On Rayner Rngs of Formal Power Series

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DOCTORAL THESIS PRESENTED TO THE
INSTITUTE OF MATHEMATICS AND STATISTICS
OF THE UNIVERSITY OF SÃO PAULO
TO ACHIEVE THE TITLE OF
DOCTOR OF SCIENCE

Program: Doctoral degree in Mathematics
Supervisor: Prof. Dr. Rogério Augusto dos Santos Fajardo

São Paulo
November 2022

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This version of the thesis contains the corrections and modifications suggested by the Thesis Committee during the defence of the original version of the work, which was held online in 09/01/2023. A copy of the original version is available at the Institute of Mathematics and Statistics of the University of São Paulo.

To my parents,
who sparked my severe thirst for knowledge.

Acknowledgements

First and foremost, I wish to thank my supervisor, Prof. Rogério Augusto dos Santos Fajardo, for his support and guidance during my doctoral program. His vast experience and keen attention to detail were immensely invaluable to the making of this work, and I am honoured to have been supervised by him.

I would like to thank the examiners of this thesis who have had the opportunity to express to me their criticism of the first submitted version. In alphabetical order, these are the Profs. Aline de Souza Lima, Dessislava Hristova Kochloukova, Hércules de Araújo Feitosa, Hugo Luiz Mariano, Raul Antonio Ferraz and Ricardo Bianconi. Their perceptive comments and suggestions were very useful for improving the exposition of the material, and such inputs have led to many revisions of the text. Special thanks to Prof. Hugo Luiz Mariano for producing a detailed commented version of the work containing many technical annotations and advice.

I must also express my gratitude and love toward my wonderful wife, Karolina, whose enduring love and tolerance provide the collectedness and self-assurance necessary to allow me to indulge my academic passion. I feel the same gratefulness toward my parents and my sister, who always stand by my side with their advice, endless patience and encouragement.

"One Ring to rule them all, One Ring to find them,
One Ring to bring them all, and in the darkness bind them".
[J. R. R. Tolkien (225)]

Abstract

Rayner rngs are rngs (rings without unity) whose elements are formal power series $\sum_{g \in G} x_g \mathrm{X}^g$, where the coefficients lie in a rng R and the exponents lie in an additive ordered group G, such that the support $\{g \in G \mid x_g \neq 0_R\}$ belongs to a predetermined ideal $\mathcal{J}$ on G constrained by a set of axioms. The work presents an inspection of the interplay between the algebraic, topological and categorical properties of the Rayner rngs, the rngs R of coefficients and the ordered groups G of exponents, studying the Rayner rngs under varied theoretical perspectives and seeking universal relations between them. Two key topologies on these structures are systematically analysed, the so-called weak and strong topologies, and a version of the Intermediate Value Theorem is obtained for the weak topology. Special attention is given to rngs of Levi-Civita, Puiseux and Hahn series, which are prominent instances of Rayner rngs.

Keywords: Intermediate Value Theorem, formal power series, Levi-Civita series, Puiseux series, Hahn series, Rayner series.

Resumo

Os rngs de Rayner são rngs (anéis sem unidade) cujos elementos são séries formais de potências $\sum_{g \in G} x_g \mathrm{X}^g$, onde os coeficientes pertencem a um rng R e os expoentes pertencem a um grupo ordenado aditivo G, tais que o suporte $\{g \in G \mid x_g \neq 0_R\}$ pertence a um predeterminado ideal $\mathcal{J}$ sobre G que satisfaz um conjunto de axiomas. O trabalho apresenta uma inspeção das relações diretas entre as propriedades algébricas, topológicas e categóricas dos rngs de Rayner, dos rngs R de coeficientes e dos grupos ordenados G de expoentes, estudando os rngs de Rayner sob diferentes perspectivas teóricas e buscando relações universais entre eles. Duas topologias essenciais nessas estruturas são sistematicamente analisadas, as topologias forte e fraca, e uma versão do Teorema do Valor Intermediário é obtida para a topologia fraca. Atenção especial é dada aos rngs de séries de Levi-Civita, Puiseux e Hahn, os quais são instâncias proeminentes de rngs de Rayner.

Descritores: Teorema do Valor Intermediário, séries de potências formais, séries de Levi-Civita, séries de Puiseux, séries de Hahn, séries de Rayner.

Résumé

Les rngs de Rayner sont des rngs (anneaux sans élément unité) dont les éléments sont des séries formelles $\sum_{g \in G} x_g \mathrm{X}^g$, où les coefficients appartiennent à un rng R et les exposants appartiennent à un groupe ordonné additif G, telle que le support $\{g \in G \mid x_g \neq 0_R\}$ appartient à un idéal prédéterminé $\mathcal{J}$ sur G qui vérifie un ensemble de propriétés. La thèse présente une étude de l'interaction entre les propriétés algébriques, topologiques et catégorielles des rngs de Rayner, les rngs R de coefficients et les groupes ordonnés G d'exposants, étudiant les rngs de Rayner sous des perspectives théoriques variées et recherchant des relations universelles entre eux. Deux topologies clés sur ces structures sont systématiquement analysées, les topologies dites faible et forte, et une version du Théorème des Valeurs Intermédiaires est obtenue pour la topologie faible. Une attention particulière est accordée aux rngs des séries de Levi-Civita, Puiseux et Hahn, qui sont des exemples importants des rngs de Rayner.

Mots clés: Théorème des Valeurs Intermédiaires, séries formelles, séries de Levi-Civita, séries de Puiseux, séries de Hahn, séries de Rayner.

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$\neg C$	**Negation** of a condition C.
$C_1 \vee C_2$ C_1 or C_2	(**Inclusive**) **disjunction** of the conditions C_1 and C_2, which is considered to be true if at least one of the conditions C_1 and C_2 is true.
$C_1 \wedge C_2$ C_1 and C_2	**Conjunction** of the conditions C_1 and C_2, which is considered to be true if both the conditions C_1 and C_2 are true.
$C_1 \Rightarrow C_2$	The condition C_1 **implies** the condition C_2, that is, C_2 is true whenever C_1 is true.
$C_1 \Leftrightarrow C_2$	The condition C_1 is **equivalent** to the condition C_2, that is, C_1 is true precisely when C_2 is true.
$A \coloneqq B$	A is **defined** as being equal to the object B.
$\in$	**Membership relation** of Set Theory.

$a_k a_{k+1} \dots a_{k+n} \in A$ Conjunction of the $n+1$ conditions

$$a_k \in A,\ a_{k+1} \in A, \dots, a_{k+n} \in A\,.$$

$C(x_1...x_n)\,(\forall x_1...x_n{\in}A)$
$(\forall x_1...x_n{\in}A)\ C(x_1...x_n)$

Universal quantification of the condition $C\,(x_1 \dots x_n)$ with its variables restricted to the set A, which is considered to be true if the condition $C\,(x_1 \dots x_n)$ is true whenever $x_1 \dots x_n \in A$. In this thesis, the **universal quantifier** $\forall$ shall appear to the right side of the condition it is refering to whenever the whole closed quantified condition is presented inline within the text, such as in $C\,(x_1 \dots x_n)\ (\forall x_1 \dots x_n \in A)$. When the closed quantified condition is displayed outside the text, the $\forall$-part is to appear on the left side, such as in

$$(\forall x_1 \dots x_n \in A)\ \ C\,(x_1 \dots x_n)\ .$$

$C(x_1...x_n)\,(\exists x_1...x_n{\in}A)$
$(\exists x_1...x_n{\in}A)\ C(x_1...x_n)$

Existential quantification of the condition $C\,(x_1 \dots x_n)$ with its variables restricted to the set A, which is considered to be true if the condition $C\,(x_1 \dots x_n)$ is true for at least one choice of the variables $x_1 \dots x_n \in A$. In this thesis, the **existential quantifier** $\exists$ shall appear to the right side of the condition it is refering to whenever the whole closed quantified condition is presented inline within the text, such as in $C\,(x_1 \dots x_n)\ (\exists x_1 \dots x_n \in A)$. When the closed quantified condition is displayed outside the text, the $\exists$-part is to appear on the left side, such as in

$$(\exists x_1 \dots x_n \in A)\ \ C\,(x_1 \dots x_n)\ .$$

$\subset$ **Non-strict inclusion relation** between sets.

$\subsetneq$ **Strict** or **proper inclusion relation** between sets.

$\notin, \not\subset, \not\subsetneq$, etc. **Negations** of the membership, non-strict inclusion and strict inclusion relations, respectively.

$\{F(x)\}_{P(x)}$
$\{F(x) \mid P(x)\}$ **Set-builder notation**. It defines a set of objects of the form $F(x)$ for objects x which satisfy the property $P(x)$. The formal systems which axiomatise Set Theory (ZF, ZFC, NBG, etc.) limit the options of properties $P(x)$ which can be used to define a set. Countable sets are sometimes described element by element via the notations $\{x_1, x_2, \ldots, x_n, \ldots\}$ or $\{x_1 x_2 \ldots x_n \ldots\}$.

$A - B$ (**Asymmetric**) **difference** between the sets A and B, given by

$$A - B := \left\{ x \in A \,\vdots\, x \notin B \right\}.$$

$\emptyset$ **Empty set**, given by

$$\emptyset := \left\{ x \,\vdots\, x \neq x \right\}.$$

$\mathbb{N}$ Set of **natural numbers**:

$$\mathbb{N} := \{1, 2, 3, 4, \dots\} .$$

$\mathbb{N}_0$ Set whose elements are the number zero and the natural numbers:

$$\mathbb{N}_0 := \{0, 1, 2, 3, 4, \dots\} .$$

$\mathbb{Z}$ Set of **integers**:

$$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\} .$$

$\mathbb{Q}$ Set of **rational numbers**:

$$\mathbb{Q} := \{{}^{p}\!/_{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} - \{0\}\} .$$

$\mathbb{R}$ Set of **real numbers**.

$\overline{\mathbb{R}}$ Set of **extended real numbers**, which is the set given by $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, where $-\infty$ and ∞ are objects that do not belong to $\mathbb{R}$, endowed with the extension of the usual order on $\mathbb{R}$ so that $-\infty < \infty$ and $-\infty < x < \infty$ $(\forall x \in \mathbb{R})$. We also set $1/\infty := 0$. The elements $-\infty$ and ∞ are said to be the **infinite elements** of $\overline{\mathbb{R}}$.

$\mathbb{C}$ Set of **complex numbers**.

$\{X_i\}_{P(i)}$
$\{X_i \mid P(i)\}$

Despite being essentially equal to the set-builder notations, these representations often denote a **family**, that is, a function whose domain or **set of indices** is $I := \left\{ i \,\vdots\, P(i) \right\}$ and whose rule of association is $i \mapsto X_i$. When inserted within symbolic mathematical expressions, in the presence of other symbols (such as $\in$, $\subset$, $\cap$, etc.), those notations convey the underlying set they represent. In all other occurrences, it will always be explicitly mentioned in the text if these notations represent a set or a family.

$\bigcup\limits_{i \in I} A_i$
$\bigcup \{A_i\}_{i \in I}$

Union of the family of sets $\{A_i\}_{i \in I}$, given by

$$\bigcup_{i \in I} A_i := \left\{ x \,\vdots\, (\exists i \in I)\, x \in A_i \right\}.$$

$\bigcap\limits_{i \in I} A_i$
$\bigcap \{A_i\}_{i \in I}$

Intersection of the family of sets $\{A_i\}_{i \in I}$, given by

$$\bigcap_{i \in I} A_i := \left\{ x \,\vdots\, (\forall i \in I)\, x \in A_i \right\}.$$

$\mathrm{Dom}(R)$ **Domain** of the binary relation R, given by

$$\mathrm{Dom}(R) \coloneqq \left\{ x \,\vdots\, (\exists y)\,(x,y) \in R \right\}.$$

$\mathrm{Im}(R)$ **Image** of the binary relation R, given by

$$\mathrm{Im}(R) = \left\{ y \,\vdots\, (\exists x)\,(x,y) \in R \right\}.$$

R^{-1} **Inverse relation** of the binary relation R, given by

$$R^{-1} \coloneqq \left\{ (x,y) \,\vdots\, (y,x) \in R \right\}.$$

$R\langle A \rangle$ **Image of** A under the binary relation R, given by

$$R\langle A \rangle \coloneqq \left\{ y \,\vdots\, (\exists x \in A)\,(x,y) \in R \right\}.$$

In particular, the image of A under the inverse of R, that is, the set $R^{-1}\langle A \rangle$, is called the **preimage of** A **under** R, and the preimage of a singleton $\{y\}$ under R is called the **fibre of** y **under** R;

$R \circ S$ **Composition** of the binary relation R with another binary relation S, given by

$$R \circ S \coloneqq \left\{ (x,z) \,\vdots\, (\exists y)\,((x,y) \in S \wedge (y,z) \in R) \right\}.$$

If $n \in \mathbb{N}$, then the iterated composition

$$\overbrace{R \circ R \circ \cdots \circ R}^{n \text{ times}}$$

shall be denoted by R^n.

$A/{\equiv}$ **Quotient** of the set A modulo the equivalence relation $\equiv$ on A, given by

$$A/{\equiv} \coloneqq \left\{ x/{\equiv} \,\vdots\, x \in A \right\}$$

where $x/{\equiv} \coloneqq {\equiv}\langle\{x\}\rangle$.

$f : A \to B$ **Functional notation**. It reads 'f is a function of type $A \to B$' and it conveys that f is a function whose domain is A and whose **codomain** is B. That means that $f \subset A \times B$ and that f satisfies the condition

$$(\forall x \in A)\,(\exists! y \in B)\,(x, y) \in f.$$

${}^{A}B$ Set of functions of type $A \to B$, where A and B are sets.

id_A **Identity function** of type $A \to A$, given by

$$(\forall x \in A)\ \ \mathrm{id}_A(x) \coloneqq x\,.$$

$\prod_{i \in I} A_i$ **Cartesian product** of the family of sets $\{A_i\}_{i \in I}$, given by

$$\prod_{i \in I} A_i \coloneqq \left\{ f : I \to \bigcup_{i \in I} A_i \, \vdots \, (\forall i \in I) \, f(i) \in A_i \right\}.$$

$a_k a_{k+1} a_{k+2} \cdots$
$\{a_n\}_{n \in [k,\infty)_{\mathbb{Z}}}$
$\{a_n\}$

(Infinite) sequence of objects, defined in Definition 1.35. It can be denoted in a term-by-term fashion, such as in $a_k a_{k+1} a_{k+2} \ldots$, or in a family-like notation, such as in $\{a_n\}_{n \in [k,\infty)_{\mathbb{Z}}}$. Regarding the term-by-term notation, we shall include commas between the objects of a sequence only if their absence affects the readability of the individual terms of the sequence. Analogously, we shall sometimes omit the commas when denoting infinitely countable sets, for instance in $\{a_k a_{k+1} a_{k+2} \ldots\}$. As for the family-like notation $\{a_n\}_{n \in [k,\infty)_{\mathbb{Z}}}$, we shall only consider the case $k = 1$ in this thesis, and the condition $n \in \mathbb{N}$ shall be dropped from the subscript of the notation $\{a_n\}_{n \in \mathbb{N}}$, producing the simplified version $\{a_n\}$.

$a_k a_{k+1} \ldots a_{k+n}$ **Finite sequence** of objects, defined in Definition 1.35. We shall include commas between the objects of a finite sequence with an indeterminate number of terms (e.g. with n terms) only if their absence affects the readability of the individual terms of the sequence. Analogously, when the number of terms is indeterminate and there is no risk of confusion, we shall omit the commas when denoting finite sets and ordered n-tuples, such as in $\{a_k a_{k+1} \ldots a_{k+n}\}$ and $(a_k a_{k+1} \ldots a_{k+n})$, respectively.

$A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots$ **Countable union** of the sequence of sets $A_1 A_2 \ldots A_n \ldots$.

$A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots$ **Countable intersection** of the sequence of sets $A_1 A_2 \ldots A_n \ldots$.

$A_1 \times A_2 \times \cdots \times A_n$ **Finite Cartesian product** of the finite sequence of sets $A_1 A_2 \ldots A_n$, given by

$$A_1 \times A_2 \times \cdots \times A_n \coloneqq \Big\{ (a_1 a_2 \ldots a_n) \vdots a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n \Big\} .$$

In the case $A \coloneqq A_1 = A_2 = \cdots = A_n$, that finite Cartesian product shall be denoted by $A^{\overset{n}{\times}}$.

$R \restriction\restriction_S$ **Restriction of the n-ary relation** R to the set S, given by

$$R \restriction\restriction_S \coloneqq R \cap S^{\overset{n}{\times}} .$$

That restriction shall be denoted by $R \restriction\restriction (S)$ when the set S is indicated by a large expression.

$f \restriction_S$ **Restriction of the n-ary function** f of type

$$f : A_1 \times A_2 \times \cdots \times A_n \to B$$

to the set S, given by

$$f \restriction_S \coloneqq f \cap (S^{\overset{n}{\times}} \times B) .$$

That restriction shall be denoted by $f \restriction (S)$ when the set S is indicated by a large expression.

$\mathrm{P}(A)$	**Power set** of the set A, given by $$\mathrm{P}(A) := \left\{x \vdots x \subset A\right\}.$$
$\lvert A\rvert$	**Cardinal** of the set A, which is the smallest ordinal α that is **equipotent** to A, that is, it is the smallest ordinal α such that there is a bijective function $f : A \to \alpha$.
α^+	**Cardinal successor** of the cardinal α, which is the smallest cardinal that is greater than α.
$\mathrm{P}_\alpha(A)$	Set of subsets S of the set A such that $\lvert S\rvert < \alpha$, where α is a cardinal.

Preface

November, 2022 Geovani Pereira Machado

At the onset of 2021, whilst the Covid-19 global pandemic was raging in full force, I initiated the efforts to take on the challenge of writing a doctoral thesis. Inspired by the momentous success of the modern theory of Real Analysis on $\mathbb{R}^n$, which has crucial applications to Physics, Probability Theory and other branches of Mathematics, I wondered if there is a non-Archimedean extension of $\mathbb{R}$ which can support an analytic theory that is somewhat analogous to Real Analysis, possibly also being applicable to other areas of study.

My previous major academic experience had been on the area of non-Archimedean Analysis through a master's dissertation on Non-standard Analysis (148), where the non-Archimedean **field of hyperreal numbers**, denoted by ${}^*\mathbb{R}$, was considered among other things. The main advantage of working with ${}^*\mathbb{R}$ stems from the Transfer Principle, which states that the same first-order sentences with bounded quantifiers are true for $\mathbb{R}$ and ${}^*\mathbb{R}$. Unfortunately, all known constructions of ${}^*\mathbb{R}$ (in ZFC) essentially depend on a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and the existence of such $\mathcal{U}$ cannot be demonstrated without the Axiom of Choice, that is, $\mathcal{U}$ cannot be constructively determined. Indeed, no matter what non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ one may "choose", there will always be infinitely many undecidable questions about the elements of ${}^*\mathbb{R}$, for one can never fully determine which subsets of $\mathbb{N}$ belong to $\mathcal{U}$. Thus, one cannot really perform computations involving hyperreal numbers in the same sense that one does on $\mathbb{R}$, since most elements of the former numeric system are shrouded in obscurity and undecidability. Progress has been made in devising a constructive sheaf-theoretic version of Non-standard Analysis (172, 173), albeit by changing the notion of model in order to regain the full Transfer Principle, but still it remains unclear if it is possible to formulate an

Analysis on the hyperreals in a way that satisfactorily extends the celebrated theory of Analysis on $\mathbb{R}^n$ and stands on its own.

Intrigued by those drawbacks concerning $^*\mathbb{R}$, I considered other non-Archimedean extensions of $\mathbb{R}$ on which appealing analytical considerations had been drawn in the literature. The first one I encountered which showed some promising analytical results is the **field of surreal numbers**, denoted by No, conceived by John Horton Conway and introduced by Donald Ervin Knuth in 1974 (113). In fact, No extends not only $\mathbb{R}$ but also $^*\mathbb{R}$, and it contains all ordinal numbers. As it happens, by defining a new non-local notion of topology on No, the new emergent notions of limits and derivatives become well-behaved enough so that the Intermediate Value Theorem and the Extreme Value Theorem hold for certain functions (196). Moreover, the usual transcendental functions have been defined on No, extending their classical counterparts. Many issues still need to be resolved in order to obtain a broad theory of Analysis on No, such as finding a consistent definition of integration of functions of type $\mathsf{No}^n \to \mathsf{No}$, but there is hope for future developments.

After having stumbled upon a few other investigations in that direction which failed to meet my expectations, including the studies on the non-commutative rational series (15) and the superreal fields (64), I learned of the existence of a non-Archimedean field extension of $\mathbb{R}$ on which a compelling and sophisticated theory of Differential and Integral Calculus has gradually been developed since the late 19th century. Furthermore, that number field has some important applications in Computer Science, mainly on the area of Computational Differentiation. That structure, formulated in 1893 by the Italian mathematician Tullio Levi-Civita (1873-1941) and currently denoted by $\mathcal{R}$, is called the **real Levi-Civita field** (139, 140) and I reckon it is the non-Archimedean field extension of $\mathbb{R}$ which is closest to having its own complete Analysis. As a matter of fact, it is known that many classical results of Analysis on $\mathbb{R}^n$ hold true for $\mathcal{R}^n$ when their assumptions are slightly modified, including the Intermediate Value Theorem, the Extreme Value Theorem, the Mean Value Theorem and Taylor's Theorem (17, 20, 24, 26, 28, 209, 210, 208). On top of that, a comprehensive Measure Theory on $\mathcal{R}^n$ has been conceived, enabling multiple integrals of functions of type $\mathcal{R}^n \to \mathcal{R}$ to be suitably defined so that both the Fundamental Theorem of Calculus and Fubini's Theorem hold true (202, 160, 205, 73, 211, 74). Those results are possible by way of exploiting the properties of two peculiar topologies on $\mathcal{R}$.

Each element of $\mathcal{R}$ is a formal power series $x = \sum_{q \in \mathbb{Q}} x_q \mathrm{X}^q$, where $x_q \in \mathbb{R}$ for each $q \in \mathbb{Q}$ and where the support $\operatorname{supp}(x)$ is left-finite in $\mathbb{Q}$. At first sight, the choices of the sets $\mathbb{R}$ and $\mathbb{Q}$ to be the respective sets of coefficients and exponents of those formal series are unjustified, and it is reasonable to wonder about what would the theoretical implications be if those choices were altered. It turns out that by replacing $\mathbb{R}$ with a rng (ring without unity) R and replacing $\mathbb{Q}$ with an ordered group G, one sets up a rng whose elements are formal power series $x = \sum_{g \in G} x_g \mathrm{X}^g$ whose coefficients x_g are elements of R and whose supports $\operatorname{supp}(x)$ are left-finite in G. I call that structure a **Levi-Civita rng**, denoting it by $\overset{\mathrm{lf}}{R}[[\mathrm{X}^G]]$. Although analytical considerations ignited the initial inspiration that gave rise to the research theme, an Analysis on $\overset{\mathrm{lf}}{R}[[\mathrm{X}^G]]$ cannot be properly contemplated before the (topological) rng structure of $\overset{\mathrm{lf}}{R}[[\mathrm{X}^G]]$ has been established and sufficiently appreciated. With that setting in mind, I commenced investigating the direct connections between the algebraic (and topological) properties of the rng $\overset{\mathrm{lf}}{R}[[\mathrm{X}^G]]$, the (topological) rng R and the ordered group G, studying the Levi-Civita rngs under varied perspectives and seeking universal relations between them. After having achieved some preliminary success in deriving a few interesting results, it was provisionally proposed that the thesis would mainly concern the fundamentals of the theory of Levi-Civita rngs, reserving the analytical explorations for subsequent works, and the writing process effectively began in March, 2021.

Most original theorems and propositions present in the final product of the thesis were originally conceived solely concerning the Levi-Civita rngs, as planned at the outset. However, as I scanned through the literature that could help me unravel the matter of algebraic closure in situations where $\overset{\mathrm{lf}}{R}[[\mathrm{X}^G]]$ is a field, I came across a couple articles by Francis J. Rayner (181, 182) which allowed me to understand that the Levi-Civita rngs actually belong to a more general class of structures, which I call **Rayner rngs**, and that realisation led me to consider expanding the objectives of the work to include a comparative study of the properties of distinct types of Rayner rngs. A close inspection of preliminary proofs of several results showed that such undertaking was feasible, and, after pondering the fact that other prominent non-Archimedean rngs are actually instances of Rayner rngs, such as the rngs of Puiseux series (178, 179, 8) and the rngs of Hahn series (92, 150, 145), it became clear that such generalisation would considerably enrich the scope of the thesis without overshadowing the remarkable properties of the Levi-Civita rngs. Accordingly, it was decided that that reformation would be performed and its implementation marked the completion of the production in November, 2022.

The method of exposition chosen for the work proceeds from the general to the particular. On these terms, Chapters 1 and 2 introduce the basic definitions and results on magmas and rngs under algebraic, topological, order-theoretic and categorical perspectives, laying the theoretical foundations on which the main subject matter of the composition, the Theory of Rayner Rngs, is developed in Chapters 3 and 4. Supplementarily, Appendix A provides a historically-grounded introduction to the Tarski-Grothendieck axiomatisation of Set Theory, which is adequate to avoid foundational issues concerning proper classes, and Appendix B serves as a reference to a number of basic definitions of Category Theory.

The logical framework of each chapter consists of definitions and theorems, and these are the parts upon which the author wishes to place emphasis. Less important results and those which can easily be deduced from the theorems appear in the text as propositions, lemmas and corollaries. Occasionally, assumptions are included in order to take the validity of some frequently recurrent hypotheses of the theory as given, effectively simplifying the phrasing of considerations. Statements of results taken from literature always appear accompanied by references to the works from which they are taken. Lastly, plural first-person pronouns (we, us, our, etc.) are employed throughout the text so as to instill upon the reader the feeling that he and the author are working together as his reading progresses, potentially enhancing his learning experience.

Introduction

Triggered by the customary use of infinitesimal and infinite quantities in the works of Cavalieri (52, 53) and Leibniz (137), inspired by the Cantor's theory of transfinite numbers (49) and instigated by Veronese's suggestions in the geometric treatise (230) on the possibility of the existence of line segments with infinitesimal and infinite lengths, the young Italian mathematician Tullio Levi-Civita (1873-1941) brought forth in (139) the system of **generalised hyperbolic numbers** (***numeri generali iperbolici***), which he denoted by I. In that work, he defines addition and multiplication operations along with an order on I, rendering it a non-Archimedean (totally) ordered field structure that extends the real number system. Furthermore, he indicates how power series with coefficients in I can be used to define transcendental functions on subsets of I, and he reveals how the derivatives of those functions can be obtained. Ever since that publication, many mathematicians independently rediscovered Levi-Civita's number system, such as Berz (16), Laugwitz (131), Neder (164) and Ostrowski (170), and many have considerably advanced the research on the theme, most notably Berz (17, 18, 19, 20) and Shamseddine (200, 207, 211, 208).

During the course of the 20th century, the ordered field at issue eventually came to be almost universally denoted by $\mathcal{R}$ and called the **real Levi-Civita field,** as its elements were called the **real Levi-Civita numbers**. Moreover, a robust Measure Theory on $\mathcal{R}^n$ was conceived (23, 202, 160, 205, 73, 211), giving rise to a definition of multiple integrals for functions of type $\mathcal{R}^n \to \mathcal{R}$ so that refurbished versions of many theorems of the classical theory of integration on $\mathbb{R}^n$ hold, such as the Fundamental Theorem of Calculus and Fubini's Theorem.

A real Levi-Civita number is a formal power series $x = \sum_{q\in\mathbb{Q}} x_q \mathrm{X}^q$ with real coefficients whose support $\operatorname{supp}(x) := \{q \in \mathbb{Q} \mid x_q \neq 0\}$ is **left-finite in** $\mathbb{Q}$, that is, it is a family $x = \{x_q\}_{q\in\mathbb{Q}}$ in $\mathbb{R}$ such that for each $q' \in \mathbb{Q}$, there is only a finite number of elements q of $\operatorname{supp}(x)$ so that $q \leqslant q'$. That implies that each support $\operatorname{supp}(x)$ for $x \in \mathcal{R}$ is a countable well-ordered subset of $\mathbb{Q}$ (cf. Proposition 3.10, Item (c)), say $\operatorname{supp}(x) = \{q_1 q_2 \ldots q_n \ldots\}$, and the number x is denoted by a countable formal sum

$$x = x_{q_1}\mathrm{X}^{q_1} + x_{q_2}\mathrm{X}^{q_2} + \cdots + x_{q_n}\mathrm{X}^{q_n} + \cdots .$$

The addition operation on $\mathcal{R}$ is given by $(x+y)_q \coloneqq x_q + y_q$ and the multiplication operation on $\mathcal{R}$ is given by $(xy)_q \coloneqq \sum\limits_{\substack{r,s\in\mathbb{Q}\\ r+s=q}} x_r y_s$. It turns out that this multiplication is well-defined, since the left-finiteness of $\operatorname{supp}(x)$ and $\operatorname{supp}(y)$ in $\mathbb{Q}$ implies that the sum $\sum\limits_{\substack{r,s\in\mathbb{Q}\\ r+s=q}} x_r y_s$ has only a finite number of non-zero terms (cf. Lemma 1.75). The careful reader shall notice that these operations are analogous to the usual addition and multiplication operations on polynomials, except that, in this case, we are dealing with a potentially infinite number of terms and with rational exponents. The left-finiteness property also implies that each support $\operatorname{supp}(x)$ has a least element $\operatorname{ms}(x) \coloneqq \overset{\mathbb{Q}}{\min}\{\operatorname{supp}(x)\}$ in $\mathbb{Q}$ whenever $x \neq 0$, and the image of x under $\operatorname{ms}(x)$ is denoted by $\operatorname{pc}(x) \coloneqq x_{\operatorname{ms}(x)}$. As a special case, we define $\operatorname{pc}(0) \coloneqq 0$. The order $<$ on $\mathcal{R}$ is defined so that for all $x, y \in \mathcal{R}$, the conditions $x < y$ and $\operatorname{pc}(y-x) > 0$ are equivalent.

Note that $\mathrm{X}, \mathrm{X}^2, \mathrm{X}^3, \ldots$ is a decreasing sequence of infinitesimal numbers in $\mathcal{R}$ (Definition 2.42), and $\mathrm{X}^{-1}, \mathrm{X}^{-2}, \mathrm{X}^{-3}, \ldots$ is an increasing sequence of infinite numbers in $\mathcal{R}$. With that setting, one may prove that $\mathcal{R}$ is a non-Archimedean ordered field (cf. Theorems 3.67 and 3.82), just as Levi-Civita did in 1893. The set $\mathbb{R}$ of real numbers is identified with a subset of $\mathcal{R}$ via injection $x \mapsto x\mathrm{X}^0$, and this injection is an immersion between ordered rings. Thus, we may write $\mathbb{R} \subset \mathcal{R}$.

The order on $\mathcal{R}$ canonically defines a topology on $\mathcal{R}$, which is called the **strong topology on** $\mathcal{R}$ and shall be denoted by $\overset{\mathcal{R}}{\mathsf{St}}$ for the sake of this introduction. Another topology of the utmost importance for the theory is generated by the basic open sets of the form

$$\mathsf{S}_r(x) \coloneqq \left\{ y \in \mathcal{R} \;\middle|\; \overset{\mathbb{R}}{\max_{q \leqslant 1/r}} |y_q - x_q| < r \right\}$$

for $x \in \mathcal{R}$ and $r \in (0, \infty)_{\mathbb{R}}$, which is called the **weak topology on** $\mathcal{R}$ and shall be denoted by $\overset{\mathcal{R}}{\mathsf{Wt}}$ for now. One may prove that $\overset{\mathcal{R}}{\mathsf{Wt}} \subset \overset{\mathcal{R}}{\mathsf{St}}$ and that $\mathcal{R}$ is a topological field when endowed with its strong topology (cf. Theorems 4.2 and 4.15).

Since each element of $\mathcal{R}$ is a family $x = \{x_q\}_{q \in \mathbb{Q}}$ in $\mathbb{R}$ with indices in $\mathbb{Q}$, it is advantageous to write the indices of a family in $\mathcal{R}$ in the lower-left corner, such as in ${}_i x$, reserving the lower-right corner to the elements of $\mathbb{Q}$. Hence, we define ${}_i x_q$ as being the q-image of the i-term of the family $\{{}_i x\}_{i \in I}$ in $\mathcal{R}$, that is, ${}_i x_q \coloneqq ({}_i x)_q$. As usual, the upper-right indices are reserved to denoting powers. A sequence $\{{}_n x\}$ in $\mathcal{R}$ is **regular** if the union $\bigcup\limits_{n \in \mathbb{N}} \operatorname{supp}({}_n x)$ is left-finite in $\mathbb{Q}$.

With these definitions and conventions, we present, without proofs, a couple of interesting analytical results concerning $\mathcal{R}$ for the sake of motivation. The reader should not worry too much about the details, for the following results are shown here just to give him or her a general sense of how the Analysis on $\mathcal{R}$ has been developed.

Theorem A (Criterion of $\overset{\mathcal{R}}{\mathsf{St}}$-convergence for power series in $\mathcal{R}$)**.** (17, 200) *Let $\{{}_n a\}$ be a sequence in $\mathcal{R}$, let*

$$\lambda \coloneqq \overset{\mathbb{R}}{\limsup_{n \to \infty}} \frac{-\operatorname{ms}({}_n a)}{n},$$

and take a fixed ${}_0 x \in \mathcal{R}$. For each $x \in \mathcal{R}$, we have:

- ▷ *If $\operatorname{ms}(x - {}_0 x) > \lambda$, then the power series $\sum\limits_{n=0}^{\infty} {}_n a \, (x - {}_0 x)^n$ is absolutely*[1] $\overset{\mathcal{R}}{\mathsf{St}}$*-convergent;*
- ▷ *If $\operatorname{ms}(x - {}_0 x) < \lambda$, or if $\operatorname{ms}(x - {}_0 x) = \lambda$ and $\frac{-\operatorname{ms}({}_n a)}{n} > \lambda$ for infinitely many natural numbers n, then the power series $\sum\limits_{n=0}^{\infty} {}_n a \, (x - {}_0 x)^n$ is* $\overset{\mathcal{R}}{\mathsf{St}}$*-divergent.*

---

[1] A series $\sum\limits_{n=1}^{\infty} {}_n a$ in $\mathcal{R}$ is **absolutely τ-convergent** (resp. **absolutely τ-divergent**) if the series $\sum\limits_{n=1}^{\infty} |{}_n a|$ is τ-convergent (resp. τ-divergent), where τ is a topology on $\mathcal{R}$.

Theorem B (Criterion of $\overset{\mathcal{R}}{\mathsf{Wt}}$-convergence for power series in $\mathcal{R}$)**.** (17, 200) *Let $\{{}_n a\}$ be a sequence in $\mathcal{R}$, let λ be defined as in Theorem A, suppose x and ${}_0 x$ are elements of $\mathcal{R}$ so that*[2] $\mathrm{ms}\,(x - {}_0 x) = \lambda$*, suppose that the sequence $\{{}_n b\}$ in $\mathcal{R}$ given by ${}_n b := {}_n a \cdot \mathrm{X}^{n\lambda}$ is regular, and let r be the positive extended real number given by*

$$r := \begin{cases} \dfrac{1}{\overset{\mathbb{R}}{\sup\limits_{q\in\mathbb{Q}}} \left(\overset{\mathbb{R}}{\limsup\limits_{n\to\infty}} |{}_n b_q|^{1/n}\right)} & \text{if the denominator is non-zero}\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

- *If $|(x - {}_0 x)_\lambda| < r$, then the power series $\sum\limits_{n=0}^{\infty} {}_n a\,(x - {}_0 x)^n$ is absolutely $\overset{\mathcal{R}}{\mathsf{Wt}}$-convergent;*
- *If $|(x - {}_0 x)_\lambda| > r$, then the power series $\sum\limits_{n=0}^{\infty} {}_n a\,(x - {}_0 x)^n$ is $\overset{\mathcal{R}}{\mathsf{Wt}}$-divergent.*

Corollary. (17, 200) *Let $\{a_n\}$ be a sequence in $\mathbb{R}$ and let $\eta \in \overline{\mathbb{R}}$ be the radius of convergence of the series $\sum\limits_{n=0}^{\infty} a_n x^n$ for $x \in \mathbb{R}$. For all $x \in \mathcal{R}$ so that*[3] $|x|$ *is less than η and is not infinitely close to η, the series $\sum\limits_{n=0}^{\infty} a_n x^n$ is absolutely $\overset{\mathcal{R}}{\mathsf{Wt}}$-convergent.*

Proof. Considering the definitions of λ, $\{{}_n b\}$ and r from Theorems A and B, in this situation we have

$$\lambda = \overset{\mathbb{R}}{\limsup_{n\to\infty}} \frac{-\mathrm{ms}\,({}_n a)}{n} = \overset{\mathbb{R}}{\limsup_{n\to\infty}} \frac{0}{n} = 0$$

and ${}_n b = {}_n a \cdot \mathrm{X}^{n\lambda} = {}_n a \cdot \mathrm{X}^0 = {}_n a$, and since the union

$$\bigcup_{n\in\mathbb{N}} \mathrm{supp}\,({}_n b) = \bigcup_{n\in\mathbb{N}} \mathrm{supp}\,({}_n a) = \{0\}$$

[2] This implicitly entails the supposition that $\lambda \in \mathbb{Q}$.
[3] We assume that $x < \infty$ $(\forall x \in \mathcal{R})$, and $x \sim \infty$ precisely when x is infinite in $\mathcal{R}$.

is left-finite in $\mathbb{Q}$, the sequence $\{{}_nb\}$ is regular. Moreover, since

$$\sup_{q\in\mathbb{Q}}^{\mathbb{R}}\left(\limsup_{n\to\infty}^{\mathbb{R}}|{}_nb_q|^{1/n}\right)=\sup_{q\in\mathbb{Q}}^{\mathbb{R}}\left(\limsup_{n\to\infty}^{\mathbb{R}}|{}_na_q|^{1/n}\right)=\limsup_{n\to\infty}^{\mathbb{R}}|{}_na_0|^{1/n}=\limsup_{n\to\infty}^{\mathbb{R}}|{}_na|^{1/n}=\frac{1}{\eta}$$

from the classical theory of power series on $\mathbb{R}^n$, we get $r=\eta$. Let x be an element of $\mathcal{R}$ so that $|x|$ is less than η and is not infinitely close to η. If $\operatorname{ms}(x)>0$, then the series $\sum_{n=0}^{\infty}a_nx^n$ is absolutely $\overset{\mathcal{R}}{\mathsf{St}}$-convergent (Theorem A), and, accordingly, it is absolutely $\overset{\mathcal{R}}{\mathsf{Wt}}$-convergent. Since x is finite, we have $\operatorname{ms}(x)\geqslant 0$ (cf. Theorem 3.82, Item (n)), and we may assume that $\operatorname{ms}(x)=0$ from now on. Hence, x is of the form

$$\begin{aligned}x&=x_0\mathrm{X}^0+x_{q_1}\mathrm{X}^{q_1}+x_{q_2}\mathrm{X}^{q_2}+\cdots\\&=x_0+x_{q_1}\mathrm{X}^{q_1}+x_{q_2}\mathrm{X}^{q_2}+\cdots\\&=x_0+(\text{infinitesimal terms})\ ,\end{aligned}$$

where $x_0\neq 0$ and where $q_1q_2\ldots$ is the finite sequence of positive elements of the support $\operatorname{supp}(x)$. Note that

$$|x|=|x_0|+(\text{infinitesimal terms})$$

and $|x_0|\neq\eta$, for otherwise $|x|$ would be infinitely close to η. If $\eta=\infty$, then $|x_0|<\eta$ by definition, and if $\eta<\infty$, then $\operatorname{ms}(\eta-|x|)=0$ and

$$\eta-|x_0|=\eta_0-|x|_0=(\eta-|x|)_0=\operatorname{pc}(\eta-|x|)>0$$

since $|x|<\eta$, implying $|x_0|<\eta$. Therefore, the power series $\sum_{n=0}^{\infty}a_nx^n$ is absolutely $\overset{\mathcal{R}}{\mathsf{Wt}}$-convergent (Theorem B).

□

The Corollary above justifies the extension of several real-analytic transcendental functions of type $\mathbb{R}\to\mathbb{R}$ to functions of type $\operatorname{Fin}(\mathcal{R})\to\operatorname{Fin}(\mathcal{R})$, where $\operatorname{Fin}(\mathcal{R})$ is the set of finite real Levi-Civita numbers. As examples, the power series

$$\mathsf{e}^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ ,\quad\cos(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\quad\text{and}\quad\cosh(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

define functions of type $\operatorname{Fin}(\mathcal{R})\to\operatorname{Fin}(\mathcal{R})$ which suitably extend their classical counterparts, where the infinite sums are taken with respect to the weak topology $\overset{\mathcal{R}}{\mathsf{Wt}}$ on $\mathcal{R}$.

A numerical system analogous to $\mathcal{R}$ is obtained by allowing the coefficients of the formal series to take complex values. That system is called the **complex Levi-Civita field**, being denoted by $\mathcal{C}$, and its elements are called the **complex Levi-Civita numbers**. Such numbers may be seen as ordered pairs of real Levi-Civita numbers via the bijection

$$\sum_{q\in\mathbb{Q}} z_q \mathrm{X}^q \longmapsto \Big(\sum_{q\in\mathbb{Q}} \mathrm{Re}\,(z_q)\,\mathrm{X}^q, \sum_{q\in\mathbb{Q}} \mathrm{Im}\,(z_q)\,\mathrm{X}^q\Big)\ .$$

Note that we have $\mathcal{R} \subset \mathcal{C}$, but, unlike $\mathcal{R}$, the field $\mathcal{C}$ is non-orderable, since the non-orderable field $\mathbb{C}$ may be identified with a subfield of $\mathcal{C}$ via the injective homomorphism $z \mapsto z\mathrm{X}^0$. Moreover, one may prove that $\mathcal{C}$ is algebraically closed (cf. Corollary 3.73), and it has been shown that a few key theorems of Complex Analysis hold for $\mathcal{C}$, for instance Cauchy's Integral Formula (20).

As a matter of fact, the property of left-finiteness is not the only constraint that one may impose upon the supports of formal series as a means to generate noteworthy non-Archimedean extensions of a division ring K, as it turns out. First considered by Newton (147, 91) and then further developed by Puiseux in 1850 (178, 179), the so-called **Puiseux series** are the formal power series $x = \sum_{n\in[n_0,\infty)_{\mathbb{Z}}} x_{n/d}\mathrm{X}^{n/d}$ with coefficients in K, where $n_0 \in \mathbb{Z}$ and $d \in \mathbb{N}$. These series constitute a division ring extension of K, and they have important applications in Complex Analysis and Algebraic Geometry, mainly the implications of the Newton-Puiseux Theorem for the study of algebraic curves (4). Note that the supports of the Puiseux series are subsets of $\mathbb{Q}$ of the form $\{n_0/d_0, n_1/d_1, n_2/d_2, \dots\}$, where $n_0 n_1 n_2 \dots$ is a sequence of integers and $d_0 d_1 d_2 \dots$ is a bounded sequence of natural numbers. Another prominent power series extension of K was introduced by Hahn in 1907 as he studied Hilbert's Second Problem on the question of the consistency of the axioms of Peano arithmetic (92). Considering a commutative ordered group G, the **Hahn series** are the formal power series $x = \sum_{g\in G} x_g \mathrm{X}^g$ with coefficients in K whose supports $\mathrm{supp}\,(x)$ are well-ordered in G, and they form a division ring. Later, Mal'cev (153) and Neumann (145) generalised Hahn's construction to the case where G is non-commutative. These approaches have a wide range of applications, but unfortunately neither of them seem to support an analytic theory that is somewhat resemblant to Real Analysis, in contrast to the Levi-Civita fields.

The same procedure that gave rise to those power series extensions of division rings can be generalised and unified into a single framework. As Rayner showed in 1968 (181, 182), if a general set $\mathcal{J}$ of subsets of a commutative ordered group G is constrained by a particular set of axioms, then the set of formal power series $x = \sum_{g \in G} x_g \mathrm{X}^g$ with coefficients in a division ring K and with support $\operatorname{supp}(x)$ in $\mathcal{J}$ forms a division ring extension of K, and if moreover K is ordered, then that formal power series field can also be ordered. In addition, that structure is algebraically closed whenever K is an algebraically closed field and G is divisible.

We shall see that Rayner's construction can be generalised even further, provided that one is not solely interested in working with division rings or fields. First off, not all of Rayner's axioms are strictly necessary so as to generate interesting power series rngs (rings without unity), and, by adjusting one of these axioms, one need not assume that the group of exponents G is commutative, matching Mal'cev's and Neumann's findings on the Hahn fields. Along these lines, in full generality, one may take a rng R, an ordered group G and a set $\mathcal{J}$ of subsets of G constrained by a fresh set of axioms, and then one may consider the set of formal power series with coefficients in R whose supports belong to $\mathcal{J}$. As we shall confirm in Chapter 3, that set is a rng when endowed with addition and multiplication operations akin to the ones defined for $\mathcal{R}$. We shall call such rng a **Rayner rng**, denoting it by $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, and we shall call the set $\mathcal{J}$ a **Rayner ideal on** G. Several variations of axioms upon $\mathcal{J}$ were studied by Krapp, Kuhlmann and Serra (117), as they correlated each type of ideal $\mathcal{J}$ to its corresponding algebraic impact on the rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

In essence, this thesis aims to determine the direct connections between the properties of the Rayner rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, the rng R, the ordered group G and the

Rayner ideal $\mathcal{J}$, confronting the task under algebraic, topological and categorical perspectives. That entails that the work must touch upon varied areas of Mathematics, most crucially Group Theory, Category Theory, Topology, Valuation Theory, Rng Theory and Field Theory. Chapters 1 and 2 and Appendices A and B are designed to get the reader up to speed on these subjects, at least on the topics that eventually play a role on the text. The main substance of the thesis takes place in Chapters 3 and 4, where the former lays the algebraic and categorical foundations of the Theory of Rayner Rngs and the latter dissects two key topologies on Rayner rngs, the so-called weak and strong topologies.

On that footing, the conceptual machinery of Krapp, Kuhlmann and Serra's Theory of Rayner Ideals is expanded (Definitions 3.1 and 3.17), some key algebraic and topological results from literature are generalised (Theorems 3.61, 3.78, 3.82, 4.2, 4.15 and 4.19), and some incipient concepts (Definitions 3.51, 3.85 and 4.26) are produced. The most original results of this work, which genuinely expand the body of knowledge for researchers in the field, are the following:

- ▷ Theorem 1.84;
- ▷ Proposition 3.11;
- ▷ Proposition 3.14;
- ▷ Proposition 3.18;
- ▷ Proposition 3.39;
- ▷ Proposition 3.52;
- ▷ Proposition 3.53;
- ▷ Theorem 3.54;
- ▷ Theorem 3.56;
- ▷ Corollary 3.57;
- ▷ Theorem 4.2, Items (j) and (k);
- ▷ Theorem 4.15, Items (i), (j) and (q);
- ▷ Theorem 4.21;
- ▷ Theorem 4.23;
- ▷ Theorem 4.28.

As a bonus, in Section 4.4 we present a generalisation of the notion of order-convexity for subsets of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, making it possible for a new version of the Intermediate Value Theorem to be obtained for the weak topology (Theorem 4.28). Most of the Theory of Rayner Rngs is presented through an unprecedented lens in this work, and that is by virtue of the recurring use of the Notational Device 3.42, which is inspired on some notations of the utmost importance in Asymptotic Analysis.

1 Magmas

Operations are rules which assign to each pair of elements of a set an element of the same set, and they are indispensable and paramount to the assessment of virtually all areas of Mathematics. In Algebra, they are sorted out and examined in depth, acting as foundational building blocks of that area of study, while in non-algebraic settings they are studied in the presence of additional structures, such as orders, topologies and valuations, where they are meant to be compatible with those structures in some sense.

In this chapter, we shall present a compendium of definitions and results concerning mathematical operations, narrowing focus to the topics that will be relevant to our study of Rayner rngs.

1.1 Magmas

A set endowed with an operation on it receives a name brought forth in 1970 by Bourbaki (32, 240) that has never really crossed into mainstream usage. Since no other suggestion has achieved that status, and since the series of textbooks entitled *Eléments de mathématique* has been highly influential since the mid-20th century, we shall work with Bourbaki's term.

Definition 1.1. A **magma** is a set M endowed with a function $f : M \times M \to M$. That function is said to be the **operation of** M, and whenever no particular notation is attributed to the operation f of M, we shall denote it in accordance with the following guidelines:

- ▷ If M is **commutative**, that is, if $f(x, y) = f(y, x)$ $(\forall x, y \in M)$, then f is denoted by $+$ or $+_M$ (**additive notation**) in parallel with denoting the images $f(x, y)$ by $x + y$;
- ▷ If M is not necessarily commutative[1], then f is denoted by $\times$ or $\times_M$

[1] That is, either M is not commutative or we do not have enough information to tell if it is commutative or not.

(**multiplicative notation**, standard) in parallel with denoting the images $f(x,y)$ by xy or $x \cdot y$.

We have the following notations and terminology:

- A **homomorphism of type** $M \to N$ is a function $\phi : M \to N$ between magmas such that
$$(\forall x, y \in M) \;\; \phi(xy) = \phi(x)\,\phi(y) \;;$$
- The **image** of a homomorphism $\phi : M \to N$ between magmas is the submagma of N whose underlying set is the image $\mathrm{Im}(\phi)$ of the function ϕ. That image might be denoted by $\overset{\mathbf{Mag}}{\mathrm{Im}}(\phi)$ to emphasise its magma structure;
- A **submagma of** M is a magma M' such that $M' \subset M$ and $\times_{M'} \subset \times_M$. That is equivalent to stating that $xy \in M'$ $(\forall x, y \in M')$;
- A magma is **trivial** if it has only one element. Otherwise, the magma is **non-trivial**;
- An element 0 of M is a **zero element in** M if $x \cdot 0 = 0 \cdot x = 0$ $(\forall x \in M)$. One notices that there can be at most one zero element in M;
- Let A and B be two subsets of M. We shall denote by AB the set
$$AB \coloneqq \{xy \mid x \in A \text{ and } y \in B\} \;,$$
and if M is denoted additively, then that set shall be denoted by $A + B$ and shall be given by
$$A + B \coloneqq \{x + y \mid x \in A \text{ and } y \in B\} \;;$$
- The **centre of** M is the set denoted[2] by $\mathrm{Z}(M)$ and given by
$$\mathrm{Z}(M) \coloneqq \{x \in M \mid (\forall y \in M) \;\; xy = yx\} \;;$$
- A magma M is **commutative** if $\mathrm{Z}(M) = M$, that is, if $xy = yx$ $(\forall x, y \in M)$.

[2] The letter ‘Z’ in this notation comes from the German word ‘**Z**entrum’, which means centre.

When applying an operation upon three or more elements of a general magma M, the result may depend on the order on which the elements are considered. That is the case, for instance, of the vector cross-product $\times$ on $\mathbb{R}^3$, which allows for two terms of the forms $x \times (y \times z)$ and $(x \times y) \times z$ to differ for infinitely many choices of $x, y, z \in \mathbb{R}^3$. However, that is not the case for the majority of magmas we will assess.

