**DCOBS: Forecasting the Term Structure of Interest Rates
with Dynamic Constrained Smoothing B-Splines**

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DCOBS: Forecasting the Term Structure of Interest Rates with Dynamic Constrained Smoothing B-Splines

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Abstract

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The Nelson-Siegel framework published by Diebold and Li a decade ago created an important benchmark and originated several works in the literature of forecasting term structure of interest rates. For instance, the Arbitrage-Free Nelson-Siegel framework improved predictive performance by imposing no-arbitrage conditions to the Nelson-Siegel framework. However, these frameworks were built on the top of a parametric curve model that may lead to poor fitting for sensible term structure shapes affecting forecast results. We propose DCOBS with no-arbitrage restrictions to forecast the term structure. It is built on the top of the nonparametric constrained smoothing B-splines yield curve model. This curve model has shown to be an optimum solution between financial integrity and respect to yield curve shapes. Even though this curve model may provide more volatile forward curves than parametric models, they are still more accurate than those from Nelson-Siegel frameworks. A software was developed with a complete implementation of yield curve fitting techniques discussed in this paper. DCOBS has been evaluated for ten years of brazilian government bond data and it has shown good consistence with stylized facts of yield curves. The results of DCOBS are promising, specially in short-term forecast, and has shown greater stability and lower root mean square errors than Arbitrage-Free Nelson-Siegel.

Keywords: numerical analysis, B-splines, time-series, term structure, interest rates, no-arbitrage, brazilian bonds.

Resumo

MINEO, E. P. **DCOBS: Previsão da Estrutura a Termo de Taxa de Juros com B-Splines restritas, suavizadas e dinâmicas**. 2017. 66 f. Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2017.

O framework Nelson-Siegel publicado por Diebold e Li uma década atrás criou um importante benchmark e originou diversos trabalhos na literatura de previsão de estrutura a termo de taxas de juros. Por exemplo, o framework Nelson-Siegel Livre de Arbitragem melhorou a performance preditiva impondo condições de não-arbitragem ao framework Nelson-Siegel. No entanto, estes frameworks foram construídos em cima do modelo de curvas paramétricas. Em casos mais sensíveis de formas de curvas, este modelo tem um desempenho muito ruim, afetando o resultado da previsão. Nós propomos o DCOBS com restrições de não-arbitragem para prever a estrutura a termo. Ele é construído em cima do modelo de curva não-paramétrico com B-Splines restritas e suavizadas. Este modelo demonstrou ser uma solução ótima entre integridade financeira e respeito às formas de curvas de juros. Embora este modelo de curva possa resultar em curvas forwards mais voláteis que os modelos paramétricos, ele é ainda mais acurado que aqueles do framework Nelson-Siegel. Um software foi desenvolvido com uma implementação completa das técnicas de ajustes de curvas de juros discutidas nesta dissertação. DCOBS foi avaliado utilizando dez anos de dados de títulos públicos do governo brasileiro e demonstrou boa consistência com os fatos estilizados das curvas de juros. Os resultados do DCOBS são promissores, especialmente na previsão de curto prazo, e demonstrou maior estabilidade e menor erro quadrático médio que o modelo Nelson-Siegel Livre de Arbitragem.

Palavras-chave: análise numérica, B-splines, séries-temporais, estrutura a termo, taxa de juros, não-arbitragem, títulos públicos.

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List of abbreviations

DCOBS	Dynamic Constrained Smoothing B-Splines
NS	Nelson-Siegel
DNS	Dynamic Nelson-Siegel
AFNS	Arbitrage-Free Nelson-Siegel
RMSE	Root Mean Square Error
GCV	Generalized Cross Validation

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Chapter 1

Introduction

Forecast methods applied to term structure of interest rates are important tools not only for banks and financial firms, or governments and policy makers, but for society itself, helping to understand the movements of markets and flows of money. Several works have been done during the past decades in order to predict the dynamics of term structure of interest rates. This paper presents a dynamic version of constrained smoothing B-splines model to forecast the yield curve with no-arbitrage restrictions.

A complete term structure of interest rates does not exist in the real world. Observable market data are discrete points that relate interest rates to maturity dates. Since it is unlikely that there will be an available contract in the market for every maturity needed by practitioners, a continuous curve model is necessary. The importance of these models is crucial for pricing securities, for instance. The first modeling technique that comes to mind is interpolation. With interpolation one can indeed obtain an adherent fit, but it can easily lead to unstable curves since market data is subject to many sources of disturbance.

The literature describes two approaches for estimating the term structure of interest rates: a statistical approach and an equilibrium approach. The equilibrium approach makes use of theories that describe the overall economy in terms of state variables and its implications on short-term interest rates. In the statistical approach, the construction of the yield curve relies on data observed in the market. This observed data can be smoothed with parametric or nonparametric methods. Parametric methods have functional forms and its parameters can have economic interpretations such as Nelson-Siegel [NS87] model or Svensson [Sve94] model. One advantage is that restrictions on parameters can be added so it copes with convenient economic theories such as the arbitrage-free set. However, its functional form makes parametric methods less flexible to fit observed data. This lack of adherence to data can make its practical usage inappropriate, specially in asset pricing and no-arbitrage applications due to misspecification [LM10]. The model can produce yield curves with theoretical integrity but without reflecting the reality. On the other hand, nonparametric methods do not assume any particular functional form and consequently they are very flexible and can be very robust if combined with appropriate conditions.

After 40 years since the publication of the first yield curve models, just recently the yield curve dynamics became an essential topic. With the publication of the Dynamic Nelson-Siegel (DNS) model by Diebold and Li [DL03], the subject became established. Even though the dynamics of term structure play a vital role in macroeconomic studies, Diebold and Li argued that until then little attention had been paid to forecasting term structures. They gave two reasons for this lack of interest. Firstly, they stated that no-arbitrage models had little to say about term structure dynamics. Secondly, based on the work of Duffee [Duf02], they assumed that affine equilibrium models¹ forecast poorly. Therefore, there was a belief that dynamics of yield curve could not be forecast with parsimonious models.

