**A FBI characterization for Gevrey vectors on hypo-analytic
structures and propagation of Gevrey singularities**

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Abstract

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In this work, we begin by studying the basic structure of Gevrey vectors on hypo-analytic structures, and we compare some basic properties of hypo-analytic functions and Gevrey vectors. Then we prove a FBI characterization for Gevrey vectors, and we point out the main role of the real structure bundle, in contrast to the characterization of hypo-analytic functions. We finish this work with an application, of the FBI characterization, for the propagation of singularities for solutions of the non-homogeneous system equations, associated with a real-analytic hypo-analytic structure of tube type.

Keywords: Gevrey vectors, FBI transform, Propagation of singularities, Hypo-analytic structures.

Resumo

Braun Rodrigues, N. **Caracterização de vetores Gevrey pela transformada FBI em estruturas hipo-analíticas e propagação de singularidades Gevrey**. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2020.

Neste trabalho exploramos a teoria dos vetores Gevrey em estruturas hipo-analíticas e mostramos as principais diferenças entre funções hipo-analíticas e vetores de Gevrey. Em seguida provamos uma caracterização via transformada FBI para vetores Gevrey e destacamos o papel principal do fibrado estrutural real nesse teorema, em contraste com a caracterização de funções hipo-analíticas. Concluimos este trabalho com uma aplicação na propagação de singularidades para soluções do sistema de equações não homogêneas, associado à uma estrutura hipo-analítica, analítica real e do tipo tubo.

Palavras-chave: Vetores Gevrey, Transformada FBI, Propagação de singularidades, Estruturas hipo-analíticas.

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Chapter 1

Introduction

The motivation for hypo-analytic structures lies in the theory of several complex variables, to be more precise, in the theory of holomorphic extendability of CR functions. It was in 1983 that M.S. Baouendi, C.H. Chang, and F. Trèves introduced in [BCT83] the theory of hypo-analytic structures and how one could use the FBI transform in such structures to characterize microlocal hypo-analyticity. Since then several authors made contributions to the theory of hypo-analytic structures, names like P.D. Cordaro, F. Trèves, N. Hanges, S. Berhanu, P. Caetano, L.F. Ragognette, to name a few.

The majority of the work done in this theory concerns the notion of hypo-analyticity, which is the generalization to hypo-analytic structures of holomorphic extendability. There are results about side extendability on finite type hypersurfaces [BT84], propagation of hypo-analytic singularities [HT83], [Tré90], local solvability of differential complex associated to hypo-analytic structures [CT91], and many others.

One of the simplest examples of a hypo-analytic structure is the structure induced by the complex vector fields $\{\partial_{z_1}, \dots, \partial_{z_m}\}$ in $\mathbb{C}^m$ on a maximally real submanifold $\mathcal{H}$. Let us write the restriction of the complex vector fields $\{\partial_{z_1}, \dots, \partial_{z_m}\}$ to $\mathcal{H}$ as $\{M_1, \dots, M_m\}$. A distribution u on $\mathcal{H}$ is hypo-analytic at a point $p \in \mathcal{H}$ if, in some neighborhood of p , we have that u is the restriction of a holomorphic function, defined on a full complex neighborhood of p , to $\mathcal{H}$. So given a maximally real submanifold $\mathcal{H}$ of $\mathbb{C}^m$ we have automatically a lot of hypo-analytic functions, namely all the restrictions of holomorphic functions. Note that this procedure is valid despite the regularity of $\mathcal{H}$.

Now one interesting fact about hypo-analyticity is that it is equivalent to a distribution u on $\mathcal{H}$ to be hypo-analytic at $p \in \mathcal{H}$ and to be an analytic vector for $\{M_1, \dots, M_m\}$, *i.e.*,

$$\sup_{z \in K} |M^\alpha u(z)| \leq C^{|\alpha|+1} \alpha!, \quad \forall \alpha \in \mathbb{Z}_+^m,$$

for some constant $C = C_K > 0$, for every $K \subset U$ compact set, and $U \subset \mathcal{H}$ some open neighborhood of p . So again, since we can restrict holomorphic functions to $\mathcal{H}$ (despite its regularity), it makes sense to talk about the space of analytic vectors on open sets of $\mathcal{H}$, $\mathcal{C}^\omega(U; M_1, \dots, M_m)$. Now that we can talk about analytic vectors it is natural to define the space of Gevrey vectors on open sets of $\mathcal{H}$. So we say that a function u is a Gevrey- s vector on U , and we write $u \in G^s(U; M_1, \dots, M_m)$ if for every compact $K \subset U$ there is a positive constant C such that

$$\sup_{z \in K} |M^\alpha u(z)| \leq C^{|\alpha|+1} \alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^m.$$

So again we can talk about the space of Gevrey vectors on $\mathcal{H}$, despite of its regularity. And this is, at least in my opinion, what makes the theory of Gevrey vectors on hypo-analytic structures interesting. For instance, if u is an analytic vector and u is flat at a given point $p \in \mathcal{H}$ then it vanishes in an open neighborhood of p on $\mathcal{H}$, simply because this is true for holomorphic functions (in view that $\mathcal{H}$ is maximally real). Now we know that in $\mathbb{R}^m$ we have Gevrey functions with compact support, so this raises the natural question: the set of Gevrey vectors with compact support on

smooth maximally real submanifolds of $\mathbb{C}^m$ is non trivial in general? Until now this question is open. This fair simple question with no trivial answer makes this theory very interesting to study.

There are some works done concerning Gevrey vectors on hypo-analytic structures, for instance [Cae00], [CC11] and [Rag19]. All these works on Gevrey vectors cover questions like representation theorems for ultradistributions with infinite order operators and solvability for the associate differential complex. Also R.D. Medrado characterized compositions of Denjoy-Carleman functions with a first integral Z in his Ph.D thesis [Med16].

In this work we consider regularity questions, such as propagation of singularities. But first, we use the FBI transform, adapted to the hypo-analytic structure, to characterize the Gevrey vectors. The FBI transform is a very useful tool when dealing with regularity problems. The adapted FBI transform has an additional difficulty, when compared with the usual one in $\mathbb{R}^m$, it has a "complex phase". Let us briefly try to explain that using maximally real submanifolds of $\mathbb{C}^m$. Locally a maximally real submanifold is the image of a map $Z : \mathbb{R}^m \rightarrow \mathbb{C}^m$ like

$$Z(x) = x + i\phi(x),$$

where $\phi(x)$ is a real-valued smooth function satisfying $\phi(0) = 0$, and $d\phi(0) = 0$. The FBI transform of a distribution with compact support is defined as

$$\mathfrak{F}[u](z, \zeta) = \int e^{i\zeta \cdot (z - Z(x')) - \langle \zeta \rangle \langle z - Z(x', t) \rangle^2} u(x') dZ(x'),$$

where $z \in \mathbb{C}^m$, $\zeta \in \mathfrak{C}_1 = \{\eta \in \mathbb{C}^m : |\operatorname{Im}\eta| < |\operatorname{Re}\eta|\}$, and $\langle w \rangle^2 = w \cdot w$, for $w \in \mathbb{C}^m$. Note that the two components of the exponent have real parts, in contrast with the exponent in the usual FBI transform

$$\mathfrak{F}[u](x, \xi) = \int e^{i\xi \cdot (x - x') - |\xi||x - x'|^2} u(x') dx'.$$

When dealing with analytic vectors, this more complex phase function is not a problem because holomorphic function theory is available in this scenario. But it is not the case for Gevrey vectors, for them we need to "stay" on the real structure bundle. We are not getting into the details of this definition now, but if (z, ζ) belongs to the real structure bundle and z' belongs to the maximally real submanifold then we have the following bound of the real part of the exponent:

$$\operatorname{Re}\{i\zeta \cdot (z - z') - \langle \zeta \rangle \langle z - z' \rangle^2\} \leq -c|\zeta||z - z'|^2,$$

for some $c > 0$. So in some sense, if we only consider (z, ζ) in this real structure bundle we can try to mimic the techniques used in the real scenario.

This thesis is organized as follows: in the first chapter we begin by showing the exact definition and providing some examples of maximally real submanifolds of $\mathbb{C}^m$. Then we recall some known results about the theory of holomorphic extendability and the relation between Gevrey vectors and almost-analytic extensions. Since every hypo-analytic structure defines a locally integrable structure, and most of the results presented here depend on this structure, we devoted a section to recall the principal features needed, and we finish the first chapter defining hypo-analytic structures, hypo-analytic functions, and we show some crucial differences between the theory of analytic vectors (hypo-analytic functions) and Gevrey vectors. Then in the second chapter, we define the FBI transform that we are actually going to use, we prove some inversion formulas, we also point out how important is the role of the real-structural bundle in the Gevrey scenario, and then we state and prove the first main theorem of this work, that is the FBI characterization for Gevrey vectors. In the last chapter we prove a propagation of singularities result for solutions of the non-homogeneous system of equations (associated with the hypo-analytic structure), but only for real-analytic structures of tube type, the second main theorem of this work, and we give some examples of such structures, or to be more precise, of systems of complex vector fields.

Chapter 2

Hypo-analytic structures

In 1983, M.S. Baouendi, C.H. Chang, and F. Trèves introduced the concept of a hypo-analytic structure in order to study the holomorphic extendability property for **CR** functions in a more general scenario. Since then, this theory has been studied by a lot of authors, such as P. Cordaro, S. Berhanu, F. Trèves, N. Hanges, to name a few. In this chapter we recall some basic definitions, and results, so, later on, we can point out the difficulties when dealing with some sort of Gevrey regularity in these structures.

Maximally real submanifolds of $\mathbb{C}^m$ are important objects in the theory of hypo-analytic structures, so we dedicate the first section of this chapter to them. Then we move to the definitions of hypo-analytic structures, hypo-analytic functions, and then we will discuss the link between hypo-analyticity and analytic vectors, and how much we lose working with Gevrey regularity. We end this chapter recalling what is known regarding Gevrey vectors on hypo-analytic structures.

2.1 Maximally real submanifolds of $\mathbb{C}^m$

The so-called **maximally real** submanifolds of $\mathbb{C}^m$ play a central role in the study of hypo-analytic structures. Before defining them we shall fix some notation. As usual we denote the real coordinates of $\mathbb{C}^m$ as $\{x_1, y_1, \dots, x_m, y_m\}$. Let $p \in \mathbb{C}^m$, so every vector at p , $X \in T_p\mathbb{C}^m$, can be written as

$$X = \sum_{j=1}^m \left\{ a_j \frac{\partial}{\partial x_j} \Big|_p + b_j \frac{\partial}{\partial y_j} \Big|_p \right\},$$

where a_j, b_j , $j = 1, \dots, m$, are real numbers. So the complexified tangent space of $\mathbb{C}^m$, at p , can be viewed as

$$\mathbb{C}T_p\mathbb{C}^m = \left\{ X = \sum_{j=1}^m \left[a_j \frac{\partial}{\partial x_j} \Big|_p + b_j \frac{\partial}{\partial y_j} \Big|_p \right] : a_j, b_j \in \mathbb{C}, j = 1, \dots, m \right\}.$$

As usual, for $j = 1, \dots, m$, we write

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

With these complex vector fields in $\mathbb{C}^m$, we can write an arbitrary complex vector $X \in \mathbb{C}T_p\mathbb{C}^m$, in the following form:

$$X = \sum_{j=1}^m \left\{ a_j \frac{\partial}{\partial z_j} \Big|_p + b_j \frac{\partial}{\partial \bar{z}_j} \Big|_p \right\}.$$

We say that X is a holomorphic vector if $b_j = 0$, $j = 1, \dots, m$, and that it is an antiholomorphic

vector if $a_j = 0$, $j = 1, \dots, m$. We are going to denote the set of all holomorphic vectors at p by $T_p^{1,0}\mathbb{C}^m$ and the set of all antiholomorphic vectors at p by $T_p^{0,1}\mathbb{C}^m$.

Let $\mathcal{H} \subset \mathbb{C}^m$ be a submanifold. Here submanifold stands for embedded submanifold. We say that $\mathcal{H}$ is **maximally real** if

$$\mathbb{C}T_p\mathbb{C}^m \simeq T_p^{0,1}\mathbb{C}^m \oplus \mathbb{C}T_p\mathcal{H}, \quad (2.1)$$

for every $p \in \mathcal{H}$. This is equivalent to the pullback map $\mathbb{C}T^*\mathbb{C}^m|_{\mathcal{H}} \rightarrow \mathbb{C}T^*\mathcal{H}$ inducing an isomorphism $T_{1,0}\mathbb{C}^m \cong \mathbb{C}T^*\mathcal{H}$, where $T_{1,0}\mathbb{C}^m$ is the space of all $(1, 0)$ forms.

Example 2.1.1. The most natural example of a maximally real submanifold of $\mathbb{C}^m$ is $\mathbb{R}^m$. To show that it is in fact maximally real, we need to exhibit the isomorphism (2.1). We can see $\mathbb{R}^m$ as

$$\{(z, \bar{z}) \in \mathbb{C}^m : z_j - \bar{z}_j = 0, j = 1, \dots, m\},$$

so a complex vector $X = \sum_j a_j \partial_{z_j} + b_j \partial_{\bar{z}_j}$ is tangent to $\mathbb{R}^m$ if, and only if,

$$X(z_j - \bar{z}_j) = 0, \quad \forall j = 1, \dots, m,$$

i.e., $a_j = b_j$, $j = 1, \dots, m$. So we can rewrite X ,

$$X = \sum_{j=1}^m a_j \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j} \right).$$

Hence $T^{0,1} \cap \mathbb{C}T\mathbb{R}^m = \emptyset$. Now for every $X \in \mathbb{C}T\mathbb{C}^m$ we can write

$$X = \sum_{j=1}^m a_j \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial \bar{z}_j} = \underbrace{\sum_{j=1}^m a_j \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j} \right)}_{\in \mathbb{C}T\mathbb{R}^m} + \underbrace{\sum_{j=1}^m (b_j - a_j) \frac{\partial}{\partial \bar{z}_j}}_{\in T^{0,1}\mathbb{C}^m}.$$

Therefore $\mathbb{C}T\mathbb{C}^m = T^{0,1}\mathbb{C}^m \oplus \mathbb{C}T\mathbb{R}^m$.

Example 2.1.2. Let $U \subset \mathbb{R}^m$ be an open neighborhood of the origin, and consider $\phi \in \mathcal{C}^\infty(U; \mathbb{R})$, such that $d\phi(0) = 0$. Set $\mathcal{H} \doteq \{x + i\phi(x) : x \in U\}$. We can rewrite $\mathcal{H}$ as

$$\mathcal{H} = \left\{ (z, \bar{z}) \in \mathbb{C}^m : \frac{z - \bar{z}}{2i} = \phi\left(\frac{z + \bar{z}}{2}\right), j = 1, \dots, m \right\}.$$

So a complex vector $X \in \mathbb{C}T\mathbb{C}^m$ is tangent to $\mathcal{H}$ if, and only if,

$$\begin{aligned} X \left(\frac{z_k - \bar{z}_k}{2i} - \phi_k \left(\frac{z + \bar{z}}{2} \right) \right) &= \left(\sum_{j=1}^m a_j \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial \bar{z}_j} \right) \left(\frac{z_k - \bar{z}_k}{2i} - \phi_k \left(\frac{z + \bar{z}}{2} \right) \right) \\ &= \frac{a_k - b_k}{2i} - \sum_{j=1}^m \left[\frac{\partial \phi_k}{\partial x_j} \left(\frac{z + \bar{z}}{2} \right) \right] \left(\frac{a_j + b_j}{2} \right) \\ &= 0, \end{aligned}$$

for every $k = 1, \dots, m$. Therefore an antiholomorphic vector X ($a_k = 0$) is tangent to Σ if

$$b_k = -i \sum_{j=1}^m \frac{\partial \phi_k}{\partial x_j}(\text{Rez}) b_j, \quad k = 1, \dots, m,$$

i.e.,

$$b = -i^t d\phi(\text{Rez})b.$$

In particular we have that

$$\begin{aligned} |b| &= |\text{d}\phi(\text{Rez})b| \\ &\leq \|\text{d}\phi(\text{Rez})\| |b| \\ &\leq |b| \sup_{j,k} \left| \frac{\partial \phi_k}{\partial x_j}(\text{Rez}) \right|, \end{aligned}$$

so we have that $1 \leq \sup_{j,k} \left| \frac{\partial \phi_k}{\partial x_j}(\text{Rez}) \right|$. Since $\text{d}\phi(0) = 0$, shrinking U if necessary, we have that $\mathcal{H}$ is maximally real.

The last example is in fact a very general one, in the sense that locally every maximally real submanifold of $\mathbb{C}^m$ is given by a graph, to be more precise:

Proposition 2.1.3. *Given any maximally real submanifold $\mathcal{H}$ of $\mathbb{C}^m$ and any point p of $\mathcal{H}$ there is a biholomorphism H of an open neighborhood $\mathcal{O}$ of p in $\mathbb{C}^m$ onto an open neighborhood of the origin, with $H(p) = 0$, and a map $Z(x) = x + i\phi(x)$ of an open neighborhood of the origin U in $\mathbb{R}^m$ into $\mathbb{C}^m$, with $\text{d}\phi(0) = 0$, such that $H(\mathcal{H} \cap \mathcal{O}) = \{Z(x) : x \in U\}$.*

A proof of this proposition can be found in the proof of the Proposition IX.2.2., on page 423 of [Tre92].

2.1.1 Analytic and Gevrey vectors

Let Ω a smooth manifold of dimension $m > 0$ and consider $\{L_1, \dots, L_n\}$, pair-wise commuting complex vector fields defined on Ω , where $n \leq m$.

Definition 2.1.4. Let $u \in \mathcal{D}'(\Omega)$ and $s \geq 1$. We say that u is a **Gevrey- s vector** for $\{L_1, \dots, L_n\}$ if $u \in \mathcal{C}^\infty(\Omega)$ and for every $K \subset \Omega$, compact set, there is a positive constant $C = C_K$ such that

$$\sup_{x \in K} |L^\alpha u(z)| \leq C^{|\alpha|+1} \alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^m,$$

and we write $u \in G^s(\Omega; L_1, \dots, L_n)$. If $s = 1$ we write $u \in \mathcal{C}^\omega(\Omega; L_1, \dots, L_n)$, and we say that u is an **analytic vector** for $\{L_1, \dots, L_n\}$.

As before let $\mathcal{H}$ be a maximally real submanifold of $\mathbb{C}^m$ and p an arbitrary point of $\mathcal{H}$. Even though $\mathcal{H}$ is only smooth we can talk about analytic regularity on $\mathcal{H}$. We say that a distribution u on $\mathcal{H}$ is **hypo-analytic** at p if there exists an open neighborhood U of p on $\mathcal{H}$ and a holomorphic function F defined on some open neighborhood $\mathcal{O}$ of p on $\mathbb{C}^m$ such that

$$u|_{U \cap \mathcal{O}} = F|_{U \cap \mathcal{O}}.$$

The aim of this work is to study some sort of Gevrey regularity. The definition of hypo-analyticity does not indicate what should be the definition of Gevrey regularity on $\mathcal{H}$. Consider M_j the restriction of the vector field ∂_{z_j} to $\mathcal{H}$, for $j = 1, \dots, m$. Since $\mathcal{H}$ is maximally real, we have that this complex vector fields does not vanish on $\mathcal{H}$.

Proposition 2.1.5. *Let $u \in \mathcal{D}'(\mathcal{H})$ and $p \in \mathcal{H}$. The following are equivalent:*

1. u is hypo-analytic at p ;
2. There exists U an open neighborhood of p on $\mathcal{H}$ such that $u|_U \in \mathcal{C}^\omega(U; M_1, \dots, M_m)$.

For a proof of this proposition see Proposition III.2.1 of [Tre92]. We also have a characterization of Gevrey vectors in terms of almost-analytic extensions (see Theorem 1.1, pg. 9 of [Cae00]).

Theorem 2.1.6. *Let u be a distribution on $\mathcal{H}$, $U \Subset \mathcal{H}$ an open set and $s > 1$. The following are equivalent:*

1. $u|_U \in G^s(U; M_1, \dots, M_m)$;
2. *There are an open neighborhood $\mathcal{O}$ of U on $\mathbb{C}^m$ and a Gevrey function $F \in G^s(\mathcal{O})$ such that*

$$\begin{cases} u(x) = F(x), & x \in \mathcal{O} \cap U \\ \partial_{\bar{z}} F \sim_{\mathcal{H}} 0, \end{cases}$$

where $\partial_{\bar{z}} F \sim_{\mathcal{H}} 0$ means that $\partial_{\bar{z}} F$ is flat on $\mathcal{H}$.

A useful consequence of this theorem, that we shall use in later on, is the following:

Corollary 2.1.7. *Let u be a distribution on $\mathcal{H}$, $s > 1$, and $U \Subset \mathcal{H}$ an open neighborhood of $p \in \mathcal{H}$. Suppose that $u|_U \in G^s(U; M_1, \dots, M_m)$. Then there are an open neighborhood $\mathcal{O}$ of U on $\mathbb{C}^m$ and a Gevrey function $F \in G^s(\mathcal{O})$ such that*

$$\begin{cases} u(x) = F(x), & x \in \mathcal{O} \cap U \\ |\partial_{\bar{z}} F(z)| \leq C^{k+1} k!^{s-1} \text{dist}(z; \mathcal{H})^k, & \forall k \in \mathbb{Z}_+, z \in \mathcal{O}, \end{cases} \quad (2.2)$$

where C is a positive constant.

