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**Guaranteed cost model predictive control approaches for
linear systems subject to multiplicative uncertainties with
applications to autonomous vehicles**

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**Abordagens de controle de custo garantido preditivo
baseado em modelos para sistemas lineares sujeitos a
incertezas multiplicativas com aplicações a veículos
autônomos**

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"There is nothing so practical as a good theory" (Ludwig Boltzmann)

ABSTRACT

MASSERA, C. M. **Guaranteed cost model predictive control approaches for linear systems subject to multiplicative uncertainties with applications to autonomous vehicles**. 2019. 134 p. Tese (Doutorado em Ciências – Ciências de Computação e Matemática Computacional) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

The Linear Quadratic Regulator (LQR) is an optimal control approach which aims to drive states of a linear system to its origin through the minimization of a quadratic cost functional. Such an approach has been widely successful for both theoretical and practical applications. However, when such controllers are subject to uncertainties, optimal closed-loop performance cannot be obtained since robustness properties are no longer guaranteed. Guaranteed Cost Controllers (GCC) presents robust asymptotic stability and provides a guaranteed upper bound to a quadratic cost function. Such method addresses the lack of performance guarantees of the LQR.

Meanwhile, Model Predictive Control (MPC) is a class of optimization-based control algorithms that use an explicit model of the controlled system to predict its future states. The MPC can be as a generalization of the LQR for constrained linear systems. Therefore, it equally suffers from a lack of robustness guarantees when the system is subject to uncertainties. Robust MPC (RMPC) approaches were proposed to address MPC's poor closed-loop performance subject to uncertainties. Its objective is to obtain a control input sequence that simultaneously minimizes a cost function and guarantees the feasibility of system states and control inputs, for a system subject to the worst-case disturbance within an uncertainty set.

Autonomous vehicles have gained increasing interest from both the industry and research communities in recent years. An essential aspect in the design of automotive control systems is to ensure the controller is stable and has acceptable performance within the entire operational envelope which it is designed to operate. In the case of autonomous vehicles, where there is no human driver as a fallback, it is of utmost importance to ensure the safe operations of the control system and its capability to avoid saturating the handling limits of the vehicle.

In this thesis, we propose Guaranteed Cost Controller approaches for both unconstrained and constrained linear systems subject to multiplicative structured norm-bounded uncertainties and present the application of such a controller to the lateral control problem of autonomous vehicles up to the tire saturation limits.

Keywords: Optimal Control, Robust Control, Linear Matrix Inequalities, Model Predictive Control, Autonomous Vehicles.

RESUMO

MASSERA, C. M. **Abordagens de controle de custo garantido preditivo baseado em modelos para sistemas lineares sujeitos a incertezas multiplicativas com aplicações a veículos autônomos**. 2019. 134 p. Tese (Doutorado em Ciências – Ciências de Computação e Matemática Computacional) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

O Regulador Quadrático Linear (Linear Quadratic Regulator, LQR) é uma abordagem de controle ótimo que visa conduzir estados de um sistema linear à sua origem através da minimização de um custo funcional quadrático. Tal abordagem tem sido amplamente bem sucedida para aplicações teóricas e práticas. No entanto, não é possível obter o desempenho ótimo de malha fechada quando esses controladores são sujeitos a incertezas no sistema em decorrência de suas propriedades de robustez não serem garantidas. Controladores de Custo Garantido (Guaranteed Cost Control, GCC) visam abordar a falta de garantia de desempenho do LQR, neste caso. Esses controladores apresentam estabilidade assintótica robusta e fornecem um custo garantido de pior caso para uma função de custo quadrático. O Controle Preditivo de Modelo (Model Predictive Control, MPC) é uma classe de algoritmos de controle baseados em otimização que usa um modelo explícito do sistema controlado para prever seus estados futuros. Uma possível interpretação do MPC é uma generalização do LQR para sistemas lineares com restrições de estado e entrada de controle. Portanto, essa abordagem sofre igualmente da falta de garantias de robustez quando o sistema é sujeito a incertezas. As abordagens de MPC Robustas (Robust MPC, RMPC) foram propostas para abordar o desempenho de malha fechada do MPC sujeito a incertezas no sistema. Seu objetivo é obter uma sequência de entrada de controle que minimize simultaneamente uma função de custo e garanta que os estados do sistema e as entradas de controle estão contidos dentro das restrições para um sistema sujeito à pior das perturbações dentro de um conjunto admissível de incertezas. Pesquisas voltadas para veículos autônomos ganharam crescente interesse nos últimos anos, tanto da indústria automobilística quanto da comunidade acadêmica. Um aspecto essencial no projeto de sistemas de controle automotivo é a garantia de estabilidade e desempenho do controlador dentro de todo o envelope operacional ao qual ele foi projetado para operar. No caso de veículos autônomos, onde não há motoristas humanos para lidar com casos de falha do sistema, é de suma importância assegurar as operações seguras do sistema de controle e sua capacidade de evitar a saturação dos limites de manuseio do veículo. Nesta tese, propomos abordagens GCC para sistemas lineares restritos e irrestritos, sujeitos a incertezas estruturadas contidas por norma e apresentamos a aplicação de tais controladores ao problema de controle lateral de veículos autônomos até os limites de saturação dos pneus.

Palavras-chave: Controle Ótimo, Controle Robusto, Desigualdade Matricial Linear, Controle Preditivo de Modelo, Veículos Autônomos.

LIST OF FIGURES

Figure 1	– Simulation results for the synthesized results of Example 1. (a) Closed-loop state x_k subject to uncertainties for (1); (b) Control inputs u_k for (1); (c) Closed-loop state x_k subject to uncertainties for (2) and (3); (d); Control inputs u_k for (2) and (3); (e) Closed-loop state x_k subject to uncertainties for (4) and (5); (f) Control inputs u_k for (4) and (5).	49
Figure 2	– Simulation results for GCC of Example 2. (a) Closed-loop state x_k subject to uncertainties; (b) Simulated uncertainty δ_1 and δ_2 ; (c) Control inputs u_k . . .	50
Figure 3	– Optiaml cost for GCC synthesis for varying ε values	64
Figure 4	– Computational time box plot of both OL-GCMPC and ERMPC for uniformly distributed states with $\ x\ _\infty \leq 0.5$	65
Figure 5	– Simulation results for OL-GCMPC and ERMPC controllers. (a) Simulated disturbance norm $\ \Delta\ _2$; (b) Disturbed closed-loop state x_k ; (c) Executed control inputs u_k	65
Figure 6	– State x_k , control input u_k and system uncertainties Δ_k for the system (4.74) controlled by T-GCMPC with a initial state $x_0 = [0.7, -0.7, 0.7]^T$	83
Figure 7	– State x_k , control input u_k and system uncertainties Δ_k for the system (4.74) controlled by HTMPC proposed by (RAKOVIĆ; CHENG, 2013) with a initial state $x_0 = [0.7, -0.7, 0.7]^T$	83
Figure 8	– Optimal cost of T-GCMPC (blue solid line) and OL-GCMPC (red dashed line) for $x_0 = \lambda[1, 1, 1]^T$ where $\lambda \in [0, 1]$. The crosses indicate the largest value of λ where both the T-GCMPC (blue) and OL-GCMPC (red) are feasible.	85
Figure 9	– Box-plot of solution time of 1000 executions for both the OL-GCMPC (left) and the T-GCMPC (right) using the Mosek SOCP solver.	85
Figure 10	– T-GCMPC optimization states and its invariant tube cross section for states x_1 and x_2 considering an initial state $x_0 = [0.6, -0.6, 0.6]$. The system nominal states z_k are shown in blue, while the red ellipsoids represent the invariant tube.	86
Figure 11	– T-GCMPC optimization states and its invariant tube cross section near the origin for states x_1 and x_2 considering an initial state $x_0 = [0.6, -0.6, 0.6]$. We can observe the crossection of the tube converging to zero once the system approaches the origin.	87

Figure 12 – Comparison of the “brush” tire model (blue solid line) with the linear tire model (dashed red line) where peak force and sliding model at high slip angles can be observed.	92
Figure 13 – Bicycle model representation.	94
Figure 14 – Representation of the proposed conic-bounded uncertain tire model (red solid region) and its relation to the “brush” tire model (blue solid line) and the slip angle with peak tire force (black dash-dot line).	95
Figure 15 – Proposed maximal RCI set shape for on the lifted state space $x = [v_x, v_y, r]^t$	99
Figure 16 – Proposed maximal RCI set (red) at $v_x = 10m/s$ with its respective feasible region (grey).	100
Figure 17 – Comparison for the proposed maximal RCI set (in blue) and Beal’s envelope from (BEAL; GERDES, 2013) (in red), at speeds $10m/s$, $15m/s$, and $20m/s$	100
Figure 18 – Route used for in-vehicle experiemnts.	106
Figure 19 – Results of simulation experiments. Where the left top graph shows the longitudinal speed v_x in m/s , the center top graph shows the crosstrack error e_y in m , the right top grapg shows the heading error e_ψ in $^\circ$, the left bottom graph shows the vehicle yaw rate (blue solid line), the reference yaw rate (dashed red line) and the yaw rate steady-state limits (dash-dot block line) in $^\circ/s$, the center bottom graph shows the commanded steering angle δ_f in degrees, and the right bottom graph shows g-g diagram with longitudinal and lateral acceleration m/s^2	107
Figure 20 – CaRINA II platform used in experiments, a Fiat Palio Adventure.	108
Figure 21 – Results of in-vehicle experiments. Where the left top graph shows the longitudinal speed v_x in m/s , the center top graph shows the crosstrack error e_y in m , the right top graph shows the heading error e_ψ in $^\circ$, the left bottom graph shows the vehicle yaw rate (blue solid line), the reference yaw rate (dashed red line) and the yaw rate steady-state limits (dash-dot block line) in $^\circ/s$, the center bottom graph shows to commanded steering angle δ_f in $^\circ$ (limits outside plot scale), and the right bottom graph shows the reference path curvature κ in $1/m$	109

LIST OF TABLES

Table 1 – Synthesis and simulation results for the comparative study	48
Table 2 – Synthesis vehicle model parameters	105
Table 3 – Controller Parameters	105

LIST OF ABBREVIATIONS AND ACRONYMS

ARE	Algebraic Riccati Equations
AV	Autonomous Vehicles
BMI	Bilinear Matrix Inequality
CL-RMPC	Closed-Loop Robust Model Predictive Control
DAS	Driver Assistance System
ERMPC	Enumeration-based Robust Model Predictive Control
ESC	Electronic Stability Control
ETMPC	Elastic Tube Model Predictive Control
GCC	Guaranteed Cost Control
HTMPC	Homothetic Tube Model Predictive Control
LMI	Linear Matrix Inequality
LP	Linear Programming
LQR	Linear Quadratic Regulator
LTI	Linear Time-Invariant
LTR	Loop-Transfer Recovery
MMMPC	Min-Max Model Predictive Control
MPC	Model Predictive Control
mRCI	minimal Robust Controllable Invariant
OL-GCMPC	Open-Loop Guaranteed Cost Model Predictive Control
OL-RMPC	Open-Loop Robust Model Predictive Control
PTMPC	Parametrized Tube Model Predictive Control
QP	Quadratic Programming
RCI	Robust Controllable Invariant
RCI	Robust Controllable Invariant
RMPC	Robust Model Predictive Control
RTMPC	Rigid Tube Model Predictive Control
SDP	Semi-Definite Programming
SOCP	Second Order Cone Programming
T-GCMPC	Tube-based Guaranteed Cost Model Predictive Control
TMPC	Tube Model Predictive Control

CONTENTS

1	INTRODUCTION	25
1.1	Thesis objective and main contributions	27
1.2	Publications	29
2	OPTIMAL GUARANTEED COST CONTROL OF DISCRETE-TIME LINEAR SYSTEMS SUBJECT TO STRUCTURED UNCERTAINTIES	33
2.1	Introduction	33
2.2	Problem statement	35
2.3	Preliminary results	37
2.3.1	<i>Generalized S-Procedure</i>	37
2.3.2	<i>Reciprocal projection lemma</i>	38
2.3.3	<i>Property of Kronecker products</i>	39
2.4	Guaranteed cost control for systems with structured uncertainties	39
2.4.1	<i>Convex condition for systems without disturbance feed-through</i>	42
2.4.2	<i>Convex condition for systems with disturbance feed-through</i>	43
2.4.3	<i>Optimal static Output-feedback guaranteed cost control synthesis</i>	45
2.5	Numerical examples	47
2.5.1	<i>Example 1</i>	47
2.5.2	<i>Example 2</i>	49
2.6	Conclusion	51
3	OPEN-LOOP GUARANTEED COST MODEL PREDICTIVE CONTROL OF UNCERTAIN LINEAR SYSTEMS	53
3.1	Introduction	53
3.2	Problem statement and preliminaries	55
3.3	Guaranteed cost model predictive control	57
3.4	Numerical example	63
3.5	Conclusion	65
4	TUBE-BASED GUARANTEED COST MODEL PREDICTIVE CONTROL OF UNCERTAIN LINEAR SYSTEMS	67
4.1	Introduction	67
4.2	Preliminary results	70

4.2.1	<i>Optimal min-max robust model predictive control</i>	70
4.2.2	<i>Guaranteed cost control</i>	71
4.2.3	<i>Robust controllable invariant sets</i>	73
4.2.4	<i>Principle of separable control policies</i>	73
4.3	Minimum robust controllable invariant level-set	74
4.4	Tube guaranteed cost model predictive control	79
4.5	Numerical examples	83
4.5.1	<i>Investigated system</i>	83
4.5.2	<i>Controller comparison</i>	84
4.5.3	<i>Timing analysis</i>	86
4.5.4	<i>T-GCMPC optimization results</i>	86
4.5.5	<i>Remarks</i>	87
4.6	Conclusion	87
5	AUTONOMOUS VEHICLE CONTROL UP TO THE LIMITS OF HANDLING UNDER TIRE-GROUND PARAMETER UNCERTAINTY	89
5.1	Introduction	89
5.2	Tire and vehicle models	91
5.2.1	<i>Tire models</i>	91
5.2.1.1	<i>"Brush" tire model</i>	92
5.2.1.2	<i>Linear tire model</i>	93
5.2.2	<i>Vehicle models</i>	93
5.2.2.1	<i>Nonlinear bicycle model</i>	94
5.2.2.2	<i>Linear bicycle model</i>	94
5.2.3	<i>Proposed uncertain bicycle model</i>	95
5.2.3.1	<i>Uncertain tire model</i>	95
5.2.3.2	<i>Uncertain bicycle model</i>	96
5.2.3.3	<i>Trajectory-relative system model</i>	97
5.3	Maximal Robust Controllable Invariant Set	98
5.3.1	<i>Maximal robust controllable invariant set computation</i>	98
5.3.2	<i>Proposed maximal robust controllable invariant set</i>	98
5.3.3	<i>Maximal RCI comparison</i>	99
5.4	Controller formulation	101
5.4.1	<i>Tube-based guaranteed cost model predictive control</i>	101
5.4.1.1	<i>Guaranteed cost controller synthesis</i>	102
5.4.1.2	<i>Approximate minimal RCI set</i>	102
5.4.1.3	<i>T-GCMPC optimization problem</i>	104
5.4.2	<i>Functional modelling and controller synthesis</i>	104
5.5	Experimental results	106
5.5.1	<i>Simulation experiments</i>	106

5.5.2	<i>In-Vehicle experiments</i>	107
5.6	Conclusion	108
6	CONCLUDING REMARKS	111
6.1	Future work	113
BIBLIOGRAPHY		115
APPENDIX A	PROOF OF LEMMA 6 FOR THE OPTIMAL GUAR- ANTEED COST CONTROL	125
APPENDIX B	THE ROBUST LINEAR QUADRATIC REGULATOR AS A SPECIAL CASE OF THE GUARANTEED COST CONTROL	129
B.1	Preliminaries	129
B.2	Guaranteed cost controller	130
B.3	Penalty-based robust linear quadratic regulator	130
B.4	Equivalence of controllers	131
APPENDIX C	ON THE INDEPENDENCE OF CONTROLLERS K AND $\tilde{K}$ FOR THE OPEN-LOOP GUARANTEED COST MODEL PREDICTIVE CONTROL	133

INTRODUCTION

The Linear Quadratic Regulator (LQR) (KALMAN *et al.*, 1960) is an optimal control approach which aims to drive states of a linear system to its origin through the minimization of a quadratic cost functional. However, when such controllers are subject to uncertainties, optimal closed-loop performance cannot be obtained since robustness properties are no longer guaranteed (DOUGLAS, 1991).

To address this problem, several authors in the last decades have proposed different approaches to solve it. Petersen (PETERSEN, 1987), for instance, proposed a Guaranteed Cost Control (GCC) for a continuous-time linear system subject to l_2 -norm-bounded uncertainties. Such method synthesizes a controller which presents robust asymptotic stability and provides a guaranteed upper bound to a quadratic cost function used as performance measurement. Xie and Soh (XIE; SOH, 1993) applied such approach to discrete-time systems, which Petersen et al. (PETERSEN; MCFARLANE; ROTE, 1998) later extended to a more general class of systems.

A conservative aspect of robust synthesis and analysis methods is the assumption of an arbitrary (unstructured) norm bounded set since uncertainties are often known to be structured. Several studies investigated analysis and synthesis methods for linear systems under structured uncertainties. Shamma (SHAMMA, 1994), Packard and Doyle (PACKARD; DOYLE, 1993), and Graham et al. (GRAHAM; OLIVEIRA; CALLAFON, 2006) addressed the problem of robust stability analysis. Meanwhile, Sznajder et al. (SZNAJDER *et al.*, 2002), and Feron (FERON, 1997) investigated robust H_2 analysis. Additionally, Wu and Lu (WU; LU, 2004) addressed H_∞ synthesis. However, according to Paganini (PAGANINI, 2013), the synthesis of a controller for systems subject to structured uncertainties results in non-convex optimizations.

The GCC can be seen as a example of Robust $\mathbb{H}_2$ control (ZHOU *et al.*, 1996) with simultaneous $\mathbb{H}_\infty$ robustness guarantees (KHARGONEKAR; PETERSEN; ZHOU, 1990). Such synthesis approach provides an upper bound a system output covariance (through the use of a cost function) concerning a full-state normally distributed input noise, while subject to norm-bounded

feedback uncertainties. Such property makes the GCC uniquely suitable for the design of Robust Model Predictive Controllers for a system with such a class of uncertainties.

Model Predictive Control (MPC) is a class of optimization-based control algorithms that use an explicit model of the controlled system to predict its future states (BADGWELL; QIN, 2015). The system dynamics are usually assumed to be known, and it does not consider model mismatches or external disturbances. However, the disregard for such uncertainties may lead to degraded closed-loop performance and violation of state and control input constraints (RAWLINGS; MEADOWS; MUSKE, 1994).

Robust Model Predictive Control (RMPC) approaches were proposed to address MPC's poor closed-loop performance subject to uncertainties (BEMPORAD; MORARI, 1999). Its objective is to obtain a control input sequence that simultaneously minimizes a cost function and guarantees the feasibility of system states and control inputs, for a system subject to the worst-case disturbance within an uncertainty set.

Initial studies focused on Open-Loop Robust Model Predictive Control (OL-RMPC) approaches (for example, (BLANCHINI, 1990; KOTHARE; BALAKRISHNAN; MORARI, 1996; LEE; YU, 1997)). However, such methodology yields controllers too conservative for most actual applications. Closed-Loop Robust Model Predictive Control (CL-RMPC) approaches (KERRIGAN; MACIEJOWSKI, 2003) were proposed to address the conservativeness of OL-RMPC policies. CL-RMPC consider different control policies based on past disturbances within the prediction horizon.

Tube Model Predictive Control (TMPC) (for example, (MAYNE; SERON; RAKOVIĆ, 2005; RAKOVIĆ *et al.*, 2012b; RAKOVIĆ *et al.*, 2012a; RAKOVIĆ; LEVINE; AÇIKMEŞE, 2016)) is a subclass of CL-RMPC which tries to ensure computational tractability while obtaining reasonably tight approximations of the exact solution. There are four categories of TMPC policies: Rigid, Homothetic, Parameterized and Elastic. Rigid Tube Model Predictive Control (RTMPC) (MAYNE; SERON; RAKOVIĆ, 2005) defines the tube as a translation of an off-line computed robust positive invariant set with a fixed cross-section. Homothetic Tube Model Predictive Control (HTMPC) (RAKOVIĆ *et al.*, 2012b) is a direct extension of RTMPC, which parametrizes the tube by its translation and the scaling of its cross-section. Parametrized Tube Model Predictive Control (PTMPC) (RAKOVIĆ *et al.*, 2012a) generalizes the concept of TMPC through superposition and separability of additive disturbances. Meanwhile, Elastic Tube Model Predictive Control (ETMPC) (RAKOVIĆ; LEVINE; AÇIKMEŞE, 2016) is a generalization of HTMPC to reduce its conservativeness when compared to PTMPC while maintaining linear complexity scaling. (MAYNE; KERRIGAN; FALUGI, 2011), one of the main limitations of TMPC is the lack of robustness to multiplicative disturbances.

Autonomous Vehicles (AV) have gained increasing interest from both the industry and research communities in recent years (PADEN *et al.*, 2016). One of the primary motivations for the introduction of such technology is the reduction in the overall number of accidents, especially

those with fatalities since road traffic crashes are the leading cause of death among young people between 10 and 24 years old (ORGANIZATION, 2007). Meanwhile, most of these accidents occur when the driver is unable to maintain the vehicle control due to fatigue or external factors resulting in loss-of-control scenarios (ADMINISTRATION *et al.*, 2008).

Systems to aid the driver to recover and maintain the vehicle within its stable operation envelope have already improved the overall safety of vehicles significantly. Electronic Stability Control (ESC) is an example of these systems, and it reduced the number of fatal crashes in single-vehicle accidents in 32% (KOIBUCHI *et al.*, 1996).

An essential aspect in the design of automotive control systems is to ensure the controller is stable and has acceptable performance within the entire operational envelope which it is designed to operate. In the case of autonomous vehicles, where there is no human driver as a fallback, it is of utmost importance to ensure the safe operations of the control system and its capability to avoid saturating the handling limits of the vehicle (KOOPMAN; WAGNER, 2016).

However, vehicle handling limits are directly defined by the tire-road contact characteristics. Such characteristics vary significantly with temperature (TÖNÜK; ÜNLÜSOY, 2001), wear (BRAGHIN *et al.*, 2006) and their manufacturing process. Therefore, the design of autonomous vehicles and driver assistance controllers cannot assume that these characteristics are constant or known. The disregard of such uncertainties can lead to the poor closed-loop performance of MPCs and, consequently, to the violation of state and control input constraints (RAWLINGS; MEADOWS; MUSKE, 1994).

Therefore, the introductions of robustness and estimators regarding tire characteristics are fundamental for the deployment of safe autonomous vehicles. Such problem has obtained interest in recent years. For example, Hsu *et al.* (HSU; LAWS; GERDES, 2010) demonstrated that a Steer-by-Wire system allowed friction estimation based on the steering torque. Furthermore, Beal and Gerdes (BEAL; GERDES, 2013) later used such estimator on its proposal of an MPC-based Driver Assistance System (DAS) for Steer-by-Wire vehicles able to ensure vehicle handling limits in coordination with a human driver.

Finally, RMPC approaches, and in particular TMPC, show great promise on ensuring both theoretical and practical safety guarantees for the application of AVs as such methods explicit consider model and measurement uncertainties and provide robustness guarantees towards them. However, current TMPC approaches are limited on its ability to provide robustness to systems subject to norm-bounded multiplicative uncertainties.

1.1 Thesis objective and main contributions

In this thesis, we propose Guaranteed Cost Control approaches for both unconstrained and constrained linear systems subject to multiplicative structured norm-bounded uncertainties

and present the application of such a controller to the trajectory-tracking up to the limits of handling problem of autonomous vehicles.

This thesis is organized as a collection of papers, the chapters and their respective contributions are:

Chapter 2: In this chapter, we propose state- and static output-feedback generalized GCC approaches for discrete-time linear systems subject to norm-bounded structured parametric uncertainties. This method enables the convex synthesis for a more general class of systems, where uncertainties are uncorrelated block diagonal, and no feed-through uncertainty is multiplicative with control input ones. It also provides necessary and sufficient conditions for state-feedback and sufficient conditions for static output-feedback. We also present a comparative study of the proposed controllers, standard Linear Quadratic Regulator, and GCC found in the literature.

Chapter 3: In this chapter, we propose the Open-Loop Guaranteed Cost Model Predictive Control (OL-GCMPC) for constrained uncertain discrete time systems. This approach enables the synthesis of RMPCs with quadratic cost functional and quadratically bound uncertainties, currently intractable by Min-Max Model Predictive Control (MMPC) techniques without polytopic approximations. This controller was developed based on a quadratic cost functional and guarantee robustness concerning quadratically bound uncertainties. Such a class of problems is currently intractable by Min-Max Robust Model Predictive Controllers without polytopic approximations of the uncertainties. The proposed technique is computationally more efficient than an enumeration-based approach and requires only a Second Order Cone Programming (SOCP) optimization, whereas LMI-based GCMPC approaches require Semi-Definite Programming (SDP) optimization.

Chapter 4: In this chapter, we propose a Tube-based Guaranteed Cost Model Predictive Control (T-GCMPC) considering a Homothetic formulation for constrained linear systems subject to multiplicative structured norm-bounded uncertainties. Contrarily to previous TMPC formulations, the proposed method provides an upper bound to the general MMPC, and the invariance property of the proposed tube holds for any arbitrary scaling. It provides an advantage over previous formulations regarding cost, and it guarantees feasibility while yielding a SOCP problem, similarly to the proposed OL-GCMPC. We also present a numerical example to develop a comparative study among the proposed T-GCMPC, the OL-GCMPC and a HTMPC for Linear Difference Inclusion systems.

Chapter 5: In this chapter, we propose a T-GCMPC synthesis for AVs able to avoid front and rear tire saturation and to track a provided reference trajectory up to the limits of handling of the vehicle. Towards that purpose, we also proposed a new conservative approximation of the nonlinear vehicle dynamics to a linear system subject to norm-bounded multiplicative uncertainties and a new maximal robust controllable invariant set for vehicle dynamics, which consists of a larger feasible state space region when compared to previously proposed invariant

sets. Finally, we present both simulation and in-vehicle results of the performance of the proposed approach.

Each chapter of this thesis contains a literature review related to its research topic and ends with concluding remarks regarding its contributions. Additionally, we also provide closing remarks for the thesis in Chapter 6 that summarize the main results and give an outlook for potential future research.

1.2 Publications

As part of the work of this thesis, we have published one journal article, have three journal articles under ongoing review, and have also published twelve conference papers. These works are related both to the content presented in this thesis, previous studies, and collaborative work.

Published journal articles:

- *"Safe optimization of highway traffic with robust model predictive control-based cooperative adaptive cruise control"*, Carlos M. Massera, Marco H Terra, Denis F Wolf. Published at IEEE Transactions on Intelligent Transportation Systems, Volume 18, Issue 11, November 2017 [**Qualis A1, Impact factor: 4.05**].

Accepted journal articles:

- *"Optimal Guaranteed Cost Control of Discrete-Time Linear Systems subject to Structured Uncertainties"*, Carlos M. Massera, Marco H Terra, Denis F Wolf. Submitted to the International Journal of Control in November 2017.

Journal articles under review:

- *"Autonomous Vehicle Control Up to the Limits of Handling Under Tire Parameter Uncertainty"*, Carlos M. Massera, Marco H Terra, Denis F Wolf. Submitted to IEEE Transactions on Automatic Control in February 2019.
- *"Tube-based Guaranteed Cost Robust Model Predictive Control for Linear Systems subject to Parametric Uncertainties"*, Carlos M. Massera, Marco H Terra, Denis F Wolf. Submitted to IEEE Transactions on Automatic Control in February 2019;

Published conference papers:

- *"Guaranteed cost approach for robust model predictive control of uncertain linear systems"*, Carlos M Massera, Marco H Terra, Denis F Wolf. Published at American Control Conference in May 2017;
- *"CaRINA dataset: An emerging-country urban scenario benchmark for road detection systems"*, Patrick Y Shinzato, Tiago C dos Santos, Luis Alberto Rosero, Daniela A Ridel, Carlos M Massera, Francisco Alencar, Marcos Paulo Batista, Alberto Y Hata, Fernando S Osório, Denis F Wolf. Published at IEEE International Conference on Intelligent Transportation Systems in November 2016;
- *"Guaranteed Cost Model Predictive Control-based Driver Assistance System for Vehicle Stabilization Under Tire Parameters Uncertainties"*, Carlos M Massera, Marco H Terra, Denis F Wolf. Published at IEEE International Conference on Intelligent Transportation Systems in November 2016;
- *"Fast metric multi-object vehicle tracking for dynamical environment comprehension"*, FA Alencar, Carlos Massera Filho, Daniela A Ridel, Denis F Wolf. Published at IEEE Latin America Robotics Symposium in October 2015;
- *"Lateral Control Based on Clothoids Curves for Heavy Category Ground Vehicles"*, Solander PL Agostinho, Marco H Terra, Valdir Grassi, Carlos Massera Filho, Denis F Wolf. Published at IEEE Latin America Robotics Symposium in October 2015;
- *"Fast metric tracking by detection system: radar blob and camera fusion"*, Francisco AR Alencar, Luis Alberto Rosero, Carlos Massera Filho, Fernando S Osório, Denis F Wolf. Published at IEEE Latin America Robotics Symposium in October 2015;
- *"Driver assistance controller for tire saturation avoidance up to the limits of handling"*, Carlos Massera Filho, Denis F Wolf. Published at IEEE Latin America Robotics Symposium in October 2015;
- *"A Simulation Framework for Multi-Vehicle Communication"*, Tiago C dos Santos, Andrés E Gómez, Carlos Massera Filho, Diego Gomes, Juan C Perafan, Denis F Wolf, Fernando Osório, Luis A Rosero. Published at IEEE Latin America Robotics Symposium in October 2015;
- *"Pedestrian classification using K-means and Random Decision Forests"*, Francisco AR Alencar, Carlos Massera Filho, Diego Gomes da Silva, Denis F Wolf. Published at IEEE Latin America Robotics Symposium in October 2014;
- *"Dynamic inversion-based control for front wheel drive autonomous ground vehicles near the limits of handling"*, Carlos Massera Filho, Denis F Wolf. Published at IEEE International Conference on Intelligent Transportation Systems in October 2014;

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- *"Simulation platform for cooperative vehicle systems"*, Andrés E Gómez, Tiago C dos Santos, Carlos Massera Filho, Diego Gomes, Juan C Perafan, Denis F Wolf. Published at IEEE International Conference on Intelligent Transportation Systems in October 2014;
 - *"Longitudinal and lateral control for autonomous ground vehicles"*, Carlos M. Massera, Denis F Wolf, Valdir Grassi, Fernando S Osorio. Published at IEEE Intelligent Vehicles Symposium in June 2014;

OPTIMAL GUARANTEED COST CONTROL OF DISCRETE-TIME LINEAR SYSTEMS SUBJECT TO STRUCTURED UNCERTAINTIES

In this chapter, we propose a state- and static output-feedback GCC synthesis formulation. Such method extends the previous studies on GCC to a more general class of systems, where disturbances may be present on the measured output.