In order to challenge this idea, Diebold and Li proposed the DNS model using Nelson-Siegel

¹The expression "affine term structure model" describes any arbitrage-free model in which bond yields are affine (constant-plus-linear) functions of some state vector x . For further reading, we recommend [Pia10].

yield curve fitting to forecast its dynamics. This model became very popular among financial market users and even central banks around the world. It is parsimonious and stable. Also, Nelson-Siegel functional imposes some desired economic properties such as discount function approaching zero as maturity evolves and its factors representing short, medium and long term behavior.

In practice, the forecast results of DNS are remarkable, but despite its both theoretical and empirical success, DNS did not consider restrictions for arbitrage opportunities. Practitioners could be exposed to critical financial risks, as the pricing of assets that depends on interest rates relies on arbitrage-free theory. In order to mitigate these risks, Christensen et al. [CDR11] introduce a class of Arbitrage-Free Nelson-Siegel (AFNS) models. They were affine term structure models that keep the DNS structure and incorporate no-arbitrage restrictions. Tourruco et al [TCMS16] list several appealing features of AFNS. They keep the desired economic properties of the three-factors model of the original structure of DNS. They also ensure lack of arbitrage opportunities with a more simple structure compared to those affine arbitrage-free models published previously by Duffie and Kan [DK96] and Duffie [Duf02]. This is achieved by adding a yield-adjustment term to Nelson-Siegel yield curve model described as an ordinary differential system of equations to ensure no-arbitrage. Tourruco et al [TCMS16] argues that, in long forecast horizons, AFNS model with uncorrelated factors delivers most accurate forecasts. Their conclusion is that no-arbitrage is indeed helpful, but only for longer forecasting horizons.

Barzanti and Corradi [BC01] published earlier works on the use of constrained smoothing B-splines to estimate term structures of interest rates. But Laurini and Moura [LM10] proposed constrained smoothing B-splines with a different methodology. This methodology was initially proposed by He and Shi [HS98] and He and Ng [HN99] as a general tool to smooth data with certain qualitative properties such as monotonicity and concavity or convexity constraints. Roughly, the methodology builds the yield curve as a L_1 projection of a smooth function into the space of B-splines. It is achieved by estimating a conditional median function as described in quantile regression theory of Koenker and Basset [KB78]. A great advantage is that, being a conditional median function, it is robust to outliers. Also, its formulation as a linear programming problem gives us the ability to impose several constraints without substantial increase in computational costs.

Our present work proposes a dynamic constrained smoothing B-splines (DCOBS) model to forecast the term structure of interest rates. DCOBS describes the coefficients of the yield curve model proposed by Laurini and Moura [LM10] as processes evolving over time. Even though constrained smoothing B-splines specification provides full automation in knot mesh selection, we could not use it in a dynamic framework setting. In order to build a common ground and observe curve shapes evolving over time, knots were fixed to capture short, medium and long term behavior according to observed data. These knots were adjusted equally in the dataset, so there was the same amount of coefficients on each daily curve and it was possible to run a statistical regression. DCOBS has shown great predictability in short-term, and remained stable on the long-term.

In Chapter 2 we give a brief introduction to the fundamental theories of dynamic Nelson-Siegel models. In Chapter 3 we introduce DCOBS model. In Chapter 4 we present the data set used for fitting and forecasting the brazilian term structure of interest rates. In Chapter 5 we study the outputs from time series of fitted yield curves. Finally, in Chapter 6 we finish the work pointing the conclusions we made.

The main contribution of this work are:

- A complete formulation of no-arbitrage constrained smoothing B-splines in terms of objective functions and linear constraint equations;
- A dynamic framework of constrained smoothing B-splines described as AR(1) processes (DCOBS);
- Evaluation of DCOBS framework compared to AFNS for ten years of brazilian government bonds data;
- Automatic selection of the best smoothing parameter λ for fitting yield curves;
- Yield curves estimated as conditional median functions robust to outliers;

- Yield curves with better adherence to data compared to Nelson-Siegel family curves.
- A software that fits several curve fitting models, including no-arbitrage constrained smoothing B-splines;

Chapter 2

Yield curve models

2.1 Term structure of interest rates

We will treat interest rates as a multidimensional variable that represents the return on investment expressed by three related information: spot rate, forward rate and the discount value.

Each of these values depends on several economical, political and social informations, such as supply and demand of money and the expectation of its future value, risk and trust perception, consequences of political acts, etc. The term structure of interest rates is a valuable tool not only for banks and financial firms, or governments and policy makers, but for society itself, helping to understand the movements of markets and flows of money.

We will assume that fixed income government bonds can be considered risk-free so we can define a special type of yield that is the *spot interest rate*, $s(\tau)$. This function is the return of a fixed income zero-coupon risk-free bond that expires in τ periods. Today's price of such financial instrument whose future value is \$1.00 and its interest rate is continuously compounded is given by the *discount function*, $d(\tau)$, represented by

$$d(\tau) = e^{-s(\tau) \times \tau}. \quad (2.1)$$

The [Equation 2.1](#) establishes the relationship between the discount value and the spot rate, which can be recovered by

$$s(\tau) = -\frac{\log(d(\tau))}{\tau}.$$

Based on the available bonds in the market with different maturities, it is possible to plan now a financial transaction that will take place in the future, starting at the maturity of the shorter bond and expiring at the maturity of the longer bond.

The interest rate of this future transaction is called the *forward rate*.

