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Mixed integer programming models for fairness based physician scheduling problem in emergency room

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Valdemar Abrão Pedro Anastácio Devesse

**Modelo de programação inteira mista para o problema de
escalonamento de médicos em salas de emergência
baseado em equidade**

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Matemáticas e de Computação – ICMC-USP,
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To my parents, Pedro Anastácio and Esperança Abrão.

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*“The task of the modern educator,
is not to cut down jungles but to irrigate deserts.”
(Clive Staples Lewis)*

RESUMO

DEVESSE, V. A. P. A. **Modelo de programação inteira mista para o problema de escalonamento de médicos em salas de emergência baseado em equidade**. 2020. 111 p. Tese (Doutorado em Ciências – Ciências de Computação e Matemática Computacional) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

A proposta desta tese é tratar o Problema de Escalonamento do Médico em salas de emergência aplicando o conceito de equidade, a fim de que as escalas de trabalho entre os médicos sejam mais justas. Este problema pode ser descrito como atribuição de tarefas em um horizonte de planejamento considerando um conjunto de regras do trabalho, preferências individuais e regras organizacionais, que muitas vezes são conflitantes entre si. A atribuição gerada exige que médico seja adequado para atender a demanda. O objetivo é encontrar um escalonamento que maximize a satisfação das restrições de preferências individuais, observando os regulamentos de trabalho e as regras organizacionais. A literatura em escalonamento de médicos é escassa e a aplicação deste conceito é de grande valia para a ergonomia do trabalho médico. Neste trabalho, estudamos modelos de Programação Inteira para tratar este problema. A definição dos modelos resulta de uma reformulação de modelos da literatura em que foram assumidas as características mais importantes. Após os experimentos e comparações com abordagens de literatura, nossa proposta apresenta uma programação mais justa e observamos que existem algumas tendências a explorar na estrutura matemática da definição do problema.

Palavras-chave: Problema de Escalonamento de Médicos, Programação Inteira Mista, Otimização Combinatória, Escalas Justas, Pesquisa Operacional.

ABSTRACT

DEVESSE, V. A. P. A. **Mixed integer programming models for fairness based physician scheduling problem in emergency room**¹. 2020. 111 p. Tese (Doutorado em Ciências – Ciências de Computação e Matemática Computacional) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

In this thesis, our purpose is to tackle the Physician Scheduling Problem in Emergency Rooms considering fairness, aiming to provide fairer work schedules between the physicians. This problem can be described as a task assignment on a planning horizon considering a set of work regulations, individual preferences, and organizational rules that are often conflicting. The generated assignment displays which physicians are suitable to comply with the demand. The main aim is to find a schedule that maximizes individual preferences requirements while observing work regulations and organizational rules. The literature on physician scheduling is scarce, including applying the fairness concept, which is of great value for the ergonomics of medical work. In this work, we study and propose methods based on Integer Programming. The model statement results from a collection of literature models in which we assumed the most important features. After experiments and comparisons to literature approaches, our proposal shows fairer scheduling, and theoretically, we observed that some tend to explore the mathematical structure of the problem.

Keywords: Physician Scheduling Problem, Mixed-Integer Programming, Combinatorial Optimization, Fair Schedules, Operations Research.

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LIST OF ABBREVIATIONS AND ACRONYMS

A&E	Accident & Emergency Department
CG	Column Generation
CP	Constraint Programming
DES	Discrete Event Simulation
ED	Emergency Department
ER	Emergency Room
EW	Emergency Ward
GA	Genetic Algorithms
GP	Goal Programming
HCPA	Hospital de Clínicas de Porto Alegre
ILS	Iterated Local Search
LS	Local Search
NSP	Nurse Scheduling Problem
PSO	Particle Swarm Optimization
PSP	Physician Scheduling Problem
PSPER	Physician Scheduling Problem in Emergency Room
RFFO	Relax-and-Fix with Fix-and-Optimize
RSP	Resident Scheduling Problem
SO	Stochastic Optimization
VNDS	Variable Neighborhood Descent Search

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INTRODUÇÃO

This thesis focuses on ways to exploit mixed integer programming to solve a Physician Scheduling Problem in Emergency Room (PSPER). The present Chapter makes a brief presentation of this thesis. The background and the motivations for the Physician Scheduling Problem in Emergency Rooms are given in Section 1.1, followed by the summarized objectives pursued by this work in Section 1.2, and the description of the thesis structure in Section 1.3.

1.1 Background and Motivation

Emergency Room (ER), also known as an Accident & Emergency Department (A&E), Emergency Department (ED) or Emergency Ward (EW), is a medical treatment department focusing in emergency medicine, the critical care of patients who present without prior appointment. This unit operates around the clock every day to provide adequate and immediate treatment and is usually the main entry point to the hospital. Besides, ER assistance is usually the only medical treatment that civilians can use without expenses (CABRERA *et al.*, 2012). In the USA, emergency rooms usually work at least 50% over the capacity, despite 24 hours of working service. Moreover, ER visits account for 11% of outpatient encounters, 28% of acute care visits, and 50% of hospital admissions (BARISH R. A., 2012).

One of the most concerning situations in the ER is the overcrowding associated with increased mortality, length of stay in the hospital, and direct consequences on uninsured patients that wait for approximately two times more than recommended. This situation leads to overuse, one of the six sources of waste in healthcare (SUN B. C., 2013; NEHI, 2010). Alternatives to tackle such problems comprise the use of co-location of a primary care clinician, doctor triage, rapid assessment, and patient streaming (EDWIN, 2016). All of these measures contribute to reducing patient time in the emergency room. However, the observance of physician preferences and predefined requirements must also be considered and satisfied, which can be solved through fair schedules of the workforce hired to assist the ward.

Therefore, coping with staff scheduling in ER is a complicated task. The person in charge of such a schedule must run into conflicts between physician preferences and work regulation. These conflicts are one of the conditions that generate an irregular program to physicians, which plays a crucial role in physician fatigue and decreases assistance quality.

The literature devoted to PSPER is scarce and less addressed comparing to another correlated problem that also occurs in hospital wards: the Nurse Scheduling Problems. We refer the reader to [Erhard et al. \(2018\)](#), who detailed the information about the state of the art of physician scheduling problem. Physicians must be properly assigned to specific days in specific shifts during the planning horizon and a re-organization of the schedule may occur while the predefined requirements are complied, ([ALFARES, 2004](#)). As mentioned, the scheduler, usually an expert member of the workforce, runs into situations of conflicts between physician preferences and work regulation. It takes several attempts to find a schedule that satisfies every condition, ([TOPALOGLU, 2006](#); [PUENTE et al., 2009](#)).

The process of physician assignment consists of coverage of shifts and duties to physicians in a planning period, taking into account a set of work regulations, individual preferences, and organizational rules, most of which conflict with each other. The fairness arises when an unbiased schedule of the workforce is taking into account for Physician Scheduling Problems. As pointed by [Gendreau et al. \(2007\)](#), besides biological and psychological effects regarded in the scheduling of physicians, fairness must be carefully observed since it involves balancing the distribution of different types of shifts among physicians concerning other criteria that often conflict.

1.2 Summary of Proposals

In this thesis, we raise some questions: *"is it possible to have physicians' fair schedules in emergency rooms?"* If so, *"is there any possibility of having a fair schedule with a high degree of satisfaction with hospital regulations and physician preferences? Can we apply our proposals in real cases?"* To have these questions answered, we aim to:

- collect features of mathematical formulations addressed in the literature to propose a new model. More specifically, we are gathering and adapting the features of mixed-integer programming models proposed by [Curtois and Qu \(2014\)](#) and [Wickert et al. \(2020\)](#);
- test the mathematical formulations on benchmark instances, using the same set of instances [Curtois and Qu \(2014\)](#) and [Wickert et al. \(2020\)](#) used to a fairer qualitative comparison;
- reformulate the MIP models proposed by [Curtois and Qu \(2014\)](#) and [Wickert et al. \(2020\)](#) and apply the least absolute deviations concepts to accommodate fairness in the workload and increase the degree of satisfaction of soft constraints;

- apply our proposal in real scenarios, more explicitly using the data from Hospital de Clínicas de Porto Alegre, made available by [Wickert *et al.* \(2020\)](#).

1.3 Structure of the thesis

Throughout the document, the reader may encounter in the beginning of each chapter a brief description of its content. In Chapter 2 we present the literature review on PSPER. The most important problems on physician scheduling, recent trends and the most common features are discussed. Chapter 3 presents the PSPER statement in which the formal description of the problem is made and the mathematical formulations carefully exposed. Chapter 4 is dedicated to heuristic methods proposals to tackle the PSPER, where we present a Relax-and-Fix and Fix-and-Optimize heuristic to tackle fairness in ER. In Chapter 5, we discuss another formulation of the PSPER with fairness in hours and weekends, estimation of fairness parameters and a linear approximation for objective function. In Chapter 6, we apply the techniques in a real scenario: the Hospital de Clínicas de Porto Alegre. Final remarks and perspectives are described in Chapter 7.

LITERATURE REVIEW

This chapter focuses on literature review of the PSPER. Our aim with this chapter is to raise awareness about the context and trends in the physician scheduling problem. Here we revise and discuss the PSPER addressed in 21st century, ranging from problem features to solution techniques, and make simple but important confrontation with the problem we're addressing.

2.1 Physician Scheduling Problem in Emergency Rooms

As a derivative of the Personnel Scheduling, we describe the PSPER as assigning tasks to a given number of physicians in a planning horizon by respecting a set of personnel preferences, organizational rules, labor union requirements, and others that may be of interest.

To the best of our knowledge, the first occurrence of studies on the Physician Scheduling Problem in Emergency Rooms dates from 1985, when [Vassilacopoulos \(1985\)](#) proposed dynamic programming for determining the number of doctors per shift weekly at Saint Peter Hospital in England. Since then, Operations Research and Computer Science practitioners undertake studies in the area aiming to mitigate the existing gaps.

The scheduling process, however, is not straightforward to address. Some particular features or stages must be considered. [Thompson \(2003\)](#) identifies four stages to scheduling staff based on a Hospitality Scheduling Problem, which can be extended to problems of different natures:

- Demand forecasting: involves balancing customer demands, employee work requests, and profitability. Nonetheless, this stage can vary according to the problem. The authors consider that eight aspects must be taken into account, with a detailed discussion in ([THOMPSON, 1998a](#));
- Knowing how many on-duty employees to schedule: consists of translating the demand

forecasting into the amount and the qualifications of the labor drivers. (THOMPSON, 1998b) discusses this stage;

- Developing a workforce schedule: involves constructing the schedules considering labor qualifications, personnel preferences, and determining who will execute a given task. In this stage, the planner has some pitfalls such as unavailability, personnel preferences dissatisfaction, skills, and cost assignment dissatisfaction. (THOMPSON, 1999a) presents a more in-depth analysis of this topic;
- Controlling workforce schedules in real-time: Unlike the three first tasks, this is not a planning stage and consists of comparisons of the operating reality and the planned schedule. In (THOMPSON, 1999b), there is a more detailed discussion about this stage.

While Thompson (2003) considers four stages, Ernst *et al.* (2004) on the other hand proposed six stages to the rostering process, namely:

- Demand modeling: determining how many people will be necessary to perform given tasks in the planning period;
- Days off scheduling: days off or leaves are scheduled in this stage. These days are interspersed among other working days;
- Shift schedule: in this stage, the shifts to be worked and the roster are selected to meet the demand;
- Line of work construction: the creation of work schedules, which span the planning horizon for each employee, is the main aim of this stage;
- Task assignment: different tasks are assigned to each shift, and each task to a specific staff category;
- Staff assignment: the staff is assigned together with the line of work construction in this stage.

The stages of scheduling can vary depending on the nature of the problem in hand. From these proposals, we can notice that some stages deemed by Ernst *et al.* (2004) can be compressed into a stage, i.e., days off, shifts, staff, and tasks are all resources. Moreover, we can make a relation between these resources and establish a single stage that we could name resource scheduling. Resources modeling is another stage that can be included in which we determine the number of tasks, staff, shifts and days off or duty available. We also consider the stages that Thompson (2003) addressed. Thus, our proposal to staff scheduling based on the literature review consists of:

- Demand modeling or forecasting: in cases of uncertainty, we can forecast the demand for service;
- Staffing: determining the necessary workforce to satisfy the demand;
- Resource modeling or forecasting: setting the necessary days off, staff categories, shift, and days that will be necessary to our schedule;
- Resource scheduling: the stage where the staff is assigned;
- Re-planning: it has the purpose of controlling the schedules, in which the scheduler must control the possible absence of a member of the staff with a specific category to a given shift.

In addition to solving the problem itself, a matter of concern amongst the practitioners is fairness. We define fairness in the context of Physician Scheduling as providing a balance between organization rules and the employees', which adds stimulating implications to the Physician Scheduling Problem. According to [Erhard *et al.* \(2018\)](#), modeling fairness is a consequence of aspects such: as physician shortage, contractual diversity, and adherence to preference schedule.

One way to address fairness is to create a particular type of schedule where staff will have the same workload at the end of it. Several hospitals deal with a particular type of schedule to adjust organizational rules for service demand and personnel preferences. There are two basic types of scheduling, namely, cyclic and non-cyclic scheduling. In the cyclic schedule, given the set of rules, staff members rotate under the schedule until assigning each staff member to each shift. In the non-cyclic schedule, there are situations in which staff members are assigned without any allocation pattern, i.e., there is not a defined pattern in the sequence of an assignment.

Figure 1 shows an instance of five employees assigned to acyclic scheduling within one week. Figure 2 shows the same instance for a non-cyclic schedule. The advantage of non-cyclic scheduling over cyclic is that the former is more flexible, accommodating situations such as leaves of physicians and respective replacements. The first does not affect largely the whole schedule, while the latter is more conditioned to leaves because it affects the whole schedule by not allowing the workforce to take outside engagements ([ROUSSEAU; PESANT; GENDREAU, 2002](#)). Most of the literature is devoted to non-cyclic schedules rather than cyclic due to the former handles' flexibility.

From the works evaluated in this section, cyclic scheduling is addressed in the literature by [Carter and Lapierre \(2001\)](#) [Ferrand *et al.* \(2011\)](#), while non-cyclic scheduling was addressed in the other works. A comprehensive literature review of approaches for personnel scheduling considering fairness aspects were addressed by [Wolbeck \(2019\)](#). Here, we make a literature review of fairness in the PSPER.

Figure 1 – A standard cyclic scheduling.

Emp/Days	Mon	Tue	Wed	Thu	Fri	Sat	Sun
E1	M	M		N	N		A
E2	A			M	M		N
E3	N		A	A			M
E4	M		N	N		A	A
E5			M	M		N	N

Source: Elaborated by the author.

Figure 2 – A non-cyclic scheduling.

Emp/Days	Mon	Tues	Wed	Thur	Fri	Sat	Sun
E1	N		N		A	A	A
E2		M	M		N	N	N
E3	M	M	M	M	M		
E4			A	A	A	M	M
E5	M	M	M	M	M		

Source: Elaborated by the author.

2.1.1 *PSPER with fairness*

Fairness is an aspect of concern in the realm of physician scheduling in emergency rooms. In the set of literature papers analyzed from 2000 to 2020, we identify 21 literature works that seem to be related to fairness. A total of eighteen out of 21 consider fairness explicitly as one of the requirements. Table 1 summarizes the literature papers on physician scheduling in emergency rooms reviewed by us, with fairness aspects and solution techniques highlighted.

To the best of our knowledge, the first effort to tackle PSPER in the 21st belongs to Beaulieu *et al.* (2000), modeling the scenario of Sacré-Cœur Hospital, a Canadian hospital of the Montreal region. The authors partitioned the model's constraints into four groups: compulsory, ergonomic, distribution, and goal constraints. Fairness is tackled in goal constraints and encompasses workload balance, balance in certain types of shifts (night and duties), and balance on the antagonist shift assignment. Besides, Beaulieu *et al.* (2000) ordered goal constraints according to their relative importance, which allows a higher satisfaction level of a specific type of preference than others.

Gathering information of six hospitals in Canada, [Carter and Lapierre \(2001\)](#) made an extensive discussion of the modeling process of PSPER and exposed its main characteristics, latter modeled it using Mixed-Integer Programming. The authors tackled fairness using two main aspects, the type of schedule and the constraints. In terms of the type of schedule, the authors consider a cyclic schedule. This schedule is fair, as the physicians will have worked approximately or the same workload when performing the schedule. Moreover, they also use constraints such as the distribution of weekend shifts, distribution of night shifts, and overall distribution of shifts. The physician cannot exceed the maximum number of weekends, night shifts, or a total number of shifts.

Solving PSPER of Santa Cabrini and Côte-des-Neiges Geriatric Hospital, both in the Montreal region, [Rousseau, Pesant and Gendreau \(2002\)](#) applied a combination of Constraint Programming (CP), Local Search (LS) and Genetic Algorithms (GA). To more assertive fairness, the authors used a non-cyclic roster due to the difficulty of assigning the personnel who often take outside engagements. Since the type of roster is an obstacle to fairness, authors handle fairness using a set of constraints named distribution constraints. These constraints stipulate a limit in the number of night shifts that a physician can work per week.

[Gendreau et al. \(2007\)](#) modeled a PSPER based on the characteristic of five Canadian hospitals, and solve using tabu-search. The constraints are classified into four categories: supply and demand, workload, fairness, and ergonomic constraints. Fairness measures consist of equal distribution of types of shifts. With the classification of constraints, one can quickly adapt to the scenario she/he wishes by adding, moving, or removing them accordingly.

Taking these four first works and establishing a relation between their proposals and ours, we can observe that: [Beaulieu et al. \(2000\)](#), [Rousseau, Pesant and Gendreau \(2002\)](#) and we use workload as a fairness measure, and more specifically, both [Beaulieu et al. \(2000\)](#) and we use a target number of hours to be accomplished with a penalization associated, if it is not achieved. On the other hand, [Carter and Lapierre \(2001\)](#), [Rousseau, Pesant and Gendreau \(2002\)](#), and [Gendreau et al. \(2007\)](#) use a specific group of shifts as a manner to control the workload. However, they do not control the fairness using a target value, yet they define a maximum of night shifts one can comply with. This measure may carry some disadvantages to personal satisfaction since one physician can have completed the maximum and other the very minimum. This type of difference in the schedule seems to be not fair.

Regarding constraints, [Beaulieu et al. \(2000\)](#) divided into four groups: compulsory, ergonomic, distribution, and goal constraints. [Rousseau, Pesant and Gendreau \(2002\)](#) used the concept of generic constraints, dividing into distribution and pattern constraints. On the other hand, our proposal is close to [Carter and Lapierre \(2001\)](#) that consider hard and soft/flexible constraints. In one of our models, we consider exactly the same weekend constraint as a hard constraint. Moreover, we applied a slight change in this set of constraints in our proposal to accommodate the least absolute deviations, and also allow fairness in weekends.

In our proposals, we have subsets of demand, workload, and ergonomic constraints present in [Gendreau *et al.* \(2007\)](#). The set of fairness constraints is different, since we do not consider the equal distribution of types of shifts, and [Gendreau *et al.* \(2007\)](#) consider it. Nevertheless, this proposal inspired us to develop a model that can be a scaffold for new models in PSPER or related problems.

[Topaloglu \(2006\)](#) tackles a variation of the PSPER, the medical residents in the ER of a Turkish University Hospital. As a solution technique, the author applied Goal Programming (GP). The author divided constraints into two sets: hard and soft constraints. In terms of fairness, this model considers fairness in the distribution of night shifts in certain days of the week, and fairness in the distribution of supervising resident positions. Our model is different in this aspect since we do not consider night shift balance or physicians supervision one another. Using a commercial optimization solver, the author better-generated solutions than the tailor-made, with soft constraints reaching higher compliance levels. The objective function is to minimize penalization for relevant deviations of soft constraints from their respective targets. The analysis of compliance for soft constraints described in [Topaloglu \(2006\)](#) gives us ideas about how to evaluate the level of compliance of our soft constraints.

[Puente *et al.* \(2009\)](#) apply Genetic Algorithms (GA) to tackle medical staff scheduling in an emergency department from a Spanish hospital. The fairness aspects that authors had to deal with are: to balance the number of different types of shifts among staff and work the same number of Saturday and Sundays per month. Regarding fairness at weekends, our approach is slightly similar to [Puente *et al.* \(2009\)](#) as we soften the weekend workload differences among the workforce throughout the planning horizon. Another aspect presented by the authors lies in defining targets for weekends, which tends to increase the workforce's level of satisfaction.

Emergency shift assignment to physicians is a particular case of PSPER. This problem consists of the assignment of physicians that are first assigned to any shift that is not on ER. After completing these hours, they must also cover some hours at the ER. [Frey, Hanne and Dornberger \(2009\)](#), working on a Swiss Hospital, considered six constraints with four fairness measures that cover: the same amount of shifts for every doctor, same amount of Sunday shifts, the same amount of Saturday shifts, and the same amount of Thursday shifts. Like [Puente *et al.* \(2009\)](#) [Rousseau, Pesant and Gendreau \(2002\)](#), and [Carter and Lapierre \(2001\)](#), authors consider a balanced distribution of weekend shifts, which is, as aforementioned, the point that we share in common. Moreover, differently from [Carter and Lapierre \(2001\)](#), here, a target is considered, thus seeking a more even distribution of the workload. Regarding the solution approach, authors applied GA and reported that the combination mutation operators must be taken into account to achieve the best solutions, which can inspire future applications of the technique on correlated problems.

As [Topaloglu \(2006\)](#) and [Gendreau *et al.* \(2007\)](#), [Topaloglu and Ozkarahan \(2011\)](#) consider a balance in the number of on-call nights as a measure of fairness among staff when dealing

with a Resident Scheduling Problem (RSP) in a Turkish University Hospital. Another measure was to establish a balance between constraints. [Topaloglu and Ozkarahan \(2011\)](#) consider a balance between the number of on-call nights and night assignments during the weekends. This kind of balance is a decent trade-off when the scheduler wants to balance personnel preferences and organization rules. This proposal provided insights on how to handle two possible conflicting constraints. For the case study reported in this thesis, we used penalization so that some constraints would have near or the same level of satisfaction. Regarding the solution approach, ([TOPALOGLU; OZKARAHAN, 2011](#)) applied Mixed-Integer Programming for small instances of the problem, while Column Generation (CG) is applied to solve larger instances. In this aspect, we differ, since we used Mixed-Integer Programming for small and large instances in one of the formulations proposed, and in others, for larger instances, we apply relax-and-fix and fix-and-optimize heuristic.

Using a Mixed-Integer Programming model, [Ferrand *et al.* \(2011\)](#) proposes cyclic schedules in a Children's Hospital in Cincinnati - USA. To obtain a fairer schedule, the assignment is repeated throughout the year. The alternative to tackle a possible schedule inflexibility is to prepare beforehand the individual schedule. Authors divided constraints into hard and soft, with most of them in the set of hard constraints that cover: regulatory constraints, work requirements, and physician preferences. Soft constraints also present regulatory constraint and physician preference. With the type of schedule, the authors report that the scheduling method provided more predictable patterns over the workweek, helping the physicians plan their activities. This measure is more similar to [Carter and Lapierre \(2001\)](#) in terms of how to deal with fairness. Regarding the definition of the constraints, both [Ferrand *et al.* \(2011\)](#) and our proposal divide constraints into hard and soft, although we use constraints of a specific group as fairness measure and the counterpart do not.

Through a Simulation model, [AlNajjar and Ali \(2011\)](#) simulates the patients' arrivals to determine the adequate number of nurses and physicians at the ER of two large public hospitals in Baghdad. The authors obtained fairness in the schedule by considering the largest team size to accommodate the highest expected patient arrival. This measure is similar to the model proposal in which we estimate the monthly workload and weekend distribution and can help the decision-makers define human resources better and decrease the expenses.

Integrated approaches to tackle the Physician Scheduling Problem (PSP) are reported in the literature and can be applied to PSPER eg., [Stolletz and Brunner \(2012\)](#) who addresses shift scheduling problem and integrated physician preferences and fairness aspects into the scheduling model. In this work, we see that PSPER can be part of a more complex problem, in which schedulers aim to respond the demand of service. To other important related ER problems we refer the reader to [Gul and Guneri \(2015\)](#) and [Aringhieri *et al.* \(2017\)](#) who present a literature review of Emergency Department simulations for normal and disaster conditions, and Emergency Medical Services systems, respectively.

[Gharbi, Louly and Azaiez \(2017\)](#) proposes a GP model and considers fairness in the allocation of weekends and night, as [Puate *et al.* \(2009\)](#), with the exception for seniority reasons. This approach is applied when preparing monthly schedules to a Saudi Arabia emergency room. The problem has two groups of constraints: hard and soft constraints. We consider some of these constraints, e.g., no more than one shift per day per physician. A feature that we borrowed from one of our models is to relax the constraint related to isolated days-off.