It demonstrates that incorporating uncertainty structure knowledge into robust controller synthesis methods for uncertain linear systems may provide significantly less conservative control laws. Towards that purpose, it proves that a generalized S-Procedure can be used to provide both necessary and sufficient conditions for the robustness and performance guarantees for the considered class of uncertainty structure.

2.1 Introduction

The Linear Quadratic Regulator (LQR) ([KALMAN *et al.*, 1960](#)) is an optimal control approach which aims to drive states of a linear system to its origin through the minimization of a quadratic cost functional. Safonov and Athans ([SAFONOV; ATHANS, 1976](#)) have shown that such controller has $6dB$ gain margin and 60° phase margin if the system dynamics is assumed to be known. However, when it is subject to uncertainties, optimal closed-loop performance cannot be obtained since robustness properties are no longer guaranteed ([DOUGLAS, 1991](#)).

To address this problem, several authors in the last decades have proposed different approaches to solve it. Petersen ([PETERSEN, 1987](#)), for instance, proposed a Guaranteed Cost Controller (GCC) for a continuous-time linear system subject to l_2 -norm-bounded uncertainties

based on the solution of Algebraic Riccati Equations (ARE). Such method synthesizes a controller which presents robust asymptotic stability and provides a guaranteed upper bound to a quadratic cost function used as performance measurement. Xie and Soh (XIE; SOH, 1993) applied such approach to discrete-time systems, which Petersen *et al.* (PETERSEN; MCFARLANE; ROTE, 1998) extended to a more general class of systems through the use of Linear Matrix Inequality (LMI). Previous studies applied this class of synthesis to several types of systems and controllers, including static and dynamic output feedback controllers. Some examples are: (GARCIA *et al.*, 2003; MOHEIMANI; PETERSEN, 1996; YU; CHU, 1999; CHEN; GUAN; LU, 2003; BOUKAS; SHI, 1998; YANG; ZHAO, 2016; QIU *et al.*, 2016; WANG *et al.*, 2016).

A challenging aspect of most robust synthesis problems is its inherent non-convexity, even when no uncertainties are considered. This topic has attracted significant research interest in the past two decades. Most of the approaches proposed for such class of problems are based on dilated variables, first proposed by Oliveira *et al.* (OLIVEIRA; BERNUSSOU; GEROMEL, 1999a) for discrete time systems and later extended by Apkarian *et al.* (APKARIAN; TUAN; BERNUSSOU, 2001) for continuous time systems. Some examples of applications of such method are: Output feedback synthesis (i.e., (LEE; LEE; KWON, 2006), (XIE, 1996), (DONG; YANG, 2007), (PEAUCELLE; EBIHARA, 2014), (ARZELIER; PEAUCELLE; SALHI, 2003)); systems subject to delays (i.e., (HE *et al.*, 2004), (LI *et al.*, 2016)); systems subject to Markovian jumps (i.e. (SOUZA, 2003), (MORAIS *et al.*, 2016), (SHEN *et al.*, 2015)); constrained optimal controllers (i.e. (CUZZOLA; GEROMEL; MORARI, 2002), (XIA *et al.*, 2008), (LU; LI; XI, 2013)); and many others. The use of such dilation variables is conservative, thus it only provides sufficient conditions for the controller existence. To partially overcome this conservativeness, several authors have proposed the use of iterative methods to locally solve the resulting non-convex Bilinear Matrix Inequality (BMI) problem (i.e. (SADEGHZADEH; MOMENI, 2016), (MOREIRA; OLIVEIRA; PERES, 2011), (SADEGHZADEH; KARIMI, 2015), (PEAUCELLE; ARZELIER, 2001), (DONKERS, 2017), (SADEGHZADEH, 2014)). However, iterative methods also only provide sufficient conditions, due to its local optimality guarantees.

Another conservative aspect of robust synthesis and analysis methods is the assumption of an arbitrary (unstructured) norm bounded set since uncertainties are often known to be structured. Several studies investigated analysis and synthesis methods for linear systems under structured uncertainties. Shamma (SHAMMA, 1994), Packard and Doyle (PACKARD; DOYLE, 1993), and Graham *et al.* (GRAHAM; OLIVEIRA; CALLAFON, 2006) addressed the problem of robust stability analysis. Sznaier *et al.* (SZNAIER *et al.*, 2002), and Feron (FERON, 1997) investigated robust H_2 analysis. While Wu and Lu (WU; LU, 2004) addressed H_∞ synthesis. However, according to Paganini (PAGANINI, 2013), the synthesis of a controller for systems subject to structured uncertainties results in non-convex optimizations.

In this chapter, we propose a generalized Guaranteed Cost Control synthesis problem for discrete-time linear systems subject to norm-bounded structured parametric uncertainties for

state-feedback and static output-feedback controllers. We consider a correlated block diagonal uncertainty structure, which is more general than previous studies (see for instance Szanaier *et al.* (SZNAIER *et al.*, 2002) and Graham *et al.* (GRAHAM; OLIVEIRA; CALLAFON, 2006)). We also address the non-convexity issue discussed by Paganini through the use of the generalized S-Procedure, as proposed by Iwasaki *et al.* (IWASAKI; MEINSMA; FU, 2000), which is proved to be both necessary and sufficient for the considered structure. Meanwhile, we address the non-convexity of the output feedback synthesis problem conservatively through the dilated variable approach, based on the Reciprocal Projection Lemma (APKARIAN; TUAN; BERNUSSOU, 2001).

This chapter has the following organization: Section 2.2 presents the problem statement; Section 2.3 discusses preliminary results; Section 2.4 derives the proposed Guaranteed Cost Controller; Section 2.5 provides a numerical example; finally, Section 2.6 addresses the final remarks.

The notation used throughout this chapter is standard. I_p is a identity matrix of size $p \times p$, $x^T M x = x^T M(\bullet)$ for brevity, $A \otimes B$ is the Kronecker product of matrices A and B, and $\text{tr}(A)$ is the trace of matrix A.

2.2 Problem statement

In this section, we present a Guaranteed Cost Controller definition with its required assumptions. We consider discrete-time linear system subject to parametric uncertainties of the form

$$\begin{aligned} x_{k+1} &= (A + \delta A(\Delta_k))x_k + (B^u + \delta B^u(\Delta_k))u_k \\ y_k &= (C_y + \delta C_y(\Delta_k))x_k, \end{aligned} \quad (2.1)$$

where $x_k \in \mathfrak{R}^{n_x}$ is the system state, $u_k \in \mathfrak{R}^{n_u}$ is the control input, $A \in \mathfrak{R}^{n_x \times n_x}$ is the state matrix, $B^u \in \mathfrak{R}^{n_x \times n_u}$ is the input matrix, $C_y \in \mathfrak{R}^{n_y \times n_x}$ is the output state matrix, and $\delta A(\Delta_k)$, $\delta B^u(\Delta_k)$ and $\delta C_y(\Delta_k)$ are, respectively, the state, input and output multiplicative uncertainty matrices, such that

$$\begin{bmatrix} \delta A(\Delta_k) & \delta B^u(\Delta_k) \\ \delta C_y(\Delta_k) & 0 \end{bmatrix} = \begin{bmatrix} B^w \\ D_y^w \end{bmatrix} \bar{\Delta}_k \begin{bmatrix} C_z & D_z^u \end{bmatrix}, \quad (2.2)$$

where $\bar{\Delta}_k = \Delta_k(I_{n_q} - D_z^w \Delta_k)^{-1}$, $\Delta_k \in \mathbb{D}^u = \{\Delta \mid \Delta \in \mathfrak{R}^{n_p \times n_q}, \|\Delta\|_2 \leq 1\}$, $B^w \in \mathfrak{R}^{n_x \times n_p}$, $D_y^w \in \mathfrak{R}^{n_y \times n_p}$, $D_z^w \in \mathfrak{R}^{n_q \times n_p}$, $C_z \in \mathfrak{R}^{n_q \times n_x}$ and $D_z^u \in \mathfrak{R}^{n_q \times n_u}$.

In the scope of this chapter, we are interested in investigating the controller synthesis when the structure of Δ_k is known. Therefore, we define $\mathbb{D} \subseteq \mathbb{D}^u$ to be the set of uncertainty matrices such that

$$\begin{aligned} \mathbb{D} &= \{\Delta \mid \forall i \in [1, s] : \Delta_i \in \mathfrak{R}^{n_{pi} \times n_{qi}}, \|\Delta_i\|_2 \leq 1, \\ &\quad \Delta = \text{diag}(I_{n_{r1}} \otimes \Delta_1, I_{n_{r2}} \otimes \Delta_2, \dots, I_{n_{rs}} \otimes \Delta_s)\}. \end{aligned} \quad (2.3)$$

where $n_p = \sum_{i=1}^s n_{ri} n_{pi}$ and $n_q = \sum_{i=1}^s n_{ri} n_{qi}$.

The uncertainty structure defined in (2.3) is more general than ones considered in previous studies. For example, the structure investigated by Sznaier *et al.* (SZNAIER *et al.*, 2002) and Graham *et al.* (GRAHAM; OLIVEIRA; CALLAFON, 2006) is square block diagonal ($n_{ri} = 1$ and $n_{pi} = n_{qi}$).

Associated with the uncertain linear system (2.1) is the performance measurement cost function

$$J(x_0) = \lim_{n \rightarrow \infty} \sum_{k=0}^n x_k^T Q x_k + 2x_k^T N u_k + u_k^T R u_k, \quad (2.4)$$

where $Q \in \mathbb{R}^{n_x \times n_x}$, $N \in \mathbb{R}^{n_x \times n_u}$, $R \in \mathbb{R}^{n_u \times n_u}$, $Q \succeq 0$, $R \succ 0$, and

$$\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} = \begin{bmatrix} C_c^T \\ D_c^{uT} \end{bmatrix} \begin{bmatrix} C_c & D_c^u \end{bmatrix} \succeq 0, \quad (2.5)$$

and $C_c \in \mathbb{R}^{n_c \times n_x}$ and $D_c^u \in \mathbb{R}^{n_c \times n_u}$ denote the factorization of the cost function.

We are now ready to define the proposed Guaranteed Cost Controller.

Definition 1. A static output feedback controller $u_k = -Ky_k$ is said to be a stabilizing Guaranteed Cost Controller for the uncertain system (2.1) if there exists a symmetric matrix $P \succ 0$ that upper-bounds the cost functional (2.4) for

$$\begin{aligned} & [A + \delta A + (B^u + \delta B^u(\Delta_k)K(C_y + \delta C_y(\Delta_k))]^T P[\bullet] - P + \\ & Q + NK(C_y + \delta C_y(\Delta_k)) + (C_y + \delta C_y(\Delta_k))^T K^T N^T + [K(C_y + \delta C_y(\Delta_k))]^T R[\bullet] \leq 0, \end{aligned}$$

such that $J(x_0) \leq J^*(x_0) = x_0^T P x_0$ for the closed loop system

$$x_{k+1} = [A + \delta A(\Delta_k) - (B^u + \delta B^u(\Delta_k))K(C_y + \delta C_y(\Delta_k))]x_k, \quad (2.6)$$

and all admissible uncertainties $\Delta_k \in \mathbb{D}$.

Definition 2 (Optimal Guaranteed Cost Control). A static output-feedback controller $u_k = -Ky_k$ is said to be optimal if it satisfies Definition 1 and $J^*(x_0)$ is minimal.

Remark 1. The static output-feedback GCC from Definition 1 reduces to the state-feedback case if $C_y = I_{n_x}$, $D_y^w = 0$, and $\delta C_y(\Delta_k) = 0$ for all $\Delta_k \in \mathbb{D}$.

The GCC problem from Definitions 1 and 2 is well posed under the following assumptions:

Assumption 1. For all $\Delta_k \in \mathbb{D}$, the pair $(A + \delta A(\Delta_k), B + \delta B(\Delta_k))$ is stabilizable.

Assumption 2. The uncertainty matrices (2.2) are well-posed. Therefore, for all $\Delta_k \in \mathbb{D}$, $D_z^w \Delta_k \prec I_{n_q}$, or equivalently, $\|D_z^w\|_2 < 1$.

Assumption 3. For all $\Delta_k \in \mathbb{D}$, the pair $(A + \delta A(\Delta_k), Q^{\frac{1}{2}})$ is observable.

Assumption 4. For all $\Delta_k \in \mathbb{D}$, $D_y^w \bar{\Delta}_k D_z^u = 0$. Therefore, the feed-through uncertainty is zero.

2.3 Preliminary results

This section presents previous results and preliminary work to formulate a convex condition for the Guaranteed Cost Controller. Such results include the generalized S-Procedure, the Reciprocal Projection Lemma, a property of the Kronecker Product, and an alternative representation of the system presented in Section 2.2.

The uncertain linear system (2.1) is equivalent to

$$\begin{aligned} x_{k+1} &= Ax_k + B^w w_k + B^u u_k \\ y_k &= C_y x_k + D_y^w w_k \\ z_k &= C_z x_k + D_z^w w_k + D_z^u u_k, \end{aligned} \quad (2.7)$$

subject to the feedback disturbance $w_k = \Delta_k z_k$, where $w_k \in \mathfrak{R}^{n_p}$ and $z_k \in \mathfrak{R}^{n_q}$. Similarly, the closed loop system with a static output-feedback controller $u_k = -Ky_k$ is equivalent to

$$\begin{aligned} x_{k+1} &= (A - B^u KC_y) x_k + (B^w - B^u KD_y^w) w_k \\ z_k &= (C_z - D_z^u KC_y) x_k + (D_z^w - D_z^u KD_y^w) w_k, \end{aligned} \quad (2.8)$$

whose matrices will be denoted as $\bar{A} = A - B^u KC_y$, $\bar{B}^w = B^w - B^u KD_y^w$, $\bar{C}_z = C_z - D_z^u KC_y$, and $\bar{D}_z^w = D_z^w - D_z^u KD_y^w$ for brevity. From (2.8), the closed loop uncertain system (2.6) is equivalent to

$$x_{k+1} = (\bar{A} - \bar{B}^w \bar{\Delta}_k \bar{C}_z) x_k, \quad (2.9)$$

and the closed-loop cost function (2.4) can be rewritten as

$$J(x_0) = \sum_{k=0}^{\infty} x_k^T (\bar{C}_c + \bar{D}_c^w \bar{\Delta}_k \bar{C}_z)^T (\bullet) x_k, \quad (2.10)$$

where $\bar{C}_c = C_c - D_c KC_y$, and $\bar{D}_c^w = -D_c KD_y^w$. Therefore, based on (2.9) and (2.10), the optimal GCC synthesis problem from Definition 2 is equivalent to

$$\begin{aligned} J^*(x_0) &\geq \max_{\Delta_0, \dots, \Delta_{\infty}} \min_K J(x_0) \\ s.t. \quad x_{k+1} &= (\bar{A} - \bar{B}^w \bar{\Delta}_k \bar{C}_z) x_k. \end{aligned} \quad (2.11)$$

2.3.1 Generalized S-Procedure

Iwasaki *et al.* (IWASAKI; MEINSMA; FU, 2000) described a generalized S-Procedure which provides both necessary and sufficient conditions for particular sets of matrices, entitled lossless sets. We now present the lossless set definition and the generalized S-Procedure theorem for completeness.

Definition 3 (Lossless Set). A subset $\mathbb{S}$ of $n \times n$ symmetric Real matrices is said to be lossless if it has the following properties:

- (i) $\mathbb{S}$ is convex;
- (ii) $S \in \mathbb{S} \Rightarrow \forall \tau > 0 : \tau S \in \mathbb{S}$;
- (iii) For each nonzero matrix $H \in \mathfrak{R}^{n \times n}, H \succeq 0$ such that $\forall S \in \mathbb{S} : \text{tr}(SH) \leq 0$, there exist vectors $\lambda_i \in \mathfrak{R}^n$ such that

$$H = \sum_{i=0}^{\text{rank}(H)} \lambda_i \lambda_i^*, \forall S \in \mathbb{S} : \lambda_i^* S \lambda_i \leq 0. \quad (2.12)$$

Theorem 1 (Generalized S-Procedure). Let $\Theta \in \mathfrak{R}^{n \times n}$ be a symmetric matrix and $\mathbb{S}$ be a lossless set. Then, the following statements are equivalent:

- (i) $\forall x \in \mathbb{X} = \{x \mid x \in \mathfrak{R}^n, x \neq 0, x^T S x \leq 0, \forall S \in \mathbb{S}\} : x^T \Theta x \leq 0$;
- (ii) $\exists S \in \mathbb{S} : \Theta - S \preceq 0$.

Proof. See Theorem 1 of (IWASAKI; MEINSMA; FU, 2000). □

2.3.2 Reciprocal projection lemma

Apkarian *et al.* (APKARIAN; TUAN; BERNUSSOU, 2001), based on Oliveira *et. al.* (OLIVEIRA; BERNUSSOU; GEROMEL, 1999b), proposed the Reciprocal Projection Lemma. Such Lemma enables several previously non-convex conditions present in synthesis problems to posed as convex conditions through the use of additional slack variables. Based on this result less conservative methods have been developed, particularly in the field of multi-objective synthesis and analysis (i.e., (KÖROĞLU; FALCONE, 2014), (ADEGAS, 2013)). We now present the Projection Lemma and the Reciprocal Projection Lemma.

Lemma 1 (Projection Lemma). Let $\Psi \in \mathfrak{R}^{n \times n}$ be a symmetric matrix and $P, Q \in \mathfrak{R}^{m \times n}$. Then, following statements are equivalent:

- (i) There exists $X \in \mathfrak{R}^{m \times m}$ such that $\Psi + P^T X^T Q + Q^T X P \preceq 0$;
- (ii) $N_P^T \Psi N_P \preceq 0$ and $N_Q^T \Psi N_Q \preceq 0$,

where N_P and N_Q are arbitrary bases of the null-space of P and Q , respectively.

Proof. See Lemma 3.1 of (GAHINET; APKARIAN, 1994). □

Lemma 2 (Reciprocal Projection Lemma). Let $\Psi \in \Re^{n \times n}$ be a symmetric matrix, $S \in \Re^{n \times n}$ and $X \in \Re^{n \times n}$ be an arbitrary positive definite matrix. Then, the following statements are equivalent:

- (i) $\Psi + S + S^T \preceq 0$;
- (ii) There exists $W \in \Re^{n \times n}$ such that

$$\begin{bmatrix} -X & S+W \\ S^T + W^T & \Psi + X - W - W^T \end{bmatrix} \preceq 0. \quad (2.13)$$

Proof. See Lemma 2.2 of (APKARIAN; TUAN; BERNUSSOU, 2001) □

2.3.3 Property of Kronecker products

We also present property of Kronecker products necessary for the proposed controller proof.

Lemma 3. Let $X \in \Re^{p \times p}$ and $Y \in \Re^{q \times r}$. Then,

$$Z = (X \otimes I_q)(I_p \otimes Y) = (I_p \otimes Y)(X \otimes I_r). \quad (2.14)$$

Proof. This property follows directly from the fact that each block of Z is given by $Z_{i,j} = X_{i,j}Y$, which is commutable since $X_{i,j}$ is a scalar. □

Corollary 1. For all $i \in [1, n]$, let $X_i \in \Re^{p_i \times p_i}$, $Y_i \in \Re^{q_i \times r_i}$, and

$$Y = \text{diag}(I_{p_1} \otimes Y_1, I_{p_2} \otimes Y_2, \dots, I_{p_n} \otimes Y_n). \quad (2.15)$$

Then,

$$\text{diag}(X_1 \otimes I_{q_1}, X_2 \otimes I_{q_2}, \dots, X_n \otimes I_{q_n})Y = Y \text{diag}(X_1 \otimes I_{r_1}, X_2 \otimes I_{r_2}, \dots, X_n \otimes I_{r_n}) \quad (2.16)$$

follows directly from Lemma 3.

2.4 Guaranteed cost control for systems with structured uncertainties

In this section, we present the proposed static output-feedback controller. Existence condition for the particular case of no measured disturbance $D_y^w = 0$ is first presented and we specialize it to the state-feedback controller. Then, we present the Optimal Static Output-Feedback Guaranteed Cost Controller developed.

Lemma 4 (Unstructured Uncertainty Set). Let $\xi_k = [x_k^T, w_k^T]^T$. Then, the region where $w_k = \Delta_k z_k$ holds for some $\Delta_k \in \mathbb{D}^u$ is

$$\xi_k \in \mathbb{X}^u = \{\xi \mid \xi \in \mathfrak{R}^{n_x+n_p}, \xi^T S \xi \leq 0, \forall S \in \mathbb{S}^u\}, \quad (2.17)$$

where $\mathbb{S}^u = \{\lambda S^u \mid \lambda \geq 0\}$ is the unstructured uncertainty set, and

$$S^u = \begin{bmatrix} \bar{C}_z^T \bar{C}_z & \bar{C}_z^T \bar{D}_z^w \\ \bar{D}_z^{wT} \bar{C}_z & \bar{D}_z^{wT} \bar{D}_z^w - I \end{bmatrix}. \quad (2.18)$$

Proof. Direct from the discrete-time Bounded Real Lemma (SOUZA; XIE, 1992). $\square$

The set $\mathbb{S}^u$, from Lemma 4, has been thoroughly used for the analysis and synthesis of robust controllers. It defines the region of the domain of ξ_k where we wish to ensure system stability and performance. In the case considered in this chapter, the structure of the uncertainty Δ_k is known and can be used to provide a less conservative set, which still preserves the losslessness property from $\mathbb{S}^u$.

Lemma 5 (Structured Uncertainty Set). Let Λ_p , and Λ_q be

$$\begin{aligned} \Lambda_p &= \text{diag}(\Lambda_1 \otimes I_{n_{p1}}, \Lambda_2 \otimes I_{n_{p2}}, \dots, \Lambda_s \otimes I_{n_{ps}}), \\ \Lambda_q &= \text{diag}(\Lambda_1 \otimes I_{n_{q1}}, \Lambda_2 \otimes I_{n_{q2}}, \dots, \Lambda_s \otimes I_{n_{qs}}), \end{aligned} \quad (2.19)$$

where for all $1 \leq i \leq s$, $\Lambda_i \in \mathfrak{R}^{n_{ri} \times n_{ri}}$ are positive semi-definite matrices. Then, the region where $w_k = \Delta_k z_k$ holds for some $\Delta_k \in \mathbb{D}$ is

$$\xi_k \in \mathbb{X} = \{\xi \mid \xi \in \mathfrak{R}^{n_x+n_p}, \xi^T S \xi \leq 0, \forall S \in \mathbb{S}\}, \quad (2.20)$$

where $\mathbb{S} = \{S^s(\Lambda_p, \Lambda_q) \mid \forall i \in [1, s], \forall \Lambda_i \succeq 0\}$ is the structured uncertainty set, and

$$S^s(\Lambda_p, \Lambda_q) = \begin{bmatrix} \bar{C}_z^T \Lambda_q \bar{C}_z & \bar{C}_z^T \Lambda_q \bar{D}_z^w \\ \bar{D}_z^{wT} \Lambda_q \bar{C}_z & \bar{D}_z^{wT} \Lambda_q \bar{D}_z^w - \Lambda_p \end{bmatrix}. \quad (2.21)$$

Proof. By left-multiplying the feedback disturbance equality $w_k = \Delta_k z_k$ by $\Lambda_p^{\frac{1}{2}}$, we obtain $\Lambda_p^{\frac{1}{2}} w_k = \Lambda_p^{\frac{1}{2}} \Delta_k z_k$. Which, based on Corollary 1, is equivalent to

$$\Lambda_p^{\frac{1}{2}} w_k = \Delta_k \Lambda_q^{\frac{1}{2}} z_k. \quad (2.22)$$

Consider the 2-norm of (2.22). Then, for all $\Delta_k \in \mathbb{D}$,

$$\|\Lambda_p^{\frac{1}{2}} w_k\|_2 = \|\Delta_k \Lambda_q^{\frac{1}{2}} z_k\|_2 \leq \|\Delta_k\|_2 \|\Lambda_q^{\frac{1}{2}} z_k\|_2 \leq \|\Lambda_q^{\frac{1}{2}} z_k\|_2, \quad (2.23)$$

which is equivalent to $w_k^T \Lambda_p w_k - z_k^T \Lambda_q z_k \leq 0$, or $\xi_k^T S^s(\Lambda_p, \Lambda_q) \xi_k \leq 0$. Therefore, (2.23) holds for all $\Lambda_p \succ 0$, $\Lambda_q \succ 0$, and $\Delta_k \in \mathbb{D}$, and the domain of ξ_k with admissible feedback disturbance is defined by

$$\xi_k \in \mathbb{X} = \{\xi \mid \xi \in \mathfrak{R}^{n_x+n_p}, \xi^T S \xi \leq 0, \forall S \in \mathbb{S}\}. \quad (2.24)$$

$\square$

Lemma 6. The Structured Uncertainty Set has the following properties:

- (i) $\mathbb{S}^u \subseteq \mathbb{S}$;
- (ii) The set $\mathbb{S}$ is lossless.

Proof. See Appendix A. □

We are now ready to investigate the necessary and sufficient conditions for the existence of a GCC that satisfies Definition 1.

Theorem 2. Consider a static output-feedback controller $u_k = -Ky_k$ regulating the System (2.1) subject to structured uncertainties of the form (2.3). Then, the following statements are equivalent:

- (i) The controller satisfies Definition 1;
- (ii) There exists a value function $V(x_k) = x_k^T P x_k$, where $P \succ 0$, such that for all $\Delta_k \in \mathbb{D}$

$$V(x_{k+1}) - V(x_k) \leq -x_k^T (\bar{C}_c + \bar{D}_c^w \bar{\Delta}_k \bar{C}_z)^T (\bullet) x_k; \quad (2.25)$$

- (iii) There exists $P \succ 0$ and $\Lambda_i \succeq 0$ for $i \in [1, s]$ such that

$$\begin{aligned} & \begin{bmatrix} \bar{A}^T P \bar{A} - P & \bar{A}^T P \bar{B}^w \\ \bar{B}^{wT} P \bar{A} & \bar{B}^{wT} P \bar{B}^w \end{bmatrix} + \begin{bmatrix} \bar{C}_c^T \bar{C}_c & \bar{C}_c^T \bar{D}_c^w \\ \bar{D}_c^{wT} \bar{C}_c & \bar{D}_c^{wT} \bar{D}_c^w \end{bmatrix} + \\ & + \begin{bmatrix} \bar{C}_z^T \Lambda_q \bar{C}_z & \bar{C}_z^T \Lambda_q \bar{D}_z^w \\ \bar{D}_z^{wT} \Lambda_q \bar{C}_z & \bar{D}_z^{wT} \Lambda_q \bar{D}_z^w - \Lambda_p \end{bmatrix} \preceq 0. \end{aligned} \quad (2.26)$$

Proof. We first prove (i) and (ii) are equivalent, then continue to prove (ii) and (iii) are equivalent.

(i) $\Rightarrow$ (ii): Let $c(x_k, u_k) = x_k^T Q x_k + 2x_k^T N u_k + u_k^T R u_k$ for brevity and consider the value function $V_k^*(x_k)$. From Bellman's optimality principle, we obtain

$$V_k^*(x_k) = \max_{\Delta_k \in \mathbb{D}} \min_{u_k} c(x_k, u_k) + V_{k+1}^*(x_{k+1}). \quad (2.27)$$

In the infinite horizon case, (2.27) becomes

$$V^*(x_k) = \max_{\Delta_k \in \mathbb{D}} \min_{u_k} c(x_k, u_k) + V^*(x_{k+1}). \quad (2.28)$$

Assume (i) holds, then from Definition 1 there exists a stabilizing controller $u_k = -Ky_k$ and a sub-optimal value function $V(x_k) = x_k^T P x_k \geq V^*(x_k)$ such that

$$V(x_k) = \max_{\Delta_k \in \mathbb{D}} c(x_k, -Ky_k) + V(x_{k+1}) \quad (2.29)$$

which implies

$$\forall \Delta_k \in \mathbb{D} : V(x_k) \geq c(x_k, -Ky_k) + V(x_{k+1}). \quad (2.30)$$

Therefore, (2.30) holds for the quadratic form and is equivalent to (ii), since $c(x_k, -Ky_k) = x_k^T (\bar{C}_c + \bar{D}_c^w \bar{\Delta}_k C_z)^T (\bullet) x_k$.

(i) $\Leftarrow$ (ii): Now assume (i) does not holds, then there isn't a sub-optimal value function with quadratic form that upper-bounds (2.28). In such case, the infinite horizon Bellman's optimality principle does not have a solution. Therefore, (ii) also doesn't hold.

(ii) $\Rightarrow$ (iii): Assume (ii) holds and consider the system model (2.8), then (2.30) is equivalent to $\xi_k^T M \xi_k \leq 0$, where

$$M = \begin{bmatrix} \bar{A}^T P \bar{A} - P & \bar{A}^T P \bar{B}^w \\ \bar{B}^{wT} P \bar{A} & \bar{B}^{wT} P \bar{B}^w \end{bmatrix} + \begin{bmatrix} \bar{C}_c^T \bar{C}_c & \bar{C}_c^T \bar{D}_c^w \\ \bar{D}_c^{wT} \bar{C}_c & \bar{D}_c^{wT} \bar{D}_c^w \end{bmatrix} \quad (2.31)$$

for all $\Delta_k \in \mathbb{D}$. From Lemma 5, (2.31) holds for all $\xi \in \mathbb{X}$. Then, based on the generalized S-Procedure, there exists $S \in \mathbb{S}$ such that $M - S \preceq 0$. Or equivalently, there exists $\Lambda_i \succeq 0$ for $i \in [1, s]$ such that (2.26) holds. Therefore, (iii) holds.

(ii) $\Leftarrow$ (iii): Conversely, if (ii) does not hold, there doesn't exist a $P \succ 0$ such that (2.31) holds. Therefore (iii) also does not hold. $\square$

2.4.1 Convex condition for systems without disturbance feed-through

If $D_y^w = 0$ is assumed, we obtain $\bar{B}^w = B^w$, $\bar{D}_z^w = D_z^w$, and $\bar{D}_c^w = \mathbf{0}_{n_c \times n_p}$. In such a case, a simplified existence condition can be obtained.