Consider a forward contract traded at the present day at $\tau_P = 0$. This contract arranges an investment in the future that starts at the settlement date at time τ . This investment will be kept until the maturity date, at time $\tau_M > \tau$. Then, the continuously compounded (implied) forward rate is related to the spot rate according to

$$f(\tau, \tau_M) = \frac{s(\tau_M) \times \tau_M - s(\tau) \times \tau}{\tau_M - \tau}.$$

If we take the limit of $\tau_M \rightarrow \tau$, we end up with the *instantaneous forward rate* or *short rate*. The *short rate* is the forward rate for a forward contract with an infinitesimal investment period after the settlement date, and it is defined as

$$f(\tau) = \lim_{\tau_M \rightarrow \tau} f(\tau, \tau_M).$$

The forward rate can be seen as the marginal increase in the total return from a marginal increase

in the length of the investment [Sve94]. The spot rate $s(\tau)$ is identical to the average of the instantaneous forward rates with settlement between the trade date 0 and the maturity date τ whose formulation is defined as

$$s(\tau) = \frac{1}{\tau} \int_0^\tau f(x) dx. \quad (2.2)$$

From Equation 2.1 and Equation 2.2 we can relate the discount function and forward rate as

$$d(\tau) = e^{-\int_0^\tau f(x) dx}$$

and

$$f(\tau) = -\frac{d'(\tau)}{d(\tau)}.$$

The relationship between the yields of bonds that differ only by their term to maturity gives us a function called *yield curve*. The relationship between **spot rates** of bonds that differ only in their term to maturity is called *term structure of interest rates* [CIR85].

Yield curves for coupon-bearing bonds should not be used as direct representations of the term structure of interest rates, since yield curves of coupon-bearing bonds are not equivalent of the yield curves of zero-coupon bonds with same maturity dates[Sve94].

On the other hand, yield curves of zero-coupon bonds are equivalent to their spot rates curve, from now on called term structure of interest rates.

Since there is no continuous term structure of interest rates in real world¹, a curve is estimated by mathematical models in order to represent the term structure of interest rates in a continuous way, providing a valuable tool to calculate the spot rate at any instant within a given interval.

As pointed by Diebold and Li[DL03], the classical approaches to model the term structure of interest rates are equilibrium models and no-arbitrage models.

Equilibrium models derive the term structure of interest rates from economic variables to model a stochastic process for the short rate dynamic, typically using affine models, after which spot rates can be derived under some risk premium assumptions. Examples of equilibrium models are [Vas77], [CIR85] and [DK96].

On the other hand, no-arbitrage models focuses on perfectly fitting the term structure of interest rate on observed market spot rates in such a way that there is no arbitrage opportunity. Major contribution to no-arbitrage models were give by [HW90] and [HJM92].

In the work of Diebold and Li[DL03], neither the equilibrium model nor the no-arbitrage model are used to model the term structure of interest rates. Instead, they use the Nelson and Siegel [NS87] exponential components framework because they claimed that this model produces encouraging results for out-of-sample forecasting, and until that time, little attention in literature of no-arbitrage and equilibrium models had been paid to dynamics and forecasting of interest rates.

2.2 Dynamic Nelson-Siegel

Diebold and Li proposed a dynamic version of Nelson-Siegel yield curve model

$$NS(\tau) = \beta_1 L_1(\tau) + \beta_2 L_2(\tau) + \beta_3 L_3(\tau). \quad (2.3)$$

There are several contracts available in the market each day. So, a yield curve is fitted according to Nelson-Siegel model to relate yields and maturities of available contracts for a specific day. We will refer to it as the Nelson-Siegel static model for yield curves. The curves are fitted by constructing a least squares program such that $\theta = \{\beta_1, \beta_2, \beta_3\}$ are the parameters that minimize the distance between $NS(\tau)$ and market data points. The coefficients $\beta_1, \beta_2, \beta_3$ are interpreted as three latent dynamic factors. The loading on β_1 is constant and do not change in the limit, then β_1 can be viewed as a long term factor. The loading on β_2 starts at 1 and decay quickly to zero, then β_2 can

¹Spot rates are discrete points observed in contracts available in the market

be viewed as a short term factor. Finally, the loading on β_3 starts at zero, increases and decay back to zero, then β_3 can be viewed as a medium term factor.

The Dynamic Nelson-Siegel model for forecasting the yield curve $s(\tau)$ describes the static model evolving over time. The Dynamic Nelson-Siegel model is

$$\begin{aligned} s_t(\tau) &= \beta_{1,t}L_1(\tau) + \beta_{2,t}L_2(\tau) + \beta_{3,t}L_3(\tau) + \epsilon_t \\ t &= 1, \dots, T \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} L_1(\tau) &= 1, \\ L_2(\tau) &= \frac{1 - e^{-\lambda\tau}}{\lambda\tau}, \\ L_3(\tau) &= \frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}. \end{aligned}$$

The parameter λ is a constant interpreted as the conductor of the curve exponential decay rate and the coefficients $\beta_{i,t}$ are AR(1) processes described as

$$\beta_{i,t} = c_i + \phi_i\beta_{i,t-1} + \eta_{i,t} \quad i = 1, 2, 3$$

where the intercepts c_i , the parameters ϕ_i and the coefficients $\beta_{i,t}$ are obtained fitting the model over a dataset of T daily market observations. Furthermore, $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ and $\eta_{i,t} \sim \mathcal{N}(0, \sigma_i^2)$ are independent errors. Since the yield curve model depends only on $\beta_{1,t}, \beta_{2,t}, \beta_{3,t}$, forecasting the yield curve is equivalent to forecasting $\beta_{1,t}, \beta_{2,t}, \beta_{3,t}$.

Conversely, the factors for long-term, short-term and medium-term can also be interpreted in terms of level, slope and curvature of the model. Diebold and Li [DL03] use these interpretations to claim that the historical stylized facts of the term structure of interest rates can be replicated by fitting the three factors, which means that the model can replicate yield curve geometric shapes.