Definition 1.2. A **semigroup** is a magma M whose operation is **associative**, that is, it is such that $x\,(yz) = (xy)\,z$ $(\forall x, y, z \in M)$. We have the following notations and terminology:

- ▷ One can easily notice that every functional composition of homomorphisms between semigroups is a homomorphism. Hence, such functions form a category whose composition operation is the usual functional composition, and that category is denoted by **SGrp**;
- ▷ Let x be an element of M and let n be a natural number. The **n-th power of x (in M)** is the element denoted by x^n and given by $x^n := \overbrace{x \cdot x \cdots x}^{n \text{ times}}$. If M is denoted additively, then that element is called the **n-th multiple of x (in M)**, being denoted by nx and given by $nx := \overbrace{x + x + \cdots + x}^{n \text{ times}}$;
- ▷ Let A be a subset of M and let n be a natural number. We shall denote by A^n the set $A^n := \{x^n \mid x \in A\}$, and if M is denoted additively, then that set shall be denoted by nA and shall be given by $nA := \{nx \mid x \in A\}$. Note that $A^n \subset \overbrace{A \cdot A \cdots A}^{n \text{ times}}$ and those two sets are not equal in general;
- ▷ Let S be a subset of M. The **subsemigroup of M generated by S** is the smallest subsemigroup of M that contains S, and it shall be denoted by $\operatorname*{span}\limits^{\mathbf{SGrp}}_{M}(S)$. It is easy to check that $\operatorname*{span}\limits^{\mathbf{SGrp}}_{M}(S)$ is given by

  $$\operatorname*{span}\limits^{\mathbf{SGrp}}_{M}(S) = \{s_1 \cdot s_2 \cdots s_n \mid s_1 s_2 \ldots s_n \text{ is an inhabited finite sequence in } S\} \;;$$
- ▷ The semigroup M is **divisible** if, for every $x \in M$ and for every $n \in \mathbb{N}$, there is a $y \in M$ such that $x = y^n$.

Example 1.3. The number sets $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are commutative semigroups when endowed with their usual addition operations or their usual multiplication operations. The number 0 is a zero element of the semigroups $(\mathbb{N}_0, \times_{\mathbb{N}_0})$, $(\mathbb{Z}, \times_{\mathbb{Z}})$, $(\mathbb{Q}, \times_{\mathbb{Q}})$, $(\mathbb{R}, \times_{\mathbb{R}})$ and $(\mathbb{C}, \times_{\mathbb{C}})$. The semigroups $(\mathbb{Q}, +_{\mathbb{Q}})$, $(\mathbb{R}, +_{\mathbb{R}})$, $(\mathbb{C}, +_{\mathbb{C}})$ and $(\mathbb{C}, \times_{\mathbb{C}})$ are divisible.

Proposition 1.4. (57) *Let M be a semigroup. The centre $\mathrm{Z}(M)$ of M is a subsemigroup of M.*

Proof. Note that if $x, y \in \mathrm{Z}(M)$, then for each $z \in M$ we get $xyz = xzy = zxy$, which gives us $xy \in \mathrm{Z}(M)$.

□

Some magmas contain a distinguished element, which is defined by the fact that it does not change the elements of the magma when operated upon them.

Definition 1.5. A magma M is **unital** if it has an **identity element**, that is, if it has an element e such that $xe = x = ex$ $(\forall x \in M)$. That being so, it is a standard exercise to prove that e is unique. Whenever no particular notation is attributed to the identity element of M, we shall denote it in accordance with the following guidelines:

- ▷ If the operation of M is denoted multiplicatively (standard case), then the identity element of M is denoted by 1_M;
- ▷ If the operation of M is denoted additively, then the identity element of M is denoted by 0_M.

We have following notations and terminology concerning unital magmas:

- ▷ A homomorphism $\phi : M \to N$ between unital magmas is **unital** if we have $\phi(1_M) = 1_N$;

- ▷ The **kernel** of a homomorphism $\phi : M \to N$ between unital magmas denoted additively is the fibre $\phi^{-1}\langle\{0_N\}\rangle$, which is denoted by $\operatorname{Ker}(\phi)$;

- ▷ Let I be a set. The **support** of a family $x = \{x_i\}_{i\in I} \in {}^{I}M$ is the subset of I denoted by $\operatorname{supp}(x)$ and given by

$$\operatorname{supp}(x) \coloneqq \{i \in I \mid x_i \neq 1_M\}\,;$$

- ▷ A **monoid** is a unital semigroup. One can easily notice that every functional composition of unital homomorphisms between monoids is a unital homomorphism. Hence, such functions form a category whose composition operation is the usual functional composition, and that category is denoted by **Mon**;

- ▷ The image $\overset{\mathbf{Mag}}{\operatorname{Im}}(\phi)$ of a unital homomorphism $\phi : M \to N$ between monoids is a submonoid of N. Thus, that image might be denoted by $\overset{\mathbf{Mon}}{\operatorname{Im}}(\phi)$ to emphasise its monoid structure;

- ▷ Suppose M is a monoid and let $x \in M$. An element y of M is a **left-inverse** (resp. **right-inverse**, **inverse**) **of** x **in** M if $yx = 1_M$ (resp. $xy = 1_M$, $xy = 1_M = yx$). The element x is **left-invertible** (resp. **right-invertible**, **invertible**) **in** M if there is a left-inverse (resp. right-inverse, inverse) of x in M. If x is invertible in M, then its inverse is unique and is denoted by x^{-1}, and if M is denoted additively, then that element is denoted by $-x$;

- ▷ Suppose M is a monoid, let $x \in M$ and let $\mathrm{P}_x \coloneqq \{n \in \mathbb{N} \mid x^n = 1_M\}$. The **order of** x (**in** M) is the extended natural number denoted by $\overset{\times_M}{\not{O}}(x)$ and given by

$$\overset{\times_M}{\not{O}}(x) \coloneqq \begin{cases} \min(\mathrm{P}_x) & \text{if } \mathrm{P}_x \neq \emptyset\,, \\ \infty & \text{if } \mathrm{P}_x = \emptyset\,; \end{cases}$$

- ▷ Suppose M is a monoid and let $\mathrm{N}_M := \left\{ n \in \mathbb{N} \,\vdots\, x^n = 1_M \ (\forall x \in M) \right\}$. The **characteristic of** M is the number denoted by $\mathrm{Char}\,(M)$ and given by

$$\mathrm{Char}\,(M) := \begin{cases} \min\,(\mathrm{N}_M) & \text{if } \mathrm{N}_M \neq \emptyset\,, \\ 0 & \text{if } \mathrm{N}_M = \emptyset\,. \end{cases}$$

Example 1.6. The number sets $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are monoids when endowed with their usual addition operations, the number 0 being their common identity element, and the number sets $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are monoids when endowed with their usual multiplication operations, the number 1 being their common identity element. All these monoids have characteristic zero, and their non-identity elements have infinite order.

Historically, by far the most important type of monoid is the one whose elements are invertible, and that concept has been closely related to the notion of symmetry ever since Évariste Galois proposed it in 1830 as he tackled the problem of solvability of algebraic equations by radicals. To us, that concept will be critical due to the fact that it generalises as far as possible the traits of the arithmetic operation of addition, especially when the interplay between positive and negative numbers is taken into consideration.

Definition 1.7. A **group** is a monoid G whose elements are invertible. We have following notations and terminology:

- ▷ One can easily notice that every functional composition of unital homomorphisms between groups is a unital homomorphism. Hence, such functions form a category whose composition operation is the usual functional composition, and that category is denoted by **Grp**;

- ▷ The **inversion function** on a group G is the function $\overset{\times_G}{\mathrm{Inv}} : G \to G$ that associates every element x to its inverse $\overset{\times_G}{\mathrm{Inv}}(x) := x^{-1}$ in G. It is easy to check that $\overset{\times_G}{\mathrm{Inv}}$ is an involution on G, that is, it is a bijective function that is its own inverse;
- ▷ For each subset S of G, we denote by S^{-1} the image of S with respect to the inversion function $\overset{\times_G}{\mathrm{Inv}}$, that is, we have

$$S^{-1} := \overset{\times_G}{\mathrm{Inv}} \langle S \rangle = \{x^{-1} \ | \ x \in S\} \ ;$$

- ▷ Let G be a group and let S be a subset of G. The **subgroup of G generated by S** is smallest subgroup of G that contains S, and it shall be denoted by $\underset{G}{\overset{\mathbf{Grp}}{\mathrm{span}}}(S)$. It is easy to check that $\underset{G}{\overset{\mathbf{Grp}}{\mathrm{span}}}(S)$ is given by (Definition 1.2)

$$\underset{G}{\overset{\mathbf{Grp}}{\mathrm{span}}}(S) = \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(S \cup S^{-1} \cup \{1_G\}) \ .$$

Example 1.8. The number sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are groups when endowed with their usual addition operations, and the number sets $\mathbb{Q} - \{0\}$, $\mathbb{R} - \{0\}$ and $\mathbb{C} - \{0\}$ are groups when endowed with their usual multiplication operations.

Proposition 1.9. (34)

(**a**) *If $\phi : G \to H$ is a homomorphism between groups, then ϕ is unital;*

(**b**) *The kernel* $\mathrm{Ker}(\phi)$ *of a homomorphism $\phi : G \to H$ between groups is a subgroup of G.*

Proof. We leave the proof of item (b) to the reader.

(**a**) Note that

$$1_H = \phi(1_G \cdot 1_G)(\phi(1_G))^{-1} = \phi(1_G)\,\phi(1_G)(\phi(1_G))^{-1} = \phi(1_G) \ .$$

□

Proposition 1.10. (183) *Let M be a semigroup. The following statements are equivalent:*

- ▷ *Every element of M is left-invertible;*
- ▷ *Every element of M is right-invertible;*
- ▷ *M is a group.*

1.2 Quotients in **Mon**

One may easily notice that the function of type **Mon** → **Set** (Example B.17; Definition 1.5) that associates each monoid to its universe set and associates each homomorphism between monoids to itself is a faithful functor (Definition B.25). Thus, **Mon** is a **Set**-concrete category (Definition B.42) when endowed with that function.

In this section, we shall appreciate how quotients are produced in the category **Mon** (Definition B.46).

Definition 1.11. Let M be a monoid. A **congruence relation on** M is an equivalence relation $\equiv$ on M such that for all $x, x', y, y' \in M$ so that $x \equiv x'$ and $y \equiv y'$, we have $xy \equiv x'y'$.

Example 1.12. Consider the monoid $\mathbb{N}_0$ of non-negative integers with its usual addition operation, let $n \in \mathbb{N}$ be fixed, and let $\overset{\mathbb{N}_0}{\underset{n}{\equiv}}$ be the relation of **congruence modulo** n, that is, $\overset{\mathbb{N}_0}{\underset{n}{\equiv}}$ is the binary relation on $\mathbb{N}_0$ defined so that for all $x, y \in \mathbb{N}_0$, the condition $x \overset{\mathbb{N}_0}{\underset{n}{\equiv}} y$ means that n divides $x - y$. One may easily check that $\overset{\mathbb{N}_0}{\underset{n}{\equiv}}$ is a congruence relation on $\mathbb{N}_0$. Analogously, one may define a relation $\overset{\mathbb{Z}}{\underset{n}{\equiv}}$ of congruence modulo n on the monoid $\mathbb{Z}$ of integers with its usual addition operation, which is also a congruence relation on $\mathbb{Z}$ in the sense of Definition 1.11.

Example 1.13. (174) Let M be a monoid and let $\equiv_1$ and $\equiv_2$ be the equivalence relations on the product monoid $M \times M$ (Example 1.18) given by

$$(x_1, y_1) \equiv_1 (x_2, y_2) \Leftrightarrow x_1 = x_2 \quad \text{and} \quad (x_1, y_1) \equiv_2 (x_2, y_2) \Leftrightarrow y_1 = y_2 \,.$$

Both $\equiv_1$ and $\equiv_2$ are congruence relations on $M \times M$.

Proposition 1.14. (65, 183) *Let M be a monoid. If $\equiv$ is a congruence relation on M, then the quotient set $M/\equiv$ is a monoid when endowed with the multiplication on $M/\equiv$ given by $(x/\equiv)(y/\equiv) := (xy)/\equiv$. That monoid is a quotient of M modulo $\equiv$ in* **Mon** *(Definition B.46).*

Proof. We leave the verification that those operations are well-defined to the reader, as well as the fact that $M/\equiv$ is a monoid. Let $\sigma : M \to M/\equiv$ be the canonical quotient function in **Set** associated to the quotient $M/\equiv$ (Example B.48), which is clearly a homomorphism between monoids. Thus, we have $\equiv\, = \overset{\sigma}{\text{eq}}$ (Definition B.45). Consider any monoid N and any homomorphism $f : M \to N$ so that $\equiv\, \subset \overset{f}{\text{eq}}$, and let $\overline{f} : M/\equiv\, \to N$ be the quotient lowering of f in **Set**, which is given by $\overline{f}(x/\equiv) := f(x)$. If x and y are elements of M, then

$$\overline{f}\left((x/\equiv)(y/\equiv)\right) = \overline{f}\left((xy)/\equiv\right) = f(xy) = f(x)\, f(y) = \overline{f}(x/\equiv)\, \overline{f}(y/\equiv) \,,$$

proving that $\overline{f}$ is a homomorphism. The uniqueness of $\overline{f}$ follows from the universal property of quotients in **Set**. Therefore, we have proved that σ is a quotient morphism in **Mon** associated to the quotient $M/\equiv$.

□

Example 1.15. Let n be a fixed natural number and take the congruence relations $\overset{\mathbb{N}_0}{\underset{n}{\equiv}}$ and $\overset{\mathbb{Z}}{\underset{n}{\equiv}}$ on the monoids $(\mathbb{N}_0, +_{\mathbb{N}_0})$ and $(\mathbb{Z}, +_{\mathbb{Z}})$, respectively (Example 1.12). The quotient monoids

$$(\mathbb{N}_0, +_{\mathbb{N}_0}) / \overset{\mathbb{N}_0}{\underset{n}{\equiv}} = \{0/\overset{\mathbb{N}_0}{\underset{n}{\equiv}}, 1/\overset{\mathbb{N}_0}{\underset{n}{\equiv}}, \ldots, (n-1)/\overset{\mathbb{N}_0}{\underset{n}{\equiv}}\}$$

and

$$(\mathbb{Z}, +_{\mathbb{Z}}) / \overset{\mathbb{Z}}{\underset{n}{\equiv}} = \{0/\overset{\mathbb{Z}}{\underset{n}{\equiv}}, 1/\overset{\mathbb{Z}}{\underset{n}{\equiv}}, \ldots, (n-1)/\overset{\mathbb{Z}}{\underset{n}{\equiv}}\}$$

are clearly isomorphic, and they are denoted by $\mathbb{Z}/n\mathbb{Z}$. It turns out that $\mathbb{Z}/n\mathbb{Z}$ is a commutative group for all $n \in \mathbb{N}$.

1.3 Limits in **Mon**

Monoids give rise to other monoids via the universal property encoded by the categorical notion of limit (Definition B.31).

Proposition 1.16. (2) *The category* **Mon** *is complete (Definition B.33).*

Proof. Let $\mathcal{F} : \boldsymbol{I} \to \mathbf{Mon}$ be a functor and let $\overset{\mathbf{Mon}}{\mathrm{U}} : \mathbf{Mon} \to \mathbf{Set}$ be the forgetful functor of **Mon** (Definition B.42). Let $\sigma = \{\sigma_i : S \xrightarrow{\mathbf{Set}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ be the canonical construction of the limit cone over $\overset{\mathbf{Mon}}{\mathrm{U}} \circ \mathcal{F} : \boldsymbol{I} \to \mathbf{Set}$ with vertex S (Example B.34, Item (a)), that is, S is defined as the set of cones $\lambda = \{\lambda_i : \{\emptyset\} \xrightarrow{\mathbf{Set}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ over $\overset{\mathbf{Mon}}{\mathrm{U}} \circ \mathcal{F} : \boldsymbol{I} \to \mathbf{Set}$, all with the common vertex $\{\emptyset\}$, and $\sigma = \{\sigma_i : S \xrightarrow{\mathbf{Set}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ is the family of morphisms in **Set** given by $\sigma_i(\lambda) := \lambda_i(\emptyset)$. Endow S with the multiplication on it given by $(\lambda\mu)_i(\emptyset) := \lambda_i(\emptyset)\,\mu_i(\emptyset)$. One can easily verify that S is a monoid and that each function σ_i is a homomorphism of type $S \xrightarrow{\mathbf{Mon}} \mathcal{F}(i)$. Finally, consider a monoid M and a cone $\mu = \{\mu_i : M \xrightarrow{\mathbf{Mon}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ over $\mathcal{F}$, which may also be seen as a cone over $\overset{\mathbf{Mon}}{\mathrm{U}} \circ \mathcal{F}$, and let $\overline{\mu} : M \xrightarrow{\mathbf{Set}} S$ be the limit lifting of μ along σ in **Set**. That means that for each $i \in \boldsymbol{I}_0$, the digraph

$$\begin{tikzcd} & S \arrow[d, "\sigma_i"] \\ M \arrow[ur, "\overline{\mu}"] \arrow[r, "\mu_i"'] & \mathcal{F}(i) \end{tikzcd}$$

in **Set** commutes, and for all $x, y \in M$, we obtain

$$\sigma_i(\overline{\mu}(xy)) = \mu_i(xy) = \mu_i(x)\,\mu_i(y) = \sigma_i(\overline{\mu}(x))\,\sigma_i(\overline{\mu}(y)) = \sigma_i(\overline{\mu}(x)\,\overline{\mu}(y)) \ ,$$

implying $\overline{\mu}(xy) = \overline{\mu}(x)\,\overline{\mu}(y)$ and proving that $\overline{\mu}$ is a homomorphism. Thus, the cone σ satisfies the desired universal property in **Mon**.

□

Analogous proofs to that of Proposition 1.16 show that many other categories of magma-like objects are complete, such as the categories **SGrp** and **Grp**. We shall see how the most notable limits take form in **Mon**.

Example 1.17. The simplest example of a limit in **Mon** is its terminal object (Definition B.37), which happens to be the trivial monoid $\mathbf{1} \coloneqq \{1\}$, since for every monoid M there is exactly one homomorphism of type $M \to \{1\}$. It is easy to check that $\mathbf{1}$ is also the initial object in **Mon**.

Example 1.18. (2) Let $\{M_i\}_{i\in I}$ be a family of monoids. The Cartesian product of sets $\prod\limits_{i\in I} M_i$ is a product of $\{M_i\}_{i\in I}$ in **Mon** (Definition B.38) when endowed with the multiplication operation on it given by $\{r_i\}_{i\in I}\{s_i\}_{i\in I} \coloneqq \{r_i s_i\}_{i\in I}$, where the projections $\chi_j : \prod\limits_{i\in I}^{\mathbf{Mon}} M_i \to M_j$ are the usual functions given by $\chi_j\left(\{r_i\}_{i\in I}\right) \coloneqq r_j$ for each $j \in I$. The monoid $\prod\limits_{i\in I} M_i$ might be denoted by $\prod\limits_{i\in I}^{\mathbf{Mon}} M_i$ to emphasise that it is a product in **Mon**. If the index set I is finite, then the product $\prod\limits_{i\in I}^{\mathbf{Mon}} M_i$ is also a coproduct of $\{M_i\}_{i\in I}$ in **Mon**.

Notation 1.19. Let $\{M_i\}_{i\in I}$ be a family of monoids and let $\{\equiv_i \subset M_i \times M_i\}_{i\in I}$ be a family of congruence relations. By abuse of language, we denote by $\prod\limits_{i\in I} \equiv_i$ the binary relation on the Cartesian product $\prod\limits_{i\in I} M_i$ given by

$$x\left(\prod_{i\in I} \equiv_i\right) y \ :\Leftrightarrow\ (\forall i \in I)\, x_i \equiv_i y_i$$

for all $x, y \in \prod\limits_{i\in I} M_i$, which is clearly a congruence relation on the product monoid $\prod\limits_{i\in I}^{\mathbf{Mon}} M_i$. In particular, if M is a monoid and if $\equiv$ is a congruence relation on M such that $M_i = M$ $(\forall i \in I)$ and $\equiv_i = \equiv$ $(\forall i \in I)$, then the congruence relation $\prod_{i \in I} \equiv_i$ on $\prod^{\mathbf{Mon}}_{i \in I} M_i = {}^I M$ shall be denoted by ${}^I\!\equiv$.

With these notations established, the following lemma shows that products preserve quotients modulo congruence relations in **Mon**. That result will turn out to be useful in Section 3.6.

Lemma 1.20. *Let* $\{M_i\}_{i \in I}$ *be a family of monoids, let* $\{\equiv_i \subset M_i \times M_i\}_{i \in I}$ *be a family of congruence relations and let* $\{\iota_i : M_i \to M_i/\equiv_i\}_{i \in I}$ *be the canonical family of quotient homomorphisms. The product* $\prod^{\mathbf{Mon}}_{i \in I} (M_i /\equiv_i)$ *is a quotient of* $\prod^{\mathbf{Mon}}_{i \in I} M_i$ *modulo* $\prod_{i \in I} \equiv_i$ *in* **Mon** *(Definition B.46) whose quotient homomorphism is the function denoted by* $\prod^{\mathbf{Mon}}_{i \in I} \iota_i : \prod^{\mathbf{Mon}}_{i \in I} M_i \to \prod^{\mathbf{Mon}}_{i \in I} (M_i /\equiv_i)$ *and given by*

$$\left(\prod^{\mathbf{Mon}}_{i \in I} \iota_i\right) \left(\{x_i\}_{i \in I}\right) \coloneqq \{\iota_i\,(x)\}_{i \in I}\ .$$

Proof. Let N be a monoid and let $f : \prod^{\mathbf{Mon}}_{i \in I} M_i \to N$ be a homomorphism so that $\prod_{i \in I} \equiv_i \subset \overset{\mathrm{U}f}{\mathrm{eq}}$ (Definition B.45). If $\overline{f} : \prod^{\mathbf{Mon}}_{i \in I} (M_i /\equiv_i) \to N$ is a homomorphism such that the digraph

$$\begin{tikzcd}
\prod^{\mathbf{Mon}}_{i \in I} M_i \arrow[r, "f"] \arrow[d, "\prod^{\mathbf{Mon}}_{i \in I} \iota_i"'] & N \\
\prod^{\mathbf{Mon}}_{i \in I} (M_i /\equiv_i) \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

commutes, then for all $\{x_i\}_{i\in I} \in \prod_{i\in I}^{\mathbf{Mon}} M_i$, we have

$$f\left(\{x_i\}_{i\in I}\right) = \overline{f}\left(\left(\prod_{i\in I}^{\mathbf{Mon}} \iota_i\right)(x)\right) = \overline{f}\left(\{x_i/\equiv_i\}_{i\in I}\right).$$

Indeed, the function $\overline{f} : \prod_{i\in I}^{\mathbf{Mon}} (M_i/\equiv_i) \to N$ given by $\overline{f}\left(\{x_i/\equiv_i\}_{i\in I}\right) \coloneqq f\left(\{x_i\}_{i\in I}\right)$ is well-defined since $\prod_{i\in I} \equiv_i \subset \overset{\mathrm{U}f}{\mathrm{eq}}$, and it is easy to see that it is a unital homomorphism between monoids.

$\square$

Example 1.21. (2) Let $f, g : M \to N$ be two unital homomorphisms between monoids. The submonoid $E \coloneqq \{x \in M \mid f(x) = g(x)\}$ of M is an equaliser of f and g in **Mon** (Definition B.40), where the equaliser morphism $\mathrm{eq}(f, g) : E \overset{\mathbf{Mon}}{\longrightarrow} M$ is the canonical inclusion between sets. The monoid E might be denoted by $\overset{\mathbf{Mon}}{\mathrm{Eq}}(f, g)$ to emphasise that it is an equaliser in **Mon**.

Example 1.22. (2) Let $f : M \to P$ and $g : N \to P$ be two unital homomorphisms between monoids. The subset

$$P \coloneqq \{(x, y) \in M \times N \mid f(x) = g(y)\}$$

of the product monoid $M \times N$ (Example 1.18) forms a submonoid of $M \times N$ and it is a pullback of f and g in **Mon** (Definition B.41), where the pullback morphisms $\overline{f} : P \overset{\mathbf{Mon}}{\longrightarrow} N$ and $\overline{g} : P \overset{\mathbf{Mon}}{\longrightarrow} M$ are given by $\overline{f}(x, y) \coloneqq y$ and $\overline{g}(x, y) \coloneqq x$. In other words, the monoid P is a fibred product of M and N with respect to f and g in **Mon**, and it might be denoted by $M \overset{\mathbf{Mon}}{\underset{f,g}{\times}} N$ to emphasise that it is a fibred product in **Mon**.

The following lemma describes a peculiar property of limit cones in **Mon**, and it will come in handy in Section 3.6.

Lemma 1.23. *Let* $\mathcal{F} : \boldsymbol{I} \to \mathbf{Mon}$ *be a functor and suppose that* $\chi = \{\chi_i : L \overset{\mathbf{Mon}}{\longrightarrow} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ *is a limit cone over* $\mathcal{F}$*. If* x *and* y *are elements of* L *so that* $\chi_i(x) = \chi_i(y)$ $(\forall i \in \boldsymbol{I}_0)$*, then* $x = y$*.*

Proof. Let $\sigma = \{\sigma_i : S \overset{\mathbf{Mon}}{\longrightarrow} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ be the limit cone over $\mathcal{F}$ constructed in the proof of Proposition 1.16. Since limits are unique up to unique isomorphism, there is a unique isomorphism $\alpha : L \overset{\mathbf{Mon}}{\longrightarrow} S$ such that the digraph

$$\begin{tikzcd}
L \arrow[dr, "\chi_i"] \arrow[dd, "\alpha"'] & \\
& \mathcal{F}(i) \\
S \arrow[ur, "\sigma_i"'] &
\end{tikzcd}$$

in **Mon** commutes for all $i \in \boldsymbol{I}_0$. By the hypothesis, we have

$$(\forall i \in \boldsymbol{I}_0) \;\; (\alpha(x))_i(\emptyset) = \sigma_i(\alpha(x)) = \chi_i(x) = \chi_i(y) = \sigma_i(\alpha(y)) = (\alpha(y))_i(\emptyset) \;,$$

implying $(\alpha(x))_i = (\alpha(y))_i$ $(\forall i \in \boldsymbol{I}_0)$, $\alpha(x) = \alpha(y)$ and $x = y$.

□

1.4 Preordered and ordered sets

Often in systematic investigations, it is imperative to possess a precise abstract characterisation of how the elements of certain sets are arranged relative to each other. This goal is achieved mainly by means of the notions of **preorder**, **partial order** and (**total**) **order**, which are special binary relations that emerge in almost every area of Mathematics with essential applications to the formal, natural and social sciences. The field of Order Theory provides a deep examination of those notions, and its most basic concepts are presented in the following definitions.

Definition 1.24. Let X be a set, let $< \subset X \times X$ be a binary relation on X and let

$$\leqslant \coloneqq < \cup \{(x,x) \mid x \in X\} \ .$$

- ▷ The relation $<$ is **irreflexive (on X)** if $x \nless x$ $(\forall x \in X)$;

- ▷ The relation $<$ is **transitive (on X)** if it satisfies the condition

$$(\forall x,y,z \in X)\ \left((x \leqslant y \text{ and } y \leqslant z) \Rightarrow x \leqslant z\right) ;$$

- ▷ The relation $<$ is a **preorder (on X)** if it is irreflexive and transitive. A **preordered set** is a set X endowed with a preorder $<$ on it. Whenever no particular notation is attributed to the underlying preorder of a preordered set X, that preorder shall be denoted by $<$ or $<_X$. An **interval** in a preordered set X is a subset of X of one of the forms:

$$\begin{aligned}
(a,b)_X &\coloneqq \{x \in X \mid a < x < b\} \ ; & (a,\rightarrow)_X &\coloneqq \{x \in X \mid x > a\} \ ; \\
[a,b]_X &\coloneqq \{x \in X \mid a \leqslant x \leqslant b\} \ ; & (\leftarrow,b)_X &\coloneqq \{x \in X \mid x < b\} \ ; \\
[a,b)_X &\coloneqq \{x \in X \mid a \leqslant x < b\} \ ; & [a,\rightarrow)_X &\coloneqq \{x \in X \mid x \geqslant a\} \ ; \\
(a,b]_X &\coloneqq \{x \in X \mid a < x \leqslant b\} \ ; & (\leftarrow,b]_X &\coloneqq \{x \in X \mid x \leqslant b\} \ ,
\end{aligned}$$

  where $a,b \in X$. In case X is an ordered subset of $(\mathbb{R},<_{\mathbb{R}})$, the intervals $(a,\rightarrow)_X$, $(\leftarrow,b)_X$, $[a,\rightarrow)_X$ and $(\leftarrow,b]_X$ are denoted by $(a,\infty)_X$, $(-\infty,b)_X$, $[a,\infty)_X$ and $(-\infty,b]_X$, respectively;

- ▷ The relation $<$ is **strictly transitive (on X)** if it satisfies the condition

$$(\forall x,y,z \in X)\ \left((x < y \text{ and } y < z) \Rightarrow x < z\right) .$$

  Note that the strict transitivity implies the regular transitivity;

- ▷ The relation $<$ is **total (on X)** if it satisfies the condition

$$(\forall x,y \in X)\ (x \leqslant y \text{ or } y \leqslant x) ;$$

- ▷ The relation $<$ is a **partial order (on X)** if it is irreflexive and strictly transitive. Note that every partial order on X is a preorder on X. A **partially ordered set** is a set X endowed with a partial order $<$ on it;

- ▷ The relation $<$ is an **order** (**on** X) if it is irreflexive, strictly transitive and total. An **ordered set** is a set X endowed with an order $<$ on it;

- ▷ The relation $<$ is **directed** (**on** X) if $X \neq \emptyset$ and for all $x, y \in X$, there is a $z \in X$ such that $x \leqslant z$ and $y \leqslant z$;

- ▷ A subset S of a preordered set X is ($<$-)**order-convex** (**in** X) if for any $x, y \in S$ so that $x < y$, we have $(x, y)_X \subset S$;

- ▷ The set X is ($<$-)**order-dense** if for all $x, y \in X$ so that $x < y$, there is a $z \in X$ such that $x < z < y$;

- ▷ A subset S of X is ($<$-)**order-dense** (**in** X) if for all $x, y \in X$ so that $x < y$, there is an $s \in S$ such that $x \leqslant s \leqslant y$. Note that X is order-dense in itself;

- ▷ A subset S of X is **strictly** ($<$-)**order-dense** (**in** X) if for all $x, y \in X$ so that $x < y$, there is an $s \in S$ such that $x < s < y$. Therefore, the set X is order-dense if, and only if, it is strictly order-dense in itself.

Example 1.25. Let X be a set and consider the set $\mathrm{P}(X)$ of subsets of X. The (**non-strict**) **dominance relation on** $\mathrm{P}(X)$ is the binary relation $\prec$ on $\mathrm{P}(X)$ such that for all $A, B \in \mathrm{P}(X)$, the condition $A \prec B$ is equivalent to the existence of an injective function of type $A \to B$, and that relation is transitive and total. The relation

$$\precneqq \coloneqq \prec - \{(A, A) \mid A \in \mathrm{P}(X)\}$$

on $\mathrm{P}(X)$ is a directed preorder relation on $\mathrm{P}(X)$, and the relation

$$\precnapprox \coloneqq \prec - \{(A, B) \mid A \text{ and } B \text{ are non-equipotent subsets of } X\}$$

is a directed partial order on $\mathrm{P}(X)$.

Structure-preserving functions between preordered sets often provide us with valuable information on the nature of the preorders under consideration:

Definition 1.26. Let X be a preordered set.

- ▷ Suppose Y is another preordered set. A function $f : X \to Y$ is **increasing** (resp. **decreasing**, **non-strictly increasing**, **non-strictly decreasing**) if the condition $x < y$ (resp. $x < y$, $x \leqslant y$, $x \leqslant y$) implies $f(x) < f(y)$ (resp. $f(x) > f(y)$, $f(x) \leqslant f(y)$, $f(x) \geqslant f(y)$) for all $x, y \in X$. A function $f : X \to Y$ is **monotone** (resp. **non-strictly monotone**) if it is increasing or decreasing (resp. non-strictly increasing or non-strictly decreasing);
- ▷ Every functional composition of two increasing functions between ordered sets is increasing. Thus, such functions form a category whose objects are the ordered sets and whose composition operation is the usual functional composition, and that category shall be denoted by **SetOrd**;
- ▷ A **preordered** (resp. **ordered**) **subset of** X is a preordered (resp. ordered) set X' such that $X' \subset X$ and $<_{X'} \subset <_X$.

Some concepts allow us to describe how the subsets of a preordered set stand in relation to the whole structure:

Definition 1.27. Let X be a preordered set and let S be a subset of X.

- ▷ An **upper bound** (resp. **lower bound**) **of** S (**in** X) is an element x of X such that $s \leqslant x$ (resp. $x \leqslant s$) for all $x \in S$. A subset S of X is **bounded above** (resp. **bounded below**) **in** X if it has an upper bound (resp. lower bound) in X, and otherwise it is **unbounded above** (resp. **unbounded below**);
- ▷ A **greatest element** (resp. **least element**) **of** S is an element x of S that is an upper bound (resp. lower bound) of S in X;

- ▷ A **supremum** (resp. **infimum**) **of** S (**in** X) is an upper bound (resp. lower bound) x of S in X such that $x \leqslant y$ (resp. $y \leqslant x$) for every upper bound (resp. lower bound) y of S in X. If X is ordered and if S has a supremum (resp. infimum), then that supremum (resp. infimum) is unique and it is denoted by $\overset{X}{\sup}(S)$ (resp. $\overset{X}{\inf}(S)$);

- ▷ A subset S of X is **cofinal** (resp. **strictly cofinal**) **in** X if for every $x \in X$ there is an $s \in S$ such that $x \leqslant s$ (resp. $x < s$). The **cofinality** of X is the cardinal number denoted by $\mathrm{cf}(X)$ and given by
$$\mathrm{cf}(X) \coloneqq \overset{\mathsf{Card}}{\min}\{|S| \mid S \text{ is a cofinal subset of } X\} ;$$

- ▷ A subset S of X is **coinitial** (resp. **strictly coinitial**) **in** X if for every $x \in X$ there is an $s \in S$ such that $s \leqslant x$ (resp. $s < x$). The **coinitiality** of X is the cardinal number denoted by $\mathrm{ci}(X)$ and given by
$$\mathrm{ci}(X) \coloneqq \overset{\mathsf{Card}}{\min}\{|S| \mid S \text{ is a coinitial subset of } X\} .$$

Some elementary types of preordered and ordered sets will turn out to be quite useful at a later point:

Definition 1.28.

- ▷ A **well-ordered set** is an ordered set X such that every inhabited subset of X has a least element;

- ▷ A subset S of an ordered set X is **well-ordered in** X if the ordered set $(S, <_X \cap (S \times S))$ is well-ordered. The set of well-ordered subsets of X shall be denoted by $\overset{\mathrm{wo}}{\mathrm{P}}(X)$;

- ▷ A preordered set X satisfies the **Supremum Property** if every inhabited subset of X that has an upper bound in X has a supremum in X;

- ▷ A **linear continuum** is an order-dense ordered set X with more than one element that satisfies the Supremum Property.

Example 1.29. The set $\mathbb{R}$ of real numbers with its usual order is the archetypal example of a linear continuum, but there are other linear continua that are not isomorphic to any interval in $\mathbb{R}$. For instance, consider the **closed long ray** (84), defined as the Cartesian product $\mathfrak{L} := \omega_1 \times [0,1)_{\mathbb{R}}$ endowed with the order $<$ on it so that for all $(\alpha, x), (\beta, y) \in \mathfrak{L}$, the condition $(\alpha, x) < (\beta, y)$ is equivalent to stating that either $\alpha < \beta$, or $\alpha = \beta$ and $x < y$. One may prove that $\mathfrak{L}$ is a linear continuum that is not isomorphic to $[0,\infty)_{\mathbb{R}}$ as an ordered set, since $\operatorname{cf}([0,\infty)_{\mathbb{R}}) = \omega$ and $\operatorname{cf}(\mathfrak{L}) = \omega_1$.

Proposition 1.30. *Every increasing function $f : X \to Y$ between ordered sets is injective.*

Proof. If x and y are two elements of X so that $x < y$, then $f(x) < f(y)$, and, in particular, we have $f(x) \neq f(y)$.

□

Proposition 1.31. (106) *Let X be a well-ordered set. There is a unique ordinal α and a unique function $f : X \to \alpha$ such that f is an isomorphism between the ordered sets X and α.*

Lemma 1.32. (176) *An ordered set X is well-ordered if, and only if, every sequence in X has a non-strictly increasing subsequence.*

Proof. If X is inhabited and not well-ordered, then there is an inhabited subset S of X that does not have a least element and one can easily obtain a decreasing sequence in X. Suppose X is well-ordered and suppose $\{x_n\}$ is a sequence in X. We shall recursively construct a non-strictly increasing subsequence of $\{x_n\}$ as follows. Let n_1 be the least natural number such that x_{n_1} is the least element of the set $S := \{x_n \mid n \in \mathbb{N}\}$, and if the finite sequence $n_1 n_2 \ldots n_k$ has already been defined, then let n_{k+1} be the least natural number in the interval $(n_k, \infty)_{\mathbb{N}}$ such that $x_{n_{k+1}}$ is the least element of the set $\{x_n \mid n \in (n_k, \infty)_{\mathbb{N}}\}$. Thus, the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}$ is non-strictly increasing by construction, and that proves the lemma.

$\square$

In some circumstances, it is convenient to artificially incorporate a greatest element to an ordered set:

Definition 1.33. Let X be an ordered set and let $\rightarrowtriangle$ be an arbitrary object that does not belong to X. The **superior extension of** X, denoted by $\breve{X}$, is the set $\breve{X} := X \cup \{\rightarrowtriangle\}$ endowed with the binary relation $<_{\breve{X}}$ on $\breve{X}$ given by $<_{\breve{X}} := <_X \cup (X \times \{\rightarrowtriangle\})$. Thus, $\breve{X}$ is an ordered set with a greatest element $\rightarrowtriangle$. In case X is an ordered subset of $(\mathbb{R}, <_{\mathbb{R}})$, the object $\rightarrowtriangle$ is usually denoted by ∞.

Example 1.34. The set $\mathbb{R}$ of real numbers has no greatest element, and its superior extension $\breve{\mathbb{R}}$ has an additional element ∞, which is the greatest element of $\breve{\mathbb{R}}$.

1.5 Nets and topological spaces

The intuitive notion of **neighbourhood of a point** appears in many branches of Mathematics, especially in geometric and analytical settings, and its roots may be traced back to antiquity. As the Renaissance period witnessed the beginnings of the Differential and Integral Calculus, the need for a compendious understanding of that concept became apparent in the mathematical community, since it is closely related to the notion of continuity. Later on, amidst a widespread desire to put Mathematics on a firm basis during the course of the 19th century, Riemann's work on algebraic functions and on the foundations of geometry led him to formulate a study program to investigate "*[...] a part of the theory of magnitudes which is independent of the theory of measurement and in which the magnitudes are considered not as existing independently of their* *position nor as expressible in terms of a unit of measurement, but as regions in a manifold*" (186). That program extended through the early 20th century and was pursued by many researchers, most notably Hilbert (99), reaching its highpoint with the work of Hausdorff (95), who formulated the general notion of neighbourhood practically as it is understood today, extracting the essence of the concept and formulating the definition of a **topological space**. The study of these spaces was first called *Analysis Situs* and eventually renamed (**General**) **Topology** (etymologically, "science of place"), and it is one of the main pillars of modern Mathematics.

In this section, we shall present a compendium of definitions and results concerning topological spaces, narrowing focus to the topics that will be relevant to our study of Rayner rngs. We begin with the definition of a **net**, developed by Moore and Smith in 1922 (161), which is a generalisation of the notion of sequence that turns out to be more appropriate for general topological considerations.

Definition 1.35. Let X be a set. A **net in** X is a family $\{x_\lambda\}_{\lambda\in\Lambda}$ in X, where Λ is a directed preordered set. We have the following notations and terminology:

- ▷ Suppose X is a preordered set. A net $\{x_\lambda\}_{\lambda\in\Lambda}$ in X is **increasing** (resp. **decreasing**, **monotone**, **non-strictly increasing**, **non-strictly decreasing**, **non-strictly monotone**) if it is so as a function of type $\Lambda \to X$ (Definition 1.26);
- ▷ A net $\{x_\lambda\}_{\lambda\in\Lambda}$ is a **subnet** of a net $\{y_\mu\}_{\mu\in M}$ in X if there is a non-strictly increasing net $\{\mu_\lambda\}_{\lambda\in\Lambda}$ in M that is cofinal in M and is such that $x_\lambda = y_{\mu_\lambda}$ $(\forall\lambda\in\Lambda)$. In other words, a subnet of $\{y_\mu\}_{\mu\in M}$ is a net of the form $\{y_{\mu_\lambda}\}_{\lambda\in\Lambda}$, where $\{\mu_\lambda\}_{\lambda\in\Lambda}$ is a non-strictly increasing net in M that is cofinal in M;
- ▷ A net $\{x_\lambda\}_{\lambda\in\Lambda}$ in X is **residual** in a subset S of X if there is a $\lambda_0\in\Lambda$ such that $x_\lambda\in S$ $(\forall\lambda\in[\lambda_0,\to)_\Lambda)$;
- ▷ A net $\{x_\lambda\}_{\lambda\in\Lambda}$ in X is **frequent** in a subset S of X if, for all $\lambda_0\in\Lambda$, there is a $\lambda\in[\lambda_0,\to)_\Lambda$ such that $x_\lambda\in S$;

▷ A **sequence in** X is a net in X of the form $\{x_n\}_{n\in[k,\rightarrow)_{\mathbb{Z}}}$, where $k \in \mathbb{Z}$. Alternative notations for sequences may be found in List of Symbols: $\{a_n\}$;

▷ A **finite sequence in** X is a net in X of the form $\{x_n\}_{n\in[k,l]_{\mathbb{Z}}}$, where $k, l \in \mathbb{Z}$ and $k \leqslant l$. Alternative notations for finite sequences may be found in List of Symbols: $a_k a_{k+1} \ldots a_{k+n}$;

▷ A sequence $\{x_n\}$ is a **subsequence** of a sequence $\{y_n\}$ in X if there is an increasing sequence $\{k_n\}$ in $\mathbb{N}$ such that $x_n = y_{k_n}$ $(\forall n \in \mathbb{N})$. In other words, a subsequence of $\{y_n\}$ is a sequence of the form $\{y_{k_n}\}$, where $\{k_n\}$ is an increasing[3] sequence in $\mathbb{N}$.

The model of excellence for Riemann's envisaged *"theory of magnitudes"* was attained by means of the following definition:

Definition 1.36. Let X be a set. A **topology on** X is a set τ of subsets of X such that the following three conditions are satisfied:

(**T0**) $\emptyset, X \in \tau$;

(**T1**) For every family $\{S_i\}_{i\in I}$ in τ, we have $\bigcup\limits_{i\in I} S_i \in \tau$;

(**T2**) For every finite sequence $S_1 S_2 \ldots S_n$ in τ, we have $S_1 \cap S_2 \cap \cdots \cap S_n \in \tau$.

A **topological space** is a set X endowed with a topology τ on X. Whenever no particular notation is ascribed in the context to the topology of X, that shall be denoted by τ_X. We have the following notations and terminology:

▷ A τ_X**-open set** is an element of τ_X, and a τ_X**-closed set** is the complement of an element of τ_X in X. A τ_X**-clopen set** is a subset of X that is both τ_X-open and τ_X-closed;

[3] Note that if $\{y_{k_n}\}$ is a subsequence of $\{y_n\}$, then the sequence $\{k_n\}$ in $\mathbb{N}$ is (strictly) increasing, implying that it is cofinal in $\mathbb{N}$. Hence, every subsequence of $\{y_n\}$ is a subnet of $\{y_n\}$. On the other hand, a subnet of $\{y_n\}$ of the form $\{y_{k_n}\}$ is not necessarily a subsequence of $\{y_n\}$, since the sequence $\{k_n\}$ in $\mathbb{N}$ is only required to be non-strictly increasing.

- ▷ A τ_X**-neighbourhood** of an element x of X is a subset S of X such that there is a τ_X-open set U so that $x \in U \subset S$;

- ▷ A τ_X**-subspace of** X is a topological space X' such that $X' \subset X$ and $\tau_{X'} = \{G \cap X' \mid G \in \tau_X\}$. The topology $\tau_{X'}$ is the **restriction of** τ_X **to** X' and it shall be denoted by $\tau_X \restriction_{X'}$ or $\tau_X \restriction (X')$;

- ▷ A function $f : X \to Y$ between topological spaces is **continuous** if $f^{-1}\langle U \rangle \in \tau_X$ $(\forall U \in \tau_Y)$. One can easily notice that every functional composition between two continuous functions is continuous. Hence, such functions form a category whose composition operation is the usual functional composition and that category is denoted by **Top**;

- ▷ The isomorphisms in **Top** are called **homeomorphisms** and they are the bijective functions $f : X \to Y$ between topological spaces such that both f and f^{-1} are continuous;

- ▷ A net $\{x_\lambda\}_{\lambda \in \Lambda}$ in X τ_X**-converges** to an element x of X, or symbolically $x_\lambda \xrightarrow[\lambda \in \Lambda]{\tau_X} x$, if $\{x_\lambda\}_{\lambda \in \Lambda}$ is residual in every τ_X-neighbourhood of x. In that case, we also say that x is a τ_X**-limit** (**point**) of the net $\{x_\lambda\}_{\lambda \in \Lambda}$;

- ▷ A net $\{x_\lambda\}_{\lambda \in \Lambda}$ in X τ_X**-clusters** at an element x of X, or symbolically $x_\lambda \overset{\tau_X}{\underset{\lambda \in \Lambda}{\dashrightarrow}} x$, if $\{x_\lambda\}_{\lambda \in \Lambda}$ is frequent in every τ_X-neighbourhood of x. That turns out to be equivalent to the existence of a subnet $\{x_{\lambda_\mu}\}_{\mu \in M}$ of $\{x_\lambda\}_{\lambda \in \Lambda}$ so that $x_{\lambda_\mu} \xrightarrow[\mu \in M]{\tau_X} x$. In that case, we also say that x is a τ_X**-cluster point** of the net $\{x_\lambda\}_{\lambda \in \Lambda}$.

We shall assume in this thesis that the reader is already familiar with a handful of basic topological terms, such as the notions of basis, subbasis, open covering, isolated point, etc.

We will make use of particular classes of topological spaces:

Definition 1.37. Let X be a topological space.

- ▷ The space X is **discrete** if every subset of X is τ_X-open;
- ▷ The space X is **T2** or **Hausdorff** if, for any two distinct points x and y in X, there are two disjoint τ_X-open sets U and V such that $x \in U$ and $y \in V$;
- ▷ The space X is **R2** or **regular** if, for every τ_X-closed set F and for every $x \in X - F$, there are disjoint τ_X-open sets U and V such that $F \subset U$ and $x \in V$;
- ▷ The space X is **T3** if it is R2 and T2;
- ▷ The space X is **R 5/2** or **completely regular** if, for every inhabited τ_X-closed set F and for every $x \in X - F$, there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f\langle F\rangle = \{1\}$;
- ▷ The space X is **T 7/2** or **Тихонов**[4] if it is R 5/2 and T2;
- ▷ The space X is **first-countable** if every point x of X has a countable local τ_X-system of neighbourhoods;
- ▷ The space X is **second-countable** if it has a countable basis;
- ▷ The space X is **separable** if it has a countable τ_X-dense subset;
- ▷ The space X is **connected** if it is not a disjoint union of two inhabited τ_X-open sets. Otherwise, X is **disconnected**;
- ▷ The space X is **totally disconnected** if every inhabited connected subspace of X is a singleton;
- ▷ The space X is **perfect** if it has no τ_X-isolated points;
- ▷ The space X is **countably compact** if every sequence in X has a τ_X-cluster point;

[4] According to the widely adopted ISO 9:1995 system, the Russian name 'Тихонов' should be transliterated into the Latin alphabet as 'Tihonov', but it may be found in mathematical publications written in the English language variously rendered as 'Tychonoff', 'Tychonov', 'Tikhonov', 'Tichonov', etc.

- ▷ The space X is **compact** if every net in X has a τ_X-cluster point;
- ▷ The space X is **locally compact** if every point in X has a τ_X-closed, compact neighbourhood;
- ▷ The space X is **Lindelöf** if every τ_X-open covering of X has a countable subcovering;
- ▷ The space X is **zero-dimensional** if it has a basis whose elements are all τ_X-clopen.

Proposition 1.38. (241) *Every T2, zero-dimensional topological space is totally disconnected.*

Proof. Let X be an inhabited, T2, zero-dimensional topological space, let S be an inhabited subspace of X and suppose there are two distinct elements x and y in S. Since X is T2, there is a τ_X-neighbourhood U of x so that $y \notin U$, and since x is zero-dimensional, there is a τ_X-clopen set A so that $x \in A \subset U$. Hence, the intersections $S \cap A$ and $S \cap (X - A)$ are τ_S-open subsets of S that cover S, the former containing x and the latter containing y, implying that S is disconnected.

□

Proposition 1.39. (241) *Every second-countable topological space is Lindelöf and separable.*

Proposition 1.40. (241) *Let X be a topological space and let $\sim$ be the binary relation on X such that for all $x, y \in X$, the condition $x \sim y$ holds if, and only if, there is a connected subspace S of X so that $x, y \in S$.*

- ▷ *The relation $\sim$ is an equivalence relation on X;*
- ▷ *All equivalence classes of $\sim$, that is, all **connected components of** X, are τ_X-closed and are maximal connected subspaces of X.*

Proposition 1.41. (241, 44) *Let X be a topological space. If $\{S_i\}_{i\in I}$ is a family of connected subspaces of X so that $S_i \cap S_j \neq \emptyset\,(\forall i, j \in I)$, then the union $\bigcup\limits_{i\in I} S_i$ is a connected subspace of X.*

Proposition 1.42. (241, 44, 162) *Let X be a connected topological space and let Y be a topological space. If $f : X \to Y$ is a continuous function, then $f\langle X\rangle$ is a connected subspace of Y.*

The Cartesian product of the universe sets of a family of topological spaces may be equipped with a natural topology:

Definition 1.43. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The **product topology on** $\prod\limits_{i\in I} X_i$ is the smallest topology τ on the Cartesian product $\prod\limits_{i\in I} X_i$ such that every canonical projection

$$\begin{cases} \pi_i : \left(\prod\limits_{i'\in I} X_{i'}, \tau\right) \to X_i \\ \pi_i\left(\{x_{i'}\}_{i'\in I}\right) := x_i \quad (\forall i \in I) \end{cases}$$

is continuous, and that topology shall be denoted by $\prod\limits_{i\in I}^{\mathbf{Top}} \tau_{X_i}$. One often assumes that $\prod\limits_{i\in I} X_i$ is endowed with its product topology, except when otherwise stated. It can be easily verified that the space $\left(\prod\limits_{i\in I} X_i, \prod\limits_{i\in I}^{\mathbf{Top}} \tau_{X_i}\right)$ is a product of the family $\{X_i\}_{i\in I}$ in **Top** (Definition B.38).

Proposition 1.44. (241) *Let $\{X_i\}_{i\in I}$ be an inhabited family of connected topological spaces. The product space $\prod\limits_{i\in I} X_i$ is connected.*

Distance measurements between pairs of points naturally give rise to topological spaces, perhaps the most noteworthy ones. In abstract topological frameworks, distances are described by a particular set of axioms:

Definition 1.45. Let X be a set.

- ▷ A **metric on** X is a function $d : X \times X \to [0, \infty)_{\mathbb{R}}$ such that for all $x, y, z \in X$, we have:

(**M1**) $d(x, y) \leqslant d(x, z) + d(z, y)$;

(**M2**) $d(x, y) = d(y, x)$;

(**M3**) $d(x, y) = 0$ if, and only if, $x = y$.

