finally how that the composition map $\pi_1(M) \rightarrow G \times \mathbb{R}^k \rightarrow G$ defines a discrete representation of $\pi_1(M)$.

5.1 Parabolic subalgebras

Lemma 5.1.1. The Lie algebra

$$\mathfrak{g} = \bigcap_{j=1}^k \ker d\chi_j$$

has rank greater or equal to k

Proof. The goal of the proof is to construct an abelian subalgebra of $\mathfrak{g}$ of hyperbolic elements and of dimension k .

From Corollary 4.2.34 $\mathfrak{g} = \mathfrak{g}'/\mathcal{I}$ and thus is semisimple. To compute the rank of $\mathfrak{g}$, we observe that

$$A_0 := \frac{L_0}{d\chi_j(L_0)} + \frac{X_1}{\mathfrak{s}_1} + \cdots + \frac{X_k}{\mathfrak{s}_k} \in \mathfrak{g}$$

Now, we go back to the construction of L_0 , we considered a direction v of the Anosov action such that v is an Anosov element and it's associated flow has periodic orbit through the point v_0 . Let's denote by X_v the associated vector field. We considered the differential of it's time one map, some finite iteration of this map belongs to the group H' , it is by construction hyperbolic, and therefore we can take it's logarithm to obtain the element L_0 on the Lie algebra $\mathfrak{h}'$. By construction, the adjoint action of the element L_0 is hyperbolic, and therefore so is the action of A_0 as defined above.

Notice that this construction uses the injection $j : H' \rightarrow \text{Aut}(T_{v_0}M)$, $j(h) = T_{v_0}h$, to identify the group H' with it's image. Without this identification, we do not need to take the differential of the time one map. We just take it's logarithm. This means that we consider L_0 not as an endomorphism of $T_{v_0}M$ but as a vector field on M . Clearly, $\exp(L_0) = \exp(X_v)$ and therefore, Baker–Campbell–Hausdorff implies that $L_0 = X_v + Z_v$ for some $Z_v \in \mathfrak{g}'$ such that $\exp(Z_v) = \text{Id}$.

If we consider a base of directions $\{v, v_2, \dots, v_k\}$ such that each v_j is an Anosov element and the associated flows has periodic orbit through v_0 (we remember that v_0 has compact orbit, that is, it's orbit is a k -dimensional torus, and thus such base of directions exists). Let us show that $L_0, L_2, \dots, L_k$ are linearly independent. In fact, each L_j is of the form $X_{v_j} + Z_{v_j}$ as the vector

fields X_{v_j} are linearly independent, it remains to show that the space generated by the Z_{v_j} does not intercept the space generated by the X_{v_j} . Let $Z \subset \mathfrak{g}$ denote the vector space generated by the Z_{v_j} and suppose that $Z \cap \mathcal{J} \neq \{0\}$, then there exists some linear combination L of L_j such that $L \in \mathcal{J} \setminus \{0\}$ which contradicts the fact that $L \in \mathfrak{h}'$ and therefore $L_{v_0} = 0$.

Now, let us show that the $[L_i, L_j] = 0$ for any i, j . As the action is abelian, the time one maps commute with each other, and as they are hyperbolic, they are diagonalizable, thus, there exists a basis of $T_{v_0}(M)$ such that the time one maps are simultaneously diagonalizable. The logarithm map is just the logarithm of each diagonal element, and therefore, the elements L_j are simultaneously diagonal. In particular, they commute with each other¹.

Thus we can build elements

$$A_j := \frac{L_j}{d\chi_j(L_j)} + \frac{X_1}{s_1} + \cdots + \frac{X_k}{s_k}; \quad 2 \leq j \leq k$$

such that, for $i, j = 0, 2, 3, \dots, k$ we have $A_j \in \mathfrak{g}$, $[A_i, A_j] = 0$ and by construction, each A_j is hyperbolic. Therefore, the Cartan subspace of $\mathfrak{g}$ is at least k dimensional, and $\mathfrak{g}$ has rank k . $\square$

Lemma 5.1.2. The group G' decomposes as a direct sum $G' = G \times \mathbb{R}^k$ where G is semisimple of rank greater or equal then k and has Lie algebra $\mathfrak{g}$.

Proof. We define the group G by taking characters $\chi_j : G' \rightarrow (\mathbb{R}, \cdot)$ defined by

$$g_* \zeta_j = \chi_j(g) \zeta_j$$

and taking $G = \bigcap_j \ker \chi_j$. It is clear that $\text{Lie}(G) = \mathfrak{g}$.

From Lemma 5.1.1 $\mathfrak{g}$ is semisimple with rank greater or equal than k .

If we denote by $\hat{\phi}^j$ the flow corresponding to $\hat{X}_j$ then $\chi_j(\hat{\phi}_t^j) = \delta_{ij} e^{s_i \cdot t}$. Thus, the map

$$\begin{aligned} f : G \times \mathbb{R}^k &\rightarrow G' \\ (g, t_1, \dots, t_k) &\mapsto g \cdot \hat{\phi}_{t_1}^1 \cdots \hat{\phi}_{t_k}^k \end{aligned}$$

is an isomorphism. $\square$

Lemma 5.1.3. The group G acts transitively on G'/H' . In particular, this means we can identify $G'/H' = G/H$.

¹ We note that we are identifying $\mathfrak{h}'$ with its image by the map $\theta_o : \mathfrak{g}' \rightarrow \text{End}(V_0) \times V_0$. We remember that the Lie bracket on this image is given by

$$[(A, a), (B, b)] = ([A, B] - R(a, b), Ab - Ba - T(a, b))$$

and thus, the restriction of the bracket to $\theta_o(\mathfrak{h}') \subset \text{End}(V_0) \times \{0\}$ is actually the usual commutator of matrices (if we fix a given basis for V_0).

Proof. We only need to prove that $\mathfrak{g}' = \mathfrak{g} + \mathfrak{h}'$.

Consider the projection map

$$\pi : \mathfrak{g}' \rightarrow \mathfrak{g}'/\mathfrak{g} = (\mathfrak{g} \oplus \mathcal{J})/\mathfrak{g} = \mathcal{J}$$

. We will show that π restricted to $\mathfrak{h}'$ is surjective.

In fact, let $\hat{\mathfrak{h}}' = \pi^{-1}(\pi(\mathfrak{h}'))$. Suppose that $\hat{\mathfrak{h}}' \subsetneq \mathfrak{g}'$, as $\mathfrak{g} \subset \hat{\mathfrak{h}}'$, this means that $\hat{\mathfrak{h}}' \cap \mathcal{J}$ has co-dimension at least 1. But $\mathcal{J}$ corresponds to the Anosov action, and the base point we choose² has compact orbit, therefore, there exists $X \in \mathcal{J} \setminus \hat{\mathfrak{h}}'$ such that $\exp t_0 X \in H'$ for some t_0 . However, this contradicts the fact that $tX \notin \mathfrak{h}'$ for $t \neq 0$. $\square$

Remark 5.1.4. In the previous proof we identified two different notions of exponential maps: The exponential map from a Lie algebra to its associated Lie group and the exponential map that associates a vector field on a manifold M to the flow it generates. This identification is not possible in general, even when the Lie group corresponds to a group of diffeomorphisms of the manifold M . However, we are considering an homogeneous manifold G'/H' , and the corresponding Lie algebras, and the vector field we considered integrates to a one parameter subgroup of G' . In this context it is easy to see that both notions of exponential coincide.

Remark 5.1.5. We have previously define on $\hat{G}'/\hat{H}'$ distributions $\hat{E}^\pm$ tangent to the foliations $\hat{\mathcal{F}}^\pm$ which corresponds to the strong stable and unstable foliations. We denote by $\hat{I} = \mathbb{R}\hat{X}_1 \oplus \dots \oplus \mathbb{R}\hat{X}_k$.

Lemma 5.1.6. Consider the map $\pi : G \rightarrow G/H$, and the distributions $\mathcal{P}^\pm$ on TG given by

$$d\pi(\mathcal{P}^\pm) \subset \hat{E}^\pm \oplus \hat{I}$$

Then, $\mathcal{P}^\pm$ is the distribution obtained by left translation of the Lie subalgebra $\mathfrak{p}^\pm = \mathfrak{p}'^\pm \cap \mathfrak{g}$.

Proof. In fact, let $\hat{Q}'^\pm \subset \hat{G}'$ be the connected sub group associated with the Lie subalgebra $\mathfrak{q}'^\pm$ the strong foliations $\hat{\mathcal{F}}^\pm$ were constructed by left translating $\hat{Q}'^\pm/H'$, and thus, the distribution $\hat{E}^\pm$ is precisely the left translation of $\mathfrak{q}'^\pm$.

If we consider the connected subgroup $\hat{P}'^\pm \subset \hat{G}'$ associated with the Lie subalgebra $\mathfrak{p}'^\pm$, we obtain that left translation of $\mathfrak{q}'^\pm$ corresponds to the distribution tangent to the weak foliations. $\square$

Corollary 5.1.7. The subalgebra $\mathfrak{p}^+$ (resp $\mathfrak{p}^-$) corresponds to the Lie subgroup of G that stabilizes the weak stable (resp. unstable) leaf of $eH \in G/H$. In particular, the Lie algebra

$$\mathfrak{g}_0 = (\mathfrak{h}' \oplus \mathcal{J}) \cap \mathfrak{g} = \mathfrak{p}^+ \cap \mathfrak{p}^-$$

² Remember that $\mathfrak{g}'$ is the Lie algebra of germs of local killing vector fields at fixed base point

is the Lie algebra of the stabilizer of the orbit (by the Anosov action on $\hat{V}$ (Remark 4.2.38)) of eH .

