**Modelos de Ising e Potts acoplados
as triangulações de Lorentz**

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Resumo

José Javier Cerda Hernández. **MODELOS DE ISING E POTTS ACOPLADOS AS TRIANGULAÇÕES DE LORENTZ**. 2014. 91 f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2014.

O objetivo principal da presente tese é pesquisar : Quais são as propriedades do modelo de Ising e Potts acoplado ao ensemble de CDT? Para estudar o modelo usamos dois métodos: (1) Matriz de transferência e Teorema de Krein-Rutman. (2) Representação FK para o modelo de Potts sobre CDT e dual de CDT.

Matriz de transferência permite obter propriedades espectrais da Matriz de transferência utilizando o Teorema de Krein-Rutman [KR48] sobre operadores que conservam o cone de funções positivas. Também obtemos propriedades assintóticas da função de partição e das medidas de Gibbs. Essas propriedades permitem obter uma região onde a energia livre converge. O segundo método permite obter uma região onde a curva crítica do modelo pode estar localizada. Além disso, também obtemos uma cota superior e inferior para a energia livre a volume infinito.

Finalmente, utilizando argumentos de dualidade em grafos e expansão em alta temperatura estudamos o modelo de Potts acoplado as triangulações causais. Essa abordagem permite generalizar o modelo, melhorar os resultados obtidos para o modelo de Ising e obter novas cotas, superior e inferior, para a energia livre e para a curva crítica. Além disso, obtemos uma aproximação do autovalor maximal do operador de transferência a baixa temperatura.

Palavras-chave: dinâmica de triangulações causais, modelo de Ising, modelo de Potts, medida de Gibbs, Teorema de Krein-Rutman, representação FK, modelo de Ising quântico.

Abstract

José Javier Cerda Hernández. **Ising and Potts model coupled to Lorentzian triangulations**. 2014. 91 f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2014.

The main objective of the present thesis is to investigate: What are the properties of the Ising and Potts model coupled to a CDT ensemble? For that objective, we used two methods: (1) transfer matrix formalism and Krein-Rutman theory. (2) FK representation of the q -state Potts model on CDTs and dual CDTs.

Transfer matrix formalism permits us to obtain spectral properties of the transfer matrix using the Krein-Rutman theorem [KR48] on operators preserving the cone of positive functions. This yields results on convergence and asymptotic properties of the partition function and the Gibbs measure and allows us to determine regions in the parameter quarter-plane where the free energy converges. Second methods permit us to determine a region in the quadrant of parameters $\beta, \mu > 0$ where the critical curve for the classical model can be located. We also provide lower and upper bounds for the infinite-volume free energy.

Finally, using arguments of duality on graph theory and high-T expansion we study the Potts model coupled to CDTs. This approach permits us to improve the results obtained for Ising model and obtain lower and upper bounds for the critical curve and free energy. Moreover, we obtain an approximation of the maximal eigenvalue of the transfer matrix at lower temperature.

Keywords: causal dynamical triangulation, Ising model, Potts model, Gibbs measure, Krein-Rutman theory, FK representation, quantum Ising model.

Contents

List of Figures	ix
1 Introduction	1
1.1 Introduction and statement results	1
2 Two-dimensional causal dynamical Triangulations	5
2.1 Definitions	5
2.2 Transfer matrix formalism for pure CDTs	7
3 Transfer matrix formalism for Ising model coupled to two-dimensional CDT	13
3.1 The model	13
3.2 The transfer-matrix $\mathbf{K}$ and its powers $\mathbf{K}^N$	17
3.3 Discussion and outlook	25
4 FK representation for the Ising model coupled to CDT	27
4.1 The quantum Ising model	27
4.2 FK representation for Ising model coupled to CDT	30
4.3 The main results	33
4.4 Proof of Theorem 4.3.1 and 4.3.2	37
4.4.1 Trivial lower bound using Griffith's Inequalities	38
4.4.2 Proof of Theorem 4.3.1	39
4.4.3 Proof of Theorem 4.3.2	44
4.5 Discussion and outlook	45
5 Potts model coupled to CDTs and FK representation	47
5.1 Introduction and main results of this chapter	47
5.2 Notations	51
5.2.1 A Potts model coupled to CDTs	51
5.2.2 The FK-Potts model on Lorentzian triangulations	52
5.2.3 The relation between the Potts model and FK-Potts model: Edwards-Sokal coupling	53

5.2.4	Duality for FK-Potts model coupled to CDTs with periodic boundary conditions	54
5.3	The proof of Theorem 5.1.1 and first bounds for the critical curve	58
5.4	High-T expansion of the Potts model and Proof of Theorem 5.1.2	64
5.5	Connection between transfer matrix and FK representation	68
5.5.1	$q = 2$ (Ising) systems	68
5.5.2	q -Potts systems	69
5.6	Discussion and outlook	74
A	The von Neumann-Schatten Classes of Operators	77
A.1	The space $\mathcal{C}_p$ and first properties	77
A.2	The trace class $\mathcal{C}_1$	78
A.3	The Banach space $\mathcal{C}_p$	78
A.4	The Hilbert-Schmidt class	80
B	Krein-Rutman Theorem	81
B.1	Krein-Rutman Theorem and the Principal Eigenvalue	81
	Bibliography	83

List of Figures

1.1	A strip of a causal triangulation of $\mathcal{S} \times [j, j + 1]$	2
2.1	(a) A strip of a causal triangulation of $\mathcal{S} \times [j, j + 1]$. (b) Geometric representation of a CDT with periodic spatial boundary condition.	7
2.2	Tree parametrization of a causal dynamical triangulation.	11
3.1	Illustration of the calculates (3.25) and (3.27).	22
3.2	$\lambda_Q = \lambda$ and λ_T are the maximal eigenvalues of the matrix Q and a related matrix T respectively. The area above the black curve is where the condition (3.20) holds true.	26
4.1	A trajectory sample associated with a realization $\xi = \{s_i\}_{i=1,\dots,n}$. Each trajectory $\varphi \in \psi_\xi$ can be continuous or not at each arrival time s . In this case, at arrival time s_{k-1} the trajectory φ do not have jump, and at arrival time s_k the trajectory φ have a jump.	31
4.2	In this figure, we show the Cluster $\mathcal{C}_t$ of a triangle t , and a graphic representation of relation $t \leftrightarrow t'$, where $\leftrightarrow$ on right side in the figure, represent arrival times.	32
4.3	The area above the minimum of the dotted curve I (graph of the function ψ defined in (4.23)) and dash-dotted line II is where the limiting Gibbs probability measure exists and is unique. The critical curve lies in the region below the dotted curve I and dash-dotted line II but above the continuous curve III and dashed line IV.	37
5.1	Illustrating the region where the critical curve for Potts model coupled CDTs and dual CDTs can be located.	49
5.2	Geometric representation of a dual Lorentzian triangulation $\mathbf{t}^*$ with periodic spatial boundary condition.	53
5.3	(a) Geometric representation of a net (b) Geometric representation of a cycle (c) None of cluster of w is a net or a cycle	55
5.4	Examples of three subgraphs of A with 8 edges. It is clear that the term $\xi(e_1, \dots, e_8)$ depends of the topology of the subgraphs.	65

5.5	Region where the critical curve of the Ising model coupled to dual CDTs can be located.	69
5.6	The blue line is the simulation of $ \mathcal{A} _2 = 1$ for $q = 4$. Black line: $\mu^* = 3 \ln 2$. Green line: $\mu^* = \frac{3}{2} \ln (e^{\beta^*} - 1) + \ln 2$. Red line: $\mu^* = \frac{3}{2} \ln (4^{2/3} + e^{\beta^*} - 1) + \ln 2$	74

Chapter 1

Introduction

1.1 Introduction and statement results

Models of planar random geometry appear in physics in the context of two-dimensional quantum gravity and provide an interplay between mathematical physics and probability theory.

Causal dynamical triangulation (CDT), introduced by Ambjørn and Loll (see [AL98]), together with its predecessor a dynamical triangulation (DT), constitute attempts to provide a meaning to formal expressions appearing in the path integral quantisation of gravity (see [ADJ97], [AJ06] for an overview). A causal triangulation is formed by triangulations of spatial strips as illustrated in Figure 2.1. Note that the left and right boundaries of the spatial strip are periodically identified. The idea is to regularise the path integral by approximating the geometries emerging in the integration by CDTs. As a result, the path integral over geometries is replaced with a sum over all possible triangulations where each configuration is weighted by a Boltzmann factor $e^{-\mu|T|}$, with $|T|$ standing for the size of the triangulation and μ being the cosmological constant. The evaluation of the partition function was reduced to a purely combinatorial problem that can be solved with the help of the early work of Tutte [Tut62, Tut63]; alternatively, more powerful techniques were proposed, based on random matrix models (see, e.g., [DFGZJ95]) and bijections to well-labelled trees (see [Sch97, BDFG02]).

From a probabilistic point of view there has recently been an increasing interest in DT, most notably through the work of Angel and Schramm on a uniform measure on infinite planar triangulations [AS03], as well as through the work of Le Gall, Miermont and collaborators on Brownian maps (see [LGM11] for a recent review).

From a physical point of view it is interesting to study various models of matter, such as the Ising and Potts model, coupled to the CDT. An interesting question is: What are the properties of the Ising and Potts model coupled to a CDT ensemble? It is still random and allows for a back-reaction of the spin system with the quantum geometry. Monte Carlo simulations [AAL99] (see also [BL07, AALP09]) give a strong evidence that critical exponents

Figure 1.1: A strip of a causal triangulation of $\mathcal{S} \times [j, j+1]$.

of the Ising model coupled to CDT are identical to the Onsager values. The calculation of the partition function in this case also reduces to a combinatorial problem.

The Ising model on a collection of all planar lattices with coordination number 4 was first solved in [Kaz86, BK87] by using random matrix models and later by using a bijection to well-labelled trees [BMS11].

It is interesting that the solution here is much simpler than in the case of a flat triangular or square lattice as given by Onsager [Ons44]. For the 2-state Potts model (Ising model) coupled to CDTs some progress has been recently made on existence of Gibbs measures and phase transitions (see [AAL99], [BL07], [HYSZ13] and [CH14] for details). Using transfer matrix methods, the Krein-Rutman theory of positivity-preserving operators and FK representation for the Ising model, [CH14] provides a region in the quadrant of parameters $\beta, \mu > 0$ where the infinite-volume free energy has a limit, providing results on convergence and asymptotic properties of the partition function and the Gibbs measure. Thus, FK-Potts models, introduced by Fortuin and Kasteleyn (see [FK72]), prove that these models have become an important tool in the study of phase transition for the Ising and q -state Potts model.

The goal of this thesis is to use Krein-Rutman theory of positivity-preserving operators, FK representation of the q -state Potts model on a fixed triangulation and duality theory of graph for study the q -state Potts model coupled to CDTs.

While recently much progress has been made in the development of analytical techniques for CDT [ALWZ07, ALW⁺08d], particularly random matrix models [ALW⁺08b, ALW⁺08a, ALW⁺08c], and their application to multi-critical CDT [AaGGS12, AZ12a, AZ12b], the causality constraints still makes it difficult to find an analytical solution of the Ising model coupled to CDT.

In this thesis we focus on study the q -state Potts model coupled to CDTs and is organised

as follows.

In Chapter 2 gives a summary of causal dynamical triangulations CDTs and we introduced the transfer matrix formalism for pure CDTs. Also, we study asymptotic properties of the partition function for pure CDTs. These properties will be used in next chapters.

In Chapter 3 we define the annealed Ising model coupled to two-dimensional CDT and develop a transfer matrix formalism. Spectral properties of the transfer matrix are rigorously analysed by using the Krein-Rutman theorem [KR48] on operators preserving the cone of positive functions. This yields results on convergence and asymptotic properties of the partition function and the Gibbs measure and allows us to determine regions in the parameter quarter-plane where the partition function converges. The main results of this chapter are Lemma 3.2.1 and Theorem 3.2.2.

In Chapter 4 we use the Fortuin-Kasteleyn (FK) representation of quantum Ising models via path integrals for determining a region in the quadrant of parameters $\beta, \mu > 0$ where the critical curve for the classical model can be located. In Section 4.1 we describe the quantum Ising model. In Section 4.2, we give the FK representation of Ising model coupled to CDTs via a path integral. This representation was originally derived in [AKN92] (see also [Aiz94] and [Iof09]). Section 4.3 we present the main results of this chapter (Theorems 4.3.1 and 4.3.2). Section 4.4.2 and 4.4.3 contains the proof of Theorems 4.3.1 and 4.3.2. We also provide lower and upper bounds for the infinite-volume free energy. This chapter extends results from Chapter 3 for the (annealed) Ising model coupled to two-dimensional causal dynamical triangulations.

In Chapter 5. In Section 5.2, we introduce notation, defined the Potts model coupled to CDTs and give a summary of the FK model, FK representation. Finally, we establish a technical proposition of duality that will be used in the next section. Section 5.3 contains the proof of the first main Theorem 5.1.1, and we find a first bounds for the critical curve. This result will play a key role in the proof of the second main Theorem 5.1.2 of this chapter. In Section 5.4, using the High-T expansion for q -state Potts model, we prove Theorem 5.1.2.

Finally, Appendix A and B provide a review of trace class operators and Krein-Rutman theory, used in Chapters 2 and 3.

Most of the novel results of this thesis have been published in research articles. In particular, the following chapters are based on the following articles:

- Chapter 2 and 3 on J.C. Hernández, Y. Suhov, A. Yambartsev, and S. Zohren, Bounds on the critical line via transfer matrix methods for an Ising model coupled to causal dynamical triangulations. J. Math. Phys. **54** 063301 (2013).
- Chapter 4 on submitted paper, J. Cerda-Hernández, “Critical region for an Ising model coupled to causal dynamical triangulations”. arxiv 1402.3251 (2014).
- Chapter 5 on preparation article, J. Cerda-Hernández, “Duality relation for Potts model coupled to causal dynamical triangulations” (2014).

Chapter 2

Two-dimensional causal dynamical Triangulations

In this chapter we introduce causal dynamical triangulations (CDTs) as a discretization of the partition function for two-dimensional quantum gravity. After giving a mathematical definition of CDT we show some asymptotical properties of the partition function using transfer matrix approach. These asymptotical properties will be used in next sections.

2.1 Definitions

We will work with rooted causal dynamic triangulations of the cylinder $C_N = \mathcal{S} \times [0, N]$, $N = 1, 2, \dots$, which has N bonds (strips) $\mathcal{S} \times [j, j+1]$. Here $\mathcal{S}$ stands for a unit circle. The definition of a causal triangulation starts by considering a connected graph G embedded in C_N with the property that all faces of G are triangles (using the convention that an edge incident to the same face on two sides counts twice, see [SYZ13] for more details). A triangulation $\mathbf{t}$ of C_N is a pair formed by a graph G with the above property and the set F of all its (triangular) faces: $\mathbf{t} = (G, F)$.

Definition 2.1.1. *A triangulation $\mathbf{t}$ of C_N is called a causal triangulation (CT) if the following conditions hold:*

- *each triangular face of $\mathbf{t}$ belongs to some strip $\mathcal{S} \times [j, j+1]$, $j = 1, \dots, N-1$, and has all vertices and exactly one edge on the boundary $(\mathcal{S} \times \{j\}) \cup (\mathcal{S} \times \{j+1\})$ of the strip $\mathcal{S} \times [j, j+1]$;*
- *if $k_j = k_j(\mathbf{t})$ is the number of edges on $\mathcal{S} \times \{j\}$, then we have $0 < k_j < \infty$ for all $j = 0, 1, \dots, N-1$.*

Definition 2.1.2. *A triangulation $\mathbf{t}$ of C_N is called rooted if it has a root. The root in the triangulation $\mathbf{t}$ is represented by a triangular face t of $\mathbf{t}$, called the root triangle, with an anticlock-wise ordering on its vertices (x, y, z) where x and y belong to $S^1 \times \{0\}$. The vertex x is identified as the root vertex and the (directed) edge from x to y as the root edge.*

Definition 2.1.3. *Two causal rooted triangulations of C_N , say $\mathbf{t} = (G, F)$ and $\mathbf{t}' = (G', F')$, are equivalent if there exists a self-homeomorphism of C_N which (i) transforms each slice $S^1 \times \{j\}$, $j = 0, \dots, N-1$ to itself and preserves its direction, (ii) induces an isomorphism of the graphs G and G' and a bijection between F and F' , and (iii) takes the root of $\mathbf{t}$ to the root of $\mathbf{t}'$.*

Let $\mathbb{LT}_N$ and $\mathbb{LT}_\infty$ denote the sets of causal triangulations on the finite cylinder C_N and infinity cylinder $C = \mathcal{S} \times [0, \infty)$.

A triangulation $\mathbf{t}$ of C_N is identified as a consistent sequence:

$$\mathbf{t} = (\mathbf{t}(0), \mathbf{t}(1), \dots, \mathbf{t}(N-1)),$$

where $\mathbf{t}(i)$ is a causal triangulation of the strip $\mathcal{S} \times [i, i+1]$. The latter means that each $\mathbf{t}(i)$ is described by a partition of $\mathcal{S} \times [i, i+1]$ into triangles where each triangle has one vertex on one of the slices $\mathcal{S} \times \{i\}$, $\mathcal{S} \times \{i+1\}$ and two on the other, together with the edge joining these two vertices. The property of consistency means that each pair $(\mathbf{t}(i), \mathbf{t}(i+1))$ is consistent, i.e., every side of a triangle from $\mathbf{t}(i)$ lying in $\mathcal{S} \times \{i+1\}$ serves as a side of a triangle from $\mathbf{t}(i+1)$, and vice versa.

The triangles forming the causal triangulation $\mathbf{t}(i)$ are denoted by $t(i, j)$, $1 \leq j \leq n(\mathbf{t}(i))$ where, $n(\mathbf{t}(i))$ stands for the number of triangles in the triangulation $\mathbf{t}(i)$. The enumeration of these triangles starts with what we call the root triangle in $\mathbf{t}(i)$; it is determined recursively as follows (see Figure 2.1(b)): First, we have the root triangle $t(0, 1)$ in $\mathbf{t}(0)$ (see Definition 2.1.2). Take the vertex of the triangle $t(0, 1)$ which lies on the slice $\mathcal{S} \times \{1\}$ and denote it by x' . This vertex is declared the root vertex for $\mathbf{t}(1)$. Next, the root edge for $\mathbf{t}(1)$ is the one incident to x' and lying on $\mathcal{S} \times \{1\}$, so that if y' is its other end and z' is the third vertex of the corresponding triangle then x', y', z' lists the three vertices anticlock-wise. Accordingly, the triangle with the vertices x', y', z' is called the root triangle for $\mathbf{t}(1)$. This construction can be iterated, determining the root vertices, root edges and root triangles for $\mathbf{t}(i)$, $0 \leq i \leq N-1$.

It is convenient to introduce the notion of “up” and “down” triangles (see Figure 2.1(a)). We call a triangle $t \in \mathbf{t}(i)$ an up-triangle if it has an edge on the slice $\mathcal{S} \times \{i\}$ and a down-triangle if it has an edge on the slice $\mathcal{S} \times \{i+1\}$. By Definition 2.1.1, every triangle is either of type up or down. Let $n_{\text{up}}(\mathbf{t}(i))$ and $n_{\text{do}}(\mathbf{t}(i))$ stand for the number of up- and down-triangles in the triangulation $\mathbf{t}(i)$.

Note that for any edge lying on the slice $\mathcal{S} \times \{i\}$ belongs to exactly two triangles: one up-triangle from $\mathbf{t}(i)$ and one down-triangle from $\mathbf{t}(i-1)$. This provides the following relation: the number of triangles in the triangulation $\mathbf{t}$ is twice the total number of edges on the slices. More precisely, let n^i be the number of edges on slice $\mathcal{S} \times \{i\}$. Then, for any $i = 0, 1, \dots, N-1$,

$$n(\mathbf{t}(i)) = n_{\text{up}}(\mathbf{t}(i)) + n_{\text{do}}(\mathbf{t}(i)) = n^i + n^{i+1}, \quad (2.1)$$

Figure 2.1: (a) A strip of a causal triangulation of $\mathcal{S} \times [j, j+1]$. (b) Geometric representation of a CDT with periodic spatial boundary condition.

implying that

$$\sum_{i=0}^{N-1} n(\mathbf{t}(i)) = 2 \sum_{i=0}^{N-1} n^i. \quad (2.2)$$

There is another useful property regarding the counting of triangulations. Let us fix the number of edges n^i and n^{i+1} in the slices $\mathcal{S} \times \{i\}$ and $\mathcal{S} \times \{i+1\}$. The number of possible rooted CTs of the slice $\mathcal{S} \times [i, i+1]$ with n^i up- and n^{i+1} down-triangles is equal to

$$\binom{n^i + n^{i+1} - 1}{n^i - 1} = \binom{n(\mathbf{t}(i)) - 1}{n_{up}(\mathbf{t}(i)) - 1}. \quad (2.3)$$

2.2 Transfer matrix formalism for pure CDTs

We begin by discussing the case of pure causal dynamical triangulations, as was first introduced in [AL98] (see also [MYZ01] for a mathematically more rigorous account). We define the subcritical, critical or supercritical region if $2\exp(-\mu) < 1$, $2\exp(-\mu) = 1$ and $2\exp(-\mu) > 1$, respectively.

The partition function for rooted CTs in the cylinder C_N with periodical spatial boundary conditions (where $\mathbf{t}(0)$ is consistent with $\mathbf{t}(N-1)$) and for the value of the cosmological constant μ is given by

$$Z_N(\mu) = \sum_{\mathbf{t}} e^{-\mu n(\mathbf{t})} = \sum_{(\mathbf{t}(0), \dots, \mathbf{t}(N-1))} \exp\left\{-\mu \sum_{i=0}^{N-1} n(\mathbf{t}(i))\right\}. \quad (2.4)$$

Using the properties (2.2) and (2.3) we can represent the partition function (2.4) in the

following way

$$Z_N(\mu) = \sum_{n^0 \geq 1, \dots, n^{N-1} \geq 1} \exp \left\{ -2\mu \sum_{i=0}^{N-1} n^i \right\} \prod_{i=0}^{N-1} \binom{n^i + n^{i+1} - 1}{n^i - 1}. \quad (2.5)$$

Moreover, $Z_N(\mu)$ admits a trace-related representation

$$Z_N(\mu) = \text{tr} (U^N). \quad (2.6)$$

This gives rise to a transfer matrix $U = \{u(n, n')\}_{n, n'=1, 2, \dots}$ describing the transition from one spatial strip to the next one. It is an infinite matrix with strictly positive entries

$$u(n, n') = g^n \frac{(n + n' - 1)!}{(n - 1)! n'!} g^{n'} = \binom{n + n' - 1}{n - 1} g^{n+n'}. \quad (2.7)$$

For notational convenience we use the parameter $g = e^{-\mu}$ (a single-triangle fugacity). The entry $u(n, n')$ yields the number of possible triangulations of a single strip (say, $\mathcal{S} \times [0, 1]$) with n lower boundary edges (on $\mathcal{S} \times \{0\}$) and n' upper boundary edges (on $\mathcal{S} \times \{1\}$). See Figure 2.1(a). The asymmetry in n and n' is due to the fact that the lower boundary is marked while the upper one is not. However, a symmetric transfer matrix $\tilde{U} = \{\tilde{u}(n, n')\}$ can be introduced, associated with a strip where both boundaries are kept unmarked:

$$\tilde{u}(n, n') = n^{-1} u(n, n'). \quad (2.8)$$

The N -strip Gibbs distribution $\mathcal{P}_N$ assigns the following probabilities to strings $(n^0, \dots, n^{N-1})$ with the number of triangles $n^i \geq 1$ for all $i = 0, \dots, N - 1$:

$$\mathcal{P}_{N, \mu}(n^0, \dots, n^{N-1}) = \frac{1}{Z_N(\mu)} \exp \left\{ -2\mu \sum_{i=0}^{N-1} n^i \right\} \prod_{i=0}^{N-1} \binom{n^i + n^{i+1} - 1}{n^i - 1}. \quad (2.9)$$

We state two lemmas featuring properties of matrix U :

Lemma 2.2.1. *For any $g > 0$ the matrix U and its transpose U^T have an eigenvalue $\Lambda = \Lambda(g)$ given by*

$$\Lambda(g) = \left[(1 - \sqrt{1 - 4g^2}) / (2g) \right]^2. \quad (2.10)$$

The corresponding eigenvectors

$$\phi = \{\phi(n)\}_{n=1, 2, \dots} \quad \text{and} \quad \phi^* = \{\phi^*(n)\}_{n=1, 2, \dots}$$

have entries

$$\phi(n) = n(\Lambda(g))^n, \quad \phi^*(n) = (\Lambda(g))^n. \quad (2.11)$$

Proof. A direct verification shows that

$$\sum_{n'} u(n, n') n' \Lambda^{n'}(g) = n \Lambda^{n+1}(g) \quad \text{and} \quad \sum_n \Lambda^n(g) u(n, n') = \Lambda^{n'+1}(g).$$

(In fact, each of these relations implies the other.) See Theorem 1 in [MYZ01]. $\square$

Lemma 2.2.2. *For any fixed n and any $g < 1$ (equivalently, $\mu > 0$) one has*

$$\sum_{n'} u(n, n') = \left(\frac{g}{1-g} \right)^n (1 - (1-g)^n). \quad (2.12)$$

Proof. The proof again follows from a straightforward verification. $\square$

A transfer-matrix formalism of Statistical Mechanics predicts that, as $N \rightarrow \infty$, the partition function is governed by the largest eigenvalue Λ of the transfer matrix:

$$Z_N(g) = \text{tr } U^N \sim \Lambda^N \quad (2.13)$$

We make this statement more precise in the statements of Lemma 2.2.3 and Theorem 2.2.1 below. Here the symbol ℓ^2 stands for the Hilbert space of square-summable complex sequences (infinite-dimensional vectors) $\psi = \{\psi(n)\}_{n=1,2,\dots}$ equipped with the standard scalar product $\langle \psi', \psi'' \rangle = \sum_n \psi'(n) \overline{\psi''(n)}$. Accordingly, the matrices U and U^T are treated as operators in ℓ^2 .

Lemma 2.2.3. *For any $g < 1/2$ (equivalently $\mu > \ln 2$), the following statements hold true:*

1. U and U^T are bounded operators in ℓ^2 preserving the cone of positive vectors;
2. The sum $\sum_{n,n'} u(n, n') < \infty$. Consequently, U and U^T have

$$\text{tr } (UU^T) = \text{tr } (U^T U) < \infty,$$

i.e., U and U^T are Hilbert-Schmidt operators. Therefore, $\forall N \geq 2$, U^N and $(U^T)^N$ are trace-class operators.

3. The maximal eigenvalue $\Lambda = \Lambda(g)$ of U in ℓ^2 is positive, coincides with the maximal eigenvalue of U^T and is given by Eqn (2.10). The corresponding eigenvectors $\phi, \phi^* \in \ell^2$ are unique up to multiplication by a constant factor and given in Eqn (2.11).
4. The following asymptotical formulas hold as $N \rightarrow \infty$:

$$\frac{1}{\Lambda^N} \text{tr } (U^N), \quad \frac{1}{\Lambda^N} \text{tr } ((U^T)^N) \rightarrow 1,$$

and, $\forall$ vectors $\psi', \psi'' \in \ell^2$,

$$\frac{1}{\Lambda^N} \langle \psi', U^N \psi'' \rangle \rightarrow \langle \psi', \phi \rangle \langle \phi^*, \psi'' \rangle,$$

where the eigenvectors ϕ and ϕ^* are normalized so that $\langle \phi, \phi^* \rangle = 1$.

Theorem 2.2.1. *For any $g < 1/2$ the following relation holds true:*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N(g) = \log \Lambda \quad (2.14)$$

with $\Lambda = \Lambda(g)$ given in (2.10). Further, the N -strip Gibbs measure $\mathcal{P}_{N,\mu}$ converges weakly to a limiting measure $\mathcal{P}_\mu$ which is represented by a positive recurrent Markov chain on $\mathbb{Z}_+ = \{1, 2, \dots\}$, with the transition matrix $P = \{P(n, n')\}_{n=1,2,\dots}$ and the invariant distribution π . Here

$$P(n, n') = \frac{u(n, n')\phi(n')}{\Lambda\phi(n)}$$

and

$$\pi(n) = \frac{\phi^*(n)\phi(n)}{\langle \phi^*, \phi \rangle}.$$

where $\phi(n)$ and $\phi^*(n)$ are as in (2.11).

