Regularity of almost minimizing sets

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Abstract

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This work was motivated by the famous Plateau's Problem which concerns the existence of a minimizing set of the area functional with prescribed boundary. In order to solve the Plateau's Problem, we make use of different theories: the theory of varifolds, currents and locally finite perimeter sets (Caccioppoli sets). Working on the Caccioppoli sets theory, it is straightforward to prove the existence of a minimizing set in some classical problems as the isoperimetric and Plateau's problems. If we switch the problem to find the regularity that we can extract of some minimizing set, we come across complicated ideas and tools. Although, the Plateau's Problem and other classical problems are well settled. Because of that, we have extensively studied the almost minimizing condition ((λ, r)-minimizing sets) considered by Maggi ([Mag12]) which subsumes some classical problems. We focused on the regularity theory extracted from this almost minimizing condition.
Keywords: Caccioppoli, almost minimizing, minimizing, geometric measure theory, regularity theory, locally finite perimeter, finite perimeter.

Contents

Symbols List

$E^c = \mathbb{R}^n / E$
$int(E)$ is the topological interior of E
$\bar{E}$ is the topological closure of E
$\partial E = \overline{E} \setminus int(E)$
$E_t = \{z \in \mathbb{R}^{n-1} : (z,t) \in E\}$
$E \subset \mathbb{R}^n$ then $1_E(x) = \begin{cases} 1, x \in E \\ 0, x \in E^c \end{cases}$
$G(u)$ denotes the graph of u
$Lip\,(u)$ is the Lipschitz constant of u
$\operatorname{supp} f = \overline{\{x \in dom(f) : f(x) \neq 0\}}$
For $U \subset \mathbb{R}^n$ open set $C^k(U, \mathbb{R}^m) =$
$= \{f : U \to \mathbb{R}^m : \textit{the i-th derivate } f^{(i)} \textit{ is continuous whenever } \ 0 \leq i \leq k\}$
For $U \subset \mathbb{R}^n$ open set $C^k_c(U, \mathbb{R}^m) = \{f \in C^k(U, \mathbb{R}^m) : \operatorname{supp} f \Subset U\}$
Jf is the Jacobian of f
$f, g : X \to Y$ then $\{x \in X : f(x) = g(x)\} = \{f = g\}$
$\mathbf{B}(x,r) = \{y \in \mathbb{R}^n : |x - y| < r\}$
$\mathbb{D}_r = \{z \in \mathbb{R}^{n-1} : |z| \leq r\}$
$\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is defined as $\mathbf{p}(x_1, ..., x_n) = (x_1, ..., x_{n-1})$
$\mathbf{q} : \mathbb{R}^n \to \mathbb{R}$ is defined as $\mathbf{q}(x_1, ..., x_n) = x_n$
$u \cdot v = \sum_{i=1}^n u_n v_n$ where $u = (u_1, ..., u_n)\,, v = (v_1, ..., v_n)$
$u, v, w \in \mathbb{R}^n$ then $(u \otimes v)w = (v \cdot w)u$
$\|z\|_\nu \doteq \max\{|proj_\nu z|, |proj_{\nu^\perp} z|\}$ where $proj$ is the euclidean orthogonal projection
$\mathbf{C}\,(x, r, \nu) = \{y \in \mathbb{R}^n : \|y - x\|_\nu < r\}$
μ a measure, then $E \overset{\mu}{\sim} F \Leftrightarrow \mu\,(E \Delta F) = 0$
μ a measure on $\mathbb{R}^n$, then $\mu \llcorner_A$ denotes the restriction to A
$spt\mu = \{x \in \mathbb{R}^n : \forall r > 0, \mu\,(\mathbf{B}(x,r)) > 0\}$ is the support of the measure μ
$\phi \sharp \mu$ is the push-forward of μ by ϕ
$|\mu|$ is the total variation of the measure μ
$|A|$ is the Lebesgue measure of A
$\omega_n = |\mathbf{B}(0,1)\,|$ is the volume of the ball $\mathbf{B}(0,1)$

Introduction

Geometric measure theory (GMT) has roots going back to ancient Greek mathematics. Ancient Greek mathematicians worked on the *isoperimetric problem* (to find the planar domain of given perimeter having greatest area) which leads naturally to questions about spatial regions and boundaries. The so-called *Plateau's problem* (to find the minimal surface with prescribed boundary), named in honor of the Belgian physicist *Joseph Plateau*, also belongs to the roots of GMT and has been studied since the 18th century. Despite its elderly roots, GMT has evolved, in the 20th century, into a modern theory which is in a confluence zone between Geometry and Mathematical Analysis. The name "geometric measure theory" owes its origins to Federer's masterpiece published in 1969 ([Fed96]).

A basic idea in GMT is to generalize the classic differential-geometric notion of *surface* in order to enlarge the set of possible solutions to Plateau's and similar variational problems[1]. According to the strategy used to generalize the notion of surface, the development of the theory has taken different paths, among which we may identify three main branches:

De Giorgi: The theory of Caccioppoli sets (or locally finite perimeter sets), named after the Italian mathematician Renato Caccioppoli, who introduced those sets in [Cac27]. In this formalism, a *hypersurface* is defined as the boundary of a Caccioppoli set. De Giorgi made important contributions to this theory, which can be found in [DG54], [DG55], [DG61a] and [DG61b]. We also refer the reader to H. Federer's book ([Fed96]) and more recent references, such as Lawrence C. Evans and Ronald F. Gariepy ([LCE92]), Leon Simon ([Sim83]), Luigi Ambrosio, Nicola Fusco and Diego Pallara ([LA00]), Fanghua Lin and Xiaoping Yang ([FL03]), Enrico Giusti ([Giu84]), Francesco Maggi ([Mag12]) and Frank Morgan ([Mor00]).

Federer and Fleming: The theory of *normal and integral currents* ([FF60], [Fed96]). Federer and Fleming defined a k-dimensional surface in $\mathbb{R}^n$ as a k-*current*, i.e. a continuous linear functional on the space of k-forms endowed with an inductive limit topology of Fréchet spaces. Those objects generalize the theory of Schwartz's distributions. This method is extensively studied in [Fed96] and it generalizes the theory of finite perimeter sets (which may be identified

[1]making it possible to apply the so-called *direct method of the Calculus of Variations*.

with a particular type of *normal currents*) to codimensions bigger than 1.

Almgren and Allard: The theory of *varifolds*, which was introduced by Almgren ([ALM65]) and Allard ([All72], [All87]). They defined a k-dimensional *varifold* as a Radon measure on $\mathbb{R}^n \times Gr(n,k)$ where $Gr(n,k)$ denotes the Grassmann manifold of the k-hyperplanes in $\mathbb{R}^n$. They have also shown that the *varifolds* are a more general concept than that of *currents*.

Each one of the above methods is adequate to study variational problems such as the classical *Plateau's problem* and the *isoperimetric problem*. In this work, we will focus on the theory of Caccioppoli sets. In this framework, one can easily show the existence of minimal solutions to the aforementioned problems by means of the *direct method* of the *calculus of variations*. The study of the regularity of such solutions, however, is really tough.

There exist certain minimizers in geometric variational problems with volume-constraints, potential-type energies and the like, that do not satisfy the usual minimality condition, i.e. without constraints. In order to encompass such problems into the framework, it arises the interest to introduce a notion of *almost minimality*, i.e. a relaxed notion of minimality. There are many ways to introduce such a relaxed notion; one of them was originally presented on a highly general context by Almgren in 1974 ([Alm76]) aiming to solve elliptic variational problems with constraints. One year after Almgren's work, Massari introduced another notion of minimality with constraints in [Mas75]. Indeed, if we set $\mathcal{P}(E,\cdot)$ the perimeter measure of the Cacciopoli set E and we define the functional T_g in the class of Caccioppoli sets depending on $g \in L^p(\Omega), p > n, \Omega \subset \mathbb{R}^n$ by

$$\mathbf{T}_g(E) = \mathcal{P}(E,K) + \int_{K\cap E} g(x)\,\mathrm{d}x \tag{0.1}$$

where K is a compact subset of Ω, then F will be a minimizer in Massari's concept ([Mas75]) if

$$\mathbf{T}_g(F) \leq \mathbf{T}_g(E)$$

for all K compact subset of Ω and all Caccioppoli set E such that $E\Delta F \subset K$. Massari's minimizers also satisfies

$$\mathcal{P}(F,K) \leq \mathcal{P}(E,K) + \|g\|_p |E\Delta F|^{\frac{p-1}{p}}$$

Another almost minimality condition was introduced by Tamanini in 1984 ([Tam84]). Tamanini defined an almost minimizing set in $\Omega \subset \mathbb{R}^n$ as a Caccioppoli set F such that for all $A \Subset \Omega$ exist $t \in (0, dist(A,\partial\Omega)$ and $\alpha : (0,t) \to [0,\infty)$ non-decreasing function with $\lim_{r\to 0^+} \alpha(r) = 0$ which satisfy

$$\mathcal{P}(E, \mathbf{B}(x,r)) \leq \mathcal{P}(F, \mathbf{B}(x,r)) + \alpha(r) r^{n-1} \tag{0.2}$$

for every $x \in A, r \in (0,t)$ and E Caccioppoli set with $E\Delta F \Subset \mathbf{B}(x,r)$. In this work, we will adopt the concept of almost minimality proposed in

[Mag12]. Instead of consider either 0.1 or 0.2, Maggi introduces a (Λ, r)-*minimality condition*, which we define rigorously in Definition 1.1.

The regularity problems which will be the main goal of this work are widely spread in the literature ([Giu84], [Mag12], [Alm76], [Mas75], [Sim83], [LA00], [Fed96], [Tam84], [DG61b]), with some minor variations on the definition of almost minimality adopted. In fact, regularity theorems can be proved under weaker minimality conditions than the (Λ, r)-minimality. For instance, Massari ([Mas75]) asserted that the minimizer of the functional defined in 0.1 has reduced boundary $\Omega \cap \partial^* F$ of class $C^{1,\frac{p-n}{4p}}$ with the restriction on the dimension $p > n \geq 8$. On the other hand, Tamanini ([Tam84]) established that the minimizer of 0.2 has reduced boundary $\Omega \cap \partial^* F$ of class C^1 under some forced conditions on α.

We aim to exploit the regularity that can be extracted from the class of minimizers defined in ([Mag12]). For this purpose, the notion of regularity will come up in many ways. The main tool that we will discuss is the *excess* which is used to measure the oscillation of the measure-theoretic outer unit normal of a Caccioppoli set. The precise definition of the *excess* can be found in Definition 2.1. The smallness of the *excess* allows us to collect technical results about the Caccioppoli set as the *Height Bound* (Theorem 2.17) and the *Lipschitz Approximation* (Theorem 3.3), which are the main tools to prove the regularity theorem that we will presented in this work. That is, the $C^{1,\gamma}$*-regularity of the almost minimizing sets Theorem.*

Prologue

The propose of this introductory chapter is to fix some notation and terminology. Furthermore, we will also state some well known results on geometric measure theory which will be used in the deployment of the almost minimizing sets theory. For more basic notation, one can check the Symbols List (see page v).

0.1 Functions of bounded variation

Let (U, Σ, μ) be a measurable space, then we define the space of *locally p-integrable* functions $L^p_{loc}(U, \mu), 1 \leq p < \infty$ as follows, if $f : U \to \mathbb{R}$ is measurable,

$$f \in L^p_{loc}(U, \mu) \Leftrightarrow \left(\int_K |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} < \infty, \quad \forall K \Subset U$$

If the right side on the inequality holds for U, i.e.

$$\left(\int_U |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} < \infty$$

we write $f \in L^p(U, \mu)$ to denote the space of such functions. The space $L^p_{loc}(U, \mu), 1 \leq p < \infty$ with the following norm, $f \in L^p_{loc}(U, \mu)$,

$$\|f\|_p \doteq \left(\int_U |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

is a Banach Space. For the sake of brevity, we set $L^p_{loc}(U, |\cdot|) = L^p_{loc}(U)$ and $L^p(U, |\cdot|) = L^p(U)$ for any $U \subset \mathbb{R}^n$ where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^n$.
We call the functions in $L^1_{loc}(U)$ (respectively, $L^1(U)$) by *locally integrable*

functions (resp., *integrable* functions). We will use the notation, for all $A, B \subset U$,

$$A \to B$$

to describe that $1_A \to 1_B$ in $L^1(U)$.
If $U \subset \mathbb{R}^n$ denotes an open set, we say that $f \in L^1_{loc}(U)$ has *locally bounded variation* in U, if

$$\sup \left\{ \int_K f(x) div\phi(x)\, \mathrm{d}x : \phi \in C^1_c(K, \mathbb{R}^n), |\phi| \leq 1 \right\} < \infty$$

for all $K \Subset U$. The space of the *locally bounded variation* functions is denoted by $BV_{loc}(U)$. We also establish the notation $BV(U)$ to denote the space of *bounded variation* functions, i.e. $f \in L^1(U)$ such that

$$\sup \left\{ \int_U f(x) div\phi(x)\, \mathrm{d}x : \phi \in C^1_c(U, \mathbb{R}^n), |\phi| \leq 1 \right\} < \infty$$

We recall the Structure Theorem for BV_{loc} (Theorem 5.1 in [LCE92]) which states, for each $f \in BV_{loc}(U)$, the existence of a Radon measure μ_f on U and a μ_f-measurable function $\nu_f : U \to \mathbb{R}^n$ such that $|\nu_f(x)| = 1$ μ_f-almost everywhere $x \in U$ and

$$\int_U f(x) div\phi(x)\, \mathrm{d}x = -\int_{\mathbb{R}^n} \phi \cdot \nu_f \, \mathrm{d}\mu_f \quad \forall \phi \in C^1_c(U, \mathbb{R}^n)$$

For *bounded variation* functions, we have the Lower Semicontinuity Theorem (Theorem 5.2 in [LCE92]) which establish that, if $\{f_i\}_{i \in \mathbb{N}} \subset BV(U)$ and $f_i \to f$ in $L^1_{loc}(U)$,

$$\mu_f(U) \leq \liminf_{i \to \infty} \mu_{f_i}(U)$$

If we consider the norm, $f \in BV(U)$,

$$\|f\|_{BV(U)} = \|f\|_1 + \mu_f(U)$$

we find that $BV(U) \cap C^\infty(U)$ is a dense subspace of $BV(U)$. If we now consider U open, bounded and with Lipschitz boundary, we can state a sort of compactness, i.e. if $\{f_i\} \subset BV(U)$ with

$$\sup_i \|f_i\|_{BV(U)} < \infty$$

we can take a subsequence $\{f_{i_k}\}$ and $f \in BV(U)$ such that $f_{i_k} \to f$ in $L^1(U)$ (Theorem 5.5 in [LCE92]).

0.2 Caccioppoli sets

If $E, U \subset \mathbb{R}^n$ with U an open set and $1_E \in BV_{loc}(U)$ (resp., $BV(U)$), we say that E is a set of *locally finite perimeter* (resp., *finite perimeter*) in U. Hereafter, we set

$$\nu_{1_E} = \nu_E$$

$$\mu_{1_E}(\cdot) = \mathcal{P}(E, \cdot)$$

$$\mu_E = \nu_{1_E}\mu_{1_E}$$

where ν_{1_E}, μ_{1_E} are given by the Structure Theorem for $BV_{loc}(U)$ which we mentioned before. We notice that μ_E is a $\mathbb{R}^n$-valued Radon measure on U. The sets of *locally finite perimeter* are also called *Caccioppoli* sets in many books of Geometric Measure Theory. Accordingly, we will use this terminology. We call by *perimeter measure* the Radon measure $\mathcal{P}(E, \cdot)$ on U and it holds that

$$\mathcal{P}(E, \cdot) = |\mu_E|(\cdot)$$

i.e. the *perimeter measure* is the total variation of the measure μ_E. We notice that

$$\mu_E = D1_E$$

where we denoted by $D1_E$ the distributional derivative of 1_E. Now, we shall prove a result that is simple calculation, but we have not seen its proof in the literature.

Proposition 0.1. *Let E be a Caccioppoli set in $\mathbb{R}^n$. If F is equivalent to E, i.e. $|E\Delta F| = 0$, it follows that*
(i) F is a Caccioppoli set;
(ii) $\mu_F = \mu_E$

Proof. *(i)* By the definition of Caccioppoli set, we have

$$\sup\left\{\int_E divT(x)\,\mathrm{d}x : T \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \operatorname{spt} T \subset K, \sup_{\mathbb{R}^n} |T| \leq 1\right\} < \infty$$

for every $K \subset \mathbb{R}^n$ compact set. Since F and E are equivalent, we find

$$\int_E divT(x)\,\mathrm{d}x = \int_F divT(x)\,\mathrm{d}x$$

Then, for every $K \subset \mathbb{R}^n$ compact set, we conclude that

$$\sup\left\{\int_F divT(x)\,\mathrm{d}x : T \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \operatorname{spt} T \subset K, \sup_{\mathbb{R}^n} |T| \leq 1\right\} < \infty$$

Therefore, F is a Caccioppoli set.
(ii) Since 1_F and 1_E are equal as elements of L^1_{loc}, we can ensure that the distributional derivatives $D1_E$ and $D1_F$ will also be equal and thus $\mu_E = \mu_F$. □

The converse of the last proposition is also true. However, we will not prove it here. The last proposition provides a type of invariance of μ_E and the *perimeter measure* under modifications on sets Lebesgue-null sets. Since for any *Caccioppoli* set E in $\mathbb{R}^n$ we have

$$\operatorname{spt}\mu_E = \{x \in \mathbb{R}^n : 0 < |E \cap \mathbf{B}(x,r)\,| < \omega_n r^n, \quad \forall r > 0\} \subset \partial E$$

we can show the existence of $F \overset{Lesbegue}{\sim} E$ such that

$$\operatorname{spt}\mu_F = \partial F$$

and then $\mu_F = \mu_E$. If E and F are *Caccioppoli* sets in $\mathbb{R}^n$ (resp., *finite perimeter*), we have that $E \cap F$ and $E \cup F$ are also *Caccioppoli* sets in $\mathbb{R}^n$ (resp., *finite perimeter*) and it holds that

$$\mathcal{P}(E \cup F, \cdot) + \mathcal{P}(E \cap F, \cdot) \leq \mathcal{P}(E, \cdot) + \mathcal{P}(F, \cdot)$$

The *isoperimetric inequality* in the euclidean space $\mathbb{R}^{n \geq 2}$, i.e.

$$n\omega^{\frac{1}{n}}|E|^{\frac{n-1}{n}} \leq \mathcal{P}(E, \mathbb{R}^n) \quad \forall E \subset \mathbb{R}^n \; with \; |E| < \infty$$

is an indispensable tool in geometric measure theory. It can be also stated in balls (Proposition 12.37 in [Mag12]) for any *Caccioppoli* set E in $\mathbb{R}^n$ such that

$$|E \cap \mathbf{B}(x,r)\,| \leq t|\mathbf{B}(x,r)\,|$$

whenever $n \geq 2, x \in \mathbb{R}^n, r > 0, t \in (0,1)$. Indeed, under these assumptions on E, exists $c(n,t)$ depending only on n and t such that

$$c(n,t)|E \cap \mathbf{B}(x,r)\,|^{\frac{n-1}{n}} \leq \mathcal{P}(E, \mathbf{B}(x,r))$$

0.3 Hausdorff measure

We now set *diam* to be the *diameter* function on $\mathbb{R}^n$. For $k \in [0, \infty)$, the k-dimensional Hausdorff measure of $\Omega \subset \mathbb{R}^n$ is defined as follows

$$\mathcal{H}_k(\Omega) = \lim_{\delta \to 0} \mathcal{H}_k^\delta(\Omega)$$

where, for $\delta \in (0, \infty]$,

$$\mathcal{H}_k^\delta(\Omega) = \frac{\omega_k}{2^k} \inf\left\{ \sum_{i \in \mathbb{N}} (diam(\Omega_i))^k : diam(\Omega_i) < \delta, \Omega \subset \bigcup_{i \in \mathbb{N}} \Omega_i \right\}$$

The k-dimensional Hausdorff measure has some good properties, for instance, the behavior under translation and homotheties. Indeed, $\forall x \in \mathbb{R}^n, \lambda > 0$, we have

$$\mathcal{H}_k(\lambda\Omega + x) = \lambda^k \mathcal{H}_k(\Omega) \quad \textit{for any } \Omega \subset \mathbb{R}^n$$

Since the k-dimensional Hausdorff measure is defined in function of the *diameter*, we can also ensure that

$$\mathcal{H}_k(f(\Omega)) \leq (Lip\,(f))^k \mathcal{H}_k(\Omega)$$

for all Lipschitz function f and $\Omega \subset \mathbb{R}^n$. One of the main results regarding the Hausdorff measure is that

$$\mathcal{H}_n(\Omega) = |\Omega| \quad \forall \Omega \subset \mathbb{R}^n \textit{ Borel set}$$

To finish this section, let us state the Coarea and Area formulas. If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ are Lipschitz functions, $n \geq m$, for each $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$ Lebesgue measurable sets, we have

$$\textbf{(Coarea Formula)} \quad \int_A Jf(x)\,\mathrm{d}x = \int_{\mathbb{R}^m} \mathcal{H}_{n-m}(A \cap f^{-1}(y))\,\mathrm{d}y$$

$$\textbf{(Area Formula)} \quad \int_B Jg(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \mathcal{H}_0(B \cap g^{-1}(x))\,\mathrm{d}x$$

0.4 The reduced boundary

The *reduced boundary* $\partial^* E$ of a Caccioppoli set E is the set of points $x \in \mathbb{R}^n$ such that $x \in \operatorname{spt} \mu_E$ and

$$\nu_E(x) = \lim_{r \to 0^+} \frac{\mu_E(\mathbf{B}(x,r))}{\mathcal{P}(E, \mathbf{B}(x,r))} \in \mathbb{S}^{n-1}$$

The vector ν_E is called the *measure-theoretic outer unit normal to* E. We use 0.1 and the definition of *reduced boundary* to be able to consider sets with

$$\overline{\partial^* E} = \operatorname{spt} \mu_E = \partial E$$

One of the main theorems in geometric measure theory is the De Giorgi's Structure Theorem (Theorem 4.4 in [Giu84]) which asserts, for a Caccioppoli set E, that

$$\mu_E = \nu_E\, \mathcal{H}_{n-1} \llcorner_{\partial^* E}$$

and

$$\mathcal{P}(E,\cdot)=|\mu_E|(\cdot)=\mathcal{H}_{n-1}\left(\partial^* E\cap\cdot\right)$$

We define the points of density $t\in[0,1]$ of a set $E\subset\mathbb{R}^n$ as follows

$$E^{(t)}=\left\{x\in\mathbb{R}^n:\lim_{t\to0^+}\frac{|E\cap\mathbf{B}(x,r)\,|}{\omega_n r^n}=t\right\}$$

Now, we recall the definition of the *measure theoretic boundary* of a set $E\subset\mathbb{R}^n$, i.e.

$$\partial^e E=\mathbb{R}^n\setminus\left(E^{(1)}\cup E^{(0)}\right)$$

For any *Caccioppoli* set $E\subset\mathbb{R}^n$, the well known *Federer's Theorem* states that

$$\partial^* E\subset E^{(1/2)}\subset\partial^e E$$

Moreover

$$\mathcal{H}_{n-1}\left(\partial^e E\setminus\partial^* E\right)=0$$

which can be found on Section 5.8 in [LCE92].

The almost minimizing sets

1.1 Definition of the almost minimizing condition

Let us precisely state the almost minimizing condition that we will consider in this work.

Definition 1.1. *Let $A \subset \mathbb{R}^n$ be open and E is a Caccioppoli set (or locally finite perimeter set) in $\mathbb{R}^n$. Then E is called a $(\Lambda, r)-minimizing$ in A if* $\operatorname{spt} \mu_E = \partial E, \Lambda \geq 0, r > 0$ *and for all Caccioppoli set F such that $E \Delta F \Subset A \cap \boldsymbol{B}(x, s)$, $s < r, x \in A$ it holds*

$$\mathcal{P}(E, \boldsymbol{B}(x, s)) \leq \mathcal{P}(F, \boldsymbol{B}(x, s)) + \Lambda |E \Delta F| \tag{1.1}$$

r is also called the scale and F is called competitor.

Note that the condition $\operatorname{spt} \mu_E = \partial E$ is not restrictive, since we can choose an $\mathcal{H}_n$-equivalent set with this property (by Proposition 12.19 in [Mag12]).
In the definition it holds $|E \Delta F| \leq \omega_n s^n$, i.e. the "error" factor is always bounded above by the volume of the balls where the competitors are different from E and it decreases faster than the scale. Recalling the De Giorgi's structure theorem (Theorem 4.4 in [Giu84]) which states $\mathcal{P}(E, \cdot) = \mathcal{H}_{n-1}(\partial^* E \cap \cdot)$, it makes sense to keep in mind that the "error" factor behaves like a higher order perturbation.
Before the examples, let us explain why it will henceforth be considered the hypothesis $\Lambda r \leq 1$ in this work. For this purpose, suppose that E is a $(\Lambda, r) - minimizing$ set in A and F is a competitor, it follows from the euclidean isoperimetric inequality that

$$|E \Delta F| = |E \Delta F|^{\frac{1}{n}} |E \Delta F|^{\frac{n-1}{n}} \leq (\omega_n s^n)^{\frac{1}{n}} \frac{\mathcal{P}(E \Delta F, \mathbb{R}^n)}{n \omega_n^{\frac{1}{n}}} =^*$$

$$=^* \frac{s}{n} \mathcal{P}(E \Delta F, \mathbf{B}(x, s)) \leq^{**} \frac{s}{n} \Big(\mathcal{P}(F, \mathbf{B}(x, s)) + \mathcal{P}(E, \mathbf{B}(x, s)) \Big)$$

where $(*)$ follows from $E \Delta F \Subset \mathbf{B}(x, s)$ and $(**)$ is consequence of the perimeter property (E_1, E_2 being Caccioppoli sets, then $\mathcal{P}(E_1 \cup E_2, \cdot) \leq$

$\mathcal{P}(E_1,\cdot)+\mathcal{P}(E_2,\cdot))$, $\mathcal{P}(E,\cdot) = \mathcal{H}_{n-1}(\partial^* E\cap\cdot) = \mathcal{H}_{n-1}(\partial^e E\cap\cdot)$ and $\partial^e(E\Delta F)\subset\partial^e E\cup\partial^e F$. Therefore

$$\mathcal{P}(E,\mathbf{B}(x,s))\leq\mathcal{P}(F,\mathbf{B}(x,s))+\Lambda|E\Delta F|\leq$$

$$\leq\mathcal{P}(F,\mathbf{B}(x,s))+\Lambda\frac{s}{n}\Big(\mathcal{P}(E,\mathbf{B}(x,s))+\mathcal{P}(F,\mathbf{B}(x,s))\Big)$$

It follows

$$\left(1-\frac{\Lambda s}{n}\right)\mathcal{P}(E,\mathbf{B}(x,s))\leq\left(1+\frac{\Lambda s}{n}\right)\mathcal{P}(F,\mathbf{B}(x,s)) \qquad (1.2)$$

which is clearly trivial if $n\leq\Lambda s$, i.e. when the scale "s" é too large. In the non-trivial case, $\Lambda s<n$, the scale s is bounded, in general, we will fix "1" as the upper bound for Λs intending to help with some proofs ahead.

1.2 Basic properties of the almost minimizing sets

Proposition 1.2. *Let E be a $(\Lambda,r)-$ minimizing set in A, then E^c is a $(\Lambda,r)-$ minimizing set in A.*

Proof. Since $1_E+1_{E^c}\equiv 1$, E^c is also a Caccioppoli set and the distributional derivatives satisfies

$$D1_E=-D1_{E^c}\Rightarrow\mu_E=-\mu_{E^c}$$

Thus

$$\operatorname{spt}\mu_{E^c}=\operatorname{spt}\mu_E=\partial E=\partial(E^c)$$

Now, we take F a competitor to E^c, i.e. $E^c\Delta F\Subset A\cap\mathbf{B}(x,s)\,,s<r,x\in A$, by the definition of the perimeter measure of E, i.e. $\mathcal{P}(E,\cdot)=|D1_E|(\cdot)$, we find that

$$\mathcal{P}(E^c,\cdot)=\mathcal{P}(E,\cdot) \qquad (1.3)$$

It is straightforward to verify that

$$E^c\Delta F=E\Delta F^c \qquad (1.4)$$

what ensures that F^c is a competitor to the almost minimality of E.

Therefore, we have that

$$\mathcal{P}(E^c, \mathbf{B}(x,s)) = \mathcal{P}(E, \mathbf{B}(x,s)) \leq \mathcal{P}(F^c, \mathbf{B}(x,s)) + \Lambda|E\Delta F^c|$$

Since 1.3 is valid for F in place of E, the last inequality becomes

$$\mathcal{P}(E^c, \mathbf{B}(x,s)) \leq \mathcal{P}(F, \mathbf{B}(x,s)) + \Lambda|E^c\Delta F|$$

□

Let us define the blow-up of a arbitrary set $X \subset \mathbb{R}^n$ at x at scale $r > 0$ as follows

$$E_{x,r} = \frac{E - x}{r}$$

Now, we aim to prove the compatibility of the blow-up with the almost minimality condition.

Proposition 1.3. *Given $y \in \mathbb{R}^n, r_0 > 0$. If E is a $(\Lambda, r) -$ minimizing set in A, we have that E_{y,r_0} is a $\left(\Lambda r, \frac{r}{r_0}\right) -$ minimizing set in A_{y,r_0} .*

Remark 1.4. *If we consider $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear isometry, we have that (Proposition 2.51 in [VOL])*

$$\mu_{T(E)} = T\sharp\mu_E \quad |\mu_{T(E)}| = T\sharp|\mu_E|$$

We call attention to the argument of this proof which can be easily adjusted to ensures that, if E is a $(\Lambda, r) -$ minimizing set in A, then $T(E)$ is a $(\Lambda, r) -$ minimizing set in $T(A)$.

Proof. Take F a Caccioppoli set with $E_{y,r_0}\Delta F \subset A_{y,r_0} \cap \mathbf{B}(x,s), x \in A_{y,r_0}, s < \frac{r}{r_0}$. We set

$$\phi(z) = r_0 z + y \quad \forall z \in \mathbb{R}^n$$

Clearly, ϕ is a diffeomorphism,

$$\phi\left(\mathbf{B}(x,s)\right) = r_0\mathbf{B}(x,s) + y \tag{1.5}$$

and

$$\phi(E_{y,r_0}) = r_0 E_{y,r_0} + y = E \tag{1.6}$$

Analogously, we prove

$$\phi(A_{y,r_0}) = A \tag{1.7}$$

Taking into account $E_{y,r_0}\Delta F \subset A_{y,r_0} \cap \mathbf{B}(x,s)$, we find that

$$\phi(E_{y,r_0})\Delta\phi(F) \Subset \phi(A_{y,r_0}) \cap \phi(\mathbf{B}(x,s))$$

Then, by 1.5, 1.6 and 1.7,

$$E\Delta\phi(F) \Subset A \cap (r_0\mathbf{B}(x,s) + y) = A \cap \mathbf{B}(\phi(x), r_0 s)$$

Since $x \in A_{y,r_0}$ and $s < \frac{r}{r_0}$, we have $\phi(x) \in A$ and $r_0 s < r$. Therefore,

$\phi(F)$ is a competitor to E, then

$$\mathcal{P}(E,\phi(\mathbf{B}(x,s))) \leq \mathcal{P}(\phi(F),\phi(\mathbf{B}(x,s))) + \Lambda|E\Delta\phi(F)| \tag{1.8}$$

We intend to turn 1.8 into 1.1 for E_{y,r_0}. To this end, since ϕ is a diffeomorphism and $|J\phi| \equiv r_0^n$, we ensure that

$$\begin{aligned} |E\Delta\phi(F)| &\overset{1.6}{=} |\phi(E_{y,r_0}\Delta F)| = \\ &= r_0^n|E_{y,r_0}\Delta F| \end{aligned} \tag{1.9}$$

Now, we focus on the term $\mathcal{P}(E, r_0\mathbf{B}(x,s)+y)$. Let us recall that

$$\frac{\partial^* E - y}{r_0} = \partial^*\left(\frac{E-y}{r_0}\right) = \partial^*(E_{y,r_0})$$

By the nice properties of the Hausdorff measure under translations and hometheties (Proposition 2.49 in [LA00]), the De Giorgi's Structure Theorem and the last inequality, we obtain that

$$\begin{aligned} \mathcal{P}(E,\phi(\mathbf{B}(x,s))) \overset{1.5}{=} \mathcal{P}(E, r_0\mathbf{B}(x,s)+y) = \\ \mathcal{H}_{n-1}\left(\partial^* E\cap(r_0\mathbf{B}(x,s)+y)\right) = r_0^{n-1}\,\mathcal{H}_{n-1}\left((\frac{\partial^*E-y}{r_0})\cap\mathbf{B}(x,s)\right) = \\ r_0^{n-1}\,\mathcal{H}_{n-1}\left(\partial^*(E_{y,r_0})\cap\mathbf{B}(x,s)\right) = r_0^{n-1}\,\mathcal{P}(E_{y,r_0},\mathbf{B}(x,s)) \end{aligned} \tag{1.10}$$

We can apply the same argument to find that

$$\mathcal{P}(\phi(F),\phi(\mathbf{B}(x,s))) = r_0^{n-1}\,\mathcal{P}(F,\mathbf{B}(x,s))$$

By the last equality, 1.10 and 1.9, we can turn 1.8 into

$$r_0^{n-1}\,\mathcal{P}(E_{y,r_0},\mathbf{B}(x,s)) \leq r_0^{n-1}\,\mathcal{P}(F,\mathbf{B}(x,s)) + r_0^n|E_{y,r_0}\Delta F|$$

that is

$$\mathcal{P}(E_{y,r_0},\mathbf{B}(x,s)) \leq \mathcal{P}(F,\mathbf{B}(x,s)) + \Lambda r_0|E_{y,r_0}\Delta F|$$

what conclude the proof. $\square$

1.3 Density estimates

Now, we will start to progress with the theory and some meaningful theorems on the theory of the almost minimizing sets. One of the main theorems is the Density Estimates which makes it possible to prove both the compactness and closure theorem and other further results as well. The density estimates, by their own, put us in position to extract some geometric information about the almost minimizing sets. Indeed, the estimates show that the almost minimizing sets, and its boundary, have a quite good behavior whereas it is not obvious.

Theorem 1.5. ***(Density estimates)*** *Let $n \geq 2$, exists $c(n)$ positive constant such that for all set E which is a $(\Lambda, r_0)-minimizing$ in A with $\Lambda r_0 \leq 1$, it follows*

$$\frac{1}{4^n} \leq \frac{|E \cap \boldsymbol{B}(x,r)|}{\omega_n r^n} \leq 1 - \frac{1}{4^n} \tag{1.11}$$

$$c(n) \leq \frac{\mathcal{P}(E, \boldsymbol{B}(x,r))}{r^{n-1}} \leq 3n\omega_n \tag{1.12}$$

whenever $x \in A \cap \partial E$, $\boldsymbol{B}(x,r) \subset A$ and $r < r_0$.