Lemma 5.1.8. There exists a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ such that $\mathfrak{c} \subset \mathfrak{g}_0$.

Proof. It is known that if $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a surjective morphism of Lie algebras, and $\mathfrak{c}_1$ is a Cartan subalgebra of $\mathfrak{g}_1$, then $f(\mathfrak{c}_1)$ is a Cartan subalgebra of $\mathfrak{g}_2$. We consider the morphism $\pi : \mathfrak{g}' \rightarrow \mathfrak{g}'/\mathcal{I} = \mathfrak{g}$. It is clearly surjective, and from Lemma 4.2.20, there exists a Cartan subalgebra $\mathfrak{c}' \subset \mathfrak{h}' \oplus \mathcal{I}$. It is clear that $\pi(\mathfrak{h}' \oplus \mathcal{I}) = \mathfrak{g}_0$, and thus, $\mathfrak{c} = \pi(\mathfrak{c}')$ is a desired Cartan subalgebra. $\square$

Theorem 5.1.9. Consider $\lambda \in \mathfrak{c}_{\mathbb{C}}^*$ and $E \subset \mathfrak{g}_{\mathbb{C}}$ a vector subspace stable under the adjoint action of $\mathfrak{c}_{\mathbb{C}}$. We denote

$$E^\lambda = \{Y \in E ; [Z, Y] = \lambda(Z)Y \quad \forall Z \in \mathfrak{c}_{\mathbb{C}}\}$$

Consider $\Delta(\mathfrak{c}_{\mathbb{C}}, E) = \{\lambda \in \mathfrak{c}_{\mathbb{C}}^* \setminus \{0\} ; E^\lambda \neq \{0\}\}$. Then, $\Delta(\mathfrak{c}_{\mathbb{C}}, \mathfrak{p}^+) = -\Delta(\mathfrak{c}_{\mathbb{C}}, \mathfrak{p}^-)$.

Proof. Consider $\Delta_0 = \Delta(\mathfrak{c}_{\mathbb{C}}, (\mathfrak{g}_0)_{\mathbb{C}})$ and $\Delta^\pm = \Delta(\mathfrak{c}_{\mathbb{C}}, \mathfrak{p}^\pm)$. As $\mathfrak{g} = \mathfrak{p}^+ + \mathfrak{p}^-$, it follows that

$$\Delta(\mathfrak{c}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}) = \Delta^+ \cup \Delta_0 \cup \Delta^-$$

where the union is disjoint.

Now, consider the complexification $\mathbb{C} \otimes TM$ of the tangent bundle. The sections of this bundle are called complex vector fields of M , and can be seen as derivations of complex functions on M , that is, we can identify

$$\mathbb{C} \otimes T_p M = \text{Der}_p(C^\infty(M, \mathbb{C}))$$

Moreover, the space of complex vector fields is a Lie algebra, explicitly, it is the complexification of the Lie algebra of real vector fields.

We also consider the complexified bundles $\Lambda^r(\mathbb{C} \otimes T^*M) = \mathbb{C} \otimes \Lambda^r T^*M$. The exterior derivative map

$$\Gamma(M, \Lambda^r T^*M) \xrightarrow{d} \Gamma(M, \Lambda^{r+1} T^*M)$$

induces a map

$$\Gamma(M, \mathbb{C} \otimes \Lambda^r T^*M) \xrightarrow{d_{\mathbb{C}}} \Gamma(M, \mathbb{C} \otimes \Lambda^{r+1} T^*M)$$

which satisfies

$$\begin{aligned} d_{\mathbb{C}} \omega(Z_0, Z_1, \dots, Z_r) &= \sum_j (-1)^j Z_j(\omega(Z_0, \dots, \hat{Z}_j, \dots, Z_r)) \\ &\quad - \sum_{i < j} (-1)^{i+j} \omega([Z_i, Z_j], Z_0, \dots, \hat{Z}_i, \dots, \hat{Z}_j, \dots, Z_r) \end{aligned}$$

For a real vector bundle E over M the inclusion map $E \rightarrow \mathbb{C} \otimes E$ induces an inclusion map $\Gamma(M, E) \rightarrow \Gamma(M, \mathbb{C} \otimes E)$, the image via this map will be called a complexified section. In particular, the 1-forms α_j can be complexified (we denote it by $\alpha_j^{\mathbb{C}}$) and it satisfies

$$d\alpha_j^{\mathbb{C}}(A, B) = A(\alpha_j^{\mathbb{C}}(B)) - B(\alpha_j^{\mathbb{C}}(A)) - \alpha_j^{\mathbb{C}}([A, B])$$

for any complex vector fields A, B .

Moreover, as $E^+ \oplus E^- \subset \ker(\alpha_j)$ then

$$\mathbb{C} \otimes (E^+ \oplus E^-) = (\mathbb{C} \otimes E^+) \oplus (\mathbb{C} \otimes E^-) \subset \ker(\alpha_j^{\mathbb{C}}) \quad (5.1)$$

We have the identification $\mathfrak{p}_{\mathbb{C}}^{\pm}/\mathfrak{g}_{0\mathbb{C}} = E_{v_0}^{\pm}$, which induces a identification

$$\mathfrak{p}^{\pm}/\mathfrak{g}_0 = (\mathfrak{p}^{\pm}/\mathfrak{g}_0)_{\mathbb{C}} = E_{v_0, \mathbb{C}}^{\pm}$$

For a fixed j , we have a non degenerate pairing $d\alpha_j : E_{v_0}^+ \oplus E_{v_0}^- \rightarrow \mathbb{R}$, which induces a non degenerate pairing

$$d\alpha_j^{\mathbb{C}} : \mathbb{C} \otimes E_{v_0}^+ \oplus (\mathbb{C} \otimes E_{v_0}^-) \rightarrow \mathbb{C}$$

Moreover, as $E_{v_0}^{\pm}$ are Lagrangian subspaces of $d\alpha_j$, then $E_{v_0, \mathbb{C}}^{\pm}$ are also Lagrangian subspaces of $d\alpha_j^{\mathbb{C}}$.

From Remark 3.3.37, $\mathfrak{g}^{\lambda}$ are at most one dimensional, and thus, there exists $\lambda_1, \dots, \lambda_n \in \Delta^+, Y_1^{\pm}, \dots, Y_n^{\pm}, Y_r^+ \in (\mathfrak{p}^+)_{\lambda_r}, Y_r^- \in \mathfrak{p}^-$, such that, for a fixed j

$$(d\alpha_j^{\mathbb{C}})_{v_0}((Y_r^+)_{v_0}, (Y_l^-)_{v_0}) = \delta_{rl} \quad (5.2)$$

Where $(Y_r^{\pm})_{v_0}$ denotes the image of $Y_r^{\pm}$ under the projection $\mathfrak{p}_{\mathbb{C}}^{\pm} \rightarrow \mathfrak{p}^{\pm}/\mathfrak{g}_0 = E_{v_0, \mathbb{C}}^{\pm}$. Moreover, such Y_r^- are unique, up to a $\mathfrak{g}_{0\mathbb{C}}$ -factor.

We can write the $ad_{\mathfrak{c}_1\mathbb{C}}$ -invariant splitting

$$\mathfrak{p}_{\mathbb{C}}^{\pm} = \mathfrak{g}_{0\mathbb{C}} \oplus \bigoplus_{\lambda \in \Delta^{\pm}} (\mathfrak{p}_{\mathbb{C}}^{\pm})^{\lambda}$$

Thus, there is a canonical lift, $E_{v_0, \mathbb{C}}^{\pm} = \mathfrak{p}_{\mathbb{C}}^{\pm}/\mathfrak{g}_{0\mathbb{C}} \rightarrow \mathfrak{p}^{\pm}$. Using this lift, we can choose unique Y_r^- (Now without the $\mathfrak{g}_{0\mathbb{C}}$ -factor) on the image of this lift.

Now, we take $Z \in \mathfrak{c}_{\mathbb{C}}$. We choose the Cartan subalgebra such that

$$\mathfrak{c} \subset (\mathfrak{h} \oplus \mathcal{J}) \cap \mathfrak{g}$$

, and thus, $\mathfrak{c}_{\mathbb{C}} \subset (\mathfrak{h}_{\mathbb{C}} \oplus \mathcal{J}_{\mathbb{C}})$, that is, $Z = Z_{\mathfrak{h}} + Z_{\mathcal{J}}$. Where $Z_{\mathfrak{h}} \in \mathfrak{h}_{\mathbb{C}}$ and $Z_{\mathcal{J}} \in \mathcal{J}_{\mathbb{C}}$. In particular, $(Z_{\mathfrak{h}})_{v_0} = 0$ and $Z_{\mathcal{J}}$ commutes with every vector field in $\mathfrak{g}_{\mathbb{C}}$.