Proof. The proof is a consequence of Lemma 2.2.1 and 2.2.3 and the Krein-Rutman theory [KR48]. $\square$

By Theorem 2.2.1, the measure on the set of infinite triangulations $\mathbb{LT}_\infty$ is then defined as a weak limit

$$\mathcal{P}_\mu = \lim_{N \rightarrow \infty} \mathcal{P}_N.$$

The follow Theorem gives the typical triangulation (typical behavior) under the limiting measure $\mathcal{P}_\mu$.

Theorem 2.2.2 (See [MYZ01], [KY12]). *The limit measure $\mathcal{P}_\mu = \lim_{N \rightarrow \infty} \mathcal{P}_{N,\mu}$ exist for all $\mu \geq \ln 2$. Moreover, let n_k be the number of vertices at k -th level in a triangulation $\mathbf{t}$ for each $k \geq 0$.*

- For $\mu > \ln 2$ under the limiting measure $\mathcal{P}_\mu$ the sequence $\{n_k\}$ is a positive recurrent Markov chain.
- For $\mu = \mu_{cr} = \ln 2$ the sequence $\{n_k\}$ is distributed as the branching process ξ_n with geometric offspring distribution with parameter $1/2$, conditioned to non-extinction at infinity.

Below we briefly sketch the proof of the second part of Theorem 2.2.2, a deeper investigation of related ideas appear in [SYZ13]. See also [Dur03, DJW10].

Given a triangulation $\mathbf{t} \in \mathbb{LT}_N$, define the subgraph $\tau \subset \mathbf{t}$ by taking, for each vertex $v \in \mathbf{t}$, the leftmost edge going from v downwards (see fig. 1). The graph thus obtained is a spanning forest of $\mathbf{t}$, and moreover, if one associates with each vertex of τ it's height in $\mathbf{t}$ then $\mathbf{t}$ can be completely reconstructed knowing τ . We call τ the tree parametrization of $\mathbf{t}$.

Figure 2.2: *Tree parametrization of a causal dynamical triangulation.*

For every vertex $v \in \tau$ denote by δ_v its out-degree, i.e. the number of edges of τ going from v upwards. Comparing the out-degrees in τ to the number of vertical edges in $\mathbf{t}$, and comparing the latter to the total number of triangles $n(\mathbf{t})$, it is not hard to obtain the identity

$$\sum_{v \in \tau \setminus \mathcal{S} \times \{N\}} (\delta_v + 1) = n(\mathbf{t}), \quad (2.15)$$

where the sum on the left runs over all vertices of τ except for the N -th level. Thus, under the measure $\mathcal{P}_{\mu_{cr}}$ the probability of a forest τ is proportional to

$$e^{-\mu_{cr} n(\mathbf{t})} = \prod_{v \in \tau \setminus \mathcal{S} \times \{N\}} \left(\frac{1}{2} \right)^{\delta_v + 1}, \quad (2.16)$$

which is exactly the probability to observe τ as a realization of a branching process with offspring distribution $\text{Geom}(1/2)$. After normalization we will obtain, on the left in (2.16), the probability $\mathcal{P}_{N, \mu_{cr}}(\tau)$ as defined by (2.9), and on the right the conditional probability to see τ as a realization of the branching process ξ given $\xi_N > 0$. So quite naturally when $N \rightarrow \infty$ the distribution of τ converges to the Galton-Watson tree, conditioned to non-extinction at infinity.

In particular, it follows from Theorem 2.2.2 that

$$\mathcal{P}_{\mu_{cr}}(n_k = m) = \Pr(\xi_k = m | \xi_\infty > 0) = m \Pr(\xi_k = m) \quad (2.17)$$

Remark 2.2.1. *The last equality in (2.17) means that the measure $\mathcal{P}_{\mu_{cr}}$ on triangulations can be considered as a Q -process defined by Athreya and Ney [AN72] for a critical Galton-Watson branching process. Such a process is exactly a critical Galton-Watson tree conditioned to survive forever.*

In the supercritical case $\exp(-\mu) > 1/2$, we have the following asymptotical property of the partition function

Proposition 2.2.1. *In the supercritical case, $\exp(-\mu) > 1/2$, the finite volume partition function $Z_N(\mu)$ (defined in (2.4)) exist only if*

$$\mu > \ln \left(2 \cos \frac{\pi}{N+1} \right). \quad (2.18)$$

Notice that, as $N \rightarrow \infty$ this region, where the partition function exists, become empty.

Remark 2.2.2. *The inequality in (2.18) means that if $\mu < \ln 2$ then there exists $N_0 \in \mathbb{N}$ such that the partition function $Z_N(\mu) = +\infty$ whenever $N > N_0$. Moreover, the Gibbs distribution $\mathcal{P}_{N,\mu}$ on triangulations with periodic boundary conditions cannot be defined by using the standard formula with $Z_N(\mu)$ as a normalising denominator, consequently, there is no limiting probability measure $\mathcal{P}_\mu$.*

Chapter 3

Transfer matrix formalism for Ising model coupled to two-dimensional CDT

In this chapter we introduce a transfer matrix formalism for the (annealed) Ising model coupled to two-dimensional CDTs. Using the Krein-Rutman theory of positivity preserving operators we study several properties of the emerging transfer matrix. In particular, we determine regions in the quadrant of parameters $\beta, \mu > 0$ where the infinite-volume free energy converges, yielding results on the convergence and asymptotic properties of the partition function and Gibbs measure. This is a first approach for study the Ising model coupled two-dimensional CDTs.

3.1 The model

Let $\mathbf{t} = (\mathbf{t}(0), \mathbf{t}(1), \dots, \mathbf{t}(N-1))$ be a triangulation of C_N , where $\mathbf{t}(i)$ is a causal triangulation of the strip $\mathcal{S} \times [i, i+1]$. The triangles forming the causal triangulation $\mathbf{t}(i)$ are denoted by $t(i, j)$, $1 \leq j \leq n(\mathbf{t}(i))$ where, $n(\mathbf{t}(i))$ stands for the number of triangles in the triangulation $\mathbf{t}(i)$. The enumeration of these triangles starts with what we call the root triangle in $\mathbf{t}(i)$ (see Chapter 2).

Now, with any triangle from a triangulation $\mathbf{t}$ we associate a spin taking values ± 1 . An N -strip configuration of spins is represented by a collection

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}(0), \boldsymbol{\sigma}(1), \dots, \boldsymbol{\sigma}(N-1))$$

where $\boldsymbol{\sigma}(i) = \boldsymbol{\sigma}(\mathbf{t}(i))$ is a configuration of spins $\sigma(i, j)$ over triangles $t(i, j)$ forming a triangulation $\mathbf{t}(i)$, $1 \leq j \leq n(\mathbf{t}(i))$. We will say that a single-strip configuration of spins $\boldsymbol{\sigma}(i)$ is supported by a triangulation $\mathbf{t}(i)$ of strip $\mathcal{S} \times [i, i+1]$. We consider a usual (ferromagnetic) Ising-type energy where two spins $\sigma(i, j)$ and $\sigma(i', j')$ interact if their supporting triangles $t(i, j)$, $t(i', j')$ share a common edge; such triangles are called nearest neighbors, and this property is reflected in the notation $\langle t(i, j), t(i', j') \rangle$, where we require $0 \leq i \leq i' \leq N-1$. Thus, in our model each spin has three neighbors. Moreover, a pair $\langle t(i, j), t(i', j') \rangle$ can only

occur for $i' - i \leq 1$ or $i = 0, i' = N - 1$. Formally, the Hamiltonian of the model reads:

$$\mathbb{H}(\boldsymbol{\sigma}) = - \sum_{\langle t(i,j), t(i',j') \rangle} \sigma(i,j) \sigma(i',j'). \quad (3.1)$$

We will use the following decomposition:

$$\mathbb{H}(\boldsymbol{\sigma}) = \sum_{i=0}^{N-1} H(\boldsymbol{\sigma}(i)) + \sum_{i=0}^{N-1} V(\boldsymbol{\sigma}(i), \boldsymbol{\sigma}(i+1)), \quad (3.2)$$

where we assume that $\boldsymbol{\sigma}(0) \equiv \boldsymbol{\sigma}(N)$ (the periodic spatial boundary condition). Here $H(\boldsymbol{\sigma}(i))$ represents the energy of the configuration $\boldsymbol{\sigma}(i)$:

$$H(\boldsymbol{\sigma}(i)) = - \sum_{\langle t(i,j), t(i,j') \rangle} \sigma(i,j) \sigma(i,j'). \quad (3.3)$$

Further, $V(\boldsymbol{\sigma}(i), \boldsymbol{\sigma}(i+1))$ is the energy of interaction between neighboring triangles belonging to the adjacent strips $\mathcal{S} \times [i, i+1]$ and $\mathcal{S} \times [i+1, i+2]$:

$$V(\boldsymbol{\sigma}(i), \boldsymbol{\sigma}(i+1)) = - \sum_{\langle t(i,j), t(i+1,j') \rangle} \sigma(i,j) \sigma(i+1,j'). \quad (3.4)$$

The partition function for the (annealed) N -strip Ising model coupled to CDT, at the inverse temperature $\beta > 0$ and for the cosmological constant μ , is given by

$$\begin{aligned} \Xi_N(\mu, \beta) &= \sum_{(\mathbf{t}(0), \dots, \mathbf{t}(N-1))} \exp \left\{ -\mu \sum_{i=0}^{N-1} n(\mathbf{t}(i)) \right\} \\ &\times \sum_{(\boldsymbol{\sigma}(0), \dots, \boldsymbol{\sigma}(N-1))} \prod_{i=0}^{N-1} \exp \left\{ -\beta H(\boldsymbol{\sigma}(i)) - \beta V(\boldsymbol{\sigma}(i), \boldsymbol{\sigma}(i+1)) \right\}. \end{aligned} \quad (3.5)$$

Here $n(\mathbf{t}(i))$ stands for the number of triangles in the triangulation $\mathbf{t}(i)$. Like before, the formula

$$\Xi_N(\mu, \beta) = \text{tr } \mathbf{K}^N \quad (3.6)$$

gives rise to a transfer matrix $\mathbf{K}$ with entries $K((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))$ labelled by pairs $(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')$ representing triangulations of a single strip (say, $\mathcal{S} \times [0, 1]$) and their supported spin configurations which are positioned next to each other. Formally,

$$\begin{aligned} K((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) &= \mathbf{1}_{\mathbf{t} \sim \mathbf{t}'} \exp \left\{ -\frac{\mu}{2} (n(\mathbf{t}) + n(\mathbf{t}')) \right\} \\ &\times \exp \left\{ -\frac{\beta}{2} (H(\boldsymbol{\sigma}) + H(\boldsymbol{\sigma}')) - \beta V(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \right\}. \end{aligned} \quad (3.7)$$

As earlier, $n(\mathbf{t})$ and $n(\mathbf{t}')$ are the numbers of triangles in the triangulations $\mathbf{t}$ and $\mathbf{t}'$. The indicator $\mathbf{1}_{\mathbf{t} \sim \mathbf{t}'}$ means that the triangulations $\mathbf{t}, \mathbf{t}'$ have to be consistent with each other in the

above sense: the number of down-triangles in $\mathbf{t}$ should equal the number of up-triangles in $\mathbf{t}'$, and an upper-marked edge in $\mathbf{t}$ should coincide with a lower-marked edge in triangulation $\mathbf{t}'$. It means that the pair $(\mathbf{t}, \mathbf{t}')$ forms a CDT for the strip $\mathcal{S} \times [0, 2]$.

We would like to stress that the trace $\text{tr } \mathbf{K}^N$ in (3.6) is understood as the *matrix trace*, i.e., as the sum $\sum_{\mathbf{t}, \boldsymbol{\sigma}} K^{(N)}((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}, \boldsymbol{\sigma}))$ of the diagonal entries $K^{(N)}((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}, \boldsymbol{\sigma}))$ of the matrix $\mathbf{K}^N$. (Indeed, in what follows, the notation “tr” is used for the matrix trace only.)

Our aim will be to verify that the matrix trace in (3.6) can be replaced with an *operator trace* invoking the eigenvalues of $\mathbf{K}$ in a suitable linear space (see next section).

As before, we can introduce the N -strip Gibbs probability distribution associated with formula (3.5):

$$\begin{aligned} \mathbb{P}_N((\mathbf{t}(0), \boldsymbol{\sigma}(0)), \dots, (\mathbf{t}(N-1), \boldsymbol{\sigma}(N-1))) \\ = \frac{1}{\Xi(\mu, \beta)} \prod_{i=0}^{N-1} \exp\left\{-\mu n(\mathbf{t}(i)) - \beta H(\boldsymbol{\sigma}(i)) - \beta V(\boldsymbol{\sigma}(i), \boldsymbol{\sigma}(i+1))\right\}. \end{aligned} \quad (3.8)$$

Consider several special cases of interest.

The case $\beta \approx 0$. This is the first term of the so-called high temperature expansion [AAL99].

Here one has

$$\begin{aligned} \Xi(\mu, 0) &= \sum_{(\mathbf{t}(0), \dots, \mathbf{t}(N-1))} \exp\left\{-\mu \sum_{i=0}^{N-1} n(\mathbf{t}(i))\right\} \sum_{(\boldsymbol{\sigma}(0), \dots, \boldsymbol{\sigma}(N-1))} 1 \\ &= \sum_{n^0 \geq 1, \dots, n^{N-1} \geq 1} \exp\left\{-2(\mu - \ln 2) \sum_{i=0}^{N-1} n^i\right\} \prod_{i=0}^{N-1} \binom{n^i + n^{i+1} - 1}{n^i - 1} \\ &= Z_N(\mu - \ln 2); \text{ cf. (2.4).} \end{aligned}$$

The condition $\mu - \ln 2 > \ln 2$ which guarantees properties listed in Lemma 2.2.3 and Theorem 2.2.1 results in

$$\mu > 2 \ln 2. \quad (3.9)$$

Thus, Eqn. (3.9) yields a sub-criticality condition when $\beta = 0$.

The case $\beta \approx \infty$. Observe that for any triangulation $\mathbf{t} = (\mathbf{t}(0), \dots, \mathbf{t}(N-1))$ there are two ground states: all spins $+1$ and all spins -1 , with the overall energy equals minus three half times the total number of triangles: $-3/2 \sum_{i=0}^{N-1} n(\mathbf{t}(i))$. Discarding all other spin configurations, we obtain that

$$\Xi(\mu, \beta) > \Xi_*(\mu, \beta)$$

where

$$\begin{aligned}
\Xi_*(\mu, \beta) &= \sum_{\mathbf{t}(0), \dots, \mathbf{t}(N-1)} 2 \exp \left\{ \left(-\mu + \frac{3}{2}\beta \right) \sum_{i=0}^{N-1} n(\mathbf{t}(i)) \right\} \\
&= 2 \sum_{n^0 \geq 1, \dots, n^{N-1} \geq 1} \exp \left\{ -2 \left(\mu - \frac{3}{2}\beta \right) \sum_{i=0}^{N-1} n^i \right\} \prod_{i=0}^{N-1} \binom{n^i + n^{i+1} - 1}{n^i - 1} \\
&= 2Z_N \left(\mu - \frac{3}{2}\beta \right)
\end{aligned}$$

where $\exp \left[\frac{3}{2}\beta \sum_i n(\mathbf{t}(i)) \right]$ is the energy of the (+)-configuration (or, equivalently, the (-)-configuration). For β large, we can expect that $\Xi(\mu, \beta) \sim \Xi_*(\mu, \beta)$. Then the critical inequality

$$\mu - \frac{3}{2}\beta > \ln 2$$

yields

$$\mu > \ln 2 + \frac{3}{2}\beta. \quad (3.10)$$

Equation (3.10) gives a necessary (and probably tight) criticality condition for the Ising model under consideration for large values of β . A similar result was obtained in [AAL99].

The case $0 < \beta < \infty$. Firstly, we note that for any fixed triangulation $\mathbf{t}$ the energy of any spin configuration σ on $\mathbf{t}$ will be bigger or equal than the energy of a pure configuration (all +s or all -s):

$$\begin{aligned}
H(\sigma) &= \sum_j H(\sigma(j)) + \sum_j V(\sigma(j), \sigma(j+1)) \\
&\geq -\frac{3}{2} \#(\text{of all triangles in } \underline{t}) = -3 \sum_{i=0}^{N-1} n^i,
\end{aligned}$$

where n^i is the number of edges in the i th level $S \times \{i\}$, $i = 0, 1, \dots, N-1$. Thus, for any $\beta > 0$ the inequality $\Xi(\mu, \beta) < \Xi^*(\mu, \beta)$ holds true, where

$$\begin{aligned}
\Xi^*(\mu, \beta) &= \sum_{(\mathbf{t}(0), \dots, \mathbf{t}(N-1))} \exp \left\{ \left(-\mu + \frac{3}{2}\beta + \ln 2 \right) \sum_{i=0}^{N-1} n(\mathbf{t}(i)) \right\} \\
&= \sum_{n^0 \geq 1, \dots, n^{N-1} \geq 1} \exp \left\{ -2 \left(\mu - \frac{3}{2}\beta - \ln 2 \right) \sum_{i=0}^{N-1} n^i \right\} \prod_{i=0}^{N-1} \binom{n^i + n^{i+1} - 1}{n^i - 1} \\
&= Z_N \left(\mu - \frac{3}{2}\beta - \ln 2 \right) .
\end{aligned}$$

Hence, the inequality

$$\mu - \frac{3}{2}\beta - \ln 2 > \ln 2 \quad \text{or} \quad \mu > 2 \ln 2 + \frac{3}{2}\beta \quad (3.11)$$

provides a sufficient condition for subcriticality of the Ising model under consideration.

3.2 The transfer-matrix $\mathbf{K}$ and its powers $\mathbf{K}^N$

The main results of this chapter are summarized in Lemma 3.2.1 and Theorems 3.2.1 and 3.2.2 below.

Let us start with a statement (see Proposition 3.2.1 below) which merely re-phrases standard definitions and explains our interest in the matrices $\mathbf{K}$, $\mathbf{K}^T$, $\mathbf{K}^T\mathbf{K}$, $\mathbf{K}\mathbf{K}^T$ and their powers. Cf. Definition 2.2.2 on p.83, Definition 2.4.1 on p.101, Lemma 2.3.1 on p.85 and Theorem 3.3.13 on p.139 in [Rin71]). See Appendix A for a short review.

We treat the transfer-matrix $\mathbf{K}$ and its transpose $\mathbf{K}^T$ as linear operators in the Hilbert space ℓ_{T-C}^2 (the subscript T-C refers to triangulations and spin-configurations). The space ℓ_{T-C}^2 is formed by functions $\psi = \{\psi(\mathbf{t}, \sigma)\}$ with the argument $(\mathbf{t}, \sigma)$ running over single-strip triangulations and supported configurations of spins, with the scalar product $\langle \psi', \psi'' \rangle_{T-C} = \sum_{\mathbf{t}, \sigma} \psi'(\mathbf{t}, \sigma) \overline{\psi''(\mathbf{t}, \sigma)}$ and the induced norm $\|\psi\|_{T-C}$. The action of $\mathbf{K}$ in ℓ_{T-C}^2 , in the basis formed by Dirac's delta-vectors $\delta_{(\mathbf{t}, \sigma)}$, is determined by

$$(\mathbf{K}\psi)(\mathbf{t}, \sigma) = \sum_{\mathbf{t}', \sigma'} K((\mathbf{t}, \sigma), (\mathbf{t}', \sigma')) \psi(\mathbf{t}', \sigma'); \quad (3.12)$$

in following we use the notation $\mathbf{K}$, $\mathbf{K}^T$, etc., for the matrices and the corresponding operators in ℓ_{T-C}^2 . Accordingly, the symbols $\|\mathbf{K}\|_{T-C}$, $\|\mathbf{K}^T\|_{T-C}$ etc. refer to norms in ℓ_{T-C}^2 .

Given $n = 1, 2, \dots$, suppose that the operator $\mathbf{K}^n$ (respectively, $(\mathbf{K}^T)^n$) is of trace class (see definition in Appendix A). Then the following series absolutely converges:

$$\sum_j \Lambda_j^{(n)} \left(\text{respectively, } \sum_j \Lambda_j^{*(n)} \right), \quad (3.13)$$

where $\Lambda_j^{(n)}$ ($\Lambda_j^{*(n)}$) runs through the eigenvalues of $\mathbf{K}^n$ ($(\mathbf{K}^T)^n$), counted with their multiplicities. In this case the sum (3.13) is called the operator trace of $\mathbf{K}^n$ (respectively, $(\mathbf{K}^T)^n$) in ℓ_{T-C}^2 . We adopt an agreement that the eigenvalues in (3.13) are listed in the decreasing order of their moduli, beginning with $\Lambda_0^{(n)}$ ($\Lambda_0^{*(n)}$).

Set $|\mathbf{K}^n| = \sqrt{(\mathbf{K}^T)^n \mathbf{K}^n}$ and $|(\mathbf{K}^T)^n| = \sqrt{\mathbf{K}^n (\mathbf{K}^T)^n}$.

Proposition 3.2.1. *For any positive integer r , the following inequalities are equivalent:*

$$\mathrm{tr} \left((\mathbf{K}^r (\mathbf{K}^T)^r) \right) = \mathrm{tr} \left((\mathbf{K}^T)^r \mathbf{K}^r \right) < \infty \quad \text{and} \quad (3.14)$$

$$\mathrm{tr} |\mathbf{K}^{2r}| = \mathrm{tr} |(\mathbf{K}^T)^{2r}| < \infty.$$

Moreover, each of the inequalities in (3.14) implies that $\forall N \geq 2r$, the operators $\mathbf{K}^N$ and $(\mathbf{K}^T)^N$ are of trace class in ℓ_{T-C}^2 . Hence, for $N \geq 2r$, the matrix traces $\mathrm{tr}(\mathbf{K}^N)$ and $\mathrm{tr}((\mathbf{K}^T)^N)$ are finite and coincide with the corresponding operator traces in ℓ_{T-C}^2 .

Theorem 3.2.1. *Suppose that the condition (3.14) is satisfied with $r = 1$. Then the following properties of transfer matrix $\mathbf{K}$ are fulfilled.*

1. *The square $\mathbf{K}^2$ and its transpose $(\mathbf{K}^T)^2$ are trace-class operators in ℓ_{T-C}^2 .*
2. *$\mathbf{K}$ and $\mathbf{K}^T$ have a common eigenvalue, $\Lambda = \Lambda_0(\beta, \mu) > 0$ such that the norms $\|\mathbf{K}\|_{T-C} = \|\mathbf{K}^T\|_{T-C} = \Lambda$. Furthermore, $\mathbf{K}^2$ and $(\mathbf{K}^T)^2$ have the common eigenvalue $\Lambda^2 = \Lambda_0^{(2)} = \Lambda_0^{*(2)}$ such that the norms $\|\mathbf{K}^2\|_{T-C} = \|(\mathbf{K}^T)^2\|_{T-C} = \Lambda^2$.*
3. *Λ is a simple eigenvalue of $\mathbf{K}$ and $\mathbf{K}^T$, i.e., the corresponding eigenvectors $\phi = \{\phi(\mathbf{t}, \sigma)\}$ and $\phi^* = \{\phi^*(\mathbf{t}, \sigma)\}$ are unique up to multiplicative constants. Moreover, ϕ and ϕ^T can be made strictly positive: $\phi(\mathbf{t}, \sigma), \phi^T(\mathbf{t}, \sigma) > 0 \forall (\mathbf{t}, \sigma)$. Furthermore, Λ is separated from the remaining singular values and the remaining eigenvalues of $\mathbf{K}$ and $\mathbf{K}^T$ by a positive gap. The same is true for Λ^2 and $\mathbf{K}^2$ and $(\mathbf{K}^T)^2$.*

Proof of Theorem 3.2.1. Because the entries $K((\mathbf{t}, \sigma), (\mathbf{t}', \sigma'))$ are non-negative, the condition (3.14) with $r = 1$ means that

$$\sum_{(\mathbf{t}, \sigma), (\mathbf{t}', \sigma')} K^2((\mathbf{t}, \sigma), (\mathbf{t}', \sigma')) < \infty, \quad (3.15)$$

that is, $\mathbf{K}$ and $\mathbf{K}^T$ are Hilbert-Schmidt operators. It means that the operator $\mathbf{K}\mathbf{K}^T$ has an orthonormal basis of eigenvectors and the series of squares of its eigenvalues (counted with multiplicities) converges and gives the trace $\mathrm{tr}_{T-C}(\mathbf{K}\mathbf{K}^T)$. Consequently, the operators $\mathbf{K}$ and $\mathbf{K}^T$ are bounded (and even completely bounded) and $\mathbf{K}^2$ and $(\mathbf{K}^T)^2$ are of trace class. The latter fact means that the matrix trace of the operator $\mathbf{K}^2$ coincides with its operator trace in ℓ_{T-C}^2 , and the same is true of $(\mathbf{K}^T)^2$. In addition, the operator $\mathbf{K}^2$ has the property that its matrix entries $K^{(2)}((\mathbf{t}, \sigma), (\mathbf{t}', \sigma'))$ are strictly positive:

$$K^{(2)}((\mathbf{t}, \sigma), (\mathbf{t}', \sigma')) = \sum_{(\tilde{\mathbf{t}}, \tilde{\sigma})} K((\mathbf{t}, \sigma), (\tilde{\mathbf{t}}, \tilde{\sigma})) K((\tilde{\mathbf{t}}, \tilde{\sigma}), (\mathbf{t}', \sigma')) > 0. \quad (3.16)$$

The Krein–Rutman theory (see [KR48], Proposition VII' or Appendix B) guarantees that both $\mathbf{K}$ and $\mathbf{K}^T$ have a maximal eigenvalue Λ that is positive and non-degenerate, or simple.

That is, the eigenvector $\boldsymbol{\phi}$ of $\mathbf{K}$ and the eigenvector $\boldsymbol{\phi}^*$ of $\mathbf{K}^T$ corresponding with $\boldsymbol{\Lambda}$ are unique up to multiplication by a constant, and all entries $\boldsymbol{\phi}(\mathbf{t}, \boldsymbol{\sigma})$ and $\boldsymbol{\phi}^*(\mathbf{t}, \boldsymbol{\sigma})$ are non-zero and have the same sign. In other words, the entries $\boldsymbol{\phi}(\mathbf{t}, \boldsymbol{\sigma})$ and $\boldsymbol{\phi}^*(\mathbf{t}, \boldsymbol{\sigma})$ can be made positive. The spectral gaps are also consequences of the above properties. $\square$

Set:

$$\lambda(\mu, \beta) = c^2 (m^2 + 1) (\cosh 2\beta) \left(1 + \sqrt{1 - \frac{1}{(\cosh 2\beta)^2} \frac{(m^2 - 1)^2}{(m^2 + 1)^2}} \right) \quad (3.17)$$

where c and m are determined by

$$c = \frac{\exp(\beta - \mu)}{e^{2\beta}(1 - \exp(\beta - \mu))^2 - e^{-2\mu}} \quad (3.18)$$

$$m = e^{2\beta} + (1 - e^{4\beta}) \exp(-(\beta + \mu)). \quad (3.19)$$

Lemma 3.2.1. *For any $\beta, \mu > 0$ such that*

$$\lambda(\mu, \beta) < 1, \quad (3.20)$$

the condition (3.14) is satisfied for $r = 1$:

$$\text{tr}(\mathbf{K}\mathbf{K}^T) = \text{tr}(\mathbf{K}^T\mathbf{K}) < \infty \quad \text{and} \quad \text{tr}|\mathbf{K}^2| = \text{tr}|(\mathbf{K}^T)^2| < \infty, \quad (3.21)$$

implying the assertions of Proposition 3.2.1 and Theorem 3.2.1. Moreover, the condition (3.14) implies (3.20)

Proof of Lemma 3.2.1. By definition the trace (3.21) we need to calculate the series

$$\begin{aligned} \text{tr}(\mathbf{K}^T\mathbf{K}) &= \sum_{(\mathbf{t}, \boldsymbol{\sigma})} \mathbf{K}^T\mathbf{K}((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}, \boldsymbol{\sigma})) \\ &= \sum_{(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')} K((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) K((\mathbf{t}', \boldsymbol{\sigma}'), (\mathbf{t}, \boldsymbol{\sigma})) \\ &= \sum_{(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')). \end{aligned} \quad (3.22)$$

A single-strip triangulation $\mathbf{t}$ consists of up- and down-triangles. Accordingly, it is convenient to employ new labels for spins: if a triangle $t(l)$ is an l th up-triangle then we denote it by t_{up}^l ; the corresponding spin $\sigma(j)$ will be denoted by σ_{up}^l . Similarly, if $t(j)$ is an l th down-triangle then we denote it by t_{do}^l ; the spin $\sigma(j)$ will be denoted by σ_{do}^l . Consequently, the triangulation $\mathbf{t}$ and its supported spin-configuration $\boldsymbol{\sigma}$ are represented as

$$\mathbf{t} := (\mathbf{t}_{\text{up}}, \mathbf{t}_{\text{do}}) \quad \text{and} \quad \boldsymbol{\sigma} := (\boldsymbol{\sigma}_{\text{up}}, \boldsymbol{\sigma}_{\text{do}}).$$

Here

$$\mathbf{t}_{\text{up}} = (t_{\text{up}}^1, \dots, t_{\text{up}}^n), \quad \mathbf{t}_{\text{do}} = (t_{\text{do}}^1, \dots, t_{\text{do}}^m),$$

and

$$\boldsymbol{\sigma}_{\text{up}} = (\sigma_{\text{up}}^1, \dots, \sigma_{\text{up}}^n), \quad \boldsymbol{\sigma}_{\text{do}} = (\sigma_{\text{do}}^1, \dots, \sigma_{\text{do}}^m),$$

assuming that the supporting single-strip triangulation $\mathbf{t}$ contains n up-triangles and m down-triangles. (The actual order of up- and down-triangles and supported spins does not matter.)