Remark 1.6. *The upper inequality in 1.12 is true for all $x \in A$, i.e. the restriction $x \in \partial E$ is not necessary. we intend to make this clear within the proof.*

Proof. The upper estimate in 1.11 is a consequence of the lower estimate in 1.11, because E^c also is $(\Lambda, r_0) - minimizing$ in A (by Proposition 1.2), thus

$$\frac{1}{4^n} \leq \frac{|E^c \cap \mathbf{B}(x,r)|}{\omega_n r^n} = 1 - \frac{|E \cap \mathbf{B}(x,r)|}{\omega_n r^n}$$

From the estimates in 1.11 and the relative isoperimetric inequality (Proposition 12.37 in [Mag12] putting $t = 1 - \frac{1}{4^n}$) that is

$$\mathcal{P}(E, \mathbf{B}(x,r)) \geq c_1(n) |E \cap \mathbf{B}(x,r)|^{\frac{n-1}{n}}$$

we can prove the lower estimate in 1.12 as follow

$$\frac{\mathcal{P}(E, \mathbf{B}(x,r))}{r^{n-1}} \geq \frac{c_1(n) |E \cap \mathbf{B}(x,r)|^{\frac{n-1}{n}}}{r^{n-1}} \geq \frac{c_1(n)}{r^{n-1}} \left(\frac{\omega_n r^n}{4^n}\right)^{\frac{n-1}{n}} =$$

$$= \frac{c_1(n)\, \omega_n^{\frac{n-1}{n}}}{4^{n-1}} \doteq c(n)$$

Therefore, it is sufficient to proof the other two inequalities. For this purpose, fix $x \in A$, define $d = \min\{r_0, dist(x, \partial A)\}$ and $m(r) = |E \cap \mathbf{B}(x,r)|, \forall r \in (0,d)$, then

$$\begin{aligned} m(r) = |E \cap \mathbf{B}(x,r)| =^* \int_0^r \mathcal{H}_{n-1}(E \cap \partial \mathbf{B}(x,t))\, \mathrm{d}t \Rightarrow \\ \Rightarrow m'(r) = \mathcal{H}_{n-1}(E \cap \partial \mathbf{B}(x,r)) \end{aligned} \tag{1.13}$$

for almost all $r \in (0,d)$, where the equality $(*)$ is a consequence of the Coarea formula (Theorem 3.12 in [LCE92]). Since $\mu_E = \mathcal{H}_{n-1 \llcorner \partial^* E}$ is a

Radon measure, it follows that, for almost all $r \in (0, d)$,

$$\mathcal{H}_{n-1}\left(\partial^* E \cap \partial \mathbf{B}(x, r)\right) = 0$$

Being r taken satisfying this equality, we aim to create a competitor of E to use the (Λ, r_0)-minimality of E. Let $F = E \setminus \mathbf{B}(x, r)$. Clearly $E \Delta F \Subset \mathbf{B}(x, s) \Subset A, \forall s \in (r, d)$, thus, by 1.2,

$$\left(1 - \frac{\Lambda r}{n}\right) \mathcal{P}(E, \mathbf{B}(x, s)) \leq \left(1 + \frac{\Lambda r}{n}\right) \mathcal{P}(F, \mathbf{B}(x, s)) \tag{1.14}$$

Moreover, from the operations with Caccioppoli sets (Theorem 16.3 in [Mag12]) and from the choice of r, it follows

$$\mathcal{P}(F, \mathbf{B}(x, s)) = \mathcal{P}(E \cap (\mathbf{B}(x, r))^c, \mathbf{B}(x, s)) = \overbrace{\mathcal{P}(E, (\mathbf{B}(x, r)^c)^{(1)} \cap \mathbf{B}(x, s))}^{*} +$$

$$+ \overbrace{\mathcal{P}(\mathbf{B}(x, r), E^{(1)} \cap \mathbf{B}(x, s))}^{**} + \overbrace{\mathcal{H}_{n-1}\left(\left\{\mu_E = \mu_{\mathbf{B}(x,r)}\right\} \cap \mathbf{B}(x, s)\right)}^{=0}$$

Note that in $(*)$, we have $\mathbf{B}(x, s) \cap (\mathbf{B}(x, r)^c)^{(1)} \subset \mathbf{B}(x, s) \setminus \overline{\mathbf{B}(x, r)}$, and in $(**)$ apply the De Giorgi's structure theorem to come up with

$$\mathcal{P}(F, \mathbf{B}(x, s)) \leq \mathcal{H}_{n-1}\left(E^{(1)} \cap \partial \mathbf{B}(x, r)\right) + \mathcal{P}(E, \mathbf{B}(x, s) \setminus \overline{\mathbf{B}(x, r)}) \tag{1.15}$$

Taking $s \to r^+$, it follows from 1.14 and 1.15 that

$$\left(1 - \frac{\Lambda r}{n}\right) \mathcal{P}(E, \mathbf{B}(x, r)) \leq \left(1 + \frac{\Lambda r}{n}\right) \mathcal{H}_{n-1}\left(E^{(1)} \cap \partial \mathbf{B}(x, r)\right) \tag{1.16}$$

Using that $\mathcal{H}_{n-1}\left(E^{(1)} \cap \partial \mathbf{B}(x, r)\right) \leq \mathcal{H}_{n-1}\left(\partial \mathbf{B}(x, r)\right) = n\omega_n r^{n-1}$, and since $\frac{1+\frac{\Lambda r}{n}}{1-\frac{\Lambda r}{n}} \leq 3$, by $n \geq 2$ and $\Lambda r_0 \leq 1$, the upper inequality in 1.12 is done for almost all $r \in (0, d)$. So, given $r \in (0, d)$, we choose a increasing sequence $\{r_i\}_{i \in \mathbb{N}}$ such that $r_i \to r$ and the upper inequality in 1.12 holds true for $r_i, \forall i \in \mathbb{N}$. By the continuity from below of the measure $\mathcal{P}(E, \cdot)$ (Theorem 1.8 in [Fol99]) and the continuity of r^{n-1}, we obtain that

$$\frac{\mathcal{P}(E, \mathbf{B}(x, r))}{r^{n-1}} = \frac{\mathcal{P}(E, \cup_{i \in \mathbb{N}} \mathbf{B}(x, r_i))}{r^{n-1}} = \frac{\lim_{i \to \infty} \mathcal{P}(E, \mathbf{B}(x, r_i))}{\lim_{i \to \infty} r_i^{n-1}} =$$

$$= \lim_{i \to \infty} \frac{\mathcal{P}(E, \mathbf{B}(x, r_i))}{r_i^{n-1}} \leq 3n\omega_n$$

Then, the upper inequality in 1.12 is validated for all $r \in (0, d)$ and, along with this proof, we have only used that $x \in A$. Thus, the Remark 1.6 is also verified. Finally, we will prove the lower inequality in 1.11. To this end, suppose that $x \in A \cap \partial E$, then the function m also satisfies

$$0 < m(r) < \omega_n r^n, \forall r \in (0, d) \tag{1.17}$$

because $x \in \partial E = \operatorname{spt} \mu_E$. Adding $\left(1 - \frac{\Lambda r}{n}\right) \mathcal{H}_{n-1}\left(E^{(1)} \cap \partial \mathbf{B}(x, r)\right)$ to both sides of 1.16 and using the operations with Caccioppoli sets (Theorem 16.3 in [Mag12], we obtain

$$\left(1-\frac{\Lambda r}{n}\right)\mathcal{P}(E\cap\mathbf{B}(x,r)\,,\mathbb{R}^n)\leq 2\,\mathcal{H}_{n-1}\left(E^{(1)}\cap\partial\mathbf{B}(x,r)\right) \tag{1.18}$$

By the isoperimetric inequality $\mathcal{P}(E\cap\mathbf{B}(x,r)\,,\mathbb{R}^n)\geq n\omega_n^{\frac{1}{n}}|E\cap\mathbf{B}(x,r)\,|^{\frac{n-1}{n}}$, since $1-\frac{\Lambda r}{n}\leq 2$, from 1.13 and 1.17, it follows that

$$\frac{n\omega_n^{\frac{1}{n}}}{4}\leq\frac{m'(r)}{\left(m(r)\right)^{1-\frac{1}{n}}} \tag{1.19}$$

Integrating 1.19 in the interval $(0,r)$, we get that (by Theorem 4.14 in [Gor94]) $\displaystyle\int_0^r\frac{m'(t)}{\left(m(t)\right)^{1-\frac{1}{n}}}\,\mathrm{d}t=\int_0^r\frac{\mathrm{d}}{\mathrm{d}t}\left(nm(t)\right)^{\frac{1}{n}}\mathrm{d}t=nm(r)^{\frac{1}{n}}$, it follows

$$\frac{rn\omega_n^{\frac{1}{n}}}{4}\leq nm(r)^{\frac{1}{n}}=n|E\cap\mathbf{B}(x,r)\,|^{\frac{1}{n}}$$

Putting up the power of n the proof is done. □

We recall the definition of the *essential boundary* of a Caccioppoli set E, also called *measure theoretic boundary*, as follows

$$x\in\partial^e E\Leftrightarrow x\in\mathbb{R}^n\setminus\left(E^{(1)}\cup E^{(0)}\right)$$

The following corollary is a stronger version of the well known *Federer's Theorem* (Lemma 5.5 in [LCE92]) in the context of almost minimizing sets. The *Federer's Theorem* guarantees that $\partial^e E$ and $\partial^* E$ are $\mathcal{H}_{n-1}$-equivalent for any Caccioppoli set E. For the almost minimizing sets this result can be refined, ensuring the equivalence between the reduced boundary and the topological boundary. Here it is possible to conclude that the almost minimizing sets possess a kind of extra regularity, i.e. $\mathcal{H}_{n-1}$-almost everywhere it is possible to define a normal vector to ∂E.

Corollary 1.7. *Let $n\geq 2$ and E be a $(\Lambda,r_0)-$minimizing set in A with $\Lambda r_0\leq 1$. Then*

$$\mathcal{H}_{n-1}\left(A\cap\left(\partial E\setminus\partial^* E\right)\right)=0$$

Proof. It is sufficient to note that 1.11 ensures that $\forall x\in A\cap\partial E,\forall r<r_0$

$$\frac{1}{4^n}\leq\frac{|E\cap\mathbf{B}(x,r)\,|}{\omega_n r^n}\leq 1-\frac{1}{4^n}$$

Thus, $0<\displaystyle\limsup_{r\to 0^+}\frac{|E\cap\mathbf{B}(x,r)\,|}{\omega_n r^n}<1$. By definition of *essential boundary*, we get that $x\in A\cap\partial^e E$. Therefore, $A\cap\partial E\subset A\cap\partial^e E$. Then, since $\partial^* E\subset\partial E$, the *Federer's Theorem* concludes the proof. □

Corollary 1.8. *Let $n \geq 2$ and E be a $(\Lambda, r_0)-$minimizing set in A, with $\Lambda r_0 \leq 1$, then*

$$c(n) \leq \frac{\mathcal{P}(E, \boldsymbol{C}(x,r,\nu))}{r^{n-1}} \leq \sqrt{2}^{n-1} 3n\omega_n$$

whenever $x \in A \cap \partial E$, $\sqrt{2}r < r_0$, $\boldsymbol{B}\big(x, \sqrt{2}r\big) \subset A$.

Proof. It follows directly from 1.12, using that $\mathbf{B}(x,r) \subset \mathbf{C}(x,r,\nu) \subset \mathbf{B}\big(x, \sqrt{2}r\big)$. □

1.4 Compactness theorems

In the excess theory and the approximation theorems, some constructions with sequences of almost minimizing sets will naturally appear for us. Hence, we shall work on it. For this purpose, we will extract a kind of compactness for the space of almost minimizing sets. As a consequence of the compactness of the sets of finite perimeter and the density estimates, the first result will ensure that for a sequence of almost minimizing sets, under some assumptions, we can find a set of finite perimeter which will be the limit set of the sequence.

Theorem 1.9. ***(Pre-compactness of the space of the almost minimizing sets)*** *Let $n \geq 2$ and $(E_h)_{h\in\mathbb{N}}$ be a sequence such that each E_h is $(\Lambda_h, r_h)-$minimizing in A with $\Lambda_h r_h \leq 1$ and $\liminf_{h\to\infty} r_h > 0$. Then $\forall A_0 \Subset A$ open set with finite perimeter, exists a set with finite perimeter $E \subset A_0$ and a subsequence $(E_{h'})_{h'\in\mathbb{N}}$ such that*

$$A_0 \cap E_{h'} \to E \quad \textit{and} \quad \mu_{A_0 \cap E_{h'}} \overset{*}{\rightharpoonup} \mu_E \tag{1.20}$$

Proof. Given $A_0 \Subset A$ open with finite perimeter, fix $x \in A_0$ and consider $\mathbf{B}(x,r) \Subset A$ and $0 < r < \liminf_{h\in\mathbb{N}} r_h$ such that $\mathcal{H}_{n-1}(\partial^* E_h \cap \partial \mathbf{B}(x,r)) = 0$, that is possible, since $\mathcal{H}_{n-1} \llcorner \partial^* E_h$ is Radon. The operations with Caccioppoli sets (Theorem 16.3 in [Mag12]) ensures the following equality

$$\mathcal{P}(E_h \cap \mathbf{B}(x,r), \mathbb{R}^n) =$$

$$= \mathcal{P}(E_h, \mathbf{B}(x,r)^{(1)}) + \mathcal{P}(\mathbf{B}(x,r), E_h^{(1)}) + \mathcal{H}_{n-1}\left(\left\{\nu_{E_h} = \nu_{\mathbf{B}(x,r)}\right\}\right) \leq$$
$$\leq^* \mathcal{P}(E_h, \mathbf{B}(x,r)) + \mathcal{P}(\mathbf{B}(x,r), \mathbb{R}^n) \leq^{**} 3n\omega_n r^{n-1} + n\omega_n r^{n-1}$$

where $(*)$ follows from $\mathbf{B}(x,r)^{(1)} = \mathbf{B}(x,r)$, $E_h^{(1)} \subset \mathbb{R}^n$ and

$$\mathcal{H}_{n-1}\left(\left\{\nu_{E_h} = \nu_{\mathbf{B}(x,r)}\right\}\right) \leq \mathcal{H}_{n-1}(\partial^* E_h \cap \partial^* \mathbf{B}(x,r)) = 0$$

while, in order to apply the density estimates (1.12) in $(**)$ and Remark 1.6, we assume that h is sufficiently large, i.e. we fixed a $M \in \mathbb{N}$ such that $r < \inf_{h \geq M} r_h$. In short, if $x \in \partial E \cap A_0, r < \liminf_{h\to\infty} r_h$, it follows

$$\sup_{h \geq M} \mathcal{P}(E_h \cap \mathbf{B}(x,r)\,, \mathbb{R}^n) < 4n\omega_n r^{n-1} < \infty \tag{1.21}$$

Since A_0 has compact closure in A, we can choose a finite family

$$\{B_j = \mathbf{B}(x_j, r_j)\}_{j=1}^N$$

which covers A_0 with $B_j \Subset A, x_j \in A_0$ and $r_j < \liminf_{h\to\infty} r_h$. From 1.21 and $E_h \cap B_j \subset B_j$, we are able to use the theorem of compactness for finite perimeter sets (Theorem 12.26 in [Mag12]), for each j, then, applying for $j = 1$, exists $F_1 \subset B_1$ finite perimeter set and E_{h_1} subsequence such that

$$B_1 \cap E_{h_1} \to F_1$$

Now, applying for the new subsequence E_{h_1}, with $j = 2$, we got another subsequence. Using this idea successively until $j = N$, we have obtained the last subsequence $E_{h'} = E_{j_N}$ of the initial sequence and finite perimeter sets $\{F_j \subset B_j\}_{j=1}^N$ with

$$B_j \cap E_{h'} \to F_j \tag{1.22}$$

when $h' \to \infty$ for each $1 \leq j \leq N$. Finally, define $E = A_0 \cap \left(\bigcup_{j=1}^N F_j\right)$. Then, by construction

$$F_i \cap B_i \cap B_j \overset{Lebesgue}{\sim} F_j \cap B_i \cap B_j$$

and the family $\{B_j\}_{j=1}^N$ covers A_0. Since E is defined by unions and intersection of sets of finite perimeter, it has finite perimeter. It is straightforward to verify the formula for the characteristic function of a finite union. So, in order to prove that $A_0 \cap E_{h'} \to E$, we note that

$$1_{A_0 \cap E_{h'}} = 1_{A_0 \cap \left(\cup_{j=1}^N B_j \cap E_{h'}\right)} = 1_{A_0} 1_{\cup_{j=1}^N B_j \cap E_{h'}} =$$

$$= 1_{A_0}\left(1 - \sum_{K \subset \{1,\ldots,N\}} (-1)^{|K|}\, 1_{\cap_{j\in K} B_j \cap E_{h'}}\right) =$$

$$\to^{1.22} 1_{A_0}\left(1 - \sum_{K \subset \{1,\ldots,N\}} (-1)^{|K|}\, 1_{\cap_{j\in K} F_j}\right) =$$

$$= 1_{A_0} 1_{\cup_{j=1}^N F_j} = 1_E$$

then $A_0 \cap E_{h'} \to E$ what implies that $\mu_{A_0 \cap E_{h'}} \overset{*}{\rightharpoonup} \mu_E$. □

The next theorem gives information about the limit set of a sequence of almost minimizing sets. Indeed, if the sequence locally converges in L^1 sense to a finite perimeter set, it is possible to state that the limit set will be an almost minimizing set as well.

Theorem 1.10. ***(Closure theorem for sequence of almost minimizing sets)*** *Let $n \geq 2$ and $\{E_h\}_{h\in\mathbb{N}}$ be a sequence with each E_h being (Λ_h, r_h) − minimizing in A, $\Lambda_h r_h \leq 1$, $\liminf_{h\to\infty} r_h > 0$. If $A_0 \Subset A$ is an open set with finite perimeter and $A_0 \cap E_h \to E$, E with finite perimeter. Then E is a (Λ, r_0) − minimizing set in A_0, where $\Lambda = \limsup_{h\to\infty} \Lambda_h$, $r_0 = \liminf_{h\to\infty} r_h$.*

Proof. Let F be a competitor of E, i.e. F is a Caccioppoli with $E\Delta F \Subset \mathbf{B}(x,r)\cap A_0, 0 < r < \liminf_{h\to\infty} r_h \doteq r_0$, we aim to construct one competitor for each E_h from F. For this purpose, let us state a claim which will be proved later

Claim 1: If $y \in A_0$, $d_y = \min\{r_0, dist\,(y, \partial A_0)\}$, for almost all $r \in (0, d_y)$, it holds

$$\mathcal{H}_{n-1}\left(\partial\mathbf{B}(y,r)\cap\partial^* F\right) = \mathcal{H}_{n-1}\left(\partial\mathbf{B}(y,r)\cap\partial^* E_h\right) = 0, \forall h \in \mathbb{N} \tag{1.23}$$

$$\liminf_{h\to\infty} \mathcal{H}_{n-1}\left(\partial\mathbf{B}(y,r)\cap\left(E^{(1)}\Delta E_h^{(1)}\right)\right) = 0 \tag{1.24}$$

Since $E\Delta F$ is compactly contained in $\mathbf{B}(x,r)\cap A_0$, it is possible to choose a finite family $\{B_j = \mathbf{B}(y_j, r_j)\}_{j=1}^N$ such that $y_j \in A_0$ and $r_j \in \left(0, d_{y_j}\right)$ satisfying 1.23 and 1.24 and also satisfying

$$E\Delta F \Subset G \doteq \bigcup_{j=1}^{N} B_j \Subset \mathbf{B}(x,r)\cap A_0 \tag{1.25}$$

Now, we are able to construct the competitor of E_h that we have looked for. Define

$$F_h = \left(E_h \setminus G\right)\bigcup\left(G\cap F\right) \quad \textit{for each } h \in \mathbb{N} \tag{1.26}$$

that is, F_h inside G is equal to F and outside G it is equal to E_h. Thus, it follows

$$E_h\Delta F_h \Subset \mathbf{B}(x,r)\cap A_0 \Subset \mathbf{B}(x,r_h)\cap A_0 \tag{1.27}$$

where h is being taken sufficiently large to guarantee $r < \inf_{h\geq M} r_h$ and ensure the second inclusion. Since the space of finite perimeter sets is closed under intersection, union and complement, we have that F_h is a set of finite perimeter and, since $\partial G \subset \bigcup_{j=1}^N \partial B_j$, from 1.23

$$\mathcal{H}_{n-1}\left(\partial G\cap\partial^* F\right) = \mathcal{H}_{n-1}\left(\partial G\cap\partial^* E_h\right) = 0 \quad \forall h \in \mathbb{N} \tag{1.28}$$

Claim 2: $E^{(1)}\cap\partial G = F^{(1)}\cap\partial G$

The claim and 1.24 ensure that

$$\liminf_{h\to\infty} \mathcal{H}_{n-1}\left(\partial G\cap\left(F^{(1)}\Delta E_h^{(1)}\right)\right) = 0 \tag{1.29}$$

Since $A_0 \Subset A$, from 1.27, for each h sufficiently large, F_h is a competitor for the (Λ_h, r_h)-minimality of E_h in A, thus

$$\mathcal{P}(E_h, \mathbf{B}(x,r)) \leq \mathcal{P}(F_h, \mathbf{B}(x,r)) + \Lambda_h |E_h \Delta F_h|$$

The equalities in 1.28 allows to apply the comparison theorems for Caccioppoli sets (Theorem 16.16 in [Mag12]). Hence, it follows the equality

$$\mathcal{P}(F_h, \mathbf{B}(x,r)) = \mathcal{P}(F,G) + \mathcal{P}(E_h, \mathbf{B}(x,r) \setminus \overline{G}) + \mathcal{H}_{n-1}\left(\partial^* G \cap \left(E_h^{(1)} \Delta F^{(1)}\right)\right)$$

Adding this equation to the minimality condition of E_h and, noting that, since $F_h \cap \left(\mathbf{B}(x,r) \setminus \overline{G}\right) = E_h \cap \left(\mathbf{B}(x,r) \setminus \overline{G}\right)$, we have $\mathcal{P}(E_h, \mathbf{B}(x,r) \setminus \overline{G}) = \mathcal{P}(F_h, \mathbf{B}(x,r) \setminus \overline{G})$, then it follows

$$\mathcal{P}(E_h, \mathbf{B}(x,r)) \leq$$

$$\leq \mathcal{P}(F,G) + \mathcal{P}(F_h, \mathbf{B}(x,r) \setminus \overline{G}) + \mathcal{H}_{n-1}\left(\partial^* G \cap \left(E_h^{(1)} \Delta F^{(1)}\right)\right) + \Lambda_h |E_h \Delta F_h|$$

thus, we find that

$$\liminf_{h\to\infty} \mathcal{P}(E_h, \mathbf{B}(x,r)) + \liminf_{h\to\infty}(-\mathcal{P}(F,G)) + \liminf_{h\to\infty}(-\mathcal{P}(F_h, \mathbf{B}(x,r) \setminus \overline{G}))$$

$$+ \liminf_{h\to\infty}(-\Lambda_h |E_h \Delta F_h|) \leq$$

$$\liminf_{h\to\infty}\Big(\mathcal{P}(E_h, \mathbf{B}(x,r)) - \mathcal{P}(F,G) - \mathcal{P}(F_h, \mathbf{B}(x,r) \setminus \overline{G}) - \Lambda_h |E_h \Delta F_h|\Big)$$

$$\leq \liminf_{h\to\infty} \mathcal{H}_{n-1}\left(\partial^* G \cap \left(E_h^{(1)} \Delta F^{(1)}\right)\right) =^* 0$$

where in (*) we have used 1.29 and $\partial^* G \subset \partial G$. Since $-\liminf\limits_{h\to\infty}(-a_h) = \limsup\limits_{h\to\infty} a_h$ for any (a_h), we obtain

$$\liminf_{h\to\infty} \mathcal{P}(E_h, \mathbf{B}(x,r)) \leq \mathcal{P}(F,G) + \limsup_{h\to\infty} \mathcal{P}(F_h, \mathbf{B}(x,r) \setminus \overline{G})$$

$$+ \limsup_{h\to\infty} \Lambda_h |E_h \Delta F_h|$$

By the definition of F_h (1.26), we have $E_h \Delta F_h = G \cap E_h \Delta F$. Hence, since $E_h \cap A_0 \to E$ and $G \Subset A_0$ (1.27), we conclude that $E_h \Delta F_h \to G \cap E \Delta F = E \Delta F$. Then, it follows that

$$\begin{aligned} &\liminf_{h\to\infty} \mathcal{P}(E_h, \mathbf{B}(x,r)) \leq \\ &\leq \mathcal{P}(F,G) + \limsup_{h\to\infty} \mathcal{P}(F_h, \mathbf{B}(x,r) \setminus \overline{G}) + \left(\limsup_{h\to\infty} \Lambda_h\right) |E \Delta F| \end{aligned} \tag{1.30}$$

We note that $F_h \to F$ follows directly by the definition of F_h and 1.25 (i.e. outside G the set F is equal to E), then $\mu_{F_h} \xrightarrow{*} \mu_F$. As a consequence of this weak-star convergence (Theorem 1.40 in [LCE92]) and the compactness of $\overline{\mathbf{B}(x,r)} \setminus G$, it holds that

$$\limsup_{h\to\infty} \mathcal{P}(F_h, \mathbf{B}(x,r) \setminus \overline{G}) \le \limsup_{h\to\infty} \mathcal{P}(F_h, \overline{\mathbf{B}(x,r)} \setminus G) \le$$

$$\le \mathcal{P}(F, \overline{\mathbf{B}(x,r)} \setminus G) =^{*} \mathcal{P}(F, \mathbf{B}(x,r) \setminus G)$$

where (*) follows from 1.23 and $\mathcal{H}_{n-1}\left(\llcorner\right)_{\partial^* F}(\cdot) = \mathcal{P}(F,\cdot)$. The last inequality together with $\mathcal{P}(E, \mathbf{B}(x,r)) \le \liminf_{h\to\infty} \mathcal{P}(E_h, \mathbf{B}(x,r))$ (lower-semicontinuity of the perimeter, Theorem 5.2.2 in [FL03]) and 1.30 implies

$$\mathcal{P}(E, \mathbf{B}(x,r)) \le \mathcal{P}(F, \mathbf{B}(x,r)) + \left(\limsup_{h\to\infty} \Lambda_h\right) |E\Delta F|$$

and the proof is done. Let me prove the claims.

Proof of Claim 1: Lets start with the proof of 1.24, to state that 1.24 is true almost everywhere in $(0, d_y)$, note that

$$0 =^{*} \lim_{h\to\infty} |\mathbf{B}(y, d_y) \cap \left(E\Delta E_h\right)| =^{**} \lim_{h\to\infty} |\mathbf{B}(y, d_y) \cap \left(E^{(1)}\Delta E_h^{(1)}\right)| =$$

$$=^{***} \lim_{h\to\infty} \int_0^{d_y} \mathcal{H}_{n-1}\left(\partial\mathbf{B}(y,r) \cap \left(E^{(1)}\Delta E_h^{(1)}\right)\right) \mathrm{d}r$$

where $(*)$ follows from the convergence of $A_0 \cap E_h \to E$ and $\mathbf{B}(y, d_y) \subset A_0$, $(**)$ is directly checkable from $|E\Delta E_h| = |E\Delta E_h^{(1)}| = |E^{(1)}\Delta E_h^{(1)}|$ and $(***)$ is a consequence of the coarea formula, then, by the Fatou's lemma

$$\int_0^{d_y} \liminf_{h\to\infty} \mathcal{H}_{n-1}\left(\partial\mathbf{B}(y,r) \cap \left(E^{(1)}\Delta E_h^{(1)}\right)\right) \le$$

$$\le \lim_{h\to\infty} \int_0^{d_y} \mathcal{H}_{n-1}\left(\partial\mathbf{B}(y,r) \cap \left(E^{(1)}\Delta E_h^{(1)}\right)\right) \mathrm{d}r = 0$$

thus, 1.24 is valid almost everywhere in the interval. Let I be the subset of $(0, d_y)$ such that (2.13) is valid and denote by m be the Lebesgue measure on the real line. Since $\mathcal{H}_{n-1}\llcorner_{\partial^*(\cdot)}$ is a Radon measure, we have $m\left(\{r \in (0, d_y) : \mathcal{H}_{n-1}\left(\partial\mathbf{B}(y,r) \cap \partial^* E_h\right) = 0\}\right) = 0$ for each $h \ge -1$, where $E_{-1} = F$. But,

$$J = \left(I \cap \left(\bigcap_{h\ge -1} \{r \in (0, d_y) : \mathcal{H}_{n-1}\left(\partial\mathbf{B}(y,r) \cap \partial^* E_h\right) = 0\}\right)\right)^c =$$

$$= I^c \cup \left(\bigcup_{h\ge -1} \{r \in (0, d_y) : \mathcal{H}_{n-1}\left(\partial\mathbf{B}(y,r) \cap \partial^* E_h\right) = 0\}^c\right) \Rightarrow$$

$$\Rightarrow m\left(J\right) = 0$$

thus, the intersection has total measure in the interval $(0, d_y)$, therefore 1.23 and 1.24 are true almost everywhere in the interval.

Proof of Claim 2: Take $x \in \partial G \cap E^{(1)}$, then $\exists U \subset \overline{(E\Delta F)}^c$ open set with $x \in U$, thus $1_E\lfloor_U \equiv 1_F\lfloor_U \Rightarrow x \in F^{(1)}$, therefore $\partial G \cap E^{(1)} \subset \partial G \cap F^{(1)}$, analogous argument for the reverse inclusion. □

Proposition 1.11. *Let $\{E_h\}_{h\in\mathbb{N}}, A_0$ and E as chosen in the previous theorem, then*

$$\mu_{A_0\cap E_h} \overset{*}{\rightharpoonup} \mu_E \tag{1.31}$$

$$|\mu_{E_h}| \overset{*}{\rightharpoonup} |\mu_E| \quad in \quad A_0 \tag{1.32}$$

Moreover, it is true that
(1) If $x_h \in A_0 \in \partial E_h, x_h \to x$ and $x \in A_0$, then $x \in A_0 \cap \partial E$;
(2) If $x \in A_0 \cap \partial E$, then exists $\{x_h\}_{h\in\mathbb{N}}$ such that $x_h \in A_0 \cap \partial E_h$ and $x_h \to x$.

Proof. Since $E_h \cap A_0 \to E$, the convergence in 1.31 follows from the Representation Theorem (Theorem 5.2.1 in [FL03]) which states that $\int 1_{A'} div\phi \, \mathrm{d}\mathcal{H}_n = -\int \phi \, \mathrm{d}(\nu_{A'}\mu_{A'})$ for all $\phi \in C_c^0(\mathbb{R}^n)$ and all A' finite perimeter set.