$NS(\tau)$ also correspond to a discount curve, transformed according to Equation 2.1, that begins at one for zero maturity and approaches zero as maturity goes to infinity.

For US market, [DL03] shows that DNS model outperforms traditional benchmarks such as the random walk model, even though Vicente et al.[VVT08] states that the model does not outperform random walk forecast for short-term forecasts (one-month ahead).

2.3 Arbitrage-Free Nelson-Siegel

Arbitrage-Free Nelson-Siegel static model for daily yield curve fitting was derived by Christensen et al.[CDR11] from the standard continuous-time affine Arbitrage-Free formulation of Duffie and Kan[DK96]. Let be a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_\tau), Q)$, where the filtration $(\mathcal{F}_\tau) = \{\mathcal{F}_\tau : \tau \geq 0\}$ satisfies the usual conditions. Also, consider the state variable X_τ assumed to be a Markov Process defined on a set $M \subset \mathbf{R}^n$ that solves the stochastic differential equation

$$dX_\tau = K^Q(\tau)[\Theta^Q(\tau) - X_\tau]d\tau + \Sigma(\tau)D(X_\tau, \tau)dW_\tau^Q,$$

where W^Q is a standard Brownian motion in $\mathbf{R}^n$ in respect to the filtration $(\mathcal{F}_\tau)$. The drifts $\Theta^Q : [0, \tau_M] \rightarrow \mathbf{R}^n$, dynamics $K^Q : [0, \tau_M] \rightarrow \mathbf{R}^{n \times n}$ and volatility matrix $\Sigma : [0, \tau_M] \rightarrow \mathbf{R}^{n \times n}$ are bounded continuous functions. Also, $D : M \times [0, \tau_M] \rightarrow \mathbf{R}^{n \times n}$ has diagonal structure with entries

$$[D]_{ii} = \sqrt{\gamma^i(\tau) + \delta_1^i(\tau)X_\tau^1 + \dots + \delta_n^i(\tau)X_\tau^n} \quad \forall i \in \{1, \dots, n\}$$

where $\gamma^i : [0, \tau_M] \rightarrow \mathbf{R}^n$ and $\delta^i : [0, \tau_M] \rightarrow \mathbf{R}^{n \times n}$ are bounded continuous functions. We will consider the instantaneous risk-free rate as an affine function of the state variables

$$r_\tau = \rho_0(\tau) + \rho_1(\tau)' X_\tau,$$

where $\rho_0 : [0, \tau_M] \rightarrow \mathbf{R}$ and $\rho_1 : [0, \tau_M] \rightarrow \mathbf{R}^n$ are bounded continuous functions.

Duffie and Kan [DK96] demonstrated that zero-coupon bond prices could be formulated with the previous framework as exponential affine functions of the state variables

$$P(\tau, \tau_M) = E_\tau^Q \left[\exp \left(- \int_\tau^{\tau_M} r_u du \right) \right] = \exp \left(B(\tau, \tau_M)' X_\tau + C(\tau, \tau_M) \right),$$

where $B(\tau, \tau_M)$ and $C(\tau, \tau_M)$ are the solutions of the system of ordinary differential equations

$$\begin{aligned} \frac{dB(\tau, \tau_M)}{d\tau} &= \rho_1 + (K^Q)' B(\tau, \tau_M) \\ &\quad - \frac{1}{2} \sum_{j=1}^n (\Sigma' B(\tau, \tau_M) B(\tau, \tau_M)' \Sigma)_{j,j} (\delta^j)' \\ \frac{dC(\tau, \tau_M)}{d\tau} &= \rho_0 - B(\tau, \tau_M)' K^Q \Theta^Q \\ &\quad - \frac{1}{2} \sum_{j=1}^n (\Sigma' B(\tau, \tau_M) B(\tau, \tau_M)' \Sigma)_{j,j} \gamma^j \end{aligned}$$

subject to the initial conditions

$$\begin{aligned} B(\tau_M, \tau_M) &= 0 \\ C(\tau_M, \tau_M) &= 0. \end{aligned}$$

Therefore, the zero-coupon yield function derived from the price function is

$$\begin{aligned} s(\tau, \tau_M) &= - \frac{1}{\tau_M - \tau} \log P(\tau, \tau_M) \\ &= - \frac{B(\tau, \tau_M)' X_\tau}{\tau_M - \tau} - \frac{C(\tau, \tau_M)}{\tau_M - \tau}. \end{aligned}$$

and we can decompose the yield function as a three-factor affine model with $X_\tau = (X_\tau^1, X_\tau^2, X_\tau^3)$ almost matching the Nelson-Siegel yield curve model as

$$\begin{aligned} s(\tau, \tau_M) &= X_\tau^1 + \frac{1 - e^{-\lambda(\tau_M - \tau)}}{\lambda(\tau_M - \tau)} X_\tau^2 \\ &\quad + \left[\frac{1 - e^{-\lambda(\tau_M - \tau)}}{\lambda(\tau_M - \tau)} - e^{-\lambda(\tau_M - \tau)} \right] X_\tau^3 - \frac{C(\tau, \tau_M)}{\tau_M - \tau}, \end{aligned} \tag{2.5}$$

with ODEs for the $B(\tau, \tau_M)$ functions that have solutions

$$\begin{aligned} B^1(\tau, \tau_M) &= -(\tau_M - \tau), \\ B^2(\tau, \tau_M) &= - \frac{1 - e^{-\lambda(\tau_M - \tau)}}{\lambda}, \\ B^3(\tau, \tau_M) &= (\tau_M - \tau) e^{-\lambda(\tau_M - \tau)} - \frac{1 - e^{-\lambda(\tau_M - \tau)}}{\lambda}. \end{aligned}$$

