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Models and algorithms for high school timetabling problems

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Modelos e algoritmos para problemas de horários escolares

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Ciências de Computação e Matemática Computacional. *VERSÃO REVISADA*

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ABSTRACT

SAVINIEC, L. **Models and algorithms for high school timetabling problems**. 2018. [157](#) p. Tese (Doutorado em Ciências – Ciências de Computação e Matemática Computacional) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

High school timetabling problems consist in assigning meetings between classes and teachers, with the goal of minimizing the violation of specific soft requisites. This category of problems has been extensively studied since the 1950s, mostly via mixed-integer programming and metaheuristic techniques. However, the computation of optimal or near-optimal solutions using mixed-integer programs or metaheuristics is still a challenge for most practical problems. In this thesis, we investigate new mixed-integer programming formulations, column generation approaches and parallel metaheuristic based algorithms to compute lower bounds and solutions for high school timetabling problems. Extensive computational experiments conducted with real-world instances demonstrate that our best formulations are competitive with best-known formulations, while our parallel algorithms present superior performance than the state-of-the-art methods.

Keywords: High school timetabling problem, Mixed-integer programming, Column generation, Parallel metaheuristics.

RESUMO

SAVINIEC, L. **Modelos e algoritmos para problemas de horários escolares**. 2018. 157 p. Tese (Doutorado em Ciências – Ciências de Computação e Matemática Computacional) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

Problemas de horários escolares consistem em alocar encontros entre turmas e professores, com objetivo de minimizar violações a requisitos qualitativos específicos. Esta categoria de problemas tem sido largamente estudada desde 1950, particularmente via técnicas de programação linear inteira mista e metaheurísticas. Entretanto, a computação de soluções ótimas ou quase ótimas usando programas inteiro-mistos ou metaheurísticas ainda é um desafio na maioria dos problemas práticos. Nesta tese, nós investigamos novas formulações inteiro-mistas, decomposições por geração de colunas e algoritmos baseados em metaheurísticas paralelas para computar limites inferiores e soluções para problemas de horários escolares. Extensivos experimentos computacionais conduzidos com instâncias reais demonstram que nossas melhores formulações são competitivas com as melhores formulações existentes, enquanto nossos algoritmos paralelos são superiores em performance computacional quando comparados com métodos que são estado-da-arte.

Palavras-chave: Problema de horários escolares, Programação linear inteira mista, Geração de colunas, Metaheurísticas paralelas.

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LIST OF ABBREVIATIONS AND ACRONYMS

3-SAT	3-satisfiability problem
B&B	Branch-and-bound
BA	Bee algorithm
C-ETP	Capacitated examination timetabling problem
CB-CTP	Curriculum-based course timetabling problem
CCG	Cut and column generation
CCGH	Constructive column generation heuristic
CCS	Classes with compact schedules
CF-MCFP	Compact formulation based on a multi-commodity flow problem
CF1	Compact formulation 1
CF2	Compact formulation 2
CF2-Cuts	Formulation CF2 with the addition of Fenchel cuts
CG	Column generation
CMB	Central memory based algorithm
CP	Constraint programming
CSO	Cat swarm optimization algorithm
CT	Collaborative teaching
DIMB	Diversification-intensification memory based algorithm
DIMB-CG	DIMB algorithm with column generation
EB-CTP	Post enrollment-based course timetabling problem
EF1	Extended formulation 1
EF2	Extended formulation 2
EF3	Extended formulation 3
ETP	Examination timetabling problem
F&O	Fix-and-optimize heuristic
GA	Genetic algorithm
GHSTP	Generalized high school timetabling problem
GOAL	Heuristic solver of Fonseca <i>et al.</i> (2016)
GRASP	Greedy randomized adaptive search procedure
HH	Hyper-heuristics
HM	Homogeneous algorithm

HSTP	High school timetabling problem
HSTP-A	The high school timetabling problem introduced by Saviniec (2013)
HSTP-B	The high school timetabling problem introduced by Souza (2000)
HSTP-C	A modified version of problem HSTP-B that is included in the ITC-2011
HT	Heterogeneous algorithm
ILS	Iterated local search
ITC-2002	First International Timetabling Competition
ITC-2007	Second International Timetabling Competition
ITC-2011	Third International Timetabling Competition
KHE	Software library for high school timetabling by Jeff Kingston (KINGSTON, 2015)
LAS	Late acceptance strategy
LB	Lower bound
LP	The optimal objective value of a linear program
LS-MaxSAT	A local search algorithm with neighborhoods defined by maxSAT problems
MIP	Mixed-integer programming
MISTA	Multidisciplinary International Scheduling Conference: Theory & Applications
NCCS	No classes with compact schedules
NoC	Number of columns
NRA	No room allocation
NTPS	No teachers preassigned to classes' subjects
PATAT	International Conference on the Practice and Theory of Automated Timetabling
PB	Population-based metaheuristics
PB-PI	Parallel islands
PB-PIE	Parallel individuals evaluation
RA	Room allocation
SA	Simulated annealing
STC	Simultaneous temporary classes
TB	Trajectory-based metaheuristics
TB-MA	Move acceleration
TB-PM	Parallel moves
TB-PMS	Parallel multi-start
TL	Time limit
TPS	Teachers preassigned to classes' subjects
TQ	Torque neighborhood operator
TS	Tabu search
TSD-MIP	Two-stage decomposition of an MIP model
U-ETP	Uncapacitated examination timetabling problem

UB	Upper bound
UCTP	University course timetabling problem
VNS	Variable neighborhood search
XHSTT	XML-based format to represent high school timetabling problems
XML	Extensible markup language

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INTRODUCTION

Educational timetabling problems consist in scheduling encounters between teachers (or exams) and students. Specific situations give origin to different problems with a richness of characteristics. The scientific literature has branched this family of problems in three main categories: **university course timetabling** (LEWIS, 2008), **examination timetabling** (QU *et al.*, 2009), and **high school timetabling** (WERRA, 1985; POST *et al.*, 2012; PILLAY, 2014). Each of these categories has their set of specific constraints.

In this thesis, we address the high school timetabling problem (HSTP). Particularly, we focus on problems motivated by Brazilian high schools. A simplified decision version of the HSTP was shown to be NP-complete by polynomial reduction of the 3-SAT problem (EVEN; ITAI; SHAMIR, 1975). Instances originating from practical contexts naturally extend the requisites of this simplified problem. Extra requisites are usually associated with pedagogical preferences and social/cultural particularities. Examples of extra requisites are the need to assign double lessons (two consecutive lessons within the same day) for some classes' subjects (BIRBAS; DASKALAKI; HOUSOS, 2009; SANTOS *et al.*, 2012), avoiding idle periods for teachers (periods during which the teacher is not busy, but is busy in earlier and later periods of the same day) (AVELLA *et al.*, 2007), and the need to obtain *compact* schedules for teachers (SOUZA; MACULAN; OCHI, 2004; BIRBAS; DASKALAKI; HOUSOS, 2009).

Apart from its scientific importance, the high school timetabling is also of great significance in practice, because the school staff spend long hours to produce timetables that are usually of low quality or not acceptable. The automation of the timetabling process gives the schools more freedom to allocate their staff to other activities, and it produces timetables that offer more quality in the events scheduled for students and teachers.

Traditional compact mixed-integer programming (MIP) models (SOUZA, 2000; SANTOS *et al.*, 2012; DORNELES; ARAÚJO; BURIOL, 2014; SØRENSEN; DAHMS, 2014; KRISTIANSEN; SØRENSEN; STIDSEN, 2015) for high school timetabling problems usually

have weak linear programming relaxations, and the computation of optimal solutions for large size instances is still a challenge for actual black-box solvers. Given these limitations, many researchers have focused on combining heuristic algorithms and relaxation methods to obtain high-performance hybrid approaches and optimality bounds. On the one hand, research on heuristic algorithms (e.g., (TASSOPOULOS; BELIGIANNIS, 2012a; DORNELES; ARAÚJO; BURIOL, 2014; FONSECA *et al.*, 2016; FONSECA; SANTOS; CARRANO, 2016; SAVINIEC; CONSTANTINO, 2017; SKOULLIS; TASSOPOULOS; BELIGIANNIS, 2017)) aims at the computation of high-quality solutions with reduced computational effort. On the other hand, the research on relaxation methods (e.g., decomposition by column generation (PAPOUTSIS; VALOUXIS; HOUSOS, 2003; SANTOS *et al.*, 2012; DORNELES; ARAÚJO; BURIOL, 2017)) aims at the computation of tight lower bounds on the optimal solutions.

In this thesis, we address all the above topics. Section 1.1 enumerates our main research questions. Section 1.2 describes how we have contributed to push the boundaries of knowledge in each of these topics, and Section 1.3 presents an outline of the thesis.

1.1 Research questions

We have identified three main challenges for research in high school timetabling problems:

1. Improve compact mixed-integer programming formulations.

As mentioned before, traditional compact MIP models for high school timetabling problems are known to have weak linear relaxations, and therefore, their programs are still unable in providing good integer solutions within feasible computational time. Stronger formulations have been proposed (DORNELES; ARAÚJO; BURIOL, 2017; FONSECA *et al.*, 2017). However, they still cannot effectively solve medium and large size instances.

2. Devise relaxation methods to compute tight lower bounds.

Strong lower bounds for high school timetabling problems can be generated by solving the linear relaxation of extended MIP formulations via column generation. In this case, such bounds can be compared with solutions found by heuristic algorithms to provide information on optimality gaps. Some extended formulations proposed to HSTP have shown strong lower bounds (PAPOUTSIS; VALOUXIS; HOUSOS, 2003; SANTOS *et al.*, 2012; DORNELES; ARAÚJO; BURIOL, 2017), and recent heuristic approaches (FONSECA; SANTOS; CARRANO, 2016; SAVINIEC; CONSTANTINO, 2017; SKOULLIS; TASSOPOULOS; BELIGIANNIS, 2017) have found good quality solutions. However, optimality gaps for medium and large size instances are still not sufficiently small to conclude that state-of-the-art formulations and heuristics are effective for many practical instances.

3. Devise parallel metaheuristic strategies that use the power of actual multiprocessor machines to exploit the search space better and improve the search quality.

Few strategies employing parallelism have been proposed and tested for educational timetabling problems. For high school timetabling problems, we found only three parallel approaches ([ABRAMSON, 1991](#); [ABRAMSON; ABELA, 1992](#); [SRNDIC *et al.*, 2009](#)).

1.2 Contributions

In this thesis, we have investigated all research questions pointed above. Below, we describe our main findings on solution methods for high school timetabling problems, which are organized into five parts.

In the first contribution, we propose two MIP formulations which have a relatively small number of variables when applied to instances of practical problems. As we can explicitly generate all of their variables, such formulations will be denoted by compact models. The first formulation is based on traditional models and has a weak linear relaxation. The second formulation is based on the concept of “patterns”, in which we enumerate all possible ways of assigning meetings to teaching periods within the days. This last formulation has a better linear relaxation, allowing for the generation of new cuts via the Fenchel enumeration cutting planes technique ([BOYD, 1994a](#); [BOYD, 1994b](#)). In many instances, the additional cuts are sufficient to provide tight or even optimal lower bounds.

In the second contribution, we propose three alternative extended formulations for high school timetabling problems. As these formulations have a massive number of variables that are impracticable to generate explicitly, we resort to the column generation technique ([DESAULNIERS; DESROSIERS; SOLOMON, 2005](#)) to solve their linear relaxations. The optimal linear relaxation solution of such formulations provides a tight lower bound on the objective value of the optimal integer solutions.

In the third contribution, we propose parallel metaheuristic based algorithms. We investigate parallelization schemes in which different solution methods (agents) run concurrently in different processor threads. We analyze strategies such as:

Agents cooperation: the effect of allowing the agents to cooperate by exchanging their best solutions.

Diversification: the effect of using all agents as search intensification mechanisms or allowing at least one agent to diversify the search.

Agents diversity: all threads execute the same metaheuristic (*homogeneous*) or they execute different metaheuristics (*heterogeneous*).

In the fourth contribution, we propose a parallel metaheuristic with an embedded column generation. The method is composed of a column generation procedure, two constructive heuris-

tics and a team of cooperative metaheuristics. While the column generation procedure computes a lower bound on the optimal solution of the problem, the constructive heuristics exploit the generated columns to build feasible solutions that are further improved by the metaheuristic threads. When the execution ends, the method provides a feasible integer solution and a lower bound on its optimal value.

In the last contribution, we propose a constructive heuristic for building initial feasible solutions. Our method is an improvement of the well-known MIP-based heuristic called relax-and-fix (DILLENBERGER *et al.*, 1994). The proposed improvement is a strategy to avoid infeasible partitions, in which a proximity problem (FISCHETTI; MONACI, 2014) is solved to find an augmented partial solution, as soon as an infeasible partition is detected. This part is presented as an appendix of this thesis.

1.3 Thesis outline

The thesis is organized as follows. Chapter 2 gives an overview of educational timetabling problems, situating the high school timetabling problem among other problems of the same family. We present a detailed literature review of high school timetabling problems, classifying different problem variants and constraints, along with a discussion of the state-of-the-art techniques. The chapter closes with the definition of the problem to be studied throughout the thesis.

Chapter 3 is dedicated to mixed-integer programming techniques. We describe the two compact formulations, the generation of Fenchel cuts, and the three extended formulations which are solved by column generation. We close the chapter with an extensive set of experiments to evaluate the proposed formulations.

Chapter 4 describes our exploratory study that investigates the design of parallel metaheuristics for high school timetabling problems. In this chapter, we present an introduction to different parallelization schemes, review the existing parallel algorithms for high school timetabling problems and then, we propose several algorithm implementations based on two parallel frameworks. We close the chapter with a thorough numerical experimentation that compares our best parallel algorithm with state-of-the-art algorithms.

Chapter 5 describes the proposed parallel metaheuristic with embedded column generation. We outline the main ideas behind each part of this algorithm and provide experiments to assess its performance and the quality of the constructive heuristics.

Chapter 6 concludes the thesis with final remarks and directions for future research.

Finally, two appendices complete this thesis. Appendix A presents details on the stand-alone metaheuristics used by our parallel algorithms and their parameter setting procedures. The thesis ends with Appendix B, which describes the improved relax-and-fix heuristic.

EDUCATIONAL TIMETABLING AND PROBLEM DEFINITION

Educational timetabling problems consist in assigning meetings between teachers (or exams) and students, considering a list of different requisites. Usually, these requisites are of two types: hard (mandatory) or soft (non-mandatory). A timetable that satisfies all hard requisites is said to be feasible, and a timetable that is feasible and also has a minimum number of violated soft requisites is said to be optimal.

In many educational organizations, the timetable construction is still made by hand, without the help of any computational tool. As frequent re-optimizations are necessary during the semester or year, the timetabling process usually consumes long working days.

The automated design of educational timetables started in the late 1950s, with methods that used to simulate manual procedures. [Junginger \(1986\)](#) presents a review of the first papers on timetabling problems, showing that simple heuristics were the most common solution methods. The article also reports on early computer programs that had been developed ([GOTLIEB, 1963](#)) using exact methods. However, they were able to address only small instances of simplified problems. Since then, timetabling problems have been extensively studied. The scientific literature has branched this family of problems in three main categories ([SCHAERF, 1999b](#)): **university course timetabling problems** (UCTP), **examination timetabling problems** (ETP), and **high school timetabling problems** (HSTP).

Educational timetabling problems have also been the subject of several international conferences and competitions. While the conferences aim at connecting researchers and promote new ideas, the competitions stimulate the field by encouraging researchers to share common datasets to validate their methods. Two international biennial conferences have been dedicated to educational timetabling, namely, the *International Conference on the Practice and Theory of Automated Timetabling* (PATAT) and the *Multidisciplinary International Scheduling Conference: Theory & Applications* (MISTA). Also, the community of timetabling researchers have organized

three international competitions:

- The First International Timetabling Competition (ITC-2002): this competition was dedicated to university course timetabling problems in 2002.
- The Second International Timetabling Competition (ITC-2007): this competition (<http://www.cs.qub.ac.uk/itc2007/>) was dedicated to university course and examination timetabling problems in 2007. The competition comprised three tracks:
 - Track 1: it was dedicated to examination timetabling problems – there are 12 instances available.
 - Track 2: it was dedicated to the post enrollment-based course timetabling problem, that is a variant of the UCTP – there are 24 instances available.
 - Track 3: it was devoted to the curriculum-based course timetabling problem, that is another variant of the UCTP – there are 21 instances available.
- The Third International Timetabling Competition (ITC-2011): this competition (<https://www.utwente.nl/ctit/hstt/itc2011/welcome/>) was dedicated to high school timetabling problems in 2011 (POST *et al.*, 2016). There are 25 instances available (not including deprecated ones).

In the remainder of this chapter, we discuss each category of educational timetabling problems. In Sections 2.1 and 2.2, we present an overview of the key features of the university course and examination timetabling problems. In Section 2.3, we discuss the different variants of high school timetabling problems and review the state-of-the-art methods. We close the chapter in Section 2.4 with a definition of the problem studied in this thesis.

2.1 University course timetabling problems

The university course timetabling problem arises when the institution offers a set of courses, each course is taught several times a week, and students can enroll in their preferred courses. As some courses will have students in common, they must be scheduled to different periods, so that the students can attend all their courses (LEWIS *et al.*, 2007; GASPERO; MCCOLLUM; SCHAERF, 2007; LEWIS, 2008). Furthermore, the use of space is another issue (e.g. capacity and availability) in this class of problems. Therefore, the use of classrooms must be optimized (SCHAERF, 1999b). Given these characteristics, the problem consists in assigning periods and classrooms to a set of courses, such that conflicting courses (courses with students in common) are scheduled to different periods.

We can also distinguish between two variants of university course timetabling problems (KRISTIANSEN; STIDSEN, 2013): **curriculum-based course timetabling problems**

(CB-CTP) and **post enrollment-based course timetabling problems** (EB-CTP). In the former, conflicting courses are grouped by curriculum, while in the latter, conflicting courses are determined after students enrollment.

2.1.1 Curriculum-based course timetabling problems

In curriculum-based course timetabling problems, courses are grouped by curriculum. A curriculum is a group of courses that may have students in common. Therefore, all courses of a curriculum must be scheduled to different periods. Let us consider the following notations:

- K : a set of courses.
- G : a set of curricula.
- H : a set of weekly periods.
- l_k : the number of weekly lectures to be scheduled for course $k \in K$.
- r_h : the number of classrooms available at period $h \in H$.
- y_{kh} : binary decision variable which indicates whether course $k \in K$ is scheduled to period $h \in H$.

The basic curriculum-based course timetabling problem can be formulated as an integer linear programming model as follows (WERRA, 1985; SCHAERF, 1999b):

$$\textbf{Find } y_{kh} \quad (2.1)$$

Subject to:

$$\sum_{h \in H} y_{kh} = l_k \quad \forall k \in K \quad (2.2)$$

$$\sum_{k \in K} y_{kh} \leq r_h \quad \forall h \in H \quad (2.3)$$

$$\sum_{k \in \hat{K}} y_{kh} \leq 1 \quad \forall h \in H; \hat{K} \in G \quad (2.4)$$

$$y_{kh} \in \{0, 1\} \quad \forall k \in K; h \in H \quad (2.5)$$

In this formulation, constraints (2.2) ensure that the lectures of each course are scheduled. Constraints (2.3) avoid schedules in which there are more courses in a given period than the number of classrooms available. Lastly, constraints (2.4) prevent courses belonging to a curriculum $\hat{K} \in G$ be scheduled to the same period.

This basic formulation can be used only for finding feasible solutions. Practical problems may involve a number of additional requisites. The reader is referred to the CB-CTP utilized in

track 3 of the ITC-2007 for a more realistic problem ([GASPERO; MCCOLLUM; SCHAERF, 2007](#)).

2.1.2 Post enrollment-based course timetabling problems

In post enrollment-based course timetabling problems, there are no curricula, and conflicting courses are determined after the students have enrolled in their preferred courses. Therefore, any pair of conflicting courses must be scheduled to different periods. Let us consider the following notations:

- K : a set of courses.
- H : a set of weekly periods.
- l_k : the number of weekly lectures to be scheduled for course $k \in K$.
- r_h : the number of classrooms available at period $h \in H$.
- $C_{|K| \times |K|}$: a matrix of conflicting courses, in which c_{ij} is the number of students in common enrolled in both courses $i, j \in K$;
- y_{kh} : binary decision variable which indicates whether course $k \in K$ is scheduled to period $h \in H$.

The basic post enrollment-based course timetabling problem can be formulated as an integer non-linear programming model as follows ([PETROVIC; BURKE, 2004](#)):

$$\text{Find } y_{kh} \quad (2.6)$$

Subject to:

$$\sum_{h \in H} y_{kh} = l_k \quad \forall k \in K \quad (2.7)$$

$$\sum_{k \in K} y_{kh} \leq r_h \quad \forall h \in H \quad (2.8)$$

$$\sum_{i \in K} \sum_{j \in K} \sum_{h \in H} c_{ij} y_{ih} y_{jh} = 0 \quad (2.9)$$

$$y_{kh} \in \{0, 1\} \quad \forall k \in K; h \in H \quad (2.10)$$

This formulation is similar to formulation (2.1) - (2.5), except that constraints (2.4) are replaced by constraints (2.9), which state that every pair of conflicting courses cannot run simultaneously. The reader is referred to track 2 of the ITC-2007 for a practical EB-CTP ([LEWIS et al., 2007](#)).

2.1.3 University course timetabling requisites

Practical situations may involve a number of other requisites. [Bonutti et al. \(2012\)](#) reviewed the literature and classified the main hard (H) and soft (S) requisites that may appear in university course timetabling problems. Their classification has two groups: the *basic group* – formed by the most common requisites found when comparing a number of different problems, which constitutes the core of the UCTP, and the *optional group* – which is the group of unusual requisites. We summarize this classification in Table 1.

Table 1 – Classification of university course timetabling requisites.

Requisite	Description	Type
Basic requisites		
Lectures	The lectures of all courses must be scheduled.	H
Conflicts	Courses that have the same teacher or students in common must be scheduled to different periods.	H
RoomOccupancy	Each classroom must host only one course per period.	H
RoomCapacity	Classrooms must not be overloaded.	H
Availability	Do not assign courses to periods in which their teachers are unavailable.	H
Optional requisites		
RoomStability	All lectures of a course should be scheduled to the same classroom.	S
MinimumWorkingDays	The lectures of each course should be spread over a minimum number of days.	S
ConsecutiveLectures	Courses of the same curriculum should be scheduled to consecutive periods.	S
DoubleLessons	Some courses may require some double lessons (two consecutive lessons within the same day).	S
StudentMinMaxLoad	For each curriculum the number of daily lectures should be within a given range (min, max).	S
RoomSuitability	Course should be scheduled to classrooms that contain required equipments (e.g. projectors, computers, etc).	S
TravelDistance	Students should have enough time to move from one building to another between two lectures.	S

The requisite “RoomCapacity” can also be soft, as in [Gaspero, McCollum and Schaerf \(2007\)](#).

2.2 Examination timetabling problems

The examination timetabling problem consists in assigning periods and classrooms to a set of exams so that students can attend all their required exams without clashes. [Abdullah \(2006\)](#) distinguishes between two variants of examination timetabling problems: **capacitated examination timetabling problems** (C-ETP), when classroom capacity is considered, and **uncapacitated examination timetabling problems** (U-ETP), when classroom capacity is not considered.

The examination timetabling problem can be formulated ([SCHAERF, 1999b](#)) similarly as made for the university course timetabling problem in the previous section. Let us consider the following notations:

- E : a set of exams that must be attended by students.
- H : a set of periods (e.g. a planning horizon distributed over one or more weeks).
- l_h : the maximum number of exams that can be scheduled at period $h \in H$.
- G : a set of conflicting exam groups $\hat{E}_1, \hat{E}_2, \dots$. In a conflicting exam group, all exams must be scheduled to different periods because they have students in common.
- y_{eh} : binary decision variable which indicates whether exam $e \in E$ is scheduled to period $h \in H$.

The basic examination timetabling problem can be formulated as an integer linear programming model as follows (SCHAERF, 1999b):

$$\text{Find } y_{eh} \quad (2.11)$$

Subject to:

$$\sum_{h \in H} y_{eh} = 1 \quad \forall e \in E \quad (2.12)$$

$$\sum_{e \in E} y_{eh} \leq l_h \quad \forall h \in H \quad (2.13)$$

$$\sum_{e \in \hat{E}} y_{eh} \leq 1 \quad \forall h \in H; \hat{E} \in G \quad (2.14)$$

$$y_{eh} \in \{0, 1\} \quad \forall e \in E; h \in H \quad (2.15)$$

In this formulation, constraints (2.12) ensure that all exams are scheduled. Constraints (2.13) ensure that no more than l_h exams are scheduled for each period h . These last constraints are useful if it is required to place only one exam per classroom, and in each period h there are only l_h classrooms available. Lastly, constraints (2.14) prevent the exams belonging to the same conflicting group be scheduled to the same period.

Although the basic formulation of the ETP is similar to the basic formulation of the CB-CTP presented in Section 2.1.1, practical problems usually include a number of other constraints that make the two problems different from each other. The reader is referred to the ETP used in track 1 of the ITC-2007 for a more detailed problem (MCCOLLUM *et al.*, 2007).

2.2.1 Examination timetabling requisites

In this section, we list the most common examination timetabling requisites found in the literature. The list is shown in Table 2. This list is mainly based on Burke *et al.* (1996) and Gaspero and Schaerf (2001). Burke *et al.* (1996) surveyed the most common requisites considered in British universities through a questionnaire sent to 95 universities. However, they

do not specify the type of each requisite, and we describe them as being hard or soft in Table 2. Gaspero and Schaerf (2001) also describe some important hard requisites in their case study, as shown in Table 2.

Table 2 – List of examination timetabling requisites.

Requisite	Description	Type
<i>Burke et al. (1996)</i>		
RoomCapacity	Classrooms must not be overloaded.	H/S
SimilarExams	Exams with questions in common must be scheduled to the same period.	H/S
Preference	Some exams may only be scheduled within a particular set of periods.	H/S
Preference	Only exams of the same duration should be placed into the same room.	H/S
Preference	Exams that have the greatest number of students should be scheduled to early periods.	H/S
RoomPreference	Some exams must be placed only into particular classrooms.	H/S
RoomPreference	Large exam halls must be scheduled in preference to smaller ones.	H/S
ExamPrecedence	Some exams must be scheduled before others.	H/S
ScatteredExams	No student is scheduled to attend exams in two consecutive periods.	H/S
ExamDailyLimit	No student is scheduled to more than one exam per day.	H/S
ScatteredExams	Students should have their exams scattered as much as possible over the planning horizon.	H/S
ScatteredExams	No student is scheduled to attend exams on two consecutive days.	H/S
Location	Exams must be scheduled to classrooms near their department.	H/S
<i>Gaspero and Schaerf (2001)</i>		
Exams	All exams must be scheduled.	H
Conflicts	Exams that have students in common must be scheduled to different periods.	H
Capacity	The number of students scheduled to attend exams at a given period must not exceed the total number of available seats.	H
Preassignments	Some exams may have to be necessarily scheduled to specific periods.	H
Availability	Some exams may not be scheduled to some predefined periods.	H

2.3 High school timetabling problems

In high school timetabling problems, students are usually grouped by class. The classes are disjoint groups of students enrolled in the same list of subjects. Each subject has a number of weekly lessons that are usually taught by a preassigned teacher. Given these characteristics, the problem consists in assigning weekly periods to a set of class/teacher meetings, such that no class or teacher is required to attend more than one meeting per period.

The HSTP has several variants. Related to the use of classrooms, we can distinguish between:

- **Problems without classroom allocation (NRA):** in this variant, each class has its fixed classroom and therefore, there is no need for considering classroom allocations (SOUSA; MACULAN; OCHI, 2004; VALOUXIS; HOUSOS, 2003; SOUSA; MORETTI; PODESTÁ, 2008).
- **Problems with classroom allocation (RA):** in this variant, the schools do not have enough space to assign a fixed classroom for each class. Therefore, classrooms must be properly

allocated during the timetable construction (JACOBSEN; BORTFELDT; GEHRING, 2006; WILKE; OSTLER, 2008; POST; AHMADI; GEERTSEMA, 2012).

Other two variants are related to the assignment of teachers to classes' subjects.

- **Problems without teachers preassigned to classes' subjects (NTPS):** in this variant, the assignment of teachers to each class' subject is made by the scheduling system during the timetable construction (ALVAREZ-VALDÉS; PARREÑO; TAMARIT, 2002).
- **Problems with teachers preassigned to classes' subjects (TPS):** in this variant, the assignment of teachers to each class' subject is previously made by the school staff (e.g. using trivial rules, such as priority due to seniority). Therefore, the scheduling system only needs to assign periods (or both, periods and classrooms) to meetings (SOUZA; MACULAN; OCHI, 2004; SOUSA; MORETTI; PODESTÁ, 2008).

We can also distinguish between:

- **Problems in which classes do not have compact schedules (NCCS):** the classes have free periods during the week, and usually, these free periods should be scheduled to the last periods of the days (JACOBSEN; BORTFELDT; GEHRING, 2006; AVELLA *et al.*, 2007).
- **Problems in which classes have compact schedules (CCS):** the classes are busy in all periods during the week (SOUZA; MACULAN; OCHI, 2004; SOUSA; MORETTI; PODESTÁ, 2008).

Moreover, some variants may also involve the creation of temporary classes and collaborative teaching (BELIGIANNIS *et al.*, 2008; BIRBAS; DASKALAKI; HOUSOS, 2009; TASSOPOULOS; BELIGIANNIS, 2012a).

- **Simultaneous temporary classes (STC):** during some periods, the students of one or more classes are re-organized into temporary classes, that must be simultaneously taught by different teachers. We may have two cases:
 - **Class split:** a class c may have a subject (e.g. computers, physics or technology) that requires lab work. If the lab room cannot fit all students, then two sub-classes c_i and c_j are formed. While the sub-class c_i attends the mentioned subject with a teacher t_i , the other sub-class c_j attends another subject with a teacher t_j , as illustrated in Figure 1.

Figure 1 – Class split.

- **Class split with reshuffle:** some subjects, such as foreign languages, may have two levels, beginners and advanced. For example, supposing that there is only one teacher of English (say t_h), then two classes c_i and c_j might be split into sub-classes c_{i1} , c_{i2} , c_{j1} and c_{j2} , so that the students in sub-classes c_{i1} and c_{j1} are joined together to form a new class (say c_h) to attend the beginners level, while the remaining students form another class (say c_k) and attend another subject (e.g. physical education), as illustrated in Figure 2.

Figure 2 – Class split with reshuffle.

- **Collaborative teaching (CT):** some subjects may require lab lectures with two teachers working together to teach a large group of students, as illustrated in Figure 3.

Figure 3 – Collaborative teaching.

Let us consider the following notations:

- C : a set of classes.
- T : a set of teachers.
- D : a set of days (usually one week).
- H : a set of teaching periods per day.
- r_{ct} : the number of weekly lectures that class $c \in C$ must attend with teacher $t \in T$.

- x_{ctdh} : binary decision variable which indicates whether class $c \in C$ and teacher $t \in T$ meet at period $h \in H$ of day $d \in D$.

The basic high school timetabling problem can be formulated as an integer linear programming model as follows (WERRA, 1971; SCHAERF, 1999b):

$$\textbf{Find } x_{ctdh} \quad (2.16)$$

Subject to:

$$\sum_{d \in D} \sum_{h \in H} x_{ctdh} = r_{ct} \quad \forall c \in C; t \in T \quad (2.17)$$

$$\sum_{t \in T} x_{ctdh} \leq 1 \quad \forall c \in C; d \in D; h \in H \quad (2.18)$$

$$\sum_{c \in C} x_{ctdh} \leq 1 \quad \forall t \in T; d \in D; h \in H \quad (2.19)$$

$$x_{ctdh} \in \{0, 1\} \quad \forall c \in C; t \in T; d \in D; h \in H \quad (2.20)$$

In this formulation, constraints (2.17) ensure that all required lectures are scheduled, and constraints (2.18) and (2.19) impose that no class or teacher is required to attend more than one meeting per period. This fundamental problem can be solved in polynomial time by a min-cost network flow algorithm (WERRA, 1971).

Werra (1985) describes a formulation to deal with pre-assignment of lectures and unavailability of teachers and classes that is NP-Complete (EVEN; ITAI; SHAMIR, 1975). Let us consider the following parameters:

- f_{ctdh} : binary constant that indicates whether class c is pre-assigned to meet teacher t at period h of day d .
- m_{cdh} : binary constant that indicates whether class c is available at period h of day d .
- q_{tdh} : binary constant that indicates whether teacher t is available at period h of day d .

The problem with pre-assignments and unavailabilities can be formulated as:

$$\textbf{Find } x_{ctdh} \quad (2.21)$$

Subject to:

$$\sum_{d \in D} \sum_{h \in H} x_{ctdh} = r_{ct} \quad \forall c \in C; t \in T \quad (2.22)$$

$$\sum_{t \in T} x_{ctdh} \leq m_{cdh} \quad \forall c \in C; d \in D; h \in H \quad (2.23)$$

$$\sum_{c \in C} x_{ctdh} \leq q_{tdh} \quad \forall t \in T; d \in D; h \in H \quad (2.24)$$

$$x_{ctdh} \geq f_{ctdh} \quad \forall c \in C; t \in T; d \in D; h \in H \quad (2.25)$$

$$x_{ctdh} \in \{0, 1\} \quad \forall c \in C; t \in T; d \in D; h \in H \quad (2.26)$$

In this formulation, constraints (2.18) and (2.19) of the previous formulation are replaced by constraints (2.23) and (2.24) to ensure that classes and teachers are only scheduled when they are available. Also, pre-assigned lectures can be managed by considering the additional constraints (2.25).