[Vermuyten *et al.* \(2018\)](#) combined Variable Neighborhood Descent Search (VNDS) and CG to tackle an integrated approach to staff scheduling on an Emergency Service Health Center in Portugal named INEM. Medical doctors and psychologists' schedules are built by the institutions they belong to, whereas the INEM builds nurses and technical personnel schedules. Fairness measures are: every person needs to work at least a predetermined number of night shifts, morning shifts, and afternoon shifts, and working hours should be met as much as possible. The similarity that we find here with ours lies in the workload. Both [Vermuyten *et al.* \(2018\)](#) and our proposal assume the desired number of hours as a target.

Fairness is a matter of concern when [Damcı-Kurt *et al.* \(2019\)](#) deals with the physician scheduling problem faced by Flagstaff Medical Center Hospitals in the United States. Authors consider a set of constraints considered equalization rules, which aim to leverage equalization of workload constraints to obtain schedules with fewer soft violations. Particularly they try to balance the number of night work, number of weekend assignments, and even distribution of other shift types between the personnel. This type of fairness is similar to our approach as we tackle the least absolute deviation to leverage the workload between the personnel. The authors use Mixed-Integer Programming model as e.g., [Beaulieu *et al.* \(2000\)](#), [Ferrand *et al.* \(2011\)](#). They apply relaxation variables to control violation, which are latter penalized in the objective function, when modeling this highly constrained problem. We use similar approach as will be described latter.

[Ghathbar, Louly and Mrad \(2019\)](#) proposed a Iterated Local Search (ILS) procedure to tackle the physician scheduling problem in emergency rooms of a Saudi Hospital University Hospital. Fairness measures cover an equal number of weekends off for all physicians and night shifts for all physicians. Different from us, they consider weekends off in the definition of the problem, while we consider the weekends on assignment. We can translate the measure as even distribution of weekends, which we apply as well. As [Gharbi, Louly and Azaiez \(2017\)](#), these constraints may suffer violations in case of seniority reasons or other priority rules.

Like [Carter and Lapierre \(2001\)](#) that compiled characteristics of a hospital and modeled using MIP, [Tan, Gan and Ren \(2019\)](#) compiled a set of management rules, physicians' preferences, between other characteristics of a Chinese hospital to build a Mixed-Integer Programming model to physician schedules in emergency rooms. Fairness in this problem is very similar to [Topaloglu \(2006\)](#), [Gendreau *et al.* \(2007\)](#), and [Topaloglu and Ozkarahan \(2011\)](#) since all of them aim to schedule fairly and reasonably the number of night shifts.

After being a subject of study in the early 2000s, Hospital Sacré-Cœur was a case of study in the late 2010s, when [Camiat *et al.* \(2019\)](#) proposed productivity-driven schedules to emergency physicians to align physician productivity with demand without losing fairness between physician. Like [Beaulieu *et al.* \(2000\)](#), fairness here is related to the distribution of shifts. Although our proposal and [Camiat *et al.* \(2019\)](#) aim to increase physician welfare, we address two different measures of fairness.

[Wickert *et al.* \(2020\)](#) tackles the physician scheduling problem in multiple location at Hospital de Clínicas de Porto Alegre (HCPA), Brazil, which between other departments covers emergency rooms. In the work, authors tackle workload fairness, adding constraint and formulating an extended model, and the results overcame the previous ones. We tackle the workload balance in our proposal by changing the way workload constraints are defined. We add the concept of the least absolute deviations. Moreover, we reformulated the cited work by adapting some constraints and objective function to achieve the balance in the workload. To solve this problem [Wickert *et al.* \(2020\)](#) proposed an integer programming model for small-sized instances and a fix-and-optimize math-heuristic for large instances, with a constructive heuristic based on the hard constraints of the problem. We also apply a similar procedure when solving a variant of our problem, but we use a relax-and-fix heuristic to construct the initial solution.

2.1.2 *PSPER without fairness*

We briefly describe here some works dealing with PSPER but without fairness. [Lo and Lin \(2011\)](#) addressed the PSPER by means of Particle Swarm Optimization (PSO) in a Taiwanese Hospital. Solving a mid-constrained problem, authors managed to find solutions without the violation of any hard constraint, limited violations of soft constraints in a relatively short execution time. There are no measures of fairness when scheduling physician in this proposal. A feature that we would suggest to fairness is adapt workload constraints. Instead of ranging in a set of values, a target number can be considered, so the majority of physicians can achieve it.

[Savage *et al.* \(2015\)](#) optimizes the physician shift schedules by minimizing the difference between physician productivity and patient demand in the emergency room. The problem consists of four main constraints based on uncertainty on patient arrival. This problem and ours do not have points in common when it is related to constraints, but resembles in the solution technique, as both use Mixed-Integer Programming model.

[Marchesi, Hamacher and Fleck \(2020\)](#) proposes an integrated approach to physician staffing and scheduling applying Stochastic Optimization (SO) and Discrete Event Simulation (DES). In the physician staffing problem, the goal is to determine the ideal amount of physician staff per time interval, while disregarding the scheduling problem. On the other hand, physician scheduling problem regards the staffing levels as known. In the set of constraints, none of them seem to express fairness, but all are common in the realm of physician scheduling problem. Most

of them are related to contractual agreements such as maximum number of consecutive hours of work and, if physicians are assigned to the night shift, the interval between assignments should be higher than when they are assigned to other shifts.

Table 1 – Literature on physician scheduling problem in emergency rooms

Author	Fairness	Country	Fairness aspect	Solution Technique
Beaulieu <i>et al.</i> (2000)	✓	CA	Workload balance; Balanced night shifts and duties; Balance on the assignment of antagonist shift.	MIP
Carter and Lapierre (2001)	✓	CA	Cyclic schedule; Distribution of weekend shifts; Distribution of night shifts; Overall distribution of shifts.	B&B
Rousseau, Pesant and Gendreau (2002)	✓	CA	Distribution constraints	CP, LS, GA
Topaloglu (2006)	✓	TR	Distribution of night shifts in certain days of the week; Distribution of supervising resident positions.	GP
Gendreau <i>et al.</i> (2007)	✓	CA	Equal distribution of types of shifts.	TS
Puente <i>et al.</i> (2009)	✓	ES	Balanced number of different types of shifts; Work the same number of Saturday and Sundays per month.	GA
Frey, Hanne and Dornberger (2009)	✓	CH	Same amount of shifts; Same amount of Sunday shifts; Same amount of Saturday shifts; Same amount of Thursday shifts.	GA
Topaloglu and Ozkarahan (2011)	✓	TR	Balance in the number of on-call night; Balance between on-call nights and night during the weekends	MIP, CG
Ferrand <i>et al.</i> (2011)	✓	US	Cyclic schedule	MIP
AlNajjar and Ali (2011)	✓	IQ	Team size	SIM
Lo and Lin (2011)	×	TW		PSO
Savage <i>et al.</i> (2015)	×	CA		MIP
Gharbi, Louly and Azaiez (2017)	✓	SA	Distribution of weekend shifts; Distribution of night shifts.	GP
Devesse, Santos and Toledo (2017)	✓	BR	Workload balance; Balanced night shifts and duties; Balance on the assignment of antagonist shift.	MIP, RFFO
Vermuyten <i>et al.</i> (2018)	✓	PT	Even distribution of shifts; Even distribution working hours.	IP, CG, VNDS
Damcı-Kurt <i>et al.</i> (2019)	✓	US	Distribution of night shifts; Distribution of weekend; Even distribution of shifts.	MIP
Ghathbar, Louly and Mrad (2019)	✓	CN	Weekends off balance; Balanced night shifts.	ILS
Tan, Gan and Ren (2019)	✓	CN	Distribution of night shifts.	MIP
Camiat <i>et al.</i> (2019)	✓	CA	Distribution of shifts	MIP
Wickert <i>et al.</i> (2020)	✓	BR	Balanced distribution of shifts	MIP, FOH
Marchesi, Hamacher and Fleck (2020)	×	BR		SP, DES

CG- Column Generation; CP - Constraint Programming; DES - Discrete Event Simulation; FOH - Fix-and-Optimize heuristic;
GA - Genetic Algorithms; GP - Goal Programming; ILS - Iterated Local Search; LS - Local Search; MIP - Mixed-Integer Programming;
PS - Particle Swarm Optimization; SIM - Simulation Model; SP - Stochastic Programming; TS - Tabu Search;
VNDS - Variable Neighborhood Descent Search; VNS - Variable Neighborhood Search; RFFO - Relax and Fix and Fix-and-Optimize heuristic

2.2 Final remarks and discussion

This chapter studied the PSPER and presented the related works on the area since the early 2000s. With this small survey, we aim to fill the literature gap that lacks exploration compared to Nurse Scheduling Problem. Most of PSPER and related problems such as Nurse Rostering are NP-hard with an inherent complex structure, even when the problem size is relatively small

([RAHIMIAN; AKARTUNALI; LEVINE, 2017](#)). The use of mathematical programming to solve this problem is also challenging and the most used approach. From the 21 collected papers, 14 are based in mathematical programming or have mathematical programming techniques involved in the resolution process.

A discussion we raise in this final section is related to fairness. Some of the works, e.g., [Carter and Lapierre \(2001\)](#), [Rousseau, Pesant and Gendreau \(2002\)](#), [Topaloglu \(2006\)](#) consider fairness the simple aspect of creating bounds to certain variables like weekends, workload, and night shifts. On the other hand, [Beaulieu *et al.* \(2000\)](#) aims to balance weekly hours, nights, and conflicting shifts. With this measure, we can infer that fairness is not only a matter related to balance between the workforce, but it can be defined in the approximation to maximum or minimum values a variable can assume. Practitioners can be deceived if they observe just one side of the coin.

The country that mostly explores this kind of problem seems to be Canada, with the [CIRRELT](#) group presenting a solid background and published papers. Besides this research group, we refer the reader to conferences of the area, such as [PATAT](#), and [MISTA](#), where the latest advances on timetabling and personnel scheduling problems are reported.

Regarding the trends on the literature of physician scheduling on emergency rooms, the integration of staffing and scheduling, as well as stochastic demand, e.g., [Vermuyten *et al.* \(2018\)](#) and [Marchesi, Hamacher and Fleck \(2020\)](#) are two directions fellow OR practitioners in the emergency room must increase attention since the variability of patient arrival is a recurrent situation on emergency rooms.

PHYSICIAN SCHEDULING PROBLEM WITH FAIRNESS IN WORKLOAD

This chapter's motivation dwells on the possibility of minimizing damages and losses derived from conflicts between the satisfaction of personal preferences and labor rules. These issues directly affect doctors' welfare, such as burnout, one of the problems that most affect emergency services professionals (PEREIRA, 2017). We study and reformulate a mathematical model from the literature to deal with fairness in physician scheduling hours. Thus, we propose and reformulate three models and evaluate them using benchmark instances. A brief description of the problem is presented in Section 3.1. Three mathematical formulations are described in Section 3.2 and their computational performance are evaluated at Section 3.3. Next, the main formulation proposed in Section 3.4, and its computational results are discussed in Section 3.5. Our conclusions about the findings at this step of the research are presented in Section 3.6.

3.1 Problem Description

Physician Scheduling consists of assigning a task to each available physician (cover shift or duty) in the planning horizon days. Each day, the physician can be assigned to a specific shift. As pointed by Osogami and Imai (2000), Physician Scheduling Problems by the association to Nurse Scheduling Problems are NP-hard, computationally challenging, and complicated even for small-sized problems.

Let $P = \{p_1, \dots, p_{|P|}\}$ be the set of physician to schedule throughout days of the planning horizon $D = \{d_1, \dots, d_{|D|}\}$. We consider that physicians can be assigned to one of the shifts of the set $S = \{s_1, \dots, s_{|S|}\}$. To design a schedule, we use a physician-day-shift view. This perspective is a direct depiction of a three-dimensional schedule. Accordingly, the decision variables can be defined for each physician, on each day, on each shift as X_{pds} . Variables are binary, as shown in Figure 3. This example illustrates a week schedule of five physicians in two shifts.

Figure 3 – The physician-day-shift view and the definition of variables X_{pds} .

	D ₁			D ₂			D ₃			D ₄			D ₅		
P ₁	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0
P ₂	1	0	0	1	0	0	0	1	0	0	1	0	0	0	0
P ₃	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0
P ₄	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1
P ₅	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0

Source: Elaborated by the author.

The problem and model we describe in this Section derives from [Curtois and Qu \(2014\)](#). In the scenario, the workforce has a given number of hired physicians of different skill categories. However, their timetable is not always enough to cover the legal minimum amount of personnel presence in every shift. Thus, the responsible person for staff management can hire part-time staff to cope with the demand. The objective is to assign physicians reducing the hiring costs measured by demand coverage and workload compliance, and maximize physicians' preferences.

The set of requirements, technically denominated the problem's constraints, are divided into hard and soft constraints. Hard constraints are mandatory and must be satisfied to have a feasible solution. According to the degree to which they are satisfied, soft or flexible constraints are those defining the quality of a generated roster. Soft constraints are those that allow violation.

The constraints of the problem are explained next following the formulation presented by ([CURTOIS; QU, 2014](#)):

• Hard Constraints

1. Physicians cannot work to more than one shift on a day.
2. Physicians cannot work on conflicting shifts on two consecutive days, e.g., if one is working on a night shift, he/she cannot work in the following morning or afternoon.
3. Each Physician has a maximum amount of shift of each type that she/he can be assigned to within the planning period.
4. Each Physician has a minimum and a maximum amount of total time in hours that can be assigned to within the planning period.
5. Each Physician has a maximum number of consecutive shifts in a stretch within the planning period.
6. Each Physician has a minimum number of consecutive shifts and consecutive days off, which are allowed to be worked within the planning period.

7. Each Physician has a maximum number of worked weekends within the planning period.
8. Physician has the right to holidays whenever she/he requires.

- **Soft/Flexible constraints**

1. Physician has preferences to be assigned in certain shifts.
2. There is a minimum and maximum number of Physician required to work on a shift of a day.

There are some conventions to avoid possible conflicts when creating the schedule. Regarding the assignment to antagonist and consecutive shifts, we assume that the last day of the previous planning period and the first day of the next planning horizon are days off. Regarding the constraints related to minimum consecutive and days-off, a physician reached the necessarily required assignment of straight at the end of the previous planning period and the start of the next planning period. Moreover, finally, all instances start on a Monday, and the planning horizon is always a multiple of 7 to have a whole number of weeks.

3.2 Mathematical formulations

We introduce three mathematical formulations in this section. The first one is the Nurse Scheduling Problem (NSP) model proposed by [Curtois and Qu \(2014\)](#). Afterward, we describe changes to include fairness has a relevant feature to be considered. First, we present the formal notation, where parameters and decision variables for all formulations are defined. Next, we explain the mathematical formulation for each variant of the Physician Scheduling Problem.

3.2.1 Nurse Scheduling Problem (NSP)

Hard constraints are defined in equations (3.1)-(3.9), and soft constraints are defined in equations (3.10) and (3.11).

H1: Single shift assignment

An employee cannot be assigned to more than one shift in a day.

$$\sum_{s \in S} x_{pds} \leq 1, \quad \forall p \in P, d \in D. \quad (3.1)$$

H2: Antagonist shift control

A minimum amount of rest is required after each shift. Therefore, some shifts cannot follow others. For example, a morning shift cannot follow a night shift.

$$x_{pds} + x_{p(d+1)i} \leq 1, \quad \forall p \in P, d \in \{1, \dots, |D| - 1\}, s \in S, i \in \Upsilon_s. \quad (3.2)$$

Sets and Parameters

P	Set of physicians.
D	Set of days in the planning horizon.
S	Set of shifts.
Υ_s	Set of shifts that cannot follow immediately shift s .
M	Set of months in the planning horizon.
W	Set of weeks in the planning horizon.
$(p, d) \in U_p$	Set containing the pairs of unavailable days d for physician p .
D_m	Set of days of month m .
D_{ds}	Number of physicians to assign to shift s on day d .
R_p^+	Maximum number of consecutive work days for physician p .
V_p^+	Maximum number of weekends physician p can work.
H_s	Length of the s^{th} shift in hours.
T_{pm}^H	Target number of monthly hours for physician p on month m .
LS_{ps}	Maximum number of shifts of the type s worked by physician p .
A_{pm}	Maximum number of days physician p can be assigned on month m .
b_p^{min}	Minimum hours a physician has to comply in the planning horizon.
b_p^{max}	Maximum hours a physician has to comply in the planning horizon.
c_p^{max}	Minimum number of consecutive days physician p can work.
c_p^{min}	Minimum number of consecutive days physician p can work.
o_p^{min}	Minimum number of consecutive days-off by physician p .
q_{pds}^+	Weights for shift on requests by physician p on day d and shift s .
q_{pds}^-	Weights for shift off requests by physician p on day d and shift s .
Variables	
$x_{pds} \in \mathbb{B}$	1 if physician p is assigned to shift s on day d , 0 otherwise.
$y_{pw} \in \mathbb{B}$	1 if physician p works weekend w , 0 otherwise.
w_{ds}	Negative deviation from the demand constraint in shift s on day d .
z_{ds}	Positive deviation from the demand constraint in shift s on day d .
σ_{pm}^H	Deviation between accomplished hours of physician p during month m and target T_{pm}^H .
$\bar{T}_{pm}^H$	Number of monthly hours for physician p on month m .

H3: Maximum number of shifts

The set of constraints (3.3) ensures a maximum number of shifts of each type that can be assigned to physicians, e.g., contractual requirements allow only a maximum number of night shifts.

$$\sum_{d \in D} x_{pds} \leq LS_{ps}, \quad \forall p \in P, s \in S. \quad (3.3)$$

H4: Maximum consecutive work days

This constraint ensures that an employee can work the maximum number of consecutive days without a day off is not violated. For example, part-time staff often work less consecutive

days than full-time staff.

$$\sum_{z=d-R_p^+}^d \sum_{s \in S} x_{pzs} \leq R_p^+, \quad \forall p \in P, d \in \{R_p^+ + 1, \dots, |D|\}. \quad (3.4)$$

H5: Minimum consecutive work days and days-off

Minimum consecutive work days and days-off are modeled in constraints (3.5) and (3.6), respectively. Regarding minimum consecutive days, if the minimum number of consecutive shifts is four then we must not allow any of the sequences: off-on-off, off-on-on-off, off-on-on-on-off where off is a day without a shift and on is a day with a shift assigned. Regarding minimum consecutive days-off, if the minimum number of consecutive days off is three then we must not allow any of the sequences: on-off-on, on-off-off-on.

$$\sum_{s \in S} x_{pds} + c - \sum_{j=d+1}^{d+c} \sum_{s \in S} x_{pjs} + \sum_{s \in S} x_{p(d+c+1)s} \geq 1, \quad \forall p \in P, c \in \{1 \dots c_p^- - 1\}, d \in \{1 \dots |D| - (c+1)\}. \quad (3.5)$$

$$1 - \sum_{s \in S} x_{pds} + \sum_{j=d+1}^{d+o} \sum_{s \in S} x_{pjs} + \sum_{s \in S} x_{p(d+o+1)s} \geq 0, \quad \forall p \in P, o \in \{1 \dots o_p^- - 1\}, d \in \{1 \dots |D| - (o+1)\}. \quad (3.6)$$

H6: Weekend assignment

A weekend is assumed to be worked if the employee has a shift on a Saturday or Sunday, as ensured by (3.7), while the maximum number of weekends is handled by (3.8).

$$\sum_{s \in S} (x_{p(7w-1)s} + x_{p(7w)s}) \leq 2y_{pw}, \quad \forall p \in P, w \in W. \quad (3.7)$$

$$\sum_{w \in W} y_{pw} \leq V_p^+, \quad \forall p \in P. \quad (3.8)$$

H7: Holidays

The constraint on equation (3.9) indicates days that employees cannot work, e.g., due to holidays, leaves or training.

$$x_{pds} = 0, \quad \forall p \in P, (d, s) \in U_p. \quad (3.9)$$

SI: Demand

We consider that the demand should be satisfied, but deviations for over- and under-staffing are allowed, which are penalized in the objective function.

$$\sum_{p \in P} x_{pds} - z_{ds} + w_{ds} = D_{ds}, \quad \forall d \in D, s \in S. \quad (3.10)$$

S2: Maximum and Minimum workload

The formulation described in [Curtois and Qu \(2014\)](#) defined hour balance constraints as in expression (3.11), where parameters $b_p^{\min}$ and $b_p^{\max}$ are, respectively, minimum and maximum hours a physician has to comply in the planning horizon.

$$b_p^{\min} \leq \sum_{d \in D} \sum_{s \in S} H_s x_{pds} \leq b_p^{\max}, \quad \forall p \in P. \quad (3.11)$$

Obj1: Objective Function

The objective function (3.12) maximizes the assignment of employee shift requests and minimizes under- and over-staffing. In the first term, parameters q_{pds} and p_{pds} are the weights for the shift on and shift off requests, respectively, e.g., an employee may request to work a specific shift-type on a particular day. The higher the weight, the more relevant the request is to the employee. If there is no request, then the parameter is zero. Since the variables w_{ds} , and z_{ds} give us the total numbers of staff below and above the preferred cover level for each shift, parameters v_{ds}^- and v_{ds}^+ are weights indicating the importance of under- and over-coverage.

$$\text{Minimize } \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} \left(q_{pds}(1 - x_{pds}) + p_{pds}x_{pds} \right) + \sum_{d \in D} \sum_{s \in S} \left(v_{ds}^- w_{ds} + v_{ds}^+ z_{ds} \right) \quad (3.12)$$

3.2.2 Physician Scheduling Problem with fairness in hours

The first reformulation we propose from the previous model considers a slight variation on hour constraint and objective function. We named such formulation as Physician Scheduling Problem with fairness in hours (PSPER-H), where the concept of least absolute deviation with hours as a target is employed. Another modification dwells in the objective function, where we add the deviation found in hour constraints. The target we define is the number of hours expected to be fulfilled by each physician. We determined by the average of the maximum and minimum working hours that each physician can accomplish.

S3: Hour balance

The next soft constraints describe fairness for workload, by measuring the overall

deviation from a target number of hours for each physician.

$$\sigma_{pm}^H \geq \sum_{d \in D_m} \sum_{s \in S} H_s x_{pds} - T_{pm}^H, \quad \forall p \in P, m \in M. \quad (3.13)$$

$$\sigma_{pm}^H \geq T_{pm}^H - \sum_{d \in D_m} \sum_{s \in S} H_s x_{pds}, \quad \forall p \in P, m \in M. \quad (3.14)$$

Thus, we replace constraints (3.11) from [Curtois and Qu \(2014\)](#) by variables σ_{pm}^H , which give us the deviation of hours for physician p during month m . The deviation is measured from a target number of hours defined in expression (3.15).

$$T_{pm}^H = \frac{b_p^{\max} + b_p^{\min}}{2 \times |M|}, \quad (3.15)$$

Obj2: Objective function

The overall deviation is minimized in the objective function, where a third term is added from objective function (3.16).

$$\text{Minimize } \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} \left(q_{pds}(1 - x_{pds}) + p_{pds}x_{pds} \right) + \sum_{d \in D} \sum_{s \in S} \left(v_{ds}^- w_{ds} + v_{ds}^+ z_{ds} \right) + \sum_{p \in P} \sum_{m \in M} \left(\sigma_{pm}^H \right) \quad (3.16)$$

Thus, the complete formulation for PSPEH-H is summarized next.

$$\begin{aligned} & \text{Minimize} \quad (3.16) \\ & \text{subject to} \quad (3.1) - (3.10) \\ & \quad \quad \quad (3.13), (3.14) \end{aligned}$$

3.2.3 Second formulation for PSPEH-H (PSPEH-H2)

The second formulation aims to create a more realistic model with features borrowed from real case scenarios, e.g.: [Topaloglu \(2006\)](#) and [Wickert et al. \(2020\)](#). Both [Topaloglu \(2006\)](#), and [Wickert et al. \(2020\)](#) deal with constraints of consecutive work days and their approach inspired us to relax such hard constraints. The new constraints have slack variables to allow violations in constraints (3.4)-(3.6).