We can reformulate Equation 2.5 using loading functions from dynamic Nelson-Siegel model in

Equation 2.4 as

$$\begin{aligned} s(\tau, \tau_M) = & X_\tau^1 L_1(\tau_M - \tau) + X_\tau^2 L_2(\tau_M - \tau) \\ & + X_\tau^3 L_3(\tau_M - \tau) - \frac{C(\tau, \tau_M)}{\tau_M - \tau}. \end{aligned} \quad (2.6)$$

The previous formulation almost match the Nelson-Siegel yield curve model except by the yield-adjustment term $-\frac{C(\tau, \tau_M)}{\tau_M - \tau}$. Christensen et al. [CDR11] identified the AFNS models fixing the mean levels of the state variable under the Q-measure at zero, i.e., $\Theta^Q = 0$. Thus, the yield-adjustment term will have the form

$$-\frac{C(\tau, \tau_M)}{\tau_M - \tau} = -\frac{1}{2} \frac{1}{\tau_M - \tau} \sum_{j=1}^3 \int_{\tau}^{\tau_M} (\Sigma' B(s, \tau_M) B(s, \tau_M)' \Sigma)_{j,j} ds.$$

Considering a general volatility matrix (not related to the dynamic model for forecasting the yield curve)

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix},$$

Christensen et al. [CDR11] shows that an analytical form of the yield-adjustment term can be derived as

$$\begin{aligned} \frac{C(\tau, \tau_M)}{\tau_M - \tau} = & \frac{1}{2} \frac{1}{\tau_M - \tau} \int_{\tau}^{\tau_M} \sum_{j=1}^3 (\Sigma' B(s, \tau_M) B(s, \tau_M)' \Sigma)_{j,j} ds \\ = & \bar{A} \frac{(\tau_M - \tau)^2}{6} \\ & + \bar{B} \left[\frac{1}{2\lambda^2} - \frac{1}{\lambda^3} \frac{1 - e^{-\lambda(\tau_M - \tau)}}{\tau_M - \tau} + \frac{1}{4\lambda^3} \frac{1 - e^{-2\lambda(\tau_M - \tau)}}{\tau_M - \tau} \right] \\ & + \bar{C} \left[\frac{1}{2\lambda^2} + \frac{1}{\lambda^2} e^{-\lambda(\tau_M - \tau)} - \frac{1}{4\lambda} (\tau_M - \tau) e^{-2\lambda(\tau_M - \tau)} \right. \\ & \left. - \frac{3}{4\lambda^2} e^{-2\lambda(\tau_M - \tau)} - \frac{2}{\lambda^3} \frac{1 - e^{-\lambda(\tau_M - \tau)}}{\tau_M - \tau} + \frac{5}{8\lambda^3} \frac{1 - e^{-2\lambda(\tau_M - \tau)}}{\tau_M - \tau} \right] \\ & + \bar{D} \left[\frac{1}{2\lambda} (\tau_M - \tau) + \frac{1}{\lambda^2} e^{-\lambda(\tau_M - \tau)} - \frac{1}{\lambda^3} \frac{1 - e^{-\lambda(\tau_M - \tau)}}{\tau_M - \tau} \right] \\ & + \bar{E} \left[\frac{3}{\lambda^2} e^{-\lambda(\tau_M - \tau)} + \frac{1}{2\lambda} (\tau_M - \tau) + \frac{1}{\lambda} (\tau_M - \tau) e^{-\lambda(\tau_M - \tau)} \right. \\ & \left. - \frac{3}{\lambda^3} \frac{1 - e^{-\lambda(\tau_M - \tau)}}{\tau_M - \tau} \right] \\ & + \bar{F} \left[\frac{1}{\lambda^2} + \frac{1}{\lambda^2} e^{-\lambda(\tau_M - \tau)} - \frac{1}{2\lambda^2} e^{-2\lambda(\tau_M - \tau)} \right. \\ & \left. - \frac{3}{\lambda^3} \frac{1 - e^{-\lambda(\tau_M - \tau)}}{\tau_M - \tau} + \frac{3}{4\lambda^3} \frac{1 - e^{-2\lambda(\tau_M - \tau)}}{\tau_M - \tau} \right], \end{aligned}$$

where

$$\begin{aligned}\overline{A} &= \sigma_{11}^2 + \sigma_{12}^2 + \sigma_{13}^2 \\ \overline{B} &= \sigma_{21}^2 + \sigma_{22}^2 + \sigma_{23}^2 \\ \overline{C} &= \sigma_{31}^2 + \sigma_{32}^2 + \sigma_{33}^2 \\ \overline{D} &= \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} + \sigma_{13}\sigma_{23} \\ \overline{E} &= \sigma_{11}\sigma_{31} + \sigma_{12}\sigma_{32} + \sigma_{13}\sigma_{33} \\ \overline{F} &= \sigma_{21}\sigma_{31} + \sigma_{22}\sigma_{32} + \sigma_{23}\sigma_{33}.\end{aligned}$$

For the Independent-Factors AFNS model, Christensen et al. estimated the general volatility matrix as

$$\hat{\Sigma} = \begin{pmatrix} 0.0051 & 0 & 0 \\ 0 & 0.0110 & 0 \\ 0 & 0 & 0.0264 \end{pmatrix}$$

and $\hat{\lambda} = 0.5975$ for maturities measured in years, solving the yield-adjustment equation for arbitrage-free condition.

Note that the adjustment-term of AFNS model is time-independent. In other words, it is a deterministic function which depends only on the maturity of the bond. So, let the function $\Gamma(\tau)$ be

$$\Gamma(\tau) = -\frac{C(0, \tau)}{\tau}.$$

As in the Dynamic Nelson-Siegel model, the Dynamic Arbitrage-Free Nelson-Siegel model describes the Arbitrage-Free Nelson-Siegel static model evolving over time. The specification is

$$\begin{aligned}s_t(\tau) &= \beta_{1,t}L_1(\tau) + \beta_{2,t}L_2(\tau) + \beta_{3,t}L_3(\tau) + \Gamma(\tau) + \epsilon_t \\ t &= 1, \dots, T\end{aligned}\tag{2.7}$$

where the loadings are the usual functions of [Equation 2.4](#).