Proof of 1.32: First of all, let us suppose that $|\mu_{A_0\cap E_h}| \overset{*}{\rightharpoonup} \mu$ in A_0, we contend that $\mu = |\mu_E|$ in A_0. Indeed, it holds $\mu(U) \geq |\mu_E|(U), \forall U$ Borel set of $\mathbb{R}^n$ (from Proposition 4.30 in [Mag12]). To show the reverse inequality, we will use the construction made in Theorem 1.10. Take $s_0 < r_0$ such that $x \in A_0, \mathbf{B}(x, s_0) \Subset A_0$ and fix M such that $s_0 < \inf_{h\geq M} r_h$. Define for each $h \geq M$

$$F_h = \Big(E \cap \mathbf{B}(x, s)\Big) \bigcup \Big(E_h \setminus \mathbf{B}(x, s)\Big) \tag{1.33}$$

where $s \in (0, s_0)$ is such that

$$\mathcal{H}_{n-1}(\partial\mathbf{B}(x, s) \cap \partial^* E) = \mathcal{H}_{n-1}(\partial\mathbf{B}(x, s) \cap \partial^* E_h) = 0 \quad h \geq M \tag{1.34}$$

$$\liminf_{h\to\infty} \mathcal{H}_{n-1}\left(\partial\mathbf{B}(x, s) \cap \left(E^{(1)} \Delta E_h^{(1)}\right)\right) = 0 \tag{1.35}$$

The fact that the conditions above hold for a.e. $s \in (0, s_0)$ follows from the same argument in the proof of claim 1 in Theorem 1.10. On the other hand, since $E_h \Delta F_h \Subset \mathbf{B}(x, s_0) \Subset A_0 \Subset A$, we have

$$\mathcal{P}(E_h, \mathbf{B}(x, s)) \leq \mathcal{P}(F_h, \mathbf{B}(x, s)) + \Lambda_h |E_h \Delta F_h| \tag{1.36}$$

Therefore, as in the proof of Theorem 1.10, by 1.34, we can use the operations of Caccioppoli sets (Theorem 16.16 in [Mag12]) for $\mathbf{B}(x, s_1) \Subset \mathbf{B}(x, s)$ where we fixed $s_1 \in (0, s_0)$ such that 1.35 holds true. Then

$$\mathcal{P}(F_h, \mathbf{B}(x, s)) = \mathcal{P}(E, \mathbf{B}(x, s_1)) + \mathcal{P}(E_h, \mathbf{B}(x, s) \setminus \overline{\mathbf{B}(x, s_1)})$$
$$+ \mathcal{H}_{n-1}\left(\partial\mathbf{B}(x, s_1) \cap \left(E^{(1)} \Delta E_h^{(1)}\right)\right)$$

Arguing as in the proof of Theorem 1.10, by 1.35, we obtain that

$$\liminf_{h\to\infty} \mathcal{P}(F_h, \mathbf{B}(x, s)) + \liminf_{h\to\infty} -\mathcal{P}(E, \mathbf{B}(x, s_1))$$

$$+\liminf_{h\to\infty} -\mathcal{P}(E_h, \mathbf{B}(x,s)\setminus\overline{\mathbf{B}(x,s_1)}) \leq$$

$$\leq \liminf_{h\to\infty} \mathcal{H}_{n-1}\left(\partial\mathbf{B}(x,s_1)\cap\left(E^{(1)}\Delta E_h^{(1)}\right)\right)$$

$$\Rightarrow$$

$$\liminf_{h\to\infty} \mathcal{P}(F_h, \mathbf{B}(x,s)) \leq \mathcal{P}(E, \mathbf{B}(x,s_1)) + \limsup_{h\to\infty} \mathcal{P}(E_h, \mathbf{B}(x,s)\setminus\overline{\mathbf{B}(x,s_1)})$$

We continue to follow through the steps made in Theorem 1.10. By $\mathbf{B}(x,s)\setminus\overline{\mathbf{B}(x,s_1)} \subset \overline{\mathbf{B}(x,s)}\setminus\mathbf{B}(x,s_1)$, the compactness of $\overline{\mathbf{B}(x,s)}\setminus\mathbf{B}(x,s_1)$, the characterization of weak-convergence of Radon measures (Theorem 1.40 in [LCE92]) and $\mathcal{P}(E, \partial\mathbf{B}(x,s)) = 0$ (that is 1.34), we find that

$$\liminf_{h\to\infty} \mathcal{P}(F_h, \mathbf{B}(x,s)) \leq \mathcal{P}(E, \mathbf{B}(x,s))$$

By the minimality condition (1.36) and the last inequality, we have that

$$\liminf_{h\to\infty} \mathcal{P}(E_h, \mathbf{B}(x,s)) + \liminf_{h\to\infty} -\Lambda_h|E_h\Delta F_h| \leq$$

$$\leq \liminf_{h\to\infty} \mathcal{P}(F_h, \mathbf{B}(x,s)) \leq \mathcal{P}(E, \mathbf{B}(x,s))$$

$$\Rightarrow$$

$$\liminf_{h\to\infty} \mathcal{P}(E_h, \mathbf{B}(x,s)) \leq \mathcal{P}(E, \mathbf{B}(x,s)) + \limsup_{h\to\infty} \Lambda_h|E_h\Delta F_h|$$

By the definition of F_h (1.33), we get $E_h\Delta F_h = E_h\Delta E\cap\mathbf{B}(x,s)$. Then, we conclude that

$$\limsup_{h\to\infty} \Lambda_h|E_h\Delta F_h| = \lim_{h\to\infty} \Lambda_h|E_h\Delta F_h| = 0$$

because $E_h\cap A_0 \to E$ and $\mathbf{B}(x,s)\subset A_0$. Thus,

$$\liminf_{h\to\infty} \mathcal{P}(E_h, \mathbf{B}(x,s)) \leq \mathcal{P}(E, \mathbf{B}(x,s)) \tag{1.37}$$

Since $\mathbf{B}(x,s) \Subset A_0$, we have that

$$\mathcal{P}(E_h, \mathbf{B}(x,s)) = |\mu_{E_h\cap A_0}|\left(\mathbf{B}(x,s)\right)$$

Taking into account $|\mu_{A_0\cap E_h}| \overset{*}{\rightharpoonup} \mu$ (again with Theorem 1.40 in [LCE92]), the last equality and 1.37, we establish that

$$\mu\left(\mathbf{B}(x,s)\right) \leq \mathcal{P}(E, \mathbf{B}(x,s)) = |\mu_E|\left(\mathbf{B}(x,s)\right)$$

Since we have already proved the reverse inequality, given $s_0 < r_0$ such that $\mathbf{B}(x,s_0) \Subset A_0$, for almost all $s\in(0,s_0)$, it follows

$$|\mu_E|\left(\mathbf{B}(x,s)\right) = \mu\left(\mathbf{B}(x,s)\right).$$

Then, for almost all $x\in \operatorname{spt}\mu\cap A_0$, the Radon-Nikodým derivative exists and, from the last equality, it is equal to 1, i.e.

$$(D_\mu|\mu_E|) \equiv 1$$

Since $\mu \geq |\mu_E|$ in all Borel sets of $\mathbb{R}^n$, it is clear that $|\mu_E| \ll \mu$, thus $|\mu_E|$ has no singular part with respect to μ. Therefore, by the Lebesgue-Besicovitch Differentiation Theorem

$$|\mu_E| = (D_\mu |\mu_E|)\, \mu = \mu \quad in \quad A_0$$

Since $\mu_{E_h} \lfloor_{A_0} = \mu_{E_h \cap A_0} \lfloor_{A_0}$ directly implies $|\mu_{E_h}| \lfloor_{A_0} = |\mu_{E_h \cap A_0}| \lfloor_{A_0}$, we find that $|\mu_{A_0 \cap E_h}| \overset{*}{\rightharpoonup} \mu = |\mu_E|$ in A_0. It remains to prove the existence of a Radon measure μ in A_0 such that $|\mu_{A_0 \cap E_h}| \overset{*}{\rightharpoonup} \mu$ in A_0. That follows directly from the compactness criterion for Radon measures (de la Vallée Poussin's theorem - Theorem 1.41 in [LCE92]). In order to apply the aforementioned theorem, we must verify that $\sup_{h \geq M} |\mu_{E_h \cap A_0}|\,(K) < \infty, \forall K$ compact subset of A_0. Indeed, given $K \subset A_0$ compact, we obtain that

$$\mathcal{P}(E_h \cap A_0, K) \leq \mathcal{P}(E_h \cap A_0, \mathbb{R}^n) \leq^{*} \mathcal{P}(E_h, A_0) + \mathcal{P}(A_0, \mathbb{R}^n) \leq$$

$$\leq^{**} \sum_{j=1}^{N} \mathcal{P}(E_h, B_j) + \mathcal{P}(A_0, \mathbb{R}^n) \leq$$

$$\leq^{***} 3n\omega_{n-1} \sum_{j=1}^{N} r_j^{n-1} + \mathcal{P}(A_0, \mathbb{R}^n) \leq 3Nn\omega_{n-1} r_0^{n-1} + \mathcal{P}(A_0, \mathbb{R}^n) \Rightarrow$$

$$\Rightarrow \sup_{h \geq M} \mathcal{P}(E_h \cap A_0, K) < \infty$$

where in $(*)$ it was used the operations with Caccioppoli sets, in $(**)$ it was taken an open cover of A_0 by balls $B_j = \mathbf{B}(x_j, r_j)$, all of them compactly contained in A with $x_j \in A_0 \subset A, r_j < s_0 < r_0$ and s_0 as previously chosen. And in $(***)$ we have used the upper inequality in 1.12 and the Remark 1.6.

Proof of (1): For each $h \in \mathbb{N}$, we take $x_h \in \partial E_h \cap A_0$ with $x_h \to x \in A_0$. Let $s \in (0, r_0), \mathbf{B}(x, s) \Subset A_0$. We set $M \in \mathbb{N}$ in view of $s < \inf_{h \geq M} r_h$ and $\mathbf{B}\big(x_h, \frac{s}{2}\big) \subset \mathbf{B}(x, s)$. Therefore,

$$\mathcal{P}(E, \overline{\mathbf{B}(x,s)}) \geq^{*} \limsup_{h \to \infty} \mathcal{P}(E_h, \overline{\mathbf{B}(x,s)}) \geq \limsup_{h \to \infty} \mathcal{P}(E_h, \mathbf{B}(x,s)) \geq$$

$$\geq \limsup_{h \to \infty} \mathcal{P}(E_h, \mathbf{B}\Big(x_h, \frac{s}{2}\Big)) \geq^{**} c\,(n) \left(\frac{s}{2}\right)^{n-1} > 0$$

where, since $\overline{\mathbf{B}(x,s)}$ is compact, $(*)$ follows from 1.32 and the characterization of weak-convergence of Radon measures (Theorem 1.40 in [LCE92]). Since $x_h \in \partial E_h$ and each E_h is a $(\Lambda, r_0) - minimizing$, $(**)$ follows from the lower density estimate in 1.12. So, by the last inequality and the arbitrariness of s, we have that $\mathcal{P}(E, \overline{\mathbf{B}(x,s)}) > 0, \forall s \in (0, r_0)$, thus, $x \in \operatorname{spt} \mu_E = \partial E$.

Proof of (2): We would like to highlight this argument because it is independent of the minimality condition of the sets E_h, i.e. we will only use the weak convergence of the Radon measures. We fix $x \in \partial E \cap A_0$

and suppose that exists $\epsilon > 0, h \geq 0$ such that $\left(\operatorname{spt} \mu_{E_{h'}} \cap A_0\right) \cap \mathbf{B}(x,\epsilon) = \emptyset, \forall h' \geq h$, i.e. x is not in the support of the measure $\mu_{E_{h'}}, \forall h' \geq h$. From 1.31 and the characterization of weak-convergence of Radon measures (Theorem 1.40 in [LCE92]), it follows that $\mu_E^j(\mathbf{B}(x,\epsilon)) \leq \liminf_{h' \to \infty} \mu^j_{E_{h'} \cap A_0}(\mathbf{B}(x,\epsilon)) = 0, \forall j \in \{1, ..., n\}$ where we have used the following notation $\mu = (\mu^1, ..., \mu^n)$. Thus, $x \notin \operatorname{spt} \mu_E = \partial E$ what is a contradiction with our choice of x. Then, exists a subsequence with the aimed properties. $\square$

Excess theory

2.1 Excess and its basic properties

In this section, we will define and prove results on the Excess theory. This theory is a fundamental point and one of the most significant tools in the regularity theory. The Geometric Measure Theory has evolved along three main branches, they are Currents, Varifolds and Caccioppoli sets. In all of them, the excess turns out to be the main tool used in the regularity theory of minimal or almost minimal sets.

Definition 2.1. *Let $E \subset \mathbb{R}^n$ be a Caccioppoli, $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $r > 0$. It is called* cylindrical excess *of E in x with direction ν in scale r the following number*

$$\boldsymbol{e}(E, x, r, \nu) = \frac{1}{r^{n-1}} \int_{\boldsymbol{C}(x,r,\nu)\cap\partial^* E} \frac{|\nu - \nu_E\,(y)\,|^2}{2}\,\mathrm{d}\mathcal{H}_{n-1}\,(y)$$

Remark 2.2. *The* cylindrical excess *of E in x with direction ν in scale $r > 0$ can be seen, intuitively, as a measure of how much the reduced boundary of E is far from being a hyperplane passing through x with normal vector ν inside a cylinder of radius r. If we consider the problem of finding the direction in which E is nearest to being a hyperplane, the* spherical excess *shows the answer*

Definition 2.3. *Let $E \subset \mathbb{R}^n$ be a Caccioppoli, $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $r > 0$. It is called* spherical excess *of E in x of scale r the following number*

$$\boldsymbol{e}(E, x, r) = \min_{\nu\in\mathbb{S}^{n-1}} \frac{1}{r^{n-1}} \int_{\boldsymbol{B}(x,r)\cap\partial^* E} \frac{|\nu - \nu_E\,(y)|^2}{2}\,\mathrm{d}\mathcal{H}_{n-1}\,(y)$$

Henceforth, we will prove and comment on some basic properties of the excess which will be used recurrently in this work. Unless otherwise explicitly stated, we will refer by *excess* the *cylindrical excess* and will make it evident and clear when talking about the *spherical excess*. The next result is the precise statement of the idea in Remark 2.2.

Proposition 2.4. *Let $E \subset \mathbb{R}^n$ be a Caccioppoli with* $\operatorname{spt}\mu_E = \partial E$, *$x \in \partial E$, $r > 0$ and $\nu \in \mathbb{S}^{n-1}$, then*

$$\boldsymbol{e}(E,x,r,\nu) = 0 \Leftrightarrow E \cap \boldsymbol{C}(x,r,\nu) \overset{Lebesgue}{\sim} \{y \in \boldsymbol{C}(x,r,\nu) : (y-x)\cdot\nu \leq 0\}$$

Proof. ($\Leftarrow$) Since both sets are equivalent, we have $\mu_E = \mu_{\{y \in \mathbf{C}(x,r,\nu):(y-x)\cdot\nu \leq 0\}}$ (by Proposition 0.1) what implies

$$\partial^* E \cap \mathbf{C}(x,r,\nu) \overset{\mathcal{H}_{n-1}}{\sim} \{y \in \mathbf{C}(x,r,\nu) : (y-x)\cdot\nu = 0\} \tag{2.1}$$

what ensures that $\nu_E = \nu_{\{y \in \mathbf{C}(x,r,\nu):(y-x)\cdot\nu=0\}} = \nu$ holds $\mathcal{H}_{n-1}$-almost everywhere in $\partial^* E \cap \{y \in \mathbf{C}(x,r,\nu) : (y-x)\cdot\nu = 0\} \cap \mathbf{C}(x,r,\nu)$, then it follows from the definition of excess that

$$\mathbf{e}(E,x,r,\nu) = \frac{1}{r^{n-1}} \int_{\mathbf{C}(x,r,\nu)\cap\partial^* E} \frac{|\nu - \nu_E(y)|^2}{2} \mathrm{d}\mathcal{H}_{n-1}(y) =$$

$$\overset{2.1}{=} \frac{1}{r^{n-1}} \int_{\{y \in \mathbf{C}(x,r,\nu):(y-x)\cdot\nu=0\}} \frac{|\nu - \nu|^2}{2} \mathrm{d}\mathcal{H}_{n-1}(y) = 0$$

($\Rightarrow$) Since $\mathbf{e}(E,x,r,\nu) = 0$ it follows that $\nu_E = \nu$ μ_E-almost everywhere in $\mathbf{C}(x,r,\nu)$. So, by the same argument used in the proof of Proposition 15.15 in [Mag12], there is $\alpha \in \mathbb{R}$ such that

$$E \cap \mathbf{C}(x,r,\nu) \overset{Lebesgue}{\sim} \{y \in \mathbf{C}(x,r,\nu) : (y-x)\cdot\nu < \alpha\}.$$

If $\alpha < 0$ (respectively, $\alpha > 0$), $\exists r_0 > 0$ such that $|E \cap \mathbf{B}(x,r_0)| = 0$ (respectively, $|E \cap \mathbf{B}(x,r_0)| = \omega_n r_0^n$) what is a contradiction with the fact that

$$x \in \operatorname{spt}\mu_E = \partial E = \{y \in \mathbb{R}^n : 0 < |E \cap \mathbf{B}(x,r)| < \omega_n r^n \ \ \textit{for all } r > 0\}$$

(Proposition 3.1 in [Giu84]). Then $\alpha = 0$. □

Proposition 2.5. ***(Excess and Changes of Scale)*** *Let $E \subset \mathbb{R}^n$ a Caccioppoli, $x \in \mathbb{R}^n, r > s > 0$ and $\nu \in \mathbb{S}^{n-1}$, then*

$$\boldsymbol{e}(E,x,s,\nu) \leq \left(\frac{r}{s}\right)^{n-1} \boldsymbol{e}(E,x,r,\nu)$$

Proof. Since $\mathbf{C}(x,s,\nu) \subset \mathbf{C}(x,r,\nu)$, we have

$$\frac{1}{s^{n-1}} \int_{\mathbf{C}(x,s,\nu)\cap\partial^* E} \frac{|\nu - \nu_E(y)|^2}{2} \mathrm{d}\mathcal{H}_{n-1}(y) \leq$$

$$\leq \frac{1}{s^{n-1}} \int_{\mathbf{C}(x,r,\nu)\cap\partial^* E} \frac{|\nu - \nu_E(y)|^2}{2} \mathrm{d}\mathcal{H}_{n-1}(y)$$

It suffices to multiply and divide by r^{n-1} on the right side of the inequality. □

Proposition 2.6. ***(Excess and Blow-up)*** *Let $E \subset \mathbb{R}^n$ be a Cacciop-*

poli, $x \in \partial E, r > 0$ and $\nu \in \mathbb{S}^{n-1}$, then, for all $s > 0$

$$\boldsymbol{e}(E,x,r,\nu) = \boldsymbol{e}\left(E_{x_0,s}, \frac{x-x_0}{s}, \frac{r}{s}, \nu\right) \qquad \boldsymbol{e}(E,x,r) = \boldsymbol{e}\left(E_{x_0,s}, \frac{x-x_0}{s}, \frac{r}{s}\right)$$

Proof. Note that $|\nu - \nu_E|^2 = (\nu - \nu_E) \cdot (\nu - \nu_E) = 2 - 2\nu \cdot \nu_E$. Let $\phi(z) = sz + x_0, \forall z \in \mathbb{R}^n$, one consequence of the definition of push-forward is

$$\mu_E = s^{n-1}\phi\sharp\mu_{E_{x_0,s}} \qquad |\mu_E| = s^{n-1}\phi\sharp|\mu_{E_{x_0,s}}| \tag{2.2}$$

Putting it all together

$$\mathbf{e}(E,x,r,\nu) = \frac{1}{r^{n-1}} \int_{\mathbf{C}(x,r,\nu)\cap\partial^* E} (1 - \nu \cdot \nu_E)\, \mathrm{d}\mathcal{H}_{n-1} =$$

$$=^* \frac{1}{r^{n-1}}\Big(|\mu_E|\left(\mathbf{C}\left(x,r,\nu\right)\right) - \nu \cdot \mu_E\left(\mathbf{C}\left(x,r,\nu\right)\right)\Big) =$$

$$=^{**} \frac{1}{r^{n-1}}\Big(s^{n-1}\phi\sharp|\mu_{E_{x_0,s}}|\left(\mathbf{C}\left(x,r,\nu\right)\right) - s^{n-1}\nu \cdot \phi\sharp\mu_{E_{x_0,s}}\left(\mathbf{C}\left(x,r,\nu\right)\right)\Big) =$$

$$= \left(\frac{s}{r}\right)^{n-1}\Big(|\mu_{E_{x_0,s}}|\left(\phi^{-1}\left(\mathbf{C}\left(x,r,\nu\right)\right)\right) - \nu \cdot \mu_{E_{x_0,s}}\left(\phi^{-1}\left(\mathbf{C}\left(x,r,\nu\right)\right)\right)\Big) =$$

$$=^{***} \left(\frac{1}{\frac{r}{s}}\right)^{n-1}\left(|\mu_{E_{x_0,s}}|\left(\mathbf{C}\left(\frac{x-x_0}{s}, \frac{r}{s}, \nu\right)\right) - \nu \cdot \mu_{E_{x_0,s}}\left(\mathbf{C}\left(\frac{x-x_0}{s}, \frac{r}{s}, \nu\right)\right)\right) =$$

$$=^{****} \mathbf{e}\left(E_{x_0,s}, \frac{x-x_0}{s}, \frac{r}{s}, \nu\right)$$

where $(*)$ is a consequence of the De Giorgi's Structure Theorem, $(**)$ follows from 2.2, $(***)$ it is easy to see that $\mathbf{C}\left(x - x_0, \frac{r}{s}, \nu\right) = \phi^{-1}\left(\mathbf{C}\left(x,r,\nu\right)\right)$ and $(****)$ it suffices to reproduce the same argument used in the firsts two equalities. To prove the second equality, note that $|\nu - \nu_E|^2$ reaches the minimum in $\nu = \nu_E$, thus

$$\mathbf{e}(E,x,r) = \min_{\nu\in\mathbb{S}^{n-1}} \frac{1}{r^{n-1}}\Big(|\mu_E|\left(\mathbf{C}\left(x,r,\nu\right)\right) - \nu \cdot \mu_E\left(\mathbf{C}\left(x,r,\nu\right)\right)\Big)$$

It holds

$$\mathbf{e}(E,x,r) = \frac{|\mu_E|\left(\mathbf{B}(x,r)\right)}{r^{n-1}}\left(1 - \frac{|\mu_E\left(\mathbf{B}(x,r)\right)|}{|\mu_E|\left(\mathbf{B}(x,r)\right)}\right) \tag{2.3}$$

Note that

$$\phi^{-1}\left(\mathbf{B}(x,r)\right) = \mathbf{B}\left(\frac{x-x_0}{s}, \frac{r}{s}\right)$$

With the same argument, we can conclude the proof of the second equality on the Proposition. □

Proposition 2.7. *Let $E \subset \mathbb{R}^n$ be a Caccioppoli, $x \in \partial E, r > 0, \nu \in \mathbb{S}^{n-1}$*

and $T:\mathbb{R}^n\to\mathbb{R}^n$ an isometry. Then

$$\boldsymbol{e}(T(E),x,r,\nu)=\boldsymbol{e}\big(E,T^{-1}(x),r,T^{-1}(\nu)\big)$$

Proof. Since

$$\mathbf{e}(E,x,r,\nu)=\frac{1}{r^{n-1}}\Bigg(|\mu_E|\left(\mathbf{C}\left(x,r,\nu\right)\right)-\nu\cdot\nu_E\,\mathcal{H}_{n-1}\left(\partial^*E\cap\mathbf{C}\left(x,r,\nu\right)\right)\Bigg)$$

$\mu_{T(E)}=T\sharp\mu_E$ and $|\mu_{T(E)}|=T\sharp|\mu_E|$ (Proposition 2.51 in [VOL]), we can verify that

$$\mathbf{e}(T(E),x,r,\nu)=$$

$$\frac{1}{r^{n-1}}\Bigg(|\mu_{T(E)}|\left(\mathbf{C}\left(x,r,\nu\right)\right)-\nu\cdot\nu_{T(E)}\,\mathcal{H}_{n-1}\left(\partial^*T(E)\cap\mathbf{C}\left(x,r,\nu\right)\right)\Bigg)=$$

$$=^*\frac{1}{r^{n-1}}\Bigg(T\sharp|\mu_E|\left(\mathbf{C}\left(x,r,\nu\right)\right)-\nu\cdot T(\nu_E\circ T^{-1})\,\mathcal{H}_{n-1}\left(\partial^*E\cap T^{-1}(\mathbf{C}\left(x,r,\nu\right))\right)\Bigg)=$$

$$\frac{1}{r^{n-1}}\Bigg(|\mu_E|(T^{-1}(\mathbf{C}\left(x,r,\nu\right)))-T^{-1}(\nu)\cdot\nu_E\circ T^{-1}\,\mathcal{H}_{n-1}\left(\partial^*E\cap T^{-1}(\mathbf{C}\left(x,r,\nu\right))\right)\Bigg)=$$

$$=\mathbf{e}\big(E,T^{-1}(x),r,T^{-1}(\nu)\big)$$

where in (*) we have used that $\mathcal{H}_{n-1}\left(T(A)\right)=\mathcal{H}_{n-1}\left(A\right)$ for any $A\subset\mathbb{R}^n$ which follows from the properties of the Hausdorff measure (Theorem 2.2 in [LCE92]) and that T is an isometry. $\square$

We pointed out that the reduced boundary can be seen as the regular part of the boundary. The next result provides a new interpretation of this fact. Indeed, if we choose a point x of the reduced boundary, the spherical excess at x tends to zero as the scale goes to zero. Then, we can find both a direction and a scale provided the excess at x is as small as we want.

Proposition 2.8. *Let $E\subset\mathbb{R}^n$ be a Caccioppoli and $x\in\partial^*E$. Then $\lim_{r\to0^+}\boldsymbol{e}(E,x,r)=0$. Moreover, $\forall\epsilon>0$, $\exists\nu\in\mathbb{S}^{n-1},\exists r>0$ such that $\boldsymbol{e}(E,x,r,\nu)\le\epsilon$.*

Proof. By the Remark of Definition 5.4 and *(iii)* of Theorem 5.14 both in [LCE92], we have that

$$\lim_{r\to0^+}\left|\frac{\mu_E\left(\mathbf{B}(x,r)\right)}{|\mu_E|\left(\mathbf{B}(x,r)\right)}\right|=|\nu_E\left(x\right)|=1$$

$$\lim_{r\to0^+}\frac{|\mu_E|\left(\mathbf{B}(x,r)\right)}{\omega_{n-1}r^{n-1}}=\lim_{r\to0^+}\frac{\mathcal{P}(E,\mathbf{B}(x,r))}{\omega_{n-1}r^{n-1}}=1$$

Joining these two limits in 2.3, it follows that $\lim_{r\to0^+}\mathbf{e}(E,x,r)=0$. Moreover, given $\epsilon>0$, take r such that $\mathbf{e}(E,x,r)\le\epsilon$. Since

$$\mathbf{C}\left(x,\frac{s}{\sqrt{2}},\nu\right)\subset\mathbf{B}(x,s)$$

is valid for all $s > 0$, in particular, for r, we obtain that

$$\mathbf{e}\left(E, x, \frac{r}{\sqrt{2}}, \nu\right) \leq \left(\frac{1}{\frac{r}{\sqrt{2}}}\right)^{n-1} \int_{\mathbf{B}(x,r)\cap\partial^* E} \frac{|\nu - \nu_E|^2}{2} \, \mathrm{d}\mathcal{H}_{n-1} \leq \sqrt{2}^{n-1}\epsilon$$

□

Up to now, we did not state anything about how the excess behaves with changes of the direction. For this purpose, we will be able to state some result only in the case of the almost minimizing sets as follows.

Proposition 2.9. ***(Change on the direction)*** *If $n \geq 2$, exists a constant $C(n)$ such that for all E $(\Lambda, r_0)-$minimizing set in A with $\Lambda r_0 \leq 1$ and for all $x \in \partial E \cap A$, $\sqrt{2}r < r_0$, $\boldsymbol{B}(x, 2r) \Subset A, \nu, \nu_0 \in \mathbb{S}^{n-1}$, it holds*

$$\boldsymbol{e}(E, x, r, \nu) \leq C_d\,(n) \left(\boldsymbol{e}\left(E, x, \sqrt{2}r, \nu_0\right) + |\nu - \nu_0|^2\right)$$

Proof. Since $f(x) = |x|^2$ is a convex function, taking $x = \nu_0 - \nu_E, y = \nu - \nu_0, t = \frac{1}{2}$ in the inequality $f(tx + (1-t)\,y) \leq tf(x) + (1-t)\,f(y)$ and $\mathbf{C}\,(x, r, \nu) \subset \mathbf{C}\,\left(x, \sqrt{2}r, \nu_0\right)$, then

$$\mathbf{e}(E, x, r, \nu) \leq$$

$$\frac{2}{r^{n-1}} \int_{\mathbf{C}\left(x,\sqrt{2}r,\nu_0\right)\cap\partial^* E} \left(\frac{|\nu_0 - \nu_E|^2}{2} + \frac{|\nu - \nu_0|^2}{2}\right) \mathrm{d}\mathcal{H}_{n-1} =$$

$$=^* 2\sqrt{2}^{n-1}\left(\mathbf{e}\left(E, x, \sqrt{2}r, \nu_0\right) + \frac{\mathcal{P}(E, \mathbf{C}\,(x, \sqrt{2}r, \nu_0))}{2\left(\sqrt{2}r\right)^{n-1}}|\nu - \nu_0|^2\right) \leq$$

$$\leq^{**} 2\sqrt{2}^{n-1}\left(\mathbf{e}\left(E, x, \sqrt{2}r, \nu_0\right) + \frac{3n\omega_{n-1}}{2}|\nu - \nu_0|^2\right)$$

where in $(*)$ it was used that the function in the integral is constant together with the De Giorgi's Structure Theorem and $(**)$ follows from Corollary 1.8. It suffices to take $C_d\,(n) = 2\sqrt{2}^{n-1}3n\omega_{n-1}$. □

The next result, shortly, state that when restricted to the almost minimizing sets, it is possible to affirm that the excess can not be small in two opposite directions.

Proposition 2.10. *Let E be a $(\Lambda, r_0) -$ minimizing in A, $x \in \partial E, \nu \in \mathbb{S}^{n-1}, r < r_0$, $\boldsymbol{C}\,(x, r, \nu) \subset A$, then*

$$\boldsymbol{e}(E, x, r, \nu) + \boldsymbol{e}(E, x, r, -\nu) \geq 2c\,(n)$$

where $c\,(n)$ is the constant from the density estimates.

Proof. Note that $(\nu_E - \nu)\cdot(\nu_E + \nu) = 0$, by Pitagoras' Theorem $|\nu_E - \nu|^2 + |\nu_E + \nu|^2 = 4|\nu|^2 = 4$, then

$$\mathbf{e}(E, x, r, \nu) + \mathbf{e}(E, x, r, -\nu) = \frac{1}{r^{n-1}} \int_{\partial^* E\cap\mathbf{C}(x,r,\nu)} \frac{4}{2} \, \mathrm{d}\mathcal{H}_{n-1} =$$

$$= \frac{2\,\mathcal{P}(E, \mathbf{C}\,(x, r, \nu))}{r^{n-1}} \geq$$

$$\geq \frac{2\,\mathcal{P}(E, \mathbf{B}(x, r))}{r^{n-1}} \geq^{*} 2c\,(n)$$

where $(*)$ it was used the lower inequality in (2.2). □

One result that will be used frequently and, usually, together with the compactness and closure theorems, is the lower semi-continuity of the excess. That is, what we can extract of the excess of a sequence of almost minimizing sets under some assumptions. For this purpose, let me state a preliminary result which deal with the continuity of the excess under some restrictive conditions.

Proposition 2.11. *Let A, A_0 open sets of $\mathbb{R}^n$ with $A_0 \Subset A, \mathcal{P}(A_0, \mathbb{R}^n) < \infty$. Suppose that $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of $(\Lambda, r_0)-$minimizing sets in A with $\Lambda r_0 \leq 1$ and such that $A_0 \cap E_h \to E$. Then, for all $\boldsymbol{C}\,(x, r, \nu) \Subset A_0$ satisfying*

$$\mathcal{H}_{n-1}\left(\partial^* E \cap \partial \boldsymbol{C}\,(x, r, \nu)\right) = 0$$

it holds

$$\boldsymbol{e}(E, x, r, \nu) = \lim_{h\to\infty} \boldsymbol{e}(E_h, x, r, \nu)$$

Proof. We have stated in Proposition 1.11 that

$$\mu_{A_0 \cap E_h} \overset{*}{\rightharpoonup} \mu_E$$

$$|\mu_{A_0 \cap E_h}| \overset{*}{\rightharpoonup} |\mu_E| \quad in \quad A_0$$

By Theorem 1.40 in [LCE92], $\mathcal{H}_{n-1}\left(\partial^* E \cap \partial \mathbf{C}\,(x, r, \nu)\right) = 0$ implies

$$|\mu_E|\left(\mathbf{C}\,(x, r, \nu)\right) = \lim_{h\to\infty} |\mu_{E_h}|\left(\mathbf{C}\,(x, r, \nu)\right)$$

and

$$\mu_E\left(\mathbf{C}\,(x, r, \nu)\right) = \lim_{h\to\infty} \mu_{E_h}\left(\mathbf{C}\,(x, r, \nu)\right)$$

Since $\mathbf{C}\,(x, r, \nu) \Subset A_0$ it holds

$$\mu_{A_0 \cap E_h}\left(\mathbf{C}\,(x, r, \nu)\right) = \mu_{E_h}\left(\mathbf{C}\,(x, r, \nu)\right)$$

Putting it all together and combining with the fact that, $\forall F$ Caccioppoli

$$\mathbf{e}(F, x, r, \nu) = \frac{1}{r^{n-1}}\Big(|\mu_F|\left(\mathbf{C}\,(x, r, \nu)\right) - \nu \cdot \mu_F\left(\mathbf{C}\,(x, r, \nu)\right)\Big)$$

the result follows. □

Theorem 2.12. ***(Lower semi-continuity of the excess)*** *Let A, A_0 opens of $\mathbb{R}^n$ with $A_0 \Subset A, \mathcal{P}(A_0, \mathbb{R}^n) < \infty$. Suppose that $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of $(\Lambda, r_0) -$ minimizing sets in A with $\Lambda r_0 \leq 1$ and such that $A_0 \cap E_h \to E$. Then, for all $\boldsymbol{C}\,(x, r, \nu) \Subset A_0$ it holds*

$$\boldsymbol{e}(E, x, r, \nu) \leq \liminf_{h\to\infty} \boldsymbol{e}(E_h, x, r, \nu) \tag{2.4}$$

Proof. Recalling the following equalities

$$\mathbf{e}(E,x,r,\nu) = \frac{1}{r^{n-1}} \int_{\mathbf{C}(x,r,\nu)\cap\partial^* E} (1-\nu\cdot\nu_E)\, \mathrm{d}\mathcal{H}_{n-1} =$$

$$= \frac{1}{r^{n-1}}\Big(|\mu_E| \left(\mathbf{C}\left(x,r,\nu\right)\right) - \nu \cdot \mu_E \left(\mathbf{C}\left(x,r,\nu\right)\right)\Big)$$

By the continuity from below of the measure $\mathcal{H}_{n-1}\llcorner_{\partial^* E}$ (Theorem 1.8 in [Fol99]) and the continuity of $r \mapsto r^{n-1}$, it is easy to see that $r \mapsto \mathbf{e}(E,x,r,\nu)$ is left-continuous, that is

$$\mathbf{e}(E,x,r,\nu) = \lim_{s\to r^-} \mathbf{e}(E,x,s,r)$$

Since $\mathcal{H}_{n-1}\llcorner_{\partial^* E}(\mathbf{C}\left(x,r,\nu\right)) < \infty$, exists a sequence $\{r_k\}_{k\in\mathbb{N}}$ such that $r_k \to r^-$ and $\mathcal{H}_{n-1}\left(\partial^* E \cap \partial \mathbf{C}\left(x,r_k,\nu\right)\right) = 0, \forall k \in \mathbb{N}$. The changing of scale of the excess (Proposition 2.5) yields

$$\mathbf{e}(E_h,x,r_k,\nu) \le \left(\frac{r}{r_k}\right)^{n-1} \mathbf{e}(E_h,x,r,\nu) \tag{2.5}$$

From the previous proposition and the choice of the sequence $\{r_k\}_{k\in\mathbb{N}}$, it holds for each k

$$\mathbf{e}(E,x,r_k,\nu) = \mathbf{e}(E_h,x,r_k,\nu)$$

The thesis is achieved by taking the $\liminf\limits_{k\to\infty}$ in the inequality (3.4), the left-continuity of $r \mapsto \mathbf{e}(E,x,r,\nu)$ and applying the last equality. $\square$

2.2 Bounded excess consequences

In this section, our goal is to describe locally (in general, up to H^n-equivalence) the almost minimizing sets under the assumption, which is crucial for the development of the regularity theory, of the boundedness of the excess. Under this assumption, how much information is it possible to infer concerning the local regularity of both topological and reduced boundaries? The first result of this section describes the almost minimizing sets with bounded excess inside a cylinder. From now on, we will *always* suppose that $n \ge 2$ and $e_n = (0,0,...,0,1) \in \mathbb{R}^n$.