H8: Maximum consecutive work days

We introduce the slack variable ∇_p^- to allow violation in the maximum number of consecutive work days as follows.

$$\sum_{z=d}^{d+c_p^{max}} \sum_{s \in S} x_{pzs} - \nabla_p^- \leq c_p^{max}, \quad \forall p \in P, d \in \{1, \dots, |D| - c_p^{max}\}. \quad (3.17)$$

H9: Minimum consecutive work days and days-off

The slack variables Φ_p^+ and Λ_p^+ will allow violation over the minimum amount of work days and days-off, respectively.

$$\sum_{s \in S} x_{pds} + c - \sum_{j=d+1}^{d+c} \sum_{s \in S} x_{pjs} + \sum_{s \in S} x_{p(d+c+1)s} + \Phi_p^+ \geq 1, \\ \forall p \in P; c \in \{1 \dots c_p^{min} - 1\}; d \in \{1 \dots |D| - (c + 1)\}. \quad (3.18)$$

$$1 - \sum_{s \in S} x_{pds} + \sum_{j=d+1}^{d+o} \sum_{s \in S} x_{pjs} + \sum_{s \in S} x_{p(d+o+1)s} + \Lambda_p^+ \geq 0, \\ \forall p \in P; o \in \{1 \dots o_p^{min} - 1\}; d \in \{1 \dots |D| - (o + 1)\}. \quad (3.19)$$

Obj3: Objective function

There are penalties associated to these variables with the same values, since all of them have the same level of importance. The parameter w_p is the weight associated to these violations in the objective function.

$$\min \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} \left(q_{pds} (1 - x_{pds}) + p_{pds} x_{pds} \right) + \sum_{d \in D} \sum_{s \in S} \left(v_{ds}^- w_{ds} + v_{ds}^+ z_{ds} \right) \\ + \sum_{p \in P} \sum_{m \in M} \left(\sigma_{pm}^H \right) + \sum_{p \in P} w_p \left(\nabla_p^- + \Phi_p^+ + \Lambda_p^+ \right) \quad (3.20)$$

The formulation for PSPER-H2 is summarized next.

$$\begin{aligned} & \text{Minimize} && (3.20) \\ & \text{subject to} && (3.1) - (3.4) \\ & && (3.7) - (3.10) \\ & && (3.13) - (3.14) \\ & && (3.17) - (3.19) \end{aligned}$$

3.3 Computational results from NSP, PSPER-H and PSPER-H2

We tested the proposed formulations using benchmark instances also evaluated by [Curtois and Qu \(2014\)](#) and [Rahimian, Akartunali and Levine \(2017\)](#). Table 2 presents the main aspects of such benchmark instances.

Table 2 – The characteristics of the set of benchmark instances. Available in [Curtois \(2007\)](#).

Instance	Days	Physicians	Shift types	Day off requests	Shift on/off requests
Instance01	14	8	1	8	26
Instance02	14	14	2	14	62
Instance03	14	20	3	20	64
Instance04	28	10	2	20	71
Instance05	28	16	2	32	106
Instance06	28	18	3	36	135
Instance07	28	20	3	40	168
Instance08	28	30	4	60	225
Instance09	28	36	4	72	232
Instance10	28	40	5	80	284
Instance11	28	50	6	100	336
Instance12	28	60	10	120	422
Instance13	28	120	18	240	841
Instance14	42	32	4	128	359
Instance15	42	45	6	180	490
Instance16	56	20	3	120	280
Instance17	56	32	4	160	480
Instance18	84	22	3	176	414
Instance19	84	40	5	320	834
Instance20	182	50	6	900	2318
Instance21	182	100	8	1800	4702
Instance22	364	50	10	1800	4638
Instance23	364	100	16	3600	9410
Instance24	364	150	32	5400	13809

The mathematical formulations were coded in Java using callable libraries from IBM ILOG CPLEX 12.8 optimization solver inside the ProOF framework proposed by [Arantes \(2014\)](#) and ran on an Intel Xeon E5-2680v2 2.8 GHz processor with 128GB of RAM. Table 3 presents the results obtained after running the formulations with a time limit of 3600 seconds. The parameters UB , LB , and GAP correspond, respectively, to the upper bound, lower bound, and gap defined in equation (3.21). Finally, TT stands for the total time spent by the solver on each instance. When finishing the total time without finding a feasible solution, we draw a hyphen in the instance's cells.

$$GAP = \frac{UB - LB}{UB} \times 100, \quad (3.21)$$

In numerical terms, there is a significant improvement in the performance solving PSPER-H and PSPER-H2 formulations compared to NSP from [Curtois and Qu \(2014\)](#). In 19 of the 24 instances, CPLEX solver found feasible solutions when executing PSPER-H and PSPER-H2, against 13 out of 24 when executing the NSP model. The inclusion of the least absolute deviation and slack variables on consecutive workday constraints facilitated finding feasible solutions. This behavior is expected since constraints (3.9) allow us to relax the NSP formulation in PSPER-H.

Table 3 – Results of experiments on *PSPER – H*, *PSPER – H2* and *NSP* (that does not consider fairness)

Instances	PSPER-H				PSPER-H2				NSP			
	UB	LB	GAP	TT	UB	LB	GAP	TT	UB	LB	GAP	TT
Instance01	716	716	0.00%	0.54	716	716	0.00%	0.32	607	607	0.00%	0.62
Instance02	2174	2029.4	6.65%	3600	2174	2038	6.26%	3600	828	828	0.00%	1.55
Instance03	2600	2390	8.08%	3600	2600	2600	0.00%	7.06	1001	1001	0.00%	205.18
Average			4.91%	2400.18			2.09%	1202.46	812.00	812.00	0.00%	69.11
Instance04	2321	2258.73	2.68%	3600	1909	1909	0.00%	74.08	1716	1716	0.00%	759.55
Instance05	2588	2583.7	0.17%	3600	2588	2588	0.00%	1993.09	1143	1143	0.00%	1064.94
Instance06	2248	2238.4	0.43%	3600	2102	2086.36	0.74%	3600	1950	1950	0.00%	2857.30
Instance07	3246	3242.3	0.11%	3600	3246	3246	0.00%	2477.4	1056	1050.52	0.52%	3600
Instance08	8008	5058.29	36.83%	3600	7950	5058.22	36.37%	3600	1315	1272.38	3.24%	3600
Instance09	4778	4321.3	9.56%	3600	4875	4352.65	10.71%	3600	439	182.5	58.43%	3600
Instance10	7763	6353.49	18.16%	3600	7106	6326.59	10.97%	3600	4631	4627.02	0.09%	3600
Instance11	6861	6492.22	5.38%	3600	6518	6491.9	0.40%	3600	3443	3443	0.00%	415.73
Instance12	11006	10522.5	4.39%	3600	10577	10522.4	0.52%	3600	4043	4035.37	0.19%	3600
Instance13	42183	6600.65	84.35%	3600	229424	6601.11	97.12%	3600	-	-	-	-
Average			16.21%	3600			15.68%	2974.46			6.94%	2566.39
Instance14	87053	83714.6	3.83%	3600	26721	26252.2	1.75%	3600	-	-	-	-
Instance15	81220	61491.8	24.29%	3600	19539	17458.9	10.65%	3600	-	-	-	-
Average			14.06%	3600	23130	21855.55	6.20%	3600			-	-
Instance16	9795	8932.25	8.81%	3600	9240	8934.03	3.31%	3600	-	-	-	-
Instance17	17699	12733.2	28.06%	3600	14343	11857.6	17.33%	3600	-	-	-	-
Average			18.43%	3600	-	-	10.32%	3600			-	-
Instance18	9795	8932	8.81%	3600	9240	8934	3.31%	3600	-	-	-	-
Instance19	17699	12733	28.06%	3600	14343	11858	17.33%	3600	4186	2942.11	29.72%	3600
Average			18.43%	3600			10.32%	3600			29.72%	3600
Instance20-24	-	-	-	-	-	-	-	-	-	-	-	-
Average			-	-			-	-			-	-

The formulation is even more relaxed when slack variables are inserted in PSPER-H2 through constraints (3.18) and (3.19), as one can observe in objective function values.

We evaluate the fairness aspect to a better quality performance among these three formulations by employing measures to evaluate fairness based on demand satisfaction and work hours. The demand violation is measured following equations (3.22)-(3.24).

$$demViol_{ds} = \begin{cases} 1, & \text{if } \sum_{p \in P} x_{pds} - z_{ds} + w_{ds} < D_{ds}, \\ 0, & \text{otherwise,} \end{cases} \quad (3.22)$$

$$\Delta_{dem} = \frac{\sum_{d,s} demViol_{ds}}{TotalDemConstr} \times 100, \quad (3.23)$$

$$GAP_{dem} = \frac{\Delta_{dem} - \Delta_{dem}^{NSP}}{\Delta_{dem}} \times 100. \quad (3.24)$$

The work hours violation is measured following equations (3.25)-(3.27).

$$hourViol_{pm} = \begin{cases} 1, & \text{if } |\sum_{d \in D_m} \sum_{s \in S} H_s x_{pds} - T_{pm}^H| > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (3.25)$$

$$\Delta_H = \frac{\sum_{p,m} \text{hourViol}_{pm}}{\text{TotalHourConstr}} \times 100, \quad (3.26)$$

$$GAP_H = \frac{\Delta_H(i) - \Delta_H^{NSP}}{\Delta_H} \times 100. \quad (3.27)$$

Table 4 shows the results for PSPER-H, PSPER-H2, and NSP.

Table 4 – Hour and demand violation rate for PSPER-H, PSPER-H2 and NSP formulations.

Instance	PSPER-H				PSPER-H2				NSP	
	Δ_{dem}	Δ_H	GAP_{dem}	GAP_H	Δ_{dem}	Δ_H	GAP_{dem}	GAP_H	Δ_{dem}	Δ_H
Instance01	14.00%	0.00%	-104.07%	∞	29.00%	25.00%	1.48%	-48%	28.57%	37.00%
Instance02	68.00%	79.00%	85.29%	27.85%	25.00%	36.00%	60.00%	-58.33%	10.00%	57.00%
Instance03	21.00%	25.00%	47.62%	-120.00%	19.00%	25.00%	42.11%	-120.00%	11.00%	55.00%
Instance04	14.00%	100.00%	0.00%	0.00%	9.00%	100.00%	-55.56%	0.00%	14.00%	100.00%
Instance05	77.00%	100.00%	40.26%	0.00%	38.00%	100.00%	-21.05%	0.00%	46.00%	100.00%
Instance06	55.00%	83.00%	72.73%	26.51%	15.00%	0.00%	0.00%	∞	15.00%	61.00%
Instance07	38.00%	100.00%	78.95%	0.00%	15.00%	100.00%	46.67%	0.00%	8.00%	100.00%
Instance08	6.00%	100.00%	-33.33%	0.00%	4.00%	100.00%	-100.00%	0.00%	8.00%	100.00%
Instance09	13.00%	72.00%	69.23%	-38.89%	10.00%	69.00%	60.00%	-44.93%	4.00%	100.00%
Instance10	15.00%	100.00%	-6.67%	0.00%	16.00%	100.00%	0.00%	0.00%	16.00%	100.00%
Instance11	11.00%	100.00%	0.00%	0.00%	11.00%	100.00%	0.00%	0.00%	11.00%	100.00%
Instance12	9.00%	100.00%	22.22%	0.00%	10.00%	100.00%	30.00%	0.00%	7.00%	100.00%
Instance13	13.00%	60.00%	-	-	10.00%	94.00%	100.00%	-	-	-
Instance14	13.69%	90.00%	-	-	9.00%	69.00%	100.00%	-	-	-
Instance15	15.00%	93.00%	-	-	10.00%	55.00%	100.00%	-	-	-
Instance16	12.00%	100.00%	-	-	17.00%	100.00%	100.00%	-	-	-
Instance17	15.00%	100.00%	-	-	13.00%	100.00%	100.00%	-	-	-
Instance18	30.00%	100.00%	-	-	24.00%	100.00%	100.00%	-	-	-
Instance19	16.00%	100.00%	31.25%	0.00%	20.00%	100.00%	45.00%	0.00%	11.00%	100.00%
Instance20-24	-	-	-	-	-	-	-	-	-	-

When solving the NSP model using CPLEX solver, we can notice better results 46% of the instances with lower values of Δ_{dem} , against 23% of PSPER-H2, and approximately 1% of PSPER-H. It is essential to notice that these values are related to the number of violations throughout the planning horizon and not the violations' magnitude. For example, if there is an under-staffing of 1 physician daily for four days, the rate is different from having an under-staffing of 2 physicians in two days.

If the relative distance between the demands is in the range of 5%, then it means the difference is above the acceptable, and the needs a repair. Otherwise, it would be deemed acceptable. With the current penalty values for hour deviation, CPLEX solver could not find solutions below 5% of violation for PSPER-H and PSPER-H2 and satisfying fairness aspects simultaneously.

Regarding hours, we can notice better results from PSPER-H2 compared to PSPER-H, and NSP, in 23% solved instances. Nonetheless, we can see that, for most instances, there was a violation rate higher than 50% when using PSPER-H and PSPER-H2, showing that the model does violate hour constraints, even though a fair distribution of hours is accomplished. Both hour and demand present undesired violation rates when applying formulations that consider fairness.

The main reason is that both try to establish a trade-off between balanced hour distribution and demand violations. For instances 13 to 18 and 20 to 24, the CPLEX solver could not find the solution in the established time limit, and consequently, we could not compute the values for GAP_{dem} and GAP_H .

Figure 4 shows the comparison in terms of workload deviation for some of the instances. We can observe that we have a fair distribution of hours between the physicians. However, a fair workload comes with demand violations, which is a significant drawback. The demand violation rate increases as the model choose to violate what is less regarded/penalized by PSPER-H and PSPER-H2, in this case, under-staffing. Thus, an alternative to cope with under-staffing over a balanced workload is to estimate the target number of hours and seek a high degree of compliance with demand constraints with the same or higher penalties. This approach is discussed in the next section.

3.4 PSPER with working hours as variable (PSPER- H_s)

A fair assignment of working hours among physicians came with a high demand violation, which is an unwanted side effect. We introduce the PSPER with working hours as a variable (PSPER- H_s) as a formulation aiming to reduce demand violations and keep fairness in workload. Thus, the parameter T_{pm}^H becomes variable $\bar{T}_{pm}^H$ in constraints (3.13) and (3.14) as follows.

S4: Hour balance

Our approach requires the estimation of the best target value for the number of hours a physician can comply within a month, denoted by $\bar{T}_{pm}^H$.

$$\sigma_{pm}^H \geq \sum_{d \in D_m} \sum_{s \in S} H_s x_{pds} - \bar{T}_{pm}^H, \quad \forall p \in P, m \in M. \quad (3.28)$$

$$\sigma_{pm}^H \geq \bar{T}_{pm}^H - \sum_{d \in D_m} \sum_{s \in S} H_s x_{pds}, \quad \forall p \in P, m \in M. \quad (3.29)$$

We note that the formulation described in [Curtois and Qu \(2014\)](#) defined hour balance constraints as in expression (3.11), where parameters b_p^{min} and b_p^{max} are, respectively, minimum and maximum hours a physician has to comply in the planning horizon. We apply a similar idea to define an upper bound for $\bar{T}_{pm}^H$.

H10: Upper bound for $\bar{T}_{pm}^H$.

$$\bar{T}_{pm}^H \leq (A_{pm} \times R_p^+) + \bar{A}_{pm}, \quad \forall p \in P, m \in M. \quad (3.30)$$

A_{pm} indicates the number of cycles in a month when a maximum number of consecutive workdays and a minimum number of consecutive days-off are used, as shown by equations (3.31)

(a) Workload deviations in instance4 for PSPER-H, PSPER-H2, and NSP

(b) Workload deviations in instance5 for PSPER-H, PSPER-H2, and NSP

(c) Workload deviations in instance7 for PSPER-H, PSPER-H2, and NSP

Figure 4 – NSP doesn't consider fairness, and consequently higher and irregular workload deviations. PSPER-H and PSPER-H2 have less and constant values on workload deviations.

and (3.32).

$$A_{pm} = \frac{\text{length}(m)}{R_p^+ + o_p^-}, \quad \forall m \in M, p \in P. \quad (3.31)$$

$$\bar{A}_{pm} = \text{length}(m) \bmod (R_p^+ + o_p^-), \quad \forall m \in M, p \in P. \quad (3.32)$$

The parameter $\text{length}(m)$ has the length of the month m and A_{pm} is rounded down in equation (3.31). In equation (3.32), $\bar{A}_{pm}$ has the number of remaining days, if such cycles are

used throughout the month. Finally, we calculate the total number of hours, $(A_{pm} \times R_p^+) + \bar{A}_{pm}$, which is an upper bound for $\bar{T}_{pm}^H$.

3.5 Computational results evaluating $PSPER - H_s$

Table 5 compares the mixed-integer formulations described in Section 3.4 and the mixed integer formulation of [Curtois and Qu \(2014\)](#), NSP model, described in Section 3.2.

Table 5 – Numerical results of experiments on $PSPER - H_s$ and NSP (that does not consider fairness)

Instance	$PSPER - H_s$				NSP			
	UB	LB	GAP	TT	UB	LB	GAP	TT
Instance01	507	507	0.00%	0.18	607	607	0.00%	0.62
Instance02	708	708	0.00%	0.20	828	828	0.00%	1.55
Instance03	1000	1000	0.00%	0.63	1001	1001	0.00%	205.18
Average			0.00%	1.19			0.00%	69.11
Instance04	1716	1716	0.00%	458.68	1716	1716	0.00%	759.55
Instance05	1142	1142	0.00%	224.34	1143	1143	0.00%	1064.94
Instance06	1924	1924	0.00%	41.77	1950	1950	0.00%	2857.30
Instance07	1022	1022	0.00%	110.33	1056	1050.52	0.52%	3600
Instance08	1237	1229.43	0.61%	3600	1315	1272.38	3.24%	3600
Instance09	35	35	0.00%	2.30	439	182.5	58.43%	3600
Instance10	4627	4624.4	0.06%	3600	4631	4627.02	0.09%	3600
Instance11	3412	3411.67	0.01%	151.67	3443	3443	0.00%	415.73
Instance12	4022	4022	0.00%	3285.98	4043	4035.37	0.19%	3600
Instance13	1341	1341	0.00%	1864.43	-	-	-	-
Average			0.07%	1378.83			6.94%	2566.39
Instance14	1235	1235	0.00%	1169.86	-	-	-	-
Instance15	4636	3757.88	18.94%	3600	-	-	-	-
Average			9.47%	2384.93				
Instance16	3241	3130.05	3.42%	3600	-	-	-	-
Instance17	6042	5636	6.72%	3600	-	-	-	-
Average			5.07%	3600				
Instance18	4850	4177.83	13.86%	3600	-	-	-	-
Instance19	3821	2864.66	25.03%	3600	4186	2942.11	29.72%	3600
Average			12.26%	3600			29.72%	3600

Our proposal solves 79.17% of the instances against 54.17% using NSP. In the first three instances, corresponding to a set of fortnight planning, both formulation found optimal solutions. In the first set of instances, the instances corresponding to fortnight assignment, the individual and overall performance is favourable to our proposal, with all instances solved to optimality in an average total time of 1.19 seconds, while NSP has an average total time of 69.11 seconds. In the second set of instances, corresponding to month assignment, 80% of the instances were solved in optimality with an average total time of 767.19 seconds against 44.44% with behalf of NSP with an average total time of 1274.38 seconds.

Summarizing, in the set of instances that both NSP and $PSPER - H_s$ found feasible solutions, $PSPER - H_s$ outperforms in terms of gap and computational time. For each set of instance, classified according to planning horizon, we also calculated the average gap and computational time. The quality of solution and computational time increases as the planning

horizon grows. This means that problem relative complexity is attached to the planning horizon and also number of physicians. For every set of instance solved by both, the average quality of upper bound is clearly inferior for $PSPER - H_s$ rather than NSP. When comparing both models, $PSPER - H_s$ objective function values are lesser than NSP in all instances what strengthens the evidence that our model is a relaxation of the original formulation.

A brief explanation can be given following the principle of relaxation introduced by Wolsey (1998). Let Curtois and Qu (2014) formulation, which we'll nominate P, be IP in the general form

$$P = \min\{cx : Ax \leq b, x \in B\}$$

Let x be the variable x_{pds} and constraints $Ax \leq b$ the feasible region and cx the cost function, respectively. If this problem is difficult to solve for the set of constraints $Ax \leq b$, there are two options: we can drop this set of constraints or we relax by creating a slack variable s , which give us the constraint $Ax - s \leq b$, the cost function will have a new term ls , and the resulting problem

$$P' = \min\{cx + ls : Ax - s \leq b, x \in X \subset \mathbb{Z}\}$$

is a relaxation of the problem. In our model, equations (3.28) and (3.29) would belong to the set of constraints $Ax - s \leq b$, a relaxation of the original constraints $Ax \leq b$, described in equations (3.11).

A fairer comparison between the $PSPER - H_s$ and NSP, encompasses a demand, hour and deviation between complied hours and target number of hours (to which a standard deviation is applied). Table 6 summarizes this comparison with fields Δ_{dem} expressing the demand compliance rate, $\bar{H}$ expressing the average hour complied by the workforce, $\overline{TH}$ representing the average target hour and $\overline{\sigma_{\sigma_H}}$ representing the mean hour deviation (between target number of hours and complied hours) standard deviation. The last measure will help us understand how dispersed will the deviation of hours be throughout the instances. Higher values mean higher dispersion, and lower values, lower dispersion. When $\overline{\sigma_{\sigma_H}}$ is zero, it means that the deviation of hours is balanced, and our objective is achieved. Boldface values for δ_{dem} in each cell indicate a better degree of compliance reached with the formulation at hand.

When we apply target hour as a variable through $PSPER - H_s$, we tackle the deviation of hours at once since the effort is now to create a new distribution of hours, and approximate the deviation variable to the target at maximum. Furthermore, workload values with balanced deviation amongst the physicians are accomplished with low or no prejudice of demand compliance, as we can see from Instance05 to Instance19. The effort is to assign the most physician as possible, so the demand coverage and the target hour are covered. When solving this problem at optimality, we do not establish a fair measure to demand; the preference is to have a solution with a lower violation rate in comparison to a higher demand violation but fair. A decent approach demand violation rate aims to distribute the under or over-staffing equitably throughout the days

of the planning horizon. In contrast, we do not consider that, since the real scenarios are mostly concerned with the highest coverage rate.

Table 6 – Average Demand, Hours, Target Hours and Hour Deviation average standard deviation for *PSPER* – H_s and *NSP* formulations.

Instance	<i>PSPER</i> – H_s				NSP			
	Δ_{dem}	$\bar{H}$	$\bar{TH}$	$\overline{\sigma_{\sigma_H}}$	Δ_{dem}	$\bar{H}$	$\bar{TH}$	$\overline{\sigma_{\sigma_H}}$
Instance01	28.57%	3960	3960	0	28.57%	3900	3840	± 400
Instance02	14.00%	3463	3463	0	10.00%	3223	3223	± 399
Instance03	14.00%	3456	3456	0	11.00%	3456	3300	± 250
Instance04	25.00%	7968	7968	0	14.00%	7968	8100	± 248
Instance05	27.00%	8340	8340	0	46.00%	7860	8100	± 653
Instance06	17.00%	7627	7627	0	15.00%	7013	7440	± 900
Instance07	7.00%	7464	7464	0	8.00%	6576	7020	± 991
Instance08	4.00%	7520	7520	0	8.00%	7520	7472	± 219
Instance09	0.00%	5707	5707	0	4.00%	5640	5190	± 138
Instance10	16.00%	7935	7935	0	16.00%	7923	7980	± 366
Instance11	10.00%	7469	7469	0	11.00%	7670	8100	± 68
Instance12	5.00%	7768	7768	0	7.00%	7432	7520	± 566
Instance13	1.00%	7820	7820	0	-	-	-	-
Instance14	2.00%	7473	7473	0	-	-	-	-
Instance15	7.00%	8947	8947	0	-	-	-	-
Instance16	10.00%	7800	7800	0	-	-	-	-
Instance17	11.00%	7740	7740	0	-	-	-	-
Instance18	26.00%	7855	7856	0	-	-	-	-
Instance19	11.00%	7523	7523	0	11.00%	7723	7841	± 339

Observing the results obtained by NSP proposed by [Curtois and Qu \(2014\)](#) (who do not consider fairness) in Figure 5, we can notice perturbations throughout the planning horizon for physicians of the same category. In contrast, with *PSPER* – H_s , illustrated in Figure 6, in addition to a new distribution of workload that first reorganizes the category of physicians, we manage to find fairer schedules, what brings to the problem the novelty in ergonomic balance and this measure do not compromise the demand satisfaction.

3.6 Conclusions

In this chapter, we proposed and evaluated three MIP reformulations for *PSPER* derived from an NSP, a formulation that does not consider fairness. In the first reformulation, we tackle the balance of the workload (working hours). We incorporated fairness by replacing primary working hours constraints with a new set of equations using the concept of least absolute deviation. Instead of having a minimum and a maximum number of hours, we established a target (consisting of the arithmetic average of minimum and maximum hours). The second

Figure 5 – Non balanced hour deviations for *instance07* established by Curtois and Qu (2014) between the physicians of the same category.

Figure 6 – New staff categories and hour deviations for *instance07* established by *PSPER* – H_s without hour deviation relatively to the target.

reformulation is an extension of the first reformation with relaxations for consecutive workdays and days-off constraints. Since these formulations came along with a high degree of violation of demand constraints we introduced a third formulation considering the target as a variable. We conducted experiments on benchmark instances, and our reformulations returned feasible solutions for instances up to three months.

From a total of 24 instances, we *PSPER* model solved 19 against 13 using the *NSP* model. This fraction corresponds to cases for up to three months of the planning horizon. This stems from the fact that reformulations are relaxations from the *NSP*.

With all the proposals, we reach a balanced distribution of hours between physicians and in the latter model formulation high degree of compliance of hour without compromising demand coverage. With this proposal, we aim to provide healthcare practitioners and decision-makers schedules with an accurate or near accurate satisfaction of constraints.

