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Simulation of groundwater flow by the analytic element method

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**Simulação do escoamento de água subterrânea pelo
método de elemento analítico**

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Ciências de Computação e Matemática Computacional. *VERSÃO REVISADA*

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This endeavour is dedicated to my beloved family (especially my parents), companionable supervisor Prof. Dr. José Alberto Cuminato and friends. All I have and will accomplish are only possible due to their continuous prayer and sacrifices. It is their unconditional love that motivates me to set my higher target.

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*“As invenções são, sobretudo,
o resultado de um trabalho de teimoso.”
(Santos Dumont)*

ABSTRACT

HUSSAIN, S. M. **Simulation of groundwater flow by the analytic element method**. 2017. 135 p. Tese (Doutorado em Ciências – Ciências de Computação e Matemática Computacional) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2017.

Groundwater studies face computational limitations when providing local detail within regional models. The researchers are concentrated on applying the numerical models to minimize the difference between the physical reality and the implemented numerical model by considering the minimum computational cost. This work consists of the study of line-elements (such as line-doublets, circles, polygons, fractures) using the Analytic Element Method (AEM) for groundwater flow. In this work, we consider the study of two-dimensional groundwater flow in fractured porous media by the Analytic Element Method. We develop a numerical solution based on a series expansion for a problem with more than one fracture. Each fracture has an influence that can be expanded in a series that satisfies Laplace's equation exactly. In the series expansion, the unknown coefficients are obtained from the discharge potentials of all other elements that are related to the expansion coefficients. Sizes, locations and conductivities for all inhomogeneities are selected arbitrarily. This work also discusses a matrix method obtained by imposing the intern boundary conditions for the Analytic Element Method. The convergence analysis of a Gauss-Seidel type iterative method is also discussed.

Keywords: Analytic Element Method, Fractures, Groundwater Flow, Matrix Method, Numerical Simulations.

RESUMO

HUSSAIN, S. M. **Simulação do escoamento de água subterrânea pelo método de elemento analítico**. 2017. 135 p. Tese (Doutorado em Ciências – Ciências de Computação e Matemática Computacional) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2017.

Estudos de águas subterrâneas enfrentam limitações computacionais ao fornecer detalhes locais em modelos regionais. Os pesquisadores estão concentrados na aplicação dos modelos numéricos para minimizar a diferença entre a realidade física e o modelo numérico implementado considerando o custo computacional mínimo. Este trabalho consiste no estudo de elementos de linha (como line-doublets, círculos, polígonos, fraturas) usando o Método de Elemento Analítico (AEM) para o fluxo de águas subterrâneas. Neste trabalho, consideramos o estudo do fluxo bidimensional de águas subterrâneas em meios porosos fraturados pelo Método dos Elementos Analíticos. Desenvolvemos uma solução numérica baseada em uma expansão em série para um problema com mais de uma fratura. Cada fratura tem uma influência que pode ser expandida em uma série que satisfaça exatamente a equação de Laplace. Na expansão da série, os coeficientes desconhecidos são obtidos a partir dos potenciais de descarga de todos os outros elementos que estão relacionados aos coeficientes de expansão. Tamanhos, locais e condutividades para todas as não-homogeneidades são arbitrariamente selecionados. Este trabalho também discute o método da matriz obtido impondo as condições de contorno do interno para o Método do Elemento Analítico. A análise de convergência de um método iterativo tipo Gauss-Seidel também é discutida.

Palavras-chave: Método do Elemento Analítico, Fraturas, Escoamento de Águas Subterrâneas, Método da Matrix, Simulações Numéricas.

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LIST OF SYMBOLS

q — Discharge

Q — Total volume of water per second

A — Cross-sectional area

v — Velocity

t — Time

P — A point below a standpipe

Z — Elevation of point P above the reference level

ρ — Density of water

g — Acceleration of gravity

ϕ — Hydraulic head

K — Hydraulic conductivity constant

L — Length

q_x — Discharge component along x -direction

q_y — Discharge component along y -direction

q_z — Discharge component along z -direction

Q_i — Discharge vector

H — Height/thickness of the aquifer

Q_x — Discharge vector along x -direction

Q_y — Discharge vector along y -direction

C_c — An arbitrary constant for confined aquifer

C_c — An arbitrary constant for unconfined aquifer

$\Phi(z)$ — Discharge potential of the complex potential

$\Psi(z)$ — Stream function of the complex potential

z — Complex variable in global coordinate system

Z — Local coordinate system using Writinger transformation

R_j — Radius of a circular inhomogeneity j
 c_j — Center of a circular inhomogeneity j
 a_n — Unknown coefficients in a series expansion
 $\Omega_{\neq j}(Z)$ — Combined complex potential of all circular inhomogeneities except j
 M_j — Control points on the boundary of an element j
 Q_{x_0} — Uniform flow field intensity parallel to the x -axis
 Φ_0 — Reference point with a known head
 Q_{uf} — Uniform flow field intensity in global coordinate system
 β_{uf} — An angle of uniform flow intensity relative with x -axis in global coordinate system
 C — Reference point constant
 Q_w — Flow intensity of a well in global coordinate system
 z_w — Position of a well
 $\mathbf{s}$ — Parallel axis of the fracture
 $\mathbf{n}$ — Normal axis of the fracture
 b — Width of the fracture
 $\bar{\phi}$ — Average head across the fracture
 Q_s — Discharge through the fracture
 s — Strength of a dipole
 r — Radius of a dipole
 α — Angle of analytic elements relative to the x -axis
 μ — Discharge distribution of a line-dipole
 δ — A point on the line/interface
 ω — Dimensionless variable
 $\lambda(Z)$ — Jump function or Potential discontinuity
 $p(Z)$ — Far-field correction polynomial/function
 T_n — Chebyshev polynomial
 N — Order of the Chebyshev series/series expansion
 Ω_T — Complex potential of all the elements
 Ω_{uf} — Complex potential for uniform flow
 Ω_w — Complex potential for a well

$a_{i,n}$ — The nth unknown coefficient for element i

M — Total number Analytical Elements

N_{elem} — Approximate order of the series expansion of the element

$\mathbf{A}$ — Matrix

$\mathbf{U}$ — Upper triangle of the matrix

$\mathbf{L}$ — Lower triangle of the matrix

$\mathbf{D}$ — Diagonal of the matrix

λ_i — Eigenvalues of the matrix

m/d — meter/day

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INTRODUCTION

The Analytic Element Method (AEM) was developed by Otto D. L. Strack in 1989, it is a numerical method used for the solution of partial differential equations modelling groundwater problems. The method is based on the principle of superposition. It uses the exact solution of the governing equations for basic fundamental features of the aquifer, such as a well, a river, an impermeable barrier, a sink, called Analytic Elements, and combines such solutions by superposition to approximate the solution of a much more complex real problem. The solutions to the governing equation obtained by AEM will be exact, but the boundary conditions will be satisfied only approximately. It does not require the specific domain to be discretized giving rise to a grid, only the boundary conditions at infinity and the internal conditions at certain points are required (MARIN, 2011).

Problems in groundwater flow have been an area of active research for the past few decades. Researchers working in these areas strived along the years for developing different numerical methods using the governing equations of the physical phenomena (MARIN, 2011; STRACK, 1989; HAITJEMA, 1995). To model groundwater systems, mathematical representations are used. Modelling plays an important role, and is the most effective method to make such estimations involving complexities and uncertainties which model the behaviour of the aquifer. Numerical modelling is a practical tool for understanding complex phenomena and obtaining information. The implementation of the numerical models is usually made on a computer, that can lead to errors, such as truncation and approximation. In recent years, researchers concentrated on developing numerical models with low computational cost for the required problems minimizing the difference between the physical reality and the implemented numerical model (MARIN, 2011). However, one of the major difficulties to be overcome in flow of groundwater is the problem of fractures, that has received much attention.

The problem of fractured media is complex, and difficult to obtain experimental data (MARIN, 2011; STRACK, 1989). Usual groundwater simulation codes use finite element and finite difference methods, which though capable of simulating fracture flow phenomena, suffers

from drawbacks such as difficulties with scale differences, dependence on domain discretization, mesh generation and solution approximations (WENDLAND; HIMMELSBACH, 2002). We shall introduce the Analytic Element Method which is more suitable for solving the problems of large aquifers due to computational costs.

The main objective of this work is to study AEM for an array of fractures and other elements like line-doublets, circular inhomogeneities. The numerical solution is based on series expansion for the solution that satisfy the governing equation exactly. The principle of superposition is then used to linearly combine the solutions with unknown coefficients. These coefficients are then found by solving a linear system. The study of efficient numerical methods for solving the linear system and the convergence analysis of a Gauss-Seidel type iterative method are the main contributions of this work.

This work is organized as follows: Chapter 2 consists of some basic concepts for obtaining the governing equations. Chapter 3 consists of the methodology for solving the problems that is based on the Analytic Element Method. Chapter 4 consists of the problems of circular inhomogeneities. Chapter 5 consists of the problems of fractures. Chapter 6 consists of the problems of polygonal inhomogeneities formed by line-elements called line-doublets and their comparison with the exact solutions of a circular inhomogeneity and a fracture. Chapter 7 consists of solution algorithm. Chapter 8 consists of results and its discussion. Chapter 9 consists of conclusion and Chapter 10 consists of future work.

BASIC EQUATIONS

This chapter presents some basic equations of groundwater flow based on (STRACK, 1989).

Specific discharge: The volume of water flowing through a unit area of soil per unit time is called the specific discharge or discharge, represented by q and is given by:

$$q = \frac{Q}{A}, \quad (2.1)$$

with unit of $[q] = \frac{[Q]}{[A]} = \frac{\text{m}^3/\text{s}}{\text{m}^2} = \text{m/s}$, where Q denotes the total volume of water per second and A denotes the cross-sectional area.

Velocity: The average velocity at a certain point of a porous medium is known as seepage velocity or velocity. It is obtained through the division of the specific discharge by the fraction of voids present in a unit area of the porous medium. The unit of velocity is also [m/s] and is represented by v , that is:

$$v = \frac{q}{n}, \quad (2.2)$$

where n shows the porosity of the medium, which is a constant.

For understanding the concept of specific discharge and seepage velocity, consider the water flowing through a horizontal cylindrical tube which is filled with water and sand and is placed in the space between the ends with caps 1 and 2 (represented by dotted lines in Figure 1). The cylinder is also connected with two reservoirs I and II having different water levels. The water will flow through the cylinder because of the difference in water levels. To maintain its level, the water must be poured into reservoir I and the level of reservoir II is controlled by the overflow. To measure the rate at which the water is poured into reservoir I, first we need to know the total amount of water Q flowing through the cylinder per unit time. The specific discharge

of water is obtained through the division of Q by the cross-sectional area of the cylinder A (STRACK, 1989).

Figure 1 – Schematic of Darcy's experiment

Source: Strack (2014).

Using eq. (2.1), the seepage velocity becomes:

$$v = \frac{q}{n} = \frac{Q}{nA}. \quad (2.3)$$

The seepage velocity is also used to find the average time t for a water particle to travel through the sample. If L is the length of the sample, then the time t is obtained using the following relation (STRACK, 1989):

$$vt = L \quad (2.4)$$

or

$$t = \frac{L}{v} = \frac{LnA}{Q}. \quad (2.5)$$

Pressure: From the weight of the water column above a point P in a standpipe, we can find the pressure of water at point P , whose unit is $[N/m^2]$ and is given by:

$$p = \rho g(\phi - Z). \quad (2.6)$$

Consider that the elevation of P above the reference level is Z , then the height of the water column above P is $\phi - Z$, where, the density of water is denoted by $\rho[kg/m^3]$ and acceleration of gravity is denoted by $g[m/s^2]$.

Head: At a certain point P in a soil body, the hydraulic head is defined as the level to which the water rises in an open standpipe as shown in Figure 2. The unit of head is $[m]$ and is denoted by ϕ , which is given by:

$$\phi = \frac{p}{\rho g} + Z. \quad (2.7)$$

Therefore the head equals to the sum of the pressure head $p/\rho g$ whose unit is $(N/m^2)/(N/m^3)$ or $[m]$ and the elevation head Z .

Figure 2 – Definition of the head ϕ Source: [Strack \(2014\)](#).

2.1 Darcy's Law

In 1856, the French engineer Darcy conducted an experiment from which he determined an empirical relation for the specific discharge in terms of the heads. The original form of this law is only applicable to rectilinear flow of a homogeneous liquid ([STRACK, 1989](#)).

2.1.1 Rectilinear Flow

Darcy's experiment consisted of a cylinder of sand having cross-sectional area A with two ends having cap 1 with head ϕ_1 and cap 2 with head ϕ_2 . From the experiment he concluded that the amount of flow through a cylinder of sand increases linearly with the difference in head at the ends of the sample as shown in [Figure 3](#).

Figure 3 – Darcy's experiment

Source: [Strack \(2014\)](#).

From this picture it can be seen that the pipe connecting reservoir I to the sample can be viewed as a standpipe. Using information from [Figure 3](#), Darcy's law is given by the following

equation as presented in (STRACK, 1989; BEAR, 2013).

$$Q = KA \left(\frac{\phi_1 - \phi_2}{L} \right), \quad (2.8)$$

where K is a constant known as the hydraulic conductivity. The dimensions of K are those of velocity [m/s], Q is the discharge [m³/s], A is the cross-sectional area of the sample [m²] and L is the length of the sample [m].

In terms of the specific discharge q using eq. (2.1), eq. (2.8) becomes:

$$q = K \left(\frac{\phi_1 - \phi_2}{L} \right). \quad (2.9)$$

By measuring the head at various points inside the sample along the x -axis, it appears to vary linearly over the sample and becomes:

$$\frac{d\phi}{dx} = \frac{\phi_2 - \phi_1}{L}. \quad (2.10)$$

Hence, eq. (2.9) may be expressed as:

$$q_x = -K \frac{d\phi}{dx}, \quad (2.11)$$

which gives the specific discharge in the x -direction. $\frac{d\phi}{dx}$ is known as the hydraulic gradient for flow in the x -direction, as presented in (STRACK, 1989).

2.1.2 General form of Darcy's Law

In practical problems, the flow is rarely rectilinear and neither the direction of flow nor the magnitude of the hydraulic gradient is known. Therefore, the simple form of Darcy's law defined by eq. (2.11) is not suitable for solving problems. In order to solve practical problems, it is necessary to use a generalized form of Darcy's law which gives a relation between the specific discharge vector and the hydraulic gradient. The direction of the specific discharge vector will vary with position and is in the direction of flow. The magnitude of this vector represents the amount of water flowing per unit time through a plane of unit area normal to the direction of flow, as described in (STRACK, 2014). In three dimensions, the specific discharge vector has three components with reference to a Cartesian coordinate system x, y, z . For three-dimensional flow using eq. (2.11), the general form of Darcy's law in terms of specific discharge becomes:

$$q_x = -K \frac{\partial \phi}{\partial x}, \quad q_y = -K \frac{\partial \phi}{\partial y} \quad \text{and} \quad q_z = -K \frac{\partial \phi}{\partial z}. \quad (2.12)$$

This is because the components of the specific discharge vector are proportional to the components of the hydraulic gradient, with K as the proportionality constant. The specific discharge vector with components (q_x, q_y, q_z) is represented briefly as q_i , where the index i stands for x, y

or z . The partial derivatives $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \phi}{\partial z}$ represent the three components of the hydraulic gradient. We may write the components of this vector as $\partial_x \phi$, $\partial_y \phi$ and $\partial_z \phi$, where the ∂ stands for differentiation with respect to the coordinate system (STRACK, 1989). The hydraulic gradient can then be written as:

$$[\partial_i \phi] = \left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right]. \quad (2.13)$$

Finally, using eq. (2.12) and eq. (2.13), the general form of Darcy's law is:

$$q_i = -K \partial_i \phi, \quad i = 1, 2, 3. \quad (2.14)$$

2.2 Continuity of Flow

Darcy's law consists of three equations of motion having four unknowns q_x , q_y , q_z and ϕ . For a complete description of groundwater flow an additional fourth equation is required, known as the mass balance equation. The mass balance equation is valid for steady flow, where it reduces to the equation of continuity. It results from the consideration that no water can be stored or produced in an elementary block of soil filled with water as shown in Figure 4. According to (STRACK, 1989), continuity of the flow is considered by the use of the concept of divergence.

Figure 4 – Continuity of flow

Source: Strack (2014).

2.2.1 Divergence

The divergence of the specific discharge vector is defined as the net outflow per unit area and per unit time at a point in the porous medium. The concept of divergence is not limited to

the specific discharge function, for example we could apply this concept to the velocity field of any substance, and it is useful in such considerations as mass balance.

As described in (STRACK, 1989; STRACK, 2014) the divergence of the specific discharge vector for the block shown in Figure 4 is determined. Consider that the sides of the block are Δx , Δy and Δz , and the center is at (x, y, z) . The cube is so small that the variation of the magnitudes of the components of q_i along the sides can be neglected with respect to their value at the center of each side. The inflow through each side then is equal to the normal component of q_i multiplied by the area of the side. The net flow $\Delta Q[\text{m}^3/\text{s}]$ leaving the block is obtained from Figure 4, where only the components of q_i normal to the sides are drawn:

$$\begin{aligned}\Delta Q = & \Delta y \Delta z [q_x(x + \frac{\Delta x}{2}, y, z) - q_x(x - \frac{\Delta x}{2}, y, z)] \\ & + \Delta x \Delta z [q_y(x, y + \frac{\Delta y}{2}, z) - q_y(x, y - \frac{\Delta y}{2}, z)] \\ & + \Delta x \Delta y [q_z(x, y, z + \frac{\Delta z}{2}) - q_z(x, y, z - \frac{\Delta z}{2})].\end{aligned}\quad (2.15)$$

Now dividing through by $\Delta x \Delta y \Delta z$, we have:

$$\begin{aligned}& \frac{q_x(x + \frac{\Delta x}{2}, y, z) - q_x(x - \frac{\Delta x}{2}, y, z)}{\Delta x} \\ & + \frac{q_y(x, y + \frac{\Delta y}{2}, z) - q_y(x, y - \frac{\Delta y}{2}, z)}{\Delta y} \\ & + \frac{q_z(x, y, z + \frac{\Delta z}{2}) - q_z(x, y, z - \frac{\Delta z}{2})}{\Delta z} = \text{div}(\vec{q}),\end{aligned}\quad (2.16)$$

where $\text{div}(\vec{q})$ represents the divergence of the specific discharge vector. Taking the limit, as $\Delta x \rightarrow 0, \Delta y \rightarrow 0, \Delta z \rightarrow 0$, then in terms of partial derivatives eq. (2.16) becomes:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = \text{div}(\vec{q}),\quad (2.17)$$

which is the expression for the divergence of the specific discharge vector.

The divergence of the discharge vector is zero for the case of steady flow, but also for the special case of transient flow where both the fluid and the porous medium are incompressible. Therefore eq. (2.17) becomes:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = 0,\quad (2.18)$$

which is the continuity equation for steady groundwater flow. The divergence of the specific discharge vector is non-zero only if the flow is transient, and if either the porous medium or the fluid is compressible.

The combined form of Darcy's law and the continuity equation give four equations for the following four unknowns q_x, q_y, q_z and ϕ . By substituting values of q_x, q_y and q_z in eq. (2.18),

it becomes:

$$\frac{\partial}{\partial x} \left[K \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[K \frac{\partial \phi}{\partial y} \right] + \frac{\partial}{\partial z} \left[K \frac{\partial \phi}{\partial z} \right] = 0, \quad (2.19)$$

which shows the governing equation for steady groundwater flow. Finally, after simplification assuming K is constant, we get:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad (2.20)$$

which is Laplace's equation for the head in the three dimensional case.

2.3 Basic Equations of Groundwater flow

In (STRACK, 1989), it is argued that for obtaining the basic equation of groundwater flow, we need to introduce the discharge vector Q_i to discuss about the discharge potential instead of the specific discharge. The discharge flowing through the aquifer per unit length is found by multiplying the specific discharge by the thickness of the aquifer, H , and is defined as $Q_i = Hq_i$. Consider a discharge vector with two components Q_x and Q_y for two-dimensional flow, defined as:

$$Q_x = Hq_x \quad \text{and} \quad Q_y = Hq_y. \quad (2.21)$$

Using eq. (2.11), eq. (2.21) takes the form:

$$\begin{aligned} Q_x &= Hq_x = H \left(-K \frac{\partial \phi}{\partial x} \right) \\ Q_y &= Hq_y = H \left(-K \frac{\partial \phi}{\partial y} \right) \end{aligned}, \quad (2.22)$$

where K is the hydraulic conductivity constant. Since K and H are constants, eq. (2.22) may be rewritten as:

$$\begin{aligned} Q_x &= -\frac{\partial (KH\phi)}{\partial x} \\ Q_y &= -\frac{\partial (KH\phi)}{\partial y} \end{aligned}. \quad (2.23)$$

It is convenient to introduce a new variable Φ (the discharge potential). The discharge potential Φ is the real part of the complex potential and is valid for both confined and unconfined flows whose applications deal with the flow in shallow aquifers, where the Dupuit-Forchheimer assumption¹ may be adopted. For confined aquifer with thickness H as shown in Figure 5 and hydraulic conductivity K , the discharge potential is given by:

$$\Phi = KH\phi + C_c \quad (\phi \geq H), \quad (2.24)$$

¹ According to Dupuit-Forchheimer assumption, the variation of the head ϕ in the vertical direction can be neglected as presented in different works (STRACK, 2014; STRACK, 2003; JONG, 1960).

and for the unconfined aquifer the discharge potential is given by:

$$\Phi = \frac{1}{2}K\phi^2 + C_u \quad (\phi < H), \quad (2.25)$$

where C_c and C_u are arbitrary constants.

Figure 5 – One-dimensional combined confined/unconfined flow

Source: [Marin \(2011\)](#).