Moreover, as we have seen, $\mathfrak{g}'$ is composed by infinitesimal affine transformations (for the connections ∇^j) and as ∇^j preserves the bundles $E^\pm$, then so do the vector fields on $\mathfrak{g}'$. That is if $W \in \mathfrak{g}'$, then, $[W, E^\pm] \subset E^\pm$. Thus, if $Z = W + iW' \in \mathfrak{g}'_{\mathbb{C}}$, then

$$[Z, E_{\mathbb{C}}^\pm] = [W + iW', E^\pm + iE^\pm] \subset E^\pm + iE^\pm = E_{\mathbb{C}}^\pm$$

From this considerations, we have, for $Y^\pm$ complex vector fields tangent to $\mathbb{C} \otimes E^\pm$:

$$d\alpha_j^{\mathbb{C}}(Y^+, ad_Z Y^-) = Y^+(\alpha_j^{\mathbb{C}}([Z, Y^-]) - [Z, Y^-](\alpha_j^{\mathbb{C}}(Y^{\mathbb{C}})) - \alpha_j^{\mathbb{C}}([Y^+, [Z, Y^-]])) \quad (5.3)$$

$$\stackrel{5.1}{=} -\alpha_j^{\mathbb{C}}([Y^+, [Z, Y^-]]) \quad (5.4)$$

$$= -\alpha_j^{\mathbb{C}}([Y^-, [Z, Y^+]]) - \alpha_j^{\mathbb{C}}([Z, [Y^+, Y^-]]) \quad (5.5)$$

Now, we write $[Y^+, Y^-] = W^+ + \sum_l f_l X_l + W^-$, where $W^\pm$ are complex vector fields tangent to $E_{\mathbb{C}}^\pm$. As we have seen, $[Z_{\mathfrak{h}}, E_{\mathbb{C}}^\pm] \subset E_{\mathbb{C}}^\pm$, and thus $\alpha_j^{\mathbb{C}}([Z, W^\pm]) = 0$. We have

$$\begin{aligned} \alpha_j^{\mathbb{C}}([Z, [Y^+, Y^-]]) &= \alpha_j^{\mathbb{C}}([Z_{\mathfrak{h}} + Z_{\mathcal{J}}, [Y^+, Y^-]]) = \alpha_j^{\mathbb{C}}([Z_{\mathfrak{h}}, [Y^+, Y^-]]) \\ &= \sum_l \alpha_j^{\mathbb{C}}([Z_{\mathfrak{h}}, f_l X_l]) = \sum_l \alpha_j^{\mathbb{C}}(Z_{\mathfrak{h}}(f_l) X_l) \\ &= Z_{\mathfrak{h}}(f_j) \end{aligned}$$

That is

$$d\alpha_j^{\mathbb{C}}(Y^+, ad_Z Y^-) = -\alpha_j^{\mathbb{C}}([Y^-, [Z, Y^+]]) - \alpha_j^{\mathbb{C}}([Z, [Y^+, Y^-]]) \quad (5.6)$$

$$= d\alpha_j^{\mathbb{C}}(Y^-, [Z, Y^+]) - Z_{\mathfrak{h}}(f_j) \quad (5.7)$$

As we are interested only in the values of $d\alpha_j^{\mathbb{C}}$ on the point v_0 , and $(Z_{\mathfrak{h}})_{v_0} = 0$, the last term vanishes from 5.6 and we obtain:

$$(d\alpha_j^{\mathbb{C}})_{v_0}((Y^+)_{v_0}, (ad_Z Y^-)_{v_0}) = -(d\alpha_j^{\mathbb{C}})_{v_0}((ad_Z Y^+)_{v_0}, (Y^-)_{v_0}) \quad (5.8)$$

Take $Y^+ = Y_r^+$ and $Y^- = Y_r^-$ choosen as (5.2). Then, as $ad_Z Y_r^+ = \lambda_r(Z) Y_r^+$ we have

$$\begin{aligned} (d\alpha_j^{\mathbb{C}})_{v_0}((Y_r^+)_{v_0}, (ad_Z Y_r^-)_{v_0}) &= -(d\alpha_j^{\mathbb{C}})_{v_0}((ad_Z Y_r^+)_{v_0}, (Y_r^-)_{v_0}) \\ &= -(d\alpha_j^{\mathbb{C}})_{v_0}((\lambda_r(Z) Y_r^+)_{v_0}, (Y_r^-)_{v_0}) \\ &= -\lambda_r(Z)((d\alpha_j^{\mathbb{C}})_{v_0}(Y_r^+)_{v_0}, (Y_r^-)_{v_0}) = -\lambda_r(Z) \end{aligned}$$

From the uniqueness of $Y_r^\pm$, if we suppose that $Z \notin \ker \lambda_r$, it follows that

$$\frac{ad_Z Y_r^-}{-\lambda_r(Z)} = Y_r^-$$

that is

$$[Z, Y_r^-] = -\lambda_r(Z) Y_r^-$$

By continuity, when $Z \in \ker \lambda_r$ we have $[Z, Y_r^-] = 0 = -\lambda_r(Z)Y_r^-$

and therefore $-\lambda \in \Delta^-$

□

Corollary 5.1.10. The subalgebra $\mathfrak{g}_0$ is the reductive component of the Lie algebras $\mathfrak{p}^\pm$

Proof. This is a consequence of the following theorem

Theorem [Bourbaki,(BOURBAKI, 2005), VIII.§3.4 Proposition 13]

Let $(\mathfrak{g}, \mathfrak{h})$ be a splitting Lie algebra and $\mathfrak{p} = \mathfrak{h} + \mathfrak{g}^P$ be a parabolic subalgebra of $(\mathfrak{g}, \mathfrak{h})$. Then, $\mathfrak{h} + \mathfrak{g}^{P \cap (-P)}$ is the reductive component of $\mathfrak{p}$.

In fact, as the Cartan subalgebra $\mathfrak{c}_{\mathbb{C}}$ is contained in $\mathfrak{g}_{0\mathbb{C}} = \mathfrak{g}^{\Delta^+ \cap \Delta^-}$, the above theorem shows that $\mathfrak{g}_{0\mathbb{C}}$ is the reductive component of $\mathfrak{p}_{\mathbb{C}}^\pm$ and therefore $\mathfrak{g}_0$ is the reductive component of $\mathfrak{p}^\pm$. □

5.2 Technical preparation

On this section we develop a technical result we will use latter on.

Consider a H' -invariant decomposition $E_{v_0}^+ = \bigoplus_{i \in J^+} E_{v_0,i}^+$ and define for every $1 \leq j \leq k$

$$E_{v_0,i,j}^- := \{Z \in E_{v_0}^- ; d\alpha_j(Z, E_{v_0,l}^+) = 0 ; \forall l \in J^+ \setminus \{i\}\}.$$

We have, for every $1 \leq j \leq k$,

$$E_{v_0}^- = \bigoplus_{i \in J^+} E_{v_0,i,j}^-$$

This is a consequence of the following lemma

Lemma 5.2.1. Let E^+ and E^- be finite dimensional real vector spaces and $B : E^+ \times E^- \rightarrow \mathbb{R}$ be a non degenerate bilinear map. Assume that E^+ admits a decomposition:

$$E^+ = \bigoplus_{i \in I} E_i^+$$

and define

$$E_i^- := \{Y \in E^- ; B(Z, Y) = 0 \ \forall Z \in E_l^+ ; l \neq i\}$$

Then, $E^- = \bigoplus_{i \in I} E_i^-$.

Proof. The proof is by induction on the cardinality $|I|$ of I . The case where $|I| = 1$ is trivial.

Now, suppose that $|I| > 1$, and consider the induced bilinear map

$$\bar{B} : E^+ / E_i^+ \times E^- / E_i^- \rightarrow \mathbb{R}$$

Let us show that $\bar{B}$ is non degenerate.

In fact, let $\bar{Y}^- \in E^-/E_i^-$ is such that

$$\bar{B}(\bar{Z}^+, \bar{Y}^-) = 0 \quad \forall \bar{Z}^+ \in E^+/E_i^+$$

It is clear that $E^+/E_i^+ = \bigoplus_{l \neq i} E_l^+$, and thus we identify $\bar{Z}^+$ with $Z^+ \in \bigoplus_{l \neq i} E_l^+$. Moreover, let $Y^- \in E^-$ be any representative of $\bar{Y}^-$. Any other such representative is of the form $Y^- + Y_i^-$ for some $Y_i^- \in E_i^-$. But, by definition, $B(Z^+, Y_i^-) = 0$, and therefore,

$$B(Z^+, Y^-) = B(Z^+, Y^- + E_i^-) = \bar{B}(Z^+, Y^-) = 0 \quad \forall Z \in \bigoplus_{l \neq i} E_l^+$$

and thus $Y^- \in E_i^-$ and $\bar{Y}^- = 0$.

A similar argument shows that

$$\bar{B}(\bar{Z}^+, \bar{Y}^-) = 0 \quad \forall \bar{Y}^- \in E^-/E_i^- \Rightarrow \bar{Z}^+ = 0$$

and thus $\bar{B}$ is non degenerate. By induction, this means

$$E^-/E_i^- = \bigoplus_{l \neq i} \bar{E}_l^-$$

. where

$$\bar{E}_l^- := \{\bar{Y} \in E^-/E_i^- ; \bar{B}(Z, \bar{Y}) = 0 \quad \forall Z \in E_j^+ ; j \in I \setminus \{i, l\}\}$$

Notice that we can identify $\bar{E}_l^- = E_l^-/E_i^-$. But, $E_l^- \cap E_i^- = 0$ for $i \neq l$ and thus we identify $\bar{E}_l^- = E_l^-$. That is

$$E^-/E_i^- = \bigoplus_{l \neq i} E_l^-$$

. and

$$E^- = E_i^- \oplus \bigoplus_{l \neq i} E_l^-.$$

The lemma is proved. □

We denote by $J = \{+, -\} \times J^+ \times \{1, \dots, k\}$, and fix $E_{v_0, i, j}^+ = E_{v_0, i}^+$. We consider, for $i \in J$, the G' -invariant subbundle $\tilde{E}^i$ of $T\tilde{M}$ with fiber $E_{v_0, i}$. We denote E^i the corresponding subbundle on TM , and by $\mathfrak{s}_i$ the entropy (Definition 4.2.12) of E^i with respect to the $\mathbb{R}^k$ action³ included in G' . We set $\sigma_i = \frac{\mathfrak{s}_i}{\dim(E_{v_0, i})}$.