This corollary is a simple consequence of Theorem 2.1.6 and the Taylor formula.

2.1.2 The real structure bundle and the FBI transform on maximally real submanifolds of $\mathbb{C}^m$

An object that plays a central role in the analysis of hypo-analytic structures is the so-called real structure bundle. It allows us to mimic some "real techniques" on this complex scenario. As before let $\mathcal{H}$ be a maximally real submanifold of $\mathbb{C}^m$, so $T_{1,0}\mathbb{C}^m \cong \mathbb{C}T^*\mathcal{H}$.

Definition 2.1.8. We define the **real structure bundle** of $\mathcal{H}$ as the image of the real cotangent bundle of $\mathcal{H}$, $T^*\mathcal{H}$, under the isomorphism $T_{1,0}\mathbb{C}^m \cong \mathbb{C}T^*\mathcal{H}$, and we shall denote it by $\mathbb{R}T'_{\mathcal{H}}$.

The real structure bundle $\mathbb{R}T'_{\mathcal{H}}$ is a real vector bundle over $\mathcal{H}$, with fiber dimension equal to m . Locally $\mathcal{H}$ is the image of a map $Z : U \rightarrow \mathbb{C}^m$, where $U \subset \mathbb{R}^m$. So if $p = Z(x_0)$ then $\zeta \in \mathbb{R}T'_{\mathcal{H}}$ if, and only if,

$$\zeta = {}^t Z_x(x_0)^{-1} \xi,$$

for some $\xi \in \mathbb{R}^m$. For every $\kappa > 0$ we write

$$\mathfrak{C}_{\kappa} \doteq \{\zeta \in \mathbb{C}^m : |\text{Im}\zeta| < \kappa |\text{Re}\zeta|\},$$

and if $\zeta \in \mathbb{C}^m$ we write $\langle \zeta \rangle^2 \doteq \zeta \cdot \zeta = \zeta_1^2 + \dots + \zeta_m^2$. Using the main branch of the square root we can define $\langle \zeta \rangle \doteq [\langle \zeta \rangle^2]^{1/2}$, for $\zeta \in \mathfrak{C}_1$.

Definition 2.1.9. We shall say that the maximally real submanifold $\mathcal{H}$ of $\mathbb{C}^m$ at one of its points, p , is well positioned, if there are a number κ , $0 < \kappa < 1$, and an open neighborhood U of p on $\mathcal{H}$ such that

$$\begin{aligned} & \forall z, z' \in U, \text{ and } \zeta \in (\mathbb{R}T'_{\mathcal{H}}|_z) \cap (\mathbb{R}T'_{\mathcal{H}}|_{z'}) \text{ then} \\ & \begin{cases} |\text{Im}\zeta| < \kappa |\text{Re}\zeta|; \\ \text{Im} \{ \zeta \cdot (z - z') + i \langle \zeta \rangle \langle z - z' \rangle^2 \} \geq (1 - \kappa) |\zeta| |z - z'|^2, \end{cases} \end{aligned} \quad (2.3)$$

We shall say that $\mathcal{H}$ is very well positioned at p if, given any $0 < \kappa < 1$, there is an open neighborhood U of p in $\mathcal{H}$ such that (2.3) is valid.

Proposition 2.1.10. *Given any maximally real submanifold $\mathcal{H}$ of $\mathbb{C}^m$ and any point p of $\mathcal{H}$ there is a biholomorphism H of an open neighborhood $\mathcal{O}$ of p in $\mathbb{C}^m$ onto an open neighborhood of the origin, with $H(p) = 0$, such that $H(\mathcal{H} \cap \mathcal{O})$ is very well positioned at 0.*

This is the Proposition IX.2.2. of [Tre92]. Now consider the map

$$\mathfrak{C}_\kappa \times \mathbb{C}^m \ni (\zeta, z) \mapsto \zeta + iz\langle\zeta\rangle. \quad (2.4)$$

Since this map is positive homogeneous in ζ and it is equal to the identity map in $\mathfrak{C}_\kappa$ when $z = 0$, for every $\kappa < \kappa' < 1$ there exists an open neighborhood of the origin $\mathcal{O}$ in $\mathbb{C}^m$ such that

$$\zeta + iz\langle\zeta\rangle \in \mathfrak{C}_{\kappa'},$$

for all $\zeta \in \mathfrak{C}_\kappa$ and $z \in \mathcal{O}$. We shall use the notation

$$\Delta(z, \zeta) = \det(\text{Id} + i(z \odot \zeta)/\langle\zeta\rangle),$$

for $\zeta \in \mathfrak{C}_\kappa \setminus 0$ and $z \in \mathbb{C}^m$, where $(z \odot \zeta) = (z_i \zeta_j)_{i,j=1,\dots,m}$. This can be seen as the Jacobian of the map (2.4).

Definition 2.1.11. Let $u \in \mathcal{E}'(\mathcal{H})$. So we define

$$\mathfrak{F}[u](z, \zeta) \doteq \left\langle u(z'), e^{i\zeta \cdot (z - z') - \langle\zeta\rangle \langle z - z' \rangle^2} \Delta(z - z', \zeta) \right\rangle,$$

for $z \in \mathbb{C}^m$ and $\zeta \in \mathfrak{C}_1$.

Proposition 2.1.12. *Let $u \in \mathcal{E}'(\mathcal{H})$. Then $\mathfrak{F}[u](z, \zeta)$ is holomorphic in $(z, \zeta) \in \mathbb{C}^m \times \mathfrak{C}_1$ and for every $K \subset \mathbb{C}^m$ compact set, and for every $0 < \kappa < 1$, there are constants $C, R > 0$ such that*

$$|\mathfrak{F}[u](z, \zeta)| \leq C e^{R|\zeta|}, \quad \forall z \in K, \zeta \in \mathfrak{C}_\kappa.$$

Now if in addition, $\mathcal{H}$ is well positioned at $p \in \mathcal{H}$, there exists an open neighborhood U of p on $\mathcal{H}$ such that, if the support of u is contained in U , there exist an integer $k > 0$ and a constant $C > 0$ such that

$$|\mathfrak{F}[u](z, \zeta)| \leq C(1 + |\zeta|)^k, \quad \forall (z, \zeta) \in \mathbb{R}T'_\mathcal{H}|_U,$$

where $(z, \zeta) \in \mathbb{R}T'_\mathcal{H}|_U$ means that $z \in U$ and $\zeta \in \mathbb{R}T'_\mathcal{H}|_z$.

For a proof of this proposition see Propositions IX.1.1. and IX.2.1. of [Tre92]. So outside the real structure bundle the best estimate one can expect, for the FBI transform of a compactly supported distribution on $\mathcal{H}$, is infraexponential (the same estimate for hyperfunctions, see [CT94]), and we only recover the same estimate for compactly supported distributions on $\mathbb{R}^m$, if we stay on the real structure bundle. So this proposition is an example of the importance of the real structure bundle if we want to recover estimates from the real scenario.

The main importance, in this context, of the FBI transform is that it can be used to characterize holomorphic extendability, and other types of extendabilities, as we shall see later on. But first we need an inversion formula for the FBI transform (Theorem IX.2.2. of [Tre92]):

Proposition 2.1.13. *Let $u \in \mathcal{E}'(\mathcal{H})$ and set for every ε*

$$u_\varepsilon(z) \doteq \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{-\varepsilon|\xi|^2} \mathfrak{F}[u](z, \xi) d\xi,$$

for every $z \in \mathbb{C}^m$. Then $u_\varepsilon \rightarrow u$ in $\mathcal{D}'(\mathcal{H})$.

With this inversion formula one can prove the following theorem (Theorem IX.3.1. of [Tre92]):

Theorem 2.1.14. *Let $u \in \mathcal{E}'(\mathcal{H})$ and $p \in \mathcal{H}$. The following are equivalent:*

1. *There exist $\mathcal{O} \subset \mathbb{C}^m$, an open neighborhood of p , F a holomorphic function on $\mathcal{O}$, such that $u|_{\mathcal{O} \cap \Sigma} = F|_{\mathcal{O} \cap \Sigma}$;*
2. *There exist $\mathcal{O} \subset \mathbb{C}^m$, an open neighborhood of p , $0 < \kappa' < 1$, and $C, \varepsilon > 0$ such that*

$$|\mathfrak{F}[u](z, \zeta)| \leq C e^{-\varepsilon|\zeta|}, \quad \forall (z, \zeta) \in \mathcal{O} \times \mathfrak{C}_{\kappa'}$$

3. *There exists $\mathcal{O} \subset \mathbb{C}^m$, an open neighborhood of p , such that $\mathfrak{F}[u](z, \xi)$ is dominated by an integrable function with respect to $\xi \in \mathbb{R}^m$, uniformly in $z \in \mathcal{O}$.*

The third equivalence of this theorem does not appear in the literature, but it is actually how one proves that (2) implies (1) using Proposition 2.1.13. Just for the sake of completeness let us exhibit the proof of (3) $\Rightarrow$ (1).

So suppose that the FBI transform of u is bounded by an integrable function $g(\xi)$, uniformly in $z \in \mathcal{O}$, i.e.,

$$|\mathfrak{F}[u](z, \xi)| \leq g(\xi), \quad \forall z \in \mathcal{O}.$$

Now define for $\varepsilon > 0$ and $z \in \mathcal{O}$ the holomorphic function

$$u_\varepsilon(z) \doteq \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{-\varepsilon|\xi|^2} \mathfrak{F}[u](z, \xi) d\xi.$$

Since the function g is integrable, we have that

$$\begin{aligned} |u_\varepsilon(z)| &\leq \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{-\varepsilon|\xi|^2} |\mathfrak{F}[u](z, \xi)| d\xi \\ &\leq \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{-\varepsilon|\xi|^2} g(\xi) d\xi \\ &\leq C \int_{\mathbb{R}^m} g(\xi) d\xi \\ &\leq \tilde{C}, \end{aligned}$$

for all $z \in \mathcal{O}$, where the constant $\tilde{C} > 0$ does not depend on ε . So by Montel's theorem, there is a subsequence $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$, with $\varepsilon_k \rightarrow 0$, such that the sequence $\{u_{\varepsilon_k}\}_{k \in \mathbb{Z}_+}$ converges uniformly to a holomorphic function on $\mathcal{O}$, so (1) follows from the uniqueness of the limit.

This simple observation illustrates the advantage of having holomorphic function theory at our disposal. In this case, the FBI decay is not very important because we have it controlled on a full neighborhood of p . Now if one wants to measure, for instance, smooth regularity with the FBI transform (Theorem IX.4.1. of [Tre92]), then the estimate, and where the estimate takes place, are both very important.

Theorem 2.1.15. *Let $u \in \mathcal{E}'(\mathcal{H})$ and $p \in \mathcal{H}$. The following are equivalent:*

1. *u is $\mathcal{C}^\infty$ near p ;*
2. *There exists U a neighborhood of p , such that for every $k \in \mathbb{Z}_+$ there is a $C_k > 0$, such that*

$$|\mathfrak{F}[u](z, \zeta)| \leq C_k (1 + |\zeta|)^{-k}, \quad \forall (z, \zeta) \in \mathbb{R}T'_{\mathcal{H}}|_U.$$

2.2 Locally integrable structures

In this section we shall recall some important definitions from the theory of locally integrable structures, and in this part of the exposition we shall follow closely the book [BCH08]. Let $\Omega \subset \mathbb{R}^N$ be an open set. By a locally integrable structure on Ω we mean a complex vector bundle $\mathcal{V} \subset \mathbb{C}T\Omega$, such that $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, and at every point $p \in \Omega$ there are $Z_1, \dots, Z_m$, smooth, complex-valued functions in some open neighborhood of p in Ω , such that

$$\begin{cases} dZ_1 \wedge \dots \wedge dZ_m \neq 0; \\ LZ_j = 0, \quad \forall L \in \mathcal{V}, j = 1, \dots, m. \end{cases}$$

We denote by $T' \subset \mathbb{C}T^*\Omega$ the orthogonal bundle, with respect to the duality between forms and vectors, of the bundle $\mathcal{V}$. As usual we set

$$T^0 \doteq T' \cap T^*\Omega,$$

the characteristic set of $\mathcal{V}$. The next theorem describes how these structures "look like" locally (Corollary I.10.2. of [BCH08]).

Theorem 2.2.1. *Let p be an arbitrary point at Ω . Then there exist a local coordinate system, vanishing at p , on some open set U , $\{x_1, \dots, x_m, t_1, \dots, t_n\}$, and m smooth, real-valued functions, $\phi_1, \dots, \phi_m$, defined on U and satisfying $\phi(0) = 0$ and $d_x\phi(0) = 0$, such that the differentials of the functions*

$$Z_k(x, t) \doteq x_k + i\phi_k(x, t), \quad k = 1, \dots, m,$$

span T' on U .

Under the hypotheses of the previous theorem we set the complex vector fields

$$M_j = \sum_{k=1}^m a_{j,k}(x, t) \frac{\partial}{\partial x_k}, \quad j = 1, \dots, m,$$

imposing that they satisfy the following relations

$$M_j Z_k = \delta_{jk}, \quad \forall j, k = 1, \dots, m.$$

We also define

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^m \frac{\partial \phi_k}{\partial t_j}(x, t) M_k, \quad j = 1, \dots, n,$$

and by the definition of such complex vector fields we see that $\{M_1, \dots, M_m, L_1, \dots, L_n\}$ are linear independent, pairwise commuting, and they span $\mathbb{C}TU$. We also have that $\{L_1, \dots, L_n\}$ span $\mathcal{V}$ in U . Now let $\mathcal{M}$ be an (embedded) submanifold of Ω . For every $p \in \mathcal{M}$ we set $\mathcal{V}(\mathcal{M})_p = \mathcal{V}_p \cap \mathbb{C}T_p\mathcal{M}$.

Definition 2.2.2. We say that $\mathcal{M}$ is compatible with $\mathcal{V}$ if the map

$$\mathcal{M} \ni p \mapsto \dim \mathcal{V}(\mathcal{M})_p$$

is constant.

Definition 2.2.3. We say that a compatible submanifold $\mathcal{M}$ is noncharacteristic if

$$N^*\mathcal{M} \cap T^0|_{\mathcal{M}} = 0,$$

where $N^*\mathcal{M} \subset T^*\Omega_{\mathcal{M}}$ is the conormal bundle, i.e., $\omega_p \in N_p^*\mathcal{M}$ if, and only if, $\omega_p(X_p) = 0$, for all $X_p \in T_p\mathcal{M}$.

Now let $(U, x_1, \dots, x_m, t_1, \dots, t_n)$ be the local coordinate system described on Theorem 2.2.1, where $U = V \times W$. For every $a = (a_1, \dots, a_n) \in W$ consider

$$\mathcal{M}_a \doteq \{(x, t) \in U : t_j = a_j, j = 1, \dots, n\}.$$

So $\mathcal{M}_a$ is a compatible submanifold of Ω , since $\mathcal{V}(\mathcal{M}_a) = 0$. We also have that $\{M_1, \dots, M_m\} \in \mathbb{C}T\mathcal{M}_a$, therefore $T'|_{\mathcal{M}_a} \subset \mathbb{C}T^*\mathcal{M}_a$, and in particular, $N^*\mathcal{M}_a \cap T^0|_{\mathcal{M}_a} = 0$, i.e., $\mathcal{M}_a$ is noncharacteristic.

Definition 2.2.4. Let p, q be integers and $u \in \mathcal{D}'(U)$. We say that u belongs to the Sobolev space $H_{loc}^{p,q}(U)$ if

$$\|\phi u\|_{H^{p,q}}^2 = \iint_{V \times W} (1 + |\xi|^2)^p (1 + |\eta|^2)^q |\widehat{\phi u}(\xi, \eta)|^2 d\xi d\eta < \infty,$$

for every $\phi \in \mathcal{C}_c^\infty(U)$. Here ξ and η are the dual variable of x and t . We also define

$$H_{loc}^{p,\infty}(U) \doteq \cup_{q \geq 0} H_{loc}^{p,q}(U).$$

Proposition 2.2.5. Let $(U, x_1, \dots, x_m, t_1, \dots, t_n)$ a local coordinate system as before. Consider u a distribution on U such that $L_j u \in \mathcal{C}^\infty(U)$, for $j = 1, \dots, n$. Then for every $U_1 \Subset U$ there exists an integer p such that $u \in H_{loc}^{p,\infty}(U_1)$. In particular we have that u is a $\mathcal{C}^\infty$ function of t valued in the space of distributions in x , here we are writing $U_1 = V_1 \times W_1$.

The proof of this proposition is analogous to the one of Proposition I.4.3 of [Tre92], using the fact that the set $\mathcal{M}_0$ is noncharacteristic. So if u is a distribution satisfying the hypothesis of the previous proposition, then whatever test-function $\chi \in \mathcal{C}_c^\infty(U')$, the function

$$t \mapsto \langle u(x, t), \chi(x) \rangle$$

is $\mathcal{C}^\infty$. In particular the trace of u to $\mathcal{M}_a$ is well defined, for every $a \in W$. We also have that for every compact sets $K_1 \subset V$ and $K_2 \subset W$ there exists a constant $C > 0$ and a integer $q > 0$ such that

$$|\langle u(x, t), \phi(x) \rangle| \leq C \sum_{|\alpha| \leq q} \sup_{x \in K} |\partial^\alpha \phi|, \quad \forall \phi \in \mathcal{C}_c^\infty(K_1), \quad (2.5)$$

for every $t \in K_2$. This is estimate (B.1) of appendix B.

2.3 Hypo-analytic structures

In this section we shall discuss some well known facts concerning hypo-analytic functions, and the difficulties that appear when dealing with Gevrey vectors on hypo-analytic structures. Let $\Omega \subset \mathbb{R}^N$ be an open set. A hypo-analytic structure on Ω is a pair $\{(U_\alpha)_{\alpha \in \Lambda}, (Z_\alpha)_{\alpha \in \Lambda}\}$ such that

- $(U_\alpha)_{\alpha \in \Lambda}$ is an open covering for Ω ;
- $Z_\alpha : U_\alpha \longrightarrow \mathbb{C}^m$ is a smooth maps, for every $\alpha \in \Lambda$;
- $dZ_{\alpha,1}, \dots, dZ_{\alpha,m}$ are $\mathbb{C}$ -linear independent on U_α , for every $\alpha \in \Lambda$;
- if $\alpha \neq \beta$, then to each $p \in U_\alpha \cap U_\beta$ there is a holomorphic map F such that $Z_\alpha = F \circ Z_\beta$, on an open neighborhood of p in $U_\alpha \cap U_\beta$;

- if $Z : U \rightarrow \mathbb{C}^m$ is a smooth function such that for every $p \in U \cap U_\alpha$ there exists a holomorphic function F such that $Z = F \circ Z_\alpha$, then $(U, Z) = (U_\beta, Z_\beta)$, for some $\beta \in \Lambda$.

We call a pair (U, Z) , as before, a hypo-analytic chart.

Definition 2.3.1. Let $p \in \Omega$. We say that a distribution $u \in \mathcal{D}'(\Omega)$ is hypo-analytic at p if for some $\alpha \in \Lambda$ such that $p \in U_\alpha$, there is a holomorphic function F , defined on a complex neighborhood of $Z_\alpha(p)$, such that $u = F \circ Z_\alpha$, in some open neighborhood of p .

Given a hypo-analytic structure on Ω we can associate a locally integrable structure $\mathcal{V}$ setting its orthogonal T' as the complex bundle defined, locally, by the differentials $dZ_1, \dots, dZ_m$. Let (U, Z) be a hypo-analytic chart. If F is a holomorphic function defined on an open neighborhood of $Z(U)$ in $\mathbb{C}^m$, and we define $u = F \circ Z$, then for every $L \in \mathcal{V}$ we have that

$$\begin{aligned} Lu &= L[F \circ Z] \\ &= \partial_z F \circ Z \cdot LZ + \partial_{\bar{z}} F \circ Z \cdot L\bar{Z} \\ &= 0. \end{aligned}$$

Note that u is only annihilated by all sections of $\mathcal{V}$ because F is holomorphic. Now let $\mathcal{H} \subset \Omega$ be a submanifold compatible with $\mathcal{V}$. We say that $\mathcal{H}$ is **maximally real** if

$$\mathbb{C}T_p\Omega = \mathcal{V}_p \oplus \mathbb{C}T_p\mathcal{H}, \quad \forall p \in \mathcal{H},$$

or equivalently,

$$\mathbb{C}T_p^*\Omega = \mathbb{C}N_p^*\mathcal{H} \oplus T'_p, \quad \forall p \in \mathcal{H}.$$

This also equivalent to: for every $p \in \mathcal{H}$ there is a hypo-analytic chart (U, Z) centered at p such that $Z(\mathcal{H} \cap U) \subset \mathbb{C}^m$ is maximally real. So let $\mathcal{H}$ be maximally real and let p be an arbitrary point of $\mathcal{H}$. Then we can find a hypo-analytic chart (U, Z) centered at p , and local coordinates $(x_1, \dots, x_m, t_1, \dots, t_n)$, vanishing at p , such that Z can be written as

$$Z(x, t) = x + i\phi(x, t),$$

where ϕ is a smooth, real-valued function satisfying $\phi(0) = 0$ and $d_x\phi(0) = 0$, and in this coordinate system

$$\mathcal{H} \cap U = \{(x, t) \in U : t = 0\}.$$

By Proposition 2.2.5 we have that every distribution u on Ω satisfying $Lu \in \mathcal{C}^\infty(\Omega)$, for every $L \in \mathcal{V}$, has a well defined trace to $\mathcal{H}$, moreover, writing $U = V \times W$ we have that $u \in \mathcal{C}^\infty(W, \mathcal{D}'(V))$. One of the most important results in this theory, at least one of the most useful, is the Baouendi-Treves approximation formula (Theorem II.3.2, pg. 89 of [Tre92]).