Since $\frac{\partial C_c}{\partial x} = \frac{\partial C_c}{\partial y} = 0$, using eq. (2.23) with eq. (2.24), we get:

$$\begin{aligned} Q_x &= -\frac{\partial}{\partial x} [KH\phi + C_c] = -\frac{\partial \Phi}{\partial x} \\ Q_y &= -\frac{\partial}{\partial y} [KH\phi + C_c] = -\frac{\partial \Phi}{\partial y} \end{aligned} \quad (2.26)$$

Therefore, the governing equation in terms of discharge potential for two-dimensional flow becomes:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \quad (2.27)$$

which is Laplace's equation in terms of discharge potential.

ANALYTIC ELEMENT METHOD

This chapter describes a brief description of the Analytic Element Method, which is based on (STRACK, 1989; MARIN, 2011).

3.1 Introduction

The Analytic Element Method (AEM) was presented by Otto D.L. Strack to simulate groundwater flow. It uses the exact solution of the governing equations for basic fundamental features of the aquifer, such as a well, a river, an impermeable barrier, a sink, called Analytic Elements, and combines such solutions by superposition to approximate the solution of a much more complex real problem. The AEM is based on analytic solutions of the groundwater governing equations and does not depend on domain discretization, being able to tackle multiscale problems where other methods have difficulties (MARIN, 2011; STRACK, 1989). This method has been applied in different fields such as wellhead protection area delineation, nuclear waste repositories, and vertical stratified aquifers (MARIN, 2011; WONG; CRAIG, 2010). In this method, each analytic solution represents an hydrogeologic feature. The hydrogeologic feature is discretized into simpler geometric features, like points, line segments or polygons. Each geometric feature is represented by an analytic function with unknown coefficients. The free coefficients on these functions must be calculated to satisfy the boundary conditions imposed. In one of the solution approaches, the coefficients are found by specifying the boundary conditions for the coordinate points on the elements, using the collocation method. There are several approaches to improve the numerical accuracy or decrease the computational time required to solve the resulting system of equations (BAKKER, 2008; BAKKER *et al.*, 1999; BAKKER; KELSON, 2009; BANDILLA; RABIDEAU; JANKOVIĆ, 2009; JANKOVIĆ; BARNES, 1999b). In recent years, AEM received attention with its use for harmonically heterogeneous aquifers (CRAIG, 2009), and the use of Wirtinger calculus (STRACK, 2009a; STRACK, 2009b). Recently, Strack (STRACK, 2014) used the Cauchy Integral formula combined with the Joukowski transform

to satisfy the boundary conditions for circular inhomogeneities and line-dipoles. Readers are encouraged to read (BARNES; JANKOVIĆ, 1999) for an introductory level discussion of the AEM.

Consider a system which consists of some wells, rivers, circular and polygonal inhomogeneities and fractures. By studying their groundwater flows, we will observe that influences would occur due to the rivers, circular and polygonal inhomogeneities and fractures. In order to solve such problems, Strack in (STRACK, 1989) proposed line-elements (such as line-sinks, line-dipoles and line-dipoles), which will discuss in Chapter 6 in detail. Strack and Haitjema used line-sinks for modelling creeks and rivers in a regional flow model, whereas the wells are used as the places where the inflow or outflow occurs and line-dipoles for fractures as described in (STRACK, 1982; STRACK, 1989).

In the following work we shall introduce the solution for basic Analytic Elements such as line-doublet, circle and fracture. Such elements will be used later to simulate internal boundary condition and inhomogeneous medium. We recall that each of those solution satisfies Laplace's equation with special boundary condition that are akin to the problem at hand.

The complex potential in the AEM may be written by using the superposition principle which is the sum of many complex potential components for each physical feature enclosed in a model. The complex potential, $\Omega(z)$, at point $z = x + iy$ in the complex plane has the form:

$$\Omega(z) = \Phi(z) + i\Psi(z), \quad (3.1)$$

where $\Phi(z)$ is the real part of the complex potential, called discharge potential, proportional to the hydraulic head and $\Psi(z)$ is the associated stream function. The gradient of the discharge potential is orthogonal to the gradient of the stream function and satisfies the Cauchy-Riemann equations:

$$\frac{\partial \Phi}{\partial z} = i \frac{\partial \Psi}{\partial z}. \quad (3.2)$$

Lines of constant discharge potential are defined as equipotentials and lines of constant values of the stream function are defined as streamlines. Similarly flow nets are created by plotting contours of $\Phi = \text{constant}$ and $\Psi = \text{constant}$ and choosing $\Delta\Phi = \Delta\Psi$. This can be seen from eq. (3.2).

In order to work with the AEM, first of all the discharge potential Φ_0 at the reference point z_0 is to be calculated using an aquifer condition (like confined or unconfined aquifer). After calculating Φ_0 , this discharge potential may be used to calculate the constant C for solving the problem, that takes the form:

$$C = \Phi_0 - \Re(\Omega_T(z_{ref})), \quad (3.3)$$

where Ω_T is the complex potential of all the elements at reference point z_0 .

3.2 Methods for satisfying the Boundary Conditions

The solution of the governing equation for each element (line-doublet, circle, fracture etc.) have degrees of freedom which satisfy the boundary conditions, according to the type of the element and the problem. One of the central fact about AEM is to understand how the boundary conditions are met for each problem. For this purpose, the following two methods: Collocation Method and Overspecification Method are widely used in the literature as described in (HAITJEMA, 1995; MARIN, 2011; JANKOVIĆ, 1997; JANKOVIĆ; BARNES, 1999b; BARNES; JANKOVIĆ, 1999; PRESS, 2007). The first was the collocation method. It uses predefined points on the element called collocation points, where the boundary conditions are enforced exactly. The second one is Overspecification method, where the number of unknown coefficients representing a number of control points over the element is higher than the number of degrees of freedom for each element, and the boundary conditions are satisfied in a least square sense over the control points (see further detail in (JANKOVIĆ; BARNES, 1999a)).

3.3 Coordinates Transformation

We will discuss two-dimensional problems with complex variables, in this work. Therefore, the functional relationship of the complex variable z and the variables x and y is given by (MARIN, 2011):

$$f(z) = f(x, y). \quad (3.4)$$

The use of Laplace's equation for complex variables open the possibility of the use of Wirtinger calculus. It is based on a transformation of variables and allows the direct integration of the Laplace equation in coordinate system, as described in (MARIN, 2011; STRACK, 2009a; STRACK, 2009b). This transformation is an interesting feature of working with complex variables and plays an important role in the development of the AEM. It is actually the transformation of a line in z -plane having length L with directional angle α relative to the x -axis. Consider the global coordinate system with point z , defined by:

$$z = x + iy, \quad (3.5)$$

and the local coordinate system with point Z , defined by:

$$Z = X + iY. \quad (3.6)$$

Let z_1 and z_2 be the two points in the z -plane. Then the transformation

$$Z = \frac{z - \frac{1}{2}(z_1 + z_2)}{\frac{1}{2}(z_2 - z_1)} \quad (3.7)$$

maps the line $[z_1, z_2]$ into the interval $[-1, 1]$, and the inverse transformation is given by:

$$z = Z \frac{1}{2}(z_2 - z_1) + \frac{1}{2}(z_1 + z_2), \quad (3.8)$$

where z_1 and z_2 represent the initial and end points of the line respectively. This transformation is shown in [Figure 6](#) and will be used in this work.

Figure 6 – The transformation from the global coordinate system z to the local one Z

Source: [Marin \(2011\)](#).

CIRCULAR INHOMOGENEITIES

This chapter discusses the problem of circular inhomogeneities and a methodology for solving it. The proposed methodology is based on the Analytic Element Method (AEM) presented in chapter 3 and more extensively at (MARIN, 2011; STRACK, 1989; JANKOVIĆ; BARNES, 1999a; BARNES; JANKOVIĆ, 1999).

4.1 Introduction

In this chapter we will deal with the problem of an aquifer compose of circular regions whose hydraulic conductivity is different from the surrounding medium. See the Figure 7.

Figure 7 – Aquifer with circular inhomogeneities of distinct hydraulic conductivities

Circular inhomogeneities are circular regions with different hydraulic conductivities. Let K_j^+ be the hydraulic conductivity inside and K be the hydraulic conductivity outside the circle j , which shows the permeability of the material. It depends upon the properties of the material like fluid and soil. It is a material constant and is different in range for different materials like soil, sand, clay, gravels etc. presented in (STRACK, 1989). The influence of circular inhomogeneities are simulated by expanding the complex potential in Laurent series outside and Taylor series

inside the inhomogeneity as presented in (JANKOVIĆ; BARNES, 1999a; STRACK, 1999). Consider the complex potential $\Omega(z)$ in the form:

$$\Omega(z) = \Phi(z) + i\Psi(z), \quad (4.1)$$

where $\Phi(z)$ is the real part of the complex potential, called discharge potential, and is proportional to the hydraulic head, $\Psi(z)$ is the imaginary part of the complex potential, called the stream function and $z = x + iy$ is the location of a point in the complex plane.

The complex potential for a circular inhomogeneity j having radius R_j and center c_j has been presented in different works (JANKOVIĆ, 1997; BARNES; JANKOVIĆ, 1999; MARIN, 2011), and is defined as:

$$\Omega_j(Z) = \Phi_j(Z) + i\Psi_j(Z) = \begin{cases} a_0 + \sum_{n=1}^{\infty} a_n Z_j^n & |Z| < 1 \\ -\sum_{n=1}^{\infty} \bar{a}_n Z_j^{-n} & |Z| \geq 1 \end{cases}, \quad (4.2)$$

for which

$$Z = \frac{z - c_j}{R_j}, \quad (4.3)$$

where the coefficient a_0 is real and all a_n are complex, z represents any position in the global coordinate system and $Z = X + iY$ is the local coordinate system which represents similar position specified in local coordinates. The boundary of the circular inhomogeneity is given by:

$$|z - c_j| = R_j, \quad (4.4)$$

which may be written using the local coordinate system as (BARNES; JANKOVIĆ, 1999):

$$Z = e^{i\theta}, \quad 0 \leq \theta < 2\pi. \quad (4.5)$$

According to (BARNES; JANKOVIĆ, 1999; MARIN, 2011), it is assumed that the circular inhomogeneity does not meet with other elements and the stream function $\Psi_j(Z)$ given by eq. (4.2) is continuous across the boundary due to $\Im(a_n e^{in\theta}) = -\Im(\bar{a}_n e^{-in\theta})$. Therefore, the complex potential of all the elements in the domain can be represented as:

$$\Omega(Z) = \Omega_j(Z) + \Omega_{\neq j}(Z), \quad (4.6)$$

where $\Omega_{\neq j}(Z)$ is the combined complex potential of all circular inhomogeneities except j . The continuity of $\Omega_{\neq j}(Z)$ and eq. (4.2) ensure the continuity of the stream function across the boundary of the circular inhomogeneity. Therefore, it is sufficient for the continuity of flow everywhere in the given domain.

4.2 Boundary Conditions

To satisfy the boundary conditions, the head must be continuous at the edge of an inhomogeneity (BARNES; JANKOVIĆ, 1999), that is:

$$\phi^+(\theta) = \phi^-(\theta). \quad (4.7)$$

The only component that jumps across the boundary of the circular inhomogeneity is the component of the discharge potential itself, which is the contribution from all the circular inhomogeneities (BARNES; JANKOVIĆ, 1999). Hence, the continuity condition of the head at the edge of an inhomogeneity can be written as:

$$\frac{\Phi^+}{K_j^+} = \frac{\Phi^-}{K}, \quad (4.8)$$

where the $+$ sign is the internal part and the $-$ sign is the external part of the inhomogeneity, whereas, K_j^+ is the hydraulic conductivity inside and K is the hydraulic conductivity outside the circle j . Due to the continuity of the head, the boundary can be parametrized by the angle θ , i.e., $0 \leq \theta \leq 2\pi$, that leads to the relation (MARIN, 2011; BARNES; JANKOVIĆ, 1999):

$$\frac{\Phi^+(\theta)}{K_j^+} = \frac{\Phi^-(\theta)}{K}. \quad (4.9)$$

By combining the discharge potentials of all the components of circular inhomogeneities, we get:

$$\frac{\Phi_{\neq j}^+(\theta) + \Phi_j^+(\theta)}{K_j^+} = \frac{\Phi_{\neq j}^-(\theta) + \Phi_j^-(\theta)}{K}, \quad (4.10)$$

or

$$K\Phi_j^+(\theta) - K_j^+\Phi_j^-(\theta) = (K_j^+ - K)\Phi_{\neq j}^-(\theta) = (K_j^+ - K)\Phi_{\neq j}(\theta), \quad (4.11)$$

where $\Phi_j^+(\theta)$ is the discharge potential inside the circle j , $\Phi_j^-(\theta)$ is the discharge potential outside the circle j and $\Phi_{\neq j}^+(\theta)$ is the discharge potential inside of all circular inhomogeneities, whereas $\Phi_{\neq j}^-(\theta)$ is the discharge potential outside of all circular inhomogeneities except j .

Since the discharge potential generated by the other elements is continuous over the edge of the j th circular inhomogeneity. Therefore, the boundary conditions may be written as:

$$K\Phi_j^+(\theta) - K_j^+\Phi_j^-(\theta) = (K_j^+ - K)\Phi_{\neq j}(\theta), \quad (4.12)$$

which is used to determine expressions for calculating the unknown coefficients.

4.3 Determining the Unknown Coefficients

The following techniques are used to determine the required expressions for calculating the unknown coefficients.

4.3.1 Orthogonality of Fourier Series

To calculate the unknown coefficients of potential expansion for an analytical element, the boundary condition eq. (4.7) for a circular inhomogeneity may be used. It may be calculated using eq. (4.2) by the use of Fourier orthogonal relationship. Substituting the real part of eq.

(4.2) in eq. (4.12), multiplying both sides by $e^{-im\theta}$, and integrating from 0 to 2π (MARIN, 2011; BARNES; JANKOVIĆ, 1999), we get:

$$a_0 = \left(\frac{K_j^+ - K}{K} \right) \frac{1}{2\pi} \int_0^{2\pi} \Phi_{\neq j}(\theta) d\theta \quad (4.13)$$

and

$$a_n = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{1}{\pi} \int_0^{2\pi} \Phi_{\neq j}(\theta) e^{-in\theta} d\theta, \quad (4.14)$$

where a_0 is real and all a_n are complex. The proof is presented in [Appendix A](#).

Using eq. (4.2) into eq. (4.14) we get an infinite linear system for the unknowns $a_0, a_1, \dots$. Obviously, in practice the infinite series must be truncated and only a finite number of coefficients have to be found.

4.3.2 Least Square Approach

The unknown coefficients given by eq. (4.13) and eq. (4.14) are exact, and determined by the use of Fourier orthogonal series. Here is another approach called least square approach, in which the unknown coefficients can be determined by minimization techniques. Regarding this approach we have:

$$a_0 = \left(\frac{K_j^+ - K}{K} \right) \frac{1}{M_j} \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) \quad (4.15)$$

and

$$a_n = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{2}{M_j} \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) e^{-in\theta_m}, \quad (4.16)$$

where M_j shows control points on the boundary of the circular inhomogeneity j . This is the discrete form of eq. (4.13) and eq. (4.14) and its proof is presented in [Appendix A](#).

4.4 Exact Solution for One Circular Inhomogeneity

A circular/cylindrical inhomogeneity in an uniform field with certain hydraulic conductivity will serve to illustrate the behavior of the complex potential along the boundary of an inhomogeneity. The complex potentials in the domain for an inhomogeneity inside and outside the boundary is denoted by Ω^+ and Ω^- as shown in [Figure 8](#).

Figure 8 – A circular/cylindrical inhomogeneity

Source: [Strack \(1989\)](#).

The exact solution of the problem of a circular inhomogeneity having radius R , centred at origin in an uniform flow field can be determined by the formulation presented in ([STRACK, 2014](#); [MARIN, 2011](#)):

$$\Omega^+(Z) = -Q_{x_0}z \frac{2K_j^+}{K + K_j^+} + \frac{K_j^+}{K} \Phi_0 \quad \text{for} \quad z\bar{z} < R^2 \quad (4.17)$$

and

$$\Omega^-(Z) = -Q_{x_0} \left(z - \frac{K_j^+ - K}{K + K_j^+} \frac{R^2}{z} \right) + \Phi_0 \quad \text{for} \quad z\bar{z} \geq R^2, \quad (4.18)$$

where Q_{x_0} is the uniform flow field intensity that is parallel to the x -axis and Φ_0 is a reference point with a known head. The proof is presented in [Appendix B](#).

4.5 The Uniform Flow

The uniform flow show the behavior of the streamlines in the domain. The definition of the uniform flow for the global coordinate system having intensity Q_{uf} , forming an angle β_{uf} relative to the x -axis, described in ([STRACK, 1989](#)), shown in [Figure 9](#) is given by:

$$\Omega_{uf} = -Q_{uf}ze^{-i\beta_{uf}} + C, \quad (4.19)$$

where C is called the reference point constant and must be calculated given the hydraulic head at a reference point. This expression is used for complex potential of the uniform flow in this work.

Figure 9 – Direction of uniform flow intensity Q_{uf} and an angle β_{uf} relative to the x -axis Source: [Marin \(2011\)](#).

4.6 The Well

The complex potential for a well having intensity Q_w , as described in ([STRACK, 1989](#)), page 279, is given by:

$$\Omega_w = \frac{Q_w}{2\pi} \ln(z - z_w), \quad (4.20)$$

where z represents any position and z_w represents the position of the well in the domain.

According to ([STRACK, 2014](#)), we can separate the expression for the complex potential of a well into real and imaginary parts and may be written as:

$$\Phi = \frac{Q_w}{2\pi} \ln(r), \quad (4.21)$$

and

$$\Psi = \frac{Q_w}{2\pi} \theta. \quad (4.22)$$

The flow net for a single well in an infinite aquifer is shown in [Figure 10](#) and the values of the stream function are shown next to the corresponding streamline. The values of the streamlines are obtained by the use of eq. (4.22).

Figure 10 – Equipotentials and streamlines for a well

Source: [Marin \(2011\)](#).

From both the [Figure 10](#) and the expression for Ψ defined by eq. (4.22), we see that the stream function is not single-valued, at least not everywhere. Along the negative x -axis, we obtain $\Phi = \frac{Q_w}{2}$ if we choose $\theta = \pi$, whereas $\Phi = -\frac{Q_w}{2}$ if we choose $\theta = -\pi$. This connects the well to infinity, called branch cut.

According to ([STRACK, 2014](#)), we can imagine that the water that flows from infinity to the well is transferred via the branch cut back to infinity, thereby satisfying the condition that there always must be continuity of flow. The jump in the stream function is indeed compatible with flow from the well through the branch cut to infinity. It is noted that this is a mathematical observation that has nothing to do with reality.

FRACTURES

This chapter deals with the problems of fractures in an uniform flow field and methodology for solving it. The proposed methodology is based on the Analytic Element Method (AEM) described in (STRACK, 1982; MARIN, 2011).

5.1 Introduction

Simulation of groundwater flow in fractured porous media is a relevant problem in different fields such as geologic repositories for nuclear wastes and groundwater contamination (BERKOWITZ, 2002; NEUMAN, 2005). Groundwater flow in fractures is fundamental in this context. It is usually present in almost all geologic formations, and can propagate groundwater (and solutes) several orders of magnitude faster than the porous media, being a contamination vector (BERKOWITZ, 2002). Geometry of a fracture is a key factor in the process occurring in it. One of the basic principle is that the fracture would have an ellipsoid shape with no flow at the ends. Fractures are usually thin and saturated with water. The walls of the fractures are symmetric with respect to the planes through the centres of the fractures. The most common conceptual model is that the fracture is formed by smooth and flat parallel plates, as considered for various experiments in different woks (MARIN, 2011; BERKOWITZ, 2002; BEAR; TSANG; MARSILY, 2012; WENDLAND; HIMMELSBACH, 2002). In general, there are two common classes to model flow in fracture networks:

5.1.1 Continuous Models

Continuous models consists of two or more continuous interacting fractures. It is applied when a continuous network of fractures is highly connected and dense, whereas, the porous medium can be ignored. In most cases it is complicated to assume a continuous model for fractured system (fractals), due to the directional dependence in the flow and the influence of

different scales for the properties of fractures for the transport see (MARIN, 2011; NEUMAN, 2005).

5.1.2 Discrete Models

Discrete models allow to describe phenomena that they are able to incorporate explicitly the properties of the individual fracture. These models consider each fracture independently, describing its geometric properties and the interconnection between them (BEAR; TSANG; MARSILY, 2012; MARIN, 2011; BERKOWITZ, 2002). Discrete-shaped fractures can be represented by uni, bi and three-dimensional elements. Parameters such as length, aperture and roughness are assigned to each fracture as presented in (CACAS *et al.*, 1990; SINGHAL; GUPTA, 2010).

Discrete fracture networks are generally generated using statistical distributions based on the properties found in the field, such as orientation, length distribution, position, density, aperture. One of the problem in the use of discrete network is to obtain the data. Experimental data may not be sufficient to determine some of these parameters, for which percentage of fracture actually conduct fluid, although the network appear to be well connected. The computational effort required for the simulation using a discrete network can be large, and simplifications such as the use of parallel plate models can be used to reduce such efforts, despite the various evidences that such an approach is not sufficient to reduce the experimental results for the field or laboratory scale (MARIN, 2011; NEUMAN, 2005; WENDLAND; HIMMELSBACH, 2002; CACAS *et al.*, 1990; BERKOWITZ; SCHER, 1995; BERKOWITZ; SCHER, 1997; CVETKOVIC *et al.*, 2010). Once the discrete network of fractures is generated then the flow in it must be determined. The flow models are dependent on the form of the network, but are generally divided into two classes: one considers that the flow occurs only in the fractures, and the other that there is also the flow in the porous medium. In the first case, since the flow is contained only in the fractures, the flow itself is simplified, and generally given by a linear relationship between the flow rate and the head gradient, with the fracture hydraulic conductivity being the proportionality constant (MARIN, 2011; CACAS *et al.*, 1990).

