Transversal families of piecewise expanding maps

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SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

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Resumo

Seja $t : [a, b] \rightarrow f_t$ uma família C^2 “boa” de transformações unimodais expansoras por pedaços com um ponto crítico c , que é transversal às classes topológicas de tais transformações. Dado um observável lipschitziano ϕ , considere a função

$$\mathcal{R}_\phi(t) = \int \phi \, d\mu_t,$$

onde μ_t é a única probabilidade invariante absolutamente contínua de f_t . Mostramos um teorema do limite central para o módulo de continuidade de $\mathcal{R}_\phi$, isto é

$$\lim_{h \rightarrow 0} m \left\{ t \in [a, b] : t + h \in [a, b] \text{ e } \frac{1}{\Psi(t) \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_\phi(t + h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\}$$

converge para

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{s^2}{2}} ds.$$

Vamos considerar agora $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ uma transformação expansora de classe $C^{2+\varepsilon}$ e $v : \mathbb{S}^1 \rightarrow \mathbb{R}$ uma função periódica de classe $C^{1+\varepsilon}$. Mostramos que a única solução limitada da equação cohomológica torcida

$$v(x) = \alpha(f(x)) - Df(x)\alpha(x)$$

ou é de classe $C^{1+\varepsilon}$ ou não possui derivada em ponto algum. Mostramos também que se α não possui derivada em ponto algum, então o módulo de continuidade de α satisfaz um teorema do limite central, isto é, existe $\sigma > 0$ tal que

$$\lim_{h \rightarrow 0} \mu \left\{ x : \frac{\alpha(x + h) - \alpha(x)}{\sigma \ell h \sqrt{-\log |h|}} \leq y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{t^2}{2}} dt,$$

onde μ é a probabilidade invariante absolutamente contínua associada a f .

Palavras chave. resposta linear, transformações unimodais, transformações expansoras, teorema do limite central.

Abstract

Let $t \in [a, b] \mapsto f_t$ be a C^2 family of “good” C^4 piecewise expanding unimodal maps, with a critical point c , that is transversal to the topological classes of such maps. Given a lipchitzian observable ϕ consider the function

$$\mathcal{R}_\phi(t) = \int \phi d\mu_t,$$

where μ_t is the unique absolutely continuous invariant probability of f_t . We show a central limit theorem for the modulus of continuity of $\mathcal{R}_\phi$, that is

$$\lim_{h \rightarrow 0} m \left\{ t \in [a, b] : t + h \in [a, b] \text{ and } \frac{1}{\Psi(t)\sqrt{-\log|h|}} \left(\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\}$$

converges to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{s^2}{2}} ds.$$

Now, let us consider a $C^{2+\varepsilon}$ expanding map $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and a $C^{1+\varepsilon}$ periodic function $v : \mathbb{S}^1 \rightarrow \mathbb{R}$. We show that the unique bounded solution of the twisted cohomological equation

$$v(x) = \alpha(f(x)) - Df(x)\alpha(x)$$

is either of class $C^{1+\varepsilon}$ or nowhere differentiable. We also prove that if α is nowhere differentiable, then the modulus of continuity of α satisfies a central limit theorem, that is, there is $\sigma > 0$ such that

$$\lim_{h \rightarrow 0} \mu \left\{ x : \frac{\alpha(x+h) - \alpha(x)}{\sigma \ell h \sqrt{-\log|h|}} \leq y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{t^2}{2}} dt,$$

where μ is the absolutely continuous invariant probability of f .

Key words. linear response, unimodal maps, expanding maps, central limit theorem.

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Introduction

Let f_t be a smooth family of piecewise smooth maps on a manifold M , and let us suppose that for each f_t there is a physical (or SBR) probability μ_t on M . Given an observable $\phi : M \rightarrow \mathbb{R}$, we can ask if the function

$$\begin{aligned} \mathcal{R}_\phi : [0, 1] &\longrightarrow \mathbb{R} \\ t &\longmapsto \int \phi d\mu_t \end{aligned}$$

is differentiable and if we can find an explicit formula for its derivative. The study of this question is the so called *linear response problem*.

D. Ruelle showed that $\mathcal{R}_\phi$ is differentiable and also gave the formula for $\mathcal{R}_\phi'$, in the case of smooth uniformly hyperbolic dynamical systems (see Ruelle in [29] and [30], and Baladi and Smania in [5] for more details).

In the setting of smooth families of piecewise expanding unimodal maps, Baladi and Smania (see [3]) proved that if we have a C^2 family of C^3 piecewise expanding unimodal maps then $\mathcal{R}_\phi$ is differentiable in t_0 , with $\phi \in C^{1+Lip}$, *provided* that the family f_t is tangent to the topological class of f_{t_0} at $t = t_0$. It turns out that the family $s \mapsto f_s$ is tangent to the topological class of f_t at the parameter t if and only if

$$J(f_t, v_t) = \sum_{k=0}^{M_t-1} \frac{v_t(f_t^k(c))}{Df_t^k(f_t(c))} = 0,$$

where $v_t = \partial_s f_s|_{s=t}$ and M_t is either the period of the critical point c if c is periodic, or ∞ , otherwise (see [4]). Now, let us consider a C^2 family of C^4 piecewise expanding unimodal maps that is *transversal* to the topological classes of piecewise unimodal maps, that is

$$J(f_t, v_t) = \sum_{k=0}^{M_t-1} \frac{v_t(f_t^k(c))}{Df_t^k(f_t(c))} \neq 0 \tag{0.0.1}$$

for every t .

Baladi and Smania (see [3] and [6]) proved that $\mathcal{R}_\phi$ is not differentiable, for most of the parameters t , even if ϕ is quite regular. One can ask what is the regularity of the function $\mathcal{R}_\phi$ in this case.

In Chapter 2, we will show a Central Limit Theorem for the modulus of continuity of the function $\mathcal{R}_\phi$ where ϕ is a lipschitzian observable, that is,

$$\lim_{h \rightarrow 0} m \left\{ t \in [a, b] : t + h \in [a, b] \text{ and } \frac{1}{\Psi(t) \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\}$$

converges to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{s^2}{2}} ds,$$

where $\Psi(t)$ is an appropriate normalization.

In particular $\mathcal{R}_\phi$ is not a lipschitzian function on any subset of $[a, b]$ with positive measure.

A central limit theorem-type was also proved for the modulus of continuity of some nowhere differentiable functions. The first example of such functions was the Weierstrass function.

In the beginning of the nineteenth century, many mathematicians believed that a continuous function is differentiable at most of its domain. In 1806, André-Marie Ampère tried to give a theoretical proof for this fact but, in 1872, Karl Weierstrass surprised the mathematical community by proving that this conjecture was false. He introduced an example of a function which was everywhere continuous but nowhere differentiable

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x),$$

where a is a real number with $0 < a < 1$, b is an odd integer and $ab > 1 + 3\pi/2$.

In 1916, G. H. Hardy [14] proved that the function W defined above is continuous and nowhere differentiable if $0 < a < 1$, $ab \geq 1$. The constant b does not need to be an integer.

After the publication of the Weierstrass function, many mathematicians gave their contributions on this subject. Among many other examples, we have the Takagi function. In 1903, T. Takagi [34] presented another example of a continuous function nowhere differentiable given by

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \inf_{m \in \mathbb{Z}} |2^k x - m|.$$

For a more detailed discussion of Takagi function we refer the reader to [1] and for more examples of nowhere differentiable functions to [35].

Observe that the Weierstrass and Takagi functions satisfy the cohomological equations

$$v(x) = W(x) - aW(bx), \quad \text{where} \quad v(x) = \cos(\pi x)$$

and

$$v(x) = T(x) - \frac{1}{2}T(2x), \quad \text{where} \quad v(x) = \inf_{m \in \mathbb{Z}} |x - m|,$$

respectively.

In 2000, Y. Heurteaux [15] presented a generalization of the Weierstrass function given by

$$F(x) = \sum_{n=0}^{\infty} b^{-n} g(b^n x),$$

where $1 < b < \infty$ and g is an almost periodic $C^{1+\varepsilon}$ function. Such function is called Weierstrass-type function.

Again, we can observe that such function satisfies a cohomological equation. Indeed,

$$g(x) = F(x) - \frac{1}{b}F(bx).$$

Heurteaux proved that there are only two mutually exclusive cases: F is either of class $C^{1+\varepsilon}$ or nowhere differentiable. We emphasize that in these three examples, the functions satisfy a cohomological equation

$$v(x) = \alpha(x) - \frac{1}{Df(x)}\alpha(f(x)).$$

However, in all cases, f is a linear function. We are interested in knowing what happens when f is a non-linear function. This subject will be discussed in Chapter 3.

Consider a C^1 family of C^1 maps of the circle $t \in (-\delta, \delta) \mapsto f_t$. If f_0 is an expanding map, then there is δ_0 such that for all $t \in (-\delta_0, \delta_0)$, f_t is also an expanding map and f_t is topologically conjugate to f_0 (see [19]), that is, there is a homeomorphism h_t such that $h_t \circ f_0(x) = f_t \circ h_t(x)$. Since for each $x \in \mathbb{S}^1$ the map $t \mapsto h_t(x)$ is of class C^1 for all $t \in (-\delta, \delta)$, we can differentiate this equation with respect to t to obtain

$$v_t(x) = \alpha_t(f_t(x)) - \partial_x f_t(x) \alpha_t(x),$$

where $\alpha_t(x) := (\partial_t h_t) \circ h_t^{-1}(x_t)$ and $v_t(x) := \partial_t f_t(x)$. Therefore, fixing t , we have the *twisted cohomological equation*

$$v(y) = \alpha(f(y)) - Df(y)\alpha(y),$$

which has a unique bounded solution given by

$$\alpha(x) = - \sum_{n=1}^{\infty} \frac{v(f^{n-1}(x))}{Df^n(x)}.$$

Following Heurteaux [15], we proved that one of the following statements holds:

- α is of class $C^{1+\varepsilon}$
- α is nowhere differentiable.

Another subject discussed in Chapter 3 is the modulus of continuity of α . In [11] and [12] N. Gamkrelidze used the probability theory to show a central limit theorem type for the modulus of continuity of Weierstrass and Takagi functions.

Motivated by these results and [25], we proved that if α is nowhere differentiable, then it also satisfies a central limit theorem, that is

$$\lim_{h \rightarrow 0} \mu \left\{ x : \frac{\alpha(x+h) - \alpha(x)}{\sigma \ell h \sqrt{-\log |h|}} \leq y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{t^2}{2}} dt,$$

where μ is the absolutely continuous invariant measure with respect to the Lebesgue measure and $\sigma \ell$ is an appropriate normalization.

Before Chapters 2 and 3, we present some preliminaries. In Chapter 1 we introduce some notations, concepts and basic results that will be useful throughout the text.

Preliminaries

We dedicate this chapter to introducing some notations and to presenting some basic results that we will use in this work.

We started by setting in section 1.1 the expanding map of the circle and deducing the twisted cohomological equation. And, in section 1.2, we state the bounded distortion lemma. Although we will deal with this subject again only in Chapter 3, we decided to introduce these concepts here because they are well-known results or basis for others.

Section 1.3 presents the main tool used to prove Theorem 2.0.7 and Theorem 3.1.1, the Ruelle-Perron-Frobenius operator.

We conclude the chapter giving a brief overview on some concepts of probability theory.

Our purpose is to make a brief introduction to the theory and therefore, the proofs will be omitted.

1.1 Expanding maps of the circle and the twisted cohomological equation

We will denote by $\mathbb{S}^1$ the unit circle, that is, $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Let us consider a continuous transformation $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$.

Let $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$ be a covering map defined by

$$\pi(x) = (\cos 2\pi x, \sin 2\pi x).$$

Given such map f , a *lift* of f is a continuous map $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\pi \circ F = f \circ \pi. \quad (1.1.1)$$

There are infinitely many different lifts for a given map f . For instance, if F is a lift of f , then $G : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$G(x) = F(x + n) + m,$$

is also a lift, for any $m, n \in \mathbb{Z}$. Indeed, any lift of f can be written this way for a suitable choice of n and m .

The *degree* of f is defined and denoted by

$$\deg f = F(x + 1) - F(x).$$

By (1.1.1), $F(x + 1) - F(x)$ is an integer which is independent of both x and F .

We say that a transformation $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is an *expanding map* of the circle if its lift F is a differentiable function and there is $1 < \lambda < \infty$ such that

$$\inf_{x \in \mathbb{R}} |F'(x)| = \inf_{x \in [0,1]} |F'(x)| \geq \lambda > 1. \quad (1.1.2)$$

Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be continuous functions, where X and Y are topological spaces. We say that f is *topologically conjugate* to g if there exists a homeomorphism $h : X \rightarrow Y$ such that $h \circ f = g \circ h$.

An important fact about the degree of f is that, two expanding maps of the circle f and g with the same degree are topologically conjugate (see [19] for more details).

Let us denote by $C^1(\mathbb{S}^1)$ the set of all transformations of class C^1 mapping $\mathbb{S}^1$ in $\mathbb{S}^1$ and let us consider a family of C^1 maps

$$t \in (-\delta, \delta) \mapsto f_t \in C^1(\mathbb{S}^1). \quad (1.1.3)$$

One can prove that if two expanding maps are sufficiently close to each other, then they have the same degree. Thus, if f_0 is an expanding map, then there is a δ_0 such that for all $t \in (-\delta_0, \delta_0)$ the function f_t is also an expanding map with the same degree of f_0 . Therefore, there exists a homeomorphism h_t such that

$$h_t \circ f_0(x) = f_t \circ h_t(x). \quad (1.1.4)$$

The following proposition was proved in [33]:

Proposition 1.1.1. *Let us consider a family of C^1 maps as defined in Eq. (1.1.3), where f_0 is an expanding map. Then there are $\delta_0 > 0$ and a homeomorphism h_t such that for all $t \in (-\delta_0, \delta_0)$ Eq. (1.1.4) is satisfied. Furthermore, for each $x \in \mathbb{S}^1$ the function*

$$t \mapsto h_t(x)$$

is of class C^1 and there is $C > 0$ such that

$$|\partial_t h_t(x)| \leq C$$

for all $t \in (-\delta_0, \delta_0)$ and $x \in \mathbb{S}^1$.

By Proposition 1.1.1, we can differentiate the Eq. (1.1.4) with respect to t

$$\partial_t h_t(f_0(x)) = \partial_t f_t(h_t(x)) + \partial_x f_t(h_t(x)) \partial_t h_t(x). \quad (1.1.5)$$

By defining $y_t = h_t(x)$, we have

$$f_0(x) = f_0(h_t^{-1}(y_t)) = h_t^{-1}(f_t(y_t)).$$

Thus, the Eq. (1.1.5) can be written as

$$\partial_t h_t(h_t^{-1}(f_t(y_t))) = \partial_t f_t(y_t) + \partial_x f_t(y_t) \cdot \partial_t h_t(h_t^{-1}(y_t)). \quad (1.1.6)$$

Now, let

$$\alpha_t(y_t) := (\partial_t h_t) \circ h_t^{-1}(y_t)$$

and

$$v_t(x) := \partial_t f_t(x).$$

Since h_t is a homeomorphism, there is a x such that $y_t = h_t(x)$ for all y_t . Then we obtain

$$v_t(y_t) = \alpha_t(f_t(y_t)) - \partial_x f_t(y_t) \alpha_t(y_t). \quad (1.1.7)$$

By fixing t , we have:

$$v(y) = \alpha(f(y)) - Df(y)\alpha(y), \quad (1.1.8)$$

where Df denotes the derivative of f with respect to y .

This equation is called *twisted cohomological equation*.

In [33] was proved that if f is an expanding map, the unique bounded function α satisfying Eq. (1.1.8) is given by the formula

$$\alpha(y) = - \sum_{n=1}^{\infty} \frac{v(f^{n-1}(y))}{Df^n(y)}. \quad (1.1.9)$$

This fact is stated in the following proposition.

Proposition 1.1.2. *Let f be a C^1 expanding map of the circle and let us suppose that v is a bounded function. Then the Eq. (1.1.8) has a unique bounded solution α satisfying*

$$\sup |\alpha| \leq \frac{\sup |v|}{1 - \sup |Df|^{-1}}. \quad (1.1.10)$$

If v is a continuous function, then α is also a continuous function.

Proof. Let

$$\alpha(y) = - \sum_{n=1}^{\infty} \frac{v(f^{n-1}(y))}{Df^n(y)}.$$

Observe that

$$\left| \frac{v(f^{n-1}(x))}{Df^n(x)} \right| \leq \sup |v| (\sup |Df|^{-1})^n.$$

Therefore, since f is an expanding map, the series is uniformly convergent in x . In particular, if v is a continuous function, then α is also a continuous function.

Note that α is a solution of the Eq. (1.1.8). Indeed

$$\begin{aligned} \alpha(f(x)) - Df(x)\alpha(x) &= - \sum_{n=1}^{\infty} \frac{v(f^{n-1}(f(x)))}{Df^n(f(x))} + Df(x) \sum_{n=1}^{\infty} \frac{v(f^{n-1}(x))}{Df^n(x)} \\ &= - \sum_{n=1}^{\infty} \frac{v(f^{n-1}(f(x)))}{Df^n(f(x))} + v(x) + Df(x) \sum_{n=2}^{\infty} \frac{v(f^{n-1}(x))}{Df^n(x)} \\ &= - \sum_{n=1}^{\infty} \frac{v(f^{n-1}(f(x)))}{Df^n(f(x))} + v(x) + \sum_{n=2}^{\infty} \frac{v(f^{n-2}(f(x)))}{Df^{n-1}(f(x))} \\ &= v(x). \end{aligned}$$

In order to prove the unicity, we assume that there exists another bounded solution of Eq. (1.1.8) called β . Then

$$\alpha(f(x)) - Df(x)\alpha(x) = \beta(f(x)) - Df(x)\beta(x).$$

Thus,

$$\alpha(x) - \beta(x) = \frac{\alpha(f(x)) - \beta(f(x))}{Df(x)}.$$

Iterating this equation n times, we obtain

$$\alpha(x) - \beta(x) = \frac{\alpha(f^n(x)) - \beta(f^n(x))}{Df^n(x)}.$$

Hence,

$$|\alpha(x) - \beta(x)| \leq \|\alpha - \beta\|_{\infty} \left(\frac{1}{\lambda} \right)^n,$$

where $0 < 1/\lambda < 1$. Since this estimate holds for all n , we conclude that $\alpha = \beta$. $\square$

In Chapter 3 we study a little more about α . If f is of class $C^{2+\varepsilon}$ and v is of class $C^{1+\varepsilon}$, we show that either α is of class $C^{1+\varepsilon}$ or α is nowhere differentiable and we also study about the modulus of continuity of α , proving that it satisfies a central limit theorem-type.

For this, we need some concepts and results from Ergodic Theory and Probability which will be dealt in the following section.

1.2 Bounded distortion

Let F be a lift of f as defined in section 1.1. We have seen that $F(x+1) = F(x) + k, k \in \mathbb{Z}$. Thus, $DF(x+1) = DF(x)$. Therefore, DF is a periodic function.

Let us define $V : \mathbb{R} \rightarrow \mathbb{R}$ by $V = v \circ \pi$. Then,

$$V(x+1) = v \circ \pi(x+1) = v \circ \pi(x) = V(x).$$

Therefore, V is also a periodic function. Hence,

$$V(F^{n-1}(x+1)) = V(F^{n-1}(x) + k) = V(F^{n-1}(x)).$$

Thus, the equation α defined in (1.1.9) is the uniform limit of periodic functions. And, therefore it is a periodic function.

Remark 1.2.1. We consider the same notation for $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and its lift ; as well as for $v : \mathbb{S}^1 \rightarrow \mathbb{R}$ and its lift.

An important property of the expanding maps of the circle is the bounded distortion, which gives us the idea that the iterates f^n of f maps an interval I to $f^n(I)$ without distorting too much its length. This property is stated in the following lemma. We will use it to deduce some inequalities that will be of great importance in Chapter 3.

Lemma 1.2.2. (Bounded Distortion) *Let f be the lift of a C^2 expanding map of the circle. Then there is a constant $C > 0$ such that for all $n \in \mathbb{N}$*

$$\left| \frac{Df^n(x)}{Df^n(y)} \right| \leq e^{C|f^n(x) - f^n(y)|}.$$

Proof. Observe that $\log Df$ is lipschitzian. Indeed

$$|\log Df(x) - \log Df(y)| = |(\log Df)'(\theta)| |x - y| = \left| \frac{D^2 f(\theta)}{Df(\theta)} \right| |x - y| \leq C_1 |x - y|,$$

$\theta \in [x, y]$. The last inequality holds because f is an expanding map of class C^2 . Therefore,

$$\begin{aligned}
\left| \log \frac{Df^n(x)}{Df^n(y)} \right| &= |\log Df^n(x) - \log Df^n(y)| \\
&= \left| \log \left(\prod_{j=0}^{n-1} Df(f^j(x)) \right) - \log \left(\prod_{j=0}^{n-1} Df(f^j(y)) \right) \right| \\
&= \left| \sum_{j=0}^{n-1} \log Df(f^j(x)) - \sum_{j=0}^{n-1} \log Df(f^j(y)) \right| \\
&= \left| \sum_{j=0}^{n-1} \log Df(f^j(x)) - \log Df(f^j(y)) \right| \\
&\leq \sum_{j=0}^{n-1} |\log Df(f^j(x)) - \log Df(f^j(y))| \\
&\leq \sum_{j=0}^{n-1} C_1 |f^j(x) - f^j(y)| \leq C_1 \sum_{j=1}^n \left(\frac{1}{\lambda} \right)^j |f^n(x) - f^n(y)| \\
&\leq C |f^n(x) - f^n(y)|.
\end{aligned}$$

Hence,

$$\left| \frac{Df^n(x)}{Df^n(y)} \right| \leq e^{C|f^n(x) - f^n(y)|}.$$

□

Corollary 1.2.3. *Under the same assumptions of Lemma 1.2.2, we have:*

$$\left| \frac{Df^n(x)}{Df^n(y)} - 1 \right| \leq C |f^n(x) - f^n(y)|.$$

Below, we follow with some inequalities obtained from Lemma 1.2.2 that will be used in Chapter 3.

Consider the interval $[x, x+h] \subset \mathbb{S}^1$. Let us first denote by $N(x, h)$ the unique integer such that

$$\frac{1}{|Df^{N(x,h)+1}(x)|} \leq |h| \leq \frac{1}{|Df^{N(x,h)}(x)|}. \quad (1.2.1)$$

Lemma 1.2.4. *For μ -a.e. $x \in \mathbb{S}^1$,*

$$\lim_{h \rightarrow 0} \frac{N(x, h)}{-\log |h|} = \frac{1}{L},$$

where $L := \int \log |Df| d\mu$ is the Lyapunov exponent of f .

Proof. By Ergodic Birkhoff Theorem, for μ -a.e. $x \in \mathbb{S}^1$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(x)| &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \prod_{j=0}^{n-1} |Df(f^j(x))| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |Df(f^j(x))| = \int \log |(Df(x))| d\mu = L. \end{aligned}$$

By Eq. (1.2.1) we have

$$|Df^{N(x,h)}(x)| \leq \frac{1}{|h|} \leq |Df^{N(x,h)+1}(x)|.$$

Therefore,

$$\frac{1}{N(x,h)} \log |Df^{N(x,h)}(x)| \leq \frac{1}{N(x,h)} \log \frac{1}{|h|} \leq \frac{1}{N(x,h)} \log |Df^{N(x,h)+1}(x)|.$$

Hence, for μ -a.e. $x \in \mathbb{S}^1$,

$$\frac{1}{N(x,h)} \log \frac{1}{|h|} \xrightarrow{N(x,h) \rightarrow \infty} \int \log |Df(x)| d\mu = L.$$

□

Lemma 1.2.5. *Let $v : \mathbb{S}^1 \rightarrow \mathbb{R}$ be a map of class $C^{1+\varepsilon}$ and $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a map of class $C^{2+\varepsilon}$. If $y \in [x, x+h]$ and $N(x,h)$ is as defined in Eq.(1.2.1), then there exists $K > 0$ such that the following statements hold*

(a)

$$\left| \sum_{n=1}^{N(x,h)} (v'(f^{n-1}(y)) - v'(f^{n-1}(x))) \frac{1}{Df(f^{n-1}(x))} \right| \leq K.$$

(b)

$$\left| \sum_{n=1}^{N(x,h)} v'(f^{n-1}(x)) \frac{1}{Df(f^{n-1}(x))} \left(\frac{Df^{n-1}(y)}{Df^{n-1}(x)} - 1 \right) \right| \leq K.$$

(c)

$$\left| \sum_{n=1}^{N(x,h)} \sum_{j=1}^n \frac{D^2 f(f^{j-1}(y)) - D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(y)) (Df(f^{j-1}(y)))^2} \right| \leq K.$$

(d)

$$\left| \sum_{n=1}^{N(x,h)} \sum_{j=1}^n \frac{1}{Df^{n-j}(f^j(x))} \left(\frac{Df^{n-j}(f^j(x))}{Df^{n-j}(f^j(y))} - 1 \right) \right| \leq K.$$

Proof. The proof of item (a) can be done by the following calculation

$$\begin{aligned}
& \left| \sum_{n=1}^{N(x,h)} (v'(f^{n-1}(y)) - v'(f^{n-1}(x))) \frac{1}{Df(f^{n-1}(x))} \right| \\
& \leq K_1 \sum_{n=1}^{N(x,h)} |f^{n-1}(y) - f^{n-1}(x)|^\varepsilon \frac{1}{|Df(f^{n-1}(x))|} \\
& \leq K_2 \sum_{n=1}^{N(x,h)} \left(\frac{1}{\lambda} \right)^{N-n+1} \leq K.
\end{aligned}$$

In the third inequality we used that v is of class $C^{1+\varepsilon}$.

In order to prove item (b), note that

$$\begin{aligned}
& \left| \sum_{n=1}^{N(x,h)} v'(f^{n-1}(x)) \frac{1}{Df(f^{n-1}(x))} \left(\frac{Df^{n-1}(y)}{Df^{n-1}(x)} - 1 \right) \right| \\
& \leq K_1 \left| \sum_{n=1}^{N(x,h)} \left(\frac{Df^{n-1}(y)}{Df^{n-1}(x)} - 1 \right) \right| \leq K_2 \sum_{n=1}^{N(x,h)} |f^{n-1}(x) - f^{n-1}(y)| \\
& \leq K_2 \sum_{n=1}^{N(x,h)} \left(\frac{1}{\lambda} \right)^{N(x,h)-n+1} < K.
\end{aligned}$$

In the first inequality we used that v' and $1/Df$ are bounded and, in the second inequality we used Corollary 1.2.3.

Now, for item (c) we have

$$\begin{aligned}
& \left| \sum_{n=1}^{N(x,h)} \sum_{j=1}^n \frac{D^2 f(f^{j-1}(y)) - D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(y)) [Df(f^{j-1}(y))]^2} \right| \\
& \leq \sum_{n=1}^{N(x,h)} \sum_{j=1}^n \frac{|D^2 f(f^{j-1}(y)) - D^2 f(f^{j-1}(x))|}{|Df^{n-j}(f^j(y))| |Df(f^{j-1}(y))|^2} \\
& \leq K_2 \sum_{n=1}^{N(x,h)} \sum_{j=1}^n \frac{|f^{j-1}(y) - f^{j-1}(x)|^\varepsilon}{\lambda^{n-j}} \\
& \leq K_2 \sum_{n=1}^{N(x,h)} \sum_{j=1}^n \left(\frac{1}{\lambda^\varepsilon} \right)^{n-j} \left(\frac{1}{\lambda^\varepsilon} \right)^{N(x,h)-j+1} \\
& \leq K_2 \sum_{n=1}^{N(x,h)} \left(\frac{1}{\lambda^\varepsilon} \right)^{N(x,h)-n+1} \sum_{j=1}^n \left(\frac{1}{\lambda^{2\varepsilon}} \right)^{n-j} \leq K.
\end{aligned}$$

In the second inequality we used that f is of class $C^{2+\varepsilon}$.

For item (d), we use Lemma 1.2.2

$$\begin{aligned}
& \left| \sum_{n=1}^{N(x,h)} \sum_{j=1}^n \frac{1}{Df^{n-j}(f^j(x))} \left(\frac{Df^{n-j}(f^j(x))}{Df^{n-j}(f^j(y))} - 1 \right) \right| \\
& \leq \sum_{n=1}^{N(x,h)} \sum_{j=1}^n \left| \frac{1}{Df^{n-j}(f^j(x))} \right| \left| \frac{Df^{n-j}(f^j(x))}{Df^{n-j}(f^j(y))} - 1 \right| \\
& \leq K_1 \sum_{n=1}^{N(x,h)} \sum_{j=1}^n \left(\frac{1}{\lambda} \right)^{n-j} |f^n(x) - f^n(y)| \leq K.
\end{aligned}$$

□

1.3 The Ruelle-Perron-Frobenius operator

In this section we define the Ruelle-Perron-Frobenius operator which will be a very useful tool in the following chapters.

A function $\varphi : [0, 1] \rightarrow \mathbb{R}$ is of *bounded variation* if

$$var(\varphi) := \sup \left\{ \sum_{i=1}^n |\varphi(x_i) - \varphi_{x_{i-1}}| : 0 = x_0 < \dots < x_n = 1 \right\} < \infty.$$

If $\varphi \in L^1(m)$, then

$$var(\varphi) = \inf_{\phi \in \bar{\varphi}} var(\phi),$$

where

$$\bar{\varphi} = \{ \phi : [0, 1] \rightarrow \mathbb{R} : \varphi = \phi - \text{a.e.} \}.$$

Now, we define a subset of $L^1(m)$

$$BV = \{ \varphi \in L^1(m) : var(\varphi) < \infty \}.$$

By equipping it with the norm

$$|\varphi|_{BV} = var(\varphi) + |\varphi|_{L^1},$$

it becomes a Banach space. It can be shown that BV is dense in $(L^1, |\cdot|_{L^1})$ (see [16] for more details).

Definition 1.3.1. (Ruelle-Perron-Frobenius operator) Let X be a compact manifold, $\psi : X \rightarrow \mathbb{R}$ be a continuous map and $f : X \rightarrow X$ be an expanding map. The Ruelle-Perron-Frobenius operator (it is also known as transfer operator)

$$\mathcal{L}_{\psi,f} : C(X) \rightarrow C(X)$$

is defined by

$$\mathcal{L}_{\psi,f}(\varphi)(x) = \sum_{f(y)=x} e^{\psi(y)} \varphi(y),$$

where $\varphi \in C(X)$ and $x \in X$.

Theorem 1.3.2. (Ruelle)

Let $f : X \rightarrow X$ be an expanding map and X be a Riemannian compact manifold. Consider $\psi : X \rightarrow \mathbb{R}$ a Holder-continuous map. Then there exist a strictly positive continuous function $\rho : X \rightarrow \mathbb{R}$, a probability ν on X and $\theta > 0$ such that

(a) $\mathcal{L}_{\psi,f}\rho = \theta\rho$;

(b) $\mathcal{L}_{\psi,f}^*\nu = \theta\nu$;

(c) $\int_X \rho d\nu = 1$;

(d) ρ is the unique positive eigenfunction of $\mathcal{L}_{\psi,f}$ except by a multiplication by a constant.

(e) The probability $\mu = \rho\nu$ is f -invariant, exact, positive on open sets and satisfies

$$\log \theta = h_\mu(f) + \int_X \psi d\mu,$$

where $h_\mu(f)$ is the entropy of f with respect to μ .

(f) For all f -invariant probability $\eta \neq \mu$,

$$\log \theta > h_\eta(f) + \int_X \psi d\eta.$$

Proof. See for instance [27].

Definition 1.3.3. (Pressure) Let $\psi : X \rightarrow \mathbb{R}$ be a continuous function where X is a compact metric space, $f : X \rightarrow X$ be an expanding map and ν be a f -invariant measure. The pressure of ψ is defined by:

$$P(f, \psi) := \sup \left\{ h_\nu(f) + \int_X \psi d\nu; \nu \in \mathcal{M}_f(X) \right\},$$

where $h_\nu(f)$ denotes the entropy of f with respect to ν and $\mathcal{M}_f(X)$ denotes the space of the f -invariant probabilities.

Definition 1.3.4. (Equilibrium state) We say that $\mu \in \mathcal{M}_f(X)$ is an equilibrium state if

$$P(f, \psi) = h_\mu(f) + \int_X \psi d\mu.$$

Observe that if $\psi(x) = -\log |Df(x)|$, then

$$\mathcal{L}_f(\varphi)(x) = \sum_{f(y)=x} \frac{\varphi(y)}{|Df(y)|}. \quad (1.3.1)$$

During this work we will consider the Ruelle-Perron-Frobenius as in Eq. (1.3.1).

The Ruelle-Perron-Frobenius operator plays an important role in the investigation into the existence of absolutely continuous invariant measures as well as in the study of the central limit theorem for dynamical systems.

In [24], Lasota and Yorke proved the existence of a function h such that $\mathcal{L}_f(h) = h$ which implies that there exists an absolutely continuous invariant measure with respect to Lebesgue measure m when f is a C^2 piecewise monotonic map on $[0, 1]$ such that $\inf |Df| > 1$.

In [38], Wong generalizes the results of [24]. Considering a class of C^1 piecewise monotonic maps, such that Df is of bounded variation and $\inf |Df| > 1$, Wong has shown that the number of ergodic absolutely continuous invariant probability (a.c.i.p) for f is finite.

In [16], lemmas 6 and 7, we see that $\mathcal{L}_f$ is an operator on $L^1(m)$ satisfying $m(\mathcal{L}_f(\varphi)) = m(\varphi)$ for $\varphi \in L^1(m)$, $|\mathcal{L}_f(\varphi)|_{L^1} \leq |\varphi|_{L^1}$ for all $\varphi \in L^1(m)$ and there are $\gamma \in (0, 1)$, $\beta \in (0, \infty)$ and $n > 0$ such that

$$\text{var}(\mathcal{L}_f^n(\varphi)) \leq \gamma \text{var}(\varphi) + \beta |\varphi|_{L^1},$$

for all $\varphi \in L^1(m)$.