Theorem 2.13. ***(Small-excess position)*** *Given $t_0 \in (0,1)$, exists a positive constant $\omega(n,t_0)$ such that, for all E being a $(\Lambda, r_0)-$minimizing* *in* $\boldsymbol{C}(0,2,e_n)$ *with* $\Lambda r_0 \le 1$, $0 \in \partial E$ *and*

$$\boldsymbol{e}(E,0,2,e_n) \le \omega(n,t_0)$$

it holds:

$$|\boldsymbol{q}x| < t_0 \quad \forall x \in \boldsymbol{C}(0,1,e_n) \cap \partial E \tag{2.6}$$

$$\left\{x \in \boldsymbol{C}(0,1,e_n) \cap E : \boldsymbol{q}x > t_0\right\} = \emptyset \tag{2.7}$$

$$\left\{x \in \boldsymbol{C}(0,1,e_n) \setminus E : \boldsymbol{q}x < -t_0\right\} = \emptyset \tag{2.8}$$

Proof. This proof will be done by contradiction. That is, given $t_0 \in (0,1)$, for each constant $\omega(n,t_0) > 0$, exists E in the conditions above (in the hypothesis of the theorem) such that E does not satisfy one of the equations 2.6, 2.7 or 2.8. Therefore, we will take a sequence of positive constants tending to zero, $\omega_h(n,t_0)$, and E_h the set in the conditions above referent to this constant. It holds:

$$\Lambda r_0 \le 1 \qquad \lim_{h\to\infty} \mathbf{e}(E_h,0,2,e_n) = 0 \qquad 0 \in \partial E_h, \forall h \in \mathbb{N}$$

at least one of the equations 2.6, 2.7 or 2.8, does not hold to an infinite number of E_h, otherwise the sequence will not be infinite. Since $\mathbf{C}\left(0,\frac{5}{3},e_n\right) \Subset \mathbf{C}(0,2,e_n)$ and $\mathcal{P}(\mathbf{C}\left(0,\frac{5}{3},e_n\right),\mathbb{R}^n) < \infty$, the pre-compactness theorem (Theorem 1.9) can be applied and ensures the existence of a set F of finite perimeter such that, passing to a subsequence if necessary:

$$E_h \cap \mathbf{C}\left(0,\frac{5}{3},e_n\right) \to F$$

note that, since $0 \in \partial E_h \cap \mathbf{C}\left(0,\frac{5}{3},e_n\right)$, it follows (from Proposition 1.11) that $0 \in \partial F$. Besides, since $\mathbf{C}\left(0,\frac{4}{3},e_n\right) \Subset \mathbf{C}\left(0,\frac{5}{3},e_n\right)$, follows from the lower semi-continuity of the excess:

$$\mathbf{e}\left(F,0,\frac{4}{3},e_n\right) \le \liminf_{h\to\infty} \mathbf{e}\left(E_h,0,\frac{4}{3},e_n\right) \le^{*} \left(\frac{3}{2}\right)^{n-1} \lim_{h\to\infty} \mathbf{e}(E_h,0,2,e_n) = 0$$

where, in (*), it was used the change of scale of the excess. It then follows from Proposition 2.4 that:

$$F \cap \mathbf{C}\left(0,\frac{4}{3},e_n\right) \overset{Lebesgue}{\sim} \left\{y \in \mathbf{C}\left(0,\frac{4}{3},e_n\right) : (y-0)\cdot e_n = \mathbf{q}y \le 0\right\} \tag{2.9}$$

An infinite quantity of E_h does not satisfy 2.6: Take a sequence $x_h \in \mathbf{C}(0,1,e_n) \cap \partial E_h$ such that $t_0 \le |\mathbf{q}x_h| \le 1$. Let $x_0 \in \overline{\mathbf{C}(0,1,e_n)} \cap \partial F$ the limit point of this sequence, it holds $x_0 \in \mathbf{C}\left(0,\frac{4}{3},e_n\right) \cap \partial F$ with $|\mathbf{q}x_0| \ge t_0$. On other hand, by the equivalence in 2.9 follows:

$$\mathbf{C}\left(0,\frac{4}{3},e_n\right) \cap \{x : \mathbf{q}x = 0\} = \mathbf{C}\left(0,\frac{4}{3},e_n\right) \cap \operatorname{spt} \mu_{\{x:\mathbf{q}x\le 0\}} =$$

$$= \mathbf{C}\left(0, \frac{4}{3}, e_n\right) \cap \operatorname{spt} \mu_F = \mathbf{C}\left(0, \frac{4}{3}, e_n\right) \cap \partial F$$

what contradicts the existence of x_0. So, there exists $\exists h_0$ such that $\forall h \geq h_0$ the set E_h satisfies 2.6. It follows that, for all A' Borel set of $\mathbb{R}^n$ (by Theorem 16.3 in [Mag12]):

$$\mathcal{P}(\mathbf{C}(0,1,e_n) \cap E_h, A') = \mathcal{P}(\mathbf{C}(0,1,e_n), E_h^{(1)} \cap A') +$$

$$+ \overbrace{\mathcal{P}(E_h, \mathbf{C}(0,1,e_n)^{(1)} \cap A')}^{=0,\ when\ A'=\{x\in\mathbf{C}(0,1,e_n):t_0<|\mathbf{q}x|<1\} by\ 2.9} + \mathcal{H}_{n-1}\left(\left\{\nu_{\mathbf{C}(0,1,e_n)} = \nu_{E_h}\right\} \cap A'\right)$$

Since

$$\mathcal{P}(\mathbf{C}(0,1,e_n), E_h^{(1)} \cap \{x \in \mathbf{C}(0,1,e_n) : t_0 < |\mathbf{q}x| < 1\}) =$$

$$= \mathcal{H}_{n-1}\left(\partial^* \mathbf{C}(0,1,e_n) \cap E_h^{(1)} \cap \{x \in \mathbf{C}(0,1,e_n) : t_0 < |\mathbf{q}x| < 1\}\right)$$

and $\partial^* \mathbf{C}(0,1,e_n) \cap \mathbf{C}(0,1,e_n) = \emptyset$, we have that:

$$\mathcal{P}(\mathbf{C}(0,1,e_n), E_h^{(1)} \cap \{x \in \mathbf{C}(0,1,e_n) : t_0 < |\mathbf{q}x| < 1\}) =$$

$$= \mathcal{H}_{n-1}\left(\left\{\nu_{\mathbf{C}(0,1,e_n)} = \nu_{E_h}\right\} \cap \{x \in \mathbf{C}(0,1,e_n) : t_0 < |\mathbf{q}x| < 1\}\right) = 0$$

. Thus, it holds $\mathcal{P}(\mathbf{C}(0,1,e_n) \cap E_h, \{x \in \mathbf{C}(0,1,e_n) : t_0 < |\mathbf{q}x| < 1\}) = 0$, what ensures (*) that comes in the next equation, where we set $\phi \in C_c^\infty(\{x \in \mathbf{C}(0,1,e_n) : t_0 < |\mathbf{q}x| < 1\})$:

$$\int 1_{\mathbf{C}(0,1,e_n)\cap E_h} \nabla\phi \,\mathrm{d}x = \int_{\partial^*(\mathbf{C}(0,1,e_n)\cap E_h)} \phi \nu_{\mathbf{C}(0,1,e_n)\cap E_h} \,\mathrm{d}\mathcal{H}_{n-1} =$$

$$= \int \phi \,\mathrm{d}\mu_{\mathbf{C}(0,1,e_n)\cap E_h} =^* 0$$

Since $\{x \in \mathbf{C}(0,1,e_n) : t_0 < |\mathbf{q}x| < 1\}$ is a connected open set, $1_{\mathbf{C}(0,1,e_n)\cap E_h}$ is constant almost everywhere in $\{x \in \mathbf{C}(0,1,e_n) : t_0 < |\mathbf{q}x| < 1\}$ (by Lemma 3.2 in [(au09]).

Analogously, it holds $1_{\mathbf{C}(0,1,e_n)\cap E_h}$ is equivalent to some constant in $\{x \in \mathbf{C}(0,1,e_n) : t_0 < |\mathbf{q}x| < 1\}$. By 2.9 follows $1_{\mathbf{C}(0,1,e_n)\cap E_h}$ is equivalent to 0, in the first case, and equivalent to 1, in the second case. Putting all together:

$$|\mathbf{q}x| \leq t_0 \quad \forall x \in \mathbf{C}(0,1,e_n) \cap \partial E \tag{2.10}$$

$$\left|\left\{x \in \mathbf{C}(0,1,e_n) \cap E : \mathbf{q}x > t_0\right\}\right| = 0 \tag{2.11}$$

$$\left|\left\{x \in \mathbf{C}(0,1,e_n) \setminus E : \mathbf{q}x < -t_0\right\}\right| = 0 \tag{2.12}$$

If exists $y \in \{x \in \mathbf{C}(0,1,e_n) \cap E : \mathbf{q}x > t_0\}$, by 2.10, it follows that $y \in int(E)$, what is a contradiction with 2.11, ensuring 2.7. The proof of 2.8 is analogous. □

From the theorem above, it was possible to show where the almost minimizing sets with bounded excess is situated or not inside a cylinder.

As a consequence of the last result, it will be possible to prove some formulas and then define the excess measure. For this purpose, we will prove how the perimeter measure of the cylinder can be written.

Lemma 2.14. *Let $r > 0$ and $a < b$. We define*

$$C = \{(x', x_n) \in \mathbb{R}^n : |x'| < r, a < x_n < b\}$$

then:

$$\mu_C = \nu\, \mathcal{H}_{n-1} \llcorner_{\partial \mathbb{D}_r \times (a,b)} + e_n\, \mathcal{H}_{n-1} \llcorner_{\mathbb{D}_r \times \{b\}} - e_n\, \mathcal{H}_{n-1} \llcorner_{\mathbb{D}_r \times \{a\}}$$

where $\nu(x) = (x'/r, 0)$.

Proof. Firstly, we note that

$$\begin{aligned} \partial C = & \overbrace{(\partial \mathbb{D}_r \times \{a\}) \dot\cup (\partial \mathbb{D}_r \times \{b\})}^{\doteq N_0} \dot\cup \\ & \dot\cup \overbrace{(\partial \mathbb{D}_r \times (a,b)) \dot\cup (\mathbb{D}_r \times \{a\}) \dot\cup (\mathbb{D}_r \times \{b\})}^{\doteq N} \end{aligned} \tag{2.13}$$

and

$$\mathcal{H}_{n-1}(N_0) = 0 \tag{2.14}$$

Since N_0 is a closed set and 2.14 holds true, we can say that C has almost C^1-boundary and N is the regular part of ∂E. From Theorem 9.6 in [Mag12], for every $\phi \in C_c^1(\mathbb{R}^n)$,

$$\int_C \nabla \phi = \int_N \phi \nu_C \, \mathrm{d}\mathcal{H}_{n-1}$$

Thus, we conclude that

$$\mu_C = \nu_C\, \mathcal{H}_{n-1} \llcorner_N \tag{2.15}$$

Accordingly with the definition of N, we obtain that

$$\mathcal{H}_{n-1} \llcorner_N = \mathcal{H}_{n-1} \llcorner_{(\partial \mathbb{D}_r \times (a,b))} + \mathcal{H}_{n-1} \llcorner_{(\mathbb{D}_r \times \{a\})} + \mathcal{H}_{n-1} \llcorner_{(\mathbb{D}_r \times \{b\})} \tag{2.16}$$

It is straightforward calculation to show that

$$\nu_C(x', x_n) = \begin{cases} (x'/r, 0), & \textit{if } x \in \partial \mathbb{D}_r \times (a,b) \\ e_n, & \textit{if } x \in \mathbb{D}_r \times \{b\} \\ -e_n, & \textit{if } x \in \mathbb{D}_r \times \{a\} \end{cases}$$

Finally, by 2.15, 2.16 and the last equality, we conclude the proof. □

Theorem 2.15. *If E is a Caccioppoli with $0 \in \partial E$ and $M \doteq \boldsymbol{C}(0,1,e_n) \cap \partial^* E$ such that $\exists t_0 \in (0,1)$ satisfying:*

$$|\boldsymbol{q}x| < t_0 \quad \forall x \in \boldsymbol{C}(0,1,e_n) \cap \partial E \tag{2.17}$$

$$\left| \left\{ x \in \boldsymbol{C}(0,1,e_n) \cap E : \boldsymbol{q}x > t_0 \right\} \right| = 0 \tag{2.18}$$

$$\left|\left\{x \in \boldsymbol{C}(0,1,e_n) \setminus E : \boldsymbol{q}x < -t_0\right\}\right| = 0 \tag{2.19}$$

Then, for all $G \subset \mathbb{D}_1$ Borel set, $\phi \in C_c^0(\mathbb{D}_1)$ and almost all $t \in (-1,1)$, it holds:

$$\mathcal{H}_{n-1}(G) \leq \mathcal{H}_{n-1}\left(M \cap \boldsymbol{p}^{-1}(G)\right) \tag{2.20}$$

$$\mathcal{H}_{n-1}(G) = \int_{M \cap \boldsymbol{p}^{-1}(G)} \nu_E \cdot e_n \, \mathrm{d}\mathcal{H}_{n-1} \tag{2.21}$$

$$\int_{\mathbb{D}_1} \phi = \int_M \phi(\boldsymbol{p}x)\, \nu_E(x) \cdot e_n \, \mathrm{d}\mathcal{H}_{n-1}(x) \tag{2.22}$$

$$\int_{\mathbb{D}_1 \cap E_t} \phi = \int_{M \cap \{\boldsymbol{q}x > t\}} \phi(\boldsymbol{p}x)\, \nu_E(x) \cdot e_n \, \mathrm{d}\mathcal{H}_{n-1}(x) \tag{2.23}$$

Proof. To prove 2.22 and 2.23, it suffices to prove for all $\phi \in C_c^1(\mathbb{D}_1)$, since it is possible to use the Dominated Converge Theorem together with the density of $C_c^1(\mathbb{D}_1)$ in $C_c^0(\mathbb{D}_1)$. Since $\mathcal{H}_{n-1} \llcorner_{\partial^* E}$ is a Radon measure and, in particular, $\mathcal{H}_{n-1}(\partial^* E \cap \mathbf{C}(0,1,e_n)) < \infty$, it follows that, for almost all $r \in (0,1)$:

$$\mathcal{H}_{n-1}(\partial^* E \cap (\partial \mathbb{D}_r \times [0,1])) = 0 \tag{2.24}$$

It also holds that:

$$\int_{[t_0,1]} \int_{\mathbb{R}^{n-1}} 1_{E \cap (\mathbb{D}_1 \times \{s\})}(y,s) \, \mathrm{d}\mathcal{H}_{n-1}(y) \, \mathrm{d}s =^* \int_{\mathbb{R}^n} 1_{E \cap (\mathbb{D}_1 \times [t_0,1])}(y) \, \mathrm{d}\mathcal{H}_n\, y =^{**} 0$$

where (*) follows from the Fubini's Theorem and (**) follows from 2.18. Therefore:

$$\mathcal{H}_{n-1}(E \cap (\mathbb{D}_1 \times \{s\})) = 0 \qquad \textit{for almost all} \quad s \in (t_0, 1) \tag{2.25}$$

Analogously:

$$\mathcal{H}_{n-1}(E \cap (\mathbb{D}_1 \times \{t\})) = \mathcal{H}_{n-1}(\mathbb{D}_1) \qquad \textit{for almost all} \quad t \in (-1, -t_0) \tag{2.26}$$

On the other hand, since $\mathcal{H}_{n-1} \llcorner_{\partial^* E}$ is a Radon measure

$$\mathcal{H}_{n-1}(\partial^* E \cap (\mathbb{D}_r \times \{s\})) = 0$$

holds for almost all $s \in \mathbb{R}$. Thus, fix $r \in (0,1)$ satisfying 2.24 and $s \in (t_0, 1)$ such that 2.25 and $\mathcal{H}_{n-1}(\partial^* E \cap (\mathbb{D}_r \times \{s\})) = 0$ are satisfied. Given $t \in (-1, s)$ such that $\mathcal{H}_{n-1}(\partial^* E \cap (\mathbb{D}_r \times \{t\})) = 0$, define $F = E \cap (\mathbb{D}_r \times (t,s))$. Since the intersection of sets with finite perimeter provides a set with finite perimeter, we have that F has finite perimeter. By Theorem 16.16 in [Mag12]:

$$\mu_F(\cdot) = \mu_E\Big((\mathbb{D}_r \times (t,s)) \cap \cdot\Big) + \mu_{(\mathbb{D}_r \times (t,s))}\left(E^{(1)} \cap \cdot\right) - \overbrace{\mathcal{H}_{n-1}\left(\left\{\nu_E = \nu_{\mathbb{D}_r \times (t,s)}\right\} \cap \cdot\right)}^{=^* 0}$$

where (*) follows from:

$$\left\{\nu_E = \nu_{\mathbb{D}_r\times(t,s)}\right\} \subset \partial^* E \cap \partial^* \left(\mathbb{D}_r \times (t,s)\right) \subset$$

$$\subset \partial^* E \cap \left(\partial\mathbb{D}_r \times [0,1] \cup \mathbb{D}_r \times \{t\} \cup \mathbb{D}_r \times \{s\}\right)$$

together with 2.24, the choice of s and t. Note that $\nu(x) = \frac{\mathbf{p}x}{|\mathbf{p}x|}$ is the normal vector to the cylinder $\mathbb{D}_1 \times \mathbb{R}$ for all $x \in \partial\mathbb{D}_1 \times \mathbb{R}$. So, by the previous lemma, it follows:

$$\mu_{(\mathbb{D}_r\times(t,s))}(\cdot) = e_n\, \mathcal{H}_{n-1}\left((\mathbb{D}_r \times \{s\}) \cap \cdot\right) +$$

$$+\nu\, \mathcal{H}_{n-1}\left((\partial\mathbb{D}_r \times (t,s)) \cap \cdot\right) - e_n\, \mathcal{H}_{n-1}\left((\mathbb{D}_r \times \{t\}) \cap \cdot\right)$$

Keeping in mind that $\nu(x) \cdot e_n = 0$, then, for almost all $t \in (-1,s)$:

$$\begin{aligned} e_n\mu_F(\cdot) &= e_n\mu_E\Big((\mathbb{D}_r \times (t,s)) \cap \cdot\Big) + e_n\mu_{(\mathbb{D}_r\times(t,s))}\left(E^{(1)} \cap \cdot\right) = \\ &= e_n\mu_E\Big((\mathbb{D}_r \times (t,s)) \cap \cdot\Big) + \overbrace{\mathcal{H}_{n-1}\left(E^{(1)} \cap (\mathbb{D}_r \times \{s\}) \cap \cdot\right)}^{=^* 0} \\ &\quad - \mathcal{H}_{n-1}\left(E^{(1)} \cap (\mathbb{D}_r \times \{t\}) \cap \cdot\right) = \\ &= (e_n \cdot \nu_E)\, \mathcal{H}_{n-1}\left(\partial^* E \cap (\mathbb{D}_r \times (t,s)) \cap \cdot\right) - \mathcal{H}_{n-1}\left(E^{(1)} \cap (\mathbb{D}_r \times \{t\}) \cap \cdot\right) \end{aligned} \tag{2.27}$$

where (*) follows from the choice of s (satisfying 2.25) together with

$$\Big\{x \in \mathbf{C}(0,1,e_n) \cap E : \mathbf{q}x > t_0\Big\} = \emptyset$$

what is directly obtained from 2.17 and 2.18, indeed, if exists y such that

$$y \in \{x \in \mathbf{C}(0,1,e_n) \cap E : \mathbf{q}x > t_0\}$$

by 2.17, it follows that $y \in int(E)$, what is a contradiction with 2.18. Given $\phi \in C^1_c(\mathbb{D}_1)$, define the following vector field $T(x) = \phi(\mathbf{p}x)\, e_n$, $x \in \mathbb{R}^n$. Clearly $divT \equiv 0$, if we choose $\varphi \in C^1_c(\mathbb{D}_1)$ such that $\varphi(x) = 1, \forall x \in U$ where U is an open neighborhood of F, we obtain that

$$\int e_n\phi \circ \mathbf{p}\, \mathrm{d}\mu_F =^* \int \varphi e_n \phi \circ \mathbf{p}\, \mathrm{d}\mu_F = \int \varphi T \cdot \mathrm{d}\mu_F =$$

$$= \int_F div(\varphi T)\, \mathrm{d}\mathcal{H}_n =^{**} \int_F divT\, \mathrm{d}\mathcal{H}_n = 0$$

where (**) follows from the choice of φ, i.e. $\varphi \equiv 1$ in $\overline{F}$, which, since $\operatorname{spt}\mu_F \subset \partial F$, also ensures (*).

Joining this equality to 2.27 ensures that, for almost all $t \in (-1,s)$:

$$0 = \int_{\partial^* E\cap(\mathbb{D}_r\times(t,s))} (e_n \cdot \nu_E)\, \phi \circ \mathbf{p}\, \mathrm{d}\mathcal{H}_{n-1} - \int_{E^{(1)}\cap(\mathbb{D}_r\times\{t\})} \phi \circ \mathbf{p}\, \mathrm{d}\mathcal{H}_{n-1}$$

By the Dominated Convergence Theorem and taking $r, s \to 1^-$ along suitable sequences follows, for almost all $t \in (-1,1)$, that:

$$\int_{\left(E^{(1)}\right)_t \cap \mathbb{D}_1} \phi = \int_{E^{(1)} \cap (\mathbb{D}_1 \times \{t\})} \phi \circ \mathbf{p} \, \mathrm{d}\mathcal{H}_{n-1} =$$

$$= \int_{\partial^* E \cap (\mathbb{D}_r \times (t,1))} (e_n \cdot \nu_E) \, \phi \circ \mathbf{p} \, \mathrm{d}\mathcal{H}_{n-1} =$$

$$= \int_{M \cap \{\mathbf{q}x > t\}} (e_n \cdot \nu_E) \, \phi \circ \mathbf{p} \, \mathrm{d}\mathcal{H}_{n-1}$$

Since, by Fubini's Theorem, $E \overset{Lesbegue}{\sim} E^{(1)} \Rightarrow E_t \overset{Lesbegue}{\sim} E_t^{(1)}$ for almost all t, the proof of 2.23 is concluded. Taking $t \in (-1, -t_0)$ follows: $M \cap \{\mathbf{q}x > t\} = M$ and, by 2.19, $\left(E^{(1)}\right)_t \overset{Lesbegue}{\sim} E_t \overset{Lesbegue}{\sim} \mathbb{D}_1$ and the proof of 2.22 is done. By the Cauchy-Schwarz Inequality, 2.21 implies 2.20. Lastly, we focus on the proof of 2.21. Firstly, we prove for $G' \Subset \mathbb{D}_1$ Borel set. For this purpose, we denote $\varphi_\epsilon = \eta_\epsilon * 1_{G'}$ where $\epsilon > 0$ and η_ϵ is the standard mollifiers. Since $1_{G'} \in L^1(\mathbb{D}_1)$, by Theorem 4.1 in [LCE92], we have that $\varphi \in C_c^\infty(\mathbb{D}_1)$, $\varphi_\epsilon \to 1_{G'}$ $\mathcal{H}_{n-1}$-almost everywhere in $\mathbb{D}_1$ as $\epsilon \to 0$ and

$$\varphi_\epsilon \to 1_{G'} \quad in \quad L^1(\mathbb{D}_1)$$

Therefore, we can apply 2.22 for φ_ϵ and ensure that

$$\begin{aligned} \mathcal{H}_{n-1}(G') &= \int_{\mathbb{D}_1} 1_{G'} \, \mathrm{d}\mathcal{H}_{n-1} = \int_{\mathbb{D}_1} \lim_{n\to\infty} \varphi_{1/n} \, \mathrm{d}\mathcal{H}_{n-1} = \\ &=^* \lim_{n\to\infty} \int_{\mathbb{D}_1} \varphi_{1/n} \, \mathrm{d}\mathcal{H}_{n-1} = \lim_{n\to\infty} \int_M (\nu_E \cdot e_n) \varphi_{1/n} \circ \mathbf{p} \, \mathrm{d}\mathcal{H}_{n-1} \end{aligned} \tag{2.28}$$

where in (*) we used the dominated convergence theorem. Since $\varphi_{1/n} \to 1_{G'}$ $\mathcal{H}_{n-1}$-almost everywhere in $\mathbb{D}_1$, by the dominated convergence theorem and the convergence $\varphi_{1/n} \circ \mathbf{p} \to 1_{G'} \circ \mathbf{p} = 1_{\mathbf{p}^{-1}(G')}$ $\mathcal{H}_{n-1}$-almost everywhere in $\mathbb{D}_1$, we turn 2.28 into

$$\mathcal{H}_{n-1}(G') = \lim_{n\to\infty} \int_M (\nu_E \cdot e_n) \phi \circ \mathbf{p} \, \mathrm{d}\mathcal{H}_{n-1} =$$

$$= \int_M (\nu_E \cdot e_n) \lim_{n\to\infty} \varphi_{1/n} \circ \mathbf{p} \, \mathrm{d}\mathcal{H}_{n-1} = \int_{M \cap \mathbf{p}^{-1}(G')} (\nu_E \cdot e_n) \, \mathrm{d}\mathcal{H}_{n-1}$$

that is 2.21 for any Borel set compactly contained in $\mathbb{D}_1$. In order to conclude the proof for any Borel set $G \subset \mathbb{D}_1$, we take $\{G_i\}_{i\in\mathbb{N}}$ a sequence of Borel sets with $G_i \Subset \mathbb{D}_1, G_i \subset G_{i+1}$ and $G_i \to G$, thus

$$\mathcal{H}_{n-1}(G) =^* \lim_{i\to\infty} \mathcal{H}_{n-1}(G_i) = \lim_{i\to\infty} \int_{M \cap \mathbf{p}^{-1}(G_i)} (\nu_E \cdot e_n) \, \mathrm{d}\mathcal{H}_{n-1} =$$

$$= \int_M \lim_{i\to\infty} 1_{\mathbf{p}^{-1}(G_i)} (\nu_E \cdot e_n) \, \mathrm{d}\mathcal{H}_{n-1} = \int_{M \cap \mathbf{p}^{-1}(G)} (\nu_E \cdot e_n) \, \mathrm{d}\mathcal{H}_{n-1}$$

in (*) we have used the continuity from below of the measure $\mathcal{H}_{n-1}$ (Theorem 1.8 in [Fol99]). $\square$

Corollary 2.16. ***(Excess measure)*** *In the conditions of the above theorem, the function*

$$\zeta : 2^{\mathbb{R}^{n-1}} \to \mathbb{R}_+$$

defined by:

$$\zeta(G) = \mathcal{P}(E, \boldsymbol{C}(0,1,e_n) \cap \boldsymbol{p}^{-1}(G)) - \mathcal{H}_{n-1}(G) =$$

$$= \mathcal{H}_{n-1}\left(M \cap \boldsymbol{p}^{-1}(G)\right) - \mathcal{H}_{n-1}(G)$$

is a Radon measure in $\mathbb{R}^{n-1}$ concentrated in $\mathbb{D}_1$ and such that $\zeta(\mathbb{D}_1) = \boldsymbol{e}(E, 0, 1, e_n)$

Proof. The equality $\zeta(\mathbb{D}_1) = \mathbf{e}(E, 0, 1, e_n)$ follows from the definition of ζ, De Giorgi's Structure Theorem, i.e. $\mathcal{P}(E, \cdot) = \mathcal{H}_{n-1 \llcorner \partial^* E}(\cdot)$, and

$$\mathbf{e}(E, x, r, \nu) = \frac{1}{r^{n-1}}\Big(|\mu_E|(\mathbf{C}(x,r,\nu)) - \nu \cdot \mu_E(\mathbf{C}(x,r,\nu))\Big) \tag{2.29}$$

Since $\mathbb{R}^{n-1}$ is locally compact Hausdorff space (LCH) and ζ is clearly a Borel measure on $\mathbb{R}^{n-1}$ finite on compact sets, by Theorem 7.8 in [Fol99], we conclude that ζ is a finite Radon measure on the Borel sets of $\mathbb{R}^{n-1}$. Then, by the Carathéodory construction, ζ induces a exterior Radon measure on $\mathbb{R}^{n-1}$. □

Starting from an almost minimizing set with bounded excess at the origin, we have been building up some control of the distance, within a cylinder centered in 0, between the topological boundary and the hyperplane defined by e_n. The control, which we have developed, assists to prove the Height Bound, which, starting from a bounded excess in a arbitrary point x_0 of the boundary , allows us to bound the distance between all points of the boundary, inside a cylinder, and x_0 by the size of the excess. In the literature, we can find the Height Bound stated in euclidean spaces of dimension bigger than 2, i.e. $n \geq 2$, in fact, the proofs, which we have checked, are not completely correct. It is due to the fact that the "Isoperimetric Inequality on Balls" is requested to be applied in dimension $n-1$, thus, it demands $n-1 \geq 2$. We are working to exhibit a proof which includes the case $n = 2$. However, we have not achieved this goal at the time of writing.

Theorem 2.17. ***(Height Bound)*** *Let $n \geq 3$. There exist constants $\epsilon_0(n), C_0(n)$ such that if E is $(\Lambda, r_0)-$minimizing set in $\boldsymbol{C}(x_0, 4r_0, e_n)$ with $\Lambda r_0 \leq 1, x_0 \in \partial E$ and*

$$\boldsymbol{e}(E, x_0, 4r_0, e_n) \leq \epsilon_0(n)$$

then

$$\sup\left\{\frac{|\boldsymbol{q}y - \boldsymbol{q}x_0|}{r_0} : y \in \partial E \cap \boldsymbol{C}(x_0, r_0, e_n)\right\} \leq C_0(n)\, \boldsymbol{e}(E, x_0, 4r_0, e_n)^{\frac{1}{2(n-1)}}$$

Proof. Let me start with a reduction in the argument. Suppose that we have proved the theorem for $r_0 = \frac{1}{2}$ and $x_0 = 0$. Given a set E which is

$(\Lambda', r)-minimizing$ set in $\mathbf{C}(x_0, 4r, e_n)$ with $x_0 \in \partial E, \Lambda' r \leq 1$ and

$$\mathbf{e}(E, x_0, 4r, e_n) \leq \epsilon_0(n)$$

By Proposition 1.3, the blow-up $E_{x_0,2r}$ of E is a $\left(\Lambda, \frac{1}{2}\right)-minimizing$ set in $\mathbf{C}(x_0, 4r, e_n)_{x_0,2r} = \mathbf{C}(0, 2, e_n)$ with $\Lambda = \Lambda' r, 0 \in \partial E_{x_0,2r}$. Moreover, by Proposition 2.6

$$\mathbf{e}(E_{x_0,2r}, 0, 2, e_n) = \mathbf{e}(E, x_0, 4r, e_n) \leq \epsilon_0(n)$$

By our reduction, since $y \in \partial E \cap \mathbf{C}(x_0, r, e_n)$ implies

$$\frac{y - x_0}{2r} \in \partial E_{x_0,2r} \cap \mathbf{C}\left(0, \frac{1}{2}, e_n\right)$$

we can affirm that

$$\sup\left\{\frac{|\mathbf{q}y - \mathbf{p}x_0|}{2r} : y \in \partial E \cap \mathbf{C}(x_0, r, e_n)\right\} \leq$$

$$\leq C_0(n)\, \mathbf{e}(E_{x_0,2r}, 0, 2, e_n)^{\frac{1}{2(n-1)}}$$

Therefore, it was possible to conclude the thesis of the Height Bound for the set E. So, we reduced the proof to the case with $r_0 = \frac{1}{2}$ and $x_0 = 0$. Briefly, we want to prove the existence of $C_0(n)$ and $\epsilon_0(n)$ such that

$$|\mathbf{q}x| \leq C_0(n)\, \mathbf{e}(E, 0, 2, e_n)^{\frac{1}{2(n-1)}} \quad \forall x \in \partial E \cap \mathbf{C}\left(0, \frac{1}{2}, e_n\right) \tag{2.30}$$

whenever $\mathbf{e}(E, 0, 2, e_n) \leq \epsilon_0(n)$.

Claim 1: Exists $t_0 \in \left(-\frac{1}{4}, \frac{1}{4}\right)$ such that, for all $x \in \mathbf{C}\left(0, \frac{1}{2}, e_n\right) \cap \partial E$,

$$|\mathbf{q}x - t_0| \leq C(n)\, \mathbf{e}(E, 0, 2, e_n)^{\frac{1}{2(n-1)}} \tag{2.31}$$

Since $0 \in \partial E$ implies $|t_0| \leq C(n)\, \mathbf{e}(E, 0, 2, e_n)^{\frac{1}{2(n-1)}}$, from 2.31, we obtain that

$$|\mathbf{q}x| \leq |\mathbf{q}x - t_0| + |t_0| \leq 2C(n)\, \mathbf{e}(E, 0, 2, e_n)^{\frac{1}{2(n-1)}}$$

To conclude, we define $C_0(n) = 2C(n)$.

Proof of Claim 1: First of all, assume that

$$\epsilon_0(n) \leq \omega\left(n, \frac{1}{4}\right) \tag{2.32}$$

where the constant $\omega\left(n, \frac{1}{4}\right)$ is from Theorem 2.13. If necessary, we will appropriately reduce $\epsilon_0(n)$. So, set $M = \partial E \cap \mathbf{C}(0, 1, e_n)$, from 2.6

$$|\mathbf{q}x| \leq \frac{1}{4} \quad \forall x \in M \tag{2.33}$$

By the change of scale in the excess (Proposition 2.5) holds $\mathbf{e}(E, 0, 1, e_n) \leq$

$2^{n-1}\mathbf{e}(E,0,2,e_n)$, then, by the properties of the excess measure (Corollary 2.16) and the $\mathcal{H}_{n-1}$-equivalence between $\partial E \cap \mathbf{C}(0,2,e_n)$ and $\partial^* E \cap \mathbf{C}(0,2,e_n)$ given by Corollary 1.7, it follows that

$$\begin{aligned} \mathcal{H}_{n-1}\left(\mathbf{C}(0,1,e_n) \cap \partial E \cap \mathbf{p}^{-1}(\mathbb{D}_1)\right) - \mathcal{H}_{n-1}(\mathbb{D}_1) = \\ = \mathcal{H}_{n-1}(M) - \mathcal{H}_{n-1}(\mathbb{D}_1) = \\ = \zeta(\mathbb{D}_1) = \mathbf{e}(E,0,1,e_n) \leq 2^{n-1}\mathbf{e}(E,0,2,e_n) \end{aligned} \tag{2.34}$$

As a consequence of $\zeta(G) \leq \mathbf{e}(E,0,1,e_n)$, we have that

$$\mathcal{H}_{n-1}\left(M \cap \mathbf{p}^{-1}(G)\right) \leq \mathcal{H}_{n-1}(G) + \mathbf{e}(E,0,1,e_n) \tag{2.35}$$

By a standard approximation argument and the triangle inequality, 2.23 ensures the first inequality that follows

$$\begin{aligned} 0 \leq \mathcal{H}_{n-1}(M \cap \{\mathbf{q}x > t\}) - \mathcal{H}_{n-1}(E_t \cap \mathbb{D}_1) \ \leq \\ \leq^* \mathbf{e}(E,0,1,e_n) \leq 2^{n-1}\mathbf{e}(E,0,2,e_n) \end{aligned} \tag{2.36}$$

where (*) follows from 2.35 taking $G = \mathbb{D}_1$. Note that 2.36 holds for almost all $t \in (-1,1)$. Now, we will define a function which will ensure the existence of t_0 as it is wished. So, define $f : (-1,1) \to [0, \mathcal{H}_{n-1}(M)]$ as

$$f(t) = \mathcal{H}_{n-1}(M \cap \{\mathbf{q}x > t\})$$

The function f is right-continuous as a consequence of the continuity from below of the measure $\mathcal{H}_{n-1} \llcorner \partial^* E$ (Theorem 1.8 in [Fol99]) and, evidently, f is decreasing. Moreover, f also satisfies

$$\begin{aligned} f(t) = \mathcal{H}_{n-1}(M) \qquad \forall t \in (-1, -\tfrac{1}{4}] \\ f(t) = 0 \qquad \forall t \in [\tfrac{1}{4}, 1) \end{aligned} \tag{2.37}$$

what follows directly from 2.33. We now set

$$t_0 \doteq \inf\left\{t \in (-1,1) : f(t) \leq \frac{\mathcal{H}_{n-1}(M)}{2}\right\}$$

Note that $t_0 \in (-1/4, 1/4)$ and t_0 satisfies

$$\begin{aligned} f(t) \leq \frac{\mathcal{H}_{n-1}(M)}{2} \qquad \forall t \in [t_0, 1) \\ f(t) \geq \frac{\mathcal{H}_{n-1}(M)}{2} \qquad \forall t \in (-1, t_0) \end{aligned} \tag{2.38}$$

Now, we contend to prove that t_0 is the desired value by the Claim. To this end, we will divide the proof into two steps.