Lemma 7. Assume $D_y^w = 0$. Then, a static output-feedback controller $u_k = -Ky_k$ is said to be of guaranteed cost, according to Definition 1, if and only if there exists $X \succ 0$ and $\Upsilon_i \succeq 0$ for $i \in [1, s]$ such that

$$\begin{bmatrix} -\Upsilon_q & 0 & 0 & C_z X - D_z^w Y C_y & D_z^w \Upsilon_p \\ \star & -I_{n_c} & 0 & C_c X - D_c^w Y C_y & 0 \\ \star & \star & -X & A X - B^u Y C_z & B^w \Upsilon_p \\ \star & \star & \star & -X & 0 \\ \star & \star & \star & \star & -\Upsilon_p \end{bmatrix} \preceq 0, \quad (2.32)$$

where $\bar{X} C_y = C_y X$ and $Y = K \bar{X}$.

Proof. We first apply the Schur complement in (2.26) to the terms dependent on P , to the cost

terms, and to the terms dependent on Λ_q , which results

$$\begin{bmatrix} -\Lambda_q^{-1} & 0 & 0 & \bar{C}_z & D_z^w \\ \star & -I_{n_c} & 0 & \bar{C}_c & 0 \\ \star & \star & -P^{-1} & \bar{A} & B^w \\ \star & \star & \star & -P & 0 \\ \star & \star & \star & \star & -\Lambda_p \end{bmatrix} \preceq 0. \quad (2.33)$$

Then, we apply the congruence transformation in (2.33) with $T = \text{diag}(I_{n_q}, I_{n_c}, I_{n_x}, P^{-1}, \Lambda_p^{-1})$, and perform the substitution $X = P^{-1}$, $\Upsilon_p = \Lambda_p^{-1}$, and $\Upsilon_q = \Lambda_q^{-1}$, obtaining

$$\begin{bmatrix} -\Upsilon_q & 0 & 0 & \bar{C}_z X & D_z^w \Upsilon_p \\ \star & -I_{n_c} & 0 & \bar{C}_c X & 0 \\ \star & \star & -X & \bar{A} X & B^w \Upsilon_p \\ \star & \star & \star & -X & 0 \\ \star & \star & \star & \star & -\Upsilon_p \end{bmatrix} \preceq 0. \quad (2.34)$$

Based on the substitutions $\bar{X}C_y = C_y X$, $Y = K\bar{X}$, $\bar{A}X = AX - B^u Y C_z$, $\bar{C}_c X = C_c X - D_c^u Y C_y$, and $\bar{C}_z X = C_z X - D_z^u Y C_y$, (2.32) and (2.34) are equivalent. $\square$

Remark 2. The control gain can be recovered from Lemma 7 by $K = Y(C_y X C_y^\dagger)^{-1}$, where $C_y^\dagger$ is the pseudo-inverse of C_y .

Remark 3. Given $C_y = I_{n_x}$, we obtain $X = \bar{X}$, $K = YX^{-1}$, and Lemma 7 reduces to the state-feedback controller synthesis condition.

Lemma 7 provides an LMI condition for the existence of the GCC, based on the multipliers Λ_i for $i \in [1, s]$, the inverse cost function matrix X , and Y . From the properties of LMIs we can conclude that (2.32) is convex.

2.4.2 Convex condition for systems with disturbance feed-through

If we consider the general case, where $D_y^w \neq 0$, the controller gain K would also be present in the fifth column of (2.32). In such a case, the substitution which yields Y , used in Lemma 7, would not make the condition linear since K would still exist in the fifth row and column.

The following Theorem is the main result of this chapter. It demonstrates that a similar substitution can be successfully used to convexify a “dilated” version of condition (2.32), based on the Reciprocal Projection Lemma.

Theorem 3. A static output-feedback controller $u_k = -Ky_k$ is said to be of guaranteed cost, according to Definition 1, if there exists $X \succ 0$, $\Lambda_i \succeq 0$ for $i \in [1, s]$, $V_{2,2} \in \Re^{n_c \times n_c}$, $V_{3,3} \in \Re^{n_x \times n_x}$,

$V_{4,4} \in \mathfrak{R}^{n_x \times n_x}$, $V_{4,5} \in \mathfrak{R}^{n_x \times n_q}$, $V_{5,4} \in \mathfrak{R}^{n_q \times n_x}$, $V_{5,5} \in \mathfrak{R}^{n_q \times n_q}$, and $Y \in \mathfrak{R}^{n_u \times n_y}$ such that

$$\begin{bmatrix} -M & 0 & V \\ \star & -M & \bar{N} + M \\ \star & \star & -V - V^T \end{bmatrix} \preceq 0 \quad (2.35)$$

where, $M = \text{diag}(\Upsilon_q, I_{n_c}, X, X, \Upsilon_q)$,

$$V = \begin{bmatrix} V_{1,1} & V_{1,2} & V_{1,3} & V_{1,4} & V_{1,5} \\ V_{2,1} & V_{2,2} & V_{2,3} & V_{2,4} & V_{2,5} \\ V_{3,1} & V_{3,2} & V_{3,3} & V_{3,4} & V_{3,5} \\ 0 & 0 & 0 & V_{4,4} & V_{4,5} \\ 0 & 0 & 0 & V_{5,4} & V_{5,5} \end{bmatrix}, \quad (2.36)$$

$$\bar{N} = \frac{1}{2} \begin{bmatrix} -V_{1,1} & -V_{1,2} & -V_{1,3} & \Phi_z & \Sigma_z \\ -V_{2,1} & -V_{2,2} & -V_{2,3} & \Phi_c & \Sigma_c \\ -V_{3,1} & -V_{3,2} & -V_{3,3} & \Phi_x & \Sigma_x \\ 0 & 0 & 0 & -V_{4,4} & -V_{4,5} \\ 0 & 0 & 0 & -V_{5,4} & -V_{5,5} \end{bmatrix}, \quad (2.37)$$

$$\begin{bmatrix} \Phi_z & \Sigma_z \\ \Phi_c & \Sigma_c \\ \Phi_x & \Sigma_x \end{bmatrix} = 2 \begin{bmatrix} C_z & D_z^w \\ C_c & 0 \\ A & B^u \end{bmatrix} \begin{bmatrix} V_{4,4} & V_{4,5} \\ V_{5,4} & V_{5,5} \end{bmatrix} + \begin{bmatrix} V_{1,4} & V_{1,5} \\ V_{2,4} & V_{2,5} \\ V_{3,4} & V_{3,5} \end{bmatrix} - 2 \begin{bmatrix} D_z^u \\ D_c^u \\ B^u \end{bmatrix} Y \begin{bmatrix} C_y & D_y^w \end{bmatrix}, \quad (2.38)$$

$$\bar{V} \begin{bmatrix} C_y & D_y^w \end{bmatrix} = \begin{bmatrix} C_y & D_y^w \end{bmatrix} \begin{bmatrix} V_{4,4} & V_{4,5} \\ V_{5,4} & V_{5,5} \end{bmatrix}, \quad (2.39)$$

$Y = K\bar{V}$, $X = P^{-1}$, $\Upsilon_p = \Lambda_p^{-1}$, and $\Upsilon_q = \Lambda_q^{-1}$.

Proof. We apply again Schur complement in (2.26) to obtain

$$\begin{bmatrix} -\Lambda_q^{-1} & 0 & 0 & \bar{C}_z & \bar{D}_z^w \\ \star & -I_{n_c} & 0 & \bar{C}_c & \bar{D}_c^w \\ \star & \star & -P^{-1} & \bar{A} & \bar{B}^w \\ \star & \star & \star & -P & 0 \\ \star & \star & \star & \star & -\Lambda_p \end{bmatrix} \preceq 0. \quad (2.40)$$

Then, we apply the congruence transformation in (2.40) with $T = \text{diag}(\lambda_q, I_{n_c}, P, I_{n_x}, I_{n_p})$, and obtain

$$E = \begin{bmatrix} -\Lambda_q & 0 & 0 & \Lambda_q \bar{C}_z & \Lambda_q \bar{D}_z^w \\ \star & -I_{n_c} & 0 & \bar{C}_c & \bar{D}_c^w \\ \star & \star & -P & P \bar{A} & P \bar{B}^w \\ \star & \star & \star & -P & 0 \\ \star & \star & \star & \star & -\Lambda_p \end{bmatrix} \preceq 0. \quad (2.41)$$

We now define S as

$$S = \begin{bmatrix} -\frac{1}{2}\Lambda_q & 0 & 0 & \Lambda_q \bar{C}_z & \Lambda_q \bar{D}_z^w \\ 0 & -\frac{1}{2}I_{n_c} & 0 & \bar{C}_c & \bar{D}_c^w \\ 0 & 0 & -\frac{1}{2}P & P\bar{A} & P\bar{B}^w \\ 0 & 0 & 0 & -\frac{1}{2}P & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\Lambda_p \end{bmatrix}, \quad (2.42)$$

such that $S + S^T = E$. Then, from the Reciprocal Projection Lemma (Lemma 2) we obtain that for any given $Y \succ 0$ there exists W , of appropriate dimensions, such that

$$\begin{bmatrix} -Y & S + W \\ \star & Y - W - W^T \end{bmatrix} \preceq 0. \quad (2.43)$$

Subsequently, we apply a congruence transformation in (2.43) with $T = \text{diag}(M, I_{n_q+n_c+n_x+n_x+n_p})$ and substitute $Y = M^{-1}$, which yields

$$\begin{bmatrix} -M & MS + MW \\ \star & M^{-1} - W - W^T \end{bmatrix} \preceq 0. \quad (2.44)$$

Let $V = W^{-1}$ and assume it takes the form (2.36). Then, we apply a congruence transformation in (2.43) with $T = \text{diag}(I_{n_q+n_c+n_x+n_x+n_p}, V)$, resulting

$$\begin{bmatrix} -M & MSV + M \\ \star & VM^{-1}V - V - V^T \end{bmatrix} \preceq 0 \quad (2.45)$$

which, with a Schur complement of the terms related to M^{-1} and $MSV = \bar{N}$, is equivalent to

$$\begin{bmatrix} -M & 0 & V \\ \star & -M & \bar{N} + M \\ \star & \star & -V - V^T \end{bmatrix} \preceq 0 \quad (2.46)$$

which, in turn, is identical to (2.35). $\square$

Theorem 3 provides a convex condition for the existence of the GCC in the general static output-feedback case. It enables the synthesis of Optimal GCCs through SDP optimization problems.

2.4.3 Optimal static Output-feedback guaranteed cost control synthesis

In this sub-section, we present the resulting optimization problem for the optimal static output feedback GCC synthesis. Although the formulated GCC conditions are state-independent, the cost function (2.4) is still dependent. Therefore, we must represent the system cost in a

state-independent manner. Two methods are often employed to address this issue: Minimizing the worst case cost for all states in the unit norm ball (PETERSEN; MCFARLANE; ROTEA, 1998), or the expected cost value for a zero mean and unit covariance initial state distribution (XIE; SOH, 1993). We have chosen the latter, since the worst case cost only minimizes the largest eigenvalue of P , while the stochastic interpretation minimizes $\text{tr}(P)$. Another advantage of such an approach is its equivalence to the LQR when the system reduces to the state-feedback case without uncertainties.

Assumption 5. The initial state x_0 is a zero mean random variable with unit covariance.

From Assumption 5, $E(J^*(x_0)) = E(x_0^T P x_0) = E(\text{tr}(x_0^T P x_0)) = \text{tr}(P E(x_0 x_0^T)) = \text{tr}(P)$.

Theorem 4. A static output feedback controller $u_k = -K y_k$ is said to be an optimal guaranteed cost, according to Definition 2, if and only if

$$\begin{aligned} K = \arg \min \quad & \text{tr}(Z) \\ \text{s.t.} \quad & \begin{bmatrix} -M & 0 & V \\ \star & -M & \bar{N} + M \\ \star & \star & -V - V^T \end{bmatrix} \preceq 0 \\ & \begin{bmatrix} -Z & I \\ \star & -X \end{bmatrix} \preceq 0 \end{aligned} \quad (2.47)$$

where M , $\bar{N}$, V and X are defined in Theorem 3, and $Z \in \Re^{n_x \times n_x}$ is the cost matrix P relaxation.

Proof. From Definition 2, a controller is of optimal guaranteed cost if it satisfies Definition 1 and $J^*(x_0)$ is minimal. Since the distribution of x_0 is assumed to be known (Assumption 5), consider $\hat{J} = E(J(x_0))$. Then, the optimal GCC synthesis is equivalent to

$$\begin{aligned} \hat{J}^* = \min \quad & \text{tr}(P) \\ \text{s.t.} \quad & \begin{bmatrix} -M & 0 & V \\ \star & -M & \bar{N} + M \\ \star & \star & -V - V^T \end{bmatrix} \preceq 0 \end{aligned} \quad (2.48)$$

which minimizes the expected cost. However, such optimization is not linear on P since the LMI depends on its inverse $X = P^{-1}$. Therefore, we introduce the relaxation variable $Z \in \Re^{n_x \times n_x}$, such that $Z \succeq P$. From Schur complement, we obtain

$$\begin{bmatrix} -Z & I \\ \star & -X \end{bmatrix} \preceq 0. \quad (2.49)$$

Notice that minimizing Z is equivalent to minimizing P , since its constraint is only dependent on X . Therefore, (2.48) is equivalent to (2.47). $\square$

Remark 4. The optimal static output feedback for the case $D_y^w = 0$, can be defined similarly to Theorem 3, with the substitution of condition (2.35) by (2.32).

2.5 Numerical examples

This section provides two numerical examples of GCC for systems subject to structured uncertainties. The first compares the proposed synthesis method with GCC for systems subject to unstructured uncertainties (XIE; SOH, 1993), and the Linear Quadratic Regulator (KALMAN *et al.*, 1960). The second presents the synthesis and simulation for a system that previously proposed methods cannot synthesize a controller. We have used the YALMIP Toolbox (LÖFBERG, 2004) and the Mosek solver (APS, 2015) for the modeling of the problem¹.

2.5.1 Example 1

Consider the uncertain linear system ((2.1) and (2.2)), from (TERRA; CERRI; ISHII-HARA, 2014), defined by

$$\begin{aligned} A &= \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 0 & 1.2 \\ -1 & 1 & 0 \end{bmatrix}, B^u = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}, B^w = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & -0.4 \\ -1 & 0 \end{bmatrix}, \\ C_z &= \begin{bmatrix} 0.41 & 0.43 & -0.5 \\ 0 & -0.32 & 0.44 \end{bmatrix}, D_z^u = \begin{bmatrix} 0.4 & -0.4 \\ 0 & 0 \end{bmatrix}, D_z^w = 0, \end{aligned} \quad (2.50)$$

measurement matrices

$$C_y = I_{n_x}, D_y^w = 0. \quad (2.51)$$

and the disturbance

$$\Delta_k = \begin{bmatrix} \delta_{1,k} & 0 \\ 0 & \delta_{2,k} \end{bmatrix} \quad (2.52)$$

where $\delta_{1,k}, \delta_{2,k} \in [-1, 1]$. In this example, we synthesize and compare controllers for five different methods.

1. LQR (KALMAN *et al.*, 1960): The LQR controller is generated based on the nominal plant with no modeled disturbances.
2. GCC (unstructured uncertainty) (XIE; SOH, 1993): Based on Xie's approach.
3. Theorem 3-based GCC (unstructured): Controller based on Theorem 3, generated with the SDP optimization of Theorem 4, while assuming no structure to Δ_k . Therefore, $\Lambda_p = \lambda I_p$ and $\Lambda_q = \lambda I_q$ where $\lambda > 0$.
4. Lemma 7-based GCC (structured): Controller based on Theorem 7, generated with the SDP optimization of Theorem 4 without the "Dilated" LMI (see Remark 4), while exploiting the diagonal uncertainty structure of Δ_k . Therefore, $\Lambda_p = \text{diag}(\lambda_1, \lambda_2)$ and $\Lambda_q = \text{diag}(\lambda_1, \lambda_2)$ where $\lambda_i > 0$ for all $i \in [1, 2]$.

¹ The source code for both numerical examples are available at: https://gitlab.com/cmasserat/gcc_structured_uncertainty

5. Theorem 3-based GCC (structured): Controller based on Theorem 3, generated with the SDP optimization of Theorem 4, while exploiting the diagonal uncertainty structure.

The cost function matrices selected for all methods were $Q = I_3$, and $R = I_2$. Different methods are compared based on the synthesis cost, $\text{tr}(P)$, and the effective cost, $E(J(x_0))$. We calculated the effective cost as the mean value of the cost function for 5000 simulations performed with a horizon of 200 time-steps. Then, we present a simulation example for the three controllers synthesized from these methods and qualitatively compare the obtained results.

Table 1 presents the results of Example 1. It is possible to see that for this particular system, the effective cost of the LQR (1) exceeds its synthesis cost by one order of magnitude demonstrating its expected lack of robustness margins, although it is still able to stabilize the system. Both unstructured uncertainty GCCs (2 and 3) yield the same control gain and cost since $D_y^w = 0$. Their synthesis cost is greater than the effective cost, which illustrates the upper-bound nature of its robustness margins. However, they yield extremely conservative bounds on cost (of an order of magnitude) due to not exploiting the uncertainty structure. Finally, both structured uncertainty GCCs (4 and 5) also yield the same control gain to each other since $D_y^w = 0$, but decrease the conservativeness of the synthesis by a factor of seven while also reducing the effective cost from 46.56 to 44.16.

Table 1 – Synthesis and simulation results for the comparative study

	Synthesis cost	Effective cost
1) LQR	22.15	490.13
2) GCC proposed by (XIE, 1996)	581.79	46.56
3) Theorem 3 GCC (unstructured)	581.79	46.56
4) Lemma 7 GCC (structured)	97.63	44.16
5) Theorem 3 GCC (structured)	97.63	44.16

Figure 1 presents results acquired by one of the 5000 simulations performed with the set of controllers considered in this example. Identical disturbances and initial states, $x_0 = [1, 1, 1]^T$, were used for all simulation. Figures 1a and 1b show the results for the standard LQR controller with effective cost 12.6707. We can observe a significant impact of the disturbances on the overall system behavior, due to the lack of robustness guarantees on its synthesis. Figures 1c and 1d show the results for the GCC considering unstructured disturbances with effective cost 10.7861; we can observe that it rejects the disturbances significantly better than the LQR controller; however, it yields an 80% overshoot before reaching steady-state. Finally, Figures 1e and 1f show the results for the GCC considering unstructured disturbances with effective cost of 7.3934. This controller was able to yield a faster settling time, smaller overshoot, and lower effective cost when compared to both other synthesized controllers.

The average cost reduction, combined by the performance improvement observed in Figure 1, demonstrates the advantages of incorporating the uncertainty structure information into

Figure 1 – Simulation results for the synthesized results of Example 1. (a) Closed-loop state x_k subject to uncertainties for (1); (b) Control inputs u_k for (1); (c) Closed-loop state x_k subject to uncertainties for (2) and (3); (d); Control inputs u_k for (2) and (3); (e) Closed-loop state x_k subject to uncertainties for (4) and (5); (f) Control inputs u_k for (4) and (5).

the controller synthesis procedure.

2.5.2 Example 2

In this example we present the synthesis and simulation for a system that was not supported by previous methods, where $D_y^w \neq 0$. Consider the system from (2.50) with measurement

Figure 2 – Simulation results for GCC of Example 2. (a) Closed-loop state x_k subject to uncertainties; (b) Simulated uncertainty δ_1 and δ_2 ; (c) Control inputs u_k .

matrices

$$C_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad D_y^w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (2.53)$$

In such a case, the LQR (1), Xie's GCC (2) and Lemma 7-based GCC (4) cannot be used as they assume $D_y^w = 0$. However, the unstructured uncertainty GCC based on Theorem 3 (3) also cannot be synthesized, since Assumption 4 ($D_y^w \Delta_k D_z^u = 0$) does not hold for the generic uncertainty case. Therefore, the proposed GCC controller based on Theorem 3 for structured uncertainties is the only capable of synthesizing a controller for this system.

The structured uncertainty GCC based on Theorem 3 synthesis resulted in a gain matrix

$$K = \begin{bmatrix} 1.1431 & 0.1282 & -0.3585 & 0.0947 \\ 0.6881 & -0.7581 & 0.4561 & 0.0596 \end{bmatrix}, \quad (2.54)$$

and cost matrix

$$P = \begin{bmatrix} 61.9182 & 3.2483 & -41.2619 \\ 3.2483 & 9.8246 & -7.1655 \\ -41.2619 & -7.1655 & 34.2462 \end{bmatrix}. \quad (2.55)$$

It provides a synthesis cost of 94.15 and an effective cost of 45.62. Figure 2 shows the simulation result for initial state $x_0 = [1, 1, 1]^T$, where we can observe that the control law is able to correctly stabilize the system subject to varying disturbances.

2.6 Conclusion

In this chapter, we have proposed a state-feedback and static output-feedback generalized Guaranteed Cost Control method for discrete-time linear systems subject to norm-bounded structured parametric uncertainties. Such controllers provide upper bounds to the quadratic cost functional and guarantee robust stability and performance to a larger class of uncertain linear systems if compared with previous methods. There are two main contributions for the class of systems considered: It relaxes the assumption from no uncertainty feed-through ($D_y^w = 0$) to an assumption that no feed-through uncertainty is multiplicative with control inputs ones ($D_y^w \Delta_k D_z^u = 0$). Moreover, it represents a more general uncertainty structure (correlated block diagonal).

Numerical simulations exemplified that exploiting the internal structure of uncertainties may provide a significant reduction in conservativeness while still yielding controllers which guarantee robust stability and performance.

Although the proposed method provides both necessary and sufficient conditions for the synthesis of GCCs with a more general uncertainty structure, it still provides only sufficient conditions for the synthesis of robust static output-feedback controllers. Besides the underlying limitation that Guaranteed Cost Controllers only provide sufficient conditions for the optimal controller robust synthesis due to the simplifying assumption $P(\Delta) = P$ on the structure of the cost function.

OPEN-LOOP GUARANTEED COST MODEL PREDICTIVE CONTROL OF UNCERTAIN LINEAR SYSTEMS

In this chapter, we propose the Open-Loop Guaranteed Cost Model Predictive Control (OL-GCMPC). It generalizes the concepts and guarantees from the GCC formulation to constrained uncertain discrete-time systems subject to norm-bounded uncertainties through an MPC framework. We extend such a concept to a more general class of systems while introducing better robustness guarantees in Chapter 4 through the use of the formulations presented in Chapter 2 and the generalization of the proposed OL-GCMPC.

3.1 Introduction

Model Predictive Control (MPC) is a class of optimization-based control algorithm that relies on explicit models of the controlled system to predict its future states (BADGWELL; QIN, 2015). Several different areas, such as refineries, food processing plants, mining, aerospace, and automotive control have had applications of such techniques (QIN; BADGWELL, 2003). MPC's objective is to minimize a cost function and maintain the system states and control inputs within a feasible set. The system dynamics are usually assumed to be known; therefore, the formulation does not consider model mismatches or external disturbances. However, the disregard of such uncertainties may lead to poor closed-loop performance and the violation of state and control input constraints (RAWLINGS; MEADOWS; MUSKE, 1994).

Robust Model Predictive Control (RMPC) addresses the poor closed-loop performance of MPC approaches subject to uncertainties (BEMPORAD; MORARI, 1999). Min-Max Model Predictive Control (MMMPC) is the main RMPC approach; its objective is to obtain a control input sequence that minimizes a cost function and to guarantee the feasibility of system states

and control inputs when the system is subject to the worst-case disturbance. Different RMPC approaches contain different assumptions and approximations; therefore comparisons among different approaches are difficult. However, most of these formulations belong to two classes (KERRIGAN; MACIEJOWSKI, 2004), namely open-loop and feedback RMPC. The open-loop RMPC is similar to the nominal MPC, which optimizes a control input sequence for the open-loop system. It is considered conservative for actual applications due to not accounting how disturbances affect the control inputs in the predictive horizon. Meanwhile, feedback RMPC optimizes a control input sequence for a disturbance-rejecting closed-loop system, which addresses the conservativeness of the open-loop approach.

RMPC schemes are most often defined by exact or approximate solutions to l_1 -norm, l_2 -norm or l_∞ -norm cost functionals subject to polytopic bounded uncertainties (e.g. bound by l_1 -norm or l_∞ -norm) (LOFBERG, 2003; SCOKAERT; MAYNE, 1998; GENCELI; NIKOLAOU, 1993). Linear Programming (LP) problems may approximate both l_1 -norm and l_∞ -norm MMMPC problems without a significant increase in the optimization problem complexity. However, l_2 -norm MMMPCs can be represented only by explicit enumeration of all vertexes of the uncertainty polytopic set (GAO; CHONG, 2012; BEMPORAD; BORRELLI; MORARI, 2003). The enumeration approach suffers from Bellman's curse of dimensionality since the number of vertexes grows exponentially with the number of dimensions. Furthermore, the case where the uncertainty set is defined by l_2 -norm further aggravates this issue, as the enumeration approach does not apply to non-polytopic sets.

Xie and Soh (XIE; SOH, 1993) designed the Guaranteed Cost Control (GCC) to address the poor performance of the Linear Quadratic Regulator (LQR) when a Linear Time-Invariant (LTI) system model is subject to l_2 -norm bounded parametric uncertainties. It provides a quadratically stable closed-form controller that guarantees an upper bound to l_2 -norm cost functionals. However, the GCC design considers only unconstrained control problem.

GCCs are obtained by either Linear Matrix Inequalities (LMI) (PETERSEN; MCFARLANE; ROTEA, 1998) or Algebraic Riccati Equations (ARE) (XIE; SOH, 1993) and the concept was extended to polytopic-bounded l_2 -norm RMPCs problems based on LMIs (ROSSI; WASCHBURGER; GALVAO, 2013; ROSINOVÁ; VESELÝ; KUČERA, 2003). However, the guaranteed cost constrained optimization requires the solution to a Semi-Definite Programming (SDP) problem, which increases computational requirements in comparison to other MPC approaches which rely on LP, Quadratic Programming (QP), or Second Order Cone Programming (SOCP).

In this chapter, we propose a Open-Loop Guaranteed Cost approach for Robust Model Predictive Control (OL-GCMPC) of linear time-invariant systems subject to bounded parametric uncertainties. This approach provides an SOCP-based solution to l_2 -norm Robust Model Predictive Controllers subject to l_2 -norm-bounded uncertainties and polytopic constraints and ensures sufficient conditions for stability, feasibility and performance robustness.

This chapter is organized as follows: Section 3.2 presents the robust constrained optimal control problem; Section 3.3 derives the Guaranteed Cost Model Predictive Controller; Section 3.4 provides a numerical example; finally, Section 3.5 addresses the final remarks.

3.2 Problem statement and preliminaries

A deterministic LTI system model (Definition 4) and a parametrically uncertain one (Definition 5) were considered for the formulation.

Definition 4. Let the discrete state space LTI model be

$$x_{k+1} = Ax_k + B^u u_k \quad (3.1)$$

where $x_k \in \mathfrak{R}^n$ is the system state, $u_k \in \mathfrak{R}^m$ is the control input, $A \in \mathfrak{R}^{n \times n}$ is the state matrix and $B^u \in \mathfrak{R}^{n \times m}$ is the input matrix.

Definition 5. (TERRA; CERRI; ISHIHARA, 2014) Let the discrete state space LTI model subject to parametric uncertainties be

$$\bar{x}_{k+1} = (A + \delta A(\Delta_k))\bar{x}_k + (B^u + \delta B^u(\Delta_k))u_k \quad (3.2)$$

where $\bar{x}_k \in \mathfrak{R}^n$ is the uncertain system state and $\delta A(\Delta_k)$ and $\delta B^u(\Delta_k)$ are, respectively, the state and input multiplicative uncertainty matrices, such that

$$\begin{bmatrix} \delta A(\Delta_k) & \delta B^u(\Delta_k) \end{bmatrix} = B^w \Delta_k \begin{bmatrix} C_y & D_y^u \end{bmatrix} \quad (3.3)$$

with $\Delta_k \in \mathbb{W} = \{\Delta \mid \Delta \in \mathfrak{R}^{p \times l}, \|\Delta\|_2 \leq 1\}$, $B^w \in \mathfrak{R}^{n \times p}$, $C_y \in \mathfrak{R}^{l \times n}$ and $D_y^u \in \mathfrak{R}^{l \times m}$.

Optimality is defined by a quadratic cost functional for LQR and l_2 -norm MPC approaches, given by

$$J_i(x_i) = x_N^T P_N x_N + \sum_{k=i}^{N-1} x_k^T Q x_k + u_k^T R u_k \quad (3.4)$$

where $P_N \succeq 0$, $Q \succeq 0$ and $R \succ 0$ are symmetric weight matrices, $\mathbf{x} = \{x_k \mid k \in [0, N]\}$, $\mathbf{u} = \{u_k \mid k \in [0, N-1]\}$, N is the prediction horizon and the system model satisfies either Definition 4, or 5.

Definition 6. (KALMAN *et al.*, 1960) Consider the deterministic system model from Definition 4. The Linear Quadratic Regulator is a feedback controller given by the optimal control problem

$$\begin{aligned} J_i^*(x_i) &= \inf_{\mathbf{u}} J_i(x_i) \\ \text{s.t. } x_{k+1} &= Ax_k + B^u u_k \end{aligned} \quad (3.5)$$

which has a closed form solution $J_i^*(x_i) = x_i^T P_i x_i$ achieved at $u_k = -K_k x_k$, where

$$\begin{aligned} K_k &= (R + B^{uT} P_{k+1} B^u)^{-1} B^{uT} P_{k+1} A \\ P_k &= A^T P_{k+1} A + Q - K_k^T (R + B^{uT} P_{k+1} B^u) K_k \end{aligned} \quad (3.6)$$

Definition 7. (CAMACHO; ALBA, 2013) Consider the deterministic system model from Definition 4. A Model Predictive Control is a feedback controller given by the constrained optimal control problem

$$\begin{aligned} J_i^*(x_i) = & \inf_{\mathbf{u}} J_i(x_i) \\ \text{s.t. } & x_{k+1} = Ax_k + B^u u_k \\ & [x_k^T, u_k^T]^T \in \mathbb{C}_k \end{aligned} \quad (3.7)$$

where $\mathbb{C}_k$ defines the feasible set of the system states and control inputs at timestep k .

MPC has no general closed-form solution. Therefore, it requires either an online evaluation with a Quadratic Programming solver (LÖFBERG, 2004) or an offline evaluation based on explicit solutions (ALESSIO; BEMPORAD, 2009).

Assumption 6. The feasible set of variables x_k and u_k at a given timestep k , given by $[x_k^T, u_k^T]^T \in \mathbb{C}_k$, is a convex non-empty polytope described by

$$\mathbb{C}_k = \{z \mid z \in \mathbb{R}^{n+m}, H_k z + g_k \leq 0\} \quad (3.8)$$

where $H_k = [H_{x,k}, H_{u,k}]$, $H_{x,k} \in \mathbb{R}^{q \times n}$, $H_{u,k} \in \mathbb{R}^{q \times m}$, and $g_k \in \mathbb{R}^q$.