Remark 5.2.2. Notice that for every j , we have a smooth splitting $TM = T\phi \oplus \bigoplus_{i \in J^+} (E^{+, i, j} \oplus E^{-, i, j})$.

³ We defined the entropy with respect to a flow, but the definition extends naturally to a definition of entropy for a subbundle invariant by an $\mathbb{R}^k$ -action.

Lemma 5.2.3. There exists a unique smooth connections ∇^j on M that satisfies

$$\nabla^j d\alpha_j = 0 ; \nabla^j \alpha_i = 0 \quad \forall i ; \nabla^j(E^i) \subset E^i ; \nabla^j T\phi \subset T\phi \quad (5.9)$$

and for, Z^i, Z^l sections of E^i, E^l

$$\nabla_{Z^i}^j Z^l = p^l([Z^i, Z^l]) \quad (5.10)$$

$$\nabla_{X_i}^j Z^l = [X_i, Z^l] + \delta_{ij} \sigma_i Z^l \quad (5.11)$$

Where p^l denotes the projection $TM \rightarrow E^l$.

Proof. Analogous to the proof of Lemma. 4.2.1 □

Lemma 5.2.4. Let Λ^i denote the determinant bundle of E^i , and, for each $j = 1, \dots, k$, let ∇^j be the induced connection on Λ^i . Then, for each j , the bundle (Λ^i, ∇^j) is flat.

Proof. The proof follows the same path as the analogous proposition for the determinant bundle $\Lambda^n(E^+)_*$:

- Consider $\Omega^i = \Omega^{ij}$ the curvature form of (Λ^i, ∇^j) , and define the linear maps $B = B^{ij}$ by

$$\Omega^{ij}(u, v) = d\alpha_j(Bu, v)$$

.

- Consider a primitive β^i of Ω^i .
- Show that $\int_M \beta^{ij}(X_j) dM_j = 0$ (same proof as Lemma 4.2.11).
- Show that for every $p \in \{1, \dots, n-1\}$ we have $(\Omega^i)^p \wedge d\alpha_j^{n-p} = 0$ (same proof as Lemma 4.2.13).
- Conclude that B is nilpotent (same proof as Theorem 4.2.10).
- Prove that B is semisimple (same proof as Lemma 4.2.22).

□

Lemma 5.2.5. For any $Y \in \mathfrak{h}$ and any $i \in J$, we have $Tr_{|_{E_{v_0}^i}}(Y) = 0$

Proof. In fact, for a fixed j the determinant bundle $(\Lambda E^i, \nabla^j)$ is flat⁴, and so is the associated bundle $(\Lambda \tilde{E}^i, \nabla^j)$ on the universal cover $\tilde{M}$. Thus, there exists a parallel section ζ_{ij} on $\tilde{M}$. This section is unique up to a constant.

⁴ Remember that we have omitted the index j on the bundle E^i . The most complete notation should be $(\Lambda E^{\sigma i, j}, \nabla^j)$ for $\sigma \in \{+, -\}$ and $i \in J^+$

We define the character $\chi_{ij} : G' \rightarrow \mathbb{R}^*$ defined by

$$g_* \zeta_{ij} = \chi_{ij}(g) \zeta_{ij} \quad \forall g \in G'$$

and its differential $d\chi_{ij} : \mathfrak{g}' \rightarrow \mathbb{R}$ given by

$$\mathcal{L}_Y \zeta_{ij} = d\chi_{ij}(Y) \zeta_{ij}$$

Once again (demonstration of Lemma 4.2.29, formula 4.18) we have that $(d\chi_{ij})|_{\mathfrak{h}'} = Tr|_{E_{v_0}^i}$. As $\mathfrak{g}$ is semisimple and $d\chi_{ij} : \mathfrak{g}' \rightarrow \mathbb{R}$ is a morphism of Lie algebras, $d\chi_{ij}(\mathfrak{g}) = 0$, then, for $Y \in \mathfrak{h} = \mathfrak{h}' \cap \mathfrak{g}$ it follows

$$0 = d\chi_{ij}(Y) = Tr|_{E_{v_0}^i}(Y)$$

□

5.3 Decomposing $\mathfrak{g}$

Let us consider $\mathfrak{n}^\pm$ the unipotent radical of $\mathfrak{p}^\pm$, $\mathfrak{z}_0$ the center of $\mathfrak{g}_0$. We define

$$\begin{aligned} \mathfrak{z}_0^E &= \{Y \in \mathfrak{z}_0; ad_{\mathfrak{g}}(Y) \text{ is elliptic}\} \\ \mathfrak{a} &= \{Y \in \mathfrak{z}_0; ad_{\mathfrak{g}}(Y) \text{ is hyperbolic}\} \end{aligned}$$

We have the equality $\mathfrak{z}_0 = \mathfrak{z}_0^E \oplus \mathfrak{a}$. Let $\mathfrak{m} = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{z}_0^E$, then the Langlands decomposition of $\mathfrak{p}^+$ is

$$\mathfrak{p}^+ = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$$

and $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$.

Theorem 5.3.1. Under the above notations, we have $\mathfrak{h} \cap \mathfrak{a} = \{0\}$.

Proof. Let $Y \in \mathfrak{h} \cap \mathfrak{a}$. As $Y \in \mathfrak{h}$ it induces an decomposition

$$E_{v_0}^+ = \bigoplus_{\mu \in \mathbb{R}} E_{v_0, \mu}^+$$

of $E_{v_0}^+$ induced by the adjoint action of Y on $\mathfrak{p}^{'+}/\mathfrak{h}' = \mathfrak{p}^+/\mathfrak{h} = E_{v_0}^+$. Because $Y \in \mathfrak{a} \in Z(\mathfrak{g}_0)$, this decomposition is H_e' -invariant. Lemma 5.2.5 implies that

$$Tr|_{E_{v_0}^\mu}(Y) = \mu \cdot \dim(E_{v_0}^\mu) = 0$$

Thus, $ad_{\mathfrak{g}}(Y)$ has eigenvalues with zero real part. But $Y \in \mathfrak{a}$ and thus $ad_{\mathfrak{g}}(Y)$ is hyperbolic. Therefore, $Y = 0$, as we desired. □

Remark 5.3.2. It is known [COLOCAR REFERENCIA] that the orbits of an Anosov action are cylinders and the Anosov action acts on this orbit by translation.

As we have already remarked (Corolary 5.1.7), the Lie algebra $\mathfrak{g}_0$ is the algebra of the group that stabilizes the orbit (by the Anosov action) of eH . Moreover, the (right) action of $\mathfrak{g}_0$ on G/H commutes with the Anosov action on G/H . This means that on the Anosov orbit of eH in G/H , $\mathfrak{g}_0$ must also acts as translations, that is, we have a linear map $d\pi^{act} : \mathfrak{g}_0 \rightarrow \mathbb{R}^k$.

Lemma 5.3.3. Under the notations above, the map $d\pi^{act}$ is a morphism of Lie algebras.

Proof. Let $\mathcal{O}$ be the orbit through eH , and $G_{\mathcal{O}} \subset G$ the stabilizer of λ . We have a group homomorphism

$$G_{\mathcal{O}} \rightarrow \text{Diff}(\mathcal{O})$$

But as we have seen, $G_{\mathcal{O}}$ must acts though translations on $\mathcal{O}$, that is, the image of $G_{\mathcal{O}}$ on $\text{Diff}(\mathcal{O})$ is isomorphic to $\mathbb{R}^k \subset$ and the induced map

$$\pi^{act} : G_{\mathcal{O}} \rightarrow \mathbb{R}^k$$

is a Lie group homomorphism. Its differential is precisely the map $d\pi^{act}$ and it is therefore a Lie algebra homomorphism. $\square$

Corollary 5.3.4. The restriction $d\pi^{act}|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathbb{R}^k$ is injective.

Proof. In fact, suppose that we have a vector field $Y \in \mathfrak{g}_0$ such that $d\pi^{act}(Y) = 0$, this means that Y acts trivially on the orbit, and in particular, it fixes the point eH , that is, $Y \in \mathfrak{h}$. Thus, $\ker(d\pi^{act}) = \mathfrak{h}$. From Theorem 5.3.1, $\mathfrak{a} \cap \ker(d\pi^{act}) = \{0\}$, and thus $d\pi^{act}|_{\mathfrak{a}}$ is injective. $\square$

Lemma 5.3.5. We have $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{h} \subset \mathfrak{m}$

Proof. As $d\pi^{act}$ is a morphism of Lie algebras, it follows that $d\pi^{act}([\mathfrak{g}_0, \mathfrak{g}_0]) = 0$ and thus, $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{h}$. Now, let $x \in \mathfrak{h}$. We consider the decomposition

$$x = x_0 + x_c$$

where $x_0 \in [\mathfrak{g}_0, \mathfrak{g}_0]$ and $x_c \in \mathfrak{z}_0^E \oplus \mathfrak{a} = Z(\mathfrak{g}_0)$.

Consider the adjoint action of x_c on $\mathfrak{g}$. As $x_c \in \mathfrak{h} \cap Z(\mathfrak{g}_0)$ it follows that the decomposition of $\mathfrak{g}$ in eigenspaces is H'_e invariant. On the other hand, $x_c \in \mathfrak{h}$ and thus, from Lemma 5.2.5 it follows that $ad(x_c)$ has no eigenvalue with non zero real part, that is x_c is elliptic, that is $\mathfrak{h} \subset [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{z}_0^E = \mathfrak{m}$ $\square$

Remark 5.3.6. As the map $d\pi^{act} : \mathfrak{g}_0 \rightarrow \mathbb{R}^k$ is clearly surjective, it follows that there exists some subspace $L' \subset \mathfrak{z}_0^E$ such that the abelian subalgebra $L = L' \oplus \mathfrak{a}$ is k dimensional and such that the anosov action is given by right multiplication of the elements of L .