Step one: Suppose that

$$f(t_0) \leq \sqrt{\mathbf{e}(E,0,2,e_n)} \tag{2.39}$$

Thus, if $x \in \mathbf{C}\left(0, \frac{1}{2}, e_n\right) \cap \partial E$ with $\mathbf{q}x > t_0$, by 2.33 and $|t_0| \leq 1/4$, it holds that $\mathbf{q}x - t_0 < \frac{1}{2}$, then $\mathbf{B}(x, \mathbf{q}x - t_0) \Subset \mathbf{C}(0,1,e_n) \Subset \mathbf{C}(0,2,e_n)$ what allows us to use the lower density estimate in 1.12 as follows

$$c(n) \leq \frac{\mathcal{P}(E, \mathbf{B}(x, \mathbf{q}x - t_0))}{(\mathbf{q}x - t_0)^{n-1}} \leq^* \frac{f(t_0)}{(\mathbf{q}x - t_0)^{n-1}}$$

where (*) follows from the inclusion $\mathbf{B}(x, \mathbf{q}x - t_0) \subset \mathbf{C}(0,1,e_n) \cap \{\mathbf{q}x > t_0\}$ and

$$\mathcal{P}(E, \mathbf{C}(0,1,e_n) \cap \{\mathbf{q}x > t_0\}) = \mathcal{H}_{n-1}(M \cap \{\mathbf{q}x > t_0\})$$

Therefore, by our assumption (2.39), we get that

$$c(n)(\mathbf{q}x - t_0)^{n-1} \leq f(t_0) \leq \sqrt{\mathbf{e}(E,0,2,e_n)} \quad \forall x \in \mathbf{C}\left(0, \frac{1}{2}, e_n\right) \cap \partial E \tag{2.40}$$

what implies

$$\mathbf{q}x - t_0 \leq \frac{1}{c(n)^{\frac{1}{n-1}}} \mathbf{e}(E,0,2,e_n)^{\frac{1}{2(n-1)}} \qquad \forall x \in \mathbf{C}\left(0, \frac{1}{2}, e_n\right) \cap \partial E \tag{2.41}$$

note that, if $\mathbf{q}x \leq t_0$, the inequality above (2.43) is trivial. Thus, if f satisfies 2.39, the proof of the Claim 1 is done.

Step two: Suppose that

$$f(t_0) > \sqrt{\mathbf{e}(E,0,2,e_n)} \tag{2.42}$$

Now, we are not able to establish the estimates in 2.40 for the constant t_0. In order to obtain an similar inequality, we shall define an auxiliary constant t_1 which will satisfy 2.40 in place of t_0. Furthermore, we will prove that t_1 and t_0 are related as follows

$$t_1 - t_0 \leq C^0(n)\, \mathbf{e}(E,0,2,e_n)^{\frac{1}{2(n-1)}}$$

Therefore, we set

$$t_1 \doteq \inf\left\{t \in (-1,1) : f(t) \leq \sqrt{\mathbf{e}(E,0,2,e_n)}\right\}$$

Thus,

$$f(t) \leq \sqrt{\mathbf{e}(E,0,2,e_n)} \qquad \forall t \in [t_1, 1)$$
$$f(t) > \sqrt{\mathbf{e}(E,0,2,e_n)} \qquad \forall t \in (-1, t_1)$$

By our assumption (2.42) and recalling that f is decreasing, we have that $t_1 > t_0$. Since $f(1/4) = 0$ (2.37), we also have $t_0 < 1/4$. Then, $t_1 \in (t_0, 1/4)$. If $x \in \mathbf{C}\left(0, \frac{1}{2}, e_n\right) \cap \partial E$ with $\mathbf{q}x > t_1$, by 2.33 and the choice of t_1, it holds $\mathbf{q}x - t_1 < \frac{1}{2}$, then

$$\mathbf{B}(x, \mathbf{q}x - t_1) \Subset \mathbf{C}(0,1,e_n) \Subset \mathbf{C}(0,2,e_n)$$

what allows to use the lower density estimate in 1.12 as follows

$$c(n) \leq \frac{\mathcal{P}(E, \mathbf{B}(x, \mathbf{q}x - t_1))}{(\mathbf{q}x - t_1)^{n-1}} \leq^* \frac{f(t_1)}{(\mathbf{q}x - t_1)^{n-1}}$$

where (*) follows from the inclusion $\mathbf{B}(x, \mathbf{q}x - t_1) \subset \mathbf{C}(0,1,e_n) \cap \{\mathbf{q}x > t_1\}$ and

$$\mathcal{P}(E, \mathbf{C}(0,1,e_n) \cap \{\mathbf{q}x > t_1\}) = \mathcal{H}_{n-1}(M \cap \{\mathbf{q}x > t_1\})$$

Therefore, by the choice of t_1

$$c(n)(\mathbf{q}x - t_1)^{n-1} \leq f(t_1) \leq \sqrt{\mathbf{e}(E,0,2,e_n)} \qquad \forall x \in \mathbf{C}\left(0, \frac{1}{2}, e_n\right) \cap \partial E$$

what implies

$$\mathbf{q}x - t_1 \leq \frac{1}{c(n)^{\frac{1}{n-1}}} \mathbf{e}(E,0,2,e_n)^{\frac{1}{2(n-1)}} \qquad \forall x \in \mathbf{C}\left(0, \frac{1}{2}, e_n\right) \cap \partial E \tag{2.43}$$

note that, when $\mathbf{q}x \leq t_1$ the inequality above is trivial. Now, we will prove that

$$t_1 - t_0 \leq C^0(n)\, \mathbf{e}(E,0,2,e_n)^{\frac{1}{2(n-1)}} \tag{2.44}$$

To this end, by Theorem 18.11 in [Mag12], for almost all $t \in \mathbb{R}$, E_t has finite perimeter and we have

$$\mathcal{H}_{n-2}\left(\partial^* E_t \Delta (\partial^* E)_t\right) = 0$$

from this equality we find that

$$\begin{aligned} \mathcal{H}_{n-2}(\mathbb{D}_1 \cap \partial^* E_t) &= \mathcal{H}_{n-2}\left(\mathbb{D}_1 \cap (\partial^* E)_t\right) = \\ &=^* \mathcal{H}_{n-2}\left((\mathbf{C}(0,1,e_n) \cap \partial^* E)_t\right) \end{aligned} \tag{2.45}$$

the equality (*) is deduced from $\mathbb{D}_1 \cap (\partial^* E)_t = (\mathbf{C}(0,1,e_n) \cap \partial^* E)_t$. From the Coarea formula for locally $(n-1)$-rectifiable (Theorem 2.93 in [LA00] taking $f = \mathbf{q}$), we deduce that

$$\int_{\partial^* E \cap \mathbf{C}(0,1,e_n)} \sqrt{1 - (\nu_E \cdot e_n)^2}\, \mathrm{d}\mathcal{H}_{n-1} = \int_{\mathbb{R}} \mathcal{H}_{n-2}\left((\mathbf{C}(0,1,e_n) \cap \partial^* E)_t\right) \mathrm{d}t \tag{2.46}$$

Thus

$$\begin{aligned} \int_{[-1,1]} \mathcal{H}_{n-2}(\mathbb{D}_1 \cap \partial^* E_t)\, \mathrm{d}t &=^{2.46}_{2.45} \int_{\mathbf{C}(0,1,e_n) \cap \partial^* E} \sqrt{1 - (\nu_E \cdot e_n)^2}\, \mathrm{d}\mathcal{H}_{n-1} \leq \\ &\leq^* \sqrt{2} \int_M \sqrt{1 - \nu_E \cdot e_n}\, \mathrm{d}\mathcal{H}_{n-1} \leq^{**} \sqrt{2\, \mathcal{H}_{n-1}(M)\, \mathbf{e}(E,0,1,e_n)} \leq \\ &\leq^{2.35} \sqrt{2\left(\mathcal{H}_{n-1}(\mathbb{D}_1) + 2^{n-1}\mathbf{e}(E,0,2,e_n)\right)} \sqrt{2^{n-1}\mathbf{e}(E,0,2,e_n)} \leq \\ &\leq^{***} \sqrt{2^n\left(\mathcal{H}_{n-1}(\mathbb{D}_1) + 2^{n-1}\epsilon_0(n)\right)} \sqrt{\mathbf{e}(E,0,2,e_n)} \end{aligned} \tag{2.47}$$

where (*) follows by $(a-1)^2 \geq 0 \Rightarrow -2a + 2 \geq -a^2 + 1$ and $M \overset{\mathcal{H}_{n-1}}{\sim}$

$\mathbf{C}(0,1,e_n)\cap\partial^* E$, (**) is directly from Holder's Inequality and $|\nu_E - e_n| = 2 - 2\nu_E \cdot e_n$, (***) is consequence of the boundedness of the excess. Define $C^1(n) = \sqrt{2^n\left(\mathcal{H}_{n-1}(\mathbb{D}_1) + 2^{n-1}\epsilon_0(n)\right)}$.

Claim 2: Exists $c^0(n) > 0$ such that, for almost all $t \in [t_0, t_1)$

$$\mathcal{H}_{n-2}\left(\mathbb{D}_1 \cap \partial^* E_t\right) \geq c^0(n)\, \mathcal{H}_{n-1}\left(E_t \cap \mathbb{D}_1\right)^{\frac{n-2}{n-1}} \tag{2.48}$$

The Claim 2 combined with 2.47 yields

$$\int_{[t_0,t_1)} \mathcal{H}_{n-1}\left(E_t \cap \mathbb{D}_1\right)^{\frac{n-2}{n-1}} \mathrm{d}t \leq \frac{C^1(n)}{c^0(n)}\sqrt{\mathbf{e}(E,0,2,e_n)}$$

By 2.36, for almost all $t \in [t_0, t_1)$

$$\mathcal{H}_{n-1}\left(E_t \cap \mathbb{D}_1\right) \geq \mathcal{H}_{n-1}\left(M \cap \{\mathbf{q}x > t\}\right) - 2^{n-1}\mathbf{e}(E,0,2,e_n) \geq$$

$$\geq^* \sqrt{\mathbf{e}(E,0,2,e_n)} - 2^{n-1}\mathbf{e}(E,0,2,e_n) \geq^{**} \sqrt{\mathbf{e}(E,0,2,e_n)}\left(1 - 2^{n-1}\sqrt{\epsilon_0(n)}\right)$$

in (*) was used the choice of t_1 and (**) is consequence of the boundedness of the excess. In order to have $1 - 2^{n-1}\sqrt{\epsilon_0(n)} > 0$, if necessary, we will reduce the size of $\epsilon_0(n)$. Putting together the last two inequalities, we find that

$$\int_{[t_0,t_1)} \left(\sqrt{\mathbf{e}(E,0,2,e_n)}\left(1 - 2^{n-1}\sqrt{\epsilon_0(n)}\right)\right)^{\frac{n-2}{n-1}} \mathrm{d}t \leq C^1(n)\sqrt{\mathbf{e}(E,0,2,e_n)}$$

what ensures that

$$t_1 - t_0 \leq \overbrace{\frac{C^1(n)}{c^0(n)\left(1 - 2^{n-1}\sqrt{\epsilon_0(n)}\right)^{\frac{n-2}{n-1}}}}^{\doteq C^0(n)} \mathbf{e}(E,0,2,e_n)^{\frac{1}{2(n-1)}}$$

thus, by 2.43 and the last inequality

$$\mathbf{q}x - t_0 \leq \left(C^0(n) + \frac{1}{c(n)^{\frac{1}{n-1}}}\right)\mathbf{e}(E,0,2,e_n)^{\frac{1}{2(n-1)}}$$

Applying the same arguments of both *Steps one* and *Step two* for E^c, it is possible to conclude the same inequality for $t_0 - \mathbf{q}x$, thus, we finish the proof.

Proof of Claim 2: By 2.34, 2.36 and the choice of t_0, for almost all $t \in [t_0, t_1)$, we have

$$\mathcal{H}_{n-1}\left(E_t \cap \mathbb{D}_1\right) \leq \mathcal{H}_{n-1}\left(M \cap \{\mathbf{q}x > t\}\right) = f(t) \leq \frac{1}{2}\mathcal{H}_{n-1}(M) \leq$$

$$\leq \frac{\mathcal{H}_{n-1}(\mathbb{D}_1) + 2^{n-1}\mathbf{e}(E,0,2,e_n)}{2} \leq \frac{3}{4}\mathcal{H}_{n-1}(\mathbb{D}_1)$$

where we reduced, if necessary, the size of $\epsilon_0(n)$ provided that $2^{n-2}\epsilon_0(n) \leq \mathcal{H}_{n-1}(\mathbb{D}_1)$. The inequalities above allow us to use the relative isoperimetric inequality (Proposition 12.37 in [Mag12]) for $E_t \cap \mathbb{D}_1$ as a Caccioppoli in $\mathbb{R}^{n-1}$, namely

$$\mathcal{P}(E_t \cap \mathbb{D}_1, \mathbb{D}_1) \geq c^0(n)\, \mathcal{H}_{n-1}(E_t \cap \mathbb{D}_1)^{\frac{n-2}{n-1}} \tag{2.49}$$

where we used both $n-1 \geq 2$ and $\mathcal{H}_{n-1}$ coincide with the Lebesgue measure in $\mathbb{R}^{n-1}$. We recall that

$$\mathcal{P}(E_t \cap \mathbb{D}_1, \mathbb{D}_1) \leq \mathcal{P}(\mathbb{D}_1, \mathbb{D}_1) + \mathcal{P}(E_t, \mathbb{D}_1) \tag{2.50}$$

Since $\partial^*\mathbb{D}_1 \cap \mathbb{D}_1 = \emptyset$ and

$$\mathcal{P}(\mathbb{D}_1, \mathbb{D}_1) = \mathcal{H}_{n-2}(\partial^*\mathbb{D}_1 \cap \mathbb{D}_1)$$

the first term on the right side of 2.50 is equal to 0. By the De Giorgi's Structure Theorem, we have that

$$\mathcal{P}(E_t, \mathbb{D}_1) = \mathcal{H}_{n-2}(\partial^* E_t \cap \mathbb{D}_1)$$

what, by 2.49, concludes the proof of the Claim. □

Approximation theorems

In this chapter, we will use the theorems that we have proved for an almost minimizing set E, as the Height Bound and the Small-Excess Position, to construct a Lipschitz function u such that the graph of u satisfies some properties. For instance, one of these properties will make it possible to locally insert the piece of the boundary of E with bounded excess into the graph of u. Moreover, we will be able to measure, with respect to $\mathcal{H}_{n-1}$, how large the portion of the boundary of E will not be contained in the graph of u. We shall prove that the function u possess one property that will be called almost harmonicness which will allow us to approximate the function u by harmonic functions and thus making it possible to prove a new estimate on the excess. Henceforth, we will use the following notations for the hypograph and epigraph of a function $u : \mathbb{R}^{n-1} \to \mathbb{R}$

$$\text{hypo}\,(u) = \{(z,t) \in \mathbb{R}^n : t > u(z)\}$$
$$\text{epi}\,(u) = \{(z,t) \in \mathbb{R}^n : t < u(z)\}$$

3.1 Lipschitz boundary criterion

The first theorem shows that, if the regular part of the boundary of a Caccioppoli set, that is, its reduced boundary, is at least of regularity Lipschitz inside a cylinder, the topological boundary has the same regularity.

Theorem 3.1. *Let E be a Caccioppoli with $\operatorname{spt}\mu_E = \partial E, 0 \in E$. If $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with $Lip\,(u) \leq 1$ and $\boldsymbol{C}(0,1,e_n) \cap \partial^* E \subset G(u) \cap \boldsymbol{C}(0,1,e_n)$, then $\boldsymbol{C}(0,1,e_n) \cap \partial E = \boldsymbol{C}(0,1,e_n) \cap G(u)$ and* *either*

$$\begin{aligned}\boldsymbol{C}(0,1,e_n)\cap(E\setminus\partial E) &= \boldsymbol{C}(0,1,e_n)\cap\operatorname{epi}(u)\\ \nu_E(z,u(z)) &= \frac{-(\nabla u(z),-1)}{\sqrt{1+|\nabla u(z)|^2}}\quad \textit{for almost all}\ \ z\in\mathbb{D}_1\end{aligned}\tag{3.1}$$

Or

$$\begin{aligned}\boldsymbol{C}(0,1,e_n)\cap(E\setminus\partial E) &= \boldsymbol{C}(0,1,e_n)\cap\operatorname{hypo}(u)\\ \nu_E(z,u(z)) &= \frac{(\nabla u(z),-1)}{\sqrt{1+|\nabla u(z)|^2}}\quad \textit{for almost all}\ \ z\in\mathbb{D}_1\end{aligned}\tag{3.2}$$

Proof. Since $G(u)\cap\mathbf{C}(0,1,e_n)$ is closed, it holds

$$\mathbf{C}(0,1,e_n)\cap\partial E=\mathbf{C}(0,1,e_n)\cap\overline{\partial^* E}\subset G(u)\cap\mathbf{C}(0,1,e_n)\tag{3.3}$$

Note that $u(0)=0$, because $(0,0)\in\partial E$. Since $\operatorname{spt}\mu_E=\partial E$, by 3.3, we have that $\mu_E(\mathbf{C}(0,1,e_n)\cap\operatorname{epi}(u)) = \mu_E(\mathbf{C}(0,1,e_n)\cap\operatorname{hypo}(u)) = 0$. Then, for all $\phi\in C_c^\infty(\operatorname{epi}(u))$

$$\int_{\operatorname{epi}(u)}1_E\nabla\phi=\int_{\operatorname{epi}(u)\cap\partial^*E}\phi\,\mathrm{d}\mu_E=0$$

analogously for $\operatorname{hypo}(u)$. By the connectedness of the epigraph and the hypograph and Lemma 3.2 in [(au09], we find that 1_E is equivalent to a constant in each one of them.

If these constants are equal, we deduce that 1_E is constant almost everywhere in $\mathbf{C}(0,1,e_n)$ what provides that $|\mu_E|(\mathbf{C}(0,1,e_n))=0$. On the other hand, since $0\in\partial E\cap\mathbf{C}(0,1,e_n)=\operatorname{spt}\mu_E\cap\mathbf{C}(0,1,e_n)$, we have $|\mu_E|(\mathbf{C}(0,1,e_n))>0$ what is an absurd. If 1_E is equivalent to 1 in $\operatorname{epi}(u)$ and equivalent to 0 in $\operatorname{hypo}(u)$, we can show that 1_E will be truly equal these constants in each respective set. Indeed, if $\exists x\in\operatorname{epi}(u)\cap\mathbf{C}(0,1,e_n)$ such that $1_E(x)=0$, we have $x\in\partial E$. But $\mathbf{C}(0,1,e_n)\cap\partial E\subset G(u)\cap\mathbf{C}(0,1,e_n)$ which is another absurd. Thus, 1_E is equal to 1 in $\operatorname{epi}(u)$ and, analogously, it is equal to 0 in $\operatorname{hypo}(u)$. It concludes the proof of the both first equalities in 3.1 and 3.2. We now turn our attention to the proof of the formulas of ν_E. Since they are mostly the same, we will show the formula on 3.1. We have proved that both

$$\mathbf{C}(0,1,e_n)\cap(E\setminus\partial E)=\mathbf{C}(0,1,e_n)\cap\operatorname{epi}(u)$$

is valid, thus, we obtain that $|(E\Delta G(u))\cap\mathbf{C}(0,1,e_n)|=0$. From 0.1, we find that

$$\mu_{E\cap\mathbf{C}(0,1,e_n)}=\mu_{G(u)\cap\mathbf{C}(0,1,e_n)}$$

Thus, since the outer unit normal of $G(u)\cap\mathbf{C}(0,1,e_n)$ exists almost everywhere, we have that

$$\nu_E(x)=\nu_{G(u)}(x)\quad\textit{for almost all}\ \ x\in\mathbf{C}(0,1,e_n)\cap G(u)$$

Therefore, our problem turn into the proof of the formula for the outer unit normal of a Lipschitz graph. To this end, if we define $f(z) = (z, u(z))$ for all $z \in \mathbb{D}_1$, by Lemma 10.4 in [Mag12], we conclude that

$$T_{f(z)}G(u) = \nabla f(z)(\mathbb{R}^{n-1})$$

Since $\nabla f(z) = \binom{id_{\mathbb{R}^{n-1}}}{\nabla u(z)}$ for almost all $z \in \mathbb{D}_1$, we show that

$$v \in T_{f(z)}G(u) \Leftrightarrow \quad \exists w \in \mathbb{R}^{n-1} : \quad v = \nabla f(z)w = (w, w \cdot \nabla u(z)) \Leftrightarrow$$

$$\Leftrightarrow v \cdot (-\nabla u(z), 1) = 0 \Leftrightarrow v \in (-\nabla u(z), 1)^{\perp}$$

Thus, $\nu_E(z) = \frac{(-\nabla u(z),1)}{\sqrt{1+|\nabla u(z)|^2}}$ for almost all $z \in \mathbb{D}_1$. □

Corollary 3.2. *In the conditions of the theorem above, given $G \subset \mathbb{D}_1$ Borel set, it holds*

$$\mathcal{P}(E, \boldsymbol{C}(0, 1, e_n) \cap \boldsymbol{p}^{-1}(G)) = \int_G \sqrt{|\nabla u(z)|^2 + 1}\,\mathrm{d}z$$

Proof. From the area formula (Theorem 3.8 in [LCE92]), we have that

$$\mathcal{H}_{n-1}(\{(z, u(z)) : z \in G\}) = \int_G \sqrt{|\nabla u(z)|^2 + 1}\,\mathrm{d}z$$

Since $\mathbf{C}(0, 1, e_n) \cap \partial E = \mathbf{C}(0, 1, e_n) \cap G(u)$, we turn the last equality into

$$\mathcal{H}_{n-1} \llcorner \partial E(\mathbf{C}(0, 1, e_n) \cap \mathbf{p}^{-1}(G)) = \int_G \sqrt{|\nabla u(z)|^2 + 1}\,\mathrm{d}z \tag{3.4}$$

In the last theorem (3.1), we have established that $\nu_E(z, u(z))$ exists for almost all $z \in \mathbb{D}_1$. As a consequence of this existence and the properties of the Hausdorff measure under Lipschitz maps (Theorem 2.3 in [LA00]), we obtain that

$$\mathcal{H}_{n-1}(f(N)) \leq Lip(f)\,\mathcal{H}_{n-1}(N) = 0$$

where we have set $f(z) = (z, u(z))$ and $N \subset \mathbb{D}_1$ as the set of points where $\nu_E(z, u(z))$ does not exist. Note that $(\partial E \setminus \partial^* E) \cap \mathbf{C}(0, 1, e_n) \subset f(N)$, therefore, $\partial E \cap \mathbf{C}(0, 1, e_n) \overset{\mathcal{H}_{n-1}}{\sim} \partial^* E \cap \mathbf{C}(0, 1, e_n)$ what, by 3.4, implies that

$$\mathcal{P}(E, \mathbf{C}(0, 1, e_n) \cap \mathbf{p}^{-1}(G)) =$$

$$= \mathcal{H}_{n-1} \llcorner \partial^* E(\mathbf{C}(0, 1, e_n) \cap \mathbf{p}^{-1}(G)) = \mathcal{H}_{n-1} \llcorner \partial E(\mathbf{C}(0, 1, e_n) \cap \mathbf{p}^{-1}(G))$$

$$= \int_G \sqrt{|\nabla u(z)|^2 + 1}\,\mathrm{d}z$$

□

3.2 Lipschitzian approximation

We have developed sufficient tools to state the first result which provides a kind of regularity to a piece of the topological boundary of an almost minimizing set. Indeed, if $x_0 \in \partial E$ is such that $\mathbf{e}(E, x_0, 25r, e_n)$ is bounded by a constant depending only on the dimension n, we will be able to show that the piece of $\partial E \cap \mathbf{C}\,(x_0, r, e_n)$ with bounded excess, i.e. M_0 is a subset of the graph of a Lipschitz function $\Gamma \cap \mathbf{C}\,(x_0, r, e_n)$. Moreover, the result will state that the size of $(\partial E \Delta \Gamma) \cap \mathbf{C}\,(x_0, r, e_n)$, i.e. the piece that is not contained in the Lipschitz graph $\Gamma \cap \mathbf{C}\,(x_0, r, e_n)$, is controlled by the size of the excess.

Theorem 3.3. ***(Lipschitzian Approximation)*** *Let $n \geq 3$, there exist constants $C_1\,(n)\,, \epsilon_1\,(n)$ and $\delta_0\,(n)$ such that if E is a $(\Lambda, r_0)-$minimizing set in $\boldsymbol{C}\,(x_0, 25r, e_n)$ with $\Lambda r_0 \leq 1, x_0 \in \partial E$ and*

$$25r < r_0, \qquad \boldsymbol{e}(E, x_0, 25r, e_n) \leq \epsilon_1\,(n)$$

there exists a Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ with $Lip\,(u) < 1$ and

$$\sup_{\mathbb{R}^n} \frac{|u|}{r} \leq C_1\,(n)\,\boldsymbol{e}(E, x_0, 25r, e_n)^{\frac{1}{2(n-1)}} \tag{3.5}$$

such that, if $M = \boldsymbol{C}\,(x_0, r, e_n) \cap \partial E$ and

$$M_0 = \left\{ y \in M : \sup_{0<s<8r} \boldsymbol{e}(E, y, s, e_n) \leq \delta_0\,(n) \right\}$$

it holds

$$M_0 \subset M \cap \Gamma \tag{3.6}$$

where $\Gamma = (x_0 + G(u)) \cap \boldsymbol{C}\,(x_0, r, e_n)$. Moreover

$$\frac{\mathcal{H}_{n-1}\,(M \Delta \Gamma)}{r^{n-1}} \leq C_1\,(n)\,\boldsymbol{e}(E, x_0, 25r, e_n) \tag{3.7}$$

Proof. Suppose that we have proved the existence of constants $C_1\,(n)\,, \epsilon_1\,(n)$ and $\delta_0\,(n)$ in the case that E is a $(\Lambda', r'_0)-$*minimizing* set in $\mathbf{C}\,(0, 25, e_n)$ with $\Lambda' r'_0 \leq 1, 0 \in \partial E$ and

$$25 < r'_0, \qquad \mathbf{e}(E, 0, 25, e_n) \leq \epsilon_1\,(n)$$

Thus, if E' is in the conditions of the theorem, we have that, by Proposition 1.3, $E = E'_{x_0,r}$ is a $(\Lambda', r'_0) -$ *minimizing* set in $\mathbf{C}\,(0, 25, e_n)$ with $\Lambda' = \Lambda r$ and $r'_0 = \frac{r_0}{r}$. Moreover, we have that $\Lambda' r'_0 \leq 1, 0 \in \partial E$ and, by Proposition 2.6, $\mathbf{e}(E, 0, 25, e_n) = \mathbf{e}(E', x_0, 25r, e_n) \leq \epsilon_1(n)$. Therefore, by our assumption, exists $u : \mathbb{R}^{n-1} \to \mathbb{R}$ with $Lip\,(u) < 1$ with the properties above. Taking $u_r(z) = ru(\frac{z}{r})$, we have

$$\sup_{\mathbb{R}^n} \frac{|u_r|}{r} = \sup_{\mathbb{R}^n} |u| \leq C_1(n)\mathbf{e}(E, 0, 25, e_n)^{\frac{1}{2(n-1)}} = C_1(n)\mathbf{e}(E', x_0, 25r, e_n)^{\frac{1}{2(n-1)}}$$

We set $M' \doteq \mathbf{C}\,(x_0, r, e_n) \cap \partial E'$ and

$$M'_0 \doteq \left\{ y \in M' : \sup_{0<s<8r} \mathbf{e}(E, y, s, e_n) \leq \delta_0\,(n) \right\}$$

By Proposition 2.6, it is straightforward to verify that $M'_{x_0,r} = M$, $(M'_0)_{x_0,r} = M_0$ and $\Gamma_{\frac{-x_0}{r},\frac{1}{r}} = (rG(u)+x_0) \cap \mathbf{C}\,(x_0, r, e_n) = (G(u_r)+x_0) \cap \mathbf{C}\,(x_0, r, e_n) \doteq \Gamma_r$. Then, since $M_0 \subset M \cap \Gamma$, we deduce that $M'_0 \subset M' \cap \Gamma_{\frac{-x_0}{r},\frac{1}{r}} = M' \cap \Gamma_r$ that is 3.6. Taking into account the properties of $\mathcal{H}_{n-1}$ with respect to translations and homotheties (Proposition 2.49 in [LA00]) and

$$M'\Delta\Gamma_r = (M' \cap \Gamma_r^c) \bigcup (M'^c \cap \Gamma_r) =$$

$$= \left((M \cap \Gamma^c) \bigcup (M^c \cap \Gamma) \right)_{\frac{-x_0}{r},\frac{1}{r}} = (M\Delta\Gamma)_{\frac{-x_0}{r},\frac{1}{r}}$$

we can conclude 3.7. From now on, we shall prove the theorem for E being a $(\Lambda', r'_0) - minimizing$ set in $\mathbf{C}\,(0, 25, e_n)$ with $\Lambda' r'_0 \leq 1, 0 \in \partial E$ and

$$25 < r'_0, \qquad \mathbf{e}(E, 0, 25, e_n) \leq \epsilon_1\,(n) \tag{3.8}$$

To this end, take the constants $\epsilon_0(n), C_0(n)$ given by the Height Bound (Theorem 2.17). Assume that $\epsilon_1(n) \leq \epsilon_0(n)$ in order to apply the Height Bound for E (taking $r_0 = \frac{25}{4}$ and $\Lambda = \Lambda'$). Therefore

$$\sup \left\{ |\mathbf{q}y| : y \in \partial E \cap \mathbf{C}\left(0, \frac{25}{4}, e_n\right) \right\} \leq \frac{25C_0(n)}{4} \mathbf{e}(E, 0, 25, e_n)^{\frac{1}{2(n-1)}} \tag{3.9}$$

where we will still denote the constant $\frac{25C_0(n)}{4}$ by $C_0(n)$. Recalling 2.32 where we asked that $\epsilon_0(n) \leq \omega(n, \frac{1}{4})$, and, if necessary, reducing $\epsilon_1(n)$ in order to have $\mathbf{e}(E, 0, 2, e_n) \leq (\frac{25}{2})^{n-1}\mathbf{e}(E, 0, 25, e_n) \leq \epsilon_1(n)$, we are able to apply the Small-excess position theorem (Theorem 2.13) which provides 2.17, 2.18 and 2.19. Thus, from Corollary 2.16, we have that

$$\begin{aligned} 0 \leq \mathcal{H}_{n-1}\left(M \cap \mathbf{p}^{-1}(G)\right) - \mathcal{H}_{n-1}\,(G) \leq \mathbf{e}(E, 0, 1, e_n) \leq \\ \leq 25^{n-1}\mathbf{e}(E, 0, 25, e_n) \quad \textit{for all} \;\; G \subset \mathbb{D}_1 \;\; \textit{borel set} \end{aligned} \tag{3.10}$$

where we used Corollary 1.7 (because, in Corollary 2.16, M is defined as $\mathbf{C}\,(0, 1, e_n) \cap \partial^* E$). Let us start the construction of the wished Lipschitz function, fix $y \in M_0, x \in M$ and let $\|v\|_{e_n} = \max\{|\mathbf{p}v|, |\mathbf{q}v|\}, \forall v \in \mathbb{R}^n$. It is straightforward to verify that $\partial E_{y,\|y-x\|_{e_n}} = \frac{\partial E - y}{\|y-x\|_{e_n}}$. Therefore, since $y \in M \subset \partial E$, we have that

$$0 \in \partial E_{y,\|y-x\|_{e_n}} \tag{3.11}$$

By Proposition 1.3, we find that $E_{y,\|y-x\|_{e_n}}$ is a $(\Lambda_{y,x}, r_{y,x}) - minimizing$

set in
$\mathbf{C}\left(-y, 25\|y-x\|_{e_n}^{-1}, e_n\right)$ with $\Lambda_{y,x} r_{y,x} \le 1$ where $\Lambda_{y,x} = \Lambda'\|y-x\|_{e_n}, r_{y,x} = \frac{r_0'}{\|y-x\|_{e_n}}$. Since $y, x \in \mathbf{C}(0,1,e_n)$, we have that $4\|y-x\|_{e_n} < 8$, thus, by the definition of M_0 and Proposition 2.6

$$\mathbf{e}\big(E_{y,\|y-x\|_{e_n}}, 0, 4, e_n\big) = \mathbf{e}(E, y, 4\|y-x\|_{e_n}, e_n) \le \delta_0(n) \tag{3.12}$$

We may assume that $\delta_0(n) \le \epsilon_0(n)$ in order to apply the Height Bound for $E_{y,\|y-x\|_{e_n}}$. Since, by 3.8

$$r_{y,x} = \frac{r_0'}{\|y-x\|_{e_n}} > \frac{25}{2} \quad \textit{and} \quad \mathbf{C}(0,4,e_n) \subset \mathbf{C}\left(-y, \frac{25}{\|y-x\|_{e_n}}, e_n\right)$$

we also have that $E_{y,\|y-x\|_{e_n}}$ is a $(\Lambda_{y,x}, 1)-\textit{minimizing}$ set in $\mathbf{C}(0,4,e_n)$. Then, applying the Height Bound (Theorem 2.17), we find that

$$\sup\left\{|\mathbf{q}v| : v \in \partial E_{y,\|y-x\|_{e_n}} \cap \mathbf{C}(0,1,e_n)\right\} \le C_0(n)\mathbf{e}\big(E_{y,\|y-x\|_{e_n}}, 0, 4, e_n\big)^{\frac{1}{2(n-1)}} \le$$
$$\le C_0(n)\delta_0(n)^{\frac{1}{2(n-1)}}$$

We want to apply this inequality for $\frac{x-y}{\|y-x\|_{e_n}}$. For this purpose, it is sufficient to recall that $x \in M = \mathbf{C}(0,1,e_n) \cap \partial E$. Then

$$|\mathbf{q}y - \mathbf{q}x| \le C_0(n)\delta_0(n)^{\frac{1}{2(n-1)}}\|y-x\|_{e_n}$$

We may consider $\delta_0(n) < \frac{1}{C_0(n)^{2(n-1)}}$ in order to have $\|y-x\|_{e_n} = |\mathbf{p}y - \mathbf{p}x|$, by the definition of $\|\cdot\|_{e_n}$. Putting it all together, we conclude that

$$|\mathbf{q}y - \mathbf{q}x| \le \overbrace{C_0(n)\delta_0(n)^{\frac{1}{2(n-1)}}}^{\doteq L_n}|\mathbf{p}y - \mathbf{p}x| \quad \textit{for all} \;\; y \in M_0, x \in M \tag{3.13}$$