Assumption 6 is common in literature since it enables the use of widely available QP solvers (e.g. Gurobi (OPTIMIZATION, 2015), CVXGEN (MATTINGLEY; BOYD, 2012) and FORCES Pro (GMBH, 2016)) and the solution of several practical control problems. However, neither LQR nor MPC provides performance, stability or feasibility robustness guarantees for the control of systems subject to multiplicative or additive uncertainties.

Definition 8. (PETERSEN; MCFARLANE; ROTE, 1998) Consider the system model from Definition 5. A control law $u_k = -K_k \bar{x}_k$ is said to be a Guaranteed Cost Controller if there exists a symmetric cost matrix sequence $\mathbb{S} = \{S_k \mid S_k \succeq 0, k \in [0, N]\}$, such that

$$\forall \Delta_k \in \mathbb{W} : [A - B^u K_k + \delta A(\Delta_k) - \delta B^u(\Delta_k) K_k]^T S_{k+1} [\bullet] - S_k + Q + K_k^T R K_k < 0. \quad (3.9)$$

By definition, GCC ensures the quadratic stability of the system for any disturbance within the admissible set $\mathbb{W}$. A closed form solution for $\mathbb{S}$ based on AREs is proposed by Xie and Soh (XIE; SOH, 1993).

Theorem 5. Consider the parametrically uncertain system model from Definition 5 and assume a control law $u_k = -K_k \bar{x}_k$ is a Guaranteed Cost Control with an associated cost matrix S_k . For some $\varepsilon_k > 0$, such that $S_k \succ 0$, the closed loop uncertain system

$$\bar{x}_{k+1} = [A - B^u K_k + \delta A(\Delta_k) - \delta B^u(\Delta_k) K_k] \bar{x}_k \quad (3.10)$$

has an upper bound cost

$$J_k(\bar{x}_k) < \bar{x}_k^T S_k \bar{x}_k \quad (3.11)$$

given by

$$S_k = A^T X_{k+1} A + Q_\varepsilon - (A^T X_{k+1} B^u + N_\varepsilon)(R_\varepsilon + B^{uT} X_{k+1} B^u)^{-1}(\bullet) \quad (3.12)$$

where $Q_\varepsilon = Q + \varepsilon_k^{-1} C_y^T C_y$, $R_\varepsilon = R + \varepsilon_k^{-1} D_y^{uT} D_y^u$, $N_\varepsilon = \varepsilon_k^{-1} C_y^T D_y^u$, $I - \varepsilon_k B^{wT} S_k B^w \succ 0$, and

$$X_k = (S_k^{-1} - \varepsilon_k^{-1} B^w B^{wT})^{-1} \succ 0 \quad (3.13)$$

$$K_k = (R_\varepsilon + B^{uT} X_{k+1} B^u)^{-1} (B^{uT} X_{k+1} A + N_\varepsilon^T). \quad (3.14)$$

Proof. See (XIE; SOH, 1993). □

However, the results from Theorem 5 are valid only for $[\bar{x}_k^T, u_k^T]^T \in \mathfrak{R}^{n+m}$ and have not been extended to the constrained quadratic optimal control case.

3.3 Guaranteed cost model predictive control

This section addresses the formulation of the constrained guaranteed cost control.

Lemma 8. Let $\varepsilon \in (0, \|B^{wT} S B^w\|)$ and

$$X = (S^{-1} - \varepsilon B^w B^{wT})^{-1} \succ 0 \quad (3.15)$$

for a given symmetric $S \succ 0$ and a non-singular B^w . Then, $X \succ S$.

Proof. We employ Woodbury's matrix inversion Lemma (HENDERSON; SEARLE, 1981) to (3.15), which results in

$$X = S + S B^w (\varepsilon^{-1} I - B^{wT} S B^w)^{-1} B^{wT} S \quad (3.16)$$

and conclude, from $\varepsilon^{-1} I - B^{wT} S B^w \succ 0$,

$$S B^w (\varepsilon^{-1} I - B^{wT} S B^w)^{-1} B^{wT} S \succ 0 \quad (3.17)$$

Therefore, $X \succ S$. □

Theorem 6. Consider the deterministic system model from Definition 4 and

$$\begin{aligned} \bar{J}_i^*(x_i) &= \inf_{\mathbf{u}} \bar{J}_i(\mathbf{x}, \mathbf{u}) \\ s.t. \quad x_{k+1} &= A x_k + B^u u_k \end{aligned} \quad (3.18)$$

where

$$\bar{J}_i(\mathbf{x}, \mathbf{u}) = x_N^T S_N x_N + \sum_{k=i}^N c(x_k, u_k) \quad (3.19)$$

$$c(x, u) = x^T Q_\varepsilon x + u^T R_\varepsilon u + 2x^T N_\varepsilon u \quad (3.20)$$

with ε chosen such that the guaranteed cost control condition is satisfied. The guaranteed cost control is equivalent to the sub-optimal solution of (3.18), given by

$$\begin{aligned} \bar{J}_N(x_N) &< x_N^T X_N x_N \\ \bar{J}_k(x_k) &< \inf_{u_k} c(x_k, u_k) + x_{k+1}^T X_{k+1} x_{k+1} = x_k^T S_k x_k \end{aligned} \quad (3.21)$$

Proof. Let the value function $\bar{J}_k^*(x)$, related to the optimization problem from (3.18), be recursively defined as

$$\bar{J}_N^*(x) = x^T S_N x \quad (3.22)$$

$$\bar{J}_k^*(x) = \inf_u c(x, u) + \bar{J}_{k+1}^*(Ax + B^u u) \quad (3.23)$$

based on Bellman's optimality principle.

From Lemma 8 and (3.22),

$$\bar{J}_N^*(x) = x^T S_N x < x^T X_N x \quad (3.24)$$

Therefore,

$$\bar{J}_{N-1}^*(x) = \inf_u c(x, u) + (Ax + B^u u)^T S_N(\bullet) \quad (3.25)$$

$$< \inf_u c(x, u) + (Ax + B^u u)^T X_N(\bullet) \quad (3.26)$$

The solution of (3.26) yields

$$u = -(R_{\varepsilon_k} + B^{uT} X_N B^u)^{-1} (B^{uT} X_N A + N_{\varepsilon_k}^T) x \quad (3.27)$$

which substituted back in (3.26) results in

$$\bar{J}_{N-1}^*(x) < x^T [A^T X_N A + Q_\varepsilon - (A^T X_N B^u + N_\varepsilon)(R_\varepsilon + B^{uT} X_N B^u)^{-1}(\bullet)] x \quad (3.28)$$

Therefore, $\bar{J}_{N-1}^*(x) < x^T S_{N-1} x$, where S_{N-1} is the guaranteed cost matrix at timestep $k = N - 1$, as stated in Theorem 5.

Assume $\exists k \in [0, N - 2] : \bar{J}_{k+1}^*(x) < x^T S_{k+1} x$, where S_{k+1} is a guaranteed cost matrix.

$$\bar{J}_k^*(x) < \inf_u c(x, u) + (Ax + B^u u)^T S_{k+1}(\bullet) \quad (3.29)$$

$$< \inf_u c(x, u) + (Ax + B^u u)^T X_{k+1}(\bullet) \quad (3.30)$$

which, analogously to $\bar{J}_{N-1}$, results in

$$\bar{J}_k(x) < x^T [A^T X_{k+1} A + Q_\varepsilon - (A^T X_{k+1} B^u + N_\varepsilon)(R_\varepsilon + B^{uT} X_{k+1} B^u)^{-1}(\bullet)] x \quad (3.31)$$

Therefore, we conclude by induction

$$\forall k \in [0, N-1]. \bar{J}_k(x) < x^T S_k x \quad (3.32)$$

hence the guaranteed cost control is a sub-optimal solution to the optimization problem in (3.18). $\square$

Theorem 6 demonstrates the equivalence of the GCC of a parametrically uncertain system to a sub-optimal solution of a Dynamic Programming problem for the deterministic system model. It also shows that the GCC bounds $J_i^*(\bar{x}_i) < x_i^T S_i x_i$ and $\bar{J}_i^*(\mathbf{x}, \mathbf{u}) < x_i^T S_i x_i$. However, the result from Theorem 6 is valid only for the unconstrained case where $x_k \in \mathfrak{R}^n$ and $u_k \in \mathfrak{R}^m$.

Lemma 9. Consider the parametrically uncertain system model from Definition 5 and let K_k be a guaranteed cost control, $u_k = -K_k \bar{x}_k + v_k$ and $\bar{R}_k = R_\varepsilon + B^{uT} X_{k+1} B^u$.

$$\begin{aligned} J_i^*(x_i) &< \inf_{\mathbf{v}} \sum_{k=i}^{N-1} v_k \bar{R}_k v_k + x_i^T S_i x_i \\ \text{s.t. } x_{k+1} &= (A - B^u K_k) x_k + B^u v_k \end{aligned} \quad (3.33)$$

is equivalent to the guaranteed cost control sub-optimal solution from Theorem 6.

Proof. Consider the value function from (3.21) and the deterministic system model from Definition 4.

$$\bar{J}_k^*(x) < \inf_u x^T (Q_\varepsilon + A^T X_{k+1} A) x + u^T (R_\varepsilon + B^{uT} X_{k+1} B^u) u + 2x^T A^T X_{k+1} B^u u \quad (3.34)$$

where the substitution of $u = K_k x_k + v_k$ yields

$$\bar{J}_k^*(x) < x^T S_k x + \inf_v v^T (R_\varepsilon + B^{uT} X_{k+1} B^u) v \quad (3.35)$$

The recursive substitution of (3.35) for $k \in [i, N-1]$ results in

$$\begin{aligned} \bar{J}_i^*(x_i) &< \inf_{\mathbf{v}} \sum_{k=i}^{N-1} v_k \bar{R}_k v_k + x_i^T S_i x_i \\ \text{s.t. } x_{k+1} &= (A - B^u K_k) x_k + B^u v_k \end{aligned} \quad (3.36)$$

Therefore, from the results of Theorem 6 and (XIE; SOH, 1993),

$$\begin{aligned} J_i^*(x_i) &< \inf_{\mathbf{v}} \sum_{k=i}^{N-1} v_k \bar{R}_k v_k + x_i^T S_i x_i \\ \text{s.t. } x_{k+1} &= (A - B^u K_k) x_k + B^u v_k \end{aligned} \quad (3.37)$$

This concludes the proof. $\square$

If the optimization domain of (3.33) is $\mathbb{C}_k \equiv \mathfrak{R}^{n+m}$, the solution is trivially $v_k \equiv 0$ and (3.33) reduces to Theorem 6. However, if $\mathbb{C}_k \subset \mathfrak{R}^{n+m}$, (3.33) provides an upper bound to the MPC cost functional.

Since (3.37) is based on the deterministic model, enforcement of the feasible set constraints $([x_k^T, u_k^T]^T \in \mathbb{C}_k)$ would ensure that only the undisturbed states and control inputs are feasible and there would be no feasibility guarantees for the parametrically uncertain ones. Feasibility robustness requires

$$\forall \Delta_k \in \mathbb{W}. [\bar{x}_k^T, u_k^T]^T \in \mathbb{C}_k. \quad (3.38)$$

Lemma 10. Consider the parametrically uncertain system model from Definition 5 and let K_k be a Guaranteed Cost Controller, $u_k = -K_k \bar{x}_k + v_k$ and $\bar{\mathbb{W}}_k = \{w \mid w \in \mathfrak{R}^{p \times l}, \|w\|_2 \leq \|\tilde{C}_y \bar{x}_k + D_y^u v_k\|_2\}$ where $\tilde{C}_y = C_y - D_y^u K_k$.

$$\bar{x}_{k+1} = (A - B^u K_k) \bar{x}_k + B^u v_k + B^w w_k \quad (3.39)$$

is an approximation of the closed loop parametrically uncertain model.

Proof. Let $w_k = \Delta_k [\tilde{C}_y \bar{x}_k + D_y^u v_k]$ with $\tilde{C}_y = C_y - D_y^u K_k$. From (3.39), we conclude

$$\begin{aligned} \bar{x}_{k+1} &= (A - B^u K_k) \bar{x}_k + B^u v_k + B^w \Delta_k (\tilde{C}_y \bar{x}_k + D_y^u v_k) \\ &= A \bar{x}_k + B^u u_k + B^w \Delta_k (C_y \bar{x}_k + D_y^u u_k) \\ &= (A + \delta A(\Delta_k)) \bar{x}_k + (B^u + \delta B^u(\Delta_k)) u_k \end{aligned} \quad (3.40)$$

and the admissible set of w_k , $\bar{\mathbb{W}}$, is given by

$$\begin{aligned} \|w_k\|_2 &= \|\Delta_k [\tilde{C}_y \bar{x}_k + D_y^u v_k]\|_2 \\ &\leq \|\Delta_k\|_2 \|\tilde{C}_y \bar{x}_k + D_y^u v_k\|_2 \\ &\leq \|\tilde{C}_y \bar{x}_k + D_y^u v_k\|_2 \end{aligned} \quad (3.41)$$

Therefore, $\bar{\mathbb{W}}_k = \{w \mid w \in \mathfrak{R}^{p \times l}, \|w\|_2 \leq \|\tilde{C}_y \bar{x}_k + D_y^u v_k\|_2\}$. Since $\bar{\mathbb{W}} \supset \mathbb{W}$, such a representation is a conservative approximation of Definition 5. $\square$

Lemma 10 provides a conservative approximation to the parametrically uncertain system. Such representation enables multiplicative uncertainties to be represented similarly to the modeling of additive uncertainties in previous RMPC studies (i.e., (SCOKAERT; MAYNE, 1998)).

Corollary 2. Consider the deterministic and parametrically uncertain system models from Definition 4 and Lemma 10, respectively, and assume the system is regulated by a feedback controller $\tilde{K}$.

$$\bar{x}_k = x_k + \sum_{i=0}^{k-1} \tilde{A}^{k-i-1} B^w w_i \quad (3.42)$$

where $\tilde{A} = A - B^u \tilde{K}$ is the closed loop system dynamics.

Corollary 2 provides a representation of the uncertain system state according to the deterministic system state and all previous disturbances and enables the externalization of disturbances from the system dynamics to the feasible set constraints. Therefore, based on (3.38) and Assumption 6, the parametrically uncertain model feasible set $\mathbb{C}_k$ with $u_k = -K_k x_k + v_k$ is

$$\forall w_k \in \tilde{\mathbb{W}} : \tilde{H}_{x,k} x_k + H_{u,k} v_k + c_k + \tilde{H}_{x,k} \sum_{i=0}^{k-1} \tilde{A}^{k-i-1} B^w w_i \leq 0. \quad (3.43)$$

where $\tilde{H}_{x,k} = H_{x,k} - H_{u,k} K_k$ and $\tilde{K}$ is arbitrarily defined¹. However, uncertainty admissible set $\tilde{\mathbb{W}}$ is still dependent on uncertain state $\bar{x}_k$.

Lemma 11. Let $\phi_k(x, v) = \|\tilde{C}_{y,k} x + D_y^u v\|_2$ and $\rho_i = \|C_y \tilde{A}^i B^w\|_2$. Then,

$$\|w_k\| \leq \phi_k(\bar{x}_k, v_k) \leq \phi_k(x_k, v_k) + \sum_{i=0}^{k-1} c(k, i) \phi_i(x_i, v_i) \quad (3.44)$$

where

$$\forall i < k. c(k, i) = \rho_{k-i-1} + \sum_{j=0}^{k-i-2} \rho_j c(k-j-1, i) \quad (3.45)$$

Proof. Consider the case $k = 1$ and $\phi_0(\bar{x}_0, v_0) = \phi_0(x_0, v_0)$, then

$$\begin{aligned} \phi_1(\bar{x}_1, v_1) &= \|\tilde{C}_{y,1} \bar{x}_1 + D_y^u v_1\|_2 \\ &= \|\tilde{C}_{y,1} x_1 + D_y^u v_1 + \tilde{C}_{y,1} B^w w_0\|_2 \\ &\leq \phi_1(x_1, v_1) + \|\tilde{C}_{y,1} B^w w_0\|_2 \\ &\leq \phi_1(x_1, v_1) + \|\tilde{C}_{y,1} B^w\|_2 \|w_0\|_2 \\ &\leq \phi_1(x_1, v_1) + \rho_{0,1} \phi_0(x_0, v_0) \end{aligned} \quad (3.46)$$

therefore, (3.44) is valid for $k = 1$.

Assuming that $\forall i \in [0, k-1]$ (3.45) holds, it is possible to represent $\phi_k(\bar{x}_k, v_k)$ as

$$\begin{aligned} \phi_k(\bar{x}_k, v_k) &= \left\| \tilde{C}_{y,k} x_k + D_y^u v_k + \sum_{i=0}^{k-1} \tilde{C}_{y,k} \tilde{A}^{k-i-1} B^w w_i \right\|_2 \\ &\leq \|\tilde{C}_{y,k} x_k + D_y^u v_k\|_2 + \sum_{i=0}^{k-1} \|\tilde{C}_{y,k} \tilde{A}^{k-i-1} B^w w_i\|_2 \\ &\leq \phi_k(x_k, v_k) + \sum_{i=0}^{k-1} \rho_{k-i-1,k} \phi_i(\bar{x}_i, v_i) \end{aligned} \quad (3.47)$$

Substituting (3.44) and (3.45) on (3.47) yields

$$\phi_k(\bar{x}_k, v_k) \leq \phi_k(x_k, v_k) + \sum_{i=0}^{k-1} \rho_{k-i-1,k} \left(\phi_i(x_i, v_i) + \sum_{j=0}^{i-1} c(i, j) \phi_j(x_j, v_j) \right), \quad (3.48)$$

which can be rearranged to

$$\begin{aligned} \phi_k(\bar{x}_k, v_k) &\leq \phi_k(x_k, v_k) + \rho_{0,1} \phi_k(x_{k-1}, v_{k-1}) + \\ &\quad + \sum_{i=0}^{k-2} \left(\rho_{k-i-1,k} + \sum_{j=0}^{k-i-2} \rho_{j,k} c(k-j-1, i) \right) \phi_i(x_i, v_i). \end{aligned} \quad (3.49)$$

¹ See Appendix C for a proof on the controllers independence.

Thus, substituting (3.45) to (3.49) results

$$\phi_k(\bar{x}_k, v_k) \leq \phi_k(x_k, v_k) + \sum_{i=0}^{k-1} c(k, i) \phi_i(x_i, v_i) \quad (3.50)$$

concluding this proof by induction. $\square$

Duality filter, enumeration filter, explicit maximization filter, Pólya filter, and elimination filter are five main approaches that devise robust counterparts of $\mathbb{C}_k$ (LÖFBERG, 2012), such that (3.38) is satisfied for an optimization based on the deterministic model. Pólya and elimination filters are relaxations of the constraints and are not sufficient conditions to satisfy (3.43). Enumeration and the duality filters cause the number of variables to grow significantly. Therefore, this study has focused on the explicit maximization filter.

Definition 9. Given an uncertain variable $w \in \{w \mid w \in \mathfrak{R}^n, \|w\|_p \leq 1\}$ and an inequality constraint

$$(A^T w + b)^T x + (c^T w + d) \leq 0 \quad (3.51)$$

The robust constraint counterpart by explicit maximization is given by

$$b^T x + d + \|Ax + c\|_{p^*} \leq 0 \quad (3.52)$$

where $\|z\|_{p^*}$ is the dual norm (BOYD; VANDENBERGHE, 2004) of $\|Ax + c\|_p$ and satisfies $1/p + 1/p^* = 1$.

Theorem 7. Let $\bar{\mathbb{C}}_k$ be the robust counterpart of $\mathbb{C}_k$ by explicit maximization and, based on Lemma 11, $\bar{\phi}_k(\mathbf{x}, \mathbf{v}) = \phi_k(x_k, v_k) + \sum_{i=0}^{k-1} c(k, i) \phi_i(x_i, v_i)$. Then,

$$\bar{\mathbb{C}}_k = \{[x_k^T, v_k^T]^T \mid x_k \in \mathfrak{R}^n, v_k \in \mathfrak{R}^m, \forall i \in [1, q], \tilde{H}_{x,k}^{(i)} x_k + H_{u,k}^{(i)} v_k + c_k^{(i)} + \Phi_{k,i}(\mathbf{x}, \mathbf{v}) \leq 0\} \quad (3.53)$$

where $\Phi_{k,i}(\mathbf{x}, \mathbf{v}) = \sum_{j=0}^{k-1} \|\tilde{H}_{x,k}^{(i)} \tilde{A}^{k-j-1} B^w\|_2 \bar{\phi}_j(\mathbf{x}, \mathbf{v})$ and $\tilde{H}_{x,k}^{(i)}$, $H_{u,k}^{(i)}$ and $c_k^{(i)}$ are the i -th row of matrices $\tilde{H}_{x,k}$, $H_{u,k}$ and vector c_k , respectively.

Proof. Consider Definition 9 and Lemma 11. The explicitly maximized counterpart of $\tilde{H}_{x,k}^{(i)} \tilde{A}^{k-i-1} B^w w_i$ is

$$\max_{\|w_i\|_2 \leq \bar{\phi}_k(\mathbf{x}, \mathbf{v})} \tilde{H}_{x,k}^{(i)} \tilde{A}^{k-i-1} B^w w_i = \|\tilde{H}_{x,k}^{(i)} \tilde{A}^{k-i-1} B^w\|_2 \bar{\phi}_k(\mathbf{x}, \mathbf{v}) \quad (3.54)$$

Therefore, the explicit maximization of (3.43) results

$$\tilde{H}_{x,k}^{(i)} x_k + H_{u,k}^{(i)} v_k + c_k^{(i)} + \sum_{j=0}^{k-1} \|\tilde{H}_{x,k}^{(i)} \tilde{A}^{k-j-1} B^w\|_2 \bar{\phi}_j(\mathbf{x}, \mathbf{v}) \leq 0 \quad (3.55)$$

which is a sufficient condition for the robust feasibility of (3.43). $\square$

The results from Theorems 6 and 7 and Lemma 9 provide the basis of the proposed method.

Corollary 3. Consider the results from Theorem 7 and Lemma 9. The guaranteed cost model predictive controller is given by

$$\begin{aligned} J_i^*(\bar{x}_i) &< \inf_{\mathbf{v}} \sum_{k=i}^{N-1} v_k \bar{R}_k v_k + x_i^T S_i x_i \\ \text{s.t. } x_{k+1} &= (A - B^u K_k) x_k + B^u v_k \\ [x_k^T, v_k^T]^T &\in \bar{\mathcal{C}}_k \end{aligned} \quad (3.56)$$

Corollary 3 defines the Guaranteed Cost Model Predictive Control, which ensures an upper bound to the linearly constrained l_2 -norm optimal control problem subject to parametric uncertainties.

3.4 Numerical example

This section provides a numerical example of OL-GCMPC and its comparison with a Enumeration-based Robust Model Predictive Control (ERMPC) approach², adapted from (SCOKAERT; MAYNE, 1998). We have used the YALMIP Toolbox (LÖFBERG, 2004) and the Mosek solver (APS, 2015) for system modeling and controller synthesis of the problem¹.

Consider the system from Definition 5 with matrices

$$\begin{aligned} A &= \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 0 & 1.2 \\ -1 & 1 & 0 \end{bmatrix} & B^u &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} \\ B^w &= \begin{bmatrix} 0.7 \\ 0.5 \\ -0.7 \end{bmatrix} & C_y &= \begin{bmatrix} 0.4 \\ 0.5 \\ -0.6 \end{bmatrix}^T & D_y^u &= \begin{bmatrix} 0.4 \\ -0.4 \end{bmatrix}^T \end{aligned} \quad (3.57)$$

subject to constraints given by

$$H_{x,k} \equiv \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad H_{u,k} \equiv 0 \quad g_k \equiv -1 \quad (3.58)$$

and cost matrices $Q = I, R = I$.

¹ The source code for the numerical example with both OL-GCMPC and ERMPC are available at: <https://github.com/cmasserat/gcmprc>

² The approximation $\|\Delta\|_\infty \leq \|\Delta\|_2 \leq 1$ was used to transform the disturbance set into a polytope and enable enumeration to be performed.

Figure 3 – Optiaml cost for GCC synthesis for varying ϵ values

An infinite horizon positive definite solution S exists to the GCC problem for $\epsilon_k \equiv \epsilon \in (0, 0.0220)$ and $\epsilon = 0.0180$ provides the optimal stabilizing solution with optimal cost 53.7109 as shown in Figure 3. Such result is given by

$$S_k \equiv S = \begin{bmatrix} 31.4751 & -0.9359 & -20.6124 \\ -0.9359 & 5.7340 & -1.3900 \\ -20.6124 & -1.3900 & 16.5017 \end{bmatrix} \quad (3.59)$$

with associated controller

$$K_k \equiv K = \begin{bmatrix} 1.1801 & 0.2151 & -0.5076 \\ 0.7401 & -0.8385 & 0.5162 \end{bmatrix} \quad (3.60)$$

whereas the OL-GCMPC cost matrix $\bar{R}_k$ is

$$\bar{R}_k \equiv \bar{R} = \begin{bmatrix} 123.22 & 133.78 \\ 133.78 & 197.26 \end{bmatrix} \quad (3.61)$$

and $\tilde{K}$, a second order nil-potent controller, is

$$\tilde{K} = \begin{bmatrix} 11.5 & -6 & -6 \\ 1.1 & 0 & 0 \end{bmatrix} \quad (3.62)$$

The resulting optimization problems required a minimum of 6.5ms and a maximum of 11.1ms for the OL-GCMPC and a minimum of 771.03ms and a maximum of 1684.3ms for the ERMPC on a dual 2.6Ghz Intel Xeon E5-2670 with 32Gb of RAM. Figure 4 shows the computational time distributions, where it is possible to see that the OL-GCMPC is on average two orders of magnitude faster than the ERMPC.

Figure 5 shows the simulation results of both controllers for an initial state $x_0 = [1, 1, 1]^T$. The controlled system was subject to a uniformly distributed disturbance $\Delta_k \in [-1, 1]$, shown in

Figure 4 – Computational time box plot of both OL-GCMPC and ERMPC for uniformly distributed states with $\|x\|_\infty \leq 0.5$

Figure 5a). The OL-GCMPC successfully stabilized the system and maintained the state within the feasible domain with an performance equivalent to optimal ERMPC, as shown in Figure 5b).

Figure 5 – Simulation results for OL-GCMPC and ERMPC controllers. (a) Simulated disturbance norm $\|\Delta\|_2$; (b) Disturbed closed-loop state x_k ; (c) Executed control inputs u_k .

3.5 Conclusion

In this chapter, we have proposed a Open-Loop Guaranteed Cost Robust Model Predictive Controller for constrained uncertain discrete-time systems. Such controller provides an upper bound to the quadratic cost functional and ensures feasibility of states and control inputs within a prediction horizon for a system subject to a quadratically bound multiplicative uncertainty. This problem class was only tractable through vertex enumeration of a polytopic approximation of the uncertainty set.

The proposed approach is computationally more efficient than enumeration-based techniques and executed on average two orders of magnitude faster for this chapter's numerical example. It also requires only a SOCP optimization, which results in lower-complexity solutions than those of LMI-based guaranteed cost approaches based on SDP optimizations.

TUBE-BASED GUARANTEED COST MODEL PREDICTIVE CONTROL OF UNCERTAIN LINEAR SYSTEMS

The OL-GCMPC proposed in Chapter 3 does not consider any Robust Controllable Invariant (RCI) sets as terminal constraints and considered the uncertainties propagated based on a nil-potent feedback model. Therefore, it does not provide strong recursive feasibility guarantees, and it yields controllers too conservative for practical applications.

In this chapter, we propose the Tube-based Guaranteed Cost Model Predictive Control (T-GCMPC), from the considerations of such limitations of the OL-GCMPC formulation. Such formulation applies the concept of homothetic tubes to yield a less conservative controller. Additionally, it generalizes the considered class of system to reduce the controller conservativeness by the use of uncertainty structures, based on the properties from Chapter 2.

4.1 Introduction

Model Predictive Control (MPC) is a class of optimization-based control algorithms that use an explicit model of the controlled system to predict its future states (BADGWELL; QIN, 2015). Several different fields have applications which use this technique, such as refineries, food processing plants, mining, aerospace, and automotive control (QIN; BADGWELL, 2003). A MPC minimizes a cost function while maintaining the system states and control inputs within a feasible set. The system dynamics are usually assumed to be known, and it does not consider model mismatches or external disturbances. However, the disregard for such uncertainties may lead to degraded closed-loop performance and violation of state and control input constraints (RAWLINGS; MEADOWS; MUSKE, 1994).

Robust MPC (RMPC) approaches were proposed to address MPC's poor closed-loop

performance subject to uncertainties (BEMPORAD; MORARI, 1999). Its objective is to obtain a control input sequence that simultaneously minimizes a cost function and guarantees the feasibility of system states and control inputs, for a system subject to the worst-case disturbance within an uncertainty set. Most RMPCs have different assumptions and approximations; therefore comparisons among different approaches are difficult. However, we can classify most of them into two categories (KERRIGAN; MACIEJOWSKI, 2004), namely open-loop and closed-loop RMPC.

Initial studies focused on open-loop min-max optimization-based RMPC (OL-RMPC) approaches (for example, (BLANCHINI, 1990), (KOTHARE; BALAKRISHNAN; MORARI, 1996), (LEE; YU, 1997)). These approaches were conservative for actual applications since future control inputs within the prediction horizon did not perform any feedback on prior disturbances or states within the predicted horizon, as would happen on a standard state-feedback controller.

Closed-loop RMPC (CL-RMPC) approaches (KERRIGAN; MACIEJOWSKI, 2003) were proposed to address the conservativeness of OL-RMPC policies. CL-RMPC consider different control policies based on past disturbances within the prediction horizon. Scokaert and Mayne (SCOKAERT; MAYNE, 1998) proposed a Dynamic Programming-based CL-RMPC which provides an exact solution for bounded uncertainties; however, this approach yields an infinite dimensional optimization in the general case and exponential dimensional for polytopic bounded uncertainties, resulting in intractable or prohibitively large optimization problems. Meanwhile, Löfberg (LOFBERG, 2003) proposed an approximate solution to CL-RMPCs with l_1 -norm and l_∞ -norm cost functions and polytopic-bounded additive disturbances. Löfberg (LÖFBERG, 2002) also investigated approximate CL-RMPC synthesis for CL-RMPCs with l_2 -norm cost function and l_2 -norm bounded multiplicative disturbances; however, the proposed approach for such system relies on online solution of Semi-Definite Programming (SDP) problems, which increases overall computational requirements in comparison to LPs, QPs and SOCPs; and limits practical application of these methods.