Considering an ergodic theorem of Ionescu-Tulcea and Marinescu [17], Hofbauer and Keller have shown the following decomposition theorem:

[16, Theorem 1]: Let $\mathcal{L}_f$ be the Ruelle-Perron-Frobenius operator and f be piecewise monotonic C^1 . Then

- (i) $\mathcal{L}_f : L^1(m) \rightarrow L^1(m)$ has a finite number of eigenvalues $\lambda_1, \dots, \lambda_m$ of modulus 1.
- (ii) If $F_i = \{\varphi \in L^1(m) : \mathcal{L}_f(\varphi) = \lambda_i(\varphi)\}$ for $1 \leq i \leq m$, then $F_i \subset BV$ and $\dim(F_i) < \infty$.

(iii) We can decompose the operator $\mathcal{L}_f$ as

$$\mathcal{L}_f = \sum_{i=1}^m \lambda_i \phi_i + \psi$$

where ϕ_i are projections onto the eigenspaces F_i , $|\phi_i|_{L^1} < 1$, and ψ is a linear operator on $L^1(m)$ with $\sup_{n \geq 1} |\psi^n|_{L^1} < \infty$. Moreover,

$$\phi_i \phi_j = 0 \quad \text{if } i \neq j \quad \text{and} \quad \phi_i \psi = 0 \quad \text{for all } i.$$

(iv) We have ψ is a linear operator on $(BV, |\cdot|_{BV})$, since $\psi(BV) \subset BV$. Furthermore, there are constants M and p satisfying $M > 0$ and $0 < p < 1$ such that $|\psi^n|_{BV} \leq Mp^n$, $n \leq 1$.

(v) We have 1 an eigenvalue of $\mathcal{L}_f$.

(vi) Denoting $\rho := \phi_1(1)$ and $\mu = \rho m$ we have

$$\rho \geq 0, \quad \int \rho \, dm = 1 \quad \text{and} \quad \mathcal{L}_f \rho = \rho.$$

Therefore, μ is a f -invariant probability on $[0, 1]$, and if ν is also a f -invariant and $\nu \ll m$, then $\nu \ll \mu$.

Therefore, $\mathcal{L}_f$ is a quasi-compact operator and if f is a piecewise expanding map, there is an absolutely continuous invariant probability and its density ρ is a function of bounded variation.

Considering the Ruelle-Perron-Frobenius operator defined in 1.3.1, if f is topologically mixing then it admits a unique absolutely continuous probability measure (see [37], corollary 3.6). Therefore, by item (e) from Theorem 1.3.2,

$$0 = \log 1 = \log \theta = h_\mu(f) + \int_X \psi \, d\mu = P(f, \psi),$$

where $\mu = \rho m$.

Remark 1.3.5. Let us consider a family of smooth deformations (1.1.3) f_t , where $f_0 = f$ and $\partial_t f_t(x)|_{t=0} = v_t(x)$ such that there are conjugations h_t where $h_t \circ f = f_t \circ h_t$ satisfying $\partial_t h_t(x)|_{t=0} = \alpha(x)$ and a family of potentials

$$\psi_t(x) = -\log |D(f_t \circ h_t)(x)|$$

Since $P(f, \psi) = P(f_t, \psi_t)$ for every t , it follows that $P(f_t, \psi_t) = 0$ for every t . Therefore,

$$\partial_t P(f_t, \psi_t)|_{t=0} = 0.$$

On the other hand, by classical arguments of thermodynamic formalism (see [27]), we have

$$\partial_t P(f_t, \psi_t)|_{t=0} = \int \partial_t \psi_t|_{t=0} d\mu = \int -\frac{v'(x) + D^2 f(x)\alpha(x)}{Df(x)} d\mu,$$

where $\mu \in \mathcal{M}_f(X)$ is the equilibrium state of f .

1.4 Some results of probability theory

In Chapters 3 and 2 we will need some concepts and results from Probability Theory. Because of this, we will devote this section to discuss these results. Here, we follow [7].

Let $(\Omega, \mathcal{B}, P)$ be a probability space and X be a map from $(\Omega, \mathcal{B}, P)$ into a metric space M . We say that X is a *random element* if X is measurable, and if $M = \mathbb{R}$ then we call X a *random variable*.

Observe that if we consider $M = C([0, 1])$, the space of the continuous functions on $[0, 1]$, then for each $\omega \in \Omega$, $X(\omega)$ is a continuous function on $[0, 1]$ and its value at $\theta \in [0, 1]$ we denote by $X(\theta, \omega)$. In this case, we call X *random function*. For fixed θ , note that $X(\theta)$ is a real function on Ω with value $X(\theta, \omega)$ at ω .

Observe that if X is a random element of $(C([0, 1]), \mathcal{C})$, that is, if $X^{-1}(\mathcal{C}) \subset \mathcal{B}$, then each $X(\theta)$ is a random variable.

1.4.1 Types of convergence

Definition 1.4.1. (Types of convergence) Let X and $(X_n)_{n \geq 1}$ be random variables.

- Assume that X and $(X_n)_{n \geq 1}$ are in L^p . We say X_n converges to X in L^p if

$$\|X_n - X\|_p = \left(\int |X_n - X|^p dP \right)^{\frac{1}{p}} \rightarrow 0 \quad \text{when } n \rightarrow \infty.$$

- Let $P, (P_n)_{n \geq 1}$ be probabilities on a metric space (M, d) . We say that P_n converges weakly to P and we denote by $P_n \Rightarrow P$, if for all bounded continuous function $f : M \rightarrow \mathbb{R}$ we have

$$\int f dP = \lim_{n \rightarrow \infty} \int f dP_n.$$

- A sequence of random elements $(X_n)_{n \geq 1}$ on the metric space (M, d) converges in probability to X if

$$\lim_{n \rightarrow \infty} P(d(X_n, X) > \varepsilon) = 0,$$

for all $\varepsilon > 0$, and we denote $X_n \xrightarrow{P} X$.

The following theorem provides useful conditions equivalent to weak convergence.

Theorem 1.4.2. (Portmanteau Theorem) *Let $(P_n)_{n \geq 1}$ and P be probability measures on M . These five conditions are equivalent:*

- (i) $P_n \Rightarrow P$;
- (ii) $\limsup_{n \rightarrow \infty} P_n(F) \leq P(F)$ for all closed $F \subset M$;
- (iii) $P(A) \leq \liminf_{n \rightarrow \infty} P_n(A)$ for all open $A \subset M$;
- (iv) $P(E) = \lim_{n \rightarrow \infty} P_n(E)$ for all $E \subset M$ such that $P(\partial E) = 0$.

Proof: The proof can be found in [7].

Let X be a random element from a probability space $(\Omega, \mathcal{B}, P)$ into a metric space M . The *distribution* of X is the probability measure μ_X on $(M, \mathcal{B}(M))$:

$$\mu_X(A) = P(X^{-1}(A)) = P(\omega : X(\omega) \in A), \forall A \in \mathcal{B}(M).$$

Definition 1.4.3. (Convergence in distribution) A sequence of random elements $(X_n)_{n \geq 1}$ on the metric space M converges in distribution to the random element X if the distributions μ_{X_n} of X_n converge weakly to the distribution μ_X of X , that is, $\mu_{X_n} \Rightarrow \mu_X$. We denote by $X_n \xrightarrow{D} X$.

Observe that when we write $X_n \xrightarrow{D} X$, we are not stating about the convergence of random elements (or random variables if $M = \mathbb{R}$) but the convergence of their distributions in $(M, \mathcal{B}(M))$. Thereby, even if X_n are defined on different probability spaces, it makes sense to talk about convergence in distribution. Furthermore, if μ is a probability in $(M, \mathcal{B}(M))$, we also denote $X_n \longrightarrow \mu$ when $\mu_{X_n} \Rightarrow \mu$.

1.4.2 Wiener measure and brownian motion

Wiener measure is a probability measure on the space of the continuous functions $(C([0, 1]), \mathcal{C})$. A random function whose distribution is the Wiener measure is called *Brownian motion*.

The Brownian motion was introduced by Robert Brown to describe the random motion of a pollen grain suspended in water.

Definition 1.4.4. (Wiener measure) A probability measure W on $(C([0, 1]), \mathcal{C})$ is called Wiener measure if the following conditions are satisfied

- If $\theta = 0$, $W(X(0) = 0) = 1$;
- For each $\theta \in [0, 1]$ the random variable $X(\theta)$ is normally distributed under W with mean 0 and variance θ :

$$W(X(\theta) \leq \alpha) = \frac{1}{\sqrt{2\pi\theta}} \int_{-\infty}^{\alpha} e^{-\frac{u^2}{2\theta}} du;$$

- The stochastic process $\{X(\theta) : 0 \leq \theta \leq 1\}$ with distribution W has independent increments under W : If $0 \leq \theta_1 < \theta_2 < \dots < \theta_k \leq 1$ the random variables $X(\theta_1) - X(\theta_0)$, $X(\theta_2) - X(\theta_1)$, $\dots$, $X(\theta_k) - X(\theta_{k-1})$ are independent under W .

In Chapter 2 and 3 we have strongly used the results below. For more details we refer the reader to [7], Chapter 3.

Let $\xi_1, \dots, \xi_n$ be random variables and $S_n = \xi_1 + \dots + \xi_n$. For each n , let ν_n be a positive integer random variable defined on the same probability space of ξ_n . We define X_n by

$$X_n(\theta, \omega) = \frac{1}{\sigma\sqrt{n}} S_{[n\theta]}(\omega) + (n\theta - [n\theta]) \frac{1}{\sigma\sqrt{n}} \xi_{[n\theta]+1}(\omega),$$

where $[n\theta] = n - 1$ and Y_n by

$$Y_n(\theta, \omega) = \frac{1}{\sigma\sqrt{\nu_n(\omega)}} S_{[\nu_n(\omega)\theta]}(\omega) + (\nu_n\theta - [\nu_n\theta]) \frac{1}{\sigma\sqrt{\nu_n(\omega)}} \xi_{[\nu_n(\omega)\theta]+1}(\omega).$$

Proposition 1.4.5. *If*

$$\frac{\nu_n}{a_n} \xrightarrow{P} \beta, \tag{1.4.1}$$

where β is a positive constant and $(a_n)_n$ is a sequence such that $a_n \rightarrow \infty$ when $n \rightarrow \infty$, then

$$X_n \xrightarrow{D} W$$

implies

$$Y_n \xrightarrow{D} W.$$

Proof. See [7], page 152.

Remark 1.4.6. In [7], page 153 the author observes that since the assumptions of theorem above requires only (1.4.1) and $X_n \xrightarrow{D} W$, it can be applied to dependent sequences (ξ_n) .

Central limit theorem for the modulus of continuity of $\mathcal{R}_\phi$

In this chapter, let us consider a $t \mapsto f_t$ a C^2 family of C^4 piecewise expanding unimodal maps that is *transversal* to the topological classes of piecewise unimodal maps, that is

$$J(f_t, v_t) = \sum_{k=0}^{M_t-1} \frac{v_t(f_t^k(c))}{Df_t^k(f_t(c))} \neq 0 \quad (2.0.1)$$

for every t , where $v_t = \partial_s f_s|_{s=t}$ and M_t is either the period of the critical point c if c is periodic, or ∞ otherwise (see [4], for more details). Note that each f_t has a unique absolutely continuous invariant probability $\mu_t = \rho_t m$, where its density ρ_t has bounded variation.

As we have seen in Introduction, Baladi and Smania proved in [3] and [6] that $\mathcal{R}_\phi$, defined by

$$\begin{aligned} \mathcal{R}_\phi : [0, 1] &\longrightarrow \mathbb{R} \\ t &\longmapsto \int \phi \, d\mu_t \end{aligned}$$

is not differentiable, for most of the parameters t , even if ϕ is quite regular. Therefore, we can ask about the regularity of the function $\mathcal{R}_\phi$ in this case.

Let $\phi : [0, 1] \rightarrow \mathbb{R}$ be a lipschitzian observable such that

$$\sigma_t^2 = \sigma_t^2(\phi) = \lim_{n \rightarrow \infty} \int \left(\frac{\sum_{j=0}^{n-1} (\phi \circ f_t^j - \int \phi d\mu_t)}{\sqrt{n}} \right)^2 d\mu_t \neq 0.$$

And let

$$L_t = \int \log |Df_t| \, d\mu_t > 0, \quad \ell_t = \frac{1}{\sqrt{L_t}}.$$

The density ρ_t is continuous except on the forward orbit $f_t^j(c)$ of the critical point (see Baladi [2]). Let S_t be the jump of ρ_t at the critical value, that is

$$S_t = \lim_{x \rightarrow f_t(c)^-} \rho_t(x) - \lim_{x \rightarrow f_t(c)^+} \rho_t(x) = \lim_{x \rightarrow f_t(c)^-} \rho_t(x) > 0.$$

We will show a central limit theorem for the modulus of continuity of the function $\mathcal{R}_\phi$.

Theorem 2.0.7. *Let*

$$t \in [a, b] \mapsto f_t,$$

be a transversal C^2 family of good C^4 piecewise expanding unimodal maps

$$f_t: [0, 1] \rightarrow [0, 1].$$

If ϕ is a lipschitzian observable satisfying $\sigma_t \neq 0$ for every $t \in [a, b]$, then for every $y \in \mathbb{R}$

$$\lim_{h \rightarrow 0} m \left\{ t \in [a, b] : t + h \in [a, b] \text{ and } \frac{1}{\Psi(t) \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_\phi(t + h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\} \quad (2.0.2)$$

converges to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{s^2}{2}} ds,$$

where

$$\Psi(t) = \sigma_t S_t J_t \ell_t.$$

and m is the Lebesgue measure normalized in such way that $m([a, b]) = 1$.

Corollary 2.0.8. *Under the same assumptions above, the function $\mathcal{R}_\phi$ is not a lipschitzian function on any subset of $[a, b]$ with positive Lebesgue measure.*

2.1 Families of piecewise expanding unimodal maps

We begin this section by setting the one-parameter family of piecewise expanding unimodal maps.

Definition 2.1.1. A C^r **piecewise expanding unimodal map** $f: [0, 1] \rightarrow [0, 1]$ is a continuous map with a critical point $c \in (0, 1)$, $f(0) = f(1) = 0$ and such that $f|_{[0, c]}$ and $f|_{[c, 1]}$ are C^r and

$$\left| \frac{1}{Df} \right|_\infty < 1.$$

A C^r piecewise expanding unimodal map f is *good* if either c is not a periodic point of f or

$$\liminf_{x \rightarrow c} |Df^p(x)| \geq 2.$$

where $p \geq 2$ is the prime period of c (see [3] and [4] for more details).

We say that f is mixing if f is topologically mixing on the interval $[f^2(c), f(c)]$.

We can see the set of all C^r piecewise expanding unimodal maps that share the same critical point $c \in (0, 1)$ as a convex subset of the affine subspace $\{f \in B^r : f(0) = f(1)\}$ of the Banach space B^r of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ that are C^r on the intervals $[0, c]$ and $[c, 1]$, with the norm

$$|f|_r = |f|_\infty + |f|_{[0,c]}|_{C^r} + |f|_{[c,1]}|_{C^r}.$$

Let $f_t : [0, 1] \rightarrow [0, 1]$, $t \in [a, b]$ be a one-parameter family of piecewise expanding C^4 unimodal maps. We assume some natural assumptions

1. For all $t \in [a, b]$ the critical point of f_t is c .
2. The maps f_t are uniformly expanding, that is, there exist constants $1 < \lambda \leq \Lambda < \infty$ such that for all $t \in [a, b]$,

$$\left| \frac{1}{Df_t} \right|_\infty < \frac{1}{\lambda} \quad \text{and} \quad |Df_t|_\infty < \Lambda.$$

3. The map

$$t \in [a, b] \mapsto f_t \in B^4$$

is of class C^2 . In particular, the map $(t, x) \mapsto f_t(x)$ is continuous on $[a, b] \times [0, 1]$, and it is C^2 on $[a, b] \times [0, c]$ and $[a, b] \times [c, 1]$.

Each f_t admits a unique absolutely continuous invariant probability measure μ_t and its density ρ_t has bounded variation (see [24]).

If f_t is good then f_t is also mixing. By Keller and Liverani (see [22]),

$$|\rho_{t+h} - \rho_t|_{L^1} \leq C|h|(\log \frac{1}{|h|} + 1). \quad (2.1.1)$$

2.2 Good transversal families

It turns out that we can cut the parameter interval of a transversal family f_t in smaller intervals in such way that the family, when restricted to each one of

those intervals satisfies stronger assumptions. Here, we introduce the notation of partitions following Schnellmann in [32]. Let us denote by $K(t) = [f_t^2(c), f_t(c)]$ the support of ρ_t .

Let $\mathcal{P}_j(t)$, $j \geq 1$ be the partition on the dynamical interval composed by the maximal open intervals of smooth monotonicity for the map $f_t^j : K(t) \rightarrow K(t)$, where t is a fixed parameter value. Therefore, $\mathcal{P}_j(t)$ is the set of open intervals $\omega \subset K(t)$ such that $f_t^j : \omega \rightarrow K(t)$ is C^4 and ω is maximal.

We can also define analogous partitions on the parameter interval $[a, b]$. Let

$$\begin{aligned} x_0 : [a, b] &\longrightarrow [0, 1] \\ t &\longmapsto f_t(c) \end{aligned}$$

be a C^2 map from the parameter interval into the dynamical interval. We will denote by

$$x_j(t) := f_t^j(x_0(t)),$$

$j \geq 0$, the orbit of the point $x_0(t)$ under the map f_t .

Consider a interval $J \subset [a, b]$. Let us denote by $\mathcal{P}_j|J$, $j \geq 1$, the partition on the parameter interval composed by all open intervals ω in J such that $x_i(t) \neq c$, for all i satisfying $0 \leq i < j$, that is

$$f_t^i(x_0(t)) = f_t^{i+1}(c) \neq c,$$

for all $t \in \omega$, and such that ω is maximal, that is, if $s \in \partial\omega$, then there exists $0 \leq i < j$ such that $x_i(s) = c$.

The intervals $\omega \in \mathcal{P}_j$ are also called cylinders.

We quote almost verbatim the definition of the Banach spaces V_α given in [32]. The spaces V_α were introduced by Keller [20].

Definition 2.2.1. (Banach space V_α) For every $\psi \in L^1(m)$ and $\gamma > 0$, we can define

$$\text{osc}(\psi, \gamma, x) = \text{ess sup } \psi|_{(x-\gamma, x+\gamma)} - \text{ess inf } \psi|_{(x-\gamma, x+\gamma)}.$$

Given $A > 0$ and $0 < \alpha \leq 1$ denote

$$|\psi|_\alpha = \sup_{0 < \gamma \leq A} \frac{1}{\gamma^\alpha} \int_0^1 \text{osc}(\psi, \gamma, x) dx.$$

The Banach space V_α is the set of all $\psi \in L^1(m)$ such that $|\psi|_\alpha < \infty$, endowed with the norm

$$\|\psi\|_\alpha = |\psi|_\alpha + |\psi|_{L^1}.$$

We quote almost verbatim the definition of the almost sure invariant principle given in [32].

Definition 2.2.2. Given a sequence of functions ξ_i on a probability space, we say that it satisfies the **almost sure invariance principle (ASIP)**, with exponent $\kappa < 1/2$ if one can construct a new probability space that has a sequence of functions ζ_i , $i \geq 1$ and a representation of the Wiener process W satisfying

- The sequences $\{\zeta_i\}_{i \geq 1}$ and $\{\xi_i\}_{i \geq 1}$ have identical distributions.
- We have

$$\left| W(n) - \sum_{i=1}^n \zeta_i \right| = O(n^\kappa),$$

almost surely as $n \rightarrow \infty$.

Definition 2.2.3. A C^2 transversal family of good C^4 piecewise expanding unimodal maps f_t , $t \in [c, d]$ is a **good transversal family** if we can extend this family to a C^2 transversal family of good C^4 piecewise expanding unimodal maps f_t , $t \in [c - \delta, d + \delta]$, for some $\delta > 0$, with the following properties

- (I) There exist $j_0 > 0$ with the following property. For every $t \in [c, d]$ and for each $j \geq j_0$ there exists a neighborhood V of t such that for all $t' \in V \setminus \{t\}$ and all $0 < i < j$, we have $f_{t'}^i(c) \neq c$. In particular the one-sided limits

$$\lim_{t' \rightarrow t^+} \frac{\partial_{t'} f_{t'}^j(c)}{D f_{t'}^{j-1}(f_{t'}(c))} \quad \text{and} \quad \lim_{t' \rightarrow t^-} \frac{\partial_{t'} f_{t'}^j(c)}{D f_{t'}^{j-1}(f_{t'}(c))}$$

exist for every $j \geq j_0$, and there is $C \geq 1$ so that

$$\frac{1}{C} \leq \left| \lim_{t' \rightarrow t^+} \frac{\partial_{t'} f_{t'}^j(c)}{D f_{t'}^{j-1}(f_{t'}(c))} \right| \leq C, \quad (2.2.1)$$

and

$$\frac{1}{C} \leq \left| \lim_{t' \rightarrow t^-} \frac{\partial_{t'} f_{t'}^j(c)}{D f_{t'}^{j-1}(f_{t'}(c))} \right| \leq C, \quad (2.2.2)$$

for all $j \geq j_0$ and $t \in [c - \delta, d + \delta]$.

- (II) The map f_t is mixing and there are constants $\delta > 0$, $L \geq 1$ and $0 < \tilde{\beta} < 1$ such that for all $\psi \in V_\alpha$

$$\|\mathcal{L}_t^n \psi\|_\alpha \leq L \tilde{\beta}^n |\psi|_\alpha + L |\psi|_{L^1}, \quad (2.2.3)$$

for all $t \in [c - \delta, d + \delta]$.

(III) There is $\delta > 0$ such that for every $\zeta > 0$ there is a constant $\tilde{C}$ satisfying

$$\sum_{\omega \in \mathcal{P}_n[[c-\delta, d+\delta]]} \frac{1}{|x'_n|_\omega|_\infty} \leq \tilde{C}e^{n^\zeta}$$

for all $n \geq 1$.

(IV) For all $\varphi \in V_\alpha$ such that $\sigma_t(\varphi) > 0$ the functions $\xi_i : [c - \delta, d + \delta] \rightarrow \mathbb{R}$ $i \geq 1$, defined by

$$\xi_i(t) = \frac{1}{\sigma_t(\varphi)} \left(\varphi(f_t^{i+1}(c)) - \int \varphi d\mu_t \right)$$

satisfy the ASIP for any error exponent $\gamma > 2/5$.

(V) There are positive constants $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4, \tilde{C}_5, \tilde{C}_6$ and $\beta \in (0, 1)$ such that for every $t \in [c - \delta, d + \delta]$ and its respective density ρ_t of the unique absolutely continuous invariant probability of f_t

A₁. The Ruelle-Perron-Frobenius operator $\mathcal{L}_t$ satisfies the Lasota-Yorke inequality in the space of bounded variation functions

$$|\mathcal{L}_t^k \phi|_{BV} \leq \tilde{C}_6 \beta^k |\phi|_{BV} + \tilde{C}_5 |\phi|_{L^1(m)}.$$

A₂. We have $\rho_t \in BV$ and $|\rho_t|_{BV} < \tilde{C}_1$.

A₃. We have $\rho'_t \in BV$ and $|\rho'_t|_{BV} < \tilde{C}_2$. Moreover

$$\rho_t(x) = \int_0^x \rho'_t(u) du + \sum_{i=1}^{M_t-1} s_k(t) H_{f_t^k(c)}(x),$$

where $H_a(x) = 0$ if $x < a$ and $H_a(x) = -1$ if $x \geq a$,

$$s_1(t) = \frac{\rho_t(c)}{|Df_t(c-)|} + \frac{\rho_t(c)}{|Df_t(c+)|}$$

and

$$s_k(t) = \frac{s_1(t)}{Df_t^{k-1}(f_t(c))}.$$

A₄. We have $\rho''_t \in BV$ and $|\rho''_t|_{BV} < \tilde{C}_3$. Moreover

$$\rho'_t(x) = \int_0^x \rho''_t(u) du + \sum_{i=1}^{M_t-1} s'_k(t) H_{f_t^k(c)},$$

where

$$|s'_k(t)| \leq \frac{\tilde{C}_4}{|Df_t^{k-1}(f_t(c))|}.$$

(VI) Let $j_0 > 0$ be the constant given by condition (I). For all i, j satisfying $0 \leq i, j \leq j_0$ and $t \in [c, d]$, such that $t + h \in [c - \delta, d + \delta]$ we have

$$c \notin I_{i,j}(t, h),$$

where $I_{i,j}(t, h)$ is the smallest interval that contains the set

$$\{f_{t+h}^{i+j+1}(c), f_t^{i+j+1}(c), f_{t+h}^i \circ f_t^{j+1}(c), f_{t+h}^{i+1} \circ f_t^j(c)\}.$$

Remark 2.2.4. Conditions (I), (II) and (III) are exactly those that appears in Schnellmann [31], with obvious cosmetic modifications.

Proposition 2.2.5. *Let $f_t, t \in [a, b]$, be a transversal C^2 family of good C^4 piecewise expanding unimodal maps. Then there is a countable family of intervals $[c_i, d_i] \subset [a, b]$, $i \in \Delta \subset \mathbb{N}$, with pairwise disjoint interior and*

$$m([a, b] \setminus \bigcup_{i \in \Delta} [c_i, d_i]) = 0,$$

such that f_t is a good transversal family on each $[c_i, d_i]$, $i \in \Delta$.

Proof. Since f_t is transversal, there is just a countable number Q of parameters where f_t has a periodic critical point. Consider $\Omega = [a, b] \setminus (Q \cup \{a, b\})$. It follows from the analysis in the proof of [5, Theorem 4.1] and [2, Proposition 3.3] that for every $t' \in \Omega$ there exists $\epsilon_1 = \epsilon_1(t')$ such that if $[c, d] \subset (t' - \epsilon_1, t' + \epsilon_1)$ then the family f_t restricted to $[c, d]$ satisfies condition (V). By Schnellmann [32] for every $t' \in \Omega$ there exists $\epsilon_2 = \epsilon_2(t')$ such that if $[c, d] \subset (t' - \epsilon_2, t' + \epsilon_2)$ then the family f_t restricted to $[c, d]$ satisfies conditions (I), (II), (III) and (IV).

We claim that for every $t' \in \Omega$ there is $\epsilon_3 = \epsilon_3(t')$ such that if $[c, d] \subset (t' - \epsilon_3, t' + \epsilon_3)$ and $\delta > 0$ is small enough then the family f_t , with $t \in [c, d]$, satisfies condition (VI). Indeed, since c is not a periodic point of $f_{t'}$, there is $\epsilon_3(t') > 0$ such that

$$\eta := \min \{|f_t^{i+j+1}(c) - c| : 0 \leq j \leq j_0 \text{ and } 0 < i \leq j_0, t \in (t' - \epsilon_3, t' + \epsilon_3)\} > 0, \quad (2.2.4)$$

Since $t \in [t' - \epsilon_3/2, t' + \epsilon_3/2] \mapsto f_t$ is a C^2 family, the map

$$(t, h) \mapsto f_{t+h}^i(f_t^j(c))$$

is continuous for every $0 < i \leq j_0$ and every j satisfying $0 \leq j \leq j_0$. Therefore there is $\gamma_1 := \gamma_1(i, j) < \epsilon_3/2$ such that, if $|h| \leq \gamma_1$ and $t \in [t' - \epsilon_3/2, t' + \epsilon_3/2]$, then

$$|f_{t+h}^{i+1}(f_t^j(c)) - f_t^{i+1}(f_t^j(c))| \leq \eta,$$

and

$$|f_{t+h}^i(f_t^{j+1}(c)) - f_t^i(f_t^{j+1}(c))| \leq \eta,$$

for all $0 \leq j \leq j_0$ and $0 < i \leq j_0$. Let $\gamma := \min\{\gamma_1(i, j) : 0 \leq j \leq j_0 \text{ and } 0 < i \leq j_0\}$. In particular if $|h| \leq \gamma$ and $t \in [t' - \epsilon_3/2, t' + \epsilon_3/2]$ then $c \notin I_{i,j}(t, h)$ for all $0 \leq j \leq j_0, 0 < i \leq j_0$.

Let $\epsilon_4(t') = \min\{\epsilon_1(t'), \epsilon_2(t'), \gamma\}$. Consider the family $\mathcal{F}$ of intervals $[c, d] \subset [a, b]$ such that $[c, d] \subset (t' - \epsilon_4(t'), t' + \epsilon_4(t'))$ for some $t' \in \Omega$. By the Vitali's covering theorem there exists a countable family of intervals $[c_i, d_i] \subset [a, b]$, $[c_i, d_i] \in \mathcal{F}$, $i \in \Delta \subset \mathbb{N}$, with pairwise disjoint interior and

$$m([a, b] \setminus \bigcup_{i \in \Delta} [c_i, d_i]) = m(\Omega \setminus \bigcup_{i \in \Delta} [c_i, d_i]) = 0.$$

□

We will also need

Lemma 2.2.6. *Let*

$$t \in [a, b] \mapsto f_t$$

be a good transversal C^2 family of good C^4 piecewise expanding unimodal maps

$$f_t : [0, 1] \rightarrow [0, 1].$$

If ϕ is a lipschitzian observable satisfying $\sigma_t \neq 0$ for every $t \in [a, b]$ then

$$\underline{J} = \inf_{t \in [a, b]} |J(f_t, v_t)|, \quad \underline{\sigma} = \inf_{t \in [a, b]} \sigma_t(\phi), \quad \underline{s} = \inf_{t \in [a, b]} |S_t|, \quad \underline{\ell} = \inf_{t \in [a, b]} |\ell_t|,$$

are positive.

Proof. The function

$$t \mapsto J(f_t, v_t)$$

is *not* continuous in a transversal family (see [4]). Indeed, its points of discontinuity lie on the parameters t where the critical point c is periodic for f_t , where this function have one-sided limits. However, in [4], Baladi and Smania showed that if v_n converges to v and f_n converges to f , and $J(f_n, v_n) \rightarrow 0$ when $n \rightarrow \infty$ then $J(f, v) = 0$. From this, follows that $\underline{J} > 0$. In [32], Schnellmann proved that $t \mapsto \sigma_t$ is Hölder continuous. Therefore, $\underline{\sigma} > 0$. Suppose that $\lim_n s_1(t_n) = 0$. Remember that (see [3] and [2]),

$$\rho_{t_n} = \rho_{abs, t_n} + \rho_{sal, t_n} = \rho_{abs, t_n} + \sum_{i=1}^{M_t-1} \frac{s_1(t_n)}{Df_{t_n}^{k-1}v(f_{t_n}(c))} H_{f_{t_n}^k(c)} \quad (2.2.5)$$

where ρ_{abs,t_n} is absolutely continuous, ρ'_{sal,t_n} has bounded variation and

$$|\rho'_{sal,t_n}|_{BV} \leq C. \quad (2.2.6)$$

Taking a subsequence, if necessary, we can assume that $\lim_n t_n = t$ and that ρ_{t_n} converges in $L^1(m)$ to ρ_t . But if $\lim_n s_1(t_n) = 0$ then by Eqs. (2.2.5) and (2.2.6) we conclude that ρ_t is a continuous function. But this is absurd since $s_1(t) \neq 0$ for every t . $\square$

Remark 2.2.7. As an example, we have the family of tent maps defined by

$$f_t(x) = \begin{cases} tx, & \text{if } x < 1/2, \\ t - tx, & \text{if } x \geq 1/2, \end{cases}$$

$t \in (1, 2)$. Tsujii [36] showed that the family of tent maps is a transversal family. We can observe that, since f_t is a piecewise linear map for all t , the density ρ_t is purely a saltus function.

2.3 Decomposition of the Newton quotient for good families

In this section we will assume that f_t is a good family. In order to prove Theorem 2.0.7 we will decompose the quotient

$$\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h}$$

in two parts which will be called the *Wild part* and the *Tame part* of the decomposition.

Proposition 2.3.1. *Assume that f_t is a family of piecewise expanding unimodal maps as defined in section 2.1 and let $\mathcal{L}_t$ be the Ruelle-Perron-Frobenius operator. Then*

$$\frac{\rho_{t+h} - \rho_t}{h} = (I - \mathcal{L}_{t+h})^{-1} \left(\frac{\mathcal{L}_{t+h}(\rho_t) - \mathcal{L}_t(\rho_t)}{h} \right).$$

Proof. We know that if $g \in BV$ is such that $\int g dm = 1$ then $\rho_t = \lim_{n \rightarrow \infty} \mathcal{L}_t^n(g)$, the limit being in the BV topology. Observe that

$$\int \frac{\mathcal{L}_{t+h}(\rho_t) - \mathcal{L}_t(\rho_t)}{h} dm = \int \frac{\mathcal{L}_{t+h}(\rho_t) - \rho_t}{h} dm = 0.$$

Then

$$\begin{aligned}
(I - \mathcal{L}_{t+h})^{-1} \left(\frac{\mathcal{L}_{t+h}(\rho_t) - \mathcal{L}_t(\rho_t)}{h} \right) &= \sum_{i=0}^{\infty} \mathcal{L}_{t+h}^i \left(\frac{\mathcal{L}_{t+h}(\rho_t) - \rho_t}{h} \right) \\
&= \frac{1}{h} \sum_{i=0}^{\infty} (\mathcal{L}_{t+h}^{i+1}(\rho_t) - \mathcal{L}_{t+h}^i(\rho_t)) \\
&= \lim_{n \rightarrow \infty} \frac{1}{h} \sum_{i=0}^n (\mathcal{L}_{t+h}^{i+1}(\rho_t) - \mathcal{L}_{t+h}^i(\rho_t)) \\
&= \lim_{n \rightarrow \infty} \frac{\mathcal{L}_{t+h}^{n+1}(\rho_t) - \rho_t}{h} \\
&= \frac{\rho_{t+h} - \rho_t}{h}.
\end{aligned}$$

□

Definition 2.3.2. Let $g : [0, 1] \rightarrow \mathbb{R}$ be a function of bounded variation and $t \in [a, b]$. We define the projection

$$\begin{aligned}
\Pi_t : BV &\longrightarrow BV \\
g &\longmapsto g - \rho_t \int g dm.
\end{aligned}$$

Note that

$$\sup_t |\Pi_t|_{BV} < \infty \text{ and } \sup_t |\Pi_t|_{L^1(m)} < \infty.$$

A function $g \in L^1(m)$ belongs to $\Pi_t(BV)$ if and only if $\int g dm = 0$. In particular the operator $(I - \mathcal{L}_t)^{-1}$ is well defined on $\Pi_t(BV)$. We are going to use the following observation quite often. If $\int g dm = 0$, and

$$g = \sum_{i=0}^{\infty} g_i,$$

with $g_i \in BV$ and the convergence of the series is in the BV norm, then

$$(I - \mathcal{L}_t)^{-1} g = \sum_{i=0}^{\infty} (I - \mathcal{L}_t)^{-1} \Pi_t(g_i).$$

Note also that

$$\Pi_t \circ \mathcal{L}_t = \mathcal{L}_t \circ \Pi_t.$$

Proposition 2.3.3. Let f_t be a C^2 family of good C^4 piecewise expanding unimodal maps that satisfies property (V) in Definition 2.2.3. There exists $C > 0$ with the following property. For every $t \in [a, b]$ such that the critical point of f_t is not periodic, we can decompose

$$\frac{\mathcal{L}_{t+h}(\rho_t) - \mathcal{L}_t(\rho_t)}{h} = \Phi_h + r_h$$

where

$$\Phi_h = \frac{1}{h} \sum_{k=0}^{\infty} s_{k+1}(t) \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right)$$

and r_h satisfies

$$\int r_h dm = 0 \quad \text{and} \quad \sup_{h \neq 0} |r_h|_{BV} < C.$$

We will prove Proposition 2.3.3 in Section 2.7. We will call $\mathcal{W}(t, h) = (I - \mathcal{L}_{t+h})^{-1} \Phi_h$ the *Wild part* and $(I - \mathcal{L}_{t+h})^{-1} r_h$ will be called the *Tame part* of the decomposition.