Proposition 5.3.7. Consider the projection $\pi : G \times \mathbb{R}^k \rightarrow G$ and the holonomy representation $\rho : \pi_1(M) \rightarrow G' = G \times \mathbb{R}^k$. Suppose that the Anosov action is faithful. Then, $\Gamma = \pi \circ \rho(\pi_1(M))$ is discrete.

Proof. Let us denote $\hat{\Gamma} = \overline{(\pi \circ \rho(\pi_1(M)))}_e$, where $(\cdot)_e$ denotes the identity component of the closure on G .

Suppose that $\hat{\Gamma}$ is non trivial.

It is known (BARBOT; MAQUERA, 2011) that an Anosov action admits only a finite number of compact orbits of fixed volume. As G acts by isometries, for a fixed compact orbit $\mathcal{O}$, sufficiently small elements of G (that is, elements on a neighborhood $U_{\mathcal{O}} \subset G$ of the identity) must send this orbit to itself. As $\hat{\Gamma}$ is connected, we have that $U_{\mathcal{O}} \cap \hat{\Gamma}$ generates $\hat{\Gamma}$ and thus $\hat{\Gamma}$ must stabilize $\mathcal{O}$.

It is known that if G is a real semisimple Lie group and P is a parabolic subgroup then P is closed in G , in particular, if $P^{\pm}$ are opposed parabolic subgroups, then $P \cap P^+$ is closed in G . If we take the parabolic subgroups $P^{\pm}$ associated with the Lie subalgebras $\mathfrak{p}^{\pm}$, then $G_{\mathcal{O}} = P \cap P^+$ has Lie algebra $\mathfrak{g}_0$ and thus, is the stabilizer of an orbit. Therefore, $G/G_{\mathcal{O}}$ is the orbit space of the Anosov action (lifted to its universal cover), and it has a manifold structure.

As $\hat{\Gamma}$ stabilizes the compact orbits, which are dense in $G/G_{\mathcal{O}}$, it follows that $\hat{\Gamma}$ stabilizes every orbit.

Then

$$\hat{\Gamma} \subset \bigcap_{g \in G} gG_{\mathcal{O}}g^{-1}$$

Let us denote $N = \bigcap_{g \in G} gG_{\mathcal{O}}g^{-1}$. Notice that $N \subset G_{\mathcal{O}}$ and, moreover, N is a normal subgroup of G . As G is semisimple, then N is also semisimple, and thus, N is on the kernel of the projection map

$$\pi^{act} : G_{\mathcal{O}} \rightarrow \mathbb{R}^k$$

This implies that $N \subset H$. The same arguments shows that N belongs to the kernel of the projections maps

$$\pi^{act} : gG_{\mathcal{O}}g^{-1} \rightarrow \mathbb{R}^k$$

for every g , that is $N \subset gHg^{-1}$ for every g . Therefore the elements of $\hat{\Gamma}$ acts trivially on G/H .

Now, we observe that the projection of $\rho(\Gamma)$ on the second coordinate

$$\Gamma \xrightarrow{\rho} G \times \mathbb{R}^k \rightarrow \mathbb{R}^k$$

acts on G/H as an element of the Anosov action, and thus, let $\gamma \in \Gamma$ such that $\rho(\gamma) = (\gamma_1, \gamma_2)$, $\gamma_1 \in \hat{\Gamma} \subset G$, then γ_1 acts trivially on G/H , and thus, γ acts on G/H precisely as an element of the anosov action, that is, if we consider $\tilde{\phi}$ the lift of the anosov action to the universal cover $\tilde{M} = G/H$, then, there exists $t(\gamma) \in \mathbb{R}^k$ such that for every $\tilde{m} \in \tilde{M}$

$$\gamma \cdot \tilde{m} = \tilde{\phi}^{t(\gamma)}(\tilde{m})$$

This means that on the manifold $M = \Gamma \backslash \tilde{M}$, the element $\gamma(t) \in \mathbb{R}^k$ acts trivially, which contradicts the faithfulness of the action. $\square$

Theorem 5.3.8. Under the hypothesis of Theorem 1.0.1, let $(G, K, \Gamma, \mathfrak{a})$ be our quasi-algebraic model. Suppose that G is an algebraic group. Then $\mathfrak{a}$ contained in the Cartan subspace of G

Lemma 5.3.9. If G is algebraic we have $\mathfrak{m} \subset \mathfrak{h}$

Proof. As $d\pi^{act}$ is a morphism of Lie algebras, $d\pi^{act}([\mathfrak{g}_0, \mathfrak{g}_0]) = 0$, and thus, $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{h} = \ker(d\pi^{act})$. It remains to show that $\mathfrak{z}_0^E \subset \mathfrak{h}$.

Take $\xi_0 \in \mathfrak{z}_0^E$. As $ad(\xi_0)$ is elliptic, its action on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ has pure imaginary eigenvalues. If we denote $G_{\mathbb{C}}$ the complexification of the group G , we have

$$\overline{\{\exp(t \, ad(x_0))\}} \subset ad(G_{\mathbb{C}}) \text{ is compact}$$

As $\xi_0 \in \mathfrak{g}$, then

$$\overline{\{\exp(t \, ad(x_0))\}} \subset ad(G) \subset ad(G_{\mathbb{C}})$$

and thus,

$$\overline{\{\exp(ad \mathfrak{z}_0^E)\}} \subset ad(G) \text{ is compact}$$

.

Let us denote $L = \overline{\{\exp(ad \mathfrak{z}_0^E)\}}$. As L is compact on $ad(G)$ it has only compact orbits, however, $\mathfrak{z}_0^E \subset \mathfrak{g}_0$ and thus, L stabilizes the orbits of the Anosov action.

The following lemma finishes the proof. $\square$

Lemma 5.3.10. Suppose that the Anosov action is faithful. Let $\mathcal{O} \subset G/H$ be an orbit of the Anosov action and $G_{\mathcal{O}} \subset G$ be the stabilizer of $\mathcal{O}$. Suppose that $K \subset G_{\mathcal{O}}$ is compact. Then, $K \subset H$.

Proof. In fact, as G_θ is the stabilizer of an orbit, and G acts transitively, then the orbits of the right action

$$G/H \curvearrowright G_\theta$$

coincide with the orbits of the $\mathbb{R}^k$ -action.

As the action is faithful we have the identification

$$G_\theta/H = \mathbb{R}^k$$

As K is compact, the projection

$$K \rightarrow G_\theta/H$$

is trivial, and thus $K \subset H$ □

Theorem 5.3.11. Under the hypothesis of Theorem 1.0.1, let $(G, H, \Gamma, \mathfrak{a})$ be our quasi-algebraic model. Suppose that H is a compact subgroup, then the action is the Weyl chamber action.

Proof. First, let us show that $\mathfrak{m} = \mathfrak{h}$.

Let us show that $\mathfrak{m} \subset \mathfrak{h}$ (we have already proved that $\mathfrak{h} \subset \mathfrak{m}$). Let $z_e = \mathfrak{h} \cap \mathfrak{m}$, it preserves a Riemannian metric on the Lie algebra $\mathfrak{g}$, in fact the metrics preserved by it forms an convex cone on the space of metrics on $\mathfrak{g}$. The group H acts on this convex cone, and as H is compact, it fixes a point ((HORNE, 1978), Theorem 5.1). This metric is therefore invariant by $\mathfrak{h}$. It can therefore be extended by left translation to G/H . This metric is also preserved by $\pi_1(M)$ for $H_1(\pi_1(M), \mathbb{R}) = 0$.

Now, suppose that there exists some element $L \in \mathfrak{m} \cap \mathfrak{h}^c$. Then L must acts via isometries (for they commute with z_e), which is not possible due to the Anosovness property.

The compacity of H implies that $\mathfrak{h}$ can not contain any other hyperbolic element and, therefore, $\mathfrak{a}$ is a maximal abelian subalgebra of hyperbolic elements of $\mathfrak{g}$, that is, it is the Cartan subspace of $\mathfrak{g}$.

Finally, $\pi_1(M)$ is a lattice, for $M = \Gamma \backslash G/H$ is compact, and thus, $\Gamma \backslash G$ is also compact, for it is a fibration over M with compact fiber H . □

CONCLUSION

We have obtained in this work a result similar to the one given by Benoist, Foulon and Labourie (BFL - (BENOIST; FOULON; LABOURIE, 1992)). It is not, however, in the strictest sense, a generalization of the result given by BFL, for the simple reason that in the case $k = 1$ our result is actually weaker for we conclude only the quasi algebraicity of the action, whereas BFL concludes it's algebraicity. Moreover, our result requires some additional hypothesis. Nonetheless, we feel that our result may represents a important step in the direction of proving the general conjecture that Anosov actions are in fact algebraic.

This feeling is powered by Klein's views on geometry, that is, that geometry is the study of things which are invariant by the action of a certain group. According to this view, algebraic Anosov actions should be associated with some kind of geometric structure.

On this work, we have shown that (under some mild hypothesis) for a specific geometric structure, that of generalized k contact, the associated Anosov action is quasi-algebraic. It is our hope that a similar strategy could be used to prove the same kind of result for more general geometric structures.

We devote the following sections to a brief study of the hypothesis on our theorem and some possible avenues for future research.