The two previous formulations are very limited in practice. Practical problems usually include a number of additional requisites that cannot be completely satisfied, because they conflict with each other. As an alternative, practical formulations seek for minimizing the number of times that conflicting requisites are violated. One of the first formulations in this direction is due to Junginger (1986). The author considers the minimization of lectures occurring in undesirable periods. The formulation is as follows:

$$\textbf{Minimize } \sum_{c \in C} \sum_{t \in T} \sum_{d \in D} \sum_{h \in H} w_{ctdh} x_{ctdh} \quad (2.27)$$

Subject to:

$$\sum_{d \in D} \sum_{h \in H} x_{ctdh} = r_{ct} \quad \forall c \in C; t \in T \quad (2.28)$$

$$\sum_{t \in T} x_{ctdh} \leq m_{cdh} \quad \forall c \in C; d \in D; h \in H \quad (2.29)$$

$$\sum_{c \in C} x_{ctdh} \leq q_{tdh} \quad \forall t \in T; d \in D; h \in H \quad (2.30)$$

$$x_{ctdh} \in \{0, 1\} \quad \forall c \in C; t \in T; d \in D; h \in H \quad (2.31)$$

In this formulation, w_{ctdh} is a parameter that expresses the preference for assigning class c to meet teacher t at period h of day d . The greater the value of w_{ctdh} , the smaller the preference for this assignment.

2.3.1 High school timetabling requisites

Apart from the basic formulations described above, practical high school timetabling problems usually involve several other specific requisites. In Table 3, we classify the most common high school timetabling requisites found in the literature. An alternative classification is also available in Pillay (2014).

Table 3 – List of high school timetabling requisites.

Requisite	Description	Type
Lectures	All lectures must be scheduled (SCHAERF, 1999b).	H
Conflicts	No class or teacher is required to attend more than one meeting per period (SCHAERF, 1999b).	H
Availability	No class or teacher is assigned to periods that they are unavailable (WERRA, 1985).	H
RoomOccupancy	Each classroom must host only one lecture per period (ABRAMSON, 1991).	H
Availability	Classrooms cannot be used in periods that they are unavailable (JACOBSEN; BORTFELDT; GEHRING, 2006; POST; AHMADI; GEERTSEMA, 2012).	H
RoomCapacity	Classrooms must not be overloaded (WILKE; OSTLER, 2008).	H
TemporaryClasses	Class splits and reshuffles must be satisfied, see Section 2.3.	H
CollaborativeTeaching	Auxiliary teachers must be assigned to lectures that require collaborative teaching, see Section 2.3.	H
WorkloadLimit	Classes should not attend more than a given number of lectures of the same subject per day (SOUZA; MACULAN; OCHI, 2004; AVELLA <i>et al.</i> , 2007).	H/S
WorkloadLimit	Classes that do not have compact schedules should have their daily workloads ranging between a (min, max) number of lectures (AVELLA <i>et al.</i> , 2007; MARTE, 2007).	H/S
WorkloadLimit	Teachers should have their daily workloads ranging between a (min, max) number of lectures (SCHAERF, 1999a; AVELLA <i>et al.</i> , 2007).	S
Preference	Some assignments are not allowed. For example, some classes' subjects must not be scheduled to some periods (AVELLA <i>et al.</i> , 2007).	H
Preference	Classes must have consecutive lectures of the same subject placed into the same classroom (STEFANO; TETTAMANZI, 2001).	H
Preference	Some class' subjects may require a minimum number of consecutive double lessons (SOUZA; MACULAN; OCHI, 2004; SOUSA; MORETTI; PODESTÁ, 2008).	S
Preference	Teachers may or may not prefer specific periods to work (SCHAERF, 1999a; FILHO; LORENA, 2001; JUNGINGER, 1986).	S
Preference	Teachers may or may not prefer specific classrooms to work (MELICIO; CALDEIRA; ROSA, 2004).	S
Preference	Some subjects should precede or succeed other subjects (STEFANO; TETTAMANZI, 2001; NURMI; KYNGÄS, 2007).	H/S
Compactness	Classes that do not have compact schedules should have as few idle periods as possible (JACOBSEN; BORTFELDT; GEHRING, 2006; ZHANG <i>et al.</i> , 2010).	H/S

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Table3 – continued from previous page

Requisite	Description	Type
Compactness	Classes that do not have compact schedules should have their free periods at the end of the days (AVELLA <i>et al.</i> , 2007; TASSOPOULOS; BELIGIANNIS, 2012a).	H/S
Compactness	Each class' subject should have consecutive lectures when it is scheduled more than once to the same day (SCHAERF, 1999a; AVELLA <i>et al.</i> , 2007).	S
Scattering	Each class' subject should not have consecutive lectures within the same day (BELIGIANNIS <i>et al.</i> , 2008; KOCHETOV; KONONOVA; PASCHENKO, 2008).	S
Scattering	Each class' subject should have their lectures uniformly distributed during the week (CALDEIRA; ROSA, 1997; TASSOPOULOS; BELIGIANNIS, 2012b).	S
Compactness	Teachers' schedules should have as few idle periods as possible (SOUZA; MACULAN; OCHI, 2004; ZHANG <i>et al.</i> , 2010).	S
Scattering	The total number of idle periods should be uniformly distributed among all teachers, while individual teachers should have their idle periods scattered during the week (BELIGIANNIS <i>et al.</i> , 2008; TASSOPOULOS; BELIGIANNIS, 2012a).	S
Scattering	The teachers' weekly workload should be uniformly distributed among the days (BELIGIANNIS <i>et al.</i> , 2008).	H/S
Compactness	The teachers' weekly workload should be distributed in a minimum number of days (SOUZA; MACULAN; OCHI, 2004; HAAN <i>et al.</i> , 2007).	H/S
Balancing	Extra working days should be balanced among teachers (SAVINIEC, 2013).	S
TravelTime	Teachers should have enough time to travel from one building to another between to consecutive lectures (SCHAERF, 1999a).	S

2.3.2 State-of-the-art techniques

Since the first studies in the late 1950s, as described by Junginger (1986), the scientific literature has reported numerous papers dealing with solution techniques for high school timetabling problems. In this section, we review those that are state-of-the-art.

Up to the 2000s, the high school timetabling research had been mostly based on independent case studies, making it difficult to compare different solution methods. According to our literature review, the first high school timetabling dataset available appeared in the early 2000s. Souza (2000) introduced a small set of instances from Brazilian high schools¹, which became a commonly used benchmark, being addressed later by both heuristics (SOUZA; MACULAN; OCHI, 2004; SANTOS; OCHI; SOUZA, 2005; SAVINIEC *et al.*, 2013; DORNELES; ARAÚJO; BURIOL, 2014; SAVINIEC; CONSTANTINO, 2017) and mixed-integer programming techniques (SANTOS *et al.*, 2012; DORNELES; ARAÚJO; BURIOL, 2014; DORNELES; ARAÚJO;

¹ Available in <<http://labic.ic.uff.br/Instance/>>

BURIOL, 2017). This dataset was extended with new real cases² proposed by Saviniec and Constantino (2017). The state-of-the-art heuristic methods for this problem are fix-and-optimize heuristics (DORNELES; ARAÚJO; BURIOL, 2014) and iterated local search metaheuristics (SAVINIEC *et al.*, 2013; SAVINIEC; CONSTANTINO, 2017).

A second high school timetabling dataset appeared in the late 2000s. Beligiannis *et al.* (2008) introduced a set of real-case instances from Greek high schools³ that also became the subject of several studies (BELIGIANNIS; MOSCHOPOULOS; LIKOTHANASSIS, 2009; ZHANG *et al.*, 2010; TASSOPOULOS; BELIGIANNIS, 2012a; SKOULLIS; TASSOPOULOS; BELIGIANNIS, 2017). The state-of-the-art heuristic method reported for the Greek problem is a cat swarm optimization based algorithm (SKOULLIS; TASSOPOULOS; BELIGIANNIS, 2017).

During the PATAT conference of 2008, the timetabling community proposed an XML-based format to represent different high school timetabling problems and create a unified set of benchmark instances (POST *et al.*, 2012). This format, called XHSTT (POST *et al.*, 2014), established rules to model the most common high school timetabling requisites found around the world. In other words, the XHSTT is a format to represent a generalized high school timetabling problem (GHSTP). After its creation, new instances from different countries have been added to the XHSTT repository⁴, including those proposed by Souza (2000). Such instances were used in the ITC-2011. The state-of-the-art method for these instances is the heuristic solver GOAL (FONSECA *et al.*, 2016), which won the ITC-2011. An updated version of GOAL is also described in Fonseca, Santos and Carrano (2016), which employs metaheuristic and matheuristic based algorithms.

Regarding the use of mixed-integer programming techniques, different compact formulations have been proposed for high school timetabling problems (SOUZA, 2000; SANTOS *et al.*, 2012; DORNELES; ARAÚJO; BURIOL, 2014; SØRENSEN; DAHMS, 2014; AL-YAKOOB; SHERALI, 2015; KRISTIANSEN; SØRENSEN; STIDSEN, 2015). However, all of these formulations have shown weak linear relaxations. The strongest formulation for high school timetabling is based on a multi-commodity flow problem. The formulation was first proposed for problems of Brazilian high schools (DORNELES; ARAÚJO; BURIOL, 2017) and adapted later to manage the GHSTP (FONSECA *et al.*, 2017). Extended formulations solved by column generation have also been proposed for high school timetabling problems. Santos *et al.* (2012) and Dorneles, Araújo and Buriol (2017) proposed different formulations for the Brazilian case, which showed similar results regarding the quality of their lower bounds. Another column generation was also proposed in Al-Yakoob and Sherahli (2015), for problems of Kuwaiti high schools.

As mentioned before, the literature has reported many other case studies dealing with

² Available in <<http://www.gpea.uem.br/benchmark.html>>

³ Available in <http://www.deapt.upatras.gr/cso_timetabling/school-timetabling.html>

⁴ Available in <<https://www.utwente.nl/ctit/hstt/>>

solution techniques for high school timetabling problems. A detailed review of these methods is available in Pillay (2014). In Table 4, we sample the most recent and relevant papers by chronological order. For each paper, we identify the problem characteristics, as discussed at the beginning of Section 2.3, and the type of technique employed (see the list of abbreviations for complete names). Table 4 shows that solution techniques for high school timetabling problems have been mostly based on genetic algorithms (GA), simulated annealing (SA), tabu search (TS) and mixed-integer programming models (MIP).

Table 4 – Solution techniques for high school timetabling.

Paper	Problem characteristics	Solution techniques
(SOUZA, 2000)	NRA, TPS, CCS	MIP, TS, GRASP, SA
(FILHO; LORENA, 2001)	NRA, TPS, CCS	GA
(STEFANO; TETTAMANZI, 2001)	RA, TPS, NCCS, CT	GA
(VALOUXIS; HOUSOS, 2003)	NRA, TPS, NCCS	CP
(SOUZA; MACULAN; OCHI, 2004)	NRA, TPS, CCS	GRASP, TS
(SANTOS; OCHI; SOUZA, 2005)	NRA, TPS, CCS	TS
(JACOBSEN; BORTFELDT; GEHRING, 2006)	RA, TPS, NCCS	TS
(MARTE, 2007)	RA, TPS	CP
(AVELLA <i>et al.</i> , 2007)	NRA, TPS, NCCS	MIP, SA
(SOUSA; MORETTI; PODESTÁ, 2008)	NRA, TPS, CCS	MIP, TS
(RAGHAVJEE; PILLAY, 2008)	RA, TPS	GA
(BELIGIANNIS <i>et al.</i> , 2008)	NRA, TPS, NCCS, STC, CT	GA
(CERDEIRA-PENA <i>et al.</i> , 2008)	NRA, TPS, STC	GA
(BELLO; RANGEL; BOERES, 2008)	NRA, TPS, CCS	TS
(WILKE; OSTLER, 2008)	RA	TS, SA, GA, B&B
(LARA; FLORES; CALDERÓN, 2008)	NRA, TPS, NCCS	BA
(KOCHETOV; KONONOVA; PASCHENKO, 2008)	NRA, TPS	VNS, TS
(BIRBAS; DASKALAKI; HOUSOS, 2009)	NRA, TPS, NCCS, STC, CT	MIP
(RAGHAVJEE; PILLAY, 2009)	NRA, TPS, NCCS, STC, CT	GA
(PILLAY, 2010)	RA, TPS	HH, GA
(RAGHAVJEE; PILLAY, 2010a)	NRA, TPS, STC	GA
(ZHANG <i>et al.</i> , 2010)	NRA, NTPS or TPS, NCCS, STC, CT	SA
(RAGHAVJEE; PILLAY, 2010b)	NRA, TPS, NCCS, STC	GA
(SANTOS <i>et al.</i> , 2012)	NRA, TPS, CCS	CCG
(DORNELES; ARAÚJO; BURIOL, 2012)	NRA, TPS, CCS	MIP
(BRITO <i>et al.</i> , 2012)	GHSTP	SA, VNS
(SAVINIEC <i>et al.</i> , 2013)	NRA, TPS, CCS	ILS
(DORNELES; ARAÚJO; BURIOL, 2014)	NRA, TPS, CCS	MIP, F&O
(FONSECA; SANTOS, 2014)	GHSTP	VNS
(SØRENSEN; STIDSEN, 2014)	GHSTP	F&O
(SØRENSEN; DAHMS, 2014)	RA	TSD-MIP

Continue on next page

Table4 – continued from previous page

Paper	Problem characteristics	Solution techniques
(KRISTIANSEN; SØRENSEN; STIDSEN, 2015)	GHSTP	MIP
(AHMED; ÖZCAN; KHEIRI, 2015)	GHSTP	HH
(AL-YAKOOB; SHERALI, 2015)	NRA, NTPS, NCCS	MIP, CG
(FONSECA <i>et al.</i> , 2016)	GHSTP	SA, ILS
(FONSECA; SANTOS; CARRANO, 2016)	GHSTP	SA, LAS
(FONSECA; SANTOS; CARRANO, 2016)	GHSTP	VNS, F&O
(DORNELES; ARAÚJO; BURIOL, 2017)	NRA, TPS, CCS	MIP, CG
(DEMIROVIĆ; MUSLIU, 2017)	GHSTP	LS-MaxSAT
(FONSECA <i>et al.</i> , 2017)	GHSTP	MIP
(SKOULLIS; TASSOPOULOS; BELIGIANNIS, 2017)	NRA, TPS, NCCS, STC, CT	CSO
(SAVINIEC; CONSTANTINO, 2017)	NRA, TPS, CCS	ILS, VNS

2.4 The problem under study

We focus on real timetabling problems motivated by rules of Brazilian high schools. As mentioned before, the first Brazilian HSTP reported in the scientific literature was introduced by Souza (2000). The author proposed a compact MIP formulation and instances that represent the main characteristics of high school timetabling problems arising in Brazilian schools. His work showed that, at that time, even small size instances were a challenge for MIP solvers. Since then, the introduced instances became a benchmark for tests, and several other studies addressed the problem.

Souza, Maculan and Ochi (2004) proposed the first metaheuristic approach to deal with the problem introduced by Souza (2000). The proposed method is a hybrid algorithm which employs a GRASP procedure to construct an initial solution that is improved by a tabu search. Following, Santos, Ochi and Souza (2005) proposed a tabu search with two memory based diversification strategies that showed to outperform the hybrid approach of Souza, Maculan and Ochi (2004).

After the attempt of Souza (2000) to solve the Brazilian HSTP via a compact MIP formulation, Santos *et al.* (2012) proposed an extended formulation that is solved by column generation. Additional cuts generated by the Fenchel cutting planes technique (BOYD, 1994a; BOYD, 1994b) are added to improve their formulation. The work of Santos *et al.* (2012) provided, for the first time, strong lower bounds and some optimal solutions to the problem when compared to previous solutions found by the tabu search of Santos, Ochi and Souza (2005).

Later, [Saviniec et al. \(2013\)](#) showed that the lower bounds of [Santos et al. \(2012\)](#) were indeed the optimal values for all instances by comparing with solutions found by iterated local search based algorithms. Afterward, [Dorneles, Araújo and Buriol \(2014\)](#) found similar results via fix-and-optimize heuristics.

The latest studies addressing the problem are the works of [Dorneles, Araújo and Buriol \(2017\)](#) and [Saviniec and Constantino \(2017\)](#). [Dorneles, Araújo and Buriol \(2017\)](#) proposed a compact and an extended formulation based on a multi-commodity flow problem. The extended formulation is solved by column generation, and the generated lower bounds were reported to be equivalent to the lower bounds found by [Santos et al. \(2012\)](#). [Saviniec and Constantino \(2017\)](#) proposed new instances and algorithms based on iterated local search and variable neighborhood search that showed better results than the previous approaches when tested on the instances proposed by [Souza \(2000\)](#).

The problem introduced by [Souza \(2000\)](#) was also addressed in the ITC-2011 with a slight modification. The ITC-2011 addressed several high school timetabling variants from other countries, which differ from the Brazilian case in timetabling rules.

One of the most important requisites in Brazilian HSTP is to obtain *compact schedules for teachers* ([SOUZA, 2000](#)). [Saviniec \(2013\)](#) considered a case with *balancing on teachers' extra worked days*. Other requisites are also present in Brazilian HSTP, such as availability of teachers, preference for double lessons, and avoiding idle periods for teachers.

In this thesis, we consider the three HSTP variants mentioned above. We will refer to these problems as HSTP-A – the problem introduced by [Saviniec \(2013\)](#), HSTP-B – the problem introduced by [Souza \(2000\)](#), and HSTP-C – the version of the problem of [Souza \(2000\)](#) included in the ITC-2011. In Section 2.4.1, we present the definition of these problems. The problems themselves and their objective functions are modeled in Chapter 3.

2.4.1 Definition

The high school timetabling problems considered in this thesis are structured as shown in Figure 4. The school has a set of classes C and a set of teachers T . Classes are disjoint groups of students enrolled in the same list of subjects (e.g., mathematics, physics, etc.) and with no free periods during the week. Each class' subject has a number of weekly meetings (or lessons) that are taught by a preassigned teacher. The set of all classes' subjects is called “Required meetings” (R) and each class' subject is called a “requirement”. Teachers may also be unavailable during some days and periods. Given these input data, the goal is to obtain weekly timetable specifying the schedule for all required meetings in a particular shift (morning, afternoon, or evening). A timetable is constructed for each independent shift, and it usually spreads over the set D of weekdays (Monday to Friday) with a set H of five periods per day. Throughout this thesis, we also refer to periods as a set of timeslots $(d, h) \in D \times H$.

Figure 4 – High school timetabling structure.

Given these definitions, the timetabling process must fulfill the following requisites:

1. **Meeting of weekly required lessons:** all required lessons must be assigned.
2. **No clashes in classes' schedules:** each class $c \in C$ must attend exactly one lesson per period.
3. **No clashes in teachers' schedules:** each teacher $t \in T$ must teach at most one lesson per period.
4. **No assignment of teachers in their unavailable periods:** teachers must not be assigned to periods in which they are unavailable.
5. **No daily workload violation for requirements:** for each requirement $r \in R$, there is a limit $\tilde{\delta}_r$ on the amount of meetings per day. Therefore, daily workloads greater than $\tilde{\delta}_r$ should be avoided or are forbidden.
6. **No holes in requirements' schedules:** holes in requirements' schedules should be avoided or are forbidden. A *hole* is a period splitting the daily schedule of a requirement into non-consecutive meetings.
7. **Meeting of double lessons for requirements:** for each requirement $r \in R$, there is a specified minimum number of consecutive double lessons $\tilde{\pi}_r$ that should be met.

8. **No idle periods in teachers' schedules:** teachers should not have idle periods in their daily schedules.
9. **Compact schedules for teachers:** teachers should be scheduled to come to school the minimum possible number of days.
10. **Balancing on teachers' extra working days:** the extra working days should be balanced among teachers. The number of extra working days for a teacher t is the difference between the number of worked days and an estimated minimum $\tilde{d}_t$, defined later in Section 3.1.

Depending on the considered variant HSTP-A, HSTP-B or HSTP-C, some of these requisites may be hard or soft, and we model them in Chapter 3.

MIXED-INTEGER PROGRAMMING MODELS

In this chapter, we propose five mixed-integer programming formulations for the Brazilian HSTP variants described in Section 2.4. For two of these formulations (Section 3.1), the variables can be explicitly generated and solved in a black-box MIP solver. For the three others (Section 3.2), there is a huge number of variables, and we resort to the column generation technique to solve their linear relaxation, which provide tight lower bounds on the optimal integer solution of problem instances.

3.1 Compact formulations

This section presents the two formulations for which the variables can be explicitly generated. Let us consider the following parameters, that represent a high school timetabling instance.

- C : a set of classes.
- T : a set of teachers.
- D : a set of weekdays.
- H : a set of teaching periods per day.
- H_{td} : a set of periods in day $d \in D$ for which teacher $t \in T$ is available.
- R : the set of required meetings (or lessons), as defined in Section 2.4.1.
- R_c : the set of requirements that belong to class $c \in C$.
- R_t : the set of requirements that belong to teacher $t \in T$.
- $\tilde{\theta}_r$: the number of weekly lessons of requirement $r \in R$.

- $\tilde{\delta}_r$: a daily limit for lessons of requirement $r \in R$.
- $\tilde{\pi}_r$: the desired number of consecutive double lessons of requirement $r \in R$.
- $\tilde{d}_t = \left\lceil \frac{\sum_{r \in R_t} \tilde{\theta}_r}{|H|} \right\rceil$: the estimated minimum number of days in which teacher $t \in T$ can be assigned and still meet his/her teaching requirements.

3.1.1 Formulation CF1

In this compact formulation, a timetable output can be represented by a set of binary variables x_{rdh} , for all $(r, d, h) \in R \times D \times H$. Table 5 describes the notation used in formulation CF1 and the HSTP requisites can be formulated as:

Table 5 – Notation used in formulation CF1.

Notation	Definition
Decision variables	
$x_{rdh} \in \{0, 1\}$	indicates whether requirement $r \in R$ is assigned to period $h \in H$ of day $d \in D$.
Auxiliary variables	
$\hat{\delta}_{rd} \in \mathbb{N}$	the number of lessons that exceeds the limit $\tilde{\delta}_r$ in day d for requirement r .
$\hat{j}_{rdh} \in \{0, 1\}$	indicates whether the schedule of requirement r has a hole at period h of day d .
$\hat{\phi}_{rdh} \in \{0, 1\}$	indicates whether requirement r has a double lesson at periods $h - 1$ and h of day d .
$\hat{\pi}_r \in \mathbb{N}$	the number of unmet weekly double lessons of requirement r .
$\hat{d}_{td} \in \{0, 1\}$	indicates whether teacher t works in day d .
$\hat{i}_{tdk} \in \{0, 1\}$	indicates whether teacher t is idle at period k of day d .
$\hat{\beta} \in \mathbb{N}$	the greatest value among all teachers' extra working days.

- **Meeting of weekly required lessons:**

$$\sum_{d \in D} \sum_{h \in H} x_{rdh} = \tilde{\theta}_r \quad \forall r \in R \quad (3.1)$$

- **No clashes in classes' schedules:**

$$\sum_{r \in R_c} x_{rdh} = 1 \quad \forall c \in C; d \in D; h \in H \quad (3.2)$$

- **No clashes in teachers' schedules:**

$$\sum_{r \in R_t} x_{rdh} \leq 1 \quad \forall t \in T; d \in D; h \in H_{td} \quad (3.3)$$

- **No assignment of teachers in their unavailable periods:**

$$\sum_{r \in R_t} x_{rdh} = 0 \quad \forall t \in T; d \in D; h \in H \setminus H_{td} \quad (3.4)$$

- **No daily workload violation for requirements:** this requisite may appear in both forms.

- **Hard requisite:** for each requirement $r \in R$, daily workloads greater than $\tilde{\delta}_r$ are forbidden (SOUZA, 2000; SOUSA; MORETTI; PODESTÁ, 2008).

$$\sum_{h \in H} x_{rdh} \leq \tilde{\delta}_r \quad \forall r \in R; d \in D \quad (3.5)$$

- **Soft requisite:** for each requirement $r \in R$, daily workloads greater than $\tilde{\delta}_r$ should be avoided (SAVINIEC, 2013). Violations of this requisite can be measured by the auxiliary variables $\hat{\delta}_{rd}$.

$$\hat{\delta}_{rd} \geq \sum_{h \in H} x_{rdh} - \tilde{\delta}_r \quad \forall r \in R; d \in D \quad (3.6)$$

$$\hat{\delta}_{rd} \geq 0 \quad \forall r \in R; d \in D \quad (3.7)$$

- **No holes in requirements' schedules:** this requisite may appear in both forms.

- **Hard requisite:** holes in requirements' schedules are forbidden (DORNELES; ARAÚJO; BURIOL, 2017).

$$\begin{aligned} x_{rdi} - x_{rdh} + x_{rdj} - 1 &\leq 0 \quad \forall r \in R; d \in D; h = 1, \dots, |H| - 2; \\ &\quad i = 0, \dots, h - 1; j = h + 1, \dots, |H| - 1 \end{aligned} \quad (3.8)$$

- **Soft requisite:** holes in requirements' schedules should be avoided (SOUSA; MORETTI; PODESTÁ, 2008; SAVINIEC, 2013). For each requirement $r \in R$, a hole in period $h \in \{1, \dots, |H| - 2\}$ of day d is flagged by the auxiliary variables $\hat{j}_{rdh}$.

$$\begin{aligned} \hat{j}_{rdh} &\geq x_{rdi} - x_{rdh} + x_{rdj} - 1 \quad \forall r \in R; d \in D; h = 1, \dots, |H| - 2; \\ &\quad i = 0, \dots, h - 1; j = h + 1, \dots, |H| - 1 \end{aligned} \quad (3.9)$$

$$\hat{j}_{rdh} \geq 0 \quad \forall r \in R; d \in D; h = 1, \dots, |H| - 2 \quad (3.10)$$

- **Meeting of double lessons for requirements:** for each requirement $r \in R$, the auxiliary variables $\hat{\phi}_{rdh}$ are set to “1” if a double lesson ends in periods $h \in \{1, \dots, |H| - 1\}$ within each day, while variables $\hat{\pi}_r$ quantify the number of unmet weekly double lessons.

$$\hat{\phi}_{rdh} \leq x_{rdh} \quad \forall r \in R; d \in D; h = 1, \dots, |H| - 1 \quad (3.11)$$

$$\hat{\phi}_{rdh} \leq x_{r,d,h-1} \quad \forall r \in R; d \in D; h = 1, \dots, |H| - 1 \quad (3.12)$$

$$\hat{\phi}_{rdh} \leq 1 - \hat{\phi}_{r,d,h-1} \quad \forall r \in R; d \in D; h = 2, \dots, |H| - 1 \quad (3.13)$$

$$\hat{\pi}_r \geq \tilde{\pi}_r - \sum_{d \in D} \sum_{h=1}^{|H|-1} \hat{\phi}_{rdh} \quad \forall r \in R \quad (3.14)$$

$$\hat{\phi}_{rdh} \geq 0 \quad \forall r \in R; d \in D; h = 1, \dots, |H| - 1 \quad (3.15)$$

$$\hat{\pi}_r \geq 0 \quad \forall r \in R \quad (3.16)$$

- **No idle periods in teachers' schedules:** For each teacher $t \in T$ and day $d \in D$, idle periods k are flagged by the auxiliary variables $\hat{i}_{tdk}$. There are two cases for this requisite.

- A teacher may be idle at period $k \in H$ even if he/she is unavailable in such period (SOUZA, 2000).

$$\begin{aligned} \hat{i}_{tdk} &\geq \sum_{r \in R_t} (x_{rdi} - x_{rdk} + x_{rdj}) - 1 & \forall t \in T; d \in D; k = 1, \dots, |H| - 2; \\ & & i = 0, \dots, k - 1; j = k + 1, \dots, |H| - 1 \end{aligned} \quad (3.17)$$

$$\hat{i}_{tdk} \geq 0 \quad \forall t \in T; d \in D; k = 1, \dots, |H| - 2 \quad (3.18)$$

- A teacher may be idle at period $k \in H$ only if he/she is available at such period (SAVINIEC, 2013).

$$\begin{aligned} \hat{i}_{tdk} &\geq \sum_{r \in R_t} (x_{r,d,h_i} - x_{r,d,h_k} + x_{r,d,h_j}) - 1 & \forall t \in T; d \in D; k = 1, \dots, |H_{td}| - 2; \\ & & i = 0, \dots, k - 1; j = k + 1, \dots, |H_{td}| - 1; \\ & & h_i, h_k, h_j \in H_{td} \end{aligned} \quad (3.19)$$

$$\hat{i}_{tdk} \geq 0 \quad \forall t \in T; d \in D; k = 1, \dots, |H_{td}| - 2 \quad (3.20)$$

Inequalities (3.19) are similar to inequalities (3.17), except that we omit variable $\hat{i}$ for periods in which the teacher is unavailable. For example, consider that the set of periods is $H = \{0, 1, 2, 3, 4\}$ and suppose that teacher $t \in T$ is not available to teach at periods 1 and 2 within day $d \in D$, then $H_{td} = \{0, 3, 4\}$. Now, only variable $\hat{i}_{t,d,3}$ and related constraints are taken into account.

- **Compact schedules for teachers:** For each teacher $t \in T$, the auxiliary variables $\hat{d}_{td}$ flag if he/she works in day $d \in D$. The sum of these variables is minimized later in the objective function to obtain compact schedules.

$$\hat{d}_{td} \geq \sum_{r \in R_t} x_{rdh} \quad \forall t \in T; d \in D; h \in H_{td} \quad (3.21)$$

$$\hat{d}_{td} \geq 0 \quad \forall t \in T; d \in D \quad (3.22)$$

- **Balancing on teachers' extra working days:** to create schedules in which the extra working days are balanced among teachers, we minimize the auxiliary variable $\hat{\beta}$ which captures the greatest value among teachers' extra worked days.

$$\hat{\beta} \geq \sum_{d \in D} \hat{d}_{td} - \tilde{d}_t \quad \forall t \in T \quad (3.23)$$

$$\hat{\beta} \geq 0 \quad (3.24)$$

3.1.1.1 Problem HSTP-A

This is the problem proposed by [Saviniec \(2013\)](#). Apart from constraints to obtain compact schedules for teachers, the problem also includes constraints to obtain timetables in which the extra worked days are balanced among the teachers. The complete mixed-integer program can be written as:

$$\begin{aligned} \text{Minimize} \quad & \alpha_5 \sum_{r \in R} \sum_{d \in D} \hat{\delta}_{rd} + \alpha_6 \sum_{r \in R} \sum_{d \in D} \sum_{h=1}^{|H|-2} \hat{j}_{rdh} + \alpha_7 \sum_{r \in R} \hat{\pi}_r \\ & + \alpha_8 \sum_{t \in T} \sum_{d \in D} \sum_{k=1}^{|H_{td}|-2} \hat{i}_{tdk} + \alpha_9 \sum_{t \in T} \sum_{d \in D} \hat{d}_{td} + \alpha_{10} \cdot \hat{\beta} \end{aligned} \quad (3.25)$$

Subject to:

- (3.1) Meeting of weekly required lessons
 - (3.2) No clashes in classes' schedules
 - (3.3) No clashes in teachers' schedules
 - (3.4) No assignment of teachers in their unavailable periods
 - (3.6) – (3.7) No daily workload violation for requirements
 - (3.9) – (3.10) No holes in requirements' schedules
 - (3.11) – (3.16) Meeting of double lessons for requirements
 - (3.19) – (3.20) No idle periods in teachers' schedules
 - (3.21) – (3.22) Compact schedules for teachers
 - (3.23) – (3.24) Balancing on teachers' extra working days
- $$x_{rdh} \in \{0, 1\} \quad \forall r \in R; d \in D; h \in H \quad (3.26)$$

The goal of this problem is to satisfy requisites 1 to 4, defined in Section 2.4.1, while minimizes the violations of requisites 5 to 10. In the objective function (3.25), the constant α_i is a penalty that expresses the importance of satisfying the i^{th} requisite, for $i = 5, \dots, 10$.

The MIP program (3.25) - (3.26) is also augmented with the additional cuts (3.27) which specify that each teacher t cannot work less than the minimum number of working days $\tilde{d}_t$ (SOUZA, 2000).

$$\sum_{d \in D} \hat{d}_{td} \geq \tilde{d}_t \quad \forall t \in T \quad (3.27)$$

3.1.1.2 Problem HSTP-B

This is the problem proposed by Souza (2000). The author does not consider the “balancing on teachers' extra working days” and the requisite “no daily workload violation for requirements” is a hard requisite. The complete mixed-integer program can be written as:

$$\text{Minimize} \quad \alpha_7 \sum_{r \in R} \hat{\pi}_r + \alpha_8 \sum_{t \in T} \sum_{d \in D} \sum_{k=1}^{|H|-2} \hat{i}_{tdk} + \alpha_9 \sum_{t \in T} \sum_{d \in D} \hat{d}_{td} \quad (3.28)$$

Subject to:

- (3.1) Meeting of weekly required lessons
 - (3.2) No clashes in classes' schedules
 - (3.3) No clashes in teachers' schedules
 - (3.4) No assignment of teachers in their unavailable periods
 - (3.5) No daily workload violation for requirements
 - (3.11) – (3.16) Meeting of double lessons for requirements
 - (3.17) – (3.18) No idle periods in teachers' schedules
 - (3.21) – (3.22) Compact schedules for teachers
- $$x_{rdh} \in \{0, 1\} \quad \forall r \in R; d \in D; h \in H \quad (3.29)$$

The goal of this problem is to satisfy requisites 1 to 5, while minimizes the violations of requisites 7 to 9. The MIP program (3.28) - (3.29) is also augmented with additional cuts to specify that teachers do not work less than a minimum number of working days (SOUZA, 2000). The cuts are given by inequalities (3.30):

$$\sum_{d \in D} \hat{d}_{td} \geq \max \left\{ \tilde{d}_t, \max_{r \in R_t} \left\{ \left\lceil \frac{\tilde{\theta}_r}{\tilde{\delta}_r} \right\rceil \right\} \right\} \quad \forall t \in T \quad (3.30)$$

3.1.1.3 Problem HSTP-C

This is the variant of the problem of Souza (2000) included in the ITC-2011. Problem HSTP-C has an additional hard requisite to ensure that there are “*no holes in requirements' schedules*”, so that the lessons are always consecutive within each day. The complete mixed-integer program can be written as:

$$\textbf{Minimize} \quad \alpha_7 \sum_{r \in R} \hat{\pi}_r + \alpha_8 \sum_{t \in T} \sum_{d \in D} \sum_{k=1}^{|H|-2} \hat{i}_{tdk} + \alpha_9 \sum_{t \in T} \sum_{d \in D} \hat{d}_{td} \quad (3.31)$$

Subject to:

- (3.1) Meeting of weekly required lessons
 - (3.2) No clashes in classes' schedules
 - (3.3) No clashes in teachers' schedules
 - (3.4) No assignment of teachers in their unavailable periods
 - (3.5) No daily workload violation for requirements
 - (3.8) No holes in requirements' schedules
 - (3.11) – (3.16) Meeting of double lessons for requirements
 - (3.17) – (3.18) No idle periods in teachers' schedules
 - (3.21) – (3.22) Compact schedules for teachers
- $$x_{rdh} \in \{0, 1\} \quad \forall r \in R; d \in D; h \in H \quad (3.32)$$

The goal of this problem is to satisfy requisites 1 to 6, while minimizes the violations of requisites 7 to 9. The MIP program (3.31) - (3.32) is also augmented by cuts (3.30).