The same can be done for the pair $(\mathbf{t}', \boldsymbol{\sigma}')$ as illustrated in (3.22). Let recall that the triangulations $\mathbf{t}$ and $\mathbf{t}'$ are consistent ($\mathbf{t} \sim \mathbf{t}'$) iff number of the down-triangles in $\mathbf{t}$ equals that of up-triangles in $\mathbf{t}'$.

To calculate the sum (3.22) we divide the summation over $(\mathbf{t}', \boldsymbol{\sigma}')$ into a summation over $(\mathbf{t}'_{\text{up}}, \boldsymbol{\sigma}'_{\text{up}})$ and $(\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}})$. Firstly, fix a pair $(\mathbf{t}'_{\text{up}}, \boldsymbol{\sigma}'_{\text{up}})$ and make the sum over $(\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}})$. Note that the term $V((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))$ depends only on $\boldsymbol{\sigma}_{\text{do}}$ and $\boldsymbol{\sigma}'_{\text{up}}$. Consequently,

$$\begin{aligned} & \sum_{\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}}} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) \\ &= e^{-\beta H(\boldsymbol{\sigma})} e^{-2\beta V((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))} e^{-\mu n(\mathbf{t})} \sum_{(\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}})} e^{-\beta H(\boldsymbol{\sigma}')} e^{-\mu n(\mathbf{t}')} . \end{aligned} \quad (3.23)$$

The sum in the right-hand side of (3.23) can be represented in a matrix form. Denote by $e_{\pm 1}$ the standard spin-1/2 unit vectors in $\mathbb{R}^2$:

$$e_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Next, let us introduce a 2×2 matrix T where

$$T = e^{-\mu} \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix} := \begin{pmatrix} t_{++} & t_{+-} \\ t_{-+} & t_{--} \end{pmatrix} . \quad (3.24)$$

Denote by $n(i), i = 1, \dots, n_{\text{up}}(\mathbf{t}')$ the number of down-triangles in $\mathbf{t}'$ which are between the

i th and $(i + 1)$ th up-triangles in $\mathbf{t}'$. Let $n_{up}(\mathbf{t}') = k$ then

$$\begin{aligned} \sum_{\mathbf{t}'_{do}, \boldsymbol{\sigma}'_{do}} e^{-\beta H(\boldsymbol{\sigma}')} e^{-\mu n(\mathbf{t}')} &= \sum_{n(i) \geq 0: \sum_i n(i) \geq 1} \prod_{l=1}^k \left(e_{\sigma'_{up}^l}^T T^{n(l)+1} e_{\sigma'^{l+1}_{up}} \right) \\ &= \prod_{l=1}^k \left(e_{\sigma'_{up}^l}^T M e_{\sigma'^{l+1}_{up}} \right) - \prod_{l=1}^k \left(e_{\sigma'_{up}^l}^T T e_{\sigma'^{l+1}_{up}} \right) \end{aligned} \quad (3.25)$$

where the matrix M is the sum of the geometric progression

$$M = \sum_{n=1}^{\infty} T^n := \begin{pmatrix} m_{++} & m_{+-} \\ m_{-+} & m_{--} \end{pmatrix}. \quad (3.26)$$

Using the same procedure we can obtain the sum over all up-triangles into the triangulation $\mathbf{t}$. The only difference is the existence of marked up-triangle in the strip: let as before $n_{up}(\mathbf{t}') = n_{do}(\mathbf{t}) = k$ then

$$\sum_{\mathbf{t}_{up}, \boldsymbol{\sigma}_{up}} e^{-\beta H(\boldsymbol{\sigma})} e^{-\mu n(\mathbf{t})} = \prod_{l=1}^{k-1} \left(e_{\sigma_{up}^l}^T M e_{\sigma_{up}^{l+1}} \right) \left(e_{\sigma_{up}^k}^T M^2 e_{\sigma_{up}^1} \right) \quad (3.27)$$

See Figure 3.1 for illustration of these calculations (3.25) and (3.27). Further, supposing the existence of the matrix M and using (3.25) and (3.27) we obtain the following:

$$\begin{aligned} \sum_{\mathbf{t}_{up}, \boldsymbol{\sigma}_{up}} \sum_{\mathbf{t}'_{do}, \boldsymbol{\sigma}'_{do}} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) &= e^{-2\beta V((\mathbf{t}_{do}, \boldsymbol{\sigma}_{do}), (\mathbf{t}'_{up}, \boldsymbol{\sigma}'_{up}))} \\ &\times \sum_{\mathbf{t}_{up}, \boldsymbol{\sigma}_{up}} e^{-\beta H(\boldsymbol{\sigma})} e^{-\mu n(\mathbf{t})} \sum_{(\mathbf{t}'_{do}, \boldsymbol{\sigma}'_{do})} e^{-\beta H(\boldsymbol{\sigma}')} e^{-\mu n(\mathbf{t}')} \\ &= e^{-2\beta V((\mathbf{t}_{do}, \boldsymbol{\sigma}_{do}), (\mathbf{t}'_{up}, \boldsymbol{\sigma}'_{up}))} \\ &\times \left[\prod_{l=1}^k \left(e_{\sigma'_{up}^l}^T M e_{\sigma'^{l+1}_{up}} \right) \prod_{l=1}^{k-1} \left(e_{\sigma'_{do}^l}^T M e_{\sigma'^{l+1}_{do}} \right) \left(e_{\sigma_{up}^k}^T M^2 e_{\sigma_{up}^1} \right) \right. \\ &\quad \left. - \prod_{l=1}^k \left(e_{\sigma'_{up}^l}^T T e_{\sigma'^{l+1}_{up}} \right) \prod_{l=1}^{k-1} \left(e_{\sigma'_{do}^l}^T M e_{\sigma'^{l+1}_{do}} \right) \left(e_{\sigma_{up}^k}^T M^2 e_{\sigma_{up}^1} \right) \right]. \end{aligned} \quad (3.28)$$

Necessary and sufficient condition for the convergence of the matrix series for M is that the maximal eigenvalue of matrix T is less than 1. The eigenvalues of T are

$$\lambda_{\pm} = e^{(\beta-\mu)} \pm e^{-(\beta+\mu)}, \quad (3.29)$$

Figure 3.1: Illustration of the calculates (3.25) and (3.27).

and the above condition means that $\lambda_+ < 1$ or, equivalently,

$$\mu > \ln(2\cosh(\beta)). \quad (3.30)$$

Under this condition (3.30), the matrix M is calculated explicitly:

$$M = \frac{e^{(\beta-\mu)}}{e^{2\beta}(1 - e^{(\beta-\mu)})^2 - e^{-2\mu}} \times \begin{pmatrix} e^{2\beta} + (1 - e^{4\beta})e^{-(\beta+\mu)} & 1 \\ 1 & e^{2\beta} + (1 - e^{4\beta})e^{-(\beta+\mu)} \end{pmatrix}. \quad (3.31)$$

We are now in a position to calculate the sum in (3.22). To this end, we again represent it through the product of transfer matrices. Pictorially, we express the above sum as the partition function of a one-dimensional Ising-type model where states are pairs of spins $(\sigma_{do}^l, \sigma_{up}^l)$ and the interaction is via the matrix T between the members of the pair and via

matrix M between neighboring pairs. More precisely, define the following 4×4 matrices:

$$Q = \begin{pmatrix} e^{2\beta} m_{++} m_{++} & m_{++} m_{+-} & m_{+-} m_{++} & e^{2\beta} m_{+-} m_{+-} \\ m_{++} m_{-+} & e^{-2\beta} m_{++} m_{--} & e^{-2\beta} m_{+-} m_{-+} & m_{+-} m_{--} \\ m_{-+} m_{++} & e^{-2\beta} m_{-+} m_{+-} & e^{-2\beta} m_{--} m_{++} & m_{--} m_{+-} \\ e^{2\beta} m_{-+} m_{-+} & m_{-+} m_{--} & m_{--} m_{-+} & e^{2\beta} m_{--} m_{--} \end{pmatrix} \quad (3.32)$$

$$Q_m = \begin{pmatrix} e^{2\beta} m_{++} m_{++}^{(2)} & m_{++} m_{+-}^{(2)} & m_{+-} m_{++}^{(2)} & e^{2\beta} m_{+-} m_{+-}^{(2)} \\ m_{++} m_{-+}^{(2)} & e^{-2\beta} m_{++} m_{++}^{(2)} & e^{-2\beta} m_{+-} m_{-+}^{(2)} & m_{+-} m_{--}^{(2)} \\ m_{-+} m_{++}^{(2)} & e^{-2\beta} m_{-+} m_{+-}^{(2)} & e^{-2\beta} m_{--} m_{++}^{(2)} & m_{++} m_{++}^{(2)} \\ e^{2\beta} m_{-+} m_{-+}^{(2)} & m_{-+} m_{--}^{(2)} & m_{--} m_{-+}^{(2)} & e^{2\beta} m_{--} m_{--}^{(2)} \end{pmatrix} \quad (3.33)$$

$$Q_t = \begin{pmatrix} e^{2\beta} t_{++} m_{++} & t_{++} m_{+-} & t_{+-} m_{++} & e^{2\beta} t_{+-} m_{+-} \\ t_{++} m_{-+} & e^{-2\beta} t_{++} m_{--} & e^{-2\beta} t_{+-} m_{-+} & t_{+-} m_{--} \\ t_{-+} m_{++} & e^{-2\beta} t_{-+} m_{+-} & e^{-2\beta} t_{--} m_{++} & t_{--} m_{+-} \\ e^{2\beta} t_{-+} m_{-+} & t_{-+} m_{--} & t_{--} m_{-+} & e^{2\beta} t_{--} m_{--} \end{pmatrix} \quad (3.34)$$

$$Q_{tm} = \begin{pmatrix} e^{2\beta} t_{++} m_{++}^{(2)} & t_{++} m_{+-}^{(2)} & t_{+-} m_{++}^{(2)} & e^{2\beta} t_{+-} m_{+-}^{(2)} \\ t_{++} m_{-+}^{(2)} & e^{-2\beta} t_{++} m_{++}^{(2)} & e^{-2\beta} t_{+-} m_{-+}^{(2)} & t_{+-} m_{--}^{(2)} \\ t_{-+} m_{++}^{(2)} & e^{-2\beta} t_{-+} m_{+-}^{(2)} & e^{-2\beta} t_{--} m_{++}^{(2)} & t_{++} m_{++}^{(2)} \\ e^{2\beta} t_{-+} m_{-+}^{(2)} & t_{-+} m_{--}^{(2)} & t_{--} m_{-+}^{(2)} & e^{2\beta} t_{--} m_{--}^{(2)} \end{pmatrix} \quad (3.35)$$

where $m_{ij}, m_{ij}^{(2)}$ and $t_{i,j}$ ($i, j \in \{-, +\}$) are elements of the matrices M, M^2 , and T respectively.

Now for the sum under consideration (3.22) we obtain using representation (5.53)

$$\begin{aligned}
 \sum_{(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) &= \sum_{(\mathbf{t}_{\text{do}}, \boldsymbol{\sigma}_{\text{do}}), (\mathbf{t}'_{\text{up}}, \boldsymbol{\sigma}'_{\text{up}})} e^{-2\beta V((\mathbf{t}_{\text{do}}, \boldsymbol{\sigma}_{\text{do}}), (\mathbf{t}'_{\text{up}}, \boldsymbol{\sigma}'_{\text{up}}))} \\
 &\times \left[\prod_{l=1}^k \left(e_{\sigma'_{\text{up}}^l}^T M e_{\sigma'_{\text{up}}^{l+1}} \right) \prod_{l=1}^{k-1} \left(e_{\sigma_{\text{do}}^l}^T M e_{\sigma_{\text{do}}^{l+1}} \right) \left(e_{\sigma_{\text{up}}^k}^T M^2 e_{\sigma_{\text{up}}^1} \right) \right. \\
 &\quad \left. - \prod_{l=1}^k \left(e_{\sigma'_{\text{up}}^l}^T T e_{\sigma'_{\text{up}}^{l+1}} \right) \prod_{l=1}^{k-1} \left(e_{\sigma_{\text{do}}^l}^T M e_{\sigma_{\text{do}}^{l+1}} \right) \left(e_{\sigma_{\text{up}}^k}^T M^2 e_{\sigma_{\text{up}}^1} \right) \right] \\
 &= \text{tr} \left(\left(\sum_{k=0}^{\infty} Q^k \right) Q_m \right) - \text{tr} \left(\left(\sum_{k=1}^{\infty} Q_t^k \right) Q_{tm} \right). \tag{3.36}
 \end{aligned}$$

By the construction the matrix Q is greater then Q_t elementwise. Thus the eigenvalue of matrix Q is greater than the eigenvalue of the matrix Q_t (it follows from the Perron-Frobenius theorem). Therefore the necessary and sufficient condition for the convergence in (3.22) is that the largest eigenvalue of Q is less than 1. It is possible to calculate its eigenvalue analytically. In order to express the eigenvalues of Q it is convinient to use notations (3.18) and (3.19). In this notations the matrix M , i.e. (3.31), is represented as following

$$M = c \begin{pmatrix} m & 1 \\ 1 & m \end{pmatrix}.$$

The equations for the eigenvalues of Q are:

$$\begin{aligned}
 \lambda_1 &= c^2 e^{\beta} (m^2 - 1) \\
 \lambda_2 &= c^2 e^{-\beta} (m^2 - 1) \\
 \lambda_3 &= c^2 (m^2 + 1) (\cosh \beta) \left(1 - \sqrt{1 - \frac{(m^2 - 1)^2}{(\cosh \beta)^2 (m^2 + 1)^2}} \right) \\
 \lambda_4 &= c^2 (m^2 + 1) (\cosh \beta) \left(1 + \sqrt{1 - \frac{(m^2 - 1)^2}{(\cosh \beta)^2 (m^2 + 1)^2}} \right)
 \end{aligned}$$

A straightforward inspection confirms that the largest eigenvalue is given by λ_4 . The condition $\lambda_4 < 1$ coincides with (3.20). Finally, using matrices Q, Q_m (see formulas (3.32) and (3.33)) with positive entries and of size 4×4 , we have the following representation of 3.22

$$\text{tr}(\mathbf{K}\mathbf{K}^T) = \text{tr} \left(\left(\sum_{k \geq 1} Q^k \right) Q_m \right) + \dots$$

The convergence of the matrix series $\sum_{k \geq 1} Q^k$ is equivalent to the condition that the maximal eigenvalue of the matrix Q is less than 1. This is exactly the condition (3.20). This completes the proof of Lemma 3.2.1. $\square$

Theorem 3.2.2. *Under condition (3.20), the following limit holds:*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \Xi_N(\beta, \mu) = \log \Lambda. \quad (3.37)$$

Moreover, as $N \rightarrow \infty$, the N -strip Gibbs measure $\mathbb{P}_N$ (see Eqn (5.9)) converges weakly to a limiting probability distribution $\mathbb{P}$ that is represented by a positive recurrent Markov chain with states $(\mathbf{t}, \boldsymbol{\sigma})$, the transition matrix

$\mathbf{P} = \{P((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))\}$ and the invariant distribution $\boldsymbol{\pi} = \{\pi(\mathbf{t}, \boldsymbol{\sigma})\}$ where

$$\begin{aligned} P((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) &= \frac{K((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) \phi(\mathbf{t}', \boldsymbol{\sigma}')}{\Lambda \phi(\mathbf{t}, \boldsymbol{\sigma})} \\ \pi(\mathbf{t}, \boldsymbol{\sigma}) &= \phi(\mathbf{t}, \boldsymbol{\sigma}) \phi^T(\mathbf{t}, \boldsymbol{\sigma}) / \langle \phi, \phi^T \rangle_{\text{T-C}} \end{aligned}$$

with the norm $\|\phi\|_{\text{T-C}}^2 = \sum_{\mathbf{t}, \boldsymbol{\sigma}} \phi(\mathbf{t}, \boldsymbol{\sigma})^2$.

Proof of Theorem 3.2.2. The spectral gap for $\mathbf{K}$ implies that $\forall \psi \in \ell_{\text{T-C}}^2$, we have the convergence

$$\lim_{N \rightarrow \infty} \frac{1}{\Lambda^N} \mathbf{K}^N \psi = (\langle \psi, \phi \rangle_{\text{T-C}}) \phi$$

in the norm of space $\ell_{\text{T-C}}^2$. Moreover, let $\boldsymbol{\Pi}$ denote the operator of projection to the subspace spanned by the eigenvectors of $\mathbf{K}$ different from ϕ . Then

$$\frac{1}{\Lambda} \|\boldsymbol{\Pi} \mathbf{K} \mathbf{P}\|_{\text{T-C}} < 1 \implies \lim_{N \rightarrow \infty} \frac{1}{\Lambda^N} \|(\boldsymbol{\Pi} \mathbf{K} \mathbf{P})^N\|_{\text{T-C}} = 0.$$

In turn, this implies that

$$\frac{1}{N} \log \Xi_N(\mu, \beta) = \frac{1}{N} \log \text{tr}_{\text{T-C}} \mathbf{K}^N \rightarrow \log \Lambda.$$

Convergence of the Gibbs measure $\mathbb{P}_N$ follows as a corollary. $\square$

3.3 Discussion and outlook

This chapter makes a step towards determining the subcriticality domain for an Ising-type model coupled to two-dimensional causal dynamical triangulations (CDT). In doing so we employ transfer-matrix techniques and in particular the Krein-Rutman theorem. We complement the discussion of the previous sections with the following two concluding remarks:

Remark 1. *It is instructive to summarise the logical structure of the argument establishing Lemma 3.2.1 and Theorems 3.2.1 and 3.2.2:*

- First, (3.21) holds iff condition (3.20) holds: see the proof of Lemma 3.2.1.
- Next, (3.21) implies that $\mathbf{K}$ is a Hilbert–Schmidt operator and $\mathbf{K}^2$ is a trace class operator in ℓ_{T-C}^2 .
- The last fact, together with the property of positivity (3.16), allow us to use the Krein–Rutman theory, deriving all assertions of Theorems 3.2.1 and 3.2.2.

On the other hand, if (3.20) fails (and therefore (3.21) fails), it does not necessarily mean that the assertions Theorems 3.2.1 and 3.2.2 fail. In other words, we do not claim that the boundary of the domain of parameters β and μ where the model exhibits uncritical behavior is given by Eqn. (3.20). Moreover, Figure 3.2 shows the result of a numerical calculation indicating that the condition (3.20) is worse than (3.11) for (moderately) large values of β .

An apparent condition closer to necessity is the pair of inequalities (3.14) for some (possibly) large r . This issue needs a further study.

Remark 2. *Physical considerations suggest that the critical curve in the (β, g) quarter-plane would have some predictable patterns of behavior: as a function of β , it would decay and exhibit a first-order singularity at a unique point $\beta = \beta_{cr} \in (0, \infty)$.*

A plausible conjecture is that the boundary of the critical domain coincides with the locus of points (β, μ) where $\mathbf{\Lambda}$ loses either the property of positivity or the property of being a simple eigenvalue. This direction also requires further research.

Figure 3.2: $\lambda_Q = \lambda$ and λ_T are the maximal eigenvalues of the matrix Q and a related matrix T respectively. The area above the black curve is where the condition (3.20) holds true.

Chapter 4

FK representation for the Ising model coupled to CDT

The main goal of this chapter is to define and find lower and upper bounds for the critical curve using old approaches of the statistical mechanics. The existence of the Gibbs measures of the model (region of subcritical behaviour of the Ising model) was shown in [HYSZ13], but the existence of phase transition is also an open question.

This chapter extends results from the previous chapter for the (annealed) classical Ising model coupled to two-dimensional causal dynamical triangulations. Using the Fortuin-Kasteleyn (FK) representation of quantum Ising models via path integrals, we determine a region in the quadrant of parameters $\beta, \mu > 0$ where the critical curve for the classical model can be located. In particular, we determine a region where the infinite-volume Gibbs measure exists and it is unique, and a region where the finite-volume Gibbs measure has no weak limit (in fact, does not exist if the volume is large enough). We also study the infinite-volume free energy and provide lower and upper bounds using the FK representation of the Ising model.

FK models were introduced by Fortuin and Kasteleyn (see [FK72]). These models have become an important tool in the study of phase transition for the Ising and Potts model. The goal of this chapter is to introduce the FK representation of a quantum Ising model coupled to CDTs via a path integral (see [Aiz94], [Iof09], [AKN92] for an overview), and use this representation to obtain information of the critical curve. The aforementioned FK representation uses a family of Poisson point processes and the Lie-Trotter product formula to interpret exponential sums of operators as random operator products. This representation was originally derived in [AKN92] and [Aiz94].

4.1 The quantum Ising model

In this section we write the classical partition function, over a given triangulation, by using ingredients of the quantum Ising model.

Henceforth, for simplicity in notation and exposure of the following chapter, we shall

denote a triangle of any triangulation $\mathbf{t}$ doing without put the indices i, j as was done in previous chapter. Thus, in this chapter, the Hamiltonian the (annealed) model is written as follow

$$\mathbb{H}(\boldsymbol{\sigma}) = - \sum_{\langle t, t' \rangle} \sigma(t) \sigma(t'). \quad (4.1)$$

Here, $\langle t, t' \rangle$ stands that triangles t, t' have a common edge. These triangles are called nearest neighbor.

Let $\mathbf{t} = (\mathbf{t}(0), \mathbf{t}(1), \dots, \mathbf{t}(N-1))$ be a causal triangulation of C_N with periodical spatial boundary condition (see Figure 2.1 (b)). Let $\Delta(\mathbf{t})$ denote the set of triangles of the triangulation $\mathbf{t}$.

We define $\Omega(\mathbf{t})$ to be the set of all spin configurations supported by the triangles of $\mathbf{t}$, i.e., $\Omega(\mathbf{t}) = \{-1, +1\}^{\Delta(\mathbf{t})}$. Let $\mathcal{Z}_N^{\beta, \mathbf{t}}$ be the partition function of the Ising model on the CDT $\mathbf{t}$, at inverse temperature $\beta > 0$

$$\mathcal{Z}_N^{\beta, \mathbf{t}} = \sum_{\boldsymbol{\sigma} \in \Omega(\mathbf{t})} \exp\{-\beta \mathbb{H}(\boldsymbol{\sigma})\}, \quad (4.2)$$

where $\mathbb{H}(\boldsymbol{\sigma})$ represents the energy of configuration $\boldsymbol{\sigma} \in \Omega(\mathbf{t})$, defined by the formula (4.1).

The quantum Ising model on a causal triangulation $\mathbf{t}$ is defined as follows.

Let

$$\hat{\boldsymbol{\sigma}}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.3)$$

be the Pauli matrix with their corresponding eigenvectors

$$\boldsymbol{\phi}_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\phi}_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.4)$$

In the quantum language spins values ± 1 are understood as eigenvalues of Pauli matrix. Notice that $\hat{\boldsymbol{\sigma}}^z \boldsymbol{\phi}_\nu = \nu \boldsymbol{\phi}_\nu$ for $\nu = \pm 1$.

To each triangle $t \in \Delta(\mathbf{t})$ we associate a spin taking values $\boldsymbol{\phi}_{+1}$ and $\boldsymbol{\phi}_{-1}$. Thus, the space of all such spin configurations on $\mathbf{t}$ is defined as the real vector space $\mathbb{X}_{\mathbf{t}} = \bigotimes_{t \in \Delta(\mathbf{t})} \mathbb{R}^2$, where $\bigotimes$ stands for the tensor product. Notice that $\mathbb{X}_{\mathbf{t}}$ is a real vector space of dimension 2 to the $n(\mathbf{t})$ power: $\dim(\mathbb{X}_{\mathbf{t}}) = 2^{n(\mathbf{t})}$.

For each classical configuration $\boldsymbol{\sigma} \in \Omega(\mathbf{t})$ we associate the quantum configuration as

tensor products

$$\phi_\sigma := \bigotimes_{t \in \Delta(\mathbf{t})} \phi_{\sigma(t)},$$

where $\sigma(t)$ is the spin supported by the triangle $t \in \Delta(\mathbf{t})$. Notice that there is a one-one correspondence between $\Omega(\mathbf{t})$ and the collection $\{\phi_\sigma\}_{\sigma \in \Omega(\mathbf{t})}$. Moreover, the collection of quantum configurations is a complete orthonormal basis of $\mathbb{X}_{\mathbf{t}}$ with respect to the following scalar product

$$\langle \phi_\sigma | \phi_{\sigma'} \rangle := \prod_{t \in \Delta(\mathbf{t})} (\phi_{\sigma(t)}, \phi_{\sigma'(t)})_2,$$

where $(\cdot, \cdot)_2$ is the usual scalar product of $\mathbb{R}^2$. With each triangle $t \in \Delta(\mathbf{t})$ we associate a linear self-adjoint operator $\hat{\sigma}_t^z : \mathbb{X}_{\mathbf{t}} \rightarrow \mathbb{X}_{\mathbf{t}}$ which acts as a copy of Pauli matrix $\hat{\sigma}^z$ on the coordinate of ϕ_σ associated to the triangle t of $\mathbf{t}$. That is, for each $\sigma \in \Omega_{\mathbf{t}}$,

$$\hat{\sigma}_t^z \phi_\sigma = \phi_{\sigma(t_1)} \otimes \cdots \otimes (\hat{\sigma}^z \phi_{\sigma(t)}) \otimes \cdots = \sigma(t) \phi_\sigma. \quad (4.5)$$

Note that operators $\hat{\sigma}_t^z, \hat{\sigma}_{t'}^z$ commute, and satisfies

$$\hat{\sigma}_t^z \hat{\sigma}_{t'}^z \phi_\sigma = \sigma(t) \sigma(t') \phi_\sigma. \quad (4.6)$$

The Hamiltonian $\mathcal{H}_{\mathbf{t}}$ of the quantum Ising model is a linear self-adjoint operator defined on $\mathbb{X}_{\mathbf{t}}$:

$$\mathcal{H}_{\mathbf{t}} = - \sum_{\langle t, t' \rangle} \hat{\sigma}_t^z \hat{\sigma}_{t'}^z, \quad (4.7)$$

where two operators $\hat{\sigma}_t^z$ and $\hat{\sigma}_{t'}^z$ interact if their supporting triangles $t, t' \in \Delta(\mathbf{t})$ are nearest neighbors.

Note that $\mathcal{H}_{\mathbf{t}} \phi_\sigma = \mathbb{H}(\sigma) \phi_\sigma$. In other words, $\mathcal{H}_{\mathbf{t}}$ is a diagonal in the $\{\phi_\sigma\}$ basis, and corresponding eigenvalues being equal to values of the classical Ising Hamiltonian on configurations σ . This allows write the classical partition function $\mathcal{Z}_N^{\beta, \mathbf{t}}$ for Ising model, at inverse temperature $\beta > 0$ associated with triangulation $\mathbf{t}$, as follows

$$\mathcal{Z}_N^{\beta, \mathbf{t}} = \sum_{\sigma \in \Omega(\mathbf{t})} \exp\{-\beta \mathbb{H}(\sigma)\} = \sum_{\sigma \in \Omega(\mathbf{t})} \langle \phi_\sigma | e^{-\beta \mathcal{H}_{\mathbf{t}}} | \phi_\sigma \rangle = \text{tr} (e^{-\beta \mathcal{H}_{\mathbf{t}}}). \quad (4.8)$$

Finally, using the quantum representation (4.8), the partition function for the N -strip Ising model coupled to CDT, at the inverse temperature $\beta > 0$ and for the cosmological constant μ , can be written as follows

$$\Xi_N(\beta, \mu) = \sum_{\mathbf{t}} e^{-\mu n(\mathbf{t})} \text{tr} (e^{-\beta \mathcal{H}_{\mathbf{t}}}). \quad (4.9)$$

4.2 FK representation for Ising model coupled to CDT

In order to calculate $\text{tr}(e^{-\beta\mathcal{H}_t})$ in (4.9) we will use the FK representation for the Ising model via path integrals, see [Aiz94, Iof09, AKN92]. By representation (4.9), the trace $\text{tr}(e^{-\beta\mathcal{H}_t})$ may be expressed in terms of a type of path integral with respect to the continuous random-cluster model on $\Delta(\mathbf{t}) \times [0, \beta]$ for any Lorentzian triangulation $\mathbf{t}$ (see Proposition 4.2.1 below).