A RELAX-AND-FIX AND FIX-AND-OPTIMIZE HEURISTIC TO TACKLE FAIRNESS IN ER

This chapter proposes a Relax-and-Fix and Fix-and-Optimize heuristic to solve the benchmark instances with a larger number of days. The exact method, running over the *PSPER* – H_s model, presented higher gaps from the lower bound in these instances. Thus, we are proposing mathematical programming based heuristic using the *PSPER* – H_s formulation. Section 4.1 describes the Relax-and-Fix and Fix-and-Optimize heuristic procedure and the results achieved are reported in Section 4.2. Our conclusions follow in Section 4.3.

4.1 Methods

Two solution approaches are applied to solve the *PSPER*: exact method and math-heuristic. The exact method is the Branch & Cut algorithm from CPLEX solver. The math-heuristic is a combination of Relax-and-Fix and Fix-and-Optimize methods. Relax-and-Fix (RF) heuristic was initially proposed by [Dillenberger et al. \(1994\)](#) as an algorithm to schedule production quantities. The method solves MIP-subproblems for a set of variables, while the remaining variables are relaxed. It is a construction heuristic that builds solutions by partitioning the problem. Fix-and-Optimize heuristic was initially proposed by [Pochet and Wolsey \(2006\)](#) with the original name of *exchange* heuristic. It is an improvement heuristic since it takes an initial solution and optimizes its MIP-subproblems to search for better solutions. Thus, a subset of variables is optimized while the remains are fixed in the current value.

We describe the Relax-and-Fix heuristic as follows: given a problem (Q), and splitting the set of integer variables into (R) disjoint subsets from which each set is (Q^r) with $r = 1$. A sub-problem MIP^r with $r = 1$ is solved in an iterative fashion beginning in the MIP^1 sub-

problem. In the subset (Q^1), variables are defined as integer while the remaining are relaxed. After solving MIP^1 , the subset of integer variables Q^1 is fixed with their obtained values. If a feasible solution is reached, the process continues for the remaining subsets Q^{2R} ; otherwise, the process is aborted. Thus, the method is always aborted whenever no feasible solution is found. We present the pseudo-code of the Relax-and-Fix heuristic proposed to solve PSPER in Algorithm 1.

Algorithm 1 – Relax and fix heuristic pseudo-code. Originally from [Toledo et al. \(2015\)](#)

```

1: procedure RELAXANDFIX(solution, windowSize, windowType, overlap, timeLimit)
2:   window  $\leftarrow$  initWindow(windowSize, windowType, solution)
3:   solfix  $\leftarrow$   $\emptyset$ 
4:   solMIP  $\leftarrow$  window
5:   solLP  $\leftarrow$  solution – window
6:   while fixed solution not reached and elapsedTime < timeLimit do
7:     solve(solfix, solMIP, solLP)
8:     window  $\leftarrow$  moveWindow(overlap, windowSize)
9:     solfix  $\leftarrow$  solfix  $\cup$  (solMIP – window)
10:    solMIP  $\leftarrow$  window
11:    solLP  $\leftarrow$  solLP – window
12:  end while
13:  solution  $\leftarrow$  solfix
14:  return solution
15: end procedure

```

The parameters in the Relax-and-Fix procedure are the set of binary variables (*solution*), in our problem, the set of variables X_{pds} and K_{pw} , the number of binary variables to be chosen by iteration (*windowSize*), the selection criteria or partitioning strategy to select the variables (*strat*), the overlap rate of binary variables to be re-optimized (*overlap*) and the execution time limit (*timeLimit*). At the beginning of the RF procedure, the variables are relaxed with their values ranging between 0.0 and 1.0.

A window is defined as a set with a determined number of variables (line 2 of algorithm 1). The variables inside the window are enforced to be an integer in the set *sol_{MIP}* (lines 4 and 10 of algorithm 1), while the others are kept relaxed in *sol_{LP}* (lines 5 and 11 of algorithm 1). The resulting MIP is then solved (line 7 of algorithm 1). Next, a new set of variables (*window*) is defined, a subset of integer variables is fixed (*sol_{fix}*), and other sets of integer and relaxed variables are optimized (lines 8 to 11 of algorithm 1). There is a subset of variables in *window* that are re-optimized, and the amount of such variables is given by parameter *overlap*.

Formally, we can describe the Fix-and-Optimize as follows. Let the feasible solution be the returned by the RF heuristic. In the first stage, the set Q^1 is set free for optimization, and the sub-problem is solved, while the variables of $Q^2 \cup \dots \cup Q^R$ are fixed from the RF solution. At the second stage, the set Q^2 is set free to be optimized with variables in $Q^1 \cup Q^3 \cup \dots \cup Q^R$ fixed. This procedure is iterative until not finding any improvement in the current solution. At

each iteration, if there is a sub-problem with a better solution than the incumbent one, it will be replaced by the sub-problem's current solution. The pseudo-code for Fix-and-Optimize is described in Figure 2.

Algorithm 2 – Fix-and-Optimize pseudo-code. Originally from [Toledo et al. \(2015\)](#)

```

1: procedure FIXANDOPTIMIZE(solution, windowSize, windowType, overlap, timeLimit)
2:   windowType  $\leftarrow$  0
3:   repeat
4:     window  $\leftarrow$  initWindow( windowSize, windowType )
5:     solMIP  $\leftarrow$  window
6:     solfix  $\leftarrow$  solution – window
7:     while window not reached end do
8:       Solve( solfix, solMIP )
9:       window  $\leftarrow$  moveWindow( overlap, windowSize )
10:      solMIP  $\leftarrow$  window
11:      solfix  $\leftarrow$  solution – window
12:      solution  $\leftarrow$  solfix  $\cup$  solMIP
13:    end while
14:    windowType  $\leftarrow$  1 – windowType
15:  until elapsedTime < timeLimit and windowType  $\neq$  0
16: end procedure

```

Besides the initial solution, Relax-and-Fix also receives as input the number of binary variables by iteration (*windowSize*), the partitioning strategy to choose variables (*strat*), the overlap rate of binary variables to be re-optimized (*overlap*), and the execution time limit (*timeLimit*). These parameters are employed to define and solve the MIP-subproblems in the same way described for Relax-and-Fix.

The method combining Relax-and-Fix with Fix-and-Optimize (RFFO) heuristic is described in Algorithm 3. If the improvement achieved by a FO solution did not satisfy a given threshold of *thr*, the window size is increased by a given *inc* variable, which the user defines in the parameter setting stage. Moreover, the MIP-subproblems become larger as an attempt to find better solutions.

We explored three partitioning strategies to define the sub-problems: row, column, and value-wise partitioning. [Toledo et al. \(2015\)](#) applied these strategies to solve multi-level lot-sizing problems. The difference between our and [Toledo et al. \(2015\)](#) is that we do not use the overlap concept in the Relax-and-Fix phase of the RFFO procedure [Toledo et al. \(2015\)](#) does it because we aimed to find a feasible solution at the beginning and the Fix-and-Optimize would improve the solution. Figure 7 shows how does RF procedure work with row, column, and value-wise partitioning. The relaxed variables are in light gray cells, the variables free to optimize are in white cells, and the variables to be fixed with the incumbent solution's value are in dark gray cells.

Figure 7 (a), Column-wise partitioning strategy: variables in the column range of P_1, D_1 to

Algorithm 3 – RFFO pseudo-code. Originally from Toledo *et al.* (2015)

```

1: procedure RFFO(rfSize, rfOverlap, foSize, foOverlap, thr, windowTypeRF, windowTypeFO, inc, timeLimit)
2:   solution  $\leftarrow \emptyset$ 
3:   sol  $\leftarrow$  RelaxAndFix(solution, rfSize, windowTypeRF, rfOverlap, timeLimit)
4:   prevCost  $\leftarrow$  sol.cost
5:   while timeElapsed < timeLimit do
6:     FixAndOptimize(sol, foSize, windowTypeFO, foOverlap, timeRemaining)
7:     if (sol.cost - prevCost) / prevCost < thr then
8:       foSize  $\leftarrow$  foSize + inc
9:     end if
10:  end while
11: end procedure

```

Figure 7 – Relax-and-Fix different types of windows: (a) column-wise; (b) row-wise; (c) value-wise. For each partitioning strategy, we present two iterations in sequence.

P_5, D_1 are incorporated in the optimization *window*. After solving and finding this sub-problem solution, variables in the next range are fixed and leave the window set. This process is repeated throughout the schedule until every column is covered.

Figure 7 (b), Row-wise/physician partitioning: variables in the range of P_1, D_1 to P_1, D_7 are added to the optimization *window*. After solving the respective sub-problem and finding the solution of the MIP, variables P_1, D_1 to w_{P_1, D_7} are fixed and leave the *windows* set, while variables w_{P_2, D_1} and P_2, D_7 are enforced to be binary variables and included in the optimization window. The procedure is repeated until every row is covered.

Figure 7 (c), Value-wise partitioning strategy: variables that are more away from value 0.5 selected to the optimization window. This strategy can help to build feasible solutions quickly, despite the low quality. When two or more variables have the same value, the tie-break criteria are the column and row. Those variables in the left-most columns and or uppermost rows are selected.

After finding a feasible solution using the Relax-and-Fix with one of the partitioning strategies, we apply the Fix-and-Optimize using two partitioning strategies depicted in figure 8. There are three main situations: variables can be free to optimization, marked in white. Variables can be fixed with the incumbent solution value, marked in dark gray. Finally, we have the newly included optimization variables, italic boldface. In figure (a) and (b), we see the row and column-wise partitioning strategies, respectively. At each iteration, variables in row or column are optimized, and all others fixed in the incumbent solution.

Figure 8 – Fix-and-Optimize different types of windows: (a) row-wise; (b) column-wise; . For each partitioning strategy, we present two iterations in sequence.

In this work we combined the strategies to obtain:

- $RFFO_{ROCO}$: Relax-and-Fix with row-wise partitioning and and Fix-and-Optimize with column-wise partitioning.
- $RFFO_{CORO}$: Relax-and-Fix with column-wise partitioning and and Fix-and-Optimize with row-wise partitioning.
- $RFFO_{VARO}$: Relax-and-Fix with value-wise partitioning and and Fix-and-Optimize with row-wise partitioning.

4.2 Computational experiments

We coded the RFFO using the ProOF framework [Arantes \(2014\)](#) with Java version of callable libraries from IBM ILOG CPLEX 12.8 optimization solver. The experiments ran on an

Intel Xeon E5-2680v2 2.8 GHz processor and 128GB of RAM.

The results regarding Relax-and-Fix and Fix-and-Optimize are obtained by solving such type of instances from the benchmark set in [Curtois and Qu \(2014\)](#). The characteristics of these instances are presented in Table 7.

Table 7 – The characteristics of the set of benchmark instances

Instance	Days	Physicians	Shift types	Days off requests	Shift on/off requests
Instance14	42	32	4	128	359
Instance15	42	45	6	180	490
Instance16	56	20	3	120	280
Instance17	56	32	4	160	480
Instance18	84	22	3	176	414
Instance19	84	40	5	320	834
Instance20	182	50	6	900	2318
Instance21	182	100	8	1800	4702
Instance22	364	50	10	1800	4638
Instance23	364	100	16	3600	9410
Instance24	364	150	32	5400	13809

We set the math-heuristic time limit of 1 hour, and the parameter configurations are summarized in Table 8. The *window* size (rf-Size and fo-Size) was set as half of the number of total binary in the instance (Inst-Size), while the addition of other variables (rf-inc and fo-inc) was a quarter of *window* size. We reach such values after some empiric evaluations over such large instances.

Table 8 – RFFO configurations

Instance	Inst-Size	rf-Size	rf-Inc	fo-Size	fo-Inc
Instance14	5568	2784	1392	2784	1392
Instance15	11610	5805	1935	5805	1935
Instance16	3520	1760	880	1760	880
Instance17	7424	3712	1856	3712	1856
Instance18	5808	2904	1452	2904	1452
Instance19	17280	8640	4320	8640	4320

Besides these configurations, Relax-and-Fix and Fix-and-Optimize strategies use a limit of gap and nodes. Therefore, in the Relax-and-Fix procedure, the branching tree cannot explore every node or even solve the sub-problems until finding an optimal solution. With these principles, we established the RF node limit to 1000, and FO node limit to 1000000. The relative gap of each sub-problem is 0.05%. These parameters allow us to find solutions from instance 14 to 19, but we could not solve instances 20 to 24. The Fix-and-Optimize step did not return a feasible solution within 1 hour. The reason for that is the huge number of variables, even when solving the relaxed version of such instances. Table 9 shows the results obtained.

To calculate the variation of upper bounds between complete model and the Heuristic we

Table 9 – $PSPER - H_s$ and RFFO performance

Instances	$PSPER - H_s$			$RFFO_{ROCO}$		$RFFO_{CORO}$		$RFFO_{VARO}$	
	UB	LB	TB	UB	TB	UB	TB	UB	TB
Instance14	1235	1235	1169.79	1235	1350.65	1235	957.62	1235	1972.69
Instance15	4636	3757.88	3593.22	4534	3599.38	5302	3599.38	4942	3600
Average			2381.51		2475.02		2278.50		2786.35
Instance16	3241	3130.05	3549.32	3245	1432	3251	1478.62	3260	1157.19
Instance17	6042	5636	3414.91	5777	2696.46	5761	1784.31	5875	3476.66
Instance18	4850	4177.83	3590.97	4619	3590.2	4729	3593.15	5101	3600.01
Instance19	3821	2864.66	3502.39	3173	3576.4	3481	3576.98	3871	3600
Average			3514.40		2823.77		2608.27		2958.47

use the Δ_{UB} metric, given by the expression in (4.1).

$$\Delta_{UB} = \frac{RFFO_{UB} - MIP_{LB}}{RFFO_{UB}} \times 100 \quad (4.1)$$

Figure 9 – Objective function relative distance comparison between $PSPER - H_s$ and RFFO strategies.

A similar metric is used for the relative distance from RFFO time (spent to find the best solutions) to MIP time using expression 4.2 and depicted in Figure 10. The average time to find the best solution by $RFFO_{ROCO}$ is higher than the average time to find solutions using $RFFO_{CORO}$ strategy, which is, in turn, the better approach in this aspect and also it has better values comparing to $PSPER - H_s$.

$$\Delta_{TB} = \frac{RFFO_{TB} - MIP_{TB}}{RFFO_{TB}} \times 100 \quad (4.2)$$

4.3 Conclusions

In this chapter, we proposed a math-heuristic to tackle physician scheduling in emergency rooms, which will run based on a mathematical formulation where the monthly target hours

Figure 10 – Time to find the best solution comparison between the rffo strategies.

for physicians become decision variables. We introduced this formulation in Chapter 3, from which we solved 79.16% of the instances, and 68.42% of them reach the optimality. However, the formulation struggles in more than four weeks spending higher computational time finding solutions. We propose a Relax-and-Fix and Fix-and-Optimize heuristic to solve such instances. A total of three strategies are applied to define MIP-subproblems, and we managed to improve both the computational time and upper bounds in 75% of the instances with eight weeks, using the best strategy. Nonetheless, this problem's bottleneck is the number of days and physicians, which increases the problem's difficulty. The large instances (instances 20 to 24) are still without a solution even with the relaxation brought by fairness constraints or applying the Relax-and-Fix and fix-optimize heuristic. We conclude that the designed math-heuristic could not handle the number of variables for those large instances.

ESTIMATION OF FAIRNESS PARAMETERS AND LINEAR APPROXIMATION

This chapter presents an extension of the $PSPER - H_s$ formulation to include weekend assignments as a decision variable and cope with fairness related to this feature. We also report results evaluating objective functions with quadratic and linear costs. We state a total of four model variations from which we obtained fair schedules in terms of hours and weekends, without compromising the high degree of demand compliance. The remainder of this chapter is organized as follows: in Section 5.1 we discuss the proposed changes, while computational experiments are presented in Section 5.2 and conclusions in Section 5.3.

5.1 Model variations from $PSPER - H_s$

We consider that a weekend is assigned whenever a physician works on a Saturday or Sunday. We model this situation in constraints (3.7), and the maximum number of weekends in constraints (3.8). However, as the number of weekends can be a negotiable resource in a real-world scenario, the decision-maker can estimate such resources. Thus, this negotiable resource can lead to better decisions for the whole assignment instead of assuming a previous maximum number for weekend assignments.

5.1.1 Weekend balance constraints

We introduce new variables to handle better weekend assignments, aiming to describe deviations and target weekends throughout the planning horizon.

S6: Weekend balance 1

The former hard constraints (3.7) and (3.8) only establish the maximum limit to the number of weekends. However, our goal is to add soft constraints for better decisions on this

Variables

σ_p^W	Deviation between the number of weekends of a physician p during the planning horizon and the target number of weekends
$\bar{T}_p^W$	Target number of weekends W for p^{th} physician in the planning horizon.

issue. Thus, constraints (3.28) and (3.29) are proposed to create fairness between physicians by measuring the deviation from the maximum number of weekends (V_p^+) a physician can work. The variables σ_p^W control such deviations during the planning horizon. It is the same procedure applied to hour constraints described in constraints (3.13) and (3.14).

$$\sigma_p^W \geq \sum_{w \in W} y_{pw} - V_p^+ \quad \forall p \in P, \quad (5.1)$$

$$\sigma_p^W \geq V_p^+ - \sum_{w \in W} y_{pw} \quad \forall p \in P, \quad (5.2)$$

S7: Weekend balance 2

In constraints (5.3) and (5.4), the target number of weekends V_p^+ becomes a decision variable $\bar{T}_p^W$.

$$\sigma_p^W \geq \sum_{w \in W} y_{pw} - \bar{T}_p^W \quad \forall p \in P, \quad (5.3)$$

$$\sigma_p^W \geq \bar{T}_p^W - \sum_{w \in W} y_{pw} \quad \forall p \in P, \quad (5.4)$$

5.1.2 Objective functions and model variations

To the former objective function in (3.12), we add the number of weekend deviations to have the new objective function (5.10). In the objective function described in expression (5.10), the deviation is measured by the total deviation $\sum_{p \in P} \sum_{m \in M} (\sigma_{pm}^H)$ which is a linear expression. The idea is to evaluate a quadratic measure to tackle greater deviations from hours and weekends. A Mixed Integer Quadratic Problem (MIQP) is NP-hard, mostly because it contains MILP as a special case (BLIEK; BONAMI; LODI, 2014). To handle the inherent complexity of the MIQP, we also introduce a linear approximation for the quadratic term in the objective function. Equation (5.5) states a function to represent the quadratic term, and we will define a piece-wise linear approximation for such function.

$$F(\sigma_{pm}^H) = \sum_{p \in P} \sum_{m \in M} ((\sigma_{pm}^H)^2) \quad (5.5)$$

N linear problems compose the domain of the function. The variables λ_{pmn} are auxiliary variables, and B_n is the growth coefficient of the straight segment that approximates the curve with $n = 0 \dots N - 1$. The term σ^{max} is the maximum value that σ_{pm} deviation variable can assume. Through linear approximation, the quadratic term in equation (5.5) can be expressed by constraints (5.6) - (5.9).

$$F(\sigma_{pm}^H) = \sum_p \sum_m \sum_n B_n \lambda_{pmn} \quad (5.6)$$

$$\sigma_{pm}^H = \sum_n \lambda_{pmn} \quad (5.7)$$

$$0 \leq \lambda_{pmn} \leq \frac{\sigma^{max}}{N} \quad (5.8)$$

$$B_n = \frac{(2n + 1) \sigma^{max}}{N} \quad (5.9)$$

The linear approximation defines values of λ_{pmn} depending on N linear segments and σ^{max} . Constraints (5.6) - (5.9) are latter included in the proposed model variations as cuts. This linear approximation provides a linear formulation of the problem which, in turn, allows the solver to use more robust algorithms such as cutting-plane algorithms. Moreover it enables an improvement of primal and dual bounds for this problem as we will report in results section latter. Next we present the model variations which include weekends deviations, as well as quadratic terms and its linear approximation in the objective function.

5.1.2.1 PSPER – $(HW)_p^2$

This formulation consist of an objective function with quadratic costs on hours and weekends, while the values of shift on and shift off requests, and demand deviations are linear. The objective function (5.10) maximizes the allocation of employee shift requests and minimizes under- and over-staffing, as well as distortions on the number of hours and weekend worked. The parameters q_{pds}^+ and q_{pds}^- represent the weights for the shift on and shift off requests, respectively. The higher the weight, the more relevant the request is to the employee. If there is no request, then the parameter is zero. The variables w_{ds} and z_{ds} are the total numbers of staff below and above the preferred cover level for each shift s on the day d . The parameters v_{ds}^- and v_{ds}^+ are weights indicating the importance of under- and over-coverage. The last terms of the objective function are the deviation on target hours and weekends, respectively.

$$\begin{aligned}
G(x_{pds}, \sigma_{pm}^H, \sigma_p^W) = & \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} \left(q_{pds}^+ (1 - x_{pds}) + q_{pds}^- x_{pds} \right) \\
& + \sum_{d \in D} \sum_{s \in S} \left(v_{ds}^- w_{ds} + v_{ds}^+ z_{ds} \right) + \sum_{p \in P} \sum_{m \in M} (\sigma_{pm}^H)^2 + \sum_{p \in P} (\sigma_p^W)^2 \quad (5.10)
\end{aligned}$$

The complete model is:

$$\begin{aligned}
& \text{Minimize} && G(x_{pds}, \sigma_{pm}^H, \sigma_p^W) \\
& \text{subject to} && (3.1) - (3.10) \\
& && (3.13), (3.14) \\
& && (3.28) \text{ and } (3.29)
\end{aligned}$$

5.1.2.2 $PSPER - (HW)_p$

The present model differs from $PSPER - (HW)_p^2$ by the way the objective function is defined. The objective function (5.10) has quadratic costs to estimate tighter deviation values. Here we apply the linear approximation to describe such deviations.

$$\begin{aligned}
H(x_{pds}, F(\sigma_{pm}^H), N(\sigma_p^W)) = & \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} \left(q_{pds}^+ (1 - x_{pds}) + q_{pds}^- x_{pds} \right) + \sum_{d \in D} \sum_{s \in S} \left(v_{ds}^- w_{ds} + v_{ds}^+ z_{ds} \right) \\
& + \sum_{p \in P} \sum_{m \in M} \left(N(\sigma_{pm}^H) \right) + \sum_{p \in P} N(\sigma_p^W) \quad (5.11)
\end{aligned}$$

The complete model is:

$$\begin{aligned}
& \text{Minimize} && H(x_{pds}, F(\sigma_{pm}^H), N(\sigma_p^W)) \\
& \text{subject to} && (3.1) - (3.10) \\
& && (3.13), (3.14) \\
& && (5.1) \text{ and } (5.2)
\end{aligned}$$

5.1.2.3 $PSPER - (HW)_s^2$

This model variation differs from the previous one by taking into account quadratic costs and the target number of hours and weekend as decision variables.

$$\begin{aligned}
& \text{Minimize} && G(x_{pds}, \sigma_{pm}^H, \sigma_p^W) \\
& \text{subject to} && (3.1) - (3.10) \\
& && (3.28), (3.29) \\
& && (5.3) \text{ and } (5.4)
\end{aligned}$$

5.1.2.4 $PSPER - (HW)_s$

This model considers the same set of constraints used in $PSPER - (HW)_s^2$ but the objective function is the same applied in $PSPER - (HW)_p$. The complete formulation is then:

$$\begin{aligned}
& \text{Minimize} && H(x_{pds}, N(\sigma_{pm}^H), N(\sigma_p^W)) \\
& \text{subject to} && (3.1) - (3.10) \\
& && (3.28), (3.29) \\
& && (5.3) \text{ and } (5.4)
\end{aligned}$$

5.1.2.5 L^1 and L^∞ norms for measuring fairness

Since we are adding the fairness into the objective function in the form of a quadratic term, an obvious question arises on whether using L^1 or L^∞ norms would be useful in order to achieve a linear model, as investigated for other problems, e.g., [Cadoux \(2010\)](#), [Akartunali et al. \(2016\)](#). The L^1 norm (or Manhattan distance) is simply the sum of the magnitudes of the vector in the space, which is the magnitude of the deviation variables' vector in our case. For L^1 norm, we have the following constraints:

$$N_1(\sigma_{pm}^H) = \sum_p \sum_m \Delta_{pm}^H \quad (5.12)$$

$$\Delta_{pm}^H \geq \sigma_{pm}^H \quad \forall p \in P, m \in M, \quad (5.13)$$

$$\Delta_{pm}^H \geq -\sigma_{pm}^H \quad \forall p \in P, m \in M, \quad (5.14)$$

$$N_1(\sigma_p^W) = \sum_p \Delta_p^W \quad (5.15)$$

$$\Delta_p^W \geq \sigma_p^W \quad \forall p \in P, \quad (5.16)$$

$$\Delta_p^W \geq -\sigma_p^W \quad \forall p \in P \quad (5.17)$$

Then, the objective function will as follows:

$$\begin{aligned} H(x_{pds}, N_1(\sigma_{pm}^H), N_1(\sigma_p^W)) &= \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} (q_{pds}^+(1 - x_{pds}) + q_{pds}^- x_{pds}) \\ &\quad + \sum_{d \in D} \sum_{s \in S} (v_{ds}^- w_{ds} + v_{ds}^+ z_{ds}) + (N_1(\sigma_{pm}^H) + N_1(\sigma_p^W)) \end{aligned} \quad (5.18)$$

The complete model with the L^1 norm is:

$$\begin{aligned} &\text{Minimize} \quad H(x_{pds}, N_1(\sigma_{pm}^H), N_1(\sigma_p^W)) \\ &\text{subject to} \quad (3.1) - (3.10) \\ &\quad \quad \quad (3.13), (3.14) \\ &\quad \quad \quad (5.1), (5.2) \\ &\quad \quad \quad (5.12) - (5.17) \end{aligned}$$

The other linear alternative, L^∞ norm, gives us the largest magnitude among all elements of a vector, which is equivalent to the largest deviations of hours and weekend assignments in our context. For L^∞ , we have the following constraints:

$$N_\infty(\sigma_{pm}^H) = \kappa \quad (5.19)$$

$$\kappa \geq \sigma_{pm}^H \quad \forall p \in P, m \in M, \quad (5.20)$$

$$\kappa \geq -\sigma_{pm}^H \quad \forall p \in P, m \in M, \quad (5.21)$$

$$N_\infty(\sigma_p^W) = \varepsilon \quad (5.22)$$

$$\varepsilon \geq \sigma_p^W \quad \forall p \in P \quad (5.23)$$

$$\varepsilon \geq -\sigma_p^W \quad \forall p \in P \quad (5.24)$$

The objective function of the problem is as follows:

$$\begin{aligned} H(x_{pds}, N_\infty(\sigma_{pm}^H), N_\infty(\sigma_p^W)) = & \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} \left(q_{pds}^+ (1 - x_{pds}) + q_{pds}^- x_{pds} \right) \\ & + \sum_{d \in D} \sum_{s \in S} \left(v_{ds}^- w_{ds} + v_{ds}^+ z_{ds} \right) + \left(N_\infty(\sigma_{pm}^H) + N_\infty(\sigma_p^W) \right) \end{aligned} \quad (5.25)$$

The complete model with the L^∞ norm is

$$\begin{aligned} \text{Minimize} \quad & H(x_{pds}, N_\infty(\sigma_{pm}^H), N_\infty(\sigma_p^W)) \\ & (3.1) - 3.10 \\ & (3.13), (3.14) \\ & (5.1), (5.2) \\ & (5.19) - (5.24) \end{aligned}$$

5.2 Computational results

In this section, we present computational experiments for the proposed *PSPER* and its variations described in this chapter. We will compare our models with the results presented in [Curtois and Qu \(2014\)](#). The authors presented results for the Nurse Scheduling Problem, without considering fairness, over the same benchmark set of instances ([CURTOIS, 2007](#)) reported in the previous chapters. We coded these approaches using the ProOF framework [Arantes \(2014\)](#) with Java version of callable libraries from IBM ILOG CPLEX 12.8 optimization solver. The experiments ran on a PC with Intel Xeon E5-2680v2 2.8 GHz processor and 128GB of RAM. We repeat below the instances characteristics to facilitate the analyses of results.