3.1.2 Formulation CF2

Table 6 – An illustration of all possible assignment layouts in a day with four teaching periods.

Integer ID	Layout	Comments
0	0000	a day off.
1	0001	an assignment at period h_3 .
2	0010	
3	0011	
4	0100	an assignment at period h_1 .
5	0101	
6	0110	
7	0111	for a requirement, this layout exceeds a daily limit of two lessons.
8	1000	an assignment at period h_0 .
9	1001	for a teacher, he/she is idle at periods h_1 and h_2 .
10	1010	for a requirement, there is a hole at period h_1 (non-consecutive lessons).
11	1011	
12	1100	a double lesson at periods h_0 and h_1 .
13	1101	
14	1110	
15	1111	a full-day assignment layout.

In this section, we propose an alternative formulation that follows the idea of *layouts*, in which daily schedules of teachers and requirements are represented by assignment layouts. A *daily assignment layout* is a binary vector of size $|H|$ in which the h^{th} position represents the h^{th} period of the day. A bit value “1” represents an assignment in the h^{th} period, while a bit value “0” represents no assignment. An illustration is shown in Table 6, for a day with four teaching periods. In this formulation, all possible number of layouts for a requirement or teacher is given by $|D| \cdot 2^{|H|}$. Since $|H|$ is usually a small constant for most high school timetabling problems, in our case $|H| = 5$, we can explicitly enumerate all variables.

Table 7 – Notation used in formulation CF2.

Notation	Definition
Sets	
L	the set of assignment layouts that a requirement r (or teacher t) can follow in a day.
Parameters	
$\tilde{a}_l \in \mathbb{N}$	the number of assignments in layout $l \in L$.
$\tilde{\omega}_l \in \mathbb{N}$	the number of consecutive double lessons in layout l .
$\tilde{h}_l \in \mathbb{N}$	the number of holes in layout l for requirements.
$\tilde{e}_{rl} \in \mathbb{N}$	the number of lessons that exceed the daily limit $\tilde{\delta}_r$ in layout l for a requirement r .
$\tilde{i}_{tdl} \in \mathbb{N}$	the number of idle periods in layout l for teacher t on day d .
$\tilde{\mu}_{lh} \in \{0, 1\}$	indicates whether layout l has an assignment at period h .
$\tilde{\phi}_l \in \{0, 1\}$	indicates whether the number of assignments in layout l is greater than zero.
Decision variables	
$x_{rdl} \in \{0, 1\}$	indicates whether requirement r follows layout l on day d .
$y_{tdl} \in \{0, 1\}$	indicates whether teacher t follows layout l on day d .
Auxiliary variables	
$\hat{\pi}_r \in \mathbb{N}$	the number of unmet weekly double lessons of requirement r .
$\hat{\beta} \in \mathbb{N}$	the greatest value among all teachers' extra working days.

Given the above definitions and the auxiliary notation presented in Table 7, a timetable output can be represented by a set of binary variables x_{rdl} and y_{tdl} . These variables indicate the daily assignment layouts $l \in \{0, 1, \dots, 2^{|H|} - 1\}$ that each requirement $r \in R$ and teacher $t \in T$ must follow in each day $d \in D$, respectively. Therefore, the HSTP requisites can be reformulated as:

- **Meeting of weekly required lessons:**

$$\sum_{d \in D} \sum_{l \in L} \tilde{a}_l x_{rdl} = \tilde{\theta}_r \quad \forall r \in R \quad (3.33)$$

- **No clashes in classes' schedules:**

$$\sum_{r \in R_c} \sum_{l \in L} \tilde{\mu}_{lh} x_{rdl} = 1 \quad \forall c \in C; d \in D; h \in H \quad (3.34)$$

- **Only one layout per day:** requirements and teachers must follow only one layout per day.

$$\sum_{l \in L} x_{rdl} = 1 \quad \forall r \in R; d \in D \quad (3.35)$$

$$\sum_{l \in L} y_{tdl} = 1 \quad \forall t \in T; d \in D \quad (3.36)$$

Constraints (3.36) also ensure that each teacher $t \in T$ teaches at most one lesson per period.

- **No assignment of teachers in their unavailable periods:** we ensure this requisite by preprocessing variables of layouts which contain assignments in periods that teachers are unavailable to teach.

$$y_{tdl} = 0 \quad \forall t \in T; d \in D; l \in L; h \in H \setminus H_{td}; \tilde{\mu}_{lh} > 0 \quad (3.37)$$

$$x_{rdl} = 0 \quad \forall t \in T; r \in R_t; d \in D; l \in L; h \in H \setminus H_{td}; \tilde{\mu}_{lh} > 0 \quad (3.38)$$

- **No daily workload violation for requirements:**

- **Hard requisite:** we ensure this requisite by setting to zero the variables of layouts in which the number of assignments $\tilde{a}_l$ exceeds the daily limit $\tilde{\delta}_r$.

$$x_{rdl} = 0 \quad \forall r \in R; d \in D; l \in L; \tilde{a}_l > \tilde{\delta}_r \quad (3.39)$$

- **Soft requisite:** the number of violations of this requisite is computed by

$$\sum_{r \in R} \sum_{d \in D} \sum_{l \in L} \tilde{e}_r l x_{rdl}.$$

- **No holes in requirements' schedules:**

- **Hard requisite:** we ensure this requisite by setting to zero the variables of layouts that contain holes.

$$x_{rdl} = 0 \quad \forall r \in R; d \in D; l \in L; \tilde{h}_l > 0 \quad (3.40)$$

- **Soft requisite:** the number of violations of this requisite is computed by

$$\sum_{r \in R} \sum_{d \in D} \sum_{l \in L} \tilde{h}_l x_{rdl}.$$

- **Meeting of double lessons for requirements:**

$$\hat{\pi}_r \geq \tilde{\pi}_r - \sum_{d \in D} \sum_{l \in L} \tilde{\omega}_l x_{rdl} \quad \forall r \in R \quad (3.41)$$

$$\hat{\pi}_r \geq 0 \quad \forall r \in R \quad (3.42)$$

- **No idle periods in teachers' schedules:** the number of violations of this requisite is computed by $\sum_{t \in T} \sum_{d \in D} \sum_{l \in L} \tilde{i}_{tdl} y_{tdl}$.

- **Compact schedules for teachers:** the number of violations of this requisite is computed by $\sum_{t \in T} \sum_{d \in D} \sum_{l \in L} \tilde{\phi}_l y_{tdl}$.

- **Balancing on teachers' extra working days:**

$$\hat{\beta} \geq \sum_{d \in D} \sum_{l \in L} \tilde{\phi}_l y_{tdl} - \tilde{d}_t \quad \forall t \in T \quad (3.43)$$

$$\hat{\beta} \geq 0 \quad (3.44)$$

- **Linking constraints:** the daily schedules of teachers must match their requirements' schedules.

$$\sum_{l \in L} \tilde{\mu}_{lh} y_{tdl} = \sum_{r \in R_t} \sum_{l \in L} \tilde{\mu}_{lh} x_{rdl} \quad \forall t \in T; d \in D; h \in H \quad (3.45)$$

3.1.2.1 Problem HSTP-A

The reformulated model for problem HSTP-A can be written as:

$$\begin{aligned} \text{Minimize} \quad & \sum_{r \in R} \sum_{d \in D} \sum_{l \in L} (\alpha_5 \tilde{e}_{rl} + \alpha_6 \tilde{h}_l) x_{rdl} + \alpha_7 \sum_{r \in R} \hat{\pi}_r \\ & + \sum_{t \in T} \sum_{d \in D} \sum_{l \in L} (\alpha_8 \tilde{i}_{tdl} + \alpha_9 \tilde{\phi}_l) y_{tdl} + \alpha_{10} \hat{\beta} \end{aligned} \quad (3.46)$$

Subject to:

(3.33) Meeting of weekly required lessons

(3.34) No clashes in classes' schedules

(3.35) Only one layout per day for requirements

(3.36) Only one layout per day for teachers

(3.37) – (3.38) No assignment of teachers in their unavailable periods

(3.41) – (3.42) Meeting of double lessons for requirements

(3.43) – (3.44) Balancing on teachers' extra working days

(3.45) Linking constraints

$$x_{rdl} \in \{0, 1\} \quad \forall r \in R; d \in D; l \in L \quad (3.47)$$

$$y_{tdl} \in \{0, 1\} \quad \forall t \in T; d \in D; l \in L \quad (3.48)$$

The cuts (3.27) become:

$$\sum_{d \in D} \sum_{l \in L} \tilde{\phi}_l y_{tdl} \geq \tilde{d}_t \quad \forall t \in T \quad (3.49)$$

3.1.2.2 Problem HSTP-B

The reformulated model for problem HSTP-B can be written as:

$$\text{Minimize} \quad \alpha_7 \sum_{r \in R} \hat{\pi}_r + \sum_{t \in T} \sum_{d \in D} \sum_{l \in L} (\alpha_8 \tilde{i}_{tdl} + \alpha_9 \tilde{\phi}_l) y_{tdl} \quad (3.50)$$

Subject to:

(3.33) Meeting of weekly required lessons

(3.34) No clashes in classes' schedules

(3.35) Only one layout per day for requirements

(3.36) Only one layout per day for teachers

(3.37) – (3.38) No assignment of teachers in their unavailable periods

(3.39) No daily workload violation for requirements

(3.41) – (3.42) Meeting of double lessons for requirements

(3.45) Linking constraints

$$x_{rdl} \in \{0, 1\} \quad \forall r \in R; d \in D; l \in L \quad (3.51)$$

$$y_{tdl} \in \{0, 1\} \quad \forall t \in T; d \in D; l \in L \quad (3.52)$$

The cuts (3.30) become:

$$\sum_{d \in D} \sum_{l \in L} \tilde{\phi}_l y_{tdl} \geq \max \left\{ \tilde{d}_t, \max_{r \in R_t} \left\{ \left\lceil \frac{\tilde{\theta}_r}{\tilde{\delta}_r} \right\rceil \right\} \right\} \quad \forall t \in T \quad (3.53)$$

3.1.2.3 Problem HSTP-C

The reformulated model for problem HSTP-C can be written as:

$$\text{Minimize} \quad \alpha_7 \sum_{r \in R} \hat{\pi}_r + \sum_{t \in T} \sum_{d \in D} \sum_{l \in L} (\alpha_8 \tilde{i}_{tdl} + \alpha_9 \tilde{\phi}_l) y_{tdl} \quad (3.54)$$

Subject to:

- (3.33) Meeting of weekly required lessons
- (3.34) No clashes in classes' schedules
- (3.35) Only one layout per day for requirements
- (3.36) Only one layout per day for teachers
- (3.37) – (3.38) No assignment of teachers in their unavailable periods
- (3.39) No daily workload violation for requirements
- (3.40) No holes in requirements' schedules
- (3.41) – (3.42) Meeting of double lessons for requirements
- (3.45) Linking constraints
- $x_{rdl} \in \{0, 1\} \quad \forall r \in R; d \in D; l \in L \quad (3.55)$
- $y_{tdl} \in \{0, 1\} \quad \forall t \in T; d \in D; l \in L \quad (3.56)$

The MIP program (3.54) - (3.56) is also augmented by cuts (3.53).

3.1.2.4 Cutting planes generation

In this section, we describe cuts that can be added to formulation CF2 to strengthen its linear relaxation. These cuts are generated by using the Fenchel enumeration cutting planes technique (BOYD, 1994a; BOYD, 1994b). The idea is adapted from Santos *et al.* (2012), which employed these cuts to strengthen their column generation.

Let us consider the following notation:

- $K_{td} = \{1, \dots, |H_{td}|\}$: the set of all possible number of lessons that teacher t can have on day d .
- $Q_{rd} = \{1, \dots, \min\{|H_{td}|, \tilde{\theta}_r\}\}$: the set of all possible number of lessons that requirement r can have on day d . In which t is the teacher of requirement r .
- A_k : the subset of layouts in L which have exactly k assignments.
- w_{tdk} : binary variable which indicates whether teacher t has exactly k lessons on day d .
- v_{rdk} : binary variable which indicates whether requirement r has exactly k lessons on day d .

To derive valid cuts, we augment formulation CF2 by introducing new binary variables and constraints. The new variables w_{tdk} (resp. v_{rdk}) are the sum of the values of all variables for teacher $t \in T$ (resp. requirement $r \in R$) on day $d \in D$, for which the corresponding layouts have exactly k assignments, i.e.:

$$\sum_{l \in A_k} y_{tdl} = w_{tdk} \quad \forall t \in T; d \in D; k \in K_{td} \quad (3.57)$$

$$\sum_{l \in A_k} x_{rdl} = v_{rdk} \quad \forall r \in R; d \in D; k \in Q_{rd} \quad (3.58)$$

Now consider each of the following valid polyhedra:

- Polyhedron Π_t for lessons of a teacher t :

$$\sum_{d \in D} \sum_{k \in K_{td}} kw_{tdk} = \sum_{r \in R_t} \tilde{\theta}_r \quad (3.59)$$

$$\sum_{k \in K_{td}} w_{tdk} \leq 1 \quad \forall d \in D \quad (3.60)$$

$$w_{tdk} \in \{0, 1\} \quad \forall d \in D; k \in K_{td} \quad (3.61)$$

Constraints (3.59) ensure that the weekly workload of teacher t is met, and constraints (3.60) impose that at most one variable w is active for each day.

- Polyhedron Π_r for lessons of a requirement r :

$$\sum_{d \in D} \sum_{k \in Q_{rd}} kv_{rdk} = \tilde{\theta}_r \quad (3.62)$$

$$\sum_{k \in Q_{rd}} v_{rdk} \leq 1 \quad \forall d \in D \quad (3.63)$$

$$v_{rdk} \in \{0, 1\} \quad \forall d \in D; k \in Q_{rd} \quad (3.64)$$

Constraints (3.62) ensure that the correct weekly workload of requirement r is met, and constraints (3.63) impose that at most one variable v is active for each day.

Let $\bar{w}$ be the fractional values of variables w in the optimal solution of the linear program of augmented formulation CF2. We can easily enumerate all integer solutions $\hat{w} \in \Pi_t$, and if there is a hyperplane which separates $\bar{w}$ from all these integer points $\hat{w} \in \Pi_t$, then it can be checked and generated by solving the following oracle linear program $\mathcal{O}_t$:

$$\textbf{Maximize} \quad z_t = \sum_{d \in D} \sum_{k \in K_{td}} \bar{w}_{tdk} a_{tdk} \quad (3.65)$$

Subject to:

$$\sum_{d \in D} \sum_{k \in K_{td}} \dot{w}_{tdk} a_{tdk} \leq 1 \quad \forall \dot{w} \in \Pi_t \quad (3.66)$$

$$a_{tdk} \geq 0 \quad \forall d \in D; k \in K_{td} \quad (3.67)$$

Let (z_t^*, a_{tdk}^*) be the optimal solution of $\mathcal{O}_t$. If $z_t^* > 1$, then the point $\bar{w}$ lies outside of Π_t and

$$\sum_{d \in D} \sum_{k \in K_{td}} a_{tdk}^* w_{tdk} \leq 1 \quad (3.68)$$

is a valid violated cut. Otherwise, $\bar{w}$ belongs to Π_t . By adapting program (3.65) - (3.67) to manage variables v_{rdk} , the same idea above can be employed to identify cuts to polyhedron Π_r .

The described cuts are iteratively identified and inserted at the root node of the integer program of CF2. At each iteration, we solve the linear program of CF2 and run the oracle program for all teachers and requirements to check the existence of new cuts. The process stops when no cut is found.

3.2 Column generation

This section presents the three extended formulations for which the variables cannot be explicitly generated. All three formulations are based on model CF2 of Section 3.1.2. As these formulations have an enormous number of variables, but in the optimal solution only a small amount of the variables are greater than zero, we use a column generation algorithm to solve their linear relaxations. These solutions usually provide strong lower bounds on the optimal integer solutions of problem instances. For simplicity, we present the three formulations only for problem HSTP-A. Their extensions to manage problems HSTP-B and HSTP-C are straightforward.

3.2.1 Formulation EF1

In formulation CF2 of Section 3.1.2, layouts are considered per day. In this formulation, the layouts extend over the whole week and the variables are decisions if a requirement $r \in R$ or teacher $t \in T$ follows a given weekly assignment layout.

A *weekly assignment layout* for a fixed requirement $\hat{r}$ is any assignment of binary values to variables $x_{\hat{r}dl}$ in model CF2, that satisfies constraints (3.33) and (3.35).

A *weekly assignment layout* for a fixed teacher $\hat{t}$ is any assignment of binary values to variables $y_{\hat{t}dl}$ in model CF2, that satisfies constraints (3.36) to (3.38) and ensures that his/her workload is completely scheduled.

In other words, a weekly assignment layout is a binary vector of size $|D| \cdot |H|$. For example, consider a toy instance with $|D| = 3$ days and $|H| = 5$ periods per day. The layout “00000 00011 00000” specifies that a given requirement (or teacher) has two consecutive lessons at the last two periods of the second day, and the layout “10001 01001 00000” specifies that a teacher has three idle periods on the first day and two at the second. Table 8 describes the notation used in formulation EF1, and the mixed-integer program can thus be written as:

Table 8 – Notation used in formulation EF1.

Notation	Definition
Sets	
L_r	the set of all possible assignment layouts that a requirement r can follow during the week.
L_t	the set of all possible assignment layouts that a teacher t can follow during the week.
Parameters	
f_{ri}	the cost of violating requisites 5 to 7 in the i^{th} weekly layout of requirement r .
g_{tj}	the cost of breaking requisites 8 and 9 in the j^{th} weekly layout of teacher t .
$\bar{x}_{ri}^{dh} \in \{0, 1\}$	constant that indicates whether requirement r is assigned to period h of day d in the i^{th} weekly layout.
$\bar{y}_{tj}^{dh} \in \{0, 1\}$	constant that indicates whether teacher t is assigned to period h of day d in the j^{th} weekly layout.
$\tilde{\phi}_l \in \mathbb{N}$	the number of days that have at least one assignment in a weekly layout l .
Decision variables	
$\lambda_{ri} \in \{0, 1\}$	indicates whether requirement r follows the i^{th} weekly layout.
$\chi_{tj} \in \{0, 1\}$	indicates whether teacher t follows the j^{th} weekly layout.
Auxiliary variables	
$\hat{\beta} \in \mathbb{N}$	the greatest value among all teachers' extra working days.

$$\text{Minimize} \quad \sum_{r \in R} \sum_{i \in L_r} f_{ri} \lambda_{ri} + \sum_{t \in T} \sum_{j \in L_t} g_{tj} \chi_{tj} + \alpha_{10} \hat{\beta} \quad (3.69)$$

Subject to:

$$(\xi_r^1) \quad \sum_{i \in L_r} \lambda_{ri} = 1 \quad \forall r \in R \quad (3.70)$$

$$(\xi_t^2) \quad \sum_{j \in L_t} \chi_{tj} = 1 \quad \forall t \in T \quad (3.71)$$

$$(\xi_{cdh}^3) \quad \sum_{r \in R_c} \sum_{i \in L_r} \bar{x}_{ri}^{dh} \lambda_{ri} = 1 \quad \forall c \in C; d \in D; h \in H \quad (3.72)$$

$$(\xi_{tdh}^4) \quad \sum_{r \in R_t} \sum_{i \in L_r} \bar{x}_{ri}^{dh} \lambda_{ri} - \sum_{j \in L_t} \bar{y}_{tj}^{dh} \chi_{tj} = 0 \quad \forall t \in T; d \in D; h \in H \quad (3.73)$$

$$(\xi_t^5) \quad \hat{\beta} - \sum_{j \in L_t} \tilde{\phi}_j \chi_{tj} \geq -\tilde{d}_t \quad \forall t \in T \quad (3.74)$$

$$\hat{\beta} \geq 0 \quad (3.75)$$

$$\lambda_{ri} \in \{0, 1\} \quad \forall r \in R; i \in L_r \quad (3.76)$$

$$\chi_{tj} \in \{0, 1\} \quad \forall t \in T; j \in L_t \quad (3.77)$$

The objective function (3.69) minimizes the cost of weekly layouts related to violating requisites 5 to 9, plus the cost of breaking requisite 10. Constraints (3.70) and (3.71) ensure that only one weekly layout is selected for each requirement and teacher. Constraints (3.72) avoid clashes in classes' schedules. Constraints (3.73) link layouts of teachers and requirements. The last constraints (3.74) model the balancing on teachers' extra working days.

For practical size instances, formulation EF1 usually has a huge number of variables that are hard to generate explicitly. Even if the explicit writing of the problem is possible, it will result in MIP programs that usually cannot be tackled by actual solvers. On the other hand, as the optimal solutions are defined by small subsets of such variables, we can use column generation (DESAULNIERS; DESROSIERS; SOLOMON, 2005) as an effective tool to solve the associated relaxed problem. The linear relaxation of EF1 usually provides tight lower bounds on the optimal solutions of its original MIP program.

Our column generation procedure works as follows. The master problem is a linear relaxation of program (3.69) - (3.77), and the columns are generated via two types of pricing subproblems. The subproblems defined by $\mathcal{P}_r$ (Section 3.2.1.1) generate the columns with reduced costs associated with weekly layouts of requirements and the subproblems $\mathcal{P}_t$ (Section 3.2.1.2) generate the columns with reduced costs associated with weekly layouts of teachers. The initial master program contains a reduced number of artificial columns, to which we assign high costs. At each iteration, every subproblem $\mathcal{P}_r$ and $\mathcal{P}_t$ is solved to obtain the columns that have the smallest reduced costs, for requirements and teachers, based on the dual values of the last master program solution. The new columns are inserted into the master program, which is solved again, and the process continues until there is no column with negative cost.

3.2.1.1 Subproblems $\mathcal{P}_r$

There are $|R|$ pricing subproblems, one for each requirement $r \in R$. Each subproblem $\mathcal{P}_r$ generates columns (weekly layouts) with costs related to requisites 5 to 7. Table 9 defines the notation used in the formulation of subproblem $\mathcal{P}_r$ and the mixed-integer program can thus be written as:

Table 9 – Notation used in subproblem $\mathcal{P}_r$.

Notation	Definition
Sets	
L	the set of all possible assignment layouts that a requirement r can follow in a day.
Parameters	
$\tilde{a}_l \in \mathbb{N}$	the number of assignments in layout $l \in L$.
$\tilde{e}_{rl} \in \mathbb{N}$	the number of lessons that exceeds the daily limit $\tilde{\delta}_r$ in layout l for a requirement r .
$\tilde{h}_l \in \mathbb{N}$	the number of holes in layout l for requirements.
$\tilde{\omega}_l \in \mathbb{N}$	the number of double lessons in layout l .
$\tilde{\mu}_{lh} \in \{0, 1\}$	indicates whether layout l has an assignment at period h .
$\rho(r)$	the class associated with requirement r .
$\sigma(r)$	the teacher associated with requirement r .
Decision variables	
$x_{dl} \in \{0, 1\}$	indicates whether requirement r follows layout l on day d .
Auxiliary variables	
$\hat{\pi} \in \mathbb{N}$	the number of unmet weekly double lessons of requirement r .

$$\begin{aligned}
\text{Minimize } \bar{c}_r = & \sum_{d \in D} \sum_{l \in L} (\alpha_5 \tilde{e}_{rl} + \alpha_6 \tilde{h}_l) x_{dl} + \alpha_7 \hat{\pi} \\
& - \xi_r^1 - \sum_{d \in D} \sum_{h \in H} \xi_{\rho(r)dh}^3 \sum_{l \in L} \tilde{\mu}_{lh} x_{dl} - \sum_{d \in D} \sum_{h \in H} \xi_{\sigma(r)dh}^4 \sum_{l \in L} \tilde{\mu}_{lh} x_{dl}
\end{aligned} \tag{3.78}$$

Subject to:

$$\sum_{d \in D} \sum_{l \in L} \tilde{a}_l x_{dl} = \tilde{\theta}_r \tag{3.79}$$

$$\sum_{l \in L} x_{dl} = 1 \quad \forall d \in D \tag{3.80}$$

$$\hat{\pi} \geq \tilde{\pi}_r - \sum_{d \in D} \sum_{l \in L} \tilde{\omega}_l x_{dl} \tag{3.81}$$

$$\hat{\pi} \geq 0 \tag{3.82}$$

$$x_{dl} \in \{0, 1\} \quad \forall d \in D; l \in L \tag{3.83}$$

The objective function (3.78) is the reduced cost $\bar{c}_r$ involving the dual values ξ_r^1 , ξ_{cdh}^3 and ξ_{tdh}^4 associated with constraints (3.70), (3.72) and (3.73) of the master problem (3.69) - (3.77). Constraints (3.79) ensure that the required number of weekly lessons of requirement r is scheduled. Constraints (3.80) ensure that requirement r follows only one layout per day, and constraints (3.81) compute its number of unmet double lessons.

3.2.1.2 Subproblems $\mathcal{P}_t$

There are $|T|$ pricing subproblems, one for each teacher $t \in T$. Each subproblem $\mathcal{P}_t$ generates columns (weekly layouts) with costs related to requisites 8 and 9. Table 10 defines the

notation used in the formulation of subproblem $\mathcal{P}_t$ and the mixed-integer program can thus be written as:

Table 10 – Notation used in subproblem $\mathcal{P}_t$.

Notation	Definition
Sets	
L	the set of all possible assignment layouts that a teacher t can follow in a day.
Parameters	
$\tilde{i}_{tdl} \in \mathbb{N}$	the number of idle periods in layout $l \in L$ for teacher t on day d .
$\tilde{\phi}_l \in \{0, 1\}$	indicates whether the number of assignments in layout l is greater than zero.
Decision variables	
$y_{dl} \in \{0, 1\}$	indicates whether teacher t follows layout l on day d .

$$\begin{aligned}
 \text{Minimize } \bar{c}_t = & \sum_{d \in D} \sum_{l \in L} (\alpha_8 \tilde{i}_{tdl} + \alpha_9 \tilde{\phi}_l) y_{dl} \\
 & - \xi_t^2 + \sum_{d \in D} \sum_{h \in H} \xi_{tdh}^4 \sum_{l \in L} \tilde{\mu}_{lh} y_{dl} + \xi_t^5 \sum_{d \in D} \sum_{l \in L} \tilde{\phi}_l y_{dl}
 \end{aligned} \tag{3.84}$$

Subject to:

$$\sum_{d \in D} \sum_{l \in L} \tilde{a}_l y_{dl} = \sum_{r \in R_t} \tilde{\theta}_r \tag{3.85}$$

$$\sum_{l \in L} y_{dl} = 1 \quad \forall d \in D \tag{3.86}$$

$$\sum_{d \in D} \sum_{l \in L} \tilde{\phi}_l y_{dl} \geq \tilde{d}_t \tag{3.87}$$

$$y_{dl} \in \{0, 1\} \quad \forall d \in D; l \in L \tag{3.88}$$

The objective function (3.84) is the reduced cost $\bar{c}_t$ involving the dual values ξ_t^2 , ξ_{tdh}^4 and ξ_t^5 associated with constraints (3.71), (3.73) and (3.74) of the master problem (3.69) - (3.77). Constraints (3.85) ensure that the weekly workload of teacher t is scheduled. Constraints (3.86) ensure that teacher t follows only one layout per day. The last constraints (3.87) are the cuts to ensure that teacher t does not work less than the minimum number of working days $\tilde{d}_t$ (SOUZA, 2000). To avoid violations in teachers' unavailable periods, we preprocess the variables as follows.

$$y_{dl} = 0 \quad \forall d \in D; l \in L; h \in H \setminus H_{td}; \tilde{\mu}_{lh} > 0 \tag{3.89}$$

3.2.2 Formulation EF2

In formulation EF1, there are columns for requirements and teachers. In this formulation, columns are generated only for teachers. A column contains the weekly assignment layouts of all requirements belonging to a given teacher. The variables are decisions if a teacher $t \in T$ follows a given weekly schedule. A *weekly schedule* for a fixed teacher t is any assignment of binary values to variables y_{idl} and x_{rdl} (for all $r \in R_t$) in model CF2, that satisfies constraints (3.33), (3.35) to (3.38), and (3.45). Table 11 describes the notation used in formulation EF2 and the mixed-integer program can thus be written as:

Table 11 – Notation used in formulation EF2.

Notation	Definition
Sets	
S_t	the set of all possible weekly schedules that a teacher $t \in T$ can follow.
T_c	the set of teachers who teach to class $c \in C$.
Parameters	
f_{ti}	the cost of violating requisites 5 to 9 in the i^{th} weekly schedule of teacher t .
$\bar{x}_{ti}^{cdh} \in \{0, 1\}$	constant that indicates whether teacher t teaches to class c in schedule $i \in S_t$ at period h of day d .
$\tilde{\phi}_{ti} \in \mathbb{N}$	the number of worked days for teacher t in schedule $i \in S_t$.
Decision variables	
$\lambda_{ti} \in \{0, 1\}$	indicates whether teacher t follows the weekly schedule $i \in S_t$.
Auxiliary variables	
$\hat{\beta} \in \mathbb{N}$	the greatest value among all teachers' extra working days.

$$\text{Minimize} \quad \sum_{t \in T} \sum_{i \in S_t} f_{ti} \lambda_{ti} + \alpha_{10} \hat{\beta} \quad (3.90)$$

Subject to:

$$(\xi_t^1) \quad \sum_{i \in S_t} \lambda_{ti} = 1 \quad \forall t \in T \quad (3.91)$$

$$(\xi_{cdh}^2) \quad \sum_{t \in T_c} \sum_{i \in S_t} \bar{x}_{ti}^{cdh} \lambda_{ti} = 1 \quad \forall c \in C; d \in D; h \in H \quad (3.92)$$

$$(\xi_t^3) \quad \hat{\beta} - \sum_{i \in S_t} \tilde{\phi}_{ti} \lambda_{ti} \geq -\tilde{d}_t \quad \forall t \in T \quad (3.93)$$

$$\hat{\beta} \geq 0 \quad (3.94)$$

$$\lambda_{ti} \in \{0, 1\} \quad \forall t \in T; i \in S_t \quad (3.95)$$

The objective function (3.90) minimizes the cost of weekly schedules related to violating requisites 5 to 9, plus the cost of breaking the requisite 10. Constraints (3.91) ensure that only

one weekly schedule is selected for each teacher. Constraints (3.92) avoid clashes in classes' schedules, and the last constraints (3.93) model the balancing on teachers' extra working days.

For this formulation, the master problem of our column generation algorithm is a linear relaxation of program (3.90) - (3.95), and the columns are generated via $|T|$ pricing subproblems $\mathcal{Q}_t$, described in Section 3.2.2.1, which select the most attractive weekly schedules (columns) for teachers. The master program starts with a reduced number of artificial columns, with high costs, and at each iteration, every subproblem $\mathcal{Q}_t$ is solved to obtain the columns that have the smallest reduced costs for teachers.

3.2.2.1 Subproblems $\mathcal{Q}_t$

Each pricing subproblem $\mathcal{Q}_t$ generates columns (weekly schedules) with costs related to requisites 5 to 9. Table 12 defines the notation used in the formulation of subproblem $\mathcal{Q}_t$ and the mixed-integer program can thus be written as:

Table 12 – Notation used in subproblem $\mathcal{Q}_t$.