Theorem 2.3.2. *Let u be a distribution on $U = V \times W$ such that $Lu = 0$, for every $L \in \mathcal{V}$. If $V_0 \Subset V$ and $W_0 \Subset W$ are sufficiently small, and $\chi \in \mathcal{C}_c^\infty(V)$ is such that $\chi \equiv 1$ in V_0 , then in $\mathcal{C}^\infty(W_0, \mathcal{D}'(V_0))$,*

$$u(x, t) = \lim_{\varepsilon \rightarrow 0^+} \left\langle u(y, 0), e^{i\xi \cdot (Z(x, t) - Z(y, 0))} \chi(y) \det Z_x(y, 0) \right\rangle. \quad (2.6)$$

An important consequence of this theorem is that if u is a solution of $Lu = 0$, for all $L \in \mathcal{V}$, and the trace, $u|_{\mathcal{H}} = 0$, then u vanishes in some open neighborhood of $\mathcal{H}$. Now let $\{L_1, \dots, L_n, M_1, \dots, M_m\}$ be such that

$$\begin{aligned} L_j Z_k &= 0 & M_j Z_k &= \delta_{j,k} \\ L_j t_k &= \delta_{j,k} & M_j t_k &= 0. \end{aligned}$$

Recall that a smooth function u in U belongs to $G^s(U; L_1, \dots, L_n, M_1, \dots, M_m)$ if for every compact $K \subset U$, there exists a constant $C > 0$ such that

$$\sup_{(x,t) \in K} |L^\alpha M^\beta u(x,t)| \leq C^{|\alpha|+|\beta|+1} \alpha!^s \beta!^s, \quad \forall \alpha \in \mathbb{Z}_+^n, \beta \in \mathbb{Z}_+^m.$$

Now we will present two typical arguments that are valid when dealing with "analytic regularity" but falls apart when dealing with "Gevrey regularity". Let $u \in C^\infty(W; \mathcal{D}'(V))$ be such that $L_j u = 0$, for $j = 1, \dots, n$ and $\mathcal{H}$ a maximally real submanifold of Ω . The hypo-analytic structure on Ω induces a hypo-analytic structure on $\mathcal{H}$, and since $\mathcal{H}$ is maximally real, the induced structure is spanned by all the differentials $dZ_1, \dots, dZ_m$ restricted to $\mathcal{H}$. Suppose that $u|_{\mathcal{H}}$ is hypo-analytic at $p \in \mathcal{H}$ with respect to the induced hypo-analytic structure. Therefore there exist $U_{\mathcal{H}}$ an open neighborhood of p on $\mathcal{H}$, and a holomorphic function F defined on $\mathcal{O}$, an open neighborhood of $Z(U_{\mathcal{H}})$ on $\mathbb{C}^m$, such that

$$u(p') = F(Z(p')), \quad \forall p' \in U_{\mathcal{H}}.$$

Now set $\tilde{u} \doteq F \circ Z$, defined in some neighborhood of p on Ω . Since F is holomorphic we have that $L\tilde{u} = 0$, for every $L \in \mathcal{V}$, so the same is valid for $u - \tilde{u}$, and $u - \tilde{u}$ vanishes on a neighborhood of p on $\mathcal{H}$, therefore it vanishes on some neighborhood of p , *i.e.*, u is hypo-analytic at p . What happens if instead of $u|_{\mathcal{H}}$ hypo-analytic at p we had that $u|_{\mathcal{H}}$ is a Gevrey vector at p ? By Caetano's result, Theorem 2.1.6, applied to the hypo-analytic structure induced on H , we have that there exists a Gevrey function F defined on an open neighborhood of $Z(U_{\mathcal{H}})$ on $\mathcal{O}$ such that

$$\begin{cases} u(p') = F(Z(p')), & \forall p' \in U_{\mathcal{H}}, \\ \partial_{\bar{z}} F \sim 0 \text{ on } Z(U_{\mathcal{H}}). \end{cases}$$

So if we define $\tilde{u} = F \circ Z$ on some open neighborhood of p on Ω , we do not have anymore that $\tilde{u}$ satisfies $L\tilde{u} = 0$, for every $L \in \mathcal{V}$, because

$$L\tilde{u} = \partial_z F \circ Z \cdot LZ + \partial_{\bar{z}} F \circ Z \cdot L\bar{Z},$$

and we only know that $\partial_{\bar{z}} F \circ Z = 0$ on $U_{\mathcal{H}}$, not on a full neighborhood on Ω . Conclusion: knowing that the restriction of a solution u , *i.e.*, $Lu = 0$, for every $L \in \mathcal{H}$, to a maximally real submanifold is a Gevrey vector does not imply that u is a Gevrey vector

Now suppose that u is a Gevrey vector on U . Since the set $\{(Z(x,t), t) : (x,t) \in U\} \subset \mathbb{C}^m$ is maximally real (if U is small enough) we can apply Caetano's characterization, Theorem 2.1.6, to the hypo-analytic structure generated by $\{Z_1, \dots, Z_m, t_1, \dots, t_n\}$, obtaining a Gevrey function $F(z, \omega)$ such that

$$\begin{cases} F(Z(x,t), t) = u(x,t), & (x,t) \in U; \\ (\partial_{\bar{z}} + \partial_{\bar{\omega}})F \sim 0 \text{ on } \{(Z(x,t), t) : (x,t) \in U\}. \end{cases}$$

If instead of a Gevrey vector, u were an analytic vector, then the function F would be actually holomorphic, and if in addition we had that u were a solution, we would have that

$$\begin{aligned} (\partial_{\omega_j} F)(Z(x,t), t) &= L_j(F(Z(x,t), t)) \\ &= 0. \end{aligned}$$

Since the function $\partial_{\omega_j} F(\omega)$ is holomorphic, and vanishes on a maximally real submanifold of $\mathbb{C}^m$ it vanishes on a full neighborhood of $\{(Z(x,t), t) : (x,t) \in U\}$, for $j = 1, \dots, n$, therefore

$$F(z, \omega) = F(z).$$

Now notice that this argument does not hold for the Gevrey regularity. These two examples indicate

that the lack of tools from holomorphic function theory in the Gevrey setting brings more difficulties than just a "power of s ".

Chapter 3

The FBI characterization of Gevrey vectors

In this chapter we show how the FBI transform can be used to characterize Gevrey vectors. The proof of the characterization in this set up is more technical than the usual one on $\mathbb{R}^N$ because of the complex nature of our set up. Let Ω be an open set of $\mathbb{R}^N$ endowed with a hypo-analytic structure. All of the results that we will present in this chapter have a local nature, so we fix an arbitrary point at Ω , and for simplicity let us call it the origin. As we saw in the previous chapter, there is a hypo-analytic chart $(U, Z_1(x, t), \dots, Z_m(x, t))$, with $0 \in U$, and we can assume that the Z s are given by

$$Z_j(x, t) = x_j + i\phi_j(x, t), \quad j = 1, \dots, m,$$

with $(x, t) \in U$, where the map $\phi(x, t) = (\phi_1(x, t), \dots, \phi_m(x, t))$ is smooth, real-valued, $\phi(0) = 0$ and $d_x\phi(0) = 0$. We can associate the complex vector fields $\{M_1, \dots, M_m, L_1, \dots, L_n\}$ with the following properties:

$$\begin{aligned} L_j Z_k &= 0 & M_j Z_k &= \delta_{j,k} \\ L_j t_k &= \delta_{j,k} & M_j t_k &= 0. \end{aligned}$$

We can also assume that

$$|\phi_j(x, t)| \leq C(|x|^3 + |t|), \quad j = 1, \dots, m, \quad (3.1)$$

for some positive constant C , and

$$|\phi(x, t) - \phi(x', t)| \leq \mu|x - x'|, \quad (3.2)$$

with $0 < \mu$ as small as we want, for instance, lesser than 1 (see pg. 433 of [Tre92]). From now on, we are going to assume that $U = V \times W$, where $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n$ are balls centered at the origin. Under these assumptions we can assume that for some $0 < \kappa < 1$ and $c > 0$ (see Appendix A):

$$\forall x, x' \in V, t \in W, \xi \in \mathbb{R}^m, \text{ if } \zeta = {}^t Z_x(x, t)^{-1} \xi \text{ then}$$

$$\begin{cases} |\operatorname{Im} \zeta| < \kappa |\operatorname{Re} \zeta|; \\ \operatorname{Im} \{ \zeta \cdot (Z(x, t) - Z(x', t)) + i \langle \zeta \rangle \langle Z(x, t) - Z(x', t) \rangle^2 \} \geq c |\zeta| |Z(x, t) - Z(x', t)|^2. \end{cases} \quad (3.3)$$

Now set, for every $t \in W$,

$$\mathfrak{W}_t \doteq \{Z(x, t) : x \in V\},$$

so the real structure bundle $\mathfrak{W}_t$ and its fiber at $Z(x, t)$, $\mathbb{R}T'_{\mathfrak{W}_t}|_{Z(x, t)}$, can be written as

$$\mathbb{RT}'_{\mathfrak{W}_t} = \{(Z(x, t), {}^t Z_x(x, t)^{-1} \xi) : x \in V, \xi \in \mathbb{R}^m\},$$

and

$$\mathbb{RT}'_{\mathfrak{W}_t}|_{Z(x, t)} = \{{}^t Z_x(x, t)^{-1} \xi : \xi \in \mathbb{R}^m\}.$$

We can also assume that

$$\operatorname{Im} \left\{ \zeta \cdot (Z(x, t) - Z(x', t)) + i \frac{1}{2} \langle \zeta \rangle \langle Z(x, t) - Z(x', t) \rangle^2 \right\} \geq c |\zeta| |Z(x, t) - Z(x', t)|^2, \quad (3.4)$$

for every $x, x' \in V, t \in W, \zeta \in \mathbb{RT}'_{\mathfrak{W}_t}|_{Z(x, t)} \cup \mathbb{RT}'_{\mathfrak{W}_t}|_{Z(x', t)}$.

Remark 3.0.1. For simplicity we shall assume that U is small enough so the image of the map

$$\mathbb{RT}'_{\mathfrak{W}_t}|_{Z(x', t)} \ni \zeta \longmapsto \tilde{\zeta} \doteq \zeta + i \langle \zeta \rangle (Z(x, t) - Z(x', t)),$$

is contained in $\mathfrak{C}_{\kappa'}$, for all $x' \in V$ and $t \in W$, for some $0 < \kappa' < 1$.

Definition 3.0.2. Let $u \in \mathcal{C}^\infty(W; \mathcal{E}'(V))$ and $\lambda > 0$. We define the FBI transform of u as

$$\mathfrak{F}^\lambda[u](t; z, \zeta) = \int_V e^{i\zeta \cdot (z - Z(x', t)) - \lambda \langle \zeta \rangle \langle z - Z(x', t) \rangle^2} u(x', t) \Delta(\lambda(z - Z(x', t)), \zeta) dZ(x', t),$$

with $z \in \mathbb{C}^m$ and $\zeta \in \mathfrak{C}_1 \setminus 0$.

If we denote by $\tilde{u}(z, t) = u(x, t)$, for $z = Z(x, t)$, then we can write

$$\mathfrak{F}^\lambda[u](t; z, \zeta) = \int_{\mathfrak{W}_t} e^{i\zeta \cdot (z - z') - \lambda \langle \zeta \rangle \langle z - z' \rangle^2} \tilde{u}(z', t) \Delta(\lambda(z - z'), \zeta) dz'.$$

Note that the integral is to be understood in the dual sense. Since u has compact support in x we have that $\mathfrak{F}^\lambda[u](t; z, \zeta)$ is holomorphic with respect to $(z, \zeta) \in \mathbb{C}^m \times \mathfrak{C}_1 \setminus 0$, and $\mathcal{C}^\infty$ with respect to t . For simplicity we write $\mathfrak{F}[u](t; z, \zeta)$, for $\lambda = 1$. Since we are dealing with distributions with compact support in V , we can derive the following bound for the FBI transform (see Theorem IX.2.1. of [Tre92]).

Lemma 3.0.3. Let $u \in \mathcal{C}^\infty(W, \mathcal{E}'(V))$. Then there exists $C > 0$ and $k \in \mathbb{Z}_+$ such that

$$|\mathfrak{F}[u](t; z, \zeta)| \leq C(1 + |\zeta|)^k, \quad \forall (z, \zeta) \in \mathbb{RT}'_{\mathfrak{W}_t}. \quad (3.5)$$

Now we shall derive the inversion formula that we will need for the applications that we are interested in. We start recalling an approximation formula used to prove the Baouendi-Treves approximation formula:

Theorem 3.0.4. Let $f \in \mathcal{C}^\infty(U)$ such that $\operatorname{supp} f \subset K \times W$, where $K \Subset V$. For $\varepsilon > 0$ set

$$\mathcal{E}_\nu f(x, t) \doteq \left(\frac{\nu}{\pi}\right)^{\frac{m}{2}} \int_K e^{-\nu \langle Z(x, t) - Z(x', t) \rangle^2} f(x', t) dZ(x', t), \quad (x, t) \in U.$$

Then $\mathcal{E}_\nu f \xrightarrow{\nu \rightarrow \infty} f$ in $\mathcal{C}^\infty(U)$.

For a proof of this theorem see Lemma II.2.3 in [Tre92]. There is also a version of this theorem for distributions (Lemma II.3.1 in [Tre92]):

Theorem 3.0.5. Let $u \in \mathcal{C}^\infty(W; \mathcal{E}'(V))$, then $\mathcal{E}_\nu u \xrightarrow{\nu \rightarrow \infty} u$ in $\mathcal{C}^\infty(W; \mathcal{D}'(V))$.

With this formula we can derive some sort of inversion Fourier formula:

Corollary 3.0.6. *Let $u \in \mathcal{C}^\infty(W; \mathcal{E}'(V))$, then*

$$u(x, t) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \left\langle u(x', t), e^{i\xi \cdot (Z(x, t) - Z(x', t)) - \varepsilon |\xi|^2} \det Z_x(x', t) \right\rangle d\xi, \quad (3.6)$$

in $\mathcal{C}^\infty(W; \mathcal{D}'(V))$.

Proof. Let $\varepsilon > 0$ be fixed. Then we can write

$$\begin{aligned} \int_{\mathbb{R}^m} \left\langle u(x', t), \det Z_x(x', t) e^{i\xi \cdot (Z(x, t) - Z(x', t)) - \varepsilon |\xi|^2} \right\rangle d\xi &= \\ &= \left\langle u(x', t), \int_{\mathbb{R}^m} e^{-\left\langle \sqrt{\varepsilon} \xi - i \frac{(Z(x, t) - Z(x', t))}{2\sqrt{\varepsilon}} \right\rangle^2} e^{-\frac{(Z(x, t) - Z(x', t))^2}{4\varepsilon}} \det Z_x(x', t) d\xi \right\rangle \\ &= \left(\frac{\pi}{\varepsilon} \right)^{\frac{m}{2}} \left\langle u(x', t), e^{-\frac{(Z(x, t) - Z(x', t))^2}{4\varepsilon}} \det Z_x(x', t) \right\rangle \\ &= (2\pi)^m E_{1/4\varepsilon} u(x, t) \\ &\xrightarrow{\varepsilon \rightarrow 0^+} (2\pi)^m u(x, t). \end{aligned}$$

□

Remark 3.0.7. Since for every $\varepsilon > 0$ the function $e^{i\zeta \cdot (Z(x, t) - Z(x', t)) - \varepsilon \langle \zeta \rangle^2}$ is holomorphic with respect to ζ in $\mathfrak{C}_1$, then we can deform the contour of integration in (3.6) and obtain

$$u(x, t) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{(2\pi)^m} \int_{\mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x, t)}} \left\langle u(x', t), e^{i\zeta \cdot (Z(x, t) - Z(x', t)) - \varepsilon \langle \zeta \rangle^2} \det Z_x(x', t) \right\rangle d\zeta, \quad (3.7)$$

where the convergence takes place in $\mathcal{C}^\infty(W; \mathcal{D}'(V))$.

With the formula (3.7) we can prove an intermediate inversion formula:

Proposition 3.0.8. *Let $u \in \mathcal{C}^\infty(W; \mathcal{E}'(V))$. For every $\varepsilon > 0$ set*

$$u_\varepsilon(x, t) \doteq \frac{1}{(2\pi)^m} \int_{\mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x, t)}} e^{-\varepsilon \langle \zeta \rangle^2} \mathfrak{F}^{\frac{1}{2}}[u](t; Z(x, t), \zeta) d\zeta. \quad (3.8)$$

Then $u_\varepsilon(x, t) \rightarrow u(x, t)$ in $\mathcal{C}^\infty(W; \mathcal{D}'(V))$.

Remark 3.0.9. The integral (3.8) is to be interpreted as

$$\lim_{\delta \rightarrow 0^+} \int_{\{\zeta \in \mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x, t)} : |\zeta| > \delta\}} e^{-\varepsilon \langle \zeta \rangle^2} \mathfrak{F}^{\frac{1}{2}}[u](t; Z(x, t), \zeta) d\zeta$$

Proof. Let $\varepsilon, \delta > 0$, $x, x' \in K$, $t \in W$, and consider the integral

$$\int_{\{\zeta \in \mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x, t)} : |\zeta| > \delta\}} e^{i\zeta \cdot (Z(x, t) - Z(x', t)) - \varepsilon \langle \zeta \rangle^2} d\zeta.$$

We shall perform the following deformation of the contour of integration:

$$\zeta \rightarrow \tilde{\zeta} \doteq \zeta + \frac{i}{2} \langle \zeta \rangle (Z(x, t) - Z(x', t)).$$

Then

$$\begin{aligned}
& \int_{\{\zeta \in \mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)} : |\zeta| > \delta\}} e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \varepsilon \langle \zeta \rangle^2} d\zeta = \\
& = \int_{\{\zeta \in \mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)} : |\zeta| > \delta\}} e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \frac{1}{2} \langle \zeta \rangle \langle Z(z,t) - Z(x',t) \rangle^2 - \varepsilon \langle \zeta + i\frac{1}{2}(Z(x,t) - Z(x',t)) \langle \zeta \rangle \rangle^2} \\
& \quad \cdot \Delta \left(\frac{1}{2} (Z(x,t) - Z(x',t)), \zeta \right) d\zeta + \\
& + \int_0^{\frac{1}{2}} \int_{\{\zeta \in \mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)} : |\zeta| = \delta\}} e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \lambda \langle \zeta \rangle \langle Z(z,t) - Z(x',t) \rangle^2 - \varepsilon \langle \zeta + i\lambda(Z(x,t) - Z(x',t)) \langle \zeta \rangle \rangle^2} \\
& \quad \cdot \Delta (\lambda(Z(x,t) - Z(x',t)), \zeta) d\zeta d\lambda.
\end{aligned}$$

Since $\Delta(z, \zeta)$ is bounded with respect to ζ for every z , we have that

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \int_{\{\zeta \in \mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)} : |\zeta| = \delta\}} e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \lambda \langle \zeta \rangle \langle Z(z,t) - Z(x',t) \rangle^2 - \varepsilon \langle \zeta + i\lambda(Z(x,t) - Z(x',t)) \langle \zeta \rangle \rangle^2} \\
& \quad \cdot \Delta (\lambda(Z(x,t) - Z(x',t)), \zeta) d\zeta d\lambda \xrightarrow{\delta \rightarrow 0^+} 0,
\end{aligned}$$

so

$$\begin{aligned}
& \int_{\mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)}} e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \varepsilon \langle \zeta \rangle^2} d\zeta = \int_{\mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)}} e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \frac{1}{2} \langle \zeta \rangle \langle Z(z,t) - Z(x',t) \rangle^2 - \varepsilon \langle \tilde{\zeta} \rangle^2} \\
& \quad \cdot \Delta \left(\frac{1}{2} (Z(x,t) - Z(x',t)), \zeta \right) d\zeta.
\end{aligned}$$

Now using (3.7) we have that

$$\begin{aligned}
& \frac{1}{(2\pi)^m} \int_{\mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)}} \left\langle u(x',t), e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \varepsilon \langle \zeta \rangle^2} \det Z_x(x',t) \right\rangle d\zeta = \\
& = \frac{1}{(2\pi)^m} \left\langle u(x',t), \int_{\mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)}} e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \varepsilon \langle \zeta \rangle^2} \det Z_x(x',t) d\zeta \right\rangle \\
& = \frac{1}{(2\pi)^m} \left\langle u(x',t), \int_{\mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)}} e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \frac{1}{2} \langle \zeta \rangle \langle Z(z,t) - Z(x',t) \rangle^2 - \varepsilon \langle \tilde{\zeta} \rangle^2} \right. \\
& \quad \left. \cdot \Delta \left(\frac{1}{2} (Z(x,t) - Z(x',t)), \zeta \right) \det Z_x(x',t) d\zeta \right\rangle.
\end{aligned}$$