5.2.1 Evaluating performances

For the approaches of both parameterized and estimated models, we show the importance of linear approximation by comparing the solutions obtained with the quadratic counterpart. And in the last, also present the results obtained by [Curtois and Qu \(2014\)](#) ran in CPLEX solver. In some tables, we have columns with UB, LB, GAP and Time values. We report gap and execution time of CPLEX after termination. In columns UB and LB we present the upper and the lower

Table 10 – The characteristics of the set of benchmark instances proposed by (CURTOIS, 2007)

Instances	Days	Nurses	Shift types	Day off requests	Shift on/off requests
Instance01	14	8	1	8	26
Instance02	14	14	2	14	62
Instance03	14	20	3	20	64
Instance04	28	10	2	20	71
Instance05	28	16	2	32	106
Instance06	28	18	3	36	135
Instance07	28	20	3	40	168
Instance08	28	30	4	60	225
Instance09	28	36	4	72	232
Instance10	28	40	5	80	284
Instance11	28	50	6	100	336
Instance12	28	60	10	120	422
Instance13	28	120	18	240	841
Instance14	42	32	4	128	359
Instance15	42	45	6	180	490
Instance16	56	20	3	120	280
Instance17	56	32	4	160	480
Instance18	84	22	3	176	414
Instance19	84	40	5	320	834
Instance20	182	50	6	900	2318
Instance21	182	100	8	1800	4702
Instance22	364	50	10	1800	4638
Instance23	364	100	16	3600	9410
Instance24	364	150	32	5400	13809

bounds reported by CPLEX on finishing. The solution gap, computed according to the formula $GAP = \frac{(UB-LB)}{UB}$ is given in column GAP. Column Time gives the computational time, in seconds, reported by CPLEX.

5.2.1.1 Points to an accurate linear approximation

When establishing the formulations, we made a linear approximation to L^2 norm, with a number of points as a parameter. To make an accurate linear approximation from a quadratic function, one must take into account the number of points. As described in expression (5.8), there is straight segment that models the curve, with $n = 0 \dots N - 1$. We tested different values for N in order to have a good description. These experiments provided us the results in table 11. For N equal to 8, 16, 32, 64 and 128 points, we have the corresponding columns with UB, LB, GAP and Time values.

Table 11 – Linear approximation with 8, 16, 32, 64 and 128 points. Experiments run in CPLEX solver for 3600 seconds

Instance	8points				16points				32points				64points				128points			
	UB	LB	GAP	Time	UB	LB	GAP	Time	UB	LB	GAP	Time	UB	LB	GAP	Time	UB	LB	GAP	Time
Instance01	716	716	0.00%	0.94	716	716	0.00%	0.41	716	716	0.00%	0.59	716	716	0.00%	0.92	716	716	0.00%	0.57
Instance02	1620	1620	0.00%	0.36	1620	1620	0.00%	0.23	1620	1620	0.00%	0.27	1620	1620	0.00%	0.19	1620	1620	0.00%	0.28
Instance03	1900	1900	0.00%	2.07	1900	1900	0.00%	1.92	1900	1900	0.00%	2.51	1900	1900	0.00%	2.1	1900	1900	0.00%	4.27
Instance04	2204	2203.78	0.01%	9.08	2204	2203.78	0.01%	8.33	2204	2203.78	0.01%	11.36	2204	2203.78	0.01%	17.27	2204	2203.78	0.01%	8.58
Instance05	3211	3210.75	0.01%	9.85	3211	3210.75	0.01%	8.93	3211	3210.75	0.01%	19.49	3211	3210.75	0.01%	8.69	3211	3210.75	0.01%	8.76
Instance06	2246	2186.39	2.65%	3600	2246	2187.73	2.59%	3600	2245	2187.7	2.55%	3600	2246	2187.76	2.59%	3600	2246	2187.78	2.59%	3600
Instance07	4030	4030	0.00%	37.14	4030	4030	0.00%	35.73	4030	4030	0.00%	39.18	4030	4030	0.00%	36.56	4030	4030	0.00%	37.46
Instance08	2966	2946.19	0.67%	3600	2967	2946.91	0.68%	3600	2968	2946.87	0.71%	3600	2966	2946.94	0.64%	3600	2967	2946.95	0.68%	3600
Instance09	3866	3866	0.00%	23.67	3866	3866	0.00%	24.34	3866	3866	0.00%	24.98	3866	3866	0.00%	23.06	3866	3866	0.00%	23.763
Instance10	6329	6328.4	0.01%	1014.41	6329	6328.4	0.01%	1779.78	6329	6328.4	0.01%	1053.62	6329	6328.4	0.01%	1054.23	6329	6328.4	0.01%	1013.84
Instance11	2310	2260.44	2.15%	3600	2310	2260.44	2.15%	3600	2310	2260.44	2.15%	3600	2310	2260.44	2.15%	3600.44	2310	2260.44	2.15%	3600
Instance12	11721	11721	0.00%	3268.92	11722	11721	0.01%	1749.2	11721	11721	0.00%	3312.74	11722	11721	0.01%	1758.01	11722	11721	0.01%	2020.61
Instance13	9181	6558.29	28.57%	3600	9067	6558.29	27.67%	3600	8699	6558.27	24.61%	3600	8069	6558.29	18.72%	3600	9464	6558.27	30.70%	3600
Instance14	1235	1234.95	0.00%	1871.39	1235	1234.95	0.00%	1900.61	1235	1234.95	0.00%	1874.13	1235	1234.95	0.00%	3315.61	1235	1234.95	0.00%	1917.12
Instance15	4013	3758.32	6.49%	3600	4130	3758.32	9.00%	3600	4185	3758.3	10.20%	3600	4027	3758.92	6.67%	3600	4055	3758.92	7.32%	3600
Instance16	2945	2834.98	7.45%	3600	2945	2835	7.45%	3600	2943	2834.98	5.75%	3600	2947	2835.03	3.60%	3600	2948	2835.03	3.60%	3600
Instance17	4886	4267.61	12.66%	3600	5181	4267.61	17.63%	3600	4813	4267.61	10.74%	3600	4813	4267.61	11.33%	3600	4903	4267.61	12.96%	3600
Instance18	4646.13	4266.95	8.16%	3600	4728.25	4266.75	9.76%	3600	4633	4266.95	7.90%	3600	4640.13	4266.95	8.04%	3600	4528.13	4266.95	8.77%	3600
Instance19	5116.38	3450.16	32.57%	3600	5078.63	3450.16	32.07%	3600	5760.38	3450.16	40.11%	3600	8063.13	3450.16	57.21%	3600	5760.38	3450.16	40.11%	3600
instance20 - 24	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

The execution time is short for three first instances, in the range of 0 to less than 5 seconds for all number of points we tested, and the average gap is 0%. These instances have the shortest planning horizons and a significant small number of physicians. The best average in terms of execution time belongs to linear approximation with 16 points, with a value near to 0.85 seconds, while the worst belongs to the variant with 128 points that has an average of 1.71 seconds. Now, we can easily infer that it is mostly related to the number of variables needed to make the curve.

For instances of 4 weeks of planning horizon, CPLEX solver proved optimality for 60% of them, and a average gap of ranges from 2.41% to 3.62%, with the linear approximation with 32 points happens to have the best averages, and 128 the worst. Regarding total time to execution, linear approximation with 64 points has the best average, a value near to 1729.85 seconds, and the worst average belongs to linear approximation with 32 points with a 1886.14 seconds. The variant of 64 points was relatively good in the instances for which the CPLEX solver proved optimality. The high average of time for 32 points variant is related to a particular instance solved in a long time compared to other variants. This delay may be associated to the calculations that CPLEX solver makes when solving this particular instances.

Regarding 6, 8 and 12 weeks, the behavior is irregular amongst the variants. The variant with 8 points had a average gap of 3.25% for 6 weeks and 2736 seconds, the best between the variants. When solving instances with 8 weeks, the best average gap is 7.20% found in the variant of 32 points, whereas for 12 weeks, the variant with 8 points presents a slightly better average gap, 20.36%. For instances of 8 and 12 weeks, CPLEX solver took all the available time to solve them.

From the tested numbers of points, the average of results show that the best number of points is $N = 8$. The number of points has a significant influence in the overall performance, since the more points to make a linear approximation the more variables needed, more constraints to deal with, and consequently more calculations to make on behalf of the solver.

5.2.1.2 L^1 and L^∞ norm

We compare now the quadratic variant of L^2 norm with $N = 8$ against L^1 and L^∞ norm. Table 12 shows the results achieved. We can see that L^1 norm takes longer than other two norms even for small instances. This has to do with the number of variables to express the L^1 norm and constraint attached to these variables. Nevertheless, L^1 norm had competitive performance in terms of gap for some instances with 4 and 8 weeks of planning horizon. L^2 norm presents a better performance for 21% of the instances, while L^1 norm presents 10.5% and L^∞ 5.26%, when there wasn't a tie in the remainder of the instances. On the other hand, considering exclusively the execution time L^∞ outperforms the other two approaches, however, this is not conclusive since other approaches also have better performances in other criteria. To a tie break, we used the average gap. The approach that has the smallest value is the L^2 norm, which will be also used

Table 12 – The benchmark results for parameterized approach. Testing linear approximation with L^1 , L^2 and L^∞ norms. Experiments run in CPLEX solver for 3600 seconds

Instance	L^1 norm				L^2 norm				L^∞ norm			
	UB	LB	GAP	Time	UB	LB	GAP	Time	UB	LB	GAP	Time
Instance01	716	716	0.00%	505.05	716	716	0.00%	0.94	716	716	0.00%	1.13
Instance02	1620	1620	0.00%	272.10	1620	1620	0.00%	0.36	1620	1620	0.00%	0.29
Instance03	1900	1900	0.00%	239.11	1900	1900	0.00%	2.07	1900	1900	0.00%	2.01
Instance04	2204	2203.78	0.01%	868.31	2204	2203.78	0.01%	9.08	2204	2203.78	0.01%	8.48
Instance05	3211	3210.75	0.01%	920.29	3211	3210.75	0.01%	9.85	3211	3210.75	0.01%	18.03
Instance06	2246	2187.77	2.59%	3600	2246	2186.39	2.65%	3600	2246	2187.77	2.59%	3600
Instance07	4030	4030	0.00%	365.07	4030	4030	0.00%	37.14	4030	4030	0.00%	35.54
Instance08	2967	2946.93	0.68%	3600	2966	2946.19	0.67%	3600	2967	2946.95	0.68%	3600
Instance09	3866	3866	0.00%	243.53	3866	3866	0.00%	23.67	3866	3866	0.00%	24.45
Instance10	6329	6328.4	0.01%	1509.26	6329	6328.4	0.01%	1014.41	6329	6328.4	0.01%	1034.88
Instance11	2310	2260.44	2.15%	3600	2310	2260.44	2.15%	3600	2310	2260.44	2.15%	3600
Instance12	11722	11721	0.01%	1666.92	11721	11721	0.00%	3268.92	11722	11721	0.01%	1541.38
Instance13	9075	6558.29	27.73%	3600	9181	6558.29	28.57%	3600	9181	6558.29	28.57%	3600
Instance14	1235	1234.95	0.00%	1832.49	1235	1234.95	0.00%	1871.39	1235	1234.95	0.00%	1772.02
Instance15	4404	3758.32	14.66%	3600	4019	3758.32	6.49%	3600	4136	3758.32	9.13%	3600
Instance16	2947	2835	3.80%	3600	2945	2834.98	3.74%	3600	2948	2835	3.83%	3600
Instance17	4806	4267.61	11.20%	3600	4886	4267.61	12.66%	3600	4813	4267.61	11.33%	3600
Instance18	4646.13	4266.95	8.16%	3600	4646.13	4266.95	8.16%	3600	4536.13	4266.95	5.93%	3600
Instance19	8365.25	3450.16	58.76%	3600	5116.38	3450.16	32.57%	3600	8265.25	3450.16	58.26%	3600
Instance20-24	-	-	-	-	-	-	-	-	-	-	-	-

in further tests.

5.2.1.3 Analyzing the $PSPER - (HW)_p$ and $PSPER - (HW)_p^2$ models

Table 14 compares results from $PSPER - (HW)_p$ and $PSPER - (HW)_p^2$ against the original model.

Table 13 – Overall results for parameterized and estimated models in comparison to [Curtois and Qu \(2014\)](#) model. Time limit = 3,600 seconds

Instances	$PSPER - (HW)_p$		$PSPER - (HW)_p^2$		$PSPER - (HW)_s$		$PSPER - (HW)_s^2$		Curtois and Qu (2014)	
	GAP	TIME	GAP	TIME	GAP	TIME	GAP	TIME	GAP	TIME
Instance01	0.00%	1.51	0.00%	1.77	0.00%	0.21	0.00%	0.77	0.00%	0.62
Instance02	0.00%	1.82	0.00%	2.59	0.00%	0.25	0.00%	2.06	0.00%	1.55
Instance03	0.00%	2.49	0.00%	143.31	0.00%	0.66	0.00%	127.45	0.00%	205.18
Instance04	0.01%	13.76	0.00%	970.29	0.00%	1.20	0.00%	3.98	0.00%	759.55
Instance05	0.01%	18.75	0.00%	1374.96	0.00%	695.91	15.63%	3600	0.00%	1064.94
Instance06	2.65%	3600	16.36%	3600	0.00%	22.16	0.00%	674.76	0.00%	2857.30
Instance07	0.00%	45.26	0.03%	3600	0.00%	13.48	0.00%	1124.75	0.52%	3600
Instance08	0.67%	3600	19.78%	3600	0.00%	148.57	0.00%	3572.58	3.24%	3600
Instance09	0.00%	34.48	0.04%	3600	0.00%	1.64	0.00%	29.76	58.43%	3600
Instance10	0.01%	1339.48	14.31%	3600	0.00%	103.61	97.18%	3600	0.09%	3600
Instance11	2.150%	3600	98.38%	3600	0.00%	20.18	0.00%	627.47	0.00%	415.73
Instance12	0.00%	2262.58	99.99%	3600	0.00%	335.99	97.95%	3600	0.19%	3600
Instance13	28.57%	3600	-	-	0.00%	1267.20	-	-	85.67%	3600
Instance14	0.00%	2616.4	98.23%	3600	0.00%	15.80	0.00%	3351.52	-	-
Instance15	6.49%	3600	-	-	3.03%	3600	-	-	-	-
Instance16	3.74%	3600	61.49%	3600	82.55%	3600	89.70%	3600	-	-
Instance17	12.66%	3600	-	-	48.15%	3600	99.94%	3600	-	-
Instance18	8.16%	3600	-	-	12.21%	3600	-	-	-	-
Instance19	32.57%	3600	-	-	31.91%	3600	-	-	29.72%	3600
Instance20	-	-	-	-	0.00%	128.27	-	-	-	-
Instance21-24	-	-	-	-	-	-	-	-	-	-

It comes that 19 out of 24 were solved using $PSPER - (HW)_p$, whereas $PSPER - (HW)_p^2$

and [Curtois and Qu \(2014\)](#) solved 14 out of 24, corresponding to 79% and 58%, respectively. These differences between the linear and quadratic approaches stems from the fact that the solver takes more time to decide which variables optimize in the quadratic problem. In respect to the literature model, these differences are in the way the model was reformulated. The relaxation of hour and weekend constraints to apply subsequently the L^1 norm for fairness enlarges the set of feasible solutions of the problem.

To have a better comprehension of the models, we build two performance profile figures, using the computational time and gap as performance measure. In these figures, the models are represented by curves. In the gap performance, each point shows the reached value of gap for each of the model approaches in the test instances. For the time performance, each point shows the computational time each model took to solve each test instance. To draw the graphs we used all the benchmark instances, but, if a model can not find a feasible solution in the defined computational time, we consider that the gap is higher than 100%, and we consider the computational time as greater than 3600 seconds.

From Figures 11 and 12, we can notice that $PSPER - (HW)_p$ outperform in 46% of the benchmark instances in terms of gap, from which 5 out of these 11 has reached the optimal, whereas [Curtois and Qu \(2014\)](#) does it for 4 instances. The behaviour of $PSPER - (HW)_p$ is consistent throughout the instances as the gap increases for significantly difficult instances such as Instance15, Instance19, Instance20 and Instance13. The same happens to $PSPER - (HW)_p^2$ and [Curtois and Qu \(2014\)](#). In terms of computational time, $PSPER - (HW)_p$ was faster in 12 of the solved instances, three times more than [Curtois and Qu \(2014\)](#), while $PSPER - (HW)_p^2$ could not find better solutions until the program terminated. From the results, we can observe that the performance of $PSPER - (HW)_p$ is better than $PSPER - (HW)_p^2$, mostly due to the application of linear approximation on quadratic terms of the objective function. This allows us to obtain feasible solutions in less computational time for medium and large-sized instances. Summarizing, $PSPER - (HW)_p$ has a better performance than the other two models, for this set of test instances.

5.2.1.4 Analyzing the $PSPER - (HW)_s$ and $PSPER - (HW)_s^2$ models

Table 15 provides the computational results for estimated models in the same fashion as in the previous subsection, along with Figures 13 and 14 providing a summary view. The remainder of the benchmark instances that were not solved within the total computational time are not included.

After reaching the time limit, a total of 20 instances were solved using the $PSPER - (HW)_s$, against 15 and 14 on behalf of $PSPER - (HW)_s^2$ and [Curtois and Qu \(2014\)](#) (corresponding to 83.33%, 62.5% and 58.33% success rates respectively).

As in the formulation with hours and weekends as parameter, the differences between the linear and quadratic are notorious. This stems from the fact that MIQP is more challenging

Table 14 – The benchmark results for $PSPER - (HW)_p$ and $PSPER - (HW)_p^2$ in comparison with the original model run in CPLEX solver for 3600 seconds

Instances	$PSPER - (HW)_p$				$PSPER - (HW)_p^2$				Curtois and Qu (2014)			
	UB	LB	GAP	TIME	UB	LB	GAP	TIME	UB	LB	GAP	TIME
Instance01	716	716	0.00%	1.51	716	716	0.00%	1.77	607	607	0.00%	0.62
Instance02	1620	1620	0.00%	1.82	1620	1620	0.00%	2.59	828	828	0.00%	1.55
Instance03	1900	1900	0.00%	2.49	1900	1900	0.00%	143.31	1001	1001	0.00%	205.18
Instance04	2204	2203.78	0.01%	13.76	2204	2204.00	0.00%	970.29	1716	1716	0.00%	759.55
Instance05	3211	3210.75	0.01%	18.75	3211	3211.00	0.00%	1374.96	1143	1143	0.00%	1064.94
Instance06	2246	2186.39	2.65%	3600	2446	2045.93	16.36%	3600	1950	1950	0.00%	2857.30
Instance07	4030	4030	0.00%	45.26	4031	4029.85	0.03%	3600	1056	1050.52	0.52%	3600
Instance08	2966	2946.19	0.67%	3600	3667	2941.51	19.78%	3600	1315	1272.38	3.24%	3600
Instance09	3866	3866	0.00%	34.48	3867	3865.41	0.04%	3600	439	182.5	58.43%	3600
Instance10	6329	6328.4	0.01%	1339.48	7381	6324.74	14.31%	3600	4631	4627.02	0.09%	3600
Instance11	2310	2260.44	2.15%	3600	85775	1387.44	98.38%	3600	3443	3443	0.00%	415.73
Instance12	11721.0	11721	0.00%	2262.58	1.35E+08	11719.20	99.99%	3600	4043	4035.37	0.19%	3600
Instance13	9181	6558.29	28.57%	3600	-	-	-	-	9398	1347	85.67%	3600
Instance14	1235	1234.95	0.00%	2616.4	69472	1232.76	98.23%	3600	-	-	-	-
Instance15	4019	3758.32	6.49%	3600	-	-	-	-	-	-	-	-
Instance16	2945	2834.98	3.74%	3600	6077	2340.06	61.49%	3600	-	-	-	-
Instance17	4886	4267.61	12.66%	3600	-	-	-	-	-	-	-	-
Instance18	4646.13	4266.95	8.16%	3600	-	-	-	-	-	-	-	-
Instance19	5116.38	3450.16	32.57%	3600	-	-	-	-	4186	2942.11	29.72%	3600
Instance20-24	-	-	-	-	-	-	-	-	-	-	-	-

Figure 11 – Gap comparison between parameterized approaches of PSP and Curtois and Qu (2014)

to solve in practice than their MILP counterparts. On the other hand, we emphasize that the performance of the model of (CURTOIS; QU, 2014) is primarily the result of how this model was reformulated to take into account fairness (and hence resulting in a larger solution space.)

To have a better perspective, we also present two performance profile graphs, using the computational time and gap as performance measure, as presented in Figures 13 and 14. We note that, though all instances are included in these graphs, for cases when a model could not solve an instance, we simply assumed them with $> 100\%$ gap and $> 3,600$ seconds.