By 3.13, we can deduce that $\mathbf{p}y = \mathbf{p}x$ implies $y = x$ for all $x, y \in M_0$. Then, the function $\overline{u} : \mathbf{p}(M_0) \to \mathbb{R}$ given by $\overline{u} = \mathbf{q} \circ \mathbf{p}^{-1}$ is well defined and also satisfy

$$|\overline{u}(\mathbf{p}y) - \overline{u}(\mathbf{p}x)| \le L_n|\mathbf{q}y - \mathbf{q}x| \quad \forall y, x \in M_0$$

From Whitney-MacShane Extension Theorem (Theorem 2.3 in [J.05]), we find an extension $u : \mathbb{R}^n \to \mathbb{R}$ of $\overline{u}$ with $Lip(u) \le L_n < 1$. For each $x \in M_0$ it holds $u(\mathbf{p}x) = \mathbf{q}x$, that is

$$M_0 \subset \Gamma = G(u) \cap \mathbf{C}(0,1,e_n) \tag{3.14}$$

what concludes the proof of 3.6 provided we prove 3.5, but 3.5 can be checked by 3.9 and truncating u. Let us prove 3.7. To this end, by the definition of M_0, we have that

$$y \in M \setminus M_0 \Leftrightarrow y \in M \quad \textit{and}\ \exists s_y \in (0,8) \;\; \textit{such that} \;\; \delta_0(n) < \mathbf{e}(E, y, s_y, e_n) \tag{3.15}$$

Applying the Besicovitch's Covering Theorem (Theorem 1.27 in [LCE92]) for the family of balls $\left\{\mathbf{B}\big(y, \sqrt{2}s_y\big)\right\}_{y \in M\setminus M_0}$, we find N_n (constant depending only of n) countable families of disjoint closed balls $\mathcal{F}_1, ..., \mathcal{F}_{N_n}$ such that

$$M \setminus M_0 \subset \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{F}_i} B$$

Thus we set k such that it maximizes $\sum_{B \in \mathcal{F}_i} \mathcal{H}_{n-1}(B)$. Denoting $\mathcal{F}_k$ by $\left\{\overline{\mathbf{B}\big(y_h, \sqrt{2}s_h\big)}\right\}_{h \in \mathbb{N}}$, we have that

$$\begin{aligned} \mathcal{H}_{n-1}(M \setminus M_0) \leq \sum_{i=1}^{N_n} \sum_{B \in \mathcal{F}_i} \mathcal{H}_{n-1}(B) \leq \\ \leq N_n \sum_{h \in \mathbb{N}} \mathcal{H}_{n-1}\left(\overline{\mathbf{B}\Big(y_h, \sqrt{2}s_h\Big)}\right) \end{aligned} \tag{3.16}$$

Since $y_h \in \mathbf{C}(0, 1, e_n)$ and $s_h < 8$, we find that $\overline{\mathbf{B}\big(y_h, \sqrt{2}s_h\big)} \subset \mathbf{C}(0, 25, e_n)$. In order to apply the density estimates (Theorem 1.5) at the inequalities below, note that $\mathbf{B}\big(y_h, (1+\frac{1}{n})\sqrt{2}s_h\big) \Subset \mathbf{C}(0, 25, e_n)$. Recalling the $\mathcal{H}_{n-1}$-equivalence between ∂E and $\partial^* E$, we have that

$$\mathcal{H}_{n-1}(M \setminus M_0) \leq^{3.16} N_n \sum_{h \in \mathbb{N}} \mathcal{H}_{n-1}\left(\overline{\mathbf{B}\Big(y_h, \sqrt{2}s_h\Big)}\right) \leq$$

$$\leq N_n \sum_{h \in \mathbb{N}} \mathcal{H}_{n-1}\left(\mathbf{B}\left(y_h, (1+\frac{1}{n})\sqrt{2}s_h\right)\right) \leq 3nN_n\omega_n\left(\sqrt{2}(1+\frac{1}{n})\right)^{n-1} \sum_{h \in \mathbb{N}} s_h^{n-1}$$

By 3.15, the last inequalities and $M \setminus \Gamma \subset M \setminus M_0$ (from **??**), we conclude that

$$\begin{aligned} \mathcal{H}_{n-1}(M \setminus \Gamma) \leq \mathcal{H}_{n-1}(M \setminus M_0) \leq 3nN_n\omega_n\left(\sqrt{2}(1+\frac{1}{n})\right)^{n-1} \sum_{h \in \mathbb{N}} s_h^{n-1} \\ \leq \frac{3nN_n\omega_n\left(\sqrt{2}(1+\frac{1}{n})\right)^{n-1}}{\delta_0(n)} \sum_{h \in \mathbb{N}} s_h^{n-1}\mathbf{e}(E, y_h, s_h, e_n) \end{aligned} \tag{3.17}$$

Since $\mathbf{C}(y_h, s_h, e_n) \subset \mathbf{C}(0, 25, e_n)$, we can state that

$$\sum_{h \in \mathbb{N}} s_h^{n-1}\mathbf{e}(E, y_h, s_h, e_n) \leq \sum_{h \in \mathbb{N}} \int_{\mathbf{C}(y_h, s_h, e_n) \cap \partial^* E} \frac{|\nu_E - e_n|^2}{2} \,\mathrm{d}\mathcal{H}_{n-1} \leq$$

$$\leq \int_{\cup_{h \in \mathbb{N}} \mathbf{C}(y_h, s_h, e_n) \cap \partial^* E} \frac{|\nu_E - e_n|^2}{2} \,\mathrm{d}\mathcal{H}_{n-1} \leq \int_{\mathbf{C}(0, 25, e_n) \cap \partial^* E} \frac{|\nu_E - e_n|^2}{2} \,\mathrm{d}\mathcal{H}_{n-1}$$

Thus, by 3.17, we can show that

$$\mathcal{H}_{n-1}(M \setminus \Gamma) \leq \overbrace{\frac{3nN_n 25^{n-1}\omega_n\left(\sqrt{2}(1+\frac{1}{n})\right)^{n-1}}{\delta_0(n)}}^{\doteq C^1(n)} \mathbf{e}(E,0,25,e_n) \tag{3.18}$$

We can calculate the area of a graph (of codimension 1) of a Lipschitz functions (Theorem 9.1 in [Mag12]) as follows

$$\mathcal{H}_{n-1}(\Gamma \setminus M) = \int_{\mathbf{p}(\Gamma\setminus M)} \sqrt{1+|\nabla u|^2}\, \mathrm{d}\mathcal{H}_{n-1}$$

Recalling that $Lip\,(u) < 1$, we can conclude that

$$\begin{aligned} \mathcal{H}_{n-1}(\Gamma \setminus M) &\leq \sqrt{2}\,\mathcal{H}_{n-1}(\mathbf{p}(\Gamma \setminus M)) \leq \\ &\leq^{3.10} \sqrt{2}\,\mathcal{H}_{n-1}\left(M \cap \mathbf{p}^{-1}(\mathbf{p}(\Gamma \setminus M))\right) \end{aligned} \tag{3.19}$$

We now notice that $x \in M \cap \mathbf{p}^{-1}(\mathbf{p}(\Gamma \setminus M)) \Rightarrow x \in M$ and $\exists z \in \Gamma \setminus M$ such that $\mathbf{p}z = \mathbf{p}x$. If we assume $x \in \Gamma$, we can write $x = (\mathbf{p}x, u(\mathbf{p}x)) = (\mathbf{p}z, u(\mathbf{p}z)) = z$ that is a contradiction, because $z \notin M$ and $x \in M$. We have then $x \in M \setminus \Gamma$ what implies

$$M \cap \mathbf{p}^{-1}(\mathbf{p}(\Gamma \setminus M)) \subset M \setminus \Gamma$$

From 3.19, the last inclusion and $M\Delta\Gamma = (\Gamma \setminus M) \cup (M \setminus \Gamma)$, we conclude the proof of 3.7 as follows

$$\mathcal{H}_{n-1}(M\Delta\Gamma) \leq \mathcal{H}_{n-1}(\Gamma \setminus M) + \mathcal{H}_{n-1}(M \setminus \Gamma) \leq$$

$$\sqrt{2}\,\mathcal{H}_{n-1}(M \setminus \Gamma) + \mathcal{H}_{n-1}(M \setminus \Gamma) \leq^{3.18} \sqrt{2}C^1(n)\mathbf{e}(E,0,25,e_n)$$

□

The next result will state that the function u, from the last theorem, has interesting estimations over ∇u which will allow us to approximate u by Harmonic Functions as a consequence of some results that we will show in the next section.

Proposition 3.4. ***(Almost harmonicness of the approximation)*** *The function u and the constant $C_1(n)$ in the last theorem also satisfy that $\forall \phi \in C_c^1(\mathbb{D}_r)$*

$$\frac{1}{r^{n-1}} \int_{\mathbb{D}_r} |\nabla u|^2 \leq C_1(n)\, \boldsymbol{e}(E, x_0, 25r, e_n) \tag{3.20}$$

$$\frac{1}{r^{n-1}} \left| \int_{\mathbb{D}_r} \nabla u \cdot \nabla \phi \right| \leq C_1(n) \sup_{\mathbb{D}_r} |\nabla \phi| \Big(\boldsymbol{e}(E, x_0, 25r, e_n) + \Lambda r \Big) \tag{3.21}$$

Proof. We will use the same reduction that we have done in the proof of the Lipschitz Approximation Theorem (Theorem 3.3). By Theorem 4.3 in [Sim83] and the fact that $\partial^* E$ and ∂E are $\mathcal{H}_{n-1}$-equivalent, we find for $\mathcal{H}_{n-1}$-almost everywhere $x \in M \cap \Gamma$ that $T_x \partial^* E = T_x \Gamma$ what implies

$$\nu_E(x) = \lambda_x \frac{(-\nabla u(\mathbf{p}x), 1)}{\sqrt{1 + |\nabla u(\mathbf{p}x)|^2}} \quad \textit{with } \lambda_x \in \{1, -1\} \tag{3.22}$$

Taking $Lip\,(u) < 1$ into account, we find out that

$$\int_{\mathbb{D}_1} |\nabla u|^2 \leq \int_{\mathbf{p}(M \cap \Gamma)} |\nabla u|^2 + \int_{\mathbf{p}(M \Delta \Gamma)} |\nabla u|^2 \leq$$

$$\leq \int_{\mathbf{p}(M \cap \Gamma)} \frac{\sqrt{2}|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} + \mathcal{H}_{n-1}\left(\mathbf{p}(M \Delta \Gamma)\right)$$

Since $Lip\,(\mathbf{p}) = 1$ and 3.7 holds true, due to the behavior of Hausdorff measure under Lipschitz maps (Theorem 2.3 in [LCE92]), we can properly estimate $\mathcal{H}_{n-1}\left(\mathbf{p}(M \Delta \Gamma)\right)$. Then, by Theorem 9.1 in [Mag12], we have that

$$\begin{aligned} \int_{\mathbb{D}_1} |\nabla u|^2 &\leq C_1(n)\mathbf{e}(E, 0, 25, e_n) + \int_{M \cap \Gamma} \frac{\sqrt{2}|\nabla u \circ \mathbf{p}|^2}{1 + |\nabla u \circ \mathbf{p}|^2} \\ &= C_1(n)\mathbf{e}(E, 0, 25, e_n) + \int_{M \cap \Gamma} \sqrt{2}|\mathbf{p}\nu_E|^2 \end{aligned} \tag{3.23}$$

Since $(a-1)^2 \geq 0$ implies $1 - a \geq \frac{1-a^2}{2}$ for all $a \in \mathbb{R}$, we can affirm that

$$\frac{|\nu_E - e_n|^2}{2} = 1 - \nu_E \cdot e_n \geq \frac{1 - (\nu_E \cdot e_n)^2}{2} =^* |\mathbf{p}\nu_E|^2$$

where (*) follows from $|\nu_E|^2 = |\mathbf{p}\nu_E|^2 + |\mathbf{q}\nu_E|^2$. Using the last inequality in 3.23 and the change of scale in the excess (Proposition 2.5), we can conclude the proof of 3.20 as follows

$$\int_{\mathbb{D}_1} |\nabla u|^2 \leq C_1(n)\mathbf{e}(E, 0, 25, e_n) + \int_{M \cap \Gamma} \sqrt{2}\frac{|\nu_E - e_n|^2}{2} \leq$$

$$\leq \max\{C_1(n), 25^{n-1}\sqrt{2}\}\mathbf{e}(E, 0, 25, e_n)$$

In order to prove 3.21, since $\frac{1}{\sqrt{1+|\nabla u|^2}} \leq 1$ and 3.7 is validated, we notice that it suffices to prove

$$\left| \int_{\mathbb{D}_1} \frac{\nabla u \cdot \nabla \phi}{\sqrt{1 + |\nabla u|^2}} \right| \leq C_1'(n) \sup_{\mathbb{D}_1} |\nabla \phi| \Big(\mathcal{H}_{n-1}\left(M \Delta \Gamma\right) + \Lambda r \Big)$$

for all $\phi \in C_c^1(\mathbb{D}_1)$. For $\mathcal{H}_{n-1}$-almost everywhere $x \in M \cap \Gamma$, notice that

$$(\nu_E \cdot \nabla \phi)(\nu_E \cdot e_n) = \frac{\nabla u(\mathbf{p}x) \cdot \nabla \phi(\mathbf{p}x)}{1 + |\nabla u(\mathbf{p}x|^2} \tag{3.24}$$

what follows from 3.22, we also note that we can suppose $\sup_{\mathbb{D}_1} |\nabla \phi| = 1$ without loss of generality. Let us state a claim which will be proved later.

Claim 1:

$$\left|\int_M (\nu_E \cdot \nabla\phi)(\nu_E \cdot e_n)\right| \leq C^c(n)\Big(\mathcal{H}_{n-1}\left(M\Delta\Gamma\right) + \Lambda r\Big)$$

By the connectedness of $\mathbf{p}^{-1}(z) \cap \mathbf{C}\left(0,1,e_n\right), \forall z \in \mathbb{D}_1$, together with Theorem 2.13, $\mathbf{p}(M)$ and $\mathbb{D}_1$ are $\mathcal{H}_{n-1}$-equivalent what ensures that

$$\left|\int_{\mathbb{D}_1} \frac{\nabla u \cdot \nabla\phi}{\sqrt{1+|\nabla u|^2}}\right| = \left|\int_{\mathbf{p}(M)} \frac{\nabla u \cdot \nabla\phi}{\sqrt{1+|\nabla u|^2}}\right| \leq$$

$$\leq \left|\int_{\mathbf{p}(M\setminus\Gamma)} \frac{\nabla u \cdot \nabla\phi}{\sqrt{1+|\nabla u|^2}}\right| + \left|\int_{\mathbf{p}(M\cap\Gamma)} \frac{\nabla u \cdot \nabla\phi}{\sqrt{1+|\nabla u|^2}}\right| \leq^*$$

$$\leq \mathcal{H}_{n-1}\left(M \setminus \Gamma\right) + \left|\int_{\mathbf{p}(M\cap\Gamma)} \frac{\nabla u \cdot \nabla\phi}{\sqrt{1+|\nabla u|^2}}\right|$$

where in (*) we have used that $|\frac{\nabla u \cdot \nabla\phi}{\sqrt{1+|\nabla u|^2}}| \leq 1$ and the behavior of the Hausdorff measure under Lipschitz maps (Theorem 2.3 in [LCE92]). From the last inequality and Theorem 9.1 in [Mag12], we have that

$$\left|\int_{\mathbb{D}_1} \frac{\nabla u \cdot \nabla\phi}{\sqrt{1+|\nabla u|^2}}\right| \leq \mathcal{H}_{n-1}\left(M \setminus \Gamma\right) + \left|\int_{M\cap\Gamma} \frac{\nabla u \circ \mathbf{p} \cdot \nabla\phi \circ \mathbf{p}}{1+|\nabla u \circ \mathbf{p}|^2}\right|$$

Then, by 3.24 and with the help of Claim 1, we obtain that

$$\left|\int_{\mathbb{D}_1} \frac{\nabla u \cdot \nabla\phi}{\sqrt{1+|\nabla u|^2}}\right| \leq \mathcal{H}_{n-1}\left(M \setminus \Gamma\right) + \left|\int_{M\cap\Gamma} (\nu_E \cdot \nabla\phi)(\nu_E \cdot e_n)\right| \leq$$

$$\leq \mathcal{H}_{n-1}\left(M\Delta\Gamma\right) + C^c(n)\Big(\mathcal{H}_{n-1}\left(M\Delta\Gamma\right) + \Lambda r\Big)$$

choosing wisely the constant depending only on the dimension n, we conclude the proof of the proposition. Now, we turn our attention to the proof of Claim 1.

Proof of Claim 1: We recall our assumption on ϕ, i.e. $\sup_{\mathbb{D}_1} |\nabla\phi| = 1$, then, the Mean Value Theorem (in several variables) yields

$$\sup_{\mathbb{D}_1} |\phi| \leq \sup_{\mathbb{D}_1} |x| \sup_{\mathbb{D}_1} |\nabla\phi| = 1$$

Now, we choose $\alpha \in C_c^\infty((-1,1))$ with

$$0 \leq \alpha(s) \leq 1, \forall s \in (-1,1)$$

and

$$\alpha(s) = 1, \forall s \in [-\frac{1}{4}, \frac{1}{4}]$$

Then, fix $t \in (-\frac{1}{5}, \frac{1}{5})$ and define

$$g_t(s) = s + t\alpha(s), \forall s \in \mathbb{R}$$

In order to g_t be invertible on the real line, we require that $|\alpha'(s)| < 5, \forall s \in \mathbb{R}$. Indeed, we have for each $t \in (-1/5, 1/5)$ that

$$g_t'(s) = 1 + t\alpha'(s) > 0, \forall s \in \mathbb{R}$$

Hence, if we define

$$f_t(x) = x + t\alpha(\mathbf{q}x)\phi(\mathbf{p}x)e_n \quad \forall x \in \mathbb{R}^n$$

and $x = (x', x_n)$, we obtain that $f_t(x', x_n) = (x', g_{t\phi(x')}(x_n))$. For any $x' \in \mathbb{R}^{n-1}, t \in (-1/5, 1/5)$, since $t\phi(x') \in (-1/5, 1/5)$, we have that $g_{t\phi(x')}$ is invertible on $\mathbb{R}$. Thus, f_t is invertible as well and we note that

$$Jf_t(x) = \det \left(\begin{array}{c|c} id_{\mathbb{R}^{n-1}} & 0 \\ \hline 0 & g'_{t\phi(x')}(x_n) \end{array} \right) = g'_{t\phi(x')}(x_n) > 0$$

Then, $\{f_t\}_{t \in (-\frac{1}{5}, \frac{1}{5})}$ is a one-parameter family of diffeomorphisms. In order to prove that $f_t(E)$ is a competitor for the almost minimality of E, we remind that, since f_t is a diffeomorphism, $f_t(E)$ is a set of finite perimeter and $f_t(\partial^* E), \partial^* f_t(E)$ are $\mathcal{H}_{n-1}$-equivalent (Proposition 17.1 in [Mag12]). Denote by supp the support of a function (i.e. $\operatorname{supp} f = \{x \in dom(f) : f(x) \neq 0\}$). By the definition of f_t, $\operatorname{supp} \alpha \Subset (-1, 1)$ and $\operatorname{supp} \phi \Subset \mathbb{D}_1$, we find that

$$\operatorname{supp}(f_t - id_{\mathbb{R}^n}) \subset \operatorname{supp}(\alpha \circ \mathbf{q}) \cap \operatorname{supp}(\phi \circ \mathbf{p}) \Subset \mathbf{C}(0, 1, e_n) \tag{3.25}$$

whenever $t \in (-\frac{1}{5}, \frac{1}{5})$. Then, we conclude that

$$f_t(E) \Delta E \subset \operatorname{supp}(f_t - id_{\mathbb{R}^n}) \Subset \mathbf{C}(0, 1, e_n)$$

Therefore, using the almost minimality condition of E, for all $\forall t \in (-\frac{1}{5}, \frac{1}{5})$,

$$\mathcal{P}(E, \mathbf{C}(0, 1, e_n)) \leq \mathcal{P}(f_t(E), \mathbf{C}(0, 1, e_n)) + \Lambda' |f_t(E) \Delta E| \tag{3.26}$$

From Lemma 17.9 in [Mag12], 3.25 and the compactness of $\operatorname{supp}(\alpha \circ \mathbf{q}) \cap \operatorname{supp}(\phi \circ \mathbf{p})$, we obtain the existence of constants $K(n)$ and $\epsilon_0 < \frac{1}{5}$ such that, for all $t \in (-\epsilon_0, \epsilon_0)$,

$$|f_t(E) \Delta E| \leq K(n)|t| \, \mathcal{P}(E, \operatorname{supp}(\alpha \circ \mathbf{q}) \cap \operatorname{supp}(\phi \circ \mathbf{p})) \leq$$

$$\leq K(n)|t| \, \mathcal{P}(E, \mathbf{C}(0, 1, e_n))$$

what, along with 3.26, provides

$$\begin{gathered} \mathcal{P}(E, \mathbf{C}(0, 1, e_n)) \leq \mathcal{P}(f_t(E), \mathbf{C}(0, 1, e_n)) + \\ + \Lambda' K(n)|t| \, \mathcal{P}(E, \mathbf{C}(0, 1, e_n)) \quad \forall t \in (-\epsilon_0, \epsilon_0) \end{gathered} \tag{3.27}$$

Putting into account $f_t(\partial^* E) \overset{\mathcal{H}_{n-1}}{\sim} \partial^* f_t(E), M \overset{\mathcal{H}_{n-1}}{\sim} \partial^* E \cap \mathbf{C}(0, 2, e_n)$ and $f_t(\mathbf{C}(0, 1, e_n)) = \mathbf{C}(0, 1, e_n)$ (because of 3.25), we get that

$$\mathcal{P}(f_t(E), \mathbf{C}(0, 1, e_n)) - \mathcal{P}(E, \mathbf{C}(0, 1, e_n)) =$$

$$= \mathcal{H}_{n-1}\left(\partial^* f_t(E) \cap \mathbf{C}\left(0,1,e_n\right)\right) - \mathcal{H}_{n-1}\left(\partial^* E \cap \mathbf{C}\left(0,1,e_n\right)\right) =$$
$$= \mathcal{H}_{n-1}\left(f_t(\partial^* E \cap \mathbf{C}\left(0,1,e_n\right))\right) - \mathcal{H}_{n-1}\left(M\right)$$

By the area formula for countably $(n-1)$-rectifiable sets (Theorem 11.6 in [Mag12]), we find that

$$\begin{aligned} &\mathcal{P}(f_t(E), \mathbf{C}\left(0,1,e_n\right)) - \mathcal{P}(E, \mathbf{C}\left(0,1,e_n\right)) = \\ &= \int_{\partial^* E \cap \mathbf{C}(0,1,e_n)} J^M f_t \,\mathrm{d}\mathcal{H}_{n-1} - \mathcal{H}_{n-1}\left(M\right) = \\ &= \int_M (J^M f_t - 1)\,\mathrm{d}\mathcal{H}_{n-1} \end{aligned} \tag{3.28}$$

whenever $t \in (-\frac{1}{5}, \frac{1}{5})$. Since $\alpha(s) = 1, \forall s \in [-\frac{1}{4}, \frac{1}{4}]$ and $\mathbf{q}x < \frac{1}{4}, \forall x \in M$ (from 2.6), we obtain that

$$f_t(x) = x + t\phi(\mathbf{p}(x))e_n$$

It is straightforward to verify that, for all $v = (v', v_n) \in \mathbb{R}^n$,

$$D(f_t)_x v = id_{\mathbb{R}^n} v + t(v' \cdot \nabla\phi \circ \mathbf{p}(x))e_n =$$

$$= id_{\mathbb{R}^n} v + t(e_n \otimes \nabla\phi \circ \mathbf{p}(x))v \quad \textit{whenever } x \in M$$

Since $0 \neq |\nabla\phi \circ \mathbf{p}(x)| \leq 1$ and $|t| < 1/5$, we have $|t| < |\nabla\phi \circ \mathbf{p}(x)|^{-1}$. Thus, if $\mathbf{p}_1$ denotes the orthogonal projection onto $(\nu_E)^\perp$, i.e.

$$\mathbf{p}_1(\nabla\phi \circ \mathbf{p}) = \nabla\phi \circ \mathbf{p} - (\nabla\phi \circ \mathbf{p} \cdot \nu_E)\nu_E$$

by Lemma 23.10 in [Mag12], we find out that

$$J^M f_t(x) - 1 = J^{(\nu_E(x))^\perp}\Big(id_{\mathbb{R}^n}(x) + te_n \otimes \phi \circ \mathbf{p}(x)\Big) - 1 \leq$$

$$\leq t\mathbf{p}_1(\nabla\phi \circ \mathbf{p}(x)) \cdot e_n + L(n)|t\nabla\phi \circ \mathbf{p}(x)|^2$$

for $\mathcal{H}_{n-1}$-a.e. $x \in M$. Note that, if $|\nabla\phi \circ \mathbf{p}(x)| = 0$, the inequality above is trivially verified because of that $J^M f_t(x) = 1$. Therefore, since $\sup_{\mathbb{D}_1} |\nabla\phi| = 1$ and $e_n \cdot \nabla\phi \circ \mathbf{p} = 0$, we have that

$$J^M f_t - 1 \leq -t(\nu_E \cdot e_n)(\nu_E \cdot \nabla\phi) + L(n)t^2$$

for $\mathcal{H}_{n-1}$-a.e. $x \in M$. The last inequality and 3.28 ensure that

$$\mathcal{P}(f_t(E), \mathbf{C}\left(0,1,e_n\right)) - \mathcal{P}(E, \mathbf{C}\left(0,1,e_n\right)) \leq$$

$$\leq -t \int_M (\nu_E \cdot e_n)(\nu_E \cdot \nabla\phi)\,\mathrm{d}\mathcal{H}_{n-1} + L(n)t^2\,\mathcal{H}_{n-1}\left(M\right)$$

We can find a upper bound $L'(n)$ of $\mathcal{H}_{n-1}\left(M\right)$ by the density estimates (Corollary 1.8), then

$$\mathcal{P}(f_t(E), \mathbf{C}\left(0,1,e_n\right)) - \mathcal{P}(E, \mathbf{C}\left(0,1,e_n\right)) \leq -t \int_M (\nu_E \cdot e_n)(\nu_E \cdot \nabla\phi)\,\mathrm{d}\mathcal{H}_{n-1} + L^1(n)t^2$$

Finally, we put this inequality into 3.27 to produce, for all $t \in (-\epsilon_0, \epsilon_0)$,

$$\mathcal{P}(E, \mathbf{C}(0,1,e_n)) \leq \mathcal{P}(E, \mathbf{C}(0,1,e_n)) - t\int_M (\nu_E \cdot e_n)(\nu_E \cdot \nabla\phi)\,\mathrm{d}\mathcal{H}_{n-1} +$$

$$+L^1(n)t^2 + \Lambda' K(n)|t|\,\mathcal{P}(E, \mathbf{C}(0,1,e_n))$$

We have already noticed that $\mathcal{H}_{n-1}(M) = \mathcal{H}_{n-1}(\partial^* E \cap \mathbf{C}(0,1,e_n)) = \mathcal{P}(E, \mathbf{C}(0,1,e_n))$ what guarantee that $L'(n)$ also bounds $\mathcal{P}(E, \mathbf{C}(0,1,e_n))$, thus

$$t\int_M (\nu_E \cdot e_n)(\nu_E \cdot \nabla\phi)\,\mathrm{d}\mathcal{H}_{n-1} \leq L^1(n)t^2 + \Lambda' K(n)L'(n)|t|$$

$$\leq \overbrace{\max\{L^1(n), K(n)L'(n)\}}^{\doteq C^c(n)}\Big(t^2 + \Lambda'|t|\Big)$$

since it holds for all $t \in (-\epsilon_0, \epsilon_0)$, we have that

$$|t|\left|\int_M (\nu_E \cdot e_n)(\nu_E \cdot \nabla\phi)\,\mathrm{d}\mathcal{H}_{n-1}\right| \leq C^c(n)\Big(t^2 + \Lambda'|t|\Big)$$

Dividing by $|t|$ on both sides and taking

$$0 < |t| < \min\{\epsilon_0, \mathcal{H}_{n-1}(M\Delta\Gamma) + \Lambda'\}$$

we conclude the proof of the Claim. □

3.3 Approximations results on harmonic functions

We will gather two technical results about Harmonic Functions that will be used together with the idea presented in the last chapter about the almost harmonicness of the Lipschitz approximation. First of all, let us remember the mean value property of harmonic functions, that is, v harmonic function on $\mathbf{B}(0,1) \subset \mathbb{R}^n$, then

$$v(x) = \fint_{\partial\mathbf{B}(x,r)} v\,\mathrm{d}\mathcal{H}_{n-1} = \fint_{\mathbf{B}(x,r)} v, \qquad \forall \mathbf{B}(x,r) \Subset \mathbf{B}(0,1) \tag{3.29}$$

the proof for this result can be found in [Fol95] (Theorem 2.8 and Corollary 2.9) .

Lemma 3.5. *If v is a harmonic function in $\boldsymbol{B}(0,1)$ and $w(x) = v(0) + x \cdot \nabla v(0), \forall x \in \boldsymbol{B}(0,1)$, then*

$$\sup_{\boldsymbol{B}\left(0,\frac{1}{2}\right)} |\nabla v| \leq H(n)\|v\|_{L^2(\boldsymbol{B}(0,1))} \tag{3.30}$$

and $\forall \alpha \in \left(0, \frac{1}{2}\right]$

$$\sup_{\boldsymbol{B}(0,\alpha)} |v - w|^2 \leq H(n)\alpha^2 \|\nabla v\|_{L^2(\boldsymbol{B}(0,1))}$$

Proof. Let v be harmonic in $\mathbf{B}(0,1)$, $x \in \mathbf{B}\left(0, \frac{1}{2}\right)$ and $\eta \in \mathbb{S}^{n-1}$. We claim that $\eta \cdot \nabla v$ is harmonic in $\mathbf{B}(0,1)$, indeed

$$div\Big(\nabla(\eta \cdot \nabla v)\Big) = div\Big((\partial_1(\eta \cdot \nabla v), ..., \partial_n(\eta \cdot \nabla v))\Big) =$$

$$= \sum_{i=1}^{n} \partial_i^2(\eta \cdot \nabla v) = \sum_{i,j=1}^{n} \eta_j \partial_j \partial_i^2 v =$$

$$= \sum_{j=1}^{n} \eta_j \partial_j (\Delta v) \equiv 0$$

We are now able to apply the mean value property for $\eta \cdot \nabla v$ (3.29), thus

$$|\eta \cdot \nabla v(x)| = \frac{1}{r^n \omega_n} \left| \int_{\mathbf{B}(0,r)} \eta \cdot \nabla v \right| = \frac{1}{r^n \omega_n} \left| \eta \cdot \int_{\mathbf{B}(0,r)} \nabla v \right| =$$

$$=^* \frac{1}{r^n \omega_n} \left| \eta \cdot \int_{\partial \mathbf{B}(0,r)} v \nu_{\mathbb{S}^{n-1}} \right| = \frac{1}{r^n \omega_n} \left| \int_{\partial \mathbf{B}(0,r)} v \eta \cdot \nu_{\mathbb{S}^{n-1}} \right| \leq$$

$$\leq \frac{1}{r^n \omega_n} \int_{\partial \mathbf{B}(0,r)} |v \eta \cdot \nu_{\mathbb{S}^{n-1}}| \leq \frac{1}{r^n \omega_n} \int_{\partial \mathbf{B}(0,r)} |v(y)| \, \mathcal{H}_{n-1}(y)$$

whenever $r < \frac{1}{4}$ what put us in position to apply in (*) the divergence theorem (Theorem 0.4 in [Fol95]). Applying the mean value property on the last inequality ensures that

$$|\eta \cdot \nabla v(x)| \leq \frac{1}{r^n \omega_n} \int_{\partial \mathbf{B}(0,r)} | \fint_{\mathbf{B}(y,r)} v(z) \, \mathcal{H}_{n-1}(z) \, | \, \mathcal{H}_{n-1}(y) \leq$$

$$\leq \frac{n \omega_n r^{n-1}}{\omega_n^2 r^{2n}} \int_{\mathbf{B}(x,2r)} |v(z)| \, \mathcal{H}_{n-1}(z) \leq^* H^1(n) \|v\|_{L^2(\mathbf{B}(0,1))}$$

where in (*) we have used the Holder's inequality (6.2 in [Fol99, p. 174]). To conclude the proof of 3.30, it is suffices to set $\eta = \frac{\nabla v}{|\nabla v|}$. As we have noticed above, $\eta \cdot \nabla v$ is harmonic on $\mathbf{B}(0,1)$, then we can apply 3.30 leading to

$$\sup_{\mathbf{B}\left(0,\frac{1}{2}\right)} |\nabla^2 v| \leq H^1(n) \|\nabla v\|_{L^2(\mathbf{B}(0,1))}$$

By Taylor's Theorem with Lagrange Remainder, we can find, for all $x \in$

$\mathbf{B}(0,\alpha)$, a number $t \in (0,1)$ such that

$$|v(x) - w(x)| \leq C|\nabla^2 v(tx)||x|^2$$

By the last two inequalities, the proof is concluded. □

Let $U \subset \mathbb{R}^n$ be a open set, then we will denote the Sobolev space by $W^{1,2}(U)$ as in Definition 4.2 in [LCE92].