For any pair $t, t' \in \Delta(\mathbf{t})$ of nearest neighbor triangles, we associate a Poisson process $\xi_{\langle t, t' \rangle}(s)$ on the time interval $[0, \beta]$ with intensity 2. We refer to the process $\xi_{\langle t, t' \rangle}$ as process of arrivals of operator $K_{\langle t, t' \rangle}$ on the interval $[0, \beta]$, where

$$K_{\langle t, t' \rangle} = \frac{I + \hat{\sigma}_t^z \hat{\sigma}_{t'}^z}{2}. \quad (4.10)$$

Let ξ be the collection of independent Poisson processes $\xi_{\langle t, t' \rangle} : \xi(s) := \{\xi_{\langle t, t' \rangle}(s)\}_{\langle t, t' \rangle \in \mathcal{E}_t}$, where $\mathcal{E}_t$ is the set of all pairs of neighbor triangles: $\mathcal{E}_t := \{\langle t, t' \rangle : t, t' \in \Delta(\mathbf{t})\}$.

Let $\mathbb{P}_{\beta, \mathbf{t}}$ denote the probability measure associated with the family of Poisson process ξ . We shall abuse notation by using ξ to denote a realization of process of arrivals $\xi(s)$, $s \in [0, \beta]$. By independence there are no simultaneous arrivals $\mathbb{P}_{\beta, \mathbf{t}}$ -a.s. Thus, a realization ξ of process of arrivals can be represented by a collection of arrival times $\{s_i\}_{i=1, \dots, N_\xi}$ contained in $[0, \beta]$ and its corresponding arrival types $L(s_i) \in \mathcal{E}_t$, $\xi \equiv \{s_i, L(s_i)\}_{i=1, \dots, N_\xi}$, where N_ξ is the total number of arrivals during the time $[0, \beta]$.

With a fixed realization ξ we associated a family of all possible piecewise constant right-continuous functions $\psi_\xi = \{\varphi : [0, \beta] \rightarrow \{\phi_\sigma\}\}$, having jumps only at arrival times of ξ . Since $\mathbb{X}_t$ is finite dimensional and there are $\mathbb{P}_{\beta, \mathbf{t}}$ -a.s. finite number of arrivals, we have $|\psi_\xi| < \infty$, $\mathbb{P}_{\beta, \mathbf{t}}$ -a.s., where $|\psi_\xi|$ is the total number of functions in the set ψ_ξ .

For each arrival time s of a realization ξ corresponding a unique arrival type $L(s)$ a.s. Suppose that $L(s) = \langle t, t' \rangle$ for $t, t' \in \Delta(\mathbf{t})$ nearest neighbor, then $K_{L(s)} = K_{\langle t, t' \rangle} : \mathbb{X}_t \rightarrow \mathbb{X}_t$. Let $\varphi \in \psi_\xi$, and denote $\varphi(s^-) = \lim_{t \rightarrow s^-} \varphi(t)$. Notice that the function $\varphi \in \psi_\xi$ can be continuous or not at each arrival time s (see Figure 4.1 below).

Using the before notation, we have the following proposition.

Proposition 4.2.1. *The matrix elements of the linear operator $e^{-\beta\mathcal{H}_t}$ with respect to the basis $\{\phi_\sigma\}$ are given by*

$$\langle \phi_\sigma | e^{-\beta\mathcal{H}_t} | \phi_{\sigma'} \rangle = \exp \left\{ \frac{3}{2} \beta n(\mathbf{t}) \right\} \int \mathbb{P}_{\beta, \mathbf{t}}(d\xi) \sum_{\substack{\varphi \in \psi_\xi \\ \varphi(0) = \phi_{\sigma'}, \varphi(\beta) = \phi_\sigma}} \prod_{s \in \xi} \langle \varphi(s^-) | K_{L(s)} | \varphi(s) \rangle, \quad (4.11)$$

for all $\mathbf{t} \in \mathbb{LT}_N$.

Figure 4.1: A trajectory sample associated with a realization $\xi = \{s_i\}_{i=1,\dots,n}$. Each trajectory $\varphi \in \psi_\xi$ can be continuous or not at each arrival time s . In this case, at arrival time s_{k-1} the trajectory φ do not have jump, and at arrival time s_k the trajectory φ have a jump.

Formula (4.11) was proved in [Aiz94] and [Iof09] for any general finite graph.

With any realization ξ we associate a graph $G_\xi = (\Delta_\xi, \mathcal{E}_\xi)$, where the set of vertices is $\Delta_\xi = \Delta(\mathbf{t})$ and the set of edges $\mathcal{E}_\xi \subseteq \mathcal{E}_\mathbf{t}$ is defined by following rule: an edge $e = \langle t, t' \rangle \in \mathcal{E}_\mathbf{t}$ belong to $\mathcal{E}_\xi$ if and only if there exist a arrival time s such that the corresponding arrival type $L(s)$ is $\langle t, t' \rangle$ into the realization ξ .

We say that two triangulations t and t' are connected, denoted by $t \leftrightarrow t'$, if and only if there exist a path in G_ξ connecting t and t' . For any $t \in \Delta(\mathbf{t})$, we suppose that $t \leftrightarrow t$. A subset $\mathcal{C} \subseteq \Delta(\mathbf{t})$ is called a cluster (maximal connected component) if for any $t, t' \in \mathcal{C}$ then $t \leftrightarrow t'$, and $t \not\leftrightarrow t'$ for any $t \in \mathcal{C}$ and $t' \notin \mathcal{C}$ (see Figure 5.1 below). Thus, any realization ξ of the Poisson process splits $\Delta(\mathbf{t})$ into the disjoint union of maximal connected components, i.e., for any realization ξ there exists $k = k(\xi) \in \{1, \dots, n(\mathbf{t})\}$ and sets $\mathcal{C}_1 = \mathcal{C}_1, \dots, \mathcal{C}_k = \mathcal{C}_k(\xi) \subseteq \Delta(\mathbf{t})$ such that

$$\Delta(\mathbf{t}) = \bigcup_{i=1}^{k(\xi)} \mathcal{C}_i,$$

and $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for $i \neq j$. Here $k(\xi)$ is the number of clusters defined by the relation $\leftrightarrow$. Additionally, we define the cluster $\mathcal{C}_t$ of a triangle t by $\mathcal{C}_t = \{t' \in \Delta(\mathbf{t}) : t \leftrightarrow t'\}$.

Let $\sigma, \sigma' \in \Omega(\mathbf{t})$ be two configurations and let $\phi_\sigma, \phi_{\sigma'}$ be the corresponding quantum configuration. Then, for any $\langle t, t' \rangle \in \mathcal{E}_\mathbf{t}$

$$\langle \phi_\sigma | K_{\langle t, t' \rangle} | \phi_{\sigma'} \rangle = \delta_{\{\sigma = \sigma'\}} \delta_{\{\sigma(t) = \sigma(t')\}}. \quad (4.12)$$

Relation (4.12) implies that, for any realization ξ only constant functions of ψ_ξ contribute to

Figure 4.2: In this figure, we show the Cluster $\mathcal{C}_t$ of a triangle t , and a graphic representation of relation $t \leftrightarrow t'$, where $\leftrightarrow$ on right side in the figure, represent arrival times.

the sum inside the integral (4.11). Additionally, an arrival of $K_{\langle t, t' \rangle}$ at arrival time $s \in [0, \beta]$ imposes an additional condition $\sigma(t) = \sigma(t')$ for contribute to the sum in (4.11). Define

$$\Omega(\mathbf{t}, \xi) = \{\sigma \in \Omega(\mathbf{t}) : \sigma \text{ has same sign in each cluster } \mathcal{C}_i\}.$$

Notice that $|\Omega(\mathbf{t}, \xi)| = 2^{k(\xi)}$, and

$$\sum_{\substack{\varphi \in \psi_\xi \\ \varphi(0) = \varphi(\beta) = \phi_\sigma}} \prod_{s \in \xi} \langle \varphi(s^-) | K_{L(s)} | \varphi(s) \rangle = \begin{cases} 1 & \text{if } \sigma \in \Omega(\mathbf{t}, \xi) \\ 0 & \text{if } \sigma \notin \Omega(\mathbf{t}, \xi) \end{cases} \quad (4.13)$$

As an elementary consequence of (4.13) the following representation for partition function $\mathcal{Z}_N^{\beta, \mathbf{t}}$ holds.

Proposition 4.2.2. *Let $\mathbf{t} \in \mathbb{LT}_N$ and $\beta > 0$. We have that*

$$\mathcal{Z}_N^{\beta, \mathbf{t}} = \text{tr}(e^{-\beta \mathcal{H}_{\mathbf{t}}}) = \exp \left\{ \frac{3}{2} \beta n(\mathbf{t}) \right\} \int 2^{k(\xi)} \mathbb{P}_{\beta, \mathbf{t}}(d\xi). \quad (4.14)$$

Proof. The proof is consequence of Proposition 4.2.1 and equation (4.13). $\square$

Using the N -strip Gibbs probability distribution $\mathcal{P}_{N, \mu}$ (introduced in Eqn (2.9)) for pure CDTs with periodical boundary condition, and substituting (4.14) on the right-hand side of (4.9) we obtain the FK representation of partition function for the N -strip Ising model

coupled to CDTs, at inverse temperature $\beta > 0$ and for the cosmological constant μ

$$\Xi_N(\beta, \mu) = Z_N(r) \sum_{\mathbf{t} \in \mathbb{LT}_N} \left\{ \int 2^{k(\xi)} \mathbb{P}_{\beta, \mathbf{t}}(d\xi) \right\} \mathcal{P}_{N, r}(\mathbf{t}), \quad (4.15)$$

where $r = \mu - \frac{3}{2}\beta$ and $Z_N(\cdot)$ is defined by (2.4).

4.3 The main results

This section contains the statement of the main theorems of the present chapter.

Understanding by *critical curve* of the model the boundary of the domain of parameters β and μ where the model exhibits subcritical behavior, this chapter makes a rigorous derivation of a subcriticality domain for an Ising model coupled to two-dimensional CDTs, and we also find a domain where the typical infinite-volume Gibbs measure does not exist. In Figure 4.3, we show a region where the critical curve of the model should be located. Formally, we define the critical curve as follow: We denote by $\mathcal{G}_{\beta, \mu}$ the set of *Gibbs measures* given by the closed convex hull of the set of weak limits:

$$\mathbb{P}^{\beta, \mu} = \lim_{N \rightarrow \infty} \mathbb{P}_N^{\beta, \mu}, \quad (4.16)$$

and define the domain of parameters where the weak limit Gibbs distribution exists

$$\Gamma = \{(\beta, \mu) \in \mathbb{R}_+^2 : \mathcal{G}_{\beta, \mu} \neq \emptyset\},$$

and domain where the weak limit Gibbs distribution exists and it is unique

$$\Gamma_1 = \{(\beta, \mu) \in \mathbb{R}_+^2 : |\mathcal{G}_{\beta, \mu}| = 1\}.$$

It is evident that $\Gamma_1 \subseteq \Gamma$. Thus, the critical curve γ_{cr} for the Ising model coupled to CDT is defined by

$$\gamma_{cr} = \partial\Gamma_1 \cap \mathbb{R}_+^2. \quad (4.17)$$

Another thermodynamic quantity, in this context, which we will use in order to study the Ising model on random CDTs, is the asymptotic *free energy*, which is defined as

$$\phi_N(\beta, \mu) = \frac{1}{N} \ln \Xi_N(\beta, \mu) \quad (4.18)$$

in the thermodynamic limit of $N \rightarrow \infty$, i.e., $\phi(\beta, \mu) = \lim_{N \rightarrow \infty} \phi_N(\beta, \mu)$ (this quantity is also sometimes called in the literature also free entropy density or pressure). The Ising model on CDT's can be consider as Ising model on random graph, in this case on CDT's, but in general, for a much wider class of random graph there is still no solution. The first Ising

model on a collection of all planar lattices (with coordination number 4) was defined and solved, using two-matrix model in the large N limit, by Kazakov in [Kaz86].

Note that, for $\beta = 0$ here one has

$$\begin{aligned}\phi_N(0, \mu) &= \frac{1}{N} \ln \left(\sum_{\mathbf{t}} \exp \left\{ -\mu \sum_{i=0}^{N-1} n(\mathbf{t}(i)) \right\} \sum_{\boldsymbol{\sigma}} 1 \right) \\ &= \frac{1}{N} \ln Z_N(\mu - \ln 2).\end{aligned}$$

Using (2.14), we show that $\phi_N(0, \mu)$ converges as $n \rightarrow \infty$ to $\ln \Lambda(\mu - \ln 2)$, i.e.,

$$\phi(0, \mu) = \lim_{N \rightarrow \infty} \phi_N(0, \mu) = \ln \Lambda(\mu - \ln 2).$$

Note also that, if $0 < \beta < \infty$, for any triangulation $\mathbf{t}$ the energy of any spin configuration $\boldsymbol{\sigma}$ on $\mathbf{t}$ will be bigger or equal than the energy of a pure configuration (all $+$'s or all $-$'s). i.e., $\mathbf{h}(\boldsymbol{\sigma}, \mathbf{t}) \geq -3/2n(\mathbf{t})$. Thus, for any $\beta > 0$ the inequality

$$\phi_N(\beta, \mu) < \frac{1}{N} \ln \left(\sum_{\mathbf{t}} e^{-(\mu - \frac{3}{2}\beta - \ln 2)n(\mathbf{t})} \right) = \frac{1}{N} \ln Z_N \left(\mu - \frac{3}{2}\beta - \ln 2 \right).$$

holds true. Therefore,

$$\phi(\beta, \mu) = \lim_{N \rightarrow \infty} \phi_N(\beta, \mu) \leq \ln \Lambda \left(\mu - \frac{3}{2}\beta - \ln 2 \right).$$

Hence, the inequality $\mu > \frac{3}{2}\beta + 2 \ln 2$ provides a sufficient condition for subcriticality of the Ising model under consideration. This bound permite to give an alternative definition of the critical curve for the model

$$\mu(\beta) = \gamma_{cr}(\beta) = \inf\{\mu > 0 : \phi(\beta, \mu) < \infty\}, \text{ for all } \beta \geq 0, \quad (4.19)$$

and give a first upper bound for the critical curve, i.e., $\gamma_{cr}(\beta) < \frac{3}{2}\beta + 2 \ln 2$. The previous bound in this paper is, of course, far from sharp, and it is of some interest to speculate on what the best-possible results might be. In [HYSZ13] was computed a better upper bound in low temperature.

Let $\lambda(\beta, \mu)$ be given by

$$\lambda(\beta, \mu) = c^2 (m^2 + 1) (\cosh 2\beta) \left(1 + \sqrt{1 - \frac{1}{(\cosh 2\beta)^2} \frac{(m^2 - 1)^2}{(m^2 + 1)^2}} \right) \quad (4.20)$$

where c and m are determined by

$$c = \frac{\exp(\beta - \mu)}{e^{2\beta}(1 - \exp(\beta - \mu))^2 - e^{-2\mu}} \quad (4.21)$$

$$m = e^{2\beta} + (1 - e^{4\beta}) \exp(-(\beta + \mu)), \quad (4.22)$$

Remember that identity (4.20) was defined in Chapter 3, Lemma 3.2.1.

We define the strictly increasing function

$$\psi(\beta) = \inf\{\mu \in \mathbb{R}^+ : \lambda(\beta, \mu) < 1\}, \quad \text{for } \beta > 0, \quad (4.23)$$

and the following set

$$\Sigma = \left\{ (\beta, \mu) \in \mathbb{R}_+^2 : \mu < \max \left\{ 2 \ln 2, \frac{3}{2} \ln(2 \sinh \beta) + \ln 2 \right\} \right\}.$$

Notice that $\psi(0) = \lim_{\beta \rightarrow 0^+} \psi(\beta) = 2 \ln 2$.

Let $\mathbf{t}_1, \dots, \mathbf{t}_k$ be triangulations of a single strip $\mathcal{S} \times [0, 1]$ and $\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_k$ be their corresponding spin configurations. Given $0 \leq i_1 < \dots < i_k \leq N - 1$ we define the finite-dimensional cylinder $\mathcal{C}_{i_1, \dots, i_k} = \mathcal{C}_{i_1, \dots, i_k}^{(\mathbf{t}_1, \boldsymbol{\sigma}_1), \dots, (\mathbf{t}_k, \boldsymbol{\sigma}_k)}$ as follows

$$\mathcal{C}_{i_1, \dots, i_k} = \{(\mathbf{t}, \boldsymbol{\sigma}) : (\mathbf{t}(i_1), \boldsymbol{\sigma}(i_1)) = (\mathbf{t}_1, \boldsymbol{\sigma}_1), \dots, (\mathbf{t}(i_k), \boldsymbol{\sigma}(i_k)) = (\mathbf{t}_k, \boldsymbol{\sigma}_k)\} \quad (4.24)$$

Theorem 4.3.1. *If $(\beta, \mu) \in \Sigma$ then there exists $N_0 \in \mathbb{N}$ such that the partition function $\Xi_N(\beta, \mu) = +\infty$ whenever $N > N_0$. Moreover, the Gibbs distribution $\mathbb{P}_N^{\beta, \mu}$ with periodic boundary conditions cannot be defined by using the standard formula with $\Xi_N(\beta, \mu)$ as a normalising denominator, consequently, there is no limiting probability measure $\mathbb{P}^{\beta, \mu}$ as $N \rightarrow \infty$. Formally, for any finite-dimensional cylinder $\mathcal{C}_{i_1, \dots, i_k}$ we obtain $\mathbb{P}_N^{\beta, \mu}(\mathcal{C}_{i_1, \dots, i_k}) = 0$ whenever $N > N_0 \geq \max\{i_1, \dots, i_k\}$.*

Let β_1^*, β_2^* be positive solution of equations

$$2 \ln 2 = \frac{3}{2} \ln(2 \sinh \beta) + \ln 2 \quad (4.25)$$

and

$$\frac{3}{2} \beta + 2 \ln 2 = \psi(\beta), \quad (4.26)$$

respectively. Together with results from previous chapter (see [HYSZ13] for more details), Theorem 4.3.1 provides two-side bounds for the critical curve.

Theorem 4.3.2. *The critical curve γ_{cr} satisfies the following inequalities.*

1. If $(\beta, \mu(\beta)) \in \gamma_{cr}$ and $0 < \beta < \beta_1^*$, then

$$2 \ln 2 \leq \mu < \psi(\beta).$$

The above bound implies that: For any sequence $\{(\beta_k, \mu_k)\} \subset \gamma_{cr}$ such that $\beta_k \rightarrow 0$, then $\lim_{k \rightarrow \infty} \mu_k = 2 \ln 2$.

2. If $(\beta, \mu) \in \gamma_{cr}$ and $\beta_1^* \leq \beta < \beta_2^*$, then

$$\frac{3}{2} \ln(2 \sinh \beta) + \ln 2 \leq \mu < \psi(\beta).$$

3. If $(\beta, \mu) \in \gamma_{cr}$ and $\beta_2^* \leq \beta < \infty$, then

$$\frac{3}{2} \ln(2 \sinh \beta) + \ln 2 \leq \mu < \frac{3}{2} \beta + 2 \ln 2.$$

As a by-product of the proof of Theorems 4.3.1 and 4.3.2, using the FK representation, we also find a lower and upper bound for the infinite-volume free energy. Define the following functions,

$$f_1(\beta) = \max \left\{ \ln 2, \frac{3}{2} \ln(2 \sinh \beta) \right\},$$

and

$$f_2(\beta) = \min \left\{ \psi(\beta) - \ln 2, \frac{3}{2} \beta + \ln 2 \right\},$$

Corollary 4.3.1. For (β, μ) such that $\mu > f_2(\beta)$, then the free energy for the infinite-volume Ising model coupled to CDTs is finite and satisfies the following inequality,

$$\ln \Lambda(\mu - f_2(\beta)) \leq \phi(\beta, \mu) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu) \leq \ln \Lambda(\mu - f_1(\beta)).$$

Here $\Lambda(s)$ is given by (2.10).

Finally, for each $N \in \mathbb{N}$, we define the follow set in $\mathbb{R}_+^2$

$$\Gamma_N = \{(\beta, \mu) \in \mathbb{R}_+^2 : \mathbf{K}^N \text{ is of trace class in } \ell_{T-C}^2\}, \quad (4.27)$$

$$\Gamma^- = \bigcap_{N \in \mathbb{N}} \Gamma_N \quad \text{and} \quad \Gamma^+ = \bigcup_{N \in \mathbb{N}} \Gamma_N. \quad (4.28)$$

Obviously, $\Gamma^- \subset \Gamma_N \subset \Gamma^+$, for any $N \geq 1$, and $\mathbb{P}_N^{\beta, \mu}$ there exist on Γ_N . In order to each $N \geq 1$, we define the N -strip functions f_N associated with the N -strip partition function for the Ising model coupled to CDTs as

$$f_N(\beta) = \inf \{\mu \in \mathbb{R}_+^2 : (\beta, \mu) \in \Gamma_N\} \quad \text{for } \beta > 0. \quad (4.29)$$

Figure 4.3: The area above the minimum of the dotted curve I (graph of the function ψ defined in (4.23)) and dash-dotted line II is where the limiting Gibbs probability measure exists and is unique. The critical curve lies in the region below the dotted curve I and dash-dotted line II but above the continuous curve III and dashed line IV.

Notice that $\lim_{\beta \rightarrow 0} f_N(\beta) = 2 \ln 2$ for all $N \leq 1$.

According to Theorem 3.2.2 in chapter 3, the Theorem 4.3.2 and Proposition 4.4.5, given in the Section 4.4.3, implies a similar version of the Theorem 3.2.2, as following.

Theorem 4.3.3. For $(\beta, \mu) \in \Gamma^+ = \{(\beta, \mu) \in \mathbb{R}_+^2 : \mu > f_{T-C}(\beta)\}$, the following limit holds:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu) = \ln \Lambda(\beta, \mu), \quad (4.30)$$

where $\Lambda(\beta, \mu)$ is the maximal eigenvalue of $\mathbf{K}$ and $\mathbf{K}^T$ in ℓ_{T-C}^2 and f_{T-C} is pointwise limit of the family of functions $\{f_N\}$. Consequently, as $N \rightarrow \infty$ the N -strip Gibbs measure $\mathbb{P}_N^{\beta, \mu}$ converges weakly to a limiting probability distribution $\mathbb{P}^{\beta, \mu}$.

4.4 Proof of Theorem 4.3.1 and 4.3.2

The proof is based on finding of upper and lower bounds for the functions f_N , introduced in (4.29), using the FK representation (4.15) and the asymptotic behaviour of the partition function $Z_N(\cdot)$ for pure CDTs with periodical boundary condition. These bounds joint with the Proposition 4.4.2, Proposition 4.4.3 and Proposition 4.4.4, established bounds for the critical curve.

First, we establish a trivial lower bound for the model. This bound appear in part of the

lower bound established on Theorem 4.3.2.

4.4.1 Trivial lower bound using Griffith's Inequalities

The aim of this section is to understand the behaviour of the function $\beta \rightarrow \Xi_N(\beta, \mu)$ for all $N \in \mathbb{N}$ and use this behaviour to obtain a don't sharp lower bound for the critical curve.

Proposition 4.4.1. *If $\beta_1 > \beta_2$, then*

$$\mathcal{Z}_N^{\beta_1, \mathbf{t}} \geq \mathcal{Z}_N^{\beta_2, \mathbf{t}},$$

for all triangulation $\mathbf{t} \in \mathbb{LT}_N$.

Proof. Note that, by definition of the partition function $\mathcal{Z}_N^{\beta, \mathbf{t}}$, we obtain

$$\frac{\partial \mathcal{Z}_N^{\beta, \mathbf{t}}}{\partial \beta} = \sum_{\boldsymbol{\sigma}} \sum_{\langle t, t' \rangle} \sigma_t \sigma_{t'} e^{\beta \sum_{\langle t, t' \rangle} \sigma_t \sigma_{t'}} = \mathcal{Z}_N^{\beta, \mathbf{t}} \sum_{\boldsymbol{\sigma}} \sum_{\langle t, t' \rangle} \sigma_t \sigma_{t'} \mu_{\beta}^{\mathbf{t}}(\boldsymbol{\sigma}).$$

Here $\mu_{\beta}^{\mathbf{t}}$ denote the Gibbs measure on $\mathbf{t}$, defined as follows

$$\mu_{\beta}^{\mathbf{t}}(\boldsymbol{\sigma}) = \frac{1}{\mathcal{Z}_N^{\beta, \mathbf{t}}} \exp \left\{ -\beta \mathbf{h}(\boldsymbol{\sigma}, \mathbf{t}) \right\},$$

where

$$\mathbf{h}(\boldsymbol{\sigma}, \mathbf{t}) = - \sum_{\langle t, t' \rangle} \sigma_t \sigma_{t'}.$$

Thus, we obtain the following identity

$$\frac{\partial \mathcal{Z}_N^{\beta, \mathbf{t}}}{\partial \beta} = \mathcal{Z}_N^{\beta, \mathbf{t}} \sum_{\langle t, t' \rangle} \int \sigma_{\{t, t'\}} \mu_{\beta}^{\mathbf{t}}(\boldsymbol{\sigma}) = \mathcal{Z}_N^{\beta, \mathbf{t}} \sum_{\langle t, t' \rangle} \mu_{\beta}^{\mathbf{t}}(\sigma_{\{t, t'\}}).$$

Proof of the Proposition 4.4.1 follows from Griffiths inequality. $\square$

Theorem 4.4.1 (Griffiths inequality [Gri72], [Lig85]). *Suppose that S is a finite. Let μ be the corresponding Gibbs measure. Then*

$$\int \sigma_A d\mu \geq 0, \text{ for all } A \subset S, \quad (4.31)$$

where $\sigma_A(\boldsymbol{\sigma}) = \prod_{i \in A} \sigma_i$.

Now, in particular, we have which $\mathcal{Z}_N^{\beta, \mathbf{t}} \geq \mathcal{Z}_N^{0, \mathbf{t}} = e^{n(\mathbf{t}) \ln 2}$. Therefore, $\Xi_N(\beta, \mu) \geq \mathcal{Z}_N(\mu - \ln 2)$, and using Proposition 2.2.1, we obtain

$$\gamma_{cr}(\beta) > 2 \ln 2, \text{ for all } \beta > 0. \quad (4.32)$$

Thus, the free energy satisfy $\phi(\beta, \mu) \geq \ln \Lambda(\mu - \ln 2)$. Remember that $\gamma_{cr}(0) = 2 \ln 2$ and $\phi(0, \mu) = \ln \Lambda(\mu - \ln 2)$.

4.4.2 Proof of Theorem 4.3.1

We need two preparatory results. Let $\mathbf{t}$ be a Lorentzian CDT on the cylinder C_N . Given $1 \leq i \leq n(\mathbf{t})$, we define the sets

$$\mathbf{\Pi}_i = \{\text{all realization } \xi \text{ of process } \{\xi_{\langle t, t' \rangle}\} \text{ such that } k(\xi) = i\}. \quad (4.33)$$

Thus, we have the following representation of (4.14)

$$\mathcal{Z}_N^{\beta, \mathbf{t}} = e^{\frac{3}{2}\beta n(\mathbf{t})} \sum_{k=1}^{n(\mathbf{t})} 2^k \mathbb{P}_{\beta, \mathbf{t}}(\mathbf{\Pi}_k) \quad (4.34)$$

Let $\xi \in \mathbf{\Pi}_k$ and let $\{\mathcal{C}_l\}_{l=1}^k$ be the corresponding cluster decomposition of the set $\Delta(\mathbf{t})$ (In fact, each cluster $\mathcal{C}_l$ is a subgraph of $\mathbf{t}$, i.e. $\mathcal{C}_l = (V_l, E_l)$). Notice that $\mathcal{C}_l = \mathcal{C}_l(\xi)$ and that this cluster decomposition $\{\mathcal{C}_l\}_{l=1}^k$ includes isolated vertices, i.e., the cluster $\mathcal{C}_l = \{t\}$, for some triangle t . Denote by $\eta_l = |V_l|$ and $\kappa_l = |E_l|$ the number of vertices (triangles) in cluster $\mathcal{C}_l$ and the number of edges in $\mathcal{C}_l$, respectively. Note that κ_l depends on the geometry of cluster $\mathcal{C}_l(\xi)$ and $\sum_{l=1}^k \eta_l = n(\mathbf{t})$.