We first note that, excluding the ties, $PSPER - (HW)_s$ outperforms the other methods in 6 of the instances in terms of gap, where in 3 out of these 6 instances, it has reached the optimal, and it is also the only method that has found solutions for 4 instances. The behaviour of

Figure 12 – Computational time comparison between parameterized approaches of PSP and [Curtois and Qu \(2014\)](#)

Table 15 – The benchmark results for $PSPER - (HW)_s$ and $PSPER - (HW)_s^2$ in comparison with the original model run in CPLEX solver for 3,600 seconds

Instance	$PSPER - (HW)_s$				$PSPER - (HW)_s^2$				Curtois and Qu (2014)			
	UB	LB	GAP	TIME	UB	LB	GAP	TIME	UB	LB	GAP	TIME
Instance01	13	13	0.00%	0.21	13	13	0.00%	0.77	607	607	0.00%	0.89
Instance02	5	5	0.00%	0.25	5	5	0.00%	2.06	828	828	0.00%	152.55
Instance03	0	0	0.00%	0.66	0	0	0.00%	127.45	1001	1001	0.00%	120.45
Instance04	1112	1112	0.00%	1.20	1112	1112	0.00%	3.98	1716	1716	0.00%	1620.95
Instance05	22	22	0.00%	695.91	22	18.56	15.63%	3600	1150	874.81	23.93%	3600
Instance06	117	117	0.00%	22.16	117	117	0.00%	674.76	1950	1950	0.00%	2938.24
Instance07	14	14	0.00%	13.48	14	14	0.00%	1124.75	1056	1050.08	0.56%	3600
Instance08	1225	1225	0.00%	148.57	1225	1225	0.00%	3572.58	1642	1268.42	22.75%	3600
Instance09	32	32	0.00%	1.64	32	32	0.00%	29.76	444	185	58.33%	3600
Instance10	7	7	0.00%	103.61	248	7	97.18%	3600	4631	4626.28	0.10%	3600
Instance11	1	1	0.00%	20.18	1	1	0.00%	627.47	3443	3443	0.00%	92.13
Instance12	15	15	0.00%	335.99	730	15	97.95%	3600	4168	4035.92	3.17%	3600
Instance13	224	224	0.00%	1267.20	-	-	-	-	-	-	-	-
Instance14	1217	1217	0.00%	15.80	1217	1217	0.00%	3351.52	-	-	-	-
Instance15	3865	3747.82	3.03%	3600	-	-	-	-	-	-	-	-
Instance16	119	20.76	82.55%	3600	219	22.56	89.70%	3600	-	-	-	-
Instance17	27	14	48.15%	3600	21847	14	99.94%	3600	-	-	-	-
Instance18	1097	963.11	12.21%	3600	-	-	-	-	-	-	-	-
Instance19	47	32	31.91%	3600	-	-	-	-	6813	2942.81	- 56.81%	3600
Instance20	240	240	0.00%	128.27	-	-	-	-	-	-	-	-
Instance21-24	-	-	-	-	-	-	-	-	-	-	-	-

$PSPER - (HW)_s$ is consistent throughout the instances, including significantly difficult instances such as Instance16, and Instance17. In terms of computational time, $PSPER - (HW)_s$ is faster in 12 of the solved instances (three times faster than [Curtois and Qu \(2014\)](#)), while $PSPER - (HW)_s^2$ has reached the time limit for 5 of the 15 instances, 3 of them in instances that $PSPER - (HW)_s$ has not. From the results, we can observe that the performance of $PSPER - (HW)_s$ performs better than $PSPER - (HW)_s^2$, in particular for medium and large-sized instances.

Figure 13 – Gap comparison between estimated approaches of PSP and Curtois and Qu (2014).

Figure 14 – Computational time comparison between estimated approaches of PSP and Curtois and Qu (2014)

Figure 15 – The evaluation of the lower and upper bound for $PSPER - (HW)_s$ for Instance05

Figure 16 – The evaluation of the lower and upper bound for $PSPER - (HW)_s^2$ for Instance05

5.2.2 Parameterized vs estimated approaches

Table 16 summarizes a comparison between proposed models in terms of demand satisfaction and workload deviations. Equations 5.26 and 5.27 show how do we establish the weekend violation, beside the former described Δ_H . Both estimated approaches present a high degree of satisfaction of demand and weekend constraints without variation of hours because the target number hour is relaxed (is a variable), and the effort is to assign as much as a physician as possible to comply with the demand. The null standard deviation on the deviation of hours indicates that the distribution is even amongst the physicians. In the case of the parameterized approach, demand and hour constraints have not complied as much as in the estimated approach due to the target number of hours as a parameter. The effort is to make a reasonable trade-off between demand violation and equitable distribution of workload.

$$wkndViol_p = \begin{cases} 1 & \text{If } |\sum_{w \in W} y_{pw} - T_p^W| > 0 \\ 0 & \text{otherwise} \end{cases} \quad p \in P, \quad (5.26)$$

$$\Delta_W = \frac{\sum_{p,m} wkndViol_p}{TotalWknConstr} \times 100 \quad (5.27)$$

Table 16 – Average demand, Hours, Target hours, Hour deviation average standard deviation and average weekend for $PSPER - (HW)_p$, $PSPER - (HW)_p^2$ formulations.

Instance	$PSPER - (HW)_p$					$PSPER - (HW)_p^2$					$PSPER - (HW)_s$					$PSPER - (HW)_s^2$				
	Δ_{dem}	$\bar{H}$	$\bar{TH}$	$\bar{\sigma}_{\sigma_H}$	Δw	Δ_{dem}	$\bar{H}$	$\bar{TH}$	$\bar{\sigma}_{\sigma_H}$	Δw	Δ_{dem}	$\bar{H}$	$\bar{TH}$	$\bar{\sigma}_{\sigma_H}$	Δw	Δ_{dem}	$\bar{H}$	$\bar{TH}$	$\bar{\sigma}_{\sigma_H}$	Δw
Instance01	36.00%	72	64	0	0.00%	43.00%	72	64	0	0.00%	0.00%	71	71	0	0.00%	0.00%	71	71	0	0.00%
Instance02	36.00%	52.57	52.57	0	0.00%	39.00%	52.57	52.57	0	0.00%	0.00%	61.7	61.7	0	0.00%	11.00%	61.7	61.7	0	0.00%
Instance03	28.00%	54	54	0	0.00%	40.00%	54	54	0	0.00%	0.00%	61.6	61.6	0	0.00%	19.00%	61.6	61.6	0	0.00%
Instance04	21.00%	121.6	128	± 11.80	0.00%	50.00%	121.6	128	± 11.8	0.00%	21.00%	136.8	136.8	0	0.00%	12.00%	136.8	136.8	0	0.00%
Instance05	39.00%	128	128	0	0.00%	68.00%	128	128	0	0.00%	<u>0.00%</u>	135	135	0	0.00%	<u>0.00%</u>	135	135	0	0.00%
Instance06	63.00%	124	124	0	0.00%	44.00%	124	124	0	11.00%	<u>1.00%</u>	135.1	135.1	0	0.00%	<u>1.00%</u>	135.1	135.1	0	0.00%
Instance07	27.00%	110	110	0	0.00%	45.00%	110	110	0	0.00%	<u>0.00%</u>	128.8	128.8	0	0.00%	<u>0.00%</u>	128.8	128.8	0	0.00%
Instance08	15.00%	120.8	120.8	0	0.00%	25.00%	120.8	120.8	0	7.00%	<u>0.00%</u>	125.6	125.6	0	0.00%	<u>5.00%</u>	125.6	125.6	0	0.00%
Instance09	20.00%	83.38	84.72	± 3.58	0.00%	27.00%	83.38	84.72	± 3.58	0.00%	0.00%	95.1	95.1	0	0.00%	0.00%	95.1	95.1	0	0.00%
Instance10	23.00%	128.15	128.15	0	0.00%	40.00%	128.15	128.15	0	3.00%	<u>0.00%</u>	141.75	141.75	0	0.00%	<u>0.00%</u>	141.75	141.75	0	0.00%
Instance11	27.00%	128	128	0	18.00%	4.00%	123.67	128	± 5.12	10.00%	<u>0.00%</u>	129.92	129.92	0	0.00%	<u>0.00%</u>	129.92	129.92	0	0.00%
Instance12	20.00%	118.67	118.67	0	0.00%	97.00%	22.53	118.67	± 26.98	2.00%	0.00%	135.10	135.07	0	0.00%	3.00%	134.8	134.8	0	0.00%
Instance13	11.00%	118.1	118.1	0	0.00%	0.00%	110.25	124.3	16.87	81.00%	0.00%	131.43	131.52	± 0.92	0.00%	52.00%	124	124	0	0.00%
Instance14	3.00%	124.3	124.3	0	0.00%	99.00%	110.25	124.3	16.87	81.00%	6.00%	124	124	0	0.00%	2.00%	124	124	0	0.00%
Instance15	7.00%	147.2	147.2	0	0.00%	0.00%	129.4	138.5	7.93	50.00%	6.00%	149.82	149.82	0	2.00%	12.00%	134.8	134.8	0	0.00%
Instance16	8.00%	142	138.5	± 3.62	0.00%	14.00%	129.4	138.5	7.93	50.00%	12.00%	134.8	134.8	0	0.00%	1.00%	134.8	134.8	0	0.00%
Instance17	8.00%	141	141.43	± 4.5	66.00%	0.00%	129.45	129.45	46.00%	0.00%	0.00%	136.25	136.25	0	0.00%	52.00%	136.25	136.25	0	0.00%
Instance18	13.00%	130.42	130.42	0	0.00%	0.00%	127.68	127.68	0	0.00%	0.00%	127.68	127.68	0	0.00%	0.00%	127.68	127.68	0	0.00%
Instance19	25.00%	130.68	130.68	0	10.00%	0.00%	138.52	138.52	0	0.00%	0.00%	138.52	138.52	0	0.00%	0.00%	138.52	138.52	0	0.00

As an example, Figure 17 illustrates the behavior of Instance07, where we can observe that for each model, there is no deviation of hours. This is due to the effort to minimize at most the values of deviations with a high-cost on the objective function and consider penalize the low-cost parts of the objective function, namely the requested shifts and demand compliance. At this aspect, these models have an equivalent performance.

Figure 17 – Comparison of hour deviations between $PSPER - (HW)_p$, $PSPER - (HW)_p^2$, $PSPER - (HW)_s$ and $PSPER - (HW)_s^2$ for Instance07

Figure 18 depicts the behavior of each model in terms of the weekend. For each physician on Instance07, one can notice that only blue dotted model has deviations, which is equivalent to the $PSPER - (HW)_p^2$. The behavior found in this test instances shows the slight disturbances in the weekend for the parameterized model with quadratic costs. The perceived solution is near to the optimal one found by the linearized model. This result corroborates with the principle that linear approximation is an alternative to find cuts and improvements on the formulation, such that improved feasible solutions can be found in less computational time. In a nutshell, the overall results show that estimated values with linear approximation are better than the parameterized.

Figure 18 – Comparison of weekend deviations between $PSPER - (HW)_p$, $PSPER - (HW)_p^2$, $PSPER - (HW)_s$ and $PSPER - (HW)_s^2$ for Instance07.

5.3 Conclusion

This chapter tackles the Physician Scheduling Problem in Emergency Rooms with fairness in hours and weekends using concepts of L^1 , L^2 , and L^∞ norms, and linear approximation of quadratic terms. Moreover, we estimated target hours and weekends' values to achieve a greater level of satisfaction of fairness in hours, weekends, and demand coverage. Our formulations are based on a literature MIP model applied to the Nurse Rostering Problem. We used a set of benchmark instances to conduct the experiments on the CPLEX optimization solver. To infer the quality of linear approximations of the quadratic terms, we tested different values of numbers of points to approximate the curve.

We performed comparisons between our model and ??) in terms of fairness and the number of solutions found in the established computational time. In both criteria, our approaches present better results. This is because our models are less constrained versions of the original model.

Regarding linear approximations on a variant of L^2 , we can infer that the number of points significantly influences overall performance. The more points to make a linear approximation, the more variables are needed.

When comparing L^1 , L^2 , and L^∞ , we could observe the performance in terms of solution quality and time. The L^2 norm outperformed in LB and L^∞ in computational time, mostly due to the number of variables to describe the feasible region.

Estimating the parameters of hours and weekends offers better results than parameterized approaches once target variables are relaxed to assume the best values to comply with and make reasonable trade-offs between coverage and workload satisfaction.

HCPA: A REAL SCENARIO OF PHYSICIAN SCHEDULING PROBLEM

In this chapter, we address the physician scheduling problem in multiple locations, the case of Hospital de Clínicas de Porto Alegre. A mathematical model reformulation is proposed, based on hospital requirements, by applying the previous concepts presented to obtain fairness in schedules.

6.1 Case Study Scenario: Hospital de Clínicas de Porto Alegre

The *Hospital de Clínicas de Porto Alegre* is a public and large university hospital affiliated to the *Universidade Federal do Rio Grande do Sul*. The hospital is one of the main pillars of public health care, offering excellent and high complexity care in a wide range of specialties such as cardiology, general surgery, oncology, etc, [HCPA \(2020\)](#). Regarding Emergency, the ward has beds registered to serve the *Sistema Único de Saúde* in the following specialties: General Clinic, Gynecology, Surgery, and Pediatrics. The Emergency is the door for the care of critically ill patients at risk of life. Patients are welcomed according to the Manchester Scale risk classification described in [Table 17](#). The most severe patients are treated with priority over the less critical patients, regardless of their arrival time.

Table 17 – The Manchester triage scale ([LINDENAU-STOCKFISCH; SEARLE; MÖCKEL, 2013](#)).

Color	Category	Situation
1 - Red	Immediate resuscitation	Patient in need of immediate treatment for preservation of life
2 - Orange	Very urgent	Seriously ill or injured patients whose lives are not in immediate danger
3 - Yellow	Urgent	Patients with serious problems, but apparently stable condition
4 - Green	Standard	Standard cases without immediate danger or distress
5 - Blue	Nonurgent	Patients whose conditions are not true accidents or emergencies

The work schedules for the Hospital de Clínicas de Porto Alegre's medical staff are very demanding since the responsible schedulers face a challenging tailor-made task given the institution's complexity. This complexity is due to some factors such as the variability and unpredictability of patient demand, the range of the services offered, the work mode (shifts, duties, or a combination of both), the type of labor contract (full time or part-time) as well as the allowance to exchange physicians between wards/departments.

Physicians' assignment to hospital wards was tailor-made until recently, approximately 2018, which compromised the qualified workforce in some situations. Moreover, the provided schedules are not guaranteed to be feasible since the number of rules to build a schedule is significantly high, and a lot of the rules are conflicting.

To tackle this problem, the authors in (WICKERT *et al.*, 2020) proposed a MIP model and solved it using IBM CPLEX and Coin-OR solvers. The hospital schedule management board considered that workload should be more accurate to provide a more efficient work plan, specifically in workload. Indeed the authors presented a relevant and satisfactory response, adding balance in workload and consecutiveness constraints to overcome this drawback. As already mentioned in the previous chapter, the responsible person for preparing the schedule considers mandatory and flexible requirements, technically referred to as hard and soft/flexible constraints. In HCPA, these constraints are:

- **Hard constraints**

- **Limited workload:** Physicians cannot be assigned to more than one shift per day during business days;
- **Maximum coverage:** There is a maximum of physicians on each shift of a day on each location;
- **Minimum coverage:** There is a minimum of physicians on each shift of a day on each location;
- **Assignments on non-business days:** When assigned to non-business days, a physician is assigned to both day shifts or a Nightshift, or he will be assigned a day off, so he complies the 12 hours duty for non-business days;
- **Forbidden assignments:** There are certain types of shifts that cannot be assigned immediately after others, so a minimum of rest can be observed.
- **Unavailability:** For personal requirement or legal rules, a physician can be unavailable for some shifts or days;
- **Same location:** If a physician works both day shifts in a day, he must be assigned to the same location;
- **Qualification:** A physician can only be assigned to a shift in a location he is qualified to;

- **Soft constraints**

- **Shift consecutiveness:** There is a maximum number of consecutive assignments to the same shift;
- **General consecutiveness:** There is a maximum number of consecutive assignments in all shifts;
- **Requested shift off:** Each physician has a set of undesired working days or shifts;
- **Weekend assignments:** When a physician is assigned to at weekends he ideally works both Saturday and Sunday or none;
- **Maximum of weekends:** There is a maximum number of working weekends;
- **Minimum assignments:** There is a minimum number of assignments over the planning horizon according to the working contract;
- **Maximum assignments:** There is a maximum number of assignments over the planning horizon according to the working contract.

Table 18 depicts in a physician-day view a weekly schedule for four fictional physicians, $\{P[1], \dots, P[4]\}$ in 3 locations/wards $\{1, 2, 3\}$, numbered accordingly. The shift $\{E\text{-Early}, L\text{-Late}, \text{or } N\text{-Night}\}$, to which a physician is assigned, is labeled in the corresponding day cell followed by the ward in brackets. A distinction is made between weekdays and weekends by the number of day shifts are assigned. If a physician works in a day shift [early or late] on weekends, she/he must work in both shifts, so she/he will comply with the third hard constraint. Moreover, each shift type has also the forbidden shift types sequences. In the illustration example, the successions Late $\rightarrow$ Early, Night $\rightarrow$ Early and Night $\rightarrow$ Late are forbidden. Observe that the Saturday shift assignment of P[4] is marked in red and italic, showing that the assignment on non-business days constraint is violated.

Table 18 – A week schedule

Physician	Mon	Tue	Wed	Thu	Fri	Sat	Sun
P ₁		N[1]	N[2]			E/L[1]	E/L[1]
P ₂	E[1]	L[2]			N[1]		
P ₃			E[1]	E[3]		N[1]	N[2]
P ₄	N[2]			L[1]		<i>E[1]</i>	E/L[2]

6.2 Model formulation

Monthly, the responsible person for preparing the schedules considers the data from the previous month to avoid undesired conflicts. This measure is essential to cope with situations similar to, if a physician has completed the maximum consecutive stretch on the last day of

the previous month, he/she cannot be assigned to the first day of the current month. Therefore, it is necessary to tackle such situations, and so, before the solving method starts, the border data from the previous month is read. It includes the total number of assignments, last assigned shift type, number of following duties of the last type shift, and number of consecutive worked days. The original model is HCPA, and the variations we address in this thesis are HCPA-II and HCPA-III, respectively, considering weekend fairness and complete assignment. In comparison to the original paper, we applied slight changes for clarity in some sets, which we cite. The sets of physicians and locations were N and K , whereas here we define as P and L , respectively. Furthermore, we change the set $\hat{S}$ by Υ_s to define the set of shifts conflicting with s , set of unavailable days, shifts, and locations for a physician was originally defined as R , and we change by U_p .

Table 19 presents the sets, parameters, and variables used in the mathematical description of the problem. In subsection 6.2.1, the formulation considering both hard and soft constraints for the Physician Scheduling Problem at HCPA is introduced. In advance, we address the reader to the Table 20 that presents, from the left to right columns, the following information: Indexes of the constraints, respective equations reference, constraint description, the identifier of the constraint, and the weights used in the objective function such that one can quickly identify the variables associated with their constraints.

6.2.1 HCPA

This sub-section is devoted to introduce the Mixed-Integer Programming model formulation of the PSPER at HCPA proposed by Wickert *et al.* (2020).

HCPAObjF: Objective function

The objective function consists of a weighted sum on deviation on soft constraints. The first factor of the objective function is the penalization of maximum consecutive assignments to the same shift and worked days. The second penalizes the undesired assignment of physicians. The third is related to the whole weekend. The last factor is associated with minimum and maximum assignments over the planning horizon and the maximum number of working weekends.

$$\text{Min} \sum_{p \in P} \sum_{d \in D} \sum_{i \in \{3,4\}} c_{pd}^i \omega^i + \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} g_{pds}^5 \omega^5 + \sum_{p \in P} \sum_{w \in W} h_{pw}^6 \omega^6 + \sum_{p \in P} \sum_{i \in \{7,8,9\}} j_p^i \omega^i \quad (6.1)$$

HCI: Weekday assignment

This constraint ensures that a physician is assigned to at most one shift per day on

Table 19 – Table containing sets, parameters, decision and auxiliary variables

Sets and Parameters	
P	Set of physicians.
D	Set of days.
$\bar{D}$	Set of all non-working days.
S	Set of shifts. s_e, s_l , and s_n are indexes for Early, Late and Night shift respectively.
L	Set of locations.
L_{pl}	Set containing the pairs of forbidden locations of physician p at location l .
Y_s	Set of shifts conflicting with s .
W	Set of all Saturday indices not including the last Saturday if it is the last day of the month. w is a Saturday index.
$\tilde{W}$	Set of all week indices.
U_p	Set containing triples with unavailable physician p on day d , and shift s
β_p^i	Limits associated to constraints with indexes 3, 4, 7, 8, 9 in Table 20. Respectively, the maximum number of consecutive assignments to the same shift (3), maximum number of consecutive working days (4), minimum/maximum number of assignments over the planning horizon (7, 8), and maximum number of working weekends (9),
α_{dsl}^i	$i \in \{1, 2\}$ that is the minimum and maximum number of physicians per day d , shift s , and location l ;
ω^i	weight for violating the lower and upper bounds of soft constraint i
Decision variables	
$x_{pds} \in \mathbb{B}$	1, If physician p is assigned to shift s , day d , and location l 0, Otherwise
$y_{pw} \in \mathbb{B}$	1, If physician p works weekend w 0, Otherwise
$z_{pd} \in \mathbb{N}$	1, If physician p works both the Early and Late shifts on day d , 0, Otherwise
$o_{pd} \in \mathbb{N}$	1, If physician p is assigned to day d 0, Otherwise
Auxiliary variables	
$c_{pd}^i \in \mathbb{N}$	Number of violations associated to constraint with indices $i \in \{3, 4\}$ in Table 20, for physician p on day d
$g_{pds}^5 \in \mathbb{N}$	Number of violations to constraint with index 5 in Table 20, for physician p on day d , shift s
$\Delta_p^w \in \mathbb{N}$	Deviation of weekends for each physician
$j^i \in \mathbb{N}$	Number of violations of the soft constraint with indexes $i \in \{1, 8, 9\}$ in Table 20, for physician p
a_{dsl}^i	Number of violation of soft constraint with indexes $i \in \{1, 2\}$ in Table 20, for day d on shift s in location l

business days.

$$\sum_{s \in S} \sum_{l \in L} x_{pds} \leq 1 \quad \forall p \in P, d \in D \setminus \bar{D}, s \in S, \quad (6.2)$$

Table 20 – Description of the constraints, equation and their respective weights

Index	Equation	Constraint description	Id	Weight
1	(6.10)	Minimum number of physician per day/shift/location	FC6	100000
2	(6.11)	Maximum number of physician per day/shift/location	FC7	100000
3	(6.12)	Maximum number of consecutive assignments to the same shift	FC1	15
4	(6.13)	Maximum number of consecutive assignments (worked days)	FC2	30
5	(6.14)	Physician undesired working day/shift	FC3	10
6	(6.15)	Complete weekend	FC4(B)	30
7	(6.17)	Minimum number of assignments over the planning horizon	FC5	20
8	(6.18)	Maximum number of assignments over the planning horizon	FC5	20
9	(6.16)	Maximum number of working weekends	FC4	30

HC2: Weekend assignment

Constraints (6.3) and (6.4) establishes that a physician is assigned to both day shifts (morning and afternoon), or one night shift, or no shift on non-business days. This constraint will avoid situations such as the depicted in the last row of Table 18.

$$\sum_{l \in L} (x_{pds_{el}} + x_{pds_{ll}}) = 2z_{pd} \quad \forall p \in P, d \in \bar{D}, \quad (6.3)$$

$$\sum_{l \in L} x_{pds_{nl}} + z_{pd} \leq 1 \quad \forall p \in P, d \in \bar{D}, \quad (6.4)$$

HC3: Assignment control

If a physician p works on d , this constraint is activated. The controlling variable o_{pd} will help detect maximum assignment of days or weekends.

$$\sum_{s \in S} \sum_{l \in L} x_{pds_l} \leq 2o_{pd} \quad \forall p \in P, d \in D \quad (6.5)$$

HC4: Invalid succession

This constraint tackles an invalid sequence of shifts between days. E.g., if a physician works at night today, tomorrow he/she cannot work Morning or Afternoon.

$$\sum_{l \in L} (x_{pds_l} + x_{pds_{fl}}) \leq 1 \quad \forall p \in P, d \in \{1, \dots, |D| - 1\}, s \in S, f \in Y_s, \quad (6.6)$$

HC5: Forbidden assignments to locations

Constraints on equation (6.7) ensures that a physician is not assigned to a location that he/she is not allowed.

$$\sum_{s \in S} \sum_{l \in L} x_{pds_l} = 0 \quad \forall (p, l) \in L_{pl}, \quad (6.7)$$

HC6: Holidays/Leaves

If a physician asks to be out of the schedule in some shifts/days/locations, he/she has the mandatory right to be out of the schedule. Thus, constraints on equation (6.8) ensures that a physician is not assigned to the days he requests to be off the schedule.

$$\sum_{l \in L} x_{pds l} = 0 \quad \forall (p, d, s) \in U_p, \quad (6.8)$$

HC7: Early and Evening shifts

This constraints ensures that a physician works both day shifts at the same location if they are worked on non-working days.

$$x_{pds_e l} = x_{pds_l l} \quad \forall p \in P, d \in D, d \in \bar{D}, l \in L, \quad (6.9)$$

HC8: Minimum and maximum coverage

Constraints on equations (6.10) and (6.11) ensure a minimum and a maximum number of physician per day/shift/location, respectively.

$$\sum_{p \in P} x_{pds l} \geq \alpha_{ds l}^1 \quad \forall d \in D, s \in S, l \in L, \quad (6.10)$$

$$\sum_{p \in P} x_{pds l} \leq \alpha_{ds l}^2 \quad \forall d \in D, s \in S, l \in L, \quad (6.11)$$

FC1: Night Consecutiveness

If a physician is assigned to a night shift, there is a maximum stretch in which she/he will be assigned throughout the scheduling horizon. This is the first in the set of flexible constraints.

$$\sum_{j=d}^{d+\beta_p^3} \sum_{l \in L} x_{pjs_n l} - c_{pd}^3 \leq \beta_p^3 \quad \forall p \in P, d \in \{1, \dots, |D| - \beta_p^3\}, \quad (6.12)$$

FC2: Consecutiveness control

There is a preference to a limit on consecutive day shifts, i.e, if a physician is assigned to a day shift (morning or afternoon), she/he has a maximum of shifts she/he can be assigned to. Due to uncertainty, unavailability of other physicians to cover specific shifts, among other factors, violations are accepted.

$$\sum_{j=d}^{d+\beta_p^4} o_{pj} - c_{pd} \leq \beta_p^4 \quad \forall p \in P, d \in \{1, \dots, |D| - \beta_p^4\}, \quad (6.13)$$

FC3: Undesired working days

This constraint calculate the undesired worked day or shift violations. To the best of our knowledge it is unique.

$$\sum_{l \in L} x_{pds l} \leq g_{pds}^5 \quad \forall (p, d, s) \in U_p, \quad (6.14)$$

FC4: Weekend assignment

Constraints in equation (6.15) calculates the complete weekend violations. As weekends are undesired by the majority of the employees, constraint set in equation (6.16) ensures the maximum number of working weekends considering violations.

$$o_{pw} + o_{p(w+1)} + h_{pw}^6 = 2y_{pw} \quad \forall p \in P, w \in W, \quad (6.15)$$

$$\sum_{w \in W} y_{pw} - j_p^9 \leq \beta_p^9 \quad \forall p \in P, w \in W, \quad (6.16)$$

FC5: Minimum and Maximum number of worked shifts

As every individual of the workforce has a minimum and maximum of shifts he/she can accomplish during the planning horizon, constraints (6.17) and (6.18) aim to control the workload of each physician during the scheduling horizon considering possible violations.

$$\sum_{d \in D} \sum_{s \in \{s_e, s_l\}} \sum_{l \in L} x_{pds l} + \sum_{d \in D} \sum_{l \in L} 2x_{pds_n l} + j_p^7 \geq \beta_p^7 \quad \forall p \in P, \quad (6.17)$$

$$\sum_{d \in D} \sum_{s \in \{s_e, s_l\}} \sum_{l \in L} x_{pds l} + \sum_{d \in D} \sum_{l \in L} 2x_{pds_n l} - j_p^8 \leq \beta_p^8 \quad \forall p \in P, \quad (6.18)$$

6.2.2 HCPA-II

One of the problems when solving the HCPA is the unbalanced distribution of weekends. We added the least absolute deviation concept on weekend constraints to cope with this issue, which we borrowed from PSPER-HW formulation. This approach will help us find a more equitable distribution of weekends, without compromising the remaining requirements. The inclusion of the least absolute deviation constraints modifies the model, and the objective function has a new term, which is the deviation. Thus, HCPA and HCPA-II differ in a set of weekend constraints and objective functions.