Definition 2.3.4. Given $h \neq 0$ and $t \in [0, 1]$, let $N := N(t, h)$ be the unique integer such that

$$\frac{1}{|Df_t^{N+1}(f_t(c))|} \leq |h| < \frac{1}{|Df_t^N(f_t(c))|}. \quad (2.3.1)$$

There is some ambiguity in the definition of $N(t, h)$ when $f_t^k(c) = c$ for some $k > 0$. But since the family is transversal, there exists just a countable number of such parameters (see [4]).

The following proposition gives us a control on the orbit of the critical point.

Proposition 2.3.5. For large $\varepsilon > 0$ and every $\gamma > 0$ there exists $\delta > 0$ such that for every small h_0 there are sets $\Gamma_{h', h_0}^\delta, \Gamma_{h_0}^\delta \subset I$, with $\Gamma_{h', h_0}^\delta \subset \Gamma_{h_0}^\delta$, for every h' satisfying $0 < h' < h_0$, with the following properties

A. $\lim_{h' \rightarrow 0} m(\Gamma_{h', h_0}^\delta) = m(\Gamma_{h_0}^\delta) > 1 - \gamma.$

B. If $t \in \Gamma_{h', h_0}^\delta$ and $|h| \leq h'$ then there exists $N_3(t, h)$ such that

$$\lfloor \frac{\varepsilon}{2} \log N(t, h) \rfloor \leq N(t, h) - N_3(t, h) \leq C_5 \varepsilon \log N(t, h) \quad (2.3.2)$$

and

$$c \notin I_{i,j}(t, h) \quad (2.3.3)$$

for all $0 \leq j < N_3(t, h)$ and $0 \leq i < N_3(t, h) - j$, where $I_{i,j}(t, h)$ is the smallest interval that contains the set

$$\{f_{t+h}^{i+j+1}(c), f_t^{i+j+1}(c), f_{t+h}^i \circ f_t^{j+1}(c), f_{t+h}^{i+1} \circ f_t^j(c)\}$$

C. For every $t \in \Gamma_{h', h_0}^\delta$ the critical point of f_t is not periodic.

D. If $0 < \hat{h} < h' \leq h_0$ then $\Gamma_{h', h_0}^\delta \subset \Gamma_{\hat{h}, h_0}^\delta$.

We will prove Proposition 2.3.5 in Section 2.5.

Proposition 2.3.6. *Let f_t be a good transversal family. Let $\phi : [0, 1] \rightarrow \mathbb{R}$ be a lipschitzian observable. If $t \in \Gamma_{h,h_0}^\delta$, where Γ_{h,h_0}^δ is the set given by Proposition 2.3.5, then*

$$\int \phi \mathcal{W}(t, h) \, dm = s_1(t) J(f_t, v_t) \sum_{j=0}^{N_3(t,h)} \left(\phi(f_t^j(c)) - \int \phi \, d\mu_t \right) + O\left(\log \log \frac{1}{|h|}\right).$$

We will prove Proposition 2.3.6 in Section 2.6.

2.4 Proof of Central Limit Theorem for the modulus of continuity of $\mathcal{R}_\phi$

To simplify the notation in this section, given a transversal family $t \mapsto f_t$ we will denote $S_t^f = s_1^f(t)$, $J_t^f = J(f_t, \partial_s f_s|_{s=t})$, $\sigma_t^f = \sigma_t^f(\phi)$. Moreover

$$L_t^f = \int \log |Df_t| \, d\mu_t^f,$$

where μ_t^f is the unique absolutely continuous invariant probability of f_t , and

$$\ell_t^f = \frac{1}{\sqrt{L_t^f}}.$$

When there are not confusion with respect to which family we are dealing with, we will omit f in the notation.

Lemma 2.4.1 (Functional Central Limit Theorem). *Let f_t be a good transversal C^2 family of C^4 unimodal maps and $\sigma_t(\phi) \neq 0$ for every t . Then*

$$\begin{aligned} & X_N(\theta, t) \\ &= \frac{1}{\sigma_t \sqrt{N}} \sum_{k=0}^{\lfloor N\theta \rfloor - 1} \left(\phi(f_t^k(c)) - \int \phi \, d\mu_t \right) + \frac{(N\theta - \lfloor N\theta \rfloor)}{\sigma_t \sqrt{N}} \left(\phi(f_t^{\lfloor N\theta \rfloor}(c)) - \int \phi \, d\mu_t \right) \end{aligned}$$

converges in distribution to the Wiener Process. We denote $X_N \xrightarrow{D} W$.

Proof. By Schnellmann [32] we know that the sequence of functions

$$\xi_i(t) = \frac{1}{\sigma_t} \left(\phi(f_t^{i+1}(c)) - \int_0^1 \phi \, d\mu_t \right)$$

satisfies the ASIP for every exponent error larger than $2/5$. By [28, Theorem E], the ASIP implies the Functional Central Limit Theorem for $X_N(\theta, t)$. $\square$

From now on we will denote

$$\mathcal{D}_\mathcal{N}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{s^2}{2}} ds.$$

The following lemma will be used many times.

Lemma 2.4.2 (A variation of Slutsky's Theorem). *Let $A_n: [0, 1] \rightarrow \mathbb{R}$ be a function and $\Omega_n \subset [0, 1]$ be such that*

$$\liminf_n m(\Omega_n) > 1 - \gamma,$$

and for every $y \in \mathbb{R}$ the sequence

$$a_n(y) = m(t \in \Omega_n: A_n(t) \leq y)$$

eventually belong to

$$O(y, \epsilon) = (\mathcal{D}_\mathcal{N}(y) - \epsilon, \mathcal{D}_\mathcal{N}(y) + \epsilon),$$

that is, there is $n_0 = n_0(y)$ such that $a_n(y) \in O(y, \epsilon)$ for every $n \geq n_0$. Then

A. There exists $\delta > 0$ such that if $B_n: [0, 1] \rightarrow \mathbb{R}$ is a function such that

$$\liminf_n m(t \in [0, 1]: |B_n(t) - 1| < \delta) > 1 - \gamma,$$

then the sequence

$$b_n(y) = m(t \in [0, 1]: A_n(t)B_n(t) \leq y)$$

eventually belong to $O(y, \epsilon + 3\gamma)$.

B. There exists $\delta > 0$ such that if $B_n: [0, 1] \rightarrow \mathbb{R}$ is a function such that

$$\liminf_n m(t \in [0, 1]: |B_n(t)| < \delta) > 1 - \gamma,$$

then the sequence

$$b_n(y) = m(t \in [0, 1]: A_n(t) + B_n(t) \leq y)$$

eventually belong to $O(y, \epsilon + 3\gamma)$.

Proof of A. Define

$$D_A^n(y) = \{t \in \Omega_n: A_n(t) \leq y\}$$

$$D_B^n = \{t \in [0, 1]: |B_n(t) - 1| < \delta\}$$

$$D_{AB}^n(y) = \{t \in [0, 1]: A_n(t)B_n(t) \leq y\}$$

Choose $\delta > 0$ such that

$$\sup_{y \in \mathbb{R}} \sup_{|\delta'| < \delta} |\mathcal{D}_\mathcal{N}(y) - \mathcal{D}_\mathcal{N}(y(1 - \delta'))| < \gamma,$$

and

$$\sup_{y \in \mathbb{R}} \sup_{|\delta'| < \delta} |\mathcal{D}_\mathcal{N}(y) - \mathcal{D}_\mathcal{N}(y(1 - \delta')^{-1})| < \gamma.$$

If $y \geq 0$

$$D_A^n((1 - \delta)y) \cap D_B^n \subset D_{AB}^n(y) \text{ and } D_{AB}^n(y) \cap D_B^n \cap \Omega_n \subset D_A^n((1 - \delta)^{-1}y),$$

Thus, if n is large

$$\begin{aligned} m(D_{AB}^n(y)) &\geq m(D_A^n((1 - \delta)y) \cap D_B^n) \\ &\geq m(D_A^n((1 - \delta)y)) - \gamma \geq \mathcal{D}_\mathcal{N}((1 - \delta)y) - \epsilon - \gamma \\ &\geq \mathcal{D}_\mathcal{N}(y) - \epsilon - 2\gamma, \end{aligned} \tag{2.4.1}$$

and

$$\begin{aligned} m(D_{AB}^n(y)) &\leq m(D_{AB}^n(y) \cap D_B^n) + \gamma \\ &\leq m(D_{AB}^n(y) \cap D_B^n \cap \Omega_n) + 2\gamma \\ &\leq m(D_A^n((1 - \delta)^{-1}y)) + 2\gamma \leq \mathcal{D}_\mathcal{N}((1 - \delta)^{-1}y) + \epsilon + 2\gamma \\ &\leq \mathcal{D}_\mathcal{N}(y) + \epsilon + 3\gamma, \end{aligned} \tag{2.4.2}$$

and if $y < 0$ we have

$$D_A^n((1 - \delta)^{-1}y) \cap D_B^n \subset D_{AB}^n(y) \text{ and } D_{AB}^n(y) \cap D_B^n \cap \Omega_n \subset D_A^n((1 - \delta)y),$$

and an analogous analysis as above gives

$$m(D_{AB}^n(y)) \in O(y, \epsilon + 3\gamma).$$

□

Proof of B. The proof is analogous to the proof of A. Define

$$D_A^n(y) = \{t \in \Omega_n: A_n(t) \leq y\}$$

$$D_B^n = \{t \in [0, 1]: |B_n(t)| < \delta\}$$

$$D_{A+B}^n(y) = \{t \in [0, 1]: A_n(t) + B_n(t) \leq y\}$$

Choose $\delta > 0$ such that

$$\sup_{y \in \mathbb{R}} \sup_{|\delta'| < \delta} |\mathcal{D}_N(y) - \mathcal{D}_N(y + \delta')| < \gamma,$$

and

$$\sup_{y \in \mathbb{R}} \sup_{|\delta'| < \delta} |\mathcal{D}_N(y) - \mathcal{D}_N(y - \delta')| < \gamma.$$

Then for every $y \in \mathbb{R}$

$$D_A^n(y - \delta) \cap D_B^n \subset D_{A+B}^n(y) \text{ and } D_{A+B}^n(y) \cap D_B^n \cap \Omega_n \subset D_A^n(y + \delta).$$

Hence, if n is large

$$\begin{aligned} m(D_{A+B}^n(y)) &\geq m(D_A^n(y - \delta) \cap D_B^n) \\ &\geq m(D_A^n(y - \delta)) - \gamma \geq \mathcal{D}_N(y - \delta) - \epsilon - \gamma \\ &\geq \mathcal{D}_N(y) - \epsilon - 2\gamma, \end{aligned} \tag{2.4.3}$$

and

$$\begin{aligned} m(D_{A+B}^n(y)) &\leq m(D_{A+B}^n(y) \cap D_B^n) + \gamma \\ &\leq m(D_{A+B}^n(y) \cap D_B^n \cap \Omega_n) + 2\gamma \\ &\leq m(D_A^n(y + \delta)) + 2\gamma \leq \mathcal{D}_N(y + \delta) + \epsilon + 2\gamma \\ &\leq \mathcal{D}_N(y) + \epsilon + 3\gamma, \end{aligned} \tag{2.4.4}$$

then

$$m(D_{A+B}^n(y)) \in O(y, \epsilon + 3\gamma).$$

□

Lemma 2.4.3. *Let $t \mapsto f_t$, $t \in [a, b]$ be a good transversal C^2 family of C^4 unimodal maps. Let $\psi: [c, d] \rightarrow [a, b]$ be an affine map, $\psi(c) = a$ and $\psi(d) = b$ and $g_\theta = f_{\psi(\theta)}$. For every small enough $h \neq 0$ we can define*

$$\Omega_g(h, y) = \left\{ \theta \in [c, d]: \frac{1}{\sigma_\theta^g \ell_\theta^g S_\theta^g J_\theta^g \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_{\phi_g}(\theta + h) - \mathcal{R}_{\phi_g}(\theta)}{h} \right) \leq y \right\}$$

and

$$\Omega_f(w, y) = \left\{ t \in [a, b]: \frac{1}{\sigma_t^f \ell_t^f S_t^f J_t^f \sqrt{-\log |w|}} \left(\frac{\mathcal{R}_{\phi_f}(t + w) - \mathcal{R}_{\phi_f}(t)}{w} \right) \leq y \right\}.$$

If

$$\frac{m(\Omega_g(h, y))}{m([c, d])}$$

eventually belong to $O(y, \gamma)$ when h converges to 0 then

$$\frac{m(\Omega_f(rh, y))}{m([a, b])}$$

eventually belong to $O(y, \gamma')$ when h converges to 0, for every $\gamma' > \gamma$. Here $r = \psi'$.

Proof. Notice that $\sigma_{\psi(\theta)}^f = \sigma_\theta^g$, $S_{\psi(\theta)}^f = S_\theta^g$, $\ell_{\psi(\theta)}^f = \ell_\theta^g$, and

$$\mathcal{R}_{\phi_f}(\psi(\theta) + rh) = \mathcal{R}_{\phi_f}(\psi(\theta + h)) = \mathcal{R}_{\phi_g}(\theta + h),$$

for every h , and furthermore $J_\theta^g = rJ_{\psi(\theta)}^f$. Thus,

$$\begin{aligned} & \frac{1}{\sigma_\theta^g \ell_\theta^g S_\theta^g J_\theta^g \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_{\phi_g}(\theta + h) - \mathcal{R}_{\phi_g}(\theta)}{h} \right) \\ &= \frac{1}{\sigma_{\psi(\theta)}^f \ell_{\psi(\theta)}^f S_{\psi(\theta)}^f J_{\psi(\theta)}^f \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_{\phi_f}(\psi(\theta) + rh) - \mathcal{R}_{\phi_f}(\psi(\theta))}{rh} \right). \end{aligned} \quad (2.4.5)$$

Observe that $\psi(\Omega_g(h, y)) = \Omega'(h, y)$, where

$$\Omega'(h, y) = \left\{ t \in [a, b] : \frac{1}{\sigma_t^f \ell_t^f S_t^f J_t^f \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_{\phi_f}(t + rh) - \mathcal{R}_{\phi_f}(t)}{rh} \right) \leq y \right\}.$$

Therefore,

$$\frac{m(\Omega'(h, y))}{m([a, b])} = \frac{m(\Omega_g(h, y))}{m([c, d])}.$$

Since

$$\lim_{h \rightarrow 0} \frac{\sqrt{-\log |rh|}}{\sqrt{-\log |h|}} = 1,$$

we can use Lemma 2.4.2.A to conclude that

$$\frac{m(\Omega_f(rh, y))}{m([a, b])}$$

eventually belong to $O(y, \gamma')$. □

Remark 2.4.4. Lemma 2.4.3 implies that it is enough to show our main theorem for families parametrized by $[0, 1]$.

Proposition 2.4.5. *For every $\gamma > 0$ there exists Q_1 with the following property. Let f_t be a good transversal C^2 family of C^4 piecewise expanding unimodal maps with $\sigma_t(\phi) \neq 0$ for every t and*

$$Q = \sup_{t, t' \in [c, d]} \left| 1 - \frac{L_{t'}}{L_t} \right| < Q_1.$$

Then for every h small enough we have

$$\frac{1}{m([c, d])} m \left\{ t \in [c, d] : \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\}$$

belongs to $O(y, 12\gamma)$.

Proof. Without loss of generality we assume that $[c, d] = [0, 1]$. It is enough to prove the following claim: For every sequence

$$h_n \rightarrow_n 0$$

and every $\gamma > 0$, the sequence

$$s_n = m \left\{ t \in [0, 1] : \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log |h_n|}} \left(\frac{\mathcal{R}_\phi(t+h_n) - \mathcal{R}_\phi(t)}{h_n} \right) \leq y \right\}$$

eventually belong to the interval $O(y, 12\gamma)$.

Fix a large $\varepsilon > 0$. By Proposition 2.3.5, for every $\gamma > 0$ there exist $\delta > 0$, $h_0 > 0$ and sets $\Gamma_{h, h_0}^\delta, \Gamma_{h_0}^\delta \subset I$, with $\Gamma_{h, h_0}^\delta \subset \Gamma_{h_0}^\delta$, for every $h \neq 0$ satisfying $|h| < h_0$, such that

$$\text{A. } \lim_{h \rightarrow 0} m(\Gamma_{h, h_0}^\delta) = m(\Gamma_{h_0}^\delta) > 1 - \gamma.$$

B. If $t \in \Gamma_{h, h_0}^\delta$ then there exists $N_3(t, h)$ such that

$$\lfloor \frac{\varepsilon}{2} \log N(t, h) \rfloor \leq N(t, h) - N_3(t, h) \leq C_5 \varepsilon \log N(t, h)$$

and

$$c \notin [f_{t+h}^i \circ f_t^{j+1}(c), f_{t+h}^{i+1} \circ f_t^j(c)]$$

for all $1 \leq j < N_3(t, h)$ and $0 \leq i < N_3(t, h) - j$.

For all $h \neq 0$ and $t \in [0, 1]$, define $N_4(t, h) = N_3(t, h)$ if $t \in \Gamma_{h, h_0}^\delta$ and $|h| < h_0$, and $N_4(t, h) = N(t, h)$, otherwise. Of course

$$N(t, h) - N_4(t, h) \leq C_5 \varepsilon \log N(t, h)$$

for every (t, h) . Since

$$\frac{1}{|Df_t^{N(t,h)+1}(f_t(c))|} \leq |h| < \frac{1}{|Df_t^{N(t,h)}(f_t(c))|},$$

we have

$$\frac{1}{N(t, h)} \sum_{k=1}^{N(t, h)} \log |Df_t(f_t^k(c))| < \frac{-\log |h|}{N(t, h)} \leq \frac{1}{N(t, h)} \sum_{k=1}^{N(t, h)+1} \log |Df_t(f_t^k(c))|.$$

By Schnellmann [31], we have for almost every t

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^N \log |Df_t(f_t^k(c))| = L_t = \int \log |Df_t| d\mu_t,$$

which implies that for almost every t

$$\lim_{h \rightarrow 0} \frac{-\log |h|}{N(t, h)} = \int \log |Df_t| d\mu_t.$$

And since

$$\frac{-\log |h|}{N(t, h)} \leq \frac{-\log |h|}{N_4(t, h)} \leq \frac{-\log |h|}{N(t, h) - C_5 \varepsilon \log N(t, h)},$$

we also have

$$\lim_{h \rightarrow 0} \frac{L_t N_4(t, h)}{-\log |h|} = 1. \quad (2.4.6)$$

for almost every $t \in [0, 1]$. Fix $t_0 \in [0, 1]$ such that $L_{t_0} = \min_{t \in [0, 1]} L_t$. Then

$$\frac{L_t / L_{t_0} N_4(t, h_n)}{-\log |h_n|} \xrightarrow{P} \frac{1}{L_{t_0}}. \quad (2.4.7)$$

By Lemma 2.4.1 and Propostion 1.4.5,

$$Y_n(\theta, t) \xrightarrow{D} W, \quad (2.4.8)$$

where $Y_n(\theta, t)$ is

$$\frac{1}{\sigma_t \sqrt{\nu_n(t)}} \sum_{k=0}^{\lfloor \nu_n \theta \rfloor - 1} (\phi(f_t^k(c)) - \int \phi d\mu_t) + \frac{(\nu_n \theta - \lfloor \nu_n \theta \rfloor)}{\sigma_t \sqrt{\nu_n(t)}} (\phi(f_t^{\lfloor \nu_n \theta \rfloor}(c)) - \int \phi d\mu_t),$$

and W is the Wiener process, with

$$\nu_n(t) = N_4(t, h_n) \frac{L_t}{L_{t_0}}.$$

Hence, taking $\theta = 1$ we conclude that

$$Y_n(1, t) \xrightarrow{D} \mathcal{N}(0, 1),$$

where $\mathcal{N}(0, 1)$ denotes the Normal distribution with average zero and variance one. Let

$$Q = \sup_{t \in [0, 1]} \left| 1 - \frac{L_{t_0}}{L_t} \right|.$$

Fix $\alpha \in (0, 1/2)$. The Lévy's modulus of continuity theorem (see for instance Karatzas and Shreve [18]) implies that for almost every function f with respect to the Wiener process there exists C_f such that

$$|f(\theta') - f(\theta)| \leq C_f |\theta' - \theta|^\alpha$$

for all $\theta', \theta \in [0, 1]$. In particular there exist $H = H(\gamma)$ and a set Ω_γ of α -Hölder continuous functions in $C([0, 1], \mathbb{R})$, whose measure with respect to the Wiener process is larger than $1 - \gamma$, such that

$$|f(\theta') - f(\theta)| \leq H |\theta' - \theta|^\alpha.$$

In particular for $f \in \Omega_\gamma$ we have

$$\max_{\theta \in [1-Q, 1]} |f(1) - f(\theta)| \leq HQ^\alpha, \quad (2.4.9)$$

Due to Eq. (2.4.8),

$$\liminf_n m\{t \in [0, 1] : \max_{\theta \in [1-Q, 1]} |Y_n(1, t) - Y_n(\theta, t)| \leq HQ^\alpha\} > 1 - \gamma.$$

In particular if

$$D_n = \{t \in [0, 1] : |Y_n(1, t) - Y_n(\frac{L_{t_0}}{L_t}, t)| \leq 2HQ^\alpha\}$$

then $\liminf_n m(D_n) > 1 - \gamma$. Applying Lemma 2.4.2.B with $\Omega_n = D_n$, $A_n(t) = Y_n(1, t)$ and $B_n(t) = Y_n(\frac{L_{t_0}}{L_t}, t) - Y_n(1, t)$, $\epsilon = 0$, then there exists $\delta > 0$ such that if $2HQ^\alpha < \delta$ we have

$$m(t \in [0, 1] : Y_n(\frac{L_{t_0}}{L_t}, t) \leq y) \quad (2.4.10)$$

eventually belongs to $O(y, 3\gamma)$. Choose $Q_0 > 0$ such that if $Q < Q_0$ then $2HQ^\alpha < \delta$. Note that

$$Y_n(\frac{L_{t_0}}{L_t}, t) = \sqrt{\frac{L_{t_0}}{L_t}} \frac{1}{\sigma_t \sqrt{N_4(t, h_n)}} \sum_{k=0}^{\lfloor N_4(t, h_n) \rfloor - 1} (\phi(f_t^k(c)) - \int \phi d\mu_t). \quad (2.4.11)$$

By Eq. (2.4.6) and Lemma 2.4.2.A, the sequence

$$m(t \in [0, 1] : \frac{\sqrt{L_{t_0}}}{\sigma_t \sqrt{-\log |h_n|}} \sum_{k=0}^{\lfloor N_4(t, h_n) \rfloor - 1} (\phi(f_t^k(c)) - \int \phi d\mu_t) \leq y)$$

eventually belongs to $O(y, 6\gamma)$. Applying again Lemma 2.4.2.A, with

$$A_n(t) = \frac{\sqrt{L_{t_0}}}{\sigma_t \sqrt{-\log |h_n|}} \sum_{k=0}^{\lfloor N_4(t, h_n) \rfloor - 1} (\phi(f_t^k(c)) - \int \phi d\mu_t),$$

$\Omega_n = [0, 1]$ and

$$B_n(t) = \sqrt{\frac{L_t}{L_{t_0}}},$$

there exists $\delta > 0$ such that if

$$\left| \sqrt{\frac{L_t}{L_{t_0}}} - 1 \right| < \delta \quad (2.4.12)$$

for every t then

$$m(t \in [0, 1]: \frac{\sqrt{L_t}}{\sigma_t \sqrt{-\log |h_n|}} \sum_{k=0}^{\lfloor N_4(t, h_n) \rfloor - 1} (\phi(f_t^k(c)) - \int \phi d\mu_t) \leq y)$$

eventually belong to $O(y, 9\gamma)$. Choose $Q_1 < Q_0$ such that $Q < Q_1$ implies Eq. (2.4.12). Finally by Propostion 2.3.6 if $0 < |h_n| \leq h_0$ and $t \in \Gamma_{h_n, h_0}^\delta$ we have

$$\frac{\mathcal{R}_\phi(t + h_n) - \mathcal{R}_\phi(t)}{S_t J_t h_n} = \sum_{j=0}^{N_3(t, h_n)} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) + O \left(\log \log \frac{1}{|h_n|} \right).$$

Since

$$\frac{\log \log \frac{1}{|h_n|}}{\sqrt{\log \frac{1}{|h_n|}}} \rightarrow_n 0,$$

we have

$$\frac{\mathcal{R}_\phi(t + h_n) - \mathcal{R}_\phi(t)}{S_t \sigma_t J_t h_n \sqrt{-\log |h_n|}} = \frac{1}{\sigma_t \sqrt{-\log |h_n|}} \sum_{j=1}^{N_3(t, h_n)} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) + r(t, h_n),$$

where

$$\lim_n \sup_{t \in \Gamma_{h_n, h_0}^\delta} |r(t, h_n)| = 0.$$

Hence, it is easy to conclude that

$$\begin{aligned} & \frac{\mathcal{R}_\phi(t + h_n) - \mathcal{R}_\phi(t)}{S_t \sigma_t \ell_t J_t h_n \sqrt{-\log |h_n|}} \\ &= \frac{1}{\ell_t \sigma_t \sqrt{-\log |h_n|}} \sum_{j=1}^{N_3(t, h_n)} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) + r'(t, h_n), \end{aligned} \quad (2.4.13)$$

for every $t \in \Gamma_{h_n, h_0}^\delta$, where

$$\ell_t = \frac{1}{\sqrt{L_t}}$$

and

$$\lim_n \sup_{t \in \Gamma_{h_n, h_0}^\delta} |r'(t, h_n)| = 0.$$

Since $m(\Gamma_{h, h_0}^\delta) > 1 - \gamma$, we can apply Lemma 2.4.2 (remember that $N_4(t, h) = N_3(t, h)$ for $t \in \Gamma_{h, h_0}^\delta$) to conclude that the sequence

$$m(t \in [0, 1]: \frac{\mathcal{R}_\phi(t + h_n) - \mathcal{R}_\phi(t)}{S_t \sigma_t \ell_t J_t h_n \sqrt{-\log |h_n|}} \leq y)$$

eventually belong to the interval $O(y, 12\gamma)$. $\square$

Remark 2.4.6. A quite important observation is that $Q_1 > 0$ depends only on $\gamma > 0$, it does not depend on the transversal family f_t .

Lemma 2.4.7. Let $[c_i, d_i] \subset [a, b]$, $i \in \Delta \subset \mathbb{N}$, be intervals with pairwise disjoint interior and such that

$$m([a, b] \setminus \bigcup_{i \in \Delta} [c_i, d_i]) = 0.$$

If $t \mapsto f_t$, with $t \in [c_i, d_i]$, is a good transversal family such that for all $i \in \Delta$ and $y \in \mathbb{R}$ we have

$$\frac{1}{m([c_i, d_i])} m \left\{ t \in [c_i, d_i]: \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_\phi(t + h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\}$$

eventually belongs to $O(y, \gamma)$, then

$$\frac{1}{m([a, b])} m \left\{ t \in [a, b]: t + h \in [a, b] \text{ and } \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_\phi(t + h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\}$$

eventually belongs to $O(y, \gamma + \epsilon)$, for every $\epsilon > 0$.

Proof. Define

$$\Omega(h, y) = \left\{ t \in [a, b]: t + h \in [a, b] \text{ and } \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_\phi(t + h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\}$$

and

$$\Omega_i(h, y) = \left\{ t \in [c_i, d_i]: t + h \in [a, b] \text{ and } \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log |h|}} \left(\frac{\mathcal{R}_\phi(t + h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\}.$$

Observe that $\Omega_i(h, y)$ are pairwise disjoint up to a countable set, $\Omega_i(h, y) \subset \Omega(h, y)$ and

$$m(\Omega(h, y) \setminus \cup_i \Omega_i(h, y)) = 0.$$

Then

$$m(\Omega(h, y)) = \sum_{i \in \Delta} m(\Omega_i(h, y)).$$

Given $\epsilon \in (0, 1)$, choose i_0 such that

$$m(\cup_{i > i_0} [c_i, d_i]) < \epsilon m([a, b]).$$

For every $i \leq i_0$ there exists $h_i > 0$ such that for every $|h| < h_i$ we have

$$\frac{m(\Omega_i(h, y))}{m([c_i, d_i])}$$

belongs to $O(y, \gamma + \epsilon)$. Let $\hat{h} = \min_{i \leq i_0} h_i$. Let

$$U_{i_0}(h, y) = \cup_{i \leq i_0} \Omega_i(h, y),$$

and

$$W_{i_0}(h, y) = \cup_{i \leq i_0} [c_i, d_i].$$

Then for $|h| < \hat{h}$ we have

$$\frac{m(U_{i_0}(h, y))}{m(W_{i_0}(h, y))} = \sum_{i \leq i_0} \frac{m([c_i, d_i])}{m(W_{i_0}(h, y))} \frac{m(\Omega_i(h, y))}{m([c_i, d_i])}$$

is a convex combination of elements of $O(y, \gamma + \epsilon)$, then it belongs to $O(y, \gamma + \epsilon)$.

We conclude that

$$\begin{aligned} m(\Omega(h, y)) &\geq m(U_{i_0}(h, y)) \\ &\geq (\mathcal{D}_{\mathcal{N}}(y) - \gamma - \epsilon) m(W_{i_0}(h, y)) \\ &\geq (\mathcal{D}_{\mathcal{N}}(y) - \gamma - \epsilon) (m([a, b]) - \epsilon m([a, b])) \\ &\geq \mathcal{D}_{\mathcal{N}}(y) - \gamma - 2\epsilon m([a, b]). \end{aligned} \tag{2.4.14}$$

On the other hand,

$$\begin{aligned} m(\Omega(h, y)) &\leq m(U_{i_0}(h, y)) + \epsilon m([a, b]) \\ &\leq (\mathcal{D}_{\mathcal{N}}(y) + \gamma + \epsilon) m(W_{i_0}(h, y)) + \epsilon m([a, b]) \\ &\leq (\mathcal{D}_{\mathcal{N}}(y) + \gamma + 2\epsilon) m([a, b]). \end{aligned} \tag{2.4.15}$$

□

Proof of Theorem 2.0.7. Remember that

$$t \mapsto L_t$$

is a continuous and positive function on $[a, b]$. Given $\gamma > 0$, let $Q_1 > 0$ be as in Proposition 2.4.5. Then there are $k > 0$ and intervals $[c_i, d_i]$, $i \leq k = k(\gamma)$, which forms a partition $\mathcal{F}$ of $[a, b]$ and

$$\sup_{t, t' \in [c_i, d_i]} \left| 1 - \frac{L_{t'}}{L_t} \right| < Q_1$$

for every $i \leq k$. Then the restrictions of the family f_t to each one of the intervals $[c_i, d_i]$ satisfy the assumptions of Proposition 2.4.5. Now it remains to apply Lemma 2.4.7 to the full family and the partition $\mathcal{F}$. Since $\gamma > 0$ is arbitrary we completed the proof of Theorem 2.0.7. $\square$

2.5 Controlling how the orbit of the critical point moves

The aim of this section is to prove Proposition 2.3.5. Let us denote by $I = [0, 1]$ the interval of parameters. Given $a \in \mathbb{R}$ define

$$\lfloor a \rfloor = \max\{k \in \mathbb{Z} : a \geq k\}.$$

Remark 2.5.1. In Schnellmann [32, Lemma 4.4] it is proven that there is $C_1 > 0$ such that if $N \geq 1$, $|t_1 - t_2| < 1/N$ and if $\omega_1 \in \mathcal{P}_N(t_1)$ and $\omega_2 \in \mathcal{P}_N(t_2)$ have the same combinatorics up to the $(N - 1)$ -th iteration then

$$\left| \frac{Df_{t_1}^N(x_1)}{Df_{t_2}^N(x_2)} \right| \leq C_1,$$

for all $x_1 \in \omega_1$ and $x_2 \in \omega_2$.