Notation	Definition
Sets	
L	the set of all possible assignment layouts that a teacher t and requirements $r \in R_t$ can follow in a day.
C_t	the set of classes taught by teacher t .
R_{ct}	the set of requirements belonging to class c which are taught by teacher t .
Parameters	
$\tilde{a}_l \in \mathbb{N}$	the number of assignments in layout $l \in L$.
$\tilde{\omega}_l \in \mathbb{N}$	the number of consecutive double lessons in layout l .
$\tilde{e}_{rl} \in \mathbb{N}$	the number of lessons that exceeds the daily limit $\tilde{\delta}_r$ in layout l for a requirement $r \in R_t$.
$\tilde{h}_l \in \mathbb{N}$	the number of holes in layout l for requirements.
$\tilde{\mu}_{lh} \in \{0, 1\}$	indicates whether layout l has an assignment at period h .
$\tilde{i}_{tdl} \in \mathbb{N}$	the number of idle periods in layout l for teacher t on day d .
$\tilde{\phi}_l \in \{0, 1\}$	indicates whether the number of assignments in layout l is greater than zero.
Decision variables	
$y_{dl} \in \{0, 1\}$	indicates whether teacher t follows layout l on day d .
$x_{rdl} \in \{0, 1\}$	indicates whether requirement $r \in R_t$ follows layout l on day d .
Auxiliary variables	
$\tilde{\pi}_r \in \mathbb{N}$	the number of unmet weekly double lessons of requirement r .

$$\begin{aligned}
 \text{Minimize } \bar{c}_t = & \sum_{r \in R_t} \sum_{d \in D} \sum_{l \in L} (\alpha_5 \tilde{e}_{rl} + \alpha_6 \tilde{h}_l) x_{rdl} + \alpha_7 \sum_{r \in R_t} \tilde{\pi}_r + \sum_{d \in D} \sum_{l \in L} (\alpha_8 \tilde{i}_{tdl} + \alpha_9 \tilde{\phi}_l) y_{dl} \\
 & - \xi_t^1 - \sum_{c \in C_t} \sum_{d \in D} \sum_{h \in H} \xi_{cdh}^2 \sum_{r \in R_{ct}} \sum_{l \in L} \tilde{\mu}_{lh} x_{rdl} + \xi_t^3 \sum_{d \in D} \sum_{l \in L} \tilde{\phi}_l y_{dl}
 \end{aligned} \tag{3.96}$$

Subject to:

$$\sum_{d \in D} \sum_{l \in L} \tilde{a}_l x_{rdl} = \tilde{\theta}_r \quad \forall r \in R_t \quad (3.97)$$

$$\sum_{r \in R_t} \sum_{l \in L} \tilde{\mu}_{lh} x_{rdl} = 1 \quad \forall c \in C_t; d \in D; h \in H \quad (3.98)$$

$$\sum_{l \in L} x_{rdl} = 1 \quad \forall r \in R_t; d \in D \quad (3.99)$$

$$\sum_{l \in L} y_{dl} = 1 \quad \forall d \in D \quad (3.100)$$

$$\sum_{r \in R_t} \sum_{l \in L} \tilde{\mu}_{lh} x_{rdl} = \sum_{l \in L} \tilde{\mu}_{lh} y_{dl} \quad \forall d \in D; h \in H \quad (3.101)$$

$$\hat{\pi}_r \geq \tilde{\pi}_r - \sum_{d \in D} \sum_{l \in L} \tilde{\omega}_l x_{rdl} \quad \forall r \in R_t \quad (3.102)$$

$$\sum_{d \in D} \sum_{l \in L} \tilde{\phi}_l y_{dl} \geq \tilde{d}_t \quad (3.103)$$

$$\hat{\pi}_r \geq 0 \quad \forall r \in R_t \quad (3.104)$$

$$x_{rdl} \in \{0, 1\} \quad \forall r \in R_t; d \in D; l \in L \quad (3.105)$$

$$y_{dl} \in \{0, 1\} \quad \forall d \in D; l \in L \quad (3.106)$$

The objective function (3.96) is the reduced cost $\bar{c}_t$ involving the dual values ξ_t^1, ξ_{cdh}^2 and ξ_t^3 associated with constraints (3.91) to (3.93) of the master problem (3.90) - (3.95). Constraints (3.97) ensure that the required number of weekly lessons is scheduled for each requirement of a teacher t . Constraints (3.98) avoid clashes in classes' schedules. We also need these constraints in the subproblems to avoid cases in which a teacher teaches to the same class in two different requirements $r_1 \in R$ and $r_2 \in R$. A frequent case is a teacher t who teaches mathematics (represented by r_1) and physics (represented by r_2) to the same class. Constraints (3.99) and (3.100) ensure that teacher t and his/her requirements $r \in R_t$ follow only one layout per day. The linking constraints (3.101) ensure that the daily schedules of teacher t match his/her requirements' schedules. Constraints (3.102) compute the number of unmet double lessons for requirements. The last constraints (3.103) are the cuts to ensure that teacher t does not work less than the minimum number of working days $\tilde{d}_t$ (SOUZA, 2000).

To avoid violating teachers' unavailable periods, we preprocess the variables as follows.

$$y_{dl} = 0 \quad \forall d \in D; l \in L; h \in H \setminus H_{td}; \tilde{\mu}_{lh} > 0 \quad (3.107)$$

$$x_{rdl} = 0 \quad \forall r \in R_t; d \in D; l \in L; h \in H \setminus H_{td}; \tilde{\mu}_{lh} > 0 \quad (3.108)$$

3.2.3 Formulation EF3

In this formulation, variables are decisions if a weekday $d \in D$ follows a given schedule. A *day schedule* for a fixed day $\hat{d}$ is any assignment of binary values to variables $x_{rd\hat{d}}$ and $y_{t\hat{d}l}$ in

model CF2, that satisfies constraints (3.34) to (3.38) and (3.45). Table 13 describes the notation used in formulation EF3 and the mixed-integer program can thus be written as:

Table 13 – Notation used in formulation EF3.

Notation	Definition
Sets	
S_d	the set of all possible schedules that a day $d \in D$ can follow.
Parameters	
f_{di}	the cost of violating soft requisites 5, 6, 8 and 9 associated with the i^{th} schedule of day d .
$\tilde{a}_{di}^r \in \mathbb{N}$	the number of lessons assigned to requirement r in schedule $i \in S_d$ of day d .
$\tilde{\omega}_{di}^r \in \mathbb{N}$	the number of double lessons assigned to requirement r in schedule $i \in S_d$ of day d .
$\tilde{\phi}_{di}^t \in \{0, 1\}$	indicates whether teacher t teaches in schedule $i \in S_d$ of day d .
Decision variables	
$\lambda_{di} \in \{0, 1\}$	indicates whether schedule $i \in S_d$ is selected for day d .
Auxiliary variables	
$\hat{\pi}_r \in \mathbb{N}$	the number of unmet weekly double lessons of requirement r .
$\hat{\beta} \in \mathbb{N}$	the greatest value among all teachers' extra working days.

$$\text{Minimize} \quad \sum_{d \in D} \sum_{i \in S_d} f_{di} \lambda_{di} + \alpha_7 \sum_{r \in R} \hat{\pi}_r + \alpha_{10} \hat{\beta} \quad (3.109)$$

Subject to:

$$(\xi_d^1) \quad \sum_{i \in S_d} \lambda_{di} = 1 \quad \forall d \in D \quad (3.110)$$

$$(\xi_r^2) \quad \sum_{d \in D} \sum_{i \in S_d} \tilde{a}_{di}^r \lambda_{di} = \tilde{\theta}_r \quad \forall r \in R \quad (3.111)$$

$$(\xi_r^3) \quad \hat{\pi}_r + \sum_{d \in D} \sum_{i \in S_d} \tilde{\omega}_{di}^r \lambda_{di} \geq \tilde{\pi}_r \quad \forall r \in R \quad (3.112)$$

$$(\xi_t^4) \quad \hat{\beta} - \sum_{d \in D} \sum_{i \in S_d} \tilde{\phi}_{di}^t \lambda_{di} \geq -\tilde{d}_t \quad \forall t \in T \quad (3.113)$$

$$(\xi_t^5) \quad \sum_{d \in D} \sum_{i \in S_d} \tilde{\phi}_{di}^t \lambda_{di} \geq \tilde{d}_t \quad \forall t \in T \quad (3.114)$$

$$\hat{\pi}_r \geq 0 \quad \forall r \in R \quad (3.115)$$

$$\hat{\beta} \geq 0 \quad (3.116)$$

$$\lambda_{di} \in \{0, 1\} \quad \forall d \in D; i \in S_d \quad (3.117)$$

The objective function (3.109) minimizes the cost of day schedules that violate requisites 5, 6, 8 and 9, plus the cost of infringing requisites 7 and 10. Constraints (3.110) ensure that only one schedule is selected for each day. Constraints (3.111) ensure that the required number of weekly lessons is scheduled for each requirement. Constraints (3.112) compute the number of

unmet double lessons for requirements. Constraints (3.113) model the balancing on teachers' extra working days. The last constraints (3.114) are the cuts to ensure that each teacher t does not work less than the minimum number of working days $\tilde{d}_t$ (SOUZA, 2000).

For this formulation, the master problem of our column generation algorithm is a linear relaxation of program (3.109) - (3.117), and the columns are generated via $|D|$ pricing subproblems $\mathcal{B}_d$ which are described in Section 3.2.3.1. The master program starts with a reduced number of artificial columns, containing high costs, and at each iteration, every subproblem $\mathcal{B}_d$ is solved to obtain the columns that have the smallest reduced costs for days.

3.2.3.1 Subproblem $\mathcal{B}_d$

Each pricing subproblem $\mathcal{B}_d$ generates columns (day schedules) with costs related to requisites 5, 6, 8 and 9. Table 14 defines the notation used in the formulation of subproblem $\mathcal{B}_d$ and the mixed-integer program can thus be written as:

Table 14 – Notation used in subproblem $\mathcal{B}_d$.

Notation	Definition
Sets	
L	the set of all possible assignment layouts that a requirement r (or teacher t) can follow on day d .
Parameters	
$\tilde{a}_l \in \mathbb{N}$	the number of assignments in layout $l \in L$.
$\tilde{\omega}_l \in \mathbb{N}$	the number of consecutive double lessons in layout l .
$\tilde{e}_{rl} \in \mathbb{N}$	the number of lessons that exceeds the daily limit $\tilde{\delta}_r$ in layout l for a requirement r .
$\tilde{h}_l \in \mathbb{N}$	the number of holes in layout l for requirements.
$\tilde{\mu}_{lh} \in \{0, 1\}$	indicates whether layout l has an assignment at period h .
$\tilde{i}_{tdl} \in \mathbb{N}$	the number of idle periods in layout l for teacher t on day d .
$\tilde{\phi}_l \in \{0, 1\}$	indicates whether the number of assignments in layout l is greater than zero.
Decision variables	
$x_{rl} \in \{0, 1\}$	indicates whether requirement r follows layout l on day d .
$y_{tl} \in \{0, 1\}$	indicates whether teacher t follows layout l on day d .

$$\begin{aligned}
 \text{Minimize } \bar{c}_d = & \sum_{r \in R} \sum_{l \in L} (\alpha_5 \tilde{e}_{rl} + \alpha_6 \tilde{h}_l) x_{rl} + \sum_{t \in T} \sum_{l \in L} (\alpha_8 \tilde{i}_{tdl} + \alpha_9 \tilde{\phi}_l) y_{tl} \\
 & - \xi_d^1 - \sum_{r \in R} \sum_{l \in L} (\xi_r^2 \tilde{a}_l + \xi_r^3 \tilde{\omega}_l) x_{rl} + \sum_{t \in T} \sum_{l \in L} (\xi_t^4 - \xi_t^5) \tilde{\phi}_l y_{tl}
 \end{aligned} \tag{3.118}$$

Subject to:

$$\sum_{l \in L} x_{rl} = 1 \quad \forall r \in R \tag{3.119}$$

$$\sum_{l \in L} y_{tl} = 1 \quad \forall t \in T \tag{3.120}$$

$$\sum_{r \in R_c} \sum_{l \in L} \tilde{\mu}_{lh} x_{rl} = 1 \quad \forall c \in C; h \in H \quad (3.121)$$

$$\sum_{r \in R_t} \sum_{l \in L} \tilde{\mu}_{lh} x_{rl} = \sum_{l \in L} \tilde{\mu}_{lh} y_{tl} \quad \forall t \in T; h \in H \quad (3.122)$$

$$x_{rl} \in \{0, 1\} \quad \forall r \in R; l \in L \quad (3.123)$$

$$y_{tl} \in \{0, 1\} \quad \forall t \in T; l \in L \quad (3.124)$$

The objective function (3.118) is the reduced cost $\bar{c}_d$ involving the dual values $\xi_d^1, \xi_r^2, \xi_r^3, \xi_t^4$ and ξ_t^5 associated with constraints (3.110) to (3.114) of the master problem (3.109) - (3.117). Constraints (3.119) and (3.120) ensure that each requirement and teacher follows only one layout per day. Constraints (3.121) avoid clashes in classes' schedules. The last constraints (3.122) link layouts of teachers and requirements on day d .

To avoid violations in teachers' unavailable periods, we preprocess the variables as follows.

$$y_{tl} = 0 \quad \forall t \in T; l \in L; h \in H \setminus H_{td}; \tilde{\mu}_{lh} > 0 \quad (3.125)$$

$$x_{rl} = 0 \quad \forall t \in T; r \in R_t; l \in L; h \in H \setminus H_{td}; \tilde{\mu}_{lh} > 0 \quad (3.126)$$

3.3 Computational experiments

In this section, we report three experiments in order to assess the proposed formulations.

Section 3.3.1: we analyze formulations CF1, CF2, EF1, EF2, and EF3 applied to problem HSTP-A.

Section 3.3.2: the best compact formulation and the best extended formulation are adjusted to problem HSTP-B and the results are compared with literature results.

Section 3.3.3: the best formulations are adjusted and applied to problem HSTP-C.

The programs of all formulations were coded in C++ using the Concert Technology Libraries from CPLEX 12.6 and compiled with GNU Compiler Collection 4.4.7. The experiments were carried out on a server with 2 CPU Intel Xeon E5-2680v2 (2.8 GHz) and 128 GB of RAM, running the operating system *Red Hat Enterprise Linux Server 6.5*.

3.3.1 Results for problem HSTP-A

This section evaluates the proposed formulations on problem HSTP-A. We used 34 instances collected from Brazilian high schools (SAVINIEC; CONSTANTINO, 2017). Table 15

Table 15 – Features of the 34 real instances from Brazilian high schools.

ID	Instance	$ C $	$ T $	$ D $	$ H $	$\sum_{r \in R} \tilde{\theta}_r$	$\sum_{r \in R} \tilde{\pi}_r$
1	CL-CEASD-2008-V-A	12	27	5	5	300	132
2	CL-CEASD-2008-V-B	12	27	5	5	300	132
3	CL-CECL-2011-M-A	13	31	5	5	325	144
4	CL-CECL-2011-M-B	13	31	5	5	325	143
5	CL-CECL-2011-N-A	9	28	5	5	225	107
6	CL-CECL-2011-V-A	14	29	5	5	350	164
7	CM-CECM-2011-M	20	51	5	5	500	234
8	CM-CECM-2011-N	8	30	5	5	200	96
9	CM-CECM-2011-V	13	34	5	5	325	142
10	CM-CEDB-2010-N	5	17	5	5	125	60
11	CM-CEUP-2008-V	16	35	5	5	400	192
12	CM-CEUP-2011-M	16	38	5	5	400	192
13	CM-CEUP-2011-N	3	15	5	5	75	36
14	CM-CEUP-2011-V	16	34	5	5	400	169
15	FA-EEF-2011-M	4	12	5	5	100	42
16	JNS-CEDPII-2011-M	8	19	5	5	200	85
17	JNS-CEDPII-2011-V	7	21	5	5	175	73
18	JNS-CEJXXIII-2011-M	5	18	5	5	125	60
19	JNS-CEJXXIII-2011-N	4	15	5	5	100	48
20	JNS-CEJXXIII-2011-V	5	18	5	5	125	60
21	MGA-CEDC-2011-M	19	37	5	5	475	210
22	MGA-CEDC-2011-V	12	31	5	5	300	131
23	MGA-CEGV-2011-M	31	62	5	5	775	352
24	MGA-CEGV-2011-V	32	75	5	5	800	357
25	MGA-CEJXXIII-2010-V	16	35	5	5	400	192
26	MGA-CEVB-2011-M	10	21	5	5	250	108
27	MGA-CEVB-2011-V	9	20	5	5	225	97
28	NE-CESVP-2011-M-A	18	45	5	5	450	212
29	NE-CESVP-2011-M-B	18	44	5	5	450	212
30	NE-CESVP-2011-M-C	18	45	5	5	450	211
31	NE-CESVP-2011-M-D	18	45	5	5	450	211
32	NE-CESVP-2011-V-A	16	44	5	5	400	183
33	NE-CESVP-2011-V-B	16	43	5	5	400	184
34	NE-CESVP-2011-V-C	16	43	5	5	400	182

presents the main features of these instances. The columns present the number of classes ($|C|$), teachers ($|T|$), days ($|D|$), periods per day ($|H|$), the total number of weekly lessons ($\sum_{r \in R} \tilde{\theta}_r$) and the total number of consecutive double lessons ($\sum_{r \in R} \tilde{\pi}_r$) required for each instance, respectively. The following weighting parameters are used in the objective function for these instances: $\alpha_5 = 100$, $\alpha_6 = 25$, $\alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = 10$. These parameter values were originated from the school's preferences for attending the considered list of soft requisites.

3.3.1.1 Comparing the linear programs of formulations CF1 and CF2

This section compares the linear relaxation of formulations CF1 and CF2. Both programs were run with CPLEX's default settings. The results are shown in Table 16. The table presents for each formulation, the execution time (*Time*) and the value of the linear program (*LP*) on the tested instances. The columns (LP_{2A}) and (LP_{2B}) are the *LP* values of CF2 before and after the addition of the Fenchel cuts described in Section 3.1.2.4, respectively. The gaps presented

on columns labeled with $(Gap_{\#})$ are calculated as $Gap_{\#} = 100 \cdot (LP2_{\#} - LPI) \div LP2_{\#}$, for $\#$ representing A or B. The results of Table 16 show that CF2 has better linear relaxation than CF1 and its linear program also runs faster. On average, the LP of CF2 is 21.09 % better than the LP of CF1. Moreover, the Fenchel cuts provided significant LP improvements in CF2 with short computational time. After the addition of such cuts, the LP of CF2 became 23.32 % better than CF1, on average.

Table 16 – Comparison of linear relaxations between formulations CF1 and CF2.

ID	CF1		CF2					
	Time (s)	LPI	Time (s)	$LP2_A$	Gap_A	Time (s)	$LP2_B$	Gap_B
1	0.90	720	0.35	885	18.64	13.64	930	22.58
2	0.85	720	0.32	885	18.64	10.37	930	22.58
3	1.55	760	0.49	925	17.84	39.45	990	23.23
4	1.67	770	0.50	925	16.76	31.01	970	20.62
5	0.90	623.34	0.30	788.34	20.93	23.01	791.67	21.26
6	1.70	830	0.51	1075	22.79	48.33	1080	23.15
7	1.96	1275	0.55	1640	22.26	22.57	1680	24.11
8	0.27	707.06	0.16	864.38	18.20	1.28	910.77	22.37
9	0.66	853.42	0.27	1065.42	19.90	3.91	1100	22.42
10	0.37	320	0.17	420	23.81	7.85	420	23.81
11	1.32	1040	0.41	1377.50	24.50	14.48	1391.67	25.27
12	1.12	1043.67	0.36	1401.12	25.51	7.09	1447	27.87
13	0.03	310	0.04	392.5	21.02	0.12	410	24.39
14	0.84	987.34	0.30	1193	17.24	7.98	1260	21.64
15	0.13	250	0.10	340	26.47	0.99	375	33.33
16	0.48	509	0.20	606.67	16.10	12.45	626.67	18.78
17	0.42	470	0.19	525	10.48	10.62	540	12.96
18	0.38	325	0.17	425	23.53	7.66	445	26.97
19	0.33	270	0.14	350	22.86	2.97	360	25.00
20	0.37	306.67	0.17	422.50	27.42	5.88	446.67	31.34
21	1.77	1120	0.54	1405	20.28	30.27	1450	22.76
22	0.61	754	0.23	1009.09	25.28	4.60	1029.29	26.75
23	5.11	1929.53	1.21	2478.34	22.14	46.72	2500	22.82
24	4.13	2062.53	0.98	2597.23	20.59	45.53	2680	23.04
25	1.16	969	0.42	1295	25.17	34.10	1305	25.75
26	0.58	620	0.25	740	16.22	6.71	740	16.22
27	0.45	584	0.19	710	17.75	4.28	730	20.00
28	1.71	1110	0.59	1440	22.92	66.74	1450	23.45
29	1.58	1110	0.57	1440	22.92	57.28	1450	23.45
30	1.65	1094	0.56	1430	23.50	32.52	1450	24.55
31	1.44	1090	0.51	1410	22.70	18.24	1420	23.24
32	1.50	1010	0.55	1280	21.09	34.39	1295	22.01
33	1.32	1000	0.51	1275	21.57	35.88	1300	23.08
34	1.19	1016	0.43	1271.67	20.11	33.65	1305	22.15
Avg.:	1.19		0.39		21.09	21.25		23.32

3.3.1.2 Comparing the integer programs of formulations CF1 and CF2

This section compares the output of the integer programs of CF1 and CF2. Both programs were run with CPLEX's default settings during a time limit of three hours. Table 17 presents for each formulation, the execution time ($Time$), the best lower bound (LB), the objective value of the best integer solution (UB), and the optimality gap (Gap) on the tested instances. For formulation

CF2, we present the versions without and with Fenchel cuts, *CF2* and *CF2-Cuts*, respectively. In column (*Time*), TL means that the integer program reached the time limit of three hours without proving the optimality. The best LB and UB are shown in bold font. The row (*Rank*) presents the number of instances in which the formulations proved the optimality or reached the best LB or UB. The results show that formulation CF2 is, in general, better than CF1 in the quality of lower and upper bounds. Except that for the version without cuts, CF2 performed a bit worse than CF1 in the quality of integer solutions. The average optimality gap of CF2 is worse than CF1. However, CF2 proved more optimal solutions than CF1. Formulation CF2 proved 11 optimal solutions against 5 of CF1. With the addition of the Fenchel cuts, CF2 proved the optimality for 18 out of the 34 tested instances.

Table 17 – Results of the integer programs of formulations CF1 and CF2 in a time limit of three hours.

ID	CF1				CF2				CF2-Cuts			
	Time (s)	LB	UB	Gap	Time (s)	LB	UB	Gap	Time (s)	LB	UB	Gap
1	TL	826.24	1290	35.95	TL	931.25	1805	48.41	TL	940	1040	9.62
2	TL	814.61	1260	35.35	TL	932.70	1705	45.30	TL	941.43	950	0.90
3	TL	839.17	1350	37.84	TL	972.15	9910	90.19	TL	1000	1040	3.85
4	TL	842.50	1430	41.08	TL	962.70	1845	47.82	TL	990	1900	47.89
5	TL	820	1020	19.61	7546.46	870	870	0.00	1054.40	870	870	0.00
6	TL	880.36	1540	42.83	TL	1075	2545	57.76	TL	1080	1340	19.40
7	TL	1647.82	2325	29.13	TL	1680	12840	86.92	TL	1690	3525	52.06
8	2258.93	950	950	0.00	77.64	950	950	0.00	77.37	950	950	0.00
9	TL	1062.59	1430	25.69	TL	1098.43	1100	0.14	83.30	1100	1100	0.00
10	4466.99	420	420	0.00	80.60	420	420	0.00	91.28	420	420	0.00
11	TL	1373.34	1885	27.14	TL	1397.15	1420	1.61	323.13	1400	1400	0.00
12	TL	1422.16	2010	29.25	1216.46	1450	1450	0.00	422.49	1450	1450	0.00
13	2.59	420	420	0.00	0.69	420	420	0.00	0.50	420	420	0.00
14	TL	1227.25	1650	25.62	TL	1249.73	1975	36.72	529.50	1270	1270	0.00
15	1566.40	395	395	0.00	15.20	395	395	0.00	9.39	395	395	0.00
16	TL	579.17	750	22.78	401.87	640	640	0.00	517.54	640	640	0.00
17	TL	531.67	660	19.44	TL	576	590	2.37	568.90	580	580	0.00
18	TL	442.50	490	9.69	TL	452.23	460	1.69	120.26	460	460	0.00
19	TL	350	360	2.78	21.63	360	360	0.00	17.79	360	360	0.00
20	TL	417.34	520	19.74	TL	462.50	470	1.60	2076.64	470	470	0.00
21	TL	1415	1990	28.89	TL	1440	11165	87.10	5801.61	1450	1450	0.00
22	TL	1018.75	1245	18.17	483.84	1035	1035	0.00	295.88	1035	1035	0.00
23	TL	2458.45	5715	56.98	TL	2482.78	22215	88.82	TL	2504.28	17790	85.92
24	TL	2604.80	5115	49.08	TL	2669.98	20330	86.87	TL	2680	16825	84.07
25	TL	1301.67	1685	22.75	TL	1310	2650	50.57	TL	1315	1320	0.38
26	TL	724.29	940	22.95	6678.12	750	750	0.00	532.73	750	750	0.00
27	6105.45	730	730	0.00	8269.66	730	730	0.00	204.87	730	730	0.00
28	TL	1457	2250	35.24	TL	1547.15	3040	49.11	TL	1565	13420	88.34
29	TL	1313.58	2160	39.19	TL	1520.56	4000	61.99	TL	1545	2040	24.26
30	TL	1444.29	2165	33.29	TL	1557.50	3615	56.92	TL	1560.84	3500	55.40
31	TL	1425.38	2070	31.14	TL	1552.50	2705	42.61	TL	1554.45	3480	55.33
32	TL	1275.56	1830	30.30	TL	1356.67	3235	58.06	TL	1360	2185	37.76
33	TL	1273.17	1920	33.69	TL	1386.67	2705	48.74	TL	1390	2755	49.55
34	TL	1269.45	1920	33.88	TL	1336	3145	57.52	TL	1380	2205	37.41
Rank:		5	16	5		11	14	11		34	24	18
Avg.:				25.28				32.61				19.18

3.3.1.3 Comparing formulation CF2 with EF1, EF2 and EF3

This section analyzes the lower bounds provided by the three extended formulations EF1, EF2 and EF3 compared to the lower bounds of the compact formulation CF2-Cuts. The column generation algorithms were run using CPLEX's default settings but with a single thread.

The results are shown in Table 18. The table presents for each formulation, the execution time (*Time*), the lower bound (*LB*) and the gap (*Gap*) to the best-known integer solution (*Best UB*) reported in Table 17. The lower bounds of CF2-Cuts were obtained with a time limit of three hours. The best gaps are shown in bold font, and shaded rows are instances for which the optimality was proven. The results demonstrate that, in general, the extended formulation EF2 provides the best lower bounds for problem HSTP-A. It generated the best lower bounds in 28 out of the 34 tested instances, against 23 ranked by the compact formulation CF2-Cuts. Also, the average gap of EF2 is smaller than the one of CF2-Cuts. The other two extended formulations, EF1 and EF3, do not compete with EF2 and CF2-Cuts in the quality of lower bounds, but they provide gaps that are close to the optimal solutions. Concerning computational time, formulation EF1 produced the best results among all formulations. On the other hand, the sequential version of the column generation algorithm of formulation EF3 is a very time-consuming process, and we had to solve the subproblems in parallel to achieve feasible computational time.

Table 18 – The lower bounds of formulations EF1, EF2 and EF3 compared to lower bounds of formulation CF2-Cuts.

ID	Best UB	CF2-Cuts (3 h)			EF1			EF2			EF3		
		Time (s)	LB	Gap	Time (s)	LB	Gap	Time (s)	LB	Gap	Time (s)	LB	Gap
1	1040	TL	940	9.62	24.41	930	10.58	47.16	950	8.65	191.84	923.34	11.22
2	950	TL	941.43	0.90	24.79	930	2.11	47.89	950	0.00	181.99	923.34	2.81
3	1040	TL	1000	3.85	41.01	990	4.81	62.40	1010	2.88	443.43	965	7.21
4	1430	TL	990	30.77	38.00	970	32.17	64.63	990	30.77	819.46	955	33.22
5	870	1054.40	870	0.00	25.72	791.67	9.00	35.82	870	0.00	156.03	865	0.57
6	1340	TL	1080	19.40	41.30	1080	19.40	75.69	1080	19.40	442.19	1075	19.78
7	2325	TL	1690	27.31	40.84	1680	27.74	118.60	1690	27.31	1168.52	1648.34	29.10
8	950	77.37	950	0.00	6.22	910.77	4.13	22.19	930	2.11	36.01	892.45	6.06
9	1100	83.30	1100	0.00	21.29	1100	0.00	92.47	1100	0.00	150.30	1070.63	2.67
10	420	91.28	420	0.00	7.62	420	0.00	19.66	420	0.00	19.15	420	0.00
11	1400	323.13	1400	0.00	46.17	1391.67	0.59	108.70	1400	0.00	407.97	1384.38	1.12
12	1450	422.49	1450	0.00	26.08	1447	0.21	88.33	1450	0.00	329.46	1405.50	3.07
13	420	0.50	420	0.00	1.81	410	2.38	3.38	417.15	0.68	1.68	400	4.76
14	1270	529.50	1270	0.00	25.77	1260	0.79	71.02	1270	0.00	270.16	1208.75	4.82
15	395	9.39	395	0.00	2.58	375	5.06	8.13	387	2.03	4.35	375	5.06
16	640	517.54	640	0.00	12.89	626.67	2.08	27.70	640	0.00	43.98	620	3.13
17	580	568.90	580	0.00	12.85	540	6.90	21.11	580	0.00	48.20	565	2.59
18	460	120.26	460	0.00	8.85	445	3.26	19.87	455	1.09	27.32	442.50	3.80
19	360	17.79	360	0.00	5.68	360	0.00	13.01	360	0.00	29.02	350	2.78
20	470	2076.64	470	0.00	8.95	446.67	4.96	20.35	470	0.00	27.31	457.50	2.66
21	1450	5801.61	1450	0.00	39.34	1450	0.00	97.66	1450	0.00	577.28	1410	2.76
22	1035	295.88	1035	0.00	16.40	1029.29	0.55	51.53	1029.29	0.55	88.19	1019.50	1.50
23	5715	TL	2504.28	56.18	158.18	2500	56.26	302.46	2505	56.17	10597.09	2478.34	56.63
24	5115	TL	2680	47.61	128.38	2680	47.61	208.42	2680	47.61	11651.14	2614.45	48.89
25	1320	TL	1315	0.38	32.70	1305	1.14	72.36	1313.34	0.50	369.13	1300	1.52
26	750	532.73	750	0.00	14.47	740	1.33	49.56	750	0.00	66.27	750	0.00
27	730	204.87	730	0.00	9.34	730	0.00	37.01	730	0.00	25.98	725	0.68
28	2250	TL	1565	30.44	63.53	1450	35.56	119.56	1570	30.22	950.67	1540	31.56
29	2040	TL	1545	24.26	59.86	1450	28.92	106.92	1570	23.04	923.66	1530.50	24.98
30	2165	TL	1560.84	27.91	62.21	1450	33.03	126.22	1590	26.56	790.74	1560	27.94
31	2070	TL	1554.45	24.91	57.02	1420	31.40	94.32	1570	24.15	545.36	1545	25.36
32	1830	TL	1360	25.68	48.47	1295	29.23	72.45	1390	24.04	884.74	1359.67	25.70
33	1920	TL	1390	27.60	54.01	1300	32.29	75.38	1410	26.56	655.71	1371.67	28.56
34	1920	TL	1380	28.13	46.06	1305	32.03	61.65	1390	27.60	948.29	1340.28	30.19
Rank:		23			7			28			2		
Avg.:		11.32			13.69			11.23			13.31		

Table 19 shows the number of generated columns (*NoC*) for each extended formulation. The last row of the table shows the average time per column (*Avg. T/C*). These numbers provide some measure on how fast each subproblem is solved. They show that EF1 and EF2 provided the best execution time performance in the tested instances.

Table 19 – Number of generated columns (*NoC*) in formulations EF1, EF2 and EF3.

ID	EF1			EF2		EF3	
	Time (s)	NoC λ	NoC χ	Time (s)	NoC λ	Time (s)	NoC λ
1	24.41	3447	728	47.16	1782	191.84	1145
2	24.79	3610	754	47.89	1770	181.99	1120
3	41.01	4490	1113	62.40	2032	443.43	1622
4	38.00	4465	1068	64.63	2065	819.46	3133
5	25.72	3182	996	35.82	1412	156.03	1353
6	41.30	4650	968	75.69	2183	442.19	1849
7	40.84	5123	866	118.60	3517	1168.52	2105
8	6.22	1653	354	22.19	1266	36.01	676
9	21.29	3428	600	92.47	2703	150.30	1075
10	7.62	1883	540	19.66	891	19.15	413
11	46.17	4802	696	108.70	3191	407.97	1495
12	26.08	4128	506	88.33	2971	329.46	1301
13	1.81	496	103	3.38	347	1.68	157
14	25.77	3727	603	71.02	2767	270.16	1161
15	2.58	834	256	8.13	589	4.35	269
16	12.89	2410	620	27.70	1195	43.98	669
17	12.85	2155	687	21.11	999	48.20	709
18	8.85	2006	616	19.87	869	27.32	571
19	5.68	1423	511	13.01	673	29.02	744
20	8.95	2067	641	20.35	933	27.31	499
21	39.34	4902	588	97.66	2933	577.28	1337
22	16.40	2883	486	51.53	1915	88.19	738
23	158.18	9879	1530	302.46	6096	10597.09	3288
24	128.38	8878	1653	208.42	5741	11651.14	4411
25	32.70	4375	688	72.36	2546	369.13	1465
26	14.47	2693	561	49.56	1686	66.27	726
27	9.34	2174	423	37.01	1409	25.98	476
28	63.53	5074	1500	119.56	3282	950.67	1920
29	59.86	5083	1467	106.92	3128	923.66	1979
30	62.21	5091	1498	126.22	3277	790.74	1844
31	57.02	5007	1430	94.32	3075	545.36	1774
32	48.47	4511	1408	72.45	2601	884.74	2082
33	54.01	4562	1424	75.38	2696	655.71	1790
34	46.06	4406	1332	61.65	2472	948.29	2062
Avg. T/C:			0.006	0.028		0.378	

3.3.2 Results for problem HSTP-B

This section reports the results of formulations CF2-Cuts and EF2 adapted to problem HSTP-B. For this problem, we consider the instances proposed by Souza (2000). The instances are shown in Table 20. As mentioned in Section 2.4, these instances have optimal solutions known, which is a contribution of previous studies using heuristic solutions (SOUZA; MACULAN; OCHI, 2004; SANTOS; OCHI; SOUZA, 2005; SAVINIEC *et al.*, 2013; DORNELES; ARAÚJO; BURIOL, 2014) and column generation lower bounds (SANTOS *et al.*, 2012; DORNELES;

ARAÚJO; BURIOL, 2017). According to Santos, Ochi and Souza (2005), the objective function weighting parameters for these instances are: $\alpha_7 = 1$, $\alpha_8 = 3$ and $\alpha_9 = 9$.