FC6: Balanced weekend assignment

$$\Delta_p^w \geq \sum_{w \in W} y_{pw} - \beta_p^9 \quad \forall p \in P, \quad (6.19)$$

$$\Delta_p^W \geq \beta_p^9 - \sum_{w \in W} y_{pw} \quad \forall p \in P, \quad (6.20)$$

FC7: Minimum and maximum coverage

Originally, Wickert *et al.* (2020) modelled minimum and maximum coverage constraints as hard constraint, described in equations (6.17) and (6.18), however, these constraints come with a high degree of infeasibility. To cope with this problem, we add slack variables on the minimum and maximum constraints, so we could consider solutions with under or over-staffing.

$$\sum_{p \in P} x_{pds} - a_{dsl}^1 \geq \alpha_{dsl}^1 \quad \forall d \in D, s \in S, l \in L, \quad (6.21)$$

$$\sum_{p \in P} x_{pds} + a_{dsl}^2 \leq \alpha_{dsl}^2 \quad \forall d \in D, s \in S, l \in L, \quad (6.22)$$

HCPAObjF2: Objective function

The new objective function is the weighted sum of soft constraints to which we add deviations related to weekend balance and remove the violations of maximum number of weekends.

$$\begin{aligned} \text{Minimize } & \sum_{d \in D} \sum_{s \in S} \sum_{l \in L} \sum_{i \in \{1,2\}} a_{dsl}^i \omega^i + \sum_{p \in P} \sum_{d \in D} \sum_{i \in \{3,4\}} c_{pd}^i \omega_p^i + \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} g_{pds}^5 \omega_p^5 + \sum_{p \in P} \sum_{w \in W} h_{pw}^6 \omega_p^6 \\ & + \sum_{p \in P} \sum_{i \in \{7,8\}} j_p^i \omega_p^i + \sum_{p \in P} \sum_{w \in W} \Delta_p^W \\ \text{subject to } & (6.2) - (6.9) \\ & (6.12) - (6.22) \end{aligned}$$

6.3 Computational experiments for HCPA- II

To evaluate the performance of our least absolute deviation approach in terms of solution time and quality, we used two sets of instances described in detail in subsection 6.3.1 which results are discussed in subsection 6.3.2. Further changes on the model are reported in subsection 6.5 that considers a mixed-integer quadratic approach to tackle higher deviations. The experiments correspond to a model implemented using Java version with callable libraries from IBM ILOG CPLEX 12.8 optimization solver running on a Intel Core i5-2410M CPU with 2.30GHz $\times$ 4 processor and 128GB of RAM running Linux Ubuntu 18.04.3 LTS 64-bit.

6.3.1 Instance generation

The generation of the instances is based on the information provided by the HCPA. Originally the hospital has a workforce of 50 physicians to assign. HCPA, HCPA-II, and HCPA-III models was tested in 20 generated instances. As HCPA faced an expansion in order of 70% at the end of 2019, the instances are generated increasing the number of physicians. The objective is to analyze whether these larger and more demanding instances can be solved using the proposed HCPA model variants. The summarized information is presented in Table 21.

Table 21 – The characteristics of the set of HCPA instances

Instance	#Weeks	#Physicians	#Shift	#Locations	Contract type
$p050_inst_ [1 - 10]$	4	50	3	3	2
$p100_inst_ [1 - 10]$	4	100	3	3	2

When tackling these instances the model has to be able to solve a single stage of the problem corresponding to one week. A history is carried out between consecutive weeks, so that the one coming from the previous week has to be taken into account by the solver. This history includes information such as the last worked shift of each nurse, and counters for cumulative data, such as total worked night shifts. Counters' value has to be checked against global thresholds, but only at the end of the planning period. The planning horizon is not rolling but fixed, in the sense that in the final week all counters are checked against their limits. The difference between the instances dwells on the number of requested shift off, and the number of unavailable shift which are random.

6.3.2 Results and discussion

When performing analyses on the schedules, we identify the soft constraints that affect the problem as well as the impact of the least absolute deviation of weekend constraints in the whole formulation. Table 22 shows the results obtained using HCPA and HCPA-II. We report the gap, expressed by the formula $GAP = \frac{UB-LB}{UB}$ in column GAP. In columns UB and LB, we present the primal and the dual bounds reported by CPLEX on finishing. We set the time limit for 10 minutes, 20 minutes, and 40 minutes to the first and second sets of instances. If a gap 0, i.e., an optimal is found, it means that the solver found it in the estimated time, or else, only a feasible solution was found.

As we observe in Table 22, both formulations were successful in finding optimal solutions for the first set of instances in the stipulated time. From the values of the objective function, it is straightforward to infer that HCPA-II is a relaxation of HCPA, once objective function values are lower. CPLEX solver found optimal solutions in the first set of instances with both formulations, but could not prove optimality for both formulations within the elapsed time in the second set of instances. Thus, we can see differences between the formulations regarding GAP, since the same time is given to both. As we formerly interpreted, the HCPA-II as a relaxation of the HCPA, we

Table 22 – Computational experiments on HCPA and HCPA-II after 10, 20, 40 minutes of execution

Instance	HCPA						HCPA-II					
	Time Limit = 10, 20 min			Time Limit = 40 min			Time Limit = 10, 20 min			Time Limit = 40 min		
	UB	LB	GAP	UB	LB	GAP	UB	LB	GAP	UB	LB	GAP
p050_inst_01	30305	30305	0.00%	30305	30305	0.00%	28655	28655	0.00%	28655	28655	0.00%
p050_inst_02	30460	30460	0.00%	30460	30460	0.00%	28810	28810	0.00%	28810	28810	0.00%
p050_inst_03	30505	30505	0.00%	30505	30505	0.00%	28915	28915	0.00%	28915	28915	0.00%
p050_inst_04	30965	30965	0.00%	30965	30965	0.00%	29375	29375	0.00%	29375	29375	0.00%
p050_inst_05	30685	30685	0.00%	30685	30685	0.00%	29095	29095	0.00%	29095	29095	0.00%
p050_inst_06	31705	31705	0.00%	31705	31705	0.00%	30025	30025	0.00%	30025	30025	0.00%
p050_inst_07	30015	30015	0.00%	30015	30015	0.00%	28425	28425	0.00%	28425	28425	0.00%
p050_inst_08	30215	30215	0.00%	30215	30215	0.00%	28565	28565	0.00%	28565	28565	0.00%
p050_inst_09	31670	31670	0.00%	31670	31670	0.00%	30020	30005	0.05%	30020	30005	0.05%
p050_inst_10	30765	30765	0.00%	30765	30765	0.00%	29085	29085	0.00%	29085	29085	0.00%
Average			0.00%			0.00%			0.00%			0.00%
p100_inst_01	26885	23630	12.11%	26885	24290	9.65%	26540	25550	3.73%	26540	25550	3.73%
p100_inst_02	118770	26050	78.07%	266550	26748	89.97%	68965	27970	59.44%	68965	27970	59.44%
p100_inst_03	542240	24342	95.51%	254775	24342	90.45%	27270	26120	4.22%	27080	26120	3.55%
p100_inst_04	313135	24149	92.29%	313135	24286	92.24%	157040	25390	83.83%	26165	25390	2.96%
p100_inst_05	27560	24542	10.95%	27560	24575	10.83%	26675	25780	3.36%	26675	25780	3.36%
p100_inst_06	28625	25212	11.92%	28625	25644	10.41%	27935	26830	3.96%	27705	26830	3.16%
p100_inst_07	26550	23225	12.52%	25105	23235	7.45%	25755	24590	4.52%	25740	24590	4.47%
p100_inst_08	27585	24590	10.86%	27655	25282	8.58%	27890	26500	4.98%	27545	26500	3.79%
p100_inst_09	255550	25287	90.10%	255550	25918	89.86%	28175	27110	3.78%	26025	24890	4.36%
p100_inst_10	27030	23770	12.06%	27030	23770	12.06%	26025	24890	4.36%	26025	24890	4.36%
Average			42.64%			42.15%			17.62%			9.28%

can also see the degree of increase of the objective function. When LB values the HCPA-II are higher than its counterpart, we can understand that the LB evolve quicker. This is mostly due to the inclusion of the least absolute deviation equations.

Moreover, these equations allow the sudden decrease of the upper bound. We can see this by the gap value. For every instance, the gap of HCPA-II was tighter. Overall results show that HCPA-II solved in comparison to HCPA 100% of the cases with 100 physicians, with tighter gaps and a low average gap.

Since the numerical values are not enough to evaluate the quality of each formulation, we compare both models using the constraint satisfaction rate and we analyse the correlation between constraints. Table 23 presents the constraint satisfaction rate. For each constraint described in Table 20, we can see that the maximum weekend constraints satisfaction rate increased with the inclusion of the least absolute deviation without any significant change in other flexible constraints. With these findings, we can infer that this measure is adequate for the type of problem in hand.

The Pearson product-moment correlation coefficient (PPMCC) statistic relationship measures the linear correlation between bi-variate data. Its values can vary between -1 and +1, indicating total negative and positive linear correlation, respectively, whereas 0 indicates no linear correlation. Thus, taking data in column FC4(B), we establish a relationship with the remaining columns. This relation is expressed by the formula:

Table 23 – Constraints' degree of satisfaction

Instance	Formulation	FC1	FC2	FC3	FC4	FC4(B)	FC5	FC6	FC7
p050_inst_01	HCPA	78.67%	0.00%	84.00%	20.00%	41.00%	0.00%	100.00%	100.00%
	HCPA-II	80.67%	0.00%	84.00%	100.00%	41.00%	0.00%	100.00%	100.00%
p050_inst_02	HCPA	77.00%	0.00%	98.00%	18.00%	40.00%	2.00%	100.00%	100.00%
	HCPA-II	77.00%	0.00%	98.00%	100.00%	40.00%	2.00%	100.00%	100.00%
p050_inst_03	HCPA	78.00%	0.00%	98.00%	24.00%	43.00%	0.00%	100.00%	100.00%
	HCPA-II	75.00%	0.00%	98.00%	100.00%	43.00%	0.00%	100.00%	100.00%
p050_inst_04	HCPA	77.00%	0.00%	100.00%	26.00%	23.00%	0.00%	100.00%	100.00%
	HCPA-II	74.00%	0.00%	100.00%	100.00%	43.00%	0.00%	100.00%	100.00%
p050_inst_05	HCPA	75.00%	0.00%	96.00%	18.00%	43.00%	0.00%	100.00%	100.00%
	HCPA-II	75.00%	0.00%	96.00%	100.00%	43.00%	0.00%	100.00%	100.00%
p050_inst_06	HCPA	77.00%	0.00%	96.00%	20.00%	39.00%	2.00%	100.00%	100.00%
	HCPA-II	77.00%	0.00%	96.00%	100.00%	38.00%	2.00%	100.00%	100.00%
p050_inst_07	HCPA	76.00%	0.00%	96.00%	20.00%	43.00%	0.00%	100.00%	100.00%
	HCPA-II	76.00%	0.00%	96.00%	100.00%	43.00%	0.00%	100.00%	100.00%
p050_inst_08	HCPA	77.00%	0.00%	84.00%	22.00%	41.00%	0.00%	100.00%	100.00%
	HCPA-II	74.00%	0.00%	84.00%	100.00%	41.00%	0.00%	100.00%	100.00%
p050_inst_09	HCPA	75.00%	0.00%	90.00%	20.00%	40.00%	2.00%	100.00%	100.00%
	HCPA-II	75.00%	0.00%	90.00%	100.00%	39.00%	2.00%	100.00%	100.00%
p050_inst_10	HCPA	74.00%	0.00%	90.00%	16.00%	39.00%	2.00%	100.00%	100.00%
	HCPA-II	77.00%	0.00%	90.00%	100.00%	38.00%	2.00%	100.00%	100.00%

$$\rho(FC4(B), FCi) = \frac{Cov(FC4(B), FCi)}{\sqrt{Var(FC4(B))Var(FCi)}}, \quad i \in \{1, \dots, 7\} \setminus \{4\}$$

As we can observe from Table 24 and Figure 19, the values from the PPMCC are close to zero, what indicates no linear correlation between the data. In consequence we infer that the inclusion of least absolute deviation do not alter the remaining constraints and fairness is established.

Table 24 – Correlation between FC4(B)- complete weekend with least absolute deviation constraint and the remaining

	FC4(B)-FC1	FC4(B)-FC2	FC4(B)-FC3	FC4-FC4(B)	FC4(B)-FC5	FC4(B)-FC6	FC4(B)-FC7
$\rho(FC4(B), FCi)$	-0.15	-	-0.14	0.17	-0.18	-	-

6.4 HCPA-III

As we can see in Table 23, there is a high degree of violation of constraints related to complete assignments and the maximum number of consecutive assignments. Thus, the question arises: *is it possible to achieve a balance in the satisfaction of all these rules?* This led us to create an alternative formulation to HCPA formulation. As studied in former chapters, the inclusion of quadratic costs and fairness approach on hours constraints allowed us to produce satisfying results, and a higher degree of satisfaction of related constraints within a reasonable computational time. Therefore, as we can interpret total assignments is the equivalent constraint

Figure 19 – Correlation between FC4B constraints and remaining

to workload, we establish a target for complete assignments through equation $T_\beta = \frac{\beta_p^7 + \beta_p^8}{2}$ and penalize in the objective function with a quadratic cost.

HCPAObjF3: Objective function

The new objective function is the weighted sum of soft constraints to which we add deviations related to the target number of worked shifts with quadratic costs and remove the violations of the maximum number of worked shifts. As in the definition of the weights, both minimum and the maximum number of assignments over the planning horizon have the same weight; we considered doable to use one or another weight.

$$\begin{aligned} \text{Min} \quad & \sum_{d \in D} \sum_{s \in S} \sum_{l \in L} \sum_{i \in \{1,2\}} a_{dsl}^i \omega^i \sum_{p \in P} \sum_{d \in D} \sum_{i \in \{3,4\}} c_{pd}^i \omega_p^i + \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} g_{pds}^5 \omega_p^5 + \sum_{p \in P} \sum_{w \in W} h_{pw}^6 \omega_p^6 \\ & + \sum_{p \in P} j_p^2 \omega_p^7 + \sum_{p \in P} \sum_{w \in W} \Delta_p^W \quad (6.23) \end{aligned}$$

We changed the former minimum and maximum number of assignments over the planning horizon constraints (6.17) and (6.18) to tackle fairness using the new constraints described in equations (6.24) and (6.25).

$$j_p \geq \sum_{d \in D} \sum_{s \in \{s_e, s_l\}} \sum_{l \in L} x_{pds} + \sum_{d \in D} \sum_{l \in L} 2x_{pds_n l} - T_\beta \quad \forall p \in P, \quad (6.24)$$

$$j_p \geq T_\beta - \sum_{d \in D} \sum_{s \in \{s_e, s_l\}} \sum_{l \in L} x_{pds} + \sum_{d \in D} \sum_{l \in L} 2x_{pds_n l} \quad \forall p \in P, \quad (6.25)$$

The complete model HCPA-III is:

$$\begin{aligned}
\text{Min} \quad & \sum_{d \in D} \sum_{s \in S} \sum_{l \in L} \sum_{i \in \{1,2\}} a_{dsl}^i \omega^i \sum_{p \in P} \sum_{d \in D} \sum_{i \in \{3,4\}} c_{pd}^i \omega_p^i + \sum_{p \in P} \sum_{d \in D} \sum_{s \in S} g_{pds}^5 \omega_p^5 + \sum_{p \in P} \sum_{w \in W} h_{pw}^6 \omega_p^6 \\
& + \sum_{p \in P} (j_p)^2 \omega_p^7 + \sum_{p \in P} \sum_{w \in W} \Delta_p^w \\
& \text{subject to} \quad (6.2) - (6.9) \\
& \quad (6.12) - (6.15) \\
& \quad (6.19) - (6.22) \\
& \quad (6.24), (6.25)
\end{aligned}$$

6.5 Results from HCPA-III

The results reported in Table 25 present Time, which wasn't reported before for the elapsed time. A time limit of 2400 second was set. At first sight, through Table 25, one notices that objective function values for HCPA-III are higher than HCPA and HCPA-II. Besides the quadratic costs, this situation stems from the fact that demand constraints were not satisfied and thus penalized. On the other hand, HCPA-III formulation provided optimal solution in lesser computational time when compared to HCPA and HCPA-II for both sets of instances, without exception.

Given the weights on constraints, preference was given to violate minimum demand, as we can also read from Table 27. Regarding average gap and time, HCPA-II and HCPA-III present similar results for the first set of instances, since both found the solver proved optimal solution in the elapsed time. For the second set of instances, HCPA-III had lower values for the gap in comparison to HCPA and HCPA-II, mostly due to the combination of least absolute deviation and quadratic costs in the objective function, and the CPLEX solver proved optimality for 90% of the instances. Regarding the elapsed time, the solver spent approximately 11% of total time to solve at optimality these instances, whereas HCPA and HCPA-II spent the whole given time.

Besides the performance tests, in Table 27, we describe the constraint degree of satisfaction for each of the three formulations, to a better qualitative analysis. With HCPA-III, we can see a great difference in the degree of satisfaction of FC2 and FC5 constraints, with its values varying from 0 to 100% and approximately 75%. Conversely to what happened in the analysis between HCPA and HCPA-II, here we can see a strong correlation between some of the constraints, as we included quadratic cost and least absolute deviation.

We also depict 3 graphs of correlation between constraints for the most affected constraint in Figures 20, 21, and 22. There is a strong positive correlation between FC4B and FC1, FC2, and FC5, after the inclusion of quadratic cost, and a strong negative correlation between FC4B

Table 25 – Computational experiments for HCPA, HCPA-II, and HCPA-III

Instance	HCPA				HCPA-II				HCPA-III			
	UB	LB	GAP	Time	UB	LB	GAP	Time	UB	LB	GAP	Time
p050_inst_01	30305	30305	0.00%	519.05	28655	28655	0.00%	464.8	5184240	5184240	0.00%	3.27
p050_inst_02	30460	30460	0.00%	508.05	28810	28810	0.00%	416.18	4784120	4784120	0.00%	3.66
p050_inst_03	30505	30505	0.00%	473.68	28915	28915	0.00%	493.19	5334120	5334120	0.00%	3.73
p050_inst_04	30965	30965	0.00%	433.89	29375	29375	0.00%	554.98	5434180	5434180	0.00%	3.17
p050_inst_05	30685	30685	0.00%	442.01	29095	29095	0.00%	384	5284120	5284120	0.00%	2.61
p050_inst_06	31705	31705	0.00%	380.22	30025	30025	0.00%	382.11	5334120	5334120	0.00%	4.61
p050_inst_07	30015	30015	0.00%	495.01	28425	28425	0.00%	336.09	5234120	5234120	0.00%	2.7
p050_inst_08	30215	30215	0.00%	429.48	28565	28565	0.00%	443.31	5184120	5184120	0.00%	2.76
p050_inst_09	31670	31670	0.00%	513.04	30020	30005	0.05%	764.78	5234120	5234120	0.00%	3.68
p050_inst_10	30765	30765	0.00%	427.83	29085	29085	0.00%	450.88	5134120	5134120	0.00%	5.02
Average			0.00%	462.23			0.00%	469.03			0.00%	3.52
p100_inst_01	26885	23630	12.11%	TL	26540	25550	3.73%	TL	6997660	6992675	0.07%	2400
p100_inst_02	118770	26050	78.07%	TL	68965	27970	59.44%	TL	7747960	7747960	0.00%	8.16
p100_inst_03	542240	24342	95.51%	TL	27270	26120	4.22%	TL	7297960	7297960	0.00%	14.72
p100_inst_04	313135	24149	92.29%	TL	157040	25390	83.83%	TL	7197960	7197960	0.00%	11.04
p100_inst_05	27560	24542	10.95%	TL	26675	25780	3.36%	TL	7017750	7017750	0.00%	16.89
p100_inst_06	28625	25212	11.92%	TL	27935	26830	3.96%	TL	7247960	7247960	0.00%	7.82
p100_inst_07	26550	23225	12.52%	TL	25755	24590	4.52%	TL	6997660	6997660	0.00%	115.23
p100_inst_08	27585	24590	10.86%	TL	27890	26500	4.98%	TL	7597960	7597960	0.00%	8.9
p100_inst_09	255550	25287	90.10%	TL	28175	27110	3.78%	TL	7397960	7397960	0.00%	9.84
p100_inst_10	27030	23770	12.06%	TL	26025	24890	4.36%	TL	7147960	7147960	0.00%	9.17
Average			42.64%	2400			17.62%	2400			0.01%	260.18

Table 26 – Correlation between FC4(B), FC6 and FC5 and the remaining constraints.

$\rho(\text{FC4(B),FCi})$	FC4(B)-FC1	FC4(B)-FC2	FC4(B)-FC3	FC4(B)-FC4(B)	FC4(B)-FC5	FC4(B)-FC6	FC4(B)-FC7
	0,92	0,93	0,50	0,52	0,93	-0,93	-
$\rho(\text{FC6,FCi})$	FC6-FC1	FC6-FC2	FC6-FC3	FC6-FC4	FC6-FC4(B)	FC6-FC5	FC6-FC7
	-0,99	-0,999	-0,58	-0,5	-0,93	-0,995	-
$\rho(\text{FC5,FCi})$	FC5-FC1	FC5-FC2	FC5-FC3	FC5-FC4	FC5-FC4(B)	FC5-FC6	FC5-FC7
	0,99	0,997	0,58	0,50	0,93	-0,995	-

and FC6, which means that the higher the degree of satisfaction of complete weekend constraints, the higher the degree of satisfaction of complete assignments, and consecutive night and worked days constraints. On the other hand, the higher the degree of satisfaction of complete weekend constraints, the lower the degree of satisfaction of minimum demand.

Regarding the complete assignments, as this constraint is more complied, the more consecutiveness and total weekend constraints have fulfilled, due to the strong positive correlation between FC5, and FC1, FC2, and FC4(B). Conversely, the less the minimum demand is satisfied.

With these two inputs, we can notice that the inclusion of the least absolute deviation and quadratic costs provide an equitable distribution of weekends and a higher degree of satisfaction with complete assignments. Nevertheless, the hospital runs into a situation of under-staffing. These results measures encourage the use of quadratic cost and least absolute deviation as an alternative to balance a schedule and aid decision-makers in terms of the degree of satisfaction of personnel preferences and hospital rules.