Tube Model Predictive Control (TMPC) ((MAYNE; SERON; RAKOVIĆ, 2005), (RAKOVIĆ *et al.*, 2012b), (RAKOVIĆ *et al.*, 2012a), (RAKOVIĆ; LEVINE; AÇIKMEŞE, 2016)) is a subclass of CL-RMPC which tries to ensure computational tractability while obtaining reasonably tight approximations of the exact solution. There are four categories of TMPC policies: Rigid, Homothetic, Parameterized and Elastic. Rigid TMPC (RTMPC) (MAYNE; SERON; RAKOVIĆ, 2005) is the first proposed type of TMPC, where the tube is a translation of an off-line computed Robust Controllable Invariant (RCI) set with a fixed cross-section. Homothetic TMPC (HTMPC) (RAKOVIĆ *et al.*, 2012b) is a direct extension of RTMPC, which parametrizes the tube by its translation and the scaling of its cross-section. Parameterized TMPC (PTMPC) (RAKOVIĆ *et al.*, 2012a) generalizes the concept of TMPC through superposition and separability of additive disturbances allowing for a significantly less conservative approximation of exact CL-RMPC at the expense of increased computational cost (PTMPC scales quadratically with prediction horizon,

while RTMPC and HTMPC scale linearly). Meanwhile, Elastic TMPC (ETMPC) (RAKOVIĆ; LEVINE; AÇIKMEŞE, 2016) are a generalization of HTMPC to reduce its conservativeness when compared to PTMPC while maintaining linear complexity scaling. The tubes in ETMPC are parametrized based on its translation and an arbitrary scaling for each of its facets.

According to Mayne *et al.* (MAYNE; KERRIGAN; FALUGI, 2011), one of the main limitations of TMPC is the lack of robustness to multiplicative disturbances. Evans *et al.* (EVANS; CANNON; KOUVARITAKIS, 2012) proposed an extended TMPC for the case of stochastic multiplicative uncertainties. Meanwhile, Flemings *et al.* (FLEMING; KOUVARITAKIS; CANNON, 2015) proposed a RTMPC for polytopic-bounded multiplicative uncertainties. However, both of these methods only consider disturbances in the state transition matrix. Raković and Cheng (RAKOVIĆ; CHENG, 2013) proposed an HTMPC approach to deal with polytopic-bounded multiplicative uncertainties. A more general class of systems subject to uncertainties is considered which encompasses state transition and control input matrices, combined with additive disturbances. Nevertheless, their proposed method is not able to completely isolate the uncertain error dynamics from the nominal system dynamics through the homothetic tube formulation. Furthermore, most TMPC methods require a particular design of a cost function specific to its formulation; therefore it does not provide robustness guarantees to a $\mathcal{H}_2$ performance metric for the overall system.

Xie and Soh (XIE; SOH, 1993) designed the Guaranteed Cost Control (GCC) to address the poor performance of the Linear Quadratic Regulator (LQR) when a linear system model is subject to l_2 -norm bounded parametric uncertainties. It provides a quadratically stable closed-form controller that guarantees an upper bound to l_2 -norm cost functionals. However, such an approach was limited to unconstrained synthesis. Posteriorly, Rossi *et al.* (ROSSI; WASCHBURGER; GALVAO, 2013) and Rosinová (ROSINOVÁ; VESELÝ; KUČERA, 2003) extended such approach for constrained RMPCs problems. However, these also require the online solution of SDP problems. To address such limitation imposed by the online solution of SDP problems, we proposed the Open-Loop Guaranteed Cost Model Predictive Controller (OL-GCMPC) in Chapter 3. This method yields an SOCP formulation; however, it does not consider uncertainties within the prediction horizon, which increases the level of conservativeness of the solution.

In this chapter, we propose a Tube-based Guaranteed Cost Model Predictive Controller (T-GCMPC) through a Homothetic formulation. The proposed controller extends the application of TMPCs to constrained linear systems subject to multiplicative structured norm-bounded uncertainties. The proposed approach differs from similar Homothetic TMPC formulations in two aspects:

1. It provides a cost function associated with the non-robust synthesis counterpart; thus, reducing to the nominal MPC case when disturbances are not present;

2. It internalizes all uncertainties into the homothetic tube formulation, providing guarantees that its cross-section volume tends to zero.

The first property is provided based on the GCC synthesis. While to support the latter property we also propose an approximate minimal Robust Controllable Invariant (mRCI) level-set for this class of systems. Finally, we present a numerical example where we compare the proposed T-GCMPC with the previously proposed OL-GCMPC and the HTMPC for polytopic-bounded multiplicative uncertainties of (RAKOVIĆ; CHENG, 2013).

This chapter is organized as follows: Section 4.2 presents the preliminary results; Section 4.3 formulates the mRCI set; Section 4.4 derives the Tube-based Guaranteed Cost Model Predictive Controller; Section 4.5 provides a numerical example; finally, Section 4.6 addresses the final remarks.

4.2 Preliminary results

We consider discrete-time linear system subject to parametric uncertainties of the form

$$x_{k+1} = (A + \delta A(\Delta_k))x_k + (B^u + \delta B^u(\Delta_k))u_k \quad (4.1)$$

where $x_k \in \mathfrak{R}^{n_x}$ is the system state, $u_k \in \mathfrak{R}^{n_u}$ is the control input, $A \in \mathfrak{R}^{n_x \times n_x}$ is the state matrix, $B^u \in \mathfrak{R}^{n_x \times n_u}$ is the input matrix, and $\delta A(\Delta_k)$, $\delta B^u(\Delta_k)$ are, respectively, the state and input multiplicative uncertainty matrices, such that

$$\begin{bmatrix} \delta A(\Delta_k) & \delta B^u(\Delta_k) \end{bmatrix} = B^w \Delta_k \begin{bmatrix} C_y & D_y^u \end{bmatrix}, \quad (4.2)$$

where $B^w \in \mathfrak{R}^{n_x \times n_p}$, $C_y \in \mathfrak{R}^{n_q \times n_x}$, $D_y^u \in \mathfrak{R}^{n_q \times n_u}$, and $\Delta_k \in \mathbb{D}$ where

$$\mathbb{D} = \{\Delta \mid \forall i \in [1, s] : \Delta_i \in \mathfrak{R}^{n_{pi} \times n_{qi}}, \|\Delta_i\|_2 \leq 1, \Delta = \text{diag}(\Delta_1, \Delta_2, \dots, \Delta_s)\}. \quad (4.3)$$

In order to evaluate the predictive approach we are dealing with in this paper, we rewrite System (4.1) in the following equivalent form

$$\begin{aligned} x_{k+1} &= Ax_k + B^u u_k + B^w w_k \\ y_k &= C_y x_k + D_y^u u_k \\ w_k &= \Delta_k y_k. \end{aligned} \quad (4.4)$$

where parametric uncertainties of (4.1) are considered as disturbances in a feedback system formulation.

4.2.1 Optimal min-max robust model predictive control

Rigid, Homothetic and Elastic TMPC formulations have taken into account cost functions based on nominal states, nominal control inputs and the tube relaxation variables. In this chapter,

we consider a cost function defined on states subject to disturbances. This results in a more general formulation which reduces to the nominal MPC problem when we consider $B_w = 0$ and/or $C_y = D_y^u = 0$. Therefore, the cost function is defined as

$$J_i(x_i, \mathbf{u}, N) = \sum_{k=i}^{N-1} x_k^T Q x_k + u_k^T R u_k + 2x_k^T N u_k \quad (4.5)$$

where $P_N \succeq 0$, $Q \succeq 0$, and $R \succ 0$ are symmetric weighting matrices, and $\mathbf{u} = \{u_k \mid k \in [0, N-1]\}$.

In the scope of this chapter, we are interested in investigating the T-GCMPC synthesis under the following assumptions:

Assumption 7. For all $\Delta_k \in \mathbb{D}$, the pair $(A + \delta A(\Delta_k), B + \delta B(\Delta_k))$ is stabilizable.

Assumption 8. For all $\Delta_k \in \mathbb{D}$, the pair $(A + \delta A(\Delta_k), Q^{\frac{1}{2}})$ is observable.

We are now ready to define the general infinite horizon Min-Max Robust Model Predictive Controller.

Definition 10. Consider the linear system model from (4.1). Then, the infinite horizon Min-Max Robust Model Predictive Control is given by

$$\begin{aligned} J^*(x_0) = & \inf_{\mathbf{u}} \sup_{\Delta} \lim_{N \rightarrow \infty} J_0(x_0, \mathbf{u}, N) \\ \text{s.t.} \quad & x_{k+1} = A x_k + B^u u_k + B^w w_k \\ & w_k = \Delta_k (C_y x_k + D_y^u u_k) \\ & [x_k^T, u_k^T]^T \in \mathbb{C} \end{aligned} \quad (4.6)$$

where $\mathbb{C} = \{[x^T, u^T]^T \mid H_x x + H_u u \leq g\}$ defines the feasible set of states and control inputs.

The formulation of Definition 10 provides an exact solution to the optimal robust controller synthesis. However, such formulation is intractable due to its infinite dimension and due to the non-convexity of the optimization. Therefore, conservative approximations can be performed to address these two issues.

4.2.2 Guaranteed cost control

The first approximation is formulating a limit horizon problem that provides sufficient conditions to Definition 1. Towards that purpose, we define the Guaranteed Cost Control approach proposed by (XIE; SOH, 1993).

Definition 11. A state feedback controller $u_k = -K x_k$ is said to be a stabilizing Guaranteed Cost Controller for the uncertain system (4.1) if there exists a symmetric matrix $P \in \mathfrak{R}^{n_x \times n_x}$, $P \succ 0$ that satisfies

$$A_{cl}(\Delta)^T P A_{cl}(\Delta) - P + Q + N K + K^T N^T + K^T R K \preceq 0, \quad (4.7)$$

such that $J_i(x_0, \mathbf{u}) \leq \bar{J}_u^*(x_0) = x_0^T P x_0$ for the closed loop system

$$A_{cl} = A + \delta A(\Delta_k) - (B^u + \delta B^u(\Delta_k)) K \quad (4.8)$$

subject to all admissible uncertainties $\Delta_k \in \mathbb{D}$.

Definition 12. A guaranteed cost feedback controller $u_k = -Kx_k$, satisfying Definition 11, with associated cost matrix P is said to be optimal if $\text{tr}(P)$ is infimal.

Lemma 12. A state feedback controller $u_k = -Kx_k$ is said to be of guaranteed cost, according to Definition 11, if and only if there exist $X \succ 0$ and $\Upsilon_i \succeq 0$ for $i \in [1, s]$ such that

$$\begin{bmatrix} -\Upsilon_q & 0 & 0 & C_y X - D_y^u Y \\ \star & -I_{n_c} & 0 & C_c X - D_c^u Y \\ \star & \star & -X + B^w \Upsilon_p B^{wT} & A X - B^u Y \\ \star & \star & \star & -X \end{bmatrix} \preceq 0, \quad (4.9)$$

where $X = P^{-1}$, $Y = K P^{-1}$, the cost matrices $C_c \in \mathfrak{R}^{n_c \times n_x}$ and $D_c^u \in \mathfrak{R}^{n_c \times n_u}$ are given by the factorization of the cost function matrices

$$\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} = \begin{bmatrix} C_c & D_c^u \end{bmatrix}^T \begin{bmatrix} C_c & D_c^u \end{bmatrix}, \quad (4.10)$$

and the generalized S-Procedure variables are given by

$$\begin{aligned} \Upsilon_p &= \text{diag}(v_1 I_{n_{p1}}, v_2 I_{n_{p2}}, \dots, v_s I_{n_{ps}}), \\ \Upsilon_q &= \text{diag}(v_1 I_{n_{q1}}, v_2 I_{n_{q2}}, \dots, v_s I_{n_{qs}}), \end{aligned} \quad (4.11)$$

Proof. Refer to Lemma 7 in Chapter 2. □

Theorem 8. A state feedback controller $u_k = -Kx_k$ is said to be an optimal Guaranteed Cost Controller, according to Definition 12, if and only if

$$\begin{aligned} K &= \arg \min_K \text{tr}(Z) \\ s.t. \quad & \begin{bmatrix} -Z & I_{n_x} \\ \star & -X \end{bmatrix} \preceq 0 \\ & \text{Condition (4.9)} \end{aligned} \quad (4.12)$$

where $Z \in \mathfrak{R}^{n_x \times n_x}$ is the relaxation of the cost matrix P .

Proof. Refer to Theorem 4 and Remark 4 in Chapter 2. □

4.2.3 Robust controllable invariant sets

We now present the Robust Controllable Invariant (RCI) Set definition.

Definition 13. The set $\mathbb{R} \subseteq \text{Proj}_x(\mathbb{C})$ is said to be Robustly Controllable Invariant (RCI) for an uncertain linear system governed by $x_{k+1} = (A + \delta A(\Delta))x_k + (B^u + \delta B^u(\Delta))u_k$ if and only if

$$\forall x \in \mathbb{R}, \forall \Delta \in \mathbb{D}, [x^T, u^T]^T \in \mathbb{C} : (A + \delta A(\Delta))x + (B^u + \delta B^u(\Delta))u \in \mathbb{R}, \quad (4.13)$$

or equivalently,

$$\forall \Delta \in \mathbb{D} : \begin{bmatrix} A + \delta A(\Delta) & B + \delta B^u(\Delta) \end{bmatrix} \left(\begin{bmatrix} I \\ 0 \end{bmatrix} \mathbb{R} \cap \mathbb{C} \right) \subseteq \mathbb{R}. \quad (4.14)$$

For diagonal disturbances, we can enumerate the vertexes of the disturbance set. Therefore, the disturbance model in the System (4.1) can be represented as a polytopic uncertain set, where matrices A and B belong to a polytopic matrix set. In such a case, methods based on infinite reachable sets or one-step sets (KERRIGAN, 2001) can be applied to some systems although they do not provide convergence guarantees. Alternatively, approximate methods have been proposed to guarantee finite time convergence of the RCI synthesis (see for instance, (KOURAMAS *et al.*, 2005), (PLUYMERS *et al.*, 2005)).

However, we cannot enumerate the vertex of the disturbance set for non-diagonal disturbances. Therefore, polytopic approaches are not applicable to the general case. Ellipsoidal RPI sets were widely studied for such cases (e.g. (KOLMANOVSKY; GILBERT, 1998), (BLANCHINI, 1999), and (ALAMO; CEPEDA; LIMON, 2005)); however, these sets hold computational disadvantages compared to polytopic representations (BLANCO; CANNON; MOOR, 2010).

In this chapter, RCI sets will be employed to guarantee the infinite time constraint for a finite horizon controller and the separability of the uncertain dynamics, similar to (RAKOVIĆ *et al.*, 2012b). Towards this purpose, the next definition presents the minimum RCI.

Definition 14. Let Γ be the set of sets that satisfy Definition 13. Then, the set $\mathbb{R}^{min} \subseteq \mathbb{C}$ is said to be minimal Robustly Controllable Invariant (mRCI) if and only if $\mathbb{R}^{min} \in \Gamma$ and $\forall \mathbb{R} \in \Gamma : \mathbb{R}^{min} \subseteq \mathbb{R}$.

4.2.4 Principle of separable control policies

The principle of separable control policies has been employed in all tube-based model predictive controls, from rigid TMPC (LANGSON *et al.*, 2004), to parametrized TMPC (RAKOVIĆ *et al.*, 2012a). It is one of the main contributions that enable such methods to be less conservative.

The separated dynamics considered in this chapter are defined as

$$\begin{aligned} z_{k+1} &= Az_k + B^u v_k \\ e_{k+1} &= Ae_k + B^u \rho_k + B^w w_k \\ y_k &= C_y(z_k + e_k) + D_u^u(v_k + \rho_k) \end{aligned} \quad (4.15)$$

where z_k defines the nominal dynamics and e_k defines the error dynamics. Eqs. (4.15) are equivalent to the System (4.1) under $x_k = z_k + e_k$ and $u_k = v_k + \rho_k$.

4.3 Minimum robust controllable invariant level-set

In this section, we propose a novel mRPI level-set for linear systems subject to multiplicative uncertainties. Such level-set will be used to relax the error dynamics for the T-GCMPC formulation.

We define the following level-sets:

$$\mathbb{Y}(\sigma) = \{x \mid \forall i \in [1, s] : x^T \bar{C}_{y,i}^T \bar{C}_{y,i} x \leq \sigma_i\}, \quad (4.16)$$

$$\mathbb{W}(\sigma) = \{w \mid \forall i \in [1, s] : w_i^T w_i \leq \sigma_i\}, \quad (4.17)$$

where $\bar{C}_y = C_y - D_y^u K$. Then, from Lemma 5 in Chapter 2 we obtain the following property for the sets $\mathbb{Y}$ and $\mathbb{W}$:

$$\forall \sigma \geq 0 : x_k \in \mathbb{Y}(\sigma) \Rightarrow w_k \in \mathbb{W}(\sigma), \quad (4.18)$$

as a sufficient condition to

$$\forall i \in [1, s] : w_i^T w_i \leq x^T \bar{C}_{y,i}^T \bar{C}_{y,i} x. \quad (4.19)$$

The σ -level-sets (4.16) and (4.17) can be interpreted as an tight conservative approximations of the multiplicative uncertainties (4.19) by an additive disturbance model (4.17), valid within a particular set of the state space (4.16).

We are now ready to formalize the existence conditions of the RPI level set.

Lemma 13. Consider the System (4.1) subject to a state feedback controller $u_k = -Kx_k$. Then, $\mathbb{R}(\sigma)$ is a Robust Controllable Invariant level set if the following properties hold:

- (i) $\forall \sigma \geq 0 : \mathbb{R}(\sigma) \subseteq \mathbb{Y}(\sigma)$;
- (ii) $\forall \sigma \geq 0 : \bar{A}\mathbb{R}(\sigma) \oplus B^w \mathbb{W}(\sigma) \subseteq \mathbb{R}(\sigma)$ where $\oplus$ is the Minkowski sum of sets and $\bar{A} = A - B^u K$.

Proof. From Definition 13, $\mathbb{R}(\sigma)$ is a RPI if and only if

$$\forall \sigma \geq 0, \forall \Delta \in \mathbb{D} : (\bar{A} - B^w \Delta \bar{C}_y) \mathbb{R}(\sigma) \subset \mathbb{R}(\sigma). \quad (4.20)$$

Assume (i) and (ii) hold. Then, (4.19) can be expressed without loss of generality as a function of the sets (4.16) and (4.17) as

$$\forall \lambda \geq 0, \forall \Delta \in \mathbb{D} : \Delta \bar{C}_y \mathbb{Y}(\sigma) \subseteq \mathbb{W}(\sigma). \quad (4.21)$$

By consequence,

$$\forall \sigma \geq 0, \forall \Delta \in \mathbb{D} : \Delta \bar{C}_y \mathbb{R}(\sigma) \subseteq \mathbb{W}(\sigma), \quad (4.22)$$

and

$$\forall \lambda \geq 0, \forall \Delta \in \mathbb{D} : (\bar{A} - B^w \Delta \bar{C}_y) \mathbb{R}(\sigma) \subseteq \bar{A} \mathbb{R}(\sigma) \oplus B^w \mathbb{W}(\sigma) \subseteq \mathbb{R}(\sigma). \quad (4.23)$$

Therefore, (ii) is a sufficient condition to ensure $\mathbb{R}(\sigma)$ is a RPI level set.

To prove necessity, we first assume (i) does not hold. By consequence, (4.22) does not hold and invalidates (4.23). Therefore, (ii) does not hold if (i) does not hold.

From the losslessness property of Lemma 6 in Chapter 2, its inequality is a tight relaxation for any admissible values of Λ_i . Therefore,

$$\mathbb{W}(\sigma) = \bigcup_{\Delta \in \mathbb{D}} \Delta \bar{C}_y \mathbb{Y}(\sigma) \quad (4.24)$$

and, consequently, (4.23) is also a tight relaxation.

Therefore, if $\mathbb{R}(\sigma)$ does not satisfy (4.23), it also does not satisfy (4.20). Therefore, (ii) is a necessary condition to ensure $\mathbb{R}(\sigma)$ is a RPI level set and (i) is a necessary condition of (ii). $\square$

Remark 5. The mRPI level set formulation is a direct result of Lemma 13 and Definition 14.

Since $\mathbb{Y}(\sigma)$ and $\mathbb{W}(\sigma)$ have ellipsoidal structures, we have chosen to investigate the construction of ellipsoidal mRPI level sets. Therefore, we consider $\mathbb{R}(\sigma) = \{x \mid x^T E_R x \leq \sigma\}$.

Theorems 9 and 10 present the exact mRPI and the approximate mRPI formulations, respectively. The exact mRPI enables the direct optimization of the set volume; however, it assumes a predefined control gain to ensure convexity of the optimization, significantly limiting its search space. Meanwhile, the approximate mRPI enables the combined search of the set parametric shape and the control gain variables; although such formulation cannot directly optimize the set volume and must rely on $\text{tr}(E_R)$ as its proxy.

Theorem 9. $\mathbb{R}^{\min}(\sigma)$ is an ellipsoidal minimal Robust Controllable Invariant level set for the System (4.1) subject to a given feedback controller $u_k = -Kx_k$ if $E_R = \Theta(a_\alpha, K)$ for some

$a_\alpha \in [0, 1]$, where $\Theta(a_\alpha, K)$ is a SDP problem given by

$$\Theta(a_\alpha, K) = \arg \inf_Z -\log \det(Z) \quad (4.25)$$

$$s.t. \begin{bmatrix} -Z & Z\bar{A} & ZB^w \\ \star & -Za_\alpha & 0 \\ \star & \star & -\Lambda \end{bmatrix} \preceq 0 \quad (4.26)$$

$$\bar{C}_{y,i}^T \bar{C}_{y,i} - Z \preceq 0 \quad (4.27)$$

$$a_\alpha + \sum_{i=1}^s a_{\sigma_i} \leq 1 \quad (4.28)$$

where $\bar{A} = A - B^u K$, $\bar{C}_y = C_y - D_y^u K$, and

$$\Lambda = \text{diag}(a_{\sigma_1} I_{n_{p1}}, a_{\sigma_2} I_{n_{p2}}, \dots, a_{\sigma_s} I_{n_{ps}}). \quad (4.29)$$

Proof. Consider $\mathbb{R}(\sigma)$ and $\mathbb{W}(\sigma)$. Then,

$$\xi \in \mathbb{L} = \left\{ \begin{bmatrix} x^T & w^T \end{bmatrix}^T \mid x^T E_R x \leq \sigma, \forall i \in [1, s], w_i^T w_i \leq \sigma \right\}. \quad (4.30)$$

We intend to obtain an mRPI level-set that satisfies

$$\bar{A}\mathbb{R}(\sigma) \oplus \mathbb{W}(\sigma) \subseteq \mathbb{R}(\sigma) \quad (4.31)$$

and,

$$\mathbb{R}(\sigma) \subseteq \mathbb{Y}(\sigma). \quad (4.32)$$

Which are equivalent to

$$\forall \xi \in \mathbb{L} : \xi^T \begin{bmatrix} \bar{A}^T E_R \bar{A} & \bar{A}^T E_R B^w \\ B_{wT} E_R \bar{A} & B_{wT} E_R B^w \end{bmatrix} \xi \leq \sigma \quad (4.33)$$

and,

$$\forall i \in [1, s] : E_R \succeq \bar{C}_{y,i}^T \bar{C}_{y,i}, \quad (4.34)$$

respectively. From the generalized S-Procedure (IWASAKI; MEINSMA; FU, 2000), the following inequalities are a sufficient condition to (4.33):

$$\begin{bmatrix} \bar{A}^T E_R \bar{A} - a_\alpha E_R & \bar{A}^T E_R B^w \\ \star & B_{wT} E_R B^w - \Lambda \end{bmatrix} \preceq 0, \quad (4.35)$$

$$a_\alpha + \sum_{i=1}^s a_{\sigma_i} \leq 1. \quad (4.36)$$

We consider (4.35). Applying Schur's complement to the positive terms related to X , yields

$$\begin{bmatrix} E_R^{-1} & \bar{A} & B^w \\ \star & -a_\alpha E_R & 0 \\ \star & \star & -\Lambda \end{bmatrix} \preceq 0, \quad (4.37)$$

which, under the symmetric transform $T = \text{diag}(E_R, I, I)$, is equivalent to

$$\begin{bmatrix} E_R & E_R \bar{A} & E_R B^w \\ \star & -a_\alpha E_R & 0 \\ \star & \star & -\Lambda \end{bmatrix} \preceq 0, \quad (4.38)$$

which is equivalent to (4.26). Meanwhile, (4.36) and (4.34) are equivalent to (4.28) and (4.27), respectively.

Based on these results, we conclude that the conditions from optimization $\Theta(a_\alpha, K)$ are sufficient for the synthesis of an mRPI level-set.

The mRPI synthesis problem is equivalent to the minimum volume RPI synthesis. The volume of $\mathbb{R}(\sigma)$ is proportional to $\log \det(X)$, which is equivalent to $-\log \det(Z)$. Therefore, we choose to use it as the function to be minimized. $\square$

Theorem 10. $\mathbb{R}^{\min}(\sigma)$ is an ellipsoidal approximately minimal Robust Controllable Invariant level set for the System (4.1) with associated control law $u_k = -Kx_k$ if $E_R = X^{*-1}$ and $K = Y^*X^{*-1}$, where X^* and Y^* are the optimal solutions to $\Theta_a(a_\alpha)$ for some $a_\alpha \in [0, 1]$, and $\Theta_a(a_\alpha)$ is a SDP problem given by

$$\Theta_a(a_\alpha) = \inf \text{tr}(X) \quad (4.39)$$

$$s.t. \quad \begin{bmatrix} -X & AX - B^u Y & B^w \Upsilon_p \\ \star & -X a_\alpha & 0 \\ \star & \star & -\Upsilon_p \end{bmatrix} \preceq 0 \quad (4.40)$$

$$\begin{bmatrix} -I_{n_{qi}} & C_{y,i} X - D_{y,i}^u Y \\ \star & -X \end{bmatrix} \preceq 0 \quad (4.41)$$

$$\begin{bmatrix} -v_i & 1 \\ \star & -a_{\sigma_i} \end{bmatrix} \preceq 0 \quad (4.42)$$

$$a_\alpha + \sum_{i=1}^s a_{\sigma_i} \leq 1 \quad (4.43)$$

where $Y = KX$.

Proof. We first apply the symmetric transform $T = \text{diag}(X, X, \Lambda_p^{-1})$ to (4.26), where $X = Z^{-1}$, to obtain (4.40). Similarly, based on the symmetric transforms $T = \text{diag}(I_{n_{qi}}, X)$ to (4.28) yields (4.41); finally, (4.42) is a direct result of the S-Procedure variable tight relaxation required on the transformation $\Upsilon_p = \Lambda^{-1}$.

However, $\log \det(X)$ is a concave function and cannot be minimized with global convergence guarantees. Therefore we choose to obtain an approximate mRPI level-set through the proxy minimization of $\text{tr}(X)$. $\square$

Remark 6. The mRPI synthesis formulations from both Theorems 9 and 10 yield Bilinear Matrix Inequality (BMI) conditions due to a_α , which are inherently non-convex. In this particular case, we can address this issue through an exhaustive search on the variable domain since it is a unidirection in the bounded set $[0, 1]$.

The separation provided in Lemma 13 enables Theorems 9 and 10 to provide RPI level-sets for each λ -parametrized uncertainty domain. If such an approach is not used, the resulting optimization will be the trivial minimal set $\mathbb{R}(\sigma) = \{0\}$ for a system without additive disturbances.

We now demonstrate the contractive properties over time of the proposed level set.

Lemma 14. Let $\mathbb{R}(\sigma)$ be a RPI level set satisfying Lemma 13; assume $\mathbb{R}(\sigma)$ is within the feasible region of $\Theta(a_\alpha)$ as defined in Theorems 9 or 10, $x_k \in \mathbb{R}(\alpha_k^2)$, and $x_k \in \mathbb{Y}(\sigma_k^2)$, where σ_k^2 is an element-wise operation. Then, the dynamic model of α_k is

$$\alpha_{k+1}^2 \geq a_\alpha \alpha_k^2 + \sum_{i=1}^s a_{\sigma_i} (\sigma_k)_i^2 \quad (4.44)$$

or, equivalently,

$$\alpha_{k+1} \geq \left\| \left[\sqrt{a_\alpha} \alpha_k, \sqrt{a_{\sigma_1}} (\sigma_k)_1, \dots, \sqrt{a_{\sigma_s}} (\sigma_k)_s \right] \right\|_2, \quad (4.45)$$

where a_α and a_{σ_i} are optimal variables from the SDP problem in Theorems 9 or 10.

Proof. For the time varying cross-section case, (4.31) is given by

$$\bar{A}\mathbb{R}(\alpha_k^2) \oplus B^w \mathbb{W}(\sigma_k^2) \subseteq \mathbb{R}(\alpha_{k+1}^2) \quad (4.46)$$

which is equivalent to the pair of conditions defined by (4.33) and (4.44). $\square$

Corollary 4. Consider the minimal α_k given by

$$\alpha_k^* = \min_{x_k \in \mathbb{R}(\alpha_k^2)} \alpha_k \quad (4.47)$$

and assume $\sigma_k \equiv \sigma$. Then,

$$\lim_{k \rightarrow \infty} \alpha_k^* \leq \sigma. \quad (4.48)$$

Proof. Omitted. $\square$

Corollary 5. Assume $\mathbb{R}(\alpha_k^2) \subseteq \varepsilon \mathbb{Y}(\sigma_k^2)$ for some $\varepsilon \in [0, 1)$ and consider the minimal α_k and σ_k . Then, the following properties hold:

1. $\alpha_{k+1}^* \leq \sqrt{\varepsilon} \alpha_k^*$;
2. $\lim_{k \rightarrow \infty} \alpha_k^* = 0$.

Similarly, consider the minimal σ_k given by

$$\sigma_k^* = \min_{x_k \in \mathbb{Y}(\sigma_k^2)} \sigma_k. \quad (4.49)$$

Then, $\|\sigma_{k+1}^*\|_2 \leq \sqrt{\varepsilon} \alpha_k^*$.

Proof. Omitted. □

The property presented in Lemma 14 will be generalized in the next section for the dynamics of the error set defined in the T-GCMPC formulation.

4.4 Tube guaranteed cost model predictive control

In this section, we present the formulation of the Tube-based Guaranteed Cost Model Predictive Controller. Towards that purpose, we first address the infinite dimensionality of Definition 10.

Lemma 15. Consider a controller $u_k = -Kx_k$ with an associated cost matrix P , such that K and P satisfy Lemma 12 and let $\mathbb{R}_N = \{x \mid x^T E_N x \leq 1\} \subseteq \mathbb{C}$ be an arbitrary RPI set for the System (4.1) subject to controller K . Then, the existence of a solution for

$$\begin{aligned} J_f^*(x_0) = & \inf_{\mathbf{u}} \sup_{\Delta} J_0(x_0, \mathbf{u}) + x_N^T P x_N \\ \text{s.t.} \quad & x_{k+1} = Ax_k + B^u u_k + B^w w_k \\ & w_k = \Delta_k(C_y x_k + D_y^u u_k) \\ & [x_k^T, u_k^T]^T \in \mathbb{C} \\ & x_N \in \mathbb{R}_N \end{aligned} \quad (4.50)$$

is a sufficient condition to the existence of a solution for Definition 10, and $J_f^*(x_0) \geq J^*(x_0)$.