Denote by $\pi(\mathbf{t})$ the set of all unordered partitions of $\Delta(\mathbf{t})$, i.e. an element of $\pi(\mathbf{t})$ is an unordered n -ple $\{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n\}$ of maximal connected subset $\mathcal{C}_i$, with $1 \leq n \leq n(\mathbf{t})$, such that, for $i, j \in I_n = \{1, 2, \dots, n\}$, $\mathcal{C}_i \subset \Delta(\mathbf{t})$, $\mathcal{C}_i \neq \emptyset$, $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ and $\cup \mathcal{C}_i = \Delta(\mathbf{t})$. Given a connected subset $\mathcal{C}$ of $\Delta(\mathbf{t})$, we define the set of spanning subgraph of $\mathcal{C}$, denoted by $Span(\mathcal{C})$ as follow

$$Span(\mathcal{C}) = \{\gamma \text{ subgraph of } \mathcal{C} : \text{supp } \gamma = \mathcal{C}\}.$$

We recall that $\text{supp } \gamma$ denotes the suport of γ , i.e. the vertices in $\Delta(\mathbf{t})$ where γ es supported. Thus, a spanning subgraph $\gamma \in Span(\mathcal{C})$ is a is a subgraph γ with $V(\gamma) = V(\mathcal{C})$, i.e. γ and $\mathcal{C}$ have exactly the same vertex set. The probability that two nearest neighbor triangles t, t' are linked is given by $\mathbb{P}_{\beta, \mathbf{t}}(t \leftrightarrow t') = 1 - e^{-2\beta}$. Then, denoting $p := 1 - e^{-2\beta}$, we can rewrite the partition function on $\mathbf{t}$ as

$$\mathcal{Z}_N^{\beta, \mathbf{t}} = e^{\frac{3}{2}\beta n(\mathbf{t})} (1-p)^{\frac{3}{2}n(\mathbf{t})} \sum_{k=1}^{n(\mathbf{t})} 2^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\mathbf{t})} \rho(\mathcal{C}_1) \cdots \rho(\mathcal{C}_k), \quad (4.35)$$

where

$$\rho(\mathcal{C}) = \sum_{\gamma \in Span(\mathcal{C})} \left(\frac{p}{1-p} \right)^{|E(\gamma)|}.$$

Here $|E(\gamma)|$ denote the number of edges in the graph γ . Notice that $|E(\gamma)| \geq 0$.

A convenient expansion parameter for our analyse is $u = \frac{p}{1-p} \in [0, \infty)$, we now look for

an expansion of the function $\rho(\mathcal{C})$

$$\rho(\mathcal{C}) = \sum_{\gamma \in \text{Span}(\mathcal{C})} u^{|E(\gamma)|},$$

for some values of u , but we first give some definitions and lemmas about abstract graphs which will be useful below.

Lemma 4.4.1. *Let d_i the degree of each vertex on a finite graph $G = (V, E)$. Thus*

$$\sum_{i=1}^{|V|} d_i = 2|E|.$$

Theorem 4.4.2. *Each connected graph has a spanning tree, that is, a spanning graph that is a tree.*

Corollary 4.4.1. *For each connected graph $G = (V, E)$, $|E| \geq |V| - 1$. Moreover, a connected graph G is a tree if and only if $|E| = |V| - 1$.*

Let $\gamma \in \text{Span}(\mathcal{C})$ for some connected subset $\mathcal{C}$ of $\Delta(\mathbf{t})$. Using Lemma 4.4.1 and Corollary 4.4.1, we obtain the lower and upper bound for any γ ,

$$|\mathcal{C}| - 1 \leq |E(\gamma)| \leq \frac{3}{2}|\mathcal{C}| - 1, \text{ if } \mathcal{C} \subsetneq \Delta(\mathbf{t}). \quad (4.36)$$

If $\mathcal{C} = \Delta(\mathbf{t})$ and we consider the subgraph $\gamma = (V(\gamma), E(\gamma))$, with $|V(\gamma)| = n(\mathbf{t})$ vertices and $|E(\gamma)| = \frac{3}{2}n(\mathbf{t})$ edges, we have the equality $|E(\gamma)| = \frac{3}{2}|\mathcal{C}| = \frac{3}{2}n(\mathbf{t})$. Notice that there exists one graph γ with these properties, the triangulation $\mathbf{t}$ with all edges.

Using the previous results, we consider two special case of interest.

The case $u = \frac{p}{1-p} > 1$. Using the inequality (4.36) and the condition $u > 1$, we obtain the triavial expression

$$\left(\frac{p}{1-p}\right)^{|\mathcal{C}|-1} \leq \left(\frac{p}{1-p}\right)^{|E(\gamma)|} \leq \left(\frac{p}{1-p}\right)^{\frac{3}{2}|\mathcal{C}|-1},$$

for $\gamma \neq \mathbf{t}$. Thus, if $\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\mathbf{t})$, with $k \geq 2$ we obtain

$$\left(\frac{p}{1-p}\right)^{n(\mathbf{t})-k} \prod_{i=1}^k f(\mathcal{C}_i) \leq \prod_{i=1}^k \rho(\mathcal{C}_i) \leq \left(\frac{p}{1-p}\right)^{\frac{3}{2}n(\mathbf{t})-k} \prod_{i=1}^k f(\mathcal{C}_i), \quad (4.37)$$

where $f(\mathcal{C}) = \sum_{\gamma \in \text{Span}(\mathcal{C})} 1$ and $k \geq 2$. The case $k = 1$ is treated separately later.

Additionally, we write the partition function $\mathcal{Z}_N^{\beta, \mathbf{t}}$ as follow

$$\mathcal{Z}_N^{\beta, \mathbf{t}} = e^{\frac{3}{2}\beta n(\mathbf{t})} (1-p)^{\frac{3}{2}n(\mathbf{t})} \left(2\rho(\Delta(\mathbf{t})) + \sum_{k=2}^{n(\mathbf{t})} 2^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\mathbf{t})} \rho(\mathcal{C}_1) \cdots \rho(\mathcal{C}_k) \right).$$

Remember that $\rho(\Delta(\mathbf{t})) = \sum_{\gamma \in \text{Span}(\Delta(\mathbf{t}))} u^{|E(\gamma)|}$. Using inequality (4.37), we obtain the upper bound for the partition function $\mathcal{Z}_N^{\beta, \mathbf{t}}$ on $\mathbf{t}$,

$$\mathcal{Z}_N^{\beta, \mathbf{t}} \leq e^{-\frac{3}{2}\beta n(\mathbf{t})} \left\{ 2\rho(\Delta(\mathbf{t})) + \left(\frac{p}{1-p} \right)^{\frac{3}{2}n(\mathbf{t})} \sum_{k=2}^{n(\mathbf{t})} \left(\frac{2(1-p)}{p} \right)^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\mathbf{t})} \prod_{i=1}^k f(\mathcal{C}_i) \right\} \quad (4.38)$$

Similarly, using inequality (4.37), we obtain the lower bound

$$e^{-\frac{3}{2}\beta n(\mathbf{t})} \left\{ 2\rho(\Delta(\mathbf{t})) + \left(\frac{p}{1-p} \right)^{n(\mathbf{t})} \sum_{k=2}^{n(\mathbf{t})} \left(\frac{2(1-p)}{p} \right)^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\mathbf{t})} \prod_{i=1}^k f(\mathcal{C}_i) \right\} \leq \mathcal{Z}_N^{\beta, \mathbf{t}} \quad (4.39)$$

Notice that in the case $k \geq 2$, each cluster $\mathcal{C}_i$ has at least two vertices with degree equal to 2. Thus, if $\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\mathbf{t})$, we have the upper bound for $f(\mathcal{C}_i)$,

$$f(\mathcal{C}_i) = \sum_{\gamma \in \text{Span}(\mathcal{C}_i)} 1 \leq \sum_{k=|\mathcal{C}_i|-1}^{\frac{3}{2}|\mathcal{C}_i|-1} \binom{\frac{3}{2}|\mathcal{C}_i|-1}{k} \leq 2^{\frac{3}{2}|\mathcal{C}_i|-1}.$$

Then,

$$\prod_{i=1}^k f(\mathcal{C}_i) \leq 2^{\frac{3}{2}n(\mathbf{t})-k}.$$

The case $u = \frac{p}{1-p} < 1$. Using the inequality (4.36) and the condition $u < 1$, we obtain the triavial expression

$$\left(\frac{p}{1-p} \right)^{\frac{3}{2}|\mathcal{C}|-1} \leq \left(\frac{p}{1-p} \right)^{|E(\gamma)|} \leq \left(\frac{p}{1-p} \right)^{|\mathcal{C}|-1},$$

for $\gamma \neq \mathbf{t}$. Thus, if $\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\mathbf{t})$, with $k \geq 2$ we obtain

$$\left(\frac{p}{1-p} \right)^{\frac{3}{2}n(\mathbf{t})-k} \prod_{i=1}^k f(\mathcal{C}_i) \leq \prod_{i=1}^k \rho(\mathcal{C}_i) \leq \left(\frac{p}{1-p} \right)^{n(\mathbf{t})-k} \prod_{i=1}^k f(\mathcal{C}_i), \quad (4.40)$$

Using inequality (4.40), we obtain the upper bound for the partition function $\mathcal{Z}_N^{\beta, \mathbf{t}}$ on

$\mathbf{t}$,

$$\mathcal{Z}_N^{\beta, \mathbf{t}} \leq e^{-\frac{3}{2}\beta n(\mathbf{t})} \left\{ 2\rho(\Delta(\mathbf{t})) + \left(\frac{p}{1-p}\right)^{n(\mathbf{t})} \sum_{k=2}^{n(\mathbf{t})} \left(\frac{2(1-p)}{p}\right)^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\mathbf{t})} \prod_{i=1}^k f(\mathcal{C}_i) \right\} \quad (4.41)$$

Similarly, using inequality (4.37), we obtain the lower bound

$$e^{-\frac{3}{2}\beta n(\mathbf{t})} \left\{ 2\rho(\Delta(\mathbf{t})) + \left(\frac{p}{1-p}\right)^{\frac{3}{2}n(\mathbf{t})} \sum_{k=2}^{n(\mathbf{t})} \left(\frac{2(1-p)}{p}\right)^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\mathbf{t})} \prod_{i=1}^k f(\mathcal{C}_i) \right\} \leq \mathcal{Z}_N^{\beta, \mathbf{t}} \quad (4.42)$$

In order to obtain *lower bounds* for the critical curve, we employ the inequalities (4.39) and (4.42), and we consider again two case:

Lower bound over condition $u = \frac{p}{1-p} > 1$. This condition is equivalent to $\beta > \frac{\ln 2}{2}$. For this case we use the inequality (4.39) and the trivial lower bound $1 \leq f(\mathcal{C})$, for all $\mathcal{C} \subset \Delta(\mathbf{t})$,

$$\begin{aligned} \mathcal{Z}_N^{\beta, \mathbf{t}} &\geq e^{\frac{3}{2}\beta n(\mathbf{t})} (1-p)^{\frac{3}{2}n(\mathbf{t})} \left\{ 2\rho(\Delta(\mathbf{t})) + \left(\frac{p}{1-p}\right)^{n(\mathbf{t})} \sum_{k=2}^{n(\mathbf{t})} \left(\frac{2(1-p)}{p}\right)^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\mathbf{t})} 1 \right\} \\ &\geq e^{\frac{3}{2}\beta n(\mathbf{t})} (1-p)^{\frac{3}{2}n(\mathbf{t})} \left\{ 2\rho(\Delta(\mathbf{t})) + \left(\frac{p}{1-p}\right)^{n(\mathbf{t})} \sum_{k=1}^{n(\mathbf{t})-1} \left(\frac{2(1-p)}{p}\right)^{k+1} \binom{n(\mathbf{t})-1}{k} \right\} \\ &\geq e^{\frac{3}{2}\beta n(\mathbf{t})} (1-p)^{\frac{3}{2}n(\mathbf{t})} \left\{ 2\rho(\Delta(\mathbf{t})) + 2 \left(\frac{p}{1-p}\right)^{n(\mathbf{t})-1} \left(\left(1 + \frac{2(1-p)}{p}\right)^{n(\mathbf{t})-1} - 1 \right) \right\} \\ &\geq e^{\frac{3}{2}\beta n(\mathbf{t})} (1-p)^{\frac{3}{2}n(\mathbf{t})} \left\{ 2\rho(\Delta(\mathbf{t})) - 2 \left(\frac{p}{1-p}\right)^{n(\mathbf{t})-1} + 2 \left(\frac{2-p}{1-p}\right)^{n(\mathbf{t})-1} \right\} \end{aligned}$$

Now, notice that

$$\begin{aligned} \rho(\Delta(\mathbf{t})) = \sum_{\gamma \in \text{Span}(\Delta(\mathbf{t}))} \left(\frac{p}{1-p} \right)^{|E(\gamma)|} &= \left(\frac{p}{1-p} \right)^{\frac{3}{2}n(\mathbf{t})} + \sum_{\substack{\gamma \in \text{Span}(\Delta(\mathbf{t})): \\ |E(\gamma)| < \frac{3}{2}n(\mathbf{t})}} \left(\frac{p}{1-p} \right)^{|E(\gamma)|} \\ &> \left(\frac{p}{1-p} \right)^{\frac{3}{2}n(\mathbf{t})}. \end{aligned}$$

Using previous inequality, we obtain the following lower bounds for the partition function of the Ising model on triangulation $\mathbf{t}$

$$\mathcal{Z}_N^{\beta, \mathbf{t}} \geq 2e^{\frac{3}{2}\beta n(\mathbf{t})} p^{\frac{3}{2}n(\mathbf{t})} - 2e^{\frac{3}{2}\beta n(\mathbf{t})} p^{n(\mathbf{t})-1} (1-p)^{\frac{1}{2}n(\mathbf{t})+1} + 2e^{\frac{3}{2}\beta n(\mathbf{t})} (2-p)^{n(\mathbf{t})-1} (1-p)^{\frac{1}{2}n(\mathbf{t})+1} \quad (4.43)$$

Using (4.15), the lower bound in (4.43) provides the following lower bound to $\Xi_N(\beta, \mu)$,

$$\begin{aligned} \Xi_N(\beta, \mu) &\geq 2Z_N \left(\mu - \frac{3}{2}\beta - \frac{3}{2}\ln p \right) - 2Z_N \left(\mu - \frac{3}{2}\beta - \ln p \sqrt{1-p} \right) + \\ &2Z_N \left(\mu - \frac{3}{2}\beta - \ln(2-p) \sqrt{1-p} \right). \end{aligned} \quad (4.44)$$

Thus, using the asymptotic property given in Proposition (2.2.1) and Remark 2.2.2, we obtain that the partition function $\Xi_N(\beta, \mu)$ there exists if

$$\mu > \frac{3}{2}\beta + \ln 2 + \frac{3}{2}\ln(1 - e^{-2\beta}) \ln \left(\cos \frac{\pi}{N+1} \right) \quad \text{and} \quad \beta > \frac{\ln 2}{2}.$$

Letting $N \rightarrow \infty$ we obtain the following proposition.

Proposition 4.4.2. *If $(\beta, \mu) \in \mathbb{R}_+^2$ such that*

$$\mu < \frac{3}{2}\beta + \ln 2 + \frac{3}{2}\ln(1 - e^{-2\beta}) \quad \text{and} \quad \beta > \frac{\ln 2}{2}$$

then there exists $N_0 \in \mathbb{N}$ such that the partition function $\Xi_N(\beta, \mu) = +\infty$ whenever $N > N_0$. Moreover, the Gibbs distribution $\mathbb{P}_N^{\beta, \mu}$ with periodic boundary conditions cannot be defined by using the standard formula with $\Xi_N(\beta, \mu)$ as a normalising denominator, consequently, there is no limiting probability measure $\mathbb{P}^{\beta, \mu}$ as $N \rightarrow \infty$. Furthermore, for any finite-dimensional cylinder $\mathcal{C}_{i_1, \dots, i_k}$ we obtain $\mathbb{P}_N^{\beta, \mu}(\mathcal{C}_{i_1, \dots, i_k}) = 0$ whenever $N > N_0 \geq \max\{i_1, \dots, i_k\}$.

Lower bound over condition $u = \frac{p}{1-p} < 1$. This condition is equivalent to $\beta < \frac{\ln 2}{2}$. Using

the same technique, we obtain the lower bound

$$\mathcal{Z}_N^{\beta, \mathbf{t}} \geq 2e^{\frac{3}{2}\beta n(\mathbf{t})} p^{\frac{3}{2}n(\mathbf{t})} - 2(1-p)e^{\frac{3}{2}\beta n(\mathbf{t})} p^{\frac{3}{2}n(\mathbf{t})-1} + 2(1-p)e^{\frac{3}{2}\beta n(\mathbf{t})} p^{\frac{1}{2}n(\mathbf{t})} (2-p)^{n(\mathbf{t})-1}. \quad (4.45)$$

Thus, a lower bound for $\Xi_N(\beta, \mu)$ on $\beta < \frac{\ln 2}{2}$ is

$$\begin{aligned} \Xi_N(\beta, \mu) \geq 2Z_N \left(\mu - \frac{3}{2}\beta - \frac{3}{2}\ln p \right) - \frac{2(1-p)}{2-p} Z_N \left(\mu - \frac{3}{2}\beta - \frac{3}{2}\ln p \right) + \\ \frac{2(1-p)}{2-p} Z_N \left(\mu - \frac{3}{2}\beta - \frac{1}{2}\ln p - \ln(2-p) \right). \end{aligned} \quad (4.46)$$

Thus, using the asymptotic property given in Proposition (2.2.1) and Remark 2.2.2, we obtain that the partition function $\Xi_N(\beta, \mu)$ there exists if

$$\mu > \frac{3}{2}\beta + \ln 2 + \frac{3}{2}\ln(1 - e^{-2\beta}) \ln \left(\cos \frac{\pi}{N+1} \right) \quad \text{and} \quad \beta > \frac{\ln 2}{2}.$$

Letting $N \rightarrow \infty$ we obtain the following proposition.

Proposition 4.4.3. *If $(\beta, \mu) \in \mathbb{R}_+^2$ such that*

$$\mu < \frac{3}{2}\beta + \ln 2 + \frac{1}{2}\ln(1 - e^{-2\beta}) + \ln(1 + e^{-2\beta}) \quad \text{and} \quad \beta < \frac{\ln 2}{2}$$

then there exists $N_0 \in \mathbb{N}$ such that the partition function $\Xi_N(\beta, \mu) = +\infty$ whenever $N > N_0$. Moreover, the Gibbs distribution $\mathbb{P}_N^{\beta, \mu}$ with periodic boundary conditions cannot be defined by using the standard formula with $\Xi_N(\beta, \mu)$ as a normalising denominator, consequently, there is no limiting probability measure $\mathbb{P}^{\beta, \mu}$ as $N \rightarrow \infty$. Furthermore, for any finite-dimensional cylinder $\mathcal{C}_{i_1, \dots, i_k}$ we obtain $\mathbb{P}_N^{\beta, \mu}(\mathcal{C}_{i_1, \dots, i_k}) = 0$ whenever $N > N_0 \geq \max\{i_1, \dots, i_k\}$.

unfortunately, Proposition 4.4.3 don't improve the lower bound given by the Griffiths inequality.

Proof of Theorem 4.3.1. The proof follows immediately from Proposition 4.4.2 and the lower bound given by the Griffiths inequality. $\square$

4.4.3 Proof of Theorem 4.3.2

The proof of Theorem 4.3.2 relies on two additional observations. These are:

- (1) Upper bounds for the functions f_N and existence of the pointwise limit $\lim_{N \rightarrow \infty} f_N = f_{T-C}$.

(2) The fact that graph of f_{T-C} provides an upper bound for the critical curve.

Consequently, as by-product of Chapter 3 (see [HYSZ13]), we obtain the following assertions.

Proposition 4.4.4. *For all $N \in \mathbb{N}$, the following property of functions f_N is fulfilled:*

1. *If $0 < \beta < \beta_2^*$, then*

$$f_N(\beta) \leq \psi(\beta), \quad (4.47)$$

where β_2^ is positive solution of Eqn (4.26) and function ψ is introduced in Eqn (4.23).*

2. *If $\beta_2^* \leq \beta < \infty$, then*

$$f_N(\beta) \leq \frac{3}{2}\beta + 2 \ln 2. \quad (4.48)$$

Proposition 4.4.5. *Functions f_N converge pointwise:*

$$f_{T-C}(\beta) := \lim_{N \rightarrow \infty} f_N(\beta) \quad \text{for } \beta > 0. \quad (4.49)$$

Combining (4.47), (4.48) with Proposition 4.4.5 and letting $N \rightarrow \infty$, we obtain the desired upper bound for the limit function f_{T-C}

$$\left\{ \begin{array}{ll} f_{T-C}(\beta) \leq \psi(\beta) & \text{if } 0 < \beta < \beta_2^* \\ f_{T-C}(\beta) \leq \frac{3}{2}\beta + 2 \ln 2 & \text{if } \beta_2^* \leq \beta < \infty. \end{array} \right. \quad (4.50)$$

Since the graph of f_{T-C} lies above the critical curve, the right-hand side of (4.50) provides an upper bound for the critical curve.

Proof of Theorem 4.3.2. The upper bound of Theorem 4.3.2 is consequence of Eqn (4.50). The lower bound is consequence of Proposition 4.4.2 and the lower bound given by the Griffiths inequality. This concludes the proof of Theorem 4.3.2. $\square$

4.5 Discussion and outlook

This chapter makes a first step towards determining the critical curve for an Ising-type model coupled to two-dimensional causal dynamical triangulations (CDT). In doing so we employ the FK-representation of the Ising model on causal triangulations.

Compute the critical curve (point) for models in statistical mechanics was always a challenge problem. In this case we have two parameters, μ being the cosmological constant,

where each triangle has weight logical constant, where each triangle has weight (Boltzman factor) $e^{-\mu}$, and β the inverse of temperature. Spin system with quantum geometry interact simultaneous and make it difficult to find an analytical solution. For example, compute the partition function $\mathcal{Z}_N^{\beta, \mathbf{t}}$ using (4.34) or (4.35) for any triangulation $\mathbf{t} \in \mathbb{LT}_N$, is very difficult. A possible solution for this problem is study the model at high and low temperature. In these two case the triangulations have same behaviour and this will permite obtain a same asymptotic formulas for the $\mathcal{Z}_N^{\beta, \mathbf{t}}$ almost surely $\mathbf{t}$. This issue needs a further study.

The previous bound in this chapter is, of course, far from sharp, and it is of some interest speculate on what the best-possible results might be.

Physical considerations suggest that the typical triangulation in the region $\mu > \lambda_{cr}(\beta)$ is the same as in the pure CDT for $\mu > \ln 2$ (triangulations with bounded diameter), and which over the critical curve, that is, for $\mu = \lambda_{cr}(\beta)$, the typical triangulation has Hausdorff dimension d_H equal to 2, and which the Ising model has a kind of phase transition at point $\beta = \beta_{cr}$. In the region $\mu < \lambda_{cr}(\beta)$ the model cannot be defined.

A plausible conjecture is that the boundary of the critical domain coincides with the locus of points (β, μ) where $\mu = \frac{3}{2}\beta + \ln 2$, for $\beta > \beta_{cr}$, and that the critical curve is non differentiable only at a unique point $\beta = \beta_{cr} \in (0, \infty)$. This direction also requires further research.

We believe also that compute the exactly critical curve for the model require developing new techniques.

Chapter 5

Potts model coupled to CDTs and FK representation

In this chapter using a natural generalization of Ising model, we extend results from before chapters for the (annealed) classical Ising model coupled to two-dimensional causal dynamical triangulations. Such generalization is called of *Potts model*. Whereas in Ising systems the spins on two different values, in the q -state Potts model q distinct values, represent by the elements of the set $\{1, \dots, q\}$, are allowed on any vertex from the triangulation $\mathbf{t}$.

In Chapter 3 and 4, the Ising model was defined putting spins on any triangles (faces), but it is equivalent to put spins on any vertex of dual triangulation, defined in Section 5.2.2. Using duality relation on a torus (periodic boundary condition), we provide a relation between the free energy of Potts model coupled to CDTs and Potts model coupled to DUAL CDTs. Additionally, using the high temperature expansion (and duality relation), we determined a region where the critical curve can be located. This bound serves for Ising model case, and improves the bounds found in the before chapters (Chapter 3: Theorem 4.3.1 and Theorem 4.3.2. Chapter 4: Lemma 3.2.1 and Theorem 3.2.2).

5.1 Introduction and main results of this chapter

A causal dynamical triangulation (CDT), introduced by Ambjørn and Loll (see [AL98]) constitute attempts to provide a meaning to formal expressions appearing in the path integral quantisation of gravity (see [ADJ97], [AJ06] for an overview). The idea is to regularise the path integral by approximating the geometries emerging in the integration by CDTs. As a result, the path integral over geometries is replaced with a sum over all possible triangulations where each configuration is weighted by a Boltzmann factor $e^{-\mu|T|}$, with $|T|$ standing for the size of the triangulation and μ being the cosmological constant.

From a physical point of view it is interesting to study various models of matter, such as the q -state Potts model, coupled to the CDT.

The goal of this chapter is define the q -state Potts model coupled to CDTs and will use the FK representation for study this model.

In this case, the calculation of the partition function also reduces to a combinatorial problem. For the 2-state Potts model (Ising model) coupled to a CDT some progress has been recently made on existence of Gibbs measures and phase transitions (see [AAL99], [BL07], [HYSZ13] and [CH14] for details). In particular, using transfer matrix methods, the Krein-Rutman theory and FK representation for the Ising model, [CH14] provides a region in the quadrant of parameters $\beta, \mu > 0$ where the infinite-volume free energy has a limit, providing results on convergence and asymptotic properties of the partition function and the Gibbs measure. Thus, FK-Potts models, introduced by Fortuin and Kasteleyn (see [FK72]), prove that these models have become an important tool in the study of phase transition for the Ising and q -state Potts model.

In general, the FK-Potts model on a finite connected graph (not necessarily planar) is a model of edges of the graphs, each edge is either closed or open. The probability of a configuration is proportional to

$$p^{\#\text{open edges}} (1 - p)^{\#\text{closed edges}} q^{\#\text{clusters}},$$

where the edge-weight $p \in [0, 1]$ and the cluster-weight $q \in (0, \infty)$ are the parameters of the model. For $q \geq 1$, this model can be extended to infinite-volume where it exhibits a phase transition at some critical parameter $p_c(q)$, that depend on the geometry of the graph. In the case of planar graphs, there is a connection between FK-Potts models on a graph and on its dual with the same cluster-weight q and appropriately related edge-weight p and $p^* = p^*(p)$ (Kramers-Wannier duality). For example, this relation leads in the particular case of $\mathbb{Z}^2$ to a natural conjecture: the critical point is the same as the so-called self-dual point satisfying $p_{sd} = p^*(p_{sd})$, proved by Beffara and Duminil-Copin in [BDC12].

In the case of a FK-Potts model defined on a causal dynamical triangulation $\mathbf{t}$ with periodic boundary condition, or equivalently defined on a torus (see Figure 2.1 for a geometric representation), its dual, defined on $\mathbf{t}^*$, is not a FK-Potts model; but will be enough for our purposes. This relation together with the Edwards-Sokal coupling, using $p = 1 - e^{-\beta}$, permits find a relation between the parameters (β, μ) of the Potts model coupled to CDT and the parameters (β^*, μ^*) of its dual for the infinite-volume (thermodynamic limit).

In the present chapter, we prove the following duality relation.

Theorem 5.1.1. *Let $q \geq 2$. The free energy of the q -state Potts model coupled to causal dynamical triangulation and its dual satisfied the following duality relation*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N^*(\beta^*, \mu^*) \quad (5.1)$$

Figure 5.1: Illustrating the region where the critical curve for Potts model coupled CDTs and dual CDTs can be located.

where Ξ_N, Ξ_N^* denote the partition function of the q -state Potts model coupled to CDT and coupled dual CDT respectively (defined in Section 5.2.1), and

$$\beta^* = \ln \left(1 + \frac{q}{e^\beta - 1} \right), \quad \mu^* = \mu - \frac{3}{2} \ln(e^\beta - 1) + \ln q. \quad (5.2)$$

Thus, (5.1) relates the free energy of the q -state Potts model coupled to CDTs and the free energy of the q -state Potts model coupled to dual CDTs, and maps the high and low temperature of the dual models onto each other.