The parameter λ will assume the estimated value $\hat{\lambda} = 0.5975$. It has the same interpretation of dynamic Nelson-Siegel model as a conductor of the curve exponential decay rate. Also, the coefficients $\beta_{i,t}$ are AR(1) processes described as

$$\beta_{i,t} = c_i + \phi_i\beta_{i,t-1} + \eta_{i,t} \quad i = 1, 2, 3$$

where the intercepts c_i , the parameters ϕ_i and the coefficients $\beta_{i,t}$ are obtained fitting the model over a dataset of T daily market observations. Furthermore, $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ and $\eta_{i,t} \sim \mathcal{N}(0, \sigma_i^2)$ are independent errors. Since the yield curve model depends only on $\beta_{1,t}, \beta_{2,t}, \beta_{3,t}$, forecasting the yield curve is equivalent to forecasting $\beta_{1,t}, \beta_{2,t}, \beta_{3,t}$.

Chapter 3

Dynamic Constrained Smoothing B-splines

Constrained Smoothing B-Splines is a methodology first proposed by He and Shi[HS98] and then formalized by He and Ng[HN99] as a proper algorithm. It extends smoothing splines to a conditional quantile function estimation and then formulates the fitting model into a linear programming problem that can incorporate constraints such as monotonicity, convexity and boundary conditions.

Laurini and Moura [LM10] applied this methodology as a static model to fit daily yield curves along with no-arbitrage constraints. The estimation of the daily term structure of interest rates is set to be a conditional median estimation that is robust to outliers. This model produces yield curves as L_1 projection into the space of B-splines. The flexible nature of B-splines by its local support and the constraints of the model can make the curve fit the data according to its financial meaning and carrying with it all possible information.

Our contribution in this paper is a Dynamic Constrained Smoothing B-splines model that describes the static model evolving over time.

The dynamic model, DCOBS, is estimated by the Penalized Least Absolute Deviation

$$\min_{\theta \in \mathbb{R}^C} \sum_{i=1}^n \left| y_i - \sum_{j=1}^C a_j B_j(\tau_i) \right| + \Lambda \max_{\tau} \sum_{i=1}^C a_j (B_j(\tau))'', \quad (3.1)$$

where n is the number of contracts available in the reference day, y_i are market yields of the contracts, $C = N + m$ is the number of coefficients, N is the number of internal knots, $m = 3$ is the order for quadratic B-Splines basis, $\theta = (a_1, \dots, a_C)$ is the coefficient vector to be estimated, B_j are the quadratic B-splines basis and τ_i are distinct maturities of the contracts. The selection of the smoothing parameter Λ is automated with generalized cross validation (Leave-One-Out GCV) method of [FNZ95]¹.

The previous formulation can be rewritten as

$$\min_{\theta \in \mathbb{R}^C} \sum_{i=1}^n \left| y_i - \sum_{j=1}^C a_j B_j(\tau_i) \right| + \Lambda \omega,$$

such that

$$-\omega \leq \sum_{j=1}^C a_j (B_j(t_k))'' \leq \omega$$

where $k = 1, \dots, N$ and t_k is an internal knot position.

¹GCV is "Leave-One-Out" cross validation technique where the curve is fitted leaving one sample out, and then calling the best parameter Λ with all samples considered.

The above static model can be implemented as an equivalent linear programming problem that minimizes the objective function z such that

$$\min z = \sum_{i=1}^n |e_i| + |\omega|.$$

Each yield observed in the market will produce five linear constraint equations: two constraints for fitting the curve, one constraint for smoothing and two constraints for no-arbitrage conditions. The fitting constraints are

$$\begin{aligned} \sum_{j=1}^C a_j B_j(\tau_i) + |e_i| &\geq y_i \\ \sum_{j=1}^C a_j B_j(\tau_i) - |e_i| &\leq y_i \end{aligned}$$

where all $B_j(\tau_i)$ are quadratic B-spline basis.

The smoothing constraint is

$$\Lambda \sum_{j=1}^C a_j (B_j(\tau_i))'' - |\omega| \leq 0.$$

Finally, the no-arbitrage constraints are

$$\begin{aligned} \sum_{j=1}^C a_j B_j(\tau_i) &> 0 \\ \sum_{j=1}^C a_j (B_j(\tau_i))' &< 0. \end{aligned}$$

The resulting fitted yield curve,

$$\hat{s}(\tau) = \sum_{j=1}^C \hat{a}_j B_j(\tau)$$

is a conditional median function represented by quadratic smoothing B-splines.

We proposed the Dynamic Constrained Smoothing B-Splines model as

$$\begin{aligned} s_t(\tau) &= \sum_{j=1}^C a_{j,t} B_j(\tau) + \epsilon_t \\ t &= 1, \dots, T \end{aligned} \tag{3.2}$$

where the coefficients $a_{j,t}$ are AR(1) processes described as

$$a_{j,t} = c_j + \phi_j a_{j,t-1} + \eta_{j,t} \quad j = 1 \dots C$$

where the intercepts c_j , the parameters ϕ_j and the coefficients $a_{j,t}$ are obtained fitting the model over a dataset of T daily market observations. Furthermore, $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ and $\eta_{j,t} \sim \mathcal{N}(0, \sigma_j^2)$ are independent errors. Since the yield curve model depends only on $a_{j,t}$, forecasting the yield curve is equivalent to forecasting $a_{j,t}$.