Proof. Assume $\forall k \geq N : u_k = -Kx_k$. Then, $x_N \in \mathbb{R}_N \Rightarrow x_k \in \mathbb{R}_N \subseteq \mathbb{C}$. Therefore the feasibility of (4.50) implies the feasibility of (4.6).

From Definition 11, the cost matrix P upper-bounds the cost function of the closed loop system for all admissible disturbances Δ . Therefore, following from the previous assumption, we obtain

$$x_N^T P x_N \geq \lim_{M \rightarrow \infty} J_N(x_N, -K\mathbf{x}) \quad (4.51)$$

and $J_f^*(x_0) \geq J^*(x_0)$. □

Another important result is the transformation of the cost function to a deviation from the GCC solution.

Lemma 16. Let $\forall k < N : u_k = -Kx_k + v_k$, where K satisfies Lemma 12 with an associated cost matrix P . Then,

$$J_0(x_0, \mathbf{u}, N) + x_N^T P x_N = x_0^T P x_0 + \sum_{k=0}^{N-1} v_k^T \bar{R} v_k, \quad (4.52)$$

where

$$\bar{R} = R + D_y^{uT} \Lambda_q D_y^u + B^u (P^{-1} - B^w \Lambda_p^{-1} B_{wT})^{-1} B^u. \quad (4.53)$$

Proof. Direct generalization of Lemma 9 in Chapter 3. $\square$

We apply the principle of separable control policies with $v_k = v_k + \rho_k$ to the closed-loop counterpart of (4.1), and obtain the equivalent system

$$\begin{aligned} z_{k+1} &= \bar{A} z_k + B^u v_k \\ e_{k+1} &= \bar{A} e_k + B^u \rho_k + B^w w_k \\ y_k &= \bar{C}_y (z_k + e_k) + D_u^u (v_k + \rho_k). \end{aligned} \quad (4.54)$$

Based on System (4.54), the approximate mRPI level-set can be used to relax the error dynamics e_k conservatively.

Lemma 17. Consider the RPI level-set $\mathbb{R}^{min}(\alpha_k^2)$ satisfying Theorems 9 or 10 with associated variables α_k , σ_k , and controller $u_k = -K_R x_k$. Then, $\forall k \geq 0 : e_k \in \mathbb{R}^{min}(\alpha_k^2)$ holds for the System (4.54) if the following properties hold:

1. $\bar{A}_R \mathbb{R}^{min}(\alpha_k^2) \oplus B^w \mathbb{W}(\sigma_k^2) \subseteq \mathbb{R}^{min}(\alpha_{k+1}^2)$, where $\bar{A}_R = A - B^u K_R$;
2. $z_k \oplus \mathbb{R}^{min}(\alpha_k^2) \subseteq \mathbb{Y}(\sigma_k^2)$;
3. $\rho_k = -(K_R - K)e_k$.

Proof. Assume $\forall k \geq 0 : e_k \in \mathbb{R}^{min}(\alpha_k^2)$, then from the System (4.54)

$$\forall e_k \in \mathbb{R}^{min}(\alpha_k^2), \forall w_k \in \mathbb{W}(\sigma_k^2) : \bar{A} e_k + B^u \rho_k + B^w w_k \subseteq \mathbb{R}^{min}(\alpha_{k+1}^2), \quad (4.55)$$

where σ_k satisfies $x_k = z_k + e_k \in \mathbb{Y}(\sigma_k^2)$. Then, (4.55) is equivalent to

$$B^u \rho_k \oplus \bar{A} \mathbb{R}^{min}(\alpha_k^2) \oplus B^w \mathbb{W}(\sigma_k^2) \subseteq \mathbb{R}^{min}(\alpha_{k+1}^2). \quad (4.56)$$

From the definition of $\mathbb{R}^{min}(\alpha_k^2)$, property (i) holds if ρ_k satisfies property (iii) and property (ii) also holds. $\square$

Remark 7. Lemma 17 provides only sufficient conditions since there may exist a $\rho_k \neq -(K_r - K)e_k$ for which property (i) still holds.

Lemma 18. The System (4.54) can be conservatively approximated by

$$z_{k+1} = \bar{A}z_k + B^u v_k \quad (4.57)$$

$$\alpha_{k+1} \geq \left\| \left[\sqrt{a_\alpha} \alpha_k, \sqrt{a_{\sigma_1}} (\sigma_k)_1, \dots, \sqrt{a_{\sigma_s}} (\sigma_k)_s \right] \right\|_2 \quad (4.58)$$

$$(\sigma_k)_i \geq \|(\bar{C}_y)_i z_k + (D_y^u)_i v_k\| + \|(C_y^\alpha)_i\|_2 \alpha_k \quad (4.59)$$

where

$$(C_y^\alpha)_i = (C_y - D_y^u K_R)_i E_R^{-\frac{1}{2}}. \quad (4.60)$$

Proof. Eq. (4.57) follows directly from (4.54), and (4.58) follows directly from Lemma 14. Therefore, we focus this proof on (4.59).

Property (ii) of Lemma 17 is equivalent to

$$\forall i \in [1, n_s], \forall e_k \in \mathbb{R}(\alpha_k^2) : \begin{bmatrix} \bullet \end{bmatrix}^T \begin{bmatrix} (\bar{C}_y)_i^T (\bar{C}_y)_i & (\bar{C}_y)_i^T D_{y,i}^u \\ \star & D_{y,i}^{uT} D_{y,i}^u \end{bmatrix} \begin{bmatrix} z_k + e_k \\ v_k \end{bmatrix} \leq (\sigma_k)_i^2. \quad (4.61)$$

From (4.61), we obtain

$$\|(\bar{C}_y)_i(z_k + e_k) + D_{y,i}^u u_k\|_2 \leq \quad (4.62)$$

$$\|(\bar{C}_y)_i z_k + D_{y,i}^u v_k\|_2 + \|(\bar{C}_{Ry})_i e_k\|_2 \leq \quad (4.63)$$

$$\|(\bar{C}_y)_i z_k + D_{y,i}^u v_k\|_2 + \max_{e_k \in \mathbb{R}(\alpha_k^2)} \|(\bar{C}_{Ry})_i e_k\|_2 = \quad (4.64)$$

$$\|(\bar{C}_y)_i z_k + D_{y,i}^u v_k\|_2 + \max_{\|p_k\|_2 \leq \alpha_k} \|(C_y^\alpha)_i p_k\|_2 = \quad (4.65)$$

$$\|(\bar{C}_y)_i z_k + (D_y^u)_i v_k\| + \|(C_y^\alpha)_i\|_2 \alpha_k \leq (\sigma_k)_i, \quad (4.66)$$

where $\bar{C}_{Ry} = C_y - D_y^u K_R$. Therefore, (4.66) is a sufficient condition for (4.61). $\square$

Lemma 18 provides a conic conservative relaxation of Eq. (4.1) that does not depend on the uncertain variable Δ_k and is compatible with a SOCP optimization formulation. However, Lemma 18 also implies that $v_k \in v_k \oplus (K - K_R) \mathbb{R}^{min}(\alpha_k^2)$. Therefore, we conservatively approximate Eq. (4.52) by

$$J_0(x_0, \mathbf{u}, N) + x_N^T P x_N \leq x_0^T P x_0 + \sum_{k=0}^{N-1} \gamma_k^2, \quad (4.67)$$

where

$$\gamma_k \geq \|\bar{R}^{\frac{1}{2}} v_k\|_2 + \|\bar{R}^{\frac{1}{2}} (K_R - K) E_R^{-\frac{1}{2}}\|_2 \alpha_k. \quad (4.68)$$

The last remaining approximation required to remove the uncertainty from the problem representation is to conservatively express the controller feasible domain $[x_k^T u_k^T]^T \in \mathbb{C}$ as a function of z_k , v_k and α_k .

Lemma 19. Consider the system from Lemma 18. Then,

$$[z_k^T, v_k^T, \alpha_k] \in \bar{\mathbb{C}} = \{[x^T, v^T, \alpha^T]^T \mid \forall i \in [1, n_c] : \bar{H}_i z + (H_u)_i v + \|(\bar{H}_R)_i X_R^{\frac{1}{2}}\|_2 \alpha \leq g_i\} \quad (4.69)$$

where $\bar{H} = H_x - H_u K$, and $\bar{H}_R = H_x - H_u K_R$, implies $[x_k^T, u_k^T]^T \subseteq \mathbb{C}$.

Proof. From the definition of $\mathbb{C}$, we obtain

$$H_x x_k + H_u u_k \leq g. \quad (4.70)$$

From the separated dynamics Eqs. (4.54) and Lemma 17, (4.70) is equivalent to

$$\forall e_k \in \mathbb{R}^{min}(\alpha_k^2) : \bar{H} z_k + H_u v_k + \bar{H}_R e_k \leq g \quad (4.71)$$

Then, the robust inequality constraint counterpart for the i -th row of (4.71), is given by

$$\bar{H}_i z_k + (H_u)_i v_k + \max_{e_k^T X^{-1} x_k \leq \alpha_k} (\bar{H}_R)_i e_k \leq g \quad (4.72)$$

which is equivalent to (4.69). $\square$

We are now ready to present the proposed T-GCMPC formulation.

Theorem 11. Consider the system from Lemma 18, subject to the guaranteed cost controller K with associated cost matrix P and a mRPI level-set $\mathbb{R}^{min}(\alpha_k^2)$ with associated controller K_R . Then, the optimization $\bar{J}^*(x_0)$, given by

$$\begin{aligned} \inf_v \quad & x_0^T P x_0 + \sum_{k=0}^{N-1} \gamma_k^2 \\ s.t. \quad & z_{k+1} = (A - B^u K) z_k + B^u v_k \\ & \alpha_{k+1} \geq \left\| \left[\sqrt{a_\alpha} \alpha_k, \sqrt{a_{\sigma_1}} (\sigma_k)_1, \dots, \sqrt{a_{\sigma_s}} (\sigma_k)_s \right] \right\|_2 \\ & (\sigma_k)_i \geq \|(\bar{C}_y)_i z_k + (D_y^u)_i v_k\| + (C_y^\alpha)_i \alpha_k \\ & \gamma_k \geq \|\bar{R}^{\frac{1}{2}} v_k\|_2 + \|\bar{R}^{\frac{1}{2}} (K_R - K) E_R^{-\frac{1}{2}}\|_2 \alpha_k \\ & \bar{H}_i z_k + (H_u)_i v_k + \|(H_R)_i E_R^{-\frac{1}{2}}\|_2 \alpha_k \leq g_i \\ & \|E_N^{\frac{1}{2}} z_N\|_2 + \|E_N^{\frac{1}{2}} E_R^{-\frac{1}{2}}\|_2 \alpha_N \leq 1 \end{aligned} \quad (4.73)$$

is a conservative approximation, both in cost and feasibility, to Lemma 15 and, by consequence, Definition 10.

Proof. The optimization problem (4.73) follows directly from Lemmas 16, 17, 19, 18, and the relaxed cost (4.67). $\square$

Theorem 11 presents the T-GCMPC formulation. It yields an SOCP optimization which provides a conservative approximation to the general intractable Robust Model Predictive Control problem from Definition 10.

Remark 8. The proofs for inequalities (4.67) and for the terminal constraint of (4.73) follow a similar procedure shown in the proof of Lemma 17 and have been omitted for brevity.

Figure 6 – State x_k , control input u_k and system uncertainties Δ_k for the system (4.74) controlled by T-GCMPC with a initial state $x_0 = [0.7, -0.7, 0.7]^T$.

Figure 7 – State x_k , control input u_k and system uncertainties Δ_k for the system (4.74) controlled by HTMPC proposed by (RAKOVIĆ; CHENG, 2013) with a initial state $x_0 = [0.7, -0.7, 0.7]^T$.

4.5 Numerical examples

In this section, we present a numerical example of the use of T-GCMPCs which compares the proposed controller from Theorem 11 to the OL-GCMPC, proposed in Chapter 3, and to the HTMPC for systems subject to polytopic-bounded multiplicative uncertainties, proposed by Raković and Cheng (RAKOVIĆ; CHENG, 2013). We have used the YALMIP Toolbox (LÖFBERG, 2004) and the Mosek solver (APS, 2015) for the modeling of the problem.

4.5.1 Investigated system

Consider the uncertain linear system (4.1) defined by

$$\begin{aligned}
 A &= \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 0 & 1.2 \\ -1 & 1 & 0 \end{bmatrix}, B^u = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}, \\
 B^w &= \begin{bmatrix} 0.17 & 0.07 \\ 0.12 & -0.1 \\ -0.17 & 0.02 \end{bmatrix}, \\
 C_y &= \begin{bmatrix} 0.41 & 0.43 & -0.5 \\ 0 & -0.32 & 0.44 \end{bmatrix}, D_y^u = \begin{bmatrix} 0.4 & -0.4 \\ 0 & 0 \end{bmatrix},
 \end{aligned} \tag{4.74}$$

with a diagonal admissible uncertainty set ($\Delta_k = \text{diag}(\Delta_{k1}, \Delta_{k2})$). Associated with the cost matrices $Q = I$, $R = I$, and $N = 0$ and subject to the constraints

$$H_x = \begin{bmatrix} I_3 \\ -I_3 \\ 0 \\ 0 \end{bmatrix}, H_u = \begin{bmatrix} 0 \\ 0 \\ I_2 \\ -I_2 \end{bmatrix}, \text{ and } g = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad (4.75)$$

The GCC synthesis yields

$$K = \begin{bmatrix} 0.05 & -0.27 & 0.46 \\ 1.89 & -0.55 & -0.43 \end{bmatrix}, \quad (4.76)$$

with associated cost matrix

$$P = \begin{bmatrix} 19.18 & -5.98 & -9.57 \\ \star & 4.42 & 2.47 \\ \star & \star & 7.21 \end{bmatrix} \quad (4.77)$$

and control input deviation cost

$$\bar{R} = \begin{bmatrix} 10.92 & 10.80 \\ \star & 22.85 \end{bmatrix}. \quad (4.78)$$

For simplifying purposes we assumed $K_R = K$, which yielded an approximate mRPI level set defined by

$$E_R = \begin{bmatrix} 1.06 & 1.44 & 1.21 \\ \star & 2.52 & 1.68 \\ \star & \star & 1.67 \end{bmatrix}^{-1}, \quad (4.79)$$

with associated cross-section variables

$$\begin{aligned} a_\alpha &= 0.48 \\ a_\sigma &= \begin{bmatrix} 0.34 & 0.17 \end{bmatrix}^T. \end{aligned} \quad (4.80)$$

4.5.2 Controller comparison

Towards an unbiased comparison, we do not consider a terminal constraint set in this example, as the OL-GCMPC formulation does not include one, and we have modified the formulation of the HTMPC from (RAKOVIĆ; CHENG, 2013) to incorporate ellipsoidal tubes. We have chosen a horizon $N = 5$ due to the limitations of the OL-GCMPC where increasingly larger horizons yield increasingly smaller feasible regions.

Figures 6 and 7 present the simulation of the T-GCMPC and HTMPC for the initial state $x_0 = [0.7, -0.7, 0.7]^T$, which is outside the feasible region for the OL-GCMPC. We can observe that both controllers can stabilize the system within the feasible state region $x_i \in [-1, 1]$ and $u_i \in [-1, 1]$ while subject to a structured norm bounded disturbance. It is worth noting that the HTMPC resulted in a larger settling time than the T-GCMPC; however, their tuning is not

Figure 8 – Optimal cost of T-GCMPC (blue solid line) and OL-GCMPC (red dashed line) for $x_0 = \lambda[1, 1, 1]^T$ where $\lambda \in [0, 1]$. The crosses indicate the largest value of λ where both the T-GCMPC (blue) and OL-GCMPC (red) are feasible.

directly comparable due to their different formulation, especially with regards to the penalization of the parameter α .

Figure 8 presents the resulting cost of both controllers for the manifold $x_0 = \lambda[1, -1, 1]^T$ where $\lambda \in [0, 0.8]$. It is possible to observe that the T-GCMPC yields a larger feasible domain $\lambda \in [0, 0.78]$ when compared to the OL-GCMPC domain $\lambda \in [0, 0.46]$ and to the HTMPC domain $\lambda \in [0, 0.75]$, while it also ensures a lower or equal cost. Note that longer horizons would yield an even larger difference among the feasible regions of the controllers. This difference is due to the conservative nature of the OL-GCMPC with respect to the uncertainties propagation through the optimization horizon.

Figure 9 – Box-plot of solution time of 1000 executions for both the OL-GCMPC (left) and the T-GCMPC (right) using the Mosek SOCP solver.

4.5.3 Timing analysis

The resulting optimization problem for the solution of the T-GCMPC required a minimum of $5.92ms$, a maximum of $19.04ms$, and on average $7.24ms$. Meanwhile, the solution of the OL-GCMPC required a minimum of $5.70ms$, a maximum of $15.27ms$, and an average $7.01ms$. Finally, the HTMPC required a minimum of $6.57ms$, a maximum of $22.44ms$, and on average $7.90ms$. These results were obtained on a dual $2.6GHz$ Intel Xeon E5-2670 with $32Gb$ of RAM. Figure 9 presents the computational time distribution box plots, where it is possible to see that all controllers have comparable execution times, where the T-GCMPC required 9.3% longer execution time than the OL-GCMPC and 2.2% shorten execution time then the HTMPC on the 99th percentile of the distribution. However, due to the vertex enumeration approach, the HTMPC requires 2^{n_p} constraints per timestep of the horizon to represent the robustified dynamics, while the T-GCMPC formulation requires $n_p + 2$ constraints per timestep. Therefore, the T-GCMPC formulation will obtain better relative performance for systems subject to a higher number of disturbance modes.

4.5.4 T-GCMPC optimization results

Figure 10 – T-GCMPC optimization states and its invariant tube cross section for states x_1 and x_2 considering an initial state $x_0 = [0.6, -0.6, 0.6]$. The system nominal states z_k are shown in blue, while the red ellipsoidals represent the invariant tube.

To illustrate the result of the T-GCMPC optimization, Figure 10 presents the nominal states and the invariant tube for the initial state $x_0 = [0.6, -0.6, 0.6]$ for a horizon $N = 20$. We can observe that the tube cross-section grows initially due to the magnitude of the disturbances when the system state is far from the origin, while once the system approaches the origin, the tube cross-section tends to zero, as shown in more details in Figure 11.

Figure 11 – T-GCMPC optimization states and its invariant tube cross section near the origin for states x_1 and x_2 considering an initial state $x_0 = [0.6, -0.6, 0.6]$. We can observe the crossection of the tube converging to zero once the system approaches the origin.

4.5.5 Remarks

The HTMPC cannot be applied for the example system if it contains non-diagonal disturbances (due to the vertex enumeration approach). Meanwhile, the T-GCMPC cannot be applied for the general polytopic uncertain system case. Therefore, based on the presented numerical example, we can observe that the proposed T-GCMPC obtains performance comparable to the HTMPC approach, and outperforms the OL-GCMPC method regarding feasibility domain and obtains a resulting cost equal or lower than the OL-GCMPC.

4.6 Conclusion

In this chapter, we have proposed a Tube-based Guaranteed Cost Model Predictive Controller (T-GCMPC) by making use of the homothetic invariant tube formulation for constrained linear systems. They are subject to multiplicative structured norm-bounded uncertainties. The proposed method provides an upper bound to the general RMPC cost function. It also internalizes all the system uncertainties through its homothetic tube formulation, which guarantees that its cross-section volume tends to zero (i.e., all uncertainties are guaranteed to be rejected). Lastly, we have also proposed a novel invariant tube definition, where any arbitrary scaling of its cross-section is an invariant set for a subset of the state-space.

Similarly to the OL-GCMPC, the proposed method provides an advantage over previous formulations as it provides cost and feasibility guarantees while yielding a Second Order Cone Programming (SOCP) problem. Meanwhile, similar methods require Semi-Definite Programming (SDP) problems, which tend to be more computationally expensive. Additionally, the T-GCMPC is less conservative than the OL-GCMPC and provides infinite horizon feasibility guarantees, which the former does not.

AUTONOMOUS VEHICLE CONTROL UP TO THE LIMITS OF HANDLING UNDER TIRE-GROUND PARAMETER UNCERTAINTY

An essential aspect in the design of automotive control systems is to ensure the controller is stable and has acceptable performance within the entire operational envelope which it is designed to operate. Therefore, RMPC approaches show great promise on ensuring both theoretical and practical safety guarantees for the application of Autonomous Vehicles (AV) since they enable the direct modeling and consideration of uncertainties; while providing robustness guarantees towards those. In this chapter, we present a practical application of the proposed T-GCMPC to the lateral control problem of autonomous vehicles. The proposed controller formulation ensures that the vehicle remains within its safe operational envelope by avoiding the saturation of both the front and rear tires.

5.1 Introduction

Road traffic crashes are the leading cause of death among young people between 10 and 24 years old ([ORGANIZATION, 2007](#)). Most of these accidents occur when the driver is unable to maintain the vehicle control due to fatigue or external factors resulting in loss-of-control scenarios ([ADMINISTRATION *et al.*, 2008](#)). In recent years, both academia and industry have devoted efforts to the development of safety systems in order to decrease the number of road accidents.

The development of control techniques to maintain vehicle stability under possible loss-of-control scenarios is of extreme importance to current driver assistance systems (DAS) and the safe control of Autonomous Ground Vehicles (AGV). New technologies are continuously

being introduced in commercial vehicles to provide more information and control capabilities for the handling limiting scenarios. One of these technologies is steer-by-wire system (ULRICH, 2013), which enables measurements of self-aligning torque and tire-road friction coefficients with higher precision (HSU; LAWS; GERDES, 2010).

Model Predictive Control (MPC) is a class of optimization-based control algorithms that use an explicit model of the controlled system to predict its future states (BADGWELL; QIN, 2015). Several different fields have applications which use this technique, such as refineries, food processing plants, mining, aerospace, and automotive control (QIN; BADGWELL, 2003). An MPC minimizes a cost function while maintaining the system states and control inputs within a predefined domain.

Controllable Invariant (CI) sets and Robust Controllable Invariant (RCI) sets (BLANCHINI, 1999) have been useful in modern control system designs whose interest has increased in the past two decades. The CI class of sets describes domains of the state space of a dynamic system where there is at least one admissible control input which maintains the future state space within the same domain. Meanwhile, RCI sets are CI sets where there is uncertainty or disturbances present in the dynamic system. MPC approaches have used CI and RCI sets to guarantee its infinite horizon stability and feasibility (BEMPORAD; MORARI, 1999). Such guarantees are fundamental in the deployment of control systems for safety-critical applications, such as AGVs.

Beal and Gerdes (BEAL; GERDES, 2013) proposed an MPC-based Driver Assistance System (DAS) for Steer-by-Wire vehicles able to ensure vehicle handling limits in coordination with a human driver. Towards that purpose, this study proposed a novel CI set based on the vehicle's maximum steady-state yaw rate and the maximum allowed rear slip angles. This work was later extended to autonomous vehicle control and obstacle avoidance by Funke et al. (FUNKE *et al.*, 2017). Carvalho et al. (CARVALHO *et al.*, 2013) designed an MPC for evasive maneuvers based on the iterative linearization of a nonlinear Ackerman model with convexified constraints. Falcone et al. (FALCONE *et al.*, 2007) developed both a nonlinear MPC and a Linear Time-Varying MPC to the reference tracking control problem of autonomous vehicles. Raffo et al. (RAFFO *et al.*, 2009) formulated a nested controller for reference tracking, which consisted of two MPC controller: a longer horizon outer-loop based on the kinematic vehicle model to reduce complexity and computational requirements; and an inner-loop which used the dynamic vehicle model to provide higher fidelity control. However, such approaches do not incorporate robustness or estimation of tire-ground parameter uncertainty.

Tire characteristics vary significantly with temperature (TÖNÜK; ÜNLÜSOY, 2001), wear (BRAGHIN *et al.*, 2006) and their manufacturing process. Furthermore, they directly impact the vehicle handling limits and the overall system behavior in a multiplicative manner (BEAL; GERDES, 2013). Therefore, the design of controllers for DAS and AGV applications cannot assume that these characteristics are constant or known if their goal is to ensure overall

operational safety. The disregard of multiplicative uncertainties, such as tire-ground parameters, can lead to the poor closed-loop performance of MPCs and, consequently, to the violation of state and control input constraints (RAWLINGS; MEADOWS; MUSKE, 1994).

Towards improving the controller performance and robustness in the case of multiplicative uncertainties (such as tire-ground parameters), we proposed in Chapter 4 a Robust MPC technique, entitled Tube-based Guaranteed Cost Model Predictive Control (T-GCMPC). It can guarantee robust stability, robust feasibility and an upper bound to an MPC optimization problem cost for a linear system with multiplicative parametric uncertainties.

In this chapter, we propose a Tube-based Guaranteed Cost Model Predictive Controller for Autonomous Vehicles able to avoid front and rear tire saturation and to track a provided reference trajectory up to the limits of handling of the vehicle. The proposed approach incorporates a novel cone-bounded uncertainty model-based conservative relaxation of tire force nonlinearities, resulting in a linear system subject to norm-bounded multiplicative uncertainties approximation of the nonlinear vehicle model. Towards that purpose, we also propose a novel maximal robust controllable invariant set for lateral vehicle control to ensure the state belongs to the domain where such approximation is valid. Finally, we present both simulation and in-vehicle tests of the proposed approach and a comparison between the proposed invariant set and the one used by (BEAL; GERDES, 2013) and (FUNKE *et al.*, 2017).

The remainder of the chapter is organized as follows: Section 5.2 presents the vehicle modeling and discusses the tire models and its uncertainties; Section 5.3 formulates the proposed Robust Controllable Invariant Set and compares it to previously proposed ones; Section 5.4 describes the proposed Tube-based Guaranteed Cost Model Predictive Controller formulation; Section 5.5 reports the simulation and experimental tests performed; and Section 5.6 provides the final remarks.

5.2 Tire and vehicle models

This section describes the linear and the “brush” tire models and the linear and nonlinear bicycle models used to approximate vehicle dynamics. Finally, it describes the proposed system representation which incorporates elements of the tire dynamics as uncertainties.

5.2.1 Tire models

There are three categories of models capable of representing the tire dynamics on saturation situations: finite element analysis (HELNWEIN *et al.*, 1993)-(KOISHI; KABE; SHIRATORI, 1998); empirical data approximation, such as the “Magic Tire Formula” (PACEJKA; BAKKER, 1992); and the dynamical approximation of the tire by a “brush” model, first proposed by Fiala (FIALA, 1987). The third technique provides a good compromise between its ability to describe tires physical properties and its complexity; since it assumes that tires are always at steady state

and tire transients are faster than chassis transients. Meanwhile, linear tire models provide a good approximation of tire forces for low slip conditions. Figure 12 presents both the brush tire model and the linear tire model forces for the tire slip angles.

Figure 12 – Comparison of the “brush” tire model (blue solid line) with the linear tire model (dashed red line) where peak force and sliding model at high slip angles can be observed.

5.2.1.1 “Brush” tire model

Brush tire model assumes a parabolic pressure distribution on the contact patch and that its rubber dynamics has reached steady-state. Under such conditions, the lateral force for a given tire i becomes a direct function of the slip angle (α_i) and four parameters: the cornering stiffness (C_i), the static friction coefficient (μ_i), the tire normal force (F_{z_i}), and the ratio between dynamical and static friction coefficients (R_{μ_i}). Then, the lateral force is given by

$$F_{y_i} = \begin{cases} a_i f_i + b_i |f_i| f_i + c_i f_i^3 & |\alpha_i| \leq \alpha_i^{sat} \\ F_{sliding,i} \text{sgn}(\alpha_i) & \text{otherwise} \end{cases}, \quad (5.1)$$

where

$$f_i = \tan(\alpha_i), \quad (5.2)$$

$$a_i = -C_i, \quad (5.3)$$

$$b_i = k_{\mu_i} C_i^2 \frac{2 - R_{\mu_i}}{3 \mu_i F_{z_i}}, \quad (5.4)$$

$$c_i = -k_{\mu_i}^2 C_i^3 \frac{1 - \frac{2}{3} R_{\mu_i}}{(3 \mu_i F_{z_i})^2}, \quad (5.5)$$

$$k_{\mu_i} = q_i - \left(\frac{2 - R_{\mu_i}}{3} - \frac{1}{9} \right) q_i^2, \quad (5.6)$$

$$q_i = \left(1 - \frac{2}{3} R_{\mu_i} \right)^{-1}, \quad (5.7)$$

$$\alpha_i^{sat} = \tan^{-1} \frac{3 \mu_i F_{z_i}}{k_{\mu_i} C_i}, \quad (5.8)$$

$$F_{sliding,i} = -\mu_i R_{\mu_i} F_{z_i}. \quad (5.9)$$

In this study, we are interested in the limits of handling performance obtained at peak forces. From (5.1), the peak lateral force and its respective slip angle are

$$F_{y_i}^{peak} = \mu F_{z_i}, \quad (5.10)$$

$$\alpha_i^{peak} = \tan^{-1} \left(\frac{q \mu_i F_{z_i}}{k_{\mu_i} C_i} \right). \quad (5.11)$$

5.2.1.2 Linear tire model

A linear approximation of the lateral tire forces is valid for low slip angle situations and enables the development of a linear model of the vehicle dynamics. From (5.1), a first-order Taylor approximation at the point $\alpha_i = 0$ yields

$$F_{y_i} \approx -C_i \alpha_i. \quad (5.12)$$

This result has been widely used by both industry and academia for decades to design controllers and estimators for DAS and AGV applications.

5.2.2 Vehicle models

The bicycle model is a simplification of the vehicle dynamics which replaces the left and right front wheels by a virtual front wheel in the center of the front axle and, analogously, the left and right rear wheels by a virtual wheel in the center of the rear axle (KRITAYAKIRANA; GERDES, 2012). Figure 13 shows a representation of a front-wheel driven vehicle on this form.

We now present both the nonlinear and the linear bicycle model formulations for such model.

Figure 13 – Bicycle model representation.

5.2.2.1 Nonlinear bicycle model

The nonlinear bicycle model is directly represented by its equations of motion with state $x = [v_x, v_y, r]^T$ and control input $u = [F_{xf}, \delta]^T$ as

$$\dot{v}_x = \frac{1}{m} (F_{xf} \cos(\delta) - F_{yf} \sin(\delta)) + r v_y, \quad (5.13)$$

$$\dot{v}_y = \frac{1}{m} (F_{xf} \sin(\delta) + F_{yf} \cos(\delta) + F_{yr}) - r v_x, \quad (5.14)$$

$$\dot{r} = \frac{1}{I_z} (a F_{xf} \sin(\delta) + a F_{yf} \cos(\delta) - b F_{yr}), \quad (5.15)$$

where v_x is the longitudinal velocity, v_y is the lateral velocity, r is the yaw rate, δ is the front wheel steering angle, a is the distance from the center of mass to the front axle, b is the distance from the center of mass to the rear axle, m is the mass, I_z is the yaw moment of inertia, and F_{yf} and F_{yr} are the lateral tire forces described by the “brush” tire model with respective parameters C_f , C_r , $R_{\mu f}$, $R_{\mu r}$, $\mu = \mu_f = \mu_r$ and normal loads $F_{zf} = mgb/(a+b)$ and $F_{zr} = mga/(a+b)$.