- ▷ An **ultrametric on** X is a metric $d : X \times X \to [0, \infty)_{\mathbb{R}}$ on X such that

$$(\forall x, y, z \in X)\ \ d(x, y) \leqslant \max\{d(x, z), d(z, y)\}\ ;$$

- ▷ A **metric space** (resp. **ultrametric space**) is a set X endowed with a metric (resp. ultrametric) on it. Whenever no particular notation is ascribed in the context to the metric of X, that shall be denoted by d_X;

- ▷ The **metric topology** induced by a metric d on X is the topology on X generated by the basic open sets

$$\mathsf{B}^d_r(x) \coloneqq \{y \in X \mid d(x, y) < r\}$$

  for $x \in X$ and $r \in (0, \infty)_{\mathbb{R}}$. That topology shall be denoted by $\mathtt{t}(d)$. A topological space X is said to be **metrizable** (resp. **ultrametrizable**) if there is a metric (resp. ultrametric) d on X such that $\tau_X = \mathtt{t}(d)$;

- ▷ Let $d : X \times X \to [0, \infty)_{\mathbb{R}}$ be a metric on X.

  – A sequence $\{x_n\}$ in X is d-**Cauchy** if for every $\epsilon \in (0, \infty)_{\mathbb{R}}$, there is an $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ $(\forall m, n \in [N, \infty)_{\mathbb{N}})$;

  – A metric $d : X \times X \to [0, \infty)_{\mathbb{R}}$ on X is **complete** if every d-Cauchy sequence on X $\mathtt{t}(d)$-converges in X. A metric space is **complete** if its metric is complete.

Example 1.46. The function $d : \mathbb{C} \times \mathbb{C} \to [0, \infty)_{\mathbb{R}}$ given by

$$d\left(x_1 + iy_1, x_2 + iy_2\right) := \sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2}$$

is a well-known complete metric on $\mathbb{C}$, and it induces the usual topology on $\mathbb{C}$, which is second-countable, T $^7/_2$, perfect and locally compact.

Proposition 1.47. (241) *Let X be a metric space. The following conditions are equivalent:*

- ▷ *X is second-countable;*
- ▷ *X is Lindelöf;*
- ▷ *X is separable.*

Theorem 1.48 (Urysohn's Metrisation Theorem)**.** (227) *A second-countable topological space is metrizable if, and only if, it is T3.*

1.6 The order topology

An order on a set naturally induces a topology on that set:

Definition 1.49. Let X be an ordered set. The **order topology on** X, which shall be denoted by $\overset{X}{\mathrm{Ordt}}$, $\mathrm{Ordt}(X)$ or $\mathrm{Ordt}(X, <_X)$, is the topology on X generated by the subbasis of intervals of the forms $(\leftarrow, x)_X$ and $(x, \rightarrow)_X$ for $x \in X$.

Note that if a and b are two elements of X, then the interval $(a, b)_X$ is $\overset{X}{\mathrm{Ordt}}$-open, since $(a, b)_X = (a, \rightarrow)_X \cap (\leftarrow, b)_X$. Intervals of the forms $[a, b]_X$, $[a, b)_X$ and $(a, b]_X$ may also be $\overset{X}{\mathrm{Ordt}}$-open depending on the situation. For instance, if X has a least element $x_{\min}$ and a greatest element $x_{\max}$, then the interval $[x_{\min}, x_{\max}]_X = X$ is clearly $\overset{X}{\mathrm{Ordt}}$-open.

Example 1.50. The order topology on $\mathbb{R}$ coincides with the usual metric topology, which is induced by the metric of Example 1.46 restricted to $\mathbb{R} \times \mathbb{R}$.

Proposition 1.51. (100, 171) *If X is an ordered set, then the order topology $\overset{X}{\mathrm{Ordt}}$ is T2.*

Proof. Let x and y be two distinct elements of X so that $x < y$. If $(x, y)_X = \emptyset$, then $(\leftarrow, y)_X \cap (x, \rightarrow)_X = (x, y)_X = \emptyset$, $x \in (\leftarrow, y)_X$ and $y \in (x, \rightarrow)_X$, and if there is a $z \in (x, y)_X$, then $(\leftarrow, z)_X \cap (z, \rightarrow)_X = \emptyset$, $x \in (\leftarrow, z)_X$ and $y \in (z, \rightarrow)_X$, thus proving the proposition.

$\square$

Example 1.52. Let X and Y be the ordered sets

$$X \coloneqq [0, 1]_{\mathbb{R}} \cup \{\sqrt{2}\} \cup [2, 3]_{\mathbb{R}} \quad \text{and} \quad Y \coloneqq [0, \sqrt{2}]_{\mathbb{R}} \cup [\sqrt{3}, 2]_{\mathbb{R}} \ ,$$

which are endowed with the induced orders inherited from the usual order on $\mathbb{R}$, and let $A \coloneqq \mathbb{Q} \cap X$ and $B \coloneqq \mathbb{Q} \cap Y$. It is easy to check that A is order-dense in X but is not $\overset{X}{\mathrm{Ordt}}$-dense in X, while B is $\overset{Y}{\mathrm{Ordt}}$-dense in Y but is not order-dense in Y.

Proposition 1.53. *Let X be an order-dense ordered set with more than one element, and let S be a subset of X. The following conditions are equivalent:*

(**a**) *S is order-dense in X;*

(**b**) *S is strictly order-dense in X;*

(**c**) *S is $\overset{X}{\mathrm{Ordt}}$-dense in X.*

Proof. Clearly, we have (b)⇒(a).

(**a**)⇒(**c**): Suppose S is an inhabited order-dense subset of X, let x be an arbitrary element of X and let U be an $\overset{X}{\mathrm{Ordt}}$-neighbourhood of x. Thus, there are finite sequences $a_1 \dots a_m$ and $b_1 \dots b_m$ in X so that $m, n \in \mathbb{N}_0$ and such that

$$x \in \bigcap_{i=1}^{m} (a_i, \rightarrow)_X \cap \bigcap_{j=1}^{n} (\leftarrow, b_i)_X \subset U \, .$$

If $U = X$, then we have $S \cap U = S \neq \emptyset$. Assume $U \neq X$, which directely implies that we cannot have $m = 0 = n$. Thus, without loss of generality, we may assume $m \neq 0$. Since $\overset{X}{\max}_{i \in [1,m]_{\mathbb{N}}} a_i < x$ and since X is order-dense, there is a $y \in X$ such that $\overset{X}{\max}_{i \in [1,m]_{\mathbb{N}}} a_i < y < x$, and, given that S is order-dense in X, there is an $s \in S$ such that $y \leqslant s \leqslant x$, which gives us

$$(\forall i \in [1,m]_{\mathbb{N}}) \;\; a_i \leqslant \overset{X}{\max_{i \in [1,m]_{\mathbb{N}}}} a_i < y \leqslant s \leqslant x \quad \text{and} \quad (\forall j \in [1,n]_{\mathbb{N}}) \;\; s \leqslant x < b_j\,,$$

implying that $s \in U$ and proving that S is $\overset{X}{\mathrm{Ordt}}$-dense in X.

(c)$\Rightarrow$**(b)**: If S is $\overset{X}{\mathrm{Ordt}}$-dense in X and if x and y are elements of X so that $x < y$, then, since the $\overset{X}{\mathrm{Ordt}}$-open interval $(x,y)_X$ is inhabited, there is an $s \in S$ that belongs to $(x,y)_X$.

$\square$

Proposition 1.54. (163, 217) *Let X be an ordered set. The order topology $\overset{X}{\mathrm{Ordt}}$ is connected if, and only if, X is a linear continuum. In that case, every order-convex subspace of X is connected.*

1.7 Topological magmas

The operation of a magma may be appointed as a continuous function when the universe set of the magma is endowed with a topology on it. That prompts a blending of two major pillars of Mathematics, viz. Algebra and Topology, yielding profound insights with far-reaching implications in a myriad of areas of study.

As strange as it may seem, a few fundamental results which appear to be purely algebraic in nature have only ever been proved by means of topological arguments, whereupon the continuity of most functions is assumed and the notion of limit is readily employed. For instance, for model-theoretic reasons it turns out that no proof of the Fundamental Theorem of Algebra that eschews the use of topological concepts can be conceived, despite the attempts of many first-class mathematicians (114). This suggests that Algebra and Topology feed off each other, and perhaps all areas of Mathematics are in such intertwined relationship.

Bearing those considerations in mind, let us proceed to the basic definitions:

Definition 1.55. A **topological magma** is a magma M endowed with a topology τ on it such that the operation

$$\times_M : (M, \tau) \times (M, \tau) \to (M, \tau)$$

is continuous. In the case of **topological groups**, one also assumes that the canonical inversion function $\overset{\times_M}{\mathrm{Inv}} : (M, \tau_M) \to (M, \tau_M)$ is continuous. Whenever no particular notation is ascribed in the context to the topology of M, that shall be denoted by τ_M. We have the following notations and terminology:

- ▷ A **topological submagma** (resp. **subgroup**) **of** M is a topological magma (resp. group) M' such that $(M', \times_M)$ is a submagma (resp. subgroup) of $(M, \times_M)$ and such that $(M', \tau_{M'})$ is a topological subspace of (M, τ_M);
- ▷ A metric $\rho : G \times G \to [0, \infty)_{\mathbb{R}}$ on a commutative group G is **invariant** if we have
$$(\forall x, y, z \in G) \;\; \rho\,(x, y) = \rho\,(x + z, y + z) \;;$$
- ▷ Let G be a commutative topological group.
  - – A net $\{x_\lambda\}_{\lambda \in \Lambda}$ in G is **τ_G-Cauchy** (**with respect to** $+_G$) if, for every τ_G-neighbourhood U of 0_G, there is a $\lambda_U \in \Lambda$ such that
$$(\forall \lambda_1, \lambda_2 \in [\lambda_U, \to)_\Lambda) \;\; x_{\lambda_1} - x_{\lambda_2} \in U \,;$$
  - – The topology τ_G is **complete**[5] (**with respect to** $+_G$), or the commutative topological group G is **complete**, if every τ_G-Cauchy net in G is τ_G-convergent;

[5] Technically speaking, the topology τ_G is induced by a **uniform structure** on G that is compatible with the underlying group structure of G, and that implied **uniform space** is **complete** (33, 111). We shall bypass the details of the general theory of uniform spaces, defining the Cauchy nets and the complete spaces only when dealing with commutative topological groups.

– The topology τ_G is **sequentially complete** (**with respect to** $+_G$), or the commutative topological group G is **sequentially complete**, if every τ_G-Cauchy sequence in G is τ_G-convergent.

Example 1.56. The group $(\mathbb{C}, +_{\mathbb{C}})$ is a topological group and the monoid $(\mathbb{C}, \times_{\mathbb{C}})$ is a topological monoid when both are endowed with the usual topology on $\mathbb{C}$ (Example 1.46).

Proposition 1.57. (33, 67) *Let G be a topological group.*

(**a**) *For each $g \in G$, the functions $x \mapsto gx$ and $x \mapsto xg$ of type $G \to G$ are homeomorphisms;*

(**b**) *The inversion function $\overset{\times_G}{\mathrm{Inv}} : G \to G$ is a homeomorphism;*

(**c**) *A homomorphism $\phi : G \to H$ between topological groups is continuous if, and only if, it is continuous at 1_G.*

Proposition 1.58. (107, 33, 241) *Let G be a commutative topological group.*

(**a**) *Every τ_G-convergent net in G is τ_G-Cauchy;*

(**b**) *If $\{x_\lambda\}_{\lambda\in\Lambda}$ is a τ_G-Cauchy net in G and if $x \in G$ is so that $x_\lambda \overset{\tau_G}{\underset{\lambda\in\Lambda}{\dashrightarrow}} x$, then $x_\lambda \xrightarrow[\lambda\in\Lambda]{\tau_G} x$ (Definition 1.36).*

Proposition 1.59. (169) *If a commutative topological group G is first-countable and sequentially complete, then it is complete.*

Proof. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a τ_G-Cauchy net in G and let $\{U_n\}$ be a countable τ_G-system of neighbourhoods of 0_G. For each $n \in \mathbb{N}$, there is a $\lambda_n \in \Lambda$ such that $x_\lambda - x_\mu \in U_n$ $(\forall \lambda, \mu \in [\lambda_n, \to)_\Lambda)$. In fact, the fact that Λ is directed implies that we may recursively choose each λ_n so that the sequence $\{\lambda_n\}$ is non-strictly increasing. Thus, it is easy to check that the sequence $\{x_{\lambda_n}\}$ is τ_G-Cauchy in G, and, by supposition, there is an $x \in G$ so that $x_{\lambda_n} \xrightarrow[n\to\infty]{\tau_G} x$. We shall prove that $x_\lambda \xrightarrow[\lambda\in\Lambda]{\tau_G} x$. Let $k \in \mathbb{N}$ be arbitrary. Since the addition operation of G is continuous, there is an $n \in \mathbb{N}$ such that $U_n + U_n \subset U_k$, and, since $x_{\lambda_n} \xrightarrow[n\to\infty]{\tau_G} x$, there is an $m \in [n,\infty)_{\mathbb{N}}$ such that $x_{\lambda_m} \in x + U_n$, that is, $x_{\lambda_m} - x \in U_n$. Finally, since $\lambda_m \geqslant \lambda_n$, we obtain

$$(\forall \lambda \in [\lambda_m, \rightarrow)_\Lambda) \;\; x_\lambda - x = (x_\lambda - x_{\lambda_m}) + (x_{\lambda_m} - x) \in U_n + U_n \subset U_k\,,$$

which proves the proposition.

□

In fact, Osborne considered Proposition 1.59 for T2 topological groups that are not necessarily commutative. Nevertheless, he assumes the space is T2 just for the sake of simplicity, and that condition is not employed at all in the argument. We took the liberty of dropping that assumption, and we presented a simplified version of his proof, designed specifically for the case in which the topological group G is commutative.

Proposition 1.60. (233) *Let G be a commutative topological group and suppose there is an invariant metric on G such that $\tau_G = \mathsf{t}(\rho)$. The topological group G is complete if, and only if, the metric ρ is complete.*

Proof. Suppose G is complete, let $\{x_n\}$ be a ρ-Cauchy sequence in G and let U be a τ_G-neighbourhood of 0_G. Since $\tau_G = \mathsf{t}(\rho)$, there is an $\epsilon \in (0,\infty)_{\mathbb{R}}$ such that $\mathsf{B}^\rho_\epsilon(0_G) \subset U$ and there is an $N \in \mathbb{N}$ such that

$$(\forall m,n \in [N,\infty)_{\mathbb{N}}) \;\; \rho\,(x_m - x_n, 0_G) = \rho\,(x_m - x_n + x_n, 0_G + x_n) = \rho\,(x_m, x_n) < \epsilon\,,$$

which gives us

$$(\forall m,n \in [N,\infty)_{\mathbb{N}}) \;\; x_m - x_n \in \mathsf{B}^\rho_\epsilon(0_G) \subset U\,,$$

implying that the sequence $\{x_n\}$ is τ_G-Cauchy. Thus, there is an $x \in G$ such that $x_n \xrightarrow[n\to\infty]{\tau_G} x$, that is, $x_n \xrightarrow[n\to\infty]{\mathrm{t}(\rho)} x$, proving that ρ is complete.

Conversely, suppose ρ is complete. Since we have assumed that G is metrizable, it is first-countable, and it suffices to prove that it is sequentially complete (Proposition 1.59). Let $\{x_n\}$ be a τ_G-Cauchy sequence in G and let $\epsilon \in (0,\infty)_{\mathbb{R}}$ be arbitrary. There is a τ_G-neighbourhood U of 0_G such that $U \subset \mathsf{B}^{\rho}_{\epsilon}(0_G)$ and there is an $N \in \mathbb{N}$ such that

$$(\forall m, n \in [N,\infty)_{\mathbb{N}})\ \ x_m - x_n \in U \subset \mathsf{B}^{\rho}_{\epsilon}(0_G)\,,$$

which gives us

$$(\forall m, n \in [N,\infty)_{\mathbb{N}})\ \ \rho\,(x_m, x_n) = \rho\,(x_m - x_n, x_n - x_n) = \rho\,(x_m - x_n, 0_G) < \epsilon\,,$$

implying that the sequence $\{x_n\}$ is ρ-Cauchy. Thus, there is an $x \in G$ such that $x_n \xrightarrow[n\to\infty]{\mathrm{t}(\rho)} x$, that is, $x_n \xrightarrow[n\to\infty]{\tau_G} x$, proving that G is sequentially complete.

□

Proposition 1.61. (175, 111) *Every topological group is $R^{5/2}$ (Definition 1.37).*

1.8 Ordered magmas

An order relation may be invariant with respect to a magma operation, and that invariance may be exploited in order to extract key information on both the order and the operation. Thus, it is natural to consider a magma endowed with an order on it, especially when these components are compatible in a sense:

Definition 1.62. An **ordered magma** is a magma M endowed with an order on it, denoted by $<$ or $<_M$, such that

$$(\forall a, b, x \in M)\ \ (a < b \Rightarrow (xa < xb \text{ and } ax < bx))\ .$$

We have the following notations and terminology:

- An **ordered submagma of** M is an ordered magma M' such that the magma $(M', \times_{M'})$ is a submagma of $(M, \times_M)$ and such that the ordered set $(M', <_{M'})$ is an ordered subset of $(M, <_M)$;

- An **ordered semigroup** (resp. **ordered monoid**, **ordered group**, etc.) is an ordered magma whose underlying magma is a semigroup (resp. monoid, group, etc.);

- Let M be an ordered magma denoted additively. The **superior extension of** M is the superior extension $\overset{\smile}{M}$ of the underlying ordered set in M (Definition 1.33) endowed with an addition operation $+_{\overset{\smile}{M}} : \overset{\smile}{M} \times \overset{\smile}{M} \to \overset{\smile}{M}$ given by

$$x +_{\overset{\smile}{M}} y := \begin{cases} x +_M y & \text{if } x, y \in M\,, \\ \rightarrowtriangle & \text{if } x = \rightarrowtriangle \text{ or } y = \rightarrowtriangle\,. \end{cases}$$

  Note that $\overset{\smile}{M}$ is *not* an ordered magma in general, for if $x, y \in \overset{\smile}{M}$ are so that $x < y$, then we have[6]

$$x + \rightarrowtriangle = \rightarrowtriangle \not< \rightarrowtriangle = y + \rightarrowtriangle\,.$$

Example 1.63.

(**a**) The semigroup $(\mathbb{N}, +_{\mathbb{N}})$ is an ordered semigroup when endowed with its usual order;

(**b**) The monoid $(\mathbb{N}_0, +_{\mathbb{N}_0})$ is an ordered monoid when endowed with its usual order, but none of the monoids $(\mathbb{N}_0, \times_{\mathbb{N}})$, $(\mathbb{Z}, \times_{\mathbb{Z}})$, $(\mathbb{Q}, \times_{\mathbb{Q}})$, $(\mathbb{R}, \times_{\mathbb{R}})$ is an ordered monoid when endowed with its usual order. Note that $1 < 2$, but $1 \cdot 0 = 0 = 2 \cdot 0$;

(**c**) The groups $(\mathbb{Z}, +_{\mathbb{Z}})$, $(\mathbb{Q}, +_{\mathbb{Q}})$ and $(\mathbb{R}, +_{\mathbb{R}})$ are ordered groups when endowed with their usual orders.

[6] However, for all $x, y, z \in \overset{\smile}{M}$, the condition $x \leqslant y$ implies $z + x \leqslant z + y$ and $x + z \leqslant y + z$. Thus, one might say that $\overset{\smile}{M}$ is a **non-strictly ordered magma**.

Example 1.64. (9, 66, 237) We shall provide an example of a non-commutative ordered group. Let $\ell \in [2, \infty)_{\mathbb{N}}$ be fixed. The **Baumslag-Solitar group** $\mathsf{BS}(1, \ell) = \mathsf{BS}_\ell$ is the set

$$\mathsf{BS}_\ell := \{f \in {}^{\mathbb{R}}\mathbb{R} \mid (\exists k \in \mathbb{Z})(\exists c \in \mathbb{Z}[1/\ell])(\forall x \in \mathbb{R})\ f(x) = \ell^k x + c\}$$

endowed with the operation of functional composition, where (cf. Definition 2.46)

$$\mathbb{Z}[1/\ell] := \{p(1/\ell) \mid p \text{ is a polynomial with integer coefficients}\} \subset \mathbb{Q}.$$

We leave to the reader the verification that BS_ℓ is a countable, non-commutative group whose identity element is the identity function $\mathrm{id}_{\mathbb{R}}$. It turns out that BS_ℓ is generated by the functions $\mathsf{t}, \mathsf{m} : \mathbb{R} \to \mathbb{R}$ given by $\mathsf{t}(x) := x + 1$ and $\mathsf{m}(x) := \ell x$, and since $\mathsf{m} \neq f^2$ $(\forall f \in \mathsf{BS}_\ell)$, the group BS_ℓ is not divisible (Definition 1.2). The Baumslag-Solitar groups play important roles in Combinatorial Group Theory and Geometric Group Theory.

We shall define an order on BS_ℓ that is invariant under functional composition. For each $f \in \mathsf{BS}_\ell$, denote by $k_f \in \mathbb{Z}$ and $c_f \in \mathbb{Z}[1/\ell]$ the numbers so that $f(x) = \ell^{k_f} x + c_f$ $(\forall x \in \mathbb{R})$. Let $\sqsubset$ be the binary relation on BS_ℓ defined so that for all $f, g \in \mathsf{BS}_\ell$, the condition $f \sqsubset g$ means that $f \neq g$ and the strict inequality[7]

$$\frac{1}{\sqrt{2}} \det \begin{pmatrix} \ell^{k_g} & c_g \\ \ell^{k_f} & c_f \end{pmatrix} < \ell^{k_g} - \ell^{k_f}$$

holds. That relation is clearly irreflexive and strictly transitive on BS_ℓ. To see that it is total, note that $\sqrt{2}$ is irrational, implying that the equality

$$\frac{1}{\sqrt{2}} \det \begin{pmatrix} \ell^{k_g} & c_g \\ \ell^{k_f} & c_f \end{pmatrix} = \ell^{k_g} - \ell^{k_f}$$

holds if, and only if, both sides are zero, and that happens precisely when $f = g$. Hence, if $f, g \in \mathsf{BS}_\ell$ are distinct, then either

$$\frac{1}{\sqrt{2}} \det \begin{pmatrix} \ell^{k_g} & c_g \\ \ell^{k_f} & c_f \end{pmatrix} < \ell^{k_g} - \ell^{k_f} \quad \text{or} \quad \frac{1}{\sqrt{2}} \det \begin{pmatrix} \ell^{k_g} & c_g \\ \ell^{k_f} & c_f \end{pmatrix} > \ell^{k_g} - \ell^{k_f},$$

[7] The choice of the factor $1/\sqrt{2}$ on the left hand side of the inequality is designed to simplify a few calculations in Example 1.71. For the purposes of the present example, that factor can be replaced by any other positive irrational number.

that is, either $f \sqsubset g$ or $g \sqsubset f$, proving that $\sqsubset$ is an order on BS_ℓ. One may straightforwardly prove that an element $f \in \mathsf{BS}_\ell$ is $\sqsubset$-positive if, and only if, $f\left(\sqrt{2}\right) > \sqrt{2}$. Accordingly, the elements t and m are $\sqsubset$-positive, and since

$$\frac{1}{\sqrt{2}} \det \begin{pmatrix} \ell & 0 \\ 1 & 1 \end{pmatrix} = \frac{\ell}{\sqrt{2}} \begin{cases} > \ell - 1 & \text{if } \ell \in \{2,3\}\ , \\ < \ell - 1 & \text{if } \ell \geqslant 4\,, \end{cases}$$

we have $\mathrm{id}_{\mathbb{R}} \sqsubset \mathsf{m} \sqsubset \mathsf{t}$ if $\ell \in \{2,3\}$, and $\mathrm{id}_{\mathbb{R}} \sqsubset \mathsf{t} \sqsubset \mathsf{m}$ if $\ell \geqslant 4$.

We are to prove that $\sqsubset$ is invariant under functional composition. Let $f, g \in \mathsf{BS}_\ell$ be so that $f \sqsubset g$, and let $h \in \mathsf{BS}_\ell$. Computing the compositions $f \circ h$ and $g \circ h$ on an argument $x \in \mathbb{R}$, we have

$$\begin{aligned} (f \circ h)(x) &= \ell^{k_f}\left(\ell^{k_h} x + c_h\right) + c_f = \ell^{k_f + k_h} x + \left(\ell^{k_f} c_h + c_f\right) \\ (g \circ h)(x) &= \ell^{k_g}\left(\ell^{k_h} x + c_h\right) + c_g = \ell^{k_g + k_h} x + \left(\ell^{k_g} c_h + c_g\right)\ , \end{aligned}$$

and thus we obtain

$$\begin{aligned} \frac{1}{\sqrt{2}} \det \begin{pmatrix} \ell^{k_{g \circ h}} & c_{g \circ h} \\ \ell^{k_{f \circ h}} & c_{f \circ h} \end{pmatrix} &= \frac{1}{\sqrt{2}} \left(\ell^{k_g + k_h}\left(\ell^{k_f} c_h + c_f\right) - \ell^{k_f + k_h}\left(\ell^{k_g} c_h + c_g\right)\right) \\ &= \frac{\ell^{k_h}}{\sqrt{2}} \det \begin{pmatrix} \ell^{k_g} & c_g \\ \ell^{k_f} & c_f \end{pmatrix} \\ &< \ell^{k_h}\left(\ell^{k_g} - \ell^{k_f}\right) = \ell^{k_{g \circ h}} - \ell^{k_{f \circ h}}\ , \end{aligned}$$

that is, $f \circ h \sqsubset g \circ h$. The proof that $h \circ f \sqsubset h \circ g$ is analogous, thus proving that BS_ℓ is a non-commutative ordered group when endowed with $\sqsubset$. This order is by no means unique, and, in fact, there are infinitely many invariant orders on BS_ℓ. In forthcoming examples, we shall denote $\sqsubset$ by $<$, by abuse of language, and we shall always assume that the group BS_ℓ is endowed with that order. Additionally, we shall denote the operation of BS_ℓ **additively**, that is, a composition $f \circ g$ between two elements f and g in BS_ℓ shall be denoted by $f + g$, and the identity function $\mathrm{id}_{\mathbb{R}}$ shall also be denoted by 0_{BS_ℓ}.

Proposition 1.65. (189) *Let M be a non-trivial ordered monoid and let $x \in M - \{1_M\}$.*

(a) *The sequence $\{x^n\}$ is monotone, being increasing if $x > 1_M$ and decreasing if $x < 1_M$. In particular, we have $\overset{\times_M}{\emptyset}(x) = \infty$ and the ordered monoid M is infinite;*

(b) *If M is an ordered group, then the sequence $\{x^n\}$ has no $\overset{M}{\mathrm{Ordt}}$-cluster point;*

(c) *For every natural number n and for every subset S of M, the function $f : S \to S^n$ given by $f(x) := x^n$ is an isomorphism between ordered sets.*

Proof.

(a) If $x \in (1_M, \to)_M$ (resp. $x \in (\leftarrow, 1_M)_M$), then we have

$$x = x \cdot 1_M < x \cdot x = x^2 = x \cdot x \cdot 1_M < x \cdot x \cdot x = x^3 < \cdots$$

$$(\text{resp. } x = x \cdot 1_M > x \cdot x = x^2 = x \cdot x \cdot 1_M > x \cdot x \cdot x = x^3 > \cdots) .$$

(b) Suppose M is an ordered group and that the sequence $\{x^n\}$ $\overset{M}{\mathrm{Ordt}}$-clusters at a point $y \in M$. Assume $x > 1_M$ without loss of generality. Hence, $x^{-1} < 1_M$, $x^{-1}y < y < xy$, and there are infinitely many natural numbers n such that $x^n \in (x^{-1}y, xy)$, that is, $x^{-1}y < x^n < xy$. Take two of those numbers, m and n, so that $n \geqslant m + 2$. Thus, we have $x^{-1}y < x^m < xy$ and

$$xy = xx\left(x^{-1}y\right) < xxx^m = x^{m+2} < x^{m+3} < x^{m+4} < \cdots < x^n < xy ,$$

which is absurd, proving the item.

(c) If x and y are elements of S so that $x < y$, then

$$f(x) = \overbrace{x \cdot x \cdots x}^{n \text{ times}} < \overbrace{y \cdot y \cdots y}^{n \text{ times}} = f(y) ,$$

proving that f is increasing, and since f is surjective, it is an isomorphism between ordered sets (Proposition 1.30).

□

Lemma 1.66. (97, 144, 188) *Let M be an ordered magma. If A and B are two well-ordered subsets of M, then AB is a well-ordered subset of M.*

Proof. The result is immediate if $A = \emptyset$ or $B = \emptyset$. Suppose A and B are inhabited, let $\{x_n\}$ be a sequence in AB and for each natural number n, let a_n be the least element of A such that there is a $b_n \in B$ so that $x_n = a_n b_n$. Since A is well-ordered, there is an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that the subsequence $\{a_{\phi(n)}\}$ of $\{a_n\}$ is non-strictly increasing (Lemma 1.32), and since B is well-ordered, there is an increasing function $\chi : \mathbb{N} \to \mathbb{N}$ such that the subsequence $\{b_{\phi(\chi(n))}\}$ of $\{b_{\phi(n)}\}$ is non-strictly increasing. Thus, the sequences $\{a_{\phi(\chi(n))}\}$ and $\{x_{\phi(\chi(n))}\}$ are non-strictly increasing, proving that AB is well-ordered.

□

Proposition 1.67. (168) *Every ordered group G is a topological group when endowed with its order topology.*

Proposition 1.68. (111) *If G is a non-trivial subgroup of $(\mathbb{R}, +_{\mathbb{R}})$, then either G is order-dense in $\mathbb{R}$ or G is of the form $r\mathbb{Z}$ for $r \in (0, \infty)_{\mathbb{R}}$.*

Consider two natural numbers, m and n so that $m < n$. We know that $n \leqslant mn$, that is, we have $n \leqslant \overbrace{m + m + \cdots + m}^{n \text{ times}}$, and, in particular, it is possible to add m to itself a finite number of times so that the new quantity is greater than n. The same happens with the positive rational numbers, since if we take ${}^{p}/_{q}, {}^{r}/_{s} \in \mathbb{Q}$ so that $p, q, r, s \in \mathbb{N}$ and ${}^{p}/_{q} < {}^{r}/_{s}$, then we have $rq \leqslant sprq$ and

$$\frac{r}{s} \leqslant \frac{p}{q} \cdot rq = \overbrace{\frac{p}{q} + \frac{p}{q} + \cdots + \frac{p}{q}}^{rq \text{ times}} \quad .$$

The oldest known work dealing with that property is Euclid's widely praised *Elements*, an enormously influential series of thirteen books on Geometry dated from 300 BC, in which the ancient Greek mathematician defines in Book 5, Definition 4 (72):

> Magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.

In the 19th century, that property came to be called the **Archimedean Property** due to the fact that Archimedes made repeated use of it to solve many problems in Geometry (96), and the mathematical systems that satisfy that property are said to be **Archimedean**. Magnitudes a and b that do not "have a ratio with respect to one another" are said to be **infinitesimal** with respect to each other, and, as it turns out, if a positive number a is infinitesimal with respect to 1, then a is smaller than every positive real number, being called an **infinitesimal** (**number**). Archimedes himself systematically used infinitesimals as a heuristic tool to "guess" the correct formulas for the volume and surface area of many geometric solids, but he did not regard these methods as rigorous proofs.

Many structures in modern Mathematics are non-Archimedean, and their study is called **Non-Archimedean Analysis**, area of which this work treats in regard to the theory of Rayner Rngs in Chapters 3 and 4.

Definition 1.69. Let M be an ordered monoid.

- ▷ The **Archimedean relation on** M is the binary relation on M denoted by $\mathscr{A}$ or $\overset{M}{\mathscr{A}}$ and defined so that for all $x, y \in M$, the condition $x \mathscr{A} y$ amounts to saying that there is an $n \in \mathbb{N}$ such that $x \leqslant y \leqslant x^n$, or $x^n \leqslant y \leqslant x$, or $y \leqslant x \leqslant y^n$ or $y^n \leqslant x \leqslant y$. One can easily check that $\mathscr{A}$ is an equivalence relation on M;
- ▷ The **Archimedean classes of** M are the equivalence classes associated to the equivalence relation $\mathscr{A}$, and, in particular, the **trivial Archimedean class of** M is the class $1_M/\mathscr{A} = \{1_M\}$ of the element 1_M. Each non-trivial Archimedean class S of M is an order-convex subsemigroup of M that is either contained in $(1_M, \rightarrow)_M$ or contained in $(\leftarrow, 1_M)_M$. It is **positive** if $S \subset (1_M, \rightarrow)_M$ and it is **negative** if $S \subset (\leftarrow, 1_M)_M$;
- ▷ An ordered monoid M is **Archimedean** if it has at most one positive Archimedean class and at most one negative Archimedean class. Otherwise, it is **non-Archimedean**;

- ▷ The **partial order of Archimedean distribution on** M is the binary relation on M denoted by $\ll$ or $\ll_M$ and defined so that for all $x, y \in M$, the condition $x \ll y$ amounts to saying that $x < y$ and the elements x and y do not belong to the same Archimedean class of M. One can directly show that $\ll$ is a partial order on M.

The function given by $S \mapsto S^{-1}$ defines a one-to-one correspondence between the positive and the negative Archimedean classes of an ordered group G. Thus, we have that G is Archimedean if, and only if, it has at most one positive Archimedean class.

Example 1.70. The ordered group $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ of real numbers is Archimedean. The ordered group $({}^*\mathbb{R}, +_{{}^*\mathbb{R}}, <_{{}^*\mathbb{R}})$ of **hyperreal numbers** (192, 148) and the ordered group $(\mathsf{No}, +_{\mathsf{No}}, <_{\mathsf{No}})$ of **surreal numbers** (113) are non-Archimedean.

Example 1.71. Consider the non-commutative ordered group BS_ℓ, denoted additively (Example 1.64). We shall prove that BS_ℓ is non-Archimedean in the case $\ell \geqslant 4$, leaving the cases $\ell = 2$ and $\ell = 3$ to the reader. Consider the element $f \in \mathsf{BS}_\ell$ given by $f(x) = \ell x - \ell$, which may also be given by $f = (-\ell\,\mathsf{t}) + \mathsf{m}$. Note that f is positive, since we have

$$f(\sqrt{2}) = \ell\sqrt{2} - \ell = \ell(\sqrt{2} - 1) > \sqrt{2}$$

for $\ell \geqslant 4$. Let n be any natural number. One may inductively show that the function nf is given by $(nf)(x) = \ell^n x - \sum_{i=1}^{n} \ell^i$, and since

$$\frac{1}{\sqrt{2}} \det \begin{pmatrix} 1 & 1 \\ \ell^n & -\sum_{i=1}^{n} \ell^i \end{pmatrix} = -\sqrt{2}\,\ell^n - \frac{1}{\sqrt{2}}\,\ell^{n-1} - \cdots - \frac{1}{\sqrt{2}}\,\ell^2 - \frac{1}{\sqrt{2}}\,\ell < 1 - \ell^n\,,$$

we have $nf < \mathsf{t}$ $(\forall n \in \mathbb{N})$, thus proving that nf and t belong to distinct positive Archimedean classes of BS_ℓ.

The outcome of Example 1.71 holds true if the Baumslag-Solitar ordered group BS_ℓ is replaced by any other non-commutative ordered group, as the following theorem attributed to Hölder reveals:

Theorem 1.72 (Hölder's Theorem)**.** (129, 115, 80) *Every Archimedean ordered group G is isomorphic to an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$. In particular, every Archimedean ordered group is commutative.*

Since the inversion function $\overset{\times_G}{\mathrm{Inv}} : G \to G$ on an ordered group G is an involution on G (Definition 1.7), and since the conditions $x > 1_G$ and $x^{-1} < 1_G$ are equivalent for all $x \in G$, it is natural to choose the positive element of the set $\{x, x^{-1}\}$ and think of it as a kind of distance between the element x and the identity element 1_G. Thus, we have the following definition:

Definition 1.73. Let G be an ordered group. The **absolute value function** $\overset{G}{\mathrm{Abs}} : G \to G$ **on** G is the function given by

$$\overset{G}{\mathrm{Abs}}(x) = |x| \coloneqq \overset{G}{\max}\{x, x^{-1}\} .$$

Thus, we have $x, x^{-1} \leqslant |x|$ $(\forall x \in G)$.

Proposition 1.74. (37) *Let G be an ordered group. For all $x, y \in G$, we have:*

- ▷ $|1_G| = 1_G$;
- ▷ $|x| = |x^{-1}|$;
- ▷ $||x|| = |x|$;
- ▷ $|x| \geqslant 1_G$;
- ▷ $|x|^{-1} \leqslant x \leqslant |x|$;
- ▷ $|x|^{-1} \leqslant x^{-1} \leqslant |x|$;
- ▷ $x \geqslant 1_G$ *if, and only if,* $|x| = x$;
- ▷ $x \leqslant 1_G$ *if, and only if,* $|x| = x^{-1}$;
- ▷ $|x| = 1_G$ *if, and only if,* $x = 1_G$.

Suppose G is commutative, thus being denoted additively.

- ▷ $|x + y| \leqslant |x| + |y|$ *(Triangular Inequality);*
- ▷ $|x + y| = |x| + |y|$ *if, and only if,* $x, y \geqslant 0_G$ *or* $x, y \leqslant 0_G$;
- ▷ $||x| - |y|| \leqslant |x - y|$.

The upcoming couple of lemmas will be quite useful in Chapter 3:

Lemma 1.75. (188, 92) *Let G be an ordered group and let A and B be two well-ordered subsets of G. For each $g \in G$, the set*

$$\mathrm{P}_g \coloneqq \{(a,b) \in A \times B \mid ab = g\}$$

is finite.

Proof. Suppose P_g is infinite and let $\{(a_n, b_n)\}$ be an injective sequence in P_g. Thus, $a_n b_n = g$ $(\forall n \in \mathbb{N})$, and if $p, q \in \mathbb{N}$ are indices so that $a_p = a_q$, then $a_p b_p = g = a_q b_q$, which implies $b_p = b_q$, $(a_p, b_p) = (a_q, b_q)$ and $p = q$, proving that the sequence $\{a_n\}$ is injective. Analogously, one may show that the sequence $\{b_n\}$ is injective as well. Since A is well-ordered, there is an increasing subsequence $\{a_{n_i}\}_{i \in \mathbb{N}}$ of $\{a_n\}$ (Lemma 1.32), and if $i, j \in \mathbb{N}$ are so that $i < j$ and $b_{n_i} \leqslant b_{n_j}$, then $a_{n_i} < a_{n_j}$ and we get

$$g = a_{n_i} b_{n_i} < a_{n_j} b_{n_i} \leqslant a_{n_j} b_{n_j} = g\,,$$

which is absurd. Thus, the sequence $\{b_{n_i}\}_{i \in \mathbb{N}}$ in B is decreasing, contradicting the fact that B is well-ordered.

□

Lemma 1.76. *Let G be a non-trivial ordered group.*

(a) $|(1_G, \rightarrow)_G| = |G|$;

(b) *If* $\operatorname{cf}(G) = \omega$, *then we have*

$$|\{\{g_n\} \in {}^{\mathbb{N}}G \mid \{g_n\} \text{ is increasing and cofinal in } G\}| = |G|^{\omega}\ .$$

Proof.

(a) The function $\overset{\times G}{\operatorname{Inv}} : G \to G$ (Definition 1.7) is a bijection such that $\overset{\times G}{\operatorname{Inv}} \langle (1_G, \rightarrow)_G \rangle = (\leftarrow, 1_G)_G$, implying that $|(1_G, \rightarrow)_G| = |(\leftarrow, 1_G)_G|$. Since the set

$$G = (1_G, \rightarrow)_G \cup (\leftarrow, 1_G)_G \cup \{1_G\}\ ,$$

is infinite, the interval $(1_G, \to)_G$ is infinite, and we have

$$\begin{aligned} |G| &= |(1_G, \to)_G \cup (\leftarrow, 1_G)_G \cup \{1_G\}| \\ &= |(1_G, \to)_G| + |(\leftarrow, 1_G)_G| + |\{1_G\}| \\ &= |(1_G, \to)_G| \ . \end{aligned}$$

(b) Since $\operatorname{cf}(G) = \omega$, there is a cofinal sequence $\{g_n\}$ in G, and since G is unbounded above, we may assume without loss of generality that the sequence $\{g_n\}$ is increasing. Taking a fixed $\{p_n\} \in {}^{\mathbb{N}}(1_G, \to)_G$, we shall check that the sequence $\{x_n\} := \{g_n p_1 p_2 \cdots p_{n-1}\}$ in G is increasing and cofinal in G. Indeed, for every natural number n, we have

$$x_n = g_n p_1 p_2 \cdots p_{n-1} = g_n p_1 p_2 \cdots p_{n-1} 1_G < g_{n+1} p_1 p_2 \cdots p_{n-1} p_n = x_{n+1} \ ,$$

proving that $\{x_n\}$ is increasing. Moreover, if g is an element of G, then there is an $n \in \mathbb{N}$ such that $g \leqslant g_n$, and we get

$$g \leqslant g_n = g_n \overbrace{1_G 1_G \cdots 1_G}^{n-1 \text{ times}} < g_n p_1 p_2 \cdots p_{n-1} = x_n \ ,$$

showing that $\{x_n\}$ is cofinal in G. Let

$$f : {}^{\mathbb{N}}(1_G, \to)_G \longrightarrow \{\{x_n\} \in {}^{\mathbb{N}}G \mid \{x_n\} \text{ is increasing and cofinal in } G\}$$

be the function given by $f(\{p_n\}) := \{g_n p_1 p_2 \cdots p_{n-1}\}$. That function is clearly injective, and, from item (a), we obtain

$$\begin{aligned} |G|^\omega &= |(1_G, \to)_G|^\omega \\ &= |{}^{\mathbb{N}}(1_G, \to)_G| \\ &\leqslant |\{\{g_n\} \in {}^{\mathbb{N}}G \mid \{g_n\} \text{ is increasing and cofinal in } G\}| \\ &\leqslant |{}^{\mathbb{N}}G| = |G|^\omega \ . \end{aligned}$$

$\square$

Proposition 1.77. *Let G be a commutative ordered group that has no least positive element*[8]*. For each $x \in (0_G, \to)_G$ and for each $n \in \mathbb{N}$, there is a $y \in (0_G, \to)_G$ such that $ny < x$.*

[8] That is, we have $\operatorname{ci}((0_G, \to)_G) \geqslant \omega$.

Proof. Suppose we have defined a finite sequence $y_1 y_2 \ldots y_n$ in $(0_G, \rightarrow)_G$ such that $ky_k < x$ $(\forall k \in [1, n]_{\mathbb{N}})$. If $(n+1)\, y_n < x$, then set $y_{n+1} \coloneqq y_n$. Otherwise, consider the case $(n+1)\, y_n \geqslant x$ and take a $y_{n+1} \in (0_G, \rightarrow)_G$ such that $y_{n+1} < x - ny_n$. Thus, if $(n+1)\, y_{n+1} \geqslant x$, then we get

$$x \leqslant (n+1)\, y_{n+1} < (n+1)\,(x - ny_n) = (n+1)\, x - n\,(n+1)\, y_n\,,$$

implying $n\,(n+1)\, y_n < nx$ and $(n+1)\, y_n < x$, which is absurd. Therefore, we have just recursively defined a sequence $\{y_n\}$ in $(0_G, \rightarrow)_G$ such that $ky_k < x$ $(\forall k \in \mathbb{N})$. □

Corollary 1.78. *If G is a commutative ordered group, then the order topology* $\overset{G}{\mathrm{Ordt}}$ *is* $T^{7/2}$.

Proof. Immediate consequence of Propositions 1.51, 1.61 and 1.67. □

1.9 Γ-Valued commutative groups

Sometimes it is convenient to endow a commutative group A with an additional function, called a **Γ-valuation on** A, introduced to represent the notion of **order of magnitude** of the elements of A, where such orders lie in an ordered set Γ. That concept was first formally defined by Kürschák in 1913 in connection to Hensel's theory of p-adic numbers (128), but long before that Kummer systematically employed $\mathbb{Z}$-valuations on the field $\mathbb{Q}(\zeta)$, where $\zeta \neq 1$ is a p-th root of unity and p is an odd prime, as he arduously attempted to prove Fermat's Last Theorem for decades (121, 122, 123, 124, 40). Valuations have found applications in many areas of Mathematics, such as Algebraic Geometry, Algebraic Number Theory and Commutative Algebra. As we shall expose later on in detail, they lie at the heart of the Theory of Rayner Rngs.

Definition 1.79. Let Γ be an ordered set and let A be a commutative group. A **Γ-valuation on** A is a surjective function $v : A \to \breve{\Gamma}$ (Definition 1.62) that satisfies the following properties for all $x, y \in A$:

(V1) $\quad v\,(x - y) \geqslant \overset{\breve{\Gamma}}{\min}\,\{v\,(x)\,, v\,(y)\}$; $\qquad$ **(V2)** $\quad v\,(x) = \rightarrowtriangle$ if, and only if, $x = 0_A$.

A **Γ-valued commutative group** is a commutative group A endowed with a Γ-valuation $v : A \to \breve{\Gamma}$ on A. When no particular notation is attributed to the Γ-valuation of A, that shall be denoted by v_A. We have the following notations and terminology:

- ▷ A **Γ-valued commutative subgroup of** A is a Γ-valued commutative group A' such that the group $(A', \times_{A'})$ is a subgroup of $(A, \times_A)$ and such that $v_{A'} \subset v_A$;
- ▷ The **value set of** A is the set Γ;
- ▷ A **ball in** A is a subset B of A such that the following condition is satisfied:

$$(\forall x, y \in B)\,(\forall z \in A)\ \ (v\,(x - z) \geqslant v\,(x - y) \Rightarrow z \in B)\ ;$$

- ▷ For each $\gamma \in \breve{\Gamma}$, we define the sets

$$\begin{aligned} \mathrm{O}^A_\gamma = \mathrm{O}^{v_A}_\gamma = \mathrm{O}_\gamma &\coloneqq \{x \in A \ \mid \ v\,(x) \geqslant \gamma\} \\ \mathrm{o}^A_\gamma = \mathrm{o}^{v_A}_\gamma = \mathrm{o}_\gamma &\coloneqq \{x \in A \ \mid \ v\,(x) > \gamma\}\ , \end{aligned}$$

  which we shall call the **big-O set of value** γ and the **little-O set of value** γ, respectively. These notations are inspired by the big-O and little-O notations from the theory of Asymptotic Analysis.

Example 1.80. (104, 112) Let p be a prime number. For each non-zero rational number x, there is a unique integer n_x such that x may be written in the form $x = p^{n_x}\,({}^a\!/\!{}_b)$, where a and b are non-zero integers coprime to p. The **p-adic $\mathbb{Z}$-valuation on** $\mathbb{Q}$ is the function $v_p \colon \mathbb{Q} \to \breve{\mathbb{Z}}$ given by

$$v_p(x) \coloneqq \begin{cases} n_x & \text{if } x \neq 0\,, \\ \infty & \text{if } x = 0\,. \end{cases}$$

We shall check that v_p is a $\mathbb{Z}$-valuation on the commutative group $(\mathbb{Q}, +_{\mathbb{Q}})$. It is clear that Axiom (V2) holds for v_p, and this function is surjective since we have $v_p\left(p^n\right) = n$ $(\forall n \in \mathbb{Z})$. Take two arbitrary non-zero rational numbers x and y, and take four non-zero integers a, b, c and d coprime to p such that $x = p^{n_x}\left({}^{a}\!/_{b}\right)$ and $y = p^{n_y}\left({}^{c}\!/_{d}\right)$. Thus, we get

$$x - y = p^{n_x}\,\frac{a}{b} - p^{n_y}\,\frac{c}{d} = p^{\min\{n_x,n_y\}}\left(\frac{p^{n_x-\min\{n_x,n_y\}}ad - p^{n_y-\min\{n_x,n_y\}}bc}{bd}\right) ,$$

and if $p^{n_x-\min\{n_x,n_y\}}ad - p^{n_y-\min\{n_x,n_y\}}bc = p^m e$, where $m \in \mathbb{N}_0$ and $e \in \mathbb{Z} - \{0\}$ so that e is coprime to p, then we obtain

$$v_p(x-y) = \min\{n_x, n_y\} + m \geqslant \min\{n_x, n_y\} = \min\{v_p(x)\,, v_p(y)\} \ ,$$

proving that Axiom (V1) holds for v_p.

Moreover, consider the function $u_p \colon \mathbb{Q} \to [0,\infty)_{\mathbb{R}}$ given by

$$u_p(x) \coloneqq \begin{cases} \mathsf{e}^{-v_p(x)} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Note that for all $x \in \mathbb{Q} - \{0\}$ and all $r \in (0,\infty)_{\mathbb{R}}$, we have

$$v_p(x) = -\log\left(u_p(x)\right) \quad \text{and} \quad \{z \in \mathbb{Q} \mid u_p(z) < r\} = \mathrm{o}^{v_p}_{-\log(r)}\,.$$

The function u_p satisfies the following properties for all $x, y \in \mathbb{Q}$:

(**N1**) $u_p(x-y) \leqslant \max\{u_p(x)\,, u_p(y)\}$; (**N2**) $u_p(x) = 0$ if, and only if, $x = 0$.

In general, taking into consideration a commutative group A, the functions $u : A \to [0,\infty)_{\mathbb{R}}$ for which Axioms (N1) and (N2) hold are called **ultranorms on** A. Thus, when Γ is an ordered subset of $\mathbb{R}$, the Γ-valuations $v : A \to \breve{\Gamma}$ on A and the ultranorms $u : A \to [0,\infty)_{\mathbb{R}}$ on A may be regarded as dual concepts, though with the caveat that one must adjust the ordered set Γ so as to ensure that the Γ-valuations are surjective. In mathematical practice, small values of $u\,(x)$ are interpreted as meaning that x is *close* to the identity element 0_A, and big values of $v\,(x)$ are interpreted as meaning that x has a *low order of magnitude*. Accordingly, although both notions are essentially interchangeable, they are used in different contexts.

Proposition 1.81. (39, 216, 229, 135, 136) *Let Γ be an ordered set and let A be a Γ-valued commutative group.*

(a) $(\forall x \in A)\ v(-x) = v(x)$;

(b) $(\forall x, y \in A)\ v(x+y) \geqslant \overset{\breve{\Gamma}}{\min}\{v(x), v(y)\}$;

(c) $(\forall x \in A)\,(\forall n \in \mathbb{Z} - \{0\})\ v(nx) \geqslant v(x)$;

(d) *If $x, y \in A$ are such that $v(x) \neq v(y)$, then $v(x-y) = \overset{\breve{\Gamma}}{\min}\{v(x), v(y)\}$;*

(e) *If $x \in A$ and $m \in \mathbb{Z} - \{0\}$ are such that $mx = 0_A$, and if $n \in \mathbb{Z} - \{0\}$ is so that $\gcd(m,n) = 1$, then we have $v(nx) = v(x)$;*

(f) *For all $\alpha, \beta \in \Gamma$, we have:*

- $\mathrm{o}_\alpha \subset \mathrm{O}_\alpha$;
- $-\mathrm{O}_\alpha = \mathrm{O}_\alpha$;
- $-\mathrm{o}_\alpha = \mathrm{o}_\alpha$;
- $\mathrm{O}_\alpha + \mathrm{O}_\beta \subset \mathrm{O}_{\min\{\alpha,\beta\}}$;
- $\mathrm{o}_\alpha + \mathrm{o}_\beta \subset \mathrm{o}_{\min\{\alpha,\beta\}}$.