The permeability in the fractured system is generally much larger than the permeability of the porous medium, which leads the velocity of the fluid in the fractures to be larger and faster carrying to solute. However, delaying effects, solute diffusion and chemical reactions can occur in porous blocks. Thus, interaction effects between the fractured and porous medium may be much more important for transport than the flow as presented in (MARIN, 2011; WILSON *et al.*, 1983; BECKER; SHAPIRO *et al.*, 2000; POTERI, 2009; CROUCH, 1976).

Strack and Jancović and Barnes used an Iterative Method to solve the linear systems resulting from the analytic elements presented in (JANKOVIĆ; BARNES, 1999a). In this work, we use the same methodology for solving different problems like circular inhomogeneities, polygonal inhomogeneities formed by line-doublets and fractures. However, we found some

cases where the Iterative Method did not converge, like the flow field with many fractures. This motivate us to a Matrix Method to satisfy the boundary conditions in this specific problem.

5.2 Boundary Conditions

The approximate equation obtained by assuming laminar flow in an elliptical fracture is given by Strack (STRACK, 1982) in terms of local Cartesian coordinates, where $\mathbf{s}$ and $\mathbf{n}$ are parallel and normal to the axis of the fracture respectively. The origin of the local coordinates system is considered at the center of the fracture. The width b of the elliptical fracture varies with position, and the length is denoted by L , as shown in Figure 11.

Figure 11 – A fracture with local coordinates $\mathbf{s}$ and $\mathbf{n}$

Source: Strack (1982).

Assuming the width b of the fracture w.r.t. its length is small, so the head along the fracture may be replaced by an average head. For the given two-dimensional case, the discharge using cubic law presented in (STRACK, 1982; LAMB, 1932; MARIN, 2011) has the form:

$$Q_s = -\beta b^3 \frac{\partial \bar{\phi}}{\partial s}, \quad (5.1)$$

where $\bar{\phi}$ represents average head across the fracture, Q_s [m^2/s] represents the discharge through the fracture and β is defined as $\beta = \frac{\rho g}{12\mu}$, whereas ρ is the unit mass of water (density of water), g is the gravity acceleration and μ is the viscosity (see further detail in (STRACK, 1982)). After transforming eq. (5.1), the linear law for the dynamics of the fracture takes the form:

$$Q_s = -K^+ b \frac{\partial \bar{\phi}}{\partial s}, \quad (5.2)$$

where the fracture is considered to be filled with a porous material with hydraulic conductivity K^+ .

Continuity of the flow in the fracture at any location $-\frac{L}{2} \leq s \leq \frac{L}{2}$, $\mathbf{n} = 0$, requires that the discharge Q_s be equal to the total inflow into the fracture, measured from the starting point, that is:

$$Q_s = \int_{-\frac{L}{2}}^s \sigma(s) dt, \quad -\frac{L}{2} \leq s \leq \frac{L}{2}, \quad \mathbf{n} = 0, \quad (5.3)$$

where $\sigma(s)$ [m/s] is the sum of the inflows from both sides, per unit length of the fracture. It can be expressed in terms of the component q_n of the specific discharge vector as shown in Figure 12, and may be written as:

$$\sigma(s) = -[q_n^+ - q_n^-]. \quad (5.4)$$

Figure 12 – Viewed for flow in the fractures (left) and in the rock (right)

Source: Strack (1982).

Assume that the average head $\bar{\phi}$ along the fracture is equal to the head outside the fracture, ϕ , in the transverse direction, that is:

$$\phi = \bar{\phi} \quad -\frac{L}{2} \leq s \leq \frac{L}{2}, \quad \mathbf{n} = 0. \quad (5.5)$$

Using Cauchy-Riemann, we have:

$$q_s = -\frac{\partial \Phi}{\partial s} = -\frac{\partial \Psi}{\partial \mathbf{n}} \quad (5.6)$$

and

$$q_n = -\frac{\partial \Phi}{\partial \mathbf{n}} = \frac{\partial \Psi}{\partial s}. \quad (5.7)$$

So, eq. (5.4) becomes:

$$\sigma(s) = -\left[\frac{\partial \Psi^+}{\partial s} - \frac{\partial \Psi^-}{\partial s} \right] = -\frac{\partial (\Psi^+ - \Psi^-)}{\partial s}, \quad (5.8)$$

and eq. (5.3) with eq. (5.8) yields:

$$Q_s = -\int_{-\frac{L}{2}}^s \frac{\partial (\Psi^+ - \Psi^-)}{\partial s} dt, \quad -\frac{L}{2} \leq s \leq \frac{L}{2}, \quad \mathbf{n} = 0. \quad (5.9)$$

The width of the fracture is taken to be zero at the ends and the flow that passes through the fracture in the direction $\mathbf{s}$ is also taken to be zero. Therefore, both Q_s and $\Psi^+ - \Psi^-$ vanishes there, and eq. (5.9) becomes:

$$Q_s = -(\Psi^+ - \Psi^-), \quad -\frac{L}{2} \leq s \leq \frac{L}{2}, \quad \mathbf{n} = 0. \quad (5.10)$$

Now using the cubic law, eq. (5.1) may be written as:

$$Q_s = -\beta b^3 \frac{\partial \phi}{\partial s}, \quad -\frac{L}{2} \leq s \leq \frac{L}{2}, \quad \mathbf{n} = 0. \quad (5.11)$$

Multiplying and dividing by K , the given equation in terms of discharge potential takes the form:

$$Q_s = -\frac{\beta b^3}{K} \frac{\partial \Phi}{\partial s}. \quad (5.12)$$

From eq. (5.10) and eq. (5.12), we have:

$$\frac{\beta b^3}{K} \frac{\partial \Phi}{\partial s} = \Psi^+ - \Psi^-, \quad (5.13)$$

and using the linear law, eq. (5.2) may be written in the form:

$$Q_s = \frac{K^+ b}{K} \frac{\partial \Phi}{\partial s} = \Psi^+ - \Psi^-. \quad (5.14)$$

In general, the discharge vector for an analytic element is given by the derivative of the complex potential, that is:

$$W(z) = -\frac{d\Omega(z)}{dz} = Q_x(z) - iQ_y(z), \quad (5.15)$$

where $Q_x(z)$ and $Q_y(z)$ are the discharge vectors in the z -plane. Using the chain rule, for the Z -plane, we get:

$$W(z) = -\frac{d\Omega(z)}{dz} = -\frac{d\Omega(z)}{dZ} \frac{dZ}{dz}. \quad (5.16)$$

Therefore,

$$W(z) = -\frac{d\Omega(z)}{dZ} \frac{2}{z_2 - z_1}. \quad (5.17)$$

The real part of the differentiated discharge potential rotating to the direction of the fracture $\mathbf{s}$ is equal to the discharge vector in the fracture:

$$Q_s = \frac{d\Phi(z)}{ds} = \Re(W(z)e^{i\alpha}), \quad (5.18)$$

and eq. (5.14) is therefore:

$$Q_s = \frac{K^+ b}{K} \Re(W(z)e^{i\alpha}) = \Psi^+ - \Psi^-, \quad (5.19)$$

which shows the final expression for the boundary conditions on the fracture along its length.

5.3 Exact Solution for One Fracture

The space between the walls of the fracture is defined as the width b . The fracture is ellipsoid as shown in [Figure 11](#). So according to ([MARIN, 2011](#); [CROUCH, 1976](#)), the parameter b is defined as:

$$b = b^* \sqrt{Z^2 - 1}. \quad (5.20)$$

where b^* is the maximum width of the fracture at the center, where $X = 0$. Using the local coordinate system Z , the width b as a function of X may be written as:

$$b = b^* \sqrt{1 - X^2}. \quad (5.21)$$

For a fracture that obey the cubic law, according to ([STRACK, 1982](#)), the given relation takes the form:

$$b = b^* (1 - X^2)^{\frac{1}{6}}. \quad (5.22)$$

Calculating the expression of the width b of the fracture, using Joukowski transformation as described in ([MARIN, 2011](#)) yields:

$$(e^{i\theta} - e^{-i\theta}) = 2\sqrt{X^2 - 1}. \quad (5.23)$$

$$2i \sin(\theta) = 2i\sqrt{1 - X^2}. \quad (5.24)$$

$$\sin(\theta) = \sqrt{1 - X^2}. \quad (5.25)$$

$$b = b^* \sqrt{1 - X^2} = b^* \sin(\theta) = b^* \sqrt{1 - \Re(Z)^2}. \quad (5.26)$$

Describing the shape of the fracture using ellipsoid width, it should be noted that inside the fracture, the shape of the fracture is given by eq. (5.26), that is described by a line, where the boundary condition is imposed. For detail description of the behaviour of the function along the element to obtain the term $2\sqrt{X^2 - 1}$ or $2i\sqrt{1 - X^2}$ see [Appendix C](#) or ([MARIN, 2011](#); [STRACK, 1982](#)).

According to Strack ([STRACK, 1982](#)), the complex potential of a fracture in local coordinates system may be written in the form:

$$\Omega(Z) = A \left(Z - \sqrt{(Z-1)(Z+1)} \right), \quad (5.27)$$

for which the dimensionless variables Z as defined in [section 3.3](#), is given by:

$$Z = \frac{z - \frac{1}{2}(z_1 + z_2)}{\frac{1}{2}(z_2 - z_1)}, \quad (5.28)$$

where z_1 and z_2 are the end points of the fracture as shown in Figure 6. This function satisfies the boundary conditions described by eq. (5.19). For the solution of the flow in a fracture in an uniform flow field the constant A must be calculated. For this purpose, the uniform flow field as described in section 4.4 must be converted to the local coordinate system, that takes the form:

$$\Omega_{uf}(Z) = -\frac{1}{2}Q_{x_0}Le^{i(\alpha-\beta_{uf})}Z + C, \quad (5.29)$$

where Q_{x_0} is the strength of the uniform flow, L is the length, C is a constant and must be calculated from the hydraulic head at a reference point, α is the angle of the fracture with respect to the x -axis and β_{uf} is the angle of the uniform flow with respect to the x -axis as shown in Figure 13.

Figure 13 – Angles α and β relative with x -axis in an uniform flow field

Substituting eq. (5.29) in eq. (5.18), after simplification, the constant A may be written in the form:

$$A = \frac{\frac{1}{2}K+b^*}{KL+K+b^*}Q_{x_0}L\cos(\alpha-\beta_{uf}). \quad (5.30)$$

This proof is presented in Appendix C.

Finally, using eq. (5.29) and eq. (5.30), the complex potential from eq. (5.27) may be written as:

$$\Omega(Z) = \frac{\frac{1}{2}K+b^*}{KL+K+b^*}Q_{x_0}L\cos(\alpha-\beta_{uf})\left(Z - \sqrt{(Z-1)(Z+1)}\right) - \frac{1}{2}Q_{x_0}Le^{i(\alpha-\beta_{uf})}Z + C, \quad (5.31)$$

or

$$\Omega(Z) = AZ - A\sqrt{(Z-1)(Z+1)} - \frac{1}{2}Q_{x_0}Le^{i(\alpha-\beta_{uf})}Z + C, \quad (5.32)$$

which is the exact solution of a fracture as described in (STRACK, 1982).

5.4 The Complex Potential

In this section inspired by the formula eq. (5.32) for one fracture, we assume the complex potential for several fractures given by eq. (5.27) can be expanded in series of power of $(Z -$

$\sqrt{Z-1}\sqrt{Z+1}$). A truncated expansion can be used as a numerical approximation for the flow field in a domain with various fractures. The complex potential of one fracture in the domain takes then the form:

$$\Omega_j(Z) = \sum_{n=1}^{N_j} a_n (Z_j - \sqrt{Z_j-1}\sqrt{Z_j+1})^n, \quad (5.33)$$

where a_n are real coefficients of the element j , Z is the dimensionless variable as defined by eq. (5.28) and N is the order of the series expansion.

Similarly for more than one fracture, the complex potential takes the form:

$$\Omega(Z) = \sum_{j=1}^M \sum_{n=1}^{N_j} a_n (Z_j - \sqrt{Z_j-1}\sqrt{Z_j+1})^n, \quad (5.34)$$

where the index j represents the complex potential of the fracture j and M represents the total number of fractures.

5.5 Determining the Unknown Coefficients

The unknown coefficients of the potential expansion for one fracture can be calculated using the orthogonality of the Fourier Series. Since the discontinuity occurs in the imaginary part of the complex potential, the discharge Q_s takes the form:

$$Q_s = \sum_{n=0}^N a_n (e^{-in\theta} - e^{in\theta}) = -2 \sum_{n=0}^N a_n \sin(n\theta), \quad (5.35)$$

which is similar to a Fourier expansion.

To calculate the coefficients a_n , multiply both sides of eq. (5.35) by $\sin(m\theta)$ and integrate in the interval $[0, \pi]$, to give:

$$\int_0^\pi Q_s \sin(m\theta) d\theta = -2 \sum_{n=0}^N a_n \int_0^\pi \sin(n\theta) \sin(m\theta) d\theta. \quad (5.36)$$

If $m = n$ and different from zero, then:

$$-2 \sum_{n=0}^N a_n \int_0^\pi \sin(n\theta) \sin(m\theta) d\theta = -2 \frac{\pi}{2} \sum_{n=0}^N a_n \delta_{mn}. \quad (5.37)$$

$$\int_0^\pi Q_s \sin(n\theta) d\theta = -\pi a_n. \quad (5.38)$$

Hence,

$$a_n = -\frac{1}{\pi} \int_0^\pi Q_s \sin(n\theta) d\theta. \quad (5.39)$$

Moreover, using the boundary conditions given by eq. (5.19), eq. (5.39) becomes:

$$a_n = -\frac{1}{\pi} \frac{K^+ b}{K} \int_0^\pi \Re(W(z) e^{i\alpha}) \sin(n\theta) d\theta. \quad (5.40)$$

Substituting b from eq. (5.26), eq. (5.40) takes the form:

$$a_n = -\frac{1}{\pi} \frac{K^+ b^*}{K} \int_0^\pi \Re(W(z) e^{i\alpha}) \sin(\theta) \sin(n\theta) d\theta. \quad (5.41)$$

This expression is used to calculate the unknown coefficients for the fractures.

LINE-DOUBLETS AND POLYGONAL INHOMOGENEITIES

This chapter describe the problem of polygonal inhomogeneities formed by line-doublets in an aquifer with uniform flow field and the methodology for solving it. The proposed methodology is based on the Analytic Element Method (AEM) presented in (STRACK, 1989; MARIN, 2011).

6.1 Introduction

The line-doublets are used to model inhomogeneities bounded by closed polygons with distinct hydraulic conductivities from that of the hydraulic medium. See the Figure 14.

Figure 14 – Aquifer consisting of polygonal inhomogeneities formed by line-doublets with different hydraulic conductivities

In the Analytic Element Method, generally groundwater problems deals with line-elements (such as line-sinks, line-dipoles and line-doublets). According to (STRACK, 2014), the

line-doublet is a continuous distribution of dipoles along a line having dipoles oriented normal to the line, as shown in Figure 15.

Figure 15 – Line-dipole and line-doublet

Source: Strack (1989).

In order to derive the complex potential for a line-doublet, first we have to derive the complex potential for a line-dipole. A line-dipole is the continuous distribution of dipoles along a line. The complex potential for a dipole at $z = z_0 = x_0 + iy_0$ is the complex function with discharge potential and stream function having strength s , as presented in (STRACK, 1989) is given by:

$$\Phi = \frac{s}{2\pi} \frac{\cos(\theta - \alpha)}{r} \quad \text{and} \quad \Psi = -\frac{s}{2\pi} \frac{\sin(\theta - \alpha)}{r}, \quad (6.1)$$

or

$$\Omega(z) = \Phi + i\Psi = \frac{s}{2\pi} \frac{\cos(\theta - \alpha) - i\sin(\theta - \alpha)}{r} = \frac{s}{2\pi} \frac{e^{i\alpha}}{re^{i\theta}}. \quad (6.2)$$

The complex number $re^{i\theta}$ represents the vector from z_0 to z , therefore, $re^{i\theta} = z - z_0$, and hence the complex potential for a dipole becomes:

$$\Omega(z) = \frac{s}{2\pi} \frac{e^{i\alpha}}{z - z_0}. \quad (6.3)$$

The complex potential for a line-dipole is obtained by integrating a dipole along a line (STRACK, 1989). Therefore, by replacing s by $-\mu$, z_0 by δ , using a line of length L , end points z_1 and z_2 , angle α relative to the x -axis as shown in Figure 6, the integral of eq. (6.3) for a line-dipole is obtained as:

$$\Omega(z) = -\frac{1}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\mu e^{i\alpha}}{z - \delta} d\xi, \quad (6.4)$$

where μ is the discharge distribution and ξ is taken to be $-\frac{L}{2}$ at z_1 and $\frac{L}{2}$ at z_2 , which are the end points of the line-element.

The complex number δ denotes a point on the line and in terms of ξ and it may be written as:

$$\delta = \xi e^{i\alpha} + \frac{1}{2}(z_1 + z_2). \quad (6.5)$$

It follows that $d\delta = e^{i\alpha}d\xi$. Hence, eq. (6.4) may be written as:

$$\Omega(z) = -\frac{1}{2\pi} \int_{z_1}^{z_2} \frac{\mu}{z - \delta} d\delta. \quad (6.6)$$

The line can be mapped into the interval $[-1, 1]$ by introducing the following dimensionless variables:

$$Z = X + iY = \frac{z - \frac{1}{2}(z_1 + z_2)}{\frac{1}{2}(z_2 - z_1)} \quad \text{and} \quad \omega = \omega_1 + i\omega_2 = \frac{\delta - \frac{1}{2}(z_1 + z_2)}{\frac{1}{2}(z_2 - z_1)}, \quad (6.7)$$

where $z_2 - z_1 = Le^{i\alpha}$, Z is a position in the local coordinate system associated with the line-element and z_1 and z_2 are the end points. From eq. (6.7) it is clear that Z and ω are real along the element and are mapped onto the real axis between $X = -1$ and $X = 1$, which shows that the complex potential is analytic in the cut plane $-1 \leq X \leq 1$. Therefore, $\frac{1}{2}(z_1 + z_2)$ corresponds to the center, and $z_2 - z_1$ represents a vector from z_1 to z_2 having length L as shown in [section 3.3](#), [Figure 6](#).

Now by introducing the function $\mu = \mu(Z)$ which is analytic for all finite Z and using the variables Z and ω defined in eq. (6.7), we get:

$$\Omega(Z) = -\frac{1}{2\pi} \int_{-1}^{+1} \frac{\mu(\omega)}{Z - \omega} d\omega. \quad (6.8)$$

By adding and subtracting $\mu(Z)$, eq. (6.8) becomes:

$$\Omega(Z) = -\frac{1}{2\pi} \int_{-1}^{+1} \frac{\mu(Z)}{Z - \omega} d\omega + \frac{1}{2\pi} \int_{-1}^{+1} \frac{\mu(Z) - \mu(\omega)}{Z - \omega} d\omega. \quad (6.9)$$

For the first integral, Z is constant and so the integration may be carried out to give:

$$\Omega(Z) = \frac{\mu(Z)}{2\pi} \ln \frac{Z-1}{Z+1} + \frac{1}{2\pi} \int_{-1}^{+1} \frac{\mu(Z) - \mu(\omega)}{Z - \omega} d\omega. \quad (6.10)$$

The integrand in the remaining integral is analytic at $Z = \omega$, which may be seen by expanding $\mu(Z)$ in a Taylor series. Since the integral of an analytic function is analytic, therefore, the second

term in eq. (6.10) is also analytic for all Z and is represented symbolically as $p(Z)$ presented in (STRACK, 1989). Finally, the complex potential for line-dipole becomes:

$$\Omega(Z) = \frac{\mu(Z)}{2\pi} \ln \frac{Z-1}{Z+1} + p(Z), \quad (6.11)$$

where $p(Z)$ can be considered as a correction function.

In the literature, the terms line-dipole and line-doublet are arbitrary used and both are used for the isolated singularity. Line-doublets are used to model features in the aquifers where the potential is discontinuous and the stream function is continuous. The hydraulic conductivity jumps across the boundary of an inhomogeneity, but the head is continuous. The potential jumps, whereas the stream function is continuous by the flow continuity (STRACK, 1989). As the line-doublet is a continuous distribution of dipoles along a line having dipoles oriented normal to the line. Therefore, the complex potential for a straight line-doublet using eq. (6.11), takes the form:

$$\Omega(Z) = \frac{1}{2\pi i} \left(\lambda(Z) \ln \frac{Z-1}{Z+1} + p(Z) \right), \quad (6.12)$$

where the function $\lambda(Z)$ called jump function which is a real polynomial along the element and $p(Z)$ is the far-field correction polynomial/function that ensures that the given expression approaches zero as $\frac{1}{Z}$ nears infinity (JANKOVIĆ; BARNES, 1999a).

From theory of approximation it is known that Chebyshev polynomials make a better basis for representing the function to be approximated (JANKOVIĆ; BARNES, 1999a). The jump function $\lambda(Z)$ will be approximated by:

$$\lambda(Z) = \sum_{n=0}^N a_n T_n(Z), \quad (6.13)$$

where a_n are real coefficients, T_n are Chebyshev polynomials and N is the order of the given Chebyshev series. As the given expression is real along the element, therefore, it may be written in the form:

$$\lambda(X) = \sum_{n=0}^N a_n T_n(X), \quad (6.14)$$

as presented in (JANKOVIĆ; BARNES, 1999a). The Chebyshev polynomials are defined by $T_n(\cos \theta) = \cos(n\theta)$ and may be used in the expansion of the jump function $\lambda(X)$ in a Fourier cosine series in terms of θ , where $X = \cos \theta$ see (JANKOVIĆ; BARNES, 1999a).