The tire slip angles required by the tire model are defined by the angle between the velocity vector of the contact patch and the tire direction. Therefore,

$$\alpha_f = \arctan\left(\frac{v_y + ar}{v_x}\right) - \delta, \quad (5.16)$$

$$\alpha_r = \arctan\left(\frac{v_y - br}{v_x}\right). \quad (5.17)$$

5.2.2.2 Linear bicycle model

The linear bicycle model is a first order Taylor approximation of its nonlinear counterpart described in (5.13)-(5.15) at $v_y = 0$ and $r = 0$ under the assumptions $\dot{v}_x \approx 0$ and $F_{xf} \approx 0$. Based on such approximation, we represent the linear bicycle model in state-space form $\dot{x}(t) = Ax(t) + B^u u(t)$ with a reduced state-space $x = [v_y, r]^T$ and input-space $u = \delta$ as

$$A = \begin{bmatrix} -\frac{C_f + C_r}{mv_x} & -\frac{aC_f - bC_r}{mv_x} - v_x \\ -\frac{aC_f - bC_r}{I_z v_x} & -\frac{a^2 C_f + b^2 C_r}{I_z v_x} \end{bmatrix}, \quad (5.18)$$

$$B^u = \begin{bmatrix} \frac{C_f}{m} \\ \frac{aC_f}{I_z} \end{bmatrix}. \quad (5.19)$$

5.2.3 Proposed uncertain bicycle model

We are now ready to formulate the proposed uncertain bicycle model. Towards that purpose, we formulate the tire-ground parameter variation as a conic set uncertainty on the force domain. Then, we extend the linear bicycle model with such conic set to obtain a linear system with norm-bounded structured multiplicative uncertainties.

5.2.3.1 Uncertain tire model

Figure 14 – Representation of the proposed conic-bounded uncertain tire model (red solid region) and its relation to the “brush” tire model (blue solid line) and the slip angle with peak tire force (black dash-dot line).

In the scope of this chapter, we are interested in ensuring that both front and rear tire slip angles remain within the peaks. Therefore,

$$-\alpha_f^{peak} \leq \alpha_f \leq \alpha_f^{peak}, \quad (5.20)$$

$$-\alpha_r^{peak} \leq \alpha_r \leq \alpha_r^{peak}, \quad (5.21)$$

where the front, rear and peak slip angles are defined by (5.16), (5.17) and (5.11), respectively.

The “brush” tire model has a concave derivative within the domain of interest. Therefore, the absolute lateral tire force is upper-bounded by the tire cornering stiffness C_i and lower-bounded by the instantaneous slope defined at the peak tire force C_i^{peak} , described by

$$C_i^{peak} = \mu F_{zi} / \alpha_i^{peak} \approx \frac{k_\mu C_i}{q}. \quad (5.22)$$

It is worth noting that (5.22) is approximately independent of friction based on $\tan^{-1}(x) \approx x$ used in the second term. Based on (5.22), we conclude that

$$\forall \|\alpha_i\| \leq \alpha_i^{peak} \exists \|\gamma_i\| \leq 1 : F_{yi}(\alpha_i) = -(\bar{C}_i + \gamma_i \delta C_i) \alpha_i, \quad (5.23)$$

where

$$\bar{C}_i = \frac{C_i + C_i^{peak}}{2}, \quad (5.24)$$

$$\delta C_i = \frac{C_i - C_i^{peak}}{2}. \quad (5.25)$$

The relaxation from (5.23) is the proposed uncertain tire model, where γ_i is the norm-bounded uncertain gain related to the tire i . An example of this model is shown in Figure 14.

5.2.3.2 Uncertain bicycle model

The proposed uncertain bicycle model is a direct application of the uncertain tire model from (5.23) to the linear bicycle model from (5.18) and (5.19), which can be expressed as a standard continuous-time linear system subject to norm-bounded uncertainties of the form

$$\dot{x}(t) = (A_b + B_b^w \Delta_k C_{by})x(t) + (B_b^u + B_b^w \Delta_k D_{by}^u)u(t) \quad (5.26)$$

where $x(t) = [v_y, r]$, v_y is the lateral velocity, and r is the yaw rate,

$$A_b = \begin{bmatrix} -\frac{\bar{C}_f + \bar{C}_r}{mv_x} & -\frac{a\bar{C}_f - b\bar{C}_r}{mv_x} - v_x \\ -\frac{a\bar{C}_f - b\bar{C}_r}{I_z v_x} & -\frac{a^2\bar{C}_f + b^2\bar{C}_r}{I_z v_x} \end{bmatrix}, \quad (5.27)$$

$$B_b^u = \begin{bmatrix} \frac{\bar{C}_f}{m} \\ \frac{a\bar{C}_f}{I_z} \end{bmatrix}, \quad (5.28)$$

$$B_b^w = \begin{bmatrix} \frac{1}{mv_x} & 0 \\ 0 & \frac{1}{I_z v_x} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -a & -b \end{bmatrix}, \quad (5.29)$$

$$C_{by} = \begin{bmatrix} \delta C_f & 0 \\ 0 & \delta C_r \end{bmatrix} \begin{bmatrix} 1 & a \\ -1 & b \end{bmatrix}, \quad (5.30)$$

$$D_{by}^u = \begin{bmatrix} -\delta C_f \\ 0 \end{bmatrix}, \quad (5.31)$$

$$\Delta_k = \begin{bmatrix} \gamma_f & 0 \\ 0 & \gamma_r \end{bmatrix}, \quad (5.32)$$

where $\|\Delta_k\| \leq 1$ and Δ_k has a known diagonal structure, which the T-GCMPC synthesis will exploit to create a less conservative controller.

The vertexes of the disturbance Δ_k can be enumerated due to its diagonal structure. This fact will be used in Section 5.3 to generate the maximal RCI set for the system. Such enumeration is done by evaluating the matrices A and B_u for the four permutations of the values $\gamma_f, \gamma_r \in \{-1, 1\}$.

5.2.3.3 Trajectory-relative system model

The main goal of the proposed controller is to track a reference trajectory. Towards that purpose, we extend the uncertain bicycle model from (5.27)-(5.32) to contain the vehicle position and heading in Frenet coordinates (FRENET, 1852) in the state space and the path curvature as the system reference (QIAN; FORTELLE; MOUTARDE, 2016). Similarly to the uncertain bicycle model, such system is also represented in the continuous-time linear system subject to norm-bounded uncertainties format with an additional reference input

$$\dot{x}(t) = (A + B^w \Delta_k C_y)x(t) + (B^u + B^w \Delta_k D_y^u)u(t) + B^r r(t) \quad (5.33)$$

where $x(t) = [e_y, e_\psi, v_y, r]$, e_y is the cross-track error, e_ψ is the heading error, v_y is the lateral velocity, and r is the yaw rate, with matrices:

$$A = \begin{bmatrix} 0 & v_x & 1 & d_m \\ 0 & 0 & 0 & 1 \\ 0 & 0 & & \\ 0 & 0 & A_b & \end{bmatrix}, \quad (5.34)$$

$$B^u = \begin{bmatrix} 0 \\ 0 \\ B_b^u \end{bmatrix}, \quad (5.35)$$

$$B^w = \begin{bmatrix} 0 \\ 0 \\ B_b^w \end{bmatrix}, \quad (5.36)$$

$$B^r = \begin{bmatrix} 0 & -v_x & 0 & 0 \end{bmatrix}^T, \quad (5.37)$$

$$C_y = \begin{bmatrix} 0 & 0 & C_{by} \\ 0 & 0 & \end{bmatrix}, \quad (5.38)$$

$$D_y^u = D_{by}^u, \quad (5.39)$$

$$\Delta_k = \Delta_{kb}, \quad (5.40)$$

where d_m is the distance along the vehicle body between the center of mass of the vehicle and the reference point where the cross-track error is measured, and the matrices with b subscripts are given by the uncertain bicycle model from (5.27)-(5.32).

The Frenet coordinate-based uncertain system from (5.33)-(5.40) enables the design of trajectory tracking controllers capable of operating up to the limits of the vehicle's tire forces. Therefore, the design of the proposed T-GCMPC in Section 5.4 will use such a formulation.

5.3 Maximal Robust Controllable Invariant Set

This section describes the proposed maximal RCI set and the algorithm for its synthesis. Its goal is to ensure that the system remains within the domain where the uncertain bicycle model is valid. Additionally, the use of RCI sets as terminal constraints on Robust MPC problems provides recursive feasibility guarantees.

5.3.1 Maximal robust controllable invariant set computation

We consider the discrete-time linear difference inclusion system $x_{k+1} = Ax_k + B^u u_k$ (BOYD *et al.*, 1994), where

$$\begin{bmatrix} A & B \end{bmatrix} \in \left\{ \sum_{i=1}^{n_s} \alpha_i \begin{bmatrix} A_i & B_i \end{bmatrix} \middle| \sum_{i=1}^{n_s} \alpha_i = 1, \alpha_i \geq 0 \right\}. \quad (5.41)$$

Also, let $\mathbb{C}$ be the feasible set defined by

$$\mathbb{C} = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \middle| H_x x + H_u u \leq g \right\} \quad (5.42)$$

with state and control input space projections $\mathbb{C}_x = \text{Proj}_x(\mathbb{C})$ and $\mathbb{C}_u = \text{Proj}_u(\mathbb{C})$, respectively.

Let $\mathbb{R}_0 = \mathbb{C}_x$ and (A_i, B_i) be the vertex matrices from the uncertain polytopic representation of the uncertain bicycle model. Then, the backward recursive approach to generate maximal RCIs (KERRIGAN, 2001) is given by

$$\mathbb{S}_{k+1,i} = [A_i B_i]^\dagger \mathbb{R}_k, \quad (5.43)$$

$$\widehat{\mathbb{S}}_{k+1} = \mathbb{C}_x \cap \left(\bigcap_{i=1}^{n_s} \mathbb{S}_{k+1,i} \right), \quad (5.44)$$

$$\mathbb{R}_{k+1} = \text{Proj}_x(\widehat{\mathbb{S}}_{k+1}). \quad (5.45)$$

Convergence is reached once $\mathbb{R}_{k+1} = \mathbb{R}_k = \mathbb{R}_\infty$ for some k . Moreover, even without finite-time convergence guarantees for such algorithm, it is still very effective for practical applications.

5.3.2 Proposed maximal robust controllable invariant set

We generate a maximal RCI set based on the exact polytopic representation of the uncertain bicycle model from (5.27)-(5.32). Such invariant set has two main goals: To ensure the tires slip angles are within the boundaries where the uncertain bicycle model approximation is valid (from (5.20) and (5.21)) and to ensure the commanded steering angle is within the mechanical limits of the system, given by the constraint

$$-\delta_f^{\max} \leq \delta_f \leq \delta_f^{\max}. \quad (5.46)$$

Since the slip angles are nonlinear functions on the system states and control inputs, we use the small angle approximation $\tan^{-1}(x) \approx x$ to linearize the constraints, which results

$$H_x = \begin{bmatrix} 1 & a \\ 1 & -b \\ 0 & 0 \\ -1 & -a \\ -1 & b \\ 0 & 0 \end{bmatrix}, \quad H_u = \begin{bmatrix} -v_x \\ 0 \\ 1 \\ v_x \\ 0 \\ -1 \end{bmatrix}, \quad g = \begin{bmatrix} v_x \alpha_f^{peak} \\ v_x \alpha_r^{peak} \\ \delta_f^{max} \\ v_x \alpha_f^{peak} \\ v_x \alpha_r^{peak} \\ \delta_f^{max} \end{bmatrix}. \quad (5.47)$$

Figure 15 – Proposed maximal RCI set shape for on the lifted state space $x = [v_x, v_y, r]^T$.

We generated the maximal RCI set for speeds from $3m/s$ to $40m/s$ and scheduled it based on the current vehicle speed since the bicycle model is parametrized by v_x . Figure 15 shows the resulting set on the space $x = [v_x, v_y, r]^T$. We can observe that at slow speeds the RCI is constrained by the actuator limits while at higher speeds it is bounded by slip limits in the tires and elongated by the slower system dynamics.

Additionally, Figure 16 shows the generated maximal RPI set for the speed $v_x = 10m/s$, where it is possible that the RCI set reaches the peak rear slip angle constraints, while it doesn't reach the peak front slip angle constraints. This is a consequence of the front slip angle being directly controllable by the steering angle, which results in a relaxed projection of the peak front slip angle constraints, since there only needs to exist one feasible control input for a given state to be deemed feasible as a consequence of the projection of the constraint into the state space.

5.3.3 Maximal RCI comparison

Beal and Gerdes (BEAL; GERDES, 2013) proposed an RCI set based on the maximum rear slip angle and the maximum achievable steady-state yaw rates, given by

$$-\alpha_r^{peak} \leq \alpha_r \leq \alpha_r^{peak}, \quad (5.48)$$

$$-r^{max} \leq r \leq r^{max}, \quad (5.49)$$

Figure 16 – Proposed maximal RCI set (red) at $v_x = 10\text{m/s}$ with its respective feasible region (grey).

Figure 17 – Comparison for the proposed maximal RCI set (in blue) and Beal's envelope from (BEAL; GERDES, 2013) (in red), at speeds 10m/s , 15m/s , and 20m/s .

where

$$r^{max} = \frac{\mu}{v_x} \frac{ab + \max(a,b)^2}{\min(a,b)(a+b)}. \quad (5.50)$$

Figure 17 shows the proposed maximal RCI set and the set proposed by Beal and Gerdes. We can observe that the peak side slip angle of the rear tire limits both sets. However, the proposed set achieves larger yaw rate values. However, such larger yaw rates do not have achievable steady-state (x, u) pairs, by the definition of Beal's envelope. Nonetheless, the proposed maximal RCI set enables the execution of highly dynamic maneuvers which momentarily exceed stable yaw rates.

Another critical factor is that both sets are directly scalable by friction, while the model used to generate the proposed maximal RCI is independent of friction. Such linear scaling enables the designed system to be independent of friction estimation performance while operating under nominal conditions and only using such information for limits of handling operation, where such information has better observability (DOUMIATI *et al.*, 2009; HSU; LAWS; GERDES, 2010).

5.4 Controller formulation

In this section, we describe the proposed controller formulation. First, we present an overview of the T-GCMPC method proposed in Chapter 4. Next, we describe its application for the trajectory tracking problem.

5.4.1 Tube-based guaranteed cost model predictive control

The Tube-based Guaranteed Cost Model Predictive Control goal is to provide robust feasibility, stability, and optimality for constrained linear systems subject to multiplicative structured norm-bounded uncertainties¹. Consider the uncertain discrete-time system given by

$$\begin{aligned} x_{k+1} &= A_d x_k + B_d^u u_k + B_d^w w_k, \\ y_k &= C_y x_k + D_y^u u_k, \\ c_k &= C_c x_k + D_c^u u_k. \end{aligned} \quad (5.51)$$

where $x_k \in \mathbb{R}^{n_x}$ is the system state, $u_k \in \mathbb{R}^{n_u}$ is the control input, $w_k = \Delta_k y_k$ is the uncertainty input, $y_k \in \mathbb{R}^{n_y}$ is the uncertainty output, $c_k \in \mathbb{R}^{n_c}$ is the cost output, $A_d \in \mathbb{R}^{n_x \times n_x}$ is the discretized system matrix, $B_d^u \in \mathbb{R}^{n_x \times n_u}$ is the discretized control input matrix, $B_d^w \in \mathbb{R}^{n_x \times n_w}$ is the discretized uncertainty input matrix, $C_y \in \mathbb{R}^{n_y \times n_x}$ is the state uncertainty matrix, $D_y^u \in \mathbb{R}^{n_y \times n_u}$ is the control input uncertainty matrix, $C_c \in \mathbb{R}^{n_c \times n_x}$ is the state cost matrix, and $D_c^u \in \mathbb{R}^{n_c \times n_u}$ is the control input cost matrix. Note that the nomenclature of output related matrices was not changed, since exact discretization methods do not modify them.

Towards that purpose, the T-GCMPC provides a conservative solution to the intractable Min-Max MPC problem

$$\begin{aligned} J^*(x_0) &= \inf_{\mathbf{u}} \sup_{\blacksquare} \lim_{N \rightarrow \infty} J_0(x_0, \mathbf{u}, N) \\ \text{s.t.} \quad &x_{k+1} = A_d x_k + B_d^u u_k + B_d^w w_k, \\ &w_k = \Delta_k (C_y x_k + D_y^u u_k), \\ &[x_k^T, u_k^T]^T \in \mathbb{C}, \end{aligned} \quad (5.52)$$

where the cost function J_i is defined by

$$J_i(x_i, \mathbf{u}, N) = \sum_{k=i}^{N-1} x_k^T Q x_k + u_k^T R u_k + 2x_k^T N u_k, \quad (5.53)$$

with symmetric weight matrices $P_N \succeq 0$, $Q \succeq 0$, and $R \succ 0$, $\mathbf{u} = \{u_k \mid k \in [0, N-1]\}$, and $\blacksquare = \{\Delta_k \mid k \in [0, N-1]\}$.

The T-GCMPC controller synthesis consists of three steps, where the first two are computed *a priori* and the last is performed during the controller execution:

¹ We refer the reader to the GCMPC toolbox at <https://gitlab.com/cmasseraj/gcmpe> for implementation details.

5.4.1.1 Guaranteed cost controller synthesis

The Optimal Guaranteed Cost Controller (GCC) is a robust version of the Linear Quadratic Regulator (LQR) (XIE; SOH, 1993). Its solution is a state-feedback controller $u_k = -Kx_k$ with associated symmetric matrix cost $P \succ 0$ with minimal trace that satisfies

$$A_{cl}(\Delta)^T P A_{cl}(\Delta) - P + Q + NK + K^T N^T + K^T R K \preceq 0, \quad (5.54)$$

for all admissible values of Δ and where

$$A_{cl}(\Delta) = A_d + B_d^w \Delta C_y - (B_d^u + B_d^w \Delta D_y^u) K. \quad (5.55)$$

Such controller synthesis can be posed as a Semi-definite Programming (SDP) problem given by

$$\min \text{trace}(Z) \quad (5.56)$$

$$s.t. \quad (5.57)$$

$$\begin{bmatrix} -Z & I \\ \star & -X \end{bmatrix} \preceq 0, \quad (5.58)$$

$$\begin{bmatrix} -\Lambda_q & 0 & 0 & C_y X - D_y^u Y \\ \star & -I & 0 & C_c X - D_c^u Y \\ \star & \star & -\bar{X} & A_d X - B_d^u Y \\ \star & \star & \star & -X \end{bmatrix} \preceq 0, \quad (5.59)$$

where $X = P^{-1}$, $Y = KP^{-1}$, $\bar{X} = X - B_d^w \Lambda_p B_d^{wT}$, the cost matrices $C_c \in \Re^{n_c \times n_x}$ and $D_c^u \in \Re^{n_c \times n_u}$ are given by the factorization of the cost function matrices

$$\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} = \begin{bmatrix} C_c & D_c^u \end{bmatrix}^T \begin{bmatrix} C_c & D_c^u \end{bmatrix}, \quad (5.60)$$

or, equivalently,

$$J_i(x_i, \mathbf{u}, N) = \sum_{k=i}^{N-1} c_k^T c_k \quad (5.61)$$

and the generalized S-Procedure variables are given by

$$\begin{aligned} \Lambda_p &= \text{diag}(\lambda_1 I_{n_{p1}}, \lambda_2 I_{n_{p2}}, \dots, \lambda_s I_{n_{ps}}), \\ \Lambda_q &= \text{diag}(\lambda_1 I_{n_{q1}}, \lambda_2 I_{n_{q2}}, \dots, \lambda_s I_{n_{qs}}). \end{aligned} \quad (5.62)$$

For more details on the synthesis of GCC controllers we refer to reader to Lemma 7 and Theorem 4 in Chapter 2.

5.4.1.2 Approximate minimal RCI set

One of the main differentiating properties of Robust MPCs is their approach towards the propagation of uncertainties. Tube-based methods are based on the principle of separable control

policies, where the following system is considered

$$\begin{aligned} z_{k+1} &= Az_k + B^u v_k, \\ e_{k+1} &= Ae_k + B^u \rho_k + B^w w_k, \\ y_k &= C_y(z_k + e_k) + D_y^u(v_k + \rho_k), \\ c_k &= C_c(z_k + e_k) + D_c^u(v_k + \rho_k), \end{aligned} \quad (5.63)$$

where z_k defines the nominal dynamics, and e_k defines the error dynamics. Based on the identities $x_k = z_k + e_k$ and $u_k = v_k + \rho_k$, the System (5.63) becomes equivalent to System (5.51).

Since e_k propagates the error, it is a unknown variable for future states. Therefore we define a RCI set $\mathbb{E}$, associated with a controller K_R and a scaling variable $\alpha_k \geq 0$, such that $e_k \in \alpha_k^2 \mathbb{E}$, and $\rho_k \in \alpha_k^2 (-K_R) \mathbb{E}$. This yields the following relaxed α dynamics

$$\alpha_{k+1}^2 \mathbb{E} \supseteq \alpha_k^2 (A_d - B_d^u K_R) \mathbb{E} \oplus \lambda_k^2 B_d^w \mathbb{W} \quad (5.64)$$

where $\mathbb{W} = \{w \mid w^T w \leq 1\}$, λ_k satisfies

$$\begin{bmatrix} z_k \oplus \alpha_k^2 \mathbb{E} \\ v_k \oplus \alpha_k^2 (-K_R) \mathbb{E} \end{bmatrix} \subseteq \lambda_k^2 \mathbb{Y}, \quad (5.65)$$

and

$$\mathbb{Y} = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \mid (\bullet)^T (C_y x + D_y^u u) \leq 1 \right\}. \quad (5.66)$$

To minimize the overall conservativeness of the controller, it is ideal to obtain the minimal RCI (mRCI) set that bounds errors dynamics. However, computing such set is often intractable. Therefore, many approximate approaches were proposed, for example (RAKOVIĆ *et al.*, 2005; KOURAMAS *et al.*, 2005). The T-GCMPC synthesis proposed a novel approximate mRCI synthesis, where any arbitrary scaling of such set is also an RCI set for a subset of the state space. The mRCI set $\mathbb{E} = \{x \mid x^T E_R x \leq 1\}$ is the result of the following SDP optimization:

$$\Theta_a(a_\alpha) = \inf \text{tr}(X) \quad (5.67)$$

$$s.t. \begin{bmatrix} -X & A_d X - B_d^u Y & B_d^w \Upsilon_p \\ \star & -X a_\alpha & 0 \\ \star & \star & -\Upsilon_p \end{bmatrix} \preceq 0, \quad (5.68)$$

$$\begin{bmatrix} -I_{n_{qi}} & C_{y,i} X - D_{y,i}^u Y \\ \star & -X \end{bmatrix} \preceq 0, \quad (5.69)$$

$$\begin{bmatrix} -v_i & 1 \\ \star & -a_{\sigma_i} \end{bmatrix} \preceq 0, \quad (5.70)$$

$$a_\alpha + \sum_{i=1}^s a_{\sigma_i} \leq 1, \quad (5.71)$$

where $E_R = X^{-1}$ and $K_R = YX^{-1}$ and the generalized S-Procedure variables are given by

$$\Upsilon_p = \text{diag}(v_1 I_{n_{p1}}, v_2 I_{n_{p2}}, \dots, v_s I_{n_{ps}}), \quad (5.72)$$

$$\Upsilon_q = \text{diag}(v_1 I_{n_{q1}}, v_2 I_{n_{q2}}, \dots, v_s I_{n_{qs}}). \quad (5.73)$$

5.4.1.3 T-GCMPC optimization problem

Based on the results of the GCC synthesis (5.59), the approximate mRPI synthesis (5.71), and a terminal RCI set $\mathbb{R}_N = \{x \mid H_N x \leq g^N\}$ to guarantee recursive feasibility, we can pose the T-GCMPC optimization problem as

$$\begin{aligned}
 \inf_v \quad & x_0^T P x_0 + \sum_{k=0}^{N-1} \gamma_k^2 \\
 \text{s.t.} \quad & z_{k+1} = (A_d - B_d^u K) z_k + B_d^u v_k, \\
 & \alpha_{k+1} \geq \left\| \left[\sqrt{a_\alpha} \alpha_k, \sqrt{a_{\sigma_1}} (\sigma_k)_1, \dots, \sqrt{a_{\sigma_s}} (\sigma_k)_s \right] \right\|_2, \\
 & (\sigma_k)_i \geq \|(\bar{C}_y)_i z_k + (D_y^u)_i v_k\| + (C_y^\alpha)_i \alpha_k \\
 & \gamma_k \geq \|\bar{R}^{\frac{1}{2}} v_k\|_2 + \|\bar{R}^{\frac{1}{2}} (K_R - K) E_R^{-\frac{1}{2}}\|_2 \alpha_k, \\
 & \bar{H}_i z_k + (H_u)_i v_k + \|(\bar{H}_R)_i E_R^{-\frac{1}{2}}\|_2 \alpha_k \leq g_i, \\
 & H_i^N z_N + \|H_i^N E_R^{-\frac{1}{2}}\|_2 \alpha_N \leq g_i^R,
 \end{aligned} \tag{5.74}$$

where $\bar{H} = H_x - H^u K$, $\bar{H}_R = H_x - H^u K_R$, and the resulting control command $u_0 = -Kx_0 + v_0$.

The T-GCMPC optimization (5.74) is a conservative approximation to the Min-Max MPC problem (5.52). It results in an SOCP optimization problem; nonetheless, the increase in complexity caused by the robustness formulation grows linearly with the number of uncertainties, which is one of the main advantages of tube-based approaches. Meanwhile, other RMPC methods may grow exponentially (RAKOVIĆ; CHENG, 2013; SCOKAERT; MAYNE, 1998) or quadratically (RAKOVIĆ *et al.*, 2012a).

5.4.2 Functional modelling and controller synthesis

One of the fundamental aspects of optimal control design is the proper choice of a cost function and weighting terms, as it directly impacts its nominal region performance (where constraints are inactive). Towards the purpose, we choose to apply the Implicit Model Following (IMF) (TYLER, 1964) approach to the cost function definition.

Consider the trajectory-relative bicycle model without uncertainties. Then, the goal of the controller is to minimize the cross-track error with a damped response. Therefore, we choose a first order model with time-constant τ , such that

$$\dot{e}_y^r(t) = -\tau^{-1} e_y^r(t), \tag{5.75}$$

which, given $\bar{C}_c = [1 \ 0 \ 0 \ 0]$ and $C_c B^u = 0$, is equivalent to

$$\bar{C}_c (A + \tau^{-1} I) x(t) + \bar{C}_c B^u u(t) = \bar{C}_c (A + \tau^{-1} I) x(t) = 0. \tag{5.76}$$

However, (5.76) only defines a manifold of the system state space. Therefore, we introduce the error variable $c(t) = C_c x(t) + D_c^u u(t)$ such that

$$C_c = \begin{bmatrix} \bar{C}_c (A + \tau^{-1} I) \\ 0 \end{bmatrix}, \quad D_c^u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{5.77}$$

where its first row, $c_1(t)$, defines the deviation from the IMF manifold from (5.76) and its second row, $c_2(t)$, defines the control input "effort".

The T-GCMPC synthesis requires discrete-time dynamics; meanwhile, the uncertain bicycle model is defined in continuous time. Therefore, we perform an exact discretization method, which results in a system in the form (5.51).

As mentioned in Section 5.3, the controller is synthesized and gain-scheduled for several longitudinal velocities, as the bicycle model is parametrized by v_x . For each synthesis a controller gain K is obtained based on the GCC (5.59), an approximated mRPI set is calculated from (5.71), and an optimization problem of the form (5.74) is posed with $x_0 = z_0$ and $\alpha_0 = 0$. Additionally, all constraints are modeled as soft due to possible effects of unmodeled disturbances.

Finally, Tables 2 and 3 show the vehicle parameters and controller parameters used for synthesis.

Table 2 – Synthesis vehicle model parameters

Parameter	Symbol	Value	Unit
Vehicle mass	m	1231	Kg
Inertia Moment z	I_z	2034.5	$\frac{Kg}{m^2}$
Front axle distance to CG	a	1.07	m
Back axle distance to CG	b	1.40	m
Front tire cornering stiffness	C_f	100000	$\frac{N}{rad}$
Rear tire cornering stiffness	C_r	130000	$\frac{N}{rad}$
Peak front tire cornering stiffness	C_f^{peak}	41171	$\frac{N}{rad}$
Peak rear tire cornering stiffness	C_r^{peak}	53522	$\frac{N}{rad}$
Tire-road friction coefficient	μ	0.8	-
Friction coefficient ratio	$R_{\mu,i}$	0.85	-

Table 3 – Controller Parameters

Parameter	Symbol	Value	Unit
Prediction horizon	N	10	—
IMF time constraint	τ	1	s
IMF manifold weight	W_{imf}	0.087	s^2/m^2
Steering angle weight	W_{δ_f}	1	$1/rad^2$
Maximum front slip angle	α_f^{peak}	7.5760	$^\circ$
Maximum rear slip angle	α_r^{peak}	4.4711	$^\circ$
Sampling Time	T_s	0.025	s

5.5 Experimental results

In this section, we present results for in-vehicle test experiments. We considered a 750m route consisting of two left turns, where the vehicle must turn right to exit and enter such turn, as shown in Figure 18.

Figure 18 – Route used for in-vehicle experiemnts.

5.5.1 Simulation experiments

The simulation was performed based on a nonlinear bicycle model for vehicle dynamics with a second order damped steering actuator model. For more details on the synthesis and simulation implementation details and for result reproduction we refer the reader to https://gitlab.com/cmasserat/robust_vehicle_control.

Figure 19 shows the results for simulations results. The route is equivalent to the in-vehicle experiments; however, the simulation results have Proportional-Integrator (PI) controller with feedforward acceleration term implemented to control the longitudinal speed of the vehicle. Similarly to the results in Figure 21, the vehicle saturates the front tire on the second turn causing the controller to react quickly to recover the car. Meanwhile, when the controller was subject to other operation conditions, the cross-track error remained bounded to $e_y \in [-0.03, 0.03]m$.