We will use the duality relation of Theorem 5.1.1 and the high-temperature expansion for the q -state Potts model for determine a region in the quadrant of parameters where the critical curve for the q -state Potts model coupled CDTs and q -state Potts model coupled dual CDTs can be located (see Figure 5.1).

Understanding by *critical curve* of the model the boundary of the domain of parameters β and μ (β^* and μ^* on its dual, respectively) where the model exhibits subcritical behavior (see definition in Section 4.3 of Chapter 4), this chapter makes a rigorous derivation of the subcriticality domain for an q -Potts model coupled to two-dimensional CDT and a domain where the typical infinite-volume Gibbs measure does not exists. The proof involve two techniques: the duality relation (Theorem 5.1.1) and high-temperature expansion for the q -state Potts model. In Figure 5.1, we show the region where the critical curve of the model should be located (gray region).

Define the sets

$$\Sigma = \left\{ (\beta, \mu) \in \mathbb{R}_+^2 : \mu < \max \left\{ \ln(2\sqrt{q}), \frac{3}{2} \ln(e^\beta - 1) + \ln 2 \right\} \right\},$$

and

$$\Sigma^* = \left\{ (\beta^*, \mu^*) \in \mathbb{R}_+^2 : \mu^* < \max \left\{ \ln(2q), \frac{3}{2} \ln(e^{\beta^*} - 1) + \ln 2 \right\} \right\}.$$

We prove the following long theorem for existence and no existence of Gibbs measure for the model.

Theorem 5.1.2. *Let $q \geq 2$.*

1. *Potts model coupled to CDTs. If $(\beta, \mu) \in \Sigma$ then there exists $N_0 \in \mathbb{N}$ such that the partition function $\Xi_N(\beta, \mu) = +\infty$ whenever $N > N_0$. Moreover, the Gibbs distribution $\mathbb{P}_N^{\beta, \mu}$ with periodic boundary conditions cannot be defined by using the standard formula with $\Xi_N(\beta, \mu)$ as a normalising denominator, consequently, there is no limiting probability measure $\mathbb{P}^{\beta, \mu}$ as $N \rightarrow \infty$. Furthermore, if (β, μ) satisfies*

$$\mu > \frac{3}{2} \ln(q + e^\beta - 1) + \ln 2 - \ln q + \frac{3}{2} \ln \left(1 + (q^{2/3} - 1) \frac{e^\beta - 1}{q + e^\beta - 1} \right), \quad (5.3)$$

the infinite-volume free energy exists, i.e. the following limit there exists:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu).$$

Moreover, as $N \rightarrow \infty$, the Gibbs distribution $\mathbb{P}_N^{\beta, \mu}$ converges weakly to a limiting probability distribution $\mathbb{P}^{\beta, \mu}$.

2. *Potts model coupled to dual CDTs. If $(\beta^*, \mu^*) \in \Sigma^*$ then we have the same conclusion for the the Gibbs distribution $\mathbb{P}_N^{\beta^*, \mu^*}$, i.e. there is no limiting probability measure $\mathbb{P}^{\beta^*, \mu^*}$ as $N \rightarrow \infty$. Furthermore, if (β^*, μ^*) satisfies*

$$\mu^* > \frac{3}{2} \beta^* + \ln 2 + \frac{3}{2} \ln \left(1 + \frac{q^{2/3} - 1}{e^{\beta^*}} \right), \quad (5.4)$$

the infinite-volume free energy exists and, as $N \rightarrow \infty$, the Gibbs distribution $\mathbb{P}_N^{\beta^, \mu^*}$ converges weakly to a limiting probability distribution $\mathbb{P}^{\beta^*, \mu^*}$.*

As a byproduct, the Theorem 5.1.2 serves to find lower and upper bounds for the infinite-volume free energy. Moreover, in the case of 2-state Potts model (Ising model), Theorem 5.1.2 extends earlier results from [CH14], [HYSZ13] and improves the approximation of the curve in high temperature given in [AAL99]. In addition, this approach allows to get a better approximation of the critical curve and check the asymptotic behavior of the critical curve given in [AAL99], and it say that critical curve is asymptotic to $\frac{3}{2}\beta + \ln 2$, for $q \geq 2$. In Theorem 5.1.2, we find a lower and upper curve that converges fast to $\frac{3}{2}\beta + \ln 2$.

5.2 Notations

In this section we first introduce notations and give a summary of q -state Potts model and we define the Potts model coupled to CDTs. Finally, we give a short review of the Edwards-Sokal coupling. We refer to [MYZ01], [Gri06], [HYSZ13], [Gri08], for more details. We attempt at establishing regions where the infinite-volume free energy converges, yielding results on the convergence and asymptotic properties of the partition function and the Gibbs measure.

5.2.1 A Potts model coupled to CDTs

Let $\mathbf{t}$ be a CDT on the cylinder C_N with periodic boundary condition. Each triangulation $\mathbf{t}$ can be view as a graph $\mathbf{t} = (V(\mathbf{t}), E(\mathbf{t}))$ embedded on a torus. Potts spin systems are generalizations of the Ising model. Whereas in Ising systems the spins on two different values, in the q -state Potts model q distinct values, represent by the elements of the set $\{1, \dots, q\}$, are allowed on any vertex from the triangulation $\mathbf{t}$. We consider the product sample space $\Omega(\mathbf{t}) = \{1, \dots, q\}^{V(\mathbf{t})}$ and we consider a usual (ferromagnetic) q -state Potts model energy, where two spins $\sigma(t)$ and $\sigma(t')$ interact if their supporting vertices t, t' are connected by an common edge; such vertices are called nearest neighbors, and this property is reflected in the notation $\langle t, t' \rangle$. Thus, the Hamiltonian used for the q -state Potts model on $\mathbf{t}$ is given by

$$\mathbf{h}(\boldsymbol{\sigma}) = - \sum_{\langle t, t' \rangle} \delta_{\sigma(t), \sigma(t')}. \quad (5.5)$$

The partition function for the q -state Potts model on $\mathbf{t}$ is define by

$$Z_P(\beta, q, \mathbf{t}) = \sum_{\boldsymbol{\sigma}} \exp\{-\beta \mathbf{h}(\boldsymbol{\sigma})\}, \quad (5.6)$$

where the summation is over any configurations $\boldsymbol{\sigma} \in \{1, \dots, q\}^{V(\mathbf{t})}$. Thus, the q -state Potts measure on $\mathbf{t}$ is define as follows

$$\mu_{\beta, q}^{\mathbf{t}}(\boldsymbol{\sigma}) = \frac{1}{Z_P(\beta, q, \mathbf{t})} \exp\{-\beta \mathbf{h}(\boldsymbol{\sigma})\}. \quad (5.7)$$

Using the partition function for the q -state Potts model on a fixed $\mathbf{t}$, we define the partition function for the q -state Potts model coupled to CDTs, at the inverse temperature $\beta > 0$ and the cosmological constant μ , as follows

$$\Xi_N(\beta, \mu) = \sum_{\mathbf{t}} \exp\{-\mu n(\mathbf{t})\} Z_P(\beta, q, \mathbf{t}) \quad (5.8)$$

where $n(\mathbf{t})$ stands for the number of triangles in the triangulation $\mathbf{t}$. Similarly, we introduce the N -strip Gibbs probability distribution associated with (5.8)

$$\mathbb{P}_N^{\beta, \mu}(\mathbf{t}, \boldsymbol{\sigma}) = \frac{1}{\Xi_N(\beta, \mu)} \exp\{-\mu n(\mathbf{t}) - \beta \mathbf{h}(\boldsymbol{\sigma})\}. \quad (5.9)$$

and we denote by $\mathcal{G}_{\beta, \mu}$ the set of *Gibbs measures* given by the closed convex hull of the set of weak limits:

$$\mathbb{P}^{\beta, \mu} = \lim_{N \rightarrow \infty} \mathbb{P}_N^{\beta, \mu}, \quad (5.10)$$

In general, the q -state Potts model can be defined on a general lattice G . Therefore, it is possible to define the q -state Potts model over the dual $\mathbf{t}^*$ of the triangulation $\mathbf{t}$ (see next section for a formal definition of $\mathbf{t}^*$). The partition function for the q -state Potts model on $\mathbf{t}^*$ will denote by $Z_P(\beta^*, q, \mathbf{t}^*)$. Finally, we define the partition function for the q -state Potts model coupled to dual CDTs, $\Xi_N^*(\beta^*, \mu^*)$ as follow

$$\Xi_N^*(\beta^*, \mu^*) = \sum_{\mathbf{t}} \exp\{-\mu^* n(\mathbf{t})\} Z_P(\beta^*, q, \mathbf{t}^*). \quad (5.11)$$

5.2.2 The FK-Potts model on Lorentzian triangulations

We now turn to the FK representation of the q -state Potts model. The random cluster model was originally introduced by Fortuin and Kasteleyn [FK72] and it can be understood as an alternative representation of the q -state Potts model. This representation will be referred to as the FK representation or FK-Potts model. We are interested in study FK-Potts model on CDTs and dual CDTs, and find a duality relation between the parameters of the model on CDTs and its dual. In [HYSZ13], [CH14], the model was defined putting spins on any triangle (faces), but it is equivalent to put spins on any vertex of dual graph, in this case, dual triangulation. In this section we work with triangulations with periodic boundary conditions, i.e., Lorentzian triangulations embedded in a torus $\mathbb{T}$ (see Figure 3.1 (b)) and its dual. In general, let $G = (V, E)$ be a graph embedded in $\mathbb{T}$, we obtain its dual graph $G^* = (V^*, E^*)$ as follows: we place a dual vertex within each face of G . For each $e \in E$ we place a dual $e^* = \langle x^*, y^* \rangle$ joining the two dual vertices lying in the two faces of G abutting e . Thus, V^* is in one-one correspondence with the set of faces of G , and E^* is a one-one correspondence with E . For each Lorentzian triangulation $\mathbf{t}$, we denote by $\mathbf{t}^*$ its dual.

Let $\mathbf{t} = (V(\mathbf{t}), E(\mathbf{t}))$ be a Lorentzian triangulation with periodic boundary condition, where $V(\mathbf{t}), E(\mathbf{t})$ denote the set of vertices and edges, respectively. The state space for the FK-Potts model is the set $\Sigma(\mathbf{t}) = \{0, 1\}^{E(\mathbf{t})}$, containing configurations that allocate 0's and 1's to the edge $e = \langle i, j \rangle \in E(\mathbf{t})$. For $w \in \Sigma(\mathbf{t})$, we call an edge e open if $w(e) = 1$, and closed if $w(e) = 0$. For $w \in \Sigma(\mathbf{t})$, let $\eta(w) = \{e \in E(\mathbf{t}) : w(e) = 1\}$ denote the set of open edges. Thus, each $w \in \Sigma(\mathbf{t})$ splits $V(\mathbf{t})$ into the disjoint union of maximal connected components, which are called the open clusters of $\Sigma(\mathbf{t})$. We denote by $k(w)$ the number of connected components (open clusters) of the graph $(V(\mathbf{t}), \eta(w))$, and note that $k(w)$ includes a count

Figure 5.2: Geometric representation of a dual Lorentzian triangulation $\mathbf{t}^*$ with periodic spatial boundary condition.

of isolated vertices. Two sites of $\mathbf{t}$ are said to be connected if one can be reached from another via a chain of open bonds.

The partition function of the FK-Potts model on $\mathbf{t}$ with parameters p and q and periodic boundary condition is defined by

$$Z_{FK}(p, q, \mathbf{t}) = \sum_{w \in \Sigma(\mathbf{t})} \left(\prod_{e \in E(\mathbf{t})} (1-p)^{1-w(e)} p^{w(e)} \right) q^{k(w)}, \quad (5.12)$$

Thus, the FK-Potts measure on $\mathbf{t}$ is define as follows

$$\Phi_{p,q}^{\mathbf{t}}(w) = \frac{1}{Z_{FK}(p, q, \mathbf{t})} \left(\prod_{e \in E(\mathbf{t})} (1-p)^{1-w(e)} p^{w(e)} \right) q^{k(w)}. \quad (5.13)$$

We will use a similarly notation for the FK-Potts model on dual triangulation $\mathbf{t}^*$. We denote by $Z_{FK}(p^*, q, \mathbf{t}^*)$ and $\Phi_{p^*,q}^{\mathbf{t}^*}$ the partition function and the FK-Potts measure on $\mathbf{t}^*$ with parameters p^* and q , respectively.

5.2.3 The relation between the Potts model and FK-Potts model: Edwards-Sokal coupling

There are several ways to make the connection between the Potts and FK-Potts model. The correspondence between the q -state Potts model and FK-Potts model was established by Fortuin and Kasteleyn [FK72] (see also [ES88], [Gri06], [Gri08]). In a modern approach, these two models are related via a coupling, i.e., coupled the two systems on a common probability space. This coupling was introduced by Edwards-Sokal in [ES88].

Let $\mathbf{t}$ be a CDT on the cylinder C_N with periodic boundary condition. We consider the product sample space $\Omega(\mathbf{t}) \times \Sigma(\mathbf{t})$ where $\Omega(\mathbf{t}) = \{1, 2, \dots, q\}^{V(\mathbf{t})}$ and $\Sigma(\mathbf{t}) = \{0, 1\}^{E(\mathbf{t})}$. The

Edwards-Sokal measure $\mathcal{Q}$ on $\Omega(\mathbf{t}) \times \Sigma(\mathbf{t})$ is define by

$$\mathcal{Q}(\sigma, w) \propto \prod_{e=\{i,j\} \in E(\mathbf{t})} \{(1-p)\delta_{w(e),0} + p\delta_{w(e),1}\delta_{\sigma_i,\sigma_j}\}$$

Theorem 5.2.1 (Edwards-Sokal [ES88]). *Let $q \in \{2, 3, \dots\}$. Let $p \in (0, 1)$ and $\mathbf{t}$ a CDT with periodic boundary condition, and suppose that $p = 1 - e^{-\beta}$. If the configuration w is distributed according to an FK-Potts measure with parameters (p, q) on $\mathbf{t}$, then σ is distributed according to a q -state Potts measure with inverse temperature β . Furthermore, the Edwards-Sokal measure provides a coupling of $\mu_{\beta,q}^{\mathbf{t}}$ and $\Phi_{p,q}^{\mathbf{t}}$, i.e.*

$$\sum_{w \in \Sigma(\mathbf{t})} \mathcal{Q}(\sigma, w) = \mu_{\beta,q}^{\mathbf{t}}(\sigma),$$

for all $\sigma \in \Omega(\mathbf{t})$, and

$$\sum_{\sigma \in \Omega(\mathbf{t})} \mathcal{Q}(\sigma, w) = \Phi_{p,q}^{\mathbf{t}}(w),$$

for all $w \in \Sigma(\mathbf{t})$. Moreover, we have the relation between partition functions

$$Z_{FK}(p, q, \mathbf{t}) = e^{-\beta|E(\mathbf{t})|} Z_P(\beta, q, \mathbf{t}). \quad (5.14)$$

5.2.4 Duality for FK-Potts model coupled to CDTs with periodic boundary conditions

In this section we obtain a relation between the partition functions of FK-Potts model on a triangulation $\mathbf{t}$ and its dual. This relation was studied by Beffara and Duminil-Copin for the FK-Potts model on $\mathbb{Z}^2$ with free, wired and periodic boundary condition (see [BDC12] for details). We will view which the dual of a FK-Potts model defined on a torus is a quasi FK-Potts model, but it is not very different from one.

Let $\mathbf{t}$ and $\mathbf{t}^*$ a CDT with periodic boundary condition and its dual. Each configuration $w \in \Sigma(\mathbf{t}) = \{0, 1\}^{E(\mathbf{t})}$ gives rise to a dual configuration $w^* \in \Sigma(\mathbf{t}^*) = \{0, 1\}^{E(\mathbf{t}^*)}$ given by $w^*(e^*) = 1 - w(e)$. That is, e^* is declared open if and only if the corresponding bond e is closed. The new configuration w^* is called the dual configuration of w , and note that there exists an one-one correspondence between $\Sigma(\mathbf{t})$ and $\Sigma(\mathbf{t}^*)$. As in the Section 5.2.2, to each configuration w^* there corresponds the set $\eta(w^*) = \{e^* \in E(\mathbf{t}^*) : w^*(e^*) = 1\}$ of its open edges.

Now, beginning of FK-Potts model on $\mathbf{t}$, we try to obtain the dual model on the dual triangulation $\mathbf{t}^*$.

Let $o(w)$ (resp. $c(w)$) denote the number of open edges (resp. closed) of w , $k(w)$ the number of connected components of w , and $f(w)$ the number of faces delimited by w , i.e.

Figure 5.3: (a) Geometric representation of a net (b) Geometric representation of a cycle (c) None of cluster of w is a net or a cycle

the number of connected components of the complement of the set of open bonds. We will now define an additional parameters $\delta(w)$.

Call a connected component of w a *net* if it contains two non-contractible simple loops γ_1, γ_2 of different homotopy classes, and a *cycle* if it contain a non-contractible simple loops γ_1 non-contractible but is not a net (see Figure 5.3). These definitions were introduced in [BDC12]. In additional, notice that every configuration w can be of one three types:

- One of the cluster of w is a net. Then no other cluster can be a net or a cycle. In that case, we let $\delta(w) = 2$;
- One of the cluster of w is a cycle. Then no other cluster can be a net, but other cluster can be cycles as well (in which case all the involved, simple loops are in the same homotopy class) We then let $\delta(w) = 1$;
- None of the cluster of w is a net or a cycle. We let $\delta(w) = 0$.

Using this definition for the parameter δ , we obtained the following version of Euler's formula.

Proposition 5.2.1 (Euler's formula). *Let $\mathbf{t}$ a CDT with periodic boundary condition and $w \in \{0, 1\}^{E(\mathbf{t})}$. Then*

$$|V(\mathbf{t})| - o(w) + f(w) = k(w) + 1 - \delta(w). \quad (5.15)$$

Using duality and Proposition 5.2.1, we have the following relations

$$o(w) + o(w^*) = |E(\mathbf{t})|, \quad f(w) = k(w^*) \quad \text{and} \quad \delta(w) + \delta(w^*) = 2. \quad (5.16)$$

Let $q \in (0, \infty)$ and $p \in (0, 1)$. The partition function of the FK-Potts model is given by

$$\begin{aligned} Z_{FK}(p, q, \mathbf{t}) &= \sum_{w \in \Sigma(\mathbf{t})} \left(\prod_{e \in E(\mathbf{t})} (1-p)^{1-w(e)} p^{w(e)} \right) q^{k(w)} \\ &= \sum_{w \in \Sigma(\mathbf{t})} p^{o(w)} (1-p)^{c(w)} q^{k(w)}. \end{aligned}$$

Using Euler's formula and relations (5.16), we rewrite the number of cluster of w in terms of its dual w^*

$$k(w) = |V(\mathbf{t})| - |E(\mathbf{t})| + o(w^*) + k(w^*) + 1 - \delta(w^*).$$

We note also that $o(w) + o(w^*) = |E(\mathbf{t})| = |E(\mathbf{t}^*)|$. Plugging before relations into the partition function of the FK-Potts model, we obtain

$$\begin{aligned} Z_{FK}(p, q, \mathbf{t}) &= \sum_{w \in \Sigma(\mathbf{t})} p^{o(w)} (1-p)^{|E(\mathbf{t})|-o(w)} q^{k(w)} \\ &= (1-p)^{|E(\mathbf{t})|} \sum_{w \in \Sigma(\mathbf{t})} \left(\frac{p}{1-p} \right)^{o(w)} q^{k(w)} \\ &= (1-p)^{|E(\mathbf{t})|} \sum_{w \in \Sigma(\mathbf{t})} \left(\frac{p}{1-p} \right)^{|E(\mathbf{t})|-o(w^*)} q^{|V(\mathbf{t})|-|E(\mathbf{t})|+o(w^*)+k(w^*)+1-\delta(w^*)} \\ &= p^{|E(\mathbf{t})|} q^{|V(\mathbf{t})|-|E(\mathbf{t})|} \sum_{w \in \Sigma(\mathbf{t})} \left(\frac{1-p}{p} \right)^{o(w^*)} q^{o(w^*)+k(w^*)+1-\delta(w^*)} \end{aligned}$$

As there exists an one-one correspondence between $\Sigma(\mathbf{t})$ and $\Sigma(\mathbf{t}^*)$, in the last equality, we change the sum in $\Sigma(\mathbf{t})$ by the sum in $\Sigma(\mathbf{t}^*)$. Thus, we obtain the following representation of the partition function in terms of the dual triangulation and dual configurations

$$Z_{FK}(p, q, \mathbf{t}) = p^{|E(\mathbf{t})|} q^{|V(\mathbf{t})|-|E(\mathbf{t})|} \sum_{w^* \in \Sigma(\mathbf{t}^*)} \left(\frac{q(1-p)}{p} \right)^{o(w^*)} q^{k(w^*)+1-\delta(w^*)} \quad (5.17)$$

Using the relation (5.17), we obtain the following lemma.

Lemma 5.2.1. *Let $\mathbf{t}$ be a CDT with periodic boundary condition. Then, the following comparison inequalities both*

$$Z_{FK}(p, q, \mathbf{t}) \leq \left(\frac{p}{1-p^*} \right)^{|E(\mathbf{t})|} q^{|V(\mathbf{t})|-|E(\mathbf{t})|+1} Z_{FK}(p^*, q, \mathbf{t}^*) \quad (5.18)$$

and

$$\left(\frac{p}{1-p^*}\right)^{|E(\mathbf{t})|} q^{|V(\mathbf{t})|-|E(\mathbf{t})|-1} Z_{FK}(p^*, q, \mathbf{t}^*) \leq Z_{FK}(p, q, \mathbf{t}) \quad (5.19)$$

hold, where $Z_{FK}(p^*, q, \mathbf{t}^*)$ is the partition function for FK-Potts model on $\mathbf{t}^*$ with parameters q and $p^* = p^*(p, q)$ satisfying

$$p^*(p, q) = \frac{(1-p)q}{(1-p)q+p}, \quad \text{or equivalently} \quad \frac{p^*}{1-p^*} \cdot \frac{p}{1-p} = q.$$

Proof. We introduce the parameter $p^* = p^*(p, q)$ as solution of the equation

$$\frac{p^*}{1-p^*} = \frac{(1-p)q}{p}.$$

Thus, the partition function can be written in the following ways

$$\begin{aligned} Z_{FK}(p, q, \mathbf{t}) &= p^{|E(\mathbf{t})|} q^{|V(\mathbf{t})|-|E(\mathbf{t})|} \sum_{w^* \in \Sigma(\mathbf{t}^*)} \left(\frac{p^*}{1-p^*}\right)^{o(w^*)} q^{k(w^*)+1-\delta(w^*)} \\ &= \frac{p^{|E(\mathbf{t})|}}{(1-p^*)^{|E(\mathbf{t}^*)|}} q^{|V(\mathbf{t})|-|E(\mathbf{t})|} (1-p^*)^{|E(\mathbf{t}^*)|} \sum_{w^* \in \Sigma(\mathbf{t}^*)} \left(\frac{p^*}{1-p^*}\right)^{o(w^*)} q^{k(w^*)+1-\delta(w^*)}. \end{aligned}$$

Notice that $-1 \leq 1 - \delta(w^*) \leq 1$, for all $w^* \in \Sigma(\mathbf{t}^*)$. We define $Z_{FK}(p^*, q, \mathbf{t}^*)$, the partition function of a FK-Potts model with parameters p^* and q . Thus, we obtain the upper bound

$$Z_{FK}(p, q, \mathbf{t}) \leq \frac{p^{|E(\mathbf{t})|}}{(1-p^*)^{|E(\mathbf{t}^*)|}} q^{|V(\mathbf{t})|-|E(\mathbf{t})|+1} Z_{FK}(p^*, q, \mathbf{t}^*),$$

and the lower bound

$$\frac{p^{|E(\mathbf{t})|}}{(1-p^*)^{|E(\mathbf{t}^*)|}} q^{|V(\mathbf{t})|-|E(\mathbf{t})|-1} Z_{FK}(p^*, q, \mathbf{t}^*) \leq Z_{FK}(p, q, \mathbf{t})$$

for the partition function of FK-Potts model on $\mathbf{t}$ with parameters p and q . Using the one-one correspondence between $E(\mathbf{t})$ and $E(\mathbf{t}^*)$, we conclude the proof. $\square$

The partition function for pure CDT's has been determined as a sum over all possible triangulations of a cylinder where each configuration is weighted by a Boltzmann factor $e^{-\mu n(\mathbf{t})}$, where $n(\mathbf{t})$ standing for the size of the triangulation and μ being the cosmological constant. Thus, in quantum gravity the volume $n(\mathbf{t})$ becomes a dynamical variable for the model. Therefore, we rewrite the duality relation (Lemma 5.2.1) for the partition function of the FK-Potts model on a triangulation $\mathbf{t}$, in terms of dynamical variable $n(\mathbf{t})$. In the Table 5.1 we show the relation among $V(\mathbf{t})$, $E(\mathbf{t})$, $V(\mathbf{t}^*)$, $E(\mathbf{t}^*)$ and the number of triangles $n(\mathbf{t})$ of a CDT $\mathbf{t}$.

$\mathbf{t} = (V(\mathbf{t}), E(\mathbf{t}))$	$\mathbf{t}^* = (V(\mathbf{t}^*), E(\mathbf{t}^*))$
$ V(\mathbf{t}) = \frac{1}{2}n(\mathbf{t})$	$ V(\mathbf{t}^*) = n(\mathbf{t})$
$ E(\mathbf{t}) = \frac{3}{2}n(\mathbf{t})$	$ E(\mathbf{t}^*) = \frac{3}{2}n(\mathbf{t})$
$ \text{faces in } \mathbf{t} = n(\mathbf{t})$	$ \text{faces in } \mathbf{t}^* = \frac{1}{2}n(\mathbf{t})$

Table 5.1: Relation between the graphs $\mathbf{t}, \mathbf{t}^*$ and $n(\mathbf{t})$

Using the relations of Table 5.1, the Lemma 5.2.1 becomes be written in terms of $n(\mathbf{t})$ as follow

Corollary 5.2.1. *Let $\mathbf{t}$ be a CDT with periodic boundary condition. Then the following comparison inequalities both*

$$\left(\frac{p}{1-p^*}\right)^{\frac{3}{2}n(\mathbf{t})} q^{-1-n(\mathbf{t})} \leq \frac{Z_{FK}(p, q, \mathbf{t})}{Z_{FK}(p^*, q, \mathbf{t}^*)} \leq \left(\frac{p}{1-p^*}\right)^{\frac{3}{2}n(\mathbf{t})} q^{1-n(\mathbf{t})} \quad (5.20)$$

and

$$\left(\frac{p^*}{1-p}\right)^{\frac{3}{2}n(\mathbf{t})} q^{-1-\frac{1}{2}n(\mathbf{t})} \leq \frac{Z_{FK}(p^*, q, \mathbf{t}^*)}{Z_{FK}(p, q, \mathbf{t})} \leq \left(\frac{p^*}{1-p}\right)^{\frac{3}{2}n(\mathbf{t})} q^{1-\frac{1}{2}n(\mathbf{t})} \quad (5.21)$$

where $Z_{FK}(p^*, q, \mathbf{t}^*)$ is the partition function for FK-Potts model on $\mathbf{t}^*$ with parameters q and $p^* = p^*(p, q)$ satisfying

$$p^*(p, q) = \frac{(1-p)q}{(1-p)q + p}, \text{ or equivalently } \frac{p^*}{1-p^*} \frac{p}{1-p} = q.$$

5.3 The proof of Theorem 5.1.1 and first bounds for the critical curve

In the previous section we found comparison inequalities between the partition function of the FK-Potts model on $\mathbf{t}$ and the partition function of the FK-Potts model on its dual $\mathbf{t}^*$. In this section we will use these comparison inequalities to prove Theorem 5.1.1. Combining inequalities (5.20), (5.21) and the Edwards Sokal coupling (Theorem 5.2.1), we obtain the following comparison inequalities between the partition function of the q -state Potts model on $\mathbf{t}$ and the partition function of the q -state Potts model on its dual $\mathbf{t}^*$.

$$\left(\frac{p}{1-p^*}\right)^{\frac{3}{2}n(\mathbf{t})} q^{-1-n(\mathbf{t})} e^{\frac{3}{2}(\beta-\beta^*)n(\mathbf{t})} \leq \frac{Z_P(\beta, q, \mathbf{t})}{Z_P(\beta^*, q, \mathbf{t}^*)} \leq \left(\frac{p}{1-p^*}\right)^{\frac{3}{2}n(\mathbf{t})} q^{1-n(\mathbf{t})} e^{\frac{3}{2}(\beta-\beta^*)n(\mathbf{t})} \quad (5.22)$$

and

$$\left(\frac{p^*}{1-p}\right)^{\frac{3}{2}n(\mathbf{t})} q^{-1-\frac{1}{2}n(\mathbf{t})} e^{\frac{3}{2}(\beta^*-\beta)n(\mathbf{t})} \leq \frac{Z_P(\beta^*, q, \mathbf{t}^*)}{Z_P(\beta, q, \mathbf{t})} \leq \left(\frac{p^*}{1-p}\right)^{\frac{3}{2}n(\mathbf{t})} q^{1-\frac{1}{2}n(\mathbf{t})} e^{\frac{3}{2}(\beta^*-\beta)n(\mathbf{t})} \quad (5.23)$$

where $(e^\beta - 1)(e^{\beta^*} - 1) = q$.