Similarly, the far-field correction function $p(Z)$ may be written in the form:

$$p(Z) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \sum_{p=0}^{n-1} a_n Z^p \left(\frac{1^{n-p} - (-1)^{n-p}}{n-p} \right), \quad (6.15)$$

see (MARIN, 2011). Hence, the complex potential for a line-doublet from eq. (6.12) using eq. (6.13) and eq. (6.15) becomes:

$$\Omega(Z) = \frac{1}{2\pi i} \left(\sum_{n=0}^N a_n T_n(Z) \ln \frac{Z-1}{Z+1} + \sum_{n=1}^{\infty} \sum_{p=0}^{n-1} a_n Z^p \left(\frac{1^{n-p} - (-1)^{n-p}}{n-p} \right) \right), \quad (6.16)$$

or

$$\Omega(Z) = \frac{1}{2\pi i} \left(\sum_{n=0}^N a_n Z^n \ln \frac{Z-1}{Z+1} + \sum_{n=1}^{\infty} \sum_{p=0}^{n-1} a_n Z^p \left(\frac{1^{n-p} - (-1)^{n-p}}{n-p} \right) \right), \quad (6.17)$$

with

$$Z = \frac{z - \frac{1}{2}(z_1 + z_2)}{\frac{1}{2}(z_2 - z_1)}. \quad (6.18)$$

6.2 Boundary Conditions

The boundary conditions for line-doublets are similar to those of circular inhomogeneities, in which the hydraulic head must be continuous at the interface, that is:

$$\phi^+(\theta) = \phi^-(\theta), \quad (6.19)$$

where $\phi^+(\theta)$ is the head inside the inhomogeneity and $\phi^-(\theta)$ is the head outside the inhomogeneity. By definition, the only component that have a jump is the discharge potential of the complex potential Ω for the element j see (BARNES; JANKOVIĆ, 1999). Therefore, the continuity condition of head at the edge of the element j in terms of discharge potential can be written as:

$$\frac{\Phi^+(\delta_l)}{K_j^+} = \frac{\Phi^-(\delta_l)}{K}, \quad (6.20)$$

where δ_l denotes a point on the interface, the $+$ and $-$ signs shows the sides of the element as shown in Figure 16, that are considered as internal and external parts for polygons. K_j^+ and K are the internal and external hydraulic conductivities of bounded polygons as shown in Figure 17.

Figure 16 – A line-doublet with $+$ and $-$ sides in local coordinate system Z

Source: Marin (2011).

The real part of the complex potential Ω is discontinuous and the imaginary part is continuous across the interface. Therefore, in order to create the jump in the complex potential across the element (line-doublet), the highest order terms of the series are used whereas all the other terms are the correction terms (JANKOVIĆ; BARNES, 1999a).

Figure 17 – A polygon representing a region (inhomogeneity) with different hydraulic conductivity from the background

Source: [Marin \(2011\)](#).

The function $\ln((Z - 1)/(Z + 1))$ on the element itself is:

$$\ln \frac{Z-1}{Z+1} = \begin{cases} \ln \left| \frac{X-1}{X+1} \right| + i\pi & \text{for } -1 < X < 1, \quad Y = 0^+ \\ \ln \left| \frac{X-1}{X+1} \right| - i\pi & \text{for } -1 < X < 1, \quad Y = 0^- \end{cases}, \quad (6.21)$$

where (+) shows the left side of the element when its going from the first point to the second and (−) shows the right side. Since, all the correction terms are real along the element and do not contribute to the discharge potential, on the element the discharge potentials for left and right sides will become:

$$\Phi(X, 0^+) = \frac{1}{2} \sum_{n=0}^N a_n T_n(X), \quad (6.22)$$

and

$$\Phi(X, 0^-) = -\frac{1}{2} \sum_{n=0}^N a_n T_n(X). \quad (6.23)$$

After simplification the jump in the discharge potential becomes:

$$\Delta\Phi(X) = \Phi(X, 0^+) - \Phi(X, 0^-) = \sum_{n=0}^N a_n T_n(X), \quad (6.24)$$

which shows the jump function $\lambda(X)$ as presented in eq. (6.14). The stream function is continuous across the element due to the function $\ln \left| \frac{X-1}{X+1} \right|$, that does not jump across the element. Hence, eq. (6.22) and eq. (6.23) are used to determine the unknown coefficients for line-doublets.

6.3 Determining the Unknown Coefficients

This section presents similar techniques to those in [section 4.3](#) for calculating the unknown coefficients.

6.3.1 Orthogonality of Fourier Series

The boundary conditions in this section are similar to those in [section 4.2](#), that is:

$$K\Phi_j^+(\theta) - K_j^+\Phi_j^-(\theta) = (K_j^+ - K)\Phi_{\neq j}(\theta). \quad (6.25)$$

As we know $\lambda(Z) = \sum_{n=0}^{\infty} a_n Z^n$, which shows that the discharge of the complex potential have a jump and is real along the element. Using eq. (6.22) and eq. (6.23), the discharge potential may be written in the form:

$$\Phi^+(\theta) = \lambda^+(\theta) = \frac{1}{2} \sum_{n=0}^{\infty} a_n \cos(n\theta) = \frac{1}{2}a_0 + \frac{1}{2} \sum_{n=1}^{\infty} a_n \cos(n\theta), \quad (6.26)$$

and

$$\Phi^-(\theta) = \lambda^-(\theta) = -\frac{1}{2} \sum_{n=0}^{\infty} a_n \cos(n\theta) = -\frac{1}{2}a_0 - \frac{1}{2} \sum_{n=1}^{\infty} a_n \cos(n\theta). \quad (6.27)$$

Substituting eq. (6.26) and eq. (6.27), eq. (6.25) becomes:

$$K \left(\frac{1}{2}a_0 + \frac{1}{2} \sum_{n=1}^{\infty} a_n \cos(n\theta) \right) - K_j^+ \left(-\frac{1}{2}a_0 - \frac{1}{2} \sum_{n=1}^{\infty} a_n \cos(n\theta) \right) = (K_j^+ - K)\Phi_{\neq j}(\theta), \quad (6.28)$$

$$\frac{1}{2}Ka_0 + \frac{1}{2} \sum_{n=1}^{\infty} Ka_n \cos(n\theta) + \frac{1}{2}K_j^+a_0 + \frac{1}{2} \sum_{n=1}^{\infty} K_j^+a_n \cos(n\theta) = (K_j^+ - K)\Phi_{\neq j}(\theta). \quad (6.29)$$

$$\frac{1}{2}Ka_0 + \frac{1}{2}K_j^+a_0 + \frac{1}{2} \sum_{n=1}^{\infty} Ka_n \cos(n\theta) + \frac{1}{2} \sum_{n=1}^{\infty} K_j^+a_n \cos(n\theta) = (K_j^+ - K)\Phi_{\neq j}(\theta). \quad (6.30)$$

Integrating from 0 to π , eq. (6.30) takes the form:

$$\frac{1}{2} \int_0^{\pi} (K + K_j^+)a_0 d\theta = \int_0^{\pi} (K_j^+ - K)\Phi_{\neq j}(\theta) d\theta. \quad (6.31)$$

$$\pi(K + K_j^+)a_0 = 2(K_j^+ - K)\Phi_{\neq j}(\theta) d\theta. \quad (6.32)$$

$$a_0 = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{2}{\pi} \int_0^{\pi} \Phi_{\neq j}(\theta) d\theta. \quad (6.33)$$

Multiplying both sides of eq. (6.30) by $\cos(m\theta)$, we may rewrite as:

$$\begin{aligned} Ka_0 \cos(m\theta) + K_j^+a_0 \cos(m\theta) + K \sum_{n=1}^{\infty} a_n \cos(n\theta) \cos(m\theta) + K_j^+ \sum_{n=1}^{\infty} a_n \cos(n\theta) \cos(m\theta) \\ = 2(K_j^+ - K)\Phi_{\neq j}(\theta) \cos(m\theta). \end{aligned} \quad (6.34)$$

Integrating from 0 to π , if $m = n$, after simplification eq. (6.34) may be written as:

$$Ka_n \frac{\pi}{2} + K_j^+a_n \frac{\pi}{2} = \int_0^{\pi} 2(K_j^+ - K)\Phi_{\neq j}(\theta) \cos(n\theta) d\theta. \quad (6.35)$$

$$(K + K_j^+)a_n \frac{\pi}{2} = 2(K_j^+ - K) \int_0^\pi \Phi_{\neq j}(\theta) \cos(n\theta) d\theta. \quad (6.36)$$

$$a_n = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{4}{\pi} \int_0^\pi \Phi_{\neq j}(\theta) \cos(n\theta) d\theta. \quad (6.37)$$

Hence,

$$a_0 = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{2}{\pi} \int_0^\pi \Phi_{\neq j}(\theta) d\theta, \quad (6.38)$$

and

$$a_n = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{4}{\pi} \int_0^\pi \Phi_{\neq j}(\theta) \cos(n\theta) d\theta. \quad (6.39)$$

6.3.2 Least Square Approach

The boundary conditions presented in [section 4.2](#) and eq. (6.25), may be written as:

$$K\Phi_j^+(\theta) - K_j^+\Phi_j^-(\theta) - (K_j^+ - K)\Phi_{\neq j}(\theta) = 0. \quad (6.40)$$

Using eq. (6.26) and eq. (6.27) we have:

$$\Phi^+(\theta) = \frac{1}{2}a_0 + \frac{1}{2} \sum_{n=1}^{\infty} a_n \cos(n\theta), \quad (6.41)$$

and

$$\Phi^-(\theta) = -\frac{1}{2}a_0 - \frac{1}{2} \sum_{n=1}^{\infty} a_n \cos(n\theta). \quad (6.42)$$

Substituting eq. (6.41) and eq. (6.42), eq. (6.40) takes the form:

$$\frac{1}{2} (Ka_0 + K_j^+a_0) + \frac{1}{2} \sum_{n=1}^{\infty} [Ka_n \cos(n\theta) + K_j^+a_n \cos(n\theta)] - (K_j^+ - K)\Phi_{\neq j}(\theta) = 0. \quad (6.43)$$

Let M_j be the control points on the boundary of the line-doublet j and $m = 0, 1, \dots, M_j - 1$. The coefficients a_n will be determined such that they minimize the sum of the squares of the errors, where the sum is taken over the M_j control points.

By minimizing, we get:

$$\begin{aligned} \frac{1}{2} \sum_{m=0}^{M_j-1} (Ka_0 + K_j^+a_0) + \sum_{m=0}^{M_j-1} \left[\frac{1}{2} \sum_{n=0}^{\infty} (K + K_j^+) (a_n \cos(n\theta)) - (K_j^+ - K)\Phi_{\neq j}(\theta_m) \right]^2 \\ \rightarrow \text{MIN}(a_0, \dots, a_n). \end{aligned} \quad (6.44)$$

Using eq. (6.44), we may write:

$$\frac{1}{2} \sum_{m=0}^{M_j-1} (K + K_j^+) a_0 - \sum_{m=0}^{M_j-1} (K_j^+ - K)\Phi_{\neq j}(\theta_m) = 0. \quad (6.45)$$

$$M_j \left(K + K_j^+ \right) a_0 = (K_j^+ - K) \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m). \quad (6.46)$$

$$a_0 = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{1}{M_j} \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m). \quad (6.47)$$

Again using eq. (6.44), we can write:

$$\sum_{m=0}^{M_j-1} \left[\frac{1}{2} \sum_{n=0}^{\infty} (K + K_j^+) (a_n \cos(n\theta)) - (K_j^+ - K) \Phi_{\neq j}(\theta_m) \right]^2 \rightarrow \text{MIN}(a_0, \dots, a_n). \quad (6.48)$$

Taking derivative w.r.t. $\Re(a_r)$, eq. (6.48) may be written as:

$$\sum_{m=0}^{M_j-1} \left[\sum_{n=0}^{N_j} (K + K_j^+) (a_n \cos(n\theta)) - (K_j^+ - K) \Phi_{\neq j}(\theta_m) \right] \cos(r\theta_m) = 0. \quad (6.49)$$

By exchanging the order of summation, we get:

$$\sum_{n=0}^{N_j} (K + K_j^+) a_n \left[\sum_{m=0}^{M_j-1} \cos(n\theta_m) \cos(r\theta_m) \right] = (K_j^+ - K) \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) \cos(r\theta_m). \quad (6.50)$$

If $n = r$, then we have the following properties:

$$\sum_{m=0}^{M_j-1} \cos(n\theta_m) \cos(r\theta_m) = \frac{M_j}{2}, \quad (6.51)$$

$$\sum_{m=0}^{M_j-1} \sin(n\theta_m) \cos(r\theta_m) = 0, \quad (6.52)$$

and eq. (6.50) becomes:

$$(K + K_j^+) a_n \frac{M_j}{2} = (K_j^+ - K) \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) \cos(r\theta_m). \quad (6.53)$$

$$a_n = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{2}{M_j} \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) \cos(r\theta_m). \quad (6.54)$$

Hence,

$$a_0 = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{1}{M_j} \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m), \quad (6.55)$$

and

$$a_n = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{2}{M_j} \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) \cos(r\theta_m). \quad (6.56)$$

6.4 Selecting the Control Points

The location of the control points is selected to avoid the singularities caused by the function $\ln((Z - 1)/(Z + 1))$ that are -1 and 1 as Chebyshev control points. Such control points can be chosen as:

$$X^m = \cos\left(\frac{\pi(m - 0.5)}{M_j}\right), \quad (6.57)$$

where $m = 1, 2, \dots, M_j$. For a large number of functions the Chebyshev control Points yield near optimal approximations ([JANKOVIĆ; BARNES, 1999a; CHENEY, .](#)).

SOLUTION ALGORITHM

Different strategies have been used to satisfy the boundary conditions in the AEM. The first was the collocation, proposed in (STRACK, 1989; HAITJEMA, 1995). It uses predefined points on the element (such as well, line-doublet, circle, fracture etc.) called collocation points, where the boundary conditions are enforced exactly. Each element has a fixed number of degrees of freedom, denoting the analytic element order: a first order element strength is given by a polynomial expansion of order one, with two degrees of freedom, and has two associated collocation points. A system of equations is constructed requiring that the boundary conditions must be satisfied exactly on the collocation points over the element. For example, given a section of a river with a specified head, it can be modelled using a line-sink element with a given contribution computed in such a way that the discharge potential of all the elements gives the specified head value at the collocation points. This method suffers when applied to high-order elements and has been extensively discussed in the literature, see (STRACK, 1989; HAITJEMA, 1995; BARNES; JANKOVIĆ, 1999; BATISTA, 2006; NETO; FRANCA, 2008). Later, Strack and Janković and Barnes developed the idea of overspecification, where the number of control points over the element is higher than the number of degrees of freedom for each element, and the boundary conditions are satisfied in a least square sense over the control points presented in (JANKOVIĆ; BARNES, 1999a; JANKOVIĆ; BARNES, 1999b; BARNES; JANKOVIĆ, 1999). They used an Iterative Method to solve the linear systems resulting from the analytic elements see (JANKOVIĆ; BARNES, 1999a). This method has the advantage that no matrix needs to be constructed explicitly, saving computer memory. In general fast convergence is achieved for the most types of elements using this method, with high order elements (up to 75) and a relative low overspecification fold. We use the same methodology for solving different problems like circular inhomogeneities, polygonal inhomogeneities formed by line-doublets and fractures. However, we found some cases where the Iterative Method did not converge, like the flow field with many fractures (MARIN; WENDLAND; STRACK, 2012; MARIN, 2011). This motivate us to develop the Matrix Method to satisfy the boundary conditions in this specific problem.

7.1 Solution Algorithm for Circular Inhomogeneities

The unknown coefficients of the analytical elements (circles) can be found from eq. (4.14), that takes the form:

$$a_{i,n} = P_{i,n} \int_0^{2\pi} \Re(\Omega_T(\theta_i)) e^{-in\theta_i} d\theta_i, \quad (7.1)$$

or

$$a_{i,n} = P_{i,n} \int_0^{2\pi} \Phi_{\neq j}(\theta_i) e^{-in\theta_i} d\theta_i, \quad (7.2)$$

which calculate the n th unknown coefficient for the circle i , where Ω_T is the complex potential of all the elements. The discharge potential $\Phi_{\neq j}$ is the real part of the sum of the complex potential of all other elements present in the system, including the uniform regional flow as given by eq. (4.19). Equation (7.1) gives rise to a set of linear equations that can be solved by two distinct techniques: The Iterative Method and the Matrix Method.

7.1.1 Iterative Method

Consider a system of circular inhomogeneities having M circles in an uniform flow field and a well. The expression to calculate the unknown coefficients for the element/circle i using eq. (7.1) becomes:

$$a_{i,n} = P_{i,n} \sum_{j \neq i}^M \int_0^{2\pi} \Re(\Omega_j(Z_j(\theta_i))) e^{-in\theta_i} d\theta_i + P_{i,n} \int_0^{2\pi} \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) e^{-in\theta_i} d\theta_i, \quad (7.3)$$

or

$$a_{i,n}^{k+1} = P_{i,n} \sum_{j \neq i}^M \int_0^{2\pi} \Re(\Omega_j(Z_j(\theta_i))) e^{-in\theta_i} d\theta_i + P_{i,n} \int_0^{2\pi} \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) e^{-in\theta_i} d\theta_i, \quad (7.4)$$

where k counts the steps of the iterative process and i shows a specific element. The regional uniform flow $\Omega_{uf}(\theta_i)$ is the same as defined in eq. (4.19), the complex potential of the well $\Omega_w(\theta_i)$ is the same as defined in eq. (4.20) and

$$P_{i,n} = \begin{cases} \frac{1}{2\pi} \left(\frac{K_j^+ - K}{K} \right) & \text{for } n = 0 \\ \frac{1}{\pi} \left(\frac{K_j^+ - K}{K + K_j^+} \right) & \text{for } n > 0 \end{cases}. \quad (7.5)$$

The coefficients $a_{j,m}^0$ are initialized with zero. The terms on the right hand side of eq. (7.2) are assumed to be known and are used for calculating coefficients $a_{i,n}$. The updated coefficients are then used for calculating the coefficients of the next element. This process continues until the variation in the coefficients between the iterations is less than a certain specified value ε . This method is described in (MARIN, 2011; JANKOVIĆ, 1997; JANKOVIĆ; BARNES, 1999b).

7.1.2 Matrix Method

Despite that the Iterative Method comes out quite naturally from eq. (7.1), is easy to understand and implement, it sometimes fail to converge. For this reason it may be necessary to form the coefficient matrix $\mathbf{A}$ of the linear system

$$\mathbf{Ax} = \mathbf{b} \quad (7.6)$$

and solve it using appropriate techniques for linear systems. The matrix $\mathbf{A}$ is constructed from the expansion in power series of the potential of circular inhomogeneities. The known vector $\mathbf{b}$ in eq. (7.6) results from the contribution of the uniform flow and the well. After simplification eq. (7.3) can be rewritten in the form:

$$\begin{aligned} \frac{a_{i,n}}{P_{i,n}} - \sum_{j \neq i}^M \int_0^{2\pi} \Re(\Omega(Z_{j,i}(\theta_i))e^{-in\theta_i}d\theta_i - \int_0^{2\pi} \Re(C)e^{-in\theta_i}d\theta_i \\ = \int_0^{2\pi} \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i))e^{-in\theta_i}d\theta_i, \end{aligned} \quad (7.7)$$

and is used to calculate the unknown coefficients in the Matrix Method, whereas the uniform flow, well and the conductivity relation are the same as in eq. (4.19), eq. (4.20) and eq. (7.5).