We also observe that if $x, y \in \omega \in \mathcal{P}_N(t)$, then by the bounded distortion lemma, there is $C_2 > 0$ such that

$$\left| \frac{Df_t^j(x)}{Df_t^j(y)} \right| \leq C_2,$$

for every $j \leq N$. Let

$$\tilde{M} = \sup_{0 \leq j \leq j_0} \sup_{t \in [a, b]} |\partial_t f_t^j(c)|,$$

and let us define

$$C_3 = \max\{C, \tilde{M}\},$$

where C is the constant given by the transversality condition (see Eqs (2.2.1) and (2.2.2)) and

$$C_4 = \sup_{t \in [a, b]} \sup_{x \in [0, 1]} |\partial_t f_t(x)|.$$

To prove Proposition 2.3.5 we will need

Lemma 2.5.2. *Let $\omega \in \mathcal{P}_{N_3}$ be such that*

$$|\omega| \leq \frac{1}{N_3}.$$

If $t \in \omega$ and

$$\text{dist}(t, \partial\omega) > (M + 1)|h|, \quad (2.5.1)$$

where

$$M > \max\{C_1 C_3 C_4, C_1^2 C_2 C_3^2\} \quad (2.5.2)$$

Then

$$c \notin I_{i,j}(t, h) \quad (2.5.3)$$

for all $0 \leq j < N_3$ and $0 \leq i < N_3 - j$, where $I_{i,j}(t, h)$ is the smallest interval that contains the set

$$\{f_{t+h}^{i+j+1}(c), f_t^{i+j+1}(c), f_{t+h}^i \circ f_t^{j+1}(c), f_{t+h}^{i+1} \circ f_t^j(c)\}.$$

Proof. If $j > j_0$ define $i_1 = 0$ and if $0 \leq j < j_0$ define $i_1 = j_0$. First of all, we observe that if $0 \leq j \leq j_0$ and $0 \leq i \leq j_0$ then Eq. (2.5.3) follows from condition (VI). In particular

$$c \notin I_{i,j}(t, h) \text{ for every } i < i_1. \quad (2.5.4)$$

Hence, it is left to consider the cases when

$$j_0 < j < N_3 \text{ and } 0 < i \leq N_3 - j$$

and

$$0 < j < j_0 \text{ and } j_0 < i \leq N_3 - j.$$

We claim that

$$c \notin I_{i_1,j}(t, h). \quad (2.5.5)$$

Indeed, if $0 \leq j \leq j_0$, it follows from condition (VI). Now, if $j_0 < j < N_3$, due to the transversality assumption the maps

$$\theta \mapsto f_\theta^k(c)$$

are diffeomorphisms for every $j_0 < k \leq N_3$ and they do not contain the critical point in its image, for all $j_0 < k < N_3$, $\theta \in \omega$. In particular if $\omega = (s_1, s_2)$ then

$$c \notin \{f_\theta^k(c) : \theta \in \omega\} = (f_{s_1}^k(c), f_{s_2}^k(c)) \quad (2.5.6)$$

for every $j_0 < k < N_3$. Therefore,

$$c \notin [f_t^k(c), f_{t+h}^k(c)].$$

By Mean Value Theorem and Remark 2.5.1, for every $j < N_3$

$$|f_t^{j+1}(c) - f_{t+h}^{j+1}(c)| = |\partial_\theta f_\theta^{j+1}(c)|_{\theta=\theta_1} |h| \leq C_3 |Df_{\theta_1}^j(f_{\theta_1}(c))| |h| \leq C_3 C_1 |Df_t^j(f_t(c))| |h|.$$

Moreover,

$$|f_{t+h}(f_t^j(c)) - f_t(f_t^j(c))| \leq |\partial_\theta f_\theta^j(f_t^j(c))|_{\theta=\theta_2} |h| \leq C_4 |h|. \quad (2.5.7)$$

By assumption, $d([t, t+h], \partial\omega) > M|h|$. Thus,

$$|\omega| \geq (2M+1)|h|. \quad (2.5.8)$$

If $\partial\omega = \{s_1, s_2\}$ and $s \in [t, t+h]$ then

$$\begin{aligned} |f_{s_i}^{k+1}(c) - f_s^{k+1}(c)| &= |\partial_\theta f_\theta^{k+1}(c)|_{\theta=\theta_3} |s_i - s| \\ &\geq \frac{1}{C_3} |Df_{\theta_3}^k(f_{\theta_3}(c))| M|h| \\ &\geq \frac{1}{C_1 C_3} |Df_t^k(f_t(c))| M|h| \end{aligned} \quad (2.5.9)$$

for every $j_0 < k < N_3$. Taking $k = j$ we obtain

$$\begin{aligned} |f_{t+h}(f_t^j(c)) - f_t(f_t^j(c))| &\leq C_4 |h| \leq \frac{M}{C_1 C_3} |h| \\ &\leq \frac{M}{C_1 C_3} |Df_t^j(f_t(c))| |h| \leq |f_{s_i}^{j+1}(c) - f_t^{j+1}(c)|. \end{aligned} \quad (2.5.10)$$

Hence,

$$[f_{t+h}(f_t^j(c)), f_t(f_t^j(c))] \subset (f_{s_1}^{j+1}(c), f_{s_2}^{j+1}(c)). \quad (2.5.11)$$

In particular

$$c \notin I_{0,j}(t, h) = I_{i_1,j}(t, h).$$

Now fix $0 \leq j < N_3$. We concluded the proof of our claim. We are going to prove by induction on i that, for every $i_1 \leq i < N_3 - j$,

$$c \notin I_{i,j}(t, h), \quad (2.5.12)$$

The case $i = i_1$ follows from Eq. (2.5.5). Now suppose that Eq. (2.5.12) holds up to i . Provided that $i \geq i_1$, we have $i + j + 1 > j_0$. Therefore, by Eq. (2.5.6), with $k = i + j + 2$, we obtain

$$f_{t+h}^{(i+1)+j}(f_{t+h}(c)) \in (f_{s_1}^{(i+1)+j+1}(c), f_{s_2}^{(i+1)+j+1}(c)).$$

And by Eq. (2.5.9)

$$|f_{s_i}^{(i+1)+j}(f_{s_i}(c)) - f_{t+h}^{(i+1)+j}(f_{t+h}(c))| \geq \frac{1}{C_3 C_1} |Df_{t+h}^{(i+1)+j}(f_{t+h}(c))| M |h| \quad (2.5.13)$$

Moreover by induction assumption and Eq. (2.5.4), we have for every $0 \leq k \leq i$

$$c \notin I_{k,j}(t, h)$$

Thus the points

$$f_{t+h}^{j+1}(c) \text{ and } f_t^{j+1}(c)$$

have the same combinatorics up to i iterations of the map f_{t+h} . Then by Remark 2.5.1

$$\begin{aligned} |f_{t+h}^{i+1}(f_t^{j+1}(c)) - f_{t+h}^{i+1}(f_{t+h}^{j+1}(c))| &\leq C_2 |Df_{t+h}^{i+1}(f_{t+h}^{j+1}(c))| |f_t^{j+1}(c) - f_{t+h}^{j+1}(c)| \\ &\leq C_2 |Df_{t+h}^{i+1}(f_{t+h}^{j+1}(c))| |\partial_\theta f_\theta^{j+1}(c)|_{\theta=\theta_4} |h| \\ &\leq C_3 C_2 |Df_{t+h}^{i+1}(f_{t+h}^{j+1}(c))| |Df_{\theta_4}^j(f_{\theta_4}(c))| |h| \\ &\leq C_1 C_2 C_3 |Df_{t+h}^{i+1}(f_{t+h}^{j+1}(c))| |Df_{t+h}^j(f_{t+h}(c))| |h| \\ &\leq C_1 C_2 C_3 |Df_{t+h}^{(i+1)+j}(f_{t+h}(c))| |h| \end{aligned} \quad (2.5.14)$$

and

$$f_t^j(c) \text{ and } f_{t+h}^j(c)$$

have the same combinatorics up to $i+1$ iterations of the map f_{t+h} . Then by Remark 2.5.1

$$\begin{aligned} |f_{t+h}^{(i+1)+1}(f_t^j(c)) - f_{t+h}^{(i+1)+1}(f_{t+h}^j(c))| &\leq C_2 |Df_{t+h}^{(i+1)+1}(f_{t+h}^j(c))| |f_t^j(c) - f_{t+h}^j(c)| \\ &\leq C_2 |Df_{t+h}^{(i+1)+1}(f_{t+h}^j(c))| |\partial_\theta f_\theta^j(c)|_{\theta=\theta_5} |h| \\ &\leq C_2 C_3 |Df_{t+h}^{(i+1)+1}(f_{t+h}^j(c))| |Df_{\theta_5}^{j-1}(f_{\theta_5}(c))| |h| \\ &\leq C_1 C_2 C_3 |Df_{t+h}^{(i+1)+1}(f_{t+h}^j(c))| |Df_{t+h}^{j-1}(f_{t+h}(c))| |h| \\ &\leq C_1 C_2 C_3 |Df_{t+h}^{(i+1)+j}(f_{t+h}(c))| |h|. \end{aligned} \quad (2.5.15)$$

Since

$$C_1 C_2 C_3 < \frac{M}{C_1 C_3},$$

Eqs. (2.5.13), (2.5.14) and (2.5.15) imply that

$$\{f_{t+h}^{(i+1)+1}(f_t^j(c)), f_{t+h}^{i+1}(f_t^{j+1}(c))\} \subset (f_{s_1}^{(i+1)+j+1}(c), f_{s_2}^{(i+1)+j+1}(c)).$$

In particular, $c \notin I_{i,j}(t, h)$ for all $0 \leq j < N_3$ and $i_1 < i < N_3 - j$. $\square$

To prove Proposition 2.3.5 we need to show that, for each given $h \neq 0$, for most of the parameters $t \in [0, 1]$ we can find a cylinder $\omega \in \mathcal{P}_{N_3(t, h)}$ where $[t, t + h]$ is deep inside ω (see Eq. (2.5.1)) and moreover $N_3(t, h)$ satisfies Eq. (2.3.2). To this end, for most t we will find ω , with $t \in \omega$, in such way that $|\omega|$ is quite large with respect to $|h|$ and $N_3(t, h)$ satisfies Eq. (2.3.2), but not necessarily the whole interval $[t, t + h]$ is deep inside ω . Then we will use a simple argument to conclude that for most of the parameters t this indeed occurs.

Let $\mathcal{P}_j$ be the partition of level $j > j_0$. Observe that for each cylinder $\omega \in \mathcal{P}_j$

$$|\omega| \leq C_3 \left(\frac{1}{\lambda} \right)^j,$$

where C_3 is the constant given by the transversality condition.

Let $N > 1$ and define $j = j(N)$ as

$$j = \left\lfloor \frac{\log(C_3 N)}{\log \lambda} \right\rfloor + 1.$$

Note that the cylinders of $\mathcal{P}_j$ divide the interval of parameters I in subintervals of length shorter than $1/N$. Let J be one of these intervals in $\mathcal{P}_j$. And we will denote by t_R the right boundary point of J .

Observe that, by definition, there is an integer i , $0 \leq i < j$ such that

$$x_i(t_R) = f_{t_R}^{i+1}(c) = c.$$

Fix an integer τ such that $2^{1/\tau} \leq \sqrt{\lambda}$.

Definition 2.5.3 (The sets $E_{N,J}$). Let $J \in \mathcal{P}_j$, $j = j(N)$. Let $\mathbf{E}_{N,J}$ be the family of all intervals $\omega \in \mathcal{P}_N$ such that for every k satisfying

$$0 \leq k \leq \left\lfloor \frac{\varepsilon \log N}{\tau} \right\rfloor$$

there is not

$$\tilde{\omega} = (a, b) \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor + q}, \text{ with } \omega \subset \tilde{\omega} \subset J,$$

where

$$q = \min\{(k+1)\tau, \lfloor \varepsilon \log N \rfloor\}$$

and for every i satisfying

$$0 \leq i < N - \lfloor \varepsilon \log N \rfloor + k\tau$$

we have

$$x_i(a) \neq c \text{ and } x_i(b) \neq c.$$

Define

$$E_N = \bigcup_{J \in \mathcal{P}_J} E_{N,J}.$$

For $\tilde{\omega} \in \mathcal{P}_n$ define

$$\delta_t := \min\{|f_t^i(c) - f_t^j(c)| : f_t^i(c) \neq f_t^j(c) \text{ } i, j \leq \tau.\}$$

$$\delta_\omega := \frac{\min_{t \in \bar{\omega}} \delta_t}{2}.$$

Notice that if $\tilde{\omega} \supset \omega$ then $\delta_{\tilde{\omega}} \leq \delta_\omega$.

Let C_L be such that

$$|f_t^i(c) - f_s^i(c)| \leq C_L |t - s|$$

for all $i \leq \tau$, $s, t \in [0, 1]$.

Lemma 2.5.4. *There is $C > 0$ such that the following holds. If $\tilde{\omega} \in \mathcal{P}_i$, with $|\tilde{\omega}| < 1/i$ and $t \in \tilde{\omega}$ then*

$$\frac{1}{C} \frac{|x_i(\tilde{\omega})|}{|Df_t^i(f_t(c))|} \leq |\tilde{\omega}| \leq C \frac{|x_i(\tilde{\omega})|}{|Df_t^i(f_t(c))|}. \quad (2.5.16)$$

Moreover, if $\omega \in \mathcal{P}_N \setminus E_N$ then there exists i satisfying

$$N - \lfloor \varepsilon \log N \rfloor \leq i \leq N$$

such that $\omega \subset \tilde{\omega} \in \mathcal{P}_i$ and if

$$C_L |\tilde{\omega}| < \delta_{\tilde{\omega}}$$

then

$$|x_i(\tilde{\omega})| \geq \delta_{\tilde{\omega}}$$

and

$$\frac{1}{C} \frac{\delta_{\tilde{\omega}}}{|Df_t^i(f_t(c))|} \leq |\tilde{\omega}| \leq C \frac{1}{|Df_t^i(f_t(c))|} \quad (2.5.17)$$

for every $t \in \tilde{\omega}$.

Proof. If $t \in \tilde{\omega} \in \mathcal{P}_k$ then by Mean Value Theorem for some $\theta_1 \in \tilde{\omega}$

$$|x_k(\tilde{\omega})| = |\partial_\theta f_\theta^{k+1}(c)|_{\theta=\theta_1} |\tilde{\omega}|,$$

then

$$\frac{|Df_t^k(f_t(c))| |\tilde{\omega}|}{C_1 C_3} \leq \frac{|Df_{\theta_1}^k(f_{\theta_1}(c))| |\tilde{\omega}|}{C_3} \leq |x_k(\tilde{\omega})|$$

and

$$|x_k(\tilde{\omega})| \leq C_3 |Df_{\theta_1}^k(f_{\theta_1}(c))| |\tilde{\omega}| \leq C_1 C_3 |Df_t^k(f_t(c))| |\tilde{\omega}|,$$

therefore, Eq. (2.5.16) holds. Now assume $\omega \in \mathcal{P}_N \setminus E_N$. Then there exists k satisfying

$$1 \leq k \leq \left\lfloor \frac{\varepsilon \log N}{\tau} \right\rfloor$$

and

$$\tilde{\omega} = (a, b) \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor + k\tau}$$

such that $x_{i_a}(a) = c = x_{i_b}(b)$, where

$$N - \lfloor \varepsilon \log N \rfloor + (k-1)\tau \leq i_a, i_b < N - \lfloor \varepsilon \log N \rfloor + k\tau,$$

in particular

$$x_{N - \lfloor \varepsilon \log N \rfloor + k\tau}(\tilde{\omega}) = (f_a^{n_a}(c), f_b^{n_b}(c)),$$

where

$$0 \leq n_a, n_b < \tau, \text{ with } n_a \neq n_b.$$

Thus,

$$\begin{aligned} |x_{N - \lfloor \varepsilon \log N \rfloor + k\tau}(\tilde{\omega})| &= |f_a^{n_a}(c) - f_b^{n_b}(c)| \\ &\geq |f_a^{n_a}(c) - f_a^{n_b}(c)| - |f_a^{n_b}(c) - f_b^{n_b}(c)| \\ &\geq 2\delta_{\tilde{\omega}} - C_L |a - b| \geq \delta_{\tilde{\omega}}. \end{aligned} \tag{2.5.18}$$

□

Since $\delta_{\tilde{\omega}} \geq 0$ depends only on a fixed finite number of iterations of the family f_t , it will be easy to give positive lower bounds to it that hold for most of the intervals $\tilde{\omega}$. Indeed define

$$\Lambda_{N_0}^\delta = \{t \in [0, 1]: \text{ for every } N \geq N_0 \text{ if } t \in \omega \in \mathcal{P}_{N - 2\lfloor \varepsilon \log N \rfloor} \text{ then } \delta_\omega > \delta\}.$$

Note that $\Lambda_{N_0}^\delta \subset \Lambda_{N_0+1}^\delta$. Moreover $\delta' < \delta$ implies $\Lambda_{N_0}^{\delta'} \supset \Lambda_{N_0}^\delta$.

Lemma 2.5.5. *Given $\gamma > 0$ there exists $\delta > 0$ such that*

$$\lim_{N_0 \rightarrow \infty} |\Lambda_{N_0}^\delta| \geq 1 - \gamma.$$

Proof. Since f_t is a transversal family, the set of parameters t such that $f_t^i(c) = f_t^j(c)$ for some $i \neq j$, with $i, j \leq \tau + 1$ is finite. Let $t_1, \dots, t_m$ be those parameters. The function $t \rightarrow \delta_t$ is positive and continuous on

$$O = [0, 1] \setminus \{t_1, \dots, t_m\}.$$

Choose N_0 large enough such that

$$\#\{\omega \in \mathcal{P}_{N_0-2\lfloor \varepsilon \log N_0 \rfloor} : \bar{\omega} \cap \{t_1, \dots, t_m\} \neq \emptyset\} \leq 2m.$$

Thus,

$$|\{\omega \in \mathcal{P}_{N_0-2\lfloor \varepsilon \log N_0 \rfloor} : \bar{\omega} \subset O\}| \geq 1 - \frac{2Cm}{\lambda^{N_0-2\lfloor \varepsilon \log N_0 \rfloor}} > 1 - \gamma,$$

provided N_0 is large enough. Let

$$\delta := \frac{1}{2} \min\{\delta_\omega : \omega \in \mathcal{P}_{N-2\lfloor \varepsilon \log N \rfloor}, \bar{\omega} \subset O\}.$$

Note that $\delta > 0$ and

$$\Lambda_N^\delta \supset \bigcup \{\omega \in \mathcal{P}_{N-2\lfloor \varepsilon \log N \rfloor} : \bar{\omega} \subset O\}$$

for every $N \geq N_0$, provided that N_0 is large. $\square$

Proposition 2.5.6. *There exist $\hat{C}_1, \hat{C}_2 > 0$, that do not depend on ε , such that for every $\varepsilon' < \varepsilon$ there exists $K = K(\varepsilon') > 0$ such that*

$$|E_N| \leq KN^{\hat{C}_2 - \hat{C}_1 \varepsilon'}. \quad (2.5.19)$$

The proof of this proposition follows from

Lemma 2.5.7. *There exists $\hat{C}_1 > 0$, that does not depend on ε , such that for every $\varepsilon' < \varepsilon$ there exists $K = K(\varepsilon') > 0$ such that if $J \in \mathcal{P}_j$, $j = j(N)$, and $E_{N,J}$ is as defined before, then*

$$|E_{N,J}| \leq KN^{-\hat{C}_1 \varepsilon'}. \quad (2.5.20)$$

We will prove Lemma 2.5.7 later in this section.

Proof of Proposition 2.5.6. We have

$$E_N = \bigcup_{J \in \mathcal{P}_j} E_{N,J}.$$

Since there are at most 2^j cylinders of level j , we have by Lemma 2.5.7 that there exist $\hat{C}_1 > 0$ and $K = K(\varepsilon')$ such that

$$|E_N| \leq 2^{\left(\frac{\log(C_3 N)}{\log \lambda}\right)} K N^{-\hat{C}_1 \varepsilon'} = K C_3^{\frac{\log 2}{\log \lambda}} N^{\frac{\log 2}{\log \lambda} - \hat{C}_1 \varepsilon'}. \quad (2.5.21)$$

□

Define

$$\Omega_{N_0} = \{t \in [0, 1] : t \in \omega \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor} \text{ satisfying } \omega \notin E_{N - \lfloor \varepsilon \log N \rfloor}, \text{ for every } N \geq N_0\}. \quad (2.5.22)$$

Note that $\Omega_{N_0} \subset \Omega_{N_0+1}$.

Corollary 2.5.8. *If $\hat{C}_2 - \hat{C}_1 \varepsilon < -1$ we have*

$$\lim_{N_0 \rightarrow \infty} |\Omega_{N_0}| = 1. \quad (2.5.23)$$

Proof. Notice that

$$\Omega_{N_0} = \bigcap_{N \geq N_0} \bigcup_{\omega \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor} \setminus E_{N - \lfloor \varepsilon \log N \rfloor}} \omega.$$

If we choose $\varepsilon' < \varepsilon$ such that $\hat{C}_2 - \hat{C}_1 \varepsilon' < -1$ we have

$$|\Omega_{N_0}^c| = \left| \bigcup_{N \geq N_0} \bigcup_{\omega \in E_{N - \lfloor \varepsilon \log N \rfloor}} \omega \right| \leq \sum_{N \geq N_0} K(N - \lfloor \varepsilon \log N \rfloor)^{\hat{C}_2 - \hat{C}_1 \varepsilon'} \xrightarrow{N_0 \rightarrow \infty} 0.$$

□

From now on we choose and fix $\varepsilon > 0$ satisfying $\hat{C}_2 - \hat{C}_1 \varepsilon < -1$.

Corollary 2.5.9. *For every $\gamma > 0$ there exists $\delta > 0$ such that*

$$\lim_{N_0 \rightarrow \infty} m(\Lambda_{N_0}^\delta \cap \Omega_{N_0}) > 1 - \gamma.$$

Definition 2.5.10. Given $\delta > 0$ and $h_0 > 0$, define

$$\Gamma_{h_0}^\delta$$

as the set of all parameters $t \in [0, 1]$ such that for every h , $0 < |h| \leq h_0$, there exists k satisfying

$$N(t, h) - 2\lfloor \epsilon \log N(t, h) \rfloor \leq k \leq N(t, h) - \lfloor \epsilon \log N(t, h) \rfloor$$

such that if $t \in \hat{\omega} \in \mathcal{P}_k$ then $|x_k(\hat{\omega})| > \delta$.

Given $t \in \Gamma_{h_0}^\delta$ and $h \neq 0$, let $N_2(t, h)$ be the largest k with this property.

Definition 2.5.11. Given h and $t \in [0, 1]$, define

$$N_1(t, h) := N(t, h) - \lfloor \epsilon \log N(t, h) \rfloor, \quad (2.5.24)$$

and for $h_0 > 0$ define

$$\hat{N}_1(h_0) := \min_{t \in I, |h| \leq h_0} N_1(t, h).$$

Since

$$\lim_{N \rightarrow \infty} \max_{t \in [0, 1]} \frac{1}{|Df_t^N(f_t(c))|} = 0,$$

we have

$$\lim_{h_0 \rightarrow 0} \hat{N}_1(h_0) = +\infty.$$

Lemma 2.5.12. For every $\gamma > 0$ there exists $\delta > 0$ such that

$$\lim_{h_0 \rightarrow 0} m(\Gamma_{h_0}^\delta) > 1 - \gamma.$$

Proof. By Corollary 2.5.9 there exist $\delta > 0$ and N_0 such that

$$m(\Lambda_{N_0}^\delta \cap \Omega_{N_0}) > 1 - \gamma.$$

Choose h_0 such

$$\hat{N}_1(h_0) > N_0.$$

Let $|h| \leq h_0$. Then

$$N(t, h) - \lfloor \epsilon \log N(t, h) \rfloor \geq N_0.$$

If $t \in \Lambda_{N_0}^\delta \cap \Omega_{N_0}$, choosing $\tilde{\omega}$ such that $t \in \tilde{\omega} \in \mathcal{P}_{N(t, h) - \lfloor \epsilon \log N(t, h) \rfloor}$ then

$$\tilde{\omega} \notin E_{N(t, h) - \lfloor \epsilon \log N(t, h) \rfloor}.$$

Hence, by Lemma 2.5.4 there exists $N_2(t, h)$ satisfying (here $N = N(t, h)$)

$$N - \lfloor \epsilon \log N \rfloor - \lfloor \epsilon \log(N - \lfloor \epsilon \log N \rfloor) \rfloor \leq N_2(t, h) \leq N - \lfloor \epsilon \log N \rfloor$$

such that if $t \in \tilde{\omega} \subset \hat{\omega} \in \mathcal{P}_{N_2(t, h)}$ then

$$|x_{N_2(t, h)}(\hat{\omega})| \geq \delta_{\hat{\omega}} > \delta$$

since $t \in \Lambda_{N_0}^\delta$. Therefore, $\Gamma_{h_0}^\delta \supset \Lambda_{N_0}^\delta \cap \Omega_{N_0}$. □

Definition 2.5.13. Given $h_0 > 0$ and $\delta > 0$, for every h such that $|h| \leq h_0$ let $\mathcal{A}_{h,h_0}^\delta$ be a covering of $\Gamma_{h_0}^\delta$ by intervals ω with the following properties

P₁. There exists $t \in \Gamma_{h_0}^\delta$ such that $t \in \omega \in \mathcal{P}_{N_2(t,h)}$.

P₂. If $t' \in \Gamma_{h_0}^\delta$ and $t' \in \omega$ then $\omega' \subset \omega$, where $t' \in \omega' \in \mathcal{P}_{N_2(t',h)}$.

P₃. There does not exist $t'' \in \Gamma_{h_0}^\delta$ such that $t'' \in \omega'' \in \mathcal{P}_{N_2(t'',h)}$ and $\omega \subsetneq \omega''$.

One can check that one such collection $\mathcal{A}_{h,h_0}^\delta$ does exist. Indeed, consider the covering of $\Gamma_{h_0}^\delta$ given by

$$\{\omega: \text{ there exists } t \in \Gamma_{h_0}^\delta \text{ such that } t \in \omega \in \mathcal{P}_{N_2(t,h)}\}.$$

Of course this covering satisfies property P₁. Remove from this covering all intervals ω that does not satisfy property P₃. Then the remaining collection is a covering of $\Gamma_{h_0}^\delta$ satisfying properties P₁, P₂ and P₃. Note also that the distinct intervals in $\mathcal{A}_{h,h_0}^\delta$ are pairwise disjoint. Indeed, if $\omega, \omega' \in \mathcal{A}_{h,h_0}^\delta$, with $\omega \neq \omega'$ and $\omega \cap \omega' \neq \emptyset$ then either $\omega \subsetneq \omega'$ or $\omega' \subsetneq \omega$, which is in contradiction with property P₃.

We note that $|\mathcal{A}_{h,h_0}^\delta| \geq m(\Gamma_{h_0}^\delta)$, since $\mathcal{A}_{h,h_0}^\delta$ covers $\Gamma_{h_0}^\delta$.

Lemma 2.5.14. *If h_0 is small enough there are $C_5 > 0$ and $C > 0$ such that the following holds. Given $t' \in \Gamma_{h_0}^\delta$, let ω be the unique interval in $\mathcal{A}_{h,h_0}^\delta$ such that $t' \in \omega$. Let $t \in \Gamma_{h_0}^\delta$ be such that $t \in \omega \in \mathcal{P}_{N_2(t,h)}$. Then*

$$\lfloor \frac{\epsilon}{2} \log N(t', h) \rfloor \leq N(t', h) - N_2(t, h) \leq C_5 \epsilon \log N(t', h) \quad (2.5.25)$$

and

$$|\omega| \geq C \delta N(t', h)^{\epsilon \frac{\log \lambda}{2}} |h|. \quad (2.5.26)$$

Proof. Consider ω' such that

$$t' \in \omega' \in \mathcal{P}_{N_2(t',h)}.$$

Then by property P₂ we have $\omega' \subset \omega$. Thus,

$$\frac{\delta}{C_1 C_3} \frac{1}{|Df_{t'}^{N_2(t',h)}(f_{t'}(c))|} \leq |\omega'| \leq |\omega| \leq \frac{C_1 C_3}{|Df_t^{N_2(t,h)}(f_t(c))|}.$$

Since $t, t' \in \omega$, there is $C_1 > 1$ such that

$$\frac{1}{C_1} \frac{1}{|Df_{t'}^i(f_{t'}(c))|} \leq \frac{1}{|Df_t^i(f_t(c))|} \leq C_1 \frac{1}{|Df_{t'}^i(f_{t'}(c))|}.$$

for every $i \leq N_2(t, h)$. Choose $\bar{C}$ such that

$$\frac{\delta}{C_3^2 C_1^3} > \frac{1}{\lambda^{\bar{C}}}. \quad (2.5.27)$$

Then

$$N_2(t', h) \geq N_2(t, h) - \bar{C},$$

otherwise

$$\begin{aligned} \frac{\delta}{C_1 C_3} \frac{1}{|Df_{t'}^{N_2(t', h)}(f_{t'}(c))|} &\leq \frac{C_1 C_3}{|Df_t^{N_2(t, h)}(f_t(c))|} \\ &\leq \frac{C_1 C_3}{|Df_t^{N_2(t, h) - N_2(t', h)}(f_t^{N_2(t', h) + 1}(c))|} \frac{1}{|Df_t^{N_2(t', h)}(f_t(c))|} \\ &\leq \frac{C_1 C_3}{\lambda^{\bar{C}}} \frac{C_1}{|Df_{t'}^{N_2(t', h)}(f_{t'}(c))|}, \end{aligned}$$

which contradicts Eq. (2.5.27). In particular

$$\begin{aligned} N(t', h) - N_2(t, h) &\geq N(t', h) - N_2(t', h) - \bar{C} \\ &\geq \lfloor \epsilon \log N(t', h) \rfloor - \bar{C} \\ &\geq \lfloor \frac{\epsilon}{2} \log N(t', h) \rfloor. \end{aligned} \quad (2.5.28)$$

Note that the lower bound holds if h_0 is small enough. Thus,

$$N(t', h) > N_2(t, h).$$

Moreover,

$$\begin{aligned} |h| &\leq \frac{1}{|Df_{t'}^{N(t', h)}(f_{t'}(c))|} \\ &\leq \frac{1}{|Df_{t'}^{N(t', h) - N_2(t, h)}(f_{t'}^{N_2(t, h) + 1}(c))|} \frac{1}{|Df_{t'}^{N_2(t, h)}(f_{t'}(c))|} \\ &\leq \frac{1}{|Df_{t'}^{N(t', h) - N_2(t, h)}(f_{t'}^{N_2(t, h) + 1}(c))|} \frac{C_1}{|Df_t^{N_2(t, h)}(f_t(c))|}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |h| &\geq \frac{1}{|Df_t^{N(t, h) + 1}(f_t(c))|} \\ &\geq \frac{1}{|Df_t^{N(t, h) - N_2(t, h)}(f_t^{N_2(t, h) + 1}(c))|} \frac{1}{|Df_t^{N_2(t, h)}(f_t(c))|}. \end{aligned} \quad (2.5.29)$$

Then

$$\log |Df_{t'}^{N(t', h) - N_2(t, h)}(f_{t'}^{N_2(t, h) + 1}(c))| - \log C_1 \leq \log |Df_t^{N(t, h) - N_2(t, h)}(f_t^{N_2(t, h) + 1}(c))|$$

and consequently

$$N(t', h) - N_2(t, h) \leq \hat{C}_3(N(t, h) - N_2(t, h)) + \hat{C}_4.$$

In a similar way, we can obtain

$$N(t, h) - N_2(t, h) \leq \hat{C}_3(N(t', h) - N_2(t, h)) + \hat{C}_4,$$

where

$$\hat{C}_3 = \frac{\log \Lambda}{\log \lambda}$$

and

$$\hat{C}_4 = \frac{\log C_1}{\log \lambda}.$$

$$\begin{aligned} N(t, h) &= N(t, h) - N_2(t, h) + N_2(t, h) \\ &\leq 2\epsilon \log N(t, h) + N(t', h) \\ &\leq \frac{N(t, h)}{N(t, h) - 2\epsilon \log N(t, h)} N(t', h) \\ &\leq 2N(t', h), \end{aligned}$$

provided that h_0 is small. Consequently

$$\begin{aligned} N(t', h) - N_2(t, h) &\leq \hat{C}_3(N(t, h) - N_2(t, h)) + \hat{C}_4 \\ &\leq \hat{C}_3 2[\epsilon \log N(t, h)] + \hat{C}_4 \\ &\leq \hat{C}_3 2\epsilon \log[2N(t', h)] + \hat{C}_4 \\ &\leq C_5 \epsilon \log N(t', h). \end{aligned} \tag{2.5.30}$$

Here the last inequality holds if h_0 is small enough. Moreover, by the transversality condition if h_0 is small enough then

$$C_L |\omega| \leq \frac{C_3 C_L}{\lambda^{N_2(t, h)}} < \delta.$$

Therefore, by Lemma 2.5.4 we get

$$|\omega| \geq \frac{1}{C} \frac{\delta_\omega}{|Df_{t'}^{N_2(t, h)}(f_{t'}(c))|}.$$

Consequently,

$$\begin{aligned} |\omega| &\geq \frac{1}{C} \frac{\delta}{|Df_{t'}^{N_2(t, h)}(f_{t'}(c))|} = \frac{\delta}{C} \frac{|Df_{t'}^{N(t', h) - N_2(t, h)}(f_{t'}^{N_2(t, h) + 1}(c))|}{|Df_{t'}^{N(t', h)}(f_{t'}(c))|} \\ &\geq \frac{\delta}{C} \frac{\lambda^{N(t', h) - N_2(t, h)}}{|Df_{t'}^{N(t, h)}(f_{t'}(c))|} \geq \frac{\delta}{C} \lambda^{\frac{\epsilon \log N(t', h)}{2} - 1} |h| = \frac{\delta}{C \lambda} N(t', h)^{\epsilon \frac{\log \lambda}{2}} |h| \end{aligned} \tag{2.5.31}$$

Hence, we obtain Eq. (2.5.26). $\square$

Choose $\epsilon > 0$ such that

$$\frac{1}{\sqrt{\lambda}} < 1 - \epsilon.$$

Lemma 2.5.15. *Given $M > 0$, define*

$$B_{h,h_0,M}^\delta = \left\{ t : t \in \omega \in \mathcal{A}_{h,h_0}^\delta \text{ and } \text{dist}(t, \partial\omega) > \frac{M+1}{1-\epsilon}|h| \right\}.$$

Let $h_i = (1 - \epsilon)^i h_0$. Given h satisfying $0 < |h| \leq h_0$, let

$$i(h) = \max\{i \in \mathbb{N} : |h| < (1 - \epsilon)^{i-1} h_0\}.$$

For every $h > 0$ define

$$\hat{\Gamma}_{h,h_0}^\delta = \Gamma_{h_0}^\delta \cap \left(\bigcap_{i \geq i(h)} B_{h_i,h_0,M}^\delta \right).$$

Then

A. If $0 < \hat{h} < h$ then $\hat{\Gamma}_{h,h_0}^\delta \subset \hat{\Gamma}_{\hat{h},h_0}^\delta$,

B. We have

$$\lim_{h \rightarrow 0} m(\hat{\Gamma}_{h,h_0}^\delta) = m(\Gamma_{h_0}^\delta).$$

Proof. Note that

$$\min_{t \in [0,1]} N(t, h_i) \geq -\frac{\log h_0}{\log \Lambda} - 1 - \frac{i \log(1 - \epsilon)}{\log \Lambda},$$

where

$$-\frac{\log h_0}{\log \Lambda} > 0 \quad \text{and} \quad -\frac{i \log(1 - \epsilon)}{\log \Lambda} > 0.$$

Therefore, if h_0 is small enough, there are $K_1, K_2 > 0$, such that

$$\min_{t \in [0,1]} N(t, h_i) \geq K_1 + iK_2.$$

Define

$$A_h = \bigcup_{\omega \in \mathcal{A}_{h,h_0}} \omega.$$

If $\omega \in \mathcal{A}_{h,h_0}^\delta$ then there is $t \in \Gamma_{h_0}^\delta$ such that $t \in \omega \in \mathcal{P}_{N_2(t,h)}$. By Lemma 2.5.14

$$\begin{aligned} m(\omega \cap (B_{h,h_0,M}^\delta)^c) &= m\{t \in \omega : \text{dist}(t, \partial\omega) \leq \frac{M+1}{1-\epsilon}|h|\} \\ &\leq 2 \frac{M+1}{1-\epsilon} |h| \\ &\leq \frac{2(M+1)|h|}{(1-\epsilon)|\omega|} |\omega| \\ &\leq \frac{2C\lambda(M+1)}{\delta(1-\epsilon)N(t,h)^{\varepsilon \frac{\log \lambda}{2}}} |\omega|. \end{aligned} \tag{2.5.32}$$

Choose ε large enough such that $\varepsilon \log \lambda > 2$. Then

$$\sum_{i=0}^{\infty} m(A_{h_i} \cap (B_{h_i, h_0, M}^\delta)^c) \leq \sum_{i=0}^{\infty} \frac{2C\lambda(M+1)\sqrt{\lambda}}{\delta(K_1 i + K_2)^{\varepsilon \frac{\log \lambda}{2}}} < \infty. \quad (2.5.33)$$

In particular

$$\begin{aligned} m\left(\Gamma_{h_0}^\delta \cap \left(\bigcap_{i \geq i(h)} B_{h_i, h_0, M}^\delta\right)\right) &= m(\Gamma_{h_0}^\delta) - m(\Gamma_{h_0}^\delta \cap \left(\bigcap_{i \geq i(h)} B_{h_i, h_0, M}^\delta\right)^c) \\ &\geq m(\Gamma_{h_0}^\delta) - \sum_{i \geq i(h)} m(\Gamma_{h_0}^\delta \cap (B_{h_i, h_0, M}^\delta)^c) \\ &\geq m(\Gamma_{h_0}^\delta) - \sum_{i \geq i(h)} m(A_{h_i} \cap (B_{h_i, h_0, M}^\delta)^c). \end{aligned} \quad (2.5.34)$$

Eq. (2.5.33) implies that

$$\lim_{h \rightarrow 0} \sum_{i \geq i(h)} m(A_{h_i} \cap (B_{h_i, h_0, M}^\delta)^c) = 0.$$

□

Proof of Proposition 2.3.5. By Lemma 2.5.12 for every $\gamma > 0$ there exists $\delta > 0$ such that for every small h_0 we have

$$m(\Gamma_{h_0}^\delta) > 1 - \gamma.$$

Choose M satisfying Eq. (2.5.2). Define

$$\Gamma_{h, h_0}^\delta = \hat{\Gamma}_{h, h_0}^\delta \setminus Q,$$

where $\hat{\Gamma}_{h, h_0}^\delta$ is the set defined in Lemma 2.5.15 and Q is the countable set of parameters where f_t has a periodic critical point. By Lemma 2.5.15 Property A. holds. Let $t' \in \Gamma_{h, h_0}^\delta$, with $|h| < h_0$. There exists $i \geq i(h)$ such that

$$h_{i+1} \leq |h| \leq h_i,$$

where $h_i = (1 - \epsilon)^i h_0$. Thus, $N(t', h) = N(t', h_j)$, for some $j \in \{i, i+1\}$, and consequently $N_2(t', h) = N_2(t', h_j)$. Then there exists a unique $\omega \in \mathcal{A}_{h_j, h_0}^\delta$ and $t \in \Gamma_{h_0}^\delta$ such that $t, t' \in \omega \in \mathcal{P}_{N_2(t, h)}$. Moreover, since $t' \in B_{h_j, h_0, M}^\delta$ we have

$$\text{dist}(t', \partial\omega) \geq \frac{M+1}{1-\epsilon} h_j \geq (M+1)|h|.$$

Define $N_3(t', h) = N_2(t, h)$. By Lemma 2.5.14 we have Eq. (2.3.2) holds. By Lemma 2.5.2, Eq. (2.3.3) holds.