(g) *For each $\gamma \in \Gamma$ and for each $x \in A$, we have*

$$(\forall y \in x + \mathrm{O}_\gamma)\ x + \mathrm{O}_\gamma = y + \mathrm{O}_\gamma \quad \text{and} \quad (\forall y \in x + \mathrm{o}_\gamma)\ x + \mathrm{o}_\gamma = y + \mathrm{o}_\gamma\,.$$

Also, the sets $x + \mathrm{O}_\gamma$ and $x + \mathrm{o}_\gamma$ are balls in A;

(h) *If B_1 and B_2 are two balls in A so that $B_1 \cap B_2 \neq \emptyset$, then $B_1 \subset B_2$ or $B_2 \subset B_1$;*

(i) *For all $\alpha, \beta \in \Gamma$ and all $x, y \in A$ such that $(x + \mathrm{O}_\alpha) \cap (y + \mathrm{O}_\beta) \neq \emptyset$, the conditions $x + \mathrm{O}_\alpha \subset y + \mathrm{O}_\beta$ and $\alpha \geqslant \beta$ are equivalent;*

(j) *For all $\alpha, \beta \in \Gamma$ and all $x, y \in A$ such that $(x + \mathrm{o}_\alpha) \cap (y + \mathrm{o}_\beta) \neq \emptyset$, the conditions $x + \mathrm{o}_\alpha \subset y + \mathrm{o}_\beta$ and $\alpha \geqslant \beta$ are equivalent;*

(k) *If B is a ball on A, if $x \in B$ and if $y \in A - B$, then $B \cap (y + \mathrm{o}_{v(x-y)}) = \emptyset$.*

Proof. These items have rather easy proofs, except for item (h) whose proof is little trickier. We present some of those arguments here so the reader can get a taste of how they look like.

(a) We have

$$\begin{aligned} v(-x) &= v(0_A - x) \\ &\geqslant \overset{\breve{\Gamma}}{\min}\{v(0_A), v(x)\} \\ &= v(0_A - (-x)) \\ &\geqslant \overset{\breve{\Gamma}}{\min}\{v(0_A), v(-x)\} = v(-x) \ . \end{aligned}$$

(d) Assume $v(y) > v(x)$ without loss of generality. We must prove that $v(x-y) = v(x)$. Suppose otherwise. Thus, since

$$v(x-y) \geqslant \overset{\breve{\Gamma}}{\min}\{v(x), v(y)\} = v(x) \ ,$$

we get

$$v(x-y) > v(x) = v(x-y+y) \geqslant \overset{\breve{\Gamma}}{\min}\{v(x-y), v(y)\} \ .$$

With that in mind, note that if $v(x-y) \leqslant v(y)$, then $v(x-y) > v(x-y)$, which is absurd. Hence, we obtain

$$v(x-y) > v(y) > v(x) \geqslant \overset{\breve{\Gamma}}{\min}\{v(x-y), v(y)\} = v(y) \ ,$$

which is absurd, proving the item.

(g) Let $y \in x + \mathrm{O}_\gamma$. Thus, we have $v(y-x) \geqslant \gamma$, and if $z \in x + \mathrm{O}_\gamma$, then $v(z-x) \geqslant \gamma$, and, since

$$v(y-z) = v((y-x) + (x-z)) \geqslant \overset{\frown}{\overset{\Gamma}{\min}}\{v(y-x), v(x-z)\} \ ,$$

we have $v(y-z) \geqslant \gamma$, proving the inclusion $x + \mathrm{O}_\gamma \subset y + \mathrm{O}_\gamma$. The proof of the opposite inclusion and the proof of the equation $x + \mathrm{o}_\gamma = y + \mathrm{o}_\gamma$ are analogous. Lastly, if $a, b \in x + \mathrm{O}_\gamma$ and $c \in A$ are so that $v(a-c) \geqslant v(a-b)$, then, since

$$v(a-b) = v((a-x) + (x-b)) \geqslant \overset{\breve{\Gamma}}{\min}\{v(a-x), v(x-b)\} \geqslant \gamma \, ,$$

we get $v(a-c) \geqslant \gamma$ and $c \in a + \mathrm{O}_\gamma = x + \mathrm{O}_\gamma$, proving that $x + \mathrm{O}_\gamma$ is a ball in A. The proof that $x + \mathrm{o}_\gamma$ is a ball in A is analogous.

(i) Suppose $\alpha \geqslant \beta$ and $x + \mathrm{O}_\alpha \not\subset y + \mathrm{O}_\beta$. Thus, we have $\mathrm{O}_\alpha \subset \mathrm{O}_\beta$ and $y + \mathrm{O}_\beta \subset x + \mathrm{O}_\alpha$ by items (g) and (h), implying $y - x \in y - x + \mathrm{O}_\alpha \subset \mathrm{O}_\beta$. If $z \in x + \mathrm{O}_\alpha$, then we get

$$z = y + (z-x) + (x-y) \in y + \mathrm{O}_\alpha + \mathrm{O}_\beta \subset y + \mathrm{O}_\beta$$

and $y + \mathrm{O}_\beta = x + \mathrm{O}_\alpha \not\subset y + \mathrm{O}_\beta$, which is absurd, proving the sufficient condition of the item.

Suppose $x + \mathrm{O}_\alpha \subset y + \mathrm{O}_\beta$ and $\alpha < \beta$. We have $x - y \in \mathrm{O}_\beta$, and, since the function v is surjective, there is a $z \in A$ such that $v(z) = \alpha$, implying $z \in \mathrm{O}_\alpha - \mathrm{O}_\beta$. Since $x + \mathrm{O}_\alpha \subset y + \mathrm{O}_\beta$, we get $x + z \in y + \mathrm{O}_\beta$ and

$$z \in y - x + \mathrm{O}_\beta \subset \mathrm{O}_\beta + \mathrm{O}_\beta \subset \mathrm{O}_\beta\,,$$

which is absurd, thus proving the item.

□

Let Γ be an ordered set and let A be a Γ-valued commutative group. We are to show that the set

$$\mathcal{B} \coloneqq \{x + \mathrm{o}_\gamma \mid x \in A \text{ and } \gamma \in \Gamma\}$$

is a (synthetic) basis of a topology on A. Let $x, y \in A$, let $\alpha, \beta \in \Gamma$ and take an arbitrary element z in the intersection $(x + \mathrm{o}_\alpha) \cap (y + \mathrm{o}_\beta)$. We may assume without loss of generality that $\alpha \geqslant \beta$. Thus, we have $x + \mathrm{o}_\alpha \subset y + \mathrm{o}_\beta$ by Item (i) of Proposition 1.81, and we have

$$(x + \mathrm{o}_\alpha) \cap (y + \mathrm{o}_\beta) = x + \mathrm{o}_\alpha = z + \mathrm{o}_\alpha$$

by Item (f) of Proposition 1.81, proving that $\mathcal{B}$ is the basis of a topology $\tau_{\mathcal{B}}$ on A. Furthermore, that fact and Item (f) of Proposition 1.81 readily imply that, for each $x \in A$, the family of sets $\{x + \mathrm{o}_\gamma\}_{\gamma \in \Gamma}$ is a local $\tau_{\mathcal{B}}$-basis of x. These considerations allow us to make the following definition:

Definition 1.82. Let Γ be an ordered set and let A be a Γ-valued commutative group. The **topology on** A **induced by the** Γ**-valuation** v_A is the topology on A generated by the basis $\{x + \mathrm{o}_\gamma \mid x \in A \text{ and } \gamma \in \Gamma\}$, which shall be denoted by $\overset{A}{\mathrm{Valt}}$ or $\overset{v_A}{\mathrm{Valt}}$.

Proposition 1.83. (39, 233, 226) *Let Γ be an ordered set and let A be a Γ-valued commutative group.*

(**a**) *Every ball in A is $\overset{A}{\mathrm{Valt}}$-closed;*

(**b**) *For each $\gamma \in \Gamma$ and each $x \in A$, the set $x + \mathrm{O}_\gamma$ is $\overset{A}{\mathrm{Valt}}$-open;*

(**c**) *If Γ has no greatest element, then $\overset{A}{\mathrm{Valt}}$ is perfect (Definition 1.37);*

(**d**) *The topology $\overset{A}{\mathrm{Valt}}$ is T2, zero-dimensional and totally disconnected.*

Proof. Item (a) is a direct consequence of Item (k) of Proposition 1.81.

(**b**) If y is an element of $x + \mathrm{O}_\gamma$, then (Proposition 1.81, Item (f))

$$y - x + \mathrm{o}_\gamma \subset \mathrm{O}_\gamma + \mathrm{o}_\gamma \subset \mathrm{O}_\gamma + \mathrm{O}_\gamma \subset \mathrm{O}_\gamma\,,$$

which implies $y \in y + \mathrm{o}_\gamma \subset x + \mathrm{O}_\gamma$.

(**c**) Consider a basic $\overset{A}{\mathrm{Valt}}$-open neighbourhood $x + \mathrm{o}_\gamma$ of an element $x \in A$, where $\gamma \in \Gamma$. Since Γ has no greatest element, there is a $\delta \in \Gamma$ so that $\delta > \gamma$, and since v_A is surjective, there is a $y \in A$ so that $v\,(y) = \delta$, which gives us $y \in x + \mathrm{o}_\gamma$ and $y \neq x$. Thus, x is not $\overset{A}{\mathrm{Valt}}$-isolated and the topology $\overset{A}{\mathrm{Valt}}$ is perfect.

(**d**) The topology $\overset{A}{\mathrm{Valt}}$ is zero-dimensional by items (a) and (b). If x and y are two distinct elements of A, then $x \in x + \mathrm{o}_{v(y-x)}$, $y \notin x + \mathrm{o}_{v(y-x)}$, and (Proposition 1.81, Item (k))

$$\left(x + \mathrm{o}_{v(y-x)}\right) \cap \left(y + \mathrm{o}_{v(y-x)}\right) = \emptyset\,,$$

proving that $\overset{A}{\mathrm{Valt}}$ is T2 and totally disconnected (Proposition 1.38).

$\square$

It turns out that the problem of metrizability of valuation topologies on Γ-valued commutative groups depends only on the cofinality of Γ:

Theorem 1.84. *Let Γ be an ordered set and let A be a Γ-valued commutative group. The following conditions are equivalent:*

(**a**) *The topology* $\overset{A}{\mathrm{Valt}}$ *is metrizable;*

(**b**) *The topology* $\overset{A}{\mathrm{Valt}}$ *is ultrametrizable;*

(**c**) $\operatorname{cf}(\Gamma) \leqslant \omega$.

Furthermore, if those conditions hold, then there is an invariant ultrametric ρ on A such that $\overset{A}{\mathrm{Valt}} = \mathsf{t}(\rho)$.

Proof. Item (b) clearly implies (a).

(**a**)⇒(**c**): Suppose the topology $\overset{A}{\mathrm{Valt}}$ is metrizable and let ρ be a metric on A compatible with it. For each $r \in (0,\infty)_{\mathbb{R}}$ there is a $\gamma_r \in \Gamma$ such that $\mathrm{o}_{\gamma_r} \subset \mathsf{B}^{\rho}_{r}(0_A)$. We shall prove that the sequence $\{\gamma_{1/n}\}$ is cofinal in Γ. Let $\gamma \in \Gamma$. Then, there is an $r_0 \in (0,\infty)_{\mathbb{R}}$ such that $\mathsf{B}^{\rho}_{r_0}(0_A) \subset \mathrm{o}_{\gamma}$, and, assigning a natural number n so that $1/n < r_0$, we have
$$\mathrm{o}_{\gamma_{1/n}} \subset \mathsf{B}^{\rho}_{1/n}(0_A) \subset \mathsf{B}^{\rho}_{r_0}(0_A) \subset \mathrm{o}_{\gamma}\,,$$
leading up to $\gamma_{1/n} \geqslant \gamma$.

(**c**)⇒(**b**): Suppose $\operatorname{cf}(\Gamma) \leqslant \omega$, let $\{\gamma_n\}$ be a non-strictly increasing, cofinal sequence in Γ, and, for each distinct $x, y \in A$, let $\langle x, y\rangle \in \mathbb{N}_0$ be the number
$$\langle x, y\rangle := \begin{cases} 0 & \text{if } v_A(x-y) < \gamma_1\,,\\ \max\{n \in \mathbb{N} \mid v_A(x-y) \geqslant \gamma_n\} & \text{otherwise.}\end{cases}$$
Define the function $\rho : A \times A \to [0,\infty)_{\mathbb{R}}$ given by
$$\rho(x,y) := \begin{cases} 0 & \text{if } x = y\,,\\ \mathsf{e}^{-\langle x,y\rangle} & \text{if } x \neq y\,.\end{cases}$$
Thus, $\rho \leqslant 1$, and, for all $x, y \in A$, we have $\rho(x,y) = \rho(y,x)$ and the conditions $\rho(x,y) = 0$ and $x = y$ are equivalent. Take arbitrary distinct elements $x, y, z \in A$. If $\rho(x,z) = 1$ or $\rho(z,y) = 1$, then we have
$$\rho(x,y) \leqslant 1 = \max\{\rho(x,z), \rho(z,y)\}\,.$$

If $\rho(x,z), \rho(z,y) < 1$, then $\langle x,z\rangle, \langle z,y\rangle > 0$ and $v_A(x-z), v_A(z-y) \geqslant \gamma_1$, resulting in

$$\begin{aligned} v_A(x-y) &= v_A(x-z+z-y) \\ &\geqslant \overset{\breve{\Gamma}}{\min}\{v_A(x-z), v_A(z-y)\} \\ &\geqslant \overset{\breve{\Gamma}}{\min}\{\gamma_{\langle x,z\rangle}, \gamma_{\langle z,y\rangle}\} \\ &= \gamma_{\min\{\langle x,z\rangle, \langle z,y\rangle\}}, \end{aligned}$$

and $\langle x,y\rangle \geqslant \min\{\langle x,z\rangle, \langle z,y\rangle\}$, which gives us

$$\rho(x,y) = \mathsf{e}^{-\langle x,y\rangle} \leqslant \exp(-\min\{\langle x,z\rangle, \langle z,y\rangle\}) = \max\{\rho(x,z), \rho(z,y)\}.$$

Hence, the function ρ is an ultrametric on A, and it is straightforward to check that it is invariant on A.

Lastly, we shall show that the identity function on A, $\mathrm{id} : (A, \overset{A}{\mathsf{Valt}}) \to (A, \mathsf{t}(\rho))$, is a homeomorphism. Take a fixed $a \in A$. Note that if $r \in (0,\infty)_{\mathbb{R}}$, if $n \in \mathbb{N}$ is so that $n > \log(r^{-1})$, and if $x \in a + \mathrm{o}_{\gamma_n}$, then we have $v_A(x-a) > \gamma_n$ and $\langle x,a\rangle \geqslant n > \log(r^{-1})$, implying

$$\rho(x,a) = \mathsf{e}^{-\langle x,a\rangle} < \mathsf{e}^{-\log(r^{-1})} = r$$

and $a + \mathrm{o}_{\gamma_n} \subset \mathsf{B}^{\rho}_{r}(a)$. On the other hand, if $\gamma \in \Gamma$, if $n \in \mathbb{N}$ is so that $\gamma \leqslant \gamma_n$, and if $x \in \mathsf{B}^{\rho}_{\mathsf{e}^{-n}}(a)$, then we have $\rho(x,a) = \mathsf{e}^{-\langle x,a\rangle} < \mathsf{e}^{-n}$, $\langle x,a\rangle > n$ and

$$v_A(x-a) \geqslant \gamma_{\langle x,a\rangle} \geqslant \gamma_n \geqslant \gamma,$$

implying $x \in a + \mathrm{o}_\gamma$ and $\mathsf{B}^{\rho}_{\mathsf{e}^{-n}}(a) \subset a + \mathrm{o}_\gamma$. Therefore, the identity function id is a homeomorphism and the ultrametric ρ is compatible with $\overset{A}{\mathsf{Valt}}$.

□

Theorem 1.84 can be generalised to the case of Γ-valued groups that are not necessarily commutative (149).

2 Rngs

The most fundamental operations of Arithmetic are the **addition** and **multiplication** operations, and they have pervaded throughout nearly all mathematical considerations since prehistoric times (197, 42). In fact, all other arithmetic operations, viz. the subtraction, division, exponentiation and n-th root operations, may be defined in terms of those two primal concepts. In the late 19th century and the early 20th century, the essence of the interplay between those two functions was drawn and extensively studied by several mathematicians, culminating in Fraenkel and Noether's definition of a **ring** as it is understood today (75, 76, 167), which is an abstract set endowed with two operations that satisfy the basic laws of Arithmetic. The study of these structures is called **Ring Theory**, and it has far-reaching applications in a wealth of areas, most crucially in Algebraic Number Theory and Algebraic Geometry. In non-algebraic settings, rings are studied in the presence of additional structures, such as orders, topologies, valuations and derivations, where they are meant to be compatible with those structures in some sense.

We will adopt Jacobson's term '**rng**'[1] (105) in this thesis to refer to the structures that are analogous to rings, except for the fact that the existence of an identity element for multiplication is not assumed. The term was derived by dropping the letter 'i', which one may consider to stand for the word '**i**dentity', from the word 'ring'. We shall consider rngs quite often, and the use of that distinguished denomination is meant to avoid the use of the lengthy and ambivalent expressions 'ring without identity' or 'ring with no identity', which can be sporadically found in mathematical works.

[1] It is pronounced rung.

In this chapter, we shall present a compendium of definitions and results concerning rngs, narrowing focus to the topics that will be relevant to our study of Rayner rngs.

2.1 Rngs

We begin with some basic definitions concerning rngs:

Definition 2.1. A **rng** is a set R endowed with two operations on it, $+ = +_R : R \times R \to R$ and $\times = \times_R : R \times R \to R$, the former always denoted additively and the latter always denoted multiplicatively, such that:

(**R0**) $(\forall x, y, z \in R)\ (x+y)\,z = xz + yz$ and $x\,(y+z) = xy + xz\,;$

(**R1**) $(R, +_R)$ is a commutative group;

(**R2**) $(R, \times_R)$ is a semigroup.

That being the case, the operation $+$ is called **the addition of** R and the operation $\times$ is called **the multiplication of** R. We have the following notations and terminology:

- A **subrng of** R is a rng R' such that the group $(R', +_{R'})$ is a subgroup of $(R, +_R)$ and such that the semigroup $(R', \times_{R'})$ is a subsemigroup of $(R, \times_R)$;
- A **homomorphism of type** $R \to S$ between rngs is a homomorphism $\phi : (R, +_R) \to (S, +_S)$ between groups that is also a homomorphism $\phi : (R, \times_R) \to (S, \times_S)$ between semigroups. Every functional composition of two homomorphisms between rngs is a homomorphism. Hence, such functions form a category whose composition operation is the canonical functional composition, and that category is denoted by **Rng**;

- ▷ The **kernel** of a homomorphism $\phi : R \to S$ is the fibre $\phi^{-1}\langle\{0_S\}\rangle$, which is denoted by $\operatorname{Ker}(\phi)$. That fibre is a subrng of R and it might be denoted by $\overset{\mathbf{Rng}}{\operatorname{Ker}}(\phi)$ to emphasise its rng structure;

- ▷ Let I be a set. The **support** of a family $x = \{x_i\}_{i\in I} \in {}^{I}R$ is the subset of I denoted by $\operatorname{supp}(x)$ and given by

$$\operatorname{supp}(x) \coloneqq \{i \in I \mid x_i \neq 0_R\};$$

- ▷ The **image** of a homomorphism $\phi : R \to S$ between rngs is the subrng of S whose underlying set is the image $\operatorname{Im}(\phi)$ of the function ϕ. That image might be denoted by $\overset{\mathbf{Rng}}{\operatorname{Im}}(\phi)$ to emphasise its rng structure;

- ▷ The **characteristic of** R is the characteristic of the underlying additive monoid $(R, +_R)$ (Definition 1.5), and it is denoted by $\operatorname{Char}(R)$;

- ▷ The **centre of** R is the centre of the semigroup $(R, \times_R)$, and it is denoted by $\mathrm{Z}(R)$;

- ▷ A rng is **trivial** if 0_R is its only element. Otherwise, it is **non-trivial**;

- ▷ A rng R **has no zero divisors** if for all $x, y \in R$ the condition $xy = 0_R$ implies $x = 0_R$ or $y = 0_R$.

Example 2.2. For each natural number n, the number set $n\mathbb{Z}$, endowed with its usual addition and multiplication operations, is a commutative rng of characteristic zero and with no zero divisors. If m and n are natural numbers so that m divides n, then $n\mathbb{Z}$ is a subrng of $m\mathbb{Z}$.

Example 2.3. Let n be a natural number, let R be a rng, let $\mathrm{M}_n(R)$ be the set of $n \times n$ matrices with entries in R and let $\mathrm{T}_n(R)$ be the set of $n \times n$

strictly-upper triangular matrices with entries in R. Thus, the elements of $\mathrm{M}_n(R)$ and $\mathrm{T}_n(R)$ are of the forms

$$\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & \cdots & \cdots & r_{nn} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0_R & r_{12} & \cdots & r_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & r_{n-1,n} \\ & & & 0_R \end{pmatrix} ,$$

respectively, where $r_{ij} \in R$ ($\forall i, j \in [1, n]_{\mathbb{N}}$) and where all empty spaces within the second matrix represent the zero element of R. Both $\mathrm{M}_n(R)$ and $\mathrm{T}_n(R)$ are rngs of characteristic $\mathrm{Char}\,(R)$ when endowed with the usual matrix addition and matrix multiplication operations. They are not commutative in general, even when the rng R is commutative, and they have zero divisors if $n \geqslant 2$, for if we take an $r \in R - \{0_R\}$, then we get

$$\begin{pmatrix} & & 0_R & r \\ & & \iddots & 0_R \\ & & & \\ \end{pmatrix} \mathrm{T}_n(R) = \{\mathbf{0}\} ,$$

where $\mathbf{0}$ is the zero matrix.

Example 2.4. Every commutative group G may be seen as a rng, provided that one considers the **trivial multiplication operation on** G, given by $xy := 0_G$ ($\forall x, y \in G$). That rng is not ring when G is non-trivial. If R is a rng, then the multiplication operation of the matrix rng $\mathrm{T}_2(R)$ (Example 2.3) is trivial.

Most rngs that are of importance contain identity elements for their multiplication operations:

Definition 2.5. A **ring** is a rng such that the semigroup $(R, \times_R)$ is a monoid. We have the following notations and terminology:

- ▷ The **multiplicative identity (element) of** R is the identity element of the monoid $(R, \times_R)$, and it is denoted by 1_R;
- ▷ A **subring** of a rng S is a subrng of S that is a ring. It is worth noting that a subrng of a rng can be a ring (Example 2.10), and a subring of a ring may contain a multiplicative identity element distinct from that of the larger ring (Example 2.11);
- ▷ A homomorphism $\phi : R \to S$ between rings is **unital** if $\phi(1_R) = 1_S$. Every functional composition of two unital homomorphisms between rings is a unital homomorphism. Hence, such functions form a category whose composition operation is the canonical functional composition, and that category is denoted by **Ring**;
- ▷ An element x of a ring R is a **unit in** R if it has a multiplicative inverse in R, that is, if there is a $y \in R$ so that $xy = 1_R = yx$. If x is a unit in R, then its multiplicative inverse is unique and is denoted by x^{-1};
- ▷ A **division ring** is a non-trivial ring K such that every non-zero element of K is a unit. That being so, one notices that $(K - \{0_K\}, \times_K \restriction_{K-\{0_K\}})$ is a group and K has no zero divisors;
- ▷ A **field** is a commutative division ring.

Example 2.6. Given a natural number n, the rng $n\mathbb{Z}$ (Example 2.2) is a ring if, and only if, $n = 1$. In that case, the number 1 is the multiplicative identity element of $1\mathbb{Z} = \mathbb{Z}$, and $\mathbb{Z}$ is not a field. For each $m \in [2, \infty)_{\mathbb{N}}$, the rng $m\mathbb{Z}$ is a subrng of $\mathbb{Z}$ that is not a ring.

Example 2.7. The number sets $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields of characteristic zero when endowed with their usual addition and multiplication operations.

Example 2.8. The **system of quaternions**, denoted by $\mathbf{H}$, is a non-commutative division ring first described by Hamilton in 1843 (93).

Example 2.9. Let n be a natural number, let R be a ring and consider the matrix rngs introduced in Example 2.3. The rng $\mathrm{T}_n(R)$ is not a ring, and the rng $\mathrm{M}_n(R)$ is a ring whose multiplicative identity is the so-called **identity matrix**:

$$\mathrm{I}_n := \begin{pmatrix} 1_R & & & \\ & 1_R & & \\ & & \ddots & \\ & & & 1_R \end{pmatrix}.$$

Example 2.10. Let R and S be the following subrngs of $\mathrm{M}_2(\mathbb{R})$:

$$R := \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x \in \mathbb{R} \right\} \quad \text{and} \quad S := \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in \mathbb{R} \right\}.$$

Thus, R is a subring of S with identity element $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, but it is easy to check that S is not a ring. In fact, R is isomorphic to the field $\mathbb{R}$ of real numbers.

Example 2.11. (228) The set

$$T := \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

forms a subring of the ring $\mathrm{M}_2(\mathbb{R})$, and its multiplicative identity element is the matrix $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, which is distinct from the 2×2 identity matrix $\mathrm{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Example 2.12. The set $\mathtt{C}$ of matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for $a, b \in \mathbb{R}$ forms a subfield of the 2×2 matrix ring $\mathrm{M}_2(\mathbb{R})$ that is isomorphic to $\mathbb{C}$ via the isomorphism $a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Thus, the multiplicative identity of $\mathtt{C}$ is the identity matrix I_2.

Proposition 2.13. (56, 103, 83)

(a) *If R is a rng, then we have*

$$(\forall x \in R)\ 0_R \cdot x = x \cdot 0_R = 0_R \ \text{ and } \quad (\forall x, y \in R)\ -(xy) = (-x)\, y = x\,(-y)\ ;$$

(b) *A ring R is trivial if, and only if, $0_R = 1_R$;*

(c) *If $\phi : R \to S$ is a homomorphism between rngs, then $\phi\,(0_R) = 0_S$;*

(d) *If $\phi : K \to L$ is a homomorphism between division rings, then ϕ is unital, that is, $\phi\,(1_K) = 1_L$.*

Proof. Items (c) and (d) are direct consequences of Item (a) of Proposition 1.9.

(a) For all $x \in R$, we have
$$x \cdot 0_R = x\ (0_R + 0_R) = x \cdot 0_R + x \cdot 0_R\,,$$
which gives us
$$0_R = x \cdot 0_R - x \cdot 0_R = x \cdot 0_R + x \cdot 0_R - x \cdot 0_R = x \cdot 0_R\,.$$
The proof of $0_R \cdot x = 0_R$ is analogous.

(b) If R is a trivial ring, then $1_R \in R = \{0_R\}$ and $0_R = 1_R$. Conversely, if $0_R = 1_R$, then
$$(\forall x \in R)\ \ x = x \cdot 1_R = x \cdot 0_R = 0_R$$
by item (a), which immediately gives us $R = \{0_R\}$.

□

We have defined that 0_R and 1_R are the respective identity elements of the addition and multiplication operations of a ring R. We shall extend those notations, providing meaning to the term x_R when x is an integer and providing meaning to x_K when x is a rational number and K is a division ring.

Definition 2.14. Let R be a ring.

▷ For each $n \in \mathbb{N}$, we define
$$n_R \coloneqq \overbrace{1_R + 1_R + \cdots + 1_R}^{n \text{ times}} = n1_R\,;$$

▷ For each $n \in (\leftarrow, 0)_{\mathbb{Z}}$, we define $n_R \coloneqq -(-n)_R$;

▷ For each subset S of $\mathbb{Z}$, we define $S_R \coloneqq \{n_R \mid n \in S\}$.

Let K be a division ring.

▷ For all $p, q \in \mathbb{Z}$ so that $q_K \neq 0_K$, we define $({}^p\!/\!_q)_K \coloneqq p_K q_K^{-1}$. Note that if $r \in \mathbb{Z} - \{0\}$, then the term $({}^{pr}\!/\!_{qr})_K$ may not be well-defined, for we may have $(qr)_K = 0_K$. Thus, one may not switch between equivalent fractions in the notation $({}^p\!/\!_q)_K$;

▷ For each subset S of $\mathbb{Q}$, we denote by S_K the set
$$S_K \coloneqq \left\{({}^p\!/\!_q)_K \mid p, q \in \mathbb{Z}, q_K \neq 0_K \text{ and } {}^p\!/\!_q \in S\right\}.$$

Example 2.15. Let n be a natural number and let S be a subset of $\mathbb{Z}$. The set $S_{\mathrm{M}_n(\mathbb{R})}$ is given by
$$S_{\mathrm{M}_n(\mathbb{R})} = \left\{\begin{pmatrix} k & & & \\ & k & & \\ & & \ddots & \\ & & & k \end{pmatrix} \mid k \in S\right\}.$$

Example 2.16. Let S be a subset of $\mathbb{Q}$ and consider the field $\mathtt{C}$ (Example 2.12). The set $S_{\mathtt{C}}$ is given by
$$S_{\mathrm{M}_n(\mathbb{R})} = \left\{\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \mid q \in S\right\}.$$

Proposition 2.17. (34, 83) *Let R be a non-trivial ring and let K be a division ring.*

(a) $\mathrm{Z}(R)$ *is a commutative subring of R;*

(b) $\mathbb{Z}_R$ *is a subring of R contained in $\mathrm{Z}(R)$;*

(c) $\mathbb{Q}_K$ *is a subfield of K contained in $\mathrm{Z}(K)$.*

Proof.

(a) We know that $\mathrm{Z}(R)$ is a subsemigroup of $(R, \times_R)$ (Proposition 1.4). If $x, y \in \mathrm{Z}(R)$, then for all $z \in R$, we get (Proposition 2.13, Item (a))

$$(-x)\,z = -(xz) = -(zx) = z\,(-x)$$

and

$$(x+y)\,z = xz + yz = zx + zy = z\,(x+y)\ ,$$

implying $-x, x+y \in \mathrm{Z}(R)$ and proving the item.

(b) Firstly, 1_R is clearly the multiplicative identity of $\mathbb{Z}_R$, and if $n \in \mathbb{Z}$, then

$$-n_R = -\left(-(-n)_R\right) = (-n)_R \in \mathbb{Z}_R\,.$$

Considering two numbers $m, n \in \mathbb{Z} - \{0\}$ so that $m \leqslant n$, then we have three cases:

1. Case $m, n > 0$: Since $+_R$ is associative on R, we get

$$m_R + n_R = \Big(\overbrace{1_R + \cdots + 1_R}^{m \text{ times}}\Big) + \Big(\overbrace{1_R + \cdots + 1_R}^{n \text{ times}}\Big) = (m+n)_R \in \mathbb{Z}_R\,,$$

and since $\times_R$ is left-distributive over $+_R$, we get

$$m_R \cdot n_R = m_R \cdot \Big(\overbrace{1_R + \cdots + 1_R}^{n \text{ times}}\Big) = \overbrace{m_R + \cdots + m_R}^{n \text{ times}} = (mn)_R \in \mathbb{Z}_R\,.$$

2. Case $m < 0 < n$: We get

$$m_R + n_R = -\Big(\overbrace{1_R + \cdots + 1_R}^{-m \text{ times}}\Big) + \Big(\overbrace{1_R + \cdots + 1_R}^{n \text{ times}}\Big) = (n - (-m))_R \in \mathbb{Z}_R\,,$$

and, by case 1, we get

$$m_R \cdot n_R = -(-m)_R \cdot n_R = -((-m)\,n)_R = -(-mn)_R = (mn)_R \in \mathbb{Z}_R\,.$$

3. Case $m, n < 0$: By case 1, we get

$$m_R + n_R = -((-m)_R + (-n)_R) = -(-(m+n))_R = (m+n)_R \in \mathbb{Z}_R\,,$$

and, by case 1 and by Item (a) of Proposition 2.13, we get

$$m_R \cdot n_R = (-(-m)_R)\,(-(-n)_R) = (-m)_R\,(-n)_R = ((-m)\,(-n))_R = (mn)_R \in \mathbb{Z}_R\,.$$

Therefore, we have shown that

$$m_R + n_R = (m+n)_R \in \mathbb{Z}_R \quad\text{and}\quad m_R \cdot n_R = (mn)_R \in \mathbb{Z}_R$$

for all $m, n \in \mathbb{Z}$, proving that $\mathbb{Z}_R$ is a subring of R. Lastly, if $n \in \mathbb{N}$ and if x is an element of R, then, drawing from the fact that $\times_R$ is left and right-distributive over $+_R$, we obtain

$$n_R \cdot x = \Big(\overbrace{1_R + \cdots + 1_R}^{n \text{ times}}\Big)\, x = \overbrace{x_R + \cdots + x_R}^{n \text{ times}} = x\,\Big(\overbrace{1_R + \cdots + 1_R}^{n \text{ times}}\Big) = x \cdot n_R$$

and

$$(-n)_R\, x = [-(-(-n))_R]\, x = -n_R \cdot x = -x \cdot n_R = x\,[-(-(-n))_R] = x\,(-n)_R\,,$$

proving that $\mathbb{Z}_R \subset \mathrm{Z}(R)$.

(c) Firstly, we show that the set $\mathbb{Q}_K$ is closed under $\times_K$, leaving the (tedious but rather straightforward) proof that it is closed under $+_K$ to the reader. Consider any two elements $({}^{p}\!/{}_{q})_K$ and $({}^{p'}\!/{}_{q'})_K$ of $\mathbb{Q}_K$. Since $q_K \neq 0_K \neq q'_K$, we have $(qq')_K = q_K q'_K \neq 0_K$ and, by item (b), we obtain

$$({}^{p}\!/{}_{q})_K\,({}^{p'}\!/{}_{q'})_K = p_K q_K^{-1} p'_K q_K'^{-1} = p_K p'_K q_K^{-1} q_K'^{-1} = (pp')_K\,(qq')_K^{-1} = ({}^{(pp')}\!/{}_{(qq')})_K \in \mathbb{Q}_K\,.$$

Thus, $\mathbb{Q}_K$ is a subring of K. Furthermore, note that if $({}^{p}/_{q})_K \neq 0_K$, then $p_K, q_K \neq 0_K$ and

$$({}^{p}/_{q})_K \, ({}^{q}/_{p})_K = p_K q_K^{-1} q_K p_K^{-1} = 1_K \,,$$

that is, $({}^{p}/_{q})_K^{-1} = ({}^{q}/_{p})_K \in \mathbb{Q}_K$, showing that $\mathbb{Q}_K$ is a division ring.

We show that $\mathbb{Q}_K \subset \mathrm{Z}\,(K)$. Let $({}^{p}/_{q})_K$ be an element of $\mathbb{Q}_K$ and let x be any element of K. By item (b), we have

$$q_K^{-1} x = q_K^{-1} x q_K q_K^{-1} = q_K^{-1} q_K x q_K^{-1} = x q_K^{-1}$$

and

$$({}^{p}/_{q})_K \, x = p_K q_K^{-1} x = p_K x q_K^{-1} = x p_K q_K^{-1} = x \, ({}^{p}/_{q})_K \,,$$

proving that $({}^{p}/_{q})_K \in \mathrm{Z}\,(K)$.

□

2.2 Quotients in **Rng**

One may effortlessly notice that the function of type **Rng** → **Mon** (Example B.17) that associates each rng R to its underlying additive monoid $(R, +_R)$ and associates each homomorphism between rngs to itself is a faithful functor (Definition B.25). Thus, the category **Rng** is a **Mon**-concrete category (Definition B.42) when endowed with that function. Moreover, since **Mon** is a **Set**-concrete category (Section 1.2), it is clear that **Rng** may also be seen as a **Set**-concrete category.

In this section, we shall appreciate how quotients are produced in the category **Rng** (Definition B.46).

Definition 2.18. Let R be a rng. A **congruence relation on** R is an equivalence relation $\equiv$ on R such that for all $x, x', y, y' \in R$ so that $x \equiv x'$ and $y \equiv y'$, we have $x + y \equiv x' + y'$ and $xy \equiv x'y'$.

Proposition 2.19. (12, 102) *Let R be a rng. If $\equiv$ is a congruence relation on R, then the quotient $R/\equiv$ in* **Set** *is a rng when endowed with the addition and multiplication operations on it given by*

$$(x/\equiv) + (y/\equiv) \coloneqq (x+y)/\equiv \quad \text{and} \quad (x/\equiv)(y/\equiv) \coloneqq (xy)/\equiv \ .$$

That rng is a quotient of R modulo $\equiv$ in **Rng** *(Definition B.46) whose quotient morphism is the canonical function $\sigma : R \xrightarrow{\mathbf{Set}} R/\equiv$.*

Proof. We leave the verification that those operations are well-defined to the reader, as well as the fact that $R/\equiv$ is a rng. Let $\sigma : R \to R/\equiv$ be the canonical quotient function of that type in **Set** (Example B.48), which is clearly a homomorphism between rngs. Thus, we have $\equiv = \overset{\sigma}{\mathrm{eq}}$ (Definition B.45). Consider any rng S and any homomorphism $f : R \to S$ so that $\equiv \subset \overset{f}{\mathrm{eq}}$, and let $\overline{f} : R/\equiv \to S$ be the quotient lowering of f in **Set**, which is given by $\overline{f}(x/\equiv) \coloneqq f(x)$. If x and y are elements of R, then

$$\overline{f}((x/\equiv) + (y/\equiv)) = \overline{f}((x+y)/\equiv) = f(x+y) = f(x) + f(y) = \overline{f}(x/\equiv) + \overline{f}(y/\equiv)$$

and

$$\overline{f}((x/\equiv)(y/\equiv)) = \overline{f}((xy)/\equiv) = f(xy) = f(x)f(y) = \overline{f}(x/\equiv)\overline{f}(y/\equiv) \ ,$$

proving that $\overline{f}$ is a homomorphism. The uniqueness of $\overline{f}$ follows from the universal property of quotients in **Set**. Therefore, we have proved that σ is a quotient morphism in **Rng** associated to the quotient $R/\equiv$.

□

A congruence relation on a rng R is usually indirectly specified by a subset of R that satisfies a few requirements:

Definition 2.20. Let R be a rng. A **left** (resp. **right**) **ideal in** R is a subgroup I of $(R, +_R)$ such that the inclusion $RI \subset I$ (resp. $IR \subset I$) holds. Note that the left (resp. right) ideals in R also happen to be subrngs of R. An **ideal in** R is a left and right ideal in R.

The following proposition shows that there is a one-to-one correspondence between the ideals in R and the congruence relations on R:

Proposition 2.21. *Let R be a rng. A subset I of R is an ideal in R if, and only if, the binary relation*

$$\underset{I}{\equiv} := \{(x, y) \in R \times R \mid y - x \in I\}$$

is a congruence relation on R. In that case, we have $I = 0_R/\underset{I}{\equiv}$.

Proof. Suppose $\underset{I}{\equiv}$ is a congruence relation on R. Hence, if i and j are elements of I and if r is an element of R, then $i, j \underset{I}{\equiv} 0_R$, $r \underset{I}{\equiv} r$ and we get

$$i \pm j \underset{I}{\equiv} 0_R \pm 0_R = 0_R\,, \ \ ri \underset{I}{\equiv} r \cdot 0_R = 0_R \ \text{ and } \ \ ir \underset{I}{\equiv} 0_R \cdot r = 0_R\,,$$

that is, $i \pm j \in I$ and $ri, ir \in I$, proving that I is an ideal in R.

Conversely, suppose the set I is an ideal in R. Thus, since $0_R \in I$, we have $x \underset{I}{\equiv} x$ $(\forall x \in R)$, and if x, y and z are three elements of R so that $x \underset{I}{\equiv} y \underset{I}{\equiv} z$, then $y - x, z - y \in I$ and

$$z - x = (z - y) + (y - x) \in I + I \subset I$$

that is, $x \underset{I}{\equiv} z$, proving that $\underset{I}{\equiv}$ is an equivalence relation on R. Moreover, if x, x', y and y' are four elements of R so that $x \underset{I}{\equiv} x'$ and $y \underset{I}{\equiv} y'$, then

$$(x' + y') - (x + y) = (x' - x) + (y' - y) \in I + I \subset I$$

and

$$\begin{aligned} x'y' - xy &= x'y' - (xy' + x'y - xy) - xy + (xy' + x'y - xy) \\ &= (x' - x)\,(y' - y) + x\,(y' - y) + (x' - x)\,y \\ &\in II + RI + IR \subset I \end{aligned}$$

that is, $x + y \underset{I}{\equiv} x' + y'$ and $xy \underset{I}{\equiv} x'y'$. Therefore, $\underset{I}{\equiv}$ is a congruence relation on R. $\square$

Example 2.22. If m and n are natural numbers so that m divides n, then $n\mathbb{Z}$ is an ideal in the commutative rng $m\mathbb{Z}$, and the congruence relation $\underset{n\mathbb{Z}}{\equiv}$ is the usual relation of **congruence modulo** n on $m\mathbb{Z}$ (cf. Example 1.12), that is, $\underset{n\mathbb{Z}}{\equiv}$ is the binary relation on $m\mathbb{Z}$ defined so that for all $x, y \in m\mathbb{Z}$, the condition $x \underset{n\mathbb{Z}}{\equiv} y$ means that n divides $x - y$. We shall examine three particular cases:

(**a**) Let p be a prime number and consider the case where $m = 1$ and $n = p$. The quotient rng $\mathbb{Z}/p\mathbb{Z}$ has p elements:

$$\mathbb{Z}/p\mathbb{Z} = \left\{0/\underset{p\mathbb{Z}}{\equiv}, 1/\underset{p\mathbb{Z}}{\equiv}, \dots, (p-1)/\underset{p\mathbb{Z}}{\equiv}\right\} ,$$

where $1/\underset{p\mathbb{Z}}{\equiv}$ is the multiplicative identity element of $\mathbb{Z}/p\mathbb{Z}$. Given that, we shall show that it is a field. If $x \in [1, p-1]_{\mathbb{Z}}$, then $\gcd(x, p) = 1$ and there are $y, q \in \mathbb{Z}$ so that $xy + pq = 1$, implying $xy \equiv 1 \mod p$, that is, $\left(x/\underset{p\mathbb{Z}}{\equiv}\right)\left(y/\underset{p\mathbb{Z}}{\equiv}\right) = \left(1/\underset{p\mathbb{Z}}{\equiv}\right)$, proving that the class $y/\underset{p\mathbb{Z}}{\equiv}$ is the multiplicative inverse of $x/\underset{p\mathbb{Z}}{\equiv}$ in $\mathbb{Z}/p\mathbb{Z}$. Thus, $\mathbb{Z}/p\mathbb{Z}$ is a field, and it is usually denoted by F_p.

(**b**) Consider the case $m = 2$ and $n = 4$. The quotient $2\mathbb{Z}/4\mathbb{Z}$ has two elements:

$$2\mathbb{Z}/4\mathbb{Z} = \left\{0/\underset{4\mathbb{Z}}{\equiv}, 2/\underset{4\mathbb{Z}}{\equiv}\right\} = \{\{\dots, -4, 0, 4, \dots\}, \{\dots, -2, 2, 6, \dots\}\} .$$

Let $A := 0/\underset{4\mathbb{Z}}{\equiv}$ and $B := 2/\underset{4\mathbb{Z}}{\equiv}$. Thus, we have

$$\begin{aligned} &A + A = A; \quad B + A = B; \qquad AA = A; \quad BA = A; \\ &A + B = B; \quad B + B = A; \qquad AB = A; \quad BB = A. \end{aligned}$$

It turns out that $2\mathbb{Z}/4\mathbb{Z}$ is isomorphic to the finite commutative group $\mathbb{Z}/2\mathbb{Z}$ (Example 1.15) endowed with its trivial multiplication operation (Example 2.4). In particular, $2\mathbb{Z}/4\mathbb{Z}$ is not a ring.

(**c**) Consider the case $m = 2$ and $n = 6$. Now, the quotient $2\mathbb{Z}/6\mathbb{Z}$ has three elements:

$$\begin{aligned} 2\mathbb{Z}/6\mathbb{Z} &= \left\{0/\underset{6\mathbb{Z}}{\equiv}, 2/\underset{6\mathbb{Z}}{\equiv}, 4/\underset{6\mathbb{Z}}{\equiv}\right\} \\ &= \{\{\dots, -6, 0, 6, \dots\}, \{\dots, -4, 2, 8, \dots\}, \{\dots, -2, 4, 10, \dots\}\} . \end{aligned}$$

Let $A \coloneqq 0/\underset{6\mathbb{Z}}{\equiv}$, $B \coloneqq 2/\underset{6\mathbb{Z}}{\equiv}$ and $C \coloneqq 4/\underset{6\mathbb{Z}}{\equiv}$. Thus, we have

$$\begin{array}{llllll} A+A=A\,; & B+A=B\,; & C+A=C\,; & AA=A\,; & BA=A\,; & CA=A\,; \\ A+B=B\,; & B+B=C\,; & C+B=A\,; & AB=A\,; & BB=C\,; & CB=B\,; \\ A+C=C\,; & B+C=A\,; & C+C=B\,; & AC=A\,; & BC=B\,; & CC=C\,. \end{array}$$

It turns out that C is a multiplicative identity element of the quotient rng $2\mathbb{Z}/6\mathbb{Z}$, and the function $f : 2\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ given by

$$f(A) \coloneqq 0/\underset{3\mathbb{Z}}{\equiv}, f(B) \coloneqq 2/\underset{3\mathbb{Z}}{\equiv} \quad \text{and} \quad f(C) \coloneqq 1/\underset{3\mathbb{Z}}{\equiv}$$

is an isomorphism between rings, implying that $2\mathbb{Z}/6\mathbb{Z}$ is a field by case (a).

The lemma below will come in handy in Section 3.6:

Lemma 2.23. *The canonical forgetful functor* $\overset{\mathbf{Rng}}{\mathrm{U}} : \mathbf{Rng} \to \mathbf{Mon}$ *of the* **Mon***-concrete category* **Rng** *sends quotients modulo ideals in* **Rng** *to quotients modulo congruence relations in* **Mon** *(Definition B.50).*

Proof. Let R be a rng, let I be an ideal in R whose associated congruence relation on R is denoted by $\underset{I}{\equiv}$ (Proposition 2.21), suppose Q is a quotient of R modulo I in **Rng** with quotient morphism $\iota : R \xrightarrow{\mathbf{Rng}} Q$ and let $\sigma : R \xrightarrow{\mathbf{Rng}} R/I$ be the canonical quotient morphism of that type (Proposition 2.19). Since quotients are unique up to unique isomorphism, there is a unique isomorphism $\alpha : Q \xrightarrow{\mathbf{Rng}} R/I$ such that the digraph

$$\begin{tikzcd} & Q \arrow[dd, "\alpha"] \\ R \arrow[ur, "\iota"] \arrow[dr, "\sigma"'] & \\ & R/I \end{tikzcd}$$

in **Rng** commutes. Note that $\underset{I}{\equiv}$ is a congruence relation on the monoid U^R, while $\mathrm{U}^{R/I}$ is precisely the canonical construction of the quotient of U^R modulo

$\underset{I}{\equiv}$ in **Mon** with quotient morphism $\overset{\mathbf{Rng}}{\mathrm{U}}(\sigma) : \mathrm{U}^R \overset{\mathbf{Mon}}{\longrightarrow} \mathrm{U}^{R/I}$ (cf. the proof of Proposition 1.14). Finally, since

$$(\forall i \in \boldsymbol{I}_0) \quad \overset{\mathbf{Rng}}{\mathrm{U}}(\sigma) = \overset{\mathbf{Rng}}{\mathrm{U}}(\alpha \circ \iota) = \overset{\mathbf{Rng}}{\mathrm{U}}(\alpha) \circ \overset{\mathbf{Rng}}{\mathrm{U}}(\iota)$$

and since the morphism $\overset{\mathbf{Rng}}{\mathrm{U}}(\alpha) : \mathrm{U}^Q \overset{\mathbf{Mon}}{\longrightarrow} \mathrm{U}^{R/I}$ is an isomorphism between monoids (Proposition B.30), we have that $\overset{\mathbf{Rng}}{\mathrm{U}}(\iota) : \mathrm{U}^R \overset{\mathbf{Mon}}{\longrightarrow} \mathrm{U}^Q$ is a quotient morphism of U^R modulo $\underset{I}{\equiv}$ in **Mon** (Remark B.47).

□

2.3 Limits in **Rng**

Rngs give rise to other rngs via the universal property encoded by the categorical notion of limit (Definition B.31).

Proposition 2.24. (2) *The categories* **Rng** *and* **Ring** *are complete.*

Proof. For each of these categories, the argument here is analogous to the proof of Proposition 1.16, only now we are dealing with two operations on each object of the category.

□

We shall see how the most notable limits take form in **Rng**.

Example 2.25. The simplest example of a limit in **Rng** is its terminal object (Definition B.37), which happens to be the trivial rng $\{0\}$, since for every rng R there is exactly one homomorphism of type $R \to \{0\}$. It is easy to check that $\{0\}$ is also the initial object in **Rng**, which is why the rng $\{0\}$ is usually denoted by $\mathbf{0}$.

Example 2.26. (2) Let $\{R_i\}_{i\in I}$ be a family of rngs. The Cartesian product of sets $\prod_{i\in I} R_i$ is a product of $\{R_i\}_{i\in I}$ in **Rng** (Definition B.38) when endowed with the addition and multiplication operations on $\prod_{i\in I} R_i$ given by

$$\{r_i\}_{i\in I} + \{s_i\}_{i\in I} := \{r_i + s_i\}_{i\in I} \quad \text{and} \quad \{r_i\}_{i\in I}\{s_i\}_{i\in I} := \{r_i s_i\}_{i\in I} \ ,$$

where the projections $\chi_j : \overset{\mathbf{Rng}}{\prod_{i\in I}} R_i \to R_j$ are the usual product projections given by $\chi_j\left(\{r_i\}_{i\in I}\right) := r_j$ for each $j \in I$. Thus, the rng $\prod_{i\in I} R_i$ might be denoted by $\overset{\mathbf{Rng}}{\prod_{i\in I}} R_i$.

Example 2.27. (2) Let $f, g : R \to S$ be two homomorphisms between rngs. The kernel $\operatorname{Ker}(f-g)$ (Definition 2.1) is an equaliser of f and g in **Rng** (Definition B.40), where the equaliser morphism $\operatorname{eq}(f,g) : \operatorname{Ker}(f-g) \xrightarrow{\mathbf{Rng}} R$ is the canonical inclusion between sets. Thus, the rng $\operatorname{Ker}(f-g)$ might be denoted by $\overset{\mathbf{Rng}}{\operatorname{Eq}}(f,g)$.

Example 2.28. (2) Let $f : R \to T$ and $g : S \to T$ be two homomorphisms between rngs. The subset

$$P := \{(x,y) \in R \times S \mid f(x) = g(y)\}$$

of the product rng $R \times S$ (Example 2.26) forms a subrng of $R \times S$ and it is a pullback of f and g in **Rng** (Definition B.41), where the pullback morphisms $\overline{f} : P \xrightarrow{\mathbf{Rng}} S$ and $\overline{g} : P \xrightarrow{\mathbf{Rng}} R$ are given by $\overline{f}(x,y) := y$ and $\overline{g}(x,y) := x$. In other words, the rng P is a fibred product of R and S with respect to f and g in **Rng**, and it might be denoted by $R \overset{\mathbf{Rng}}{\underset{f,g}{\times}} S$.