Since $Z = \left| \frac{z - c_j}{r_j} \right| > 1$,

$$\begin{aligned} \sum_{j \neq i}^M \sum_{m=1}^N \int_0^{2\pi} \Re \left[\bar{a}_{j,m} \left(\frac{c_i + r_i e^{i\theta_i} - c_j}{r_j} \right)^{-m} \right] e^{-in\theta_i} d\theta_i + \frac{a_{i,n}}{P_{i,n}} - \int_0^{2\pi} \Re(C) e^{-in\theta_i} d\theta_i \\ = \int_0^{2\pi} \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) e^{-in\theta_i} d\theta_i. \end{aligned} \quad (7.8)$$

As we know that, $\Re(a.b) = \Re(a)\Re(b) - \Im(a)\Im(b)$, and write, $a_{i,n} = x_{i,n} + iy_{i,n}$ and $\bar{a}_{i,n} = x_{i,n} - iy_{i,n}$, we obtain

$$\begin{aligned} \sum_{j \neq i}^M \sum_{m=1}^N \int_0^{2\pi} \Re \left[\Re(\bar{a}_{j,m}) \Re(Z_{j,i}^{-m}) - \Im(\bar{a}_{j,m}) \Im(Z_{j,i}^{-m}) \right] e^{-in\theta_i} d\theta_i + \frac{a_{i,n}}{P_{i,n}} - \int_0^{2\pi} \Re(C) e^{-in\theta_i} d\theta_i \\ = \int_0^{2\pi} \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) e^{-in\theta_i} d\theta_i, \end{aligned} \quad (7.9)$$

or

$$\begin{aligned} \sum_{j \neq i}^M \sum_{m=1}^N \left[x_{j,m} \int_0^{2\pi} \Re(Z_{j,i}^{-m}) e^{-in\theta_i} d\theta_i + y_{j,m} \int_0^{2\pi} \Im(Z_{j,i}^{-m}) e^{-in\theta_i} d\theta_i \right] + \frac{a_{i,n}}{P_{i,n}} - \int_0^{2\pi} \Re(C) e^{-in\theta_i} d\theta_i \\ = \int_0^{2\pi} \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) e^{-in\theta_i} d\theta_i. \end{aligned} \quad (7.10)$$

Now by taking the real part, we get:

$$\sum_{j \neq i}^M \sum_{m=1}^N \left[x_{j,m} \Re \left(\int_0^{2\pi} \Re \left(Z_{j,i}^{-m} \right) e^{-in\theta_i} d\theta_i \right) + y_{j,m} \Re \left(\int_0^{2\pi} \Im \left(Z_{j,i}^{-m} \right) e^{-in\theta_i} d\theta_i \right) \right] + \frac{x_{i,n}}{P_{i,n}} - \int_0^{2\pi} \Re(C) e^{-in\theta_i} d\theta_i = \int_0^{2\pi} \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) e^{-in\theta_i} d\theta_i, \quad (7.11)$$

and by taking imaginary part, we get:

$$\sum_{j \neq i}^M \sum_{m=1}^N \left[x_{j,m} \Im \left(\int_0^{2\pi} \Re \left(Z_{j,i}^{-m} \right) e^{-in\theta_i} d\theta_i \right) + y_{j,m} \Im \left(\int_0^{2\pi} \Im \left(Z_{j,i}^{-m} \right) e^{-in\theta_i} d\theta_i \right) \right] + \frac{y_{i,n}}{P_{i,n}} - \int_0^{2\pi} \Im(C) e^{-in\theta_i} d\theta_i = \int_0^{2\pi} \Im(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) e^{-in\theta_i} d\theta_i. \quad (7.12)$$

In general, using the Matrix Method we would generate the matrix $\mathbf{A}$ with dimension $M * N_{elem} + 1$, i.e. $\mathbf{A}_{(M*N_{elem}+1) \times (M*N_{elem}+1)}$, where M is the total number of analytical elements, N_{elem} is the approximate order of the series expansion of the element and the additional term is necessary for calculating the constant. The last row and column are for calculating the constant. It is noticed that in case of circular inhomogeneities the unknown coefficients are complex. Therefore, using eq. (7.11) and eq. (7.12), we can get the matrix in the form $\mathbf{A}_{(2(M*N_{elem})+1) \times (2(M*N_{elem})+1)}$. This is due to the separation of real and imaginary parts. For the case of 2 analytical elements (circles) with an approximated series expansion of order 3, the general form of the matrix $\mathbf{A}$ is:

The matrix **A** and vectors **x** and **b** for linear system $\mathbf{Ax} = \mathbf{b}$ of the problems of circular inhomogeneities with two elements has the form:

$$\mathbf{A} = \begin{bmatrix} \frac{1}{P_{1,0}} - \int_0^{2\pi} F_{1,0}(\theta_1) d\theta_1 & \int_0^{2\pi} F_{1,1}(\theta_1) d\theta_1 & \int_0^{2\pi} F_{1,2}(\theta_1) d\theta_1 & \int_0^{2\pi} F_{2,0}(\theta_1) d\theta_1 & \int_0^{2\pi} F_{2,1}(\theta_1) d\theta_1 & \int_0^{2\pi} F_{2,2}(\theta_1) d\theta_1 & \int_0^{2\pi} \Re(C) d\theta_1 \\ \int_0^{2\pi} F_{1,0}(\theta_1) e^{-i\theta_1} d\theta_1 & \frac{1}{P_{1,1}} - \int_0^{2\pi} F_{1,1}(\theta_1) e^{-i\theta_1} d\theta_1 & \int_0^{2\pi} F_{1,2}(\theta_1) e^{-i\theta_1} d\theta_1 & \int_0^{2\pi} F_{2,0}(\theta_1) e^{-i\theta_1} d\theta_1 & \int_0^{2\pi} F_{2,1}(\theta_1) e^{-i\theta_1} d\theta_1 & \int_0^{2\pi} F_{2,2}(\theta_1) e^{-i\theta_1} d\theta_1 & \int_0^{2\pi} \Re(C) e^{-i\theta_1} d\theta_1 \\ \int_0^{2\pi} F_{1,0}(\theta_1) e^{-i2\theta_1} d\theta_1 & \int_0^{2\pi} F_{1,1}(\theta_1) e^{-i2\theta_1} d\theta_1 & \frac{1}{P_{1,2}} - \int_0^{2\pi} F_{1,2}(\theta_1) e^{-i2\theta_1} d\theta_1 & \int_0^{2\pi} F_{2,0}(\theta_1) e^{-i2\theta_1} d\theta_1 & \int_0^{2\pi} F_{2,1}(\theta_1) e^{-i2\theta_1} d\theta_1 & \int_0^{2\pi} F_{2,2}(\theta_1) e^{-i2\theta_1} d\theta_1 & \int_0^{2\pi} \Re(C) e^{-i2\theta_1} d\theta_1 \\ \int_0^{2\pi} F_{1,0}(\theta_2) d\theta_2 & \int_0^{2\pi} F_{1,1}(\theta_2) d\theta_2 & \int_0^{2\pi} F_{1,2}(\theta_2) d\theta_2 & \frac{1}{P_{2,0}} - \int_0^{2\pi} F_{2,0}(\theta_2) d\theta_2 & \int_0^{2\pi} F_{2,1}(\theta_2) d\theta_2 & \int_0^{2\pi} F_{2,2}(\theta_2) d\theta_2 & \int_0^{2\pi} \Re(C) d\theta_2 \\ \int_0^{2\pi} F_{1,0}(\theta_2) e^{-i\theta_2} d\theta_2 & \int_0^{2\pi} F_{1,1}(\theta_2) e^{-i\theta_2} d\theta_2 & \int_0^{2\pi} F_{1,2}(\theta_2) e^{-i\theta_2} d\theta_2 & \int_0^{2\pi} F_{2,0}(\theta_2) e^{-i\theta_2} d\theta_2 & \frac{1}{P_{2,1}} - \int_0^{2\pi} F_{2,1}(\theta_2) e^{-i\theta_2} d\theta_2 & \int_0^{2\pi} F_{2,2}(\theta_2) e^{-i\theta_2} d\theta_2 & \int_0^{2\pi} \Re(C) e^{-i\theta_2} d\theta_2 \\ \int_0^{2\pi} F_{1,0}(\theta_2) e^{-i2\theta_2} d\theta_2 & \int_0^{2\pi} F_{1,1}(\theta_2) e^{-i2\theta_2} d\theta_2 & \int_0^{2\pi} F_{1,2}(\theta_2) e^{-i2\theta_2} d\theta_2 & \int_0^{2\pi} F_{2,0}(\theta_2) e^{-i2\theta_2} d\theta_2 & \int_0^{2\pi} F_{2,1}(\theta_2) e^{-i2\theta_2} d\theta_2 & \frac{1}{P_{2,2}} - \int_0^{2\pi} F_{2,2}(\theta_2) e^{-i2\theta_2} d\theta_2 & \int_0^{2\pi} \Re(C) e^{-i2\theta_2} d\theta_2 \\ F_{1,0}(z_0) & F_{1,1}(z_0) & F_{1,2}(z_0) & F_{2,0}(z_0) & F_{2,1}(z_0) & F_{2,2}(z_0) & 1 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} a_{1,0} \\ a_{1,1} \\ a_{1,2} \\ a_{2,0} \\ a_{2,1} \\ a_{2,2} \\ C \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \int_0^{2\pi} \Re(\Omega_{uf} + \Omega_w) d\theta_1 \\ \int_0^{2\pi} \Re(\Omega_{uf} + \Omega_w) e^{-i\theta_1} d\theta_1 \\ \int_0^{2\pi} \Re(\Omega_{uf} + \Omega_w) e^{-i2\theta_1} d\theta_1 \\ \int_0^{2\pi} \Re(\Omega_{uf} + \Omega_w) d\theta_2 \\ \int_0^{2\pi} \Re(\Omega_{uf} + \Omega_w) e^{-i\theta_2} d\theta_2 \\ \int_0^{2\pi} \Re(\Omega_{uf} + \Omega_w) e^{-i2\theta_2} d\theta_2 \\ \Phi_0 \end{bmatrix},$$

where $F_{i,n}(\theta_i) = \Re(\Omega_i(Z_i(\theta_i)))$ is the complex potential of the circular inhomogeneity i , n is the approximate order of series expansion, a_0 are real coefficients and all the remaining are complex, Φ_0 is the hydraulic head at reference point, z_0 is the position of the reference point, C is a constant, Ω_{uf} is the complex potential of the uniform flow and Ω_w is the complex potential of a well.

7.2 Solution Algorithm for Fractures

This section adopt the same methodology of solution algorithm as described in [section 7.1](#). The unknown coefficients of the analytical elements (fractures) can be found from eq. (5.39), that take the form:

$$a_{i,n} = P_{i,n} \int_0^\pi \Re(\Omega_T(\theta_i)) \sin(n\theta_i) d\theta_i, \quad (7.13)$$

or

$$a_{i,n} = P_{i,n} \int_0^\pi \Phi_{\neq j}(\theta_i) \sin(n\theta_i) d\theta_i, \quad (7.14)$$

which calculates the n th unknown coefficient for fracture i , where Ω_T is the combination of the complex potential of all the elements.

Equation (7.13) also gives rise to a set of linear equations that can be solved by two distinct techniques: The Iterative Method and the Matrix Method. The general convergence of these methods for analytic elements has been analysed in this work. This convergence is analysed by the use of spectral radius as presented in ([GENE; CHARLES, 1996](#)) for the Iterative Method.

7.2.1 Iterative Method

Consider a system of M fractures in an uniform flow field. The expression to calculate the unknown coefficients for the element i using eq. (5.41), eq. (7.13) becomes:

$$\begin{aligned} a_{i,n} = & P_{i,n} \sum_{j \neq i}^M \int_0^\pi \Re(W_j(\theta_i) e^{i\alpha_i}) \sin(\theta_i) \sin(n\theta_i) d\theta_i \\ & + P_{i,n} \int_0^\pi \Re((\Omega_{uf}(\theta_i)) e^{i\alpha_i}) \sin(\theta_i) \sin(n\theta_i) d\theta_i, \end{aligned} \quad (7.15)$$

or

$$\begin{aligned} a_{i,n}^{k+1} = & P_{i,n} \sum_{j \neq i}^M \int_0^\pi \Re(W_j(\theta_i) e^{i\alpha_i}) \sin(\theta_i) \sin(n\theta_i) d\theta_i \\ & + P_{i,n} \int_0^\pi \Re((\Omega_{uf}(\theta_i)) e^{i\alpha_i}) \sin(\theta_i) \sin(n\theta_i) d\theta_i, \end{aligned} \quad (7.16)$$

where k counts the steps of the iterative process and i shows a specific element. The regional uniform flow $\Omega_{uf}(\theta_i)$ is as defined in eq. (4.19) and

$$P_{i,n} = -\frac{1}{\pi} \frac{K_j^+ b_j^*}{K}, \quad \forall n. \quad (7.17)$$

Therefore,

$$\begin{aligned} a_{i,n}^{k+1} = & -P_{i,n} \sum_{j \neq i}^M \int_0^\pi \Re\left(\frac{d\Omega(Z_j)}{dZ_j} \frac{dZ_j}{dz} e^{i\alpha_i}\right) \sin(\theta_i) \sin(n\theta_i) d\theta_i \\ & + P_{i,n} \int_0^\pi \Re\left(\frac{d(\Omega_{uf}(\theta_i))}{dz} e^{i\alpha_i}\right) \sin(\theta_i) \sin(n\theta_i) d\theta_i, \end{aligned} \quad (7.18)$$

or

$$\begin{aligned}
 a_{i,n}^{k+1} = & P_{i,n} \sum_{j \neq i}^M \sum_{m=1}^N \int_0^\pi \Re \left[\left(\frac{ma_{j,m}^k (Z_j - \sqrt{Z_j - 1} \sqrt{Z_j + 1})^m}{\sqrt{Z_j - 1} \sqrt{Z_j + 1}} \right) \frac{2e^{i\alpha_i}}{z_2 - z_1} \right] \sin(\theta_i) \sin(n\theta_i) d\theta_i \\
 & - P_{i,n} \int_0^\pi \Re \left[\left(Q_{uf} e^{-i\beta_{uf}} \right) e^{i\alpha_i} \right] \sin(\theta_i) \sin(n\theta_i) d\theta_i,
 \end{aligned} \tag{7.19}$$

which is obtained by taking the derivative of eq. (5.17), for which

$$\Omega_j(Z) = \sum_{n=1}^N a_{j,n} (Z_j - \sqrt{Z_j - 1} \sqrt{Z_j + 1})^n. \tag{7.20}$$

7.2.2 Matrix Method

To calculate the unknown coefficients for the element i , after simplification eq. (7.13) can be rewritten in the form:

$$\begin{aligned}
 \frac{a_{i,n}}{P_{i,n}} - \sum_{j \neq i}^M \int_0^\pi \Re \left(\frac{d\Omega(Z_j)}{dZ_j} \frac{dZ_j}{dz} e^{i\alpha_i} \right) \sin(\theta_i) \sin(n\theta_i) d\theta_i \\
 = \int_0^\pi \Re \left(\frac{d(\Omega_{uf}(\theta_i))}{dz} e^{i\alpha_i} \right) \sin(\theta_i) \sin(n\theta_i) d\theta_i,
 \end{aligned} \tag{7.21}$$

or from eq. (7.19)

$$\begin{aligned}
 \frac{a_{i,n}}{P_{i,n}} - \sum_{j \neq i}^M \sum_{m=1}^N \int_0^\pi \Re \left[\left(\frac{ma_{j,m} (Z_j - \sqrt{Z_j - 1} \sqrt{Z_j + 1})^m}{\sqrt{Z_j - 1} \sqrt{Z_j + 1}} \right) \frac{2e^{i\alpha_i}}{z_2 - z_1} \right] \sin(\theta_i) \sin(n\theta_i) d\theta_i \\
 = \int_0^\pi \Re \left(-Q_{uf} e^{i(\alpha_i - \beta_{uf})} \right) \sin(\theta_i) \sin(n\theta_i) d\theta_i.
 \end{aligned} \tag{7.22}$$

This expression is used to calculate the unknown coefficients in the Matrix Method.

Using eq. (7.22), we would generate the matrix $\mathbf{A}$ with dimension $M * N_{elem}$, that takes the form $\mathbf{A}_{(M * N_{elem}) \times (M * N_{elem})}$, where M is the total number of analytical elements and N_{elem} is the order of the series expansion of each element. For the case of 2 analytical elements (fractures) with an approximated series expansion of order 3, the general form of the matrix $\mathbf{A}$ is:

The matrix $\mathbf{A}$ and vectors $\mathbf{x}$ and $\mathbf{b}$ for linear system $\mathbf{Ax} = \mathbf{b}$ of the problems of fractures with two elements has the form:

$$\mathbf{A} = \begin{bmatrix} \frac{1}{P_{1,1}} - \int_0^\pi \int_0^\pi F_{1,1}(\theta_1) \sin(\theta_1) \sin(\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{1,2}(\theta_1) \sin(\theta_1) \sin(\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{1,3}(\theta_1) \sin(\theta_1) \sin(\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_1) \sin(\theta_1) \sin(\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,2}(\theta_1) \sin(\theta_1) \sin(\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,3}(\theta_1) \sin(\theta_1) \sin(\theta_1) d\theta_1 \\ \int_0^\pi \int_0^\pi F_{1,1}(\theta_1) \sin(\theta_1) \sin(\theta_1) d\theta_1 & \frac{1}{P_{1,2}} - \int_0^\pi \int_0^\pi F_{1,2}(\theta_1) \sin(\theta_1) \sin(2\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{1,3}(\theta_1) \sin(\theta_1) \sin(2\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_1) \sin(\theta_1) \sin(2\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,2}(\theta_1) \sin(\theta_1) \sin(2\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,3}(\theta_1) \sin(\theta_1) \sin(2\theta_1) d\theta_1 \\ \int_0^\pi \int_0^\pi F_{1,1}(\theta_1) \sin(\theta_1) \sin(3\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{1,2}(\theta_1) \sin(\theta_1) \sin(3\theta_1) d\theta_1 & \frac{1}{P_{1,3}} - \int_0^\pi \int_0^\pi F_{1,3}(\theta_1) \sin(\theta_1) \sin(3\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_1) \sin(\theta_1) \sin(3\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,2}(\theta_1) \sin(\theta_1) \sin(3\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,3}(\theta_1) \sin(\theta_1) \sin(3\theta_1) d\theta_1 \\ \int_0^\pi \int_0^\pi F_{1,1}(\theta_2) \sin(\theta_2) \sin(\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,2}(\theta_2) \sin(\theta_2) \sin(\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,3}(\theta_2) \sin(\theta_2) \sin(\theta_2) d\theta_2 & \frac{1}{P_{2,1}} - \int_0^\pi \int_0^\pi F_{2,1}(\theta_2) \sin(\theta_2) \sin(\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,2}(\theta_2) \sin(\theta_2) \sin(\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,3}(\theta_2) \sin(\theta_2) \sin(\theta_2) d\theta_2 \\ \int_0^\pi \int_0^\pi F_{1,1}(\theta_2) \sin(\theta_2) \sin(2\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,2}(\theta_2) \sin(\theta_2) \sin(2\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,3}(\theta_2) \sin(\theta_2) \sin(2\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_2) \sin(\theta_2) \sin(2\theta_2) d\theta_2 & \frac{1}{P_{2,2}} - \int_0^\pi \int_0^\pi F_{2,2}(\theta_2) \sin(\theta_2) \sin(2\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,3}(\theta_2) \sin(\theta_2) \sin(2\theta_2) d\theta_2 \\ \int_0^\pi \int_0^\pi F_{1,1}(\theta_2) \sin(\theta_2) \sin(3\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,2}(\theta_2) \sin(\theta_2) \sin(3\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,3}(\theta_2) \sin(\theta_2) \sin(3\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_2) \sin(\theta_2) \sin(3\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,2}(\theta_2) \sin(\theta_2) \sin(3\theta_2) d\theta_2 & \frac{1}{P_{2,3}} - \int_0^\pi \int_0^\pi F_{2,3}(\theta_2) \sin(\theta_2) \sin(3\theta_2) d\theta_2 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ a_{1,3} \\ a_{2,1} \\ a_{2,2} \\ a_{2,3} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \int_0^\pi \Re((\Omega_{urf}) e^{i\alpha_1}) \sin(\theta_1) \sin(\theta_1) d\theta_1 \\ \int_0^\pi \Re((\Omega_{urf}) e^{i\alpha_1}) \sin(\theta_1) \sin(2\theta_1) d\theta_1 \\ \int_0^\pi \Re((\Omega_{urf}) e^{i\alpha_1}) \sin(\theta_1) \sin(3\theta_1) d\theta_1 \\ \int_0^\pi \Re((\Omega_{urf}) e^{i\alpha_2}) \sin(\theta_2) \sin(\theta_2) d\theta_2 \\ \int_0^\pi \Re((\Omega_{urf}) e^{i\alpha_2}) \sin(\theta_2) \sin(2\theta_2) d\theta_2 \\ \int_0^\pi \Re((\Omega_{urf}) e^{i\alpha_2}) \sin(\theta_2) \sin(3\theta_2) d\theta_2 \end{bmatrix},$$

where $F_{i,n}(\theta_i) = \Re(\Omega_i(Z_i(\theta_i)))$ is the potential of the inhomogeneity i due to the presence of fractures, n is the approximate order of series expansion, Φ_0 is the hydraulic head at reference point, z_0 is the position of the reference point, C is a constant and Ω_{urf} is the complex potential of the uniform flow.

7.2.3 Convergence Test

Consider the matrix $\mathbf{A}$ and its decomposition $\mathbf{A} = \mathbf{D} + \mathbf{U} + \mathbf{L}$, where $\mathbf{U}$ is the upper triangle matrix, $\mathbf{L}$ the lower triangle matrix, and $\mathbf{D}$ the diagonal.

1. The Jacobi matrix partition:

$$\mathbf{R} = \mathbf{D}, \quad (7.23)$$

$$\mathbf{T} = -(\mathbf{U} + \mathbf{L}). \quad (7.24)$$

2. The Gauss-Seidel (GS) matrix partition:

$$\mathbf{R} = \mathbf{D} + \mathbf{L}, \quad (7.25)$$

$$\mathbf{T} = -\mathbf{U}, \quad (7.26)$$

with the system solution obtained using the standard iterative scheme:

$$x^{k+1} = \mathbf{R}^{-1}\mathbf{T}x^k + \mathbf{R}^{-1}x^k. \quad (7.27)$$

This iterative scheme will always converge for any initial vector x^0 given that the spectral radius, defined in (GENE; CHARLES, 1996):

$$\rho(\mathbf{G}) = \max(|\lambda_i|), \quad (7.28)$$

is less than 1, where λ_i are the eigenvalues of the matrix $\mathbf{G}$, where $\mathbf{G} = \mathbf{R}^{-1}\mathbf{T}$. In this work we used the Gauss-Seidel matrix partition to check the convergence.

7.3 Solution Algorithm for Line-doublets

The solution algorithm in this section for determining the unknown coefficients of the analytic elements (line-doublets) follows the same procedure as described in previous sections. The unknown coefficients of the analytical elements can be found from eq. (6.39), that takes the form:

$$a_{i,n} = P_{i,n} \int_0^\pi \Re(\Omega_T(\theta_i)) \cos(n\theta) d\theta_i, \quad (7.29)$$

which calculate the n th unknown coefficient for line-doublet i , where Ω_T is the combination of the complex potential of all the elements.