□

2.5.1 Proof of Lemma 2.5.7

The sets $E_{N,J}$ ‘live’ in the parameter space. To estimate its measures we will compare them, following [31], with the measures of similarly defined sets in the phase space of the map f_{t_R} .

Definition 2.5.16 (The sets $\hat{E}_{N,t_R}$). Denote by $\hat{E}_{N,t_R}$ the set of all

$$\eta \in \mathcal{P}_N(t_R)$$

such that for all k satisfying

$$0 \leq k \leq \left\lfloor \frac{\varepsilon \log N}{\tau} \right\rfloor$$

there is not

$$\tilde{\eta} \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor + j}(t_R), \quad \eta \subset \tilde{\eta},$$

where

$$j = \min\{(k+1)\tau, \lfloor \varepsilon \log N \rfloor\},$$

such that

$$f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor + k\tau}(\tilde{\eta}) \in \mathcal{P}_{j - k\tau}(t_R).$$

Using a strategy similar to the one applied in [31], we estimate the measure $|E_{N,J}|$ in terms of the measure $|\hat{E}_{N,t_R}|$. To this end we need to define the map $\mathcal{U}_J$.

Definition 2.5.17 (The map $\mathcal{U}_J$). Let $J = (t_L, t_R)$. Consider the map $\mathcal{U}_J$

$$\mathcal{U}_J : \mathcal{P}_N|_J \rightarrow \mathcal{P}_N(t_R)$$

defined by Schnellmann [31, proof of Lemma 3.2] in the following way. Let $\omega \in \mathcal{P}_N|_J$ and choose $t \in \omega$. Since ω is a cylinder, it follows that $x_j(t) \neq c$ for all $0 \leq j < N$. Therefore, there is a cylinder $\omega(x_0(t))$ in the partition $\mathcal{P}_N(t)$ such that $x_0(t) \subset \omega(x_0(t))$.

Let

$$\mathcal{U}_J(\omega) = \mathcal{U}_{t,t_R,N}(\omega(x_0(t))),$$

where $\mathcal{U}_{t,t_R,N} : \mathcal{P}_N(t) \rightarrow \mathcal{P}_N(t_R)$ is such that for all $\eta \in \mathcal{P}_N(t)$, the elements η and $\mathcal{U}_J(\eta)$ have the same combinatorics.

$$\text{symp}_t(f_t^i(\eta)) = \text{symp}_{t_R}(f_{t_R}^i(\mathcal{U}_{t,t_R,N}(\eta))),$$

for $0 \leq i < N$. Schnellmann [31] proved that $\mathcal{U}_{t,t_R,N}$ is well defined when f_t is a family of piecewise expanding unimodal maps satisfying our assumptions.

Therefore, the cylinder $\omega' = \mathcal{U}_J(\omega) = \mathcal{U}_{t,t_R,N}(\omega(x_0(t)))$ has the same combinatorics as ω , that is,

$$\text{symb}(x_j(\omega)) = \text{symb}_{t_R}(f_{t_R}^j(\omega')),$$

when $0 \leq j < N$. Since there are not two cylinders in $\mathcal{P}_N(t_R)$ with the same combinatorics, the element ω' does not depend on the choice of $t \in \omega$. Therefore, $\mathcal{U}_J$ is well defined.

Lemma 2.5.18. *If $\omega \in E_{N,J}$, then $\mathcal{U}_J(\omega) \in \hat{E}_{N,t_R}$. Moreover, there exists $C' \geq 1$ such that*

$$|\omega| \leq C' |\mathcal{U}_J(\omega)|. \quad (2.5.35)$$

In particular

$$|E_{N,J}| \leq C' |\hat{E}_{N,t_R}|. \quad (2.5.36)$$

Proof. Note that $\mathcal{U}_J(\omega) \in \hat{E}_{N,t_R}$ follows from the fact that ω and $\mathcal{U}_J(\omega)$ have the same combinatorics [31]. By [31, Lemma 3.2], there exists a constant $C' \geq 1$ such that

$$|\omega| \leq C' |\mathcal{U}_J(\omega)|.$$

Thus,

$$|E_{N,J}| \leq \sum_{\omega \in E_{N,J}} |\omega| \leq \sum_{\omega \in E_{N,J}} C' |\mathcal{U}_J(\omega)| \leq C' |\hat{E}_{N,t_R}|. \quad (2.5.37)$$

□

Definition 2.5.19. For each $\eta' \in \mathcal{P}_{N-\lfloor \varepsilon \log N \rfloor}(t_R)$, define the set

$$\hat{E}_{N,t_R,\eta'} = \left\{ \eta \in \mathcal{P}_N(t_R) : \eta \in \hat{E}_{N,t_R} \text{ and } \eta \subset \eta' \right\}.$$

Lemma 2.5.20. *Let $\eta' \in \mathcal{P}_{N-\lfloor \varepsilon \log N \rfloor}(t_R)$. Then*

$$\#\hat{E}_{N,t_R,\eta'} \leq 2^{\lfloor \frac{\varepsilon \log N}{\tau} \rfloor + 1}. \quad (2.5.38)$$

Proof. Define

$$k_0 = \left\lfloor \frac{\varepsilon \log N}{\tau} \right\rfloor.$$

Notice that

$$N \geq N - \lfloor \varepsilon \log N \rfloor + k_0 \tau > N - \tau.$$

If $N = N - \lfloor \varepsilon \log N \rfloor + k_0 \tau$ define $k_1 = k_0$. Otherwise define $k_1 = k_0 + 1$. For every k satisfying

$$0 \leq k \leq k_1,$$

define families of intervals $\mathcal{F}_k$ in the following way. If $k \leq k_0$ define

$$\mathcal{F}_k = \{\hat{\eta} \subset \eta' : \hat{\eta} \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor + k\tau}(t_R) \text{ and there is } \eta \in \hat{E}_{N, t_R, \eta'} \text{ with } \eta \subset \hat{\eta}\} \quad (2.5.39)$$

otherwise $k = k_1 = k_0 + 1$ and

$$\mathcal{F}_{k_1} = \hat{E}_{N, t_R, \eta'}. \quad (2.5.40)$$

Note that if $k_1 = k_0$ then we also have $\mathcal{F}_{k_1} = \hat{E}_{N, t_R, \eta'}$. We claim that

$$\#\mathcal{F}_k \leq 2^k. \quad (2.5.41)$$

We observe that, taking $k = k_1$ in Eq. (2.5.41) we obtain Eq. (2.5.38). Note that either $\mathcal{F}_0$ is the empty set or $\mathcal{F}_0 = \{\eta'\}$. Then $\#\mathcal{F}_0 \leq 1$. Moreover, it is easy to see that if $\hat{\eta}_{k+1} \in \mathcal{F}_k$, with $k < k_1$, then there exists a unique $\hat{\eta}_k \in \mathcal{F}_k$ such that $\hat{\eta}_{k+1} \subset \hat{\eta}_k$. Therefore, it is enough to show that for each $\hat{\eta}_k \in \mathcal{F}_k$, with $k < k_1$, there are at most two intervals $\hat{\eta}_{k+1} \in \mathcal{F}_{k+1}$ such that $\hat{\eta}_{k+1} \subset \hat{\eta}_k$. Indeed, given $k < k_1$, for every $\hat{\eta}_k \in \mathcal{F}_k$ we have $\hat{\eta}_k \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor + k\tau}(t_R)$. Moreover, there is j such that for every $\hat{\eta}_{k+1} \in \mathcal{F}_{k+1}$ we have $\hat{\eta}_{k+1} \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor + j}(t_R)$, with $k\tau < j \leq \lfloor \varepsilon \log N \rfloor$, and $j \leq k\tau + \tau$. Note that if the closure of $\hat{\eta}_{k+1} = (a, b)$ is contained in the interior of $\hat{\eta}_k$, then for every $x \in \overline{\hat{\eta}_{k+1}}$ we have $f_{t_R}^p(x) \neq c$, for every $p < N - \lfloor \varepsilon \log N \rfloor + k\tau$. Furthermore, there are n_a, n_b such that

$$f_{t_R}^{n_a}(a) = c = f_{t_R}^{n_b}(b),$$

where

$$N - \lfloor \varepsilon \log N \rfloor + k\tau \leq n_a, n_b < N - \lfloor \varepsilon \log N \rfloor + j.$$

We conclude that

$$f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor + k\tau}(\hat{\eta}_{k+1}) \in \mathcal{P}_{j - k\tau}(t_R).$$

where $j - k\tau \leq \tau$. Therefore, if $\eta \subset \hat{\eta}_{k+1}$, with $\eta \in \mathcal{P}_N(t_R)$, then $\eta \notin \hat{E}_{N, t_R, \eta'}$ and consequently $\hat{\eta}_{k+1} \notin \mathcal{F}_{k+1}$. Since there are at most two intervals $\mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor + j}(t_R)$ whose closure is not contained in the interior of $\hat{\eta}_k$, we conclude that there are at most two intervals in $\mathcal{F}_{k+1}$ that are contained in $\hat{\eta}_k$. □

Lemma 2.5.21. *Let $\eta', \eta'' \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor}(t_R)$ such that*

$$f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\eta') = f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\eta'').$$

Then

$$f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\hat{E}_{N, t_R, \eta'}) = f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\hat{E}_{N, t_R, \eta'').$$

Proof. Let $\omega' = (y'_1, y'_2) \in \mathcal{P}_N(t_R)$, with $\omega' \subset \eta'$, be a cylinder in $\hat{E}_{N, t_R, \eta'}$. Then

$$f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\omega') \subset f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\eta') = f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\eta''). \quad (2.5.42)$$

Remember that since $\omega' \in \mathcal{P}_N(t_R)$, it follows that for all $x \in \omega'$

$$f_{t_R}^i(x) \neq c \quad \text{for all } 0 \leq i < N, \quad (2.5.43)$$

and if $y \in \partial\omega'$, then there exists j , $0 \leq j < N$ such that $f_{t_R}^j(y) = c$. Define

$$a_i = f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(y'_i).$$

Then $f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\omega') = (a_1, a_2)$ is an open interval and, by Eq. (2.5.42), we have $(a_1, a_2) \subset f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\eta'')$. Therefore, there is an open interval $\omega'' = (y''_1, y''_2) \subset \eta''$ such that $f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\omega'') = (a_1, a_2)$ with

$$a_i = f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(y''_i).$$

We claim that ω'' is also a cylinder. Indeed, let $x \in \omega''$. Then, since $\omega'' \subset \eta''$ and η'' is a cylinder of level $N - \lfloor \varepsilon \log N \rfloor$, it follows that

$$f_{t_R}^i(x) \neq c,$$

for all $1 \leq i < N - \lfloor \varepsilon \log N \rfloor$. On the other hand,

$$f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\omega'') = f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\omega'),$$

and by Eq. (2.5.43), we can conclude that $f_{t_R}^i(x) \neq c$ for all i satisfying $N - \lfloor \varepsilon \log N \rfloor \leq i < N$. Therefore, for all $x \in \omega''$, we have $f_{t_R}^i(x) \neq c$ for all $0 \leq i < N$. Now, let $y''_i \in \partial\omega''$. Since $\omega'' \subset \eta''$, we have two cases.

Case 1: $y''_i \in \partial\eta''$. In this case, there is an integer j , $0 \leq j < N - \lfloor \varepsilon \log N \rfloor$, such that $f_{t_R}^j(y''_i) = c$.

Case 2: $y_i'' \notin \partial\eta''$. In this case, $f_{t_R}^j(y_i'') \neq c$ for all $0 \leq j < N - \lfloor \varepsilon \log N \rfloor$. Then $f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(y_i'') = a_i = f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(y_i')$ belongs to the interior of $f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\eta'') = f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\eta')$. Thus, y_i' belongs to the interior of η' , which implies that there exists j such that $N - \lfloor \varepsilon \log N \rfloor \leq j < N$ such that $f_{t_R}^j(y_i') = f_{t_R}^j(y_i'') = c$.

Therefore $\omega'' \in \mathcal{P}_N(t_R)$.

By assumption, $\omega' \in \hat{E}_{N, t_R, \eta'}$. Then for all $0 \leq k \leq \lfloor \frac{\varepsilon \log N}{\tau} \rfloor$, if

$$\tilde{\omega}_k \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor + j(k)}(t_R),$$

where $\omega' \subset \tilde{\omega}_k \subset \eta'$ and

$$j(k) = \min\{(k+1)\tau, \lfloor \varepsilon \log N \rfloor\},$$

then there is $z'_k \in \partial\tilde{\omega}_k$ satisfying

$$f_{t_R}^{q'_k}(z'_k) = c, \text{ for some } q'_k, 0 \leq q'_k < N - \lfloor \varepsilon \log N \rfloor + k\tau. \quad (2.5.44)$$

In the same manner as for ω' , there exists a unique cylinder $\hat{\omega}_k \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor + j(k)}$, $\hat{\omega}_k \subset \eta''$, such $f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\tilde{\omega}_k) = f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\hat{\omega}_k)$. Note that $\omega'' \subset \hat{\omega}_k$. Let $z''_k \in \partial\hat{\omega}_k$ such that

$$f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(z'_k) = f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(z''_k).$$

If $z''_k \in \partial\eta''$ then there exists $i < N - \lfloor \varepsilon \log N \rfloor$ such that $f_{t_R}^i(z''_k) = c$. Define $q''_k = i$. If $z''_k \notin \partial\eta''$ then $z'_k \notin \partial\eta'$. Thus, $f_{t_R}^q(z'_k) \neq c$ for every $q < N - \lfloor \varepsilon \log N \rfloor$, which implies that

$$N - \lfloor \varepsilon \log N \rfloor \leq q'_k < N - \lfloor \varepsilon \log N \rfloor + k\tau.$$

Then $f_{t_R}^{q'_k}(z''_k) = f_{t_R}^{q'_k}(z'_k) = c$. Define $q''_k = q'_k$.

In both cases we have $0 \leq q''_k < N - \lfloor \varepsilon \log N \rfloor + k\tau$, then $\omega'' \in \hat{E}_{N, t_R, \eta''}$.

$$f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\hat{E}_{N, t_R, \eta''}) \subset f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\hat{E}_{N, t_R, \eta'}).$$

A similar argument shows that

$$f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\hat{E}_{N, t_R, \eta'}) \subset f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\hat{E}_{N, t_R, \eta''}).$$

□

Proof of Lemma 2.5.7. Due to Lemma 2.5.18 it is enough to show that for every $\varepsilon' < \varepsilon$ there exists $C > 0$ and $K = K(\varepsilon') > 0$ such that if $J \in \mathcal{P}_j$, $j = j(N)$ then

$$|\hat{E}_{N,t_R}| \leq KN^{-C\varepsilon'}. \quad (2.5.45)$$

By Lemma 2.5.20 we have

$$\#\hat{E}_{N,t_R,\eta'} \leq 2^{\lfloor \frac{\varepsilon \log N}{\tau} \rfloor + 1}.$$

Let us define the set

$$\Omega = \bigcup_{\eta' \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor}(t_R)} f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\hat{E}_{N,t_R,\eta'}).$$

Note that

$$\hat{E}_{N,t_R} \subset f_{t_R}^{-(N - \lfloor \varepsilon \log N \rfloor)}(\Omega).$$

Therefore, if μ_{t_R} is the acip for f_{t_R} we have

$$\mu_{t_R}(\hat{E}_{N,t_R}) \leq \mu_{t_R}(\Omega). \quad (2.5.46)$$

In [31, Section 6.2] it is shown that there is $C'_1 \geq 1$ such that for every density ρ_t of the unique acip of f_t

$$\frac{1}{C'_1} \leq \rho_t(x) \leq C'_1,$$

for μ_t -almost every $x \in [0, 1]$, then

$$|\hat{E}_{N,t_R}| \leq C_1'^2 |\Omega|.$$

Since $J \in \mathcal{P}_j$, $j = j(N)$, there exists an integer p , $0 \leq p < j$ such that $x_p(t_R) = f_{t_R}^p(f_{t_R}(c)) = c$. In particular

$$\#\{f_{t_R}^i(c)\}_{i \geq 0} = p + 1.$$

Thus,

$$\#\{f_{t_R}^{N - \lfloor \varepsilon \log N \rfloor}(\eta'), \eta' \in \mathcal{P}_{N - \lfloor \varepsilon \log N \rfloor}(t_R)\} \leq (p + 1)^2.$$

Therefore, by Lemma 2.5.21,

$$\begin{aligned}
|\hat{E}_{N,t_R}| &\leq C_1'^2 |\Omega| = C_1'^2 |\cup_{\eta' \in \mathcal{P}_{N-\lfloor \varepsilon \log N \rfloor}(t_R)} f_{t_R}^{N-\lfloor \varepsilon \log N \rfloor}(\hat{E}_{\eta'})| \\
&\leq C_1'^2 (p+1)^2 \max_{\eta' \in \mathcal{P}_{N-\lfloor \varepsilon \log N \rfloor}(t_R)} |f_{t_R}^{N-\lfloor \varepsilon \log N \rfloor}(\hat{E}_{\eta'})| \\
&\leq C_1'^2 (p+1)^2 \left(\frac{1}{\lambda}\right)^{\lfloor \varepsilon \log N \rfloor} \# \left\{ \eta \in \mathcal{P}_N(t_R) |_{\hat{E}_{\eta'}} \right\} \\
&\leq C_1'^2 (p+1)^2 \left(\frac{1}{\lambda}\right)^{\lfloor \varepsilon \log N \rfloor} 2^{\lfloor \frac{\varepsilon \log N}{\tau} \rfloor + 1} \leq C_1'^2 (p+1)^2 \left(\frac{1}{\lambda}\right)^{\frac{\lfloor \varepsilon \log N \rfloor}{2}} \\
&\leq C_1'^2 j^2 \left(\frac{1}{\lambda}\right)^{\frac{\lfloor \varepsilon \log N \rfloor}{2}} \leq C_1'^2 \left(\left\lfloor \frac{\log(C_3 N)}{\log \lambda} \right\rfloor \right)^2 \left(\frac{1}{\lambda}\right)^{\frac{\lfloor \varepsilon \log N \rfloor}{2}} \\
&\leq K N^{-\frac{\log \lambda}{2} \epsilon'}
\end{aligned}$$

where $K = K(\epsilon')$. □

2.6 Estimates for the Wild part

We start this section with a technical lemma.

Lemma 2.6.1. *Given a good transversal family f_t there are constants L_1 and L_2 such that the following holds. Let $\varphi : [0, 1] \rightarrow \mathbb{R}$, $|\varphi|_{L^1(m)} > 0$, be a function of bounded variation such that*

$$\int \varphi dm = 0.$$

Then

$$|(I - \mathcal{L}_t)^{-1}(\varphi)|_{L^1} \leq \left(L_1 \log \frac{|\varphi|_{BV}}{|\varphi|_{L^1}} + L_2 \right) |\varphi|_{L^1}.$$

Proof. We have

$$(I - \mathcal{L}_t)^{-1}(\varphi) = \sum_{i=0}^{\infty} \mathcal{L}_t^i(\varphi). \quad (2.6.1)$$

Hence, by Lasota-Yorke inequality and condition (V) in Definition 2.2.3, there are $0 < \tilde{\beta} < 1$ and $L > 1$ such that

$$|\mathcal{L}_t^i(\varphi)|_{BV} \leq L \tilde{\beta}^i |\varphi|_{BV} + L |\varphi|_{L^1}.$$

Let $\tilde{j} > 0$ such that

$$L \tilde{\beta}^{\tilde{j}} |\varphi|_{BV} = |\varphi|_{L^1}.$$

Then

$$\tilde{j} = \frac{\log |\varphi|_{L^1} - \log |\varphi|_{BV} - \log L}{\log \tilde{\beta}}.$$

Note that, since $|\varphi|_{L^1} \leq |\varphi|_{BV}$ and $L > 1$, it follows that

$$\log |\varphi|_{L^1} - \log |\varphi|_{BV} - \log L < 0.$$

On the other hand, since $\tilde{\beta} < 1$, we have $\log \tilde{\beta} < 0$. In particular $\tilde{j} \geq 0$ and

$$\tilde{j} \leq K_1 (\log |\varphi|_{BV} + \log L - \log |\varphi|_{L^1})$$

for some $K_1 > 0$. Let $j_0 > 0$ be the smallest integer greater than $\tilde{j}$. Let us divide the series in Eq. (2.6.1) as follows

$$(I - \mathcal{L}_t)^{-1}(\varphi) = \underbrace{\sum_{i=0}^{j_0} \mathcal{L}_t^i(\varphi)}_{A_1} + \underbrace{\sum_{i=j_0+1}^{\infty} \mathcal{L}_t^i(\varphi)}_{A_2}.$$

Thus,

$$|(I - \mathcal{L}_t)^{-1}(\varphi)|_{L^1} \leq |A_1|_{L^1} + |A_2|_{L^1}.$$

In A_1 we use the fact that $|\mathcal{L}_t \varphi|_{L^1} \leq |\varphi|_{L^1}$.

$$|A_1|_{L^1} \leq \sum_{i=0}^{j_0} |\varphi|_{L^1} \leq j_0 |\varphi|_{L^1}.$$

Since $j_0 = \tilde{j} + \xi$, where $\xi \in (0, 1)$, it follows that

$$|A_1|_{L^1} \leq (\tilde{j} + \xi) |\varphi|_{L^1} \leq (K_1 (\log |\varphi|_{BV} + \log L - \log |\varphi|_{L^1}) + \xi) |\varphi|_{L^1}.$$

In A_2 we use that if

$$\int \varphi dm = 0,$$

then $|\mathcal{L}_t^l \varphi|_{BV} \leq \tilde{L} \theta^l |\varphi|_{BV}$, where $\theta \in (0, 1)$ and the fact that $|\varphi|_{L^1} \leq |\varphi|_{BV}$. Hence,

$$\begin{aligned} |A_2|_{L^1} &\leq \sum_{i=j_0+1}^{\infty} |\mathcal{L}_t^i \varphi|_{L^1} \leq \sum_{l=1}^{\infty} |\mathcal{L}_t^l(\mathcal{L}_t^{j_0} \varphi)|_{BV} \leq \sum_{l=1}^{\infty} \tilde{L} \theta^l |\mathcal{L}_t^{j_0} \varphi|_{BV} \\ &\leq \sum_{l=1}^{\infty} \tilde{L} \theta^l \left(L \tilde{\beta}^{j_0} |\varphi|_{BV} + L |\varphi|_{L^1} \right) \leq K_2 |\varphi|_{L^1}. \end{aligned}$$

Therefore,

$$\begin{aligned} |(I - \mathcal{L}_t)^{-1}(\varphi)|_{L^1} &\leq (K_1 (\log |\varphi|_{BV} - \log |\varphi|_{L^1}) + K_1 \log L + \xi + K_2) |\varphi|_{L^1} \\ &\leq \left(L_1 \log \frac{|\varphi|_{BV}}{|\varphi|_{L^1}} + L_2 \right) |\varphi|_{L^1}. \end{aligned}$$

□

The following proposition will be quite important to study the Wild part of the decomposition. Denote

$$\text{supp}(\psi) = \overline{\{x \in [0, 1] : \psi(x) \neq 0\}}.$$

Proposition 2.6.2. *There exist $K, K'_1, K'_2 > 0$ such that the following holds. For all $i, k \geq 0$, $t \in [0, 1]$ and $h \neq 0$, let*

$$\varphi_{k,i,h} = \frac{1}{h} \mathcal{L}_{t+h}^i \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right).$$

Then

$$|\varphi_{k,i,h}|_{L^1} \leq K, \quad (2.6.2)$$

and

$$|\varphi_{k,i,h}|_{BV} \leq \frac{K}{|h|}. \quad (2.6.3)$$

Furthermore,

$$|(I - \mathcal{L}_{t+h})^{-1} \Pi_{t+h}(\varphi_{k,i,h})|_{L^1} \leq K'_1 \max\{0, \log |\varphi_{k,i,h}|_{BV}\} + K'_2. \quad (2.6.4)$$

Proof. Note that

$$\begin{aligned} & |\mathcal{L}_{t+h}^i \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right)|_{L^1} \\ & \leq |H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}|_{L^1} \\ & \leq (\sup_t |v_t|) |h|, \end{aligned} \quad (2.6.5)$$

and, by Assumption (V) in Definition 2.2.3

$$\begin{aligned} & |\mathcal{L}_{t+h}^i \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right)|_{BV} \\ & \leq 2\tilde{C}_6 \beta^i + \tilde{C}_5 (\sup_t |v_t|) |h| \leq \hat{C}. \end{aligned} \quad (2.6.6)$$

Thus, we have Eqs. (2.6.2) and (2.6.3). In particular

$$|\Pi_{t+h}(\varphi_{k,i,h})|_{L^1(m)} \leq 2|\varphi_{k,i,h}|_{L^1(m)} \leq 2K,$$

and if h is small

$$|\Pi_{t+h}(\varphi_{k,i,h})|_{BV} \leq |\varphi_{k,i,h}|_{BV} + |\varphi_{k,i,h}|_{BV} \sup_{t \in [0,1]} |\rho_t|_{BV} \leq C |\varphi_{k,i,h}|_{BV},$$

where $C \geq 1$.

Obeserve that if $\gamma \in (0, 1)$, then

$$|\gamma \log \gamma| < 1.$$

Therefore, if $|\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} \in (0, 1)$, then by Lemma 2.6.1,

$$\begin{aligned} & |(I - \mathcal{L}_{t+h})^{-1} \Pi_{t+h}(\varphi_{k,i,h})|_{L^1} \\ & \leq (L_1 \log |\Pi_{t+h}(\varphi_{k,i,h})|_{BV} - L_1 \log |\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} + L_2) |\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} \\ & \leq L_1 \log C |\varphi_{k,i,h}|_{BV} |\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} \\ & \quad - L_1 |\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} \log |\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} + L_2 |\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} \\ & \leq 2KL_1(\log C + \log |\varphi_{k,i,h}|_{BV}) + L_1 + 2KL_2 \\ & \leq K'_1 \max\{0, \log |\varphi_{k,i,h}|_{BV}\} + K'_2. \end{aligned}$$

And if $|\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} > 1$, then,

$$\begin{aligned} & |(I - \mathcal{L}_{t+h})^{-1} \Pi_{t+h}(\varphi_{k,i,h})|_{L^1} \\ & \leq (L_1 \log |\Pi_{t+h}(\varphi_{k,i,h})|_{BV} - L_1 \log |\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} + L_2) |\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} \\ & \leq L_1 \log C |\varphi_{k,i,h}|_{BV} |\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} + L_2 |\Pi_{t+h}(\varphi_{k,i,h})|_{L^1} \\ & \leq 2KL_1(\log C + \log |\varphi_{k,i,h}|_{BV}) + 2KL_2 \\ & \leq K'_1 \max\{0, \log |\varphi_{k,i,h}|_{BV}\} + K'_2. \end{aligned}$$

□

Proposition 2.6.3. *There exists $K > 0$ such that the following holds. Let $t \in \Gamma_{h',h_0}^\delta$ and $0 < |h| \leq h'$. Then*

$$\text{var}\left(\frac{1}{h} \mathcal{L}_{t+h}^i \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right)\right) \leq \frac{K}{|h| |Df_t^i(f_t^{k+1}(c))|}, \quad (2.6.7)$$

and

$$\begin{aligned} & \int \phi(x) \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) (x) dx \\ & = \phi(f_t^{i+k+1}(c)) v_t(f_t^k(c)) + O(|Df_t^i(f_t^{k+1}(c))| |h|), \end{aligned} \quad (2.6.8)$$

where $0 \leq k \leq N_3(t, h)$ and $i < N_3(t, h) - k$.