The following lemma will be useful in Section 3.6:

Lemma 2.29. *The canonical forgetful functor* $\overset{\mathbf{Rng}}{\mathrm{U}} : \mathbf{Rng} \to \mathbf{Mon}$ *of the* **Mon**-*concrete category* **Rng** *sends limits in* **Rng** *to limits in* **Mon** *(Definition B.36).*

Proof. Let $\mathcal{F} : \boldsymbol{I} \to \mathbf{Rng}$ be a functor, suppose $\chi = \{\chi_i : L \xrightarrow{\mathbf{Rng}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ is a limit cone over $\mathcal{F}$ with vertex L, and let $\sigma = \{\sigma_i : S \xrightarrow{\mathbf{Rng}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ be the canonical construction of the limit cone over $\mathcal{F}$ with vertex S (Theorem 2.24). Since limits are unique up to unique isomorphism, there is a unique isomorphism $\alpha : L \xrightarrow{\mathbf{Rng}} S$ such that the digraph

$$\begin{tikzcd} L \arrow[dr, "\chi_i"] \arrow[dd, "\alpha"'] & \\ & \mathcal{F}(i) \\ S \arrow[ur, "\sigma_i"'] & \end{tikzcd}$$

in $\mathbf{Rng}$ commutes for all $i \in \boldsymbol{I}_0$. Note that the cone

$$\overset{\mathbf{Rng}}{\mathrm{U}}(\sigma) := \left\{ \overset{\mathbf{Rng}}{\mathrm{U}}(\sigma_i) : \mathrm{U}^S \xrightarrow{\mathbf{Mon}} \mathrm{U}^{\mathcal{F}(i)} \right\}_{i \in \boldsymbol{I}_0}$$

is precisely the canonical construction of the limit cone of $\overset{\mathbf{Rng}}{\mathrm{U}} \circ \mathcal{F} : \boldsymbol{I} \to \mathbf{Mon}$ (cf. the proof of Proposition 1.16). Finally, since

$$(\forall i \in \boldsymbol{I}_0) \quad \overset{\mathbf{Rng}}{\mathrm{U}}(\chi_i) = \overset{\mathbf{Rng}}{\mathrm{U}}(\sigma_i \circ \alpha) = \overset{\mathbf{Rng}}{\mathrm{U}}(\sigma_i) \circ \overset{\mathbf{Rng}}{\mathrm{U}}(\alpha)$$

and since $\overset{\mathbf{Rng}}{\mathrm{U}}(\alpha) : \mathrm{U}^L \xrightarrow{\mathbf{Mon}} \mathrm{U}^S$ is an isomorphism (Proposition B.30), the cone

$$\overset{\mathbf{Rng}}{\mathrm{U}}(\chi) = \left\{ \overset{\mathbf{Rng}}{\mathrm{U}}(\chi_i) : \mathrm{U}^L \xrightarrow{\mathbf{Mon}} \mathrm{U}^{\mathcal{F}(i)} \right\}_{i \in \boldsymbol{I}_0}$$

is a limit cone over $\overset{\mathbf{Rng}}{\mathrm{U}} \circ \mathcal{F} : \boldsymbol{I} \to \mathbf{Mon}$ with vertex U^L (Remark B.32).

□

2.4 Topological rngs

The operations of a rng may be appointed as continuous functions when the universe set of the rng is endowed with a topology on it.

Definition 2.30. A **topological rng** R is a rng endowed with a topology τ on it such that $(R, +_R, \tau)$ is a topological group and $(R, \times_R, \tau)$ is a topological semigroup. That amounts to saying that the operations

$$+_R : (R, \tau) \times (R, \tau) \to (R, \tau) \ , \ \overset{+_R}{\mathrm{Inv}} : (R, \tau) \to (R, \tau) \quad \text{and} \quad \times_M : (R, \tau) \times (R, \tau) \to (R, \tau)$$

are continuous. Whenever no particular notation is ascribed in the context to the topology of R, that shall be denoted by τ_R. We have the following terminology:

- ▷ A **topological subrng of** R is a topological rng R' such that the rng $(R', +_{R'}, \times_{R'})$ is a subrng of $(R, +_R, \times_R)$ and such that the topological space $(R', \tau_{R'})$ is a subspace of (R, τ_R);
- ▷ A **topological ring** is a topological rng whose underlying rng is a ring;
- ▷ A **topological division ring** is a topological ring R whose underlying ring is a division ring and whose multiplicative inversion operation

  $$\overset{\times_R}{\mathrm{Inv}} : (R - \{0_R\}, \tau \upharpoonright (R - \{0_R\})) \to (R, - \{0_R\}, \tau \upharpoonright (R - \{0_R\}))$$

  is continuous;
- ▷ A **topological field** is a topological division ring whose underlying ring is a field.

Example 2.31. The number fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are topological fields when endowed with their usual topologies.

Example 2.32. Let $\{R_i\}_{i\in I}$ be a family of topological rngs and let $\overset{\mathbf{Rng}}{\prod_{i\in I}} R_i$ be its product rng (Example 2.26). Consider the product topology $\tau := \overset{\mathbf{Top}}{\prod_{i\in I}} \tau_{R_i}$ on $\overset{\mathbf{Rng}}{\prod_{i\in I}} R_i$ (Definition 1.43). If $f, g \in \overset{\mathbf{Rng}}{\prod_{i\in I}} R_i$ and if $\{f_\lambda\}_{\lambda\in\Lambda}$ and $\{g_\lambda\}_{\lambda\in\Lambda}$ are two nets in $\overset{\mathbf{Rng}}{\prod_{i\in I}} R_i$ so that $f_\lambda \xrightarrow[\lambda\in\Lambda]{\tau} f$ and $g_\lambda \xrightarrow[\lambda\in\Lambda]{\tau} g$, then we have

$$(\forall i \in I) \;\; f_\lambda(i) \xrightarrow[\lambda\in\Lambda]{\tau_{R_i}} f(i) \quad \text{and} \quad g_\lambda(i) \xrightarrow[\lambda\in\Lambda]{\tau_{R_i}} g(i) \; ,$$

and since for each $i \in I$ the operations $+_{R_i} : R_i \times R_i \to R_i$ and $\times_{R_i} : R_i \times R_i \to R_i$ are continuous with respect to τ_{R_i}, we get

$$(\forall i \in I) \;\; f_\lambda(i) + g_\lambda(i) \xrightarrow[\lambda\in\Lambda]{\tau_{R_i}} f(i) + g(i) \quad \text{and} \quad f_\lambda(i)\, g_\lambda(i) \xrightarrow[\lambda\in\Lambda]{\tau_{R_i}} f(i)\, g(i) \; ,$$

implying $f_\lambda + g_\lambda \xrightarrow[\lambda\in\Lambda]{\tau} f+g$ and $f_\lambda g_\lambda \xrightarrow[\lambda\in\Lambda]{\tau} fg$ and proving that the operations of $\prod_{i\in I}^{\mathbf{Rng}} R_i$ are continuous with respect to τ. Thus, $\prod_{i\in I}^{\mathbf{Rng}} R_i$ is a topological rng when endowed with τ.

2.5 Ordered rngs

An order relation may be invariant with respect to the addition operation of a rng, but invariance is not expected for the multiplication operation whatsoever. For instance, in the case of the ring $\mathbb{Z}$ endowed with its usual order, we note that $1 < 2$ and $1 \cdot (-1) \not< 2 \cdot (-1)$. Alternatively, a wide range of arithmetic-like considerations arise from the assumption that any product of two positive elements of a rng is positive.

Definition 2.33. An **ordered rng** is a rng R endowed with an order $<$ on it such that $(R, +_R, <)$ is an ordered commutative group and such that $(0_R, \rightarrow)_{(R,<)}$ is a subsemigroup of $(R, \times_R)$. Whenever no particular notation is attributed to the order $<$, it shall be denoted by $<_R$. We have the following notations and terminology:

- ▷ An **ordered subrng of** R is an ordered rng R' such that the rng $(R', +_{R'}, \times_{R'})$ is a subrng of $(R, +_R, \times_R)$ and such that the ordered set $(R', <_{R'})$ is an ordered subset of $(R, <_R)$;
- ▷ A rng is **orderable** if there is an order on it that turns it into an ordered rng. Otherwise, it is **non-orderable**;
- ▷ The **positive cone of** R is the interval $(0_R, \rightarrow)_R$;
- ▷ The **absolute value function on** R is the absolute value function $\overset{(R,+_R)}{\mathrm{Abs}}$ on the ordered group $(R, +_R)$ (Definition 1.73) and it shall be denoted by $\overset{R}{\mathrm{Abs}}$. Thus, we have $\overset{R}{\mathrm{Abs}}(x) = |x| = \overset{G}{\max}\{x, -x\}$ and $x, -x \leqslant |x|$ for all $x \in R$.

Proposition 2.34. (234, 29, 59) *Let R be an ordered rng.*

(**a**) $(\forall x, y \in R)\ |xy| = |x|\,|y|$;

(**b**) *R has no zero divisors;*

(**c**) $(\forall x \in R - \{0_R\})\ x^2 > 0_R$;

(**d**) *If R is non-trivial and if $x \in R - \{0_R\}$, then the sequence $\{nx\}$ is monotone, being increasing if $x > 0_R$ and decreasing if $x < 0_R$. In particular, we have* $\mathrm{Char}\,(R) = 0$ *(Definition 2.1) and the rng R is infinite and unbounded above and below in that case;*

(**e**) *If R is a non-trivial ordered ring, then $n_R > 0_R$ $(\forall n \in \mathbb{N})$.*

Proof.

(**a**) If $x, y \geqslant 0_R$, then $xy \geqslant 0_R$ and $|xy| = xy = |x|\,|y|$. If $x \geqslant 0_R$ and $y < 0_R$, then $-y > 0_R$ and $-(xy) = x(-y) \geqslant 0_R$ (Proposition 2.13, Item (a)), which gives us
$$|xy| = -(xy) = x(-y) = |x|\,|y| \ .$$
Finally, if $x, y < 0_R$, then $-x, -y > 0_R$ and
$$xy = -(-(xy)) = -((-x)\,y) = (-x)\,(-y) > 0_R\,,$$
which gives us
$$|xy| = xy = (-x)\,(-y) = |x|\,|y| \ .$$

(**b**) If x and y are two non-zero elements of R, then $|x|\,, |y| > 0_R$ and $|xy| = |x|\,|y| > 0_R$ by item (a), which gives us $xy \neq 0_R$ and proves the item.

(**c**) If $x > 0_R$, then clearly $x^2 > 0_R$, and if $x < 0_R$, then $-x > 0_R$ and
$$x^2 = (-x)\,(-x) > 0_R\,.$$

(**d**) Since the ordered monoid $(R, +_R, <_R)$ is non-trivial, the result follows from Item (a) of Proposition 1.65.

(**e**) We have $0_R \neq 1_R$ (Proposition 2.13, Item (b)) and $1_R = 1_R^2 > 0_R$ by item (c). Moreover, the sequence $\{n1_R\} = \{n_R\}$ is increasing by item (d) and we have $n_R \geqslant 1_R > 0_R$ $(\forall n \in \mathbb{N})$.

□

Example 2.35. The number rings $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are ordered rings when endowed with their usual orders. The number field $\mathbb{C}$ is well-known to be non-orderable. In fact, if $<$ is an order on $\mathbb{C}$ so that $(\mathbb{C}, <)$ is an ordered field, then $1 = 1^2 > 0$ and $-1 = i^2 > 0$ (Proposition 2.34, Item (c)), implying $1 > 0 = 1 - 1 > 1 + 0 = 1$, which is absurd.

Example 2.36. Let p be a prime number. The finite field F_p (Example 2.22) is non-orderable. As a matter of fact, if $<$ is an order on F_p so that $(\mathrm{F}_p, <)$ is an ordered field, then $1/\underset{p\mathbb{Z}}{\equiv} = \big(1/\underset{p\mathbb{Z}}{\equiv}\big)^2 > 0/\underset{p\mathbb{Z}}{\equiv}$ (Proposition 2.34, Item (c)), and we get

$$0/\underset{p\mathbb{Z}}{\equiv} = p/\underset{p\mathbb{Z}}{\equiv} = \overbrace{1/\underset{p\mathbb{Z}}{\equiv} + 1/\underset{p\mathbb{Z}}{\equiv} + \cdots + \ 1/\underset{p\mathbb{Z}}{\equiv}}^{p \text{ times}} > 0/\underset{p\mathbb{Z}}{\equiv},$$

which is absurd.

Example 2.37. Let n be a natural number and let R be a rng. The matrix rngs $\mathrm{M}_n(R)$ and $\mathrm{T}_n(R)$ (Example 2.3) are non-orderable since they have zero divisors (Proposition 2.34, Item (b)).

It is possible to define a compatible order on a rng by specifying its set of positive elements, provided that such set meets certain criteria:

Definition 2.38. A **synthetic positive cone** in a rng R is a subset P of R that is closed under addition, closed under multiplication and is such that the sets $-P$, P and $\{0_R\}$ form a partition of R.

Proposition 2.39. (58) *Let R be a rng. The function given by*

$$P \mapsto \{(x, y) \in R \times R \ | \ y - x \in P\}$$

is a one-to-one correspondence between the set of synthetic positive cones in R and the set of orders on R that are compatible with the rng structure of R. The inverse of that correspondence is given by $< \mapsto (0_R, \to)_{(R,<)}$.

The terminology introduced in Definition 1.69, which relates to the Archimedean Property on ordered monoids, naturally extends to the theory of ordered rngs R, where all definitions are considered with respect to the underlying additive ordered monoids $(R, +_R, <_R)$:

Definition 2.40. Let R be an ordered rng.

- ▷ The **Archimedean relation on** R is the Archimedean relation $\overset{(R,+_R,<)}{\mathscr{A}}$ on the ordered group $(R, +_R, <)$ (Definition 1.69) and it shall be denoted by $\mathscr{A}$ or $\overset{R}{\mathscr{A}}$. Accordingly, for all $x, y \in R$, the condition $x \mathscr{A} y$ amounts to saying that there is an $n \in \mathbb{N}$ such that

$$x \leqslant y \leqslant nx \text{ , or } nx \leqslant y \leqslant x \text{ , or } y \leqslant x \leqslant ny \text{ or } ny \leqslant x \leqslant y\,;$$

- ▷ The **Archimedean classes of** R are the equivalence classes associated to the equivalence relation $\mathscr{A}$, and, in particular, the **trivial Archimedean class of** R is the class $0_R/\mathscr{A} = \{0_R\}$ of the element 0_R. Each non-trivial Archimedean class S of R is an order-convex subsemigroup of $(R, +_R, <)$ that is either contained in $(0_R, \rightarrow)_R$ or contained in $(\leftarrow, 0_R)_R$. It is **positive** if $S \subset (0_R, \rightarrow)_R$ and it is **negative** if $S \subset (\leftarrow, 0_R)_R$;
- ▷ An ordered rng R is **Archimedean** if it has at most one positive Archimedean class. Otherwise, it is **non-Archimedean**;
- ▷ The **partial order of Archimedean distribution on** R is the partial order of Archimedean distribution $\ll_{(R,+_R,<)}$ on the ordered group $(R, +_R, <)$ (Definition 1.69) and this partial order shall be denoted by $\ll$ or $\ll_R$.

Example 2.41. The ordered rings $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are the archetypal examples of Archimedean ordered rings.

In the case of an ordered ring R, it is useful to classify the elements of R according to where they stand relative to the Archimedean class of the multiplicative identity 1_R of R:

Definition 2.42. Let R be a non-trivial ordered ring.

- ▷ An element $x \in R$ is **infinitesimal in** R if $|x| \ll 1_R$. Note that 0_R is infinitesimal in R;
- ▷ An element $x \in R$ is **infinite in** R if $|x| \gg 1_R$. Otherwise, it is **finite in** R;
- ▷ An element $x \in R$ is **appreciable in** R if x is neither infinitesimal nor infinite in R;
- ▷ The **relation of infinite proximity on** R is the binary relation on R denoted by $\sim$ or $\overset{R}{\sim}$ and defined so that for all $x, y \in R$, the condition $x \sim y$ is equivalent to saying that the difference $x - y$ is infinitesimal in R. That binary relation is an equivalence relation on R;
- ▷ Two elements $x, y \in R$ are **infinitely close in** R if $x \sim y$;
- ▷ The **monad** of an element $x \in R$ is the equivalence class $x/\sim$.

Example 2.43. Consonant with Example 1.70, we have that the ordered field $({}^*\mathbb{R}, +_{{}^*\mathbb{R}}, \times_{{}^*\mathbb{R}}, <_{{}^*\mathbb{R}})$ of hyperreal numbers and the ordered field $(\mathsf{No}, +_{\mathsf{No}}, \times_{\mathsf{No}}, <_{\mathsf{No}})$ of surreal numbers are non-Archimedean. Thus, they contain infinitesimal and infinite elements.

2.6 Γ-Pseudovalued rngs

Γ-Valuations (Section 1.9) find their greatest application in Mathematics when they are considered on rings and fields. In fact, in most publications, one usually assumes that Γ is a commutative ordered group, and, taking R to be a ring, one considers Γ-valuations $v : R \to \breve{\Gamma}$ that are Γ-valuations on the commutative group $(R, +_R)$ in the sense of Definition 1.79 and that translate products in R to sums in Γ. We shall work with a slightly more general setting:

Definition 2.44. Let Γ be an ordered magma denoted additively and let R be a rng. A **Γ-pseudovaluation** (resp. **Γ-valuation**) **on** R is a Γ-valuation $v : R \to \breve{\Gamma}$ on the commutative group $(R, +_R)$ (Definition 1.79) such that

$$(\forall x, y \in R) \;\; v(xy) \geqslant v(x) + v(y) \quad (\text{resp. } v(xy) = v(x) + v(y)) \; .$$

A **Γ-pseudovalued** (resp. **Γ-valued**) **rng** is a rng R endowed with a Γ-pseudovaluation (resp. Γ-valuation) on R. Whenever no particular notation is attributed to the Γ-pseudovaluation of R, it shall be denoted by v_R. The **value set of** R is the set Γ (Definition 1.33).

Example 2.45. Let p be a prime number and consider the $\mathbb{Z}$-valued commutative group $(\mathbb{Q}, +_{\mathbb{Q}}, v_p)$ of Example 1.80. We shall prove that this structure becomes a $\mathbb{Z}$-valued field when the usual multiplication operation on $\mathbb{Q}$ is considered. Effectivelly, if $x = p^{n_x}\left({}^{a}\!/_{b}\right)$ and $y = p^{n_y}\left({}^{c}\!/_{d}\right)$ are two non-zero rational numbers, then $xy = p^{n_x+n_y}\frac{ac}{bd}$, and, since ac and bd are non-zero integers coprime to p, we obtain

$$v_p(xy) = n_x + n_y = v_p(x) + v_p(y) \; ,$$

proving that $(\mathbb{Q}, +_{\mathbb{Q}}, \times_{\mathbb{Q}}, v_p)$ is a $\mathbb{Z}$-valued field.

2.7 Polynomial Rngs and algebraically closed fields

An **algebraic equation** is an equation of the form $p(x) = q(x)$, where p and q are any two polynomials in an unknown variable x, and they were, almost exclusively, the object of study of all algebraists up until the 19th century. The long history of their study is fascinating, most crucially including ancient Greek philosophers who struggled to accept the nature of irrational numbers, a number of medieval Hindu and Arab mathematicians who developed the theory of quadratic equations, a couple of Italian mathematicians who solved the cubic in the 16th century and an underrated French prodigy in the early 19th century who revealed an upper bound for the degree of algebraic equations that have general solutions by radicals (40).

Since the dawn of modern Mathematics, the notion of a polynomial has been significantly generalised, so that the coefficients of a polynomial need not be numbers of any kind as long as they belong to a ring-like structure:

Definition 2.46. Let R be a rng. The **polynomial rng with coefficients in** R is the set denoted by $R[\mathrm{X}]$ and given by

$$R[\mathrm{X}] \coloneqq \{p \in {}^{\mathbb{N}_0}R \mid \operatorname{supp}(p) \text{ is finite}\} ,$$

being endowed with the addition and multiplication operations on it given by

$$(p+q)_n \coloneqq p_n + q_n \quad \text{and} \quad (pq)_n \coloneqq \sum_{\substack{u,v\in\mathbb{N}_0 \\ u+v=n}} p_u q_v \,.$$

Clearly, $R[\mathrm{X}]$ is a rng that is a ring if, and only if, R is a ring. We have the following notations and terminology:

- ▷ Each polynomial $p \in R[\mathrm{X}]$ is almost invariably denoted by a finite formal sum $p = \sum_{i=0}^{n} p_i \mathrm{X}^i$, where n is any upper bound of $\operatorname{supp}(p)$ in $\mathbb{N}_0$;
- ▷ The **n-th degree coefficient** of a polynomial $p \in R[\mathrm{X}]$ is the element p_i of R. A polynomial may only have a finite number of non-zero coefficients;
- ▷ The **independent coefficient of a polynomial** $p \in R[\mathrm{X}]$ is its 0-th degree coefficient, p_0. The **constant term of** p is the polynomial $p_0\mathrm{X}^0$;
- ▷ For each $n \in \mathbb{N}_0$ and each $r \in R$, the polynomial $p \in R[\mathrm{X}]$ given by

  $$p_m \coloneqq \begin{cases} r & \text{if } m = n\,, \\ 0_R & \text{if } m \in \mathbb{N}_0 - \{n\} \end{cases}$$

  is denoted by $r\mathrm{X}^g$, and if $n \neq 0$ and $r \neq 0_R$, then that polynomial p is called the **n-th degree monomial with coefficient** r. A non-zero polynomial p is the actual sum of the finite sequence of monomials $p_{i_1}\mathrm{X}^{i_1}, p_{i_2}\mathrm{X}^{i_2}, \ldots, p_{i_n}\mathrm{X}^{i_n}$ with respect to the addition on $R[\mathrm{X}]$ defined above, where $\{i_1 i_2 \ldots i_n\}$ is the support of p;
- ▷ If R is a ring, then for each $n \in \mathbb{N}_0$, the element $1_R\mathrm{X}^n$ of $R[\mathrm{X}]$ is denoted by X^n. The element X^1 is denoted by X;

- ▷ For each $r \in R$, the **constant polynomial with value** r is the element $r\mathrm{X}^0$ of $R[\mathrm{X}]$ and it is denoted by r, by abuse of language. In particular, the element $0_R\mathrm{X}^0$ is called the **zero polynomial in** $R[\mathrm{X}]$ and it is denoted by 0_R, and if R is a ring, then the element $1_R\mathrm{X}^0$ is denoted by 1_R;
- ▷ The **degree** of a non-zero polynomial $p \in R[\mathrm{X}]$ is the greatest element of $\operatorname{supp}(p)$, and it is denoted by $\deg(p)$;
- ▷ Let $r \in R$ and let $p \in R[\mathrm{X}]$. The r**-image** of a polynomial $p = \sum_{i=0}^{n} p_i\mathrm{X}^i$ in $R[\mathrm{X}]$ is the element of R denoted by $p(r)$ and given by the finite sum $p(r) := \sum_{i=0}^{n} p_i r^i$. The element r is a **root of** p **in** R if $p(r) = 0_R$.

Example 2.47. Let R be a rng, let $x \in R$ and let $n \in \mathbb{N}$. The n**-th roots of** x **(in** R**)** are the elements $y \in R$ so that $y^n = x$. They are called **square roots of** x in the case $n = 2$, and **cubic roots of** x in the case $n = 3$. If R is a ring, the n-th roots of x in R are precisely the roots of the polynomial $\mathrm{X}^n - x$ in R. The number -1 has no square roots in $\mathbb{R}$, but it has two square roots in $\mathbb{C}$, viz. i and $-i$.

Example 2.48. Let $k \in \mathbb{N}_0$ and let p be a prime number. It follows from Fermat's Little Theorem (141, 62) that all elements of the finite field F_p (Example 2.22) are roots of the p^k-th degree polynomial $\mathrm{X}^{p^k} - \mathrm{X} \in \mathrm{F}_p[\mathrm{X}]$.

Definition 2.49. A field K is **algebraically closed** if every polynomial in $K[\mathrm{X}]$ of positive degree has at least one root in K.

Example 2.50. The number fields $\mathbb{Q}$ and $\mathbb{R}$ are not algebraically closed, but the field $\mathbb{C}$ is.

Example 2.51. The field ${}^*\mathbb{R}$ of hyperreal numbers (192, 148) is not algebraically closed, but the field ${}^*\mathbb{C}$ of hypercomplex numbers is.

Example 2.52. The field No of surreal numbers (113) is not algebraically closed.

Example 2.53. No finite field is algebraically closed. As a matter of fact, if $K = \{x_1 x_2 \ldots x_n\}$ is a finite field, then the polynomial

$$(\mathrm{X} - x_1)(\mathrm{X} - x_2) \cdots (\mathrm{X} - x_n) + 1_K \in K[\mathrm{X}]$$

has no roots in K.

2.8 Differential rings

Given a ring R, an endomorphism on R may possess certain features in common with the derivative operator on the ring $\mathsf{C}^\infty(\mathbb{R})$ of infinitely differentiable functions. Those endomorphisms, called **derivations**, were first researched by Ritt in 1932 (191), as he made significant breakthroughs in the study of the geometry of differential equations on fields of characteristic zero, founding the powerful area known today as **Differential Algebra**. This area brought on profound implications on a number of different parts of Mathematics, being further specialised in a handful of fascinating subareas such as Differential Galois Theory and Differential Algebraic Geometry, and it has found a few applications in modern Computer Algebra.

Definition 2.54. Let R be a ring. A **derivation of** R is an endomorphism $\partial : (R, +_R) \xrightarrow{\mathbf{Grp}} (R, +_R)$ such that **Leibniz's Product Rule** is satisfied:

$$(\forall x, y \in R) \ \ \partial(xy) = (\partial x)\, y + x\,(\partial y) \ .$$

A **differential ring** is a ring R endowed with a finite number of non-zero derivations of R. We shall assume that R has only one underlying derivation, which will be denoted by ∂_R.

Example 2.55. The classical **differential operator** ${}^{\mathrm{d}}\!/_{\mathrm{d}x} : \mathsf{C}^{\infty}(\mathbb{R}) \to \mathsf{C}^{\infty}(\mathbb{R})$ on the ring $\mathsf{C}^{\infty}(\mathbb{R})$ of infinitely differentiable functions of type $\mathbb{R} \to \mathbb{R}$ is the archetypal example of a derivation (of $\mathsf{C}^{\infty}(\mathbb{R})$).

Example 2.56. Let R be a ring. The function $\mathrm{D} : R\,[\mathrm{X}] \to R\,[\mathrm{X}]$ given by

$$\mathrm{D}\left(\sum_{i=0}^{n} p_i \mathrm{X}^i\right) := \sum_{i=1}^{n} i p_i \mathrm{X}^{i-1}$$

is a derivation of the polynomial ring $R\,[\mathrm{X}]$. Note that that function artificially mimics the formula for calculating derivatives of polynomial functions of type $\mathbb{R} \to \mathbb{R}$.

Proposition 2.57. (195, 11) *Let A be a ring and let $\partial : A \to A$ be a derivation of A.*

(**a**) $\partial_A\,(1_A) = 0_A$*;*

(**b**) *If $x_1 \ldots x_n \in A$, then*

$$\partial_A\,(x_1 \cdots x_n) = \sum_{i=1}^{n} x_1 \cdots x_{i-1} \partial_A\,(x_i)\, x_{i+1} \cdots x_n\,;$$

(**c**) *If A is commutative, then*

$$(\forall x \in A)\,(\forall n \in \mathbb{N})\;\; \partial_A\,(x^n) = n x^{n-1} \partial_A\,(x)$$

and

$$(\forall x, y \in A)\,(\forall n \in \mathbb{N})\;\; \partial_A^n\,(xy) = \sum_{k=0}^{n} \binom{n}{k} \partial_A^{n-k}\,(x)\, \partial_A^k\,(y)\;.$$

3 Rayner Rngs of Formal Power Series

With the theoretical apparatus presented in Chapters 1 and 2 available, we are ready to address our Theory of Rayner Rngs, which is the main subject of this thesis. In this chapter, we shall formally establish the framework of that theory, defining the notion of a **Rayner ideal** on an ordered group and the notion of a **Rayner rng**, and then proving a set of algebraic, categorical, order-theoretical and differential-algebraic results related to these notions. Moreover, we shall consider special types of Rayner ideals which lead to a few prominent structures in Non-Archimedean Mathematics.

3.1 Rayner ideals

As mentioned in the Introduction, the elements of a Rayner rng will be defined to be formal power series $x = \sum_{g \in G} x_g \mathrm{X}^g$ with coefficients in a rng R and with exponents in an ordered group G (Definition 3.26), so that the addition and multiplication operations on the set of these series are to be defined as

$$x + y = \sum_{g \in G} x_g \mathrm{X}^g + \sum_{g \in G} y_g \mathrm{X}^g \coloneqq \sum_{g \in G} (x_g + y_g)\, \mathrm{X}^g$$

and

$$xy = \Big(\sum_{g \in G} x_g \mathrm{X}^g\Big) \Big(\sum_{g \in G} y_g \mathrm{X}^g\Big) \coloneqq \sum_{p \in G} \Bigg(\sum_{\substack{g,h \in G \\ g+h=p}} x_g y_h \Bigg) \mathrm{X}^p .$$

As Rayner pointed out in 1968, for the coefficients $\sum_{\substack{g,h \in G \\ g+h=p}} x_g y_h$ of a product xy of x and y to be well-defined as finite sums in R, one has to assume that the **set of possible supports** of these power series, which we shall denote by $\mathcal{J}$, contains only well-ordered subsets of G, and if one is to expect that the set of formal power series with supports in $\mathcal{J}$ to be closed under those operations, then further constraints must be imposed upon $\mathcal{J}$. We call the set $\mathcal{J}$ a **Rayner ideal**[1] **on** G, and, in this section, we shall examine its fundamental properties.

Since the elements of the ordered group G are meant to appear as exponents of the formal variable X in the power series that define the elements of Rayner rngs, and since it is convenient to render the traditional Product of Powers Rule $\mathrm{X}^g\mathrm{X}^h = \mathrm{X}^{g+h}$ as valid, the ordered group G shall always be denoted additively whenever one or more ideals on G are taken into consideration.

Definition 3.1.

- ▷ An **ideal on a set** J is an inhabited set $\mathcal{J}$ of subsets of J such that the following two conditions are satisfied:

  (I1) $\quad (\forall A \in \mathcal{J})\,(\forall B \in \mathrm{P}\,(J))\,(B \subset A \Rightarrow B \in \mathcal{J})$;

  (I2) $\quad (\forall A, B \in \mathcal{J})\,(A \cup B \in \mathcal{J})$.

  An ideal $\mathcal{J}$ on J is **proper** if $\mathcal{J} \subsetneq \mathrm{P}\,(J)$, otherwise it is **improper**. It is easy to check that the set of $\mathrm{P}_\omega(J)$ of finite subsets of J is an ideal on J, and, if J is an ordered set, then the set $\overset{\mathrm{wo}}{\mathrm{P}}(J)$ of well-ordered subsets of J (Definition 1.28) is an ideal on J;

- ▷ Let $\mathcal{J}$ be an ideal on a set J. A **subideal** (resp. **proper subideal**) **of** $\mathcal{J}$ is an ideal $\mathcal{S}$ on J such that $\mathcal{S} \subset \mathcal{J}$ (resp. $\mathcal{S} \subsetneq \mathcal{J}$);

- ▷ An ideal $\mathcal{J}$ on an ordered set J is **incremental** if every subset S of J such that $S \cap (\leftarrow, j]_J \in \mathcal{J}$ $(\forall j \in J)$ is an element of $\mathcal{J}$;

- ▷ Let κ be a cardinal number. An ideal $\mathcal{J}$ on a set J is **κ-dominated** if $|A| < \kappa$ $(\forall A \in \mathcal{J})$;

[1] Rayner himself called that set a **field family with respect to** G. We took the liberty of renaming the concept in his honor.

- ▷ An ideal $\mathcal{J}$ on an ordered set J is **cofinal** if every infinite element of $\mathcal{J}$ is cofinal in J;
- ▷ An ideal on a group G is **spanning** if $\overset{\mathbf{Grp}}{\underset{G}{\operatorname{span}}}(\bigcup \mathcal{J}) = G$ (Definition 1.7);
- ▷ An ideal on a group G is **well-balanced** if for all $A \in \mathcal{J}$ and all $g \in A$, there is a $B \in \mathcal{J}$ such that $-g \in B$;
- ▷ An **arithmetic Rayner ideal** on an ordered group G is an ideal $\mathcal{J}$ on G that is a subideal of $\overset{\mathrm{wo}}{\mathrm{P}}(G)$ and is such that $A + B \in \mathcal{J}$ $(\forall A, B \in \mathcal{J})$;
- ▷ A **full Rayner ideal** on an ordered group G is an ideal $\mathcal{J}$ on G that is a subideal of $\overset{\mathrm{wo}}{\mathrm{P}}(G)$ and is such that the following two conditions are satisfied:

  (**F1**) $\quad (\forall A \in \mathcal{J})\,(\forall g \in G)\ \ A + g, g + A \in \mathcal{J}\,;$

  (**F2**) $\quad (\forall A \in \mathcal{J})\ \ A \subset [0_G, \rightarrow)_G \Rightarrow \overset{\mathbf{SGrp}}{\underset{G}{\operatorname{span}}}(A) \in \mathcal{J}$ (Definition 1.2).

Rayner considered solely what we call here a **spanning full Rayner ideal on** G (181, 182), and Krapp, Kuhlmann and Serra (117) took into account several variations of axioms upon $\mathcal{J}$. Both assessments dealt exclusively with the case in which the ordered group G is commutative. The abstraction of the scenario so as to include non-commutative groups G, as well as the notions of incremental, κ-dominated, cofinal, well-balanced and arithmetic Rayner ideals on G, are novel in this work.

Proposition 3.2. *Let J be an ordered set and let G be an ordered group.*

(**a**) *If $\mathcal{J}$ is a cofinal ideal on J, then $\mathcal{J}$ is $\mathrm{cf}\,(J)^+$-dominated;*

(**b**) *If there is a cofinal ideal $\mathcal{J}$ on J that has an infinite element, then $\mathrm{cf}\,(J) \leqslant \omega$;*

(**c**) *An arithmetic Rayner ideal $\mathcal{J}$ on G is spanning and well-balanced if, and only if, $\{g\} \in \mathcal{J}$ $(\forall g \in G)$. In that case, Axiom (F1) holds for $\mathcal{J}$;*

(**d**) *(117) Every full Rayner ideal $\mathcal{J}$ on G is arithmetic.*

Proof. Item (a) is straightforward. We shall prove the remaining items.

(b) Let S be an infinite element of $\mathcal{J}$ and let C be an infinitely countable subset of S. Thus, we have $C \in \mathcal{J}$, and, since the ideal $\mathcal{J}$ is cofinal, the set C is cofinal in J, which gives us $\operatorname{cf}(J) \leqslant \omega$.

(c) If $\{g\} \in \mathcal{J}$ $(\forall g \in G)$, then the ideal $\mathcal{J}$ is clearly well-balanced and we have $\underset{G}{\overset{\mathbf{Grp}}{\operatorname{span}}}(\bigcup \mathcal{J}) = \underset{G}{\overset{\mathbf{Grp}}{\operatorname{span}}}(G) = G$, that is, $\mathcal{J}$ is spanning. Conversely, suppose $\mathcal{J}$ is spanning and let g be an element of G. Thus, there is a finite sequence $A_1 A_2 \ldots A_n$ in $\mathcal{J}$ and also a finite sequence $g_1 g_2 \ldots g_n$ in G such that $\pm g_i \in A_i$ $(\forall i \in [1,n]_{\mathbb{N}})$ and $g = g_1 + g_2 + \cdots + g_n$. Since $\mathcal{J}$ is well-balanced, we may assume that $g_i \in A_i$ $(\forall i \in [1,n]_{\mathbb{N}})$, and we have

$$\{g\} \subset A_1 + A_2 + \cdots + A_n \in \mathcal{J}\,,$$

which implies $\{g\} \in \mathcal{J}$. Finally, if A is an element of $\mathcal{J}$, then we get

$$A + g = A + \{g\} \in \mathcal{J} \quad \text{and} \quad g + A = \{g\} + A \in \mathcal{J}\,.$$

(d) Suppose A and B are inhabited elements of $\mathcal{J}$, let g_A be the least element of A and let g_B be the least element of B. Then, $(-g_A) + A, B + (-g_B) \in \mathcal{J}$ (Axiom (F1)) and

$$((-g_A) + A) \cup (B + (-g_B)) \in \mathcal{J}$$

implying (Axiom (F2))

$$S \coloneqq \underset{G}{\overset{\mathbf{SGrp}}{\operatorname{span}}}((-g_A) + A) \cup (B + (-g_B)) \in \mathcal{J}$$

since $(-g_A) + A, B + (-g_B) \subset [0_G, \rightarrow)_G$. If $x \in A$ and $y \in B$, then we have

$$(-g_A) + (x + y) + (-g_B) = ((-g_A) + x) + (y + (-g_B)) \in S$$

and $x + y \in g_A + S + g_B$, proving the inclusion $A + B \subset g_A + S + g_B \in \mathcal{J}$ (Axiom (F1)) and $A + B \in \mathcal{J}$.

□

The proofs of several upcoming results depend upon the requirement that every element of G must be allowed to appear as an exponent of the formal variable X in the construction of the elements of a Rayner rng. Thus, the author reckons that it is reasonable to assume the validity of that condition for all Rayner ideals considered in our discussions:

Assumption 3.3. *From now on, we are only to consider ideals $\mathcal{J}$ on a set J that satisfy $\mathrm{P}_\omega(J) \subset \mathcal{J}$. That implies that all arithmetic Rayner ideals on a group G are assumed to be spanning and well-balanced on G.*

Proposition 3.4. *Let G be a non-trivial ordered group.*

(**a**) *The set $\mathrm{P}_\omega(G)$ of finite subsets of G is an ω-dominated arithmetic Rayner ideal on G that is neither incremental nor full;*

(**b**) *The set $\overset{\mathrm{wo}}{\mathrm{P}}(G)$ of well-ordered subsets of G is an incremental full Rayner ideal on G.*

Proof. Item (a) is elementary and its proof shall be omitted.

(**b**) We leave to the reader the simple proof that $\overset{\mathrm{wo}}{\mathrm{P}}(G)$ is an ideal on G. Taking that into account, we know that $\overset{\mathrm{wo}}{\mathrm{P}}(G)$ is an arithmetic Rayner ideal on G (Lemma 1.66) and Axiom (F1) of Definition 3.1 holds for $\overset{\mathrm{wo}}{\mathrm{P}}(G)$ (Proposition 3.2, Item (c)). If S is a subset of G such that the condition $S \cap (\leftarrow, g]_G \in \overset{\mathrm{wo}}{\mathrm{P}}(G)$ $(\forall g \in G)$ holds, if U is an inhabited subset of S and if u is an element of U, then $(\leftarrow, u]_U$ is an inhabited subset of $S \cap (\leftarrow, u]_G$ and it is clear that the least element of $(\leftarrow, u]_U$ is also the least element of U. Thus, the ideal $\overset{\mathrm{wo}}{\mathrm{P}}(G)$ on G is incremental.

It remains to prove that $\overset{\mathrm{wo}}{\mathrm{P}}(G)$ satisfies Axiom (F2). Suppose A is a well-ordered subset of G contained in $[0_G, \rightarrow)_G$. We are to prove that the semigroup $\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(A)$ is well-ordered in G. Since $\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(A) \subset \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(A \cup \{0_G\})$, we may assume, without loss of generality, that we have $0_G \in A$. By Proposition 1.31, there is a unique ordinal γ and there is a unique isomorphism $\phi : A \to \gamma$ between ordered sets. We shall prove the desired result by transfinite induction on γ. It is trivially true for $\gamma = 0$. Suppose it holds true for every well-ordered subset of G contained in $[0_G, \to)_G$ that is isomorphic in **OrdSet** to an ordinal less than γ. Let S be an inhabited subset of $\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(A)$, take a fixed element s of S, let $a_1 a_2 \ldots a_n$ be a finite sequence in A such that $s = a_1 + a_2 + \cdots + a_n$ and let $g \coloneqq \overset{G}{\underset{i \in [1,n]_{\mathbb{N}}}{\max}} a_i$. Thus, $s \leqslant ng$ and note that the restriction

$$\phi \restriction (\leftarrow, g)_A : (\leftarrow, g)_A \to \phi(g)$$

is an isomorphism between ordered sets. Since $\phi(g) < \gamma$, the semigroup $\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}((\leftarrow, g)_A)$ is well-ordered in G by the inductive hypothesis.

We are to show that S has a least element. If $g = 0_G$, then $s = 0_G$ is clearly the least element of S. Let us assume that $g > 0_G$. Consider an element x of the interval $(\leftarrow, s]_S$ and let $p_1 p_2 \ldots p_k$ be a finite sequence in A such that $x = p_1 + p_2 + \cdots + p_k$. If $i_1 i_2 \ldots i_m$ are the indices in $[1, k]_{\mathbb{N}}$ such that $p_{i_u} \geqslant g$ $(\forall u \in [1, m]_{\mathbb{N}})$, then $m \leqslant n$, for otherwise we would have $m > n$ and

$$\begin{aligned} x &= \Big(\sum_{i=1}^{i_1-1} p_i\Big) + p_{i_1} + \Big(\sum_{i=i_1+1}^{i_2-1} p_i\Big) + p_{i_2} + \cdots + p_{i_m} + \Big(\sum_{i=i_m+1}^{k} p_i\Big) \\ &\geqslant 0_G + g + 0_G + g + \cdots + g + 0_G \\ &= mg > ng \geqslant s\,, \end{aligned}$$

which is absurd. Therefore, since

$$(\forall i \in [1, k]_{\mathbb{N}} - \{i_1 i_2 \ldots i_m\}) \;\; p_i < g\,,$$

we obtain [2]

$$\begin{aligned} x &= \Big(\sum_{i=1}^{i_1-1} p_i\Big) + p_{i_1} + \Big(\sum_{i=i_1+1}^{i_2-1} p_i\Big) + p_{i_2} + \cdots + p_{i_m} + \Big(\sum_{i=i_m+1}^{k} p_i\Big) \\ &\in \overbrace{\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}((\leftarrow, g)_A) + A + \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}((\leftarrow, g)_A) + A + \cdots + A + \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}((\leftarrow, g)_A)}^{n \text{ occurrences of } A}\,. \end{aligned}$$

[2] The fact $0_G \in (\leftarrow, g]_A \subset \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}((\leftarrow, g]_A)$ ensures that the conclusions that follow hold true if $m < n - 1$, or $i_1 = 1$, or $i_m = n$ or even if $i_j + 1 = i_{j+1}$ for some $j \in [1, m-1]_{\mathbb{N}}$.

That last set above, which we denote by P, is well-ordered in G since the Rayner ideal $\overset{\text{wo}}{\mathrm{P}}(G)$ on G is arithmetic. We have proved that $(\leftarrow, s]_S \subset P$, and, thus, the interval $(\leftarrow, s]_S$ has a least element which is clearly the least element of S, proving that $\underset{G}{\overset{\mathbf{SGrp}}{\operatorname{span}}}(A)$ is well-ordered and concluding our induction.

$\square$

Example 3.5. The sets $\mathrm{P}_\omega(\mathbb{Z}, +_\mathbb{Z}, <_\mathbb{Z})$, $\mathrm{P}_\omega(\mathbb{Q}, +_\mathbb{Q}, <_\mathbb{Q})$, $\mathrm{P}_\omega(\mathbb{R}, +_\mathbb{R}, <_\mathbb{R})$ and $\mathrm{P}_\omega(\mathsf{BS}_\ell)$ (Example 1.64) are ω-dominated arithmetic Rayner ideals on the respective ordered groups $(\mathbb{Z}, +_\mathbb{Z}, <_\mathbb{Z})$, $(\mathbb{Q}, +_\mathbb{Q}, <_\mathbb{Q})$, $(\mathbb{R}, +_\mathbb{R}, <_\mathbb{R})$ and BS_ℓ, and they are neither incremental nor full (Proposition 3.4, Item (a)). They are trivially cofinal, for they contain no infinite elements.

Example 3.6. The set $\overset{\text{wo}}{\mathrm{P}}(\mathbb{Z}, +_\mathbb{Z}, <_\mathbb{Z})$ is an incremental full Rayner ideal on the ordered group $(\mathbb{Z}, +_\mathbb{Z}, <_\mathbb{Z})$ (Proposition 3.4, Item (b)), and, given that every infinite well-ordered subset of $\mathbb{Z}$ is cofinal in $\mathbb{Z}$, this Rayner ideal is cofinal.

Example 3.7. The sets $\overset{\text{wo}}{\mathrm{P}}(\mathbb{Q}, +_\mathbb{Q}, <_\mathbb{Q})$, $\overset{\text{wo}}{\mathrm{P}}(\mathbb{R}, +_\mathbb{R}, <_\mathbb{R})$ and $\overset{\text{wo}}{\mathrm{P}}(\mathsf{BS}_\ell)$ are incremental full Rayner ideals on the respective ordered groups $(\mathbb{Q}, +_\mathbb{Q}, <_\mathbb{Q})$, $(\mathbb{R}, +_\mathbb{R}, <_\mathbb{R})$ and BS_ℓ (Proposition 3.4, Item (b)). These ideals are not cofinal, since, for instance, the sets (cf. Example 1.71)

$$\{{-1}/n \mid n \in \mathbb{N}\} \subset \mathbb{Q} \subset \mathbb{R} \quad \text{and} \quad \{n\,((-\ell\,\mathsf{t}) + \mathsf{m}) \mid n \in \mathbb{N}\} \subset \mathsf{BS}_\ell$$

are infinite well-ordered subsets that are not cofinal in their respective ordered groups. Moreover, the ideals $\overset{\text{wo}}{\mathrm{P}}(\mathbb{Q}, +_\mathbb{Q}, <_\mathbb{Q})$ and $\overset{\text{wo}}{\mathrm{P}}(\mathsf{BS}_\ell)$ are ω_1-dominated as $\mathbb{Q}$ and BS_ℓ are countable, and the ideal $\overset{\text{wo}}{\mathrm{P}}(\mathbb{R}, +_\mathbb{R}, <_\mathbb{R})$ is ω_1-dominated since every well-ordered subset of $\mathbb{R}$ is countable (33).

Example 3.8. Let α be an ordinal number and consider the set No_α of surreal numbers with birthdays less than α. One can easily show that $\operatorname{cf}(\mathsf{No}_\alpha) = \operatorname{cf}(\alpha)$, and a result proved by Dries and Ehrlich (69) reveals that if α is of the form [3] ω^β,

[3] Ordinals of this form are said to be **additively indecomposable**.

where β is any ordinal, then No_α forms a commutative ordered group when endowed with its usual addition operation and its usual order. In particular, we have $\mathrm{cf}\,(\mathsf{No}_{\omega_1}) = \mathrm{cf}\,(\omega_1) = \omega_1$, and, given that $\omega_1 = \omega^{\omega_1}$ (38, 89), it turns out that the ideal $\mathrm{P}_\omega(\mathsf{No}_{\omega_1})$ is the only cofinal ideal on the ordered group No_{ω_1} (Proposition 3.2, Item (b)), and it is an ω-dominated arithmetic Rayner ideal on No_{ω_1} that is neither incremental nor full (Proposition 3.4, Item (a)).

3.2 Left-finiteness of subsets and ideals

As first pointed out by Levi-Civita in 1893 (139), a particular class of subsets of $\mathbb{Q}$ is closely related to some arithmetic systems of formal power series that exhibit many striking properties of analytical character, and, in fact, many classical results of Analysis on $\mathbb{R}^n$ hold true for those systems when their assumptions are slightly modified. Such class of subsets of ordered sets is of the utmost importance to our Theory of Rayner Rngs, and it shall be introduced in this section.

Definition 3.9. A subset S of an ordered set X is **left-finite in** X if we have

$$(\forall x \in X) \quad |S \cap (\leftarrow, x]_X| < \omega\,.$$

The set of left-finite subsets of X shall be denoted[4] by $\overset{\mathrm{lf}}{\mathrm{P}}(X)$. An ideal on X is **left-finite** if it is contained in $\overset{\mathrm{lf}}{\mathrm{P}}(X)$.

Proposition 3.10. (17, 200) *Let X be an ordered set.*

(**a**) *If S is a left-finite subset of X, then S is left-finite in itself;*

(**b**) *The set $\overset{\mathrm{lf}}{\mathrm{P}}(X)$ of left-finite subsets of X is an ideal on X (Definition 3.1) that contains $\mathrm{P}_\omega(X)$ as a subideal;*

[4] The letters 'lf' in this notation stand for '**l**eft-**f**inite'.

(**c**) *A subset S of X is left-finite in X if, and only if, it is either finite or isomophic to ω and (strictly) cofinal in X;*

(**d**) *An ideal $\mathcal{J}$ on X is left-finite if, and only if, it is cofinal. In that case, $\mathcal{J}$ is a subideal of $\overset{\mathrm{wo}}{\mathrm{P}}(X)$ that is ω_1-dominated, and if $\mathcal{J}$ has an infinite element, then* $\mathrm{cf}\,(X) = \omega$ *(Definition 1.27);*

(**e**) *If $\{x_n\}$ is a sequence in X whose image $\{x_n \mid n \in \mathbb{N}\}$ is infinite and left-finite in X, then the sequence $\{x_n\}$ is not* $\overset{X}{\mathrm{O}}\mathrm{rdt}$*-convergent;*

(**f**) *If G is a non-Archimedean ordered group, and if $\mathcal{J}$ is a left-finite ideal on G, then Axiom (F2) (Definition 3.1) does not hold for $\mathcal{J}$.*

Proof. All claims are trivially obvious in the case $X = \emptyset$. Assume $X \neq \emptyset$ in the following proofs.

(**a**) Note that

$$(\forall x \in S)\;\; S \cap (\leftarrow, x]_S = (\leftarrow, x]_S \subset S \cap (\leftarrow, x]_X \;,$$

implying that $S \cap (\leftarrow, x]_S$ is finite for all $x \in S$.

(**b**) Clearly, we have $\mathrm{P}_\omega(X) \subset \overset{\mathrm{lf}}{\mathrm{P}}(X)$ and the set $\overset{\mathrm{lf}}{\mathrm{P}}(X)$ is closed under subsets. Take $A, B \in \overset{\mathrm{lf}}{\mathrm{P}}(X)$. For all $x \in X$, we have

$$(A \cup B) \cap (\leftarrow, x]_X = (A \cap (\leftarrow, x]_X) \cup (B \cap (\leftarrow, x]_X) \;,$$

that is, the set $(A \cup B) \cap (\leftarrow, x]_X$ is the union of two finite sets, and, therefore, it is finite, proving that $A \cup B \in \overset{\mathrm{lf}}{\mathrm{P}}(X)$.

(**c**) Suppose S is a left-finite subset of X. Firstly, we shall prove that S is well-ordered. Let W be an inhabited subset of S and let w be an element of W. Since S is left-finite in X, the set W is left-finite in X (item (b)), and if $w_1 \ldots w_n$ is the increasing finite sequence of elements of the intersection $W \cap (\leftarrow, w]_X$, then one can easily conclude that w_1 is the least element of W. Hence, S is well-ordered, and, by Proposition 1.31, there is an ordinal κ such that S is isomorphic to κ. Since S is left-finite in itself (item (a)), the ordinal κ is also left-finite in itself, implying $\kappa \leqslant \omega$, because otherwise we would have $\omega < \kappa$ and

$$\kappa \cap (\leftarrow, \omega]_\kappa = [0, \omega]_\kappa = \omega + 1\,,$$

which is not a finite set. Finally, if S is infinite and bounded above in X, then it is contained in an interval of the form $(\leftarrow, b]_X$ for $b \in X$ and the intersection $S \cap (\leftarrow, b]_X = S$ is infinite, contradicting the left-finiteness of S in X.

Conversely, suppose S is a subset of X that is isomophic to ω and is cofinal in X. Thus, we may identify S with ω, for instance writing $\omega \subset X$, and, since ω is cofinal in X, for every $x \in X$ there is a finite ordinal n such that $x \leqslant n$, which implies that the intersection $\omega \cap (\leftarrow, x]_X$ is contained in the finite interval $[0, n]_\omega$, proving that S is left-finite in X.

(d) By item (c), we know that every left-finite ideal on X is a subideal of $\overset{\text{wo}}{\mathrm{P}}(X)$ that is cofinal and ω_1-dominated, and if it has an infinite element, then $\mathrm{cf}\,(X) = \omega$. Suppose $\mathcal{J}$ is a cofinal ideal on X and suppose S is an infinite element of $\mathcal{J}$. Thus, S is cofinal in X, and if X has a greatest element $x_{\max}$, then $x_{\max} \in S$ and $S - \{x_{\max}\}$ is an infinite element of $\mathcal{J}$ that is not cofinal in X, which is absurd. Thus, X has no greatest element. If x is an arbitrary element of X and if the intersection $S \cap (\leftarrow, x]_X$ is infinite, then $S \cap (\leftarrow, x]_X$ is an infinite element of $\mathcal{J}$, and, since $\mathcal{J}$ is cofinal, $S \cap (\leftarrow, x]_X$ is cofinal in X and x is the greatest element of X, which is absurd. Therefore, for all $x \in X$ the intersection $S \cap (\leftarrow, x]_X$ is finite, proving that S is left-finite in X and the ideal $\mathcal{J}$ is left-finite.