The Dynamic Constrained Smoothing B-Splines model describes the static model evolving over time. In other words, the dynamic model extrapolates the temporal axis creating a surface of fitted

curves.

Figure 3.1 gives us a visual idea of the differences between each yield curve model.

Figure 3.1: Yield curves for term structure at 2009-10-20

Chapter 4

Descriptive Analysis

This work used 10 years (2006-2016) of public historical data of brazilian government bonds LTN and NTN-F. The buying prices were used in this paper. The period of 2006-2015 was designed as sample data and 2016 was designed as test data.

LTN is a short-term fixed income zero-coupon bond considered risk-free. NTN-F is a medium-term fixed income coupon-bearing bond that pays 10% of the principal semiannually. We extracted the zero-coupon equivalent rates from NTN-F through financial bootstrapping procedure¹ so it was possible to merge data points from the two bonds and get a larger yield curve horizon.

The sample data is constituted of 20.604 data points in 2.500 reference dates spanning between 2006-01-02 and 2015-12-31. The term structure horizon spans between 1 business day and 2.744 business days. The sample data set originated 5.000 yield curves, being 2.500 AFNS and 2.500 DCOBS. By the nature of parametric models, there is no challenge in relating the coefficients in a time-dependent process. So, Arbitrage-Free Nelson-Siegel yield curves are fitted using the raw sample data.

On the other hand, nonparametric B-Splines model depends on its knots and data points position. Therefore, a two-step normalization procedure was applied so the coefficients could be related in a time-dependent process. The first step normalizes the horizon length. The internal knots are defined as $K_{int} = (126, 1260)$ and the boundary knots as $K_{bnd} = (0, 2745)$, based on observed data and the overall fitting quality it produced. This definition implied a term structure of 5 knots². Since B-Splines can not be used for extrapolation, since its basis are zero outside their definitions, we used Nelson-Siegel model to extrapolate the horizon and calculate the yields on the boundaries of the term structure. The second step normalizes the data points position. This time, we build a curve with DCOBS with knots being equally distributed across the horizon. With the resulting fitted curve, we calculate the final normalized term structure at previously selected points $(0, s(126), s(567), s(1260), s(2745))$.

For instance, consider the term structure of interest rates as in 2013-01-02 in Table 4.1.

Fitting this term structure with Nelson-Siegel model produces a term structure with yields at the boundaries as shown in Table 4.2.

Finally, fitting the resulting term structure with constrained smoothing B-Splines and knots distributed equally across the horizon, a normalized term structure is obtained as shown in Table 4.3.

¹Bootstrapping is an iterative procedure that creates an equivalent zero-coupon bond from a coupon-bearing bond. This is achieved by looking in the market for zero-coupon bonds with maturities equal to the maturities of future coupon payments. The zero-coupon bond interest rates is then related to the future coupon payments of the coupon-bearing bonds. The resulting zero-coupon bonds will be equivalent to the coupon-bearing bonds with its coupon payments translated and summed to one single final payment.

²For more information: <https://goo.gl/z4C9Dz>

Business Days	Yield
253	0.0715751
506	0.0775
756	0.0825
1007	0.0867064
2010	0.0975515
2512	0.100645

Table 4.1: *Term Structure as in 2013-01-02.*

Business Days	Yield
1	0.06495709
253	0.0715751
506	0.0775
756	0.0825
1007	0.0867064
2010	0.0975515
2512	0.100645
2745	0.101758

Table 4.2: *Term structure with yields at the boundaries.*

Business Days	Yield
0	0
126	0.0686091
567	0.0788766
1260	0.0903086
2745	0.101758

Table 4.3: *Normalized term structure.*

Chapter 5

Results

We have generated a normalized data set with 12.500 data points which are the inputs for the yield curves. The procedure we proposed in DCOBS to normalize daily term structures and create a temporal relationship between coefficients certainly imply loss of information. However, the resulting yield curves had a better performance compared to AFNS considering RMSE¹ in every year of sample data set as shown in Table 5.1. The difference of fitting between both methods can be seen in Figure 5.1.

Figure 5.1: *DCOBS compared to AFNS*

We fitted 2.500 yield curves for both methods AFNS and DCOBS. For each curve, the method AFNS produced three coefficients and DCOBS produced five coefficients.

The first coefficient of AFNS model shows a stable behaviour, however the second and third keep its stability until the business day 1.000, where it becomes unstable. In one day, the second coefficient goes from -0.0203354 to 4.73094 between 2011-12-30 and 2012-01-02. Analogously, the third coefficient goes from -0.155503 to -5.0988, compensating the jump of second coefficient. This behaviour repeats for reference dates 2012-12-28, 2013-12-27 and 2014-12-31. Figure 5.2 shows the three series of AFNS coefficients.

We concluded that this behaviour reveals a sensibility of AFNS approach to the number of contracts available in the market. Taking into account the reference date 2011-12-30 (1.497 business days), there were six contracts available against five contracts at 2012-01-02, the next business day.

¹Root Mean Square Error

Year	DCOBS RMSE	AFNS RMSE
2006	0.012265	0.021373
2007	0.016608	0.054421
2008	0.029025	0.045070
2009	0.042418	0.047167
2010	0.015881	0.047804
2011	0.010699	0.033103
2012	0.008795	0.059564
2013	0.013845	0.026148
2014	0.011540	0.040362
2015	0.009979	0.025593
Total	0.171055	0.400605

Table 5.1: RMSE fitting errors.

Figure 5.2: AFNS Coefficient Series

Parameter	test-statistic
β_1	-36.3622
β_2	-35.555
β_3	-35.5626

Table 5.2: AFNS ADF Test for coefficients

Using the Augmented Dickey-Fuller Test [DF79], we see that the three series are $I(1)$, or in other words, applying one order of differentiation the series become stationary, considering a 1% critical value of -2.58 . Since all tests are lower than the critical value, we then reject the unitary root null hypothesis $\pi = 0$. Table 5.2 shows ADF test statistics for AFNS coefficients with one order of differentiation.