In this case eq. (7.29) gives rise to a set of linear equation that can be solved by the same two distinct techniques:

7.3.1 Iterative Method

Consider a system of polygonal inhomogeneity having M line-doublets forming a closed polygon, an uniform flow field and a well. The expression to calculate the unknown coefficients for element i using eq. (7.29) becomes:

$$a_{i,n} = P_{i,n} \sum_{j \neq i}^M \int_0^\pi \Re(\Omega_j(Z_j(\theta_i))) \cos(n\theta_i) d\theta_i + P_{i,n} \int_0^\pi \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) \cos(n\theta_i) d\theta_i, \quad (7.30)$$

or

$$a_{i,n}^{k+1} = P_{i,n} \sum_{j \neq i}^M \int_0^\pi \Re(\Omega_j(Z_j(\theta_i))) \cos(n\theta_i) d\theta_i + P_{i,n} \int_0^\pi \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) \cos(n\theta_i) d\theta_i, \quad (7.31)$$

where k counts the steps of the iterative process and i shows a specific element. Moreover the elements Ω_{uf} and Ω_w are the same as defined in eq. (4.19) and eq. (4.20), whereas

$$P_{i,n} = \begin{cases} \frac{2}{\pi} \left(\frac{K_j^+ - K}{K + K_j^+} \right) & \text{for } n = 0 \\ \frac{4}{\pi} \left(\frac{K_j^+ - K}{K + K_j^+} \right) & \text{for } n > 0 \end{cases}. \quad (7.32)$$

7.3.2 Matrix Method

This section follows the same procedure as described in subsection 7.1.2. After simplification eq. (7.30) can be rewritten in the form:

$$\begin{aligned} \frac{a_{i,n}}{P_{i,n}} - \sum_{j \neq i}^M \int_0^\pi \Re(\Omega(Z_{j,i}(\theta_i))) \cos(n\theta_i) d\theta_i - \int_0^\pi \Re(C) \cos(n\theta_i) d\theta_i \\ = \int_0^\pi \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) \cos(n\theta_i) d\theta_i. \end{aligned} \quad (7.33)$$

Hence,

$$\begin{aligned} \frac{a_{i,n}}{P_{i,n}} - \sum_{j \neq i}^M \int_0^\pi \Re \left[\frac{1}{2\pi i} \left(\sum_{m=0}^N a_{j,m} Z^m \ln \frac{Z-1}{Z+1} + \sum_{m=1}^\infty \sum_{p=0}^{m-1} a_{j,m} Z^p \left(\frac{1^{m-p} - (-1)^{m-p}}{m-p} \right) \right) \right] \cos(n\theta_i) d\theta_i \\ - \int_0^\pi \Re(C) \cos(n\theta_i) d\theta_i = \int_0^\pi \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) \cos(n\theta_i) d\theta_i, \end{aligned} \quad (7.34)$$

or

$$\begin{aligned}
& \frac{a_{i,n}}{P_{i,n}} - \sum_{j \neq i}^M a_{j,0} \int_0^\pi \Re \left(\frac{1}{2\pi i} \ln \frac{Z-1}{Z+1} \right) \cos(n\theta_i) d\theta_i \\
& - \sum_{j \neq i}^M \sum_{m=1}^N a_{j,m} \int_0^\pi \Re \left[\frac{1}{2\pi i} Z^m \ln \frac{Z-1}{Z+1} + \sum_{p=0}^{m-1} \frac{1}{2\pi i} Z^p \left(\frac{1^{m-p} - (-1)^{m-p}}{m-p} \right) \right] \cos(n\theta_i) d\theta_i \\
& - \int_0^\pi \Re(C) \cos(n\theta_i) d\theta_i = \int_0^\pi \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) \cos(n\theta_i) d\theta_i.
\end{aligned} \tag{7.35}$$

The matrix $\mathbf{A}$ has dimension $M * N_{elem} + 1$ rows using eq. (7.35), where M is the total number of analytical elements (line-doublets), N_{elem} is the approximate order of the series expansion and the additional term is necessary for calculating the constant as described in subsection 7.1.2. The general form of the matrix is:

The matrix $\mathbf{A}$ and vectors $\mathbf{x}$ and $\mathbf{b}$ for linear system $\mathbf{Ax} = \mathbf{b}$ of the problems of polygonal inhomogeneities with two elements has the form:

$$\mathbf{A} = \begin{bmatrix} \frac{1}{P_{1,0}} - \int_0^\pi \int_0^\pi F_{1,0}(\theta_1) \cos(0) d\theta_1 & \int_0^\pi \int_0^\pi F_{1,1}(\theta_1) \cos(0) d\theta_1 & \int_0^\pi \int_0^\pi F_{1,2}(\theta_1) \cos(0) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,0}(\theta_1) \cos(0) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_1) \cos(0) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,2}(\theta_1) \cos(0) d\theta_1 & \int_0^\pi \int_0^\pi \Re(C) \cos(0) d\theta_1 \\ \int_0^\pi \int_0^\pi F_{1,0}(\theta_1) \cos(\theta_1) d\theta_1 & \frac{1}{P_{1,1}} - \int_0^\pi \int_0^\pi F_{1,1}(\theta_1) \cos(\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{1,2}(\theta_1) \cos(\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,0}(\theta_1) \cos(\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_1) \cos(\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,2}(\theta_1) \cos(\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi \Re(C) \cos(\theta_1) d\theta_1 \\ \int_0^\pi \int_0^\pi F_{1,0}(\theta_1) \cos(2\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{1,1}(\theta_1) \cos(2\theta_1) d\theta_1 & \frac{1}{P_{1,2}} - \int_0^\pi \int_0^\pi F_{1,2}(\theta_1) \cos(2\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,0}(\theta_1) \cos(2\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_1) \cos(2\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi F_{2,2}(\theta_1) \cos(2\theta_1) d\theta_1 & \int_0^\pi \int_0^\pi \Re(C) \cos(2\theta_1) d\theta_1 \\ \int_0^\pi \int_0^\pi F_{1,0}(\theta_2) \cos(0) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,1}(\theta_2) \cos(0) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,2}(\theta_2) \cos(0) d\theta_2 & \frac{1}{P_{2,0}} - \int_0^\pi \int_0^\pi F_{2,0}(\theta_2) \cos(0) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_2) \cos(0) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,2}(\theta_2) \cos(0) d\theta_2 & \int_0^\pi \int_0^\pi \Re(C) \cos(0) d\theta_2 \\ \int_0^\pi \int_0^\pi F_{1,0}(\theta_2) \cos(\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,1}(\theta_2) \cos(\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,2}(\theta_2) \cos(\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,0}(\theta_2) \cos(\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_2) \cos(\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,2}(\theta_2) \cos(\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi \Re(C) \cos(\theta_2) d\theta_2 \\ \int_0^\pi \int_0^\pi F_{1,0}(\theta_2) \cos(2\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,1}(\theta_2) \cos(2\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{1,2}(\theta_2) \cos(2\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,0}(\theta_2) \cos(2\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi F_{2,1}(\theta_2) \cos(2\theta_2) d\theta_2 & \frac{1}{P_{2,2}} - \int_0^\pi \int_0^\pi F_{2,2}(\theta_2) \cos(2\theta_2) d\theta_2 & \int_0^\pi \int_0^\pi \Re(C) \cos(2\theta_2) d\theta_2 \\ F_{1,0}(z_0) & F_{1,1}(z_0) & F_{1,2}(z_0) & F_{2,0}(z_0) & F_{2,1}(z_0) & F_{2,2}(z_0) & 1 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} a_{1,0} \\ a_{1,1} \\ a_{1,2} \\ a_{2,0} \\ a_{2,1} \\ a_{2,2} \\ C \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \int_0^\pi \int_0^\pi \Re(\Omega_{arf} + \Omega_w) \cos(0) d\theta_1 \\ \int_0^\pi \int_0^\pi \Re(\Omega_{arf} + \Omega_w) \cos(\theta_1) d\theta_1 \\ \int_0^\pi \int_0^\pi \Re(\Omega_{arf} + \Omega_w) \cos(2\theta_1) d\theta_1 \\ \int_0^\pi \int_0^\pi \Re(\Omega_{arf} + \Omega_w) \cos(0) d\theta_2 \\ \int_0^\pi \int_0^\pi \Re(\Omega_{arf} + \Omega_w) \cos(\theta_2) d\theta_2 \\ \int_0^\pi \int_0^\pi \Re(\Omega_{arf} + \Omega_w) \cos(2\theta_2) d\theta_2 \\ \Phi_0 \end{bmatrix},$$

where $F_{i,n}(\theta_i) = \Re(Z_i(Z_i(\theta_i)))$ is the complex potential of the inhomogeneity i with polygonal geometry, n is the approximate order of series expansion, Φ_0 is the hydraulic head at reference point, z_0 is the position of the reference point, C is a constant, Ω_{arf} is the complex potential of the uniform flow and Ω_w is the complex potential of a well.

RESULTS AND DISCUSSION

The results presented in this chapter were obtained by using the AEM to solve ground-water flow problems of different complexities on domain with different geometries.

We start by presenting some tables with the parameters that define the problem and will be used in the simulations. The length parameters are chosen in meters and all the simulations presented in this chapter use the following data, as shown in [Figure 18](#):

- Reference point: $z_0 = (0, 0)$
- Hydraulic reference head at reference point: $\phi_0 = 50$ m
- Background hydraulic conductivity: $K = 1$ m/day
- Well at point: $z_w = (100, 100)$

Figure 18 – Initial simulation parameters of an aquifer.

8.1 Circular Inhomogeneities

The results presented in this section are for problems composed of circular inhomogeneities in an uniform flow field with different hydraulic conductivities and radii. All the problems presented below were solved by both the Iterative and the Matrix Method.

8.1.1 Comparison with the Analytical Solution

This section presents comparison between the analytical and the numerical solution of a circular inhomogeneity in an uniform flow field. The analytical solution known from (STRACK, 1989) is shown in Figure 19 and its comparison with the numerical solution is shown in Figure 20, which is obtained with parameters given in Table 1. By comparing the analytical solution defined in section 4.4 with the numerical solution of the problem, we found out that the relative error (see Appendix D) is 9.5×10^{-17} , which is the order of machine epsilon.

Figure 19 – Analytic solution of a circular inhomogeneity in an uniform flow field with larger hydraulic conductivity than the background. Streamlines are horizontal and piezometric contours (hydraulic head contours) are in the vertical direction.

Table 1 – Hydraulic conductivity, radius and order of the series expansion for a circular inhomogeneity centred at the origin in the flow field shown in Figure 19.

Circle	K^+ m/d	Radius	Order
1	10	50	10

Figure 20 – Comparison between the analytical and the numerical solution.

(a) Streamlines

(b) Piezometric Contours

The numerical solution of the problem for a circular inhomogeneity is obtained by both the Iterative and the Matrix Method with parameters given in Table 1, which coincide, as shown in Figure 21. This indicates that the Iterative Method, as well as the matrix solution, using numerical integration to obtain the coefficients is able to obtain almost identical results.

Figure 21 – Comparison between the Iterative and the Matrix Method for a circular inhomogeneity centered at the origin, with parameters given in Table 1.

(a) Streamlines

(b) Piezometric Contours

Figure 22 shows the streamlines and piezometric contours. The horizontal lines as shown in Figure 22a are the streamlines that are continuous at the boundary and is the imaginary part of the complex potential, whereas the vertical lines as shown in Figure 22b are the discharge potentials which are discontinuous at the boundary of the inhomogeneity and is the real part of the complex potential. Figure 22b also show the variation of the hydraulic head values in the aquifer.

Figure 22 – Streamlines and values of hydraulic head obtained from the simulation of a circular inhomogeneity by both the Iterative and the Matrix Method with parameters given in [Table 1](#).

8.1.2 Simulation for a cluster of Circular Inhomogeneities

Figure 23 consists of 5 circular inhomogeneities with different radii and hydraulic conductivities in an uniform flow field. The circles parameters are given in [Table 2](#). The uniform flow intensity is equal to $Q_{uf} = 0.5$ m/day with orientation 30° along the horizontal direction.

Figure 23 – Streamlines and piezometric contours obtained from the simulation of 5 circular inhomogeneities in an uniform flow field, with parameters given in Table 2.

(a) Streamlines

(b) Piezometric contours

Table 2 – Simulation parameters for the circles in the flow field shown in Figure 23.

Circle	K^+ m/d	Radius	Order
1	0.0008	50	50
2	0.001	60	50
3	0.0005	50	50
4	0.00007	30	50
5	0.0003	20	50

Figure 23 illustrates the behaviour of the streamlines and piezometric contours. Note

that the streamlines avoid to enter the inhomogeneities, whereas the piezometric contours do. This occurs due to the lower hydraulic conductivity inside the inhomogeneities. It means that the permeability inside the inhomogeneities is lower than the medium.

The problem shown in Figure 24 consists of 5 circular inhomogeneities randomly distributed with parameters given in Table 3, and a well of discharge $Q_w = 100 \text{ m}^3/\text{day}$ located at point z_w instead of uniform flow in the domain. It shows that the well draws all of its water from everywhere (infinity) and is transferred via the branch cut back to infinity as defined in section 4.6.

Figure 24 – 5 circular inhomogeneities with parameters given in Table 3 and a well in the domain instead of uniform flow field. Streamlines goes towards the well from everywhere and is transferred via the branch cut back to infinity, whereas the piezometric contours, shown in dotted lines.

Table 3 – Hydraulic conductivity, radii and order of the series expansion for the circles shown in Figure 24.

Circle	K^+ m/d	Radius	Order
1	0.0008	50	50
2	50	60	50
3	0.0005	50	50
4	20	30	50
5	10	20	50

In order to demonstrate the capabilities of our methods, further simulations were made for a more complex problem consisting of 10 circular inhomogeneities distributed randomly close to each other in the domain, with an uniform flow field and a well. The uniform flow intensity is $Q_{uf} = 0.5 \text{ m/day}$ along the horizontal direction as shown in Figure 25, with parameters given

in Table 4, and the well has discharge $Q_w = 100 \text{ m}^3/\text{day}$ is located at point z_w as shown in Figures 25 and 26. Although these test cases are more challenging due to the proximity of inhomogeneities (4 and 7) and a well between them, the Matrix Method as well as the Iterative Method are able to solve adequately the expected flow field. The details demonstrate the ability of the methods to simulate the flow field at different scales.

Figure 25 – Streamlines and piezometric contours obtained from the simulation of 10 circular inhomogeneities in a uniform flow field and a well, with parameters given in Table 4.

Figure 26 – System consists of 10 circular inhomogeneities and a well in the domain without the uniform flow field intensity. The resulting flow field was obtained for the parameters given in Table 4.

Table 4 – Hydraulic conductivity, radii and order of the series expansion for the circles shown in Figures 25 and 26.

Circle	K^+ m/d	Radius	Order
1	10	50	50
2	0.001	20	50
3	5	50	50
4	0.0008	30	50
5	0.0005	60	50
6	20	70	50
7	50	40	50
8	0.00007	70	50
9	15	60	50
10	0.0003	50	50

All the results show the behaviour of the streamlines and piezometric contours and it is noticed that when the hydraulic conductivity inside the inhomogeneities is greater than the medium, the streamlines enter the inhomogeneities whereas the piezometric contours avoid to enter due to high permeability or vice versa. During simulations it is observed that the Iterative Method as well as the Matrix Method gives the same results in all the examined cases.

8.2 Fractures

In this section different configurations of fracture systems submitted to uniform flow are evaluated. The fractures present different hydraulic conductivities and the resulting system of equations is solved both by the Iterative and the Matrix Method.

8.2.1 Comparison with Analytical Solution

We start with the case that simulates the influence of a single fracture oriented parallel to the uniform flow. This problem has an analytical solution as proposed by (STRACK, 1982). The uniform flow is of 0.5 m /day and the background conductivity of the aquifer is 1 m/day. The elliptical fracture with an aperture of 0.01 meter has a hydraulic conductivity of 1000 m/day, which is 3 orders of magnitude larger than that of the aquifer. The order of expansion for the numerical solutions is $N = 10$.

Figure 27 – Analytic solution of a fracture centered at origin in an uniform flow along horizontal direction obtained by the matrix method. Horizontal lines are streamlines and vertical lines are hydraulic head contours.

Figure 27 shows the streamlines and the hydraulic head given by the analytic solution. Numerical solutions of the problem obtained by both the Iterative and the Matrix Methods present

the same behaviour as the analytical one, as shown in Figure 28. The relative error, calculated as the difference between analytical and numerical solution normalized by the hydraulic head at each point is shown in Figure 29. By comparing the exact and numerical solutions, we found that the maximal relative error is 7.2×10^{-05} .

Figure 28 – Comparison between the Iterative and the Matrix Method for a fracture, centered at origin in an uniform flow field along horizontal direction.

(a) Streamlines

(b) Piezometric Contours

Although the problem is symmetric the relative error is not, since the error is normalized by the hydraulic head, which reduces in the flow direction. Obviously the error is larger close to the fracture, where the influence of the expansion approximation is larger. The error is also

influenced by the fracture geometry. In the analytical solution the geometry of the fracture is elliptic, while in the numerical expansion, the fracture presents constant width. Nevertheless this comparison demonstrates that the numerical expansion seems to approximate well analytic solution.

Figure 29 – Absolute and Relative errors. Relative error calculated as the difference between analytical and numerical solution normalized by the hydraulic head at each point for a single fracture.

(a) Absolute Error

(b) Relative Error

8.2.2 Simulation for a cluster of Fractures

In order to evaluate the convergence behaviour of the numerical solutions, several system of fractures in an uniform flow were simulated with different orientations. In each case, the

background hydraulic conductivity is equal to $K = 1$ m/day and the intensity of the uniform regional flow is equal to $Q_{uf} = 0.5$ m/day. In all the cases, the Iterative Method failed to converge and the problem was solved by the Matrix Method proposed in this work.

In the first example we have 3 fractures parallel to each other and oriented at an angle of 30° to the uniform flow. The fracture parameters are given in Table 5. In this case there is no analytical solution for verification of the results. However the resulting flow field is sound and resembles the expected streamlines behaviour. The uniform flow is disturbed by the high conductive fractures leading the flow through its narrow widths.

Figure 30 – 3 parallel fractures of different hydraulic conductivities and lengths in an uniform flow field oriented with an angle of 30° to the horizontal direction. The resulting flow field was obtained by the Matrix Method with parameters given in Table 5.

Table 5 – Hydraulic conductivity, width and order of the series expansion for the fractures in the flow field shown in Figure 30.

Fracture	K^+ m/d	b(m)	Order
1	10000	0.01	10
2	5000	0.01	10
3	7000	0.01	10

In the second example we have 2 groups with 3 fractures each that are orthogonal to each other and oriented at an angle of 30° to the uniform flow. The fracture parameters are given in Table 6. Although this test case is more challenging due to the proximity (distance = 5 m) of orthogonal fractures (1 and 2), the Matrix Method is able to solve adequately the expected flow field. Again the high conductive fractures control the streamline path and hydraulic head distribution. The details demonstrate the ability of the method to simulate the flow field at different scales. Despite the short distance between orthogonal fractures (1 and 2) the piezometric contours are continuous through the fractures and the streamlines demonstrate the mass conservation.

Figure 31 – System of fractures solved by the Matrix Method in an uniform flow field, with parameters given in Table 6.

Table 6 – Hydraulic conductivities for the fractures in the flow field shown in Figure 31.

Fracture	K^+ m/d
1	7000
2	1000
3	1500
4	1500
5	2500
6	2500

Figure 32 – 10 fractures of different hydraulic conductivities and lengths in an uniform flow field. The streamlines go from left to right with orientation at 30° along horizontal direction, whereas the piezometric contours, shown in dotted lines, are perpendicular to the streamlines.

Table 7 – Hydraulic conductivity and length for the fractures in the flow field shown in Figure 32.

Fracture	K^+ m/d	Length(m)
1	10000	300
2	5000	200
3	7000	200
4	4500	200
5	2200	200
6	8000	200
7	5300	200
8	1500	100
9	1000	100
10	500	200

Figure 32 presents a more general flow field controlled by a system of 10 fractures randomly distributed in the domain. The fractures parameters are presented in Table 7. The uniform flow is oriented 30° to the horizontal and is strongly affected by the fractures. Nevertheless the resulting streamlines are continuous and describe adequately the expected flow in this domain.

8.2.3 Convergence Analysis

In the simulations presented in Figures 30 to 32, it was observed that the Iterative Method diverged. The Matrix Method, using a direct solver, managed to find a suitable solution for the flow field for a system of fractures. Following the observed behaviour a convergence analysis has been done by the criteria presented in subsection 7.2.3 in this work.

8.2.3.1 Increasing Fracture Hydraulic Conductivity

In order to evaluate the matrix eigenvalues, 3 different geometries of fracture systems in uniform flow were evaluated. These geometries were randomly distributed fractures (geometries 1 and 2) as well as parallel oriented fractures (geometry 3). In the first simulation all the fractures have the same hydraulic conductivity $K^+ = 1000$ m/day and width $b = 0.01$ m. The background aquifer conductivity is 1 m/day and the uniform flow is along the horizontal direction. In the following simulations the hydraulic conductivity of one fracture (fracture 4 in each geometry) is successively increased.

The maximal eigenvalue obtained for the matrix in each case is presented as a function of the growing factor P , calculated as the product of $P_{i,n}$ and G_c . $P_{i,n}$ is defined in eq. (7.17) and G_c is a geometric constant, obtained from the integral function of the fractures, as defined in eq. (7.22).

Figure 34 shows the maximum eigenvalue obtained for the proposed geometries and increasing hydraulic conductivity. The hydraulic conductivity increased from $K^+ = 10^2$ m/day to $K^+ = 10^6$ m/day only for one fracture. In the first case, $P = 5 \times 10^{-2}$; whereas for the last one $P = 5 \times 10^2$.

The analysis of the eigenvalues of the matrix $\mathbf{G}$, presented in subsection 7.2.3, shows that the maximum eigenvalue becomes larger than 1, even in the first case for geometry 2. This is the reason for convergence failure of the Iterative Method.