6.1 Theorem 4.0.1 hypothesis

On this section we shall make some brief comments on the importance and reasonability of the hypothesis used on Theorem 4.0.1, explicitly, the smoothness of the invariant bundles, the faithfulness of the Anosov action and the nullity of the first cohomology of the fundamental group of the base manifold.

- Smoothness

The smoothness of the invariant bundles was used to prove that the adapted connection (Lemma 4.2.1) was smooth (Remark 4.2.2). This smoothness was of crucial importance on some of our calculations, for example, the results about killing vector fields and compatible connections on subsection 2.3.3 require the smoothness of the connection.

It is known that algebraic Anosov actions have smooth invariant bundles, and therefore, the smoothness hypothesis is in accordance with the standing conjecture that every Anosov action of higher rank is algebraic. This means that should the conjecture be proven true, this hypothesis is not actually necessary. However, we must proceed with caution, it has been proved by Foulon, P. and Hasselblatt, B. ((FOULON; HASSELBLATT, 2013)) that the smoothness is a necessary condition for the case of flows, that is, they had shown that there exists a contact Anosov flow (on dimension 3) without smooth invariant bundles, which is not algebraic.

- Faithfulness and $H^1(\pi_1(M), \mathbb{R}) = 0$

Faithfulness is an irreducibility hypothesis, if the action is not faithful, then, the base manifold M is a fiber bundle $M \rightarrow B$ whose standard fiber is a torus $\mathbb{T}^l$ and there is a $\mathbb{R}^{k-l}$ action transverse to the fibres, which induces a faithful action on B . This induced action is Anosov if, and only if the original action is Anosov, moreover, it is easy to see that, a generalized k -contact structure compatible with the Anosov action on M induces a generalized $(k-l)$ -contact structure on B which is compatible with the induced action. This means that there is a canonical way to build a faithful action from a non faithful one.

The cohomology condition is more subtle, it is a technical condition, and it seems that there is no reason to suppose that a manifold supporting an Anosov action should satisfy this condition. In fact, this condition was not used to prove the quasi-algebraicity, but to prove that the fundamental group $\pi_1(M)$, represented as a discrete subgroup of $G' = G \times \mathbb{R}^k$, can actually be represented as a discrete subgroup of G , this allowed us to reduce our (G', X) -structure to a (G, X) -structure, where G is a semisimple Lie group.

On BFL's work, they did not use this hypothesis, but the flexibility of dimension one allowed them to re-parametrize the flow in such a way that $\pi_1(M)$ acts trivially on the abelian factor of $G' = G \times \mathbb{R}$, but the rigidity of higher ranked actions means that we could not re-parametrize in our case.

6.2 Further work

Under very mild conditions, we have obtained that quasi-algebraicity of the action, it is therefore, a rather natural to question

Question 1: Under which conditions a quasi-algebraic action is an algebraic action?

This is an open question, a purely algebraic one, and a rather difficult problem. It is related to another active area of research that is the question of when does an homogeneous space G/H admits a compact quotient $\Gamma \backslash G/H$. This is a well understood problem when H is compact, but for general closed H this is an open problem. A small survey on this area was given by F. Kassel on (KASSEL, 2009). Given the difficulty of this question, let us pose some similar question assuming stronger hypothesis.

Under a few extra hypothesis (faithfulness of the action and nullity of the first cohomology group of $\pi_1(M)$), we obtained a reduction of the quasi-algebraic action to a semisimple Lie group. Moreover, at least part of the action is given by right translation of an abelian subalgebra of hyperbolic elements 5.3.6. This raises the following question.

Question 2: Which additional dynamical or topological hypothesis on M can be used to remove the algebricity hypothesis from Theorem 5.3.8? And to remove the compact hypothesis from Theorem 5.3.11

Another possibility for further research is to try to generalize the proof for nilpotent group actions. More precisely

Question 3: Consider an Anosov action of a nilpotent Lie group associated with a transverse pluricontact structure (Definition 3.2.50). is this action quasi algebraic?

On another direction, we can pose questions when we have prior knowledge about the ambient manifold

Question 4: Which algebraic Anosov action admits a compatible generalized k -structure, or more generally, a transverse pluricontact structure?

There are also open questions about the geometric structure we have defined

Question 5: Does the generalized k -contact structure admits some kind of Darboux theorem?

This question is motivated by the fact that there is a Darboux theorem for both contact pairs and for k -contact structures. Moreover, consider $\mathbb{R}^{2n+k}$ with coordinates $(z_1, \dots, z_k, x_1, y_1, \dots, x_n, y_n)$, then it is easy to see that for any symplectic matrix $S_1, \dots, S_k$, $S_l = (s_{ij}^l)_{ij}$, the 1-forms

$$dz_l + \sum_{1 \leq i < j \leq n} s_{ij}^l x_i dy_j \quad l = 1, \dots, k$$

define a generalized k -contact structure on $\mathbb{R}^{2n+k}$ with splitting $\mathbb{R}^k \times \mathbb{R}^{2n}$. It is natural to question if every generalized k -contact structure is locally of this form.

Finally, another possibility for new investigations is inspired by the following two results by Y. Fang:

Theorem 6.2.1 (Theorem 2: (FANG, 2003)). Anosov flows on dimension 5 with smooth invariant bundles which preserves a non degenerated pseudo-metric are (smoothly conjugated to) algebraic.

Theorem 6.2.2 (Theorem 1: (FANG, 2007)). Uniformly quasi-conformal Anosov flows of co-dimension at least 2 which preserves a volume form and such that $E^+ \oplus E^-$ is smooth are (smoothly conjugated to) algebraic.

Another such result also proved by Y. Fang shows that the uniformly quasi-conformal Anosov flows of co-dimension at least 2 such that $E^+ \oplus E^-$ is smooth than, is algebraic.

On the heart of the proof of both of this theorems lies BFL's result. That is, Y. Fang uses the imposed dynamical conditions to, at least on some cases, recover the hypothesis of BFL's theorem. This motivate us to believe that it is possible to give generalizations for both of this theorems.

(G, X) -STRUCTURES

A.0.1 Geometric structures

Definition A.0.1. Consider a category $\mathcal{C}$ such that every morphism is an isomorphisms. Such category is called a groupoid. In set theoretical terms, a groupoid is a set $\mathcal{G}$ with a "sort of" binary operation that is not actually defined for every pair $g, h \in \mathcal{G}$.

To translate from the categorical view to the set theoretical view, we take the set $\mathcal{G}$ as the set of all morphisms of the category $\mathcal{C}$. If $\mathcal{C}$ has only one object, the groupoid we obtain is actually the group of every morphism of this object. If $\mathcal{C}$ has more then one object, then not every morphism can be composed, but when the composition is defined, it is associative and always have a unique unit.

The most common example of groupoid (and the one we are interested in) arise from local isomorphisms. Consider a category $\mathcal{E}$ and an object X of $\mathcal{E}$. Consider a subcategory $\mathcal{C}$ of sub-objects of X (that is monomorphisms ¹ $Y \rightarrow X$) and all isomorphisms between them. $\mathcal{C}$ is by definition a groupoid. In our case, we want $\mathcal{E}$ the category of topological spaces, X a topological manifold, and $\mathcal{G}_X$ the category whose objects are open sets of X and whose morphisms are homeomorphisms between them. We call $\mathcal{G}_X$ the groupoid of local homeomorphisms of X . This groupoid has an additional structure which makes it into a Pseudo-group, that is, it is endowed with a restriction map.

Definition A.0.2. Let X be a topological space. Then, a pseudo-group $\mathcal{G}$ in X will mean subcategory of the groupoid $\mathcal{G}_X$ such that

¹ a morphisms f is a monomorphisms if for every pair of morphisms g_1, g_2 we have

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

- The objects of $\mathcal{G}$ cover X
- For every local homeomorphism g in $\mathcal{G}$, its restriction to any open set of X is also an element of $\mathcal{G}$
- If g is an element of $\mathcal{G}$, then so is its inverse.
- Consider a family $\{U_i\}$ of open sets and a homeomorphism $g : U = \cup U_i \rightarrow V \subset X$ such that every $g|_{U_i}$ is an element of $\mathcal{G}$, then so is g .

The pseudo-groups $\mathcal{G}_X$ can be used to define manifolds, and more generally (we shall bellow), geometric structures on the manifolds.

Definition A.0.3 ($\mathcal{G}$ -Manifold). Consider $\mathcal{G}$ a pseudo-group in $\mathbb{R}^n$, a $\mathcal{G}$ -manifold is a topological, Hausdorff space M , and an atlas $\{U_i, \varphi_i : U_i \rightarrow V_i \subset \mathbb{R}^n\}$, such that the transition maps $\varphi_i \circ \varphi_j^{-1}$ are in $\mathcal{G}$.

If we take $\mathcal{G}$ the pseudo-group of local diffeomorphisms of class C^k , $k = 1, \dots, \infty, \omega$, we obtain a smooth manifold in the traditional sense.

Definition A.0.4 ($(\mathcal{G}, X)$ -Manifold). We take X to be an arbitrary manifold and $\mathcal{G}$ a pseudo-group on X . A $(\mathcal{G}, X)$ -manifold M is a topological Hausdorff space M with an atlas with values on X such that the transition maps are in $\mathcal{G}$. A $(\mathcal{G}, X)$ structure on M is a maximal $\mathcal{G}$ -atlas.