Table 27 – Constraints’ degree of satisfaction for HCPA, HCPA-II and HCPA-III

Instance	Formulation	FC1	FC2	FC3	FC4	FC4(B)	FC5	FC6	FC7
p050_inst_01	HCPA	78.67%	0.00%	84.00%	20.00%	41.00%	0.00%	100.00%	100.00%
	HCPA-II	80.67%	0.00%	84.00%	100.00%	41.00%	0.00%	100.00%	100.00%
	HCPA-III	100.00%	100.00%	100.00%	100.00%	59.50%	70.00%	48.02%	100.00%
p050_inst_02	HCPA	77.00%	0.00%	98.00%	18.00%	40.00%	2.00%	100.00%	100.00%
	HCPA-II	77.00%	0.00%	98.00%	100.00%	40.00%	2.00%	100.00%	100.00%
	HCPA-III	100.00%	100.00%	100.00%	100.00%	59.50%	74.00%	51.19%	100.00%
p050_inst_03	HCPA	78.00%	0.00%	98.00%	24.00%	43.00%	0.00%	100.00%	100.00%
	HCPA-II	75.00%	0.00%	98.00%	100.00%	43.00%	0.00%	100.00%	100.00%
	HCPA-III	100.00%	100.00%	100.00%	100.00%	60.50%	70.00%	46.43%	100.00%
p050_inst_04	HCPA	77.00%	0.00%	100.00%	26.00%	23.00%	0.00%	100.00%	100.00%
	HCPA-II	74.00%	0.00%	100.00%	100.00%	43.00%	0.00%	100.00%	100.00%
	HCPA-III	100.00%	100.00%	100.00%	100.00%	58.50%	70.00%	47.22%	100.00%
p050_inst_05	HCPA	75.00%	0.00%	96.00%	18.00%	43.00%	0.00%	100.00%	100.00%
	HCPA-II	75.00%	0.00%	96.00%	100.00%	43.00%	0.00%	100.00%	100.00%
	HCPA-III	100.00%	100.00%	100.00%	100.00%	59.00%	74.00%	46.43%	100.00%
p050_inst_06	HCPA	77.00%	0.00%	96.00%	20.00%	39.00%	2.00%	100.00%	100.00%
	HCPA-II	77.00%	0.00%	96.00%	100.00%	38.00%	2.00%	100.00%	100.00%
	HCPA-III	100.00%	100.00%	100.00%	100.00%	59.00%	76.00%	45.24%	100.00%
p050_inst_07	HCPA	76.00%	0.00%	96.00%	20.00%	43.00%	0.00%	100.00%	100.00%
	HCPA-II	76.00%	0.00%	96.00%	100.00%	43.00%	0.00%	100.00%	100.00%
	HCPA-III	100.00%	100.00%	100.00%	100.00%	58.50%	76.00%	48.02%	100.00%
p050_inst_08	HCPA	77.00%	0.00%	84.00%	22.00%	41.00%	0.00%	100.00%	100.00%
	HCPA-II	74.00%	0.00%	84.00%	100.00%	41.00%	0.00%	100.00%	100.00%
	HCPA-III	100.00%	100.00%	100.00%	100.00%	60.00%	76.00%	48.81%	100.00%
p050_inst_09	HCPA	75.00%	0.00%	90.00%	20.00%	40.00%	2.00%	100.00%	100.00%
	HCPA-II	75.00%	0.00%	90.00%	100.00%	39.00%	2.00%	100.00%	100.00%
	HCPA-III	100.00%	100.00%	100.00%	100.00%	57.50%	62.00%	44.84%	100.00%
p050_inst_10	HCPA	74.00%	0.00%	90.00%	16.00%	39.00%	2.00%	100.00%	100.00%
	HCPA-II	77.00%	0.00%	90.00%	100.00%	38.00%	2.00%	100.00%	100.00%
	HCPA-III	100.00%	100.00%	100.00%	100.00%	61.00%	72.00%	49.21%	100.00%

Figure 20 – Correlation between FC4B constraints and remaining

6.6 Conclusion

In this chapter, we address the physician scheduling problem in multiple locations, the case of HCPA. Moreover, we propose a mathematical model, combining hospital requirements and comparing to literature models on physician scheduling problems that share similarities with this problem and using the concepts applied to former models to obtain fairness in schedules.

Figure 21 – Correlation between FC5 constraints and remaining

Figure 22 – Correlation between FC6 constraints and remaining

With the reformulations proposed, we managed to find a feasible and high-quality solution in terms of equity. After performing computational experiments, the analyzed results through tables and figures show that the inclusion of fairness in weekends increased the level of balance amongst the physician without compromising other requirements. Moreover, this inclusion came with a higher degree of satisfaction on weekend assignments, a double-winning.

Regarding the inclusion of quadratic costs to soften the discrepancies on assignments, it allowed us to find a solution with a higher degree of satisfaction of minimum and maximum allocations and consecutiveness constraints. In contrast, the main drawback is that this came with a higher degree of over-staffing, which implies expenses on extra-wages in practical terms.

Summarizing, the proposals we offer the decision-makers a valuable alternative to the basic model with a high degree of satisfaction of hospital requirements combined with new features such as fairness and the thesis's scope.

CONCLUSIONS AND FUTURE WORK

In this thesis, we addressed the physician scheduling problem in emergency rooms, considering fairness (PSPER). We report two problem descriptions and reformulate papers discussed in the literature. One is derived from nurse scheduling proposed by [Curtois and Qu \(2014\)](#) and another one from a real scenario, based on the Hospital de Clínicas de Porto Alegre, harnessed by [Wickert *et al.* \(2020\)](#).

To a more reliable contextualize, we made a literature review on physician scheduling problems in the emergency room since the early 2000s in Chapter 2, collecting characteristics of the issues yet explored and extracting points to enhance our model. In this endeavor, fairness proved to be a topic of interest definitively and has to be dealt with sharpness. In this thesis, we proposed a new concept in physician scheduling to provide a high level of compliance of rules with the balance between the workforce preferences and the organization rules.

From the NSP model proposed by [Curtois and Qu \(2014\)](#), we reformulated models to accommodate fairness and other more realistic features such as relaxing consecutive shift constraints. The reformulations are:

- *PSPER – H*: a model with adapted working hours as flexible constraint, modeled as least absolute deviation. To hold the least absolute deviation, the target number of hours is the arithmetic average of the maximum and minimum hours.
- *PSPER – H2*: Roughly, the *PSPER – H* with relaxed consecutiveness constraints.
- *PSPER – H_s*: A *PSPER – H* with target hours as a variable.
- *PSPER – (HW)_s* and *PSPER – (HW)_s²* a model with the target number of weekends and hours as variable and modeled with least absolute deviation. The second variant has quadratic costs on hours and weekends.

- $PSPER - (HW)_p$ and $PSPER - (HW)_p^2$ a model with the target number of weekends and hours as parameters and the least absolute deviation. The second variant has quadratic costs on hours and weekends.

These formulations were run CPLEX commercial optimization solver with a time limit of one hour using a set of 24 instances of the benchmark available in [Curtois \(2007\)](#). These instances are an adaptation of nurse and physician scheduling problems found in the literature. We rectified the high degree of violation of demand on $PSPER - H$ and $PSPER - H2$ using $PSPER - H_s$, and beyond the increase of demand compliance rate, we achieved higher rates of hour compliance due to the inclusion of hours as a variable.

We developed a relax-and-fix and fix-and-optimize heuristic procedure to cope with the challenges with large-sized instances solved by $PSPER - H_s$. In three decomposition strategies, we significantly decreased the average time to solve the most challenging instances and improve the primal bounds. MIP based heuristics show then, apparently, an excellent gateway toward solving adequately such problems up to a given size.

As quadratic variants of our models struggle to find reasonable solutions in reasonable computational time, we proposed linear approximation to $L^2 - norm$ and compared with its variants on L^1 , and $L^\infty - norms$. The overall results show that the linear approximation we introduced with $L^2 - norm$ outperform in terms of quality solution, especially when the number of points to mimic the parable is restricted. We estimated the target values of hours and weekends using $L^2 - norm$ to distribute the workload better and increase workforce coverage demand. The leverage of workload is then highly effective. The coverage two can aid the decision-makers in the scheduling procedure instead of establishing predefined workloads and estimating the workforce.

A real scenario has a world of complexity, and dealing with them is not an easy task. In this thesis, we borrowed concepts used to solve the PSPER at HCPA, and the results are satisfying. We applied the least absolute deviation concept separately to leverage the distribution of workforce weekends (HCPA-II), and the least absolute deviation to leverage of assignments (HCPA-III) throughout the planning horizon and quadratic costs. Both proposals offer a high degree of satisfaction with flexible constraints, emphasizing HCPA-II, which had a balanced weekend assignment without compromising the demand and improving the HCPA. Another gain with our reformulation is HCPA-III that allowed us to provide a high degree of satisfaction of constraints that were not satisfied using HCPA or HCPA-II.

As future work, one of the several perspectives is to solve the most challenging instances of the problem using a decomposition technique such as column generation and obtain better lower bounds, especially in the cases with a relatively high gap. We also intend to apply our model in real scenarios and expand/adapt it to other staff scheduling related problems in practical terms.

Moreover, as the PSPER is a multi-objective optimization problem, were some of the objectives are optimized separately, we intend to solve it through a multi-objective approach to obtain a set of Pareto optimal solutions instead, besides the single-objective.

Regarding new trends in the literature of physician scheduling in emergency rooms, the integration of staffing and scheduling and stochastic demand is a direction we intend to work in the upcoming years, once the variability of patient arrival is a recurrent situation in emergency rooms.

BIBLIOGRAPHY

AKARTUNALI, K.; FRAGKOS, I.; MILLER, A.; WU, T. Local cuts and two-period convex hull closures for big-bucket lot-sizing problems. **INFORMS J Comput.**, v. 28, n. 4, p. 766–780, 2016. Citation on page 71.

ALNAJJAR, S. M.; ALI, S. H. Staffing and scheduling emergency rooms in two public hospitals: A case study. **International Journal of Business Administration**, v. 2, n. 2, p. 137 – 148, 2011. Citations on pages 35 and 38.

ALFARES, H. K. Survey, categorization and comparison of recent tour scheduling literature. **Annals of Operations Research**, Kluwer Academic Publishers, v. 127, n. 1-4, p. 145–175, 2004. Citation on page 26.

ARANTES, M. da S. **Ambiente para desenvolvimento de métodos aplicados a problemas de otimização**. Phd Thesis (PhD Thesis) — Instituto de Ciências Matemáticas e de Computação, Março 2014. Available: <<http://www.teses.usp.br/teses/disponiveis/55/55134/tde-03062014-164724/pt-br.php>>. Citations on pages 49, 63, and 73.

ARINGHIERI, R.; BRUNI, M.; KHODAPARASTI, S.; ESSEN], J. [van. Emergency medical services and beyond: Addressing new challenges through a wide literature review. **Computers & Operations Research**, v. 78, p. 349 – 368, 2017. ISSN 0305-0548. Available: <<http://www.sciencedirect.com/science/article/pii/S0305054816302362>>. Citation on page 35.

BARISH R. A., M. P. L. A. T. C. Emergency room crowding: A marker of hospital health. **Transactions of the American Clinical and Climatological Association**, v. 123, n. 6, p. 304–311, 2012. Citation on page 25.

BEAULIEU, H.; FERLAND, J. A.; GENDRON, B.; MICHELON, P. A mathematical programming approach for scheduling physicians in the emergency room. **Health Care Management Science**, n. 2, p. 193–200, 2000. Citations on pages 32, 33, 36, 37, 38, and 39.

BLIEK, C.; BONAMI, P.; LODI, A. Solving mixed-integer quadratic programming problems with ibm-cplex : a progress report. In: **Proceedings of the Twenty-Sixth RAMP Symposium Hosei University, Tokyo**. [S.l.: s.n.], 2014. p. 171–180. Citation on page 68.

CABRERA, E.; TABOADA, M.; IGLESIAS, M. L.; EPELDE, F.; LUQUE, E. Simulation optimization for healthcare emergency departments. **Procedia Computer Science**, v. 9, p. 1464 – 1473, 2012. Citation on page 25.

CADOUX, F. Computing deep facet-defining disjunctive cuts for mixed-integer programming. **Math. Program.**, v. 122, n. 2, p. 197–223, 2010. Citation on page 71.

CAMIAT, F.; RESTREPO, M. I.; CHAUNY, J.-M.; LAHRICHI, N.; ROUSSEAU, L.-M. Productivity-driven physician scheduling in emergency departments. **Health Systems**, Taylor & Francis, v. 0, n. 0, p. 1–14, 2019. Available: <<https://doi.org/10.1080/20476965.2019.1666036>>. Citations on pages 37 and 38.

CARTER, M. W.; LAPIERRE, S. D. Scheduling emergency room physicians. **Health Care Management Science**, v. 4, n. 4, p. 347–360, 2001. Citations on pages 31, 33, 34, 35, 36, 38, and 39.

CURTOIS, T. **Employee scheduling benchmark data sets**. 2007. Citations on pages 19, 49, 73, 74, and 104.

CURTOIS, T.; QU, R. **Computational results on new staff scheduling benchmark instances**. NG8 1BB, Nottingham, UK, 2014. 5 p. Available: <http://www.schedulingbenchmarks.org/papers/computational_results_on_new_staff_scheduling_benchmark_instances.pdf>. Citations on pages 15, 19, 26, 42, 43, 46, 47, 49, 52, 54, 55, 56, 57, 64, 73, 76, 77, 78, 79, 80, and 103.

DAMC₁-KURT, P.; ZHANG, M.; MARENTAY, B.; GOVIND, N. Improving physician schedules by leveraging equalization: Cases from hospitals in u.s. **Omega**, v. 85, p. 182 – 193, 2019. ISSN 0305-0483. Available: <<http://www.sciencedirect.com/science/article/pii/S0305048317309726>>. Citations on pages 36 and 38.

DEVESSE, V. A. P. A. D.; SANTOS, M. O. d.; TOLEDO, C. F. M. Fairness in physician scheduling problem in emergency rooms. **Revista de Sistemas de Informação da FSMA**, n. 18, p. 9–20, 2017. Citation on page 38.

DILLENBERGER, C.; ESCUDERO, L. F.; WOLLENSAK, A.; ZHANG, W. On practical resource allocation for production planning and scheduling with period overlapping setups. **European Journal of Operational Research**, v. 75, n. 2, p. 275 – 286, 1994. ISSN 0377-2217. Lotsizing models for production planning. Citation on page 59.

EDWIN, J. P. R. Improving emergency department patient flow. **Clin Exp Emerg Med**, v. 3, n. 2, p. 63–68, 2016. Available: <<http://ceemjournal.org/journal/view.php?number=80>>. Citation on page 25.

ERHARD, M.; SCHOENFELDER, J.; FÜGENER, A.; BRUNNER, J. O. State of the art in physician scheduling. **European Journal of Operational Research**, v. 265, n. 1, p. 1 – 18, 2018. ISSN 0377-2217. Available: <<http://www.sciencedirect.com/science/article/pii/S0377221717305787>>. Citations on pages 26 and 31.

ERNST, A.; JIANG, H.; KRISHNAMOORTHY, M.; OWENS, B.; SIER, D. An annotated bibliography of personnel scheduling and rostering. **Annals of Operations Research**, v. 127, n. 1, p. 21–144, 2004. Citation on page 30.

FERRAND, Y.; MAGAZINE, M. J.; RAO, U. S.; GLASS, T. F. Building cyclic schedules for emergency department physicians. **Interfaces**, v. 41, n. 6, p. 521–533, 2011. Citations on pages 31, 35, 36, and 38.

FREY, L.; HANNE, T.; DORNBERGER, R. Optimizing staff rosters for emergency shifts for doctors. In: **2009 IEEE Congress on Evolutionary Computation**. [S.l.: s.n.], 2009. p. 2540–2546. ISSN 1089-778X. Citations on pages 34 and 38.

GENDREAU, M.; FERLAND, J.; GENDRON, B.; HAIL, N.; JAUMARD, B.; LAPIERRE, S.; PESANT, G.; SORIANO, P. Physician scheduling in emergency room. In: _____. **Practice and Theory of Automated Timetabling VI: 6th International Conference, PATAT 2006 Brno, Czech Republic, August 30–September 1, 2006 Revised Selected Papers**. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007. p. 53–66. ISBN 978-3-540-77345-0. Citations on pages 26, 33, 34, 36, and 38.

Gharbi, A.; Louly, M.; Azaiez, M. N. Physician scheduling using goal programming-an application to a large hospital in saudi arabia. In: **2017 4th International Conference on Control, Decision and Information Technologies (CoDIT)**. [S.l.: s.n.], 2017. p. 0922–0925. Citations on pages 36 and 38.

GHATHBAR, K. A.; LOULY, M. A.; MRAD, M. Iterated local search in physician scheduling problem. In: . [S.l.: s.n.], 2019. p. 888–894. Citations on pages 36 and 38.

GUL, M.; GUNERI, A. F. A comprehensive review of emergency department simulation applications for normal and disaster conditions. **Computers & Industrial Engineering**, v. 83, p. 327 – 344, 2015. Citation on page 35.

HCPA. **HCPA Description**. 2020. <<https://www.hcpa.edu.br/assistencia-apresentacao/assistencia-servicos-medicos>>. Accessed: 2020-02-18. Citation on page 85.

LINDENAU-STOCKFISCH, V.; SEARLE, J.; MöCKEL, M. Poct in emergency rooms: One key factor for process streamlining with lean management. **Conference Papers in Engineering**, v. 2013, p. 1–6, 01 2013. Citations on pages 19 and 85.

Lo, C.; Lin, T. A particle swarm optimization approach for physician scheduling in a hospital emergency department. In: **2011 Seventh International Conference on Natural Computation**. [S.l.: s.n.], 2011. v. 4, p. 1929–1933. Citations on pages 37 and 38.

MARCHESI, J. F.; HAMACHER, S.; FLECK, J. L. A stochastic programming approach to the physician staffing and scheduling problem. **Computers & Industrial Engineering**, v. 142, p. 106281, 2020. ISSN 0360-8352. Available: <<http://www.sciencedirect.com/science/article/pii/S0360835220300152>>. Citations on pages 37, 38, and 39.

NEHI. A matter of urgency: Reducing emergency department overuse. **New England Healthcare Institute**, n. 6, p. 1–15, 2010. Citation on page 25.

OSOGAMI, T.; IMAI, H. Classification of various neighborhood operations for the nurse scheduling problem. In: GOOS, G.; HARTMANIS, J.; LEEUWEN, J. van; LEE, D. T.; TENG, S.-H. (Ed.). **Algorithms and Computation**. Berlin, Heidelberg: Springer Berlin Heidelberg, 2000. p. 72–83. ISBN 978-3-540-40996-0. Citation on page 41.

PEREIRA, S. de S. **Variáveis mediadoras do Burnout em profissionais de serviços de urgência e emergência: aplicabilidade do Maslach Burnout Inventory - Human Services Survey (MBI-HSS)**. Phd Thesis (PhD Thesis) — Escola de Enfermagem de Ribeirão Preto, Maio 2017. Available: <<http://www.teses.usp.br/teses/disponiveis/22/22131/tde-17082017-130158/pt-br.php>>. Citation on page 41.

POCHET, Y.; WOLSEY, L. A. **Production Planning by Mixed Integer Programming**. [S.l.]: Springer-Verlag New York, 2006. (Springer Series in Operations Research and Financial Engineering). ISBN 978-0-387-29959-4. Citation on page 59.

PUENTE, J.; GÓMEZ, A.; FERNÁNDEZ, I.; PAOLO, P. Medical doctor rostering problem in a hospital emergency department by means of genetic algorithms. **European Journal of Operational Research**, v. 56, n. 2, p. 1232–1242, 2009. Citations on pages 26, 34, 36, and 38.

RAHIMIAN, E.; AKARTUNALI, K.; LEVINE, J. A hybrid integer programming and variable neighborhood search algorithm to solve nurse rostering problems. **European Journal of Operational Research**, v. 258, n. 2, p. 411–423, 4 2017. ISSN 0377-2217. Citations on pages 39 and 49.

ROUSSEAU, L.-M.; PESANT, G.; GENDREAU, M. A general approach to the physician rostering problem. **Annals of Operations Research**, Kluwer Academic Publishers, v. 115, n. 1-4, p. 193–205, 2002. ISSN 0254-5330. Citations on pages 31, 33, 34, 38, and 39.

SAVAGE, D. W.; WOOLFORD, D. G.; WEAVER, B.; WOOD, D. Developing emergency department physician shift schedules optimized to meet patient demand. **Canadian Journal of Emergency Medicine**, Cambridge Univ Press, v. 17, n. 01, p. 3–12, 2015. Citations on pages 37 and 38.

STOLLETZ, R.; BRUNNER, J. O. Fair optimization of fortnightly physician schedules with flexible shifts. **European Journal of Operational Research**, v. 219, n. 3, p. 622 – 629, 2012. ISSN 0377-2217. Feature Clusters. Available: <<http://www.sciencedirect.com/science/article/pii/S0377221711009787>>. Citation on page 35.

SUN B. C., H. R. Y. W. R. E. Z. D. L. L.-J. H. W. A. S. M. Effect of emergency department crowding on outcomes of admitted patients. **Annals of Emergency Medicine**, v. 61, n. 6, p. 605–611, 2013. Citation on page 25.

TAN, M.; GAN, J.; REN, Q. Scheduling emergency physicians based on a multiobjective programming approach: A case study of west china hospital of sichuan university. **Journal of Healthcare Engineering**, v. 2019, p. 1–9, 08 2019. Citations on pages 36 and 38.

THOMPSON, G. Labor scheduling, part 1: forecasting demand. **The Cornell Hotel and Restaurant Administration Quarterly**, v. 39, n. 2, p. 22–31, 1998. Citation on page 29.

_____. Labor scheduling, part 2: Knowing how many on-duty employees to schedule. **The Cornell Hotel and Restaurant Administration Quarterly**, v. 39, n. 2, p. 26–37, 1998. Citation on page 30.

_____. Labor scheduling, part 3: Developing a workforce schedule. **The Cornell Hotel and Restaurant Administration Quarterly**, v. 40, n. 2, p. 86–96, 1999. Citation on page 30.

_____. Labor scheduling, part 4: Controlling workforce schedules in real time. **The Cornell Hotel and Restaurant Administration Quarterly**, v. 40, n. 2, p. 85–96, 1999. Citation on page 30.

_____. Labor scheduling: A commentary. **The Cornell Hotel and Restaurant Administration Quarterly**, v. 44, n. 2, p. 149–155, 2003. Citations on pages 29 and 30.

TOLEDO, C. F. M.; ARANTES, M. da S.; HOSSOMI, M. Y. B.; FRANÇA, P. M.; AKAR-TUNALI, K. A relax-and-fix with fix-and-optimize heuristic applied to multi-level lot-sizing problems. **Journal of Heuristics**, v. 21, n. 5, p. 687–717, Oct 2015. ISSN 1572-9397. Available: <<https://doi.org/10.1007/s10732-015-9295-0>>. Citations on pages 17, 60, 61, and 62.

TOPALOGLU, S. A multi-objective programming model for scheduling emergency medicine residents. **Computers & Industrial Engineering**, v. 51, n. 3, p. 375 – 388, 2006. Citations on pages 26, 34, 36, 38, 39, and 47.

TOPALOGLU, S.; OZKARAHAN, I. A constraint programming-based solution approach for medical resident scheduling problems. **Computers & Operations Research**, v. 38, n. 1, p. 246 – 255, 2011. Project Management and Scheduling. Citations on pages 34, 35, 36, and 38.

VASSILACOPOULOS, G. Allocating doctors to shifts in an accident and emergency department. **The Journal of the Operational Research Society**, v. 36, n. 6, p. 517–523, 1985. Citation on page 29.

VERMUYTEN, H.; ROSA], J. N.; MARQUES, I.; BELIËN, J.; BARBOSA-PÓVOA, A. Integrated staff scheduling at a medical emergency service: An optimisation approach. **Expert Systems with Applications**, v. 112, p. 62 – 76, 2018. ISSN 0957-4174. Available: <http://www.sciencedirect.com/science/article/pii/S0957417418303634>. Citations on pages 36, 38, and 39.

WICKERT, T. I.; NETO, A. F. K.; BONIATTI, M. M.; BURIOL, L. S. An integer programming approach for the physician rostering problem. **Annals of Operations Research**, Springer, p. 1–28, 2020. Citations on pages 26, 27, 37, 38, 47, 86, 88, 93, and 103.

WOLBECK, L. Fairness aspects in personnel scheduling. Discussion paper, School of Business & Economics, Freie Universität Berlin. 2019. Available: <http://dx.doi.org/10.17169/refubium-26050>. Citation on page 31.

WOLSEY, L. **Integer Programming**. Wiley, 1998. (Wiley Series in Discrete Mathematics and Optimization). ISBN 9780471283669. Available: <https://books.google.com.br/books?id=x7RvQgAACAAJ>. Citation on page 55.