Table 20 – Features of the instances proposed by Souza (2000).

ID	$ C $	$ T $	$ D $	$ H $	$\sum_{r \in R} \tilde{\theta}_r$	$\sum_{r \in R} \tilde{\pi}_r$	Optimal value
1	3	8	5	5	75	21	202
2	6	14	5	5	150	29	333
3	8	16	5	5	200	4	423
4	12	23	5	5	300	41	652
5	13	31	5	5	325	71	762
6	14	30	5	5	350	63	756
7	20	33	5	5	500	84	1017

Table 21 – The lower bounds generated by formulation EF2 on the instances of Souza (2000).

ID	Time (s)	NoC λ	LB	Gap
1	14.66	371	202	0.00
2	24.60	848	333	0.00
3	22.54	904	423	0.00
4	35.08	1364	652	0.00
5	39.73	1677	762	0.00
6	89.80	2025	756	0.00
7	82.39	2216	1017	0.00

Table 21 presents the results of EF2 to the instances of Souza (2000). The CPLEX was run with a single thread, and the remaining parameters were set to default. The column (*Gap*) presents the gap between the lower bounds (*LB*) of EF2 and the optimal solutions. The results show that EF2 reached the optimal lower bounds for all instances. These results are competitive with Santos *et al.* (2012) and Dorneles, Araújo and Buriol (2017), which reported the same results with their column generation approaches.

Dorneles, Araújo and Buriol (2017) also reported results of a compact formulation based on a multi-commodity flow problem (CF-MCFP). In table 22, we compare these results with results of our compact formulation CF2-Cuts. The best results are shown in bold font. The program of CF2-Cuts was run with the same parameter settings used by Dorneles, Araújo and Buriol (2017), that are: CPLEX single thread and a time limit of two hours. The columns (LP_A) and (LP) show that the two formulations generated the same linear relaxation values. Regarding the integer programs, our formulation provided better lower bounds than CF-MCFP after two hours. On the other hand, the program of CF-MCFP found better integer solutions than the program of CF2-Cuts. While CF-MCFP ranked six best solutions, CF2-Cuts ranked only four best solutions. The CF2-Cuts also failed in finding a feasible solution to instance 7 during two hours. However, the lower bound found to instance 7 is optimal. We also observed that for instances 4 and 7, the Fenchel cuts led to optimal lower bounds (column LP_B).

Table 22 – Formulation CF2-Cuts compared to formulation CF-MCFP on problem HSTP-B.

ID	CF2-Cuts								CF-MCFP					
	Time (s)	LP_A	Time (s)	LP_B	Time (s)	LB	UB	Gap	Time (s)	LP	Time (s)	LB*	UB	Gap
1	0.08	189	1.66	190	126.96	202	202	0.00	0.10	189	TL	190.24	202	5.82
2	0.67	333	18.84	333	1482.61	333	333	0.00	0.40	333	196	333	333	0.00
3	0.39	414	19.29	414	TL	418	444	5.86	0.20	414	TL	413.99	429	3.50
4	2.08	643	83.92	652	1057.08	652	652	0.00	0.80	643	988	652	652	0.00
5	4.05	756	273.95	757	TL	757.50	763	0.72	2.30	756	TL	756.02	777	2.70
6	6.06	738	375.48	750	TL	750	935	19.79	2.80	738	TL	739.52	804	8.02
7	12.69	999	842.92	1017	TL	1017	–	–	7.80	999	TL	1006.58	1645	38.81
Rank:						7	4					2	6	

* [Dorneles, Araújo and Buriol \(2017\)](#) did not report the lower bounds (LB) of their integer programs after two hours. We calculated these values by the formula $LB = UB(1 - Gap \div 100)$, where Gap was reported to be the CPLEX's optimality gap.

3.3.3 Results for problem HSTP-C

This section reports the results of formulations CF2-Cuts and EF2 adapted to problem HSTP-C. For this problem, we consider the instances of Table 15. The objective function weighting parameters were set to: $\alpha_7 = 1$, $\alpha_8 = 3$ and $\alpha_9 = 9$.

The results are shown in Table 23. For this problem, instances 15 and 22 were proven to be infeasible. For the remaining instances, the table compares the lower bounds of formulations CF2-Cuts and EF2. The columns (*Gap*) show the gap between the lower bound of each formulation and the best integer solution (*UB*) of CF2-Cuts. The best gaps are shown in bold font, and shaded rows are instances for which the optimality was proven. For this problem, the compact formulation CF2-Cuts performed slightly better than EF2 in the quality of lower bounds. Both formulations ranked the same number of best lower bounds, and we could prove the optimality for 25 out of the 32 instances that are not infeasible.

Table 23 – Results of formulations CF2-Cuts (TL = 3 h) and EF2 on problem HSTP-C.

ID	CF2-Cuts								EF2			
	Time (s)	LP_A	Time (s)	LP_B	Time (s)	LB	UB	Gap	Time (s)	NoC λ	LB	Gap
1	0.37	682.50	7.50	689	1446.54	699	699	0.00	49.06	1828	699	0.00
2	0.35	682.50	11.75	689	1689.78	699	699	0.00	49.83	1817	699	0.00
3	0.54	727.50	11.70	735	3900.23	742	742	0.00	58.25	2114	742	0.00
4	0.51	726.50	22.04	731	3977.16	734	734	0.00	59.99	2131	734	0.00
5	0.35	627	39.46	627	729.99	631	631	0.00	32.32	1427	631	0.00
6	0.56	771.50	24.53	772	2942.94	772	772	0.00	71.08	2282	772	0.00
7	0.58	1197.50	14.15	1209	10799.32	1212	1212	0.00	95.15	3281	1212	0.00
8	0.17	637.86	1.09	664.80	66.33	675	675	0.00	21.86	1293	669	0.89
9	0.35	797.18	6.84	805.25	247.16	807	807	0.00	59.83	2200	806.25	0.09
10	0.18	298	3.70	298	72.79	298	298	0.00	18.17	884	298	0.00
11	0.45	942.19	9.01	948.50	1972.01	961	961	0.00	76.99	2908	959.87	0.12
12	0.37	996.13	5.01	1014.74	498.25	1018	1018	0.00	82.87	2946	1014.74	0.32
13	0.04	261.50	0.05	265	0.72	273	273	0.00	2.51	302	270.67	0.85
14	0.36	911.23	8.33	930	761.33	933	933	0.00	48.35	2476	933	0.00
15	Infeasible											
16	0.24	472.67	5.39	477	485.21	481	481	0.00	22.55	1196	481	0.00
17	0.22	455.50	10.20	457	308.88	457	457	0.00	20.56	1078	457	0.00
18	0.18	309.25	3.81	316	100.77	319	319	0.00	19.62	859	319	0.00
19	0.15	251	3.09	254	14.75	254	254	0.00	15.60	738	254	0.00
20	0.19	310.50	3.79	317	224.26	325	325	0.00	24.83	1034	325	0.00
21	0.62	1036	13.41	1050	6163.85	1050	1050	0.00	73.70	2760	1050	0.00
22	Infeasible											
23	1.51	1789.19	39.77	1812	TL	1813	2357	23.08	273.51	6107	1813	23.08
24	1.03	1932.79	30.19	1983.50	TL	1988	2535	21.58	201.99	5832	1988	21.58
25	0.47	904	27.97	909	343.97	912	912	0.00	56.54	2408	912	0.00
26	0.29	570	8.94	570	245.98	570	570	0.00	37.02	1594	570	0.00
27	0.21	549.75	4.55	552	140.12	552	552	0.00	27.67	1322	552	0.00
28	0.63	1104	64.61	1105	TL	1122.13	1190	5.70	95.61	3067	1124	5.55
29	0.68	1095	43.33	1096	TL	1112.94	1306	14.78	115.67	3136	1116	14.55
30	0.61	1113	86.22	1114	TL	1131.40	1703	33.56	107.54	3154	1133	33.47
31	0.56	1112	60.81	1113	2618.76	1127	1127	0.00	79.59	2966	1127	0.00
32	0.57	999	53.11	999	TL	1010	1740	41.95	78.22	2658	1014	41.72
33	0.54	999.50	51.71	1001	TL	1016	1081	6.01	67.91	2613	1018	5.83
34	0.53	980	37.11	981	9355.78	1005	1005	0.00	73.85	2557	1005	0.00
Rank:									27			
Avg.:									4.58			

3.4 Final remarks

In this chapter, we conducted a study with mixed-integer programming formulations for high school timetabling problems. We proposed two compact formulations CF1 and CF2, and three extended formulations EF1, EF2, and EF3. The first formulation is based on traditional models. The other four formulations are alternative models based on the concept of “patterns”, for which variables represent all possible layouts of assigning requirements or teachers to periods within days or during the week. The alternative compact formulation CF2 allows the generation of new cuts, based on the Fenchel cuts technique, to strengthen its linear relaxation. Our three extended formulations EF1, EF2, and EF3 are based on this new model. As the extended formulations have huge numbers of variables that cannot be explicitly generated, we resort to column generation to solve their linear relaxations and provide lower bounds to the problem. We tested these formulations in three different HSTP problems. Regarding the compact formulations, we observed that:

- Formulation CF2 has a better linear relaxation than CF1. However, for large size instances, CF1 obtain better quality integer solutions than CF2.
- The Fenchel enumeration cutting planes technique provides cheap and strong cuts to CF2 on the considered problems. In some instances, the addition of such cuts was sufficient to reach the optimal lower bounds.
- While the classical formulation CF1 proved optimality only for small size instances, the alternative compact formulation CF2, with the addition of the Fenchel cuts, proved the optimality for small and also for medium size instances.
- In comparison with the state-of-the-art compact formulation ([DORNELES; ARAÚJO; BURIOL, 2017](#)) to problem HSTP-B, our compact formulation CF2 (with cuts) performs better in the quality of lower bounds, but it is a bit worse in the quality of integer solutions.
- Formulation CF2 (with cuts) performs better for problems that have a more restricted feasible region, as in problem HSTP-C.

Regarding the proposed extended formulations, we noted that:

- The best formulation EF2 provides a cheap way of obtaining strong lower bounds to the three tested problems. In most of the cases, it provided optimal lower bounds.
- The best formulation EF2 is competitive in the quality of lower bounds with previous extended formulations proposed in the literature ([SANTOS *et al.*, 2012](#); [DORNELES; ARAÚJO; BURIOL, 2017](#)).

Lastly, the alternative formulations helped to prove optimality for most of the tested instances for which the traditional formulation CF1 failed.

Future extensions of this research can be done in order to devise a branch-and-price method that exploits the column generations proposed in this chapter.

PARALLEL METAHEURISTICS

This chapter is strongly based on the paper “Parallel local search algorithms for high school timetabling problems” (SAVINIEC; SANTOS; COSTA, 2018) which is part of this thesis.

Due to the difficulty of devising MIP formulations for high school timetabling problems that can be efficiently solved by MIP solvers, most of the literature has been concerned with metaheuristic techniques (PILLAY, 2014). The primary goal of such studies is to develop strategies to generate good quality solutions within reasonable computational effort. Recently proposed metaheuristics for the HSTP can be found in Fonseca and Santos (2014), Dorneles, Araújo and Buriol (2014), Fonseca *et al.* (2016), Fonseca, Santos and Carrano (2016), Saviniec and Constantino (2017) and Skoullis, Tassopoulos and Beligiannis (2017). These methods are shown to be efficient in the sense that near-optimal solutions are consistently found for different input instances.

In this chapter, we investigate parallel metaheuristics for the HSTP, in which different solution methods (agents) are run concurrently in different processor threads. The motivation is two-fold. On the one hand, the availability of multi-processor machines and appropriate coding schemes make the use of parallel algorithms more accessible than ever. On the other hand, although the literature is abundant in parallel solution methods for similar complex problems (SUBRAMANIAN *et al.*, 2010; BOŽEJKO; PEMPERA; SMUTNICKI, 2013; SÁNCHEZ-ORO *et al.*, 2015; LUQUE; ALBA, 2015), very little attention has been given to the design of parallel methods for the HSTP. The reader is referred to Section 4.1 for a closer look at these contributions (ABRAMSON, 1991; ABRAMSON; ABELA, 1992; SRNDIC *et al.*, 2009).

This chapter provides an exploratory study that investigates some questions related to the design of parallel metaheuristics in the context of the HSTP. Our central research questions can be summarized as: (1) What are good parallel strategies for the HSTP? (2) Can these strategies improve the performance of stand-alone metaheuristics? (3) Are the proposed parallel metaheuristics competitive with state-of-the-art methods?

To provide some insight into these questions, we carry out a thorough computational study that tests all parallelization schemes resulting from combinations of the following characteristics:

Agents cooperation: we wish to test the effect of allowing the agents to cooperate, by sharing good solutions among threads.

Diversification: we wish to examine the effect of using all agents as search intensification mechanisms or allowing at least one agent to diversify the search.

Agents diversity: we wish to test homogeneous and heterogeneous algorithms. An algorithm is *homogeneous* when all threads execute the same metaheuristic, and *heterogeneous* otherwise.

As stand-alone metaheuristics, we use the iterated local search from Saviniec *et al.* (2013) and propose versions of tabu search, simulated annealing, and late acceptance strategy. We compare the performance of the different parallel algorithm variants against each other, against the stand-alone metaheuristics and two state-of-the-art algorithms for problems HSTP-B and HSTP-C (DORNELES; ARAÚJO; BURIOL, 2014; FONSECA *et al.*, 2016).

The chapter is organized as follows. Section 4.1 gives an introduction to different parallelization schemes and reviews the existing parallel algorithms for the HSTP. Section 4.2 explains our parallel metaheuristics. Section 4.3 presents our computational experiments and Section 4.4 concludes the chapter with final remarks and suggestions for further investigations. Appendix A completes this chapter with details on the stand-alone metaheuristics and their parameter setting procedures.

4.1 Existing parallel algorithms for high school timetabling

Metaheuristics can be classified into *population-based (PB)* and *trajectory-based (TB)* methods (ALBA; LUQUE; NESMACHNOW, 2013). Population-based metaheuristics are characterized by keeping a pool of solutions (e.g., genetic algorithms, ant colony optimization and particle swarm optimization). The method starts with an initial population and employs, at each step, stochastic operators to evolve toward better quality populations. According to Alba, Luque and Nesmachnow (2013), two classical parallel algorithm frameworks have been used for population-based metaheuristics:

- **Parallel individuals evaluation (PB-PIE):** each individual in the population can be evaluated in parallel.
- **Parallel islands (PB-PI):** the initial population is split into a set of subpopulations (islands) in which a sequential algorithm can be employed. In this framework, the islands often exchange individuals among each other to diversify the subpopulations and prevent early convergence.

Trajectory-based metaheuristics are characterized by methods which employ a single current solution. The method starts with an initial solution and at each step, goes through neighborhoods of the current solution tracing trajectories in the search space of the problem. Some well-known metaheuristics in this class are: simulated annealing, tabu search, iterated local search and variable neighborhood search. Three classical parallel algorithm frameworks have been used in the literature for trajectory-based metaheuristics according to [Alba, Luque and Nesmachnow \(2013\)](#):

- **Parallel moves (TB-PM)**: the neighborhood of the current solution is explored in parallel. This framework is usually implemented by a manager/workers technique. At each step, the manager replicates the current solution to a set of parallel workers, each worker then computes the objective function of one of the neighboring solutions and returns it to the manager, who takes decisions.
- **Move acceleration (TB-MA)**: the objective function of a single solution is evaluated in parallel. This framework is an attractive approach when the sequential evaluation of the objective function is both time-consuming and can be decomposed into small parts to be computed in parallel.
- **Parallel multi-start (TB-PMS)**: several asynchronous threads of trajectory-based methods run simultaneously to compute high-quality solutions. These threads may be homogeneous (same method) or heterogeneous (distinct methods). They may or may not cooperate with each other. They may also start with the same solution seed or from different ones.

[Luque and Alba \(2015\)](#) state that parallel strategies for trajectory-based methods have been less studied than for population-based ones. For HSTP, we have found only three studies related to parallel metaheuristics. [Abramson \(1991\)](#) proposed a simulated annealing which employs a series of locks to enable several processes to perform simultaneous moves in the same timetable data structure (TB-PM). [Abramson and Abela \(1992\)](#) proposed a genetic algorithm that, at each step, generates and evaluates individuals in parallel (PB-PIE). [Srndic et al. \(2009\)](#) proposed a genetic algorithm approach in which the global population is divided into small subpopulations that are managed by different processes (PB-PI).

In this chapter, we focus on trajectory-based methods. Particularly, we propose parallel multi-start methods for the HSTP using as agents sequential iterated local search ([LOURENÇO; MARTIN; STÜTZLE, 2003](#)), tabu search ([GLOVER, 1990](#)), simulated annealing ([KIRKPATRICK; GELATT; VECCHI, 1983](#)) and late acceptance strategy ([BURKE; BYKOV, 2008](#)) metaheuristics. Details of these metaheuristics are presented in Appendix A. In the next section, we concentrate on the parallel algorithms.

4.2 Proposed parallel metaheuristics

The proposed algorithms follow the trajectory-based parallel multi-start (TB-PMS) scheme described in Section 4.1 and employ manager/workers strategies with shared memories. We propose two approaches. The first is based on central memory (CRAINIC *et al.*, 2004) and the second is based on diversification and intensification memories (JIN; CRAINIC; LØKKETANGEN, 2014).

(a) Central memory based: a parallel multi-start framework using a manager/workers strategy and a central memory of solutions.

(b) Diversification-intensification memory based: a parallel multi-start framework with diversification and intensification memories.

Figure 5 – Parallel multi-start metaheuristic frameworks.

4.2.1 Central memory based algorithms

The central memory based (CMB) framework, shown in Figure 5a, has a central memory which keeps up to X solutions, a manager procedure, N metaheuristic worker agents and two synchronization buffers, input and output. The main idea in this framework is that a group of metaheuristic agents can execute concurrently while possibly cooperating with each other through an exchange of current solutions. This exchange is done via the central memory, which keeps a set of elite solutions.

Each agent pulls a solution from the input buffer, operates on it for a parameter-defined amount of time, and then returns its best solution to the output buffer. The manager is responsible for implementing the policies that will determine the exchange of solutions between the two intermediate buffers (input and output) and the central memory. In our implementations with this framework, the following policies are adopted:

- *Memory initialization*: the central memory is initialized with solutions generated by the constructive heuristic described in Algorithm 7, see Appendix A.

- *Input selection*: a solution is randomly selected from the central memory and put into the input buffer whenever requested by an agent.
- *Output acceptance*: when the manager retrieves a solution Z_r from the output buffer, the solution Z_r is compared sequentially with solutions in the central memory. A first-improvement strategy is adopted, i.e., the first visited solution that is worse than or equal to Z_r is replaced. If no solution meets this criterion, then Z_r is discarded.
- *Agents' execution time*: each of the N agents is allowed to run for Y seconds before pushing its best solution into the output buffer and requesting a new solution from the input buffer. Short execution time defines cooperative agents, while long execution time defines independent agents.

4.2.2 Diversification-intensification memory based algorithms

The diversification-intensification memory based (DIMB) framework, shown in Figure 5b, is an extension of the CMB framework. The central memory is now divided into a 'diversification memory' and an 'intensification memory'. The first memory keeps a set of non-elite solutions to diversify the search while the second maintains a set of elite solutions.

One agent is responsible for providing diversified solutions to the diversification memory while all others search for better quality solutions. The details of our implementation are as follows:

- *Memory initialization*: the two memories are initialized with solutions generated by the constructive heuristic described in Algorithm 7.
- *Input selection*: whenever requested by an agent, a solution is selected from the diversification memory with probability ρ and from the intensification memory with probability $(1 - \rho)$.
- *Output acceptance*: when the manager retrieves a solution Z_r from the output buffer, it accepts the solution into the intensification memory if it is better than or equal to a solution randomly chosen in the intensification memory. Otherwise, Z_r is discarded.
- *Agents' execution time*: each of the $N - 1$ intensification agents is allowed to run for Y seconds before pushing its best solution into the output buffer and requesting a new solution from the input buffer.
- *Diversification memory update*: The solutions of this memory are continuously replaced by solutions with better or equivalent quality generated by the diversifier agent during run-time. The diversifier is, usually, a metaheuristic with slow convergence or that quickly converges to poor local optima and keeps performing plain moves (moving to other

solutions with the same quality). In our implementation, the diversifier agent is a late acceptance strategy metaheuristic with plain moves and set with parameters that lead to premature convergence. Whenever the late acceptance strategy updates its current solution Z_c , the solution Z_c is compared with a random solution Z_d in the diversification memory. If Z_c is better or equal to Z_d in terms of objective value, the metaheuristic replaces Z_d by Z_c .

In the next section, we test these two frameworks with several parameter configurations. We investigate cases in which the agents are homogeneous or heterogeneous, and cooperative or independent.

4.3 Computational experiments

In this section, we conduct a thorough computational study to evaluate the performance of the proposed parallel methods. All algorithms were coded in C++ and compiled with GNU Compiler Collection 4.4.7. The experiments were done on a server running *Red Hat Enterprise Linux 6.5*. The hardware is composed of two CPU Intel Xeon E5-2680v2 (2.8 GHz) and 128 GB of RAM. To implement parallelism, we employed the POSIX Threads Library (Pthreads) available in the GNU Compiler Collection. In all experiments reported, the metaheuristics are allowed to execute during a time limit of 625 seconds, and 25 trials (different seeds) are used for each instance. The seeds were generated by standard C++ libraries, and we used the same seeds for every metaheuristic configuration.

The experiments have been set up in two phases: in Section 4.3.1, we compare the different proposed methods against each other on problem HSTP-A. The following tests are executed:

Section 4.3.1.1: tests with the CMB parallel framework proposed in Section 4.2.1.

Section 4.3.1.2: tests with the DIMB parallel framework proposed in Section 4.2.2.

Section 4.3.1.3: tests comparing the best sequential and parallel algorithm configurations.

Section 4.3.1.4: tests comparing our best parallel algorithm with the MIP program of formulation CF2 proposed in Chapter 3.

Then, in Section 4.3.2, the best parallel algorithm is also adapted to problems HSTP-B and HSTP-C and compared with state-of-the-art literature methods:

Section 4.3.2.1: tests comparing our best parallel algorithm with methods reported to problem HSTP-B.

Section 4.3.2.2: tests comparing our best parallel algorithm with the solver winner of the ITC-2011 on problem HSTP-C.

The interested reader is referred to Appendix A, where the used stand-alone metaheuristics are described along with the procedure we used for setting their best parameters.

4.3.1 Comparisons among proposed methods

For tests 4.3.1.1 to 4.3.1.2, we use the reduced set of six instances (6, 7, 10, 13, 24, and 26) from Table 15, which were used to calibrate the sequential metaheuristics in Section A.6. For tests 4.3.1.3 to 4.3.1.4, we use all 34 instances described in Table 15.

4.3.1.1 Results for central memory based algorithms

The two parallel frameworks CMB and DIMB, outlined in Section 4.2, generate different algorithms for different combinations of metaheuristic agents. Also, agents with short execution time (parameter Y) produce cooperative searches while agents with long execution time lead to independent searches. In this section, we analyze algorithms which include homogeneous, heterogeneous, cooperative and independent agents. We assess how much these parallel algorithms are better than the sequential ones and how cooperation can play a role in the quality of solutions.

We first analyze four cases of CMB parallel metaheuristics. In each case, the agents are homogeneous threads of one of the metaheuristics ILS, TS, SA or LAS described in Section A.5. For each case, we tested configurations in which the central memory contains $X = (1, 4, 8, 16, 32)$ solutions and agents time $Y = (1, 2, 5, 10, 30, 90, 315, 625)$ seconds. Cases with homogeneous (HM) ILS showed the overall best results.

Figure 6 plots the best configurations with homogeneous ILS. In this figure and the remaining figures in this chapter, the y-axis represents normalized objective values obtained as follows. Let Z_j be a timetable solution for the j^{th} instance, its normalized objective value is given by $NOV(Z_j) = f(Z_j)/f(Z_j^*)$. Where $f(Z_j^*)$ is a benchmark objective value for instance j , which can be a lower bound or the best-known solution. As at each point x each metaheuristic collects 25 samples for each instance, the final y-value in a point x is the median of the instances' median sample. In these charts, the reference line ($y = 1$) represents the benchmark objective value $f(Z_j^*)$. In other words, the charts summarize the results for all instances and indicate, for each time point, how close or far the algorithm' median solutions are from the benchmark value, providing insights on the algorithm convergence rates.

Figure 6 plots the two best configurations with cooperative agents (*thin lines*) which were HM-ILS-1-30 with $(X, Y) = (1, 30)$ and HM-ILS-4-30 with $(X, Y) = (4, 30)$, the best configuration with independent agents (*blue-thick line*) which is HM-ILS-4-625 with $(X, Y) = (4, 625)$ and the best sequential metaheuristic, the stand-alone ILS (*red-thick line*). These results indicate

Figure 6 – Best results for CMB with homogeneous ILS agents (4 threads).

that parallel versions with homogeneous agents provide better quality solutions than sequential versions. We also observed this fact in the experiments using TS, SA and LAS as agents. In Figure 6, we also observe that after 400 seconds, configurations with cooperative ILS agents perform slightly better than their independent counterparts.

Figure 7 – Best results for CMB with homogeneous TS and LAS agents (4 threads).

Figure 7 plots the best configurations with homogeneous TS (*black lines*) and LAS (*blue lines*). We also plot the results of the stand-alone ILS in order to provide a common benchmark among different charts. The figure shows that configurations with cooperative agents (*thin lines*) perform significantly better than configurations with independent agents (*thick lines*).

On the other hand, for SA, cooperative agents performed worse than independent agents. This is shown in Figure 8, where the *thin line* plots the best configuration with cooperative agents, the *blue-thick line* represents the best configuration with independent agents and the *black-thick line* is the best sequential SA. These results probably indicate that more sophisticated tuning procedures need to be used for the parallel SA, in order to balance the diversification provided by the parallel contribution and the natural diversification structure of the SA for high temperatures.

Figure 8 – Best results for CMB with homogeneous SA agents (4 threads).

Figure 9 – Best results for CMB with heterogeneous agents (4 threads).

For CMB algorithms with heterogeneous (HT) agents, we tested the case in which there is one agent thread for each metaheuristic ILS, TS, SA and LAS. The TS agent was set with

parameters $(stSize, tbTime) = (100, 100)$, SA with $(T_0, \alpha, stSize, \epsilon) = (100, 0.9999, 25, 1)$ and LAS with $(lsSize, stSize) = (100, 100)$. These parameters were chosen based on our sequential experiments and considering a trade-off between fast convergence and quality of solutions. As before, we tested configurations with central memory sizes $X = (1, 4, 8, 16, 32)$ and agents time $Y = (1, 2, 5, 10, 30, 90, 315, 625)$. In Figure 9, we plot the two best configurations with cooperative agents (*thin lines*), which were $(X, Y) = (1, 10)$ and $(1, 30)$, and the best configuration with independent agents (*blue-thick line*), which is $(X, Y) = (1, 625)$. We observe that, configurations with cooperative agents (*thin lines*) perform better than configurations with independent agents (*blue-thick line*). Nevertheless, no heterogeneous configuration presented better results than the homogeneous ILS configuration *HM-ILS-1-30*, plotted in *black-thick line* in Figure 9.

Finally, it is interesting to see how all best versions used the central memory with a single solution, indicating a strong elitism characteristic.

4.3.1.2 Results for diversification-intensification memory based algorithms

In this section, we analyze a case of DIMB parallel metaheuristics in which the intensifier agents are several threads of ILS, which proved earlier to be the best sequential intensifier metaheuristic. As a diversifier agent, we use the LAS with parameters $(lsSize, stSize) = (1000, 25)$. Our preliminary tests showed that this metaheuristic provides diversified solutions to the diversification memory, because it quickly converges to local optima with quality near to 10 % worse than the benchmark values, and it keeps moving to other solutions with the same quality.

Figure 10 – Best results for DIMB parallel metaheuristics.

The DIMB algorithm was tested with four intensifiers (with a single solution in the intensification memory) and diversification memories with sizes $X_D = (1, 4, 8, 16, 32)$ combined with probabilities $\rho = (0.05, 0.10, 0.20, 0.25, 0.50)$. The best results were obtained with configu-

ration $X_D = 8$ and $\rho = 0.05$. We also tested this configuration with nineteen intensifier threads. In Figure 10, we show these two configurations (algorithms DIMB-4I1D-8-0.05 and DIMB-19I1D-8-0.05) and the best configuration of the CMB obtained earlier (algorithm HM-ILS-1-30). Comparing algorithms DIMB-4I1D-8-0.05 and HM-ILS-1-30, we observe that the scheme with diversification-intensification memories (DIMB) presents better performance than the scheme with central memory (CMB). Figure 10 also shows that increasing the number of threads has a beneficial effect on performance.

4.3.1.3 Sequential versus parallel algorithms

Figure 11 plots the distribution of the instances' median samples for parallel and sequential algorithms in the 34 instances from Table 15. The asterisks (*) represent outliers (best or worst solutions). This analysis provides some insights on the algorithms' robustness throughout the whole set of instances.

Figure 11 – Results of the best parallel algorithm DIMB-19I1D-8-0.05 compared to results of the best sequential ILS in all 34 instances from Table 15.

The best solutions of the sequential ILS were used as benchmark values. We observe that the parallel algorithm solutions after 81 seconds are statistically equivalent (supported by a Kruskal-Wallis test for the equality of medians) to those of the sequential ILS algorithm after 400 seconds, which is the time when the sequential ILS stagnates. Apart from being faster, the parallel algorithm is also more robust than the sequential version in all instances, as it presents less scattered distributions.

4.3.1.4 Comparisons with formulation CF2

In this section, we compare the results of the best parallel algorithm DIMB-19I1D-8-0.05 with results of the best formulation CF2 in the 34 instances described in Table 15.

Table 24 – Results of algorithm DIMB-19I1D-8-0.05 compared to results of formulation CF2 in all 34 instances of Table 15.

ID	Best LB	CF2-Cuts (3 h)		DIMB-19I1D-8-0.05 (625 s)			
		UB	Gap	Median	Gap	Best	Gap
1	950	1040	8.65	970	2.06	950	0.00
2	950	950	0.00	970	2.06	950	0.00
3	1010	1040	2.88	1020	0.98	1010	0.00
4	990	1900	47.89	1000	1.00	990	0.00
5	870	870	0.00	870	0.00	870	0.00
6	1080	1340	19.40	1100	1.82	1080	0.00
7	1690	3525	52.06	1880	10.11	1810	6.63
8	950	950	0.00	1030	7.77	980	3.06
9	1100	1100	0.00	1170	5.98	1130	2.65
10	420	420	0.00	420	0.00	420	0.00
11	1400	1400	0.00	1530	8.50	1450	3.45
12	1450	1450	0.00	1600	9.38	1530	5.23
13	420	420	0.00	420	0.00	420	0.00
14	1270	1270	0.00	1360	6.62	1300	2.31
15	395	395	0.00	395	0.00	395	0.00
16	640	640	0.00	650	1.54	640	0.00
17	580	580	0.00	590	1.69	580	0.00
18	460	460	0.00	460	0.00	460	0.00
19	360	360	0.00	360	0.00	360	0.00
20	470	470	0.00	470	0.00	470	0.00
21	1450	1450	0.00	1550	6.45	1480	2.03
22	1035	1035	0.00	1095	5.48	1055	1.90
23	2505	17790	85.92	2790	10.22	2670	6.18
24	2680	16825	84.07	2970	9.76	2860	6.29
25	1315	1320	0.38	1400	6.07	1340	1.87
26	750	750	0.00	780	3.85	750	0.00
27	730	730	0.00	750	2.67	730	0.00
28	1570	13420	88.30	1620	3.09	1590	1.26
29	1570	2040	23.04	1620	3.09	1590	1.26
30	1590	3500	54.57	1640	3.05	1600	0.63
31	1570	3480	54.89	1640	4.27	1590	1.26
32	1390	2185	36.38	1440	3.47	1410	1.42
33	1410	2755	48.82	1460	3.42	1430	1.40
34	1390	2205	36.96	1450	4.14	1410	1.42
Rank:		20		7		26	
Avg.:		18.95		3.78		1.48	

The results are shown in Table 24. The table presents the median and best solutions (out of 25 runs) of the DIMB best configuration found earlier compared to the best solutions of formulation CF2. The best solutions are shown in bold font. The results show that although the program of CF2 has proven optimality for most of the tested instances, it did not find good quality solutions for 12 out of the 34 tested instances. Our parallel algorithm, on the other hand, provided optimal and near-optimal solutions in all instances, including those in which CF2 provided poor quality solutions. In average, the gaps (computed using the best lower bounds

reported for these instances in Chapter 3, shown in column “Best LB”) of the parallel algorithm are much smaller than those of formulation CF2.

4.3.2 Comparisons with state-of-the-art algorithms

In this section, we provide evidence that the proposed parallel algorithm DIMB-19I1D-8-0.05 presents state-of-the-art results not only for problem HSTP-A but also for problems HSTP-B and HSTP-C, which have already been studied in the literature.

4.3.2.1 Comparisons with methods for problem HSTP-B

In this section, we report results of algorithm DIMB-19I1D-8-0.05 adapted to deal with problem HSTP-B. For these experiments, we consider the instances of Table 20, which have been used in the literature. The state-of-the-art algorithms for this problem are based on fix-and-optimize heuristics (DORNELES; ARAÚJO; BURIOL, 2014) and sequential ILS based metaheuristics (SAVINIEC *et al.*, 2013). The fix-and-optimize heuristics were reported to find optimal solutions in each instance, on time limits within 10 to 60 minutes in a computer with processor Intel Core i5-2300 (2.8 GHz) and 4 GB of RAM, running Linux OS. The ILS metaheuristics were reported to find optimal solutions in each instance, on a time limit of 15 minutes in a computer with processor Intel Xeon E7-4860 (2.26 GHz) and 30 GB of RAM, running Windows OS.

Figure 12 – Results of the algorithm DIMB-19I1D-8-0.05 on problem HSTP-B with instances of Table 20.