By (3.4) and the Lebesgue dominated convergence theorem (for every derivative in x' and t) we have that

$$\begin{aligned}
& \frac{1}{(2\pi)^m} \int_{\mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)}} e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \frac{1}{2} \langle \zeta \rangle \langle Z(z,t) - Z(x',t) \rangle^2} \left\{ e^{-\varepsilon \langle \zeta \rangle^2} - e^{-\varepsilon \langle \tilde{\zeta} \rangle^2} \right\} \\
& \quad \cdot \Delta \left(\frac{1}{2} (Z(x,t) - Z(x',t)), \zeta \right) \det Z_x(x',t) d\zeta =
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i {}^t Z_x(x,t)^{-1} \xi \cdot (Z(x,t) - Z(x',t)) - \frac{1}{2} \langle {}^t Z_x(x,t)^{-1} \xi \rangle \langle Z(x,t) - Z(x',t) \rangle^2} \left\{ e^{-\varepsilon \langle {}^t Z_x(x,t)^{-1} \xi \rangle^2} - \right. \\
& \quad \left. - e^{-\varepsilon \left\langle {}^t Z_x(x,t)^{-1} \xi + \frac{1}{2} \langle {}^t Z_x(x,t)^{-1} \xi \rangle \langle Z(x,t) - Z(x',t) \rangle \right\rangle^2} \right\} \cdot \\
& \quad \cdot \Delta \left(\frac{1}{2} (Z(x,t) - Z(x',t)), {}^t Z_x(x,t)^{-1} \xi \right) \det Z_x(x',t) \det {}^t Z_x(x,t)^{-1} d\xi \\
& \quad \xrightarrow[\varepsilon \rightarrow 0^+]{\mathcal{C}_{(x',t)}^\infty} 0.
\end{aligned}$$

Therefore $u_\varepsilon(x, t)$ has the same limit when $\varepsilon \rightarrow 0^+$ as

$$\frac{1}{(2\pi)^m} \int_{\mathbb{RT}'_{\mathfrak{W}_t} \big|_{Z(x,t)}} \left\langle u(x', t), e^{i\zeta \cdot (Z(x,t) - Z(x',t)) - \varepsilon \langle \zeta \rangle^2} \det Z_x(x', t) \right\rangle d\zeta,$$

so by (3.7) we have that $u_\varepsilon(x, t) \xrightarrow[\varepsilon \rightarrow 0^+]{} u(x, t)$ in $\mathcal{C}^\infty(W; \mathcal{D}'(V))$. □

Before we prove the inversion formula that we will actually use, we need to extend the function $Z(x, t)$, with respect to the variable x , to the whole $\mathbb{R}^m$:

Let $V_1 \Subset V$. Let $\psi \in \mathcal{C}_c^\infty(V)$ satisfying

$$\begin{aligned}
0 & \leq \psi \leq 1 \\
\psi & \equiv 1 \quad \text{in } V_1.
\end{aligned}$$

Define $\tilde{Z}(x, t) \doteq x + i\psi(x)\phi(x, t)$. Then $\tilde{Z}$ defines the hypo-analytic structure in $V_1 \times W$, but $\tilde{Z}(x, t)$ is defined for all $x \in \mathbb{R}^m$. Also note that ${}^t \tilde{Z}_x(x, t)^{-1} = (\text{Id} - i {}^t(\psi\phi)_x(x, t))(\text{Id} + {}^t(\psi\phi)_x(x, t)^2)^{-1}$. We can choose V_1, W_1 small enough so that ${}^t \tilde{Z}_x(x, t)$ is invertible for all $x \in \mathbb{R}^m$ and $t \in W_1$ (see Appendix C). From now on we shall write $Z(x, t)$ instead of $\tilde{Z}(x, t)$, and V and W instead of V_1 and W_1 . So now

$$\mathbb{RT}'_{\mathfrak{W}_t} = \{(z, \zeta) \in \mathbb{C}^m \times \mathfrak{C}_1 : z = Z(x, t), \zeta = {}^t Z_x(x, t)^{-1} \xi \text{ for some } (x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m\},$$

and we can also assume that the inequality (3.2) is valid for all $x \in \mathbb{R}^m$. Note that (3.3) is still valid for $(x, t) \in V \times W$. Now we can prove the following inversion formula for the FBI transform:

Theorem 3.0.10. *Let $u \in \mathcal{C}^\infty(W; \mathcal{E}'(V))$. Then*

$$u(x, t) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{(2\pi^3)^{\frac{m}{2}}} \iint_{\mathbb{RT}'_{\mathfrak{W}_t}} e^{i\zeta \cdot (Z(x,t) - z') - \langle \zeta \rangle \langle Z(x,t) - z' \rangle^2 - \varepsilon \langle \zeta \rangle^2} \mathfrak{F}[u](t; z', \zeta) \langle \zeta \rangle^{\frac{m}{2}} dz' \wedge d\zeta, \quad (3.9)$$

where the convergence takes place in $\mathcal{C}^\infty(W; \mathcal{D}'(V))$.

The proof of this theorem is very close the proof of the Lemma IX.4.1. of [Tre92].

Proof. For simplicity let us assume that u is a continuous function. Then for every $\varepsilon > 0$ we must deal with the integral

$$\begin{aligned}
& \frac{1}{(2\pi^3)^{\frac{m}{2}}} \iint_{\mathbb{RT}'_{\mathfrak{W}_t}} \int_V e^{i\zeta \cdot (z' - Z(x'',t)) - \langle \zeta \rangle \langle z' - Z(x'',t) \rangle^2} e^{i\zeta \cdot (Z(x,t) - z') - \langle \zeta \rangle \langle Z(x,t) - z' \rangle^2 - \varepsilon \langle \zeta \rangle^2} \cdot \\
& \cdot u(x'', t) \langle \zeta \rangle^{\frac{m}{2}} \Delta(z' - Z(x'', t), \zeta) dZ(x'', t) dz' \wedge d\zeta =
\end{aligned}$$

$$= \frac{1}{(2\pi^3)^{\frac{m}{2}}} \iiint_{V \times \mathbb{RT}'_{\mathfrak{W}_t}|_{Z(x',t)} \times \mathbb{R}^m} e^{i\zeta \cdot (Z(x,t) - Z(x'',t)) - \langle \zeta \rangle \langle Z(x',t) - Z(x'',t) \rangle^2 - \langle \zeta \rangle \langle Z(x,t) - Z(x',t) \rangle^2 - \varepsilon \langle \zeta \rangle^2} \cdot u(x'',t) \langle \zeta \rangle^{\frac{m}{2}} \Delta(Z(x',t) - Z(x'',t), \zeta) dZ(x'',t) d\zeta dZ(x',t)$$

First we shall change the domain of the integration in the variable ζ from $\mathbb{RT}'_{\mathfrak{W}_t}|_{Z(x',t)}$ to $\mathbb{RT}'_{\mathfrak{W}_t}|_{Z(x,t)}$. So we can change the order of integration and integrate in $Z(x',t)$ first, so we shall calculate

$$\frac{\langle \zeta \rangle^{\frac{m}{2}}}{\pi^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-\langle \zeta \rangle [\langle Z(x',t) - Z(x'',t) \rangle^2 + \langle Z(x,t) - Z(x',t) \rangle^2]} \Delta(Z(x',t) - Z(x'',t), \zeta) dZ(x',t). \quad (3.10)$$

To do so we start noticing that

$$\frac{\omega^{\frac{m}{2}}}{\pi^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-\omega \langle Z(x',t) - z \rangle^2} dZ(x',t) = 1,$$

for every $\omega \in \mathbb{C}$, with $\operatorname{Re} \omega > 0$, and $z \in \mathbb{C}^m$, here note that the imaginary part of $Z(x,t)$ has compact support, and also that

$$\frac{\omega^{\frac{m}{2}}}{\pi^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-\omega \langle Z(x',t) \rangle^2} P(Z(x',t)) dZ(x',t) = 0,$$

for every $P(z)$ polynomial such that it has degree one (exactly one) when viewed as a polynomial in each variable separately (in view of Fubini's Theorem). Therefore

$$\frac{\omega^{\frac{m}{2}}}{\pi^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-\omega \langle Z(x',t) - \tilde{z} \rangle^2} \Delta(Z(x',t) - \tilde{z}, \zeta) dZ(x',t) = \Delta(z - \tilde{z}, \zeta),$$

for every $z, \tilde{z} \in \mathbb{C}^m$. To use this identity we must rewrite $\langle Z(x',t) - Z(x'',t) \rangle^2 + \langle Z(x,t) - Z(x',t) \rangle^2$. We start noticing that

$$\begin{aligned} \left\langle Z(x',t) - \frac{(Z(x,t) + Z(x'',t))}{2} \right\rangle^2 &= \left\langle \frac{Z(x',t) - Z(x,t)}{2} + \frac{Z(x',t) - Z(x'',t)}{2} \right\rangle^2 \\ &= \frac{1}{4} \langle Z(x',t) - Z(x,t) \rangle^2 + \frac{1}{4} \langle Z(x',t) - Z(x'',t) \rangle^2 + \\ &\quad + \frac{1}{2} (Z(x',t) - Z(x,t)) \cdot (Z(x',t) - Z(x'',t)) \\ &= \frac{1}{4} \langle Z(x',t) - Z(x,t) \rangle^2 + \frac{1}{4} \langle Z(x',t) - Z(x'',t) \rangle^2 - \\ &\quad - \frac{1}{2} (Z(x,t) - Z(x',t)) \cdot (Z(x',t) - Z(x'',t)). \end{aligned}$$

So we have obtained the following identity

$$\begin{aligned} \langle Z(x',t) - Z(x,t) \rangle^2 + \langle Z(x',t) - Z(x'',t) \rangle^2 &= 4 \left\langle Z(x',t) - \frac{(Z(x,t) + Z(x'',t))}{2} \right\rangle^2 + \\ &\quad + 2(Z(x,t) - Z(x',t)) \cdot (Z(x',t) - Z(x'',t)). \end{aligned}$$

Also note that

$$\begin{aligned} \langle Z(x,t) - Z(x'',t) \rangle^2 &= \langle Z(x,t) - Z(x',t) + Z(x',t) - Z(x'',t) \rangle^2 \\ &= \langle Z(x',t) - Z(x,t) \rangle^2 + \langle Z(x',t) - Z(x'',t) \rangle^2 + \end{aligned}$$

$$+ 2(Z(x, t) - Z(x', t)) \cdot (Z(x', t) - Z(x'', t)).$$

Summing up these two identities we have that

$$\begin{aligned} \langle Z(x', t) - Z(x, t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2 &= \\ &= 2 \left\langle Z(x', t) - \frac{(Z(x, t) + Z(x'', t))}{2} \right\rangle^2 + \\ &+ \frac{1}{2} \langle Z(x, t) - Z(x'', t) \rangle^2. \end{aligned} \quad (3.11)$$

Now we can calculate (3.10):

$$\begin{aligned} \frac{\langle \zeta \rangle^{\frac{m}{2}}}{\pi^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-\langle \zeta \rangle [\langle Z(x', t) - Z(x'', t) \rangle^2 + \langle Z(x, t) - Z(x', t) \rangle^2]} \Delta(Z(x', t) - Z(x'', t), \zeta) dZ(x', t) &= \\ &= e^{-\frac{1}{2} \langle \zeta \rangle \langle Z(x, t) - Z(x'', t) \rangle^2} \frac{\langle \zeta \rangle^{\frac{m}{2}}}{\pi^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-2 \langle \zeta \rangle \left\langle Z(x', t) - \frac{(Z(x, t) + Z(x'', t))}{2} \right\rangle^2} \cdot \\ &\cdot \Delta(Z(x', t) - Z(x'', t), \zeta) dZ(x', t) \\ &= \frac{e^{-\frac{1}{2} \langle \zeta \rangle \langle Z(x, t) - Z(x'', t) \rangle^2}}{2^{\frac{m}{2}}} \Delta \left(\left(\frac{Z(x, t) - Z(x'', t)}{2} \right), \zeta \right). \end{aligned}$$

So let $\varepsilon > 0$. We have that

$$\begin{aligned} \frac{1}{(2\pi^3)^{\frac{m}{2}}} \iiint_{V \times \mathbb{R} T'_{\mathfrak{W}_t} \big|_{Z(x', t)} \times \mathbb{R}^m} e^{i\zeta \cdot (Z(x, t) - Z(x'', t)) - \langle \zeta \rangle \langle Z(x', t) - Z(x'', t) \rangle^2 - \langle \zeta \rangle \langle Z(x, t) - Z(x', t) \rangle^2 - \varepsilon \langle \zeta \rangle^2} \cdot \\ \cdot u(x'', t) \langle \zeta \rangle^{\frac{m}{2}} \Delta(Z(x', t) - Z(x'', t), \zeta) dZ(x'', t) d\zeta dZ(x', t) &= \\ &= \frac{1}{(4\pi^2)^{\frac{m}{2}}} \int_{\mathbb{R} T'_{\mathfrak{W}_t} \big|_{Z(x', t)}} \int_V e^{i\zeta \cdot (Z(x, t) - Z(x'', t)) - \frac{1}{2} \langle \zeta \rangle \langle Z(x, t) - Z(x'', t) \rangle^2 - \varepsilon \langle \zeta \rangle^2} u(x'', t) \cdot \\ \cdot \Delta \left(\left(\frac{Z(x, t) - Z(x'', t)}{2} \right), \zeta \right) dZ(x'', t) d\zeta &= \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R} T'_{\mathfrak{W}_t} \big|_{Z(x', t)}} e^{-\varepsilon \langle \zeta \rangle^2} \mathfrak{F}^{\frac{1}{2}}[u](t; Z(x, t), \zeta) d\zeta \\ &\xrightarrow{\varepsilon \rightarrow 0^+} u(x, t). \end{aligned}$$

□

Now we can state the main theorem of this chapter:

Theorem 3.0.11. *Let $\mathcal{V}$ be a locally integrable structure on $\Omega \subset \mathbb{R}^N$, and let $p \in \Omega$ be an arbitrary point. Consider $(V \times W, x_1, \dots, x_m, t_1, \dots, t_n)$ an local coordinate system vanishing at p , as described above. Let $u \in \mathcal{C}^\infty(W; \mathcal{D}'(V))$ be a solution of*

$$\begin{cases} L_1 u = f_1, \\ \vdots \\ L_n u = f_n, \end{cases}$$

where $f_j \in G^s(U; L_1, \dots, L_n, M_1, \dots, M_m)$, $j = 1, \dots, n$. The following are equivalent

1. There exist $V_0 \subset V$, $W_0 \subset W$ open balls containing the origin such that $u|_{V_0 \times W_0} \in G^s(V_0 \times$

$W_0; L_1, \dots, L_n, M_1, \dots, M_m);$

2. There exists $V_1 \Subset V$ an open ball centered at the origin such that for every $\chi \in \mathcal{C}_c^\infty(V_1)$, with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in some open neighborhood of the origin, there exist $\widetilde{V} \subset V_1$, $\widetilde{W} \subset W$, open balls centered at the origin, and constants $C, \varepsilon > 0$ such that

$$|\mathfrak{F}[\chi u](t; z, \zeta)| \leq C e^{-\varepsilon|\zeta|^{\frac{1}{s}}}, \quad \forall t \in \widetilde{W}, (z, \zeta) \in \mathbb{R}T'_{\mathfrak{W}_t}|_{\widetilde{V}} \setminus 0,$$

where $(z, \zeta) \in \mathbb{R}T'_{\mathfrak{W}_t}|_{\widetilde{V}}$ means that $z = Z(x, t)$, $\zeta = {}^tZ_x(x, t)^{-1}\xi$, $\xi \in \mathbb{R}^m \setminus 0$ and $x \in \widetilde{V}$;

3. For every $\chi \in \mathcal{C}_c^\infty(V)$, with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in some open neighborhood of the origin, there exist $\widetilde{V} \subset V$, $\widetilde{W} \subset W$, open balls centered at the origin, constants $C, \varepsilon > 0$ such that

$$|\mathfrak{F}[\chi u](t; z, \zeta)| \leq C e^{-\varepsilon|\zeta|^{\frac{1}{s}}}, \quad \forall t \in \widetilde{W}, (z, \zeta) \in \mathbb{R}T'_{\mathfrak{W}_t}|_{\widetilde{V}} \setminus 0, \quad (3.12)$$

where $(z, \zeta) \in \mathbb{R}T'_{\mathfrak{W}_t}|_{\widetilde{V}}$ means that $z = Z(x, t)$, $\zeta = {}^tZ_x(x, t)^{-1}\xi$, $\xi \in \mathbb{R}^m \setminus 0$ and $x \in \widetilde{V}$

Before proving this theorem we shall derive a formula for the derivatives of the Gaussian:

Lemma 3.0.12. *Let $\lambda > 0$ and $\alpha \in \mathbb{Z}_+^m$. Then*

$$\partial_x^\alpha e^{-\lambda|x|^2} = \sum_{l_1^1+2l_2^1=\alpha_1} \cdots \sum_{l_1^m+2l_2^m=\alpha_m} \frac{\alpha!}{l_1^1!l_2^1! \cdots l_1^m!l_2^m!} (-\lambda)^{l_1^1+l_2^1+\cdots+l_1^m+l_2^m} (2x_1)^{l_1^1} \cdots (2x_m)^{l_1^m} e^{-\lambda|x|^2}. \quad (3.13)$$

Proof. Let $x \in \mathbb{R}^m$ and $j = 1, \dots, m$. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(t) = e^{-\lambda\{x_1^2+\cdots+x_{j-1}^2+t^2+x_{j+1}^2+\cdots+x_m^2\}}.$$

So $f = g \circ h(t)$, where $g(t) = e^{-\lambda t}$, and $h(t) = x_1^2 + \cdots + x_{j-1}^2 + t^2 + x_{j+1}^2 + \cdots + x_m^2$. By Faà di Bruno's formula (see for instance [BM04]) we have that

$$\begin{aligned} \partial_{x_j}^{\alpha_j} e^{-\lambda|x|^2} &= f^{(\alpha_j)}(x_j) \\ &= \sum_{\{l_1+2l_2+\cdots+\alpha_j l_{\alpha_j}=\alpha_j\}} \frac{\alpha_j!}{l_1! \cdots l_{\alpha_j}!} g^{(l_1+\cdots+l_{\alpha_j})}(h(x_j)) \prod_{i=1}^{\alpha_j} \left(\frac{h^{(i)}(x_j)}{i!} \right)^{l_i} \\ &= \sum_{l_1+2l_2=\alpha_j} \frac{\alpha_j!}{l_1!l_2!} (-\lambda)^{l_1+l_2} e^{-\lambda|x|^2} (2x_j)^{l_1}. \end{aligned}$$

Since the only term in the sum above that depends on the other variables $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m$ is the Gaussian, we can apply this identity for each variable separately, obtaining (3.13). $\square$

Proof of the Theorem. 1. $\Rightarrow$ 2.:

By Theorem 2.1.6 and its corollary applied to the hypo-analytic structure generated by $(Z(x, t), t)$ on U we have that there exist $\mathcal{O} \subset \mathbb{C}^m$ an open neighborhood of $\{Z(x, t) : x \in V_0, t \in W_0\}$ on $\mathbb{C}^m$, and $F(z, t) \in \mathcal{C}^\infty(\mathcal{O} \times W_0)$ such that

$$\begin{cases} F(Z(x, t), t) = u(x, t), & \forall (x, t) \in V_0 \times W_0; \\ |\partial_z F(z, t)| \leq C^{k+1} k!^{s-1} \text{dist}(z, \mathfrak{M}_t)^k, & \forall k > 0, z \in \mathcal{O}, t \in W_0, \end{cases}$$

where C is a positive constant, as in (2.2). Set $V_1 = V_0$ and let $\chi \in \mathcal{C}_c^\infty(V_1)$ be such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in $V_2 \Subset V_1$, an open ball centered at the origin. We shall estimate

$$\mathfrak{F}[\chi u](t; z, \zeta) = \int_{V_1} e^{i\zeta \cdot (z - Z(x, t)) - \langle \zeta \rangle \langle z - Z(x, t) \rangle^2} \chi(x) u(x, t) \Delta(z - Z(x, t), \zeta) dZ(x, t).$$