Proof of Theorem 5.1.1. Using the comparison inequalities (5.22) and (5.23), we will find comparison inequalities for the partition functions of the Potts model coupled CDTs with parameters β, μ and dual CDTs with parameters $\beta^* = \beta^*(\beta)$, $\mu^* = \mu^*(\beta, \mu)$. Remember that $p^* = 1 - e^{-\beta^*}$ and $p = 1 - e^{-\beta}$. Thus,

$$\frac{p^*}{1-p} = (1 - e^{-\beta^*}) + qe^{-\beta^*} = \frac{q}{(1 - e^{-\beta}) + qe^{-\beta}}$$

and

$$\frac{p}{1-p^*} = \frac{q}{(1 - e^{-\beta^*}) + qe^{-\beta^*}} = (1 - e^{-\beta}) + qe^{-\beta}.$$

Multiplying by the Boltzmann factor $e^{-\mu n(\mathbf{t})}$ in (5.22) and (5.23), and sum over all possible CDTs of the cylinder C_N , we obtain the following comparison inequalities

$$\frac{1}{q} \Xi_N^*(\beta^*, \mu^*) \leq \Xi_N(\beta, \mu) \leq q \Xi_N^*(\beta^*, \mu^*) \quad (5.24)$$

where Ξ_N^* stands the partition function of the q -state Potts model coupled to dual CDTs with periodic boundary condition, Ξ_N stands the partition function of the q -state Potts model coupled to CDTs with periodic boundary condition, and

$$\beta^* = \ln \left(1 + \frac{q}{e^\beta - 1} \right), \quad \mu^* = \mu - \frac{3}{2} \ln(e^\beta - 1) + \ln q.$$

Similarly, we have

$$\frac{1}{q} \Xi_N(\beta, \mu) \leq \Xi_N^*(\beta^*, \mu^*) \leq q \Xi_N(\beta, \mu) \quad (5.25)$$

where

$$\beta = \ln \left(1 + \frac{q}{e^{\beta^*} - 1} \right), \quad \mu = \mu^* - \frac{3}{2} \ln(e^{\beta^*} - 1) + \frac{1}{2} \ln q.$$

Take the natural logarithm in inequalities (5.24) and (5.25), divide both sides of the above inequalities by N and let $N \rightarrow \infty$. This concludes the proof of Theorem 5.1.1. $\square$

Theorem 5.1.1 provide an interesting reformulation in terms of free energy of the q -state Potts model coupled to CDTs and its dual. This theorem relates the free energy of the q -state Potts model coupled CDTs and the free energy of the q -state Potts model coupled dual CDTs, and maps the high and low temperature of the dual models onto each other.

Using Edwards-Sokal coupling for the partition functions (5.14), duality relation found in Theorem 5.1.1 and asymptotic properties (2.14), (3.20) of the partition function for pure CDTs (see [MYZ01] for more details), we will obtain the first bounds for the critical curve of the q -state Potts model coupled to CDTs and dual CDTs. Let $\mathbf{t}$ be a CDT with periodic boundary condition. We define the set Π_i of configurations in $\Sigma(\mathbf{t})$ which splits $V(\mathbf{t})$ in i maximal connected components, i.e.

$$\Pi_i = \{w \in \Sigma(\mathbf{t}) : k(w) = i\}.$$

Similarly, we denote Π_i^* the set of configurations in $\Sigma(\mathbf{t}^*)$ which splits $V(\mathbf{t}^*)$ in i maximal connected components. Thus, we have the following representation for the partition function of q -state Potts model on $\mathbf{t}$

$$Z_P(\beta, q, \mathbf{t}) = e^{\beta|E(\mathbf{t})|} \phi_p^{\mathbf{t}}(q^{k(w)}) = e^{\beta|E(\mathbf{t})|} \sum_{i=1}^{|V(\mathbf{t})|} q^i \phi_p^{\mathbf{t}}(\Pi_i), \quad (5.26)$$

and on $\mathbf{t}^*$

$$Z_P(\beta^*, q, \mathbf{t}^*) = e^{\beta|E(\mathbf{t}^*)|} \phi_p^{\mathbf{t}^*}(q^{k(w^*)}) = e^{\beta|E(\mathbf{t}^*)|} \sum_{i=1}^{|V(\mathbf{t}^*)|} q^i \phi_p^{\mathbf{t}^*}(\Pi_i^*), \quad (5.27)$$

where $\phi_p^{\mathbf{t}}, \phi_p^{\mathbf{t}^*}$ denotes product measures on $\Sigma(\mathbf{t})$ and $\Sigma(\mathbf{t}^*)$, respectively.

Using Table 5.1, we write the representations (5.26), (5.27) for the partition function in terms of the dynamical variable $n(\mathbf{t})$. For that, we consider two cases of interest separately.

1. *The model on CDTs $\mathbf{t}$* : In this case, we can write the partition function in terms of volume $n(\mathbf{t})$ of the triangulation as follow

$$Z_P(\beta, q, \mathbf{t}) = e^{\frac{3}{2}\beta n(\mathbf{t})} \sum_{i=1}^{\frac{1}{2}n(\mathbf{t})} q^i \phi_p^{\mathbf{t}}(\Pi_i). \quad (5.28)$$

Using the first and latter term on the right-hand side of (5.28), we obtain two lower bounds for the partition function of the Potts model on $\mathbf{t}$

$$Z_P(\beta, q, \mathbf{t}) \geq q(e^\beta - 1)^{\frac{3}{2}n(\mathbf{t})}, \quad Z_P(\beta, q, \mathbf{t}) \geq q^{\frac{1}{2}n(\mathbf{t})}. \quad (5.29)$$

These lower bounds for the q -state Potts model on $\mathbf{t}$ permit to obtain a lower barrier for parameters where the model can be defined, and the partition function of the model coupled to CDTs could no explode in finite volume. These lower bounds serves to obtain information of the Gibbs measure for q -state Potts model coupled to CDTs.

Proposition 5.3.1. *If $(\beta, \mu) \in \mathbb{R}^+$ such that*

$$\mu < \frac{1}{2} \ln q + \ln 2 \quad \text{or} \quad \mu < \frac{3}{2} \ln(e^\beta - 1) + \ln 2,$$

then there exists $N_0 \in \mathbb{N}$ such that the partition function $\Xi_N(\beta, \mu) = \infty$ whenever $N > N_0$. Moreover, the Gibbs distribution $\mathbb{P}_N^{\beta, \mu}$ cannot be defined by using the standard formula with $\Xi_N(\beta, \mu)$ as a normalising denominator, consequently, there is no limiting probability measure $\mathbb{P}^{\beta, \mu}$ as $N \rightarrow \infty$.

Proof. The lower bounds in (5.29) to $Z_P(\beta, q, \mathbf{t})$ provide the following lower bounds to $\Xi_N(\beta, \mu)$,

$$\Xi_N(\beta, \mu) \geq q \sum_{\mathbf{t}} e^{-\{\mu - \frac{3}{2} \ln(e^\beta - 1)\}n(\mathbf{t})}, \quad \Xi_N(\beta, \mu) \geq q \sum_{\mathbf{t}} e^{-\{\mu - \frac{1}{2} \ln q\}n(\mathbf{t})}. \quad (5.30)$$

Using asymptotic properties of Proposition (2.2.1), we have that the partition function $\Xi_N(\beta, \mu)$ there is no exist if

$$\mu \leq \frac{1}{2} \ln q + \ln \left(2 \cos \frac{\pi}{N+1} \right) \quad \text{or} \quad \mu \leq \frac{3}{2} \ln(e^\beta - 1) + \ln \left(2 \cos \frac{\pi}{N+1} \right).$$

Letting $N \rightarrow \infty$, we conclude the proof. $\square$

Now, notice that

$$\sum_{i=1}^{\frac{1}{2}n(\mathbf{t})} q^i \phi_p^{\mathbf{t}}(\Pi_i) \leq q^{\frac{1}{2}n(\mathbf{t})},$$

for any triangulation $\mathbf{t}$. Thus, we obtain a upper bound for the partition function of the q -state Potts model on $\mathbf{t}$,

$$Z_P(\beta, q, \mathbf{t}) \leq e^{\frac{3}{2}\beta n(\mathbf{t})} q^{\frac{1}{2}n(\mathbf{t})} = e^{(\frac{3}{2}\beta + \frac{1}{2} \ln q)n(\mathbf{t})}. \quad (5.31)$$

This upper bound for the q -state Potts model on $\mathbf{t}$, permit to obtain a first upper barrier for the critical curve of the q -state Potts model coupled to CDTs, and above of that upper bound the model exhibits subcritical behavior. Moreover, this upper bound for the critical curve of the model coupled serves to obtain information of the Gibbs measure for the q -state Potts model coupled to CDTs. We get the following result.

Proposition 5.3.2. *Under condition $\mu > \frac{3}{2}\beta + \frac{1}{2} \ln q + \ln 2$, the infinite-volume free energy exists, i.e. the following limit there exists:*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu).$$

Moreover, as $N \rightarrow \infty$, the Gibbs distribution $\mathbb{P}_N^{\beta, \mu}$ converges weakly to a limiting probability distribution $\mathbb{P}^{\beta, \mu}$.

Proof. Using inequality (5.31), we get to

$$\Xi_N(\beta, \mu) \leq \sum_{\mathbf{t}} e^{-\{\mu - \frac{3}{2}\beta - \frac{1}{2}\ln q\}n(\mathbf{t})}$$

By Proposition 2.2.1, we have that the free energy there exists if $\mu - \frac{3}{2}\beta - \frac{1}{2}\ln q > \ln 2$. This concludes the proof. $\square$

Finally, using the duality relation (Theorem 5.1.1) and bounds found before propositions, we obtain bounds for the critical curve for the q -state Potts model coupled to dual CDTs.

from CDTs $\mathbf{t}$	by duality $\rightarrow$	to dual CDTs $\mathbf{t}^*$
$\mu < \frac{1}{2}\ln q + \ln 2$	$\rightarrow$	$\mu^* < \frac{3}{2}\ln(e^{\beta^*} - 1) + \ln 2$
$\mu < \frac{3}{2}\ln(e^{\beta} - 1) + \ln 2$	$\rightarrow$	$\mu^* < \ln q + \ln 2$
$\mu > \frac{3}{2}\beta + \frac{1}{2}\ln q + \ln 2$	$\rightarrow$	$\mu^* > \frac{3}{2}\ln(q + e^{\beta^*} - 1) + \ln 2$

Table 5.2: Bounds for the critical curve of the q -state Potts model on CDTs will generate bounds on its dual.

In Table 5.2 the parameters (β, μ) and (β^*, μ^*) satisfied the duality relation (5.2).

2. *The model on dual CDTs $\mathbf{t}^*$:* Similarly, we write the partition function in terms of volume $n(\mathbf{t})$ of the triangulation as follow

$$Z_P(\beta^*, q, \mathbf{t}^*) = e^{\frac{3}{2}\beta^*n(\mathbf{t})} \sum_{i=1}^{n(\mathbf{t})} q^i \phi_{p^*}^{\mathbf{t}^*}(\Pi_i^*). \quad (5.32)$$

Using the first and latter term on the right-hand side of (5.32), we obtain two lower bounds for the partition function of the q -state Potts model on $\mathbf{t}^*$

$$Z_P(\beta^*, q, \mathbf{t}^*) \geq q(e^{\beta^*} - 1)^{\frac{3}{2}n(\mathbf{t})}, \quad Z_P(\beta^*, q, \mathbf{t}^*) \geq q^{n(\mathbf{t})}, \quad (5.33)$$

and an upper bound

$$Z_P(\beta^*, q, \mathbf{t}^*) \leq e^{\frac{3}{2}\beta^*n(\mathbf{t})} q^{n(\mathbf{t})} = e^{(\frac{3}{2}\beta^* + \ln q)n(\mathbf{t})}. \quad (5.34)$$

Using asymptotic properties of Theorem 2.2.1 and Proposition 2.2.1, bounds (5.33) and (5.34) provide bounds for the critical curve of the q -state Potts model coupled dual CDTs. As in previous case, we have the following proposition on existence and non existence of Gibbs measures of the model.

Proposition 5.3.3. *For q -state Potts model coupled to dual CDTs, we have the following assertions:*

1. *If $(\beta^*, \mu^*) \in \mathbb{R}^+$ such that*

$$\mu^* < \ln q + \ln 2 \text{ or } \mu^* < \frac{3}{2} \ln(e^{\beta^*} - 1) + \ln 2,$$

then there exists $N_0 \in \mathbb{N}$ such that the partition function $\Xi_N^(\beta^*, \mu^*) = \infty$ whenever $N > N_0$. Moreover, the Gibbs distribution $\mathbb{P}_N^{\beta^*, \mu^*}$ cannot be defined by using the standard formula with $\Xi_N^*(\beta^*, \mu^*)$ as a normalising denominator, consequently, there is no limiting probability measure $\mathbb{P}^{\beta^*, \mu^*}$ as $N \rightarrow \infty$.*

2. *Under condition $\mu^* > \frac{3}{2}\beta^* + \ln q + \ln 2$, the infinite-volume free energy exists, i.e. the following limit there exists:*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N^*(\beta^*, \mu^*).$$

Moreover, as $N \rightarrow \infty$, the Gibbs distribution $\mathbb{P}_N^{\beta^, \mu^*}$ converges weakly to a limiting probability distribution $\mathbb{P}^{\beta^*, \mu^*}$.*

Finally, using the duality relation (Theorem 5.1.1), and bounds found in the before proposition, we obtain bounds for the critical curve for the Potts model coupled to CDTs. In Table

from dual CDTs $\mathbf{t}^*$	by duality $\rightarrow$	to CDTs $\mathbf{t}$
$\mu^* < \ln q + \ln 2$	$\rightarrow$	$\mu < \frac{3}{2} \ln(e^\beta - 1) + \ln 2$
$\mu^* < \frac{3}{2} \ln(e^{\beta^*} - 1) + \ln 2$	$\rightarrow$	$\mu < \frac{1}{2} \ln q + \ln 2$
$\mu^* > \frac{3}{2}\beta^* + \ln q + \ln 2$	$\rightarrow$	$\mu > \frac{3}{2} \ln(q + e^\beta - 1) + \ln 2$

Table 5.3: *Bounds for the critical curve of the q -state Potts model coupled to dual CDTs will generate bounds for the the critical curve of the q -state Potts model coupled to CDTs.*

5.3, parameters (β, μ) and (β^*, μ^*) satisfy the duality relation (5.2).

Tables 5.2 and 5.3 show that to find bounds for the critical curve for the model on CDTs provide bounds for the model on dual CDTs, and viceversa. Thus, in the next section we improves the bounds obtained for the critical curve of the Potts model on CDTs. In additional, this approach allows to get a asymptotic behavior of the critical curve for the model on CDTs and its dual.

5.4 High-T expansion of the Potts model and Proof of Theorem 5.1.2

Let $\mathbf{t}$ be a CDT with periodic boundary condition. The partition function for the Potts model on $\mathbf{t}$ is write in the usual high-T expansion as

$$Z_P(\beta, q, \mathbf{t}) = \left(\frac{q+h}{q} \right)^{|E(\mathbf{t})|} \sum_{\sigma} \prod_{\langle i,j \rangle} (1 + f_{ij}) \quad (5.35)$$

where $h = e^{\beta} - 1$ and $f_{ij} = \frac{h}{q+h}(-1 + q\delta_{\sigma_i, \sigma_j})$. It can be readily verified that $\sum_{\sigma} f_{ij} = 0$ for all $\{i, j\} \in E(\mathbf{t})$, consequently, all subgraphs with one or more vertices of degree 1 give rise to zero contributions. Thus, the partition function can be written as follow

$$Z_P(\beta, q, \mathbf{t}) = \left(\frac{q+h}{q} \right)^{|E(\mathbf{t})|} \sum_{\sigma} \sum_{A \in G(\mathbf{t})} \prod_{\{i,j\} \in A} f_{ij},$$

where $G(\mathbf{t})$ is the set of families of edges of $\mathbf{t}$ without vertices of degree 1. Therefore, we can rewrite the partition function as

$$Z_P(\beta, q, \mathbf{t}) = \left(\frac{q+h}{q} \right)^{|E(\mathbf{t})|} \sum_{A \in G(\mathbf{t})} w(A)$$

where $w(A) = \sum_{\sigma} \prod_{\{i,j\} \in A} f_{ij}$ is a weight factor associated with the subset A . We then proceeded to determine $w(A)$. An expression of $w(A)$ for general A can be obtained by further expanding in $w(A)$ the product $\sum_{\sigma} \prod_{\{i,j\} \in A} f_{ij}$. This procedure leads to

$$w(A) = \left(\frac{h}{q+h} \right)^{|A|} \sum_{\sigma} \mathcal{P}(A)(\sigma),$$

where $\mathcal{P}(A)(\sigma) = \prod_{e \in A} (-1 + q\delta_e(\sigma))$, and if $e = \{i, j\}$ then $\delta_e(\sigma) = \delta_{\sigma_i, \sigma_j}$. Expanding $\mathcal{P}(A)(\sigma)$, we have the following representation

$$\begin{aligned} \mathcal{P}(A)(\sigma) &= (-1)^{|A|} + (-1)^{|A|-1} q \sum_{e \in A} \delta_e(\sigma) + (-1)^{|A|-2} q^2 \sum_{e_1, e_2 \in A} \delta_{e_1}(\sigma) \delta_{e_2}(\sigma) \\ &\quad + \cdots + (-1) q^{|A|-1} \sum_{e_1, \dots, e_{|A|-1} \in A} \delta_{e_1}(\sigma) \cdots \delta_{e_{|A|-1}}(\sigma) \\ &\quad + q^{|A|} \delta_{e_1}(\sigma) \cdots \delta_{e_{|A|}}(\sigma). \end{aligned}$$

Figure 5.4: Examples of three subgraphs of A with 8 edges. It is clear that the term $\xi(e_1, \dots, e_8)$ depends of the topology of the subgraphs.

We choose k edges $\{e_1, \dots, e_k\}$ of A . These edges form a subgraph of A . Thus, we obtain

$$\sum_{\sigma} \delta_{e_1}(\sigma) \dots \delta_{e_k}(\sigma) = q^{|V(\mathbf{t})|-k+\xi(e_1, \dots, e_k)}$$

where $\xi(e_1, \dots, e_k)$ stands the total numbers of internal faces in each maximal connected component of $\{e_1, \dots, e_k\}$ (number of independent circuits in $\{e_1, \dots, e_k\}$). Note that this terms depends essentially on the topology of $\{e_1, \dots, e_k\}$ (see Figure 5.4). But $\xi(e_1, \dots, e_k) \leq \frac{2}{3}(k+1)$ for all k . Thus, we obtain the estimate

$$\sum_{\sigma} \delta_{e_1}(\sigma) \dots \delta_{e_k}(\sigma) \leq q^{|V(\mathbf{t})|-k+\frac{2}{3}(k+1)} = q^{|V(\mathbf{t})|-\frac{k}{3}+\frac{2}{3}}$$

and

$$\sum_{\sigma} \sum_{e_1, \dots, e_k \in A} \delta_{e_1}(\sigma) \dots \delta_{e_k}(\sigma) \leq \binom{|A|}{k} q^{|V(\mathbf{t})|-\frac{k}{3}+\frac{2}{3}}.$$

Therefore,

$$\sum_{\sigma} \mathcal{P}(A)(\sigma) \leq q^{|V(\mathbf{t})| + \frac{2}{3}} \sum_{k=0}^{|A|} \binom{|A|}{k} (-1)^{|A|-k} (\sqrt[3]{q^2})^k = q^{|V(\mathbf{t})| + \frac{2}{3}} (\sqrt[3]{q^2} - 1)^{|A|},$$

and

$$\begin{aligned} Z_P(\beta, q, \mathbf{t}) &\leq \left(\frac{q+h}{q} \right)^{|E(\mathbf{t})|} q^{|V(\mathbf{t})| + \frac{2}{3}} \sum_{A \in G(\mathbf{t})} \left((\sqrt[3]{q^2} - 1) \frac{h}{q+h} \right)^{|A|} \\ &\leq \left(\frac{q+h}{q} \right)^{|E(\mathbf{t})|} q^{|V(\mathbf{t})| + \frac{2}{3}} \left(1 + \sum_{k \geq 1} \Omega_k(\mathbf{t}) u^k \right), \end{aligned}$$

where $\Omega_k(\mathbf{t}) = |\{A \in G(\mathbf{t}) : |A| = k\}|$ and $u = (\sqrt[3]{q^2} - 1) \frac{h}{q+h}$. But $\Omega_k(\mathbf{t}) \leq \binom{|E(\mathbf{t})|}{k}$. Thus, we get the estimate

$$Z_P(\beta, q, \mathbf{t}) \leq \left(\frac{q+h}{q} \right)^{|E(\mathbf{t})|} q^{|V(\mathbf{t})| + \frac{2}{3}} (1+u)^{|E(\mathbf{t})|}. \quad (5.36)$$

Proof of Theorem 5.1.2. Using estimate (5.36) and Table 5.1, we write the new bound (5.36) for the partition function of the Potts model on $\mathbf{t}$ in terms of the dynamical variable $n(\mathbf{t})$, and make similar computations for the dual case. For that, we consider two cases of interest separately.

1. *The model on CDTs $\mathbf{t}$:* In this case, we can write the new bound for the partition function in terms of volume $n(\mathbf{t})$ of the triangulation as follow

$$Z_P(\beta, q, \mathbf{t}) \leq \left(\frac{q+h}{q} \right)^{\frac{3}{2}n(\mathbf{t})} q^{\frac{1}{2}n(\mathbf{t}) + \frac{2}{3}} (1+u)^{\frac{3}{2}n(\mathbf{t})}. \quad (5.37)$$

Using estimate (5.37), we obtain a new upper bound for the partition function of the q -state Potts model coupled to CDTs

$$\Xi_N(\beta, \mu) \leq q^{\frac{2}{3}} \sum_{\mathbf{t}} \exp \{-\tilde{\mu} n(\mathbf{t})\} = q^{\frac{2}{3}} Z_N(\tilde{\mu}), \quad (5.38)$$

where $\tilde{\mu} = \mu - \frac{3}{2} \ln \left(\frac{q+h}{q} \right) - \frac{1}{2} \ln q - \frac{3}{2} \ln(1+u)$ and $Z_N(\tilde{\mu})$ is the partition function for pure CDTs (defined in (2.4)) in the cylinder C_N with periodical spatial boundary conditions and for the value of the cosmological constant $\tilde{\mu}$. Hence, the inequality

$$\mu > \frac{3}{2} \ln(q + e^\beta - 1) + \ln 2 - \ln q + \frac{3}{2} \ln \left(1 + (q^{2/3} - 1) \frac{e^\beta - 1}{q + e^\beta - 1} \right) \quad (5.39)$$

provides a sufficient condition for subcriticality of the q -state Potts model coupled to CDTs

(summation is over all Lorentzian triangulation $\mathbf{t}$).

Comparing the new upper bound (5.39) with bounds show in Tables 5.2 and 5.3 for the model on CDTs, we observe which the condition (5.39) is better than conditions show in Tables 5.2 and 5.3 for subcriticality behavior of model. Thus, using High-T expansion for q -state Potts model we obtained a better approximation of the critical curve.

Using the duality relation proved in Theorem 5.1.1 and bound (5.39), we obtain a new condition for subcriticality of the Potts model coupled to dual CDTs

$$\begin{aligned} \mu^* &> \frac{3}{2} \ln(e^{\beta^*} - 1) + \ln 2 + \frac{3}{2} \ln \left(q + \frac{q}{e^{\beta^*} - 1} \right) - \frac{3}{2} \ln q + \frac{3}{2} \ln \left(1 + \frac{q^{2/3} - 1}{e^{\beta^*}} \right) \\ &> \frac{3}{2} \beta^* + \ln 2 + \frac{3}{2} \ln \left(1 + \frac{q^{2/3} - 1}{e^{\beta^*}} \right) \end{aligned} \quad (5.40)$$

We will see that this same approach on dual triangulations does not improve the curves obtained.

2. *The model on dual CDTs $\mathbf{t}^*$* : Similarly as Eq. (5.37), using Table 5.1 we get the estimate on a dual triangulation $\mathbf{t}^*$

$$Z_P(\beta^*, q, \mathbf{t}^*) \leq \left(\frac{q+h}{q} \right)^{\frac{3}{2}n(\mathbf{t})} q^{n(\mathbf{t}) + \frac{2}{3}} (1+u)^{\frac{3}{2}n(\mathbf{t})}. \quad (5.41)$$

Using (5.41), we obtain an upper bound for the partition function of the q -state Potts model coupled to dual CDTs

$$\Xi_N^*(\beta^*, \mu^*) \leq q^{\frac{2}{3}} \sum_{\mathbf{t}} \exp \{ -\tilde{\mu} n(\mathbf{t}) \} = q^{\frac{2}{3}} Z_N(\hat{\mu}), \quad (5.42)$$

where $\hat{\mu} = \mu^* - \frac{3}{2} \ln \left(\frac{q+h}{q} \right) - \ln q - \frac{3}{2} \ln(1+u)$ and $Z_N(\hat{\mu})$ is the partition function for pure CDTs in the cylinder C_N with periodical spatial boundary conditions and for the value of the cosmological constant $\hat{\mu}$. Hence, we obtain the inequality

$$\mu^* > \frac{3}{2} \ln(q + e^{\beta^*} - 1) + \ln 2 - \frac{1}{2} \ln q + \frac{3}{2} \ln \left(1 + (q^{2/3} - 1) \frac{e^{\beta^*} - 1}{q + e^{\beta^*} - 1} \right), \quad (5.43)$$

that provide a sufficient condition for subcriticality of the Potts model coupled to dual CDTs (summation is over all dual Lorentzian triangulation $\mathbf{t}^*$)

Finally, using the duality relation proved in Theorem 5.1.1 and bound (5.43), we obtain

a new condition for subcriticality of the Potts model coupled to CDTs

$$\begin{aligned} \mu &> \frac{3}{2} \ln(e^\beta - 1) + \ln 2 + \frac{3}{2} \ln \left(q + \frac{q}{e^\beta - 1} \right) - \frac{3}{2} \ln q + \frac{3}{2} \ln \left(1 + \frac{q^{2/3} - 1}{e^\beta} \right) \\ &> \frac{3}{2} \beta + \ln 2 + \frac{3}{2} \ln \left(1 + \frac{q^{2/3} - 1}{e^\beta} \right). \end{aligned} \quad (5.44)$$

It is easy to see that (5.43) and (5.44) does not improve the curves obtained in (5.40) and (5.39), respectively.

Convergence of the Gibbs measure $\mathbb{P}_N^{\beta, \mu}$ follows as a corollary. This concludes the proof. $\square$

5.5 Connection between transfer matrix and FK representation

In this section, we find a connection between transfer matrix approach and FK representation for the Ising model coupled to CDTs, comparing the curve obtained by transfer matrix approach and the curves obtained by FK representation. In this section we work with Potts model coupled to DUAL CDTs and will use notations of previous chapters.

5.5.1 $q = 2$ (Ising) systems

The transfer matrix method provides a curve $\mu^* = \psi(\beta^*)$ (blue line in Figure 5.5), defined in (4.23), that satisfies

$$\frac{d\psi}{d\beta^*}(0^+) = 0.$$

Therefore, we expect that critical curve satisfied the same property. Additionally, conditions $\text{tr}(\mathbf{K}^N) < \infty$ generate curves $\mu^* = \gamma_N(\beta^*)$ in the quadrant of parameters β^*, μ^* , such that $\gamma_{N+1} \leq \gamma_N$ and $\frac{d\gamma_N}{d\beta^*}(0^+) = 0$ for all N (see Proposition 4.4.4 and 4.4.5 in Chapter 4).