(e) Suppose x is an element of X such that $x_n \xrightarrow[n\to\infty]{\overset{X}{\mathrm{Ordt}}} x$. If x is the greatest element of X, then the intersection

$$\{x_n \mid n \in \mathbb{N}\} \cap (\leftarrow, x]_X = \{x_n \mid n \in \mathbb{N}\}$$

is finite, which is absurd. Hence, x is not the greatest element of X and there is a $y \in X$ such that $x < y$, implying that the interval $(\leftarrow, y)_X$ is a bounded above $\overset{X}{\mathrm{Ordt}}$-neighbourhood of x. Since $x_n \xrightarrow[n\to\infty]{\overset{X}{\mathrm{Ordt}}} x$, the sequence $\{x_n\}$ is bounded above in X, which is absurd by item (c).

(**f**) Take $g, h \in (0_G, \rightarrow)_G$ such that $ng < h$ ($\forall n \in \mathbb{N}$). Since the set

$$\operatorname*{span}_G^{\mathbf{SGrp}}(\{g\}) \cap (\leftarrow, h]_G = \{ng \mid n \in \mathbb{N}\} \cap (\leftarrow, h]_G = \{ng \mid n \in \mathbb{N}\}$$

is infinite, the span $\operatorname*{span}_G^{\mathbf{SGrp}}(\{g\})$ is not left-finite in G. Thus, $\operatorname*{span}_G^{\mathbf{SGrp}}(\{g\}) \notin \mathcal{J}$ since $\mathcal{J}$ is left-finite, proving that Axiom (F2) does not hold for $\mathcal{J}$.

$\square$

Proposition 3.11. *Let G be a non-trivial ordered group.*

(**a**) *The equation $\mathrm{P}_\omega(G) = \overset{\mathrm{lf}}{\mathrm{P}}(G)$ holds if, and only if, $\mathrm{cf}(G) > \omega$;*

(**b**) *The equation $\overset{\mathrm{lf}}{\mathrm{P}}(G) = \overset{\mathrm{wo}}{\mathrm{P}}(G)$ holds if, and only if, G is isomorphic to the ordered group $(\mathbb{Z}, +_\mathbb{Z}, <_\mathbb{Z})$.*

Proof.

(**a**) The negation of that equation is equivalent to the existence of an infinite left-finite subset of G, which, in turn, is equivalent to $\mathrm{cf}(G) = \omega$ (Proposition 3.10, Item (d)).

(**b**) If S is an infinite well-ordered subset of $\mathbb{Z}$ whose least element is m and if n is an arbitrary integer, then $S \cap (\leftarrow, n]_\mathbb{Z} \subset [m, n]_\mathbb{Z}$, implying that the set S is left-finite in $\mathbb{Z}$ and proving the sufficient condition of the item. Suppose that $\overset{\mathrm{lf}}{\mathrm{P}}(G) = \overset{\mathrm{wo}}{\mathrm{P}}(G)$. If G is non-Archimedean, then there are two positive elements g and h of G such that $ng < h$ ($\forall n \in \mathbb{N}$). In that case, the set $\{ng \mid n \in \mathbb{N}\}$ is well-ordered in G and is not left-finite in G, contradicting the supposition. Hence, the ordered group G is Archimedean and can be identified with an ordered subgroup of $(\mathbb{R}, +_\mathbb{R}, <_\mathbb{R})$ (Theorem 1.72). Lastly, if G has no greatest negative element, then there is an increasing sequence $\{g_n\}$ in $(-\infty, 0)_G$, which is absurd since in that case the countable set $\{g_n \mid n \in \mathbb{N}\}$ would be well-ordered in G and would not be left-finite in G. Thus, G is not order-dense in $\mathbb{R}$ (Proposition 1.53) and is isomorphic to $(\mathbb{Z}, +_\mathbb{Z}, <_\mathbb{Z})$ (Proposition 1.68).

$\square$

Example 3.12. $\overset{\mathrm{lf}}{\mathrm{P}}(\mathbb{Z}, +_{\mathbb{Z}}, <_{\mathbb{Z}}) = \overset{\mathrm{wo}}{\mathrm{P}}(\mathbb{Z}, +_{\mathbb{Z}}, <_{\mathbb{Z}})$ (Proposition 3.11, Item (b)).

Example 3.13. Taking into account the fact that the ordered group No_{ω_1} (Example 3.8) has uncountable cofinality ω_1, we obtain $\mathrm{P}_{\omega}(\mathsf{No}_{\omega_1}) = \overset{\mathrm{lf}}{\mathrm{P}}(\mathsf{No}_{\omega_1})$ (Proposition 3.11, Item (a)).

Proposition 3.14. *Let G be a non-trivial ordered group.*

(**a**) *The set $\overset{\mathrm{lf}}{\mathrm{P}}(G)$ of left-finite subsets of G is an incremental arithmetic Rayner ideal on G (Definition 3.1);*

(**b**) *The set $\overset{\mathrm{lf}}{\mathrm{P}}(G)$ is a full Rayner ideal on G if, and only if, G is isomorphic to an ordered subgroup of* $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$.

Proof.

(a) If S is a subset of G such that $S \cap (\leftarrow, g]_G \in \overset{\mathrm{lf}}{\mathrm{P}}(G)$ $(\forall g \in G)$, then, since each intersection $S \cap (\leftarrow, g]_G$ is left-finite and bounded above in G, then each $S \cap (\leftarrow, g]_G$ is finite (Proposition 3.10, Item (c)), proving that the ideal $\overset{\mathrm{lf}}{\mathrm{P}}(G)$ on G is incremental.

Suppose A and B are two inhabited left-finite subsets of G, let a_0 and b_0 be the least elements of A and B, respectively, and let g be any element of G. If $x \in A + B$ is so that $x \leqslant g$, and if $a \in A$ and $b \in B$ are such that $x = a + b$, then $a + b_0 \leqslant a + b \leqslant g$ and $a_0 + b \leqslant a + b \leqslant g$, implying

$$a \in A \cap (\leftarrow, g + (-b_0)]_G \quad \text{and} \quad b \in B \cap (\leftarrow, (-a_0) + g]_G \, .$$

Thus, that proves the inclusion

$$(A + B) \cap (\leftarrow, g]_G \subset (A \cap (\leftarrow, g + (-b_0)]_G) \times (B \cap (\leftarrow, (-a_0) + g]_G) \, ,$$

and, since the set on right-hand side is finite, the set $(A + B) \cap (\leftarrow, g]_G$ is finite, showing that $A + B \in \overset{\mathrm{lf}}{\mathrm{P}}(G)$.

(**b**) If G is not isomorphic to an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$, then G is non-Archimedean (Theorem 1.72) and the necessary condition of the item follows from Item (f) of Proposition 3.10. Suppose G is an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$. By item (a), the set $\overset{\text{lf}}{\mathrm{P}}(G)$ is a subideal of $\overset{\text{wo}}{\mathrm{P}}(G)$ on G that satisfies Axiom (F1) of Definition 3.1 (Proposition 3.2, Item (c)). Hence, it remains to show that Axiom (F2) also holds for $\overset{\text{lf}}{\mathrm{P}}(G)$. Let A be an inhabited left-finite subset of G contained in $(0_G, \rightarrow)_G$, let a_0 be the least element of the set A and let g be a fixed element of G. Since G is Archimedean, there is a least natural number n_0 such that $g \leqslant n_0 a_0$. If $a_1 a_2 \dots a_k$ is any finite sequence in A such that $a_1 + a_2 + \cdots + a_k \leqslant g$, then we get

$$ka_0 = \overbrace{a_0 + a_0 + \cdots + a_0}^{k \text{ times}} \leqslant a_1 + a_2 + \cdots + a_k \leqslant g \leqslant n_0 a_0 \,,$$

which implies $k \leqslant n_0$. Thus, we obtain

$$\begin{aligned}\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(A) \cap (-\infty, g]_G &\subset \left(\bigcup_{k'=1}^{n_0} \overbrace{A + A + \cdots + A}^{k' \text{ times}}\right) \cap (-\infty, g]_G \\ &= \bigcup_{k'=1}^{n_0} \left(\left(\overbrace{A + A + \cdots + A}^{k' \text{ times}}\right) \cap (-\infty, g]_G\right) ,\end{aligned}$$

proving that $\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(A) \cap (-\infty, g]_G$ is finite and proving that $\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(A) \in \overset{\text{lf}}{\mathrm{P}}(G)$.

$\square$

Example 3.15. The sets $\overset{\text{lf}}{\mathrm{P}}(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ and $\overset{\text{lf}}{\mathrm{P}}(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ are incremental full Rayner ideals on the respective ordered groups $(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ and $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ (Proposition 3.14, Item (b)), and they are cofinal and ω_1-dominated subideals of the respective ideals $\overset{\text{wo}}{\mathrm{P}}(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ and $\overset{\text{wo}}{\mathrm{P}}(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ (Proposition 3.10, Item (d)).

Example 3.16. The set $\overset{\text{lf}}{\mathrm{P}}(\mathsf{BS}_\ell)$ is an incremental arithmetic Rayner ideal on BS_ℓ (Proposition 3.14, Item (a)). That ideal is not full, given that BS_ℓ is non-Archimedean (Proposition 3.10, Item (f)).

3.3 Puiseux ideals

A class of Rayner ideals that is closely connected to the left-finite ideals and has several applications in Complex Analysis and Algebraic Geometry shall be discussed here, and we shall do so with a high degree of generality.

Definition 3.17. Let G be an ordered group.

- ▷ A **Puiseux ordered subgroup of** G is an ordered subgroup H of G that is cofinal in G and is such that the following condition holds:

$$(\forall g \in G)\,(\exists n \in \mathbb{N})\ \ ng \in H\,.$$

Let H be a Puiseux ordered subgroup of G.

- ▷ For every natural number n, we denote by $\overset{1/n}{\mathrm{P}}_H(G)$ the set of subsets S of G such that nS is a left-finite subset of H;
- ▷ The **Puiseux ideal on** G **over** H is the union $\bigcup_{n\in\mathbb{N}} \overset{1/n}{\mathrm{P}}_H(G)$, and it shall be denoted[5] by $\overset{\mathrm{bd}}{\mathrm{P}}_H(G)$.

The literature concerning Puiseux series draws attention solely to the case in which the set of exponents G is equal to $\mathbb{Q}$, and the author believes that that is due to two reasons: all known applications of those series stem from that case, and it is not so straightforward to see how a generalisation to an arbitrary group G can be achieved. Our Definition 3.17 explores the fact that the support of a classical Puiseux series is a set of the form $\{{}^{n_0}/_{d_0}, {}^{n_1}/_{d_1}, {}^{n_2}/_{d_2}, \ldots\} \subset \mathbb{Q}$, where $n_0n_1n_2\ldots$ is a sequence of integers and $d_0d_1d_2\ldots$ is a bounded sequence of natural numbers so that the sequence ${}^{n_0}/_{d_0}, {}^{n_1}/_{d_1}, {}^{n_2}/_{d_2}, \ldots$ is increasing. Thus, by multiplying all elements of that set by a common multiple of all numbers in the bounded sequence $d_0d_1d_2\ldots$, one obtains a set that is clearly

[5] The letters ‘bd’ in this notation stand for ‘**b**ounded **d**enominators’.

left-finite in $\mathbb{Z}$, and that shows that the classical Puiseux series are defined by the Puiseux ideal $\overset{\text{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ over $\mathbb{Z}$, in our terminology. Other Puiseux ideals (Examples 3.20 and 3.21) also give rise to interesting Rayner rng structures, as we shall attest in due course (Examples 3.69, 3.70 and 3.81).

Proposition 3.18. *Let G be an ordered group and let H be a Puiseux ordered subgroup of G.*

(a) *Each set $\overset{1/n}{\mathrm{P}}_H(G)$ for $n \in \mathbb{N}$ is a subideal of $\overset{\text{lf}}{\mathrm{P}}(G)$ on G. In particular, $\overset{1/n}{\mathrm{P}}_H(G)$ is ω_1-dominated and cofinal;*

(b) $\left(\forall A \in \overset{1/n}{\mathrm{P}}_H(G)\right) (\forall m \in \mathbb{N})\ (mn)\, A \in \overset{\text{lf}}{\mathrm{P}}(H)$;

(c) *The set $\overset{\text{bd}}{\mathrm{P}}_H(G)$ is a subideal of $\overset{\text{lf}}{\mathrm{P}}(G)$ on G. In particular, $\overset{\text{bd}}{\mathrm{P}}_H(G)$ is ω_1-dominated and cofinal.*

Suppose G is commutative in the following items.

(d) *Each set $\overset{1/n}{\mathrm{P}}_H(G)$ for $n \in \mathbb{N}$ is an arithmetic Rayner ideal on G;*

(e) *The set $\overset{\text{bd}}{\mathrm{P}}_H(G)$ is an arithmetic Rayner ideal on G;*

(f) *Each set $\overset{1/n}{\mathrm{P}}_H(G)$ for $n \in \mathbb{N}$ and the set $\overset{\text{bd}}{\mathrm{P}}_H(G)$ are full Rayner ideals on G if, and only if, G is isomorphic to an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$.*

Proof.

(a) We leave the proof that $\overset{1/n}{\mathrm{P}}_H(G)$ is an ideal[6] on G to the reader. We shall prove that $\overset{1/n}{\mathrm{P}}_H(G) \subset \overset{\text{lf}}{\mathrm{P}}(G)$. Take an element $A \in \overset{1/n}{\mathrm{P}}_H(G)$. The function $f : A \to nA$ given by $f(x) := nx$ is an isomorphism between ordered sets

[6] Recall that Assumption 3.3 is in force.

(Proposition 1.65, Item (c)), implying that A is either finite or isomorphic to ω (Proposition 3.10, Item (c)). Moreover, since nA is cofinal in H and since H is cofinal in G, the set nA is cofinal in G, and if g is an arbitrary element of G, then there is an $s \in A$ so that $ng \leqslant ns$, which gives us $g \leqslant s$ and proves that A is cofinal in G. Hence, A is left-finite in G.

(b) Note that (Proposition 3.14, Item (a))

$$(mn)\,A = m\,(nA) \subset \overbrace{nA + nA + \cdots + nA}^{m \text{ times}} \in \overset{\mathrm{lf}}{\mathrm{P}}\,(H)\ .$$

(c) By item (a), we know that $\overset{\mathrm{bd}}{\mathrm{P}}_H(G)$ is closed under subsets and is contained in $\overset{\mathrm{lf}}{\mathrm{P}}\,(G)$. Let m and n be two natural numbers and take two elements $A \in \overset{1/m}{\mathrm{P}}_H(G)$ and $B \in \overset{1/n}{\mathrm{P}}_H(G)$. We shall prove that $A \cup B \in \overset{\mathrm{bd}}{\mathrm{P}}_H(G)$. We have $mA, nB \in \overset{\mathrm{lf}}{\mathrm{P}}\,(H)$ and $(mn)\,A, (mn)\,B \in \overset{\mathrm{lf}}{\mathrm{P}}\,(H)$ by item (b), implying

$$(mn)\,(A \cup B) = (mn)\,A \cup (mn)\,B \in \overset{\mathrm{lf}}{\mathrm{P}}\,(H)$$

and proving the item.

(d) Take two elements A and B of $\overset{1/n}{\mathrm{P}}_H(G)$. Then $nA, nB \in \overset{\mathrm{lf}}{\mathrm{P}}\,(H)$, and, since (Proposition 3.14, Item (a))

$$n\,(A + B) = nA + nB \in \overset{\mathrm{lf}}{\mathrm{P}}\,(H)$$

we have $A + B \in \overset{1/n}{\mathrm{P}}_H(G)$.

(e) Take two natural numbers m and n and two elements $A \in \overset{1/m}{\mathrm{P}}_H(G)$ and $B \in \overset{1/n}{\mathrm{P}}_H(G)$. Then $mA, nB \in \overset{\mathrm{lf}}{\mathrm{P}}\,(H)$ and $(mn)\,A, (mn)\,B \in \overset{\mathrm{lf}}{\mathrm{P}}\,(H)$ by item (b), implying (Proposition 3.14, Item (a))

$$(mn)\,(A + B) = (mn)\,A + (mn)\,B \in \overset{\mathrm{lf}}{\mathrm{P}}\,(H)$$

and $A + B \in \overset{1/(mn)}{\mathrm{P}}_H(G)$.

(f) If G is not isomorphic to an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$, then G is non-Archimedean (Theorem 1.72) and the necessary condition of the item follows from item (a) and Item (f) of Proposition 3.10. Suppose G is an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$. By items (d) and (e), the sets $\overset{1/n}{\mathrm{P}}_H(G)$ and $\overset{\mathrm{bd}}{\mathrm{P}}_H(G)$ are subideals of $\overset{\mathrm{wo}}{\mathrm{P}}(G)$ on G that satisfy Axiom (F1) of Definition 3.1 (Proposition 3.2, Item (c)). Hence, it remains to show that Axiom (F2) also holds for $\overset{1/n}{\mathrm{P}}_H(G)$ and $\overset{\mathrm{bd}}{\mathrm{P}}_H(G)$, and given that $\overset{\mathrm{bd}}{\mathrm{P}}_H(G) = \bigcup_{n\in\mathbb{N}} \overset{1/n}{\mathrm{P}}_H(G)$, it suffices to prove the result for $\overset{1/n}{\mathrm{P}}_H(G)$. Take an element A of $\overset{1/n}{\mathrm{P}}_H(G)$ contained in $[0_G, \rightarrow)_G$. Thus, $nA \in \overset{\mathrm{lf}}{\mathrm{P}}(H)$ and $nA \subset [0_G, \rightarrow)_H$. Since $\overset{\mathrm{lf}}{\mathrm{P}}(H)$ is a full Rayner ideal on H (Proposition 3.14, Item (b)), we have

$$n \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(A) = \underset{H}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(nA) \in \overset{\mathrm{lf}}{\mathrm{P}}(H) ,$$

which gives us $\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(A) \in \overset{1/n}{\mathrm{P}}_H(G)$ and proves the item.

□

Example 3.19. The classical Puiseux ideal is the ideal $\overset{\mathrm{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ over $\mathbb{Z}$. It is a full Rayner ideal on the ordered group $(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ that is ω_1-dominated and cofinal (Proposition 3.18).

In general, Puiseux ideals are not incremental. For instance, for each $q \in (0, \infty)_{\mathbb{Q}}$ let S_q be the finite set

$$S_q \coloneqq \left\{n + \frac{1}{n} \;\middle|\; n \in \mathbb{N} \text{ and } n + \frac{1}{n} \leqslant q\right\}$$

and let $S \coloneqq \bigcup_{q\in(0,\infty)_{\mathbb{Q}}} S_q$. Since one may not express the elements of S so that the set of their denominators is bounded, the set S is not an element of $\overset{\mathrm{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$, but clearly we have

$$(\forall q \in \mathbb{Q}) \;\; S \cap (\leftarrow, q]_{\mathbb{Q}} = S_q \in \mathrm{P}_{\omega}(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}}) \subset \overset{\mathrm{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}}) ,$$

proving that the classical Puiseux ideal $\overset{\mathrm{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ is not incremental.

Example 3.20. Let $d \coloneqq d_1 d_2 \dots d_k$ be a fixed finite sequence of natural numbers, and let $\mathcal{D}_d$ be the set

$$\mathcal{D}_d \coloneqq \left\{ \frac{n}{d_1^{e_1} d_2^{e_2} \cdots d_k^{e_k}} \mid n \in \mathbb{Z} \text{ and } e_1 e_2 \dots e_k \in \mathbb{N}_0 \right\} .$$

It is easy to check that $\mathcal{D}_d$ is an ordered subgroup of $(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ in which $\mathbb{Z}$ is a Puiseux ordered subgroup. Thus, the set $\overset{\text{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathcal{D}_d)$ is a full Rayner ideal on $\mathcal{D}_d$ that is ω_1-dominated and cofinal (Proposition 3.18), and its elements are of the form $\{{}^{n_0}/_{h_0}, {}^{n_1}/_{h_1}, {}^{n_2}/_{h_2}, \dots\}$, where $n_0 n_1 n_2 \dots$ is a sequence of integers and $h_0 h_1 h_2 \dots$ is a bounded sequence of natural numbers so that each h_i is of the form $h_i = d_1^{e_1} d_2^{e_2} \cdots d_k^{e_k}$ for $e_1 e_2 \dots e_k \in \mathbb{N}_0$ and so that the sequence ${}^{n_0}/_{h_0}, {}^{n_1}/_{h_1}, {}^{n_2}/_{h_2}, \dots$ is increasing.

Example 3.21. Let $\mathcal{P}$ be the set

$$\mathcal{P} \coloneqq \left\{ \frac{n}{p_1 p_2 \cdots p_k} \mid n \in \mathbb{Z} \text{ and } p_1 p_2 \dots p_k \text{ is a finite sequence of distinct primes} \right\} ,$$

where the product $p_1 p_2 \cdots p_k$ equals 1 in the case $k = 0$, as usual. It is easy to check that $\mathcal{P}$ is an ordered subgroup of $(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ in which $\mathbb{Z}$ is a Puiseux ordered subgroup. Thus, the set $\overset{\text{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathcal{P})$ is a full Rayner ideal on $\mathcal{P}$ that is ω_1-dominated and cofinal (Proposition 3.18), and its elements are of the form $\{{}^{n_0}/_{h_0}, {}^{n_1}/_{h_1}, {}^{n_2}/_{h_2}, \dots\}$, where $n_0 n_1 n_2 \dots$ is a sequence of integers and $h_0 h_1 h_2 \dots$ is a bounded sequence of natural numbers so that each h_i is of the form $h_i = p_1 p_2 \cdots p_k$ for distinct primes $p_1 p_2 \dots p_k$ and so that the sequence ${}^{n_0}/_{h_0}, {}^{n_1}/_{h_1}, {}^{n_2}/_{h_2}, \dots$ is increasing.

From now on, in the examples, the addition operation and the order of the ordered groups $G = (\mathbb{Z}, +_{\mathbb{Z}}, <_{\mathbb{Z}})$, $G = (\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ and $G = (\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ will be left implicit whenever we write the Rayner ideals $\mathrm{P}_\omega(G)$, $\overset{\text{wo}}{\mathrm{P}}(G)$, $\overset{\text{lf}}{\mathrm{P}}(G)$ and $\overset{\text{bd}}{\mathrm{P}}_H(G)$ (where H is a Puiseux ordered subgroup of G). Thus, for instance, the Rayner ideals $\mathrm{P}_\omega(\mathbb{Z}, +_{\mathbb{Z}}, <_{\mathbb{Z}})$, $\overset{\text{wo}}{\mathrm{P}}(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$, $\overset{\text{lf}}{\mathrm{P}}(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ and $\overset{\text{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathbb{Q}, +_{\mathbb{Q}}, <_{\mathbb{Q}})$ will be denoted simply by $\mathrm{P}_\omega(\mathbb{Z})$, $\overset{\text{wo}}{\mathrm{P}}(\mathbb{Q})$, $\overset{\text{lf}}{\mathrm{P}}(\mathbb{R})$ and $\overset{\text{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathbb{Q})$, respectively.

3.4 Rayner monoids and Rayner rngs

Consider a monoid M, an ordered set J, a subideal $\mathcal{J}$ of $\overset{\text{wo}}{\mathrm{P}}(J)$ on J and the set of families

$$\mathcal{L} := \{x \in {}^{J}M \mid \operatorname{supp}(x) \in \mathcal{J}\} ,$$

where the support $\operatorname{supp}(x)$ is defined in Definition 1.5. The following proposition shows that $\mathcal{L}$ naturally inherits a monoid structure from M whose operation is defined pointwise.

Proposition 3.22. *Let M be a monoid, let J be an ordered set, let $\mathcal{J}$ be a subideal of $\overset{\text{wo}}{\mathrm{P}}(J)$ on J and consider the function $\times_{\mathcal{L}} \colon \mathcal{L} \times \mathcal{L} \to {}^{J}M$ given by $(x \times_{\mathcal{L}} y)_j := x_j y_j$.*

(a) $(\forall x, y \in \mathcal{L})\ \operatorname{supp}(x \times_{\mathcal{L}} y) \subset \operatorname{supp}(x) \cup \operatorname{supp}(y)$;

(b) *$\times_{\mathcal{L}}$ is of type $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$;*

(c) *The set $\mathcal{L}$ is a submonoid of the power monoid ${}^{J}M$ when endowed with the operation $\times_{\mathcal{L}}$.*

Proof. We leave the proof of item (c) to the reader.

(a) If $j \notin \operatorname{supp}(x) \cup \operatorname{supp}(y)$, then $(x \times_{\mathcal{L}} y)_j = x_j y_j = 1_M \cdot 1_M = 1_M$, that is, $j \notin \operatorname{supp}(x \times_{\mathcal{L}} y)$.

(b) For all $x, y \in \mathcal{L}$, we have $\operatorname{supp}(x) \cup \operatorname{supp}(y) \in \mathcal{J}$ and $\operatorname{supp}(x \times_{\mathcal{L}} y) \in \mathcal{J}$ by item (a), resulting in $x \times_{\mathcal{L}} y \in \mathcal{L}$.

$\square$

Proposition 3.22 sets the scene for the following definition:

Definition 3.23. Let M be a monoid, let J be an ordered set and let $\mathcal{J}$ be a subideal of $\overset{\mathrm{wo}}{\mathrm{P}}(J)$ on J. The ($\mathcal{J}$-)**Rayner monoid with coefficients in** M **and indices in** J is the monoid denoted by $\overset{\mathcal{J}}{M}[[\mathrm{X}^J]]$ and given by

$$\overset{\mathcal{J}}{M}[[\mathrm{X}^J]] \coloneqq \{x \in {}^J M \ | \ \operatorname{supp}(x) \in \mathcal{J}\} \ ,$$

whose multiplication operation is given by $(xy)_j \coloneqq x_j y_j$. Note that the constant family with value 1_M and with indices in J is the identity element of $\overset{\mathcal{J}}{M}[[\mathrm{X}^J]]$. If M is denoted additively, then $\overset{\mathcal{J}}{M}[[\mathrm{X}^J]]$ shall also be denoted additively, resulting that the operation of $\overset{\mathcal{J}}{M}[[\mathrm{X}^J]]$ is given by $(x+y)_j \coloneqq x_j + y_j$ in that case.

Even though the elements x of $\overset{\mathcal{J}}{M}[[\mathrm{X}^J]]$ are technically functions of type $J \to M$, they shall always be regarded as families $x = \{x_j\}_{j \in J}$ in M in this work. Thus, if j is an element of J, then the j-image of x shall not be denoted by $x(j)$, but instead by x_j.

Example 3.24. The elements of the Rayner monoid $\overset{\mathrm{P}_\omega(J)}{M}[[\mathrm{X}^J]]$ are the families $x = \{x_j\}_{j \in J}$ in M whose supports $\operatorname{supp}(x) = \{j \in J \ | \ x_j \neq 1_M\}$ are finite.

If R is a rng, if G is an ordered group and if $\mathcal{J}$ is an arithmetic Rayner ideal on G, then we shall denote the Rayner monoid $\overset{\mathcal{J}}{(R, +_R)}[[\mathrm{X}^G]]$ simply by $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. One can easily observe that $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is a commutative group whose identity element is the constant family $\{0_R\}_{g \in G}$. Furthermore, since $\mathcal{J} \subset \overset{\mathrm{wo}}{\mathrm{P}}(G)$, note that for all $x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and all $p \in G$, the sum $\sum\limits_{\substack{g,h \in G \\ g+h=p}} x_g y_h$ in R has only a finite number of non-zero summands (Lemma 1.75). It turns out that that fact and the assumption that $\mathcal{J}$ is arithmetic can be explored to define a multiplication operation on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ so that it becomes a rng. The following proposition shows how that can be accomplished.

Proposition 3.25. (181, 117) *Let R be a rng, let G be an ordered group, let $\mathcal{J}$ be an arithmetic Rayner ideal on G and consider the function*

$$\overset{\mathcal{J}}{\times} : \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \times \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to {}^G R$$

given by

$$(x \overset{\mathcal{J}}{\times} y)_p := \sum_{\substack{g,h \in G \\ g+h=p}} x_g y_h \,.$$

(a) $(\forall x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]])\ \operatorname{supp}(x \overset{\mathcal{J}}{\times} y) \subset \operatorname{supp}(x) + \operatorname{supp}(y)$;

(b) $\overset{\mathcal{J}}{\times}$ *is of type* $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \times \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$*;*

(c) *The Rayner monoid* $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ *is a rng when endowed with the multiplication operation* $\overset{\mathcal{J}}{\times}$*.*

Proof.

(a) If $p \notin \operatorname{supp}(x) + \operatorname{supp}(y)$, then

$$(x \overset{\mathcal{J}}{\times} y)_p = \sum_{\substack{g,h \in G \\ g+h=p}} x_g y_h = \sum_{\substack{g,h \in G \\ g+h=p}} 0_R = 0_R \,,$$

that is, $p \notin \operatorname{supp}(x \overset{\mathcal{J}}{\times} y)$.

(b) Let $x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. Since $\mathcal{J}$ is arithmetic, we have $\operatorname{supp}(x) + \operatorname{supp}(y) \in \mathcal{J}$ and $\operatorname{supp}(x \overset{\mathcal{J}}{\times} y) \in \mathcal{J}$ by item (a).

(c) From now on, let us denote a term of the form $x \overset{\mathcal{J}}{\times} y$ simply by xy, let $x, y, z \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and let $p \in G$. The multiplication operation $\overset{\mathcal{J}}{\times}$ is associative, since we have

$$\begin{aligned}
(x\,(yz))_p &= \sum_{\substack{a,h \in G \\ a+h=p}} x_a \left(\sum_{\substack{b,c \in G \\ b+c=h}} y_b z_c \right) \\
&= \sum_{\substack{a,b,c \in G \\ a+b+c=p}} x_a y_b z_c \\
&= \sum_{\substack{g,c \in G \\ g+c=p}} \left(\sum_{\substack{a,b \in G \\ a+b=g}} x_a y_b \right) z_c = ((xy)\,z)_p \,,
\end{aligned}$$

and it is also left-distributive over the addition operation of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, since we have

$$(x\,(y+z))_p = \sum_{\substack{g,h\in G\\ g+h=p}} x_g\,(y_h+z_h) = \sum_{\substack{g,h\in G\\ g+h=p}} (x_g y_h + x_g z_h) = (xy+xz)_p\ .$$

The proof of the right-distributivity is analogous.

$\square$

The rng structure discussed in Item (c) of Proposition 3.25 is the main object of study of this thesis, and it can finally be formally defined:

Definition 3.26. Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. The ($\mathcal{J}$-)**Rayner rng with coefficients in R and exponents in G** is the Rayner monoid $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ endowed with the multiplication operation given by $(xy)_p \coloneqq \sum_{\substack{g,h\in G\\ g+h=p}} x_g y_h$. Each element of the rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is said to be a ($\mathcal{J}$-)**Rayner power series with coefficients in R and exponents in G**. We have the following notations and terminology:

- ▷ The **generalised polynomial rng with coefficients in R and exponents in G** is the $\mathrm{P}_\omega(G)$-Rayner rng

$$\overset{\mathrm{P}_\omega(G)}{R}[[\mathrm{X}^G]] = \{x \in {}^G R \mid \ \operatorname{supp}(x) \ \text{ is finite}\}\ ,$$

  and it is denoted by $R\,[\mathrm{X}^G]$. Note that $R\,[\mathrm{X}^G]$ is a subrng of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, and the polynomial rng $R\,[\mathrm{X}]$ (Definition 2.46) can be canonically identified with a subrng of $R\,[\mathrm{X}^{\mathbb{Z}}]$;

- ▷ Suppose G is commutative and let H be a Puiseux ordered subgroup of G. The **Puiseux rng with coefficients in R and exponents in G over H** is the $\overset{\mathrm{bd}}{\mathrm{P}}_H(G)$-Rayner rng

$$\overset{\overset{\mathrm{bd}}{\mathrm{P}}_H(G)}{R}[[\mathrm{X}^G]] = \{x \in {}^G R \mid \ (\exists n \in \mathbb{N})\,(n \operatorname{supp}(x) \ \text{ is left-finite in } H)\}\ ,$$

  and it is denoted by $\overset{\mathrm{bd}}{R}_H[[\mathrm{X}^G]]$;

▷ The **Levi-Civita rng with coefficients in R and exponents in G** is the $\overset{\mathrm{lf}}{\mathrm{P}}(G)$-Rayner rng

$$\overset{\overset{\mathrm{lf}}{\mathrm{P}}(G)}{R}[[\mathrm{X}^G]] = \{x \in {}^G R \mid \operatorname{supp}(x) \text{ is left-finite in } G\},$$

and it is denoted by $\overset{\mathrm{lf}}{R}[[\mathrm{X}^G]]$;

▷ The **Hahn rng with coefficients in R and exponents in G** is the $\overset{\mathrm{wo}}{\mathrm{P}}(G)$-Rayner rng

$$\overset{\overset{\mathrm{wo}}{\mathrm{P}}(G)}{R}[[\mathrm{X}^G]] = \{x \in {}^G R \mid \operatorname{supp}(x) \text{ is well-ordered in } G\},$$

and it is denoted by $R[[\mathrm{X}^G]]$. Note that $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is a subrng of $R[[\mathrm{X}^G]]$.

The main kinds of Rayner ideals shown in this chapter, along with their properties and their corresponding Rayner rngs, are summarised in the following table:

Table 3.1: Rayner ideals, their properties and their corresponding Rayner rngs.

Rayner ideal	Is it arithmetic?	Is it full?	Is it incremental?	Its corresponding Rayner rng
Polynomial ideal $\mathrm{P}_\omega(G)$	Yes	No	No	Generalised polynomial rng $R[\mathrm{X}^G]$
Puiseux ideal $\overset{\mathrm{bd}}{\mathrm{P}}_H(G)$	Yes if G is commutative	Yes if, and only if, G is isomorphic to an ordered subgroup of the reals	Not necessarily	Puiseux rng $\overset{\mathrm{bd}}{R}_H[[\mathrm{X}^G]]$
Levi-Civita ideal $\overset{\mathrm{lf}}{\mathrm{P}}(G)$	Yes	Yes if, and only if, G is isomorphic to an ordered subgroup of the reals	Yes	Levi-Civita rng $\overset{\mathrm{lf}}{R}[[\mathrm{X}^G]]$
Hahn ideal $\overset{\mathrm{wo}}{\mathrm{P}}(G)$	Yes	Yes	Yes	Hahn rng $R[[\mathrm{X}^G]]$

Example 3.27. The classical **Puiseux field** is the Puiseux rng $\overset{\mathrm{bd}}{\mathbb{C}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]$. Thus, it is a Rayner rng, and we will verify that it is a field in Section 3.8.

Example 3.28. The **real Levi-Civita field** is the Levi-Civita rng $\overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]$, and, similarly, the **complex Levi-Civita field** is the Levi-Civita rng $\overset{\mathrm{lf}}{\mathbb{C}}[[\mathrm{X}^{\mathbb{Q}}]]$ (cf. Introduction). Thus, they are both Rayner rngs, and we will verify that they are fields in Section 3.8.

Example 3.29. The **field of Hahn series** with coefficients in a field K and exponents in a commutative group G is the Hahn rng $K\,[[\mathrm{X}^{G}]]$ which was first studied by Hahn in 1907 (92). We will verify that this Rayner rng is a field in Section 3.8.

Example 3.30. The **field of Laurent series** is the Hahn rng $\mathbb{C}\,[[\mathrm{X}^{\mathbb{Z}}]]$ which was (officially[7]) discovered by Laurent in 1843 (132), and it is fundamental to the study of Complex Analysis due to the fact that holomorphic functions on open regions in $\mathbb{C}$ have Laurent series expansions centred on their non-essential singularities. That Rayner rng structure is also a Levi-Civita rng since we have $\overset{\mathrm{lf}}{\mathbb{C}}[[\mathrm{X}^{\mathbb{Z}}]] = \mathbb{C}\,[[\mathrm{X}^{\mathbb{Z}}]]$ (Example 3.12).

A handful of additional notations are quite helpful in dealing with Rayner rngs and their elements:

List of Notations 3.31. Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G.

- ▹ For each subset S of R containing the zero element 0_R, we denote the set $\{x \in {}^{G}S \mid \operatorname{supp}(x) \in \mathcal{J}\}$ by $\overset{\mathcal{J}}{S}[[\mathrm{X}^{G}]]$. As special cases, we denote the sets

$$\overset{\mathrm{P}_\omega(G)}{S}[[\mathrm{X}^{G}]]\,,\quad \overset{\overset{\mathrm{lf}}{\mathrm{P}}(G)}{S}[[\mathrm{X}^{G}]]\,,\quad \overset{\overset{\mathrm{bd}}{\mathrm{P}}_{H}(G)}{S}[[\mathrm{X}^{G}]] \quad \text{and} \quad \overset{\overset{\mathrm{wo}}{\mathrm{P}}(G)}{S}[[\mathrm{X}^{G}]]$$

  by $S\,[\mathrm{X}^{G}]$, $\overset{\mathrm{lf}}{S}[[\mathrm{X}^{G}]]$, $\overset{\mathrm{bd}}{S}_{H}[[\mathrm{X}^{G}]]$ and $S\,[[\mathrm{X}^{G}]]$, respectively, where H is a Puiseux ordered subgroup of G;

[7] Weierstraß may have already studied it in 1841 but did not publish his findings at the time.

▷ Given $g \in G$ and $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, we shall sometimes denote a term of the form x_g by $x\,[\![g]\!]$, especially when the group element g is indicated by a large expression. In these situations, the standard subscript notation x_g looks rather cumbersome and should be avoided;

▷ Each element x of the Rayner rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is also denoted by a formal power series $x = \sum_{g \in G} x_g \mathrm{X}^g$;

▷ Let I be any set, let $f : I \to G$ be an injective function so that $\mathrm{Im}\,(f) \in \mathcal{J}$, and let $\{c_g\}_{g \in \mathrm{Im}(f)}$ be a family in R. The family $x = \{x_g\}_{g \in G}$ in R given by

$$x_g := \begin{cases} c_g & \text{if } g \in \mathrm{Im}\,(f)\ , \\ 0_R & \text{if } g \in G - \mathrm{Im}\,(f) \end{cases}$$

is denoted by $\sum_{i \in I} c_{f(i)} \mathrm{X}^{f(i)}$. The support of that power series is clearly contained in the image $\mathrm{Im}\,(f)$, thus being an element of $\mathcal{J}$ (Definition 3.1, Axiom (I1)), and we have $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. There is an additional twist to this notation: quite often the condition $i \in I$ is expressed by an equivalent condition $P\,(i)$, and that alternative form often leaves the set I implicit. Thus, the newly defined element x of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ may also be denoted by $\sum_{P(i)} c_{f(i)} \mathrm{X}^{f(i)}$, where $P\,(i) \Leftrightarrow i \in I$ (cf. Example 3.33);

▷ For each $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, we define $x_{\multimap} := 0_R$, where $\multimap$ is the greatest element of $\breve{G}$ (Definition 1.62). Note that, although one may write the term $x_{\multimap}$, that does not mean that $\multimap$ is in the domain $\mathrm{Dom}\,(x) = G$. Indeed, it is not. That non-mandatory supplemental definition of $x_{\multimap}$ is merely pragmatical and it comes in handy in a few situations to reduce the number of logical cases in some proofs;

- ▷ For all $g \in G$ and all $r \in R$, the element x of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ given by

$$x_{g'} \coloneqq \begin{cases} r & \text{if } g' = g\,, \\ 0_R & \text{if } g' \in G - \{g\} \end{cases}$$

is denoted by $r\mathrm{X}^g$ (see Assumption 3.3);

- ▷ For each subset S of R and for each subset H of G, we denote the set $\{s\mathrm{X}^h \mid s \in S \text{ and } h \in H\}$ by $S\,\{\mathrm{X}^h\}_{h\in H}$;

- ▷ If R is a ring, then for each $g \in G$, the element $1_R\mathrm{X}^g$ of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is denoted by X^g;

- ▷ For each $r \in R$, the element $r\mathrm{X}^{0_G}$ of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is denoted by r, by abuse of language. In particular, the element $0_R X^{0_G}$ is denoted by 0_R, and, if R is a ring, then the element $1_R\mathrm{X}^{0_G}$ is denoted by 1_R.

Example 3.32. Take the three Rayner rngs $\overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]$ (Example 3.28), $\mathbb{R}\,[[\mathrm{X}^{\mathbb{Q}}]]$ and $\mathbb{C}\,[[\mathrm{X}^{\mathbb{Z}}]]$ (Example 3.30), and consider the following three formal power series:

$$\begin{aligned} x &\coloneqq 1 + 2\,\mathrm{X}^{1/2} + (\sqrt{2}+1)\,\mathrm{X} + (\sqrt{3}+1)\,\mathrm{X}^{3/2} + \cdots + (\sqrt{n}+1)\,\mathrm{X}^{n/2} + \cdots \\ y &\coloneqq (\mathrm{X}^{-1} + \mathrm{X}^{-1/2} + \mathrm{X}^{-1/3} + \mathrm{X}^{-1/4} + \cdots + \mathrm{X}^{-1/n} + \cdots) + \mathrm{X}^3 \\ z &\coloneqq i\,\mathrm{X}^{-3} - \mathrm{X}^2 - i\,\mathrm{X}^5 \end{aligned}$$

Their supports are the subsets of $\mathbb{Q}$ given by $\operatorname{supp}(x) = \{n/2 \mid n \in \mathbb{N}_0\}$, $\operatorname{supp}(y) = \{-1/n \mid n \in \mathbb{N}\} \cup \{3\}$ and $\operatorname{supp}(z) = \{-3, 2, 5\}$, which happen to be well-ordered subsets of $\mathbb{Q}$. Thus, we have $x, y \in \mathbb{R}\,[[\mathrm{X}^{\mathbb{Q}}]]$ and $z \in \mathbb{C}\,[[\mathrm{X}^{\mathbb{Z}}]]$, while $x, y \notin \mathbb{C}\,[[\mathrm{X}^{\mathbb{Z}}]]$ and $z \notin \mathbb{R}\,[[\mathrm{X}^{\mathbb{Q}}]]$, since $\operatorname{supp}(x)$ and $\operatorname{supp}(y)$ have non-integer elements and since z has imaginary coefficients. The ordered set $\operatorname{supp}(x)$ is isomorphic to the ordinal ω and is cofinal in $\mathbb{Q}$, implying that it is left-finite in $\mathbb{Q}$ (Proposition 3.10, Item (c)) and $x \in \overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]$. Regarding y, the ordered set $\operatorname{supp}(y)$ is neither isomorphic to ω nor cofinal in $\mathbb{Q}$, which gives us that $\operatorname{supp}(y)$ is not left-finite in $\mathbb{Q}$ and $y \notin \overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]$. Finally, we have $z \notin \overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]$ since it has imaginary coefficients.

Example 3.33. Consider the functions

$$f : \{a^3 + b^3 + c^3 \mid a, b, c \in \mathbb{N}\} \to \mathbb{Q}$$

and

$$g : \{p \text{ prime} \mid p = 4n + 1 \ (\exists n \in \mathbb{N})\} \to \mathbb{Z}$$

given by $f(n) := n/3$ and $g(p) := p^2$, which are clearly injective, and it is easy to verify that $\mathrm{Im}(f) \in \overset{\mathrm{lf}}{\mathrm{P}}(\mathbb{Q})$ and $\mathrm{Im}(g) \in \overset{\mathrm{wo}}{\mathrm{P}}(\mathbb{Z})$. Thus, we can employ an item of our List of Notations 3.31 to define the following power series

$$x := \sum_{\substack{n \text{ is a sum}\\ \text{of three cubes}}} n\mathrm{X}^{n/3} = 3\,\mathrm{X} + 10\,\mathrm{X}^{10/3} + 17\,\mathrm{X}^{17/3} + 24\,\mathrm{X}^8 + \cdots \in \overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]$$

and

$$y := \sum_{\substack{p \text{ is a prime of}\\ \text{the form } 4n+1}} \sqrt{p}\,\mathrm{X}^{p^2} = \sqrt{5}\,\mathrm{X}^{25} + \sqrt{13}\,\mathrm{X}^{169} + \sqrt{17}\,\mathrm{X}^{289} + \cdots \in \mathbb{R}[[\mathrm{X}^{\mathbb{Z}}]]\,,$$

where the expressions for the coefficients of x and y were arbitrarily chosen.

Proposition 3.34. (181, 117) *Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G.*

(a) *The function of type $R \to \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ given by $r \mapsto r\mathrm{X}^{0_G} = r$ is an injective homomorphism. Thus, we shall identify R with a subrng of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ via that function, writing $R \subset \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$;*

(b) $\overset{\mathcal{J}}{\{0_R\}}[[\mathrm{X}^G]] = \{0_R\}$ *and* $\overset{\mathcal{J}}{R}[[\mathrm{X}^{\{0_G\}}]] \overset{\mathbf{Rng}}{\cong} R$;

(c) *The products rx and xr, between an element r of R and an element x of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, are computed pointwise. That is, we have*

$$rx = \sum_{x \in G} r x_g \mathrm{X}^g \quad \text{and} \quad xr = \sum_{x \in G} x_g r \mathrm{X}^g\,;$$

(d) *If R is a ring, then $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is an R-module when endowed with the left R-action $\odot : R \times \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ given by $\odot(r, x) := rx$.*

Proof. The proofs are straightforward and are left to the reader.

□

Item (b) of Proposition 3.34 shows that the cases in which R or G are trivial add no insight to the purposes of our discussion. From now on, we make the following assumption:

Assumption 3.35. *The rng R and the ordered group G are non-trivial, that is, they have more than one element.*

Thus, by Proposition 1.65, the ordered group G is infinite and unbounded above and below. In particular, we have $\mathrm{cf}\,(G) \geqslant \omega$.

Corollary 3.36. *Let R be a rng and let G be an ordered group.*

(**a**) *The Levi-Civita rng $\overset{\mathrm{lf}}{R}[[\mathrm{X}^G]]$ is equal to the generalised polynomial rng $R\,[\mathrm{X}^G]$ if, and only if, $\mathrm{cf}\,(G) > \omega$;*

(**b**) *The Levi-Civita rng $\overset{\mathrm{lf}}{R}[[\mathrm{X}^G]]$ is equal to the Hahn rng $R\,[[\mathrm{X}^G]]$ if, and only if, G is isomorphic to the ordered group $(\mathbb{Z}, +_{\mathbb{Z}}, <_{\mathbb{Z}})$.*

Proof. Since R and G are non-trivial, the corollary is nothing but a rewording of Proposition 3.11.

$\square$

Example 3.37. We have $R\,[\mathrm{X}^{\mathsf{No}_{\omega_1}}] = \overset{\mathrm{lf}}{R}[[\mathrm{X}^{\mathsf{No}_{\omega_1}}]] \subsetneq R\,[[\mathrm{X}^{\mathsf{No}_{\omega_1}}]]$ for all rngs R, since $\mathrm{cf}\,(\mathsf{No}_{\omega_1}) = \omega_1 > \omega$ and since No_{ω_1} is not isomorphic to the ordered group $(\mathbb{Z}, +_{\mathbb{Z}}, <_{\mathbb{Z}})$ (Example 3.8; Corollary 3.36). Hence, every element of $\overset{\mathrm{lf}}{R}[[\mathrm{X}^{\mathsf{No}_{\omega_1}}]]$ is a generalised polynomial, but infinitely many elements of $R\,[[\mathrm{X}^{\mathsf{No}_{\omega_1}}]]$ are not.

Example 3.38. We have $R\,[\mathrm{X}^{\mathsf{BS}_\ell}] \subsetneq \overset{\mathrm{lf}}{R}[[\mathrm{X}^{\mathsf{BS}_\ell}]] \subsetneq R\,[[\mathrm{X}^{\mathsf{BS}_\ell}]]$ for all rngs R, since $\mathrm{cf}\,(\mathsf{BS}_\ell) = \omega$ and since BS_ℓ is not isomorphic to the ordered group $(\mathbb{Z}, +_{\mathbb{Z}}, <_{\mathbb{Z}})$.

Proposition 3.39. *Let R be a rng, let S be a subset of R containing 0_R and let G be an ordered group. We have:*

$$|S[\mathrm{X}^G]| = \overset{\mathsf{Card}}{\max}\{|S|, |G|\} \quad \text{and} \quad |\overset{\mathrm{lf}}{S}[[\mathrm{X}^G]]| = \begin{cases} (\overset{\mathsf{Card}}{\max}\{|S|, |G|\})^{\omega} \geqslant 2^{\omega} & \text{if } \mathrm{cf}(G) = \omega\,, \\ \overset{\mathsf{Card}}{\max}\{|S|, |G|\} \geqslant \omega_1 & \text{if } \mathrm{cf}(G) > \omega\,. \end{cases}$$

Proof. Since $|\mathrm{P}_\omega(G)| = |G|$ (38), we get

$$\begin{aligned} |S[\mathrm{X}^G]| &= \Big|\bigcup_{F \in \mathrm{P}_\omega(G)} \{x \in {}^G S \mid \mathrm{supp}(x) = F\}\Big| \\ &= |S| \cdot |\mathrm{P}_\omega(G)| \\ &= \overset{\mathsf{Card}}{\max}\{|S|, |G|\}\ , \end{aligned}$$

and if $\mathrm{cf}(G) > \omega$, then the supports of the elements of $\overset{\mathrm{lf}}{S}[[\mathrm{X}^G]]$ are finite (Corollary 3.36, Item (a)), and we get

$$|\overset{\mathrm{lf}}{S}[[\mathrm{X}^G]]| = |S[\mathrm{X}^G]| = \overset{\mathsf{Card}}{\max}\{|S|, |G|\}\ .$$

This being the case, we have $|G| \geqslant \mathrm{cf}(G) > \omega$, which conveys $|G| \geqslant \omega_1$ and implies $|\overset{\mathrm{lf}}{S}[[\mathrm{X}^G]]| \geqslant \omega_1$. Suppose $\mathrm{cf}(G) = \omega$, and let

$$f : \{\{g_n\} \in {}^{\mathbb{N}}G \mid \{g_n\} \text{ is increasing and cofinal in } G\} \times {}^{\mathbb{N}}S \to \overset{\mathrm{lf}}{S}[[\mathrm{X}^G]]$$

be the function given by

$$(f(\{g_n\}, \{r_n\}))_g \coloneqq \begin{cases} r_n & \text{if } g = g_n\,, \\ 0_R & \text{if } g \in G - \{g_n\}\ . \end{cases}$$

Then, f is bijective, and, by Lemma 1.76, we get

$$\begin{aligned} |\overset{\mathrm{lf}}{S}[[\mathrm{X}^G]]| &= |\{\{g_n\} \in {}^{\mathbb{N}}G \mid \{g_n\} \text{ is increasing and cofinal in } G\}| \cdot |{}^{\mathbb{N}}S| \\ &= (\overset{\mathsf{Card}}{\max}\{|S|, |G|\})^{\omega} \\ &\geqslant \omega^{\omega} = 2^{\omega}\,. \end{aligned}$$

□

3.5 $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ as a G-pseudovalued rng

Given a Rayner rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, the least element of the support $\mathrm{supp}\,(x)$ of each $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ has a special significance to the theory, and so does the coefficient of x associated to that least element. The reason why that is so is the subject of this section.

We have the following notations and terminology:

Definition 3.40. Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G.

- ▷ The **min-supp function on** $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is the function $\overset{\mathcal{J}}{\mathrm{ms}} : \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to \breve{G}$ given by

$$\overset{\mathcal{J}}{\mathrm{ms}}\,(x) := \begin{cases} \overset{G}{\min}\,(\mathrm{supp}\,(x)) & \text{if } x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] - \{0_R\}\ , \\ \twoheadrightarrow & \text{if } x = 0_R\,; \end{cases}$$

- ▷ The **primary coefficient** of an element x of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is the element of R denoted by $\mathrm{pc}\,(x)$ and given by $\mathrm{pc}\,(x) := x_{\overset{\mathcal{J}}{\mathrm{ms}}(x)}$. Note that the conditions $\overset{\mathcal{J}}{\mathrm{ms}}\,(x) = \twoheadrightarrow$, $\mathrm{pc}\,(x) = 0_R$ and $x = 0_R$ are equivalent.