In Figure 12, we analyze the distribution of the instances’ median samples generated by our parallel algorithm DIMB-19I1D-8-0.05. The instances optimal values described in Table 20 were used as benchmark values. We observe that, in the median case, the algorithm converges to optimal solutions in any instance within a time limit of 49 seconds. These results show that

our algorithm is very robust to these instances and it is competitive with the state-of-the-art approaches.

4.3.2.2 Comparisons with methods for problem HSTP-C

Table 25 – Results of algorithm DIMB-19I1D-8-0.05 compared to results of GOAL on problem HSTP-C with instances of Table 15.

ID	Best LB	DIMB-19I1D-8-0.05 (625 s)				GOAL (625 s)			
		Median	Gap	Best	Gap	Median	Gap	Best	Gap
1	699	710	1.55	700	0.14	730	4.25	716	2.37
2	699	711	1.69	701	0.29	730	4.25	716	2.37
3	742	749	0.93	743	0.13	770	3.64	762	2.62
4	734	743	1.21	735	0.14	768	4.43	749	2.00
5	631	631	0.00	631	0.00	648	2.62	642	1.71
6	772	777	0.64	772	0.00	814	5.16	800	3.50
7	1212	1270	4.57	1239	2.18	1312	7.62	1297	6.55
8	675	698	3.30	686	1.60	682	1.03	676	0.15
9	807	827	2.42	814	0.86	838	3.70	829	2.65
10	298	298	0.00	298	0.00	300	0.67	298	0.00
11	961	1005	4.38	985	2.44	1052	8.65	1024	6.15
12	1018	1044	2.49	1030	1.17	1048	2.86	1034	1.55
13	273	273	0.00	273	0.00	273	0.00	273	0.00
14	933	954	2.20	939	0.64	971	3.91	949	1.69
15				Infeasible					
16	481	484	0.62	481	0.00	498	3.41	490	1.84
17	457	461	0.87	457	0.00	468	2.35	462	1.08
18	319	319	0.00	319	0.00	324	1.54	319	0.00
19	254	254	0.00	254	0.00	254	0.00	254	0.00
20	325	325	0.00	325	0.00	328	0.91	325	0.00
21	1050	1074	2.23	1058	0.76	1107	5.15	1091	3.76
22				Infeasible					
23	1813	1910	5.08	1867	2.89	2013	9.94	1971	8.02
24	1988	2079	4.38	2038	2.45	2171	8.43	2128	6.58
25	912	944	3.39	926	1.51	987	7.60	966	5.59
26	570	576	1.04	570	0.00	603	5.47	584	2.40
27	552	556	0.72	552	0.00	566	2.47	560	1.43
28	1124	1150	2.26	1127	0.27	1183	4.99	1167	3.68
29	1116	1145	2.53	1127	0.98	1187	5.98	1168	4.45
30	1133	1164	2.66	1142	0.79	1203	5.82	1175	3.57
31	1127	1165	3.26	1137	0.88	1195	5.69	1181	4.57
32	1014	1043	2.78	1024	0.98	1069	5.14	1063	4.61
33	1018	1047	2.77	1034	1.55	1078	5.57	1060	3.96
34	1005	1041	3.46	1015	0.99	1070	6.07	1056	4.83
Avg.:			1.98		0.74		4.35		2.93

Lastly, we compare our parallel algorithm DIMB-19I1D-8-0.05 with the heuristic solver GOAL (FONSECA *et al.*, 2016) on problem HSTP-C. This solver was the winner of the ITC-2011. It provides a pool of methods to solve timetabling problems from different countries. The latest version of this solver¹ (FONSECA; SANTOS; CARRANO, 2016), which was used in this comparison, is a three-phase approach. At the first phase, the KHE software library (KINGSTON, 2015) is employed to obtain a starting solution. This solution is refined by a

¹ Available in <<http://www.goal.ufop.br/software/hstt/>>

parallel VNS metaheuristic at the second phase, and lastly, the output of VNS is further improved with fix-and-optimize MIP heuristics.

Table 25 presents the results of 25 runs of both algorithms on instances of Table 15. Both algorithms were set with 20 threads and a time limit of 625 seconds. As suggested by the authors of GOAL, additional parameters of their solver were set to default values (alg-timelimit = 62, form-timelimit = 62, initial-soln = KHE, algorithm = SVNS, formulation = FIXOPT, formfixopt-nresources = 5 and formfixopt-optinarow = 5). In Table 25, bold font highlights the best solutions found. The results show that our algorithm outperforms GOAL, both in terms of median and best solutions. Our algorithm only performed worse than GOAL in instance 8. The probable reason is that this instance has a large number of teachers' unavailable periods, in which it is expected that MIP neighborhoods (such as those employed in GOAL) will perform better than conventional neighborhood approaches (such as the one employed in our algorithm). We also note that the average gap (computed using the best lower bounds reported for problem HSTP-C in Chapter 3, shown in column "Best LB") of our median solutions (1.98 %) is smaller than the average gap of the best solutions of GOAL (2.93 %).

Figure 13 – Results of algorithm DIMB-19I1D-8-0.05 after 25 and 625 seconds compared to results of GOAL after 625 seconds on instances of Table 15.

In Figure 13, we focus on the computational time spent by both algorithms. The chart shows the distribution of the instances' median solutions generated by DIMB-19I1D-8-0.05 after 25 and 625 seconds and the results of GOAL after 625 seconds. A Kruskal-Wallis test for the equality of medians was run and indicated that the solutions generated by our parallel algorithm after 25 seconds are of better quality than those of GOAL after 625 seconds.

4.4 Final remarks

In this chapter, we conducted an extensive study with parallel trajectory-based metaheuristics for high school timetabling problems. Our study investigated several aspects related to the design of such algorithms. In particular, we analyzed the homogeneity/heterogeneity of agents, the influence of intensification and diversification memories, and the frequency of information exchange among agents.

Our most efficient algorithm was obtained with the inclusion of a diversification memory, a very elitist central memory (only the best current solution was kept), short agents time which favored the exchange of information among threads with frequent restarts and multiple copies of the same metaheuristic agent (an iterated local search procedure). This configuration was able to consistently obtain good quality solutions for problem HSTP-A and also outperformed state-of-the-art algorithms on the other two variants, HSTP-B and HSTP-C. These last variants ignored specific requisites or considered some of them as hard instead of soft requisites.

These results suggest that the proposed method is efficient and robust concerning the enabling/disabling of constraints. This latter characteristic is particularly envisaged for HSTP algorithms since practical problems usually contain specific requisites. Our exploratory study limited itself to simulations with only four metaheuristic agents in which the threads had static execution time. Further research is aimed at exploring more flexible frameworks, in which other metaheuristics can be included and the agents' execution time can be self-calibrated according to the search history.

PARALLEL METAHEURISTIC WITH COLUMN GENERATION

In the previous chapters, we have investigated mixed-integer programming models and parallel metaheuristic based algorithms for high school timetabling problems. As shown, the computation of solutions and lower bounds is time-consuming. In this chapter, we propose a DIMB algorithm that exploits column generation into a cooperative parallel framework to simultaneously generate high-quality solutions and lower bounds for high school timetabling problems. We explain this algorithm in Section 5.1 and assess its performance in Section 5.2.

5.1 A DIMB algorithm with column generation

The proposed DIMB algorithm with column generation (DIMB-CG) is an extension of the DIMB framework described in Section 4.2.2. It is illustrated in Figure 14 and has the following components:

- *Three shared memories*: a diversification memory of solutions, an intensification memory of solutions, and a pool of columns.
- *Four communication buffers*: a buffer of input solutions, a buffer of output solutions, a buffer of dual values, and a buffer of columns.
- *A set of algorithm agents*: a column generation procedure (CG), a constructive column generation based heuristic (CCGH), a manager procedure, a diversifier procedure, and N intensifier metaheuristics.

Algorithm DIMB-CG is based on three main ideas. First, it incorporates a column generation into a parallel framework to compute lower bounds. Second, it exploits the generated columns to build new feasible solutions that are used to diversify the search. Third, it takes

advantage of cooperative parallel metaheuristics to improve the constructed solutions. Below, we explain the details of each component of this algorithm.

Figure 14 – An illustration of the algorithm DIMB-CG.

Algorithm 1 – Pseudo-code of the CG procedure.

CG()

```

1  Initialize the master program of formulation EF2 with high-cost artificial columns
2  do {
3      Solve the master program
4       $DV$  = the dual values of the updated master
5      if (the buffer Dual Values is not full) {
6          Insert  $DV$  into Dual Values
7      }
8       $newColumns = 0$ 
9      for each ( $t \in T$ ) {
10         Update the objective function of the pricing program  $\mathcal{Q}_t$  using the dual values  $DV$ 
11         Solve  $\mathcal{Q}_t$ 
12         if (reduced cost  $\bar{c}_t < 0$ ) {
13              $Col$  = the generated column
14             Insert a copy of  $Col$  into the master program
15             Insert a copy of  $Col$  into the pool of columns
16              $newColumns = newColumns + 1$ 
17         }
18     }
19     if ( $newColumns > 0$ ) {
20         Pull all columns from the buffer of columns
21         Insert a copy of the retrieved columns into the master program
22         Insert a copy of the retrieved columns into the pool of columns
23     }
24 } while ( $newColumns > 0$ )
25 return the objective value of the master program

```

Memories and buffers

The diversification memory keeps a set of non-elite solutions that are used to diversify the search, while the intensification memory maintains a set of elite solutions which are used to

intensify the search. Both memories have static sizes. The pool of columns is a dynamic memory which stores all columns generated by the column generation and the constructive column generation based heuristic. Lastly, the buffers are static and act as communication mechanisms.

Column generation procedure

The column generation procedure (CG), shown in Algorithm 1, employs formulation EF2 described in Section 3.2.2. It is a standard column generation algorithm, except that at each iteration, it interacts with the constructive heuristic CCGH (Algorithm 2) by sending dual values (lines 5 to 7) and receiving columns generated by the constructive heuristic (line 20). Also, the CG procedure stores a copy of all produced columns into the pool of columns (lines 15 and 22), so that the columns can be further exploited by the diversifier agent (Algorithm 4).

Timetable encodings

The remaining agents handle two types of timetable encoding. The timetable types X and Z , that are illustrated in Figure 15. Both of them are based on the data structure proposed in Section A.1. In timetable type X (Figure 15a), there is an array of timeslots, and each timeslot points to an array of pointers. These pointers point to requirement tuples, which represent the requirements assigned to teachers. In timetable type Z (Figure 15b), each timeslot points to an array of pointers, that point to requirement tuples assigned to classes. In the remainder of this chapter, to improve the presentation, instead of showing a pointer to a requirement tuple itself, we will display only the teacher ID or class ID belonging to the tuple being pointed by the pointer.

Figure 15 – Timetable encodings.

Neighboring solutions are generated via the TQ operator (SAVINIEC; CONSTANTINO, 2017). This operator identifies chains of two-swap moves that must be performed together so that new clashes in classes or teachers' schedules are avoided. Depending on the timetable encoding used, a *two-swap* move consists in exchanging the requirement tuples of a class $c \in C$, or teacher

Figure 16 – Construction of solutions with information provided by columns.

$t \in T$, that are assigned to two different timeslots. Chains of two-swap moves are identified by conflict graphs, similarly as made in Section A.2.

Timetable solutions are evaluated by the objective function (A.1), described in Section A.4. For a timetable solution of type Z, hard requisites 1 and 2 are left out of the objective function, because they are maintained by the TQ moves and the timetable encoding. The same is valid for hard requisites 1 and 3 in a timetable solution of type X. For both timetable encodings, all other requisites are penalized in the objective function according to the problem being handled.

In Figure 16, we explain how the columns of the column generation are interpreted and used along with these timetable encodings to construct solutions. As shown in Figures 16a and 16b, a column provided by a pricing subproblem $\mathcal{Q}_t$ (see Section 3.2.2.1) is a complete weekly schedule for a teacher $t \in T$. Columns are also represented by requirement tuples, but we will show only the associated classes to make the presentation easier. Supposing that a subset of columns is available for each teacher, it is sufficient to choose one column at random for each teacher to build an initial solution. Two situations might arise. On the one hand, if the chosen columns are conflict-free columns, as in Figure 16a, then we are able to construct solutions with both timetable encodings, as shown in Figures 16c and 16d. On the other hand, if the chosen columns have conflicting columns, as in Figure 16b, then we can still build (infeasible) solutions with timetable encoding X, as shown in Figure 16e. However, we cannot produce solutions with timetable encoding Z, as exemplified in Figure 16f.

Constructive column generation heuristic

The constructive column generation heuristic (CCGH) is shown in Algorithm 2. This heuristic tries to construct a feasible solution (lines 6 to 22) every time a new vector of dual values is received from the CG algorithm (line 5). If it succeeds, then the constructed solution is further improved by the ILS described in Algorithm 3 (line 24), a copy of its columns is sent back to the CG algorithm (line 25), and the final solution is sent to the diversification memory (lines 26 to 30). The ILS in Algorithm 3 is similar to the one described in Algorithm 8 of Section A.5.

To create feasible solutions, we use two strategies. First, we solve each subproblem by fixing some of its variables based on schedules already defined in previous subproblems (SANTOS *et al.*, 2011) (lines 8 to 15). This strategy is expected to fail, because, in the last subproblems, most of the variables will be fixed, which might result in infeasible subproblems. If it fails at a given subproblem with a partial solution Z_p , then we employ the second strategy, which consists in solving a proximity problem (FISCHETTI; MONACI, 2014) to find a complete feasible solution Z that minimizes the Hamming distance to Z_p (lines 19 to 21).

Now, we explain how the variables of a subproblem $\mathcal{Q}_t$ are preprocessed in line 10, and how the proximity problem is created and solved in line 20.

In Figure 17, we illustrate the preprocessing of variables. After solving the subproblems of teachers t_0 to t_3 , we can forbid class c_2 of being assigned to teacher t_4 at timeslots s_0, s_1, s_3

Algorithm 2 – Pseudo-code of the CCGH procedure.CCGH(*impTimeOut*, *impIterOut*)

```

1  Create a list of teachers  $Q$ 
2   $Q = T$ 
3  while (not signaled to stop) {
4      Wait while the buffer Dual Values is empty
5       $DV$  = a vector of dual values pulled from Dual Values
6      Shuffle list  $Q$ 
7       $TempCols = \emptyset$  // a temporary list of columns
8      for each ( $t \in Q$ ) {
9          Update the objective function of the pricing program  $\mathcal{Q}_t$  using the dual values  $DV$ 
10         FIXVARIABLES( $\mathcal{Q}_t$ )
11         Solve  $\mathcal{Q}_t$ 
12         if ( $\mathcal{Q}_t$  is feasible) {
13             Store the generated column into the temporary list  $TempCols$ 
14         }
15     }
16     if ( $|TempCols| = |T|$ ) {
17          $Z$  = convert all columns of  $TempCols$  into a complete feasible solution
18     } else {
19          $Z_p$  = convert all columns of  $TempCols$  into a partial solution
20         Solve a proximity problem, program 5.1 - 5.2, to find a complete feasible solution that
            minimizes its Hamming distance to  $Z_p$ 
21          $Z$  = the solution found by running the proximity program during a given time limit
22     }
23     if ( $Z \neq \emptyset$ ) {
24          $Z = \text{ILS}(Z, \text{impTimeOut}, \text{impIterOut})$  // call Algorithm 3 to improve solution  $Z$ 
25         As the schedule of each teacher in solution  $Z$  defines one column, create one column for each
            teacher and insert them into the buffer of columns
26         if (the diversification memory is full) {
27             Replace a randomly chosen solution  $Z_r$  from the diversification memory by  $Z$ 
28         } else {
29             Insert  $Z$  into the diversification memory
30         }
31     }
32 }

```

and s_4 , because it is already assigned to the same timeslots in the previous subproblems. The same idea is employed for the remaining classes (c_0 and c_4) that have meetings with teacher t_4 , shown between parenthesis in Figure 17. We apply this preprocessing strategy before solving each subproblem in line 11 of the CCGH heuristic.

Now, let us suppose that teacher t_4 is unavailable at timeslot s_2 in the example of Figure 17. In this case, the subproblem is infeasible, because class c_2 cannot be assigned to any timeslot. When we reach such infeasibility, we solve the proximity problem, in line 20, considering formulation CF1 proposed in Section 3.1.1.

Let J be the subset of variables x_{rdh} , in model CF1, which belong to teachers with schedules already fixed. For example, teachers t_0 , t_1 , t_2 and t_3 in Figure 17. Based on these fixed schedules, let $J_0 \subset J$ be the subset of variables that have values set to zero, and $J_1 \subset J$ be the subset of variables that have values set to one. The proximity problem can thus be written as:

Algorithm 3 – Pseudo-code of the sequential ILS used by algorithm DIMB-CG.ILS(Z_0 , $timeOut$, $iterOut$)

```

1   $Z^* = Z_0$ 
2   $k = 0$ 
3  while ( $f(Z^*) > 0$  and  $elapsedTime < timeOut$  and  $k < iterOut$ ) {
4       $Z = \text{PERTURBATION}(Z^*)$  // a random TQ move
5      do {
6           $best = f(Z)$ 
7           $i = \text{take a random integer in the range } 0 \leq i \leq |SP| - 1$ 
8          for ( $j = 1$  to  $|SP|$ ) {
9              Compute the connected components for tuples assigned to timeslots  $sp_i$  of  $Z$ 
10             for (each connected component) {
11                 Generate a new neighboring solution  $Z'$  from  $Z$ 
12                 if ( $f(Z') \leq f(Z)$ ) {
13                      $Z = Z'$ 
14                 }
15             }
16              $i = (i + 1) \bmod |SP|$ 
17         }
18     } while ( $f(Z) < best$ )
19     if ( $f(Z) \leq f(Z^*)$ ) {
20          $Z^* = Z$ 
21     }
22      $k = k + 1$ 
23 }
24 return  $Z^*$ 

```

	Timeslots					
	s_0	s_1	s_2	s_3	s_4	
subproblem of teacher t_0	1	2		1	2	solved
subproblem of teacher t_1	2	1	3	3		solved
subproblem of teacher t_2	4		0	2	3	solved
subproblem of teacher t_3		3	4	4	0	solved
subproblem of teacher t_4						unsolved (0, 0, 2, 4)
subproblem of teacher t_5						unsolved (0, 1, 1, 3, 4)

Figure 17 – Preprocessing of variables belonging to class c_2 in subproblem t_4 .

$$\text{Minimize} \quad \sum_{x_{rdh} \in J_0} x_{rdh} + \sum_{x_{rdh} \in J_1} (1 - x_{rdh}) \quad (5.1)$$

Subject to:*Constraints of formulation CF1*

$$x_{rdh} \in \{0, 1\} \quad \forall r \in R; d \in D; h \in H \quad (5.2)$$

The linear relaxation of a proximity problem is usually very tight (FISCHETTI; MONACI,

2014). In our case, model CF1 worked better than model CF2 to formulate the proximity problem. Program (5.1) - (5.2) can be quickly solved to optimality, which makes it an effective tool to recover the feasibility of a solution under construction.

Diversifier

The diversifier procedure is shown in Algorithm 4. This procedure is responsible for feeding the diversification memory with new solutions that are constructed by exploiting the pool of columns. At each iteration of the main loop (lines 2 to 17), the algorithm constructs a solution X in line 8 (possibly infeasible), that is repaired in line 9, converted to a timetable Z in line 10, further improved in line 11, and sent to the diversification memory.

Algorithm 4 – Pseudo-code of the diversifier procedure.

DIVERSIFIER(*repTimeOut*, *repIterOut*, *impTimeOut*, *impIterOut*)

```

1  Wait until there is at least one column of each teacher available in the pool of columns
2  while (not signaled to stop) {
3      Let PoolOfColumns[ $t$ ] be the set of columns belonging to teacher  $t \in T$  in the pool of columns
4      for each ( $t \in T$ ) {
5           $Col$  = a column chosen at random from PoolOfColumns[ $t$ ]
6          Insert the column  $Col$  into a temporary list TempCols
7      }
8       $X$  = convert all columns of TempCols into a complete solution (possibly infeasible)
9       $X$  = ILSX( $X$ , repTimeOut, repIterOut)      // call algorithm ILSX to repair clashes in
                                                classes' schedules, only  $\alpha_2$  and  $\alpha_4$  are set greater than zero in (A.1), all others are set to zero
10     Convert the solution  $X$  into a solution  $Z$ 
11      $Z$  = ILS( $Z$ , impTimeOut, impIterOut)      // call Algorithm 3 to improve solution  $Z$ 
12     if (the diversification memory is full) {
13         Replace a solution  $Z_r$  randomly chosen from the diversification memory by  $Z$ 
14     } else {
15         Insert  $Z$  into the diversification memory
16     }
17 }
```

Note that we use the timetable encoding type X to repair solutions, and the timetable encoding type Z to improve solutions. The reason why we do not use the timetable encoding X in both phases is that it is computationally less efficient than the timetable encoding Z . The improvement algorithm ILSX (line 9) is an extension of the Algorithm 3 to manage timetables of type X . Its extension from Algorithm 3 is simple and straightforward.

Manager

The manager procedure is shown in Algorithm 5. This procedure is responsible for implementing the policy to coordinate the search performed by the intensifier agents. It first sends one solution from the diversification memory to each intensifier, through the input buffer (lines 4 to 7), and then it enters into a loop (lines 8 to 27). At each iteration of the loop, it pulls one solution from the output buffer, returned by the intensifiers (line 10), decides if the solution

is accepted in the intensification memory or discarded (lines 12 to 14), and then it puts another solution into the input buffer (lines 18 to 25). In this last statements, the solution is chosen from the diversification memory with probability ρ and from the intensification memory with probability $(1 - \rho)$.

Algorithm 5 – Pseudo-code of the manager procedure.

MANAGER(*timeOut*, *N*, ρ)

```

1  Wait while the diversification memory is empty
2  Fill the intensification memory with solutions chosen at random from the diversification memory
3  added = 0
4  for (k = 1 to N) {
5      Pick a solution at random from the diversification memory and insert it into the input buffer
6      added = added + 1
7  }
8  while (added > 0) {
9      Wait while the output buffer is empty
10     Z = a solution that is pulled from the output buffer
11     Z' = a solution that is chosen at random from the intensification memory
12     if ( $f(Z) \leq f(Z')$ ) {
13         Replace Z' by Z in the intensification memory
14     }
15     if (elapsedTime > timeOut) {
16         added = added - 1
17     } else {
18         rnd = take a random value in the range  $0 \leq rnd \leq 1$ 
19         if ( $rnd \leq \rho$ ) {
20             Z'' = a solution that is chosen at random from the diversification memory
21         } else {
22             Z'' = a solution that is chosen at random from the intensification memory
23         }
24         Wait while the input buffer is full
25         Insert Z'' into the input buffer
26     }
27 }
```

Intensifiers

The pseudo-code of an intensifier procedure is shown in Algorithm 6. The main idea is that a team of *N* intensifier metaheuristics run concurrently, cooperating with each other via exchanging of best solutions, to intensify the search around the solutions received from the manager. Each intensifier agent is an ILS (Algorithm 3) that pulls a solution from the input buffer (line 3), explores its neighborhood during a parameter-defined amount of time (line 4), and then returns its best solution to the output buffer (line 6).

Table 26 describes the complete list of parameter values that we used in the implementation of our algorithm DIMB-CG.

Algorithm 6 – Pseudo-code of an intensifier procedure.

INTENSIFIER(*syncTimeOut*, *syncIterOut*)

```

1  while (not signaled to stop) {
2      Wait while the input buffer is empty
3      Z = a solution that is pulled from the input buffer
4      Z = ILS(Z, syncTimeOut, syncIterOut)      // call Algorithm 3 to explore the neighborhood
        of solution Z
5      Wait while the output buffer is full
6      Insert Z into the output buffer
7  }
```

Table 26 – List of parameter values used in the algorithm DIMB-CG.

Component	Parameter	Value
Diversification memory	Size	8
Intensification memory	Size	1
Input buffer	Size	19
Output buffer	Size	19
Buffer of dual values	Size	3
Buffer of columns	Size	$2 \cdot T $
CCGH heuristic	Proximity program time limit	30 seconds
	ILS time limit (<i>impTimeOut</i>)	10 seconds
	ILS iterations (<i>impIterOut</i>)	3 iterations
	ILSX time limit (<i>repTimeOut</i>)	1000 iterations
	ILSX iterations (<i>repIterOut</i>)	100 seconds
	ILS time limit (<i>impTimeOut</i>)	10 seconds
Diversifier	ILS iterations (<i>impIterOut</i>)	3 iterations
	Time limit (<i>timeOut</i>)	625 seconds
	Number of intensifiers (<i>N</i>)	19
Manager	Diversification probability (ρ)	0.05
	Synchronization time (<i>syncTimeOut</i>)	2 seconds
Intensifiers	Synchronization iterations (<i>syncIterOut</i>)	10000000 iterations

5.2 Computational experiments

In this section, we report two experiment sets to assess our parallel algorithm DIMB-CG.

Section 5.2.1: we test the effect of inserting the CCGH extra columns into the master program of the CG procedure. We also analyze the behavior of the two strategies used by our constructive heuristic CCGH and the quality of its solutions.

Section 5.2.2: we compare the algorithm DIMB-CG with the best parallel algorithm of Chapter 4.

For problems HSTP-A and HSTP-C, we report experiments with instances of Table 15, and for problem HSTP-B, we use the instances of Table 20.

The proposed algorithm was coded in C++ and compiled with GNU Compiler Collection 4.4.7. The experiments were performed on a server with two cores Intel Xeon E5-2680v2 (2.8 GHz) and 128 GB of RAM, running *Red Hat Enterprise Linux 6.5*. In all experiments,

the algorithm DIMB-CG was allowed to execute during a time limit of 625 seconds, and 25 independent trials were carried out for each instance.

5.2.1 Analysis of the CCGH heuristic

We start this section by analyzing the speed-up obtained by the CG procedure with the addition of extra columns into its master program.

Table 27 – Comparison of convergence time (in seconds) between SCG and CG+CCGH.

ID	HSTP-A			HSTP-B			HSTP-C		
	SCG	CG+CCGH	Sp	SCG	CG+CCGH	Sp	SCG	CG+CCGH	Sp
1	47.16	30.92	1.53	14.66	8.93	1.64	49.06	51.49	0.95
2	47.89	30.81	1.55	24.60	10.20	2.41	49.83	50.72	0.98
3	62.40	34.00	1.84	22.54	12.86	1.75	58.25	36.94	1.58
4	64.63	31.70	2.04	35.08	19.33	1.81	59.99	35.29	1.70
5	35.82	19.42	1.84	39.73	16.19	2.45	32.32	20.84	1.55
6	75.69	41.71	1.81	89.80	26.68	3.37	71.08	45.25	1.57
7	118.60	89.52	1.32	82.39	31.73	2.60	95.15	99.07	0.96
8	22.19	20.28	1.09				21.86	26.18	0.83
9	92.47	74.04	1.25				59.83	65.53	0.91
10	19.66	11.11	1.77				18.17	10.13	1.79
11	108.70	82.54	1.32				76.99	83.06	0.93
12	88.33	72.73	1.21				82.87	81.83	1.01
13	3.38	3.52	0.96				2.51	2.74	0.92
14	71.02	48.58	1.46				48.35	49.57	0.98
15	8.13	6.43	1.26					Infeasible	
16	27.70	16.56	1.67				22.55	17.26	1.31
17	21.11	14.19	1.49				20.56	14.42	1.43
18	19.87	11.81	1.68				19.62	11.50	1.71
19	13.01	7.53	1.73				15.60	8.08	1.93
20	20.35	9.69	2.10				24.83	12.09	2.05
21	97.66	69.72	1.40				73.70	74.36	0.99
22	51.53	36.89	1.40					Infeasible	
23	302.46	189.38	1.60				273.51	280.97	0.97
24	208.42	149.69	1.39				201.99	204.41	0.99
25	72.36	52.94	1.37				56.54	61.29	0.92
26	49.56	31.78	1.56				37.02	33.03	1.12
27	37.01	25.95	1.43				27.67	29.22	0.95
28	119.56	70.20	1.70				95.61	77.83	1.23
29	106.92	59.89	1.79				115.67	89.65	1.29
30	126.22	68.76	1.84				107.54	86.26	1.25
31	94.32	63.15	1.49				79.59	68.76	1.16
32	72.45	47.77	1.52				78.22	67.08	1.17
33	75.38	50.19	1.50				67.91	56.06	1.21
34	61.65	40.10	1.54				73.85	74.17	1.00
Avg.:			1.54			2.29			1.23

Table 27 shows the convergence time of the stand-alone CG algorithm (SCG) compared to the time required when the CCGH extra columns are also inserted into the master program (CG+CCGH). This table shows that the column generation process is accelerated when the extra columns are also added to the master program, and the column (Sp) is the median speed-up obtained. We observe that for problems HSTP-A and HSTP-B, the use of the constructive

heuristic CCGH introduced reasonable speed-ups on average. However, for problem HSTP-C, it was less efficient. Given that problem HSTP-C is more constrained than the other problems, we have observed that the proximity program takes longer to be solved. Therefore, very few extra columns are generated, which are insufficient to provide reasonable speed-ups.

Table 28 presents results regarding the number of columns necessary to converge the CG algorithm. Column (PoAC) is the percentage of additional columns generated when the CCGH heuristic is employed. This table shows that for problems HSTP-A and HSTP-C, the master program required, on average, more columns to converge when using our CG+CCGH strategy, than when using the stand-alone column generation. However, for problem HSTP-B, the master program required fewer columns when using the CG+CCGH strategy.

Table 28 – Number of columns generated by the SCG compared to CG+CCGH.

ID	HSTP-A			HSTP-B			HSTP-C		
	SCG	CG+CCGH	PoAC	SCG	CG+CCGH	PoAC	SCG	CG+CCGH	PoAC
1	1782	1885	5.78	371	355	-4.31	1828	2069	13.18
2	1770	1863	5.25	848	640	-24.53	1817	2005	10.35
3	2032	1795	-11.66	904	823	-8.96	2114	2106	-0.38
4	2065	1733	-16.08	1364	1156	-15.25	2131	2033	-4.60
5	1412	1266	-10.34	1677	1238	-26.18	1427	1539	7.85
6	2183	2046	-6.28	2025	1338	-33.93	2282	2280	-0.09
7	3517	4685	33.21	2216	1588	-28.34	3281	3470	5.76
8	1266	2094	65.40				1293	1388	7.35
9	2703	4255	57.42				2200	2315	5.23
10	891	845	-5.16				884	857	-3.05
11	3191	4194	31.43				2908	2967	2.03
12	2971	3198	7.64				2946	2998	1.77
13	347	568	63.69				302	399	32.12
14	2767	2951	6.65				2476	2525	1.98
15	589	710	20.54					Infeasible	
16	1195	1147	-4.02				1196	1285	7.44
17	999	1059	6.01				1078	1226	13.73
18	869	870	0.12				859	893	3.96
19	673	661	-1.78				738	667	-9.62
20	933	784	-15.97				1034	923	-10.74
21	2933	3027	3.20				2760	2767	0.25
22	1915	2534	32.32					Infeasible	
23	6096	7061	15.83				6107	6547	7.20
24	5741	7439	29.58				5832	6206	6.41
25	2546	2954	16.03				2408	2552	5.98
26	1686	1887	11.92				1594	1726	8.28
27	1409	1664	18.10				1322	1431	8.25
28	3282	3569	8.74				3067	3679	19.95
29	3128	3211	2.65				3136	3908	24.62
30	3277	3428	4.61				3154	3859	22.35
31	3075	3594	16.88				2966	3190	7.55
32	2601	2794	7.42				2658	3459	30.14
33	2696	2820	4.60				2613	3183	21.81
34	2472	2623	6.11				2557	3573	39.73
Avg.:			12.05			-20.21			8.96

In Table 29, we analyze the effectiveness of the two strategies used in our CCGH heuristic. Column (PoS1) is the percentage of success in constructing feasible solutions using

the first strategy, which solves each subproblem by fixing some of its variables. Column (PoSS) is the percentage of solved subproblems before reaching infeasibility. Column (PoS2) is the percentage of success in finding feasible solutions using the second strategy, which solves the proximity problem. This table shows that the fixing of variables is ineffective to construct complete solutions. However, it defines most of the teachers' schedules without clashes (more than 85.8 % on average). This fact may explain the high rate of success in which the proximity program succeeded in finding feasible solutions, more than 99.5 % of the trials on average.

Table 29 – Effectiveness of the two strategies used by the CCGH heuristic.