Proof. By Eq. (2.3.3), the points $f_{t+h}^{k+1}(c)$, $f_{t+h}(f_t^k(c))$, $f_t(f_t^k(c))$ belong to the same interval of monotonicity of f_{t+h}^i . Let

$$\phi: \text{Dom}(\phi) \rightarrow \text{Im}(\phi)$$

be an inverse branch associated to such interval of monotonicity, that is, ϕ is a diffeomorphism such that $f_{t+h}^i(\phi(y)) = y$ for every $y \in \text{Dom}(\phi)$ and

$$\{f_{t+h}^{k+1}(c), f_{t+h}(f_t^k(c)), f_t(f_t^k(c))\} \subset \text{Im}(\phi).$$

Hence,

$$\begin{aligned} & \mathcal{L}_{t+h}^i \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) (x) \\ &= \frac{1}{Df_{t+h}^i(\phi(x))} \mathbb{1}_{\text{dom}(\phi)}(x) \left(H_{f_{t+h}(f_t^k(c))}(\phi(x)) - H_{f_t(f_t^k(c))}(\phi(x)) \right). \end{aligned} \quad (2.6.9)$$

There is a constant $K \geq 1$ such that for all $t \in [0, 1]$, h , and i , and every interval of monotonicity Q of f_{t+h}^i we have

$$\frac{1}{K} \leq \left| \frac{Df_{t+h}^i(y_1)}{Df_{t+h}^i(y_2)} \right| \leq K$$

for all $y_1, y_2 \in Q$. Now we can estimate the variation of the function in Eq. (2.6.7) using familiar properties of the variation of functions (see Viana [37], for instance)

$$\begin{aligned} & \text{var}_{[0,1]} \left(\mathcal{L}_{t+h}^i \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \right) \\ &= \text{var}_{[0,1]} \left(\frac{1}{Df_{t+h}^i(\phi(x))} \mathbb{1}_{\text{dom}(\phi)}(x) \left(H_{f_{t+h}(f_t^k(c))}(\phi(x)) - H_{f_t(f_t^k(c))}(\phi(x)) \right) \right) \\ &= \text{var}_{\text{Dom}(\phi)} \left(\frac{1}{Df_{t+h}^i(\phi(x))} \right) \sup_{[0,1]} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \\ &+ 2 \sup_{[0,1]} \left(\frac{1}{Df_{t+h}^i(\phi(x))} \mathbb{1}_{\text{dom}(\phi)}(x) \right) \sup_{[0,1]} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \\ &+ \sup_{[0,1]} \left(\frac{1}{Df_{t+h}^i(\phi(x))} \mathbb{1}_{\text{dom}(\phi)}(x) \right) \text{var}_{[0,1]} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \\ &\leq 2 \text{var}_{\text{Dom}(\phi)} \left(\frac{1}{Df_{t+h}^i(\phi(x))} \right) + \frac{6K}{|Df_{t+h}^i(f_{t+h}^{k+1}(c))|}. \end{aligned}$$

Now, note that since ϕ is a diffeomorphism, it follows that

$$\begin{aligned} & \text{var}_{\text{dom}(\phi)} \left(\frac{1}{Df_{t+h}^i(\phi(x))} \right) = \text{var}_{\text{Im}(\phi)} \left(\frac{1}{Df_{t+h}^i(y)} \right) \\ &= \int_{\text{Im}(\phi)} \left| D \left(\frac{1}{Df_{t+h}^i(y)} \right) \right| dy \\ &= \int_{\text{Im}(\phi)} \left| \sum_{j=1}^i - \frac{D^2 f_{t+h}(f_{t+h}^{j-1}(y))}{Df_{t+h}^{i-j}(f_{t+h}^j(y)) (Df_{t+h}(f_{t+h}^{j-1}(y)))^2} \right| dy \\ &\leq K_1 |\text{Im}(\phi)| \leq K_1 \frac{|\text{Dom}(\phi)|}{|Df_{t+h}^i(f_{t+h}^{k+1}(c))|} \\ &\leq \frac{CK_2}{|Df_{t+h}^i(f_{t+h}^{k+1}(c))|}. \end{aligned}$$

Here we used that

$$\begin{aligned} & \left| \sum_{j=1}^i - \frac{D^2 f_{t+h}(f_{t+h}^{j-1}(y))}{D f_{t+h}^{i-j}(f_{t+h}^j(y)) (D f_{t+h}(f_{t+h}^{j-1}(y)))^2} \right| \\ & \leq \sum_{j=1}^i \frac{C}{\lambda^{i-j}} \\ & \leq K_1, \end{aligned} \quad (2.6.10)$$

and that

$$|Im(\phi)| \leq K \frac{|Dom(\phi)|}{|D f_{t+h}^i(f_{t+h}^{k+1}(c))|} \leq K \frac{1}{|D f_{t+h}^i(f_{t+h}^{k+1}(c))|}.$$

Therefore,

$$var_{[0,1]} \left(\mathcal{L}_{t+h}^i \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \right) \leq \frac{K_3}{|D f_{t+h}^i(f_{t+h}^{k+1}(c))|}. \quad (2.6.11)$$

Finally, by Eq. (2.3.3) note that the combinatorics up to i iterations of $f_{t+h}^{k+1}(c)$ by the map f_{t+h} is the same as the combinatorics up to i iterations of $f_t^{k+1}(c)$ by the map f_t . By Remark 2.5.1 we obtain

$$\frac{1}{|D f_{t+h}^i(f_{t+h}^{k+1}(c))|} \leq C_1 \frac{1}{|D f_t^i(f_t^{k+1}(c))|}. \quad (2.6.12)$$

Eqs. (2.6.12) and (2.6.11) give us Eq. (2.6.7). Since

$$\supp \frac{1}{h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) = [f_{t+h}(f_t^k(c)), f_t(f_t^k(c))],$$

by Eq. (2.6.9) we conclude that

$$Z_{i,k} = \supp \frac{1}{h} \mathcal{L}_{t+h}^i \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) = [f_{t+h}^{i+1}(f_t^k(c)), f_{t+h}^i(f_t^{k+1}(c))].$$

By Eq. (2.3.3), the points $f_{t+h}^{k+1}(c), f_{t+h}(f_t^k(c)), f_t(f_t^k(c))$ belong to the same interval of monotonicity of f_{t+h}^i . Hence,

$$\begin{aligned} & diam \supp \frac{1}{h} \mathcal{L}_{t+h}^i \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \\ & = diam [f_{t+h}^{i+1}(f_t^k(c)), f_{t+h}^i(f_t^{k+1}(c))] \\ & = |f_{t+h}^{i+1}(f_t^k(c)) - f_{t+h}^i(f_t^{k+1}(c))| \\ & \leq K |D f_{t+h}^i(f_{t+h}^{k+1}(c))| |f_{t+h}(f_t^k(c)) - f_t(f_t^k(c))| \\ & \leq K |D f_{t+h}^i(f_{t+h}^{k+1}(c))| \sup_t v_t |h| \\ & \leq C_1 K |D f_t^i(f_t^{k+1}(c))| \sup_t v_t |h|. \end{aligned} \quad (2.6.13)$$

Therefore,

$$\begin{aligned}
& \int \phi(x) \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) (x) dx \\
&= \phi(f_t^{i+k+1}(c)) \int \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) (x) dx \\
&+ \int (\phi(x) - \phi(f_t^{i+k+1}(c))) \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) (x) dx. \quad (2.6.14)
\end{aligned}$$

Note that

$$\begin{aligned}
& \int \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) (x) dx \\
&= \int \frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} (x) dx = v_t(f_t^k(c)) + O(|h|). \quad (2.6.15)
\end{aligned}$$

Due to Eq (2.6.13) and the fact that ϕ is a lipschitzian function with Lipschitz constant L , and that $f_t^{i+k+1}(c) \in Z_{i,k}$

$$\begin{aligned}
& \left| \int (\phi(x) - \phi(f_t^{i+k+1}(c))) \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) (x) dx \right| \\
&\leq \int_{Z_{i,k}} |\phi(x) - \phi(f_t^{i+k+1}(c))| \left| \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) (x) \right| dx \\
&\leq LC_1 K |Df_t^i(f_t^{k+1}(c))| \sup_t v_t |h| \left| \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) \right|_{L^1} \\
&\leq LC_1 K |Df_t^i(f_t^{k+1}(c))| \sup_t v_t^2 |h|. \quad (2.6.16)
\end{aligned}$$

□

Proof of Proposition 2.3.6. Let Φ_h be as in Proposition 2.3.3, that is

$$\Phi_h = \frac{1}{h} \sum_{k=0}^{\infty} s_{k+1}(t) \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right).$$

Given $t \in \Gamma_{h,h_0}^\delta$. Let $N_3(t, h)$ be as in Proposition 2.3.5. Since t and h are fixed throughout this proof, we will write N_3 instead of $N_3(t, h)$. Let us divide Φ_h as follows

$$\Phi_h = S_1 + S_2.$$

Where

$$S_1 = \frac{1}{h} \sum_{k=0}^{N_3} s_{k+1}(t) \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right)$$

and

$$S_2 = \frac{1}{h} \sum_{k=N_3+1}^{\infty} s_{k+1}(t) \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right).$$

Let us first estimate S_2 .

$$(I - \mathcal{L}_{t+h})^{-1} S_2 = \frac{1}{h} \sum_{k=N_3+1}^{\infty} s_{k+1}(t) (I - \mathcal{L}_{t+h})^{-1} \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right).$$

Thus,

$$\begin{aligned} & |(I - \mathcal{L}_{t+h})^{-1} S_2|_{L^1} \\ & \leq \sum_{k=N_3+1}^{\infty} |s_{k+1}(t)| \left| \frac{1}{h} (I - \mathcal{L}_{t+h})^{-1} \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \right|_{L^1}. \end{aligned}$$

By Proposition 2.6.3 and Lemma 2.6.1, taking

$$\varphi = \frac{1}{h} \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right),$$

we have,

$$\begin{aligned} & \left| (I - \mathcal{L}_{t+h})^{-1} \frac{1}{h} \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \right|_{L^1} \\ & \leq K_1 \log \frac{1}{|h|} + K_2 \leq K_1 \log \Lambda^{N+1} + K_2 \\ & \leq K_1(N+1) \log \Lambda + K_2 \leq K_3 N + K_4. \end{aligned}$$

Therefore,

$$\begin{aligned} |(I - \mathcal{L}_{t+h})^{-1} S_2|_{L^1} & \leq \sum_{k=N_3+1}^{\infty} \frac{1}{\lambda^k} (K_3 N + K_4) \leq \frac{K_5 N}{\lambda^{N_3}} + K_6 \\ & \leq \frac{K_5 N}{\lambda^{N-C_5 \varepsilon \log N}} + K_6 \leq K_7 h^{K_8 \log \lambda} \left(\log \frac{1}{|h|} \right)^{1+C_5 \varepsilon \log \lambda} + K_6. \end{aligned}$$

It is left to analyze S_1 . Applying the operator $(I - \mathcal{L}_{t+h})^{-1}$,

$$(I - \mathcal{L}_{t+h})^{-1} (S_1) = \frac{1}{h} \sum_{i=0}^{\infty} \mathcal{L}_{t+h}^i \sum_{k=0}^{N_3} s_{k+1}(t) \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right).$$

Then

$$\begin{aligned} (I - \mathcal{L}_{t+h})^{-1} (S_1) & = \frac{1}{h} \sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{\infty} \mathcal{L}_{t+h}^i \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \\ & = S_{11} + S_{12}. \end{aligned}$$

Where

$$S_{11} = \sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} \frac{1}{h} \mathcal{L}_{t+h}^i \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right)$$

and

$$\begin{aligned} S_{12} &= \sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=N_3-k+1}^{\infty} \frac{1}{h} \mathcal{L}_{t+h}^i \Pi_{t+h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \\ &= \sum_{k=0}^{N_3} s_{k+1}(t) \frac{1}{h} \mathcal{L}_{t+h} \circ (I - \mathcal{L}_{t+h})^{-1} \circ \Pi_{t+h} \circ \mathcal{L}_{t+h}^{N_3-k} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right). \end{aligned}$$

We observe that

$$|S_{12}|_{L^1} \leq C \sum_{k=0}^{N_3} |s_{k+1}(t)| \|(I - \mathcal{L}_{t+h})^{-1} \circ \Pi_{t+h} \circ \mathcal{L}_{t+h}^{N_3-k} \frac{1}{h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right)\|_{L^1}.$$

Let

$$\varphi_k = \frac{1}{h} \mathcal{L}_{t+h}^{N_3-k} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right),$$

By Proposition 2.6.3 it follows that

$$|\varphi_k|_{BV} = \text{var}(\varphi_k) + |\varphi_k|_{L^1} \tag{2.6.17}$$

$$\leq \frac{C}{|h| |Df_t^{N_3(t,h)-k}(f_t^{k+1}(c))|} + K_1 \tag{2.6.18}$$

$$\begin{aligned} &\leq C \frac{|Df_t^{N(t,h)+1}(f_t(c))|}{|Df_t^{N_3(t,h)-k}(f_t^{k+1}(c))|} + K_1 \\ &\leq C |Df_t^{N(t,h)+1-N_3(t,h)}(f_t^{N_3(t,h)+1}(c))| |Df_t^k(f_t(c))| + K_1 \\ &\leq C \Lambda^{N(t,h)+1-N_3(t,h)+k} + K_1. \end{aligned} \tag{2.6.19}$$

By Lemma 2.6.2 we have

$$\begin{aligned} &|(I - \mathcal{L}_{t+h})^{-1} \circ \Pi_{t+h}(\varphi_k)|_{L^1} \\ &= |(I - \mathcal{L}_{t+h})^{-1} \circ \Pi_{t+h} \circ \mathcal{L}_{t+h}^{N_3-k} \frac{1}{h} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right)|_{L^1} \\ &\leq K'_1 \log(C \Lambda^{N(t,h)+1-N_3(t,h)+k} + K_1) + K'_2 \\ &\leq K'_1 \log(K_2 \Lambda^{N(t,h)+1-N_3(t,h)+k}) + K'_2 \\ &\leq K_3(N - N_3 + k + 1). \end{aligned}$$

Therefore,

$$\begin{aligned}
|S_{12}|_{L^1} &\leq K_3 \sum_{k=0}^{N_3} |s_{k+1}(t)| (N - N_3 + k + 1) \\
&\leq K_3 (N - N_3) \sum_{k=0}^{N_3} \frac{1}{\lambda^k} + K_3 \left(\sum_{k=0}^{N_3} \frac{k}{\lambda^k} + \sum_{k=0}^{N_3} \frac{1}{\lambda^k} \right) \\
&\leq K_4 \varepsilon \log N + K_5 \leq K \left(\log \log \frac{1}{|h|} + 1 \right).
\end{aligned}$$

We proceed to examine S_{11} .

$$\begin{aligned}
S_{11} &= \underbrace{\sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right)}_{S_{111}} \\
&\quad - \underbrace{\sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} \mathcal{L}_{t+h}^i \left(\rho_{t+h} \int \frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} dm \right)}_{S_{112}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
S_{112} &= - \sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} \rho_{t+h} \int \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) dm \\
&= - \sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} (v_t(f_t^k(c)) + O(h)) \rho_{t+h}.
\end{aligned}$$

Adding and subtracting the sum

$$\sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} v_t(f_t^k(c)) \rho_t,$$

we obtain

$$S_{112} = S_{1121} + S_{1122},$$

where

$$S_{1121} = - \sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} v_t(f_t^k(c)) \rho_t$$

and

$$S_{1122} = -(\rho_{t+h} - \rho_t) \sum_{k=0}^{N_3} s_{k+1}(t) (N_3 - k) v_t(f_t^k(c)) - O(h) \sum_{k=0}^{N_3} s_{k+1}(t) (N_3 - k) \rho_{t+h}.$$

By Eq. (2.1.1)

$$\begin{aligned}
|S_{1122}|_{L^1} &\leq K_1 |v_t|_{L^1} |h| \log \left(\frac{1}{|h|} \right) \sum_{k=0}^{N_3} |s_{k+1}(t)| (N_3 - k) \\
&\quad + |\rho_{t+h}|_{L^1} |O(h)| \sum_{k=0}^{N_3} |s_{k+1}(t)| (N_3 - k) \\
&\leq \left(K_2 |h| \log \frac{1}{|h|} + K_3 |O(h)| \right) N_3 \sum_{k=0}^{N_3} \frac{1}{\lambda^k} \\
&\leq K_4 N \left(|h| \log \frac{1}{|h|} + |O(h)| \right) \leq K \log \frac{1}{|h|} \left(|h| \log \left(\frac{1}{|h|} \right) + |O(h)| \right).
\end{aligned}$$

Therefore, taking $\phi : [0, 1] \rightarrow \mathbb{R}$ a lipschitzian observable,

$$\begin{aligned}
\int \phi(x) \mathcal{W}(x) dx &= \int \phi(x) (I - \mathcal{L}_{t+h})^{-1} \Phi_h(x) dx \\
&= \int \phi(x) (S_{111} + S_{1121})(x) dx + O \left(\log \log \frac{1}{|h|} \right) \\
&= \sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} \int \phi(x) \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) (x) dx \\
&\quad - \sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} v_t(f_t^k(c)) \int \phi(x) \rho_t(x) dx + O \left(\log \log \frac{1}{|h|} \right).
\end{aligned}$$

By Eq. (2.6.8) we have

$$\begin{aligned}
&\int \phi(x) \mathcal{L}_{t+h}^i \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h} \right) (x) dx \\
&= \phi(f_t^{i+k+1}(c)) v_t(f_t^k(c)) + O(|Df_t^i(f_t^{k+1}(c))| |h|) \\
&= \phi(f_t^{i+k+1}(c)) v_t(f_t^k(c)) + O\left(\frac{|Df_t^i(f_t^{k+1}(c))|}{|Df_t^N(f_t(c))|}\right).
\end{aligned}$$

Since

$$\begin{aligned}
& \left| \sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} O\left(\frac{|Df_t^i(f_t^{k+1}(c))|}{|Df_t^N(f_t(c))|}\right) \right| \\
& \leq K_1 \sum_{k=0}^{N_3} \sum_{i=0}^{N_3-k} \frac{1}{|Df_t^k(f_t(c))|} \frac{|Df_t^i(f_t^{k+1}(c))|}{|Df_t^N(f_t(c))|} \\
& \leq K_1 \sum_{k=0}^{N_3} \sum_{i=0}^{N_3-k} \frac{1}{|Df_t^k(f_t(c))|} \frac{1}{|Df_t^{N-i-k}(f_t^{i+k+1}(c))|} \frac{1}{|Df_t^k(f_t(c))|} \\
& \leq K_1 \sum_{k=0}^{N_3} \sum_{i=0}^{N_3-k} \frac{1}{|Df_t^k(f_t(c))|^2} \frac{1}{|Df_t^{N-i-k}(f_t^{i+k+1}(c))|} \\
& \leq K_1 \sum_{k=0}^{N_3} \left(\frac{1}{\lambda}\right)^{2k} \sum_{i=0}^{N_3-k} \left(\frac{1}{\lambda}\right)^{N-i-k} \\
& \leq K_1 \sum_{k=0}^{N_3} \left(\frac{1}{\lambda}\right)^k \sum_{i=0}^N \left(\frac{1}{\lambda}\right)^{N-i} < K,
\end{aligned} \tag{2.6.20}$$

it follows that

$$\begin{aligned}
& \int \phi(x) \mathcal{W}(x) dx \\
& = \sum_{k=0}^{N_3} s_{k+1}(t) v_t(f_t^k(c)) \sum_{i=0}^{N_3-k} \left(\phi(f_t^{i+k+1}(c)) - \int \phi d\mu_t \right) + O\left(\log \log \frac{1}{|h|}\right) \\
& = \sum_{k=0}^{N_3} s_{k+1}(t) v_t(f_t^k(c)) \sum_{j=k+1}^{N_3+1} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) + O\left(\log \log \frac{1}{|h|}\right) \\
& = \sum_{j=1}^{N_3+1} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) \sum_{k=0}^{j-1} s_{k+1}(t) v_t(f_t^k(c)) + O\left(\log \log \frac{1}{|h|}\right).
\end{aligned}$$

Adding and subtracting the series

$$\sum_{j=1}^{N_3+1} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) \sum_{k=j}^{\infty} s_{k+1}(t) v_t(f_t^k(c)),$$

we obtain

$$\begin{aligned}
\int \phi(x) \mathcal{W}(x) dx & = \sum_{j=1}^{N_3+1} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) \sum_{k=0}^{\infty} \frac{s_1(t)}{Df_t^k(f_t(c))} v_t(f_t^k(c)) \\
& \quad - \underbrace{\sum_{j=1}^{N_3+1} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) \sum_{k=j}^{\infty} \frac{s_1(t)}{Df_t^k(f_t(c))} v_t(f_t^k(c))}_{I_1} \\
& \quad + O\left(\log \log \frac{1}{|h|}\right).
\end{aligned}$$

Note that $|I_1| < \infty$. Indeed,

$$\begin{aligned} |I_1| &\leq K_1 \sum_{j=1}^{N_3+1} \left| \phi(f_t^j(c)) - \int \phi d\mu_t \right| \sum_{k=j}^{\infty} \left(\frac{1}{\lambda} \right)^k \\ &\leq K_2 \sum_{j=1}^{N_3+1} \left(\frac{1}{\lambda} \right)^j \leq K. \end{aligned}$$

Therefore,

$$\begin{aligned} \int \phi(x) \mathcal{W}(x) dx &= s_1(t) J(f_t, v_t) \sum_{j=1}^{N_3+1} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) + O \left(\log \log \frac{1}{|h|} \right) \\ &= s_1(t) J(f_t, v_t) \sum_{j=0}^{N_3} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) + O \left(\log \log \frac{1}{|h|} \right). \end{aligned} \tag{2.6.21}$$

□

2.7 Estimates for the Tame part

Let ν be a signed, finite and borelian measure on $[0, 1]$. Denote by $|\nu|$ the variation measure of ν and by $\|\nu\|$ the total variation of ν . Define the push-forward of ν by f_t as the borelian measure

$$(f_t^* \nu)(A) = \nu(f_t^{-1}(A)).$$

Note that for every bounded borelian function $g: [0, 1] \rightarrow \mathbb{R}$

$$\int g d(f_t^* \nu) = \int g \circ f_t d\nu.$$

It is also easy to see that

$$|f_t^* \nu| = f_t^* |\nu|.$$

Suppose that ν has the form

$$\nu = \pi m + \sum_{x \in \hat{\Delta}} q_x \delta_x, \tag{2.7.1}$$

where $\pi \in L^\infty(m)$ with support on $[0, 1]$, m is the Lebesgue measure, $\hat{\Delta} \subset [0, 1]$ is a countable subset, $q_x \in \mathbb{R}$, with

$$\sum_{x \in \hat{\Delta}} |q_x| < \infty,$$

and δ_x is the Dirac measure supported on $\{x\}$. Then

$$|\nu| = |\pi|m + \sum_{x \in \hat{\Delta}} |q_x| \delta_x,$$

$$\|\nu\| = |\pi|_{L^1(m)} + \sum_{x \in \hat{\Delta}} |q_x|.$$

Furthermore, $f_t^* \nu$ has the form

$$f_t^* \nu = \mathcal{L}_t(\pi)m + \sum_{x \in \hat{\Delta}} q_x \delta_{f_t(x)}.$$

Proposition 2.7.1. *Let f_t be a C^1 family of C^1 piecewise expanding unimodal maps. Let ν be a signed, finite and borelian measure. Let $\psi_t : [0, 1] \mapsto \mathbb{R}$, $t \in [0, 1]$ be such that $\psi_t \in L^\infty(\nu)$ and $t \rightarrow \psi_t$ is a lipschitzian function with respect to the $L^\infty(\nu)$ norm, that is, there exists L such that for all t, h we have*

$$|\psi_{t+h} - \psi_t|_{L^\infty(\nu)} \leq L|h|.$$

Define

$$\Delta_{t,h}(x) = \int_0^x df_{t+h}^*(\psi_{t+h}\nu) - \int_0^x df_t^*(\psi_t\nu).$$

Then there exist positive constants K_1, K_2 such that

$$|\Delta_{t,h}|_{L^1(m)} \leq (L + K_1 K_2) \|\nu\| |h|$$

for all $t \in [0, 1]$, h , where

$$K_1 = \sup_t |\psi_t|_{L^\infty(\nu)} \text{ and } K_2 = \sup_{t,x} |\partial_t f_t(x)|.$$

Proof. Observe that

$$\begin{aligned} \Delta_{t,h}(x) &= \int_0^x df_{t+h}^*(\psi_{t+h}\nu) - \int_0^x df_t^*(\psi_t\nu) \\ &= \underbrace{\int_0^x df_{t+h}^*(\psi_{t+h}\nu) - \int_0^x df_{t+h}^*(\psi_t\nu)}_{\Delta_1} \\ &\quad + \underbrace{\int_0^x df_{t+h}^*(\psi_t\nu) - \int_0^x df_t^*(\psi_t\nu)}_{\Delta_2}. \end{aligned}$$

Therefore,

$$|\Delta_{t,h}(x)| \leq |\Delta_1(x)| + |\Delta_2(x)|.$$

We first estimate Δ_1 .

$$\begin{aligned} |\Delta_1(x)| &\leq \int \mathbb{1}_{[0,x]} d|f_{t+h}^*(\psi_{t+h}\nu - \psi_t\nu)| = \int \mathbb{1}_{[0,x]} d(f_{t+h}^*(|\psi_{t+h} - \psi_t||\nu|)) \\ &\leq \int \mathbb{1}_{[0,x]} \circ f_{t+h} |\psi_{t+h} - \psi_t| d|\nu| \leq |\psi_{t+h} - \psi_t|_{L^\infty(\nu)} \|\nu\| \leq L\|\nu\||h|. \end{aligned}$$

In particular

$$|\Delta_1|_{L^1(m)} \leq L\|\nu\||h|.$$

We now estimate Δ_2 .

$$\begin{aligned} \Delta_2(x) &= \int \mathbb{1}_{[0,x]} df_{t+h}^*(\psi_t\nu) - \int \mathbb{1}_{[0,x]} df_t^*(\psi_t\nu) \\ &= \int \mathbb{1}_{[0,x]} \circ f_{t+h} d(\psi_t\nu) - \int \mathbb{1}_{[0,x]} \circ f_t d(\psi_t\nu) \\ &= \int (\mathbb{1}_{f_{t+h}^{-1}([0,x])} - \mathbb{1}_{f_t^{-1}([0,x])}) d(\psi_t\nu). \end{aligned}$$

Therefore,

$$|\Delta_2(x)| \leq \int |\mathbb{1}_{f_{t+h}^{-1}([0,x])} - \mathbb{1}_{f_t^{-1}([0,x])}| |\psi_t| d|\nu| \leq K_1 \int |\mathbb{1}_{f_{t+h}^{-1}([0,x])} - \mathbb{1}_{f_t^{-1}([0,x])}| d|\nu|$$

where

$$K_1 = \sup_t |\psi_t|_{L^\infty(\nu)}.$$

By Fubini's Theorem

$$\begin{aligned} |\Delta_2|_{L^1(m)} &\leq K \int \int |\mathbb{1}_{f_{t+h}^{-1}([0,x])}(y) - \mathbb{1}_{f_t^{-1}([0,x])}(y)| d|\nu|(y) dm(x) \\ &\leq K \int \int |\mathbb{1}_{f_{t+h}^{-1}([0,x])}(y) - \mathbb{1}_{f_t^{-1}([0,x])}(y)| dm(x) d|\nu|(y) \quad (2.7.2) \end{aligned}$$

Note that

$$|\mathbb{1}_{f_{t+h}^{-1}([0,x])}(y) - \mathbb{1}_{f_t^{-1}([0,x])}(y)| = \mathbb{1}_{U_y}(x),$$

where

$$U_y = \{x \in [0, 1] : f_{t+h}(y) < x \leq f_t(y) \text{ or } f_t(y) < x \leq f_{t+h}(y)\}.$$

Observe that

$$m(U_y) = |f_{t+h}(y) - f_t(y)| \leq K_2|h|.$$

Thus,

$$\begin{aligned} |\Delta_2|_{L^1(m)} &\leq K_1 \int \int \mathbb{1}_{U_y}(x) dm(x) d|\nu|(y) \\ &\leq K_1 K_2 \|\nu\||h|. \end{aligned} \quad (2.7.3)$$

□

Remark 2.7.2. To avoid a cumbersome notation, in the Proof of Proposition 2.3.3 we will use the following notation. Whenever we take the supremum over all $t \in [0, 1]$ we actually take the supremum over all $t \in [0, 1]$ such that f_t do not have a periodic critical point. And whenever we take the supremum over all $h \neq 0$ we indeed mean taking the supremum over all $h \neq 0$ such that $0 < |h| < \delta$, where $\delta > 0$ is given by Definition 2.2.3.

Proof of Proposition 2.3.3. We first examine

$$\frac{1}{h}(\mathcal{L}_{t+h}\rho_t - \mathcal{L}_t\rho_t).$$

As we have seen, the density ρ_t can be decomposed as

$$\rho_t = (\rho_t)_{abs} + (\rho_t)_{sal}.$$

We also have $\mathcal{L}_{t+h}\rho_t \in BV$ and

$$\mathcal{L}_{t+h}\rho_t = (\mathcal{L}_{t+h}\rho_t)_{abs} + (\mathcal{L}_{t+h}\rho_t)_{sal}.$$

Therefore,

$$(\mathcal{L}_{t+h}\rho_t - \mathcal{L}_t\rho_t) = ((\mathcal{L}_{t+h}\rho_t)_{abs} - (\mathcal{L}_t\rho_t)_{abs}) + ((\mathcal{L}_{t+h}\rho_t)_{sal} - (\mathcal{L}_t\rho_t)_{sal}).$$

Let us examine the absolutely continuous term

$$\frac{1}{h}((\mathcal{L}_{t+h}\rho_t)_{abs} - (\mathcal{L}_t\rho_t)_{abs}).$$

Observe that for every t

$$(\mathcal{L}_t\rho_t)(x) = (\mathcal{L}_t\rho_t)_{abs}(x) + (\mathcal{L}_t\rho_t)_{sal}(x).$$

Differentiating with respect to x ,

$$\begin{aligned} ((\mathcal{L}_t\rho_t)_{abs})'(x) &= (\mathcal{L}_t\rho_t)'(x) \\ &= (\mathcal{L}_t\rho_t)'_{abs}(x) + (\mathcal{L}_t\rho_t)'_{sal}(x). \end{aligned}$$

Then

$$(\mathcal{L}_t\rho_t)_{abs}(x) = \int_0^x (\mathcal{L}_t\rho_t)' dm.$$

Similarly

$$(\mathcal{L}_{t+h}\rho_t)_{abs}(x) = \int_0^x (\mathcal{L}_{t+h}\rho_t)'(y) dm.$$

Therefore,

$$\begin{aligned} (\mathcal{L}_{t+h}\rho_t)_{abs}(x) - (\mathcal{L}_t\rho_t)_{abs}(x) &= \int_0^x (\mathcal{L}_{t+h}\rho_t)' - (\mathcal{L}_t\rho_t)' dm \\ &= \int_0^x (\mathcal{L}_{t+h}\rho_t)'_{abs} - (\mathcal{L}_t\rho_t)'_{abs} dm \\ &\quad + \int_0^x (\mathcal{L}_{t+h}\rho_t)'_{sal} - (\mathcal{L}_t\rho_t)'_{sal} dm. \end{aligned}$$

We define

$$A_{t,h}(x) = \int_0^x (\mathcal{L}_{t+h}\rho_t)'_{abs} - (\mathcal{L}_t\rho_t)'_{abs} dm, \quad (2.7.4)$$

and

$$B_{t,h}(x) = \int_0^x (\mathcal{L}_{t+h}\rho_t)'_{sal} - (\mathcal{L}_t\rho_t)'_{sal} dm. \quad (2.7.5)$$

Our goal is to prove that

$$\sup_{t \in [0,1]} \sup_{h \neq 0} \left| \frac{A_{t,h}}{h} \right|_{BV} < \infty \quad \text{and} \quad \sup_{t \in [0,1]} \sup_{h \neq 0} \left| \frac{B_{t,h}}{h} \right|_{BV} < \infty.$$

Since $A_{t,h}$ is absolutely continuous, it follows that

$$var(A_{t,h}) = \int |A'_{t,h}| dm.$$

Hence, to prove that

$$\sup_{t \in [0,1]} \sup_{h \neq 0} \left| \frac{A_{t,h}}{h} \right|_{BV} < \infty,$$

it is enough to prove that

$$\sup_{t \in [0,1]} \sup_{h \neq 0} \left| \frac{A'_{t,h}}{h} \right|_{L^1(m)} dm < \infty \quad \text{and} \quad \sup_{t \in [0,1]} \sup_{h \neq 0} \left| \frac{A_{t,h}}{h} \right|_{L^1} < \infty. \quad (2.7.6)$$

According to Eq. (2.7.4),

$$A'_{t,h}(x) = (\mathcal{L}_{t+h}\rho_t)'_{abs}(x) - (\mathcal{L}_t\rho_t)'_{abs}(x).$$

Differentiating $(\mathcal{L}_{t+h}\rho_t)'$, we have, for every h ,

$$((\mathcal{L}_{t+h}\rho_t)'_{abs})'(x) = (\mathcal{L}_{t+h}\rho_t)''(x).$$

In particular

$$A''_{t,h}(y) = (\mathcal{L}_{t+h}\rho_t)''(y) - (\mathcal{L}_t\rho_t)''(y),$$

and

$$A'_{t,h}(x) = \int_0^x (\mathcal{L}_{t+h}\rho_t)'' - (\mathcal{L}_t\rho_t)'' dm. \quad (2.7.7)$$

As we have seen the Ruelle-Perron-Frobenius operator for f_{t+h} is given by

$$(\mathcal{L}_{t+h}\rho_t)(x) = \sum_{f_{t+h}(y)=x} \frac{\rho_t(y)}{|Df_{t+h}(y)|}. \quad (2.7.8)$$

Differentiating the equation (2.7.8) with respect to x we obtain

$$(\mathcal{L}_{t+h}\rho_t)'(x) = \sum_{f_{t+h}(y)=x} \frac{\rho_t'(y)}{Df_{t+h}(y)|Df_{t+h}(y)|} - \frac{\rho_t(y)D^2f_{t+h}(y)}{|Df_{t+h}(y)|^3}. \quad (2.7.9)$$

Now, differentiating the equation (2.7.9) with respect to x we obtain

$$\begin{aligned} (\mathcal{L}_{t+h}\rho_t)''(x) &= \sum_{f_{t+h}(y)=x} \left(\frac{\rho_t''(y)}{|Df_{t+h}(y)||Df_{t+h}(y)|^2} - 3 \frac{\rho_t'(y)D^2f_{t+h}(y)}{|Df_{t+h}(y)||Df_{t+h}(y)|^3} \right) \\ &+ \sum_{f_{t+h}(y)=x} \left(- \frac{\rho_t(y)D^3f_{t+h}(y)}{|Df_{t+h}(y)||Df_{t+h}(y)|^3} + 3 \frac{\rho_t(y)(D^2f_{t+h}(y))^2}{|Df_{t+h}(y)||Df_{t+h}(y)|^4} \right). \end{aligned}$$

Observe that we can rewrite $(\mathcal{L}_{t+h}\rho_t)''$ as follows

$$\begin{aligned} (\mathcal{L}_{t+h}\rho_t)'' &= \mathcal{L}_{t+h} \left(\frac{\rho_t''}{|Df_{t+h}|^2} \right) - 3\mathcal{L}_{t+h} \left(\frac{\rho_t' D^2 f_{t+h}}{(Df_{t+h})^3} \right) \\ &- \mathcal{L}_{t+h} \left(\frac{\rho_t D^3 f_{t+h}}{(Df_{t+h})^3} \right) + 3\mathcal{L}_{t+h} \left(\frac{\rho_t (D^2 f_{t+h})^2}{|Df_{t+h}|^4} \right). \end{aligned} \quad (2.7.10)$$

We obtain a similar expression for $(\mathcal{L}_t\rho_t)''$.