In Figure 5.2 we see that β_2 and β_3 may be cointegrated, so we run a two-step Engle-Granger

cointegration test [EG87]. The linear regression of β_3 explained by β_2 returned an intercept of -0.144433 and a coefficient of -1.079191 . The cointegration test over linear regression residuals returned a statistic of -3.6481 , which in turn confronted with the critical values of Table 2 in [EY87] leads us to reject the unitary root hypothesis because the residuals are stationary. The conclusion is that there is a cointegration between β_2 and β_3 . Figure 5.3 shows β_3 estimated by β_2 through linear regression.

Figure 5.3: β_3 estimated by β_2

Figure 5.4: AFNS coefficient time series

The time series for β_1 and β_2 are modeled as AR(1) processes. The coefficient ϕ of the time series for β_1 is 0.0302 and for β_2 is -0.0095 . Figure 5.4 shows the forecast for both β_1 and β_2 .

The first coefficient of DCOBS model keeps constant at 1.0 for all fitted curve. Therefore, we did not create a time series for this coefficient. Except for a few outliers, the time series for all DCOBS coefficients have a very stable behaviour. The time series for DCOBS coefficients can be seen in Figure 5.5.

Figure 5.5: *DCOBS Coefficient Series*

The Augmented Dickey-Fuller Test shows that the four series are $I(1)$, or in other words, applying one order of differentiation the series become stationary, considering a 1% critical value of -2.58 . Since all tests are lower than the critical value, we then reject the unitary root null hypothesis $\pi = 0$. Table 5.3 shows ADF test statistics for DCOBS coefficients with one order of differentiation.

Parameter	test-statistic
a_2	-41.9694
a_3	-35.8711
a_4	-39.9662
a_5	-35.5342

Table 5.3: *DCOBS ADF Test for coefficients*

As seen in AFNS coefficients time series, DCOBS coefficients a_4 and a_5 seems to cointegrate, so we run a two-step Engle-Granger cointegration test. The linear regression of a_5 explained by a_4 returned an intercept of -0.018051 and a coefficient of 0.803629 . The cointegration test over linear regression residuals returned a statistic of -7.7799 , which in turn confronted with the critical values of Table 2 in [EY87] leads us to reject the unitary root hypothesis because the residuals are stationary. The conclusion is that there is a cointegration between a_4 and a_5 . Figure 5.6 shows a_5 estimated by a_4 .

The time series for a_2 , a_3 and a_4 are modeled as AR(1) processes. The coefficient ϕ of the time series for a_2 is 0.9991 , a_3 is 0.9986 and for a_4 is 0.9932 . Figure 5.7 shows the forecast for all coefficients.

The time series above modeled as AR(1) processes were used to make the out-of-sample forecast

Figure 5.6: a_5 estimated by a_4

Figure 5.7: DCOBS coefficient time series

with horizon of 249 business days, the amount of business days in test dataset of 2016. Three reference dates were considered to evaluation: 1 month, 6 months and 12 months. Figure 5.8 shows the forecast curves.

In 1-month forecast, DCOBS performs a good fit to the term structure, specially in long-term horizon, as the curve follows the data points. AFNS shows a heavy instability in short-term horizon, although in long-term it regains stability. In both 6-months and 12-months forecasts, AFNS and

Figure 5.8: *Forecast results*

DCOBS methods remained between 12.5% and 14%, while the term structure decreased from 14% to 11% in long-term horizon. AFNS also diverged in 12-months forecast at short-term and produced almost 30% of root mean square error.

We compared both forecast techniques through Diebold-Mariano accuracy test[DM95] with alternative hypothesis being DCOBS outperforming AFNS prediction. As we stated before, DCOBS outperforms AFNS in the short-term prediction. The absolute value of Diebold-Mariano statistics for 1-month forecast is greater than 1.96, so we reject the null hypothesis that both techniques have the same accuracy. On the other hand, for 6-months and 12-months forecast, the absolute value of Diebold-Mariano statistics keeps lower than 1.96, which means that we do not reject the null hypothesis and both techniques may have the same predictive accuracy. Table 5.4 shows forecasts root mean square errors and Table 5.5 shows Diebold-Mariano statistics results.

Method	1-month	6-months	12-months
AFNS	0.044101	0.065274	0.294319
DCOBS	0.029812	0.075561	0.060160

Table 5.4: *Forecast root mean square errors*

Method	1-month	6-months	12-months
DM	2.4671	-1.3599	1.0003
p-value	0.02432	0.8886	0.1779

Table 5.5: *Diebold-Mariano Test Statistics*

Chapter 6

Conclusion

In this work, we have proposed DCOBS, a methodology for forecasting the dynamics of term structure of interest rates based on Constrained Smoothing B-Splines curve fitting. This methodology was composed by five steps:

- Selection of knot positions based on historical data;
- Term structure normalization creating a temporal relationship between DCOBS coefficients;
- Yield curves fitting;
- Time series for curves coefficients;
- Forecast coefficients.

B-Splines knots were fixed at $(0, 126, 1260, 2718)$ based on the distribution of maturity dates on observed data and the overall fitting quality.

The normalization procedure of term structures created a temporal relationship between coefficients with small autocorrelation, making forecast possible. Even though the normalization implied loss of data, the quality of DCOBS fitting was still better than AFNS.

The results have shown a great predictability of DCOBS model on the short-term. This is coherent with Tourrucô's conclusion that, in long forecast horizons, AFNS model with uncorrelated factors delivers most accurate forecasts [TCMS16]. Even though the accuracy of DCOBS in the long-term are statistically equivalent with AFNS, the stability of the model can be certainly used for future works to improve its predictability quality.

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