With that, if x is an element of the Rayner rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ with support $\mathrm{supp}\,(x) = \{g_1 g_2 g_3 \dots\}$ so that $g_1 < g_2 < g_3 < \cdots$, then we have

$$\overset{\mathcal{J}}{\mathrm{ms}}\,(x) = \overset{\mathcal{J}}{\mathrm{ms}}\,(x_{g_1}\mathrm{X}^{g_1} + x_{g_2}\mathrm{X}^{g_2} + x_{g_3}\mathrm{X}^{g_3} + \cdots) = g_1$$

and

$$\mathrm{pc}\,(x) = \mathrm{pc}\,(x_{g_1}\mathrm{X}^{g_1} + x_{g_2}\mathrm{X}^{g_2} + x_{g_3}\mathrm{X}^{g_3} + \cdots) = x_{g_1}\ .$$

The min-supp function $\overset{\mathcal{J}}{\mathrm{ms}} : \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to \breve{G}$ turns out to be the primary connection between Valuation Theory and the Theory of Rayner Rngs, and it is the most important function of the latter theory.

Theorem 3.41. (181) *Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. The min-supp function $\overset{\mathcal{J}}{\mathrm{ms}} : \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to \breve{G}$ is a G-pseudovaluation on the rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ (Definition 2.44).*

Proof. By definition, the conditions $\overset{\mathcal{J}}{\mathrm{ms}}(x) = \multimap$ and $x = 0_R$ are equivalent for all $x \in R$. Let $x, y \in R - \{0_R\}$. To begin with, suppose we have

$$\overset{\mathcal{J}}{\mathrm{ms}}(x-y) < \overset{\breve{G}}{\min}\left\{\overset{\mathcal{J}}{\mathrm{ms}}(x), \overset{\mathcal{J}}{\mathrm{ms}}(y)\right\} .$$

Thus, we get $x - y \neq 0_R$, $\overset{\mathcal{J}}{\mathrm{ms}}(x-y) < \overset{\mathcal{J}}{\mathrm{ms}}(x)$, $\overset{\mathcal{J}}{\mathrm{ms}}(x-y) < \overset{\mathcal{J}}{\mathrm{ms}}(y)$, and

$$0_R \neq (x-y)_{\overset{\mathcal{J}}{\mathrm{ms}}(x-y)} = x_{\overset{\mathcal{J}}{\mathrm{ms}}(x-y)} - y_{\overset{\mathcal{J}}{\mathrm{ms}}(x-y)} = 0_R - 0_R = 0_R ,$$

which is absurd, proving the inequality

$$\overset{\mathcal{J}}{\mathrm{ms}}(x-y) \geqslant \overset{\breve{G}}{\min}\left\{\overset{\mathcal{J}}{\mathrm{ms}}(x), \overset{\mathcal{J}}{\mathrm{ms}}(y)\right\} .$$

Now, suppose we have $\overset{\mathcal{J}}{\mathrm{ms}}(xy) < \overset{\mathcal{J}}{\mathrm{ms}}(x) + \overset{\mathcal{J}}{\mathrm{ms}}(y)$. If g and h are elements of G so that $g + h = \overset{\mathcal{J}}{\mathrm{ms}}(xy)$ and $x_g\, y_h \neq 0_R$, then we obtain $x_g \neq 0_R \neq y_h$, $\overset{\mathcal{J}}{\mathrm{ms}}(x) \leqslant g$ and $\overset{\mathcal{J}}{\mathrm{ms}}(y) \leqslant h$, implying

$$\overset{\mathcal{J}}{\mathrm{ms}}(x) + \overset{\mathcal{J}}{\mathrm{ms}}(y) \leqslant g + h = \overset{\mathcal{J}}{\mathrm{ms}}(xy) < \overset{\mathcal{J}}{\mathrm{ms}}(x) + \overset{\mathcal{J}}{\mathrm{ms}}(y) ,$$

which is absurd, proving that the condition $g + h = \overset{\mathcal{J}}{\mathrm{ms}}(xy)$ implies $x_g\, y_h = 0_R$. Therefore, we obtain

$$0_R \neq (xy)_{\overset{\mathcal{J}}{\mathrm{ms}}(xy)} = \sum_{\substack{g,h \in G \\ g+h = \overset{\mathcal{J}}{\mathrm{ms}}(xy)}} x_g\, y_h = 0_R ,$$

which is also absurd, establishing the inequality $\overset{\mathcal{J}}{\mathrm{ms}}(xy) \geqslant \overset{\mathcal{J}}{\mathrm{ms}}(x) + \overset{\mathcal{J}}{\mathrm{ms}}(y)$.

□

Since $\overset{\mathcal{J}}{\mathrm{ms}} : \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to \breve{G}$ is a G-pseudovaluation on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, big values of $\overset{\mathcal{J}}{\mathrm{ms}}(x) \in \breve{G}$ are intuitively interpreted as meaning that the element $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ has a *low order of magnitude*, as usual in Valuation Theory. In particular, if r is a fixed element of R, then note that $\overset{\mathcal{J}}{\mathrm{ms}}(r\mathrm{X}^g) = g$ ($\forall g \in G$), implying that the higher the exponent g of the formal variable X, the lower the order of magnitude of the term $r\mathrm{X}^g$ becomes. That heuristic interpretation shall agree with the results of Section 3.10, where an order on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ will be defined whenever R is an ordered rng.

Notational Device 3.42. Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. We introduce a couple notational devices which make our lines of reasoning a little more flexible and versatile:

(**a**) Let g be an element of $\breve{G}$. With the intention of shifting the notations $\mathrm{O}_g^{\overset{\mathcal{J}}{\mathrm{ms}}}$ and $\mathrm{o}_g^{\overset{\mathcal{J}}{\mathrm{ms}}}$ (Definition 1.79) toward the usual big-O and little-O notations from the theory of Asymptotic Analysis, from now on those sets shall be denoted by $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ and $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$, respectively. Thus, we have

$$\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g) := \{x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \mid \overset{\mathcal{J}}{\mathrm{ms}}(x) \geqslant g\} = \{x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \mid (\forall h \in (\leftarrow, g)_G)\ x_h = 0_R\}$$

and

$$\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g) := \{x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \mid \overset{\mathcal{J}}{\mathrm{ms}}(x) > g\} = \{x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \mid (\forall h \in (\leftarrow, g]_G)\ x_h = 0_R\}\ ;$$

(**b**) (**∀∃-notation**) Let X be a set, let $\mathcal{R}$ be a binary relation on X, take any $m+n$ subsets $O_1O_2\ldots O_m, P_1P_2\ldots P_n$ subsets of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ of the forms $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ or $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$, and take two functions:

$$u_1 : O_1 \times O_2 \times \cdots \times O_m \to X \quad \text{and} \quad u_2 : P_1 \times P_2 \times \cdots \times P_n \to X$$

The condition

$$(\forall_1 x \in O_1) \cdots (\forall_m x \in O_m)\,(\exists_1 y \in P_1) \cdots (\exists_n y \in P_n)\ u_1\,({}_1x\ \ldots\ {}_mx)\,\mathcal{R}\,u_2\,({}_1y\ \ldots\ {}_ny)$$

shall often be denoted by

$$u_1\,(O_1O_2\ldots O_m)\,\dot{\mathcal{R}}\,u_2\,(P_1P_2\ldots P_n)\ ,$$

where the images $u_1\,(O_1O_2\ldots O_m)$ and $u_2\,(P_1P_2\ldots P_n)$ above are computed according to the particular definitions of the functions u_1 and u_2 as if the n-tuples $(O_1O_2\ldots O_m)$ and $(P_1P_2\ldots P_n)$ were elements of the Cartesian products $O_1 \times O_2 \times \cdots \times O_m$ and $P_1 \times P_2 \times \cdots \times P_n$, respectively. We are not to swap the left-hand side with the right-hand side of this notation, even if the relation $\mathcal{R}$ is symmetric on X. If the function u_1 (resp. u_2) is constant with value $a \in X$, then this notation is written simply as

$$a\dot{\mathcal{R}}\, u_2\,(P_1P_2\ldots P_n) \ \text{(resp. } u_1\,(O_1O_2\ldots O_m)\,\dot{\mathcal{R}}a) \ ,$$

and it actually means

$$(\exists_1 y \in P_1)\cdots(\exists_n y \in P_n) \ \ a\mathcal{R}\, u_2\,({}_1y \ \ldots \ {}_ny)$$

$$\text{(resp. } (\forall_1 x \in O_1)\cdots(\forall_m x \in O_m) \ \ u_1\,({}_1x \ \ldots \ {}_mx)\,\mathcal{R}a) \ .$$

Lastly, if both functions u_1 and u_2 are constant with respective values a and b, then we may write $a\dot{\mathcal{R}}b$, which just means $a\mathcal{R}b$.

Typically, the functions u_1 and u_2 in item (b) are not explicitely mentioned in the context and it is up to the reader to figure out what they are.

In order to get the reader up to speed on how Notational Device 3.42 is to be employed in practice, we provide a few examples:

Example 3.43. Let g be an element of $\breve{G}$ and take two elements x and y of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. We may write the condition $x \doteq y + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$, which can also be expressed as $x \doteq u\left(\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)\right)$ where the function $u : \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g) \to \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is given by $u(z) \coloneqq y + z$, and that would mean that there is a $z \in \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ such that $x = y + z$, that is, $x - y \in \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$. Note that in the case $y = 0_R$, we have that the conditions $x \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ and $x \in \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ are equivalent, and they may be written interchangeably.

Example 3.44. Let $g, h \in \breve{G}$ so that $g < h$. We may write the condition $\overset{\mathcal{J}}{\mathrm{ms}}\left(\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g) + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^h)\right) \dot{>} g$, which can also be expressed as $u\left(\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g), \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^h)\right) \dot{>} g$ where the function $u : \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g) \times \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^h) \to G$ is given by $u({}_1x, {}_2x) \coloneqq \overset{\mathcal{J}}{\mathrm{ms}}({}_1x + {}_2x)$, and that would mean

$$(\forall_1 x \in \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g))\,\big(\forall_2 x \in \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h)\big)\ \ \overset{\mathcal{J}}{\mathrm{ms}}\,({}_1x + {}_2x) > g\,,$$

which actually holds true.

Example 3.45. Let $g, h \in \breve{G}$ so that $g < h$, let $x, y, z \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and let $r \in R$. We may write the compound condition

$$x_g \,\dot{<}\, (y + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g))_g \doteq \big(z + \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h) + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)\big)_g \,\dot{\leqslant}\, r\,,$$

which can broken up into three parts:

- ▷ $x_g \,\dot{<}\, (y + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g))_g$: This can also be expressed as $x_g \,\dot{<}\, u\,(\overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g))$, where the function $u : \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) \to R$ is given by $w\,(s) := (y+s)_g$, and that would mean $x_g < (y+s)_g$ $(\exists s \in \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g))$. That is true if, and only if, $x_g < y_g$;
- ▷ $(y + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g))_g \doteq \big(z + \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h) + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)\big)_g$: Note that this condition can also be expressed as $u_1(\overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)) \doteq u_2\big(\overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h)\,, \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)\big)$, where the functions

  $$u_1 : \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) \to R \quad\text{and}\quad u_2 : \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h) \times \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) \to R$$

  are given by $u_1(s) := (y+s)_g$ and $u_2({}_1s, {}_2s) := (z + {}_1s + {}_2s)_g$, and that would mean

  $$(\forall s \in \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g))\,\big(\exists_1 s \in \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h)\big)\,(\exists_2 s \in \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g))\ \ (y+s)_g = (z + {}_1s + {}_2s)_g\ .$$

  That is true if, and only if, $y_g = z_g$;
- ▷ $\big(z + \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h) + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)\big)_g \,\dot{\leqslant}\, r$: Note that this condition can also be expressed as $u_2\big(\overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h)\,, \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)\big) \,\dot{\leqslant}\, r$, where u_2 was defined above, and that would mean

  $$\big(\forall_1 s \in \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h)\big)\,(\forall_2 s \in \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g))\ \ (z + {}_1s + {}_2s)_g \leqslant r\,.$$

  That is true if, and only if, $z_g \leqslant r$.

Therefore, the whole compound condition is equivalent to $x_g < y_g = z_g \leqslant r$.

The following proposition shows that the ∀∃-notation is transitive in a manner of speaking:

Proposition 3.46. *Let X be a set, let $\mathcal{R}_1\mathcal{R}_2\ldots\mathcal{R}_k, \mathcal{R}$ be binary relations on X so that $\mathcal{R}_k\circ\cdots\circ\mathcal{R}_2\circ\mathcal{R}_1\subset\mathcal{R}$, let $m_0m_1\ldots m_k$ be numbers in $\mathbb{N}_0$ and let $\{O_i^j\}_{\substack{j\in[0,k]_{\mathbb{Z}}\\ i\in[1,m_j]_{\mathbb{N}}}}$ be a double family of subsets of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ of the forms $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ or $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$. If*[8] *$\{u_j : O_1^j\times\cdots\times O_{m_j}^j\to X\}_{j\in[0,k]_{\mathbb{Z}}}$ is a finite sequence of functions and if the compound condition*

$$u_0(O_1^0\ldots O_{m_0}^0)\,\dot{\mathcal{R}}_1 u_1(O_1^1\ldots O_{m_1}^1)\,\dot{\mathcal{R}}_2\cdots\dot{\mathcal{R}}_k u_k(O_1^k\ldots O_{m_k}^k)$$

is true, then $u_0(O_1^0\ldots O_{m_0}^0)\,\dot{\mathcal{R}}\,u_k(O_1^k\ldots O_{m_k}^k)$.

Proof. Straightforward induction on k.

□

We shall not make specific reference to Proposition 3.46 when it is invoked in the rest of this work. For instance, still in this section, that proposition is essential in the proofs of Subitems 2, 5 and 7 of Item (b) of Theorem 3.47.

If x is any element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, if $\{g_\alpha\}_{\alpha<\gamma}$ is the increasing ordinal sequence of elements of the well-ordered set $\operatorname{supp}(x)$, and if we define $g_\gamma :=\, \multimap$, then we have $x \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)$ and

$$(\forall\beta<\gamma)\;\; x \doteq \left(\sum_{\alpha\leqslant\beta} x_{g_\alpha}\mathrm{X}^{g_\alpha}\right) + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g_{\beta+1}}\right) \doteq \left(\sum_{\alpha\leqslant\beta} x_{g_\alpha}\mathrm{X}^{g_\alpha}\right) + \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{g_\beta}\right)\,.$$

In particular, by taking $\beta=0$ we obtain

$$x \doteq \operatorname{pc}(x)\,\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)} + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g_1}\right) \doteq \operatorname{pc}(x)\,\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)} + \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)\,.$$

[8] In the case $m_j=0$, we assume that $O_1^j\times\cdots\times O_{m_j}^j := \{\emptyset\}$ (**empty product**), $O_1^j\ldots O_{m_j}^j=\emptyset$ and $u_j\,(O_1^j\ldots O_{m_j}^j) = u_j(\emptyset)\in X$.

Theorem 3.47. *Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G.*

(a) *For all $x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, we have*

$$(xy)\,[\![\overset{\mathcal{J}}{\mathrm{ms}}(x) + \overset{\mathcal{J}}{\mathrm{ms}}(y)]\!] = \mathrm{pc}(x)\ \mathrm{pc}(y)\ ,$$

and if $\overset{\mathcal{J}}{\mathrm{ms}}(x) \leqslant \overset{\mathcal{J}}{\mathrm{ms}}(y)$ and $(x \pm y)_{\overset{\mathcal{J}}{\mathrm{ms}}(x)} \neq 0_R$, then we have

$$\overset{\mathcal{J}}{\mathrm{ms}}(x \pm y) = \overset{\mathcal{J}}{\mathrm{ms}}(x) \quad \text{and} \quad \mathrm{pc}(x \pm y) = \mathrm{pc}(x) \pm y_{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\ ;$$

(b) *For all $g, h \in G$, for all $r, s \in R - \{0_R\}$ and for all $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, we have:*

1. $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g) \subsetneq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$;

2. $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)\ \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^h) \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g+h})$;

3. $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)\ \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^h) \doteq \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g+h})$;

4. $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)\ \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^h) \doteq \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g+h})$
and
$\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)\ \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^h) \doteq \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g+h})$;

5. $x\ \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g) \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)+g}\right)$
and
$\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)\ x \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)$;

6. $x\ \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g) \doteq \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)+g}\right)$
and
$\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)\ x \doteq \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{g+\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)$;

7. $r\ \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g) \pm s\ \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^h) \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\overset{\smile}{G}}{\min}\{g,h\}}\right)$
and
$\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)\ r \pm \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^h)\ s \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\overset{\smile}{G}}{\min}\{g,h\}}\right)$;

8. $r\ \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g) \pm s\ \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^h) \doteq \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{\overset{\overset{\smile}{G}}{\min}\{g,h\}}\right)$
and
$\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)\ r \pm \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^h)\ s \doteq \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{\overset{\overset{\smile}{G}}{\min}\{g,h\}}\right)$;

9. *The sets $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ and $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ are subrngs of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ if, and only if, we have $g \geqslant 0_G$.*

Proof.

(a) One can easily ascertain that all results are valid in the particular cases in which $x = 0_R$ or $y = 0_R$. Assume $x, y \neq 0_R$. If g and h are elements of G so that $g + h = \overset{\mathcal{J}}{\mathrm{ms}}(x) + \overset{\mathcal{J}}{\mathrm{ms}}(y)$ and $x_g\, y_h \neq 0_R$, then we have $g \geqslant \overset{\mathcal{J}}{\mathrm{ms}}(x)$

and $h \geqslant \overset{\mathcal{J}}{\mathrm{ms}}\,(y)$. Moreover, if $g > \overset{\mathcal{J}}{\mathrm{ms}}\,(x)$ or $h > \overset{\mathcal{J}}{\mathrm{ms}}\,(y)$, then we get

$$\overset{\mathcal{J}}{\mathrm{ms}}\,(x) + \overset{\mathcal{J}}{\mathrm{ms}}\,(y) = g + h > \overset{\mathcal{J}}{\mathrm{ms}}\,(x) + \overset{\mathcal{J}}{\mathrm{ms}}\,(y)\ ,$$

which is absurd, proving that the conditions $g + h = \overset{\mathcal{J}}{\mathrm{ms}}\,(x) + \overset{\mathcal{J}}{\mathrm{ms}}\,(y)$ and $x_g\, y_h \neq 0_G$ imply $g = \overset{\mathcal{J}}{\mathrm{ms}}\,(x)$ and $h = \overset{\mathcal{J}}{\mathrm{ms}}\,(y)$. Hence, we obtain

$$(xy)\,[\![\overset{\mathcal{J}}{\mathrm{ms}}\,(x) + \overset{\mathcal{J}}{\mathrm{ms}}\,(y)]\!] \;=\; \sum_{\substack{g,h\in G\\ g+h=\overset{\mathcal{J}}{\mathrm{ms}}(x)+\overset{\mathcal{J}}{\mathrm{ms}}(y)}} x_g y_h \;=\; x_{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\, y_{\overset{\mathcal{J}}{\mathrm{ms}}(y)} \;=\; \mathrm{pc}\,(x)\ \mathrm{pc}\,(y)\ .$$

Lastly, if conditions $\overset{\mathcal{J}}{\mathrm{ms}}\,(x) \leqslant \overset{\mathcal{J}}{\mathrm{ms}}\,(y)$ and $(x \pm y)_{\overset{\mathcal{J}}{\mathrm{ms}}(x)} \neq 0_R$ are satisfied, then we get $\overset{\mathcal{J}}{\mathrm{ms}}\,(x) \leqslant \overset{\mathcal{J}}{\mathrm{ms}}\,(x \pm y) \leqslant \overset{\mathcal{J}}{\mathrm{ms}}\,(x)$ (Proposition 1.81, Item (b)), implying $\overset{\mathcal{J}}{\mathrm{ms}}\,(x \pm y) = \overset{\mathcal{J}}{\mathrm{ms}}\,(x)$ and

$$\mathrm{pc}\,(x \pm y) = (x \pm y)_{\overset{\mathcal{J}}{\mathrm{ms}}(x)} = x_{\overset{\mathcal{J}}{\mathrm{ms}}(x)} \pm y_{\overset{\mathcal{J}}{\mathrm{ms}}(x)} = \mathrm{pc}\,(x) \pm y_{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\,.$$

(b) We prove the subitems 1, 2, 5 and 7, leaving the proofs of the remnants to the reader. Keep in mind that $\overset{\mathcal{J}}{\mathrm{ms}}$ is a G-pseudovaluation on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ (Theorem 3.41).

1. If $x \doteq \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$, then $\overset{\mathcal{J}}{\mathrm{ms}}\,(x) > g$, and, in particular, we have $\overset{\mathcal{J}}{\mathrm{ms}}\,(x) \geqslant g$ and $x \doteq \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g)$, giving us the non-strict inclusion $\overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) \subset \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g)$. Since R is non-trivial (Assumption 3.35), there is an $r \in R - \{0_R\}$. Hence, we have $r\mathrm{X}^g \in \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g) - \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$ and $\overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) \subsetneq \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g)$.

2. We have

$$\overset{\mathcal{J}}{\mathrm{ms}}\left(\overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g)\ \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h)\right) \mathrel{\dot{\geqslant}} \overset{\mathcal{J}}{\mathrm{ms}}\left(\overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g)\right) + \overset{\mathcal{J}}{\mathrm{ms}}\left(\overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h)\right) \mathrel{\dot{\geqslant}} g + h\,,$$

which implies $\overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g)\ \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h) \doteq \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^{g+h})$.

5. By subitem 2, we obtain

$$\overset{\mathcal{J}}{\mathrm{ms}}\left(x\ \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g)\right) \doteq \overset{\mathcal{J}}{\mathrm{ms}}\left(\overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)\ \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g)\right) \mathrel{\dot{\geqslant}} \overset{\mathcal{J}}{\mathrm{ms}}\,(x) + g\,,$$

which implies $x\ \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g) \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)+g}\right)$. The proof of the condition $\overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g)\ x \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)$ is analogous.

7. Since $\overset{\mathcal{J}}{\mathrm{ms}}(r) = \overset{\mathcal{J}}{\mathrm{ms}}(s) = 0_G$, we obtain

$$\overset{\mathcal{J}}{\mathrm{ms}}\left(r \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g) \pm s \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^h)\right) \dot{\geqslant} \overset{\breve{G}}{\min}\left\{\overset{\mathcal{J}}{\mathrm{ms}}\left(r \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)\right), \overset{\mathcal{J}}{\mathrm{ms}}\left(s \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^h)\right)\right\}$$
$$\doteq \overset{\breve{G}}{\min}\left\{\overset{\mathcal{J}}{\mathrm{ms}}\left(\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{0_G+g})\right), \overset{\mathcal{J}}{\mathrm{ms}}\left(\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{0_G+h})\right)\right\}$$

by subitems 2 and 5, which gives us the first desired result. The proof of $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)\, r \pm \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^h)\, s \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\breve{G}}{\min}\{g,h\}}\right)$ is analogous.

□

Proposition 3.48. (181, 117) *Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G.*

(**a**) *R is a ring if, and only if, $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is a ring. In that case, $1_R = 1_R\mathrm{X}^{0_G}$ is the multiplicative identity of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$;*

(**b**) *If R and G are commutative, then $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is commutative. The converse holds when the multiplication operation of R is non-trivial (Example 2.4);*

(**c**) *R has no zero divisors if, and only if, $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ has no zero divisors. In that case, we have*

$$(\forall x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]])\ \overset{\mathcal{J}}{\mathrm{ms}}(xy) = \overset{\mathcal{J}}{\mathrm{ms}}(x) + \overset{\mathcal{J}}{\mathrm{ms}}(y)\ ,$$

that is, $\overset{\mathcal{J}}{\mathrm{ms}}$ is a G-valuation on the rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

(**d**) *If $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is a division ring, then so is R.*

Proof.

(**a**) If R is a ring, then $1_R = 1_R\mathrm{X}^{0_G}$ is clearly the multiplicative identity of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. Conversely, if u is the multiplicative identity of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, then we have (Proposition 3.34, Item (c))

$$(\forall r \in R)\ r = ru = \sum_{g\in G} ru_g\mathrm{X}^g \text{ and } \quad r = ur = \sum_{g\in G} u_g r\mathrm{X}^g\ ,$$

implying $ru_{0_G} = r = u_{0_G} r$ $(\forall r \in R)$ and, thus, proving that u_{0_G} is the multiplicative identity of R.

(b) The proof of the necessary condition is straightforward. If $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is commutative and if r and s are two elements of R so that $rs \neq 0_R$, then R is also commutative since it is identified with a subrng of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ (Proposition 3.34, Item (a)), and, taking any two elements g and h in G, we get

$$rs\mathrm{X}^{g+h} = (r\mathrm{X}^g)\,(s\mathrm{X}^h) = (s\mathrm{X}^h)\,(r\mathrm{X}^g) = sr\mathrm{X}^{h+g}\,,$$

implying $g + h = h + g$ and proving that G is commutative.

(c) If $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ has no zero divisors, then R has no zero divisors since it is identified with a subrng of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ (Proposition 3.34, Item (a)). Suppose R has no zero divisors and let x and y be two non-zero elements of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. Thus, we have (Theorem 3.47, Item (a))

$$(xy)\,[\![\overset{\mathcal{J}}{\mathrm{ms}}\,(x) + \overset{\mathcal{J}}{\mathrm{ms}}\,(y)]\!] = \mathrm{pc}\,(x)\;\mathrm{pc}\,(y) \neq 0_R\,,$$

which implies that $\mathrm{ms}\,(xy) \leqslant \overset{\mathcal{J}}{\mathrm{ms}}\,(x) + \overset{\mathcal{J}}{\mathrm{ms}}\,(y)$ and $xy \neq 0_R$. The opposite inequality arises from the fact that $\overset{\mathcal{J}}{\mathrm{ms}}$ is a G-pseudovaluation on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ (Definition 2.44).

(d) If $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is a division ring and if r is a non-zero element of R, then R is a ring by item (a), and r has an inverse x in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, which gives us $rx = 1_R = xr$ and (Proposition 3.34, Item (c))

$$rx_{0_G} = \Big(\sum_{g\in G} rx_g\mathrm{X}^g\Big)_{0_G} = (rx)_{0_G} = 1_R = (xr)_{0_G} = \Big(\sum_{g\in G} x_g r\mathrm{X}^g\Big)_{0_G} = x_{0_G} r\,,$$

that is, x_{0_G} is the multiplicative inverse of r in R, proving that R is a division ring.

□

Example 3.49. The classical Rayner rngs $\overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]$, $\overset{\mathrm{bd}}{\mathbb{C}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]$ and $\mathbb{C}\,[[\mathrm{X}^{\mathbb{Z}}]]$ are commutative rings with no zero divisors (Proposition 3.48, Items (a), (b) and (c)).

Example 3.50. Let R be a rng and let $\mathcal{J}$ be an arithmetic Rayner ideal on BS_ℓ. The Rayner rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^{\mathsf{BS}_\ell}]]$ is non-commutative, even if the rng R is commutative (Proposition 3.48, Item (b)). For instance, if r is a non-zero element of R, then we have

$$(r\mathrm{X}^{\mathsf{t}})\,(r\mathrm{X}^{\mathsf{m}}) = r^2\mathrm{X}^{\mathsf{t}+\mathsf{m}} \neq r^2\mathrm{X}^{\mathsf{m}+\mathsf{t}} = (r\mathrm{X}^{\mathsf{m}})\,(r\mathrm{X}^{\mathsf{t}})$$

since $\mathsf{t} + \mathsf{m} \neq \mathsf{m} + \mathsf{t}$.

3.6 Generating functors

The process of construction of the Rayner rngs gives rise to a few functors, and the properties of these functors are discussed in this section.

The reader who is not comfortable with the employment of large categories (Example B.17) is referred to Appendix A, and Appendix B is designed to help the reader not acquainted with the basic definitions of Category Theory, especially with the notions of limit, colimit and quotient.

Definition 3.51. Let $\boldsymbol{S}$ be a subcategory of **SetOrd**. A **system of ideals on** $\boldsymbol{S}$ is a family $\{\mathcal{J}_J\}_{J\in\boldsymbol{S}_0}$ of sets such that the following axioms hold:

(**S1**) For each ordered set J in $\boldsymbol{S}_0$, the set $\mathcal{J}_J$ is an ideal on J;

(**S2**) For every morphism $r : J \xrightarrow{\boldsymbol{S}} K$ and every $S \in \mathcal{J}_K$, we have $r^{-1}\langle S\rangle \in \mathcal{J}_J$.

If $\boldsymbol{S}$ and $\boldsymbol{S}'$ are subcategories of **SetOrd** so that $\boldsymbol{S}$ is a subcategory of $\boldsymbol{S}'$, and if $\{\mathcal{J}_J\}_{J\in\boldsymbol{S}'_0}$ is a system of ideals on $\boldsymbol{S}'$, then the subfamily $\{\mathcal{J}_J\}_{J\in\boldsymbol{S}_0}$ is a system of ideals on $\boldsymbol{S}$.

Proposition 3.52. *The families*

$$\{\mathrm{P}_\omega(J)\}_{J\in\mathbf{SetOrd}_0}\,,\{\overset{\mathrm{wo}}{\mathrm{P}}(J)\}_{J\in\mathbf{SetOrd}_0}\ \text{ and }\ \{\overset{\mathrm{lf}}{\mathrm{P}}(J)\}_{J\in\mathbf{SetOrd}_0}$$

are systems of ideals on **SetOrd**.

Proof. Let $r : J \to K$ be an increasing function between ordered sets. Thus, r is injective (Proposition 1.30) and so is the restriction

$$r \restriction (r^{-1}\langle S\rangle) : r^{-1}\langle S\rangle \to r\langle J\rangle \cap S$$

for every subset S of K. If S is a finite (resp. well-ordered) subset of K, then $r\langle J\rangle \cap S$ is finite (resp. well-ordered), which implies that $r^{-1}\langle S\rangle$ is a finite (resp. well-ordered) subset of J. If S is a left-finite subset of K and if j is a fixed element of J, then note that

$$r^{-1}\langle S\rangle \cap (\leftarrow, j]_J \subset r^{-1}\langle S \cap (\leftarrow, r(j)]_K\rangle \ ,$$

and since $S \cap (\leftarrow, r(j)]_K$ is finite, the preimage $r^{-1}\langle S \cap (\leftarrow, r(j)]_K\rangle$ and the set $r^{-1}\langle S\rangle \cap (\leftarrow, j]_J$ are finite, proving that $r^{-1}\langle S\rangle$ is left-finite in J.

□

Proposition 3.53. *Let $\boldsymbol{S}$ be a subcategory of* **SetOrd** *and let $\{\mathcal{J}_J\}_{J\in\mathbf{SetOrd}_0}$ be a system of ideals on $\boldsymbol{S}$. The function $\mathcal{U} : \boldsymbol{S}^{\mathrm{op}} \times \mathbf{Mon} \to \mathbf{Mon}$ (Definition B.21) given by*[9]

$$\begin{cases} \mathcal{U}(J, M) := \overset{\mathcal{J}_J}{M}[[\mathrm{X}^J]] \\ \left(\mathcal{U}\left(r : K \overset{\mathbf{SetOrd}^{\mathrm{op}}}{\longrightarrow} J, \phi : M \overset{\mathbf{Mon}}{\longrightarrow} N\right)(x)\right)_j := \phi\left(x_{r(j)}\right) \end{cases}$$

is a functor (Definition B.25).

Proof. In order to show that $\mathcal{U}$ is well-defined, first we need to prove that the family $\{\phi(x_{r(j)})\}_{j\in J}$ is in the $\mathcal{J}_J$-Rayner monoid $\overset{\mathcal{J}_J}{N}[[\mathrm{X}^J]]$, where $x \in \overset{\mathcal{J}_K}{M}[[\mathrm{X}^K]]$ and where $r : K \overset{\boldsymbol{S}^{\mathrm{op}}}{\longrightarrow} J$ and $\phi : M \overset{\mathbf{Mon}}{\longrightarrow} N$ are morphisms. If j is an element of the support $\mathrm{supp}(\{\phi(x_{r(j)})\}_{j\in J})$, then $\phi(x_{r(j)}) \neq 1_N$, $x_{r(j)} \neq 1_M$ and $r(j) \in \mathrm{supp}(x)$, proving the inclusion

$$\mathrm{supp}(\{\phi(x_{r(j)})\}_{j\in J}) \subset r^{-1}\langle \mathrm{supp}(x)\rangle \ .$$

[9] Note that the ordered pair (r, ϕ) is a morphism of type $(K, M) \to (J, N)$ in the product category $\boldsymbol{S}^{\mathrm{op}} \times \mathbf{Mon}$, which implies that the image $\mathcal{U}(r, \phi)$ is a morphism of type $\overset{\mathcal{J}_K}{M}[[\mathrm{X}^K]] \overset{\mathbf{Mon}}{\longrightarrow} \overset{\mathcal{J}_J}{N}[[\mathrm{X}^J]]$.

Hence, since we have $r^{-1}\langle \mathrm{supp}\,(x)\rangle \in \mathcal{J}_J$ (Definition 3.51, Axiom (S2)), we get $\mathrm{supp}\,(\{\phi\,(x_{r(j)})\}_{j\in J}) \in \mathcal{J}_J$ and $\{\phi\,(x_{r(j)})\}_{j\in J} \in \overset{\mathcal{J}_J}{N}[[\mathrm{X}^J]]$, as intended.

Additionally, we must show that the function $\mathcal{U}\,(r,\phi) : \overset{\mathcal{J}_K}{M}[[\mathrm{X}^K]] \to \overset{\mathcal{J}_J}{N}[[\mathrm{X}^J]]$ is a unital homomorphism. Indeed, for all $x, y \in \overset{\mathcal{J}_K}{M}[[\mathrm{X}^K]]$ and all $j \in J$, we have

$$(\mathcal{U}\,(r,\phi)\,(1_M))_j = \phi\,((1_M)_{r(j)}) = \phi\,(1_M) = 1_N = (1_N)_j$$

and

$$(\mathcal{U}\,(r,\phi)\,(xy))_j = \phi\,((xy)_{r(j)}) = \phi\,(x_{r(j)}y_{r(j)}) = (\mathcal{U}\,(r,\phi)\,(x))_j\,(\mathcal{U}\,(r,\phi)\,(y))_j\ ,$$

that is, $\mathcal{U}\,(r,\phi)\,(1_M) = 1_N$ and $\mathcal{U}\,(r,\phi)\,(xy) = \mathcal{U}\,(r,\phi)\,(x)\;\mathcal{U}\,(r,\phi)\,(y)$, proving that $\mathcal{U}\,(r,\phi)$ is a unital homomorphism between monoids. Therefore, the function $\mathcal{U}$ is well-defined.

Consider morphisms $K \xrightarrow{r} J \xrightarrow{s} F$ and $M \xrightarrow{\phi} N \xrightarrow{\chi} P$ in $\boldsymbol{S}^{\mathrm{op}}$ and $\mathbf{Mon}$, respectively. For all $x \in \overset{\mathcal{J}_K}{M}[[\mathrm{X}^K]]$ and all $f \in F$, we have

$$[(\mathcal{U}\,(s,\chi)\circ\mathcal{U}\,(r,\phi))\,(x)]_f = \chi\,((\mathcal{U}\,(r,\phi)\,(x))_{s(f)}) = \chi\,(\phi\,(x_{r(s(f))})) = (\mathcal{U}\,(s\circ r,\chi\circ\phi)\,(x))_f\ .$$

Hence, $\mathcal{U}\,(s\circ r,\chi\circ\phi) = \mathcal{U}\,(s,\chi)\circ\mathcal{U}\,(r,\phi)$ and $\mathcal{U}$ is a functor.

$\square$

Theorem 3.54. *Let J be an ordered set, let $\mathcal{J}$ be an ideal on J and let $\mathcal{M} : \mathbf{Mon} \to \mathbf{Mon}$ be the function given by*

$$\begin{cases}\mathcal{M}\,(M) := \overset{\mathcal{J}}{M}[[\mathrm{X}^J]]\\ (\mathcal{M}\,(\phi : M \xrightarrow{\mathbf{Mon}} N)\,(x))_j := \phi\,(x_j)\,.\end{cases}$$

(a) *The function $\mathcal{M}$ is a functor;*

(b) *The functor $\mathcal{M}$ preserves object-finite limits in* **Mon** *(Definition B.36). In particular, $\mathcal{M}$ is left-exact;*

(c) *The functor $\mathcal{M}$ preserves quotients modulo congruence relations in* **Mon** *(Definition B.50).*

Proof.

(**a**) Let $\boldsymbol{S}$ be the subcategory of **SetOrd** whose only object is J and whose only morphism is the identity function $\mathrm{id}_J : J \to J$. Thus, the singleton family $\{\mathcal{J}\}_{J\in\boldsymbol{S}_0}$ is a system of ideals on $\boldsymbol{S}$, and, taking the functor $\mathcal{U} : \boldsymbol{S}^{\mathrm{op}} \times \mathbf{Mon} \to \mathbf{Mon}$ described in Proposition 3.53, it is easy to check that the function $\mathcal{M}$ defined in the statement of the theorem is actually given by $\mathcal{M}(M) = \mathcal{U}(J, M)$ and $\mathcal{M}(\phi) = \mathcal{U}(\mathrm{id}_J, \phi)$, implying that $\mathcal{M}$ is a functor.

(**b**) Let $\boldsymbol{I}$ be an object-finite category, let $\mathcal{F} : \boldsymbol{I} \to \mathbf{Mon}$ be a functor and suppose $\chi = \{\chi_i : L \xrightarrow{\mathbf{Mon}} \mathcal{F}(i)\}_{i\in\boldsymbol{I}_0}$ is a limit cone over $\mathcal{F}$ (Definition B.31). We shall prove that the cone

$$\begin{aligned}\mathcal{M}(\chi) &\coloneqq \{\mathcal{M}(\chi_i) : \mathcal{M}(L) \xrightarrow{\mathbf{Mon}} \mathcal{M}(\mathcal{F}(i))\}_{i\in\boldsymbol{I}_0}\\ &= \left\{\mathcal{M}(\chi_i) : \overset{\mathcal{J}}{L}[[\mathrm{X}^J]] \xrightarrow{\mathbf{Mon}} \overset{\mathcal{J}}{\mathcal{F}}(i)[[\mathrm{X}^J]]\right\}_{i\in\boldsymbol{I}_0}\end{aligned}$$

over the composition $\mathcal{M} \circ \mathcal{F} : \boldsymbol{I} \to \mathbf{Mon}$ is a limit cone. Suppose that $\lambda = \left\{\lambda_i : V \xrightarrow{\mathbf{Mon}} \overset{\mathcal{J}}{\mathcal{F}}(i)[[\mathrm{X}^J]]\right\}_{i\in\boldsymbol{I}_0}$ is another cone over the functor $\mathcal{M} \circ \mathcal{F}$, and, for each $j \in J$ and each $i \in \boldsymbol{I}_0$, let $\lambda_{i,j} : V \to \mathcal{F}(i)$ be the function given by $\lambda_{i,j}(x) \coloneqq (\lambda_i(x))_j$. It is easy to verify that each function $\lambda_{i,j}$ is a homomorphism between monoids, implying that each family $\lambda_j \coloneqq \{\lambda_{i,j} : V \xrightarrow{\mathbf{Mon}} \mathcal{F}(i)\}_{i\in\boldsymbol{I}_0}$ is a cone over $\mathcal{F}$. With that, for each $j \in J$, let $\overline{\lambda_j} : V \xrightarrow{\mathbf{Mon}} L$ be the limit lifting of λ_j along χ. That means that for each $j \in J$ and for each $i \in \boldsymbol{I}_0$, the digraph

$$\begin{tikzcd} & L \arrow[d, "\chi_i"] \\ V \arrow[ur, "\overline{\lambda_j}"] \arrow[r, "\lambda_{i,j}"'] & \mathcal{F}(i) \end{tikzcd}$$

in **Mon** commutes. If there is a homomorphism $\overline{\lambda} : V \xrightarrow{\mathbf{Mon}} \overset{\mathcal{J}}{L}[[\mathrm{X}^J]]$ such that the digraph

$$\begin{tikzcd} & \overset{\mathcal{J}}{L}[[\mathrm{X}^J]] \arrow[d, "\mathcal{M}(\chi_i)"] \\ V \arrow[ur, "\overline{\lambda}"] \arrow[r, "\lambda_i"'] & \overset{\mathcal{J}}{\mathcal{F}}(i)[[\mathrm{X}^J]] \end{tikzcd}$$

in **Mon** commutes for all $i \in \boldsymbol{I}_0$, then for all $x \in V$ and all $j \in J$ we get

$$(\forall i \in \boldsymbol{I}_0) \;\; \chi_i\left(\overline{\lambda_j}(x)\right) = \lambda_{i,j}(x) = \left(\lambda_i(x)\right)_j = \left[\mathcal{M}(\chi_i)\left(\overline{\lambda}(x)\right)\right]_j = \chi_i\left(\left(\overline{\lambda}(x)\right)_j\right)$$

implying the equation $\left(\overline{\lambda}(x)\right)_j = \overline{\lambda_j}(x)$ by Lemma 1.23. Indeed, the unital homomorphism $\overline{\lambda} : V \xrightarrow{\mathbf{Mon}} {}^J L$ given by $\overline{\lambda}(x) \coloneqq \{\overline{\lambda_j}(x)\}_{j\in J}$ is such that $\mathcal{M}(\chi_i) \circ \overline{\lambda} = \lambda_i$. It only remains to prove that $\overline{\lambda}$ is of type $V \xrightarrow{\mathbf{Mon}} \overset{\mathcal{J}}{L}[[\mathrm{X}^J]]$. Let x be an element of V and suppose j is an element of J such that $\left(\overline{\lambda}(x)\right)_j = \overline{\lambda_j}(x) \neq 1_L$. Finally, again by Lemma 1.23 there is an $i \in \boldsymbol{I}_0$ so that

$$\left(\lambda_i(x)\right)_j = \lambda_{i,j}(x) = \chi_i\left(\overline{\lambda_j}(x)\right) \neq \chi_i(1_L) = 1_{\mathcal{F}(i)}\,,$$

proving that $\operatorname{supp}\left(\overline{\lambda}(x)\right) \subset \bigcup\limits_{i\in \boldsymbol{I}_0} \operatorname{supp}\left(\lambda_i(x)\right)$ and $\operatorname{supp}\left(\overline{\lambda}(x)\right) \in \mathcal{J}$.

(**c**) Let $\equiv$ be a congruence relation on a monoid M, let $\iota : M \xrightarrow{\mathbf{Mon}} M/\equiv$ be the canonical quotient homomorphism of that type and let $\approx$ be the binary relation on $\overset{\mathcal{J}}{M}[[\mathrm{X}^J]]$ given by $\approx \coloneqq {}^J\!\equiv \restriction\restriction \left(\overset{\mathcal{J}}{M}[[\mathrm{X}^J]]\right)$ (Notation 1.19; List of Symbols: $R \restriction\restriction_S$). It is easy to check that $\approx$ is a congruence relation on $\overset{\mathcal{J}}{M}[[\mathrm{X}^J]]$ so that $\approx = \overset{\mathcal{M}(\iota)}{\mathrm{eq}}$. We shall prove that $(\overset{\mathcal{J}}{M/}\equiv)[[\mathrm{X}^J]]$ is a quotient of $\overset{\mathcal{J}}{M}[[\mathrm{X}^J]]$ modulo $\approx$ in **Mon** with quotient homomorphism

$$\mathcal{M}(\iota) : \overset{\mathcal{J}}{M}[[\mathrm{X}^J]] \xrightarrow{\mathbf{Mon}} (\overset{\mathcal{J}}{M/}\equiv)[[\mathrm{X}^J]]\,.$$

Consider another monoid N and a homomorphism $f : \overset{\mathcal{J}}{M}[[\mathrm{X}^J]] \xrightarrow{\mathbf{Mon}} N$ so that $\approx \subset \overset{f}{\mathrm{eq}}$. For each $S \in \mathcal{J}$, let $f_S : {}^J M \xrightarrow{\mathbf{Mon}} N$ be the homomorphism given by $f_S\left(\{x_j\}_{j\in J}\right) \coloneqq f\left(\{y_j\}_{j\in J}\right)$, where $\{y_j\}_{j\in J}$ is the family in M given by

$$y_j \coloneqq \begin{cases} x_j & \text{if } j \in S\,, \\ 1_M & \text{if } j \in J - S\,. \end{cases}$$

The function f_S clearly satisfies ${}^J\!\equiv \subset \overset{f_S}{\mathrm{eq}}$, and, thus, we may take its quotient lowering $\overline{f_S} : {}^J(M/\equiv) \xrightarrow{\mathbf{Mon}} N$ modulo ${}^J\!\equiv$ along the quotient morphism ${}^J\iota$

(Lemma 1.20). That means that for each $S \in \mathcal{J}$, the digraph

$$\begin{tikzcd} {}^{J}M \arrow[rr, "f_S"] \arrow[d, "{}^{J}\iota"'] & & N \\ {}^{J}(M/\equiv) \arrow[urr, "\overline{f_S}"'] & & \end{tikzcd}$$

in **Mon** commutes. If $\overline{f} : (M\overset{\mathcal{J}}{/}\equiv)[[\mathrm{X}^J]] \xrightarrow{\mathbf{Mon}} N$ is a homomorphism such that the digraph

$$\begin{tikzcd} \overset{\mathcal{J}}{M}[[\mathrm{X}^J]] \arrow[rr, "f"] \arrow[d, "\mathcal{M}(\iota)"'] & & N \\ (M\overset{\mathcal{J}}{/}\equiv)[[\mathrm{X}^J]] \arrow[urr, "\overline{f}"'] & & \end{tikzcd}$$

in **Mon** commutes, then for all $X \in (M\overset{\mathcal{J}}{/}\equiv)[[\mathrm{X}^J]]$ and all $x \in \overset{\mathcal{J}}{M}[[\mathrm{X}^J]]$ so that $X_j = x_j/\equiv$ $(\forall j \in J)$ and $\mathrm{supp}(X) = \mathrm{supp}(x)$, we have

$$\overline{f}(X) = \overline{f}(\mathcal{M}(\iota)(x)) = f(x) = f_{\mathrm{supp}(X)}(x) = \overline{f_{\mathrm{supp}(X)}}({}^{J}\iota(x)) = \overline{f_{\mathrm{supp}(X)}}(X) \ .$$

Indeed, the function $\overline{f} : (M\overset{\mathcal{J}}{/}\equiv)[[\mathrm{X}^J]] \to N$ given by $\overline{f}(X) \coloneqq \overline{f_{\mathrm{supp}(X)}}(X)$ is a homomorphism, and that proves the item.

□

In the proof of Item (c) above, we provided an abstract category-theoretic definition of the quotient lowering $\overline{f} : (M\overset{\mathcal{J}}{/}\equiv)[[\mathrm{X}^J]] \xrightarrow{\mathbf{Mon}} N$ of f modulo $\approx$, but it is quite easy to describe how that function actually operates on its arguments. Take an $X \in (M\overset{\mathcal{J}}{/}\equiv)[[\mathrm{X}^J]]$. Each X_j for $j \in J$ is an equivalence class $X_j = x_j/\equiv$, where x_j is an element of X_j, and it is clear that $\mathrm{supp}(X) \subset \mathrm{supp}(x)$, where $x \coloneqq \{x_j\}_{j\in J}$. Thus, since $\approx \subset \overset{f}{\mathrm{eq}}$, we have $f_{\mathrm{supp}(X)}(x) = f(x)$ and

$$\overline{f}(X) = \overline{f_{\mathrm{supp}(X)}}(X) = \overline{f_{\mathrm{supp}(X)}}({}^{J}\iota(x)) = f_{\mathrm{supp}(X)}(x) = f(x) \ ,$$

that is, the function $\overline{f}$ stands for the transformation $\{x_j/\equiv\}_{j\in J} \mapsto f(\{x_j\}_{j\in J})$. The advantage of working with the more abstract category-theoretic approach in defining $\overline{f}$ is twofold: such procedure leaves no doubt that $\overline{f}$ is well-defined, and it reveals information related to $\overline{f}$ which shall be useful in the proof of Item (c) of Theorem 3.56, such as the equations

$$(\forall S \in \mathcal{J})\ \overline{f_S} \circ {}^J\iota = f_S \quad \text{and} \quad \overline{f} \circ \mathcal{M}\,(\iota) = f\,.$$

Proposition 3.55. *Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. If I is an ideal in R, then the set $\overset{\mathcal{J}}{I}[[\mathrm{X}^G]]$ is an ideal in the rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.*

Proof. The set $\overset{\mathcal{J}}{I}[[\mathrm{X}^G]]$ is clearly a subgroup of $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]\,,+)$. Take any two elements $x \in \overset{\mathcal{J}}{I}[[\mathrm{X}^G]]$ and $y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. Thus, for all $p \in G$, we get

$$(xy)_p = \sum_{\substack{g,h\in G\\ g+h=p}} x_g y_h \in \sum_{\substack{g\in \mathrm{supp}(x)\\ h\in\mathrm{supp}(y)\\ g+h=p}} IR \subset \sum_{\substack{g\in \mathrm{supp}(x)\\ h\in\mathrm{supp}(y)\\ g+h=p}} I \subset I\,,$$

implying $xy \in \overset{\mathcal{J}}{I}[[\mathrm{X}^G]]$ and $\overset{\mathcal{J}}{I}[[\mathrm{X}^G]] \cdot \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \subset \overset{\mathcal{J}}{I}[[\mathrm{X}^G]]$. The proof of the inclusion $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \cdot \overset{\mathcal{J}}{I}[[\mathrm{X}^G]] \subset \overset{\mathcal{J}}{I}[[\mathrm{X}^G]]$ is analogous, showing that $\overset{\mathcal{J}}{I}[[\mathrm{X}^G]]$ is an ideal in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

□

Theorem 3.56. *Let G be an ordered group, let $\mathcal{J}$ be an arithmetic Rayner ideal on G and let $\mathcal{R} : \mathbf{Rng} \to \mathbf{Rng}$ be the function given by*

$$\begin{cases} \mathcal{R}\,(R) \coloneqq \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \\ (\mathcal{R}\,(\phi : R \xrightarrow{\mathbf{Rng}} S)\,(x))_g \coloneqq \phi\,(x_g)\,. \end{cases}$$

(a) *The function $\mathcal{R}$ is a functor of type* **Rng** $\to$ **Rng***;*

(b) *The functor $\mathcal{R}$ preserves object-finite limits in* **Rng** *(Definition B.36). In particular, $\mathcal{R}$ is left-exact;*

(c) *The functor $\mathcal{R}$ preserves quotients modulo ideals in* **Rng** *(Definition B.50).*

Proof.