Lemma 3.6. *For every $\tau > 0$ exists $\sigma(\tau) > 0$ such that if $u \in W^{1,2}(\boldsymbol{B}(0,1))$ satisfy both $\|u\|_{L^2(\boldsymbol{B}(0,1))} \leq 1$ and*

$$\left|\int_{\boldsymbol{B}(0,1)} \nabla u \cdot \nabla \phi\right| \leq \sigma(\tau) \sup_{\boldsymbol{B}(0,1)} |\nabla \phi| \quad \forall \phi \in C_c^\infty(\boldsymbol{B}(0,1)) \tag{3.31}$$

there exists v harmonic function on $\boldsymbol{B}(0,1)$ such that $\|\nabla v\|_{L^2(\boldsymbol{B}(0,1))} \leq 1$ and

$$\int_{\boldsymbol{B}(0,1)} |v-u|^2 \leq \tau$$

Proof. By contradiction, suppose that exist $\tau > 0$ and a sequence $\{u_h\}_{h\in\mathbb{N}} \subset W^{1,2}(\mathbf{B}(0,1))$ in the conditions above with $\sigma_h(\tau) = \frac{1}{h}$ such that for every v harmonic function with $\|\nabla v\|_{L^2(\mathbf{B}(0,1))} \leq 1$ it holds that

$$\int_{\mathbf{B}(0,1)} |u_h - v|^2 > \tau > 0 \tag{3.32}$$

From the classical Poincaré inequality (Theorem 4.9 in [LCE92]), we have that

$$\|u_h - \fint_{\mathbf{B}(0,1)} u_h\|_{L^{2^*}(\mathbf{B}(0,1))} \leq \overbrace{\omega_n^{\frac{1}{2^*}-\frac{1}{2}} C_2(n)}^{\doteq C_2'(n)} \|\nabla u_h\|_{L^2(\mathbf{B}(0,1))}$$

Since $\mathbf{B}(0,1)$ has finite $\mathcal{H}_n$ measure and $2 < 2^*$, we have that $\|\cdot\|_{L^2(\mathbf{B}(0,1))} \leq \|\cdot\|_{L^{2^*}(\mathbf{B}(0,1))}$. Then we can conclude that

$$\|u_h - \fint_{\mathbf{B}(0,1)} u_h\|_{L^2(\mathbf{B}(0,1))} \leq C_2'(n)\|\nabla u_h\|_{L^2(\mathbf{B}(0,1))} \leq^* C_2'(n) \tag{3.33}$$

where (*) follows from our assumptions on the $L^2(\mathbf{B}(0,1))$-norm of u_h. Since $\|\nabla u_h\|_{L^2(\mathbf{B}(0,2))} \leq 1$ and

$$\nabla(u_h - \fint_{\mathbf{B}(0,1)} u_h) = \nabla u_h \tag{3.34}$$

we find that $\{u_h - \fint_{\mathbf{B}(0,1)} u_h\}_{h\in\mathbb{N}}$ is bounded in $W^{1,2}(\mathbf{B}(0,1))$. From the compactness of the inclusion (Theorem 4.11 in [LCE92])

$$W^{1,2}(\mathbf{B}(0,1)) \hookrightarrow L^2(\mathbf{B}(0,1))$$

we can extract a subsequence of

$$u_h - \fint_{\mathbf{B}(0,1)} u_h$$

which converges, in the $L^2(\mathbf{B}(0,1))$ sense, to a function $u \in L^2(\mathbf{B}(0,1))$. Since $\|\nabla u_h\|_{L^2(\mathbf{B}(0,2))} \leq 1$ for all $h \in \mathbb{N}$, i.e. $\{\nabla u_h\}_{h\in\mathbb{N}}$ is bounded in $L^2(\mathbf{B}(0,1))$, we can apply the Banach-Alaoglu Theorem (Theorem 3.17 in [Rud91]) and thus, up to extract a subsequence, we find $v \in L^2(\mathbf{B}(0,1))$ such that ∇u_h convergences in the weak-topology of $L^2(\mathbf{B}(0,1))$ to v. Since the weak-convergence directly implies the convergence in the distributional sense, we have that $\nabla u = v$ and then $u \in W^{1,2}(\mathbf{B}(0,1))$. We also have that the norm is lower-semicontinuous, then

$$\|\nabla u\|_{L^2(\mathbf{B}(0,1))} \leq \liminf_{h\to\infty} \|\nabla u_h\|_{L^2(\mathbf{B}(0,1))} \leq 1$$

We want to prove that $-u + \fint_{\mathbf{B}(0,1)} u_h$ is harmonic, thereby obtaining a contradiction from $u_h - \fint_{\mathbf{B}(0,1)} u_h \to u$ in $L^2(\mathbf{B}(0,1))$ and 3.32. Since $\fint_{\mathbf{B}(0,1)} u_h$ is constant, it is sufficient to prove that u is harmonic. To this end, for all $\phi \in C_c^\infty(\mathbf{B}(0,1))$ note that

$$\left|\int_{\mathbf{B}(0,1)} \nabla u \cdot \nabla\phi\right| \leq \left|\int_{\mathbf{B}(0,1)} \nabla u \cdot \nabla\phi - \int_{\mathbf{B}(0,1)} \nabla(u_h - \fint_{\mathbf{B}(0,1)} u_h) \cdot \nabla\phi\right| +$$
$$+ \left|\int_{\mathbf{B}(0,1)} \nabla(u_h - \fint_{\mathbf{B}(0,1)} u_h) \cdot \nabla\phi\right|$$

By 3.31 and 3.34, we get that $\left|\int_{\mathbf{B}(0,1)} \nabla(u_h - \fint_{\mathbf{B}(0,1)} u_h) \cdot \nabla\phi\right|$ is less or equal than $\frac{1}{h}\sup_{\mathbf{B}(0,1)} |\nabla\phi|$. We recall that

$$\int_{\mathbf{B}(0,1)} \nabla(u - (u_h - \fint_{\mathbf{B}(0,1)} u_h)) \cdot \nabla\phi = \int_{\mathbf{B}(0,1)} (u - (u_h - \fint_{\mathbf{B}(0,1)} u_h))\Delta\phi$$

whenever $\phi \in C_c^\infty(\mathbf{B}(0,1))$. Then, putting the last equations into account, we conclude that

$$\left|\int_{\mathbf{B}(0,1)} \nabla u \cdot \nabla\phi\right| \leq \left|\int_{\mathbf{B}(0,1)} (u - (u_h - \fint_{\mathbf{B}(0,1)} u_h)) \cdot \nabla\phi\right| + \frac{1}{h}\sup_{\mathbf{B}(0,1)} |\nabla\phi|$$

By $u_h - \fint_{\mathbf{B}(0,1)} u_h \to u$ in $L^2(\mathbf{B}(0,1))$, the Holder's inequality (6.2 in [Fol99, p. 174]) and letting $h \to \infty$ in the last inequality, we find that $|\int_{\mathbf{B}(0,1)} \nabla u \cdot \nabla\phi| = 0$ for any $\phi \in C_c^\infty(\mathbf{B}(0,1))$. Thus, we have proved that u is harmonic what is sufficient to conclude the proof of the lemma as noticed before. $\square$

Regularity theory

We now aim to refine some estimates on the excess of an almost minimizing set. The reverse Poincaré inequality will be required to the refinements that we intend to do, because of that, we shall enunciate it. Although, the proof of reverse Poincaré inequality will not be done in this work.
The main theorem of this chapter is, of course, the $C^{1,\gamma}$-regularity theorem which states, for each $\gamma \in (0, 1/2)$, that the boundary of an almost minimizing set which satisfies a boundedness condition on the excess is, in fact, the graph of a function $u \in C^{1,\gamma}$. To prove the $C^{1,\gamma}$-regularity theorem, we will state two results on the excess which improves what we have done in Chapter 2. Furthermore, this results will allow us to show that M_0 is equal to M with M and M_0 as they were defined in the Lipschitz Approximation (Theorem 3.3) and hence equal to the graph of a Lipschitz function u. The $C^{1,\gamma}$-regularity of u will outcome of the Excess Improvement (Theorem 4.4).

4.1 Reverse Poincaré inequality

Let us introduce another concept similar with the excess that we will call by *flatness*. The *cylindrical flatness* of a Caccioppoli set at $x \in \mathbb{R}^n$ with respect to $\nu \in \mathbb{S}^{n-1}$ at the scale $r > 0$ is given by

$$\mathbf{f}(E, x, r, \nu) = \inf_{c \in \mathbb{R}} \frac{1}{r^{n-1}} \int_{\mathbf{C}(x,r,\nu) \cap \partial^* E} \frac{|(y - x) \cdot \nu - c|}{r^2} \, \mathrm{d}\mathcal{H}_{n-1}(y) \qquad (4.1)$$

The flatness provides one way to measure how far, in L^2 distance, the reduced boundary $\partial^* E$ of a Caccioppoli set is from the family of hyperplanes $\{y : (y-x) \cdot \nu = c\}$ inside the cylinder $\mathbf{C}(x, r, \nu)$. According to the requirements of this work, we shall prove only one property of the flatness despite it has some properties akin to the properties of the excess.

Lemma 4.1. ***(Flatness and Changes of Scale)*** *If $E \subset \mathbb{R}^n$ is a Caccioppoli, $x \in \mathbb{R}^n, r > s > 0$ and $\nu \in \mathbb{S}^{n-1}$, then*

$$\boldsymbol{f}(E, x, s, \nu) \leq \left(\frac{r}{s}\right)^{n-1} \boldsymbol{f}(E, x, r, \nu)$$

Proof. Since $\mathbf{C}(x, r, \nu) \subset \mathbf{C}(x, s, \nu)$ for all $r > s$, we have that

$$\inf_{c \in \mathbb{R}} \frac{1}{s^{n-1}} \int_{\mathbf{C}(x,s,\nu) \cap \partial^* E} \frac{|(y-x) \cdot \nu - c|}{r^2} \, \mathrm{d}\mathcal{H}_{n-1}(y) \leq$$

$$\leq \inf_{c \in \mathbb{R}} \frac{1}{s^{n-1}} \int_{\mathbf{C}(x,r,\nu) \cap \partial^* E} \frac{|(y-x) \cdot \nu - c|}{r^2} \, \mathrm{d}\mathcal{H}_{n-1}(y)$$

Multiplying the right side by $\frac{r^{n-1}}{r^{n-1}}$, we conclude the proof. □

We will not prove the reverse Poincaré inequality by virtue of its very extensive proof. In the statement of the next result, the constant $\omega(n, t)$ is the constant of the Small-excess position (Theorem 2.13).

Theorem 4.2. ***(Reverse Poincaré Inequality)*** *There exists a positive constant $C_p(n)$ such that if E is a (Λ, r_0) − minimizing in $\boldsymbol{C}(x_0, 4r, \nu)$ with $\Lambda r_0 \leq 1$, $x_0 \in \partial E, 4r < r_0$ and*

$$\boldsymbol{e}(E, x_0, 4r, \nu) \leq \omega\left(n, \frac{1}{8}\right)$$

then

$$\boldsymbol{e}(E, x_0, r, \nu) \leq C_p(n) \left(\boldsymbol{f}(E, x_0, 2r, \nu) + \Lambda r\right) \tag{4.2}$$

Proof. See Theorem 24.1 in [Mag12]. □

4.2 Excess revisited

In this section, we will prove the Excess Improvement by Tilting what gives a new estimate on the excess of a almost minimizing set. In short, under the assumption that the excess of E at x_0 with direction ν is bounded, the theorem provides, for each $0 < \alpha < \frac{1}{200}$, a direction ν_0 such that the excess at x_0 with direction ν_0 in scales reduced by α is bounded in terms of α and the excess at x_0 with direction ν. This result is one of the crucial steps in the proofs of the regularity theory as we will show.

Lemma 4.3. ***(Excess Improvement by Tilting)*** *Let $n \geq 3$. For all $\alpha \in \left(0, \frac{1}{200}\right)$, there exist constants $\epsilon_2(n,\alpha)$ and $C_2(n)$ such that if E is a $(\Lambda, r_0)-minimizing$ set in $\boldsymbol{C}(x_0, r, \nu)$ with $\Lambda r_0 \leq 1, r < r_0, x_0 \in \partial E$ and*

$$\boldsymbol{e}(E, x_0, r, \nu) + \Lambda r \leq \epsilon_2(n, \alpha)$$

then exists $\nu_0 \in \mathbb{S}^{n-1}$ such that

$$\boldsymbol{e}(E, x_0, \alpha r, \nu_0) \leq C_2(n)\left(\alpha^2 \boldsymbol{e}(E, x_0, r, \nu) + \alpha\Lambda r\right) \tag{4.3}$$

Proof. Let us suppose that the Lemma is proved $x_0 = 0, \nu = e_n, r = 25$. If we take E a $(\Lambda, r_0) - minimizing$ set in $\mathbf{C}(x_0, r, \nu)$ with $\Lambda r_0 \leq 1, r < r_0, x_0 \in \partial E$ and

$$\mathbf{e}(E, x_0, r, \nu) + \Lambda r \leq \epsilon_2(n, \alpha)$$

If T is the linear isometry which takes ν into e_n, by 1.3 and 1.4, we know that $T(E_{x_0, r/25})$ is a $\left(\frac{\Lambda r}{25}, \frac{25 r_0}{r}\right) - minimizing$ set in $\mathbf{C}(0, 25, e_n)$. From 2.6 and 2.7, we have that

$$\mathbf{e}\big(T(E_{x_0,\frac{r}{25}}), 0, 25, e_n\big) + \frac{\Lambda r}{25}25 = \mathbf{e}\big(E_{x_0,\frac{r}{25}}, 0, 25, \nu\big) + \Lambda r =$$

$$\mathbf{e}(E, x_0, r, \nu) + \Lambda r \leq \epsilon_2(n, \alpha)$$

Therefore, exists $\nu_0 \in \mathbb{S}^{n-1}$ such that

$$\mathbf{e}\big(T(E_{x_0,\frac{r}{25}}), 0, 25\alpha, \nu_0\big) \leq C_2(n)\Big(\alpha^2 \mathbf{e}\big(T(E_{x_0,\frac{r}{25}}), 0, 25, e_n\big) + \alpha\Lambda r\Big)$$

Thus,

$$\mathbf{e}\big(E, x_0, \alpha r, T^{-1}(\nu_0)\big) = \mathbf{e}\big(T(E_{x_0,\frac{r}{25}}), 0, 25\alpha, \nu_0\big) \leq$$

$$\leq C_2(n)\Big(\alpha^2 \mathbf{e}\big(T(E_{x_0,\frac{r}{25}}), 0, 25, e_n\big) + \alpha\Lambda r\Big) =$$

$$= C_2(n)\Big(\alpha^2 \mathbf{e}(E, x_0, r, \nu) + \alpha\Lambda r\Big)$$

Then, the reduction that we have made allows us to prove the lemma only considering that $x_0 = 0, \nu = e_n, r_0 > 25$ and E a $(\Lambda, r_0) - minimizing$ set in $\mathbf{C}(0, 25, e_n)$ with $\Lambda r_0 \leq 1, 25 < r_0$. Since $\frac{25}{4} < r_0$, provided we assume $\epsilon(n, \alpha) \leq \epsilon_0(n)$ and

$$\mathbf{e}(E, 0, 25, e_n) + 25\Lambda \leq \epsilon_2(n, \alpha) \tag{4.4}$$

we are able to use the Height Bound (Theorem 2.17). Setting $M = \partial E \cap \mathbf{C}(0, 1, e_n)$, we infer that

$$\sup\{|\mathbf{q}y| : y \in M\} \leq C_0(n)\mathbf{e}(E, 0, 25, e_n)^{\frac{1}{2(n-1)}} \tag{4.5}$$

If necessary, we reduce the size of $\epsilon_2(n, \alpha)$ to be less or equal than $\min\{\epsilon_0(n), \epsilon_1(n)\}$ in order to apply the Lipschtzian Approximation Theorem (Theorem 3.3 with $r = 1$). Therefore, we find the existence of a

Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}^n$ with $Lip\,(u) < 1$ such that satisfies

$$\mathcal{H}_{n-1}\,(M\Delta\Gamma) \le C_1(n)\mathbf{e}(E,0,25,e_n) \tag{4.6}$$

where $\Gamma = G(u) \cap \mathbf{C}\,(0,1,e_n)$. Moreover, from Proposition 3.4, we also find that

$$\int_{\mathbb{D}_1} |\nabla u|^2 \le C_1(n)\mathbf{e}(E,0,25,e_n)^{\frac{1}{2(n-1)}} \tag{4.7}$$

$$\left|\int_{\mathbb{D}_1} \nabla u \cdot \nabla\phi\right| \le C_1(n)\sup_{\mathbb{D}_1}|\nabla\phi|\Big(\mathbf{e}(E,0,25,e_n) + 25\Lambda\Big) \tag{4.8}$$

whenever $\phi \in C_c^\infty(\mathbb{D}_1)$. Setting

$$K = C_1(n)(\mathbf{e}(E,0,25,e_n) + 25\Lambda) \quad and \quad u_0 = \frac{u}{\sqrt{K}}$$

we obtain a Lipschitz function u_0 which, by 4.7 and 4.8, satisfies both

$$\|u_0\|^2_{L^2(\mathbb{D}_1)} = \int_{\mathbb{D}_1} |\nabla u_0|^2 \le \frac{C_1(n)\mathbf{e}(E,0,25,e_n)}{K} \le 1$$

and

$$\left|\int_{\mathbb{D}_1} \nabla u_0 \cdot \nabla\phi\right| \le \sup_{\mathbb{D}_1}|\nabla\phi|\frac{C_1(n)\Big(\mathbf{e}(E,0,25,e_n) + 25\Lambda\Big)}{\sqrt{K}} \le$$

$$\le \sqrt{K}\sup_{\mathbb{D}_1}|\nabla\phi|$$

Then $u_0 \in W^{1,2}(\mathbb{D}_1)$ and we can approximate u_0, in $L^2(\mathbb{D}_1)$-norm, by a harmonic function v, i.e. u_0 is in the conditions of Lemma 3.6 (setting $\tau = \alpha^{n+3}$) provided we assume

$$\sqrt{K} \le^{4.4} \sqrt{C_1(n)\epsilon_2(n,\alpha)} \le \sigma(\alpha^{n+3})$$

Thus there exists v harmonic function on $\mathbb{D}_1$ such that $\|v\|_{L^2(\mathbb{D}_1)} \le 1$ and

$$\int_{\mathbb{D}_1} |v - u_0|^2 \le \alpha^{n+3}$$

If we set $v_0 = \sqrt{K}v$, we get a harmonic function v_0 on $\mathbb{D}_1$ with $\|v_0\|_{L^2(\mathbb{D}_1)} \le \sqrt{K}$ and

$$\int_{\mathbb{D}_1} |v_0 - u|^2 \le K\alpha^{n+3} \tag{4.9}$$

Since $100\alpha < \frac{1}{2}$, from Lemma 3.5, we have that

$$\sup_{\mathbb{D}_{100\alpha}} |v_0 - w| \le \frac{(100\alpha)^2}{\omega_n}\overbrace{\|\nabla v_0\|_{L^2(\mathbb{D}_1)}}^{\le\sqrt{K}} \le$$

$$\le \sqrt{K}\frac{(100\alpha)^2}{\omega_n}$$

where $w(z) = v_0(0) + \nabla v_0(0) \cdot z, z \in \mathbb{D}_1$. By the last inequality, 4.9 and $(a+b)^2 \leq 2a^2 + 2b^2, \forall a, b > 0$, we conclude that

$$\int_{\mathbb{D}_{100\alpha}} |u-w|^2 = \|u-w\|^2_{L^2(\mathbb{D}_{100\alpha})} \leq \Big(\|u-v_0\|_{L^2(\mathbb{D}_{100\alpha})} + \|v_0-w\|_{L^2(\mathbb{D}_{100\alpha})} \Big)^2 \leq$$

$$\leq 2\|u-v_0\|^2_{L^2(\mathbb{D}_{100\alpha})} + 2\|v_0-w\|^2_{L^2(\mathbb{D}_{100\alpha})} \leq$$

$$\leq 2K\alpha^{n+3} + 2\,\mathcal{H}_{n-1}\left(\mathbb{D}_{100\alpha}\right) \left(\sqrt{K}\frac{(100\alpha)^2}{\omega_n}\right)^2 =$$

$$= 2K\left(\alpha^{n+3} + \frac{2\omega_{n-1}100^{n+3}\alpha^{n+3}}{\omega_n^2}\right)$$

Therefore

$$\frac{1}{\alpha^{n+1}} \int_{\mathbb{D}_{100\alpha}} |u-w|^2 \leq A(n)K\alpha^2 \tag{4.10}$$

where we set $A(n) = 2(1 + \frac{2\omega_{n-1}100^{n+3}}{\omega_n^2})$. We aim to prove that

$$\nu_0 = \frac{(-\nabla v_0(0), 1)}{\sqrt{1 + |\nabla v_0(0)|^2}}$$

is the direction we have searched for. For this purpose, let us state two claims which will be proved later.

Claim 1: If $K^{\frac{1}{n-1}} \leq \alpha^{n+3}$, we have that

$$\mathbf{f}\left(E, 0, 100\alpha, \nu_0\right) \leq A_1(n)K\alpha^2 \tag{4.11}$$

Claim 2: If we take $K^{\frac{1}{n-1}} \leq \alpha^{n+3}$ and $\epsilon_2(n,\alpha)$ suitably small, we find that

$$\mathbf{e}(E, 0, 100\alpha, \nu_0) \leq \omega(n, \frac{1}{8}) \tag{4.12}$$

We assume that K and $\epsilon_2(n,\alpha)$ are satisfying the requests in the Claims. Since $100\alpha < r_0$ and $\mathbf{C}\left(0, 100\alpha, \nu_0\right) \subset \mathbf{C}\left(0, 25, e_n\right)$, E is a $(\Lambda, r_0) - $ *minimizing* set in $\mathbf{C}\left(0, 100\alpha, \nu_0\right)$. Then, from 4.12, we are in the conditions to apply the reverse Poincaré inequality (Theorem 4.2) and deduce that

$$\mathbf{e}(E, 0, 25\alpha, \nu_0) \leq C_p(n)\Big(\mathbf{f}\left(E, 0, 50\alpha, \nu_0\right) + 25\Lambda\alpha\Big)$$

By 4.11, the change of scale on the *flatness* (Proposition 4.1) and the last inequality

$$\mathbf{e}(E, 0, 25\alpha, \nu_0) \leq$$

$$\leq C_p(n)\Big(\mathbf{f}\left(E, 0, 50\alpha, \nu_0\right) + 25\Lambda\alpha\Big) \leq$$

$$\leq C_p(n)\Big(2^{n-1}\mathbf{f}\left(E, 0, 100\alpha, \nu_0\right) + 25\Lambda\alpha\Big) \leq C_p(n)\Big(2^{n-1}A_1(n)K\alpha^2 + 25\Lambda\alpha\Big)$$

Recalling the definition of K and $\alpha < \frac{1}{200} < 25$, we get that

$$\mathbf{e}(E,0,25\alpha,\nu_0) \leq$$

$$\leq \overbrace{\max\{C_p(n)2^{n-1}A_1(n)C_1(n), C_p(n)\}}^{\doteq C^2(n)}\Big(\alpha^2\mathbf{e}(E,0,25,e_n)+\Lambda\alpha^2+25\Lambda\alpha\Big) \leq$$

$$\leq 2C^2(n)\Big(\alpha^2\mathbf{e}(E,0,25,e_n)+25\Lambda\alpha\Big)$$

what gives the wished direction and conclude the proof of the lemma. Let us prove the claims.

Proof of Claim 1: Suppose that $K^{\frac{1}{n-1}} \leq \alpha^{n+3}$ and set

$$c_0 = \frac{v_0(0)}{\sqrt{1+|\nabla v_0(0)|^2}}$$

By the $\mathcal{H}_{n-1}$-equivalence of ∂E and $\partial^* E$ in $\mathbf{C}\,(0,25,e_n)$ and the definition of *flatness*, we deduce that

$$\mathbf{f}\,(E,0,100\alpha,\nu_0) \leq \frac{1}{(100\alpha)^{n+1}}\int_{M\cap\mathbf{C}(0,100\alpha,\nu_0)} |y\cdot\nu_0 - c_0|^2\,\mathcal{H}_{n-1}\,(y)$$

We want to estimate the right side of the inequality in order to prove the claim. For this purpose, we will estimate the integral in both inside and outside the *graph* of u, i.e. in $M\cap\Gamma\cap\mathbf{C}\,(0,100\alpha,\nu_0)$ and $(M\setminus\Gamma)\cap\mathbf{C}\,(0,100\alpha,\nu_0)$.

Step 1: If $y=(z,u(z))\in\Gamma$, putting the definitions of ν_0, c_0 and w into account we find that

$$|y\cdot\nu_0 - c_0|^2 = \frac{|-z\cdot\nabla v_0(0)+u(z)-v_0(0)|^2}{1+|\nabla v_0(0)|^2} =$$

$$= \frac{|u(z)-w(z)|^2}{1+|\nabla v_0(0)|^2} \leq |u(z)-w(z)|^2$$

Therefore

$$\frac{1}{(100\alpha)^{n+1}}\int_{M\cap\Gamma\cap\mathbf{C}(0,100\alpha,\nu_0)} |y\cdot\nu_0 - c_0|^2\,\mathcal{H}_{n-1}\,(y) \leq$$

$$\frac{1}{(100\alpha)^{n+1}}\int_{M\cap\Gamma\cap\mathbf{C}(0,100\alpha,\nu_0)} |u(z)-w(z)|^2\,\mathcal{H}_{n-1}\,(y) \leq$$

$$\leq^{4.10} A(n)K\alpha^2$$

Step 2: We shall provided an estimate for $(M\setminus\Gamma)\cap\mathbf{C}\,(0,100\alpha,\nu_0)$. Keeping in mind the definitions of ν_0 and c_0, we find that

$$\frac{1}{(100\alpha)^{n+1}}\int_{M\setminus\Gamma\cap\mathbf{C}(0,100\alpha,\nu_0)} |y\cdot\nu_0 - c_0|^2\,\mathcal{H}_{n-1}\,(y) =$$

$$= \frac{1}{(100\alpha)^{n+1}} \int_{M\setminus\Gamma\cap\mathbf{C}(0,100\alpha,\nu_0)} \frac{|-\mathbf{p}y\cdot\nabla v_0(0)+\mathbf{q}y+v_0(0)|^2}{1+|\nabla v_0(0)|^2}\,\mathcal{H}_{n-1}(y) \leq$$

$$\leq^* \frac{1}{(100\alpha)^{n+1}} \int_{M\setminus\Gamma\cap\mathbf{C}(0,100\alpha,\nu_0)} |\mathbf{p}y\cdot\nabla v_0(0)|^2+|\mathbf{q}y|^2+|v_0(0)|^2\,\mathcal{H}_{n-1}(y) \leq$$

$$\leq \frac{1}{(100\alpha)^{n+1}}\mathcal{H}_{n-1}(M\setminus\Gamma)\left(\sup_{y\in M}\left(|\mathbf{q}y|^2\right)+\sup_{y\in M}\left(|\mathbf{p}y\cdot\nabla v_0(0)|^2\right)+|v_0(0)|^2\right)$$

where in (*) we have used that $\frac{1}{1+|\nabla v_0(0)|^2}\leq 1$, the triangle inequality and $(a+b+c)^2\leq 2a^2+2b^2+2c^2, \forall a,b,c>0$. By 4.5, 4.6 and the definition of K, we have that

$$\begin{aligned}\frac{1}{(100\alpha)^{n+1}} \int_{M\setminus\Gamma\cap\mathbf{C}(0,100\alpha,\nu_0)} |y\cdot\nu_0-c_0|^2\,\mathcal{H}_{n-1}(y) \leq \\ \frac{C_1(n)\mathbf{e}(E,0,25,e_n)}{(100\alpha)^{n+1}}\Big(C_0(n)^2\mathbf{e}(E,0,25,e_n)^{\frac{1}{(n-1)}}+ \\ +\sup_{y\in M}\left(|\mathbf{p}y\cdot\nabla v_0(0)|^2\right)+|v_0(0)|^2\Big)\leq \\ \leq \frac{A^1(n)K}{\alpha^{n+1}}\left(K^{\frac{1}{n-1}}+\sup_{y\in M}\left(|\mathbf{p}y\cdot\nabla v_0(0)|^2\right)+|v_0(0)|^2\right)\end{aligned} \tag{4.13}$$

where we have set $A^1(n)=\frac{C_1(n)}{100^{n+1}}\max\{C_0(n)^2,1\}$. In order to finish this *step*, we shall estimate $\sup_{y\in M}(|\mathbf{p}y\cdot\nabla v_0(0)|^2)+|v_0(0)|^2$. Note that $|\mathbf{p}y\cdot\nabla v_0(0)|^2\leq|\mathbf{p}y|^2|\nabla v_0(0)|^2\leq|\nabla v_0(0)|^2$, then we can reduce our effort to estimate $|\nabla v_0(0)|^2+|v_0(0)|^2$. From the mean value property of harmonic functions (3.29), we obtain

$$|v_0(0)|^2 = |\fint_{\mathbb{D}_1} v_0|^2 \leq \left(\fint_{\mathbb{D}_1}|v_0|\right)^2 \leq$$

$$\leq^* \frac{1}{\omega_{n-1}^2}\|v_0\|^2_{L^2(\mathbb{D}_1)}\left(\int_{\mathbb{D}_1}1\right) = \frac{1}{\omega_{n-1}}\|v_0\|^2_{L^2(\mathbb{D}_1)}$$

where in (*) we have used Holder's inequality (6.2 in [Fol99, p. 174]). From 3.30, we get

$$|\nabla v_0(0)|^2 \leq H(n)\|v_0\|^2_{L^2(\mathbb{D}_1)}$$

The last two inequalities ensure that

$$|v_0(0)|^2+|\nabla v_0(0)|^2 \leq \overbrace{2\max\{\frac{1}{\omega_n},\frac{1}{\omega_{n-1}}\}}^{\doteq A'(n)}\|v_0\|^2_{L^2(\mathbb{D}_1)} \leq$$

$$\leq A'(n)\left(\|u-v_0\|^2_{L^2(\mathbb{D}_1)}+\|u\|^2_{L^2(\mathbb{D}_1)}\right)\leq$$

$$\leq^{4.9}_{4.7} A'(n)(K\alpha^{n+3}+C_1(n)\mathbf{e}(E,0,25,e_n)^{\frac{1}{n-1}}) \leq A''(n)(K\alpha^{n+3}+K^{\frac{1}{n-1}})$$

Finally, by the last inequality, 4.13 and $K^{\frac{1}{n-1}}\leq\alpha^{n+3}$, we conclude the proof of this step as follows

$$\frac{1}{(100\alpha)^{n+1}}\int_{M\setminus\Gamma\cap\mathbf{C}(0,100\alpha,\nu_0)}|y\cdot\nu_0-c_0|^2\,\mathcal{H}_{n-1}(y)\leq$$

$$\leq\frac{A^1(n)K}{\alpha^{n+1}}\left((1+A''(n))K^{\frac{1}{n-1}}+A''(n)K\alpha^{n+3}\right)\leq$$

$$\leq\overbrace{A^1(n)(1+A''(n))}^{\doteq A_1(n)}K\alpha^2$$

Proof of Claim 2: Note that $100\sqrt{2}\alpha<r_0$ and $\mathbf{B}(0,200\alpha)\Subset\mathbf{C}(0,25,e_n)$, then, from Proposition 2.9, we find that

$$\mathbf{e}(E,0,100\alpha,\nu_0)\leq C_d(n)\left(\mathbf{e}\left(E,0,100\sqrt{2}\alpha,e_n\right)+|\nu_0-e_n|^2\right)$$

By the definition of ν_0, we find that

$$|\nu_0-e_n|^2=\frac{|\nabla v_0(0)|^2+(1-\sqrt{1+|\nabla v_0(0)|^2})^2}{1+|\nabla v_0(0)|^2}\leq$$

$$\leq^{*}|\nabla v_0(0)|^2+\frac{|\nabla v_0(0)|^4}{4}\leq^{**}\frac{5}{4}|\nabla v_0(0)|^2\leq^{***}C'(n)K$$

where in (*) we have used that $\frac{1}{1+|\nabla v_0(0)|^2}\leq 1$ and $\sqrt{1+s}-1\leq\frac{s}{2},\forall s>0$ with $|\nabla v_0(0)|^2$ in place of s, in (**) and (***) we took $\|v_0\|^2_{L^2(\mathbb{D}_1)}\leq K\leq\alpha^{(n-3)(n-1)}<1$ into account. Using the change of scale of the excess (Proposition 2.5) we produce $\mathbf{e}\big(E,0,100\sqrt{2}\alpha,e_n\big)\leq(\frac{25}{100\sqrt{2}\alpha})^2\mathbf{e}(E,0,25,e_n)$. Putting it all together and recalling that $\mathbf{e}(E,0,25,e_n)\leq\frac{K}{C_1(n)}$ (from the definition of K), we get that

$$\mathbf{e}(E,0,100\alpha,\nu_0)\leq C_d(n)\left((\frac{25}{100\sqrt{2}\alpha})^2\mathbf{e}(E,0,25,e_n)+C'(n)K\right)\leq$$

$$\leq C_d(n)\left((\frac{25}{100\sqrt{2}\alpha})^2\frac{K}{C_1(n)}+C'(n)K\right)=KC(n,\alpha)$$

what ensures the proof of the Claim 2 provided we take $\epsilon_2(n,\alpha)$ small enough in order to have $KC(n,\alpha)\leq\omega(n,\frac{1}{8})$. □

We aim to refine the Excess Improvement by Tilting providing a result such that ensures the existence of a direction ν_0 with certain properties as exhibited before and, furthermore, also enables to control the distance of ν_0 from the direction ν which has the bounded excess (that is 4.15. To achieve this goal, we must replace the exponent 2 in 4.3 with a smaller one 2γ $(0<\gamma<\frac{1}{2})$. For that reason, the constant $\alpha^{2\gamma}$ will be bigger than α^2, therefore, it makes the estimate 4.3 weaker than 4.14.