To do so we shall deform the contour of integration. So let $(z, \zeta) \in \mathbb{RT}'_{\mathfrak{M}_t}|_{\widetilde{V}}$ be fixed, where $t \in \widetilde{W}$, and $\widetilde{V} \Subset V$, $\widetilde{W} \Subset W$ are open balls centered at the origin to be chosen latter. Let $\lambda > 0$ such that the image of the map

$$V_1 \times W_0 \ni (y, t) \mapsto \Theta_\lambda(y, t) \doteq Z(y, t) - i\lambda \mathbb{E}_{V_2}(y) \frac{\zeta}{\langle \zeta \rangle},$$

is contained in $\mathcal{O}$, where $\mathbb{E}_{V_2}$ is characteristic function of V_2 . By Stokes theorem we obtain

$$\begin{aligned} \mathfrak{F}[\chi u](t; z, \zeta) &= \int_{V_1 \setminus V_2} e^{i\zeta \cdot (z - Z(x, t)) - \langle \zeta \rangle \langle z - Z(x, t) \rangle^2} \chi(x) u(x, t) \Delta(z - Z(x, t), \zeta) dZ(x, t) \\ &\quad + \int_{V_2} e^{i\zeta \cdot (z - Z(x, t)) - \langle \zeta \rangle \langle z - Z(x, t) \rangle^2} F(Z(x, t), t) \Delta(z - Z(x, t), \zeta) dZ(x, t) \\ &= \underbrace{\int_{V_1 \setminus V_2} e^{i\zeta \cdot (z - Z(x, t)) - \langle \zeta \rangle \langle z - Z(x, t) \rangle^2} \chi(x) u(x, t) \Delta(z - Z(x, t), \zeta) dZ(x, t)}_{(1)} \\ &\quad + \underbrace{\int_{V_2} e^{i\zeta \cdot (z - \Theta_\lambda(x, t)) - \langle \zeta \rangle \langle z - \Theta_\lambda(x, t) \rangle^2} F(\Theta_\lambda(x, t), t) \Delta(z - \Theta_\lambda(x, t), \zeta) dZ(x, t)}_{(2)} \\ &\quad + \underbrace{(-1)^{m-1} 2i \int_0^\lambda \int_{V_2} e^{i\zeta \cdot (z - \Theta_\sigma(x, t)) - \langle \zeta \rangle \langle z - \Theta_\sigma(x, t) \rangle^2} \overline{\partial_z F(\Theta_\sigma(x, t), t)} \cdot \frac{\zeta}{\langle \zeta \rangle} dZ(x, t) \cdot \Delta(z - \Theta_\sigma(x, t), \zeta) d\sigma}_{(3)} \\ &\quad - \underbrace{\int_0^\lambda \int_{\partial V_2} e^{i\zeta \cdot (z - \Theta_\sigma(x, t)) - \langle \zeta \rangle \langle z - \Theta_\sigma(x, t) \rangle^2} F(\Theta_\sigma(x, t), t) \Delta(z - \Theta_\sigma(x, t), \zeta) dS_{\mathfrak{M}_t} d\sigma}_{(4)}, \end{aligned}$$

where $dS_{\mathfrak{M}_t}$ is the surface measure in $\{Z(x, t) : x \in \partial V_2\}$. We shall estimate these four integrals separately. Since the estimate for (1) and (4) are very similar, we will estimate them first. We start writing $\widetilde{V} = B_r(0)$, so $z = Z(x_0, t)$ for some $x_0 \in B_r(0)$. In view of (3.3) and

$$|z - Z(x, t)| \geq |x_0 - x|,$$

for every x , we have that

$$\text{Im}\{\zeta \cdot (z - Z(x, t)) + i\langle \zeta \rangle \langle z - Z(x, t) \rangle^2\} \geq c(r_2 - r)^2 |\zeta|,$$

for every $x \in V_1 \setminus V_2$, where $V_2 = B_{r_2}(0)$, and we are choosing $r < r_2$. Therefore

$$\left| \int_{V_1 \setminus V_2} e^{i\zeta \cdot (z - Z(x, t)) - \langle \zeta \rangle \langle z - Z(x, t) \rangle^2} \chi(x) u(x, t) \Delta(z - Z(x, t), \zeta) dZ(x, t) \right| \leq C e^{-c(r_2 - r)^2 |\zeta|}.$$

Now the exponent of (4) can be written as

$$i\zeta \cdot (z - \Theta_\sigma(x, t)) - \langle \zeta \rangle \langle z - \Theta_\sigma(x, t) \rangle^2 = i\zeta \cdot (z - Z(x, t)) - \sigma \langle \zeta \rangle - \langle \zeta \rangle \langle z - Z(x, t) \rangle^2 + \sigma^2 \langle \zeta \rangle - 2i\sigma(z - Z(x, t)) \cdot \zeta.$$

Now recall that in (4) we are integrating in σ from 0 to λ , so $\sigma < \lambda$, and using (3.3) and (A.2) we have that

$$\begin{aligned} \operatorname{Im}\{\zeta \cdot (z - \Theta_\sigma(x, t)) + i\langle \zeta \rangle \langle z - \Theta_\sigma(x, t) \rangle^2\} &= \operatorname{Im}\{\zeta \cdot (z - Z(x, t)) + i\langle \zeta \rangle \langle z - Z(x, t) \rangle^2\} \\ &\quad + \sigma \operatorname{Im}\{i\langle \zeta \rangle (1 - \sigma) - 2(z - Z(x, t)) \cdot \zeta\} \\ &\geq c|z - Z(x, t)|^2 |\zeta| + \sigma \operatorname{Re}\langle \zeta \rangle (1 - \sigma) - \\ &\quad - 2\sigma \operatorname{Im}\{\zeta \cdot (z - Z(x, t))\} \\ &\geq c|z - Z(x, t)|^2 |\zeta| + \sigma \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} (1 - \sigma) |\zeta| - \\ &\quad - 2\sigma |\zeta| |z - Z(x, t)| \\ &\geq |\zeta| |z - Z(x, t)| \{c|z - Z(x, t)| - 2\lambda\} \\ &\geq |\zeta| |z - Z(x, t)| [c(r_2 - r) - 2\lambda] \\ &\geq |\zeta| (r_2 - r) [c(r_2 - r) - 2\lambda], \end{aligned}$$

where we are choosing λ satisfying $2\lambda < c(r_2 - r)$. Therefore

$$\left| \int_0^\lambda \int_{\partial V_1} e^{i\zeta \cdot (z - \Theta_\sigma(x, t)) - \langle \zeta \rangle \langle z - \Theta_\sigma(x, t) \rangle^2} F(\Theta_\sigma(x, t), t) \Delta(z - \Theta_\sigma(x, t), \zeta) dS_{\mathfrak{M}_t} \sigma \right| \leq C e^{-\varepsilon_1 |\zeta|},$$

where $\varepsilon_1 = (r_2 - r)[c(r_2 - r) - 2\lambda]$. Before estimating (2) and (3), note that the exponent that appears in each of them is similar to the one that we have just estimated. In (2) we have that $x \in V_2$, i.e., $|x| < r_2$, so the exponential have the following estimate:

$$\begin{aligned} \left| e^{i\zeta \cdot (z - \Theta_\lambda(x, t)) - \langle \zeta \rangle \langle z - \Theta_\lambda(x, t) \rangle^2} \right| &\leq e^{-\left\{ c|z - Z(x, t)|^2 |\zeta| + \lambda \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} (1 - \lambda) |\zeta| - 2\lambda |\zeta| |z - Z(x, t)| \right\}} \\ &\leq e^{-|\zeta| \left\{ \lambda \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} (1 - \lambda) + |z - Z(x, t)| [c|z - Z(x, t)| - 2\lambda] \right\}}. \end{aligned}$$

When $c|z - Z(x, t)| \geq 2\lambda$ we have that

$$\left| e^{i\zeta \cdot (z - \Theta_\lambda(x, t)) - \langle \zeta \rangle \langle z - \Theta_\lambda(x, t) \rangle^2} \right| \leq e^{-|\zeta| \lambda \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} (1 - \lambda)},$$

and when $c|z - Z(x, t)| \leq 2\lambda$,

$$\begin{aligned} \left| e^{i\zeta \cdot (z - \Theta_\lambda(x, t)) - \langle \zeta \rangle \langle z - \Theta_\lambda(x, t) \rangle^2} \right| &\leq e^{-|\zeta| \left\{ \lambda \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} (1 - \lambda) - 2\lambda |z - Z(x, t)| \right\}} \\ &\leq e^{-|\zeta| \left\{ \lambda \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} (1 - \lambda) - \frac{4\lambda^2}{c} \right\}} \\ &\leq e^{-|\zeta| \lambda \left\{ \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} (1 - \lambda) - \frac{4\lambda}{c} \right\}}. \end{aligned}$$

Combining these two estimates we conclude that

$$\left| \int_{V_1} e^{i\zeta \cdot (z - \Theta_\lambda(x, t)) - \langle \zeta \rangle \langle z - \Theta_\lambda(x, t) \rangle^2} F(\Theta_\lambda(x, t), t) \Delta(z - \Theta_\lambda(x, t), \zeta) dZ(x, t) \right| \leq C e^{-\lambda \varepsilon_2 |\zeta|},$$

where $\varepsilon_2 = \sqrt{\frac{1-\kappa^2}{1+\kappa^2}}(1-\lambda) - \frac{4\lambda}{c} > 0$, decreasing λ if necessary. To estimate (3) we reason as before, so for each $0 < \sigma \leq \lambda$ we have that if $|z - Z(x, t)| \geq 2\sigma/c$ then

$$\left| e^{i\zeta \cdot (z - \Theta_\sigma(x, t)) - \langle \zeta \rangle \langle z - \Theta_\sigma(x, t) \rangle^2} \right| \leq e^{-|\zeta| \sigma \sqrt{\frac{1-\kappa^2}{1+\kappa^2}}(1-\sigma)},$$

and if $|z - Z(x, t)| \leq 2\sigma/c$,

$$\begin{aligned} \left| e^{i\zeta \cdot (z - \Theta_\sigma(x, t)) - \langle \zeta \rangle \langle z - \Theta_\sigma(x, t) \rangle^2} \right| &\leq e^{-|\zeta| \left\{ \sigma \sqrt{\frac{1-\kappa^2}{1+\kappa^2}}(1-\sigma) - 2\sigma |z - Z(x, t)| \right\}} \\ &\leq e^{-|\zeta| \left\{ \sigma \sqrt{\frac{1-\kappa^2}{1+\kappa^2}}(1-\sigma) - \frac{4\sigma^2}{1-\kappa} \right\}} \\ &\leq e^{-|\zeta| \sigma \left\{ \sqrt{\frac{1-\kappa^2}{1+\kappa^2}}(1-\sigma) - \frac{4\sigma}{1-\kappa} \right\}}, \end{aligned}$$

and since $\sqrt{\frac{1-\kappa^2}{1+\kappa^2}}(1-\sigma) - \frac{4\sigma}{c} \geq \varepsilon_2$, for $\sigma < \lambda$, we have that

$$\left| e^{i\zeta \cdot (z - \Theta_\sigma(x, t)) - \langle \zeta \rangle \langle z - \Theta_\sigma(x, t) \rangle^2} \right| \leq e^{-\sigma \varepsilon_2 |\zeta|},$$

for every $x \in V_1$. So for every $k > 0$ we have that

$$\begin{aligned} \left| (-1)^{m-1} 2i \int_0^\lambda \int_{V_1} e^{i\zeta \cdot (z - \Theta_\sigma(x, t)) - \langle \zeta \rangle \langle z - \Theta_\sigma(x, t) \rangle^2} \overline{\partial}_z F(\Theta_\sigma(x, t), t) \cdot \frac{\zeta}{\langle \zeta \rangle} \Delta(z - \Theta_\sigma(x, t), \zeta) dZ(x, t) d\sigma \right| &\leq \\ &\leq \int_0^\lambda e^{-\sigma \varepsilon_2 |\zeta|} \sup_{(x, t) \in V_0 \times W_0} \left| \overline{\partial}_z F(\Theta_\sigma(x, t), t) \Delta(z - \Theta_\sigma(x, t), \zeta) \right| d\sigma \cdot \\ &\quad \cdot 2 \left| \frac{|\zeta|}{\langle \zeta \rangle} \right| \left| \int_{V_1} |dZ(x, t)| \right| \\ &\leq C \int_0^\lambda e^{-\sigma \varepsilon_2 |\zeta|} C^{k+1} k! s^{-1} \text{dist}(\Theta_\sigma(x, t), \mathfrak{M}_t)^k d\sigma \\ &\leq C^{k+1} k! s^{-1} \int_0^\infty e^{-\sigma \varepsilon_2 |\zeta|} \left| \frac{\sigma |\zeta|}{\langle \zeta \rangle} \right|^k d\sigma \\ &\leq C^{k+1} k! s^{-1} \int_0^\infty e^{-y} \left(\frac{y}{\varepsilon_2 |\zeta|} \right)^k \frac{1}{\varepsilon_2 |\zeta|} dy \\ &\leq C^{k+1} \frac{k! s}{(\varepsilon_2 |\zeta|)^{k+1}}. \end{aligned}$$

Since the constant $C > 0$ does not depend on k , and the above estimate holds for every $k > 0$, we have that

$$\left| \int_0^\lambda \int_{V_1} e^{i\zeta \cdot (z - \Theta_\sigma(x, t)) - \langle \zeta \rangle \langle z - \Theta_\sigma(x, t) \rangle^2} \overline{\partial}_z F(\Theta_\sigma(x, t), t) \cdot \frac{\zeta}{\langle \zeta \rangle} \Delta(z - \Theta_\sigma(x, t), \zeta) dZ(x, t) d\sigma \right| \leq C e^{-\varepsilon_3 |\zeta|^{\frac{1}{s}}},$$

for some constants $C, \varepsilon_3 > 0$. Summing up we have obtained the required estimate (3.12), with $\widetilde{W} = W_0$, and $\widetilde{V} = B_r(0)$, where $r > 0$ is any positive number less than r_2 , the radius of V_2 .

2. $\Rightarrow$ 3.:

Let $\chi \in \mathcal{C}_c^\infty(V)$, and $\chi_1 \in \mathcal{C}_c^\infty(V_1)$ as in 2. and 3.. Since $\chi\chi_1 \in \mathcal{C}_c^\infty(V_1)$, and $\chi\chi_1 \equiv 1$ in some open neighborhood of the origin. So we have that

$$|\mathfrak{F}[\chi\chi_1 u](t; z, \zeta)| \leq C e^{-\varepsilon|\zeta|^{\frac{1}{s}}}, \quad t \in \widetilde{W}, (z, \zeta) \in \mathbb{RT}'_{\mathfrak{W}_t}|_{\widetilde{V}}.$$

Now note that $\chi - \chi\chi_1 \equiv 0$ in some open neighborhood of the origin $V_2 \Subset V_1$. Write $V_2 = B_\rho(0)$. So if $x' \in B_{\frac{\rho}{2}}(0)$, $x \in V \setminus V_2$, $t \in W$, and $\zeta \in \mathbb{RT}'_{\mathfrak{W}_t}|_x$, we have that

$$\begin{aligned} \operatorname{Im}\{\zeta \cdot (Z(x, t) - Z(x', t)) + i\langle \zeta \rangle \langle Z(x, t) - Z(x', t) \rangle^2\} &\geq c|Z(x, t) - Z(x', t)|^2|\zeta| \\ &\geq c\frac{\rho^2}{4}|\zeta|. \end{aligned}$$

Therefore if we set $V_3 = B_{\frac{\rho}{2}}(0) \cap \widetilde{V}$ we have that

$$|\mathfrak{F}[(\chi - \chi\chi_1)u](t; z, \zeta)| \leq C e^{-\varepsilon'|\zeta|}, \quad t \in W, (z, \zeta) \in \mathbb{RT}'_{\mathfrak{W}_t}|_{V_3}.$$

Combining these two decays we obtain

$$|\mathfrak{F}[\chi u](t; z, \zeta)| \leq C e^{-\tilde{\varepsilon}|\zeta|^{\frac{1}{s}}}, \quad t \in \widetilde{W}, (z, \zeta) \in \mathbb{RT}'_{\mathfrak{W}_t}|_{V_3}.$$

3. $\Rightarrow$ 1.:

Let $\widetilde{V} \subset V$ and $\widetilde{W}$ be open balls centered at the origin, $\chi \in \mathcal{C}_c^\infty(V)$ with $0 \leq \chi \leq 1$, and $\chi \equiv 1$ in an open ball centered at the origin, and $C, \tilde{\varepsilon} > 0$ for which the following estimate holds

$$|\mathfrak{F}[\chi u](t; z, \zeta)| \leq C e^{-\tilde{\varepsilon}|\zeta|^{\frac{1}{s}}},$$

for every $z = Z(x, t)$ and $\zeta = {}^t Z_x(x, t)^{-1}\xi$, where $x \in \widetilde{V}$, $t \in \widetilde{W}$ and $\xi \in \mathbb{R}^m \setminus 0$. Note that we can choose $\operatorname{supp} \chi$ as small as we want, keeping in mind that $\widetilde{V}$ depends on χ . Since we already have that $L_j u \in G^s(U; L_1, \dots, L_n, M_1, \dots, M_m)$, we only have to prove that there exist $V_0 \subset V$ and $W_0 \subset W$, open balls centered at the origin, such that, writing $U_0 = V_0 \times W_0$, $u|_{U_0} \in G^s(U_0; M_1, \dots, M_m)$, since the complex vector fields $\{L_1, \dots, L_n, M_1, \dots, M_m\}$ are pair-wise commuting. We write $V_0 = B_r(0)$ and $W_0 = B_\delta(0)$. By (3.9) we have that

$$\chi(x)u(x, t) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{(2\pi^3)^{\frac{m}{2}}} \iint_{\mathbb{RT}'_{\mathfrak{W}_t}} e^{i\zeta \cdot (Z(x, t) - z') - \langle \zeta \rangle \langle Z(x, t) - z' \rangle^2 - \varepsilon \langle \zeta \rangle^2} \mathfrak{F}[\chi u](t; z', \zeta) \langle \zeta \rangle^{\frac{m}{2}} dz' \wedge d\zeta.$$

We shall split this integral in three regions:

$$\begin{aligned} Q_t^1 &\doteq \{(z', \zeta) : z = Z(x', t), \zeta = {}^t Z_x(x', t)^{-1}\xi, \text{ for some } |x'| < \tilde{r} \text{ and } \xi \in \mathbb{R}^m\} \\ Q_t^2 &\doteq \{(z', \zeta) : z = Z(x', t), \zeta = {}^t Z_x(x', t)^{-1}\xi, \text{ for some } \tilde{r} \leq |x'| < r_0 \text{ and } \xi \in \mathbb{R}^m\} \\ Q_t^3 &\doteq \{(z', \zeta) : z = Z(x', t), \zeta = {}^t Z_x(x', t)^{-1}\xi, \text{ for some } r_0 \leq |x'| \text{ and } \xi \in \mathbb{R}^m\}, \end{aligned}$$

where $\tilde{r}$ and r_0 are the radii of $\widetilde{V}$ and V . For $\varepsilon > 0$ and $j = 1, 2, 3$, we set

$$I_j^\varepsilon(x, t) \doteq \frac{1}{(2\pi^3)^{\frac{m}{2}}} \iint_{Q_t^j} e^{i\zeta \cdot (Z(x, t) - z') - \langle \zeta \rangle \langle Z(x, t) - z' \rangle^2 - \varepsilon \langle \zeta \rangle^2} \mathfrak{F}[\chi u](t; z', \zeta) \langle \zeta \rangle^{\frac{m}{2}} dz' \wedge d\zeta,$$

so we can write

$$\chi(x)u(x, t) = \lim_{\varepsilon \rightarrow 0^+} I_1^\varepsilon(x, t) + I_2^\varepsilon(x, t) + I_3^\varepsilon(x, t)$$

To prove 1. it is enough to prove the following: there exists a sequence $\{\varepsilon_j\}_{j \in \mathbb{Z}_+}$ with $\varepsilon_j \rightarrow 0$ such that $I_2^{\varepsilon_j}$ and $I_3^{\varepsilon_j}$ converge to analytic vectors for $M_1, \dots, M_m$, and that I_1^ε converges to a Gevrey vector for $M_1, \dots, M_m$. To do so we shall prove that there exist $G_2^\varepsilon(z, t)$, $G_3^\varepsilon(z, t)$, $G_2(z, t)$ and $G_3(z, t)$, holomorphic functions in some open neighborhood of the origin such that $I_2^\varepsilon(x, t) = G_2^\varepsilon(Z(x, t), t)$, $I_3^\varepsilon(x, t) = G_3^\varepsilon(Z(x, t), t)$, and $G_2^{\varepsilon_j}(z, t) \rightarrow G_2(z, t)$ and $G_3^{\varepsilon_j}(z, t) \rightarrow G_3(z, t)$ uniformly in z , for some sequence $\{\varepsilon_j\}_{j \in \mathbb{Z}_+}$ satisfying $\varepsilon_j \rightarrow 0$, and we shall also prove that there exists a positive constant C such that

$$|M^\alpha I_1^\varepsilon(x, t)| \leq C^{|\alpha|+1} \alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^m,$$

for all $(x, t) \in U_0$ and $\varepsilon > 0$.