Thurston proved that if $\mathcal{G}$ acts transitively², then this definition does not give rises to new types of manifolds, however, such structures can reveal additional geometric properties of the manifold. For example

Example A.0.5. Consider $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, and consider local diffeomorphisms φ of $\mathbb{R}^n$ of the form

$$\varphi = (\varphi_1(x, y), \varphi_2(x))$$

This is a pseudo-group on $\mathbb{R}^n$ and a manifold with this kind of structure, is actually a foliation of co-dimension k

Definition A.0.6. A map $\xi : M \rightarrow N$ between $(\mathcal{G}, X)$ manifolds is called a $(\mathcal{G}, X)$ map if, ξ is continuous and for each charts $\varphi : U \rightarrow X$ and $\psi : V \rightarrow X$ of M and N respectively, we have

$$\psi \xi \varphi^{-1} : \varphi(U \cap \xi^{-1}(V)) \rightarrow \psi(\xi(U) \cap V)$$

is an element of $\mathcal{G}$.

Obs.: A.0.7. A function $\xi : M \rightarrow N$ between $(\mathcal{G}, X)$ manifolds is a $(\mathcal{G}, X)$ -map if, and only if, for each point $u \in M$ there exists a chart $\varphi : U \rightarrow X$, $U \ni x$, such that $\xi|_U$ is an homeomorphism onto it's image, and $\varphi \xi^{-1} : \xi(U) \rightarrow X$ is a chart of N .

² For every pair of points in M , there exists a local homeomorphism in $\mathcal{G}$ that takes one point to another

Definition A.0.8. A pseudo-group $\mathcal{G}$ on X is said to acts analytically on X if every local homeomorphisms defined on a connected open set, can be extended to the whole connected component of it's domain, and moreover, this extension is unique, that is, if two local homeomorphism that coincide on a open set, then, they coincides on the connected component of this set. In particular, if X is connected, then the pseudo-group $\mathcal{G}$ is actually a group G acting on X , and the pseudo-group can be recovered from he restrictions of G to open subsets of X .

A.0.2 Developing map

The goal of this section is to prove the following proposition

Proposition A.0.9. Consider X a connected topological space and G a group acting analytically on X . Consider M a (G, X) manifold. Then, for every simple connected open subset U of X , there exists a (G, X) map from $\theta : U \rightarrow X$, moreover, any other such map is of the form $g \circ \theta$ where g denotes an element of G .

Lemma A.0.10. Consider N a connected (G, X) manifold, and (G, X) maps $f, f' : N \rightarrow X$. If f coincides with f' on a open, non empty, set, then they are equal.

Proof. Consider

$$W = \{p \in N ; \exists U_p \subset N \ f(q) = f'(q) \ \forall q \in U_p\}$$

where U_p denotes some open neighbourhood of p . By hypothesis, this set is non empty, and by definition, it is open. Let us show that it is closed.

Let w be an accumulation point in the closure $\overline{W}$. And let $\varphi : U_w \rightarrow X$ be a (G, X) -chart around w . By definition of (G, X) -map, the compositions $f \circ \varphi^{-1}$ and $f' \circ \varphi$ are given by elements g and g' of G . Moreover, as w is an accumulation point, there exists $w' \in U_w \cap W$, and thus, there exists an open set $w' \in V \subset U_w$ such that f and f' coincides on V . Therefore, g and g' coincides on $\varphi(V)$.

As G acts analytically on X , this means that $g = g'$ on $\varphi(U_w)$ and thus $f = f'$ on U_w . That is $w \in W$. □

Proof of Proposition A.0.9. Le us fix a point $v_0 \in U$, and $\varphi_0 : V_0 \rightarrow X$ a chart around v_0 . We shall define the developing map $\theta : U \rightarrow X$ in such a way that $\theta|_{V_0} = \varphi_0$.

We consider local charts $\{\varphi_l : U_l \rightarrow X\}_{l \in \mathbb{Z}^+}$ such that $U = \cup_l U_l$ and $U_l \cap U_{l+1} \neq \emptyset$ for every l .

The definition of (G, X) -structure implies that for every $l \neq 0$ there exists $g_l \in G$ such that $\varphi_{l-1} = g_l \circ \varphi_l$.

We define, for $v \in V_l$

$$\theta(v) = g_1 \circ \cdots \circ g_l \circ \varphi_l(v)$$

Let's see that this application is well defined. We have $g_{l+1} \circ \cdots \circ g_{l+k} \circ \varphi_{l+k} = \varphi_l$ and thus, if $v \in V_l \cap V_{l+k}$, it follows that

$$g_1 \circ \cdots \circ g_l \circ \varphi_l(v) = g_1 \circ \cdots \circ g_{l+k} \circ \varphi_{l+k}(v)$$

as we wanted.

The map θ we defined is of course a (G, X) map, and the previous lemma (Lemma A.0.10), shows that the prolongation θ is unique.

Moreover, suppose that θ and θ' are two possible developing maps, then, for a fixed $v_0 \in U$, there exists g such that $\theta = g \circ \theta'$ on a neighbourhood of v_0 . The Lemma A.0.10 implies that $\theta = g \circ \theta'$ on U . □

CONNECTIONS ON VECTOR BUNDLES

Definition B.0.1. Let $E \rightarrow M$ be a vector bundle with rank r . We shall denote by $\mathcal{A}^p(E)$ the space of p -forms on M with values on E , that is, a section of the tensor product (of vector bundles with the same base space) $E \otimes \Lambda^p T^*M$. for convenience, we set $\mathcal{A}^0(E)$ the space of sections of E . A connection on E is a homomorphism

$$\nabla : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^1(E)$$

such that

$$\nabla(f\zeta) = \zeta df + f\nabla\zeta$$

Let $s = (s_1, \dots, s_r)$ be a local frame field of E over an open subset $U \subset M$, that is,

1. $s_i \in \mathcal{A}^0(E|_U)$, for $i = 1, \dots, r$
2. $(s_1(x), \dots, s_r(x))$ is an ordered basis of E_x , for each $x \in U$

Thus, given a connection ∇ on E , we can write

$$\nabla s_i = \sum_j s_j \omega_i^j \quad ; \quad \omega_i^j \in \mathcal{A}^1|_U$$

The matrix of 1-forms $\omega = (\omega_i^j)_{ij}$ is called the connection form of ∇ with respect to the frame s . Making use of the matrix notation we write:

$$\nabla s = s \cdot \omega$$

Therefore, if we set $\zeta = \sum \zeta^i s_i$, $\zeta^i \in \mathcal{A}_U^0$, an arbitrary section of E over U , then

$$\nabla \zeta = \sum_i \nabla \zeta^i s_i = \sum_i (s_i d\zeta^i + \zeta^i \nabla s_i) \quad (\text{B.1})$$

$$= \sum_i (s_i d\zeta^i + \zeta^i \sum_j s_j \omega_i^j) \quad (\text{B.2})$$

$$= \sum_j s_j (d\zeta^j + \sum_i \zeta^i \omega_i^j) = d\zeta + \omega \zeta \quad (\text{B.3})$$

We call $\nabla \zeta$ the covariant derivative of ζ . We can evaluate this at a tangent vector X of M and we obtain:

$$\nabla_X \zeta := (\nabla \zeta)(X)$$

If we take another local frame $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_r)$ of U , they are related by

$$s = \tilde{s} \cdot a$$

where $a : U \rightarrow GL(n, \mathbb{R})$. If $\tilde{\omega}$ is the connection form with respect to $\tilde{s}$, we have:

$$\begin{aligned} s\omega &= \nabla s = \nabla(\tilde{s} \cdot a) = \nabla(\tilde{s})a + \tilde{s} \cdot da \\ &= (\tilde{s}\omega)a + \tilde{s}da = \tilde{s}(\omega a + da) = s(a^{-1}\omega a + a^{-1}da) \end{aligned}$$

There exists a unique way to extend this connection, to a linear map

$$\nabla : \mathcal{A}^p(E) \rightarrow \mathcal{A}^{p+1}(E) \quad ; \quad p \geq 0$$

that satisfies the Leibniz condition

$$\nabla(\sigma \cdot \varphi) = (\nabla \sigma) \wedge \varphi + \sigma \cdot d\varphi \quad ; \quad \sigma \in \mathcal{A}^0(E) \quad \varphi \in \mathcal{A}^p$$

The above formula is precisely the way to define this extension. Now, using this extension we define

$$R = \nabla \circ \nabla \mathcal{A}^0(E) \rightarrow \mathcal{A}^2(E)$$

R is called the curvature of the bundle E . Note that

$$\nabla^2(f\zeta) = \nabla(\zeta df + f\nabla \zeta) = (\nabla \zeta) \wedge df + \zeta d^2 f + df \wedge \nabla \zeta + f\nabla^2 \zeta = f\nabla^2 \zeta$$

thus ∇^2 is actually $\mathcal{A}^0$ -linear, thus, R is actually a 2-form with values on $\text{End}(E)$, if we once again, use the matrix notation, the curvature form Ω with respect to a local frame s , is defined by

$$s\Omega = \nabla^2 s$$

Thus

$$s\Omega = \nabla(\nabla s) = \nabla(s\omega) = (\nabla s) \wedge \omega + s d\omega = s\omega \wedge \omega + s d\omega$$

that is

$$\Omega = d\omega + \omega \wedge \omega$$

Choosing another local frame $\tilde{s}$, the corresponding curvature form $\tilde{\Omega}$ is related to Ω by

$$\Omega = a^{-1}\tilde{\Omega}a$$

In fact:

$$\begin{aligned} s\Omega &= \nabla^2 s = \nabla^2(\tilde{s}a) = \nabla((\nabla\tilde{s}) \cdot a + \tilde{s} \cdot da) \\ &= \nabla^2\tilde{s} \cdot a - (\nabla\tilde{s}) \wedge da + (\nabla\tilde{s}) \wedge da + \tilde{s} \cdot d^2a = \tilde{s}\tilde{\Omega}a = sa^{-1}\tilde{\Omega}a \end{aligned}$$