Figure 34 also shows that all the systems depend on the internal hydraulic conductivities as well as the orientations of the fractures.

Figure 33 – Each system consists of 4 fractures randomly distributed in the domain (geometries 1 and 2) or parallel oriented (geometry 3). The flow field was obtained by the Matrix Method. Horizontal lines are streamlines and the vertical lines are piezometric contours.

(a) Geometry 1

(b) Geometry 2

(c) Geometry 3

Figure 34 – Maximal matrix eigenvalues obtained with different fracture systems, as shown in Figure 33

Further analysis was done with different geometries as shown in Figure 35 by varying fracture's lengths. The curves for the maximum eigenvalues obtained for these geometries are shown in Figure 36. This shows that keeping hydraulic conductivities constant increasing fracture's lengths decrease the eigenvalues.

Figure 35 – Each system consists of 2 fractures randomly distributed in the domain (geometries 4 and 5). The flow field was obtained by the Matrix Method. Horizontal lines are streamlines and the vertical lines are piezometric contours.

Figure 36 – Eigenvalues obtained by increase in length of fractures.

8.3 Line-doublets and Polygonal Inhomogeneities

The results presented in this section were obtained for problems of groundwater flow with polygonal inhomogeneities formed by line-doublets in an uniform flow field using the Analytic

Element Method. Several problems with different parameters of hydraulic conductivities, radii, lengths and order of series expansion were simulated by both the Iterative and the Matrix Method for comparing the results due to the failure of the Iterative Method, as discussed in case of fractures. This section is further divided in two subsections:

8.3.1 Comparison with Analytical Solution of a Circle

This subsection presents a comparison of the analytical solution of a circle proposed by (STRACK, 1989), as presented in section 4.4, with the numerical solution of a circular polygonal inhomogeneity formed by 64 line-doublets in an uniform flow field. The uniform flow is of 0.5 m/day and the background conductivity of the aquifer is 1 m/day. Figure 37 shows the results for the streamlines and the hydraulic head for the parameters given in Table 8, which coincide.

Table 8 – Hydraulic conductivity, number of line-doublets, radius and order of the series expansion for circular polygonal inhomogeneity in the flow field shown in Figures 37, 38 and 40.

Polygon	K^+ m/d	Line-doublets	Order	Radius
1	100	64	7	50

Figure 37 – Numerical solution of a circular polygonal inhomogeneity formed by 64 line-doublets centered at origin in an uniform flow field with parameters given in Table 8. Horizontal lines are streamlines and vertical lines are hydraulic head contours.

Figure 38 shows comparison between the streamlines of the analytical and numerical solution of the problem. The relative error, calculated as the difference between analytical and numerical solution normalized by the hydraulic head at each point is shown in Figure 39. By comparing the analytical and numerical solutions, we found out that the relative error is 6.2×10^{-07} .

Figure 38 – Comparison between the analytical and numerical solution of a circular inhomogeneity formed by 64 line-dipoles. The circle parameters are given in [Table 8](#).

(a) Streamlines

(b) Piezometric contours

Figure 39 – Relative error calculated as the difference between analytical and numerical solution normalized by the hydraulic head at each point for a circular polygonal inhomogeneity.

(a)

(b)

Figure 40 illustrates the behaviour of the streamlines and the hydraulic head contours. Again the streamlines enter the inhomogeneity, whereas the hydraulic head contours avoid it. This is because, the permeability inside the polygon is higher than the background as observed in the case of circular inhomogeneities. Figure 40b also shows the variation of the hydraulic head values in the aquifer.

Figure 40 – Streamlines and values of hydraulic head obtained from the simulation of a circular polygonal inhomogeneity formed by 64 line-doublets for the parameters given in Table 8.

(a) Streamlines

(b) Values of hydraulic head

Figure 41 shows the comparison between the streamlines and piezometric contours of a circular polygonal inhomogeneity formed by 64 line-doublets, for the parameters given in Table 8. The problem was simulated by both the Iterative and the Matrix Method, providing the same results. This indicates that in case of line-doublets, the Iterative Method as well as the Matrix Method produced almost identical results, as observed in the case of circular inhomogeneities.

Figure 41 – Comparison between the Iterative and the Matrix Method for a circular polygonal inhomogeneity formed by 64 line-doublets centered at origin, for the parameters in Table 8.

(a) Streamlines

(b) Piezometric Contours

8.3.2 Simulation for a cluster of Polygonal Inhomogeneities

The system depicted in Figure 42 has six inhomogeneities randomly distributed around the background hydraulic conductivity: two lower than the background hydraulic conductivity and the others higher, with simulation parameters given in Table 9. The background hydraulic conductivity is considered as $K = 1$ m/day, and the regional uniform flow has the intensity $Q_{uf} = 0.5$ m/day.

Figure 42 – 6 polygonal inhomogeneities simulated for different hydraulic conductivities and radii in a uniform flow field oriented with an angle of 30° to the horizontal direction, for the parameters in Table 9.

Figure 43 – 6 polygonal inhomogeneities and a well in the domain instead of uniform flow. The resulting flow field was obtained for the parameters in Table 9.

Table 9 – Hydraulic conductivities, no. of line-doublets, radii and order of series expansion for polygonal inhomogeneities for the flow field shown in Figure 42 and Figure 43.

Polygon	K^+ m/d	Line-doublets	Order	Radius
1	10	40	7	50
2	20	5	7	30
3	30	20	7	20
4	0.01	120	7	50
5	0.005	100	7	50
6	50	12	7	60

The problem depicted in Figure 43 consists of the same polygonal inhomogeneities with parameters given in Table 9, and a well of discharge $Q_w = 100 \text{ m}^3/\text{day}$ located at point z_w in place of the uniform flow. It shows the same behaviour as observed in the case of circular inhomogeneities: the well draws water from everywhere passing through inhomogeneities and goes back to infinity through a branch cut.

8.3.3 Comparison with Analytical solution of a Fracture

This subsection presents comparison of the analytical solution of a fracture presented in (STRACK, 1982), defined by eq. (5.31), with the numerical solution of a fracture formed by 60 line-doublets in an uniform flow field. Figure 44 shows the results of the streamlines and hydraulic head. In this case, the internal hydraulic conductivity is taken as $K^+ = 100 \text{ m/day}$ and the order of series expansion as $N = 5$. By comparing the analytical and numerical solutions, we found out that the relative error is 9.0×10^{-07} . The relative error is shown in Figure 45.

Figure 44 – Numerical solution for a fracture formed by 60 line-doublets, centered at the origin in an uniform flow field along horizontal direction. Horizontal lines are streamlines and the vertical lines are the hydraulic head contours.

Figure 45 – Absolute and Relative errors. Relative error calculated as the difference between analytical and numerical solution normalized by the hydraulic head at each point for a fracture formed by line-doublets.

(a) Absolute Error

(b) Relative Error

8.3.4 Simulation for a cluster of Fractures

Figure 46 consists of 3 horizontal, parallel fractures each formed by 20 line-doublets oriented at an angle of 30° to the uniform flow. The fracture parameters are given in Table 10. The resulting flow field is sound and resembles the expected streamlines behaviour. The uniform flow is disturbed by the high conductive fractures leading the flow through its narrow widths.

Figure 46 – 3 parallel fractures (20 line-doublets each) with different hydraulic conductivities and lengths in an uniform flow field oriented with an angle of 30° to the horizontal direction.

Table 10 – Simulation parameters for the fractures formed by line-doublets shown in Figure 46.

Fracture	K^+ m/d	Order
1	100000	7
2	50000	7
3	70000	7

Figure 47 consists of 6 fractures: 2 groups with 3 fractures each are orthogonal to each other and oriented at an angle of 30° to the uniform flow. The fracture parameters are given in Table 11. Although this test case is also more challenging due to the proximity of orthogonal fractures (1 and 2), but both the Iterative and the Matrix Methods are able to describe adequately the expected flow field. In this case we found almost the same results as mentioned in subsection 8.2.2, whereas the hydraulic conductivities inside the fractures is 10 times greater than the previous ones. Again it is noticed that the high conductive fractures control the streamline path and hydraulic head distribution.

Table 11 – Simulation parameters for the fractures, each formed by 20 line-doublers shown in Figure 47.

Fracture	K^+ m/d	Order
1	70000	5
2	10000	5
3	15000	5
4	15000	5
5	25000	5
6	25000	5

Figure 47 – 6 parallel fractures (each formed by 20 line-doublers) with different hydraulic conductivities, and lengths in an uniform flow field oriented with an angle of 30° to the horizontal direction. The resulting flow field was obtained for the parameters given in Table 11.

From all the figures the behaviour of the streamlines and piezometric contours is observed and it is noticed that when the hydraulic conductivity inside the inhomogeneities is larger than the medium, the streamlines enter to the inhomogeneities whereas the piezometric contours avoid to enter due to high permeability or vice versa as observed in the case of circular inhomogeneities. Moreover, after simulating the problems using line-doublers to form different polygonal inhomogeneities and fractures, it is observed that the Iterative Method converges in all the cases and gives the same results by comparing with the Matrix Method.

8.3.5 Error Analysis

An error analysis has been done for both the polygonal inhomogeneities and fractures. It consists of different grids which are gradually refined by twice the previous ones. This analysis

was done using a single circular polygon and fracture, centered at the origin. In both cases, a comparison was made between the line-doublets and the relative error as shown in Figures 48 and 49. The legends show the refinement of the control point in the domain.

Figure 48 – Comparison between the line-doublets and relative error of a circular polygonal inhomogeneity formed by line-doublets, centred at origin.

Figure 49 – Comparison between the line-doublets and relative error of a fracture formed by line-doublets, centred at origin.

The analysis shows that the relative error decreases like a potential law with increase in number of line-doublets.

8.4 Computational Performance

Different problems such as circular inhomogeneities, polygonal inhomogeneities and fractures were solved by the AEM to compare the processing time for both the Iterative and the Matrix Method. In such cases, it is observed that the Iterative Method takes more processing time than the Matrix Method with high order of series expansion using the same systems. In our simulation both the Matrix and the Iterative algorithm codes are shared, with only the difference in the system solver, so the comparison was made using only our solver for both cases.

Some comparisons were made between the processing times of both the Iterative and the Matrix Method in this work. Figure 50 illustrate the processing time of both the Iterative and the Matrix Method for the problem consists of 10 circular inhomogeneities as shown in Figure 25, and Figure 51 illustrate the processing time of the Matrix Method (because the Iterative Method did not converge for more than one fracture) for the problem consists of 10 fractures as shown in Figure 32. After simulating the problems, it is noticed that the Matrix Method is much faster than the Iterative Method as illustrated in Figures 50 and 51. This means that the computational cost of the Matrix Method is lower. This is because the Iterative solver takes a large number of steps to satisfy the convergence criterion. The matrix solver needs to fully construct the coefficients matrix in the first step, and then need to solve the system once.

Figure 50 – Comparison between the processing times, obtained from simulation of the system of circular inhomogeneities shown in Figure 25, by both the Iterative and Matrix Method.

Figure 51 – Comparison between the processing times obtained from system of fractures shown in Figure 32, by both the Iterative and Matrix Method.

It must be noted that both the Iterative and the Matrix Method have not been optimized for performance or memory use, and several improvements in both performance and accuracy can be implemented. The library solvers were also not tuned for optimal performance, so the results shown here can still be improved by at least one order of magnitude or more.

CONCLUSION

This work proposes to simulate two-dimensional groundwater flow using the Analytic Element Method. Several problems were simulated and described in detail such as: circular inhomogeneities, polygonal inhomogeneities and fractures. All the simulations are programmed in Matlab R2014a and processed on a Core i7 Intel Processor computer.

Circular inhomogeneities were simulated using both the Iterative and the Matrix Method and almost identical results were obtained. The agreement of the numerical solution with the exact solution was excellent, with relative error smaller than machine precision. The complex potential for circular inhomogeneities as well as the iterative solution are known in the literature ([JANKOVIĆ, 1997](#); [BARNES](#); [JANKOVIĆ, 1999](#); [MARIN, 2011](#)). The implementation and results were compared to the existing literature. The results for circular inhomogeneities served as the basis for the comparison and verification of the polygonal inhomogeneities using line-doublets.

Polygonal inhomogeneities were implemented using line-doublets as analytical elements. The complex potential for the line-doublet and the implementation for the iterative algorithm was based on ([JANKOVIĆ](#); [BARNES, 1999a](#); [MARIN, 2011](#)). In this case, the numerical solution of a polygonal circular inhomogeneity was compared with the exact solution of a circular inhomogeneity, whereas a polygonal ellipsoidal inhomogeneity using line-doublets was compared with the exact solution of a single fracture as described in ([STRACK, 1982](#)). Such comparisons were found excellent with satisfactory results. All the systems resulting from line-doublets were solved by both the Iterative and the Matrix Method and almost identical results were obtained.

The complex potential for a discrete fracture system in porous medium was developed in this work, based on the exact solution for one single fracture ([STRACK, 1982](#)). This complex potential for one fracture was analysed and implemented by both the Iterative and the Matrix Method and the agreement of the numerical solution with the exact solution was found excellent.

However, during simulations it is observed that the Iterative Method did not converge for some cases, as for instance the flow field with many fractures. This fact led to the Matrix Method to solve the linear system.

The main contributions in this work are the development of the Matrix Method and the convergence analysis of the Iterative Method. The Matrix Method was developed by imposing the intern boundary conditions for the Analytic Element Method. This method, using a direct solver, managed to find the correct solutions for the system of fractures. The Matrix Method takes advantage of the results of the Cauchy's integral, using the orthogonality of the basic functions of the expansion of the analytical elements for the construction of the matrix.

Moreover, the divergence of the Iterative Method motivated the study of the behaviour of the iterative solutions to the proposed problems using Gauss-Seidel partition. The spectral radius of the Iterative system was analysed, and cases were observed where the spectral radius is larger than 1, which indicates that the convergence of the system to an initial solution vector is not guaranteed. The spectral radius being larger than 1 for the Iterative Method indicated non-convergence of some systems, especially in the case of fractures, and cannot be applied to all analytic element methods in all cases.

In general, it is common in numerical simulation and computation that there is a compromise between memory usage and processing time. The computational cost of the Iterative Method is higher in processing time and less in memory use, whereas in the Matrix Method the inverse occurs. Higher memory usage leads to lower processing costs, and vice versa. This relation was observed in the comparison of the solution of the same problem to the Iterative and the Matrix Method.

FUTURE WORK

In this thesis we developed a numerical solution for discrete models of fracture networks, as presented in [Chapter 5](#). In the series expansion, the unknown coefficients are obtained from the discharge potentials of all other elements that are related to the expansion coefficients. The procedure is applied to two-dimensional groundwater flow in fractured porous media by the Analytic Element Method (AEM) and it is observed that the resulting flow field looks correct and resembles the expected streamlines behaviour in all the examined cases. This motivates us to work with more complex problems such as crossing fractures as shown in [Figure 52](#).

In the literature we found some of the formulation for the crossing fractures as presented in ([STRACK, 1982](#)). This formulation is modelled by the complex potential of the line-doublets (line-segments). If we follow the method developed in ([STRACK, 1982](#)), we need to simulate minimum eight line-doublets to approximate two crossing fractures as shown in [Figure 52](#), which will be computationally costly as observed in this thesis. Thus, we try to generalize the approach presented in this thesis for crossing fractures using the conditions prescribed at the crossing as described in ([STRACK, 1982](#)).

Figure 52 – Two crossing fractures.

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DETERMINING THE UNKNOWN COEFFICIENTS

A.1 Orthogonality of Fourier Series

The boundary conditions for circular inhomogeneities presented in [section 4.2](#) are:

$$K\Phi_j^+(\theta) - K_j^+\Phi_j^-(\theta) = (K_j^+ - K)\Phi_{\neq j}(\theta). \quad (\text{A.1})$$

The discharge potential for a circular inhomogeneity j using eq. (4.2), assumes that:

$$\Phi_j^+(\theta) = a_0 + \Re\left(\sum_{n=1}^{\infty} a_n e^{in\theta}\right) \quad \text{and} \quad \Phi_j^-(\theta) = \Re\left(-\sum_{n=1}^{\infty} \bar{a}_n e^{-in\theta}\right), \quad (\text{A.2})$$

which is the real part of the complex potential.

Let,

$$\Phi_j^+(\theta) = a_0 + \sum_{n=1}^{\infty} \Re[(\Re(a_n) + i\Im(a_n))(\cos(n\theta) + i\sin(n\theta))]. \quad (\text{A.3})$$

$$\Phi_j^+(\theta) = a_0 + \sum_{n=1}^{\infty} \Re[(\Re(a_n)\cos(n\theta) - \Im(a_n)\sin(n\theta)) + i(\Re(a_n)\sin(n\theta) + \Im(a_n)\cos(n\theta))]. \quad (\text{A.4})$$

By considering the real part we get:

$$\Phi_j^+(\theta) = a_0 + \sum_{n=1}^{\infty} [\Re(a_n)\cos(n\theta) - \Im(a_n)\sin(n\theta)]. \quad (\text{A.5})$$

Similarly,

$$\Phi_j^-(\theta) = -\sum_{n=1}^{\infty} \Re[(\Re(a_n) - i\Im(a_n))(\cos(n\theta) - i\sin(n\theta))]. \quad (\text{A.6})$$

$$\Phi_j^-(\theta) = -\sum_{n=1}^{\infty} \Re[(\Re(a_n)\cos(n\theta) - \Im(a_n)\sin(n\theta)) - i(\Re(a_n)\sin(n\theta) + \Im(a_n)\cos(n\theta))], \quad (\text{A.7})$$

and by considering the real part we get:

$$\Phi_j^-(\theta) = - \sum_{n=1}^{\infty} [\Re(a_n) \cos(n\theta) - \Im(a_n) \sin(n\theta)]. \quad (\text{A.8})$$

Substituting eq. (A.2) into eq. (A.1) gives:

$$K \left(a_0 + \Re \left(\sum_{n=1}^{\infty} a_n e^{in\theta} \right) \right) - K_j^+ \Re \left(- \sum_{n=1}^{\infty} \bar{a}_n e^{-in\theta} \right) = (K_j^+ - K) \Phi_{\neq j}(\theta). \quad (\text{A.9})$$

Multiplying both sides by $e^{-im\theta} = \cos(m\theta) - i \sin(m\theta)$, yields:

$$K a_0 e^{-im\theta} + \left[K \Re \left(\sum_{n=1}^{\infty} a_n e^{in\theta} \right) + K_j^+ \Re \left(\sum_{n=1}^{\infty} \bar{a}_n e^{-in\theta} \right) \right] e^{-im\theta} = (K_j^+ - K) \Phi_{\neq j}(\theta) e^{-im\theta}. \quad (\text{A.10})$$

Now substituting eq. (A.5) and eq. (A.8) into eq. (A.10) produces:

$$\begin{aligned} K a_0 e^{-im\theta} + K \sum_{n=1}^{\infty} \left[\Re(a_n) \cos(n\theta) - \Im(a_n) \sin(n\theta) \right] \left[\cos(m\theta) - i \sin(m\theta) \right] \\ + K_j^+ \sum_{n=1}^{\infty} \left[\Re(a_n) \cos(n\theta) - \Im(a_n) \sin(n\theta) \right] \left[\cos(m\theta) - i \sin(m\theta) \right] \\ = (K_j^+ - K) \Phi_{\neq j}(\theta) e^{-im\theta}. \end{aligned} \quad (\text{A.11})$$

If $n = m$, then we have the following properties:

$$\int_0^{2\pi} \sin(nx) \sin(mx) dx = \int_0^{2\pi} \sin^2(nx) dx = \pi \quad (\text{A.12})$$

$$\int_0^{2\pi} \cos(nx) \cos(mx) dx = \int_0^{2\pi} \cos^2(nx) dx = \pi \quad (\text{A.13})$$

$$\int_0^{2\pi} \sin(nx) \cos(mx) dx = \frac{1}{2} \int_0^{2\pi} \sin(2nx) dx = 0 \quad (\text{A.14})$$

Integrating from 0 to 2π , if $m = n$, eq. (A.11) becomes:

$$K \pi [\Re(a_n) + i \Im(a_n)] + K_j^+ \pi [\Re(a_n) + i \Im(a_n)] = \int_0^{2\pi} (K_j^+ - K) \Phi_{\neq j}(\theta) e^{-in\theta} d\theta. \quad (\text{A.15})$$

$$K \pi a_n + K_j^+ \pi a_n = (K_j^+ - K) \int_0^{2\pi} \Phi_{\neq j}(\theta) e^{-in\theta} d\theta. \quad (\text{A.16})$$

$$\pi a_n (K + K_j^+) = (K_j^+ - K) \int_0^{2\pi} \Phi_{\neq j}(\theta) e^{-in\theta} d\theta. \quad (\text{A.17})$$

$$a_n = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{1}{\pi} \int_0^{2\pi} \Phi_{\neq j}(\theta) e^{-in\theta} d\theta. \quad (\text{A.18})$$

Similarly integrating from 0 to 2π , using eq. (A.11), we get:

$$\int_0^{2\pi} K a_0 d\theta = \int_0^{2\pi} (K_j^+ - K) \Phi_{\neq j}(\theta) d\theta. \quad (\text{A.19})$$

$$2\pi K a_0 = (K_j^+ - K) \int_0^{2\pi} \Phi_{\neq j}(\theta) d\theta. \quad (\text{A.20})$$

$$a_0 = \left(\frac{K_j^+ - K}{K} \right) \frac{1}{2\pi} \int_0^{2\pi} \Phi_{\neq j}(\theta) d\theta. \quad (\text{A.21})$$

Hence,

$$a_0 = \left(\frac{K_j^+ - K}{K} \right) \frac{1}{2\pi} \int_0^{2\pi} \Phi_{\neq j}(\theta) d\theta. \quad (\text{A.22})$$

and

$$a_n = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{1}{\pi} \int_0^{2\pi} \Phi_{\neq j}(\theta) e^{-in\theta} d\theta. \quad (\text{A.23})$$

which completes the proof, as described in [subsection 4.3.1](#).