$I_2^\varepsilon(x, t)$:

Let $(z', \zeta) \in Q_t^2$. Since $z' = Z(x', t)$, with $x' \in V$, we can use (3.3) and (3.2) to obtain

$$\begin{aligned} \operatorname{Im}\{\zeta \cdot (Z(0, t) - Z(x', t)) + i\langle \zeta \rangle \langle Z(0, t) - Z(x', t) \rangle^2\} &\geq c|\zeta| |Z(0, t) - Z(x', t)|^2 \\ &\geq c|\zeta|(1 - \mu^2)|x'|^2 \\ &\geq c(1 - \mu^2)\tilde{r}|\zeta|, \end{aligned}$$

in other words

$$\sup_{(z', \zeta) \in Q_t^2} \frac{\operatorname{Im}\{\zeta \cdot (Z(0, t) - z') + i\langle \zeta \rangle \langle Z(0, t) - z' \rangle^2\}}{|\zeta|} \geq c(1 - \mu^2)\tilde{r},$$

and this is valid for every $t \in W$. So there are $\mathcal{O}_1 \subset \mathbb{C}^m$ an open neighborhood of the origin and $W_1 \Subset W$ an open neighborhood of the origin, such that

$$\sup_{(z', \zeta) \in Q_t^2} \frac{\operatorname{Im}\{\zeta \cdot (z - z') + i\langle \zeta \rangle \langle z - z' \rangle^2\}}{|\zeta|} \geq \frac{c(1 - \mu^2)\tilde{r}}{2}, \quad \forall z \in \mathcal{O}_1, t \in W_1.$$

Now using (3.5) we obtain

$$\left| e^{i\zeta \cdot (z - z') - \langle \zeta \rangle \langle z - z' \rangle^2} \mathfrak{F}[\chi u](t; z', \zeta) \langle \zeta \rangle^{\frac{m}{2}} \right| \leq C(1 + |\zeta|)^{k + \frac{m}{2}} e^{-\frac{c(1 - \mu^2)\tilde{r}}{2}|\zeta|}, \quad (3.14)$$

for some $k \geq 0$ and for all $z \in \mathcal{O}_1$, $(z', \zeta) \in Q_t^2$, and $t \in W_1$. Now set

$$G_2^\varepsilon(z, t) \doteq \frac{1}{(2\pi^3)^{\frac{m}{2}}} \iint_{Q_t^2} e^{i\zeta \cdot (z - z') - \langle \zeta \rangle \langle z - z' \rangle^2 - \varepsilon \langle \zeta \rangle^2} \mathfrak{F}[\chi u](t; z', \zeta) \langle \zeta \rangle^{\frac{m}{2}} dz' \wedge d\zeta,$$

and

$$G_2(z, t) \doteq \frac{1}{(2\pi^3)^{\frac{m}{2}}} \iint_{Q_t^2} e^{i\zeta \cdot (z - z') - \langle \zeta \rangle \langle z - z' \rangle^2} \mathfrak{F}[\chi u](t; z', \zeta) \langle \zeta \rangle^{\frac{m}{2}} dz' \wedge d\zeta,$$

for $\varepsilon > 0$, $z \in \mathcal{O}_1$, and $t \in W_1$. Let $V_1 \Subset V$ and $W_2 \Subset W_1$ such that $\{Z(x, t) : (x, t) \in V_1 \times W_2\} \subset \mathcal{O}_1$, so $G_2^\varepsilon(Z(x, t), t) = I_2^\varepsilon(x, t)$ for every $(x, t) \in V_1 \times W_2$. Define $I_2(x, t) \doteq G_2(Z(x, t), t)$, for $(x, t) \in V_1 \times W_2$. In view of (3.14) we have that $G_2^\varepsilon(z, t)$ and $G_2(z, t)$ are holomorphic with respect to z , and $G_2^\varepsilon(z, t) \rightarrow G_2(z, t)$ uniformly on $\mathcal{O}_1 \times W_1$.

$I_3^\varepsilon(x, t)$:

We can deform the domain of integration with respect to the variable ζ , moving the contour of the integration from $\mathbb{R}T'_{\mathfrak{W}_t}|_{Z(x', t)}$ to $\mathbb{R}^m$, obtaining

$$\begin{aligned}
I_3^\varepsilon(x, t) &= \frac{1}{(2\pi^3)^{\frac{m}{2}}} \iint_{Q_t^3} e^{i\zeta \cdot (Z(x, t) - z') - \langle \zeta \rangle \langle Z(x, t) - z' \rangle^2 - \varepsilon \langle \zeta \rangle^2} \mathfrak{F}[\chi u](t; z', \zeta) \langle \zeta \rangle^{\frac{m}{2}} dz' \wedge d\zeta \\
&= \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \int_{r_0 \leq |x'|} e^{i\xi \cdot (Z(x, t) - Z(x', t)) - |\xi| \langle Z(x, t) - Z(x', t) \rangle^2 - \varepsilon |\xi|^2} \mathfrak{F}[\chi u](t; Z(x', t), \xi) |\xi|^{\frac{m}{2}} dZ(x', t) d\xi \\
&= \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \int_{r_0 \leq |x'|} \left\langle u(x'', t), \chi(x'') e^{i\xi \cdot (Z(x, t) - Z(x'', t)) - |\xi| [\langle Z(x, t) - Z(x', t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2]} \right. \\
&\quad \cdot e^{-\varepsilon |\xi|^2} |\xi|^{\frac{m}{2}} \Delta(Z(x', t) - Z(x'', t), \xi) \det Z_x(x'', t) \Big\rangle dZ(x', t) d\xi.
\end{aligned}$$

Now for every $\varepsilon > 0$ we set

$$\begin{aligned}
G_3^\varepsilon(z, t) &\doteq \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \int_{r_0 \leq |x'|} \left\langle u(x'', t), \chi(x'') |\xi|^{\frac{m}{2}} \Delta(Z(x', t) - Z(x'', t), \xi) \right. \\
&\quad \cdot e^{i\xi \cdot (z - Z(x'', t)) - |\xi| [\langle z - Z(x', t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2] - \varepsilon |\xi|^2} \det Z_x(x'', t) \Big\rangle dZ(x', t) d\xi
\end{aligned} \tag{3.15}$$

for $z \in \mathbb{C}^m$, and $t \in W_2$. As usual, we begin estimating the exponential, but first for $z = Z(0, t)$:

$$\begin{aligned}
\left| e^{i\xi \cdot (Z(0, t) - Z(x'', t)) - |\xi| [\langle Z(0, t) - Z(x', t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2]} \right| &\leq e^{|\xi| |\phi(0, t) - \phi(x'', t)| - |\xi| [|x'|^2 - |\phi(0, t) - \phi(x', t)|^2]} \\
&\quad \cdot e^{-|\xi| [|x' - x''|^2 - |\phi(x', t) - \phi(x'', t)|^2]} \\
&\leq e^{|\xi| \mu |x''| - |\xi| [|x'|^2 - \mu^2 |x'| + |x' - x''|^2 - \mu^2 |x' - x''|^2]} \\
&\leq e^{-|\xi| [(1 - \mu^2) |x' - x''|^2 + (1 - \mu^2) |x'|^2 - \mu |x''|]},
\end{aligned}$$

where $x'' \in \text{supp } \chi$, and $r_0 \leq |x'|$. Note that the previous argument (for I_2^ε) does not depend on the "size" of $\text{supp } \chi$, therefore we can shrink it as we want to. So we can assume that $|x''|$ is small enough so

$$\left| e^{i\xi \cdot (Z(0, t) - Z(x'', t)) - |\xi| [\langle Z(0, t) - Z(x', t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2]} \right| \leq e^{-|\xi| (1 - \mu^2) |x'|^2}.$$

Now, for $z \in \mathbb{C}^m$ we have that

$$\begin{aligned}
&\left| e^{i\xi \cdot (z - Z(x'', t)) - |\xi| [\langle z - Z(x', t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2]} \right| = \\
&= \left| e^{i\xi \cdot (Z(0, t) - Z(x'', t)) - |\xi| [\langle Z(0, t) - Z(x', t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2]} \right| \\
&\quad \cdot \left| e^{i\xi \cdot (z - Z(0, t)) - |\xi| [\langle z - Z(0, t) \rangle^2 + 2i(z - Z(0, t)) \cdot (Z(0, t) - Z(x', t))]} \right| \\
&\leq e^{-|\xi| (1 - \mu^2) |x'|^2} e^{|\xi| |z - Z(0, t)| [1 + |z - Z(0, t)| + 2|Z(0, t) - Z(x', t)|]} \\
&\leq e^{-|\xi| (1 - \mu^2) |x'|^2} e^{|\xi| |z - Z(0, t)| [1 + |z - Z(0, t)| + 2(1 + \mu) |x'|]}.
\end{aligned}$$

By continuity we can choose $\rho > 0$ such that if $|z - Z(0, t)| < \rho$, then

$$\frac{(1 - \mu^2)}{2} |x'|^2 - |z - Z(0, t)| [1 + |z - Z(0, t)| + 2(1 + \mu) |x'|] \geq 0, \quad \forall |x'| \geq r_0.$$

We can shrink, if necessary, W_2 , such that $\sup_{t \in W_2} |Z(0, t)| < \rho$. So if we define $\mathcal{O}_2 \subset \mathbb{C}^m$ as

$$\mathcal{O}_2 \doteq \left\{ z \in \mathbb{C}^m : \sup_{t \in W_2} |z - Z(0, t)| < \rho \right\},$$

then for every $z \in \mathcal{O}_2$, $t \in W_2$, and $r_0 \leq |x'|$, we have that

$$\left| e^{i\xi \cdot (z - Z(x'', t)) - |\xi| \left[\langle z - Z(x', t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2 \right]} \right| \leq e^{-|\xi| \frac{(1-\mu^2)}{2} |x'|^2}.$$

Since $\text{supp } \chi$ and $\overline{W_2}$ are compact sets, there exist $k \in \mathbb{Z}_+$ and $C > 0$, such that

$$\begin{aligned} & \left| \left\langle u(x'', t), \chi(x'') \right| \xi \right|^{\frac{m}{2}} \Delta(Z(x', t) - Z(x'', t), \xi) \det Z_x(x'', t) \cdot \\ & \quad \cdot e^{i\xi \cdot (z - Z(x'', t)) - |\xi| \left[\langle z - Z(x', t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2 \right] - \varepsilon |\xi|^2} \right| \leq \\ & \leq C \sum_{|\alpha| \leq k} \sup_{x'' \in \text{supp } \chi} \left| \partial_{x''}^\alpha \left\{ \chi(x'') \right| \xi \right|^{\frac{m}{2}} \Delta(Z(x', t) - Z(x'', t), \xi) \det Z_x(x'', t) \cdot \right. \\ & \quad \cdot e^{i\xi \cdot (z - Z(x'', t)) - |\xi| \left[\langle z - Z(x', t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2 \right] - \varepsilon |\xi|^2} \left. \right| \\ & \leq C_1 |\xi|^{k + \frac{m}{2}} e^{-|\xi| \frac{(1-\mu^2)}{2} |x'|^2}, \end{aligned}$$

for every $z \in \mathcal{O}_2$, and $t \in W_2$, where the constant $C_1 > 0$ depends on $\text{supp } \chi$, and k . Therefore the integrand in (3.15) dominated by

$$C_1 |\xi|^{k + \frac{m}{2}} e^{-|\xi| \frac{(1-\mu^2)}{4} |x'|^2} e^{-|\xi| \frac{(1-\mu^2)}{4} r_0^2}. \quad (3.16)$$

Now since the integral of $e^{-|\xi| \frac{(1-\mu^2)}{4} |x'|^2}$, with respect to x' , is bounded by a constant times $|\xi|^{-\frac{m}{2}}$, we have that (3.16) is an integrable function with respect to (x', ξ) in $\mathbb{R}^m \times \mathbb{R}^m$. Therefore by Montel's Theorem, we have that there exists a sequence $\{\varepsilon_j\}_{j \in \mathbb{Z}_+}$, with $\varepsilon_j \rightarrow 0$, such that $G_3^{\varepsilon_j}(z, t) \rightarrow G_3(z, t)$ uniformly in $\mathcal{O}_2 \times W_2$, and $G_3(z, t)$ is holomorphic with respect to z , and it is given by

$$\begin{aligned} G_3(z, t) & \doteq \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \int_{r_0 \leq |x'|} \left\langle u(x'', t), \chi(x'') \right| \xi \right|^{\frac{m}{2}} \Delta(Z(x', t) - Z(x'', t), \xi) \cdot \\ & \quad \cdot e^{i\xi \cdot (z - Z(x'', t)) - |\xi| \left[\langle z - Z(x', t) \rangle^2 + \langle Z(x', t) - Z(x'', t) \rangle^2 \right] \det Z_x(x'', t)} dx' d\xi. \end{aligned}$$

So if we take $V_2 \subset V_1$ and $W_3 \subset W_2$ neighborhoods of the origin, such that

$$\{Z(x, t) : x \in V_2, t \in W_3\} \subset \mathcal{O}_2,$$

we have that $I_3^{\varepsilon_j}(x, t) \rightarrow G_3(Z(x, t), t)$, for every $(x, t) \in V_2 \times W_3$.

$$I_1^\varepsilon(x, t):$$

Let $(x, t) \in B_r(0) \times B_\delta(0)$ and $\alpha \in \mathbb{Z}_+^m$. Then

$$M^\alpha I_1^\varepsilon(x, t) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \iint_{Q_1^1} M^\alpha \left\{ e^{i\zeta \cdot (Z(x, t) - z') - \langle \zeta \rangle \langle Z(x, t) - z' \rangle^2} \right\} e^{-\varepsilon \langle \zeta \rangle^2} \mathfrak{F}[\chi u](t; z', \zeta) \langle \zeta \rangle^{\frac{m}{2}} d\zeta dz'.$$

Since the vectors fields $\{M_1, \dots, M_m\}$ are pairwise commuting and $M_j Z_k(x, t) = \delta_{j,k}$, we can use formula (3.13) to calculate $M^\alpha \left\{ e^{i\zeta \cdot (Z(x, t) - z') - \langle \zeta \rangle \langle Z(x, t) - z' \rangle^2} \right\}$, obtaining

$$\begin{aligned}
M^\alpha I_1^\varepsilon(x, t) &= \frac{1}{(2\pi^3)^{\frac{m}{2}}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \iint_{Q_t^1} M^{\alpha-\beta} e^{i\zeta \cdot (Z(x, t) - z')} M^\beta e^{-\langle \zeta \rangle \langle Z(x, t) - z' \rangle^2} \\
&\quad \cdot e^{-\varepsilon \langle \zeta \rangle^2} \mathfrak{F}[\chi u](t; z', \zeta) \langle \zeta \rangle^{\frac{m}{2}} d\zeta dz' \\
&= \frac{1}{(2\pi^3)^{\frac{m}{2}}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{l_1^1 + 2l_2^1 = \beta_1} \cdots \sum_{l_1^m + 2l_2^m = \beta_m} \frac{\beta!}{l_1^1! l_2^1! \cdots l_1^m! l_2^m!} \\
&\quad \cdot \iint_{Q_t^1} e^{i\zeta \cdot (Z(x, t) - z') - \langle \zeta \rangle \langle Z(x, t) - z' \rangle^2 - \varepsilon \langle \zeta \rangle^2} \mathfrak{F}[\chi u](t; z', \zeta) \langle \zeta \rangle^{\frac{m}{2}} \\
&\quad \cdot (-\langle \zeta \rangle)^{l_1^1 + l_2^1 + \cdots + l_1^m + l_2^m} (i\zeta)^{\alpha-\beta} (2(Z_1(x, t) - z'_1))^{l_1^1} \cdots (2(Z_m(x, t) - z'_m))^{l_1^m} d\zeta dz'.
\end{aligned}$$

Therefore by (3.12) there exists $\tilde{\varepsilon} > 0$ such that

$$\begin{aligned}
|M^\alpha I_1^\varepsilon(x, t)| &\leq \frac{1}{(2\pi^3)^{\frac{m}{2}}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{l_1^1 + 2l_2^1 = \beta_1} \cdots \sum_{l_1^m + 2l_2^m = \beta_m} \frac{\beta!}{l_1^1! l_2^1! \cdots l_1^m! l_2^m!} \iint_{Q_t^1} e^{-(1-\kappa)|\zeta| |Z(x, t) - z'|^2} \\
&\quad \cdot |\zeta|^{\alpha-\beta + l_1^1 + l_2^1 + \cdots + l_1^m + l_2^m + \frac{m}{2}} |\mathfrak{F}[\chi u](t; z', \zeta)| d\zeta dz' \\
&\leq C_1^{|\alpha|+1} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{l_1^1 + 2l_2^1 = \beta_1} \cdots \sum_{l_1^m + 2l_2^m = \beta_m} \frac{\beta!}{l_1^1! l_2^1! \cdots l_1^m! l_2^m!} \iint_{Q_t^1} e^{-\tilde{\varepsilon} |\zeta|^{\frac{1}{s}}} \\
&\quad \cdot |\zeta|^{\alpha-\beta + l_1^1 + l_2^1 + \cdots + l_1^m + l_2^m + \frac{m}{2}} d\zeta dz' \\
&\leq C_2^{|\alpha|+1} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{l_1^1 + 2l_2^1 = \beta_1} \cdots \sum_{l_1^m + 2l_2^m = \beta_m} \frac{\beta!}{l_1^1! l_2^1! \cdots l_1^m! l_2^m!} \int_0^\infty e^{-\tilde{\varepsilon} \rho^{\frac{1}{s}}} \\
&\quad \cdot \rho^{\alpha-\beta + l_1^1 + l_2^1 + \cdots + l_1^m + l_2^m + \frac{m}{2} + m-1} d\rho \\
&\leq C_3^{|\alpha|+1} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{l_1^1 + 2l_2^1 = \beta_1} \cdots \sum_{l_1^m + 2l_2^m = \beta_m} \frac{\beta!}{l_1^1! l_2^1! \cdots l_1^m! l_2^m!} \frac{\alpha!^s}{\beta!^s} (l_1^1 + l_2^1)^s \cdots (l_1^m + l_2^m)^s \\
&= C_3^{|\alpha|+1} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{l_1^1 + 2l_2^1 = \beta_1} \cdots \sum_{l_1^m + 2l_2^m = \beta_m} \frac{\beta!}{l_1^1! (2l_2^1)! \cdots l_1^m! (2l_2^m)!} \frac{\alpha!^s}{\beta!^s} (l_1^1 + l_2^1)^s \frac{(2l_2^1)!}{l_2^1!} \cdots \\
&\quad \cdots (l_1^m + l_2^m)^s \frac{(2l_2^m)!}{l_2^m!} \\
&\leq C_4^{|\alpha|+1} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{l_1^1 + 2l_2^1 = \beta_1} \cdots \sum_{l_1^m + 2l_2^m = \beta_m} \frac{\beta!}{l_1^1! (2l_2^1)! \cdots l_1^m! (2l_2^m)!} \frac{\alpha!^s}{\beta!^s} (l_1^1 + l_2^1)^s l_2^1! \cdots \\
&\quad \cdots (l_1^m + l_2^m)^s l_2^m! \\
&\leq C_5^{|\alpha|+1} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{l_1^1 + 2l_2^1 = \beta_1} \cdots \sum_{l_1^m + 2l_2^m = \beta_m} \frac{\beta!}{l_1^1! (2l_2^1)! \cdots l_1^m! (2l_2^m)!} \frac{\alpha!^s}{\beta!^s} (l_1^1 + 2l_2^1)^s \cdots \\
&\quad \cdots (l_1^m + 2l_2^m)^s \\
&= C_5^{|\alpha|+1} \alpha!^s \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{l_1^1 + 2l_2^1 = \beta_1} \cdots \sum_{l_1^m + 2l_2^m = \beta_m} \frac{\beta!}{l_1^1! (2l_2^1)! \cdots l_1^m! (2l_2^m)!} \\
&\leq C_5^{|\alpha|+1} \alpha!^s \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{l_1^1 + l_2^1 = \beta_1} \cdots \sum_{l_1^m + l_2^m = \beta_m} \frac{\beta!}{l_1^1! l_2^1! \cdots l_1^m! l_2^m!} \\
&\leq C_6^{|\alpha|+1} \alpha!^s,
\end{aligned}$$

where the constant C_6 does not depend on ε . The constant C_6 can be taken as $3C_5$, in view of Lemma 4.2. of [BM04].

□

Remark 3.0.13. Note that for the implications $1. \Rightarrow 2. \Rightarrow 3.$ we can take $\widetilde{W} = W_0$. Also by a closer inspection on the proof of $1. \Rightarrow 2.$ we can take V_1 as V , so that if $\chi \in \mathcal{C}_c^\infty(V)$ such that $\chi \equiv 1$ on V_2 an open ball centered at the origin, then the inequality (3.12) is valid for every open ball $\widetilde{V} \Subset V_2$, centered at the origin.

Chapter 4

Propagation of singularities

In 1983 N. Hanges and F. Treves ([HT83]) proved that hypo-analytic regularity propagates along elliptic submanifolds, and in their proof they actually showed that the decay of the FBI transform propagates. But since then all the propagation of singularities results, concerning systems of complex vector fields, were obtained in the setting of CR geometry, for instance holomorphic extendability of CR functions, propagation along CR orbits, sector extendability, (see [Tum97], [Tré90] and [BZ05]) and so on. We did not find in the literature any other result concerning propagation of Gevrey singularities in this set up.