Theorem 4.4. ***(Excess Improvement)*** *Let $n\geq 3$. For every $\gamma\in(0,\frac{1}{2})$, there exist positive constants $\alpha_0(n,\gamma)<1$, $\epsilon_3(n,\gamma)$ and $C_3(n,\gamma)$ such that if E is a $(\Lambda,r_0)-$minimizing set in $\boldsymbol{C}(x_0,r,\nu)$ with $\Lambda r_0\leq$*

$1, r < r_0, x_0 \in \partial E$ *and*
$$\mathcal{E}_{x_0,\nu}(r) \leq \epsilon_3(n,\gamma)$$
where we have set $\mathcal{E}_{x,\eta}(s) \doteq \max\{\boldsymbol{e}(E,x,s,\nu), \frac{\Lambda s}{\alpha_0^{n-1+2\gamma}}\}$ *for all* $x \in \mathbb{R}^n, s > 0$ *and* $\eta \in \mathbb{S}^{n-1}$*, then there exists* $\nu_0 \in \mathbb{S}^{n-1}$ *such that*

$$\mathcal{E}_{x_0,\nu_0}(\alpha_1(n,\gamma)r) \leq \alpha_0(n,\gamma)^{2\gamma}\mathcal{E}_{x_0,\nu}(r) \tag{4.14}$$

$$|\nu_0 - \nu|^2 \leq C_3(n,\gamma)\mathcal{E}_{x_0,\nu}(r) \tag{4.15}$$

Proof. Let $C_2(n)$ be the constant provided by the Excess Improvement by Tilting (Lemma 4.3), we define $\alpha_0(n,\gamma) \doteq \min\{\frac{1}{200}, (\frac{1}{2C_2(n)})^{\frac{1}{1-2\gamma}}\}$. For the sake of brevity denote $\alpha_0 = \alpha_0(n,\gamma)$, then Lemma 4.3 also provides a constant $\epsilon_2(n,\alpha_0)$ which we use to define $\epsilon_3(n,\gamma) = \frac{\epsilon_2(n,\alpha_0)}{2}$. Taking E in the conditions above, since $2\gamma < 1$, we obtain that

$$\alpha_0 \frac{\Lambda r}{\alpha_0^{n-1+2\gamma}} \leq \alpha_0 \mathcal{E}_{x_0,\nu}(r) \leq \alpha_0^{2\gamma}\mathcal{E}_{x_0,\nu}(r)$$

In order to prove 4.14, it remains to exhibit ν_0 such that $\mathbf{e}(E,x_0,\alpha_0 r,\nu_0) \leq \alpha_0^{2\gamma}\mathcal{E}_{x_0,\nu}(r)$. If $\mathbf{e}(E,x_0,r,\nu) \leq \Lambda r$, ν, which clearly satisfies 4.15, is the desired direction. Indeed, by the Excess and Changes of Scale (Proposition 2.5), we find that

$$\mathbf{e}(E,x_0,\alpha_0 r,\nu) \leq \frac{1}{\alpha_0^{n-1}}\mathbf{e}(E,x_0,r,\nu) \leq \frac{\alpha_0^{2\gamma}}{\alpha_0^{n-1+2\gamma}}\Lambda r \leq$$
$$\leq \alpha_0^{2\gamma}\mathcal{E}_{x_0,\nu}(r)$$

Now, we suppose that $\Lambda r \leq \mathbf{e}(E,x_0,r,\nu)$, then

$$\mathbf{e}(E,x_0,r,\nu) + \Lambda r \leq 2\mathbf{e}(E,x_0,r,\nu) \leq 2\mathcal{E}_{x_0,\nu}(r) \leq$$
$$\leq 2\epsilon_3(n,\gamma) \leq^* \epsilon_2(n,\alpha_0)$$

where (*) follows from our choice of $\epsilon_3(n,\gamma)$. Thus, we are in position to apply the Excess Improvement by Tilting. Since $2C_2(n) \leq \frac{1}{\alpha_0^{1-2\gamma}}$, by our assumption, we can conclude the proof of 4.14 as follows

$$\begin{gathered}\mathbf{e}(E,x_0,\alpha_0 r,\nu_0) \leq^{4.3} C_2(n)\Big(\alpha_0^2\mathbf{e}(E,x_0,r,\nu) + \alpha_0\Lambda r\Big) \leq \\ \leq^{\alpha_0<1} 2C_2(n)\alpha_0\mathbf{e}(E,x_0,r,\nu) \leq 2C_2(n)\alpha_0\mathcal{E}_{x_0,\nu}(r) \leq \alpha_0^{2\gamma}\mathcal{E}_{x_0,\nu}(r)\end{gathered} \tag{4.16}$$

Let us prove that 4.15 holds true for the direction ν_0 provided by the Excess Improvement by Tilting. Taking into account that $(a+b)^2 \leq 2(a^2+b^2)$, the triangle inequality furnishes

$$|\nu - \nu_0|^2 \leq 2(|\nu - \nu_E|^2 + |\nu_0 - \nu_E|^2)$$

Thus, by definition of the excess,

$$\begin{aligned} &|\nu-\nu_0|^2\,\mathcal{H}_{n-1}\left(\partial^* E\cap \mathbf{C}\left(x_0,\alpha_0 r,\nu_0\right)\right)\le \\ &\le 2\int_{\partial^* E\cap\mathbf{C}(x_0,\alpha_0 r,\nu_0)}\left(|\nu_0-\nu_E|^2+|\nu-\nu_E|^2\right)\mathrm{d}\mathcal{H}_{n-1}= \\ &=2(\alpha_0 r)^{n-1}\mathbf{e}(E,x_0,\alpha_0 r,\nu_0)+2\int_{\partial^* E\cap\mathbf{C}(x_0,\alpha_0 r,\nu_0)}|\nu-\nu_E|^2\,\mathrm{d}\mathcal{H}_{n-1}\end{aligned} \tag{4.17}$$

We note that $\sqrt{2}\alpha_0 r\le\frac{\sqrt{2}}{200}r<r<r_0$ implies

$$\mathbf{C}\left(x_0,\alpha_0 r,\nu_0\right)\subset \mathbf{B}\left(x_0,\sqrt{2}\alpha_0 r\right)\subset\mathbf{B}(x_0,r)\subset\mathbf{C}\left(x_0,r,\nu\right) \tag{4.18}$$

The first inclusion in 4.18 makes possible to apply the Density Estimates for Cylinders (Corollary 1.8). Since

$$\mathcal{H}_{n-1}\left(\partial^* E\cap\mathbf{C}\left(x_0,\alpha_0 r,\nu_0\right)\right)=\mathcal{P}(E,\mathbf{C}\left(x_0,\alpha_0 r,\nu_0\right))$$

the density estimate for cylinders ensures that

$$c(n)(\alpha_0 r)^{n-1}\le\mathcal{P}(E,\mathbf{C}\left(x_0,\alpha_0 r,\nu\right))$$

By 4.18 and the last inequality, the inequality 4.17 became

$$\begin{aligned} &c(n)(\alpha_0 r)^{n-1}|\nu-\nu_0|^2\le \\ &\le 2(\alpha_0 r)^{n-1}\mathbf{e}(E,x_0,\alpha_0 r,\nu_0)+2\int_{\partial^* E\cap\mathbf{C}(x_0,r,\nu)}|\nu-\nu_E|^2\,\mathrm{d}\mathcal{H}_{n-1}= \\ &=2(\alpha_0 r)^{n-1}\mathbf{e}(E,x_0,\alpha_0 r,\nu_0)+2r^{n-1}\mathbf{e}(E,x_0,r,\nu)\end{aligned}$$

By the definition of $\mathcal{E}_{x_0,\nu}\left(r\right)$ and 4.16, we find that

$$\begin{aligned} c(n)(\alpha_0 r)^{n-1}|\nu-\nu_0|^2 &\le 2(\alpha_0 r)^{n-1}\alpha_0^{2\gamma}\mathcal{E}_{x_0,\nu}\left(r\right)+2r^{n-1}\mathbf{e}(E,x_0,r,\nu)\le \\ &\le 2r^{n-1}\left(\alpha_0^{n-1}\alpha_0^{2\gamma}+1\right)\mathcal{E}_{x_0,\nu}\left(r\right)\end{aligned}$$

Set $C_3(n,\gamma)\doteq\frac{2(\alpha_0^{2\gamma}+\alpha_0^{1-n})}{c(n)}$ to conclude the proof of 4.15. $\square$

4.3 $C^{1,\gamma}$-regularity of the almost minimizing sets

We finally have evolved all the needed tools to reach the main goal of this work. For any $\gamma \in (0, 1/2)$, we take E an almost minimizing set and $x_0 \in \partial E$, if we suppose that E content a boundedness condition (depending on γ and the dimension n) on the excess at x_0 at scale $25r$, we can exhibit a function $u \in C^{1,\gamma}$ whose graph coincides with ∂E in $\mathbf{C}(x_0, r, e_n)$. Moreover, the Holder constant of u will depend on the excess at x_0 at scale $25r$.

Theorem 4.5. *($C^{1,\gamma}$**-regularity of the almost minimizing sets**) Let $n \geq 3$. For every $\gamma \in (0, \frac{1}{2})$, there exist $\epsilon_4(n,\gamma)$ and $C_4(n,\gamma)$ such that if E is a (Λ, r_0) − minimizing set in $\boldsymbol{C}(x_0, 25r, e_n)$ with $\Lambda r_0 \leq 1, x_0 \in \partial E, 25r < r_0$ and*

$$\boldsymbol{e}(E, x_0, 25r, e_n) + \Lambda r \leq \epsilon_4(n, \gamma) \tag{4.19}$$

then there exists a Lipschitz function $u : \mathbb{R}^n \to \mathbb{R}$ with $Lip\,(u) < 1$ such that $u \in C^{1,\gamma}(\boldsymbol{p}x_0 + \mathbb{D}_r)$,

$$\sup \frac{|u|}{r} \leq C_1(n)\boldsymbol{e}(E, x_0, 25r, e_n)^{\frac{1}{2(n-1)}} \tag{4.20}$$

$$\boldsymbol{C}(x_0, r, e_n) \cap \partial E = (x_0 + G(u)) \cap \boldsymbol{C}(x_0, r, e_n) \tag{4.21}$$

and either

$$\boldsymbol{C}(x_0, r, e_n) \cap (E \setminus \partial E) = \boldsymbol{C}(x_0, r, e_n) \cap (x_0 + \text{hypo}\,(u)) \tag{4.22}$$

or

$$\boldsymbol{C}(x_0, r, e_n) \cap (E \setminus \partial E) = \boldsymbol{C}(x_0, r, e_n) \cap (x_0 + \text{epi}\,(u)) \tag{4.23}$$

Furthermore, ν_E is a Holder function in $\boldsymbol{C}(x_0, r, e_n) \cap \partial E$ and both ν_E and ∇u have Holder constant equal to $\frac{C_4(n,\gamma)}{r^\gamma}\sqrt{\boldsymbol{e}(E, x_0, 25r, e_n) + \Lambda r}$.

Proof. We should use the same notation presented in the Excess Improvement Theorem (Theorem 4.4), precisely $\mathcal{E}_{x,\eta}(s) \doteq \max\{\mathbf{e}(E, x, s, \nu), \frac{\Lambda s}{\alpha_0^{n-1+2\gamma}}\}$ for all $x \in \mathbb{R}^n, s > 0$ and $\eta \in \mathbb{S}^{n-1}$.

Claim 1: For any $x \in \mathbf{C}(x_0, r, e_n) \cap \partial E$, there exists $\nu(x) \in \mathbb{S}^{n-1}$ such that

$$\mathcal{E}_{x,\nu(x)}(s) \leq C_3'(n,\gamma)\left(\frac{s}{r}\right)^{2\gamma} \mathcal{E}_{x_0,e_n}(25r) \quad \forall s \in (0, 12r) \tag{4.24}$$

$$|\nu(x) - e_n|^2 \leq C_3'(n,\gamma)\mathcal{E}_{x_0,e_n}(25r) \tag{4.25}$$

$$\mathcal{E}_{x,e_n}(s) \leq C_3'(n,\gamma)\mathcal{E}_{x_0,e_n}(25r) \quad \forall s \in (0,24r) \tag{4.26}$$

Recalling the Lipschitz Approximation Theorem (Theorem 3.3), we have

$$M_0 = \left\{ x \in \mathbf{C}(x_0,r,e_n) \cap \partial E : \sup_{0<s<8r} \mathbf{e}(E,x,s,e_n) \leq \delta_0(n) \right\}$$

We require $\epsilon_4(n,\gamma)$ to be smaller than both $\epsilon_1(n)$ and $\delta_0(n)$, thus exists a Lipschitz functions $u : \mathbb{R}^n \to \mathbb{R}$ with $Lip(u) < 1$ such that 4.20 holds true and, by 4.26,

$$M_0 = \mathbf{C}(x_0,r,e_n) \cap \partial E \subset G(u) \cap \mathbf{C}(x_0,r,e_n)$$

what put us in position to apply the Lipschitz Boundary Criterion (Theorem 3.1), then

$$\mathbf{C}(x_0,r,e_n) \cap \partial E = G(u) \cap \mathbf{C}(x_0,r,e_n)$$

and either

$$\mathbf{C}(x_0,r,e_n) \cap (E \setminus \partial E) = \mathbf{C}(x_0,r,e_n) \cap (x_0 + \mathrm{hypo}(u))$$

or

$$\mathbf{C}(x_0,r,e_n) \cap (E \setminus \partial E) = \mathbf{C}(x_0,r,e_n) \cap (x_0 + \mathrm{epi}(u))$$

holds true, that is either 4.22 or 4.23. In order to prove the Holder continuity of ∇u and ν_E, we set

$$(\nabla u)_{z,s} = \frac{1}{|z + \mathbb{D}_s|} \int_{z+\mathbb{D}_s} \nabla u$$

and we establish the last claim of this proof.

Claim 2: For any $s \in (0,r)$ and $z \in (\mathbf{p}x_0 + \mathbb{D}_r)$, it holds that

$$\frac{1}{s^{n-1}} \int_{z+\mathbb{D}_s} |\nabla u - (\nabla u)_{z,s}|^2 \leq K(n,\gamma)(\frac{s}{r})^{2\gamma} \mathcal{E}_{x_0,e_n}(25r) \tag{4.27}$$

The Claim 2 put us in position to apply the Campanato's Criterion (Theorem 7.51 in [LA00]) and thus exists $K'(n,\gamma)$ such that

$$|u(z) - u(z')| \leq K'(n\gamma)\sqrt{\mathcal{E}_{x_0,e_n}(25r)} \left(\frac{|z - z'|}{r} \right)^{\gamma} \quad \forall z, z' \in (\mathbf{p}x_0 + \mathbb{D}_r)$$

By the very definition of $\mathcal{E}$, we set $C_4'(n,\gamma) = K'(n,\gamma)\max\{1, \frac{25}{\alpha_0^{n-1+2\gamma}}\}$ and then, $\forall z, z' \in (\mathbf{p}x_0 + \mathbb{D}_r)$, we have

$$|u(z) - u(z')| \leq \left(\frac{C_4'(n\gamma)\sqrt{\mathbf{e}(E,x_0,25r,e_n) + \Lambda r}}{r^{\gamma}} \right) |z - z'|^{\gamma} \tag{4.28}$$

what infers that $u \in C^{1,\gamma}(\mathbf{p}x_0 + \mathbb{D}_r)$. From 4.21 and the Lipschitz Boundary Criterion (Theorem 3.1), we obtain

$$\nu_E(y) = \pm \frac{(\nabla u(\mathbf{p}y), -1)}{\sqrt{1 + |\nabla u(\mathbf{p}y)|^2}} \quad \forall y \in \mathbf{C}\left(x_0, r, e_n\right) \cap \partial E$$

Since

$$F : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$$
$$w \longmapsto \frac{(-w, 1)}{\sqrt{1 + |w|^2}}$$

defines a Lipschitz function, if $x, y \in \mathbf{C}\left(x_0, r, e_n\right) \cap \partial E$, we immediately deduce that

$$|\nu_E(x) - \nu_E(y)| \leq Lip\,(F)\,|\nabla u(\mathbf{p}x) - \nabla u(\mathbf{p}y)| \leq$$

$$\leq^{4.28} Lip\,(F) \left(\frac{C_4'(n\gamma)\sqrt{\mathbf{e}(E, x_0, 25r, e_n) + \Lambda r}}{r^\gamma}\right)|\mathbf{p}x - \mathbf{p}y|^\gamma \leq$$

$$\leq \left(\frac{C_4(n\gamma)\sqrt{\mathbf{e}(E, x_0, 25r, e_n) + \Lambda r}}{r^\gamma}\right)|x - y|^\gamma$$

where we have chosen $C_4(n, \gamma) \doteq Lip\,(F)\, C_4'(n, \gamma)$.

Proof of the Claim 1: We fix $x \in \mathbf{C}\left(x_0, r, e_n\right) \cap \partial E$, then $\mathbf{C}\left(x, 24r, e_n\right) \subset \mathbf{C}\left(x_0, 25r, e_n\right)$ implies

$$\mathbf{e}(E, x, 24r, e_n) \leq \left(\frac{25}{24}\right)^{n-1} \mathbf{e}(E, x_0, 25r, e_n)$$

Furthermore, by the definition of $\mathcal{E}$, we have

$$\mathcal{E}_{x,e_n}\left(24r\right) \leq \left(\frac{25}{24}\right)^{n-1} \mathcal{E}_{x_0,e_n}\left(25r\right) \tag{4.29}$$

Since $\alpha_0 < 1$ ensures $1 < \frac{1}{\alpha_0^{n-1+2\gamma}}$, we find that

$$\mathcal{E}_{x,e_n}\left(24r\right) \leq \mathbf{e}(E, x, 24r, e_n) + \frac{24\Lambda r}{\alpha_0^{n-1+2\gamma}} \leq$$

$$\leq \max\left\{\left(\frac{25}{24}\right)^{n-1}, \frac{24}{\alpha_0^{n-1+2\gamma}}\right\}\left(\mathbf{e}(E, x_0, 25r, e_n) + \Lambda r\right)$$

In order to apply the Excess Improvement Theorem, we choose $\epsilon_4(n, \gamma) \leq \max\{(\frac{25}{24})^{n-1}, \frac{24}{\alpha_0^{n-1+2\gamma}}\}\epsilon_3(n, \gamma)$, then, by 4.19,

$$\mathcal{E}_{x,e_n}\left(24r\right) \leq \epsilon_3(n, \gamma) \tag{4.30}$$

what, from Theorem 4.4, provides the existence of a direction $\nu_1(x) \in \mathbb{S}^{n-1}$ such that 4.14 and 4.15 holds true, namely

$$\mathcal{E}_{x,\nu_1(x)}\left(24\alpha_0 r\right) \leq \alpha_0^{2\gamma} \mathcal{E}_{x,e_n}\left(24r\right)$$

$$|\nu_1(x) - e_n|^2 \leq C_3(n, \gamma) \mathcal{E}_{x,e_n}\left(24r\right)$$

Since $\alpha_0 < 1$, by 4.30, we get that

$$\mathcal{E}_{x,\nu_1(x)}\left(24\alpha_0 r\right) \le \mathcal{E}_{x,e_n}\left(24r\right) \le \epsilon_3(n,\gamma) \tag{4.31}$$

Note that $\mathbf{C}\left(x, 24\alpha_0 r, \nu_1(x)\right) \subset \mathbf{C}\left(x, 25r, e_n\right)$ and thus E is $(\Lambda, r_0)-$*minimizing* in $\mathbf{C}\left(x, 24\alpha_0 r, \nu_1(x)\right)$. Therefore, 4.31 furnishes the conditions to apply the Excess Improvement at the smaller scale $24\alpha_0 r$ and direction $\nu_1(x)$. Iterating this process, we can show the existence of a sequence $\nu_h \doteq \nu_h(x) \in \mathbb{S}^{n-1}$ such that

$$\mathcal{E}_{x,\nu_h}\left(24\alpha_0^h r\right) \le \alpha_0^{2\gamma h}\mathcal{E}_{x,e_n}\left(24r\right) \tag{4.32}$$

$$|\nu_h - \nu_{h-1}|^2 \le C_3(n,\gamma)\mathcal{E}_{x,\nu_{h-1}}\left(24\alpha_0^{h-1} r\right)$$

The second inequality combined with 4.32 implies that

$$|\nu_h - \nu_{h-1}|^2 \le C_3(n,\gamma)\alpha_0^{2\gamma(h-1)}\mathcal{E}_{x,e_n}\left(24r\right) \tag{4.33}$$

If we set $\nu_0 = e_n$, the inequalities 4.32 and 4.33 are valid for all $h \ge 1$. Let us prove the existence of the limit $\lim_{h\to\infty} \nu_h$ by showing that $\{\nu_h\}_{h\in\mathbb{N}}$ is a Cauchy sequence. Indeed, by 4.33, for all $j \ge h \ge 1$,

$$|\nu_j - \nu_h| \le \sum_{k=h}^{j} |\nu_k - \nu_{k-1}| \le \sum_{k=h}^{j} \sqrt{C_3(n,\gamma)\alpha_0^{2\gamma(h-1)}\mathcal{E}_{x,e_n}\left(24r\right)} \le$$

$$\le \sqrt{C_3(n,\gamma)\mathcal{E}_{x,e_n}\left(24r\right)} \sum_{k=h}^{j} \alpha_0^{\gamma(h-1)} = \frac{\sqrt{C_3(n,\gamma)\mathcal{E}_{x,e_n}\left(24r\right)}}{1-\alpha_0^{\gamma}}\alpha_0^{\gamma(h-1)}$$

Thus, $\alpha_0 < 1$ implies that $\{\nu_h\}_{h\in\mathbb{N}}$ is a Cauchy sequence. Therefore, we can set $\nu(x) = \lim_{h\to\infty} \nu_h$. Since

$$|\nu_j - \nu_h| \le \frac{\sqrt{C_3(n,\gamma)\mathcal{E}_{x,e_n}\left(24r\right)}}{1-\alpha_0^{\gamma}}\alpha_0^{\gamma(h-1)} \tag{4.34}$$

we let $h = 1$ and j goes to ∞ and thus, by 4.29, it becomes

$$|\nu(x) - e_n|^2 \le \frac{25^{n-1}C_3(n,\gamma)}{24^{n-1}(1-\alpha_0^{\gamma})}\mathcal{E}_{x,e_n}\left(25r\right)$$

for a suitable choice of the constant, that is exactly 4.25. Let us turn to the proof of 4.24, we take $s \in (0, 12r)$, since $\alpha_0^0 24r = 24t$ and $\alpha_0^h 24r \to 0$, we can find $h_0 \ge 0$ such that $24r\alpha_0^{h_0+1} \le 2s < 24r\alpha_0^{h_0}$. Evidently, $12\sqrt{2}r < r_0$, then, from Proposition 2.9,

$$\begin{aligned} \mathbf{e}(E,x,s,\nu(x)) &\le C_d(n)\left(\mathbf{e}\left(E,x,\sqrt{2}s,\nu_{h_0}\right) + |\nu(x)-\nu_{h_0}|^2\right) \le \\ &\overset{2.5}{\le} C_d(n)\left(\left(\frac{24r\alpha_0^{h_0}}{\sqrt{2}s}\right)^{n-1}\mathbf{e}\left(E,x,24r\alpha_0^{h_0},\nu_{h_0}\right) + |\nu(x)-\nu_{h_0}|^2\right) \le \\ &\le C_d'(n)\left(\left(\frac{24r\alpha_0^{h_0}}{s}\right)^{n-1}\mathcal{E}_{x,\nu_{h_0}}\left(24r\alpha_0^{h_0}\right) + |\nu(x)-\nu_{h_0}|^2\right) \end{aligned} \tag{4.35}$$

By our choice of h_0, we have $\frac{24r\alpha_0^{h_0}}{s} \le \frac{2}{\alpha_0}$ and thus, by 4.32,

$$(\frac{24r\alpha_0^{h_0}}{s})^{n-1}\mathcal{E}_{x,\nu_{h_0}}\left(24r\alpha_0^{h_0}\right) \le \frac{2^{n-1}}{\alpha_0^{n-1}}\alpha_0^{2\gamma h_0}\mathcal{E}_{x,e_n}(24r)$$

$$\le \frac{2^{n-1}}{\alpha_0^{n-1}}(\frac{2s}{24r\alpha_0})^{2\gamma}\mathcal{E}_{x,e_n}(24r) \le^{4.29} C_4(n,\gamma)(\frac{s}{r})^{2\gamma}\mathcal{E}_{x_0,e_n}(25r)$$

where we defined $C_4(n,\gamma) = C_d'(n)\max\{(\frac{2^{2\gamma}25}{24^{2\gamma}\alpha_0^{2\gamma}})^{n-1}, 1\}$. Combining 4.35 and the last inequality, we find that

$$\mathbf{e}(E, x, s, \nu(x)) \le C_4(n,\gamma)\left((\frac{s}{r})^{2\gamma}\mathcal{E}_{x_0,e_n}(25r) + |\nu(x) - \nu_{h_0}|^2\right) \tag{4.36}$$

To control the second term on the right side of the last inequality, we let $j \to \infty$ in 4.34 and recall the choice of h_0, then

$$|\nu(x) - \nu_{h_0}|^2 \le \frac{C_3(n,\gamma)\mathcal{E}_{x,e_n}(24r)}{(1-\alpha_0^{\gamma})^2}\alpha_0^{2\gamma h_0} \le$$

$$\le \frac{C_3(n,\gamma)}{(1-\alpha_0^{\gamma})^2}(\frac{2s}{24r\alpha_0})^{2\gamma}\mathcal{E}_{x,e_n}(24r) \le$$

$$\le^{4.29} \overbrace{\frac{25^{n-1}C_3(n,\gamma)2^{2\gamma}}{(1-\alpha_0^{\gamma})^2 24^{n-1+2\gamma}\alpha_0^{2\gamma}}}^{\doteq C_4'(n,\gamma)}(\frac{s}{r})^{2\gamma}\mathcal{E}_{x_0,e_n}(25r)$$

Taking $C_5(n,\gamma)$ equal to the maximum between $C_4'(n,\gamma)$ and $C_4(n,\gamma)$, by 4.36 and the last inequalities, it follows that

$$\mathbf{e}(E, x, s, \nu(x)) \le C_5(n,\gamma)(\frac{s}{r})^{2\gamma}\mathcal{E}_{x_0,e_n}(25r) \quad \forall s \in (0, 12r) \tag{4.37}$$

Since

$$\frac{\Lambda s}{\alpha_0^{n-1+2\gamma}} \le \frac{25\Lambda r}{\alpha_0^{n-1+2\gamma}} \tag{4.38}$$

by the last inequality, the definition of $\mathcal{E}$ and 4.37, we conclude that

$$\mathcal{E}_{x,\nu(x)}(s) \le C_5(n,\gamma)(\frac{s}{r})^{2\gamma}\mathcal{E}_{x_0,e_n}(25r) \quad \forall s \in (0, 12r)$$

Finally, we will prove 4.26. To this end, we note that 4.38 holds true for $24r$ in place of s and thus it suffices to prove

$$\mathbf{e}(E, x, s, e_n) \le C_3'(n,\gamma)\mathbf{e}(E, x_0, e_n, 25r) \quad \forall s \in (0, 24r) \tag{4.39}$$

Firstly, suppose that $s \in [6r, 24r)$, then $\frac{r}{s} \le \frac{1}{6}$ and $\mathbf{C}(x, s, e_n) \subset \mathbf{C}(x_0, 25r, e_n)$ ensure that

$$\mathbf{e}(E, x, s, e_n) \le (\frac{25r}{s})^{n-1}\mathbf{e}(E, x_0, 25r, e_n) \le (\frac{25}{6})^{n-1}\mathbf{e}(E, x_0, 25r, e_n)$$

If, otherwise, $s \in (0, 6r)$, by Proposition 2.9, we have that

$$\mathbf{e}(E, x, s, e_n) \leq C_d(n)\Big(\mathbf{e}\Big(E, x, \sqrt{2}s, \nu(x)\Big) + |\nu(x) - e_n|^2\Big)$$

We note that $\sqrt{2}s < 12r$. Therefore, we can take into account 4.24 and 4.25. Then

$$\mathbf{e}(E, x, s, e_n) \leq C_d(n)C_3'(n, \gamma)\left(\left(\frac{\sqrt{2}s}{r}\right)^{2\gamma} \mathcal{E}_{x_0, e_n}(25r) + \mathcal{E}_{x_0, e_n}(25r)\right)$$

In order to finish the proof of the Claim, note that $s \in (0, 6r)$ implies $\frac{s}{r} < 6$.

Proof of the Claim 2: We fix $s \in (0, r)$ and $z \in (\mathbf{p}x_0 + \mathbb{D}_r)$, by 4.25, up to decreasing $\epsilon_4(n, \gamma)$, we can henceforth assume that

$$\frac{1}{\sqrt{2}} \leq \mathbf{q}\nu(x) \quad \forall x \in \mathbf{C}(x_0, r, e_n) \cap \partial E \tag{4.40}$$

Therefore, it is straightforward to conclude that

$$\mathbf{C}(x, s, e_n) \subset \mathbf{B}\Big(x, \sqrt{2}s\Big) \subset \mathbf{C}(x, 2s, \nu(x))$$

Thus, by the set inclusions above and the definition of excess, we can affirm that

$$\int_{\mathbf{C}(x,s,e_n)\cap\partial^* E} \frac{|\nu_E - \nu(x)|^2}{2} \, d\mathcal{H}_{n-1} \leq (2s)^{n-1}\mathbf{e}(E, x, 2s, \nu(x)) \tag{4.41}$$

whenever $x \in \mathbf{C}(x_0, r, e_n) \cap \partial E$. From 4.40, for any $x \in \mathbf{C}(x_0, r, e_n) \cap \partial E$ we have $\mathbf{q}\nu(x) > 0$. Then, we can define

$$\begin{aligned} X : \mathbf{C}(x_0, r, e_n) \cap \partial E &\longrightarrow \mathbb{R}^{n-1} \\ x &\longmapsto -\frac{\mathbf{p}\nu(x)}{\mathbf{q}\nu(x)} \end{aligned}$$

We note that

$$\sqrt{1 + |X(x)|^2} = \frac{\sqrt{|\mathbf{p}\nu(x)|^2 + |\mathbf{q}\nu(x)|^2}}{|\mathbf{q}\nu(x)|} =^{\nu(x)\in\mathbb{S}^{n-1}} \frac{1}{|\mathbf{q}\nu(x)|}$$

Thus, we immediately deduce that

$$\begin{gathered} |X(x)| \leq 1 \\ \mathbf{p}\nu(x) = -\frac{X(x)}{\sqrt{1 + |X(x)|^2}} \\ \mathbf{q}\nu(x) = \frac{1}{\sqrt{1 + |X(x)|^2}} \end{gathered} \tag{4.42}$$

To infer that $|X(x)| \leq 1$, we required the help of 4.40 and $1 = |\nu(x)|^2 = |\mathbf{p}\nu(x)|^2 + |\mathbf{q}\nu(x)|^2$. Henceforth, we assume that 4.23 holds true, if not, it suffices to work with $-X$ in place of X. Since $y \in \mathbf{C}(x_0, r, e_n) \cap \partial E$ implies $y = (\mathbf{p}y, \mathbf{q}y) \in \mathbf{C}(x_0, r, e_n) \cap (x_0 + G(u))$, the Lipschitz Boundary Criterion (Theorem 3.1) ensures that

$$\nu_E(y) = \nu_E(\mathbf{p}y, u(\mathbf{q}y)) = -\frac{(\nabla u(\mathbf{p}y), -1)}{\sqrt{1 + |\nabla u(\mathbf{p}y)|^2}}$$

and thus, by 4.42, we find that

$$\int_{\mathbf{C}(x,s,e_n)\cap\partial^* E} \frac{|\nu_E(y) - \nu(x)|^2}{2} \, \mathrm{d}\mathcal{H}_{n-1}(y) =$$

$$\frac{1}{2}\int_{\mathbf{C}(x,s,e_n)\cap\partial^* E} \left| -\frac{\nabla u(\mathbf{p}y)}{\sqrt{1+|\nabla u(\mathbf{p}y)|^2}} + \frac{X(x)}{\sqrt{1+|X(x)|^2}} \right|^2 \mathrm{d}\mathcal{H}_{n-1}(y) +$$
$$+\frac{1}{2}\int_{\mathbf{C}(x,s,e_n)\cap\partial^* E} \left| \frac{1}{\sqrt{1+|\nabla u(\mathbf{p}y)|^2}} - \frac{1}{\sqrt{1+|X(x)|^2}} \right|^2 \mathrm{d}\mathcal{H}_{n-1}(y)$$

From 4.21, we obtain that $x = (z, u(z)) \in \partial E \cap \mathbf{C}(x_0, r, e_n)$. By Theorem 9.1 in [Mag12] and the last inequalities, we deduce that

$$\begin{aligned} &\int_{\mathbf{C}(x,s,e_n)\cap\partial^* E} \frac{|\nu_E(y) - \nu(x)|^2}{2} \, \mathrm{d}\mathcal{H}_{n-1}\, y = \\ &= \frac{1}{2}\int_{z+\mathbb{D}_s} \left| -\frac{\nabla u(w)}{\sqrt{1+|\nabla u(w)|^2}} + \frac{X(x)}{\sqrt{1+|X(x)|^2}} \right|^2 \sqrt{1+|\nabla u(w)|^2}\, \mathrm{d}\mathcal{H}_{n-1}(w) + \\ &+\frac{1}{2}\int_{z+\mathbb{D}_s} \left| \frac{1}{\sqrt{1+|\nabla u(w)|^2}} - \frac{1}{\sqrt{1+|X(x)|^2}} \right|^2 \sqrt{1+|\nabla u(w)|^2}\, \mathrm{d}\mathcal{H}_{n-1}(w) \end{aligned} \tag{4.43}$$

We claim that

$$\int_{z+\mathbb{D}_s} |\nabla u - (\nabla u)_{z,s}|^2 = \min_{v\in\mathbb{R}^n} \int_{z+\mathbb{D}_s} |\nabla u - v|^2 \tag{4.44}$$

Since $|\nabla u|^2$ and $|X(x)|^2$ both are less or equal than 1, by 4.44, we have that

$$\int_{z+\mathbb{D}_s} |\nabla u - (\nabla u)_{z,s}|^2 \le \int_{z+\mathbb{D}_s} |\nabla u - X(x)|^2 \le$$
$$\le \int_{z+\mathbb{D}_s} \left| \frac{\nabla u - X(x)}{\sqrt{1+|X(x)|^2}} \right|^2 \sqrt{1+|\nabla u|^2} \le$$
$$\int_{z+\mathbb{D}_s} \left| \frac{\nabla u}{\sqrt{1+|X(x)|^2}} - \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} + \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{X(x)}{\sqrt{1+|X(x)|^2}} \right|^2 \sqrt{1+|\nabla u|^2}$$
$$\le \int_{z+\mathbb{D}_s} \left| \frac{1}{\sqrt{1+|X(x)|^2}} - \frac{1}{\sqrt{1+|\nabla u|^2}} \right|^2 \sqrt{1+|\nabla u|^2} +$$
$$+ \int_{z+\mathbb{D}_s} \left| -\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} + \frac{X(x)}{\sqrt{1+|X(x)|^2}} \right|^2 \sqrt{1+|\nabla u|^2}$$

Therefore, by 4.43, we obtain that

$$\int_{z+\mathbb{D}_s} |\nabla u - (\nabla u)_{z,s}|^2 \leq 2 \int_{\mathbf{C}(x,s,e_n)\cap\partial^* E} \frac{|\nu_E(y) - \nu(x)|^2}{2} \, \mathrm{d}\mathcal{H}_{n-1}\, y$$

Then, by 4.24, 4.41 and the last inequality, we finally obtain that

$$\frac{1}{s^{n-1}} \int_{z+\mathbb{D}_s} |\nabla u - (\nabla u)_{z,s}|^2 \leq 2^n \mathbf{e}(E, x, 2s, \nu(x)) \leq$$

$$\leq 2^n \mathcal{E}_{x,\nu(x)}\,(2s) \leq 2^{n-1} C_3'(n,\gamma) \left(\frac{s}{r}\right)^{2\gamma} \mathcal{E}_{x_0,e_n}\,(25r)$$

for all $s \in (0, 6r)$, what concludes the proof of the Claim 2 (4.27). In order to prove 4.44, we note that $F(v) = \int_{z+\mathbb{D}_s} |\nabla u - v|^2, \forall v \in \mathbb{R}^{n-1}$ is a differentiable function and it is straightforward to calculate that

$$\nabla F(v) = -2 \int_{z+\mathbb{D}_s} \nabla u - v = 2|z + \mathbb{D}_s| v - 2 \int_{z+\mathbb{D}_s} \nabla u$$

what implies that $\nabla F(v) = 0$ if, and only if, $v = (\nabla u)_{z,s}$. Then $(\nabla u)_{z,s}$ is the unique critical point of F. Since F is a convex function, we conclude that $(\nabla u)_{z,s}$ is the minimum point of F which ensures 4.44. $\square$

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