A.2 Least Square Approach

In this section we use the same boundary conditions as mentioned in eq. (A.1) for circular inhomogeneities, that takes the form:

$$K \Phi_j^+(\theta) - K_j^+ \Phi_j^-(\theta) = (K_j^+ - K) \Phi_{\neq j}(\theta). \quad (\text{A.24})$$

Using eq. (A.5) and eq. (A.8), eq. (A.24) becomes:

$$\begin{aligned} K a_0 + \sum_{n=1}^{\infty} \left[K (\Re(a_n) \cos(n\theta) - \Im(a_n) \sin(n\theta)) + K_j^+ (\Re(a_n) \cos(n\theta) - \Im(a_n) \sin(n\theta)) \right] \\ = (K_j^+ K) \Phi_{\neq j}(\theta). \end{aligned} \quad (\text{A.25})$$

Let M_j be the control points on the boundary of the circular inhomogeneity j :

$$Z_m = e^{i\theta_m} \quad (\text{A.26})$$

where $m = 0, 1, \dots, M_j - 1$. The coefficients a_n will be determined such that they minimize the sum of the squares of the errors, where the sum is taken over the M_j control points, and the errors are defined by the difference between the left-hand side and the right-hand side of eq. (A.25), that is, solving the problem:

$$\begin{aligned} \sum_{m=0}^{M_j-1} K a_0 + \sum_{m=0}^{M_j-1} \left[\sum_{n=1}^{\infty} (K + K_j^+) [\Re(a_n) \cos(n\theta_m) - \Im(a_n) \sin(n\theta_m)] - (K_j^+ - K) \Phi_{\neq j}(\theta_m) \right]^2 \\ \rightarrow \text{MIN}(a_0, \dots, a_n). \end{aligned} \quad (\text{A.27})$$

Using eq. (A.27), we can write:

$$\sum_{m=0}^{M_j-1} K a_0 = \sum_{m=0}^{M_j-1} (K_j^+ - K) \Phi_{\neq j}(\theta_m). \quad (\text{A.28})$$

$$M_j K a_0 = (K_j^+ - K) \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m). \quad (\text{A.29})$$

$$a_0 = \left(\frac{K_j^+ - K}{K} \right) \frac{1}{M_j} \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m), \quad (\text{A.30})$$

which is the discrete version of eq. (A.22).

Taking derivative w.r.t. $\Re(a_r)$, eq. (A.27) becomes:

$$2 \sum_{m=0}^{M_j-1} \left[\sum_{n=1}^{N_j} (K + K_j^+) \left[\Re(a_n) \cos(n\theta_m) - \Im(a_n) \sin(n\theta_m) \right] - (K_j^+ - K) \Phi_{\neq j}(\theta_m) \right] (K + K_j^+) \cos(r\theta_m) = 0. \quad (\text{A.31})$$

$$\sum_{m=0}^{M_j-1} \left[\sum_{n=1}^{N_j} (K + K_j^+) \left[\Re(a_n) \cos(n\theta_m) - \Im(a_n) \sin(n\theta_m) \right] - (K_j^+ - K) \Phi_{\neq j}(\theta_m) \right] \cos(r\theta_m) = 0. \quad (\text{A.32})$$

By exchanging the summations, we get:

$$\begin{aligned} \sum_{n=1}^{N_j} (K + K_j^+) \left[\Re(a_n) \sum_{m=0}^{M_j-1} \cos(n\theta_m) \cos(r\theta_m) - \Im(a_n) \sum_{m=0}^{M_j-1} \sin(n\theta_m) \cos(r\theta_m) \right] \\ = (K_j^+ - K) \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) \cos(r\theta_m). \end{aligned} \quad (\text{A.33})$$

If $n = r$, then we have the following properties:

$$\sum_{m=0}^{M_j-1} \cos(n\theta_m) \cos(r\theta_m) = \frac{M_j}{2} \quad (\text{A.34})$$

$$\sum_{m=0}^{M_j-1} \sin(n\theta_m) \cos(r\theta_m) = 0 \quad (\text{A.35})$$

and eq. (A.33) becomes:

$$(K + K_j^+) \Re(a_n) \frac{M_j}{2} = (K_j^+ - K) \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) \cos(n\theta_m). \quad (\text{A.36})$$

Now by taking derivative w.r.t. $\Im(a_r)$, eq. (A.27) becomes:

$$\sum_{m=0}^{M_j-1} \left[\sum_{n=1}^{N_j} (K + K_j^+) \left[\Re(a_n) \cos(n\theta_m) - \Im(a_n) \sin(n\theta_m) \right] - (K_j^+ - K) \Phi_{\neq j}(\theta_m) \right] \sin(r\theta_m) = 0. \quad (\text{A.37})$$

By inverting summation, we get:

$$\begin{aligned} \sum_{n=1}^{N_j} (K + K_j^+) \left[\Re(a_n) \sum_{m=0}^{M_j-1} \cos(n\theta_m) \sin(r\theta_m) - \Im(a_n) \sum_{m=0}^{M_j-1} \sin(n\theta_m) \sin(r\theta_m) \right] \\ = (K_j^+ - K) \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) \sin(r\theta_m). \end{aligned} \quad (\text{A.38})$$

Using eq. (A.34) and eq. (A.35), eq. (A.38) becomes:

$$(K + K_j^+) \Im(a_n) \frac{M_j}{2} = -(K_j^+ - K) \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) \sin(n\theta_m). \quad (\text{A.39})$$

By adding eq. (A.36) and eq. (A.39), we get:

$$a_n = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{2}{M_j} \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) e^{-in\theta_m}, \quad (\text{A.40})$$

which is the discrete version of eq. (A.23).

Hence,

$$a_0 = \left(\frac{K_j^+ - K}{K} \right) \frac{1}{M_j} \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m). \quad (\text{A.41})$$

and

$$a_n = \left(\frac{K_j^+ - K}{K + K_j^+} \right) \frac{2}{M_j} \sum_{m=0}^{M_j-1} \Phi_{\neq j}(\theta_m) e^{-in\theta_m}, \quad (\text{A.42})$$

which completes the proof, as described in [subsection 4.3.2](#).

EXACT SOLUTION OF A CIRCLE

B.1 Exact Solution

This section presents the proof of the exact solution for a circular inhomogeneity in a field of uniform flow.

Assume that the discharge potential outside the circle j for a confined aquifer is given by:

$$\Phi_c = KH\phi - \frac{1}{2}KH^2 \quad (\text{B.1})$$

and inside the circle it is given by:

$$\Phi_1 = K_j^+ H\phi - \frac{1}{2}K_j^+ H^2. \quad (\text{B.2})$$

Similarly for an unconfined aquifer the discharge potential outside the circle j is given by:

$$\Phi_{uc} = \frac{1}{2}K\phi^2 \quad (\text{B.3})$$

and inside the circle it is given by:

$$\Phi_1 = \frac{1}{2}K_j^+ \phi^2. \quad (\text{B.4})$$

Let the total discharge potential be:

$$\Phi = Kf(\phi). \quad (\text{B.5})$$

where K is the hydraulic conductivity and $f(\phi)$ is some function of the head. The head is continuous across the boundary of the inhomogeneity and therefore the condition is given by:

$$\frac{\Phi^+}{\Phi^-} = \frac{Kf(\phi^+)}{Kf(\phi^-)} = \frac{K_j^+}{K} \quad z\bar{z} = R^2, \quad (\text{B.6})$$

where K_j^+ is the hydraulic conductivity inside the circle j , K is outside and R is the radius at the boundary. There is a jump in potential across the boundary, which we represent as λ and

$$\lambda = \Phi^+ - \Phi^- = \frac{K_j^+ - K}{K} \Phi^- \quad z\bar{z} = R^2. \quad (\text{B.7})$$

According to (STRACK, 2014), we can use the complex potential of an impermeable cylindrical object in a field of uniform flow, which is similar to the case of the circular lake/inhomogeneity to obtain the required results, that is:

$$\Omega = -Q_{x_0} \left(z + \frac{R^2}{z} \right) + C. \quad (\text{B.8})$$

Suppose that,

$$\Omega_1 = -aQ_{x_0} \left(z + \frac{R^2}{z} \right) \quad \text{if} \quad z\bar{z} \geq R^2 \quad (\text{B.9})$$

and

$$\Omega_1 = C_1 \quad \text{if} \quad z\bar{z} < R^2, \quad \Im C_1 = 0, \quad (\text{B.10})$$

where Q_{x_0} is the intensity of the uniform flow along the horizontal direction, $z = x + iy$ is the position of a point in the complex plane, R is the radius of the circle, a and C_1 are constants.

We add a second complex potential Ω_2 throughout the flow domain as described in (STRACK, 2014), i.e.

$$\Omega_2 = (a-1)Q_{x_0}z + \Phi_0 \quad \forall z, \quad (\text{B.11})$$

where Φ_0 is the discharge potential at a reference point with a known head. The potential on the outside is $\Omega_1 + \Omega_2$, therefore,

$$\Omega = \Omega_1 + \Omega_2. \quad (\text{B.12})$$

Using eq. (B.9) and eq. (B.11), eq. (B.12) takes the form:

$$\Omega = -aQ_{x_0} \left(z + \frac{R^2}{z} \right) + (a-1)Q_{x_0}z + \Phi_0 \quad \text{for} \quad z\bar{z} \geq R^2. \quad (\text{B.13})$$

$$\Omega = -Q_{x_0} \left(z + a\frac{R^2}{z} \right) + \Phi_0. \quad (\text{B.14})$$

Now using the boundary conditions, let $r = R$ and $z = Re^{i\theta}$. Suppose that,

$$\lambda_1 = \Phi_1^+ - \Phi_1^-. \quad (\text{B.15})$$

Using eq. (B.9) and eq. (B.10), eq. (B.15) takes the form:

$$\lambda_1 = C_1 + aQ_{x_0} \left(Re^{i\theta} + \frac{R^2}{R}e^{-i\theta} \right). \quad (\text{B.16})$$

or

$$\lambda_1 = C_1 + 2aQ_{x_0}R\cos\theta. \quad (\text{B.17})$$

As there is no jump in the uniform flow, therefore:

$$\lambda_2 = 0, \quad (\text{B.18})$$

and

$$\lambda = \lambda_1 + \lambda_2 = C_1 + 2aQ_{x_0}R\cos\theta. \quad (\text{B.19})$$

At the boundary using eq. (B.7), eq. (B.14) becomes:

$$\Omega^- = -Q_{x_0} \left(R e^{i\theta} + a \frac{R^2}{R} e^{-i\theta} \right) + \Phi_0. \quad (\text{B.20})$$

Let, $\Omega^- = \Phi^- + i\Psi^-$. Therefore,

$$\Phi^- = -Q_{x_0} (R \cos \theta + a R \cos \theta) \Phi_0, \quad (\text{B.21})$$

which is the real part of the complex potential.

Substituting eq. (B.21) into eq. (B.7), we get:

$$\lambda = \frac{K_j^+ - K}{K} [-Q_{x_0} R \cos \theta (1 + a) + \Phi_0]. \quad (\text{B.22})$$

$$\lambda = -\frac{K_j^+ - K}{K} Q_{x_0} R \cos \theta (1 + a) + \frac{K_j^+ - K}{K} \Phi_0. \quad (\text{B.23})$$

By comparing the values of λ from eq. (B.19) and eq. (B.23), we may write:

$$C_1 + 2a Q_{x_0} R \cos \theta = \frac{K_j^+ - K}{K} \Phi_0 - \frac{K_j^+ - K}{K} Q_{x_0} R \cos \theta (1 + a). \quad (\text{B.24})$$

From eq. (B.24), we can also write:

$$C_1 = \frac{K_j^+ - K}{K} \Phi_0, \quad (\text{B.25})$$

and

$$2a Q_{x_0} R \cos \theta = -\frac{K_j^+ - K}{K} Q_{x_0} R \cos \theta (1 + a). \quad (\text{B.26})$$

$$a = -\frac{K_j^+ - K}{K_j^+ + K}. \quad (\text{B.27})$$

Substituting value of a into eq. (B.14) yields:

$$\Omega = -Q_{x_0} \left(z - \frac{K_j^+ - K}{K_j^+ + K} \frac{R^2}{z} \right) + \Phi_0 \quad \text{for} \quad z\bar{z} \geq R^2. \quad (\text{B.28})$$

Using eq. (B.10) and eq. (B.11), eq. (B.12) takes the form:

$$\Omega = C_1 + (a - 1) Q_{x_0} z + \Phi_0. \quad (\text{B.29})$$

Since,

$$a = -\frac{K_j^+ - K}{K_j^+ + K}. \quad (\text{B.30})$$

$$a - 1 = -\frac{K_j^+ - K}{K_j^+ + K} - 1. \quad (\text{B.31})$$

$$a - 1 = \frac{-K_j^+ + K - K_j^+ - K}{K_j^+ + K}. \quad (\text{B.32})$$

$$a - 1 = \frac{2K_j^+}{K_j^+ + K}. \quad (\text{B.33})$$

Substituting the values of C_1 and $(a - 1)$ from eq. (B.25) and eq.(B.33) in eq. (B.29) we get:

$$\Omega = -\frac{2K_j^+}{K_j^+ + K} Q_{x_0} z + \frac{K_j^+}{K} \Phi_0 \quad \text{for} \quad z\bar{z} < R^2. \quad (\text{B.34})$$

Hence,

$$\Omega^+ = -Q_{x_0} z \frac{2K_j^+}{K + K_j^+} + \frac{K_j^+}{K} \Phi_0 \quad \text{for} \quad z\bar{z} < R^2, \quad (\text{B.35})$$

and

$$\Omega^- = -Q_{x_0} \left(z - \frac{K_j^+ - K}{K + K_j^+} \frac{R^2}{z} \right) + \Phi_0 \quad \text{for} \quad z\bar{z} \geq R^2. \quad (\text{B.36})$$

that completes the proof, as presented in [section 4.4](#).

THE CONSTANT FOR THE EXACT SOLUTION OF A FRACTURE

C.1 The Constant A

This section presents the calculation of the constant A , which appears in the exact solution of a fracture in an uniform flow field, see (STRACK, 1982).

Consider the arguments θ_1 and θ_2 , defined as:

$$\theta_1 = \arg(Z - 1) \quad \text{in} \quad [+ \pi, - \pi], \quad (\text{C.1})$$

and

$$\theta_2 = \arg(Z + 1) \quad \text{in} \quad [- \pi, + \pi], \quad (\text{C.2})$$

as shown in Figure 53.

To see the behaviour of the function

$$\Omega(Z) = \Phi + i\Psi = \sqrt{(Z - 1)(Z + 1)} = \sqrt{(Z - 1)}\sqrt{(Z + 1)}. \quad (\text{C.3})$$

along the real axis, assumes that:

if $-1 \leq X \leq 1$ and $Y = 0^+$, then for $\sqrt{Z - 1}$ with $Z = X + iY$, we have:

$$Z - 1 = X - 1 + iY = \rho e^{i\theta}. \quad (\text{C.4})$$

where

$$\rho = \sqrt{(X - 1)^2 + (Y)^2}. \quad (\text{C.5})$$

Therefore,

$$\sqrt{Z - 1} = \sqrt{\rho} e^{i\theta/2}. \quad (\text{C.6})$$

For $Y \rightarrow 0$, we get:

$$\sqrt{Z - 1} = i\sqrt{1 - X}. \quad (\text{C.7})$$

Figure 53 – Branch cuts for θ_1 and θ_2 . Source: [Strack \(1982\)](#).For $\sqrt{Z+1}$,

$$Z = X + iY. \quad (\text{C.8})$$

$$Z + 1 = X + 1 + iY = \rho e^{i\theta}. \quad (\text{C.9})$$

$$\sqrt{Z+1} = \sqrt{\rho} e^{i\theta/2}. \quad (\text{C.10})$$

For $Y \rightarrow 0$, we get:

$$\sqrt{Z+1} = (\sqrt{1+X}) \times 1 = \sqrt{1+X}. \quad (\text{C.11})$$

Using eq. (C.7) and eq. (C.11), eq. (C.3) becomes:

$$\Omega(Z) = (i\sqrt{1-X}) \times (\sqrt{1+X}) = i\sqrt{1-X^2}, \quad (\text{C.12})$$

which is the purely imaginary.

Similarly if $-1 \leq X \leq 1$ and $Y = 0^-$, then

$$\sqrt{Z-1} = -i\sqrt{1-X}, \quad (\text{C.13})$$

and

$$\sqrt{Z+1} = \sqrt{1+X}. \quad (\text{C.14})$$

Using eq. (C.13) and eq. (C.14), eq. (C.3) becomes:

$$\Omega(Z) = (-i\sqrt{1-X}) \times (\sqrt{1+X}) = -i\sqrt{1-X^2}, \quad (\text{C.15})$$

which is also purely imaginary.

Now if $-\infty < X \leq -1$ and $Y = 0$, then

$$\sqrt{Z-1} = i\sqrt{X-1}, \quad (\text{C.16})$$

and

$$\sqrt{Z+1} = i\sqrt{X+1}. \quad (\text{C.17})$$

Using eq. (C.16) and eq. (C.17), eq. (C.3) becomes:

$$\Omega(Z) = i\sqrt{X-1} \times i\sqrt{X+1} = -\sqrt{1-X^2}, \quad (\text{C.18})$$

which is real.

Similarly if $1 \leq X < \infty$ and $Y = 0$, then

$$\sqrt{Z-1} = \sqrt{X-1}, \quad (\text{C.19})$$

and

$$\sqrt{Z+1} = \sqrt{X+1}. \quad (\text{C.20})$$

Using eq. (C.19) and eq. (C.20), eq. (C.3) becomes:

$$\Omega(Z) = \sqrt{X-1} \times \sqrt{X+1} = \sqrt{X^2-1}, \quad (\text{C.21})$$

which is also real.

According to (STRACK, 1982), the complex potential for single fracture in an uniform flow field is given by:

$$\Omega(Z) = A\sqrt{(Z-1)(Z+1)} + AZ - \frac{1}{2}q_{x_0}Le^{i(\alpha-\beta)}Z + C. \quad (\text{C.22})$$

and the boundary conditions using the linear law is given by:

$$\Psi^+ - \Psi^- = \frac{2K^+b}{KL} \frac{\partial \Phi}{\partial X}. \quad (\text{C.23})$$

The expressions for $\frac{\partial \Phi}{\partial X}$ and $\Psi^+ - \Psi^-$ along the element using eq. (C.22) becomes:

$$\frac{\partial \Phi}{\partial X} = A - \frac{1}{2}q_{x_0}L\cos(\alpha - \beta), \quad (\text{C.24})$$

and

$$\Psi^+ - \Psi^- = -Ai\sqrt{1-X^2} - Ai\sqrt{1-X^2} = -2Ai\sqrt{1-X^2} = -2A\sqrt{1-X^2}, \quad (\text{C.25})$$

where Φ is the real part of the complex potential, Q_{x_0} is the uniform flow intensity, L is the length of the fracture, α is the angle of the fracture with respect to the x -axis, β is the angle of the uniform flow, C is a constant and must be calculated from the hydraulic head value at a reference point and A is a constant.

Substituting eq. (C.24) and eq. (C.25), eq. (C.23) takes the form:

$$-2A\sqrt{1-X^2} = \frac{2K^+b}{KL} \left(A - \frac{1}{2}q_{x_0}L\cos(\alpha - \beta) \right). \quad (\text{C.26})$$

$$-2A\sqrt{1-X^2} = \frac{2K^+b^*\sqrt{1-X^2}}{KL} \left(A - \frac{1}{2}q_{x_0}L\cos(\alpha - \beta) \right), \quad (\text{C.27})$$

where $b = b^*\sqrt{1-X^2}$ as described in [section 5.3](#).

$$-A = \frac{K^+b^*}{KL} \left(A - \frac{1}{2}q_{x_0}L\cos(\alpha - \beta) \right). \quad (\text{C.28})$$

Hence,

$$A = \frac{\frac{1}{2}K^+b^*}{KL + K^+b^*} q_{x_0}L\cos(\alpha - \beta), \quad (\text{C.29})$$

which completes the proof, as proposed in [section 5.3](#).

RELATIVE ERROR

D.1 Error Analysis

Groundwater flow simulations consists of a set of computational tools designed for the solution of the governing equations modelling the physical phenomena (MARIN, 2011; GRAND, 2003). The processing of the numerical algorithm is generally made on a computer. Computers due to their limitations have the restriction that they only work with discrete and finite representations of continuous quantities, like mathematical and physical models which may include continuous and discrete quantities. This limitation is one of the origin of the error associated to numerical models (MARIN, 2011; PRESS, 2007). Other source of the error is the truncation error, associated to the numerical model itself. The truncation error arises from the finite quantity of memory of a computer to use of finite series and require to compute discrete quantities of continuous variables (MARIN, 2011; PRESS, 2007). An example of a truncation error is the error that arises when calculating an integral, computing the function to be integrated in a discrete set of points.

There are two ways to calculate the error: the absolute error and the relative error. The absolute error is defined as the magnitude of the difference between the values of the exact and the numerical solution, and is given by:

$$Absolute_{error} = \sum abs(Exact_{solution}(x_i) - Numerical_{solution}(x_i)). \quad (D.1)$$

The relative error is defined as the absolute error divided by the magnitude of the exact value for defined points, and is given by:

$$Relative_{error} = \sum abs\left(\frac{Exact_{solution}(x_i) - Numerical_{solution}(x_i)}{Exact_{solution}(x_i)}\right). \quad (D.2)$$

Mapping of error values are obtained by evaluating the absolute and the relative error for the control points of the given domain. For each point of the domain, the absolute or relative error is calculated and the results plot by the graphs.