We shall consider only analytic tube structures, *i.e.*, locally the hypo-analytic structure is given by $Z(x, t) = x + i\phi(t)$, defined on $U = V \times W$, and $\phi(t)$ is analytic. One of the reasons we are only dealing with tube structures is that the real structure bundle, $\mathbb{RT}'_{\mathfrak{M}_t}$, is trivial for every t , *i.e.*, it is equal to $Z(U) \times \mathbb{R}^m$. Now we will recall a simple comparison result for the FBI transform for solutions (see proposition IX.5.3., pg 436 of [Tre92]):

Proposition 4.0.1. *There are open balls $V_0 \Subset V_1 \Subset V$ in $\mathbb{R}^m$ and $W_0 \Subset W$ in $\mathbb{R}^n$, all centered at the origin, and constants $r, \kappa, R > 0$ such that, if $\chi \in \mathcal{C}_c^\infty(V_1)$ is equal to 1 in V_0 , then, to every solution u in $U = V \times W$, there is a constant $C > 0$ such that*

$$|\mathfrak{F}[\chi u](t; z, \zeta) - \mathfrak{F}[\chi u](t'; z, \zeta)| \leq Ce^{-|\zeta|/R},$$

in the region

$$t, t' \in W_0, z \in \mathbb{C}^m, |z| < r, \zeta \in \mathfrak{C}_\kappa.$$

This proposition can be used to show that hypo-analyticity propagates along connected fibers (recall that a fiber is locally a level set of the map $Z(x, t)$). Let us just indicate how it is done. Suppose that u is hypo-analytic at the origin and let $t_0 \in W_0$ be such that $Z(0, t_0) = 0$. To show that u is hypo-analytic at $(0, t_0)$ it is enough to show that $u|_{\mathcal{H}_{t_0}}$ is hypo-analytic at $(0, t_0)$, where $\mathcal{H}_{t_0} = \{(x, t_0) : x \in V_0\}$, but this is equivalent to

$$|\mathfrak{F}[\chi u](t_0; z, \zeta)| \leq Ce^{-\varepsilon|\zeta|},$$

for some $C, \varepsilon > 0$, and z in some open neighborhood of the origin and $\zeta \in \mathfrak{C}_\kappa$, for some $0 < \kappa < 1$. But since u is hypo-analytic at the origin, we have that

$$|\mathfrak{F}[\chi u](0; z, \zeta)| \leq Ce^{-\varepsilon|\zeta|},$$

for some $C, \varepsilon > 0$, and z in some open neighborhood of the origin and $\zeta \in \mathfrak{C}_\kappa$, for some $0 < \kappa < 1$, therefore we have the desired decay at t_0 in view of Proposition 4.0.1. One can follow the end of the proof of the Theorem 4.1.2 to globalize this argument to connected fibers. So why we can not use this same argument for Gevrey vectors? First, we do not have the property that ensures the desired regularity by only looking to restrictions on maximally real submanifolds. And the second reason is that for Gevrey regularity, to use a FBI transform argument, (z, ζ) must belong to the real structure

bundle $\mathbb{RT}'_{\mathfrak{M}_t}$, that depends on t . So to avoid this dependence we are restringing ourselves to tube structures. To deal with "restringing to maximally real submanifolds is not enough" problem we shall put some hypothesis on our "propagators", and it is in those additional hypotheses where is important for the structure to be analytic.

4.1 Propagation of Gevrey regularity for solutions of the non homogeneous system

In this section we will define the sets that will propagate the Gevrey regularity, the "propagators", and then exhibit the proof of the main theorem of this thesis. Let $\Sigma \subset \Omega$ be a connected subset of Ω , satisfying the following properties:

1. For every $p \in \Sigma$ there is (U, Z) , a hypo-analytic chart, with $p \in U$, such that $\Sigma \cap U \subset Z^{-1}(0)$;
2. In the same situation as above, for every $q \in \Sigma \cap U$, and $\tilde{U}_1 \Subset U$, an open neighborhood of p , there is $\tilde{U}_2 \Subset U$, an open neighborhood of q , such that the connected component of the fiber $Z^{-1}(Z(q'))$ that contains q' intersects $\tilde{U}_1$, for every $q' \in \tilde{U}_2$;
3. the map $\Sigma \ni p \mapsto \sup\{r > 0 : B_r(p) \subset U\}$ is continuous.

Condition 3. is not exactly a hypothesis, because we can always shrink the open set U for each p , therefore we can choose U to be a ball with radius varying continuously on p . Condition 2. implies that for every $q' \in \tilde{U}_2$ there is a curve $\gamma_{q'} : [0, 1] \rightarrow U$ satisfying

- $\gamma_{q'}(0) = q'$;
- $Z(\gamma_{q'}(\sigma)) = Z(q')$, for every $0 \leq \sigma \leq 1$;
- $\gamma_{q'}(1) \in \tilde{U}_1$;

Since the structure is analytic, the level sets of $Z(x, t)$ are subanalytic sets, therefore the curves $\{\gamma_{q'}\}_{q' \in \tilde{U}_2}$ have bounded length, see for instance section 8 of [Har82] or pg. 39 of [Tre83] (in the appendix wrote by B. Teissier). Let $p \in \Sigma$, and let (U, Z) be the hypo-analytic chart described above. Take local coordinates in $(U, x_1, \dots, x_m, t_1, \dots, t_n)$, such that in this coordinates $p = 0$, $U = V \times W$, and the real structure bundle on V , $\mathbb{RT}'_{\mathfrak{M}_t}|_V$, is well positioned for every $t \in W$, i.e., there exists $c_0 > 0$ such that

$$\text{Im}\{\xi \cdot (Z(x, t) - Z(y, t)) + i|\xi| \langle Z(x, t) - Z(y, t) \rangle^2\} \geq c_0 |\xi| |Z(x, t) - Z(y, t)|^2,$$

for every $x, y \in V$, $t \in W$, and $\xi \in \mathbb{R}^m$.

Lemma 4.1.1. *Let $\rho > 0$ be such that $B_\rho(0) \Subset V$, and let $f \in C^\infty(W; \mathcal{E}'(K \setminus B_\rho(0)))$, where $B_\rho(0) \subset K \Subset V$, is a compact set. Then*

$$|\mathfrak{F}[f](t; Z(x, t), \xi)| \leq C e^{-\varepsilon|\xi|}, \quad \forall x \in B_{\rho/2}(0), t \in W, \xi \in \mathbb{R}^m.$$

Proof. Let $x \in B_{\rho/2}(0)$ and $y \in K \setminus B_\rho(0)$, then

$$\begin{aligned} \text{Im}\{\xi \cdot (Z(x, t) - Z(y, t)) + i|\xi| \langle Z(x, t) - Z(y, t) \rangle^2\} &\geq c_0 |\xi| |Z(x, t) - Z(y, t)|^2 \\ &\geq c_0 |\xi| |x - y|^2 \end{aligned}$$

$$\geq c_0 |\xi| \frac{\rho^2}{4},$$

for every $t \in W$ and $\xi \in \mathbb{R}^m$. Therefore

$$\begin{aligned} |\mathfrak{F}[f](t; Z(x, t), \xi)| &= \left| \left\langle f(y, t), e^{i\xi \cdot (Z(x, t) - Z(y, t)) - |\xi| \langle Z(x, t) - Z(y, t) \rangle^2} \Delta(Z(x, t) - Z(y, t), \xi) \cdot \right. \right. \\ &\quad \left. \left. \cdot \det Z_x(y, t) \right\rangle \right| \\ &\leq C \sum_{|\alpha| \leq \lambda} \sup_{y \in K \setminus B_{\rho/2}(0)} \left| \partial_y^\alpha \left\{ e^{i\xi \cdot (Z(x, t) - Z(y, t)) - |\xi| \langle Z(x, t) - Z(y, t) \rangle^2} \cdot \right. \right. \\ &\quad \left. \left. \cdot \Delta(Z(x, t) - Z(y, t), \xi) \det Z_x(y, t) \right\} \right| \\ &\leq C |\xi|^\lambda e^{-c_0 |\xi| \frac{\rho^2}{4}} \\ &\leq C e^{-\frac{c_0 \rho^2}{8} |\xi|}, \end{aligned}$$

for every $x \in B_{\rho/2}(0)$, $t \in W$, and $\xi \in \mathbb{R}^m$. □

Theorem 4.1.2. *Let $\Omega \subset \mathbb{R}^{n+m}$ be an open set endowed with an analytic hypo-analytic structure of tube type. Let $\Sigma \subset \Omega$ be a connected submanifold as described above. If $u \in \mathcal{D}'(\Omega)$ is such that $\mathbb{L}u \in \mathcal{G}^s(\Omega)$, then $\text{singsupp}_s u \cap \Sigma = \emptyset$ or $\Sigma \subset \text{singsupp}_s u$.*

Proof. Let $p \in \Sigma$, and suppose that $p \notin \text{singsupp}_s u$. Let (U, Z) be the hypo-analytic chart described before. Consider in U the local coordinates $(x_1, \dots, x_m, t_1, \dots, t_n)$, and the vector fields $\{M_1, \dots, M_m, L_1, \dots, L_n\}$, as in the previous chapter. In this coordinates system $p = 0$, and we write $U = V \times W$, where $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ are both open neighborhoods of the origin. We also have that

$$Z_k(x, t) = x_k + i\phi_k(t), \quad k = 1, \dots, m,$$

and

$$\begin{aligned} L_j Z_k &= 0 & M_l Z_k &= \delta_{l,k} \\ L_j t_i &= \delta_{j,i} & M_l t_i &= 0. \end{aligned}$$

We can also assume that the real structure bundle $\mathbb{R}T'_{\mathfrak{M}_t}$ is well-positioned, for every $t \in W$. Now let $\rho > 0$ be such that $B_\rho(0) \subset V$, and $\chi \in \mathcal{C}_c^\infty(V)$ be such that $\chi \equiv 1$ on $B_\rho(0)$. Since $\mathbb{L}u \in \mathcal{G}^s(\Omega)$, we have that $L_j u \in \mathcal{G}^s(U; L_1, \dots, L_n, M_1, \dots, M_m)$, then $u \in \mathcal{C}^\infty(W; \mathcal{D}'(V))$. By Theorem 3.0.11 and Remark 3.0.13 we have that

$$|\mathfrak{F}[\chi L_j u](t; Z(x, t), \xi)| \leq C e^{-\varepsilon_1 |\xi|^\frac{1}{s}}, \quad \forall x \in B_{\rho'}(0), \forall t \in W, \forall \xi \in \mathbb{R}^m, j = 1, \dots, n, \quad (4.1)$$

for some $C, \varepsilon_1 > 0$, where $\rho/2 \leq \rho' < \rho$. We are assuming that $u|_{U_0} \in \mathcal{G}^s(U_0; L_1, \dots, L_n, M_1, \dots, M_m)$, for some open neighborhood of the origin, $U_0 = V_0 \times W_0$. Then by Theorem 3.0.11 and Remark 3.0.13 there exist $V_1 \Subset V$, an open neighborhood of the origin, and positive constants C, ε_2 , such that

$$|\mathfrak{F}[\chi u](t; Z(x, t), \xi)| \leq C e^{-\varepsilon_2 |\xi|^\frac{1}{s}}, \quad \forall (x, t) \in V_1 \times W_0, \forall \xi \in \mathbb{R}^m. \quad (4.2)$$

By condition 2., for every $(x_0, t_0) \in \Sigma \cap (B_{\rho/2}(0) \times W)$ there exists $\tilde{V} \times \tilde{W} \subset B_{\rho/2} \times W$, an open neighborhood of (x_0, t_0) , such that, for every $(x', t') \in \tilde{V} \times \tilde{W}$ there is a curve $\gamma_{(x', t')} : [0, 1] \rightarrow U$, satisfying:

- $\gamma_{(x',t')}(0) = (x', t')$;
- $Z(\gamma_{(x',t')}(\sigma)) = Z(x', t')$, for every $0 \leq \sigma \leq 1$;
- $\gamma_{(x',t')}(1) \in V_1 \times W_0$;
- There exists $C_1 > 0$ such that

$$\int_0^1 \|\gamma'_{(x',t')}(\sigma)\| d\sigma \leq C_1,$$

for every $(x', t') \in \widetilde{V} \times \widetilde{W}$.

We write $\gamma_{(x',t')}(\sigma) = (\gamma_{(x',t')}^{(1)}(\sigma), \gamma_{(x',t')}^{(2)}(\sigma))$. By Stokes theorem we have that

$$\begin{aligned} \mathfrak{F}[\chi u](t'; Z(x', t'), \xi) - \mathfrak{F}[\chi u](\gamma_{(x',t')}^{(2)}(1), Z(\gamma_{(x',t')}(1)), \xi) &= \\ &= \int_0^1 \frac{\partial}{\partial \sigma} \mathfrak{F}[\chi u](\gamma_{(x',t')}^{(2)}(\sigma); Z(x', t'), \xi) d\sigma \\ &= \int_0^1 \sum_{j=1}^n \mathfrak{F}[L_j(\chi u)](\gamma_{(x',t')}^{(2)}(\sigma); Z(x', t'), \xi) \frac{d}{d\sigma} \gamma_{(x',t')_j}^{(2)}(\sigma) d\sigma \\ &= \int_0^1 \sum_{j=1}^n \mathfrak{F}[u L_j \chi](\gamma_{(x',t')}^{(2)}(\sigma); Z(x', t'), \xi) \frac{d}{d\sigma} \gamma_{(x',t')_j}^{(2)}(\sigma) d\sigma + \\ &+ \int_0^1 \sum_{j=1}^n \mathfrak{F}[\chi L_j u](\gamma_{(x',t')}^{(2)}(\sigma); Z(x', t'), \xi) \frac{d}{d\sigma} \gamma_{(x',t')_j}^{(2)}(\sigma) d\sigma. \end{aligned}$$

Now we analyse these two terms separately. First we note that $L_j \chi$ vanishes on $B_\rho(0)$, for $j = 1, \dots, n$ therefore, by the previous lemma, we have that there exist $C, \varepsilon_3 > 0$ such that

$$|\mathfrak{F}[u L_j \chi](t; Z(x, t), \xi)| \leq C e^{-\varepsilon_3 |\xi|}, \quad \forall x \in B_{\rho/2}(0), t \in W, \xi \in \mathbb{R}^m, j = 1, \dots, n.$$

Therefore

$$|\mathfrak{F}[u L_j \chi](\gamma_{(x',t')}^{(2)}(\sigma); Z(x', t'), \xi)| \leq C e^{-\varepsilon_3 |\xi|}, \quad 0 \leq \sigma \leq 1, \forall \xi \in \mathbb{R}^m, j = 1, \dots, n.$$

In view of (4.1) we also have that

$$|\mathfrak{F}[\chi L_j u](\gamma_{(x',t')}^{(2)}(\sigma); Z(x', t'), \xi)| \leq C e^{-\varepsilon_1 |\xi|^{\frac{1}{s}}}, \quad 0 \leq \sigma \leq 1, \forall \xi \in \mathbb{R}^m, j = 1, \dots, n.$$

Summing up we have obtained

$$\begin{aligned} |\mathfrak{F}[\chi u](t'; Z(x', t'), \xi)| &\leq |\mathfrak{F}[\chi u](\gamma_{(x',t')}^{(2)}(1), Z(\gamma_{(x',t')}(1)), \xi)| + C e^{-\varepsilon_4 |\xi|^{\frac{1}{s}}} \\ &\leq C e^{-\varepsilon |\xi|^{\frac{1}{s}}} \end{aligned}$$

for every $(x', t') \in \widetilde{V} \times \widetilde{W}$, since $\gamma_{(x',t')}(1) \in V_1 \times W_0$, where $\varepsilon_4 = \min\{\varepsilon_1, \varepsilon_3\}$ and $\varepsilon = \min\{\varepsilon_2, \varepsilon_4\}$. So we conclude that for $p \in \Sigma$ there exists a neighborhood $\mathcal{U} \Subset U$, such that if $p \notin \text{singsupp}_s u$, then $\Sigma \cap \mathcal{U} \subset \mathbb{C} \text{singsupp}_s u$. Moreover, since the map $\Sigma \ni p \mapsto \sup\{r > 0 : B_r(p) \subset U\}$ is continuous, the same can be assumed for the map $\Sigma \ni p \mapsto \sup\{r > 0 : B_r(p) \subset \mathcal{U}\}$. To indicate

the dependence of p in $\mathcal{U}$, we shall write $\mathcal{U} = \mathcal{U}(p)$. Now we claim that $\Sigma \cap \text{singsupp}_s u$ is an open set. So take $\{p_k\}_{k \in \mathbb{Z}_+}$ a sequence on $\Sigma \cap \mathbb{C} \text{singsupp}_s u$, such that $p_k \rightarrow p \in \Sigma$. Now there exists $\delta > 0$ such that the open set $\mathcal{U}(p_k)$, as described above, contains a ball, centered at p_k , of radius at least δ , for every k . So there exists $k > 0$ such that $p \in \mathcal{U}(p_k)$. Since $p_k \notin \text{singsupp}_s u$ we have that $\Sigma \cap \mathcal{U}(p_k) \subset \mathbb{C} \text{singsupp}_s u$, *i.e.*, $p \notin \text{singsupp}_s u$. Clearly $\Sigma \cap \text{singsupp}_s u$ is closed. Therefore $\Sigma \cap \text{singsupp}_s u = \emptyset$ or $\Sigma \subset \text{singsupp}_s u$. $\square$

4.2 Examples

Consider in $\mathbb{R}^2$ a real valued, real-analytic function ϕ satisfying $\phi(0) = 0$, and consider in $\mathbb{R}^3$ the structure $\mathcal{V}$ defined by the complex vector fields

$$\begin{aligned} L_1 &= \frac{\partial}{\partial t_1} - i2\phi(t_1, t_2) \frac{\partial \phi}{\partial t_1}(t_1, t_2) \frac{\partial}{\partial x}, \\ L_2 &= \frac{\partial}{\partial t_2} - i2\phi(t_1, t_2) \frac{\partial \phi}{\partial t_2}(t_1, t_2) \frac{\partial}{\partial x}. \end{aligned}$$

The first integral for theses complex vector fields is given by

$$Z(x, t_1, t_2) = x + i\phi(t_1, t_2)^2,$$

and the characteristic set T^0 is equal to

$$T^0 = \{(x, t_1, t_2, \xi, \eta_1, \eta_2) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus 0) : \phi(t_1, t_2) = 0 \text{ or } \nabla \phi(t_1, t_2) = 0\}.$$

Now suppose that the collection $\Sigma_\alpha = \phi^{-1}(\alpha)$ forms a foliation of a neighborhood of Σ_0 on $\mathbb{R}^2$ by connected, smooth curves. So we can apply our Theorem 4.1.2 for this structure $\mathcal{V}$ and for $\{x_0\} \times \Sigma_0$, for every $x_0 \in \mathbb{R}$. Note that for this example it is important that $\{x_0\} \times \Sigma_0$ is contained in the base projection of T^0 , otherwise the structure would be elliptic on $\{x_0\} \times \Sigma_0$, and we would not need to use our theorem in this case. Now we give some simple examples of such functions ϕ :

1. $\phi(t_1, t_2) = (t_1 - 1)^2 + (t_2 - 1)^2 - 2$;
2. $\phi(t_1, t_2) = t_1 - t_2$;
3. $\phi(t_1, t_2) = (t_1 + 1)(t_2 + 1) - 1$.

Appendix A

Well positionedness

In this section we shall prove that if we shrink $U = V \times W$ around the origin then we can assume that the real structure bundle $\mathbb{R}T'_{\mathfrak{W}_t}$ is well positioned for all $t \in W$. We are assuming that the function $Z(x, t)$ is given by

$$Z(x, t) = x + i\phi(x, t),$$

and the function $\phi(x, t)$ is real and

$$|\phi(x, t)| \leq C_1(|x|^3 + |t|),$$

and

$$|\phi(x', t) - \phi(x, t)| \leq \mu|x' - x|,$$

where $0 < \mu < 1$. From now on we shall write $V = B_r(0)$ and $W = B_\delta(0)$. So let $t \in W$, $x, x' \in V$ and $\xi \in \mathbb{R}^m$. We shall estimate

$$\text{Im} \left\{ {}^t Z_x(x, t)^{-1} \xi \cdot (Z(x', t) - Z(x, t)) + i \langle {}^t Z_x(x, t)^{-1} \xi \rangle \langle Z(x', t) - Z(x, t) \rangle^2 \right\}. \quad (\text{A.1})$$

Before we estimate the quantity above let us derive some estimates for the real and imaginary part of $\langle \zeta \rangle^2$ and $\langle \zeta \rangle$ when $\zeta \in \mathfrak{C}_\kappa$. We start writing

$$\zeta \cdot \zeta = |\text{Re}\zeta|^2 - |\text{Im}\zeta|^2 + 2i\text{Re}\zeta \cdot \text{Im}\zeta,$$

so

$$\begin{aligned} |\langle \zeta \rangle^2|^2 &= |\zeta \cdot \zeta|^2 \\ &= \left| |\text{Re}\zeta|^2 - |\text{Im}\zeta|^2 + 2i\text{Re}\zeta \cdot \text{Im}\zeta \right|^2 \\ &= |\text{Re}\zeta|^4 - 2|\text{Re}\zeta|^2|\text{Im}\zeta|^2 + |\text{Im}\zeta|^4 + 4|\text{Re}\zeta \cdot \text{Im}\zeta|^2 \\ &\leq |\text{Re}\zeta|^4 - 2|\text{Re}\zeta|^2|\text{Im}\zeta|^2 + |\text{Im}\zeta|^4 + 4|\text{Re}\zeta|^2|\text{Im}\zeta|^2 \\ &= \left[|\text{Re}\zeta|^2 + |\text{Im}\zeta|^2 \right]^2, \end{aligned}$$

therefore $|\langle \zeta \rangle^2| \leq |\zeta|^2$. We also have that

$$\begin{aligned} \text{Re}\langle \zeta \rangle^2 &= |\text{Re}\zeta|^2 - |\text{Im}\zeta|^2 \\ &\geq (1 - \kappa^2)|\text{Re}\zeta|^2 \end{aligned}$$

$$\geq \frac{(1 - \kappa^2)}{(1 + \kappa^2)} |\zeta|^2,$$

since $|\zeta|^2 = |\operatorname{Re}\zeta|^2 + |\operatorname{Im}\zeta|^2 < (1 + \kappa^2)|\operatorname{Re}\zeta|^2$. Now if a complex number ω is such that $\operatorname{Re}\omega > 0$, using polar coordinates we can write $\omega = |\omega|e^{i\theta}$, with $-\pi/2 < \theta < \pi/2$, so $\sqrt{\omega} = \sqrt{|\omega|}e^{i\theta/2}$ and then

$$\operatorname{Re}\sqrt{\omega} = \sqrt{|\omega|} \cos(\theta/2), \quad \operatorname{Im}\sqrt{\omega} = \sqrt{|\omega|} \sin(\theta/2).$$