Substituting Eq. (2.7.10) into Eq. (2.7.7) we obtain

$$\begin{aligned} A'_{t,h}(x) &= \underbrace{\int_0^x df_{t+h}^* \left(\frac{\rho_t''}{|Df_{t+h}|^2} m \right) - \int_0^x df_t^* \left(\frac{\rho_t''}{|Df_t|^2} m \right)}_{A_1} \\ &+ \underbrace{\int_0^x df_{t+h}^* \left(\frac{-3\rho_t' D^2 f_{t+h}}{(Df_{t+h})^3} m \right) - \int_0^x df_t^* \left(\frac{-3\rho_t' D^2 f_t}{(Df_t)^3} m \right)}_{A_2} \\ &+ \underbrace{\int_0^x df_{t+h}^* \left(\frac{-\rho_t D^3 f_{t+h}}{(Df_{t+h})^3} m \right) - \int_0^x df_t^* \left(\frac{-\rho_t D^3 f_t}{(Df_t)^3} m \right)}_{A_3} \\ &+ \underbrace{\int_0^x df_{t+h}^* \left(\frac{3\rho_t (D^2 f_{t+h})^2}{|Df_{t+h}|^4} m \right) - \int_0^x df_t^* \left(\frac{3\rho_t (D^2 f_t)^2}{|Df_t|^4} m \right)}_{A_4}. \end{aligned}$$

Observe that A_i , $1 \leq i \leq 4$, satisfy the assumptions of Proposition 2.7.1 and the total variation of each one of the measures that appears above has a upper bound that depends on the constants in Assumption (V) of Definition 2.2.3. Therefore,

$$\sup_{t \in [0,1]} \sup_{h \neq 0} \left| \frac{A'_{t,h}}{h} \right|_{L^1(m)} < \infty$$

and, consequently

$$\sup_{t \in [0,1]} \sup_{h \neq 0} \text{var} \left(\frac{A_{t,h}}{h} \right) = \sup_{t \in [0,1]} \sup_{h \neq 0} \int \left| \frac{A'_{t,h}}{h} \right| dm < \infty. \quad (2.7.11)$$

It remains to verify that the second part of Eq. (2.7.6). Note that

$$\left| \frac{A_{t,h}}{h} \right|_{L^1} = \int \left| \frac{A_{t,h}}{h} \right| dm = \int \left| \int_0^x \frac{A'_{t,h}(y)}{h} dy \right| dm \leq \left| \frac{A'_{t,h}}{h} \right|_{L^1(m)}.$$

Hence, by Eq. (2.7.11), Eq. (2.7.6) holds. It remains to show that

$$\sup_{t \in [0,1]} \sup_{h \neq 0} \left| \frac{B_{t,h}}{h} \right|_{BV} < \infty.$$

By Eq. (2.7.9) and Property (V) in Definition 2.2.3 we have

$$\begin{aligned} (\mathcal{L}_{t+h}\rho_t)'_{sal}(x) &= \sum_{k=1}^{\infty} \left(\frac{s'_k(t)H_{f_{t+h}(f_t^k(c))}(x)}{Df_{t+h}(f_t^k(c))|Df_{t+h}(f_t^k(c))|} - \frac{s_k(t)H_{f_{t+h}(f_t^k(c))}(x)}{|Df_{t+h}(f_t^k(c))|^3} D^2 f_{t+h}(f_t^k(c)) \right) \\ &\quad + \left(\frac{\rho'_t(c)}{Df_{t+h}(c-)|Df_{t+h}(c-)|} + \frac{\rho'_t(c)}{Df_{t+h}(c+)|Df_{t+h}(c+)|} \right) H_{f_{t+h}(c)}(x) \\ &\quad - \left(\frac{\rho_t(c)D^2 f_{t+h}(c-)}{|Df_{t+h}(c-)|^3} + \frac{\rho_t(c)D^2 f_{t+h}(c+)}{|Df_{t+h}(c+)|^3} \right) H_{f_{t+h}(c)}(x). \end{aligned}$$

Since for every $a \in [0, 1]$ we have

$$H_a(x) = \int_0^x d(-\delta_a),$$

we can write

$$B_{t,h}(x) = \int_0^x \sum_{i=1}^4 B_i(y) dm(y),$$

with functions B_i given by

$$B_1(x) = \int_0^x df_{t+h}^* \left(\frac{1}{Df_{t+h}|Df_{t+h}|} \nu_1 \right) - \int_0^x df_t^* \left(\frac{1}{Df_t|Df_t|} \nu_1 \right)$$

where

$$\nu_1 = \sum_{k=1}^{\infty} s'_k(t)(-\delta_{f_t^k(c)}),$$

$$B_2(x) = - \int_0^x df_{t+h}^* \left(\frac{D^2 f_{t+h}}{|Df_{t+h}|^3} \nu_2 \right) + \int_0^x df_t^* \left(\frac{D^2 f_t}{|Df_t|^3} \nu_2 \right),$$

where

$$\nu_2 = \sum_{k=1}^{\infty} s_k(t)(-\delta_{f_t^k(c)}).$$

Let $\hat{\psi}$ be the constant borelian function $\hat{\psi}: [0, 1] \rightarrow \mathbb{R}$ given by

$$\hat{\psi}_t(y) = \frac{1}{Df_t(c-)|Df_t(c-)|} + \frac{1}{Df_t(c+)|Df_t(c+)|}.$$

Then

$$B_3(x) = \int_0^x df_{t+h}^*(\hat{\psi}_{t+h}\nu_3) - \int_0^x df_t^*(\hat{\psi}_t\nu_3).$$

where

$$\nu_3 = -\rho'_t(c)\delta_c.$$

Let $\tilde{\psi}$ be the constant borelian function $\tilde{\psi}: [0, 1] \rightarrow \mathbb{R}$ given by

$$\tilde{\psi}_t(y) = \frac{D^2 f_t(c-)}{|Df_t(c-)|^3} + \frac{D^2 f_t(c+)}{|Df_t(c+)|^3}.$$

then

$$B_4(x) = - \int_0^x df_{t+h}^*(\tilde{\psi}_{t+h}\nu_4) + \int_0^x df_t^*(\tilde{\psi}_t\nu_4).$$

Here

$$\nu_4 = -\rho_t(c)\delta_c.$$

We can apply Proposition 2.7.1 on each one of the pairs (B_i, ν_i) . Moreover, by property (V) of Definition 2.2.3 there is a upper bound for the total variation of the measures ν_i , $i = 1, 2, 3, 4$, that holds for every $t \in [0, 1]$. Hence,

$$\sup_{t \in [0, 1]} \sup_{h \neq 0} \left| \frac{B_i}{h} \right|_{L^1(m)} < \infty,$$

and consequently

$$\sup_{t \in [0, 1]} \sup_{h \neq 0} \text{var} \left(\frac{B_{t,h}}{h} \right) < \infty.$$

Since

$$\left| \frac{B_{t,h}}{h} \right|_{L^1} = \int \left| \frac{B_{t,h}}{h} \right| dm = \int \left| \int_0^x \sum_{i=1}^4 \frac{B_i(y)}{h} dy \right| dm \leq \sum_{i=1}^4 \left| \frac{B_i}{h} \right|_{L^1(m)},$$

we obtain

$$\sup_{t \in [0,1]} \sup_{h \neq 0} \left| \frac{B_{t,h}}{h} \right|_{BV} < \infty.$$

Therefore,

$$\sup_{t \in [0,1]} \sup_{h \neq 0} \left| \frac{(\mathcal{L}_{t+h}\rho_t)_{abs} - (\mathcal{L}_t\rho_t)_{abs}}{h} \right|_{BV} < \infty.$$

It remains to examine the saltus.

$$\begin{aligned} & \frac{(\mathcal{L}_{t+h}\rho_t)_{sal} - (\mathcal{L}_t\rho_t)_{sal}}{h} \\ &= \frac{1}{h} \sum_{k=1}^{\infty} \underbrace{\left(\frac{s_k(t)}{Df_{t+h}(f_t^k(c))} H_{f_{t+h}(f_t^k(c))} - \frac{s_k(t)}{Df_t(f_t^k(c))} H_{f_t(f_t^k(c))} \right)}_{\tilde{S}_1} \\ &+ \frac{1}{h} \left(\left(\frac{\rho_t(c)}{|Df_{t+h}(c-)|} + \frac{\rho_t(c)}{|Df_{t+h}(c+)|} \right) H_{f_{t+h}(c)} - \left(\frac{\rho_t(c)}{|Df_t(c-)|} + \frac{\rho_t(c)}{|Df_t(c+)|} \right) H_{f_t(c)} \right). \end{aligned}$$

Let us analyze $\tilde{S}_1$. Notice that

$$\begin{aligned} \tilde{S}_1 &= \frac{1}{h} \sum_{k=1}^{\infty} \frac{s_k(t)}{Df_t(f_t^k(c))} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) \\ &+ \frac{1}{h} \sum_{k=1}^{\infty} \underbrace{\left(\frac{s_k(t)}{Df_{t+h}(f_t^k(c))} - \frac{s_k(t)}{Df_t(f_t^k(c))} \right)}_{\tilde{S}_{11}} H_{f_{t+h}(f_t^k(c))}. \end{aligned}$$

Note that

$$\begin{aligned} \left| \tilde{S}_{11} \right|_{BV} &\leq \frac{1}{|h|} \sum_{k=1}^{\infty} |s_k(t)| \left| \frac{1}{Df_{t+h}(f_t^k(c))} - \frac{1}{Df_t(f_t^k(c))} \right| \left| H_{f_{t+h}(f_t^k(c))} \right|_{BV} \\ &\leq \frac{2}{|h|} \sum_{k=1}^{\infty} |s_k(t)| \frac{|Df_{t+h}(f_t^k(c)) - Df_t(f_t^k(c))|}{|Df_{t+h}(f_t^k(c)) Df_t(f_t^k(c))|} \\ &\leq \frac{K_1}{|h|} \sum_{k=1}^{\infty} \frac{1}{|Df_t^{k-1}(f_t(c))|} |(\partial_s Df_s(f_t^k(c)))|_{s=\theta_{t,h,k}} |h| \leq K. \end{aligned}$$

Hence, $\sup_h \left| \tilde{S}_{11} \right|_{BV} < \infty$. Therefore,

$$\begin{aligned}
& \frac{(\mathcal{L}_{t+h}\rho_t)_{sal} - (\mathcal{L}_t\rho_t)_{sal}}{h} \\
&= \frac{1}{h} \sum_{k=1}^{\infty} \frac{s_k(t)}{Df_t(f_t^k(c))} \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right) + \tilde{S}_{11} \\
&+ \frac{1}{h} \left(\left(\frac{\rho_t(c)}{|Df_{t+h}(c-)|} + \frac{\rho_t(c)}{|Df_{t+h}(c+)|} \right) H_{f_{t+h}(c)} - \left(\frac{\rho_t(c)}{|Df_t(c-)|} + \frac{\rho_t(c)}{|Df_t(c+)|} \right) H_{f_t(c)} \right) \\
&= \underbrace{\frac{1}{h} \sum_{k=0}^{\infty} s_{k+1}(t) \left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))} \right)}_{\tilde{S}} + \tilde{S}_{11} \\
&+ \underbrace{\frac{1}{h} \left(\frac{\rho_t(c)}{|Df_{t+h}(c-)|} - \frac{\rho_t(c)}{|Df_t(c-)|} \right) H_{f_{t+h}(c)}}_{\tilde{S}_2} \\
&+ \underbrace{\frac{1}{h} \left(\frac{\rho_t(c)}{|Df_{t+h}(c+)|} - \frac{\rho_t(c)}{|Df_t(c+)|} \right) H_{f_{t+h}(c)}}_{\tilde{S}_3}.
\end{aligned}$$

We will analyze only $\tilde{S}_2$, the term $\tilde{S}_3$ is analogous.

$$\left| \tilde{S}_2 \right|_{BV} \leq K_1 \frac{1}{|h|} \left| \frac{1}{|Df_{t+h}(c-)|} - \frac{1}{|Df_t(c-)|} \right| \leq \frac{K_2}{|h|} |h| \leq K.$$

Hence,

$$\sup_{h \neq 0} \left| \tilde{S}_2 \right|_{BV} < \infty \text{ and } \sup_{h \neq 0} \left| \tilde{S}_3 \right|_{BV} < \infty.$$

We can write

$$\begin{aligned}
\frac{\mathcal{L}_{t+h}(\rho_t) - \mathcal{L}_t(\rho_t)}{h} &= \Pi_{t+h} \left(\frac{\mathcal{L}_{t+h}(\rho_t) - \mathcal{L}_t(\rho_t)}{h} \right) \\
&= \underbrace{\Pi_{t+h}(\tilde{S})}_{\Phi_h} + \underbrace{\Pi_{t+h} \left(\frac{A}{h} + \frac{B}{h} + \tilde{S}_{11} + \tilde{S}_2 + \tilde{S}_3 \right)}_{r_h}.
\end{aligned}$$

Therefore

$$\int r_h dm = 0 \text{ and } \sup_{t \in [0,1]} \sup_{h \neq 0} |r_h|_{BV} < \infty.$$

This finishes the proof. $\square$

2.8 The function $\mathcal{R}_\phi$ is not Lipchitz on any subset of positive measure

We give two interesting and simple consequences of our main result. They tell us that, under the assumptions of our main result, the function $\mathcal{R}_\phi$ is *not* very regular in *any* subset of the parameter space with positive Lebesgue measure. This show that there is not way to make $\mathcal{R}_\phi$ more regular using some "parameter exclusion" strategy.

Corollary 2.8.1. *Under the same assumptions of our main result, for every set $\Omega \subset [a, b]$, with $m(\Omega) > 0$, we have for almost every $t \in \Omega$*

$$\limsup_{h \rightarrow 0+} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h\sqrt{-\log|h|}} \mathbb{1}_\Omega(t+h) = +\infty \quad (2.8.1)$$

and

$$\liminf_{h \rightarrow 0+} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h\sqrt{-\log|h|}} \mathbb{1}_\Omega(t+h) = -\infty, \quad (2.8.2)$$

where $\mathbb{1}_\Omega$ denotes the indicator function of Ω .

Proof. Due Proposition 2.2.5, it is enough to prove Corollary 2.8.1 for good transversal families. We are going to prove that Eq. (2.8.1) holds for almost every $t \in \Omega$. The proof that Eq. (2.8.2) holds for almost every $t \in \Omega$ is similar.

If Eq. (2.8.1) fails for t in a subset of Ω with positive Lebesgue measure, then there exist $\hat{\Omega} \subset \Omega$, with $m(\hat{\Omega}) > 0$ and $K_1 > 0$ such that for every $t \in \hat{\Omega}$ we have

$$\limsup_{h \rightarrow 0+} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h\sqrt{-\log|h|}} \mathbb{1}_\Omega(t+h) \leq K_1.$$

Since f_t is a good transversal family, without loss of generality we can assume $\inf_t \Psi(t) > 0$, there exists $K_2 > 0$ such that

$$\limsup_{h \rightarrow 0+} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{\Psi(t)h\sqrt{-\log|h|}} \mathbb{1}_\Omega(t+h) \leq K_2$$

for every $t \in \hat{\Omega}$. Then there exists $h_0 > 0$ and a set $S \subset \hat{\Omega}$ such that for every $t \in S$ we have

$$\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{\Psi(t)h\sqrt{-\log|h|}} \mathbb{1}_\Omega(t+h) \leq K_2 + 1$$

for every h satisfying $0 < h \leq h_0$. Let $t_0 \in (a, b)$ be a Lebesgue density point of S . Choose $\delta > 0$ such that

$$\mathcal{D}_N(K_2 + 1) + \delta < 1.$$

Then for every $\epsilon > 0$ small enough,

$$\frac{m(S \cap I_\epsilon)}{m(I_\epsilon)} > \mathcal{D}_\mathcal{N}(K_2 + 1) + \delta,$$

where $I_\epsilon = [t_0 - \epsilon, t_0 + \epsilon]$. Let $S_\epsilon = S \cap I_\epsilon$. It is a well-known fact that if

$$S_\epsilon - h = \{t - h : t \in S_\epsilon\}$$

then

$$\lim_{h \rightarrow 0} m(S_\epsilon \cap (S_\epsilon - h)) = m(S_\epsilon) > 0.$$

Note that for every $t \in S_\epsilon \cap (S_\epsilon - h)$, we have $t, t + h \in S_\epsilon \subset S \subset \Omega$, then

$$\frac{\mathcal{R}_\phi(t + h) - \mathcal{R}_\phi(t)}{\Psi(t)h\sqrt{-\log|h|}} \leq K_2 + 1$$

for every $0 < h \leq h_0$. In particular

$$\begin{aligned} \limsup_{h \rightarrow 0+} \frac{1}{m(I_\epsilon)} m \left(t \in I_\epsilon : \frac{1}{\Psi(t)h\sqrt{-\log|h|}} \frac{\mathcal{R}_\phi(t + h) - \mathcal{R}_\phi(t)}{h} \leq K_2 + 1 \right) \\ \geq \frac{m(S_\epsilon)}{m(I_\epsilon)} \geq \mathcal{D}_\mathcal{N}(K_2 + 1) + \delta. \end{aligned} \quad (2.8.3)$$

On the other hand the restriction of f_t to the interval I_ϵ is a transversal family, then by Theorem 2.0.7 we obtain

$$\begin{aligned} \lim_{h \rightarrow 0+} \frac{1}{m(I_\epsilon)} m \left(t \in I_\epsilon : \frac{1}{\Psi(t)h\sqrt{-\log|h|}} \frac{\mathcal{R}_\phi(t + h) - \mathcal{R}_\phi(t)}{h} \leq K_2 + 1 \right) \\ = \mathcal{D}_\mathcal{N}(K_2 + 1), \end{aligned}$$

which contradicts Eq.(2.8.3). □

Proof of Corollary 2.0.8. It follows from Corollary 2.8.1. □

Remark 2.8.2. In Baladi and Smania [3] [6] it is proven that for almost every $t \in [a, b]$ there exists a sequence $h_n \rightarrow 0$ such that

$$\frac{\mathcal{R}_\phi(t + h_n) - \mathcal{R}_\phi(t)}{h_n}$$

is not bounded. In particular $\mathcal{R}_\phi$ is not a lipschitzian function on the whole interval $[a, b]$. Naturally Corollaries 2.8.1 and 2.0.8 do not follow from this when Ω is not an interval.

Remark 2.8.3. Two weeks before this work be completed, Fabián Contreras sent us his Ph. D. Thesis [9] where he proves a result sharper than Corollary 2.8.1 when $\Omega = [a, b]$ and ϕ is a C^1 generic observable. He proves that for almost every $t \in [a, b]$ the limit

$$\lim_{h \rightarrow 0} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h \sqrt{-\log |h| \log \log |\log |h|}} \quad (2.8.4)$$

exists and it is non zero. Note again that Corollaries 2.8.1 and 2.0.8 do not seem to follow from his result when Ω is not an interval.

Properties of α

In this chapter we will consider $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ a $C^{2+\varepsilon}$ expanding map, $v : \mathbb{S}^1 \rightarrow \mathbb{R}$ a $C^{1+\varepsilon}$ map, and

$$\alpha(x) = - \sum_{n=1}^{\infty} \frac{v(f^{n-1}(x))}{Df^n(x)}. \quad (3.0.1)$$

Let

$$L = \int \log |Df| \, d\mu > 0, \quad \ell = \frac{1}{\sqrt{L}},$$

$$\phi(x) = - \left(\frac{v'(x) + \alpha(x) D^2 f(x)}{Df(x)} \right)$$

and

$$\sigma^2(\phi) = \lim_{n \rightarrow \infty} \int \left(\frac{\sum_{j=0}^n \phi \circ f^j}{\sqrt{n}} \right)^2 d\mu.$$

The goal of this chapter is to show Theorem 3.1.1 and Theorem 3.2.3.

In Theorem 3.1.1 we show a central limit theorem for the modulus of continuity of α , if α is nowhere differentiable. That is, if $\sigma(\phi) \neq 0$, then

$$\lim_{h \rightarrow 0} \mu \left\{ x : \frac{\alpha(x+h) - \alpha(x)}{\sigma(\phi) \ell h \sqrt{-\log |h|}} \leq y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{t^2}{2}} dt,$$

where μ is the absolutely continuous invariant measure with respect to the Lebesgue measure of f .

In Theorem 3.2.3 we show that the function α satisfies one of the following statements

- α is of class $C^{1+\varepsilon}$;
- α is nowhere differentiable.

3.1 Modulus of continuity of α

This section is devoted to the study of the modulus of continuity of α . Based on Gamkrelidze [11], and also using the Probability theory, we prove the following

Theorem 3.1.1. (Modulus of continuity of α) *If $\sigma(\phi) \neq 0$, then*

$$\lim_{h \rightarrow 0} \mu \left\{ x : \frac{\alpha(x+h) - \alpha(x)}{\sigma(\phi) \ell h \sqrt{-\log|h|}} \leq y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{t^2}{2}} dt.$$

where μ is the absolutely continuous invariant measure with respect to the Lebesgue measure.

In order to prove Theorem 3.1.1 we will need some preliminaries results.

Proposition 3.1.2. *Let α , v and f be as defined in 3.0.1. For fixed $x \in \mathbb{S}^1$ and $h \neq 0$, let $N(x, h)$ be the unique integer such that*

$$\frac{1}{|Df^{N(x,h)+1}(x)|} \leq |h| \leq \frac{1}{|Df^{N(x,h)}(x)|}.$$

Then

$$\alpha(x+h) - \alpha(x) = h \sum_{i=0}^{N(x,h)} \phi(f^i(x)) + O(h),$$

where

$$\phi(x) = - \left(\frac{v'(x)}{Df(x)} + \frac{\alpha(x) D^2 f(x)}{Df(x)} \right).$$

We will prove Proposition 3.1.2 later.

Lemma 3.1.3 (Functional Central Limit Theorem). *Let $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a $C^{2+\varepsilon}$ expanding map, $v : \mathbb{S}^1 \rightarrow \mathbb{R}$ be a $C^{1+\varepsilon}$ function and $\sigma > 0$. Let*

$$\phi(x) = - \left(\frac{v'(x)}{Df(x)} + \frac{\alpha(x) D^2 f(x)}{Df(x)} \right).$$

Then

$$X_N(\theta, x) = \frac{1}{\sigma\sqrt{N}} \sum_{k=0}^{\lfloor N\theta \rfloor - 1} \phi(f^k(x)) + \frac{(N\theta - \lfloor N\theta \rfloor)}{\sigma\sqrt{N}} \phi(f^{\lfloor N\theta \rfloor}(x))$$

converges in distribution to the Wiener Process. We denote $X_N \xrightarrow{D} W$.

Proof. Observe that by remark 1.3.5 we have,

$$\int \phi d\mu = 0.$$

Therefore, by Hofbauer and Keller (see [16]), if $\sigma \neq 0$, we know that the sequence

$$\xi_i(x) = \frac{1}{\sigma}(\phi \circ f^i(x))$$

satisfies the ASIP with exponent error $1/2 - \lambda$, for some $\lambda > 0$. Thus, by [28, Theorem E], we have the Functional Central Limit Theorem for $X_N(\theta, x)$. $\square$

Proof of Theorem 3.1.1. It is enough to prove that for every sequence

$$h_n \xrightarrow{n \rightarrow \infty} 0,$$

$$\lim_{n \rightarrow \infty} \mu \left\{ x : \frac{\alpha(x + h_n) - \alpha(x)}{\sigma \ell h_n \sqrt{-\log |h_n|}} \leq y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{t^2}{2}} dt.$$

By Proposition 3.1.2,

$$\alpha(x + h_n) - \alpha(x) = h_n \sum_{i=1}^{N(x, h_n)} \phi(f^{i-1}(x)) + O(h_n).$$

Since

$$\frac{1}{\sigma \ell h_n \sqrt{-\log |h_n|}} O(h_n) \xrightarrow{n \rightarrow \infty} 0,$$

we have

$$\frac{\alpha(x + h_n) - \alpha(x)}{\sigma \ell h_n \sqrt{-\log |h_n|}} = \frac{1}{\sigma \ell \sqrt{-\log |h_n|}} \sum_{i=1}^{N(x, h_n)} \phi(f^{i-1}(x)) + r(x, h_n),$$

where

$$\lim_n \sup_{x \in [0, 1]} |r(x, h_n)| = 0.$$

Let us define

$$X_N(\theta, x) = \frac{1}{\sigma \sqrt{N}} \sum_{k=0}^{\lfloor N\theta \rfloor - 1} \phi(f^k(x)) + \frac{(N\theta - \lfloor N\theta \rfloor)}{\sigma \sqrt{N}} \phi(f^{\lfloor N\theta \rfloor}(x)),$$

and Y_n by:

$$Y_n(\theta, x) = \frac{1}{\sigma \sqrt{\nu_n(x)}} \sum_{k=0}^{\lfloor \nu_n \theta \rfloor - 1} \phi(f^k(x)) + \frac{(\nu_n(x)\theta - \lfloor \nu_n(x)\theta \rfloor)}{\sigma \sqrt{\nu_n(x)}} \phi(f^{\lfloor \nu_n(x)\theta \rfloor}(x))$$

where $\nu_n(x) = N(x, h_n)$. By Lemma 1.2.4,

$$\frac{N(x, h)}{-\log |h|} \xrightarrow{h \rightarrow 0} \frac{1}{\gamma},$$

then

$$\frac{N(x, h_n)}{-\log |h_n|} \xrightarrow{P} \frac{1}{\gamma}.$$

By Lemma 3.1.3, we have

$$X_N(\theta, x) \xrightarrow{D}_N W.$$

Then, by Theorem 1.4.5 and remark 1.4.6, we conclude that

$$Y_n(\theta, x) \xrightarrow{D}_N W,$$

where W is the Wiener process, with

$$\nu_n(x) = N(x, h_n).$$

Hence, taking $\theta = 1$ we conclude that

$$Y_n(1, x) \xrightarrow{D}_n \mathcal{N}(0, 1),$$

where $\mathcal{N}(0, 1)$ denotes the Normal distribution with average zero and variance one.

Observe that

$$Y(1, x) = \frac{1}{\sigma \sqrt{\nu_n(x)}} \sum_{k=0}^{N(x, h)-1} \phi(f^k(x)).$$

Therefore, considering

$$Z_n(x) = \frac{\sqrt{N(x, h_n)}}{\sqrt{-\log |h_n|}},$$

by Slutsky's theorem (see [13]), since

$$Z_n \xrightarrow{P}_n \ell,$$

we can conclude that

$$\tilde{Y}_n(x) = Y_n(1, x) \cdot Z_n(x) = \frac{1}{\sigma \sqrt{-\log |h_n|}} \sum_{k=0}^{N(x, h)-1} \phi(f^k(x)) \xrightarrow{D}_n \ell \mathcal{N}(0, 1).$$

Hence, taking $R_n(x) = r(x, h_n)$ and using Slutsky's theorem one more time, we have

$$\frac{1}{\ell} \tilde{Y}_n(x) + R_n(x) = \frac{\alpha(x + h_n) - \alpha(x)}{\sigma \ell h_n \sqrt{-\log |h_n|}} \xrightarrow{D}_n \mathcal{N}(0, 1).$$

□

Proof of Proposition 3.1.2. Let us start by rewriting $\alpha(x+h) - \alpha(x)$. Adding and subtracting the series

$$\sum_{n=1}^{\infty} \frac{v(f^{n-1}(x+h))}{Df^n(x)}$$

we obtain

$$\begin{aligned} \alpha(x+h) - \alpha(x) = & - \underbrace{\left(\sum_{n=1}^{\infty} \frac{1}{Df^n(x)} (v(f^{n-1}(x+h)) - v(f^{n-1}(x))) \right)}_{A_1} \\ & + \underbrace{\sum_{n=1}^{\infty} v(f^{n-1}(x+h)) \left(\frac{1}{Df^n(x+h)} - \frac{1}{Df^n(x)} \right)}_{A_2}. \end{aligned}$$

Let $N(x, h)$ be as defined in Eq. (1.2.1). Since x and h are fixed throughout this proof, we will write N instead of $N(x, h)$. We can divide the series A_1 as follows:

$$\begin{aligned} A_1 = & \underbrace{\sum_{n=1}^N \frac{1}{Df^n(x)} (v(f^{n-1}(x+h)) - v(f^{n-1}(x)))}_{A_{11}} \\ & + \underbrace{\sum_{n=N+1}^{\infty} \frac{1}{Df^n(x)} (v(f^{n-1}(x+h)) - v(f^{n-1}(x)))}_{A_{12}}. \end{aligned}$$

Let us first estimate A_{12} . Since $|v|_{\infty} < \infty$ and f is an expanding map, we have

$$\begin{aligned} |A_{12}| &= \left| \sum_{n=N+1}^{\infty} \frac{1}{Df^n(x)} (v(f^{n-1}(x+h)) - v(f^{n-1}(x))) \right| \\ &\leq \sum_{n=N+1}^{\infty} \frac{1}{|Df^n(x)|} (|v(f^{n-1}(x+h))| + |v(f^{n-1}(x))|) \\ &\leq 2|v|_{\infty} \sum_{n=N+1}^{\infty} \frac{1}{|Df^{n-(N+1)}(f^{N+1}(x)) Df^{N+1}(x)|} \\ &\leq 2|v|_{\infty} |h| \sum_{n=N+1}^{\infty} \left(\frac{1}{\lambda} \right)^{n-(N+1)} < K|h|, \end{aligned}$$

where $1 < \lambda < \infty$ is the constant defined in Eq.(1.1.2). In the third inequality we used Eq.(1.2.1).

Therefore,

$$A_{12} = O(h).$$

We now examine A_{11} . By the Mean Value Theorem, we obtain

$$v(f^{n-1}(x+h)) - v(f^{n-1}(x)) = (v \circ f^{n-1})'(\theta)h = v'(f^{n-1}(\theta))Df^{n-1}(\theta)h,$$

$\theta \in (x, x+h)$. Hence,

$$\begin{aligned} A_{11} &= \sum_{n=1}^N \frac{1}{Df^n(x)} (v(f^{n-1}(x+h)) - v(f^{n-1}(x))) \\ &= \sum_{n=1}^N \frac{1}{Df^n(x)} v'(f^{n-1}(\theta)) Df^{n-1}(\theta) h. \end{aligned}$$

We now rewrite A_{11} depending only on x . To get this, we first add and subtract the term

$$\sum_{n=1}^N \frac{v'(f^{n-1}(\theta))}{Df^n(x)} Df^{n-1}(x) h.$$

Then

$$\begin{aligned} A_{11} &= \sum_{n=1}^N \frac{v'(f^{n-1}(\theta))}{Df^n(x)} \left(Df^{n-1}(x) + Df^{n-1}(x) \left(\frac{Df^{n-1}(\theta)}{Df^{n-1}(x)} - 1 \right) \right) h \\ &= \underbrace{h \sum_{n=1}^N v'(f^{n-1}(\theta)) \frac{Df^{n-1}(x)}{Df^n(x)}}_{A_{111}} + \underbrace{h \sum_{n=1}^N v'(f^{n-1}(\theta)) \frac{Df^{n-1}(x)}{Df^n(x)} \left(\frac{Df^{n-1}(\theta)}{Df^{n-1}(x)} - 1 \right)}_{A_{112}}. \end{aligned}$$

By item (b) of Lemma 1.2.5, we also have

$$|A_{112}| \leq K|h|.$$

In A_{111} , adding and subtracting the sum

$$\sum_{n=1}^N v'(f^{n-1}(x)) \frac{Df^{n-1}(x)}{Df^n(x)} h,$$

we obtain

$$A_{111} = \underbrace{\sum_{n=1}^N (v'(f^{n-1}(\theta)) - v'(f^{n-1}(x))) \frac{Df^{n-1}(x)}{Df^n(x)} h}_{(A_{1111})} + \sum_{n=1}^N v'(f^{n-1}(x)) \frac{Df^{n-1}(x)}{Df^n(x)} h.$$

By item (a) of Lemma 1.2.5, we have

$$|A_{1111}| \leq K|h|.$$

From this, we conclude

$$A_{11} = \sum_{n=1}^N v'(f^{n-1}(x)) \frac{Df^{n-1}(x)}{Df^n(x)} h + O(h) = \sum_{n=1}^N \frac{v'(f^{n-1}(x))}{Df(f^{n-1}(x))} h + O(h).$$

Therefore, A_1 can be written as follows

$$A_1 = \sum_{n=1}^N \left(\frac{v'(f^{n-1}(x))}{Df(f^{n-1}(x))} \right) h + O(h).$$

We now analyse the term A_2 . In the same manner we can divide A_2 as follows

$$\begin{aligned} A_2 &= \underbrace{\sum_{n=1}^N v(f^{n-1}(x+h)) \left(\frac{1}{Df^n(x+h)} - \frac{1}{Df^n(x)} \right)}_{A_{21}} \\ &\quad + \underbrace{\sum_{n=N+1}^{\infty} v(f^{n-1}(x+h)) \left(\frac{1}{Df^n(x+h)} - \frac{1}{Df^n(x)} \right)}_{A_{22}}. \end{aligned}$$

We start by estimating A_{22} . Since f is an expanding map and $|v|_{\infty} < \infty$, we have

$$\begin{aligned} |A_{22}| &\leq \sum_{n=N+1}^{\infty} |v(f^{n-1}(x+h))| \left| \frac{1}{Df^n(x+h)} - \frac{1}{Df^n(x)} \right| \\ &\leq |v|_{\infty} \sum_{n=N+1}^{\infty} \frac{1}{|Df^n(x+h)|} + \frac{1}{|Df^n(x)|} \\ &\leq |v|_{\infty} \sum_{n=N+1}^{\infty} \left(\left| \frac{Df^n(x)}{Df^n(x+h)} \right| \frac{1}{|Df^n(x)|} + \frac{1}{|Df^n(x)|} \right) \\ &\leq K_1 \sum_{n=N+1}^{\infty} \frac{1}{|Df^{N+1}(x)|} \frac{1}{|Df^{n-(N+1)}(f^{N+1}(x))|} \\ &\leq K_1 |h| \sum_{n=N+1}^{\infty} \left(\frac{1}{\lambda} \right)^{n-(N+1)} \leq K |h|, \end{aligned}$$

where $1 < \lambda < \infty$ is the constant defined in Eq.(1.1.2). In the fourth inequality we used Lemma 1.2.2 and Eq.(1.2.1).