Three main sources of disturbance contribute to the observed performance difference:

1. In-vehicle tests use state observers to obtain lateral velocity and yaw rate estimates, while the simulation has noise-free full state measurement;

Figure 19 – Results of simulation experiments. Where the left top graph shows the longitudinal speed v_x in m/s , the center top graph shows the crosstrack error e_y in m , the right top graph shows the heading error e_ψ in $^\circ$, the left bottom graph shows the vehicle yaw rate (blue solid line), the reference yaw rate (dashed red line) and the yaw rate steady-state limits (dash-dot block line) in $^\circ/s$, the center bottom graph shows the commanded steering angle δ_f in degrees, and the right bottom graph shows g-g diagram with longitudinal and lateral acceleration m/s^2 .

2. Intrinsic actuation and computational delays when operating in the vehicle; and
3. Unaccounted additive disturbances, such as steering actuation misalignments, heading misalignment, banking angle influence, and others.

Nonetheless, the simulation to in-vehicle performance degradation was $\approx 0.15m$.

5.5.2 In-Vehicle experiments

We performed the in-vehicle experiments in the CaRINA II platform (SHINZATO *et al.*, 2016), shown in Figure 20. This platform is a 2011 Fiat Palio Adventure modified for computational control of steering, throttle, and brake. Cameras and a 3D LIDAR sensors are installed and used by the onboard computer running Ubuntu Linux and ROS (Robot Operating System). A Septentrio GPS System provides vehicle localization and speed information with RTK correction and IMU integration operating at 10Hz, while lateral velocity and yaw rate information is estimated based on a fixed gain-scheduled observer in Frenet coordinates. Such estimator was synthesized based on the Guaranteed Cost Estimator (PETERSEN; MCFARLANE, 1996) formulation dual to the Guaranteed Cost Controller proposed in Chapter 2 using system and

measurement covariance matrices based on Loop-Transfer Recovery (LTR) (DOYLE; STEIN, 1981).

Figure 20 – CaRINA II platform used in experiments, a Fiat Palio Adventure.

Figure 21 shows the results for the in-vehicle execution. The driver manually controlled the vehicle velocity while the proposed T-GCMPC method controlled the steering actuator. The vehicle achieved maximum lateral acceleration of $8.3m/s^2$ around 113s where the vehicle exceeded the maximum steady-state yaw rate, which resulted in a fast steering correction to recover vehicle stability. Such saturation also resulted in a peak cross-track error of 0.35m and heading error of 4.2 degrees. The turn where such peak occurred is an off-camber turn, which is a source of unmodeled additive disturbance that contributes to such saturation.

Another high lateral acceleration conditions happened between $t \in [83, 90]s$ which resulted in cross-track errors in the range $e_y \in [-0.07, 0.22]m$, where the vehicle maintained itself within stability limits. Finally, during non-limits of handling conditions the vehicle maintained cross-track errors within $e_y \in [-0.04, 0.13]m$.

5.6 Conclusion

In this chapter, we have proposed a Tube-based Guaranteed Cost Model Predictive Controller for Autonomous Vehicles able to avoid front and rear tire saturation and to track a provided reference trajectory up to the limits of handling of the vehicle. The main contributions of this chapter are

1. A novel conservative approximation of the nonlinear vehicle dynamics to a linear system subject to norm-bounded multiplicative uncertainties;
2. A maximal RCI set for vehicle dynamics, which contains a larger region of the state space when compared to other state-of-the-art invariant sets;

Figure 21 – Results of in-vehicle experiments. Where the left top graph shows the longitudinal speed v_x in m/s , the center top graph shows the crosstrack error e_y in m , the right top graph shows the heading error e_ψ in $^\circ$, the left bottom graph shows the vehicle yaw rate (blue solid line), the reference yaw rate (dashed red line) and the yaw rate steady-state limits (dash-dot block line) in $^\circ/s$, the center bottom graph shows to commanded steering angle δ_f in $^\circ$ (limits outside plot scale), and the right bottom graph shows the reference path curvature κ in $1/m$.

3. An application of Tube MPC approach proposed in Chapter 4 to the trajectory-tracking problem of autonomous vehicles.

We have also presented results for both simulation and in-vehicle experiments, where we observed both low error performance and tire saturation avoidance.

CONCLUDING REMARKS

In this thesis, we have proposed Guaranteed Cost Controller (GCC) approaches for both unconstrained and constrained linear systems subject to multiplicative structured norm-bounded uncertainties and present the application of such a controller to the trajectory-tracking up to the limits of handling problem of autonomous vehicles. Specifically, the contributions of this thesis are:

- A state- and static output-feedback Guaranteed Cost Control formulation for a more general class of linear systems subject to norm-bounded structured parametric uncertainties which, additionally, relaxes the assumption of no feed-through uncertainty;
- A constrained Open Loop Guaranteed Cost Robust Model Predictive Controller (OL-GCMPC) for uncertain discrete-time systems which provide an upper bound to the quadratic cost functional and ensures feasibility of states and control inputs within a prediction horizon for a system subject to a norm-bounded multiplicative uncertainty;
- A Tube-based Guaranteed Cost Model Predictive Controller (T-GCMPC) by making use of the homothetic invariant tube formulation for constrained linear systems subject to multiplicative structured norm-bounded uncertainties. It improves upon the GCMPC synthesis by providing infinite horizon feasibility, stability, and performance guarantees;
- An application of T-GCMPC synthesis to the trajectory tracking problem of autonomous vehicles capable of avoiding front and rear tire saturation up to the vehicle limits of handling. Additionally, we also propose a larger operational envelope for the unsaturated vehicle dynamics and a novel conservative approximation of the nonlinear vehicle dynamics.

We performed a numerical comparison between the proposed state- and static output-feedback GCC from Chapter 2 with the discrete-time GCC proposed by (XIE; SOH, 1993)

and the undisturbed LQR from (KALMAN *et al.*, 1960) in the first example of Section 2.5. In such a case, it was possible to observe that incorporating known uncertainty structures into synthesis methods for uncertain linear systems may provide significantly less conservative control laws. In the particular example considered, the proposed GCC provided an 83% reduction in the worst-case cost bounds. Additionally, the second example in Section 2.5 demonstrated the capability of the proposed method to handle a more general class of systems, where the feedback disturbance is present in the measured system output. However, it still contains an assumption that the uncertainty modes, present in the system output, do not affect the control input matrix. Nonetheless, in such cases, a conservative approach can be taken where the correlated disturbances modes are considered uncorrelated.

Meanwhile, we demonstrated in Chapter 3 that the concepts and guarantees from the GCC formulation can be applied to the constrained uncertain discrete-time systems through an MPC framework; resulting in the proposed OL-GCMPC. However, such an approach did not consider any robust controllable invariant sets as terminal constraints and considered the uncertainties propagated based on a nil-potent feedback model. Therefore, it resulted in a controller too conservative for practical applications that simultaneously did not provide strong recursive feasibility guarantees. Nonetheless, it provided good computational complexity and yielded a formulation of tractable dimension.

Considering the limitations of the OL-GCMPC formulation, we proposed the T-GCMPC in Chapter 4. Such formulation uses the concept of homothetic tubes, proposed by (RAKOVIĆ *et al.*, 2012b), to yield a less conservative controller for the same class of systems that the OL-GCMPC considers while also providing recursive feasibility guarantees, as it is a conservative approximation of the infinite horizon Min-Max MPC. Additionally, it also generalizes such class of system to incorporate the potential advantages of exploiting the uncertainty structure, as demonstrated in the properties of the proposed controller from Chapter 2. The numerical examples explored in Section 4.5 demonstrate that the T-GCMPC has a significantly large feasible region when compared with the OL-GCMPC. Additionally, it also yielded a feasible region larger than the state-of-the-art Homothetic Tube MPC for Linear Difference Inclusion systems, proposed in (RAKOVIĆ; CHENG, 2013).

Finally, we present a practical application of the proposed T-GCMPC to the lateral control problem of autonomous vehicles in Chapter 5. The proposed controller formulation ensures that the vehicle remains within its safe operational envelope by avoiding the saturation of both the front and rear tires. Such a property is fundamental for the safe operation and deployment of such systems in urban applications.

In the remainder of this chapter, we present future work topics and the list of publications associated with this thesis.

6.1 Future work

Possible future works regarding the state- and static output-feedback Guaranteed Cost Controller includes:

- Generalization of the method for the uncertain linear descriptor system case;
- Generalization of the method for uncertain linear system subject to Markovian jumps;
- Generalization of the method for piecewise linear systems;
- Application of the proposed Linear Matrix Inequality (LMI) framework for the dynamic output-feedback case;
- Investigation of the use of relaxations based on Lasserre's Methods of Moments (MoM) ([LASSERRE, 2001](#)) to approximate the resulting Bilinear Matrix Inequality as a LMI;
- Application of the generalized Kalman-Yakubovich-Popov lemma ([IWASAKI; HARA, 2005](#)) to handle uncertainties with known bounded frequency domain.

Meanwhile, possible research topics regarding extensions to the OL-GCMPC and T-GCMPC include:

- Study of polytope-represented mRPI to reduce the resulting optimization problem complexity from SOCP to a QP or QCQP;
- Generalization of the proposed controller to reference-tracking problems;
- Generalization of the proposed controller for the uncertain linear descriptor system case;
- Investigation of alternative parametrizations of the mRCI tube, analogous to elastic tubes, parametric tubes, or a possible novel formulation unique to systems subject to norm-bounded uncertainties.

Finally, regarding the application of such class of controllers to automotive systems, future work includes:

- Extension of the proposed controller to consider differential braking capabilities;
- Integrate longitudinal and lateral dynamic coupling through multiplicative uncertainties;
- Internalize state estimation uncertainty into the tube formulation;
- Incorporate road bank, road slope, and steering actuator dynamics uncertainties into formulation.

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PROOF OF LEMMA 6 FOR THE OPTIMAL GUARANTEED COST CONTROL

In this appendix, we prove Lemma 6. Towards this goal, we first present supporting theory required for the proof.

Lemma 20. For all $i \in [1, n]$, let $\mathbb{S}_i$ be lossless sets. Then,

$$\mathbb{H}\left(\text{conv}\left(\bigcup_{i=1}^n \mathbb{S}_i\right)\right) = \mathbb{H}\left(\bigcup_{i=1}^n \mathbb{S}_i\right) = \bigcap_{i=1}^n \mathbb{H}(\mathbb{S}_i) \quad (\text{A.1})$$

where $\mathbb{H}(\mathbb{S})$ is the set of matrices H that satisfy property (iii) of Definition 3.

Proof. Due to the associative property of the involved operations, we will restrict the prove, without loss of generality, to the case $n = 2$. We will first prove the second equality, then proceed to prove the first.

Let $H \in \mathbb{H}(\mathbb{S}_0) \cap \mathbb{H}(\mathbb{S}_1)$. Then,

$$\forall S \in \mathbb{S}_0 \cup \mathbb{S}_1 : \text{tr}(SH) \leq 0 \quad (\text{A.2})$$

and

$$H = \sum_{i=0}^{\text{rank}(H)} \lambda_i \lambda_i^*, \forall S \in \mathbb{S}_0 \cup \mathbb{S}_1 : \lambda_i^* S \lambda_i \leq 0. \quad (\text{A.3})$$

Therefore, $H \in \mathbb{H}(\mathbb{S}_0 \cup \mathbb{S}_1)$ and $\mathbb{H}(\mathbb{S}_0 \cup \mathbb{S}_1) = \mathbb{H}(\mathbb{S}_0) \cap \mathbb{H}(\mathbb{S}_1)$.

Consider now the following set:

$$\mathbb{S}_2 = \text{conv}(\mathbb{S}_0 \cup \mathbb{S}_1) = \{\alpha_0 S_0 + \alpha_1 S_1 \mid S_0 \in \mathbb{S}_0, S_1 \in \mathbb{S}_1, \alpha_0 \geq 0, \alpha_1 \geq 0\}, \quad (\text{A.4})$$

and $H \in \mathbb{H}(\mathbb{S}_2)$. Then, from property (iii), we obtain

$$\forall \alpha_0 \geq 0, \alpha_1 \geq 0, S_0 \in \mathbb{S}_0, S_1 \in \mathbb{S}_1, \alpha_0 \text{tr}(S_0 H) + \alpha_1 \text{tr}(S_1 H) \leq 0 \quad (\text{A.5})$$

and

$$H = \sum_{i=0}^{\text{rank}(H)} \lambda_i \lambda_i^*, \quad \alpha_0 \lambda_i^* S_0 \lambda_i + \alpha_1 \lambda_i^* S_1 \lambda_i \leq 0. \quad (\text{A.6})$$

If H is chosen such that $H \in \mathbb{H}(\mathbb{S}_0 \cup \mathbb{S}_1)$, then (A.5) and (A.6) are immediately satisfied. However, if we choose H such that $H \notin \mathbb{H}(\mathbb{S}_0 \cup \mathbb{S}_1)$, it is impossible to satisfy (A.5) and (A.6) for cases $\alpha_0 = 1, \alpha_1 = 0$ and $\alpha_0 = 0, \alpha_1 = 1$. Therefore, $\mathbb{H}(\text{Conv}(\mathbb{S}_0 \cup \mathbb{S}_1)) = \mathbb{H}(\mathbb{S}_0 \cup \mathbb{S}_1)$. $\square$

Corollary 6. The convex hull of unions of lossless sets is also lossless, since it is a convex cone (properties (i) and (ii) hold) and property (iii) holds from Lemma 20.

Lemma 21. Let $C \in \mathfrak{R}^{r(p+q) \times n}$, and $Q \in \mathfrak{R}^{r \times r}$ be a positive semi-definite matrix. Then, the set

$$\mathbb{S} = \left\{ C^T \begin{bmatrix} Q \otimes I_p & 0 \\ 0 & -Q \otimes I_q \end{bmatrix} C \mid \forall Q \succ 0 \right\} \quad (\text{A.7})$$

is lossless.

Proof. (Based on proof of Lemma 3 of (IWASAKI; MEINSMA; FU, 2000)) Properties (i) and (ii) from Definition 3 are trivial. Therefore, we focus on proving property (iii) holds.

Let

$$S = C^T \begin{bmatrix} Q \otimes I_p & 0 \\ 0 & -Q \otimes I_q \end{bmatrix} C \quad (\text{A.8})$$

for brevity. Then, from the properties of Kronecker product, there exists a self-similar transformation T such that

$$S = C^T T^T \begin{bmatrix} I_p \otimes Q & 0 \\ 0 & -I_q \otimes Q \end{bmatrix} TC. \quad (\text{A.9})$$

Let $H \in \mathfrak{R}^{n \times n}$ be a positive semi-definite matrix such that $\text{tr}(HS) \leq 0$ for all $S \in \mathbb{S}$. Then, H admits a full rank factorization $H = GG^T$ where $G \in \mathfrak{R}^{n \times r}$ where $r = \text{rank}(H)$. We define

$$\begin{bmatrix} W \\ Z \end{bmatrix} = TCG, \quad (\text{A.10})$$

$W = [W_1^T, W_2^T, \dots, W_p^T]^T$, and $Z = [Z_1^T, Z_2^T, \dots, Z_q^T]^T$. Then,

$$\begin{aligned} \text{tr}(HS) &= \text{tr}(GG^T S) = \text{tr}(G^T S G) \\ &= \text{tr} \left(\begin{bmatrix} W \\ Z \end{bmatrix}^T \begin{bmatrix} I_p \otimes Q & 0 \\ 0 & -I_q \otimes Q \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} \right) \\ &= \text{tr} \left(\sum_{i=1}^p W_i^T Q W_i - \sum_{i=1}^q Z_i^T Q Z_i \right) \\ &= \sum_{i=1}^p \text{tr}(W_i^T Q W_i) - \sum_{i=1}^q \text{tr}(Z_i^T Q Z_i) \\ &= \sum_{i=1}^p \text{tr}(W_i W_i^T Q) - \sum_{i=1}^q \text{tr}(Z_i Z_i^T Q) \leq 0. \end{aligned} \quad (\text{A.11})$$

Notice that (A.11) must hold for all $Q \succeq 0$. Then, since $W^T W = \sum_{i=1}^p W_i^T W_i$ and $Z^T Z = \sum_{i=1}^p Z_i^T Z_i$, (A.11) is equivalent to

$$W^T W - Z^T Z \preceq 0, \quad (\text{A.12})$$

which also implies that $W^T (I_p \otimes Q) W - Z^T (I_q \otimes Q) Z \preceq 0$, since $I \otimes Q \succeq 0$.

Assume that (iii) doesn't hold, then there exists $\xi_i \in \Re^n$ such that $H = \sum_{i=1}^r \xi_i \xi_i^T$, $\text{tr}(HS) \leq 0 \forall S \in \mathbb{S}$, and $\xi_i^T S \xi_i > 0$ for some i and $S \in \mathbb{S}$.

Let $\xi_i = Gu_i$ where $u_i \in \Re^r$, $\|u_i\|_2 = 1$ and $\sum_{i=1}^r u_i u_i^T = I$. Then,

$$\xi_i^T S \xi_i = u_i^T G^T S G u_i = u_i^T (W^T (I_p \otimes Q) W - Z^T (I_q \otimes Q) Z) u_i > 0 \quad (\text{A.13})$$

and $\text{tr}(HS) > 0$, which contradicts the initial assumption. Therefore, (iii) holds and $\mathbb{S}$ is lossless. $\square$

We are now ready to prove Lemma 6.

Proof. We first prove property (i) ($\mathbb{S}^u \subseteq \mathbb{S}$), then continue to prove property (ii) ($\mathbb{S}$ is lossless).

(i): Let $\Lambda_p = \tau I_{n_p}$ and $\Lambda_q = \tau I_{n_q}$. Then, $\mathbb{S}$ reduces to $\mathbb{S}^u$. Therefore, $\mathbb{S}^u \subseteq \mathbb{S}$.

(ii): For all $i \in [1, s]$, let $\mathbb{S}_i = \{S_i^s(\Lambda_i) \mid \forall \Lambda_i \succeq 0\}$, where

$$S_i^s(\Lambda_i) = \begin{bmatrix} \bar{C}_{z,i} & \bar{D}_{z,i}^w \\ 0 & I_{n_{ri}n_{qi}} \end{bmatrix}^T \begin{bmatrix} \Lambda_i \otimes I_{n_{pi}} & 0 \\ 0 & -\Lambda_i \otimes I_{n_{qi}} \end{bmatrix} (\bullet). \quad (\text{A.14})$$

Then, we conclude that

$$\mathbb{S} = \text{conv} \left(\bigcup_{i=1}^s \mathbb{S}_i \right). \quad (\text{A.15})$$

Based on Lemma 21 and Corollary 6, S_i is lossless for all $i \in [1, s]$ and $\mathbb{S}$ is lossless since it is the convex hull of the union of lossless sets. $\square$

THE ROBUST LINEAR QUADRATIC REGULATOR AS A SPECIAL CASE OF THE GUARANTEED COST CONTROL

In this appendix, we demonstrate that penalty-based robust linear quadratic regulators (RLQR), as proposed in (TERRA; CERRI; ISHIHARA, 2014), are equivalent to a particular case of state-feedback Guaranteed Cost Controllers (GCC), as formulated in Chapter 2.

B.1 Preliminaries

We consider discrete-time linear system subject to parametric uncertainties of the form

$$x_{k+1} = (A + \delta A(\Delta_k))x_k + (B + \delta B(\Delta_k))u_k \quad (\text{B.1})$$

where $x_k \in \mathfrak{R}^{n_x}$ is the system state, $u_k \in \mathfrak{R}^{n_u}$ is the control input, $A \in \mathfrak{R}^{n_x \times n_x}$ is the state matrix, $B \in \mathfrak{R}^{n_x \times n_u}$ is the input matrix and $\delta A(\Delta_k)$ and $\delta B(\Delta_k)$ are, respectively, the state and input multiplicative uncertainty matrices, such that

$$\begin{bmatrix} \delta A(\Delta_k) & \delta B(\Delta_k) \end{bmatrix} = B^w \Delta_k \begin{bmatrix} C_z^T & D_z^u \end{bmatrix} \text{ s.t. } \|\Delta_k\|_2 \leq 1 \quad (\text{B.2})$$

with $\Delta_k \in \mathfrak{R}^{n_p \times n_q}$, $B^w \in \mathfrak{R}^{n_x \times n_p}$, $C_z \in \mathfrak{R}^{n_q \times n_x}$ and $D_z^u \in \mathfrak{R}^{n_q \times n_u}$.

Both penalty-based RLQR and GCC have the objective of synthesizing a controller of the form $u_k = -K_k x_k$ such that a quadratic cost functional of the form

$$J_i(x_i) = x_N^T P_N x_N + \sum_{k=i}^{N-1} x_k^T Q x_k + u_k^T R u_k, \quad (\text{B.3})$$

is upper bounded by a cost matrix $P_i \succ 0$ for all admissible disturbances, with $N \in \mathfrak{R}_{>0} \cup \{\infty\}$, $Q = C_c^T C_c \succeq 0$, $R = D_c^u T D_c^u \succeq 0$, and $C_c^T D_c^u = 0$ therefore

$$\forall \Delta_k \in \{\Delta \mid \|\Delta\|_2 \leq 1\} : J_i(x_i) \leq x_i^T P_i x_i. \quad (\text{B.4})$$

Such robust optimal controllers are well-posed under the following assumptions:

Assumption 9. For all $\Delta_k \in \{\Delta \mid \|\Delta\|_2 \leq 1\}$, the pair $(A + \delta A(\Delta_k), B + \delta B(\Delta_k))$ is controllable.

Assumption 10. For all $\Delta_k \in \{\Delta \mid \|\Delta\|_2 \leq 1\}$, the pair $(A + \delta A(\Delta_k), Q)$ is observable.

And the following additional assumption for the penalty-based RLQR:

Assumption 11. (TERRA; CERRI; ISHIHARA, 2014) There exists a matrix K such that $C_z - D_z^u K = 0$.

B.2 Guaranteed cost controller

Note that the uncertain system (B.1) is equivalent to the system

$$x_{k+1} = Ax_k + B^u u_k + B^w w_k \quad (\text{B.5})$$

with a disturbance feedback $w_k = \Delta_k(C_z x_k + D_z^u u_k)$.

Consider the unstructured uncertainty state-feedback case of Lemma 7, where $C_y = I$, $D_y^u = 0$, $D_z^u = 0$, $Y_q = \varepsilon I$, and $Y_p = \varepsilon I$. Then, we obtain

$$M(X, Y, \varepsilon) = \begin{bmatrix} -\varepsilon I & 0 & 0 & C_z X - D_z^u Y & 0 \\ \star & -I_{n_c} & 0 & C_c X - D_c^u Y & 0 \\ \star & \star & -X & AX - B^u Y & \varepsilon B^w \\ \star & \star & \star & -X & 0 \\ \star & \star & \star & \star & -\varepsilon I \end{bmatrix} \preceq 0, \quad (\text{B.6})$$

where $P = X^{-1}$ and $K = YP$. We obtain the associated optimal GCC by means of Theorem 4 subject to (B.6), resulting in the following optimization problem

$$\begin{aligned} \Theta_{gcc}^*(\varepsilon) = \min_{S, Y} \quad & \text{tr}(P_k) \\ \text{s.t.} \quad & M(X, Y, \varepsilon) \preceq 0 \\ & \begin{bmatrix} -X & 0 \\ \star & -S \end{bmatrix} \preceq 0 \end{aligned} \quad (\text{B.7})$$

B.3 Penalty-based robust linear quadratic regulator

Cerri et al. presented their proposed controller in terms of the Dynamic Programming (DP) solution of the minimization of cost function B.3. From (38) and (39) of (TERRA; CERRI; ISHIHARA, 2014) the recursive optimization is

$$\begin{aligned} J_k^*(x_k) = \lim_{\mu \rightarrow 0} \min_{u_k, x_{k+1}} \quad & x_k^T Q x_k + u_k^T R u_k + x_{k+1}^T P_{k+1} x_{k+1} + \\ & + (x_{k+1} - Ax_k - B^u u_k)^T W^{-1} (x_{k+1} - Ax_k - B^u u_k) + \lambda \|C_z x_k + D_z^u u_k\|_2^2 \end{aligned} \quad (\text{B.8})$$

where, $W = \mu^2 I - \lambda^{-1} B^w B^{wT}$. From the results of (SAYED; NASCIMENTO, 1999) and (TERRA; CERRI; ISHIHARA, 2014), $\lambda \geq \mu^{-2} \|B^{wT} B^w\|_2$. Therefore,

$$W = \mu^2 I - \lambda^{-1} B^w B^{wT} \succeq \mu^2 \left(I - \frac{B^w B^{wT}}{\|B^{wT} B^w\|_2} \right) \succeq 0 \quad (\text{B.9})$$

and $\lim_{\mu \rightarrow 0} \lambda \rightarrow \infty$ and $\lim_{\mu \rightarrow 0} W^{-1} \rightarrow \infty$.

As the penalty weighting (i.e. W^{-1} and λ) approximate infinity, the weighted expressions in optimization (B.8) can be replaced by equality constraints. Therefore, it becomes equivalent to

$$\begin{aligned} J_k^*(x_k) &= \min_{u_k, x_{k+1}} x_k^T Q x_k + u_k^T R u_k + x_{k+1}^T P_{k+1} x_{k+1} \\ \text{s.t. } x_{k+1} &= A x_k + B^u u_k \\ C_z x_k + D_z^u u_k &= 0 \end{aligned} \quad (\text{B.10})$$

Based on the knowledge that such synthesis yields linear controllers and quadratic cost, let $u_k = -K_k x_k$ and $J_k^*(x_k) \leq x_k^T P_k x_k$, then the optimization (B.10) is equivalent to

$$\begin{aligned} \Theta_{k,rlqr}^* &= \min_{P_k, K_k} \text{trace}(P_k) \\ \text{s.t. } Q &+ K^T R K + (A - B^u K_k)^T P_{k+1} (A - B^u K_k) \preceq P_k \\ C_z - D_z^u K_k &= 0 \\ P_k &\succ 0 \end{aligned} \quad (\text{B.11})$$

Which implies that there exists an infinite horizon cost matrix P and controller K such that

$$\begin{aligned} \Theta_{rlqr}^* &= \min_{X, Y} \text{tr}(S) \\ \text{s.t. } \begin{bmatrix} -I & 0 & C_c X - D_c^u Y \\ \star & -X & A X - B^u Y \\ \star & \star & -X \end{bmatrix} &\preceq 0 \\ C_z - D_z^u K_k &= 0 \\ \begin{bmatrix} -X & 0 \\ \star & -S \end{bmatrix} &\preceq 0 \end{aligned} \quad (\text{B.12})$$

based on Assumption 11 and the generalization of the SDP-based synthesis of the LQR controller.

B.4 Equivalence of controllers

We now prove that the penalty-based RLQR is equivalent to GCC for the case $\varepsilon = 0$. First, we present a supporting Lemma.

Lemma 22. Let $C \in \Re^{n \times n}$ be a symmetric matrix and $B \in \Re^{m \times n}$. Then, the following statements are equivalent:

- (i) $\begin{bmatrix} 0 & B \\ B^T & C \end{bmatrix} \preceq 0$;
- (ii) $C \preceq 0$ and $B = 0$.

Proof. From Schur complement,

$$\begin{bmatrix} -\lambda I & B \\ B^T & C \end{bmatrix} \preceq 0 \Leftrightarrow \lambda > 0, C + \lambda^{-1} B^T B \preceq 0 \quad (\text{B.13})$$

and assume without loss of generality that there exists a symmetric matrix $M \succ 0$ such that $C \succeq -M$. Then, $-M \preceq C \preceq -\lambda^{-1} B^T B$ can only be satisfied if $B^T B \preceq M\lambda$. Therefore, as $\lambda \rightarrow 0$, $B \rightarrow 0$. Hence, in such limit case, (B.13) reduces to

$$\begin{bmatrix} 0 & B \\ B^T & C \end{bmatrix} \preceq 0 \Leftrightarrow B = 0, C \preceq 0 \quad (\text{B.14})$$

□

This concludes the proof.

Lemma 23. The Guaranteed Cost Control synthesis optimization (B.7), subject to $\varepsilon = 0$, is equivalent to the Penalty-based Robust Linear Quadratic Regulator, or equivalently $\Theta_{gcc}^*(0) = \Theta_{rlqr}^*$.

Proof. From the GCC optimization (B.7) and RLQR optimization B.12 we observe that they are equivalent for $\varepsilon = 0$ if and only if

$$M(P_k, K_k, 0) \Leftrightarrow \begin{bmatrix} -R^{-1} & 0 & K_k \\ \star & -P_{k+1}^{-1} & A - B^u K_k \\ \star & \star & -P_k + Q \end{bmatrix} \preceq 0, C_z - D_z^u K_k = 0 \quad (\text{B.15})$$

Applying Lemma 22 to the first row and column of (B.6) and disregarding the fifth row and column as they are identically zero, we conclude that such conditions are immediately equivalent.

□

Remark 9. The proofs in this annex also hold for the finite horizon GCC and RLQR, where the proof is obtained using with similar methods when considering stage-dependent variables P_k and K_k .

ON THE INDEPENDENCE OF CONTROLLERS K AND $\tilde{K}$ FOR THE OPEN-LOOP GUARANTEED COST MODEL PREDICTIVE CONTROL

Equation (3.43), and by consequence Lemma 11, assume that the controllers K and $\tilde{K}$ can be independently defined. This Appendix presents shows that such independence comes from the property that the nil-potent controller $\tilde{K}$ is used to stabilize the uncertain system dynamics while the guaranteed cost controller K is used to stabilize the deterministic system.

Consider the system

$$\bar{x}_{k+1} = A\bar{x}_k + B^u u_k + B^w w_k, \quad (\text{C.1})$$

and its deterministic ($w_k \equiv 0$) counterpart

$$x_{k+1} = Ax_k + B^u u_k. \quad (\text{C.2})$$

Consider a nil-potent controller $\tilde{K}$ and let $u_k = -\tilde{K}\bar{x}_k + p_k$, then the relation between the systems can be expressed as

$$\bar{x}_k = x_k + \sum_{i=0}^{k-1} (A - B^u \tilde{K})^{k-i-1} B^w w_i \quad (\text{C.3})$$

and the closed loop deterministic system is

$$x_{k+1} = (A - B^u \tilde{K})x_k + B^u p_k. \quad (\text{C.4})$$

Note that the control law $\tilde{K}$ is used w.r.t. the uncertain state $\bar{x}_k$ in Equation (C.3).

Consider now the Guaranteed Cost Controller K_k at timestep k and let $p_k = -(K_k - \tilde{K})x_k + v_k$, then the closed loop deterministic system becomes

$$\begin{aligned} x_{k+1} &= (A - B^u \tilde{K})x_k - B^u (K_k - \tilde{K})x_k + B^u v_k \\ &= (A - B^u K_k)x_k + B^u v_k. \end{aligned} \tag{C.5}$$

Therefore, the control input u_k is given by

$$\begin{aligned} u_k &= -K_k x_k - \tilde{K}(\bar{x}_k - x_k) + v_k \\ &= -K_k x_k - \tilde{K} \left(\sum_{i=0}^{k-1} (A - B^u \tilde{K})^{k-i-1} B^w w_i \right) + v_k \end{aligned} \tag{C.6}$$