We define the functions

$$\varphi_{inf}(\beta^*) = \max \left\{ 2 \ln 2, \frac{3}{2} \ln(e^{\beta^*} - 1) + \ln 2 \right\},$$

and

$$\varphi_{sup}(\beta^*) = \min \left\{ \psi(\beta^*), \frac{3}{2} \ln(2^{2/3} + e^{\beta^*} - 1) + \ln 2 \right\}.$$

Graphs of function φ_{inf} is a lower barrier for the graph of γ_N for all N , and the critical curve of the model. Further, graphs of function φ_{sup} provides a better upper bound for the critical curve of Ising model coupled to dual CDTs. Furthermore, in low temperature, the

free energy satisfy

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta^*, \mu^*) \approx \ln \Lambda \left(\mu^* - \frac{3}{2} \beta^* \right).$$

Therefore, maximal eigenvalue Λ of operator $\mathbf{K}$ can be approximated by

$$\Lambda(\beta^*, \mu^*) \approx \Lambda \left(\mu^* - \frac{3}{2} \beta^* \right),$$

where Λ is defined in Chapter 2 in Eq. (2.10).

Figure 5.5: Region where the critical curve of the Ising model coupled to dual CDTs can be located.

5.5.2 q -Potts systems

As in Ising model case, (See Chapter 3), the transfer-matrix formalism suggests rewrite the partition function as

$$\Xi_N(\beta, \mu) = \text{tr } \mathbf{K}^N. \quad (5.45)$$

where we assume periodic spatial boundary condition and the operator $\mathbf{K}$ is defined by

$$\begin{aligned} K((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) &= \mathbf{1}_{\mathbf{t} \sim \mathbf{t}'} \exp \left\{ -\frac{\mu}{2} (n(\mathbf{t}) + n(\mathbf{t}')) \right\} \\ &\times \exp \left\{ -\frac{\beta}{2} (h(\boldsymbol{\sigma}) + h(\boldsymbol{\sigma}')) - \beta v(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \right\}. \end{aligned} \quad (5.46)$$

Theorem 3.2.1 and Proposition 3.2.1 given conditions for existence of Gibbs measures for the model in terms of the trace of operator $\mathbf{K}$.

Estimating $\text{tr}(\mathbf{K}\mathbf{K}^T)$: The condition $\text{tr}(\mathbf{K}\mathbf{K}^T) < \infty$ guarantees that $\mathbf{K}$ and $\mathbf{K}^T$ are Hilbert-Schmidt operators. Consequently, the operators $\mathbf{K}$ and $\mathbf{K}^T$ are bounded and $\mathbf{K}^2$ and $(\mathbf{K}^T)^2$

are of trace class. In particular, $\mathbf{K}^2$ and $(\mathbf{K}^T)^2$ belong to space $\mathcal{C}_p$ for all $p \geq 2$ (see Appendix A).

By definition the trace (A.2.1), we need to calculate the series

$$\mathrm{tr}(\mathbf{K}^T \mathbf{K}) = \sum_{(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')). \quad (5.47)$$

As in Chapter 3, we represent the triangulation $\mathbf{t}$ and its supported spin-configuration $\boldsymbol{\sigma}$ as

$$\mathbf{t} := (\mathbf{t}_{\mathrm{up}}, \mathbf{t}_{\mathrm{do}}) \quad \text{and} \quad \boldsymbol{\sigma} := (\boldsymbol{\sigma}_{\mathrm{up}}, \boldsymbol{\sigma}_{\mathrm{do}}).$$

Here

$$\mathbf{t}_{\mathrm{up}} = (t_{\mathrm{up}}^1, \dots, t_{\mathrm{up}}^n), \quad \mathbf{t}_{\mathrm{do}} = (t_{\mathrm{do}}^1, \dots, t_{\mathrm{do}}^m),$$

and

$$\boldsymbol{\sigma}_{\mathrm{up}} = (\sigma_{\mathrm{up}}^1, \dots, \sigma_{\mathrm{up}}^n), \quad \boldsymbol{\sigma}_{\mathrm{do}} = (\sigma_{\mathrm{do}}^1, \dots, \sigma_{\mathrm{do}}^m),$$

assuming that the supporting single-strip triangulation $\mathbf{t}$ contains n up-triangles and m down-triangles. (The actual order of up- and down-triangles and supported spins does not matter.)

The same can be done for the pair $(\mathbf{t}', \boldsymbol{\sigma}')$ (see proof of Lemma 3.2.1). Let recall that the triangulations $\mathbf{t}$ and $\mathbf{t}'$ are consistent ($\mathbf{t} \sim \mathbf{t}'$) iff number of the down-triangles in $\mathbf{t}$ equals that of up-triangles in $\mathbf{t}'$.

To calculate the sum (5.47) we divide the summation over $(\mathbf{t}', \boldsymbol{\sigma}')$ into a summation over $(\mathbf{t}'_{\mathrm{up}}, \boldsymbol{\sigma}'_{\mathrm{up}})$ and $(\mathbf{t}'_{\mathrm{do}}, \boldsymbol{\sigma}'_{\mathrm{do}})$. Firstly, fix a pair $(\mathbf{t}'_{\mathrm{up}}, \boldsymbol{\sigma}'_{\mathrm{up}})$ and make the sum over $(\mathbf{t}'_{\mathrm{do}}, \boldsymbol{\sigma}'_{\mathrm{do}})$. Note that the term $V((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))$ depends only on $\boldsymbol{\sigma}_{\mathrm{do}}$ and $\boldsymbol{\sigma}'_{\mathrm{up}}$. Consequently,

$$\begin{aligned} & \sum_{\mathbf{t}'_{\mathrm{do}}, \boldsymbol{\sigma}'_{\mathrm{do}}} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) \\ &= e^{-\beta H(\boldsymbol{\sigma})} e^{-2\beta V((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))} e^{-\mu n(\mathbf{t})} \sum_{(\mathbf{t}'_{\mathrm{do}}, \boldsymbol{\sigma}'_{\mathrm{do}})} e^{-\beta H(\boldsymbol{\sigma}')} e^{-\mu n(\mathbf{t}')}. \end{aligned} \quad (5.48)$$

The sum in the right-hand side of (5.48) can be represented in a matrix form. Denote by e_k the unit vectors in $\mathbb{R}^q$: $e_k = (0, \dots, 1, \dots, 0)^T$. Next, let us introduce a $q \times q$ matrix T where

$$T = e^{-\mu} \begin{pmatrix} e^\beta & 1 & \cdots & 1 \\ 1 & e^\beta & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & e^\beta \end{pmatrix} \quad (5.49)$$

Denote by $n(i), i = 1, \dots, n_{up}(\mathbf{t}')$ the number of down-triangles in $\mathbf{t}'$ which are between the i th and $(i + 1)$ th up-triangles in $\mathbf{t}'$. Let $n_{up}(\mathbf{t}') = k$ then

$$\begin{aligned} \sum_{\mathbf{t}'_{do}, \boldsymbol{\sigma}'_{do}} e^{-\beta H(\boldsymbol{\sigma}')} e^{-\mu n(\mathbf{t}')} &= \sum_{n(i) \geq 0: \sum_i n(i) \geq 1} \prod_{l=1}^k \left(e_{\sigma'_{up}^l}^T T^{n(l)+1} e_{\sigma'^{l+1}_{up}} \right) \\ &= \prod_{l=1}^k \left(e_{\sigma'^l_{up}}^T M e_{\sigma'^{l+1}_{up}} \right) - \prod_{l=1}^k \left(e_{\sigma'^l_{up}}^T T e_{\sigma'^{l+1}_{up}} \right) \end{aligned} \quad (5.50)$$

where the matrix M is the sum of the geometric progression

$$M = \sum_{n=1}^{\infty} T^n \quad (5.51)$$

Using the same procedure we can obtain the sum over all up-triangles into the triangulation $\mathbf{t}$. The only difference is the existence of marked up-triangle in the strip: let as before $n_{up}(\mathbf{t}') = n_{do}(\mathbf{t}) = k$ then

$$\sum_{\mathbf{t}_{up}, \boldsymbol{\sigma}_{up}} e^{-\beta H(\boldsymbol{\sigma})} e^{-\mu n(\mathbf{t})} = \prod_{l=1}^{k-1} \left(e_{\sigma_{up}^l}^T M e_{\sigma_{up}^{l+1}} \right) \left(e_{\sigma_{up}^k}^T M^2 e_{\sigma_{up}^1} \right) \quad (5.52)$$

Supposing the existence of the matrix M and using (5.50) and (5.52) we obtain the following:

$$\begin{aligned} \sum_{\mathbf{t}_{up}, \boldsymbol{\sigma}_{up}} \sum_{\mathbf{t}'_{do}, \boldsymbol{\sigma}'_{do}} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) &= e^{-2\beta V((\mathbf{t}_{do}, \boldsymbol{\sigma}_{do}), (\mathbf{t}'_{up}, \boldsymbol{\sigma}'_{up}))} \\ &\times \sum_{\mathbf{t}_{up}, \boldsymbol{\sigma}_{up}} e^{-\beta H(\boldsymbol{\sigma})} e^{-\mu n(\mathbf{t})} \sum_{(\mathbf{t}'_{do}, \boldsymbol{\sigma}'_{do})} e^{-\beta H(\boldsymbol{\sigma}')} e^{-\mu n(\mathbf{t}')} \\ &= e^{-2\beta V((\mathbf{t}_{do}, \boldsymbol{\sigma}_{do}), (\mathbf{t}'_{up}, \boldsymbol{\sigma}'_{up}))} \\ &\times \left[\prod_{l=1}^k \left(e_{\sigma'^l_{up}}^T M e_{\sigma'^{l+1}_{up}} \right) \prod_{l=1}^{k-1} \left(e_{\sigma_{do}^l}^T M e_{\sigma_{do}^{l+1}} \right) \left(e_{\sigma_{up}^k}^T M^2 e_{\sigma_{up}^1} \right) \right. \\ &\quad \left. - \prod_{l=1}^k \left(e_{\sigma'^l_{up}}^T T e_{\sigma'^{l+1}_{up}} \right) \prod_{l=1}^{k-1} \left(e_{\sigma_{do}^l}^T M e_{\sigma_{do}^{l+1}} \right) \left(e_{\sigma_{up}^k}^T M^2 e_{\sigma_{up}^1} \right) \right]. \end{aligned} \quad (5.53)$$

Necessary and sufficient condition for the convergence of the matrix series for M is that the maximal eigenvalue of matrix T is less than 1. The eigenvalues of T are

$$\lambda_1 = e^{-\mu}(q + e^\beta - 1), \quad \lambda_2 = e^{-\mu}(e^\beta - 1), \quad (5.54)$$

and the above condition means that $\lambda_1 < 1$ or, equivalently,

$$\mu > \ln(q + e^\beta - 1). \quad (5.55)$$

Under this condition (5.55), the matrix M is calculated explicitly:

$$M = g(\beta, \mu) \times \begin{pmatrix} 1 + f(\beta, \mu) & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 + f(\beta, \mu) \end{pmatrix}_{q \times q}, \quad (5.56)$$

where

$$g(\beta, \mu) = \frac{e^{-\mu}}{(1 - e^{-\mu}(e^\beta - 1))(1 - e^{-\mu}(q + e^\beta - 1))}$$

and

$$f(\beta, \mu) = (e^\beta - 1)(1 - e^{-\mu}(q + e^\beta - 1)).$$

Now, we express the above sum (5.47) as the partition function of a one-dimensional q -state Potts model where states are pairs of spins $(\sigma_{\text{do}}^l, \sigma_{\text{up}}^l)$ and the interaction is via the matrix T between the members of the pair and via matrix M between neighboring pairs. More precisely, introducing the lexicographic order in the set $\{(i, j) : 1 \leq i, j \leq q\}$, we define the following $q^2 \times q^2$ matrices:

$$\mathcal{A} = \begin{cases} e^{2\beta}(e_n^T M e_l) \cdot (e_m^T M e_k) & , \quad n = m \text{ and } l = k \\ (e_n^T M e_l) \cdot (e_m^T M e_k) & , \quad n \neq m \text{ and } l \neq k \\ e^\beta(e_n^T M e_l) \cdot (e_m^T M e_k) & , \quad \text{either } n = m \text{ or } l = k, \end{cases} \quad (5.57)$$

$$\mathcal{A}_m = \begin{cases} e^{2\beta}(e_n^T M e_l) \cdot (e_m^T M^2 e_k) & , \quad n = m \text{ and } l = k \\ (e_n^T M e_l) \cdot (e_m^T M^2 e_k) & , \quad n \neq m \text{ and } l \neq k \\ e^\beta(e_n^T M e_l) \cdot (e_m^T M^2 e_k) & , \quad \text{either } n = m \text{ or } l = k, \end{cases} \quad (5.58)$$

$$\mathcal{A}_t = \begin{cases} e^{2\beta}(e_n^T T e_l) \cdot (e_m^T M e_k) & , \quad n = m \text{ and } l = k \\ (e_n^T T e_l) \cdot (e_m^T M e_k) & , \quad n \neq m \text{ and } l \neq k \\ e^\beta(e_n^T T e_l) \cdot (e_m^T M e_k) & , \quad \text{either } n = m \text{ or } l = k, \end{cases} \quad (5.59)$$

$$\mathcal{A}_{tm} = \begin{cases} e^{2\beta}(e_n^T T e_l) \cdot (e_m^T M^2 e_k) & , \quad n = m \text{ and } l = k \\ (e_n^T T e_l) \cdot (e_m^T M^2 e_k) & , \quad n \neq m \text{ and } l \neq k \\ e^\beta(e_n^T T e_l) \cdot (e_m^T M^2 e_k) & , \quad \text{either } n = m \text{ or } l = k. \end{cases} \quad (5.60)$$

Now for the sum under consideration (5.47) we obtain using representation (5.53)

$$\text{tr}(\mathbf{K}^T \mathbf{K}) = \sum_{(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) = \text{tr}\left(\left(\sum_{k=0}^{\infty} \mathcal{A}^k\right) \mathcal{A}_m\right) - \text{tr}\left(\left(\sum_{k=1}^{\infty} \mathcal{A}_t^k\right) \mathcal{A}_{tm}\right).$$

By the construction the matrix $\mathcal{A}$ is greater then $\mathcal{A}_t$ elementwise. Thus the eigenvalue of matrix $\mathcal{A}$ is greater than the eigenvalue of the matrix Q_t (it follows from the Perron-Frobenius theorem). Therefore the necessary and sufficient condition for the convergence in (5.47) is that the largest eigenvalue of $\mathcal{A}$ is less than 1. In general, it is impossible to calculate its eigenvalue analytically. For case $q = 2$ (Ising model) was possible calculated its eigenvalue (see Chapter 3 for review), but $q > 2$ it very difficult. In the case of $q = 4$, we make a comparison between curves obtained by duality relation and high-T expansion in Section 5.4, and curve obtained using numerical simulation. See below graph.

Finally as byproduct of Theorem 5.1.2, we have the following assertions for the free energy for q -state Potts model.

Corollary 5.5.1. *The free energy for q -state Potts model coupled to dual CDTs satisfy*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta^*, \mu^*) \leq \ln \Lambda \left(\mu^* - \frac{3}{2} \beta^* - r(\beta^*) \right),$$

Figure 5.6: The blue line is the simulation of $\|\mathcal{A}\|_2 = 1$ for $q = 4$. Black line: $\mu^* = 3 \ln 2$. Green line: $\mu^* = \frac{3}{2} \ln(e^{\beta^*} - 1) + \ln 2$. Red line: $\mu^* = \frac{3}{2} \ln(4^{2/3} + e^{\beta^*} - 1) + \ln 2$.

where $r(\beta^*) = \frac{3}{2} \ln \left(1 + \frac{q^{2/3} - 1}{e^{\beta^*}} \right)$. Moreover, in low temperature, we have that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta^*, \mu^*) \approx \ln \Lambda \left(\mu^* - \frac{3}{2} \beta^* \right).$$

Therefore, maximal eigenvalue Λ of operator $\mathbf{K}$ can be approximated by

$$\Lambda(\beta^*, \mu^*) \approx \Lambda \left(\mu^* - \frac{3}{2} \beta^* \right),$$

where Λ is defined in Chapter 2 in Eq. (2.10).

5.6 Discussion and outlook

In this chapter we define and study a Potts model coupled to CDTs and dual CDTs employing FK representation and duality theory on graphs. The results obtained serve for the Ising model (see Section 5.5.1). In particular, we find a better region where the free energy exists and can be extended analytically (line I and I' in Figure 5.1), and this region depend on analyticity of maximal eigenvalue Λ of operator $\mathbf{K}$ and eigenvalue Λ , defined in Eqn (2.10). This remark permite us to give the conjecture that the boundary of the critical domain coincides with the locus of points (β, μ) where Λ loses either the property of positivity or the property of being a simple eigenvalue.

Notice that, if (β, μ) satisfy hypothesis of Theorem 5.1.2, we don't have information on either $\mathbf{K}$ belong to $\mathcal{C}_p$ or not, for some $p > 2$. This issue needs a further study.

We can state that follows assertion: If (β, μ) satisfy the subcritical behavior of Theorem 5.1.2, the limiting probability distribution $\mathbb{P}^{\beta, \mu}$ is represented by a positive recurrent Markov

chain with states $(\mathbf{t}, \boldsymbol{\sigma})$ as Theorem 3.2.2. In the subcritical region of Theorem 5.1.2, the typical triangulation for annealed Potts model coupled to CDTs is the same as subcritical case in pure CDT (see Theorem 2.2.2).

The new bounds for the critical curve for arbitrary q suggests that the Potts model coupled to CDTs exhibit a phase transition only on the critical curve and a first-order singularity at a unique point $\beta_{cr} \in (0, \infty)$. Additionally, the triangulations in the annealed model exhibit critical behavior as critical case in pure CDT (see Theorem 2.2.2) only on the critical curve. This direction also requires further research.

Appendix A

The von Neumann-Schatten Classes of Operators

This appendix is concerned with certain classes $\mathcal{C}_p$ ($1 \leq p < \infty$) of linear operators on a separable Hilbert space $\mathcal{H}$. These classes are important for to study the transfer operator in statistical mechanics because that operator encodes information and to study the behavior of the statistical mechanics system.

A.1 The space $\mathcal{C}_p$ and first properties

In this section we define the space $\mathcal{C}_p$ and given some properties.

Definition A.1.1. When $1 \leq p < \infty$, $\mathcal{C}_p$ is the set of all operators T in $\mathcal{B}(\mathcal{H})$ which satisfy the following condition: for each orthonormal system $\{\phi_k : k \in K\}$ in $\mathcal{H}$,

$$\sum_{k \in K} |\langle T\phi_k, \phi_k \rangle|^p < \infty.$$

We shall adopt the convention that $\mathcal{C}_\infty$ is $\mathcal{B}(\mathcal{H})$. Each $\mathcal{C}_p$ is a linear subspace of $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}_p \subseteq \mathcal{C}_q$ if $1 \leq p \leq q \leq \infty$. We can see that $T^* \in \mathcal{C}_p$ (adjoint operator) whenever $T \in \mathcal{C}_p$, and that, if $1 \leq p < \infty$, then each element of $\mathcal{C}_p$ is a compact operator.

Lemma A.1.1. Suppose that $1 \leq p < \infty$, T is a compact self-adjoint operator on $\mathcal{H}$, and $\{\lambda_n\}$ is the sequence of non-zero eigenvalues of T , counted according to their multiplicities.

1. If $T \in \mathcal{C}_p$, then $\sum_n |\lambda_n|^p < \infty$.
2. If $\sum_n |\lambda_n|^p < \infty$, then $T \in \mathcal{C}_p$ and, for each orthonormal system $\{\phi_k : k \in K\}$ in $\mathcal{H}$,

$$\sum_{k \in K} |\langle T\phi_k, \phi_k \rangle|^p \leq \sum_n |\lambda_n|^p.$$

A.2 The trace class $\mathcal{C}_1$

In this section we will define the trace of an operator on the class $\mathcal{C}_1$.

Lemma A.2.1. *Let $T \in \mathcal{C}_1$ and suppose that $\{\phi_k : k \in K\}$ is an orthonormal basis in $\mathcal{H}$.*

1. *The sum $\sum_k \langle T\phi_k, \phi_k \rangle$ exist, and does not depend on the particular choise of the orthonormal basis $\{\phi_k : k \in K\}$.*
2. *If $T = T^*$, then $\sum_k \langle T\phi_k, \phi_k \rangle = \sum_k \lambda_k$, where $\{\lambda_k\}$ is the sequence of non-zero eigenvalues of T , counted according to their multiplicities.*

Definition A.2.1. *The ideal $\mathcal{C}_1$ in $\mathcal{B}(\mathcal{H})$ is called trace class of operators on $\mathcal{H}$. If $T \in \mathcal{C}_1$ and $\{\phi_k : k \in K\}$ is an orthonormal basis in $\mathcal{H}$, then the trace of T , denoted by $\text{tr}(T)$, is defined by the equation*

$$\text{tr}(T) = \sum_k \langle T\phi_k, \phi_k \rangle.$$

Lemma A.2.1 shows that $\text{tr}(T)$ depends only on T (not on the choice of the orthonormal basis), and that $\text{tr}(T)$ is the sum of the eigenvalues of T when $T = T^*$.

The main algebraic properties of tr are the following.

Theorem A.2.1. *Suppose that $S, T \in \mathcal{C}_1$, $A \in \mathcal{B}(\mathcal{H})$ and α, β are scalars.*

1. $\text{tr}(\alpha S + \beta T) = \alpha \text{tr}(S) + \beta \text{tr}(T)$.
2. $\text{tr}(S^*) = \overline{\text{tr}(S)}$.
3. $\text{tr}(S) > 0$ if $S > 0$.
4. $\text{tr}(AS) = \text{tr}(SA)$.

The main result of this section, used in Chapter 3, is the following.

Theorem A.2.2. *Suppose that T is a trace class operator acting on a Hilbert space $\mathcal{H}$, and $\{\lambda_k\}$ is the sequence of non-zero eigenvalues of T , counted according to their algebraic multiplicities. Then*

$$\text{tr}(T) = \sum_k \lambda_k. \tag{A.1}$$

A.3 The Banach space $\mathcal{C}_p$

Suppose that T is a compact operator acting on $\mathcal{H}$, and denote by $V_T H_T$ the polar decomposition of T . Then $T = V_T H_T$ and $H_T = (T^* T)^{1/2}$. Remember that V_T is a partial isometry on the closed range $\mathcal{R}_H$ of H . It is well know that there exist a decreasing sequence $\{\mu_n\}$ of

positive real numbers (the eigenvalues of H_T , counted according to their multiplicities), and orthonormal sequence $\{\phi_n\}$, $\{\psi_n\}$, such that

$$H_T(x) = \sum_n \mu_n \langle x, \phi_n \rangle \phi_n,$$

$$T(x) = \sum_n \mu_n \langle x, \phi_n \rangle \psi_n.$$

Given $p \leq 1$ the function $f_p(t) = t^p$ is continuous on the non-negative real axis, and hence also on the spectrum of the positive operator H_T . The operator $f_p(H_T)$ will be denoted by H_T^p . The operator H_T^p is compact and can be represented by

$$H_T^p(x) = \sum_n \mu_n^p \langle x, \phi_n \rangle \phi_n.$$

Lemma A.3.1. *Suppose $1 \leq q \leq p < \infty$ and $T \in \mathcal{B}(\mathcal{H})$. Then the following three conditions are equivalent*

1. $T \in \mathcal{C}_p$,
2. $H_T \in \mathcal{C}_p$,
3. $H_T^{p/q} \in \mathcal{C}_q$.

From Lemma A.3.1 and the equivalence of conditions (i) and (ii) in the before lemma, it follows that a compact operator T on $\mathcal{H}$ lies $\mathcal{C}_p$ if only if the sequence $\{\mu_n\}$ of non-zero eigenvalues of $H_T = (T^*T)^{1/2}$ satisfies $\sum_n \mu_n^p < \infty$.

Definition A.3.1. *Suppose $1 \leq p < \infty$ and $T \in \mathcal{C}_p$. Then, we define*

$$\|T\|_p = [\text{tr}(H_T^p)]^{1/p} = \left(\sum_n \mu_n^p \right)^{1/p}.$$

It is not immediately obvious that $\|\cdot\|_p$ is a norm on $\mathcal{C}_p$. Since $\|T\| = \|H_T\| = \mu_1$ (maximal eigenvalue), we have

$$\|T\| \leq \|T\|_p, \quad \text{for } T \in \mathcal{C}_p.$$

Lemma A.3.2. *For each $T \in \mathcal{C}_1$, $|\text{tr}(T)| \leq \|T\|_1$.*

Lemma A.3.3. *For each $T \in \mathcal{C}_p$, $\|T^*\|_p = \|T\|_p$.*

Lemma A.3.4. *Suppose $1 \leq p < \infty$, $T \in \mathcal{C}_p$, and $\{\lambda_n\}$ is the sequence of non-zero eigenvalues of T , counted according to their algebraic multiplicities. Then*

$$\left(\sum_n |\lambda_n|^p \right)^{1/p} \leq \|T\|_p.$$

A.4 The Hilbert-Schmidt class

Definition A.4.1. *The ideal $\mathcal{C}_2$ in $\mathcal{B}(\mathcal{H})$ is called the Hilbert-Schmidt class of operators on $\mathcal{H}$.*

If $V_T H_T$ is the polar decomposition of an element T of $\mathcal{C}_2$, then

$$(\operatorname{tr}(T^* T))^{1/2} = (\operatorname{tr}(H_T^2))^{1/2} = \|T\|_2.$$

Theorem A.4.1. *Suppose that $T \in \mathcal{B}(\mathcal{H})$, $\{\phi_k\}$ and $\{\psi_k\}$ are orthonormal bases in $\mathcal{H}$. The the following three conditions are equivalent.*

1. $\sum_k \|T\phi_k\|^2 < \infty$,
2. $\sum_{j,k} |\langle T\phi_j, \psi_k \rangle|^2 < \infty$,
3. $T \in \mathcal{C}_2$.

When these conditions are satisfied, the sums occurring in (i) and (ii) are both equal to $\|T\|_2^2$.

Appendix B

Krein-Rutman Theorem

The Krein-Rutman theorem plays a very important role in linear functional analysis, as it provides the abstract basis for the proof of the existence of various principal eigenvalues, which in turn are crucial in transfer matrix formalism of statistical mechanics system (and another areas as nonlinear partial differential equations, bifurcation theory, etc). In this appendix, we will give the well-known Krein-Rutman theorem.

B.1 Krein-Rutman Theorem and the Principal Eigenvalue

Let X a Banach space. By *cone* $K \subset X$ we mean a closed convex set such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. A cone K in X induce a *partial ordering* $\leq$ by the rule: $u \leq v$ if and only if $v - u \in K$. A Banach space with such an ordering is usually called a partially ordered Banach space and the cone generating the partial ordering is called the positive cone of the space. If $\overline{K - K} = X$, i.e., the set $\{u - v : u, v \in K\}$ is dense in X , then K is called a *total cone*. If $K - K = X$, K is called a *reproducing cone*. If a cone has nonempty interior K° , then it is called a *solid cone*. Any solid cone has the property that $K - K = X$, in particular, it is total. We write $u > v$ if $u - v \in K \setminus \{0\}$, and $u \gg v$ if $u - v \in K^\circ$.

The main results of this appendix, used in Chapter 3, are the following.

Theorem B.1.1 (The Krein-Rutman Theorem [KR48]). *Let X a Banach space, $K \subset X$ a total cone and $T : X \rightarrow X$ a compact linear operator that is positive, i.e., $T(K) \subset K$, with positive spectral radius $r(T)$. Then $r(T)$ is an eigenvalue with an eigenvector $u \in K \setminus \{0\}$: $Tu = r(T)u$. Moreover, $r(T^*) = r(T)$ is an eigenvalue of T^* .*

Let us now use Theorem B.1.1 to derive the following useful result.

Theorem B.1.2. *Let X a Banach space, $K \subset X$ a solid cone, $T : X \rightarrow X$ a compact linear operator which is strongly positive, i.e., $Tu \gg 0$ if $u > 0$. Then*

1. $r(T) > 0$, and $r(T)$ is a simple eigenvalue with an eigenvector $v \in K^o$; ; there is no other eigenvalue with a positive eigenvector.
2. $|\lambda| < r(T)$ for all eigenvalues $\lambda \neq r(T)$.

Let us recall that λ is a simple eigenvalue of T if there exists $v \neq 0$ such that $Tv = \lambda v$ and $(\lambda I - T)^n w = 0$ for some $n \geq 1$ implies $w \in \text{span}\{v\}$.

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