We now study the term A_{21} . Let us denote

$$G(x) = \frac{1}{Df^n(x)}.$$

By Mean Value Theorem,

$$G(x+h) - G(x) = G'(\theta)h,$$

where $\theta \in (x, x + h)$. Calculating G' , we obtain

$$\begin{aligned} G'(\theta) &= D \left(\frac{1}{Df^n(\theta)} \right) = - \frac{D(Df^n(\theta))}{(Df^n(\theta))^2} \\ &= - \sum_{j=1}^n \frac{D^2 f(f^{j-1}(\theta))}{Df^{n-j}(f^j(\theta)) (Df(f^{j-1}(\theta)))^2}. \end{aligned}$$

Therefore,

$$A_{21} = \sum_{n=1}^N v(f^{n-1}(x+h)) \left(\sum_{j=1}^n - \frac{D^2 f(f^{j-1}(\theta))}{Df^{n-j}(f^j(\theta)) (Df(f^{j-1}(\theta)))^2} h \right).$$

Our purpose now is to write A_{21} depending only on x . We begin by adding and subtracting the term

$$\sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n - \frac{D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(\theta)) (Df(f^{j-1}(\theta)))^2} h.$$

Then

$$\begin{aligned} A_{21} &= \sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n - \frac{D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(\theta)) (Df(f^{j-1}(\theta)))^2} h \\ &\quad + \underbrace{\sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n \frac{-D^2 f(f^{j-1}(\theta)) + D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(\theta)) (Df(f^{j-1}(\theta)))^2} h}_{A_{211}}. \end{aligned}$$

By item (c) of Lemma 1.2.5, we have $|A_{211}| \leq K|h|$.

In this way,

$$A_{21} = \sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n - \frac{D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(\theta)) (Df(f^{j-1}(\theta)))^2} h + O(h).$$

We now add and subtract the term

$$\sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n - \frac{D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(x)) (Df(f^{j-1}(\theta)))^2} h,$$

and we obtain

$$\begin{aligned} A_{21} &= \sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n - \frac{D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(x)) (Df(f^{j-1}(\theta)))^2} h \\ &\quad + \underbrace{\sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n - \frac{D^2 f(f^{j-1}(x))}{(Df(f^{j-1}(\theta)))^2} \frac{1}{Df^{n-j}(f^j(x))} \left(\frac{Df^{n-j}(f^j(x))}{Df^{n-j}(f^j(\theta))} - 1 \right) h}_{A_{212}} \\ &\quad + O(h). \end{aligned}$$

We conclude from item (d) of Lemma 1.2.5 that $|A_{212}| \leq K|h|$. Hence,

$$A_{21} = \sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n -\frac{D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(x)) (Df(f^{j-1}(\theta)))^2} h + O(h).$$

Now, adding and subtracting the term

$$\sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n -\frac{D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(x)) (Df(f^{j-1}(x)))^2} h,$$

we obtain

$$\begin{aligned} A_{21} &= \sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n -\frac{D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(x)) (Df(f^{j-1}(x)))^2} h \\ &+ \underbrace{\sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n -\frac{D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(x)) (Df(f^{j-1}(x)))^2} \left(\frac{(Df(f^{j-1}(x)))^2}{(Df(f^{j-1}(\theta)))^2} - 1 \right) h}_{A_{213}} \\ &+ O(h) \end{aligned}$$

Similar arguments apply to estimate A_{213}

$$\begin{aligned} |A_{213}| &\leq \sum_{n=1}^N |v(f^{n-1}(x+h))| \sum_{j=1}^n \frac{|D^2 f(f^{j-1}(x))|}{|Df^{n-j}(f^j(x))| (Df(f^{j-1}(x)))^2} \left| \frac{(Df(f^{j-1}(x)))^2}{(Df(f^{j-1}(\theta)))^2} - 1 \right| |h| \\ &\leq K_1 |h| \sum_{n=1}^N \sum_{j=1}^n \left(\frac{1}{\lambda} \right)^{n-j} \left| \frac{(Df(f^{j-1}(x)))^2}{(Df(f^{j-1}(\theta)))^2} - 1 \right| \\ &= K_1 |h| \sum_{n=1}^N \sum_{j=1}^n \left(\frac{1}{\lambda} \right)^{n-j} \left| \frac{Df(f^{j-1}(x))}{Df(f^{j-1}(\theta))} - 1 \right| \left| \frac{Df(f^{j-1}(x))}{Df(f^{j-1}(\theta))} + 1 \right| \\ &\leq K_2 |h| \sum_{n=1}^N \sum_{j=1}^n \left(\frac{1}{\lambda} \right)^{n-j} \left| \frac{Df(f^{j-1}(x))}{Df(f^{j-1}(\theta))} - 1 \right| \\ &\leq K_3 |h| \sum_{n=1}^N \sum_{j=1}^n \left(\frac{1}{\lambda} \right)^{n-j} |f^j(x) - f^j(\theta)| \leq K|h|. \end{aligned}$$

Thus,

$$A_{21} = \sum_{n=1}^N v(f^{n-1}(x+h)) \sum_{j=1}^n -\frac{D^2 f(f^{j-1}(x))}{Df^{n-j}(f^j(x)) (Df(f^{j-1}(x)))^2} h + O(h).$$

Finally, adding and subtracting the term

$$\sum_{n=1}^N v(f^{n-1}(x)) \sum_{j=1}^n -\frac{D^2 f(f^{j-1}(x))}{D f^{n-j}(f^j(x)) (D f(f^{j-1}(x)))^2} h,$$

we found

$$\begin{aligned} A_{21} &= \sum_{n=1}^N v(f^{n-1}(x)) \sum_{j=1}^n -\frac{D^2 f(f^{j-1}(x))}{D f^{n-j}(f^j(x)) (D f(f^{j-1}(x)))^2} h \\ &+ \underbrace{\sum_{n=1}^N (v(f^{n-1}(x+h)) - v(f^{n-1}(x))) \sum_{j=1}^n -\frac{D^2 f(f^{j-1}(x))}{D f^{n-j}(f^j(x)) (D f(f^{j-1}(x)))^2} h}_{(A_{214})} \\ &+ O(h). \end{aligned}$$

Since f is an expanding map and by Mean Value Theorem, we can estimate A_{214} as follows

$$\begin{aligned} |A_{214}| &\leq \sum_{n=1}^N |v(f^{n-1}(x+h)) - v(f^{n-1}(x))| \sum_{j=1}^n \left| \frac{D^2 f(f^{j-1}(x))}{D f^{n-j}(f^j(x)) (D f(f^{j-1}(x)))^2} h \right| \\ &\leq K_1 |h| \sum_{n=1}^N |v'(\theta)| |f^{n-1}(x+h) - f^{n-1}(x)| \\ &\leq K_2 |h| \sum_{j=0}^N \left(\frac{1}{\lambda} \right)^j \leq K |h|, \end{aligned}$$

where $0 < 1/\lambda < 1$ is the constant defined in Eq.(1.1.2).

Hence,

$$A_{21} = \sum_{n=1}^N \sum_{j=1}^n -v(f^{n-1}(x)) \frac{D^2 f(f^{j-1}(x))}{D f^{n-j}(f^j(x)) (D f(f^{j-1}(x)))^2} h + O(h).$$

If we invert the order of the sums, then

$$A_{21} = - \sum_{j=1}^N \sum_{n=j}^N \frac{v(f^{n-1}(x)) D^2 f(f^{j-1}(x))}{D f^{n-j}(f^j(x)) (D f(f^{j-1}(x)))^2} h + O(h).$$

That is,

$$\begin{aligned}
A_{21} &= - \sum_{j=1}^N \frac{D^2 f(f^{j-1}(x))}{Df(f^{j-1}(x))} \sum_{n=j}^N \frac{v(f^{n-j}(f^{j-1}(x)))}{Df^{n-j}(f^j(x)) Df(f^{j-1}(x))} h + O(h) \\
&= - \sum_{j=1}^N \frac{D^2 f(f^{j-1}(x))}{Df(f^{j-1}(x))} \sum_{n=j}^N \frac{v(f^{n-j}(f^{j-1}(x)))}{Df^{n-j+1}(f^{j-1}(x))} h + O(h) \\
&= - \sum_{j=1}^N \frac{D^2 f(f^{j-1}(x))}{Df(f^{j-1}(x))} \sum_{k=0}^{N-j} \frac{v(f^k(f^{j-1}(x)))}{Df^{k+1}(f^{j-1}(x))} h + O(h) \\
&= - \sum_{j=1}^N \frac{D^2 f(f^{j-1}(x))}{Df(f^{j-1}(x))} \sum_{k=1}^{\infty} \frac{v(f^{k-1}(f^{j-1}(x)))}{Df^k(f^{j-1}(x))} h \\
&\quad + \sum_{j=1}^N \frac{D^2 f(f^{j-1}(x))}{Df(f^{j-1}(x))} \sum_{k=N-j+1}^{\infty} \frac{v(f^k(f^{j-1}(x)))}{Df^{k+1}(f^{j-1}(x))} h + O(h) \\
&= \sum_{j=1}^N \alpha(f^{j-1}(x)) \frac{D^2 f(f^{j-1}(x))}{Df(f^{j-1}(x))} h \\
&\quad + \underbrace{\sum_{j=1}^N \frac{D^2 f(f^{j-1}(x))}{Df(f^{j-1}(x))} \sum_{k=N-j+1}^{\infty} \frac{v(f^k(f^{j-1}(x)))}{Df^{k+1}(f^{j-1}(x))} h}_{A_{215}} + O(h).
\end{aligned}$$

In the fourth equality we sum and subtract the term

$$\sum_{j=1}^N \frac{D^2 f(f^{j-1}(x))}{Df(f^{j-1}(x))} \sum_{k=N-j+1}^{\infty} \frac{v(f^k(f^{j-1}(x)))}{Df^{k+1}(f^{j-1}(x))} h.$$

The estimate of A_{215} follows analogously to the previous cases

$$\begin{aligned}
|A_{215}| &\leq K_1 \left| \sum_{j=1}^N \sum_{k=N-j+1}^{\infty} \frac{v(f^k(f^{j-1}(x)))}{Df^{k+1}(f^{j-1}(x))} h \right| \\
&\leq K_2 \sum_{j=1}^N \sum_{k=N-j+1}^{\infty} \left(\frac{1}{\lambda} \right)^{k+1} |h| \leq K_3 \sum_{j=0}^N \left(\frac{1}{\lambda} \right)^{N-j+2} |h| \\
&\leq K|h|,
\end{aligned}$$

where $0 < 1/\lambda < 1$ is the constant defined in Eq.(1.1.2).

Therefore,

$$A_2 = \sum_{j=1}^N \alpha(f^{j-1}(x)) \frac{D^2 f(f^{j-1}(x))}{Df(f^{j-1}(x))} h + O(h).$$

In this way, we obtain

$$\begin{aligned} \alpha(x+h) - \alpha(x) &= -(A_1 + A_2) \\ &= - \left(\sum_{n=1}^N \frac{v'(f^{n-1}(x))}{Df(f^{n-1}(x))} + \frac{\alpha(f^{n-1}(x))D^2f(f^{n-1}(x))}{Df(f^{n-1}(x))} \right) h + O(h). \end{aligned}$$

□

Considering $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ an expanding map of class $C^{2+\varepsilon}$ and $\phi : \mathbb{S}^1 \rightarrow \mathbb{R}$ defined by

$$\phi(x) = - \frac{v'(x) + D^2f(x)\alpha(x)}{Df(x)},$$

we can define the variance σ^2 by

$$\sigma^2 = \lim_{n \rightarrow \infty} \int_0^1 \left(\frac{\sum_{j=0}^{n-1} \phi \circ f^j}{\sqrt{n}} \right)^2 d\mu, \quad (3.1.1)$$

where μ is the f -absolutely continuous invariant measure (see [8]).

Then we have the following proposition

Proposition 3.1.4. *Considering σ^2 as defined above, then $\sigma^2 = 0$ if and only if α is of class $C^{1+\varepsilon}$.*

Proof. If $\sigma^2 = 0$ then as we can see in Proposition 6.1 from [8], there is a C^ε -function $u : \mathbb{S}^1 \rightarrow \mathbb{R}$ such that

$$\phi = u \circ f - u.$$

Then there is a constant $C > 0$ such that

$$\sup_{n,x} \left| \sum_{i=0}^n \phi(f^i(x)) \right| \leq C.$$

And, as we have seen in Proposition 3.1.2,

$$|\alpha(x+h) - \alpha(x)| \leq \left| \sum_{i=0}^{N(x,h)} \phi(f^i(x))h \right| + O(h) \leq C|h|.$$

Therefore, α is a lipschitzian function and we can derive it in almost all points.

Then, deriving Eq.(1.1.8), we obtain:

$$v'(x) = \alpha'(f(x))Df(x) - D^2f(x)\alpha(x) - Df(x)\alpha'(x).$$

Therefore,

$$\alpha'(f(x)) - \alpha'(x) = \frac{v'(x) + D^2f(x)\alpha(x)}{Df(x)} = \phi(x).$$

Since, $\phi(x) = u \circ f - u$, it follows that

$$(\alpha' - u) \circ f(x) = (\alpha' - u)(x).$$

Since f is ergodic, we can conclude that there is a constant k such that $\alpha' = u + k$.

Therefore, α is of class $C^{1+\varepsilon}$.

Reciprocally, if α is of class $C^{1+\varepsilon}$, the equation (1.1.8) can be derived. Thus,

$$v'(x) = \alpha'(f(x))Df(x) - D^2f(x)\alpha(x) - Df(x)\alpha'(x).$$

Therefore,

$$\alpha'(f(x)) - \alpha'(x) = \frac{v'(x) + D^2f(x)\alpha(x)}{Df(x)}.$$

Let p be a periodic point of f , that is, $f^n(p) = p$. Then

$$\sum_{j=0}^{n-1} \phi(f^j(p)) = \alpha'(f^n(p)) - \alpha'(p) = \alpha'(p) - \alpha'(p) = 0.$$

Therefore, by the Livsic Theorem (see [26] and the references given there), there is a ε -Hölder function u such that

$$\phi = u \circ f - u.$$

Hence, as we can see in Proposition 6.1 from [8], $\sigma^2 = 0$. □

3.2 Smoothness properties of α

We start this section by showing that α is in Zygmund class Λ_1 , that is,

$$|\alpha(x+h) + \alpha(x-h) - 2\alpha(x)| \leq C|h|, \text{ for all } x \in [0, 1] \text{ and } |h| \leq 1.$$

And we conclude it by proving that either α is of class $C^{1+\varepsilon}$ or α is nowhere differentiable.

For this, we need to introduce some notations and definitions, following Heurteaux [15].

The space of the functions $C^{1+\varepsilon}(\mathbb{R})$ is endowed with the natural norm

$$|g|_{1+\varepsilon} = |g|_{\infty} + N_{1+\varepsilon}(g),$$

where

$$N_{1+\varepsilon}(g) = \sup \left(\frac{|g(x+h) + g(x-h) - 2g(x)|}{|h|^{1+\varepsilon}}, x \in \mathbb{R}, h > 0 \right).$$

We denote the oscillation of g by

$$\text{osc}(g) = \sup(|g(x) - g(y)|, x, y \in \mathbb{R}).$$

If I is a closed interval, we denote the second-order oscillation of the function g in I by

$$\text{osc}_2(g, I) = \sup(|g(x+h) + g(x-h) - 2g(x)|, [x-h, x+h] \subset I).$$

And, finally, we denote the second-order difference of some function g by

$$\Delta_h^2 g(x) = g(x+h) + g(x-h) - 2g(x).$$

Now we can proceed to show that α is in Zygmund class Λ_1 .

Proposition 3.2.1. *Consider f and α as defined before. Then α is in Zygmund class Λ_1 .*

Proof. By Proposition 3.1.2,

$$\alpha(x+h) - \alpha(x) = - \sum_{n=1}^{N(x,h)} \frac{v'(f^{n-1}(x))}{Df(f^{n-1}(x))} + \frac{\alpha(f^{n-1}(x))D^2f(f^{n-1}(x))}{Df(f^{n-1}(x))} h + O(h).$$

And

$$\alpha(x-h) - \alpha(x) = \sum_{n=1}^{N(x,h)} \frac{v'(f^{n-1}(x))}{Df(f^{n-1}(x))} + \frac{\alpha(f^{n-1}(x))D^2f(f^{n-1}(x))}{Df(f^{n-1}(x))} h + O(h).$$

Therefore,

$$|\alpha(x+h) + \alpha(x-h) - 2\alpha(x)| \leq K|h|,$$

which completes the proof. $\square$

Our next goal is to show that α is of class $C^{1+\varepsilon}$ or nowhere differentiable.

It is well-known that a function in Zygmund class is β -Hölder for any $0 < \beta < 1$ (see [10] and the references given there). We will use this fact to show lemma below.

Lemma 3.2.2. *Let us consider α and v as defined previously and let*

$$g(x) = \frac{1}{Df(x)}v(x).$$

Then

$$\Delta_h^2 \alpha(x) = \frac{1}{Df(x)} \Delta_{Df(x)h}^2 \alpha(f(x)) - \Delta_h^2 g(x) + O(|h|^{1+\varepsilon}).$$

Proof. Since α satisfies the twisted cohomological equation (1.1.8), it follows that

$$\alpha(x) = \frac{1}{Df(x)}\alpha(f(x)) - \frac{1}{Df(x)}v(x).$$

Hence,

$$\begin{aligned} \Delta_h^2 \alpha(x) &= \alpha(x+h) + \alpha(x-h) - 2\alpha(x) \\ &= \frac{1}{Df(x+h)}\alpha(f(x+h)) + \frac{1}{Df(x-h)}\alpha(f(x-h)) - 2\frac{1}{Df(x)}\alpha(f(x)) \\ &\quad - \left(\frac{1}{Df(x+h)}v(x+h) + \frac{1}{Df(x-h)}v(x-h) - 2\frac{1}{Df(x)}v(x) \right) \\ &= \frac{1}{Df(x+h)}\alpha(f(x+h)) - \frac{1}{Df(x)}\alpha(f(x+h)) \\ &\quad + \frac{1}{Df(x-h)}\alpha(f(x-h)) - \frac{1}{Df(x)}\alpha(f(x-h)) \\ &\quad + \frac{1}{Df(x)}\alpha(f(x+h)) + \frac{1}{Df(x)}\alpha(f(x-h)) - 2\frac{1}{Df(x)}\alpha(f(x)) \\ &\quad - \Delta_h^2 g(x). \end{aligned}$$

In the third equality we added and subtracted the terms

$$\frac{1}{Df(x)}\alpha(f(x+h)) \quad \text{and} \quad \frac{1}{Df(x)}\alpha(f(x-h)).$$

By Mean Value Theorem and the Taylor expansion:

$$\begin{aligned} \Delta_h^2 \alpha(x) &= \alpha(f(x+h)) \left(\frac{1}{Df(x+h)} - \frac{1}{Df(x)} \right) \\ &\quad + \alpha(f(x-h)) \left(\frac{1}{Df(x-h)} - \frac{1}{Df(x)} \right) \\ &\quad + \frac{1}{Df(x)}\alpha(f(x) + Df(x)h + O(h^{2+\varepsilon})) + \frac{1}{Df(x)}\alpha(f(x) - Df(x)h + O(h^{2+\varepsilon})) \\ &\quad - 2\frac{1}{Df(x)}\alpha(f(x)) - \Delta_h^2 g(x) \\ &= \alpha(f(x+h)) \left(\left(\frac{1}{Df} \right)'(\theta_1)h \right) - \alpha(f(x-h)) \left(\left(\frac{1}{Df} \right)'(\theta_2)h \right) \\ &\quad + \frac{1}{Df(x)} \left(\alpha(f(x) + Df(x)h + O(h^{2+\varepsilon})) - \alpha(f(x) + Df(x)h) \right) \\ &\quad + \frac{1}{Df(x)} \left(\alpha(f(x) - Df(x)h + O(h^{2+\varepsilon})) - \alpha(f(x) - Df(x)h) \right) \\ &\quad + \frac{1}{Df(x)}\alpha(f(x) + Df(x)h) + \frac{1}{Df(x)}\alpha(f(x) - Df(x)h) \\ &\quad - 2\frac{1}{Df(x)}\alpha(f(x)) - \Delta_h^2 g(x). \end{aligned}$$

In the second equality we added and subtracted the terms

$$\frac{1}{Df(x)}\alpha(f(x) + Df(x)h) \quad \text{and} \quad \frac{1}{Df(x)}\alpha(f(x) - Df(x)h).$$

Then adding and subtracting

$$\alpha(f(x+h)) \left(\left(\frac{1}{Df} \right)' (\theta_2)h \right)$$

we obtain

$$\begin{aligned} \Delta_h^2 \alpha(x) &= \underbrace{\alpha(f(x+h)) \left(\left(\frac{1}{Df} \right)' (\theta_1)h - \left(\frac{1}{Df} \right)' (\theta_2)h \right)}_{B_1} \\ &\quad + \underbrace{(\alpha(f(x+h)) - \alpha(f(x-h))) \left(\left(\frac{1}{Df} \right)' (\theta_2)h \right)}_{B_2} \\ &\quad + \underbrace{\frac{1}{Df(x)} (\alpha(f(x) + Df(x)h + O(h^{2+\varepsilon})) - \alpha(f(x) + Df(x)h))}_{B_3} \\ &\quad + \underbrace{\frac{1}{Df(x)} (\alpha(f(x) - Df(x)h + O(h^{2+\varepsilon})) - \alpha(f(x) - Df(x)h))}_{B_4} \\ &\quad + \frac{1}{Df(x)}\alpha(f(x) + Df(x)h) + \frac{1}{Df(x)}\alpha(f(x) - Df(x)h) \\ &\quad - 2\frac{1}{Df(x)}\alpha(f(x)) - \Delta_h^2 g(x). \end{aligned}$$

Now, since f is of class $C^{2+\varepsilon}$, it follows that

$$|(B_1)| \leq K|h|^{1+\varepsilon}.$$

And, since α is β -Hölder for $0 < \beta < 1$, considering $\beta \sim 1$, we have

$$|(B_2)| \leq K|h|^{1+\varepsilon},$$

$$|(B_3)| \leq KO(|h^{2+\varepsilon}|)^\beta \leq KO(|h^{1+\varepsilon}|),$$

and

$$|(B_4)| \leq KO(|h^{2+\varepsilon}|)^\beta \leq KO(|h^{1+\varepsilon}|).$$

Thus,

$$\Delta_h^2 \alpha(x) = \frac{1}{Df(x)}\Delta_{Df(x)h}^2 \alpha(f(x)) - \Delta_h^2 g(x) + O(|h^{1+\varepsilon}|),$$

as we wanted. □

Therefore,

$$|\Delta_h^2 \alpha(x)| \geq \frac{1}{|Df(x)|} |\Delta_{Df(x)h}^2 \alpha(f(x))| - |\Delta_h^2 g(x)| - K|h|^{1+\varepsilon}, \quad (3.2.1)$$

where K is a positive constant such that $|O(|h|^{1+\varepsilon})| \leq K|h|^{1+\varepsilon}$.

We will denote by

$$b := \inf\{|Df(x)| : x \in \mathbb{R}\} \quad (3.2.2)$$

and

$$B := \sup\{|Df(x)| : x \in \mathbb{R}\}. \quad (3.2.3)$$

We can now state the main result of this section.

Theorem 3.2.3. *Let v , f and α as defined in 3.0.1. Then one of the following statements holds:*

(i) α is of class $C^{1+\varepsilon}$;

or

(ii) α is nowhere differentiable.

Before proving the theorem, we will need some preliminaries results.

Proposition 3.2.4. *Let v , f and α as defined in Eq.(3.0.1). Consider*

$$g(x) = \frac{1}{Df(x)} v(x),$$

and let $K > 0$ be the constant defined in Eq. (3.2.1). If

$$N_{1+\varepsilon}(\alpha) > \frac{N_{1+\varepsilon}(g) + K}{(b^\varepsilon - 1)},$$

then there exists a constant $C > 0$ such that for all closed interval I of length $|I| \leq 1$,

$$\text{osc}_2(\alpha, I) \geq C|I|.$$

We will prove Proposition 3.2.4 later.

Lemma 3.2.5. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a map and let $y_0 \in \mathbb{R}$ such that ϕ is differentiable at y_0 . Then*

$$\lim_{J \rightarrow \{y_0\}} \frac{\text{osc}_2(\phi, J)}{|J|} = 0.$$

Proof. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\phi(y_0 + h) = \phi(y_0) + h\phi'(y_0) + h\varepsilon(h)$$

where $|\varepsilon(h)| < \varepsilon$ when $|h| < \delta$.

Consider a interval J such that $|J| < \delta$ containing y_0 . Let us assume that that $[y - h', y + h'] \subset J$. Hence,

$$\begin{aligned} |\Delta_{h'}^2 \phi(y)| &= |\phi(y_0 + (y + h' - y_0)) + \phi(y_0 + (y - h' - y_0)) - 2\phi(y_0 + (y - y_0))| \\ &= |(y + h' - y_0)\varepsilon(y + h' - y_0) + (y - h' - y_0)\varepsilon(y - h' - y_0) - 2(y - y_0)\varepsilon(y - y_0)| \\ &\leq 4\varepsilon|J|, \end{aligned}$$

which concludes the proofs of lemma. $\square$

Now we can prove Theorem 3.2.3.

Proof of Theorem 3.2.3. A continuous bounded function ϕ is of class $C^{1+\varepsilon}$ if and only if

$$N_{1+\varepsilon}(\phi) < \infty,$$

(see [23], Lemma 5.4 page 207 for more details).

Suppose that α is of class $C^{1+\varepsilon}$. By Eq.(3.2.1), we have

$$|\Delta_h^2 \alpha(x)| \geq \frac{1}{|Df(x)|} |\Delta_{Df(x)h}^2 \alpha(f(x))| - |\Delta_h^2 g(x)| - K|h|^{1+\varepsilon}.$$

Which implies after some calculations that

$$N_{1+\varepsilon}(\alpha) \leq \frac{N_{1+\varepsilon}(g) + K}{(b^\varepsilon - 1)}.$$

Therefore, α is of class $C^{1+\varepsilon}$ if and only if

$$N_{1+\varepsilon}(\alpha) \leq \frac{N_{1+\varepsilon}(g) + K}{(b^\varepsilon - 1)}. \quad (3.2.4)$$

Suppose that α is differentiable at x_0 . Then, by Lemma 3.2.5, for all $\varepsilon > 0$ there is $\delta > 0$ such that if $|I| < \delta$, then

$$\frac{\text{osc}_2(\alpha, I)}{|I|} < \varepsilon.$$

Therefore, by Proposition 3.2.4, we have

$$N_{1+\varepsilon}(\alpha) \leq \frac{N_{1+\varepsilon}(g) + K}{b^\varepsilon - 1}.$$

Hence, by Eq.(3.2.4), α is of class $C^{1+\varepsilon}$. $\square$

In order to prove Proposition 3.2.4, we will start with a lemma giving a minorization of second-order differences of α .

Lemma 3.2.6. *Let g and K be as defined in Proposition 3.2.4 and*

$$N_{1+\varepsilon}(\alpha) > \frac{N_{1+\varepsilon}(g) + K}{(b^\varepsilon - 1)}. \quad (3.2.5)$$

Then there are real numbers $\ell \geq 1$, $L > 0$ and $h > 0$ such that all closed interval J of length $|J| = \ell$ contains a point x_J satisfying:

$$[x_J - h, x_J + h] \subset J$$

and

$$|\Delta_h^2 \alpha(x_J)| \geq \frac{N_{1+\varepsilon}(g) + L}{(b^\varepsilon - 1)} |h|^{1+\varepsilon} + \frac{K}{(b^\varepsilon - 1)} |h|^{1+\varepsilon}.$$

Proof. By Eq.(3.2.5), there are $x_0 \in \mathbb{R}$, $L > 0$ and $h > 0$ such that

$$|\Delta_h^2 \alpha(x_0)| \geq \frac{N_{1+\varepsilon}(g) + L}{(b^\varepsilon - 1)} |h|^{1+\varepsilon} + \frac{K}{(b^\varepsilon - 1)} |h|^{1+\varepsilon}. \quad (3.2.6)$$

Since α is a periodic function, there is $\ell \geq 1$ such that for all closed interval J of length $|J| > \ell$ there is $x_J \in J$ such that $|\Delta_h^2 \alpha(x_J)|$ satisfies Eq.(3.2.6). $\square$

Lemma 3.2.7. *Let $x, h \in \mathbb{R}$ satisfying:*

$$|\Delta_h^2 \alpha(x)| \geq \frac{N_{1+\varepsilon}(g) + L}{(b^\varepsilon - 1)} |h|^{1+\varepsilon} + \frac{K}{(b^\varepsilon - 1)} |h|^{1+\varepsilon},$$

for some $L > 0$ and K as in Eq.(3.2.1). Then

$$\left| \Delta_{\frac{h}{Df(y)}}^2 \alpha(y) \right| \geq \frac{N_{1+\varepsilon}(g) + L |Df(y)|^\varepsilon}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} + \frac{K}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon},$$

where $y \in \mathbb{R}$ is such that $f(y) = x$.

Proof. By Lemma 3.2.2,

$$\Delta_{\frac{h}{Df(y)}}^2 \alpha(y) = \frac{1}{Df(y)} \Delta_h^2 \alpha(x) - \Delta_{\frac{h}{Df(y)}}^2 g(y) + O\left(\left| \frac{h}{Df(y)} \right|^{1+\varepsilon}\right).$$

Therefore,

$$\begin{aligned}
\left| \Delta_{\frac{h}{Df(y)}}^2 \alpha(y) \right| &\geq \frac{1}{|Df(y)|} |\Delta_h^2 \alpha(x)| - \left| \Delta_{\frac{h}{Df(y)}}^2 g(y) \right| - \left| O\left(\left| \frac{h}{Df(y)} \right|^{1+\varepsilon} \right) \right| \\
&\geq \frac{1}{|Df(y)|} \left(\frac{N_{1+\varepsilon}(g) + L}{(b^\varepsilon - 1)} |h|^{1+\varepsilon} + \frac{K}{(b^\varepsilon - 1)} |h|^{1+\varepsilon} \right) \\
&\quad - N_{1+\varepsilon}(g) \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} - K \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} \\
&\geq \frac{N_{1+\varepsilon}(g) |Df(y)|^\varepsilon}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} + \frac{L |Df(y)|^\varepsilon}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} \\
&\quad + \frac{K |Df(y)|^\varepsilon}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} - \frac{N_{1+\varepsilon}(g) b^\varepsilon}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} \\
&\quad + \frac{N_{1+\varepsilon}(g)}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} - \frac{K b^\varepsilon}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} + \frac{K}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} \\
&\geq \frac{N_{1+\varepsilon}(g) + L |Df(y)|^\varepsilon}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon} + \frac{K}{(b^\varepsilon - 1)} \left| \frac{h}{Df(y)} \right|^{1+\varepsilon}.
\end{aligned}$$

In the fourth inequality we use the fact that $b^\varepsilon \leq |Df(y)|^\varepsilon$. □

Proof of Proposition 3.2.4. Suppose that

$$N_{1+\varepsilon}(\alpha) > \frac{N_{1+\varepsilon}(g) + K}{(b^\varepsilon - 1)},$$

and consider $\ell \geq 1$, $L > 0$ and $h > 0$ given by Lemma 3.2.6.

Let I be an interval of length $|I| \leq 1$ and take k the smallest integer such that $|f^k(I)| \geq \ell$. Therefore, there exists $\tilde{x} \in f^k(I)$ such that

$$|\Delta_h^2 \alpha(\tilde{x})| \geq \frac{N_{1+\varepsilon}(g) + L}{(b^\varepsilon - 1)} |h|^{1+\varepsilon} + \frac{K}{(b^\varepsilon - 1)} |h|^{1+\varepsilon}.$$

On the other hand, since $|f^k(I)| \geq \ell$, it follows that $|f^{k-1}(I)| < \ell$. Hence, by the Mean Value Theorem, there is $\theta \in I$ such that

$$|f^{k-1}(I)| = |Df^{k-1}(\theta)| |I| < \ell.$$

Let $y \in I$ such that $f^k(y) = \tilde{x}$. Hence,

$$|I| < \frac{\ell}{|Df^{k-1}(\theta)|} < \frac{\ell}{|Df^{k-1}(y)|} \frac{|Df^{k-1}(y)|}{|Df^{k-1}(\theta)|} < M \frac{\ell}{|Df^{k-1}(y)|},$$

where $M > 0$ is a constant provided by the Bounded Distortion Lemma 1.2.2.

Then by Lemma 3.2.7, we obtain inductively

$$\begin{aligned}
\left| \Delta_{\frac{h}{Df^k(y)}}^2 \alpha(y) \right| &\geq \frac{N_{1+\varepsilon}(g) + L|Df^k(y)|^\varepsilon + K}{(b^\varepsilon - 1)} \left| \frac{h}{Df^k(y)} \right|^{1+\varepsilon} \\
&\geq \frac{L|Df^k(y)|^\varepsilon}{(b^\varepsilon - 1)} \frac{|h|^{1+\varepsilon}}{|Df^k(y)|^\varepsilon} \frac{1}{|Df^k(y)|} \\
&\geq \frac{L|h|^{1+\varepsilon}}{(b^\varepsilon - 1)} \frac{1}{|Df^k(y)|} \frac{|Df(f^{k-1}(y))|}{|Df(f^{k-1}(y))|} \\
&\geq \frac{L|h|^{1+\varepsilon}}{(b^\varepsilon - 1)} \frac{1}{|Df(f^{k-1}(y))|} \frac{1}{|Df^{k-1}(y)|} \\
&\geq \frac{L|h|^{1+\varepsilon}}{(b^\varepsilon - 1)B} \frac{1}{|Df^{k-1}(y)|},
\end{aligned}$$

where B was defined in Eq.(3.2.3).

Taking

$$C = \frac{L|h|^{1+\varepsilon}}{B(b^\varepsilon - 1)\ell M},$$

we obtain

$$\left| \Delta_{\frac{h}{Df^k(y)}}^2 \alpha(y) \right| \geq \frac{L|h|^{1+\varepsilon}}{B(b^\varepsilon - 1)\ell M} \frac{\ell M}{|Df^{k-1}(y)|} \geq C|I|.$$

From this,

$$\text{osc}_2(\alpha, I) \geq \left| \Delta_{\frac{h}{Df^k(y)}}^2 \alpha(y) \right| \geq C|I|,$$

as we wanted. □

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