ID	HSTP-A			HSTP-B			HSTP-C		
	PoS1	PoSS	PoS2	PoS1	PoSS	PoS2	PoS1	PoSS	PoS2
1	0.0	87.4	100.0	2.7	86.7	100.0	0.0	87.7	100.0
2	0.0	87.6	100.0	7.4	92.5	100.0	0.0	87.8	100.0
3	0.3	93.5	100.0	0.0	86.1	100.0	0.3	93.6	100.0
4	1.4	94.1	100.0	0.0	84.4	100.0	1.6	94.3	100.0
5	3.3	94.0	100.0	0.3	94.2	100.0	4.1	94.2	100.0
6	0.0	91.9	100.0	0.3	92.9	100.0	0.1	91.9	100.0
7	0.0	78.0	100.0	0.0	91.4	100.0	0.0	77.7	100.0
8	0.0	77.6	100.0				0.0	78.1	100.0
9	0.0	76.7	100.0				0.0	74.7	100.0
10	7.5	94.2	100.0				6.7	94.2	100.0
11	0.0	81.5	100.0				0.0	80.5	100.0
12	0.0	76.1	100.0				0.0	76.5	99.4
13	0.6	80.8	100.0				0.4	79.0	100.0
14	0.0	77.3	100.0				0.0	75.2	100.0
15	0.0	81.1	100.0				Infeasible		
16	0.1	86.7	100.0				0.0	86.6	100.0
17	0.7	89.9	100.0				0.6	89.6	100.0
18	14.0	93.9	100.0				11.3	93.7	100.0
19	21.3	94.7	100.0				18.5	94.5	100.0
20	8.3	94.1	100.0				4.7	93.6	100.0
21	0.0	78.5	100.0				0.0	77.2	85.2
22	0.0	78.6	100.0				Infeasible		
23	0.0	81.5	100.0				0.0	81.2	100.0
24	0.0	79.4	100.0				0.0	79.0	100.0
25	0.0	82.6	100.0				0.0	81.9	100.0
26	0.0	81.7	100.0				0.0	82.5	100.0
27	0.0	80.1	100.0				0.0	79.6	100.0
28	0.0	89.5	100.0				0.0	89.2	100.0
29	0.0	89.0	100.0				0.0	88.7	100.0
30	0.0	89.2	100.0				0.0	89.0	100.0
31	0.0	87.0	100.0				0.0	86.6	100.0
32	0.0	89.7	100.0				0.0	89.7	100.0
33	0.0	89.6	100.0				0.0	89.3	100.0
34	0.0	88.2	100.0				0.0	88.1	100.0
Avg.:	1.7	85.8	100.0	1.5	89.7	100.0	1.5	85.8	99.5

In Table 30, we analyze the quality of the solutions constructed by the CCGH heuristic. This table shows that the proposed heuristic is very effective indeed. It builds feasible solutions with an average optimality gap of 19.17 % for problem HSTP-A and 10.57 % for problem HSTP-C. For problem HSTP-B, the produced solutions are almost optimal, with an average gap of only 2.74 %.

Table 30 – Quality of the CCGH solutions.

ID	HSTP-A			HSTP-B			HSTP-C		
	Best LB	Mean	Gap	Best LB	Mean	Gap	Best LB	Mean	Gap
1	950	1170	18.80	202	209	3.35	699	787	11.18
2	950	1170	18.80	333	342	2.63	699	786	11.07
3	1010	1210	16.53	423	462	8.44	742	836	11.24
4	990	1200	17.50	652	656	0.61	734	832	11.78
5	870	970	10.31	762	772	1.30	631	682	7.48
6	1080	1320	18.18	756	768	1.56	772	864	10.65
7	1690	2310	26.84	1017	1030	1.26	1212	1372	11.66
8	950	1285	26.07				675	735	8.16
9	1100	1370	19.71				807	890	9.33
10	420	500	16.00				298	331	9.97
11	1400	1880	25.53				961	1124	14.50
12	1450	2120	31.60				1018	1104	7.79
13	420	460	8.70				273	273	0.00
14	1270	1755	27.64				933	1016	8.17
15	395	465	15.05				Infeasible		
16	640	760	15.79				481	547	12.07
17	580	700	17.14				457	512	10.74
18	460	530	13.21				319	360	11.39
19	360	410	12.20				254	277	8.30
20	470	540	12.96				325	362	10.22
21	1450	1990	27.14				1050	1148	8.54
22	1035	1350	23.33				Infeasible		
23	2505	3350	25.22				1813	2069	12.37
24	2680	3625	26.07				1988	2256	11.88
25	1315	1775	25.92				912	1057	13.72
26	750	940	20.21				570	640	10.94
27	730	910	19.78				552	591	6.60
28	1570	1850	15.14				1124	1271	11.57
29	1570	1870	16.04				1116	1267	11.92
30	1590	1890	15.87				1133	1287	11.97
31	1570	1910	17.80				1127	1299	13.24
32	1390	1670	16.77				1014	1170	13.33
33	1410	1680	16.07				1018	1170	12.99
34	1390	1690	17.75				1005	1163	13.59
Avg.:			19.17			2.74			10.57

5.2.2 Analysis of the DIMB-CG algorithm

In this section, we compare the proposed algorithm DIMB-CG with the parallel algorithm DIMB-19I1D-8-0.05 of Chapter 4.

In Table 31, we present the results for problem HSTP-A. Column (Best LB) shows the best lower bounds reported in Section 3.3. The best solutions are shown in bold font. This table shows that our algorithm performs better than DIMB-19I1D-8-0.05, both in terms of median and best solutions. In column (ToBS), we show the median computational time of the best solution found by algorithm DIMB-CG, which is the time when the algorithm made the last improvement in its global best solution. This column shows that, in the median case, the algorithm takes less than 500 seconds to find near-optimal solutions, even for the largest instances 23 and 24. Also, for some small instances, such as 10, 15 and 18 to 20, the algorithm only takes a few seconds to

find the optimal solutions.

Table 31 – Results of the algorithm DIMB-CG compared to DIMB-19I1D-8-0.05 on problem HSTP-A.

ID	Best LB	DIMB-CG					DIMB-19I1D-8-0.05			
		ToBS (s)	Median	Gap	Best	Gap	Median	Gap	Best	Gap
1	950	306.90	960	1.04	950	0.00	970	2.06	950	0.00
2	950	349.62	960	1.04	950	0.00	970	2.06	950	0.00
3	1010	204.64	1010	0.00	1010	0.00	1020	0.98	1010	0.00
4	990	148.24	990	0.00	990	0.00	1000	1.00	990	0.00
5	870	29.34	870	0.00	870	0.00	870	0.00	870	0.00
6	1080	220.62	1080	0.00	1080	0.00	1100	1.82	1080	0.00
7	1690	455.17	1830	7.65	1770	4.52	1880	10.11	1810	6.63
8	950	38.78	1045	9.09	980	3.06	1030	7.77	980	3.06
9	1100	269.50	1160	5.17	1130	2.65	1170	5.98	1130	2.65
10	420	8.42	420	0.00	420	0.00	420	0.00	420	0.00
11	1400	378.52	1520	7.89	1470	4.76	1530	8.50	1450	3.45
12	1450	332.04	1580	8.23	1500	3.33	1600	9.38	1530	5.23
13	420	46.25	420	0.00	420	0.00	420	0.00	420	0.00
14	1270	313.59	1340	5.22	1290	1.55	1360	6.62	1300	2.31
15	395	11.92	395	0.00	395	0.00	395	0.00	395	0.00
16	640	77.04	640	0.00	640	0.00	650	1.54	640	0.00
17	580	147.30	580	0.00	580	0.00	590	1.69	580	0.00
18	460	6.45	460	0.00	460	0.00	460	0.00	460	0.00
19	360	4.29	360	0.00	360	0.00	360	0.00	360	0.00
20	470	8.45	470	0.00	470	0.00	470	0.00	470	0.00
21	1450	432.95	1520	4.61	1480	2.03	1550	6.45	1480	2.03
22	1035	251.71	1075	3.72	1045	0.96	1095	5.48	1055	1.90
23	2505	490.41	2740	8.58	2670	6.18	2790	10.22	2670	6.18
24	2680	477.62	2920	8.22	2840	5.63	2970	9.76	2860	6.29
25	1315	401.50	1380	4.71	1340	1.87	1400	6.07	1340	1.87
26	750	284.05	770	2.60	750	0.00	780	3.85	750	0.00
27	730	249.81	740	1.35	730	0.00	750	2.67	730	0.00
28	1570	319.46	1600	1.88	1590	1.26	1620	3.09	1590	1.26
29	1570	444.11	1600	1.88	1580	0.63	1620	3.09	1590	1.26
30	1590	343.04	1620	1.85	1610	1.24	1640	3.05	1600	0.63
31	1570	421.71	1620	3.09	1600	1.88	1640	4.27	1590	1.26
32	1390	437.21	1420	2.11	1410	1.42	1440	3.47	1410	1.42
33	1410	307.89	1450	2.76	1440	2.08	1460	3.42	1430	1.40
34	1390	427.91	1430	2.80	1420	2.11	1450	4.14	1410	1.42
Rank:			33		29		8		28	
Avg.:				2.81		1.39		3.78		1.48

In Table 32, we present the results for problem HSTP-B. This table shows that both algorithms were able to find the optimal solutions for all instances in the median case. Column (ToBS) shows that the algorithm DIMB-CG is almost instantaneous to find optimal solutions for these instances, except for instance 3, in which it takes around 30.55 seconds to reach the optimal solutions.

Table 33 presents the results for problem HSTP-C. This table shows that the algorithm DIMB-CG performs better than the algorithm DIMB-19I1D-8-0.05 in median solutions, but it is less effective in best solutions. As the problem HSTP-C is more constrained than the problem HSTP-A, the algorithm DIMB-CG takes longer to make the last improvement, see column (ToBS). However, it is still able to reach near-optimal solutions expending running time much

smaller than the specified time limit for most of the tested instances.

Table 32 – Results of the algorithm DIMB-CG compared to DIMB-19I1D-8-0.05 on problem HSTP-B.

ID	Best LB	DIMB-CG					DIMB-19I1D-8-0.05			
		ToBS (s)	Median	Gap	Best	Gap	Median	Gap	Best	Gap
1	202	2.17	202	0.00	202	0.00	202	0.00	202	0.00
2	333	2.44	333	0.00	333	0.00	333	0.00	333	0.00
3	423	30.55	423	0.00	423	0.00	423	0.00	423	0.00
4	652	4.67	652	0.00	652	0.00	652	0.00	652	0.00
5	762	2.94	762	0.00	762	0.00	762	0.00	762	0.00
6	756	3.00	756	0.00	756	0.00	756	0.00	756	0.00
7	1017	3.62	1017	0.00	1017	0.00	1017	0.00	1017	0.00
Rank:		7			7			7		
Avg.:		0.00			0.00			0.00		

Table 33 – Results of the algorithm DIMB-CG compared to DIMB-19I1D-8-0.05 on problem HSTP-C.

ID	Best LB	DIMB-CG					DIMB-19I1D-8-0.05			
		ToBS (s)	Median	Gap	Best	Gap	Median	Gap	Best	Gap
1	699	364.74	708	1.27	701	0.29	710	1.55	700	0.14
2	699	409.47	707	1.13	701	0.29	711	1.69	701	0.29
3	742	389.30	747	0.67	743	0.13	749	0.93	743	0.13
4	734	419.58	738	0.54	735	0.14	743	1.21	735	0.14
5	631	93.26	631	0.00	631	0.00	631	0.00	631	0.00
6	772	403.62	772	0.00	772	0.00	777	0.64	772	0.00
7	1212	495.65	1259	3.73	1237	2.02	1270	4.57	1239	2.18
8	675	58.63	697	3.16	679	0.59	698	3.30	686	1.60
9	807	357.22	821	1.71	812	0.62	827	2.42	814	0.86
10	298	10.42	298	0.00	298	0.00	298	0.00	298	0.00
11	961	467.90	998	3.71	987	2.63	1005	4.38	985	2.44
12	1018	334.13	1039	2.02	1032	1.36	1044	2.49	1030	1.17
13	273	4.11	273	0.00	273	0.00	273	0.00	273	0.00
14	933	440.61	946	1.37	940	0.74	954	2.20	939	0.64
15					Infeasible					
16	481	173.54	483	0.41	481	0.00	484	0.62	481	0.00
17	457	309.41	460	0.65	457	0.00	461	0.87	457	0.00
18	319	22.51	319	0.00	319	0.00	319	0.00	319	0.00
19	254	2.32	254	0.00	254	0.00	254	0.00	254	0.00
20	325	18.41	325	0.00	325	0.00	325	0.00	325	0.00
21	1050	452.70	1066	1.50	1059	0.85	1074	2.23	1058	0.76
22					Infeasible					
23	1813	534.02	1887	3.92	1874	3.26	1910	5.08	1867	2.89
24	1988	565.76	2059	3.45	2042	2.64	2079	4.38	2038	2.45
25	912	485.79	937	2.67	920	0.87	944	3.39	926	1.51
26	570	219.81	575	0.87	571	0.18	576	1.04	570	0.00
27	552	306.29	555	0.54	552	0.00	556	0.72	552	0.00
28	1124	459.30	1142	1.58	1132	0.71	1150	2.26	1127	0.27
29	1116	511.92	1137	1.85	1130	1.24	1145	2.53	1127	0.98
30	1133	426.27	1153	1.73	1142	0.79	1164	2.66	1142	0.79
31	1127	511.13	1152	2.17	1140	1.14	1165	3.26	1137	0.88
32	1014	506.90	1036	2.12	1026	1.17	1043	2.78	1024	0.98
33	1018	352.11	1039	2.02	1028	0.97	1047	2.77	1034	1.55
34	1005	417.23	1032	2.62	1020	1.47	1041	3.46	1015	0.99
Rank:			32		19		6		27	
Avg.:				1.48		0.75		1.98		0.74

5.3 Final remarks

In this chapter, we investigated the effect of exploiting column generation into a cooperative parallel metaheuristic framework to generate solutions and lower bounds for high school timetabling problems. We proposed a parallel algorithm in which a column generation procedure computes lower bounds on the optimal solutions, while two constructive heuristics exploit the generated columns to construct initial feasible solutions that are further improved by a team of cooperative metaheuristics.

As a contribution, we developed new strategies based on mixed-integer programming and metaheuristic techniques, which exploit the columns of the column generation algorithm to create feasible solutions. The computational experiments demonstrated that these strategies are quite effective in constructing good quality feasible solutions. When such heuristic strategies are embedded into the parallel DIMB framework of Chapter 4, the computational experiments suggest that the resulting algorithm performs better than the best algorithm of Chapter 4. As an overall result, the new algorithm was able to find near-optimal solutions for all three studied problems by expending relatively short execution time.

In this study, we have investigated the construction of solutions by i) converting a subset of columns into a non-clash-free solution that is repaired by a metaheuristic, and ii) solving subproblems by fixing some of their variables based on past solved subproblems. Further research can be done to investigate new constructive strategies to exploit the column generation subproblems.

CONCLUSIONS

In this thesis, we addressed the high school timetabling problem, which consists in assigning meetings between classes and teachers, with the goal of minimizing the violation of specific soft requisites. To tackle the problem, we proposed mixed-integer programming models, column generation approaches and parallel metaheuristic based algorithms.

In Chapter 3, we proposed five mixed-integer programming formulations CF1, CF2, EF1, EF2, and EF3. Formulations CF1 and CF2 can have their variables explicitly generated and solved in a black-box solver, while EF1, EF2, and EF3 cannot, because they have an exponential number of variables. Formulation CF2 has better linear relaxation than CF1, and we could also strengthen its root node with Fenchel cuts. The improved formulation CF2 outperformed the state-of-the-art formulation in the quality of lower bounds, but it is less effective in the quality of integer solutions. Formulations EF1, EF2, and EF3 are extended formulations solved by column generation, and their pricing subproblems are formulated with decompositions of formulation CF2. The best formulation EF2 provided strong lower bounds for the tested problems by expending cheap computational time. Also, it is competitive in the quality of lower bounds with extended formulations previously proposed in the literature. Although the formulations of Chapter 3 have worked quite well with Brazilian high school timetabling problems, they have a serious issue. As the number of binary variables is exponential on the number of periods per day, such formulations may be inefficient to deal with problems that require more than five periods per day.

In Chapter 4, we investigated several parallelization schemes to develop cooperative parallel metaheuristic based algorithms for high school timetabling problems. Such kind of algorithm has not been well studied in the context of educational timetabling. In our study, we analyzed several design aspects, such as homogeneity/heterogeneity of agents, the influence of intensification and diversification memories of solutions, and the frequency of information exchange among threads. The study showed that the exchange of information plays an important rule in the design of high-performance parallel metaheuristics. By running multiples copies of a

cooperative metaheuristic in different processor threads of a multi-core machine, we can explore the total machine processing capacity and obtain consistent gains in performance when compared to the use of sequential metaheuristics. Our most efficient algorithm was obtained by including a diversification memory, a very elitist intensification memory and frequent exchange of solutions among threads. This algorithm was able to find near-optimal solutions for all tested problems and also outperformed state-of-the-art algorithms. This chapter resulted in the following research paper:

- Saviniec, L.; Santos, M. O.; Costa, A. M. **Parallel local search algorithms for high school timetabling problems.** *European Journal of Operational Research*, 265 (2018) 81 – 98.

In Chapter 5, we combined the best column generation of Chapter 3 with the best strategies of Chapter 4 to develop a new parallel algorithm. This algorithm is able to compute lower bounds at the same time that it seeks for good quality solutions. The main feature of this algorithm is two embedded constructive heuristics, which exploit column generation information to construct feasible solutions that are further improved by a team of cooperative metaheuristics. Our experiments showed that these heuristics are quite effective in constructing feasible solutions, and by embedding them into our parallel framework, it resulted in an algorithm that is more effective than the best algorithm of Chapter 4.

Chapters 3 and 5 resulted in a working paper. This thesis also generated two other conference papers that were published in the proceedings of the Brazilian Symposium on Operations Research, in which the paper “A *Parallel Multi-start Iterated Local Search and a Proximity Relax-and-fix heuristic for High School Timetabling Problem*” was select among the five best papers of the conference.

- Saviniec, L.; Santos, M. O.; Costa, A. M.; Constantino, A. A. **Multithreading Iterated Local Search aplicado ao Problema de Horários Escolares.** *Brazilian Symposium on Operations Research*, Sobrapo, Rio de Janeiro, Brazil (2015) 826 – 837. (In Portuguese)
- Saviniec, L.; Santos, M. O.; Costa, A. M. **A Parallel Multi-start Iterated Local Search and a Proximity Relax-and-fix heuristic for High School Timetabling Problem.** *Brazilian Symposium on Operations Research*, Sobrapo, Rio de Janeiro, Brazil (2017) 1 – 12.

6.1 Directions for future research

Below, we list some topics that can be further investigated in future studies, in order to extend this research.

In the line of exact methods for high school timetabling problems, a branch-and-price method can be devised, so that formulation EF2 of Chapter 3 is exploited to find the optimal

solutions or improve the lower bounds for open instances. Moreover, as discussed before, formulation CF2 suffers a serious issue related to the exponential growth of binary variables, which depends on the number of periods per day. Therefore, the development of alternative formulations is also another possibility for future investigations. During this research, we have also investigated Benders decomposition for high school timetabling problems by using both compact formulations CF1 and CF2 of Chapter 3, but no exciting results were obtained. However, further investigations can be done if new formulations are developed.

Parallel metaheuristics is also a promising line of research in high school timetabling problems. In this thesis, we have developed parallel algorithms that exploit only trajectory-based metaheuristics as cooperative agents. Further studies to investigate cooperation among trajectory and population-based metaheuristics are also aimed. Another interesting research topic is the extension of the parallel algorithms proposed in Chapters 4 and 5 to run in a distributed environment. Adjust them to solve other timetabling problems, such as the university course and examination timetabling problems, is also another possible topic.

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DESCRIPTION AND TUNING OF THE STAND-ALONE METAHEURISTICS

In this appendix, we describe the sequential metaheuristics used in Chapter 4. We describe the data structure proposed to encode timetables, the neighborhood structure, the heuristic to construct initial solutions and the objective function evaluation. The pseudo-code for each metaheuristic and tuning procedures are also discussed.

A.1 Timetable encoding

Figure 18 – Proposed timetable encoding for the HSTP.

Trajectory-based metaheuristics frequently modify copies of the current or global best solution. However, some algorithms do not modify the whole copy, keeping parts that are identical to that of the previous solution. To avoid the duplication of unnecessary data (parts of the data structure that are copied but not modified), we designed a new timetable encoding to HSTP based on *persistent data structures* (KAPLAN, 2005). In a persistent data structure, each modification generates a new version of the structure, keeping the previous version accessible. There are several consolidated methods to make a data structure persistent (DRISCOLL *et al.*,

1989). One of these methods is based on *structural sharing*. The idea is to share with the new copy, parts of the original structure that remain unchanged.

To apply this approach, we designed the encoding shown in Figure 18, in which S represents an array of all timeslots $(d, h) \in D \times H$. Each position of S points to an array of integer values. In each of these arrays, the first position stores the ID of the teacher assigned to class 0, the second position stores the teacher assigned to class 1, and so on. The next section explains how this timetable encoding works with neighborhoods.

A.2 Neighborhood structure

Figure 19 – The TQ neighborhood generation with the proposed timetable encoding.

We use a two-swap operator which, for a given class $c \in C$, exchanges the teachers assigned to two different timeslots. Feasibility in hard requisites 1 (meeting of weekly required lessons), 2 (no clashes in classes' schedules) and 3 (no clashes in teachers' schedules) is maintained with the use of the Torque Neighborhood Operator (TQ) proposed in Saviniec and Constantino (2017). The TQ acts as a *Kempe Chain Interchange* (LÜ; HAO; GLOVER, 2011)

that uses the idea of conflict graph and effects a series of consecutive swaps until there are no clashes in teachers nor classes' schedules.

Figure 19 shows examples of TQ moves with the encoding scheme proposed in Section A.1. The TQ operator is applied to a partial solution Z shown in Figure 19a. The conflict graph in Figure 19b identifies valid moves (connected components) in order to avoid clashes. The teachers assigned to classes c_0 and c_1 are swapped between timeslots s_0 and s_1 . In the generated neighboring solution Z_1 , timeslots s_0 and s_1 point to new arrays while remaining timeslots share the current arrays. Note that the arrays not changed in the move are shared between both solutions. Figures 19c and 19d show that to operate or modify only timeslots s_0 , s_1 and s_2 , we do not need to copy the whole solution Z .

We employ this neighborhood generation scheme in all sequential and parallel metaheuristics proposed in this Thesis.

A.3 Initial solutions

Our initial solutions are constructed by the randomized heuristic described in Algorithm 7. This heuristic receives an input list L of pairs $(c \in C, t \in T)$, in which each entry represents one lesson to be scheduled. At each iteration, the heuristic selects a lesson from the list and assigns it to a timeslot that is randomly selected among the free timeslots. This heuristic always generates solutions that are feasible for hard requisites 1 and 2.

Algorithm 7 – Pseudo-code of the constructive heuristic.

SOLUTION-GENERATOR(L)

```

1  Initialize an empty solution  $Z$ 
2  for each  $(e \in L)$  {
3       $i =$  Take an integer in  $[0, |S| - 1]$ 
4      while (class  $e.c$  is busy in timeslot  $s_i$ ) {
5           $i = (i + 1) \bmod |S|$ 
6      }
7      Assign teacher  $e.t$  to class  $e.c$  in timeslot  $s_i$ 
8  }
9  return  $Z$ 
```

A.4 Objective function evaluation

Our metaheuristics are allowed to work in an infeasible space by relaxing the hard requisites, except 1 and 2, and penalizing them in the objective function with high penalties to enforce feasibility. Let $R_\#$ bet the set of requisites (not including 1 and 2) of problem $\#$, for $\#$

representing HSTP-A, HSTP-B or HSTP-C. The objective function of problem # is given by:

$$\text{Minimize } f(Z) = \sum_{i \in R_{\#}} \alpha_i V_i \quad (\text{A.1})$$

Where α_i and V_i are the penalty and the number of violations made on requisite $i \in R_{\#}$, respectively.

As the constructive heuristic (Algorithm 7) always generates feasible solutions to hard requisites 1 and 2, and this feasibility is maintained by the neighborhood operator TQ, then only the remaining requisites are explicitly considered in the objective function. We used the following penalty values for violations of the hard requisites. Problem HSTP-A ($\alpha_3 = \alpha_4 = 100000$), problem HSTP-B ($\alpha_3 = \alpha_4 = 100000$, $\alpha_5 = 10000$), and problem HSTP-C ($\alpha_3 = \alpha_4 = 100000$, $\alpha_5 = 10000$, $\alpha_6 = 5000$).

In our implementation, we use incremental evaluation of solutions. The algorithms are designed to evaluate only the changed parts whenever a new move is performed.

Algorithm 8 – Pseudo-code of the sequential ILS.

ILS($Z_0, timeOut$)

```

1   $Z^* = Z_0$ 
2  while ( $elapsedTime < timeOut$ ) {
3       $Z = \text{PERTURBATION}(Z^*)$ 
4      do {
5           $best = f(Z)$ 
6           $i = \text{Take an integer in } [0, |SP| - 1]$ 
7          for ( $j = 1$  to  $|SP|$ ) {
8              Compute the connected components for lessons in timeslots pair  $sp_i$  of  $Z$ 
9              for (each connected component) {
10                 Generate a new neighboring solution  $Z'$  from  $Z$ 
11                 if ( $f(Z') \leq f(Z)$ ) {
12                      $Z = Z'$ 
13                 }
14             }
15              $i = (i + 1) \bmod |SP|$ 
16         }
17     } while ( $f(Z) < best$ )
18     if ( $f(Z) \leq f(Z^*)$ ) {
19          $Z^* = Z$ 
20     }
21 }
22 return  $Z^*$ 

```

A.5 Sequential metaheuristics for high school timetabling

Our sequential metaheuristics to the HSTP include versions of iterated local search (ILS), tabu search (TS), simulated annealing (SA) and late acceptance strategy (LAS).

We define SP as the array of all timeslot pairs (s_i, s_j) , for $i, j = 0, \dots, |S| - 1$ and $i \neq j$. Therefore, neighborhoods are defined by the way in which the timeslot pairs are explored in SP . Each of our algorithms may search this array using a different strategy.

Algorithm 9 – Pseudo-code of the sequential TS.

```

TS( $Z_0, timeOut, tbTime, stSize$ )
1   $Z = Z_0$ 
2   $Z^* = Z_0$ 
3   $iter = 0$ 
4  while ( $elapsedTime < timeOut$ ) {
5       $i = \text{Take an integer in } [0, |SP| - 1]$ 
6       $f_{best} = \infty$ 
7       $i_{best} = -1$ 
8       $Z_{best} = \emptyset$ 
9      for ( $j = 1$  to  $stSize$ ) {
10         Compute the connected components for lessons in timeslots pair  $sp_i$  of  $Z$ 
11         for (each connected component) {
12             Generate a new neighboring solution  $Z'$  from  $Z$ 
13             if ( $f(Z') \leq f_{best}$  and ISNOTTABU( $f(Z'), i, iter, tbTime$ )) {
14                  $f_{best} = f(Z')$ 
15                  $i_{best} = i$ 
16                  $Z_{best} = Z'$ 
17             }
18         }
19          $i = (i + 1) \bmod |SP|$ 
20     }
21     Insert the tuple  $(f(Z), i_{best}, iter)$  into the tabu list
22     if ( $f_{best} \leq f(Z^*)$ ) {
23          $Z^* = Z_{best}$ 
24     }
25      $Z = Z_{best}$ 
26      $iter = iter + 1$ 
27 }
28 return  $Z^*$ 
  
```

A.5.1 Iterated local search

The pseudo-code of the implemented sequential iterated local search is shown in Algorithm 8. At each iteration of the outer loop (from line 2 to line 21), it makes a random perturbation (line 3) in the current solution, followed by a local search (loop from line 4 to line 17). The local search starts in a random index i of SP and evaluates the next $|SP|$ consecutive timeslot pairs. For each index, it generates the connected components according to what was discussed in Section A.2. If it finds a connected component that generates an improved solution, then the new solution is accepted. These steps are repeated while an improved solution is found after exploring $|SP|$ consecutive indexes¹. The PERTURBATION procedure, in line 3, returns a random neighboring solution of the global best solution Z^* . This procedure selects a random index k of

¹ The algorithm iterates through the indexes of SP in a circular fashion as in a circular linked list.

SP , constructs the conflict graph for lessons assigned in timeslots sp_k of the solution Z^* , and makes a random TQ move with the associated connected components.

A.5.2 Tabu search

The pseudo-code of the implemented sequential tabu search is shown in Algorithm 9. At each iteration of the outer loop (from line 4 to 27), the algorithm searches the best neighbor of the current solution Z . The best neighbor is accepted if it is better or equal to the global best solution Z^* . The search starts in a random index i of SP , as in the ILS, and evaluates a strip of SP defined by the next $stSize$ consecutive timeslot pairs. Only moves that are not tabu are accepted. A tabu move is a tuple $(f, index, iter)$, where f is the objective function value before the move, $index$ is the index of SP in which the move was done, and $iter$ is the iteration in which the move was performed. The parameter $tbTime$ is the number of iterations in which a move remains tabu (the aspiration criterion). The procedure ISNOTTABU (line 13) checks if a move is tabu and also, removes those moves in which the tabu time has expired.

Algorithm 10 – Pseudo-code of the sequential SA.

```

SA( $Z_0, timeOut, T_0, \alpha, stSize, \epsilon$ )
1   $Z = Z_0$ 
2   $Z^* = Z_0$ 
3   $T = T_0$ 
4  while ( $T \geq \epsilon$  and  $elapsedTime < timeOut$ ) {
5       $i = \text{Take an integer in } [0, |SP| - 1]$ 
6      for ( $j = 1$  to  $stSize$ ) {
7          Compute the connected components for lessons in timeslots pair  $sp_i$ 
8          for (each connected component) {
9              Generate a new neighboring solution  $Z'$  from  $Z$ 
10              $\Delta = f(Z) - f(Z')$ 
11             if ( $\Delta \geq 0$ ) {
12                  $Z = Z'$ 
13                 if ( $f(Z') < f(Z^*)$ ) {
14                      $Z^* = Z'$ 
15                 }
16             } else {
17                  $k = \text{Take a continuous value in } [0, 1]$ 
18                 if ( $k \leq e^{\Delta/T}$ ) {
19                      $Z = Z'$ 
20                 }
21             }
22         }
23          $i = (i + 1) \bmod |SP|$ 
24     }
25      $T = \alpha \cdot T$ 
26 }
27 return  $Z^*$ 

```

A.5.3 Simulated annealing

The pseudo-code of the implemented sequential simulated annealing is shown in Algorithm 10. The code follows a standard SA framework. However, instead of selecting each new neighboring solution randomly, as in standard SA, the algorithm chooses a random index of SP , as a starting point, and searches the next $stSize$ consecutive timeslot pairs. The outer loop (from line 4 to 26) controls the temperature level and stops after a time-out is reached or the temperature T reaches a small value given by parameter ε . The remaining steps are those from a standard SA framework. Parameters T_0 and α are the initial temperature and the cooling rate, respectively.

Algorithm 11 – Pseudo-code of the sequential LAS.

LAS($Z_0, timeOut, lsSize, stSize$)

```

1   $Z = Z_0$ 
2   $Z^* = Z_0$ 
3  for ( $j = 0$  to  $lsSize - 1$ ) {
4       $L_j = f(Z_0)$ 
5  }
6   $v = 0$ 
7  while ( $elapsedTime < timeOut$ ) {
8       $i = \text{Take an integer in } [0, |SP| - 1]$ 
9      for ( $j = 1$  to  $stSize$ ) {
10         Compute the connected components for lessons in timeslots pair  $sp_i$ 
11         for (each connected component) {
12             Generate a new neighboring solution  $Z'$  from  $Z$ 
13             if ( $f(Z') \leq f(Z)$  or  $f(Z') \leq L_v$ ) {
14                  $Z = Z'$ 
15                 if ( $f(Z') < L_v$ ) {
16                      $L_v = f(Z')$ 
17                 }
18                 if ( $f(Z') < f(Z^*)$ ) {
19                      $Z^* = Z'$ 
20                 }
21             }
22              $v = (v + 1) \bmod lsSize$ 
23         }
24          $i = (i + 1) \bmod |SP|$ 
25     }
26 }
27 return  $Z^*$ 

```

A.5.4 Late acceptance strategy

The last sequential metaheuristic proposed is a late acceptance strategy, for which a pseudo-code is presented in Algorithm 11. The LAS is a recent metaheuristic approach (BURKE; BYKOV, 2008) that maintains a list L of size $lsSize$ which records the objective values of the last $lsSize$ accepted solutions. The main idea of the LAS is to compare the current solution with

a previous current solution recorded in L (BURKE; BYKOV, 2008). In our version of the LAS, the neighborhood is searched by strips, as in our versions of TS and SA described above.

A.6 Parameter tuning

In this section, we report the computational experiments carried out to calibrate the sequential metaheuristics described in Section A.5.

To these experiments, we selected the problem HSTP-A and a subset of the 34 instances from Table 15 to analyze and tune the sequential metaheuristics. The reduced set of instances comprises two small (10 and 13), two medium (6 and 26) and two large (7 and 24) instances.

Figures 20–23 present charts of normalized objective values over time for different algorithm configurations. The x -axis (time in seconds) is in quadratic scale. Objective values were collected for each time $x = i^2$ seconds, for $i = 1, \dots, 25$. The y -axis are normalized objective values obtained as follows. Let Z_j be a timetable solution for the j^{th} instance, its normalized objective value is given by $NOV(Z_j) = f(Z_j)/f(Z_j^*)$. Where $f(Z_j^*)$ is a benchmark objective value for instance j , which can be a lower bound or the best-known solution. As at each point x , each metaheuristic collects 25 samples for each instance, the final y -value in a point x is the median of the instances' median sample. In these charts, the reference line ($y = 1$) represents the benchmark objective value $f(Z_j^*)$. In other words, the charts summarize the results for all instances and indicate, for each time point, how close or far the algorithm' median solutions are from the benchmark value, providing insights on the algorithm convergence rates.

A.6.1 Results

We tested a number of different values to calibrate parameters of TS, SA and LAS. The following values were tested:

- TS: $tbTime = (5, 10, 25, 50, 100, 200, 400, 600)$ and $stSize = (5, 10, 25, 50, 100, 300, 600)$.
- SA: $T_0 = (100, 500, 1000)$, $\alpha = (0.99, 0.9999)$, $stSize = (1, 5, 10, 25, 50, 100, 300, 600)$ and $\epsilon = 0$.
- LAS: $lsSize = (100, 500, 1000, 5000, 10000)$ and $stSize = (1, 5, 10, 25, 50, 100, 300, 600)$.

We analyze the behavior presented by different combinations of the above parameter values. Figure 20 shows the TS configurations obtained. The results indicate that the TS is very sensitive to the size of the neighborhood explored (parameter $stSize$). Small and large values for $stSize$ lead the TS to converge to poor quality solutions. An appropriate value is around 100. Also, the best values for the tabu aspiration criterion (parameter $tbTime$) were observed to be

between 10 and 100. These results suggest that the TS metaheuristic must be carefully calibrated concerning neighborhood sizes.

Figure 20 – Results of sequential TS for different parameter configurations.