B.0.1 Connections on Associated Vector Bundles

Let E be a vector bundle over M , let E^* denote the dual vector bundle, bilinear map induced by the duality pairing

$$\langle \cdot, \cdot \rangle : E_x^* \times E_x \rightarrow \mathbb{R}$$

induces a dual pairing

$$\langle \cdot, \cdot \rangle : \mathcal{A}^0(E^*) \times \mathcal{A}^0(E) \rightarrow \mathcal{A}^0$$

We can use this induced pairing to define a connection on E^* , also called ∇ , by the following formula

$$d\langle \xi, \eta \rangle = \langle \nabla \xi, \eta \rangle + \langle \xi, \nabla \eta \rangle \quad ; \quad \xi \in \mathcal{A}^0(E) \quad \eta \in \mathcal{A}^0(E^*)$$

Let's see how this relates to the connection form and the curvature. If $s = (s_1, \dots, s_r)$ is a local frame of E , let $t = (t^1, \dots, t^r)$ a local frame of E^* that is dual to s , that is

$$\langle t^i, s_j \rangle = \delta_{ij} \quad \text{or} \quad \langle t, s \rangle = I_r$$

where s is a row vector and t is a column vector. If $\omega = (\omega_j^i)_{ij}$ is the matrix of the connection form, then:

$$0 = d\delta_{ij} = d\langle t^i, s_j \rangle = \langle \nabla t^i, s_j \rangle + \langle t^i, \nabla s_j \rangle = \langle \nabla t^i, s_j \rangle + \langle t^i, \sum_k s_k \omega_j^k \rangle = \langle \nabla t^i, s_j \rangle + \omega_j^i$$

that is,

$$\nabla t^i = - \sum_j \omega_j^i t^j \quad \text{or} \quad \nabla t = -\omega t$$

Thus, if $\eta = \sum \eta_i t^i$ is an arbitrary section of E^* , we have (following the same calculations of (B.1)):

$$\nabla \eta = d\eta - \eta \omega$$

Analogous calculations also show that, if Ω is the curvature form of E with respect to a local frame s , then, with respect to the dual frame t , we have

$$\nabla^2 t = -\Omega t$$

To summarize, if s is a local frame on E , and t is the dual frame on E^* , if ω_j^i is the connection form of E with respect to s and Ω is the curvature form, we have:

$$\begin{aligned}\nabla s &= s \cdot \omega \quad ; \quad \nabla^2 s = s \Omega \\ \nabla t &= -\omega \cdot t \quad ; \quad \nabla^2 t = -\Omega t\end{aligned}$$

Now we consider two vector bundles E, F over the same base space M , with connections ∇^E and ∇^F , then we can define natural connections $\nabla^E \oplus \nabla^F$ on the Whitney sum $E \oplus F$ and $\nabla^{E \otimes F}$ on the tensor product $E \otimes F$. The former is trivial and the latter is given by

$$\nabla^{E \otimes F} = \nabla^E \otimes I_F + I_E \otimes \nabla^F$$

It is easy to see that the associated curvatures are

$$\begin{aligned}R_{E \oplus F} &= R_E \oplus R_F \\ R_{E \otimes F} &= R_E \otimes I_F + I_E \otimes R_F\end{aligned}$$

If we chose local frames $s = (s_1, \dots, s_r)$ and $t = (t_1, \dots, t_l)$ of E and F , and denote by $\omega_E, \omega_F, \Omega_E, \Omega_F$ the connection and curvature forms of E and F with respect to this local frames, we have

$$\omega_{E \oplus F} = \begin{pmatrix} \omega_E & 0 \\ 0 & \omega_F \end{pmatrix} \quad \text{and} \quad \Omega_{E \oplus F} = \begin{pmatrix} \Omega_E & 0 \\ 0 & \Omega_F \end{pmatrix}$$

also

$$\omega_{E \otimes F} = \omega_E \otimes I_F + I_E \otimes \omega_F \quad \text{and} \quad \Omega_{E \otimes F} = \Omega_E \otimes I_F + I_E \otimes \Omega_F$$

It is clear that this construction generalizes to finite tensor products and finite sums (or a finite combination of both)¹. For example, lets consider $E \otimes F \otimes W$, with local frames s, t and u .

$$\begin{aligned}\nabla^{E \otimes F \otimes W} &= \nabla^E \otimes I_{F \otimes W} + I_E \otimes \nabla^{F \otimes W} \\ &= \nabla^E \otimes I_F \otimes I_W + I_E \otimes \nabla^F \otimes I_W + I_E \otimes I_F \otimes \nabla^W\end{aligned}$$

An interesting case is $E^{\otimes p}$, on which we have the connection

$$\nabla^{E^{\otimes p}} = \sum_{j=1}^p I_E^{\otimes j-1} \otimes \nabla^E \otimes I_E^{\otimes (p-j)}$$

¹ It is of special interest to consider the case $\text{End}(E) = E \otimes E^*$. Consider a local frame field s on E and it's dual t on E^* . Let ξ be $\text{End}(E)$ valued p -form, that is

$$\xi = \sum \xi_j^i s_i \otimes t^j \quad ; \quad \xi_j^i \in \mathcal{A}^p$$

Let's remember that the exterior power $\Lambda^k(E)$ is obtained by taking the quotient of $E^{\otimes k}$ by the two sided ideal I given by

$$I = \{x_1 \otimes \cdots \otimes x_k \in E^{\otimes k} ; x_i = x_j \text{ for some } i \neq j\}$$

With those tools at hand, we can calculate the connection of the exterior power $\Lambda^k E$. If we chose a local frame s

$$\nabla^{\Lambda^k E}(s_{i_1} \wedge \cdots \wedge s_{i_k}) = \sum_j (-1)^{j+1} s_{i_1} \wedge \cdots \wedge \nabla^E s_{i_j} \wedge \cdots \wedge s_{i_k}$$

Now, if, we suppose that k is maximal, that is $k = \text{rank}(E) = r$, then, given a local frame s , the only element on $\Lambda^k E$ modulo $\mathcal{A}^0$ is $s_1 \wedge \cdots \wedge s_r$. Also we have

$$\begin{aligned} s_1 \wedge \cdots \wedge \nabla^E s_j \wedge \cdots \wedge s_r &= s_1 \wedge \cdots \wedge \left(\sum_i \omega_j^i s_i \right) \wedge \cdots \wedge s_r \\ &= (-1)^{j-1} \omega_j^j s_1 \wedge \cdots \wedge s_r \end{aligned}$$

thus

$$\nabla^{\Lambda^r E} = \sum_j \omega_j^j s_1 \wedge \cdots \wedge s_r$$

that is, the connection form on $\Lambda^r E$ is the trace of the original connection form on E

$$\omega^{\Lambda^r E} = \text{Tr}(\omega^E)$$

Then

$$\begin{aligned} \nabla \xi &= \sum_{ij} \nabla(\xi_j^i s_i \otimes t^j) = \sum_{ij} (((\nabla s_i) \otimes t^j + s_i \otimes (\nabla t^j)) \wedge \xi_j^i + d\xi_j^i(s_i \otimes t^j)) \\ &= \sum_{ij} ((-1)^p \xi_j^i \wedge ((\nabla s_i) \otimes t^j + s_i \otimes (\nabla t^j)) + d\xi_j^i(s_i \otimes t^j)) \\ &= \sum_{ij} (d\xi_j^i(s_i \otimes t^j) + (-1)^p \xi_j^i \wedge ((\nabla s_i) \otimes t^j + s_i \otimes (\nabla t^j))) \\ &= \sum_{ij} (d\xi_j^i(s_i \otimes t^j) + (-1)^p \xi_j^i \wedge ((\sum_k s_k \omega_i^k) \otimes t^j + s_i \otimes (-\sum_l \omega_l^j t^l))) \\ &= \sum_{ij} \left(d\xi_j^i + (-1)^p \sum_k (\xi_j^k \wedge \omega_k^i - \xi_k^i \wedge \omega_j^k) \right) s_i \otimes t^j \end{aligned}$$

The curvature R of ∇ on E is in fact a $\text{End}(E)$ valued 2-form and we write,

$$R = \sum \Omega_j^i s_i \otimes t^j$$

On the other hand, as $\Omega = d\omega + \omega \wedge \omega$ we have:

$$\begin{aligned}\Omega^{\wedge^r E} &= d\omega^{\wedge^r E} + \omega^{\wedge^r E} \wedge \omega^{\wedge^r E} \\ &= d(\text{Tr}(\omega^E) + \text{Tr}(\omega^E) \wedge \text{Tr}(\omega^E)) \\ &= \text{Tr}(d\omega^E) + \text{Tr}(\omega^E \wedge \omega^E) = \text{Tr}(\Omega^E)\end{aligned}$$

Now, a local frame $\tilde{s}$ of $\Lambda^r E$ is just a volume form, that is, we can choose $s = (s_1, \dots, s_r)$ is a local frame of E such that $\tilde{s} = f \cdot \det(s) = f \cdot s_1 \wedge \dots \wedge s_r$, thus

$$\tilde{s}\Omega^{\wedge^r E} = \text{Tr}(s\Omega^E)$$

In particular, if E is an integrable sub-bundle of TM , we have

$$\Omega^{\wedge^r E^*} = \text{Tr}(\Omega^{E^*}) = \text{Tr}(\Omega_{|E}^{T^*M}) \quad (\text{B.4})$$

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