We also have that

$$\sqrt{\operatorname{Re}\omega} = \sqrt{|\omega|} \sqrt{\cos\theta}, \quad \sqrt{|\operatorname{Im}\omega|} = \sqrt{|\omega|} \sqrt{|\sin\theta|},$$

and that $\cos(\theta/2) \geq \sqrt{\cos(\theta/2)^2 - \sin(\theta/2)^2} \geq \sqrt{\cos\theta}$. So

$$\operatorname{Re}\sqrt{\omega} \geq \sqrt{\operatorname{Re}\omega}, \quad \text{and} \quad \operatorname{Im}\sqrt{\omega} \leq \sqrt{|\omega|}.$$

Now returning to ζ , we have that

$$\operatorname{Re}\langle\zeta\rangle \geq \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} |\zeta| \quad \text{and} \quad \operatorname{Im}\langle\zeta\rangle \leq |\zeta| \quad (\text{A.2})$$

Now let us begin with the first part of (A.1). We can write

$$Z(x', t) - Z(x, t) = Z_x(x, t)(x' - x) + R(x, x', t),$$

where $|R(x, x', t)| \leq C_2(|x| + |x'| + |t|)|x - x'|^2$. Therefore

$$\begin{aligned} \operatorname{Im}\{ {}^t Z_x(x, t)^{-1} \xi \cdot (Z(x', t) - Z(x, t)) \} &= \operatorname{Im}\{ \xi \cdot Z_x(x, t)^{-1} (Z(x', t) - Z(x, t)) \} \\ &= \operatorname{Im}\{ \xi \cdot (x' - x + Z_x(x, t)^{-1} R(x, x', t)) \} \\ &= \operatorname{Im}\{ \xi \cdot Z_x(x, t)^{-1} R(x, x', t) \} \\ &\geq -|\xi| \|Z_x(x, t)^{-1}\| |R(x, x', t)| \\ &\geq -C_3 |\xi| |x - x'|^2, \end{aligned}$$

where $C_3 > 0$ can be made as small as want (depending on r and δ). Now to analyze the second part we must be very careful. First we break the imaginary part of the product, *i.e.*,

$$\begin{aligned} \operatorname{Im}\{ i \langle {}^t Z_x(x, t)^{-1} \xi \rangle \langle Z(x', t) - Z(x, t) \rangle^2 \} &= \operatorname{Re}\{ \langle {}^t Z_x(x, t)^{-1} \xi \rangle \langle Z(x', t) - Z(x, t) \rangle^2 \} \\ &= \operatorname{Re}\langle {}^t Z_x(x, t)^{-1} \xi \rangle \operatorname{Re}\langle Z(x', t) - Z(x, t) \rangle^2 - \\ &\quad - \operatorname{Im}\langle {}^t Z_x(x, t)^{-1} \xi \rangle \operatorname{Im}\langle Z(x', t) - Z(x, t) \rangle^2. \end{aligned}$$

Before we proceed we note that

$$Z_x(x, t)^{-1} = (\operatorname{Id} - i\phi_x(x, t))(\operatorname{Id} + \phi_x(x, t)^2)^{-1},$$

therefore

$$\operatorname{Re} {}^t Z_x(x, t)^{-1} \xi = (\operatorname{Id} + {}^t \phi_x(x, t)^2)^{-1} \xi, \quad \text{and} \quad \operatorname{Im} {}^t Z_x(x, t)^{-1} \xi = -{}^t \phi_x(x, t) (\operatorname{Id} + {}^t \phi_x(x, t)^2)^{-1} \xi.$$

So we have that

$$|\operatorname{Im} {}^t Z_x(x, t)^{-1} \xi| = |{}^t \phi_x(x, t) (\operatorname{Id} + {}^t \phi_x(x, t)^2)^{-1} \xi|$$

$$\begin{aligned}
&\leq \|{}^t\phi_x(x, t)\| \left| (\text{Id} + {}^t\phi_x(x, t)^2)^{-1} \xi \right| \\
&\leq C_4(|x|^2 + |t|) \left| (\text{Id} + {}^t\phi_x(x, t)^2)^{-1} \xi \right| \\
&= C_4(|x|^2 + |t|) \left| \text{Re} {}^tZ_x(x, t)^{-1} \xi \right|,
\end{aligned}$$

and if r and δ are small enough, we have that $|\text{Im} {}^tZ_x(x, t)^{-1} \xi| < \kappa |\text{Re} {}^tZ_x(x, t)^{-1} \xi|$, for every $|x| < r$, $|t| < \delta$ and $\xi \in \mathbb{R}^m$. So

$$\text{Re} \langle {}^tZ_x(x, t)^{-1} \xi \rangle \geq \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} |{}^tZ_x(x, t)^{-1} \xi|, \quad \text{and} \quad \text{Im} \langle {}^tZ_x(x, t)^{-1} \xi \rangle \leq |{}^tZ_x(x, t)^{-1} \xi|,$$

and we also have that

$$\begin{aligned}
\text{Re} \langle Z(x', t) - Z(x, t) \rangle^2 &= |x' - x|^2 - |\phi(x', t) - \phi(x, t)|^2 \\
&\geq (1 - \mu^2) |x' - x|^2,
\end{aligned}$$

and

$$\begin{aligned}
\text{Im} \langle Z(x', t) - Z(x, t) \rangle^2 &= 2(x' - x) \cdot (\phi(x', t) - \phi(x, t)) \\
&\leq 2|x' - x| |\phi(x', t) - \phi(x, t)| \\
&\leq 2\mu |x' - x|^2.
\end{aligned}$$

Summing up we have that

$$\begin{aligned}
\text{Im} \{ i \langle {}^tZ_x(x, t)^{-1} \xi \rangle \langle Z(x', t) - Z(x, t) \rangle^2 \} &= \text{Re} \langle {}^tZ_x(x, t)^{-1} \xi \rangle \text{Re} \langle Z(x', t) - Z(x, t) \rangle^2 - \\
&\quad - \text{Im} \langle {}^tZ_x(x, t)^{-1} \xi \rangle \text{Im} \langle Z(x', t) - Z(x, t) \rangle^2 \\
&\geq (1 - \mu^2) \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} |{}^tZ_x(x, t)^{-1} \xi| |x' - x|^2 \\
&\quad - 2\mu |{}^tZ_x(x, t)^{-1} \xi| |x' - x|^2,
\end{aligned}$$

and then

$$\begin{aligned}
\text{Im} \{ {}^tZ_x(x, t)^{-1} \xi \cdot (Z(x', t) - Z(x, t)) + i \langle {}^tZ_x(x, t)^{-1} \xi \rangle \langle Z(x', t) - Z(x, t) \rangle^2 \} &\geq \\
&\geq (1 - \mu^2) \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} |{}^tZ_x(x, t)^{-1} \xi| |x' - x|^2 \\
&\quad - 2\mu |{}^tZ_x(x, t)^{-1} \xi| |x' - x|^2 - C_3 |\xi| |x - x'|^2.
\end{aligned}$$

Now writing $\zeta = {}^tZ_x(x, t)^{-1} \xi$ we have that

$$|{}^tZ_x(x, t)^{-1} \xi| \leq |\xi| \inf_{|x| < r, |t| < \delta, |\eta|=1} |{}^tZ_x(x, t)^{-1} \eta|,$$

and therefore

$$\begin{aligned}
\text{Im} \{ {}^tZ_x(x, t)^{-1} \xi \cdot (Z(x', t) - Z(x, t)) + i \langle {}^tZ_x(x, t)^{-1} \xi \rangle \langle Z(x', t) - Z(x, t) \rangle^2 \} &\geq \\
&\geq |\zeta| |x' - x|^2 \left\{ (1 - \mu^2) \sqrt{\frac{1 - \kappa^2}{1 + \kappa^2}} - 2\mu - C_4 \right\}.
\end{aligned}$$

Since κ , μ and C_4 depend on r and δ in such a way that they all goes to 0 when $(r, \delta) \rightarrow 0$, we can choose r and δ small enough so

$$\operatorname{Im}\{ {}^t Z_x(x, t)^{-1} \xi \cdot (Z(x', t) - Z(x, t)) + i \langle {}^t Z_x(x, t)^{-1} \xi \rangle \langle Z(x', t) - Z(x, t) \rangle^2 \} \geq c|\zeta| |x' - x|^2,$$

and since

$$|x' - x|^2 \geq \frac{1}{1 + \mu^2} |Z(x', t) - Z(x, t)|^2,$$

we have obtained the desired estimate

$$\operatorname{Im}\{ {}^t Z_x(x, t)^{-1} \xi \cdot (Z(x', t) - Z(x, t)) + i \langle {}^t Z_x(x, t)^{-1} \xi \rangle \langle Z(x', t) - Z(x, t) \rangle^2 \} \geq \tilde{c}|\zeta| |Z(x', t) - Z(x, t)|^2.$$

Just to summarize the properties on the function $Z(x, t) = x + i\phi(x, t)$ that were needed in order to prove the well positionedness for the real structure bundle:

- $|\phi(x, t) - \phi(x', t)| \leq \mu |x' - x|$;
- $\|\phi_x(x, t)\| \leq \tilde{\mu}$,

for all $x \in V$ and $t \in W$ where $0 < \mu, \tilde{\mu}$ can be chosen as small as we want, shrinking if necessary V and W around the origin.

Appendix B

The continuity estimate for distributions in $H_{\text{loc}}^{p,\infty}$

Let $U \subset \mathbb{R}^{n+m}$ be an open neighborhood of the origin of the following form $U = V \times W$, where V and W are open neighborhoods of the origin on $\mathbb{R}^m$ and $\mathbb{R}^n$. Let p and q be integers, and u be a distribution on U . We say that $u \in H_{\text{loc}}^{p,q}(U)$ if

$$\|\phi u\|_{H^{p,q}}^2 = \iint_{\mathbb{R}^{m+n}} (1 + |\xi|^2)^p (1 + |\eta|^2)^q |\widehat{(\phi u)}(\xi, \eta)|^2 d\xi d\eta < \infty,$$

for every $\phi \in \mathcal{C}_c^\infty(U)$. Now suppose that $u \in H_{\text{loc}}^{p,\infty}(U) = \cap_{q \geq 0} H_{\text{loc}}^{p,q}(U)$. Let $\chi \in \mathcal{C}_c^\infty(V)$ be a test-function and define $v \in \mathcal{D}'(W)$ as

$$\langle v, \phi \rangle \doteq \langle u, \chi \otimes \phi \rangle,$$

where $\phi \in \mathcal{C}_c^\infty(W)$. Now we claim that $v \in \mathcal{C}^\infty(W)$. So let $\phi \in \mathcal{C}_c^\infty(W)$ be fixed, we shall prove that $\phi v \in \cap_{k \geq 0} H^k(W)$. First note that

$$\begin{aligned} \widehat{\phi v}(\eta) &= \langle v(t), \phi(t) e^{-it \cdot \eta} \rangle \\ &= \langle u(x, t), \chi(x) \phi(t) e^{-it \cdot \eta} \rangle \\ &= \langle u(x, t), \chi(x) \phi(t) e^{-ix \cdot 0 - it \cdot \eta} \rangle \\ &= \widehat{\chi \otimes \phi u}(0, \eta), \end{aligned}$$

and since

$$\iint_{\mathbb{R}^{m+n}} (1 + |\xi|^2)^p (1 + |\eta|^2)^k |\widehat{(\chi \otimes \phi u)}(\xi, \eta)|^2 d\xi d\eta < \infty,$$

for every $k \geq 0$, we have that

$$\int_{\mathbb{R}^n} (1 + |\eta|^2)^k |\widehat{\phi v}(\eta)|^2 d\eta = \int_{\mathbb{R}^n} (1 + |\eta|^2)^k |\widehat{\chi \otimes \phi u}(0, \eta)|^2 d\eta < \infty,$$

i.e., $\phi v \in H^k$, for every $k \geq 0$. Since $\phi \in \mathcal{C}_c^\infty(W)$ is arbitrary we have that $v \in \mathcal{C}^\infty(W)$. So we conclude that for every $\chi \in \mathcal{C}_c^\infty(V)$ the map

$$W \ni t \mapsto \langle u(x, t), \chi(x) \rangle,$$

is $\mathcal{C}^\infty$. Now let $K \subset V$ be a compact set. Consider $u_K(x, t)$ the restriction of u to $\mathcal{C}_c^\infty(K)$, for every $t \in W$. Let $W_1 \subset W$ be an open set with compact closure on W . So for every $\chi \in \mathcal{C}_c^\infty(K)$ the set

$$\{\langle u(x, t), \chi(x) \rangle : t \in W_1\} \subset \mathbb{C}$$

is bounded. Since $\mathcal{C}_c^\infty(K)$ is a Fréchet space, we can apply the uniform boundedness principle and obtain that the set

$$\{u_K(\bullet, t) \in L(\mathcal{C}_c^\infty(K), \mathbb{C}) : t \in W_1\}$$

is equicontinuous, therefore there exist a constant $C > 0$ and a positive integer q such that

$$|\langle u_K(x, t), \chi(x) \rangle| \leq C \sum_{|\alpha| \leq q} \sup_{x \in K} |\partial_x^\alpha \chi(x)|, \quad \forall \chi \in \mathcal{C}_c^\infty(K).$$

Since the compact set $K \subset V$ and the open set W_1 are arbitrary, we have that for every compact sets $K \subset V$ and $\overline{W_1} \subset W$ there exists $C > 0$ and a positive integer q such that

$$|\langle u(x, t), \chi(x) \rangle| \leq C \sum_{|\alpha| \leq q} \sup_{x \in K} |\partial_x^\alpha \chi(x)|, \quad \forall \chi \in \mathcal{C}_c^\infty(K), \text{ and } t \in W_1. \quad (\text{B.1})$$

Appendix C

Extending the function Z

In this section we shall prove that we can extend the function $Z(x, t)$ with respect to the variable x to the whole $\mathbb{R}^m$, while maintaining the map $Z_x(x, t)$ invertible for all $x \in \mathbb{R}^m$. We start recalling the construction of an important cut-off function, and then obtaining an estimate for its derivative. So we define

$$g(x) = \begin{cases} e^{-\frac{1}{(x-1)(x+1)}}, & -1 < x < 1; \\ 0, & x \leq -1 \text{ or } 1 \leq x, \end{cases}$$

and $G(x) = g(x) / \int g$. Note that $G \in \mathcal{C}_c^\infty(\mathbb{R})$. For $b > 0$, we denote by $\mathbb{E}_b$ the characteristic function of the set $[-b, b]$, and, as usual, we write

$$G_\varepsilon(x) = \frac{1}{\varepsilon} G\left(\frac{x}{\varepsilon}\right),$$

for every $0 < \varepsilon < b$. So we have that $\text{supp } G_\varepsilon = [-\varepsilon, \varepsilon]$, therefore $\text{supp } G_\varepsilon * \mathbb{E}_b \subset [-b - \varepsilon, b + \varepsilon]$. We also have that $G_\varepsilon * \mathbb{E}_b(x) = 1$, for $-b + \varepsilon < x < b - \varepsilon$. Now to estimate the derivative $\partial_x(G_\varepsilon * \mathbb{E}_b)$ we start estimating $G'_\varepsilon(x)$. Let $-\varepsilon \leq x \leq \varepsilon$, then

$$\begin{aligned} G'_\varepsilon(x) &= \frac{d}{dx} \left(\frac{e^{-\frac{\varepsilon^2}{(x-\varepsilon)(x+\varepsilon)}}}{\int g} \right) \\ &= \left(\int g \right)^{-1} e^{-\frac{\varepsilon^2}{(x-\varepsilon)(x+\varepsilon)}} \frac{\varepsilon^2 2x}{(x-\varepsilon)^2(x+\varepsilon)^2}. \end{aligned}$$

Therefore we can bound $\|G'_\varepsilon\|_{L^2}$ by

$$\begin{aligned} \|G'_\varepsilon\|_{L^2}^2 &= \left(\int g \right)^{-2} 4\varepsilon^4 \int_{-\varepsilon}^{\varepsilon} e^{-\frac{2\varepsilon^2}{(x-\varepsilon)(x+\varepsilon)}} \frac{x^2}{(x-\varepsilon)^4(x+\varepsilon)^4} dx \\ &= \left(\int g \right)^{-2} 4\varepsilon^4 \int_{-1}^1 e^{-\frac{2}{(y-1)(y+1)}} \frac{\varepsilon^2 y^2}{\varepsilon^8 (y-1)^4 (y+1)^4} \varepsilon dy \\ &= \left(\int g \right)^{-2} \frac{4}{\varepsilon} \int_{-1}^1 e^{-\frac{2}{(y-1)(y+1)}} \frac{y^2}{(y-1)^4 (y+1)^4} dy \\ &\leq \frac{C}{\varepsilon}. \end{aligned}$$

Now applying Young's inequality one obtains

$$\|(G_\varepsilon * \mathbb{E}_b)'\|_{L^\infty} \leq \sqrt{\frac{2bC}{\varepsilon}}.$$

Now let $0 < r_1 < r_2$. Take $\varepsilon = (r_2 - r_1)/2$, and write $b = r_1 + \varepsilon$. Set $\chi(x) = G_\varepsilon * \mathbb{E}_b(x)$. Then $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfies the following:

$$\begin{cases} 0 \leq \chi \leq 1; \\ \text{supp } \chi \subset [-r_2, r_2]; \\ \chi \equiv 1 \quad \text{at } [r_1, r_1]; \\ \|\chi'\|_{L^\infty} \leq \sqrt{\frac{2C(r_1+r_2)}{(r_2-r_1)}}. \end{cases}$$

In the multidimensional scenario, it is enough to take $\psi(x) = \chi(|x|^2)$, and we obtain the same properties. Now we are able to extend the function $Z(x, t)$. Recall that $Z(x, t) = x + i\phi(x, t)$, where $x \in V$, $t \in W$, the function $\phi(x, t)$ is real-valued, $\phi(0) = 0$, and $d_x\phi(0) = 0$. Let $V_1 \Subset V$ be an open neighborhood of the origin, $\chi \in \mathcal{C}_c^\infty(V)$ such that

$$\begin{cases} 0 \leq \chi \leq 1; \\ \chi \equiv 1 \quad \text{at } V_1. \end{cases}$$

We define $\tilde{Z}(x, t) \doteq x + i\chi(x)\phi(x, t)$, for $x \in \mathbb{R}^m$ and $t \in W$. Then we have that

$$\tilde{Z}_x(x, t) = \text{Id} + i(\chi_x(x)\phi(x, t) + \chi(x)\phi_x(x, t)).$$

In order to $\tilde{Z}_x(x, t)$ be invertible, for all $(x, t) \in \mathbb{R}^n \times W_1$, for some $W_1 \subset W$, neighborhood of the origin, it is enough that the quantity $\|\chi_x(x)\phi(x, t) + \chi(x)\phi_x(x, t)\|$ is "small". Since we are assuming that

$$|\phi(x, t)| \leq C(|x|^3 + |t|),$$

we have

$$\|\phi_x(x, t)\| \leq C(|x|^2 + |t|).$$

Suppose the $\text{supp } \chi \subset B_r(0)$, and $V_1 = B_{\frac{r}{2}}(0)$. So if we take $\varepsilon = r/2$ and χ as before, we have that

$$\begin{aligned} \|\chi_x(x)\phi(x, t) + \chi(x)\phi_x(x, t)\| &\leq C \left(\sqrt{\frac{2C(r+r/2)}{(r-r/2)}}(r^3 + |t|) + (r^2 + |t|) \right) \\ &= C \left(\sqrt{6C}(r^3 + |t|) + (r^2 + |t|) \right), \end{aligned}$$

so if we write $\psi(x, t) = \chi(x)\phi(x, t)$ we have that

$$\|\psi_x(x, t)\| \leq C \left(\sqrt{6C}(r^3 + |t|) + (r^2 + |t|) \right), \quad (\text{C.1})$$

therefore if we choose r and $\delta > 0$ small enough we have that $\tilde{Z}_x(x, t)$ is invertible for all $x \in \mathbb{R}^m$ and $|t| \leq \delta$. Now also note that

$$|\psi(x, t) - \psi(x', t)| \leq \mu|x - x'|, \quad \forall x \in \mathbb{R}^m, \forall |t| < \delta \quad (\text{C.2})$$

where $\mu > 0$ can be as small as we want, choosing $r > 0$ and δ small enough. In view of the discussion in the end of Appendix A and combining the estimates (C.1) and (C.2), we have that the real structure bundle

$$\mathbb{RT}'_{\mathfrak{M}_t} = \{(\tilde{Z}(x, t), {}^t\tilde{Z}_x(x, t)^{-1}\xi) : x \in \mathbb{R}^m, |t| < \delta, \xi \in \mathbb{R}^m\},$$

satisfies

$$\begin{cases} |\operatorname{Im}\zeta| < \kappa|\operatorname{Re}\zeta|; \\ \operatorname{Im}\{\zeta \cdot (\tilde{Z}(x, t) - \tilde{Z}(x', t)) + i\langle\zeta\rangle\langle\tilde{Z}(x, t) - \tilde{Z}(x', t)\rangle^2\} \geq c|\zeta||\tilde{Z}(x, t) - \tilde{Z}(x', t)|^2, \end{cases}$$

for all $x \in \mathbb{R}^m$, $|t| < \delta$ and $\zeta = {}^t\tilde{Z}_x(x, t)^{-1}\xi$, where $\xi \in \mathbb{R}^m$, and c, κ are positive constants (depending on r and δ) with $\kappa < 1$.

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