Figure 21 – Results of sequential SA for different parameter configurations.

Figure 21 shows the behavior of the SA for different configurations. The algorithm seems to be most sensitive to the cooling rate parameter. A good choice was $\alpha = 0.9999$. We observed that even small negative perturbations in this value could worsen convergence rates. Also, we observed that when the SA tended to be purely random, with small values for parameter *stSize*, the SA converged to poor quality solutions. On the other hand, when *stSize* tended to large values, the SA converged to better solutions, although with a slower convergence rate. In Figure

21 we plotted only configurations for $\alpha = 0.9999$. Configurations for $\alpha = 0.99$ performed as badly as purely random configurations.

Figure 22 shows the LAS configurations. The results show that LAS is not sensitive to the way in which the neighborhood is explored, parameter $stSize$. On the other hand, its convergence strongly depends on the length of the list of late accepted solutions, parameter $lsSize$. Large size lists lead the algorithm to converge to better quality solutions. However, the convergence is slow.

Figure 22 – Results of sequential LAS for different parameter configurations.

Figure 23 – Best configurations for sequential metaheuristics.

Figure 23 plots the best observed configurations for TS with $(tbTime, stSize) = (25, 100)$, SA with $(T_0, \alpha, stSize) = (100, 0.9999, 300)$ and LAS with $(lsSize, stSize) = (5000, 300)$ together

with the ILS results. The benchmark value for an instance j is the objective value $f(Z_j^*)$ of the best solution Z_j^* found in the experiments with these sequential algorithms. The chart in Figure 23 shows that ILS and TS presented better results than SA and LAS, in general. The median objective values of the four algorithms after 625 seconds of execution are 2.2% (ILS), 3.6% (TS), 6.8% (SA) and 8.3% (LAS) worse than the benchmark values. While SA and LAS take 625 seconds to reach solutions that are around 7 to 8% worse than the benchmark values, ILS and TS compute solutions with the same quality in less than 80 seconds. Also, we note that ILS perform slightly better in quality than TS.

THE IMPROVED RELAX-AND-FIX HEURISTIC

A Parallel Multi-Start Iterated Local Search and a Proximity Relax-and-Fix heuristic for High School Timetabling Problem

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ABSTRACT

The high school timetabling problem has been frequently considered in the literature, but few studies employing parallel strategies have been proposed. In this paper, we propose a parallel cooperative multi-start approach. The method constructs an initial feasible solution via a Proximity Relax-and-Fix heuristic and applies a Parallel Multi-Start Iterated Local Search to further improve the initial solution. Computational experiments with practical instances show that our approach outperforms the state-of-the-art algorithms for variants of the problem. Our results clearly demonstrate the power of this type of approach to deal with complex combinatorial problems, such as the high school timetabling problem.

KEYWORDS. High School Timetabling Problem. Proximity Relax-and-Fix. Parallel Multi-Start Iterated Local Search.

Paper topics: Parallel Algorithms in CPU & GPU. Metaheuristics. Combinatorial Optimization.

Introduction

The scientific literature has branched educational timetabling problems in three main families: University course timetabling [Lewis et al., 2007; Di Gaspero et al., 2007], high school timetabling [Pillay, 2014] and examination timetabling [McCollum et al., 2007]. Each category contains a rich number of specific constraints and particularities.

In this paper, we investigate parallel metaheuristics for the high school timetabling problem (HSTP). Due to its combinatorial characteristics, the HSTP resolution has been mainly based on approximations by sequential metaheuristics. According to our knowledge, only four studies have addressed the design of parallel metaheuristics for HSTP. Abramson [1991] proposed a Simulated Annealing which analyzes several simultaneous moves. Abramson and Abela [1992] proposed a Genetic Algorithm that evaluates several individuals of the population in parallel. Srndic et al. [2009] proposed a Genetic Algorithm which splits the global population into small islands that are managed in parallel. Saviniec et al. [2015] proposed an Iterated Local Search that, at each iteration, chooses the best local minimum among a set of local minima returned by multiples local search procedures running simultaneously.

Alba et al. [2013] reviewed the literature of parallel metaheuristics and identified a fast grown on these techniques in the last few years. The authors also classified the parallel frameworks and enumerated a few challenge topics that will be the main lines of research in the upcoming years.

According to Alba et al. [2013], parallel metaheuristics can be classified into *population-based* and *trajectory-based* methods. Population-based metaheuristics are methods that keep a pool of solutions (e.g. Genetic Algorithms and Ant Colony Optimization). Trajectory-based metaheuristics are methods that keep a single current solution (e.g. Simulated Annealing and Tabu Search). These two categories can also, be classified by the type of framework implementation [Alba et al., 2013].

Population-based metaheuristics can be classified into i) *Parallel individuals evaluation* – several individuals in the population are evaluated in parallel, and ii) *Parallel islands* – the population is split into subpopulations that can be managed in parallel.

Trajectory-based metaheuristics can be classified into i) *Parallel moves* – the neighboring solutions of the current solution are searched in parallel, ii) *Move acceleration* – the objective function of a single solution may be decomposed and evaluated in parallel, and iii) *Parallel multi-start* – several asynchronous threads, cooperative or independent, run simultaneously to search different regions of the solution space.

Among the hot topics for future research in parallel metaheuristics, Alba et al. [2013] point out the design of parallel versions of well-known metaheuristics that could benefit from specific hardware architectures. In this paper, we propose a parallel multi-start approach based on Iterated Local Search that is suitable for multi-core machines. The algorithm is designed to exploit diversification, intensification, and cooperation among asynchronous threads during the search. We also propose a constructive heuristic based on Relax-and-Fix [Dillenberger et al., 1994] and Proximity Search [Fischetti and Monaci, 2014] to provide initial solutions to our parallel approach.

The remainder of the paper is organized as following. Section 2 describes the addressed problem by a mixed-integer programming formulation. Section 3 explains our parallel approach. Section 4 presents and discusses our computational experiments. The paper ends in Section 5 with some final remarks.

The high school timetabling problem

We focus on an HSTP motivated by rules of Brazilian high schools. In this context, the schools have a set of classes and a set of teachers. Classes are disjoint groups of students enrolled in the same set of subjects (e.g.: mathematics, physics and etc.). Each subject has a pre-assigned teacher and a number of meetings that must be scheduled to run during the week (e.g. Monday to Friday). The goal of the problem is to obtain a weekly timetable for these meetings. An input of the problem is described by the following definitions:

Notation	Definition
Sets	
C	a set of classes.
T	a set of teachers.
D	a set of weekdays.
H	a set of periods per day, which is equal to every day.
H_{td}	the set of periods for which teacher $t \in T$ is available during day $d \in D$.
R	a set of subjects' requirements. A requirement specifies that a teacher $t \in T$ and a class $c \in C$ must meet a specified number of times during the week.
R_c	the set of requirements that belongs to class $c \in C$.
R_t	the set of requirements that belongs to teacher $t \in T$.
Parameters	
$\tilde{\theta}_r \in \mathbb{N}$	the number of weekly meetings specified in requirement $r \in R$.
$\tilde{\delta}_r \in \mathbb{N}$	a daily limit for lessons involving requirement $r \in R$.
$\tilde{\pi}_r \in \mathbb{N}$	a desired number of double lessons for requirement $r \in R$. Double lessons are two lessons in consecutive periods of the same day.

An output of the problem is represented by a set of binary variables x_{rdh} which indicate whether the r -th requirement is scheduled to period $h \in H$ of day $d \in D$. A feasible solution is an assignment of values to variables x_{rdh} that respects the following hard requirements:

1. **Meeting of weekly required lessons:** each requirement must be scheduled.

$$\sum_{d \in D} \sum_{h \in H} x_{rdh} = \tilde{\theta}_r \quad \forall r \in R \quad (1)$$

2. **No clashes in classes' schedules:** each class must attend exactly one meeting per period.

$$\sum_{r \in R_c} x_{rdh} = 1 \quad \forall c \in C; d \in D; h \in H \quad (2)$$

3. **No clashes in teachers' schedules:** each teacher must teach at most one lesson per period.

$$\sum_{r \in R_t} x_{rdh} \leq 1 \quad \forall t \in T; d \in D; h \in H_{td} \quad (3)$$

4. **No assignment of teachers in their unavailable periods:** teachers must not be assigned to their unavailable periods.

$$\sum_{r \in R_t} x_{rdh} = 0 \quad \forall t \in T; d \in D; h \in H \setminus H_{td} \quad (4)$$

5. **No daily workload violation for requirements:** each requirement $r \in R$ must not be involved in more than $\tilde{\delta}_r$ meetings per day.

$$\sum_{h \in H} x_{rdh} \leq \tilde{\delta}_r \quad \forall r \in R; d \in D \quad (5)$$

6. **No holes in requirements' schedules:** the schedule of each requirement must be consecutive within the same day.

$$x_{rdi} - x_{rdh} + x_{rdj} - 1 \leq 0 \quad \forall r \in R; d \in D; h = 1, \dots, |H| - 2; \\ i = 0, \dots, h - 1; j = h + 1, \dots, |H| - 1 \quad (6)$$

An optimal feasible solution minimizes the penalties associated with the following soft requirements:

7. **Meeting of double lessons for requirements:** for each requirement $r \in R$, the minimum number of $\tilde{\pi}_r$ consecutive double lessons should be met. The auxiliary variables $\hat{\phi}_{rdh}$ are set

to 1 if a double lesson ends in periods $h = 1, \dots, |H| - 1$ within each day, while variables $\hat{\pi}_r$ quantify the number of unmet weekly double lessons.

$$\hat{\phi}_{rdh} \leq x_{rdh} \quad \forall r \in R; d \in D; h = 1, \dots, |H| - 1 \quad (7)$$

$$\hat{\phi}_{rdh} \leq x_{r,d,h-1} \quad \forall r \in R; d \in D; h = 1, \dots, |H| - 1 \quad (8)$$

$$\hat{\phi}_{rdh} \leq 1 - \hat{\phi}_{r,d,h-1} \quad \forall r \in R; d \in D; h = 2, \dots, |H| - 1 \quad (9)$$

$$\hat{\pi}_r \geq \tilde{\pi}_r - \sum_{d \in D} \sum_{h=1}^{|H|-1} \hat{\phi}_{rdh} \quad \forall r \in R \quad (10)$$

$$\hat{\phi}_{rdh} \geq 0 \quad \forall r \in R; d \in D; h = 1, \dots, |H| - 1 \quad (11)$$

$$\hat{\pi}_r \geq 0 \quad \forall r \in R \quad (12)$$

8. **No idle periods in teachers' schedules:** teachers should not have idle periods in their daily schedules. A teacher t is idle in periods $h = 1, \dots, |H| - 2$ of a day d if he/she is not busy, but is busy in earlier and later periods of the same day. Idles periods are flagged by the auxiliary variables $\hat{j}_{tdh}$.

$$\hat{j}_{tdh} \geq \sum_{r \in R_t} (x_{rdi} - x_{rdh} + x_{rdj}) - 1 \quad \forall t \in T; d \in D; h = 1, \dots, |H| - 2; \quad (13)$$

$$\hat{j}_{tdh} \geq 0 \quad \forall t \in T; d \in D; h = 1, \dots, |H| - 2 \quad (14)$$

9. **Compact schedules for teachers:** for each teacher, the weekly schedule should encompass a minimum number of working days.

$$\hat{d}_{td} \geq \sum_{r \in R_t} x_{rdh} \quad \forall t \in T; d \in D; h \in H_{td} \quad (15)$$

$$\hat{d}_{td} \geq 0 \quad \forall t \in T; d \in D \quad (16)$$

The complete problem can thus, be described by the following mixed-integer programming (MIP) formulation.

$$\textbf{Minimize} \quad f(x) = \sum_{r \in R} \alpha_7 \hat{\pi}_r + \sum_{t \in T} \sum_{d \in D} \alpha_9 \hat{d}_{td} + \sum_{h=1}^{|H|-2} \alpha_8 \hat{j}_{tdh} \quad (17)$$

Subject to:

$$(1) - (16) \quad (18)$$

$$x_{rdh} \in \{0, 1\} \quad \forall r \in R; d \in D; h \in H \quad (19)$$

The objective function (17) minimizes the number of violations in requirements $i = 7, \dots, 9$. The constant α_i is an associated penalty that express the importance of the i -th requirement.

This formulation can also be augmented by the cuts (20), proposed by Souza [2000], which specify that a teacher cannot work less than a minimum number of working days.

$$\sum_{d \in D} \hat{d}_{td} \geq \max \left\{ \left\lceil \frac{\sum_{r \in R_t} \tilde{\theta}_r}{|H|} \right\rceil, \max_{r \in R_t} \left\{ \left\lceil \frac{\tilde{\theta}_r}{\tilde{\delta}_r} \right\rceil \right\} \right\} \quad \forall t \in T \quad (20)$$

The proposed approach

We propose an approach with constructive and improvement phases. The approach applies a heuristic based on Relax-and-Fix and Proximity Search (Section 3.1) to construct an initial feasible solution that is further improved by a Parallel Multi-Start Iterated Local Search (Section 3.2).

The constructive Proximity Relax-and-Fix heuristic (PRF)

The original Relax-and-Fix (RF) [Dillenberger et al., 1994] is an MIP heuristic that constructs integer solutions by solving a series of relaxed sub-problems. Let's assume that X is the set of integer variables of an MIP model. The RF heuristic relaxes the integrality of all variables of X and splits them in n subsets X^k , such that $X = \bigcup_{k=0}^{n-1} X^k$. At each iteration k , the variables belonging to subset X^k are converted to integer and the resulting problem is solved. If it results in a feasible solution, then the variables of X^k are fixed to their current values and the heuristic is repeated to iteration $k + 1$. Otherwise, no solution is possible for the chosen partition and the heuristic stops. If the RF heuristic does not stop prematurely, it provides an integer solution at the end.

Algorithm 1 Pseudo-code of the constructive PRF heuristic.

PRF(m_0, m)

```

1  Let  $m_0$  be the size of the first partition and  $m$  be the size of the remaining partitions.
2  Let  $P$  be the linear program of formulation (17) – (19).
3  Initialize  $P$  with constraints (1) – (6) and set  $f(x) = \emptyset$ .
4  Let  $LTS$  be the list of time-slots  $(d, h) \in D \times H$ .
5  Shuffle the list  $LTS$ .
6  for  $i = 0$  to  $m_0 - 1$  do
7      Insert the variables associated with time-slot  $LTS[i]$  into partition  $X^0$ .
8   $n = 1$ 
9  for  $j = m_0$  to  $|LTS| - 1$  by  $m$  do                                // compute the remaining partitions
10      $n = n + 1$ 
11     for  $i = j$  to  $\min\{j + m - 1, |LTS| - 1\}$  do
12         Insert the variables associated with time-slot  $LTS[i]$  into partition  $X^{n-1}$ .
13  Convert the variables of partition  $X^0$  to integer.
14  Solve  $P$ .
15  Record the partial integer solution  $S^0 = X^0$ .
16  Fix the variables of partition  $X^0$  to their current values.
17  Insert constraints (15) – (16) and the cuts (20) into  $P$  and set  $f(x) = \alpha_9 \sum_{t \in T} \sum_{d \in D} \hat{d}_{td}$ .
18  for  $k = 1$  to  $n - 1$  do
19      Convert the variables of partition  $X^k$  to integer.
20      Solve  $P$ .
21      if  $P$  is feasible then
22          Record  $S^k = S^{k-1} \cup X^k$ .
23          Fix the variables of partition  $X^k$  to their current values.
24      else
25          Estimate the cost of  $S^k$ , given by  $f_k = (f(S^{k-1}) \div (m_0 + (k - 1)m)) \cdot (m_0 + km)$ .
26          Release the variables of partition  $\bigcup_{i=0}^{k-1} X^i$ .
27          Set  $P$  with the proximity function  $\Delta(x, \tilde{x}) = \sum_{j \in S^{k-1}: \tilde{x}_j = 0} x_j + \sum_{j \in S^{k-1}: \tilde{x}_j = 1} (1 - x_j)$ .
28          Insert the cut-off constraint  $f(x) \leq f_k$  into  $P$ .
29          Solve  $P$ .
30          if  $P$  is infeasible then
31              return  $\emptyset$ .
32          Record  $S^k = \bigcup_{i=0}^k X^i$ .
33          Fix the variables of partition  $\bigcup_{i=0}^k X^i$ .
34          Remove the cut-off constraint and set  $P$  with the objective function  $f(x) = \alpha_9 \sum_{t \in T} \sum_{d \in D} \hat{d}_{td}$ .
35  return  $S^{n-1}$ .
```

The drawback of the original RF heuristic, when applied to the HSTP described in Section 2, is that the fixation of a partition X^k , at iteration k , very often leads to an infeasible problem at iteration $k + 1$. To overcome this limitation, we propose a variant of the RF heuristic which solves a proximity problem [Fischetti and Monaci, 2014] to escape from these infeasible partitions. Silva [2013] proposed a similar strategy to deal with this limitation of the RF heuristic in lot-sizing problems. The author employs a local branch constraint instead of a proximity model to find new

feasible partitions and escape from infeasibilities.

The proposed heuristic (PRF) is shown in Algorithm 1. The algorithm is a relax-and-fix strategy with the addition of lines 25 to 34, that solve the proximity problem to escape from infeasible partitions. At each iteration k , every time that the algorithm detects an infeasible partition, the line 25 estimates the cost f_k of the partial solution S^k , based on the previous solution S^{k-1} . The line 26 releases all fixed variables and lines 27 to 29 set and solve the proximity problem. The proximity problem tries to find a partial solution S^k that is not greater than f_k and minimizes the changes in the values previously fixed for variables of S^{k-1} . If the proximity problem is feasible, a new partial solution S^k is found, all variables in the partition $\bigcup_{i=0}^k X^i$ are fixed to their current values and the algorithm continues in the next iteration. Otherwise, it stops without any solution.

In our implementation, each partition contains the variables associated with a subset of time-slots $(d, h) \in D \times H$. We defined the first partition with size $m_0 = 10$ time-slots and the rest with $m = 5$ time-slots. Also, every time that the program P is solved in lines 14, 20 and 29, we stop it at the first solution found, to speed up the execution.

The Parallel Multi-Start Iterated Local Search (PMILS)

The proposed Parallel Multi-Start Iterated Local Search (PMILS), shown in Figure 1, employs a shared-memory manager/works strategy. The central idea of the PMILS is to exploit diversification/intensification and cooperation among threads. The algorithm has a manager thread (the diversifier agent) that shares information with N worker threads (the intensifier agents) via two communication buffers, “*InputBuffer*” and “*OutputBuffer*”. All threads are Iterated Local Search [Lourenço et al., 2003] metaheuristics.

Figure 1: Diagram of communication among the agent threads of the PMILS algorithm.

The PMILS pseudo-code is shown in Algorithm 2. The diversifier agent starts at an initial solution and at each iteration, puts its local minimum into the input buffer (lines 10–14) to be further explored by the intensifier agents. The intensifiers get solutions from the input buffer (line 3) and intensify the search in their nearby neighborhoods (line 6). When an intensifier reaches its time-limit $intTimeLim$, it puts its best solution into the output buffer (line 7) to be analyzed by the diversifier. When the diversifier retrieves a solution S_r from the output buffer (line 15), it checks for acceptance. If S_r improves its best solution S^* , then the diversifier restarts its search from S_r (line 19). Otherwise, S_r is discarded.

The difference between the diversifier and the intensifier threads is that, while the diversifier walks through the solution space (lines 6–7) by only restarting from the global best solution when an improved solution is found (line 19). The intensifiers explore the nearby neighborhood of the local minima found by the diversifier, by always restarting from their private best solutions, see line 3 of the ILS procedure.

The timetable solutions used by the PMILS algorithm are represented by non-negative integer arrays S , where S_{cdh} stores the teacher $t \in T$ assigned to teach to class $c \in C$ on period $h \in H$ of day $d \in D$. The neighborhood operator is the TQ operator proposed by Saviniec et al. [2013], which is based on the idea of *Kempe Chain Interchanges* [Lü et al., 2011] and explores moves by exchanging conflicting teachers between two time-slots $(d, h) \in TS$, with $TS = D \times$

H. The local search procedure is a first improvement strategy that explores all pairs of time-slots $(p, q) \in TS \times TS$, for $p \neq q$.

Algorithm 2 Pseudo-code of the PMILS algorithm.

DIVERSIFIER($S, divTimeLim, N$)

```

1  Initialize two buffers InputBuffer and OutputBuffer of size  $N$ .
2  Initialize  $N$  intensifier threads.
3   $S^* = S, S_r = \emptyset$ 
4   $time = 0, added = 0$ 
5  while ( $time < divTimeLim$ ) do
6       $S = PERTURBATION(S, 2)$  // Apply two random moves
7       $S = LOCALSEARCH(S)$ 
8      if  $f(S) \leq f(S^*)$  then
9           $S^* = S$ 
10      $gap = 1 - \max\{f(S) - f(S^*), 0\} \div f(S^*)$ 
11     if  $GETRANDOM(0, 1) \leq gap$  then
12          $added = added + InputBuffer.TryAdd(S)$ 
13     else
14          $added = added + InputBuffer.TryAdd(S^*)$ 
15          $removed = OutputBuffer.TryRemove(S_r)$ 
16          $added = added - removed$ 
17     if  $removed$  then
18         if  $f(S_r) < f(S^*)$  then
19              $S = S_r$  // Restart from the global best solution
20         if  $f(S_r) \leq f(S^*)$  then
21              $S^* = S_r$ 
22      $time = CPU\ TIME()$ 
23 while ( $added > 0$ ) do
24      $OutputBuffer.Remove(S_r)$ 
25      $added = added - 1$ 
26     if  $f(S_r) < f(S^*)$  then
27          $S^* = S_r$ 
28 for  $k = 1$  to  $N$  do
29      $InputBuffer.Add(NULL)$ 
30 BARRIERWAIT()
31 return  $S^*$ 

```

INTENSIFIER($intTimeLim$)

```

1   $S = \emptyset$ 
2  while ( $true$ ) do
3       $InputBuffer.Remove(S)$ 
4      if  $S = NULL$  then
5          break
6       $S = ILS(S, intTimeLim)$ 
7       $OutputBuffer.Add(S)$ 
8  return

```

ILS($S_0, ilsTimeLim$)

```

1   $S^* = S_0, time = 0$ 
2  while ( $time < ilsTimeLim$ ) do
3       $S = PERTURBATION(S^*, 1)$  // Apply one random move
4       $S = LOCALSEARCH(S)$ 
5      if  $f(S) \leq f(S^*)$  then
6           $S^* = S$ 
7       $time = CPU\ TIME()$ 
8  return  $S^*$ 

```

Computational experiments

We tested the proposed approach with two variants of the problem described in Section 2. The first variant HSTP-A is the problem introduced by Souza [2000], for which we show the results in Section 4.1. The second variant HSTP-B is a modification of the problem proposed by Souza [2000], for which we show the results in Section 4.2. The algorithms were coded in C++ and compiled with GCC on Linux operating system. To implement parallelism, we employed the Pthreads library available in the GCC compiler. The MIP model and the PRF heuristic were implemented with callable libraries of CPLEX 12.6.

Results in the HSTP-A

This problem was introduced by Souza [2000]. The problem is a relaxation of the problem described in Section 2, for which hard constraints (6) are dropped.

In this experiment, we tested the proposed approach in the set of seven instances¹ proposed by Souza [2000]. The instances, shown in Table 1, have optimal solutions known, which is a contribution of several previous researches using heuristic solutions [Souza et al., 2003; Santos et al., 2005; Saviniec et al., 2013; Dorneles et al., 2014] and column generation lower bounds [Santos et al., 2012]. According to Santos et al. [2005], the objective function weighting parameters for these instances are: $\alpha_7 = 1$, $\alpha_8 = 3$, $\alpha_9 = 9$. For this experiment, we set the algorithm PMILS with parameters $N = 3$ intensifiers, $intTimeLim = 2$ seconds and total execution time (the two phases) of 625 seconds.

Table 1: Features of the instances proposed by Souza [2000].

ID	$ C $	$ T $	$ D $	$ H $	$\sum_{r \in R} \bar{\theta}_r$	$\sum_{r \in R} \bar{\pi}_r$	Optimal values
1	3	8	5	5	75	21	202
2	6	14	5	5	150	29	333
3	8	16	5	5	200	4	423
4	12	23	5	5	300	41	652
5	13	31	5	5	325	71	762
6	14	30	5	5	350	63	756
7	20	33	5	5	500	84	1017

Table 2: Results of 25 runs of our approach compared to previous methods in the instances of Souza [2000].

ID	PRF		PMILS			ILS		F8	
	Time (s)	Median	Time (s)	Median	Best	Time (s)	Best	Time (s)	Best
1	0.12	358	7.00	202	202	900	202	600	202
2	0.56	668	11.93	333	333	900	333	600	333
3	0.77	700	529.01	426	423	900	423	21600	423
4	6.04	824	19.00	652	652	900	652	600	652
5	4.27	1345	9.87	762	762	900	762	600	762
6	3.08	1425	13.80	756	756	900	756	1800	759
7	13.29	1872	18.23	1017	1017	900	1017	1800	1017

The results are shown in Table 2. The table compares the results of our approach with best results of two previous approaches. The Iterated Local Search (ILS) proposed by Saviniec et al. [2013] and the best version (F8) of the Fix-and-Optimize heuristics proposed by Dorneles et al. [2014]. The best solutions are shown in bold font. We observe that our PRF heuristic is very fast to construct initial solutions. The constructed solutions have an average optimality gap of 41.45 %. The PMILS algorithm found similar results in quality of solutions compared with the two other approaches. However, PMILS outperforms the previous approaches in computational time. It finds the same quality of solutions much faster than the previous approaches².

Results in the HSTP-B

This is the problem described in Section 2. In this problem, we used the 34 instances proposed in Saviniec et al. [2015]. The instances are described in Table 3. The objective function weighting parameters were set to $\alpha_7 = 1$, $\alpha_8 = 3$, $\alpha_9 = 9$.

In this experiment, our approach is compared with the solver GOAL [Fonseca et al., 2014], that is the winner of the Third International Timetabling Competition [Post et al., 2016] devoted to high school timetabling problems, ITC2011³. We compare our approach with the latest version of GOAL [Fonseca et al., 2016], which employs a three-phase approach. The first phase constructs an initial solution via the KHE software libraries [Kingston, 2015]. The second phase

¹Available at the website “<http://labic.ic.uff.br/Instance/index.php?dir=SchoolTimetabling>”

²Our approach was run in a machine with inferior hardware resources than the machines described by Saviniec et al. [2013] and Dorneles et al. [2014]. We used a Notebook with CPU Intel Core i3 (2.3 GHz) and 2 GB of RAM, running *Linux Mint 17.1*.

³<https://www.utwente.nl/ctit/hstt/>

employs a parallel Variable Neighborhood Search (PVNS) metaheuristic and the third phase refines the output of PVNS with Fix-and-Optimize MIP heuristics. The experiment was made on a server with 2 CPU Intel Xeon E5-2680v2 (2.8 GHz) and 128 GB of RAM, running *Red Hat Enterprise Linux 6.5*. We set the algorithm PMILS with $N = 19$ intensifiers and the parallel VNS of GOAL with 20 threads. The additional parameters of GOAL were set with recommended values (alg-timelimit = 62, form-timelimit = 62, initial-soln = KHE, algorithm = SVNS, formulation = FIXOPT, formfixopt-nresources = 5 and formfixopt-optinarow = 5).

The results are shown in Table 4. The best solutions are shown in bold font. We observe that our approach outperforms GOAL in quality of solutions for most of the tested instances. The algorithm PMILS only performed worse than GOAL for instance 8. This instance has a large number of teachers' unavailable periods, which considerably reduces the feasible region and it is expected that the Fix-and-Optimize MIP heuristics implemented in GOAL perform better than our local search. The columns LB and UB of Table 4 show the results of CPLEX after a time-limit of 3 hours. The CPLEX was initialized with a solution found by running the PMILS algorithm during 625 seconds. The CPLEX was able to improve the solution of PMILS only for the restricted instance 8. The last column of Table 4 presents the best gaps found for these instances.

In this experiment, the PRF heuristic was also able to construct initial feasible solutions very fast. The constructed solutions have an average optimality gap of 37.89 %.

Table 3: Features of the instances proposed by Saviniec et al. [2015].

ID	Instance	$ C $	$ T $	$ D $	$ H $	$\sum_{r \in R} \theta_r$	$\sum_{r \in R} \tilde{\pi}_r$
1	CL-CEASD-2008-V-A	12	27	5	5	300	132
2	CL-CEASD-2008-V-B	12	27	5	5	300	132
3	CL-CECL-2011-M-A	13	31	5	5	325	144
4	CL-CECL-2011-M-B	13	31	5	5	325	143
5	CL-CECL-2011-N-A	9	28	5	5	225	107
6	CL-CECL-2011-V-A	14	29	5	5	350	164
7	CM-CECM-2011-M	20	51	5	5	500	234
8	CM-CECM-2011-N	8	30	5	5	200	96
9	CM-CECM-2011-V	13	34	5	5	325	142
10	CM-CEDB-2010-N	5	17	5	5	125	60
11	CM-CEUP-2008-V	16	35	5	5	400	192
12	CM-CEUP-2011-M	16	38	5	5	400	192
13	CM-CEUP-2011-N	3	15	5	5	75	36
14	CM-CEUP-2011-V	16	34	5	5	400	169
15	FA-EEF-2011-M	4	12	5	5	100	42
16	JNS-CEDPII-2011-M	8	19	5	5	200	85
17	JNS-CEDPII-2011-V	7	21	5	5	175	73
18	JNS-CEJXXIII-2011-M	5	18	5	5	125	60
19	JNS-CEJXXIII-2011-N	4	15	5	5	100	48
20	JNS-CEJXXIII-2011-V	5	18	5	5	125	60
21	MGA-CEDC-2011-M	19	37	5	5	475	210
22	MGA-CEDC-2011-V	12	31	5	5	300	131
23	MGA-CEGV-2011-M	31	62	5	5	775	352
24	MGA-CEGV-2011-V	32	75	5	5	800	357
25	MGA-CEJXXIII-2010-V	16	35	5	5	400	192
26	MGA-CEVB-2011-M	10	21	5	5	250	108
27	MGA-CEVB-2011-V	9	20	5	5	225	97
28	NE-CESVP-2011-M-A	18	45	5	5	450	212
29	NE-CESVP-2011-M-B	18	44	5	5	450	212
30	NE-CESVP-2011-M-C	18	45	5	5	450	211
31	NE-CESVP-2011-M-D	18	45	5	5	450	211
32	NE-CESVP-2011-V-A	16	44	5	5	400	183
33	NE-CESVP-2011-V-B	16	43	5	5	400	184
34	NE-CESVP-2011-V-C	16	43	5	5	400	182

Table 4: Comparison between our approach and the solver GOAL [Fonseca et al., 2016] in 25 runs of 625 seconds.

ID	PRF		PMILS		GOAL		CPLEX (3 h)		Gap (%)
	Time (s)	Median	Median	Best	Median	Best	LB	UB	
1	1.74	1158	710	703	730	716	682	—	2.99
2	1.94	1146	713	706	730	716	682	—	3.40
3	1.54	1514	751	746	770	762	726	—	2.68
4	1.19	1583	743	738	768	749	725	—	1.76
5	0.65	1253	631	631	648	642	629	—	0.32
6	1.84	1521	777	773	814	800	771	—	0.26
7	6.34	1583	1256	1245	1312	1297	1203	—	3.37
8	1.53	805	697	678	682	676	675	675	0.00
9	2.53	1002	822	816	838	829	804	—	1.47
10	0.29	688	298	298	300	298	298	—	0.00
11	3.64	1391	999	978	1052	1024	953	—	2.56
12	5.39	1184	1037	1028	1048	1034	1014	—	1.36
13	0.36	281	273	273	273	273	273	—	0.00
14	4.64	1098	950	943	971	949	929	—	1.48
15				Infeasible					
16	0.61	870	483	481	498	490	475	—	1.25
17	0.4	911	461	458	468	462	455	—	0.66
18	0.31	762	319	319	324	319	316	—	0.94
19	0.21	591	254	254	254	254	251	—	1.18
20	0.3	784	325	325	328	325	312	—	4.00
21	7.65	1266	1067	1061	1107	1091	1044	—	1.60
22				Infeasible					
23	11.8	2519	1893	1865	2013	1971	1793	—	3.86
24	11.5	2665	2065	2047	2171	2128	1965	—	4.01
25	4.02	1333	940	924	987	966	909	—	1.62
26	1.34	860	575	572	603	584	570	—	0.35
27	1.27	675	554	552	566	560	551	—	0.18
28	2.63	2009	1153	1138	1183	1167	1103	—	3.08
29	2.66	1958	1145	1133	1187	1168	1094	—	3.44
30	2.82	2041	1166	1153	1203	1175	1112	—	3.56
31	3.11	1853	1164	1147	1195	1181	1111	—	3.14
32	1.97	1928	1043	1033	1069	1063	998	—	3.39
33	2.12	1901	1050	1038	1078	1060	997	—	3.95
34	2.32	1811	1041	1029	1070	1056	979	—	4.86

Conclusions

In this paper, we proposed an approach with constructive and improvement phases to solve high school timetabling problems. The approach applies a Proximity Relax-and-Fix heuristic to construct an initial feasible solution that is further improved with a Parallel Multi-Start Iterated Local Search. Our constructive heuristic solves a series of relaxed sub-problems to achieve a completely feasible solution. When infeasible partitions are chosen, the heuristic is able to step over by solving a proximity problem to find another feasible partition. Our improvement heuristic is designed to exploit the parallelism of multi-core machines. The algorithm is a system of cooperative threads that exploit diversification and intensification to achieve high-quality solutions during the search. The proposed approach showed to find good quality solutions and outperform the state-of-the-art algorithms for two variants of the problem at hand. These results show that parallel cooperative multi-start approaches, such the one proposed here, are promising tools for handling combinatorial problems such as the HSTP. Also, these approaches are scalable algorithms that can exploit the power of machines with a large number of cores.

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