(**a**) If one were to ignore the multiplication operations within the rngs, then $\mathcal{R}$ essentially would become the functor $\mathcal{M}$ described in Theorem 3.54. Thus, the only possible barrier to the desired result is that for each homomorphism $\phi : R \xrightarrow{\mathbf{Rng}} S$ between rngs, the function $\mathcal{R}(\phi) : \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to \overset{\mathcal{J}}{S}[[\mathrm{X}^G]]$ must be multiplication-preserving. Indeed, for all $x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and all $p \in G$, we have

$$\begin{aligned}
(\mathcal{R}(\phi)(xy))_p &= \phi\left((xy)_p\right) \\
&= \phi\left(\sum_{\substack{g,h\in G\\ g+h=p}} x_g y_h\right) \\
&= \sum_{\substack{g,h\in G\\ g+h=p}} (\mathcal{R}(\phi)(x))_g \cdot (\mathcal{R}(\phi)(y))_h \\
&= (\mathcal{R}(\phi)(x) \cdot \mathcal{R}(\phi)(y))_p \,.
\end{aligned}$$

(**b**) Let $\boldsymbol{I}$ be an object-finite category, let $\mathcal{F} : \boldsymbol{I} \to \mathbf{Rng}$ be a functor and suppose $\chi = \{\chi_i : L \xrightarrow{\mathbf{Rng}} \mathcal{F}(i)\}_{i\in \boldsymbol{I}_0}$ is a limit cone over $\mathcal{F}$ (Definition B.31). We shall prove that the cone

$$\begin{aligned}
\mathcal{R}(\chi) &\coloneqq \{\mathcal{R}(\chi_i) : \mathcal{R}(L) \xrightarrow{\mathbf{Rng}} \mathcal{R}(\mathcal{F}(i))\}_{i\in \boldsymbol{I}_0} \\
&= \left\{\mathcal{R}(\chi_i) : \overset{\mathcal{J}}{L}[[\mathrm{X}^G]] \xrightarrow{\mathbf{Rng}} \overset{\mathcal{J}}{\mathcal{F}}(i)[[\mathrm{X}^G]]\right\}_{i\in \boldsymbol{I}_0}
\end{aligned}$$

over the composition $\mathcal{R} \circ \mathcal{F} : \boldsymbol{I} \to \mathbf{Rng}$ is a limit cone. Suppose that $\lambda = \left\{\lambda_i : V \xrightarrow{\mathbf{Rng}} \overset{\mathcal{J}}{\mathcal{F}}(i)[[\mathrm{X}^G]]\right\}_{i\in \boldsymbol{I}_0}$ is another cone over $\mathcal{R} \circ \mathcal{F}$. Disregarding the multiplication operations on all rngs for now, note that χ is a limit cone over the composition $\overset{\mathbf{Rng}}{\mathrm{U}} \circ \mathcal{F} : \boldsymbol{I} \to \mathbf{Mon}$ (Lemma 2.29), and the cone

$$\mathcal{M}(\chi) \coloneqq \left\{\mathcal{M}(\chi_i) : \overset{\mathcal{J}}{L}[[\mathrm{X}^G]] \xrightarrow{\mathbf{Mon}} \overset{\mathcal{J}}{\mathcal{F}}(i)[[\mathrm{X}^G]]\right\}_{i\in \boldsymbol{I}_0}$$

is a limit cone over $\mathcal{M} \circ \overset{\mathbf{Rng}}{\mathrm{U}} \circ \mathcal{F} : \boldsymbol{I} \to \mathbf{Mon}$ (Theorem 3.54, Item (b)), where $\overset{\mathbf{Rng}}{\mathrm{U}} : \mathbf{Rng} \to \mathbf{Mon}$ is the canonical forgetful functor of that type. For each $g \in G$ and each $i \in \boldsymbol{I}_0$, let $\lambda_{i,g} : V \xrightarrow{\mathbf{Mon}} \mathcal{F}(i)$ be the morphism given by $\lambda_{i,g}(x) \coloneqq (\lambda_i(x))_g$, let $\overline{\lambda_g} : V \xrightarrow{\mathbf{Mon}} L$ be the limit lifting of the cone

$\lambda_g \coloneqq \{\lambda_{i,g} : V \xrightarrow{\mathbf{Mon}} \mathcal{F}(i)\}_{i\in \boldsymbol{I}_0}$ along χ and let $\overline{\lambda} : V \xrightarrow{\mathbf{Mon}} \overset{\mathcal{J}}{L}[[\mathrm{X}^G]]$ be the limit lifting of λ along $\mathcal{M}(\chi)$. That implies that for all $i \in \boldsymbol{I}_0$ and all $g \in G$, the digraphs

$$\begin{tikzcd} & L \arrow[d, "\chi_i"] \\ V \arrow[ur, "\overline{\lambda_g}"] \arrow[r, "\lambda_{i,g}"'] & \mathcal{F}(i) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \overset{\mathcal{J}}{L}[[\mathrm{X}^G]] \arrow[d, "\mathcal{M}(\chi_i)=\mathcal{R}(\chi_i)"] \\ V \arrow[ur, "\overline{\lambda}"] \arrow[r, "\lambda_i"'] & \overset{\mathcal{J}}{\mathcal{F}}(i)[[\mathrm{X}^G]] \end{tikzcd}$$

in **Mon** commute, and, by the proof of Item (b) of Theorem 3.54, the morphism $\overline{\lambda}$ may be given by $\overline{\lambda}(x) \coloneqq \{\overline{\lambda_g}(x)\}_{g\in G}$. Note that for all $x \in V$ and all $g \in G$, we have

$$(\forall i \in \boldsymbol{I}_0)\ \ \chi_i\left(\left(\overline{\lambda}(x)\right)_g\right) = \chi_i\left(\overline{\lambda}_g(x)\right) = \lambda_{i,g}(x) = \left(\lambda_i(x)\right)_g .$$

Taking the multiplication operations on all rngs into account again, it happens that the proof of the item hinges on whether the function $\overline{\lambda}$ is multiplication-preserving or not. For all $i \in \boldsymbol{I}_0$, all $p \in G$ and all $x, y \in V$, we get

$$\begin{aligned} \chi_i\left(\left(\overline{\lambda}(xy)\right)_p\right) &= \left(\lambda_i(xy)\right)_p \\ &= \left(\lambda_i(x)\,\lambda_i(y)\right)_p \\ &= \sum_{\substack{g,h\in G \\ g+h=p}} \left(\lambda_i(x)\right)_g \left(\lambda_i(y)\right)_h \\ &= \sum_{\substack{g,h\in G \\ g+h=p}} \chi_i\left(\left(\overline{\lambda}(x)\right)_g\right) \chi_i\left(\left(\overline{\lambda}(y)\right)_h\right) \\ &= \chi_i\left(\left(\overline{\lambda}(x)\,\overline{\lambda}(y)\right)_p\right) , \end{aligned}$$

implying $\overline{\lambda}(xy) = \overline{\lambda}(x)\,\overline{\lambda}(y)$ (Lemma 1.23) and proving the item.

(c) Let I be an ideal in a rng R, let $\iota : R \xrightarrow{\mathbf{Rng}} R/I$ be the canonical quotient homomorphism and let $\overset{\mathcal{J}}{\underset{I}{\equiv}}$ be the congruence relation on the rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ induced by the ideal $\overset{\mathcal{J}}{I}[[\mathrm{X}^G]]$ (Proposition 3.55). It is easy to verify that we have $\overset{\mathcal{J}}{\underset{I}{\equiv}} = \overset{\mathcal{R}(\iota)}{\mathrm{eq}}$. We shall prove that $(\overset{\mathcal{J}}{R/I})\,[[\mathrm{X}^G]]$ is a quotient of $\overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]]$ modulo $\overset{\mathcal{J}}{I}\,[[\mathrm{X}^G]]$ in **Rng** with quotient homomorphism

$$\mathcal{R}(\iota) : \overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\longrightarrow} (\overset{\mathcal{J}}{R/I})\,[[\mathrm{X}^G]] \ .$$

Consider another rng R' and a homomorphism $f : \overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\longrightarrow} R'$ so that $\overset{\mathcal{J}}{\underset{I}{\equiv}} \subset \overset{f}{\mathrm{eq}}$. Disregarding the multiplication operations on all rngs for now, note that the morphisms $\iota : R \overset{\mathbf{Mon}}{\longrightarrow} R/I$ and

$$\mathcal{M}(\iota) : \overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]] \overset{\mathbf{Mon}}{\longrightarrow} (\overset{\mathcal{J}}{R/I})\,[[\mathrm{X}^G]]$$

are quotient morphisms in **Mon** (Lemma 2.23; Theorem 3.54, Item (c)). For each $S \in \mathcal{J}$, let $f_S : {}^G R \overset{\mathbf{Mon}}{\longrightarrow} R'$ be the homomorphism given by $f_S\left(\{x_g\}_{g\in G}\right) \coloneqq f\left(\sum_{g\in S} x_g \mathrm{X}^g\right)$, let $\overline{f_S} : {}^G(R/I) \overset{\mathbf{Mon}}{\longrightarrow} R'$ be the quotient lowering of f_S modulo ${}^G I$ along the quotient morphism ${}^G\iota$ (Lemma 1.20) and let $\overline{f} : (\overset{\mathcal{J}}{R/I})\,[[\mathrm{X}^G]] \overset{\mathbf{Mon}}{\longrightarrow} R'$ be the quotient lowering of f along $\mathcal{M}(\iota)$. That means that for each $S \in \mathcal{J}$, the digraphs

$$\begin{tikzcd} {}^G R \arrow[r, "f_S"] \arrow[d, "{}^G\iota"'] & R' \\ {}^G(R/I) \arrow[ur, "\overline{f_S}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]] \arrow[r, "f"] \arrow[d, "\mathcal{M}(\iota)"'] & R' \\ (\overset{\mathcal{J}}{R/I})\,[[\mathrm{X}^G]] \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

in **Mon** commute, and, by the proof of Item (c) of Theorem 3.54, the morphism $\overline{f}$ may be given by $\overline{f}(X) \coloneqq \overline{f_{\mathrm{supp}(X)}}(X)$. Note that for all $X \in (\overset{\mathcal{J}}{R/I})\,[[\mathrm{X}^G]]$ and all $x \in \overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]]$ so that $X_g = x_g/\equiv$ $(\forall g \in G)$, we have

$$\overline{f}(X) = \overline{f_{\mathrm{supp}(X)}}(X) = \overline{f_{\mathrm{supp}(X)}}({}^G\iota(x)) = f_{\mathrm{supp}(X)}(x) \ ,$$

and if $\mathrm{supp}(X) = \mathrm{supp}(x)$, then we obtain $\overline{f}(X) = f_{\mathrm{supp}(x)}(x) = f(x)$. Taking the multiplication operations on all rngs into account again, it happens that the proof of the item hinges on whether the function $\overline{f}$ is multiplication-preserving or not. Consider $X, Y \in (\overset{\mathcal{J}}{R/I})\,[[\mathrm{X}^G]]$ and take any two $x, y \in \overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]]$ so that $X_g = \iota(x_g)$ $(\forall g \in G)$, $Y_g = \iota(y_g)$ $(\forall g \in G)$, $\mathrm{supp}(X) = \mathrm{supp}(x)$ and $\mathrm{supp}(Y) = \mathrm{supp}(y)$. Thus, we have the inclusion

$\operatorname{supp}(XY) \subset \operatorname{supp}(xy)$, and if p is an element of the difference set $\operatorname{supp}(xy) - \operatorname{supp}(XY)$, then

$$\iota\left((xy)_p\right) = \iota\left(\sum_{\substack{g,h\in G\\ g+h=p}} x_g y_h\right) = \sum_{\substack{g,h\in G\\ g+h=p}} \iota(x_g)\,\iota(y_h) = \sum_{\substack{g,h\in G\\ g+h=p}} X_g Y_h = (XY)_p = 0_{R/I}$$

and $(xy)_p \underset{I}{\equiv} 0_R$, which gives us the conditions $xy \overset{\mathcal{J}}{\underset{I}{\equiv}} \sum_{p\in\operatorname{supp}(XY)} (xy)_p\, \mathrm{X}^p$ and $f(xy) = f\Big(\sum_{p\in\operatorname{supp}(XY)} (xy)_p\, \mathrm{X}^p\Big)$ since $\overset{\mathcal{J}}{\underset{I}{\equiv}} \subset \overset{f}{\mathrm{eq}}$. Lastly, we have

$$\begin{aligned}
\overline{f}(XY) &= f_{\operatorname{supp}(XY)}(xy)\\
&= f\Big(\sum_{p\in\operatorname{supp}(XY)} (xy)_p\, \mathrm{X}^p\Big)\\
&= f(xy)\\
&= f(x)\, f(y)\\
&= f_{\operatorname{supp}(X)}(x)\, f_{\operatorname{supp}(Y)}(y)\\
&= \overline{f}(X)\, \overline{f}(Y)\ ,
\end{aligned}$$

and that proves the item.

□

Corollary 3.57. *Let G be an ordered group, let $\mathcal{J}$ be an arithmetic Rayner ideal on G, let $R, S, T, R_1 \dots R_n$ be rngs, let $f, g : R \xrightarrow{\mathbf{Rng}} S$, $p : R \xrightarrow{\mathbf{Rng}} T$ and $q : S \xrightarrow{\mathbf{Rng}} T$ be homomorphisms and let I be an ideal in R. Regarding the functor $\mathcal{R} : \mathbf{Rng} \to \mathbf{Rng}$ defined in the statement of Theorem 3.56, we have:*

(a) $\Big(\overset{\mathcal{J}}{\underset{i\in[1,n]_{\mathbb{N}}}{\overset{\mathbf{Rng}}{\prod}}} R_i\Big)[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\cong} \underset{i\in[1,n]_{\mathbb{N}}}{\overset{\mathbf{Rng}}{\prod}} \overset{\mathcal{J}}{R_i}[[\mathrm{X}^G]]$;

(b) $\big(\overset{\mathcal{J}}{\overset{\mathbf{Rng}}{\mathrm{Eq}}}(f,g)\big)[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\cong} \overset{\mathbf{Rng}}{\mathrm{Eq}}(\mathcal{R}(f), \mathcal{R}(g))$;

(c) $\big(\overset{\mathcal{J}}{\overset{\mathbf{Rng}}{\mathrm{Ker}}}(f)\big)[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\cong} \overset{\mathbf{Rng}}{\mathrm{Ker}}(\mathcal{R}(f))$;

(d) $\big(R \overset{\mathcal{J}}{\underset{p,q}{\overset{\mathbf{Rng}}{\times}}} S\big)[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\cong} \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \underset{\mathcal{R}(p),\mathcal{R}(q)}{\overset{\mathbf{Rng}}{\times}} \overset{\mathcal{J}}{S}[[\mathrm{X}^G]]$;

(e) $\big(R \overset{\mathcal{J}}{\overset{\mathbf{Rng}}{/}} I\big)[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\cong} \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{/} \overset{\mathcal{J}}{I}[[\mathrm{X}^G]]$;

(f) $\big(\overset{\mathcal{J}}{\overset{\mathbf{Rng}}{\mathrm{Im}}}(f)\big)[[\mathrm{X}^G]] \supset \overset{\mathbf{Rng}}{\mathrm{Im}}(\mathcal{R}(f))$, and equality holds if f is injective.

Proof. Items (a)-(e) are direct consequences of Theorem 3.54, and item (f) is the only one that still needs to be proved.

(f) The proof of the inclusion $\overset{\mathcal{J}}{\left(\overset{\mathbf{Rng}}{\mathrm{Im}}(f)\right)}[[\mathrm{X}^G]] \supset \overset{\mathbf{Rng}}{\mathrm{Im}}(\mathcal{R}(f))$ is straightforward. We shall prove the opposite inclusion. Suppose f is injective and take an element $y \in \overset{\mathcal{J}}{\left(\overset{\mathbf{Rng}}{\mathrm{Im}}(f)\right)}[[\mathrm{X}^G]]$. Thus, for each $g \in G$, there is a $x_g \in R$ such that $y_g = f(x_g)$, and that defines a family $x := \{x_g\}_{g\in G}$ in R. If $g \in G$ is such that $x_g \neq 0_R$, then $y_g = f(x_g) \neq 0_S = f(0_R)$ since f is injective, implying $\mathrm{supp}(x) \subset \mathrm{supp}(y)$, $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and $y = \mathcal{R}(f)(x) \in \overset{\mathbf{Rng}}{\mathrm{Im}}(\mathcal{R}(f))$.

□

Taking into consideration the fact that the notion of a kernel of a homomorphism between rngs is defined uniquely and not up to isomorphism (Definition 2.1), we leave to the reader the verification that the isomorphism relation of Item (c) of Corollary 3.57 can be replaced by an equality relation.

Example 3.58. A **short exact sequence in Rng** is a digraph in **Rng** of the form

$$\mathbf{0} \longrightarrow R \xrightarrow{f} S \xrightarrow{g} T \longrightarrow \mathbf{0}\ ,$$

where the outer morphisms are the constant zero morphisms (Example 2.25), $f : R \to S$ is an injective homomorphism, and $g : S \to T$ is a surjective homomorphism such that $\mathrm{Im}(f) = \mathrm{Ker}(g)$. Since the functor $\mathcal{R} : \mathbf{Rng} \to \mathbf{Rng}$ preserves kernels and injective images (Corollary 3.57, Items (c) and (f)), the digraph

$$\mathbf{0} \longrightarrow \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \xrightarrow{\mathcal{R}(f)} \overset{\mathcal{J}}{S}[[\mathrm{X}^G]] \xrightarrow{\mathcal{R}(g)} \overset{\mathcal{J}}{T}[[\mathrm{X}^G]] \longrightarrow \mathbf{0}$$

is also a short exact sequence in **Rng**. For instance, take an ideal I in a rng R (Definition 2.20), and consider the digraph

$$\mathbf{0} \longrightarrow I \xrightarrow{f} R \xrightarrow{g} R/I \longrightarrow \mathbf{0}$$

in **Rng**, where $f : I \to R$ is the canonical immersion given by $f(x) := x$ and where $g : R \to R/I$ is the canonical quotient function given by $g(x) := x/\underset{I}{\equiv}$

(Propositions 2.19 and 2.21). Thus, we have $\mathrm{Im}(f) = I = \mathrm{Ker}(g)$, showing that that digraph is a short exact sequence, and so is the digraph

$$\mathbf{0} \longrightarrow \overset{\mathcal{J}}{I}[[\mathrm{X}^G]] \xrightarrow{\mathcal{R}(f)} \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \xrightarrow{\mathcal{R}(g)} (\overset{\mathcal{J}}{R/I})[[\mathrm{X}^G]] \longrightarrow \mathbf{0} \ .$$

It is easy to verify that $\mathcal{R}(f)$ is the canonical immersion of that type, and $\mathcal{R}(g)$ is the quotient function of that type (Theorem 3.56, Item (c)).

In particular, taking $R = \mathbb{Z}$ and $I = p\mathbb{Z}$, where p is a prime (Example 2.22, Case (a)), we get the following short exact sequence in **Rng**:

$$\mathbf{0} \longrightarrow (\overset{\mathcal{J}}{p\mathbb{Z}})[[\mathrm{X}^G]] \xrightarrow{\mathcal{R}(f)} \overset{\mathcal{J}}{\mathbb{Z}}[[\mathrm{X}^G]] \xrightarrow{\mathcal{R}(g)} \overset{\mathcal{J}}{\mathrm{F}}_p[[\mathrm{X}^G]] \longrightarrow \mathbf{0} \ .$$

Example 3.59. Let G be an ordered group, let $\mathcal{J}$ be an arithmetic Rayner ideal on G, let $n_1 n_2 \dots n_k$ be a finite sequence of pairwise coprime numbers in $[2, \infty)_{\mathbb{N}}$, and let $N \coloneqq n_1 n_2 \cdots n_k$. By the Chinese Remainder Theorem (37, 14), we have

$$\mathbb{Z}/N\mathbb{Z} \overset{\mathbf{Rng}}{\cong} (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_k\mathbb{Z}) \ ,$$

and by Item (a) of Corollary 3.57, we obtain

$$(\overset{\mathcal{J}}{\mathbb{Z}/N\mathbb{Z}})[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\cong} (\overset{\mathcal{J}}{\mathbb{Z}/n_1\mathbb{Z}})[[\mathrm{X}^G]] \times (\overset{\mathcal{J}}{\mathbb{Z}/n_2\mathbb{Z}})[[\mathrm{X}^G]] \times \cdots \times (\overset{\mathcal{J}}{\mathbb{Z}/n_k\mathbb{Z}})[[\mathrm{X}^G]] \ .$$

3.7 The fixed point theorem

We shall address a fixed point theorem that is a valuable tool for proving some theorems concerning $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ when the Rayner ideal $\mathcal{J}$ is incremental (Definition 3.1). It was first posited by Shamseddine regarding the real Levi-Civita field (200), and the proof that we shall present here is nothing but a generalisation of his proof written in different notations.

We begin by introducing an efficient way of denoting families in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$:

Notation 3.60. Since each element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is a family $x = \{x_g\}_{g\in G}$ in R with indices in G, it is advantageous to write the indices of a family in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ in the lower-left corner, such as in ${}_ix$, reserving the lower-right corner to the elements of G. Hence, we define ${}_ix_g$ as being the g-image of the i-term of the family $\{{}_ix\}_{i\in I}$ in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, that is, ${}_ix_g \coloneqq ({}_ix)_g$. As usual, the upper-right indices are reserved to denoting powers.

Theorem 3.61 (Fixed Point Theorem)**.** *Let R be a rng, let G be an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$, let $\mathcal{J}$ be an incremental arithmetic Rayner ideal on G, let $g_0 \in G$ and let $k \in (0,\infty)_G$. If $f : \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0}) \to \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$ is a function such that*

$$\left(\forall {}_1x, {}_2x \in \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})\right)(\forall g \in [g_0,\infty)_G)\ \left({}_1x \doteq {}_2x + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g}) \Rightarrow f({}_1x) \doteq f({}_2x) + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g+k})\right),$$

then f has a unique fixed point in $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$.

Proof. Let ${}_0a \in \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$ be arbitrary and let $\{{}_na\}$ be the recursively defined sequence in $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$ given by ${}_na \coloneqq f({}_{n-1}a)$. Employing the hypothesis and Subitem 7 of Item (b) of Proposition 3.47, one inductively obtains the results

$$(\forall m \in \mathbb{N})\ {}_ma \doteq {}_{m-1}a + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0+(m-1)k})$$

and

$$(\forall m \in \mathbb{N})(\forall n \in [0,m]_{\mathbb{N}_0})\ {}_ma \doteq {}_na + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0+nk}).$$

Let $x = \{x_g\}_{g\in G}$ be the family in R given by $x_g \coloneqq {}_{m_g}a_g$, where m_g is the smallest natural number so that $g \leqslant g_0 + (m_g - 1)k$. Note that the number m_g exists inasmuch as G is Archimedean. Also, we have $m_{g_0} = 1$ and $g < g_0 + m_gk$ $(\forall g \in G)$. If p and g are two elements of G so that $p \leqslant g$, then we get $m_p \leqslant m_g$ and

$${}_{m_g}a_p \doteq \left({}_{m_p}a + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0+m_pk})\right)_p \doteq {}_{m_p}a_p = x_p.$$

In particular, since for each $g \in G$ we have ${}_{m_g}a \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and

$$\operatorname{supp}(x) \cap (-\infty, g]_G \subset \operatorname{supp}({}_{m_g}a) \in \mathcal{J},$$

and since the ideal $\mathcal{J}$ is incremental, we obtain $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, $x \doteq {}_{m_g}a + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ and $x \doteq {}_1a + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0}) \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$. Furthermore, for each $g \in [g_0, \infty)_G$, we get

$$f(x) \doteq f({}_{m_g}a) + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g+k}) \doteq {}_{m_g+1}a + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g+k}) \doteq {}_{m_g}a + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0+m_gk}) + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g+k}) \ ,$$

leading us to

$$(f(x))_g \doteq \left({}_{m_g}a + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0+m_gk}) + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g+k})\right)_g \doteq {}_{m_g}a_g = x_g \ ,$$

proving that $f(x) = x$.

It remains to show that x is unique. Suppose y is another fixed point of f in $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$. Clearly, we have $\overset{\mathcal{J}}{\mathrm{ms}}(x-y) \geqslant g_0$, and, since $x \doteq y + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x-y)}\right)$, we get

$$x = f(x) \doteq f(y) + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x-y)+k}\right) \doteq y + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x-y)+k}\right) \ ,$$

implying $\overset{\mathcal{J}}{\mathrm{ms}}(x-y) \geqslant \overset{\mathcal{J}}{\mathrm{ms}}(x-y) + k$, $\overset{\mathcal{J}}{\mathrm{ms}}(x-y) = \infty$ and $x = y$.

□

The reader is invited to check that the statement and proof of the Fixed Point Theorem, as presented above, work well when all big-O sets are replaced by little-O sets.

Note that the Fixed Point Theorem is valid for both the Hahn and Levi-Civita rngs, in particular, since the Rayner ideals associated to those structures are incremental (Proposition 3.4, Item (b); Proposition 3.14, Item (a)).

Example 3.62. Consider the setting of Theorem 3.61, additionally assuming that R is a ring, and consider the function $h : \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0}) \to \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ given by

$$h(x) := x \cdot \mathrm{X}^k + \mathrm{X}^{g_0} = \mathrm{X}^k \cdot x + \mathrm{X}^{g_0} \, .$$

Note that for all $x \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$ we have (Theorem 3.47, Item (b))

$$h(x) = x \cdot \mathrm{X}^k + \mathrm{X}^{g_0} \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0+k}) + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0}) \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0}) + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0}) \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0}) \ ,$$

proving that the function h is of type $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0}) \to \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$, and if ${}_1x, {}_2x \in \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$ and $g \in [g_0, \infty)_G$ are so that ${}_1x \doteq {}_2x + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g})$, then

$$h({}_1x) - h({}_2x) = ({}_1x \cdot \mathrm{X}^k + \mathrm{X}^{g_0}) - ({}_2x \cdot \mathrm{X}^k + \mathrm{X}^{g_0}) = ({}_1x - {}_2x) \cdot \mathrm{X}^k \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g+k}) \ ,$$

that is, $h({}_1x) \doteq h({}_2x) + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g+k})$. By the Fixed Point Theorem, the function h has a unique fixed point u in $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$, that is, there is a unique $u \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$ so that

$$u = u \cdot \mathrm{X}^k + \mathrm{X}^{g_0} = \mathrm{X}^k \cdot u + \mathrm{X}^{g_0} \, ,$$

which, by rearranging the terms and multiplying by X^{-g_0}, gives us

$$u\,(\mathrm{X}^{-g_0} - \mathrm{X}^{-g_0+k}) = 1_R = (\mathrm{X}^{-g_0} - \mathrm{X}^{-g_0+k})\,u \, .$$

Therefore, the unique fixed point of h turns out to be the inverse of $\mathrm{X}^{-g_0} - \mathrm{X}^{-g_0+k}$ in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

3.8 Conditions for $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ to be a division ring or an algebraically closed field

We know that a ring R is a division ring whenever the Rayner ring $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is a division ring (Proposition 3.48, Item (d)), but the converse of that implication does not always hold (Example 3.71). We shall employ a couple of powerful results available in literature to ascertain that the converse statement actually does hold in an imperative case, and to establish sufficient conditions for $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ to be an algebraically closed field (Definition 2.49).

The following proposition was first proved by Hahn in 1907 for commutative ordered groups G, and it was later generalised by Mal'cev and Neumann in the late 1940s:

Proposition 3.63. (153, 145, 92) *If K is a division ring (resp. a field) and if G is an ordered group, then the Hahn ring $K[[\mathrm{X}^G]]$ is a division ring (resp. a field).*

Example 3.64. The Hahn rings $\mathbf{H}[[\mathrm{X}^{\mathbb{Z}}]]$, $\mathbf{H}[[\mathrm{X}^{\mathbb{Q}}]]$, $\mathbf{H}[[\mathrm{X}^{\mathbb{R}}]]$ and $\mathbf{H}[[\mathrm{X}^{\mathsf{BS}_\ell}]]$ are division rings (Example 2.8).

Example 3.65. The Hahn rings

$$\mathbb{Q}[[\mathrm{X}^{\mathbb{Z}}]], \mathbb{Q}[[\mathrm{X}^{\mathbb{Q}}]], \mathbb{Q}[[\mathrm{X}^{\mathbb{R}}]], \mathbb{Q}[[\mathrm{X}^{\mathsf{BS}_\ell}]], \mathbb{R}[[\mathrm{X}^{\mathbb{Z}}]], \mathbb{R}[[\mathrm{X}^{\mathbb{Q}}]], \mathbb{R}[[\mathrm{X}^{\mathbb{R}}]], \mathbb{R}[[\mathrm{X}^{\mathsf{BS}_\ell}]]$$
$$\mathbb{C}[[\mathrm{X}^{\mathbb{Z}}]], \mathbb{C}[[\mathrm{X}^{\mathbb{Q}}]], \mathbb{C}[[\mathrm{X}^{\mathbb{R}}]] \text{ and } \mathbb{C}[[\mathrm{X}^{\mathsf{BS}_\ell}]]$$

are all fields.

We shall show that the statement of Proposition 3.63 is valid for all Rayner rings generated by full Rayner ideals, not just for the Hahn rings. For that, we need the following lemma:

Lemma 3.66. (145, 92) *Let K be a division ring and let G be an ordered group. If x is a unit in the Hahn ring $K[[\mathrm{X}^G]]$ such that $x_{0_G} = 1_K$ and $\mathrm{supp}(x) \subset [0_G, \rightarrow)_G$, then $(x^{-1})_{0_G} = 1_K$ and $\mathrm{supp}(x^{-1}) \subset \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(\mathrm{supp}(x))$.*

Proof. Since $\mathrm{supp}(x^{-1})$ is well-ordered in G, there is a unique ordinal γ and a unique increasing family $\{p_\alpha\}_{\alpha<\gamma}$ whose image is $\mathrm{supp}(x^{-1})$ (Proposition 1.31). We shall prove that $p_\alpha \in \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(\mathrm{supp}(x))$ $(\forall \alpha < \gamma)$ by transfinite induction on α. Firstly, since $\mathrm{supp}(x) \subset [0_G, \rightarrow)_G$, note that

$$(1_K)_{p_0} = (xx^{-1})_{p_0} = \sum_{\substack{g,h\in G\\ g+h=p_0}} x_g\,(x^{-1})_h = x_{0_G}\,(x^{-1})_{p_0} = 1_K \cdot (x^{-1})_{p_0} = (x^{-1})_{p_0} \neq 0_K\,,$$

implying $p_0 = 0_G \in \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(\mathrm{supp}(x))$, $\mathrm{supp}(x^{-1}) \subset [0_G, \rightarrow)_G$ and $(x^{-1})_{0_G} = 1_K$. Suppose that α is an ordinal number in the interval $(0,\gamma)_{\mathsf{On}}$ such that $p_\beta \in \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(\mathrm{supp}(x))$ $(\forall \beta < \alpha)$ and take the increasing finite sequence of ordinal numbers $\alpha_1\alpha_2\ldots\alpha_n$ such that $p_{\alpha_1}p_{\alpha_2}\ldots p_{\alpha_n}$ are the elements of the finite image set (Lemma 1.75)

$$\mathrm{pr}_2\left\langle\{(g,h)\in\mathrm{supp}\,(x)\times\mathrm{supp}\,(x^{-1})\ \mid\ g+h=p_\alpha\}\right\rangle\ ,$$

where $\mathrm{pr}_2:\mathrm{supp}\,(x)\times\mathrm{supp}\,(x^{-1})\to\mathrm{supp}\,(x^{-1})$ is the canonical projection given by $\mathrm{pr}_2\,(g,h):=h$. Since $\alpha_n=\alpha$, we have

$$0_K=(1_K)_{p_\alpha}=(xx^{-1})_{p_\alpha}=\sum_{\substack{g,h\in G\\ g+h=p_\alpha}}x_g\,(x^{-1})_h=x_{0_G}\,(x^{-1})_{p_\alpha}+\sum_{i=1}^{n-1}x\,[\![p_\alpha+(-p_{\alpha_i})]\!]\ (x^{-1})_{p_{\alpha_i}}\ ,$$

which gives us

$$\sum_{i=1}^{n-1}x\,[\![p_\alpha+(-p_{\alpha_i})]\!]\ (x^{-1})_{p_{\alpha_i}}=\ -\,(x^{-1})_{p_\alpha}\neq 0_K\ ,$$

implying that there is an $i\in[1,n-1]_{\mathbb{N}}$ such that $p_\alpha+(-p_{\alpha_i})\in\mathrm{supp}\,(x)$. Finally, we obtain

$$p_\alpha\in\mathrm{supp}\,(x)+p_{\alpha_i}\subset\mathrm{supp}\,(x)+\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(\mathrm{supp}\,(x))\subset\underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(\mathrm{supp}\,(x))\ ,$$

and that concludes our induction.

□

Theorem 3.67. (181, 117) *If K is a division ring (resp. a field), if G is an ordered group and if $\mathcal{J}$ is a full Rayner ideal on G, then the Rayner ring $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ is a division ring (resp. a field).*

Proof. By Proposition 3.48, we know that $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ is a ring with no zero divisors. Let $x\in\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]-\{0_K\}$ and let

$$\begin{aligned}x^*&:=\left((\mathrm{pc}\,(x))^{-1}\,\mathrm{X}^{-\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)x\\&\doteq\left((\mathrm{pc}\,(x))^{-1}\,\mathrm{X}^{-\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)\left(\mathrm{pc}\,(x)\,\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)}+\overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)\right)\\&\doteq 1_K+\overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{-\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)\overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)\\&\doteq 1_K+\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G})\,.\end{aligned}$$

If x^* is a unit in $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$, then, since

$$x=\left((\mathrm{pc}\,(x))\,\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)x^*\ ,$$

the element
$$(x^*)^{-1}\left((\mathrm{pc}\,(x))^{-1}\,\mathrm{X}^{-\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)$$
is the multiplicative (bilateral) inverse of x in $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$. Hence, we may assume, without loss of generality, that $x \doteq 1_K + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^{0_G})$. Since the Hahn ring $K\,[[\mathrm{X}^G]]$ is a division ring (Proposition 3.63), the element x has an inverse x^{-1} in $K\,[[\mathrm{X}^G]]$, and, by Lemma 3.66, we have $(x^{-1})_{0_G} = 1_K$ and
$$\mathrm{supp}\,(x^{-1}) \subset \underset{G}{\overset{\mathbf{SGrp}}{\mathrm{span}}}(\mathrm{supp}\,(x)) \in \mathcal{J}\,,$$
implying $\mathrm{supp}\,(x^{-1}) \in \mathcal{J}$ and $x^{-1} \in \overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$. □

Corollary 3.68. *Let K be a division ring (resp. a field), let G be an ordered group and let H be a Puiseux ordered subgroup of G.*

(**a**) *If G is isomorphic to an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$, then the Puiseux ring $\overset{\mathrm{bd}}{K}_H[[\mathrm{X}^G]]$ is a division ring (resp. a field);*

(**b**) *The Levi-Civita ring $\overset{\mathrm{lf}}{K}\,[[\mathrm{X}^G]]$ is a division ring (resp. a field) if, and only if, the ordered group G is isomorphic to an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$.*

Proof. If G is isomorphic to an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$, then the arithmetic Rayner ideals $\overset{\mathrm{lf}}{\mathrm{P}}\,(G)$ and $\overset{\mathrm{bd}}{\mathrm{P}}_H(G)$ on G are full (Proposition 3.14, Item (b); Proposition 3.18, Item (f)), implying that $\overset{\mathrm{lf}}{K}\,[[\mathrm{X}^G]]$ and $\overset{\mathrm{bd}}{K}_H[[\mathrm{X}^G]]$ are division rings (Theorem 3.67). It remains to prove the necessary condition of item (b):

(**b**) Suppose G is not isomorphic to an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$, that is, suppose G is non-Archimedean (Theorem 1.72), take two positive elements g_0 and h_0 of G so that $ng_0 < h_0$ $(\forall n \in \mathbb{N})$, and consider the element
$$x \coloneqq 1_K - \mathrm{X}^{g_0} \in \overset{\mathrm{lf}}{K}\,[[\mathrm{X}^G]] \subset K\,[[\mathrm{X}^G]]\ .$$
Since the Hahn ring $K\,[[\mathrm{X}^G]]$ is a division ring (Proposition 3.63), we know that x has an inverse in $K\,[[\mathrm{X}^G]]$, which happens to be the Hahn series
$$y \coloneqq \sum_{n\in\mathbb{N}_0} \mathrm{X}^{ng_0} = 1_K + \mathrm{X}^{g_0} + \mathrm{X}^{2g_0} + \mathrm{X}^{3g_0} + \cdots + \mathrm{X}^{ng_0} + \cdots\ .$$

One may verify that $y = x^{-1}$ simply by computing the products xy and yx, and we leave that simple task to the reader. Note that the support $\operatorname{supp}(y) = \{0_G, g_0, 2g_0, 3g_0, \dots\}$ is not cofinal in G, and, therefore, is not left-finite in G, implying that $y \notin \overset{\mathrm{lf}}{K}[[\mathrm{X}^G]]$. Lastly, since the inverse of x is unique in the division ring $K[[\mathrm{X}^G]]$, it follows that the element x is not a unit in $\overset{\mathrm{lf}}{K}[[\mathrm{X}^G]]$, proving that $\overset{\mathrm{lf}}{K}[[\mathrm{X}^G]]$ is not a division ring.

□

Example 3.69. Considering the division ring $\mathbf{H}$ of quaternions as the ring of coefficients (Example 2.8), we have that the Levi-Civita rings $\overset{\mathrm{lf}}{\mathbf{H}}[[\mathrm{X}^{\mathbb{Z}}]]$, $\overset{\mathrm{lf}}{\mathbf{H}}[[\mathrm{X}^{\mathbb{Q}}]]$ and $\overset{\mathrm{lf}}{\mathbf{H}}[[\mathrm{X}^{\mathbb{R}}]]$, and the Puiseux rings $\overset{\mathrm{bd}}{\mathbf{H}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]$, $\overset{\mathrm{bd}}{\mathbf{H}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{D}_d}]]$ and $\overset{\mathrm{bd}}{\mathbf{H}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]$ (Examples 3.20 and 3.21), are all division rings.

Example 3.70. The Levi-Civita rings

$$\overset{\mathrm{lf}}{\mathbb{Q}}[[\mathrm{X}^{\mathbb{Z}}]]\,,\overset{\mathrm{lf}}{\mathbb{Q}}[[\mathrm{X}^{\mathbb{Q}}]]\,,\overset{\mathrm{lf}}{\mathbb{Q}}[[\mathrm{X}^{\mathbb{R}}]]\,,\overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Z}}]]\,,\overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]\,,\overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{R}}]]\,,$$
$$\overset{\mathrm{lf}}{\mathbb{C}}[[\mathrm{X}^{\mathbb{Z}}]]\,,\overset{\mathrm{lf}}{\mathbb{C}}[[\mathrm{X}^{\mathbb{Q}}]]\,,\text{ and }\ \overset{\mathrm{lf}}{\mathbb{C}}[[\mathrm{X}^{\mathbb{R}}]]\ ,$$

and the Puiseux rings

$$\overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]\,,\overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{D}_d}]]\,,\overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]\,,\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]\,,\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{D}_d}]]\,,\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]$$
$$\overset{\mathrm{bd}}{\mathbb{C}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]\,,\overset{\mathrm{bd}}{\mathbb{C}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{D}_d}]]\ \text{ and }\ \overset{\mathrm{bd}}{\mathbb{C}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]\ ,$$

are all fields.

Example 3.71. Let α be an ordinal of the form ω^β, where β is a countable ordinal greater than 1, and consider the ordered group No_α of surreal numbers with birthdays less than α (Example 3.8). Since we have

$$\operatorname{cf}(\mathsf{No}_\alpha) = \operatorname{cf}(\alpha) \leqslant |\alpha| = \omega\,,$$

we get the strict inclusions (Proposition 3.11)

$$\mathrm{P}_\omega(\mathsf{No}_\alpha) \subsetneq \overset{\mathrm{lf}}{\mathrm{P}}(\mathsf{No}_\alpha) \subsetneq \overset{\mathrm{wo}}{\mathrm{P}}(\mathsf{No}_\alpha) \quad \text{and} \quad \mathbb{Q}[\mathrm{X}^{\mathsf{No}_\alpha}] \subsetneq \overset{\mathrm{lf}}{\mathbb{Q}}[[\mathrm{X}^{\mathsf{No}_\alpha}]] \subsetneq \mathbb{Q}[[\mathrm{X}^{\mathsf{No}_\alpha}]]\ .$$

Thus, the ring $\overset{\mathrm{lf}}{\mathbb{Q}}[[\mathrm{X}^{\mathsf{No}_\alpha}]]$ is an instance of a non-polynomial Levi-Civita ring that is not a field but has coefficients in a field (Corollary 3.68, Item (b)).

Making use of the Fixed Point Theorem (Theorem 3.61), we exhibit an alternative proof of Theorem 3.67 in the special case in which G is an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ and the Rayner ideal $\mathcal{J}$ is incremental. That proof is largely inspired by Shamseddine's proof that the ring $\mathcal{R} = \overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]$ is a field (200).

Proof of Theorem 3.67 for a special case. Let K be a division ring, let G be an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ and let $\mathcal{J}$ be an incremental arithmetic Rayner ideal on G. Just as we noted in the proof of Theorem 3.67, we may assume, without loss of generality, that $x \doteq 1_K + \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{0_G}\right)$. If $x = 1_K$, then x is its own inverse. Suppose $x \neq 1_K$ and let $y \coloneqq x - 1_K \doteq \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{0}\right)$. By Proposition 1.10, it is enough to prove that there is a $z \in \overset{\mathcal{J}}{K}[[\mathrm{X}^G]] - \{0_K\}$ such that $(1_K + z)(1_K + y) = 1_K$. This is equivalent to the equation $z = -zy - y$. Let $f : \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right) \to \overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ be the function given by $f(z) \coloneqq -zy - y$. Accordingly, we must find a fixed point z of the function f. For any $z \in \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right)$, we have

$$-f(z) = zy + y \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(z)}\right)\overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right) + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right) \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right),$$

which points to the inclusion $f\left\langle \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right)\right\rangle \subset \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right)$. If $g \in [\overset{\mathcal{J}}{\mathrm{ms}}(y), \infty)_G$ and if ${}_1z, {}_2z \in \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right)$ are so that ${}_1z \doteq {}_2z + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g}\right)$, then we get

$${}_1zy \doteq \left({}_2z + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g}\right)\right) y \doteq {}_2zy + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g + \overset{\mathcal{J}}{\mathrm{ms}}(y)}\right)$$

and $f({}_1z) \doteq f({}_2z) + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g + \overset{\mathcal{J}}{\mathrm{ms}}(y)}\right)$. Therefore, by the Fixed Point Theorem (Theorem 3.61), there is a unique $z \in \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right)$ such that $f(z) = z$, implying that $1_K + z$ is the multiplicative left inverse of x in $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ and proving that $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ is a division ring.

□

The following theorem was first proved by Puiseux in 1850 for the Puiseux fields, then proved by Mac Lane in 1939 for the Hahn fields, and it was later generalised by Rayner in 1968:

Theorem 3.72. (178, 150, 181) *If K is an algebraically closed field of characteristic zero, if G is a divisible, commutative ordered group and if $\mathcal{J}$ is a full Rayner ideal on G, then the Rayner field $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ is algebraically closed.*

Corollary 3.73. (178, 150, 181) *Let K be an algebraically closed field of characteristic zero.*

(**a**) *If G is a divisible, commutative ordered group, then the Hahn field $K\,[[\mathrm{X}^G]]$ is algebraically closed;*

(**b**) *If G is isomorphic to a divisible ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ and if H is a Puiseux ordered subgroup of G, then the Levi-Civita field $\overset{\mathrm{lf}}{K}[[\mathrm{X}^G]]$ and the Puiseux field $\overset{\mathrm{bd}}{K}_H[[\mathrm{X}^G]]$ are algebraically closed.*

Proof. The Rayner ideal $\overset{\mathrm{wo}}{\mathrm{P}}(G)$ on G is always full (Proposition 3.4, Item (b)), and the Rayner ideals $\overset{\mathrm{lf}}{\mathrm{P}}(G)$ and $\overset{\mathrm{bd}}{\mathrm{P}}_H(G)$ on G are full when G is isomorphic to an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ (Proposition 3.14, Item (b); Proposition 3.18, Item (f)).

$\square$

Example 3.74. The Hahn fields $\mathbb{C}\,[[\mathrm{X}^{\mathbb{Q}}]]$ and $\mathbb{C}\,[[\mathrm{X}^{\mathbb{R}}]]$, the Levi-Civita fields $\overset{\mathrm{lf}}{\mathbb{C}}\,[[\mathrm{X}^{\mathbb{Q}}]]$ and $\overset{\mathrm{lf}}{\mathbb{C}}\,[[\mathrm{X}^{\mathbb{R}}]]$, and the Puiseux field $\overset{\mathrm{bd}}{\mathbb{C}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]$ are algebraically closed.

3.9 Conditions for $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ to have n-th roots

In this section, we shall discuss a property regarding n-th powers in Rayner rngs, and we shall establish conditions under which an element of a Rayner field $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ has n-th roots (Example 2.47).

Proposition 3.75. *Let R be a rng, let G be a commutative ordered group, let $\mathcal{J}$ be an arithmetic Rayner ideal on G and let $g_0 \in G$ be fixed. If ${}_1x, {}_2x \in \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g_0}\right)$ and $g \in [g_0, \rightarrow)_G$ are so that ${}_1x \doteq {}_2x + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g}\right)$, then we have*

$$(\forall n \in \mathbb{N}) \;\; {}_1x^n \doteq {}_2x^n + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+(n-1)g_0}\right) .$$

Proof. The statement is certainly true for $n = 1$. Suppose it is true for a natural number n. Since $g_0 \leqslant g$ and since G is commutative, we have

$$g + ((n+1) - 1)\, g_0 = g + ng_0 = g + g_0 + (n-1)\, g_0 \leqslant 2g + (n-1)\, g_0 ,$$

resulting in

$$\begin{aligned}
{}_1x^{n+1} &= {}_1x^n \, {}_1x \\
&\doteq \left({}_2x^n + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+(n-1)g_0}\right)\right)\left({}_2x + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g}\right)\right) \\
&\doteq {}_2x^{n+1} + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+ng_0}\right) + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+ng_0}\right) + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{2g+(n-1)g_0}\right) \\
&\doteq {}_2x^{n+1} + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+((n+1)-1)g_0}\right)
\end{aligned}$$

inasmuch as ${}_2x^n \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{ng_0}\right)$ (Proposition 3.47, Item (b), Subitem 2), proving the lemma by induction.

□

Corollary 3.76. *Let R be a rng, let G be a commutative ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. If ${}_1x, {}_2x \in \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{0_G}\right)$ are so that ${}_1x \doteq {}_2x + \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{0_G}\right)$, then ${}_1x^n \doteq {}_2x^n + \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{0_G}\right)$ for all $n \in \mathbb{N}$.*

Proof. The condition ${}_1x \doteq {}_2x + \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{0_G}\right)$ implies $g \coloneqq \overset{\mathcal{J}}{\mathrm{ms}}\left({}_1x - {}_2x\right) > 0_G$ and ${}_1x \doteq {}_2x + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g}\right)$, entailing

$$(\forall n \in \mathbb{N}) \;\; {}_1x^n \doteq {}_2x^n + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+(n-1)\cdot 0}\right) \doteq {}_2x^n + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g}\right) \doteq {}_2x^n + \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{0}\right) .$$

□

The proofs of Lemma 3.77 and Theorem 3.78 below are largely inspired by Shamseddine's proofs of similar results concerning the real Levi-Civita field (200).

Lemma 3.77. *Let K be a field, let G be an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$, let $\mathcal{J}$ be an incremental arithmetic Rayner ideal on G and let $n \in \mathbb{N}$. For all $c \doteq \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{0_G}\right)$, there is a unique solution $x \doteq \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{0_G}\right)$ for the algebraic equation $1_K + c = (1_K + x)^n$. Furthermore, we have $\overset{\mathcal{J}}{\mathrm{ms}}(c) = \overset{\mathcal{J}}{\mathrm{ms}}(x)$.*

Proof. That algebraic equation may be rewritten as $c = \sum_{k=1}^{n} \binom{n}{k} x^k$ and as $x = \left(c - \sum_{k=2}^{n} \binom{n}{k} x^k\right)/n$. Thus, we can regard x as a fixed point of the function $f : \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right) \to \overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ given by $f(u) := \left(c - \sum_{k=2}^{n} \binom{n}{k} u^k\right)/n$. We shall employ the Fixed Point Theorem (Theorem 3.61) to prove that such element x exists. For each $u \in \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right)$, we have

$$c - \sum_{k=2}^{n} \binom{n}{k} u^k \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right) + \sum_{k=2}^{n} \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{k\,\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right) \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right) ,$$

implying the conditions $f(u) \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right)$ and $f\left\langle \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right)\right\rangle \subset \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right)$. If ${}_1u, {}_2u \in \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right)$ and $g \in [\overset{\mathcal{J}}{\mathrm{ms}}(c), \infty)_G$ are so that ${}_1u \doteq {}_2u + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g}\right)$, then we get (Proposition 3.75)

$$\sum_{k=2}^{n} \binom{n}{k} {}_1u^k \doteq \sum_{k=2}^{n} \binom{n}{k} \left({}_2u^k + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+(k-1)\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right)\right) \doteq \sum_{k=2}^{n} \binom{n}{k} {}_2u^k + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right) ,$$

which straightforwardly entails $f({}_1u) \doteq f({}_2u) + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{g+\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right)$. Hence, by the Fixed Point Theorem, there is a unique fixed point $x \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right)$ of f. If $y \doteq \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{0_G}\right)$ is another solution for the algebraic equation in the statement of the lemma, then, since

$$c = \sum_{k=1}^{n} \binom{n}{k} y^k \doteq y\left(n_K + \sum_{k=1}^{n-1} \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{k\,\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right)\right) \doteq y\left(n_K + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right)\right) ,$$

we obtain (Proposition 3.48, Item (c))

$$\overset{\mathcal{J}}{\mathrm{ms}}(c) \doteq \overset{\mathcal{J}}{\mathrm{ms}}(y) + \overset{\mathcal{J}}{\mathrm{ms}}\left(n_K + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right)\right) \doteq \overset{\mathcal{J}}{\mathrm{ms}}(y)$$

and $y \doteq \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(c)}\right)$, which implies $x = y$ and proves that x is the only solution for the equation in $\overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{0_G}\right)$.

$\square$

Theorem 3.78. *Let K be a field, let G be an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$, let $\mathcal{J}$ be an incremental arithmetic Rayner ideal on G and let $n \in \mathbb{N}$. A non-zero element x of $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ has an n-th root in $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ if, and only if, we have $\overset{\mathcal{J}}{\mathrm{ms}}(x)/n \in G$ and $\mathrm{pc}\,(x)$ has an n-th root in K. In that case, there is a unique $y \doteq \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^0)$ such that the set of n-th roots of x in $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ is given by*

$$\left\{k\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)/n}\,(1_K + y) \;\mid\; k \in K \text{ and } k^n = \mathrm{pc}\,(x)\right\} .$$

Proof. Since K is a field, the element x may be written in the form

$$x = \mathrm{pc}\,(x)\ \mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\,(1_K + c)\ ,$$

where we have $c \doteq \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^0)$. Let $y \doteq \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^0)$ be the unique element such that $1_K + c = (1_K + y)^n$ (Lemma 3.77). If $\overset{\mathcal{J}}{\mathrm{ms}}(x)/n \in G$ and if k is an n-th root of $\mathrm{pc}\,(x)$ in K, then the element $k\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)/n}\,(1_K + y)$ is an apparent n-th root of x in $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$.

Conversely, suppose w is an n-th root of x in $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$. Thus, $w \neq 0_K$, and we have (Proposition 3.48, Item (c))

$$\overset{\mathcal{J}}{\mathrm{ms}}\,(x) = \overset{\mathcal{J}}{\mathrm{ms}}\,(w^n) = n\,\overset{\mathcal{J}}{\mathrm{ms}}\,(w)\ ,$$

implying $\overset{\mathcal{J}}{\mathrm{ms}}(x)/n = \overset{\mathcal{J}}{\mathrm{ms}}\,(w) \in G$ and resulting that we may write w in the form

$$w = \mathrm{pc}\,(w)\ \mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)/n}\,(1_K + y_1)\ ,$$

where $y_1 \doteq \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^0)$. Being that the ring $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ is commutative (Theorem 3.48), from $x = w^n$ we obtain

$$\mathrm{pc}\,(x)\,(1_K + c) = (\mathrm{pc}\,(w))^n\,(1_K + y_1)^n\ .$$

On account of

$$\mathrm{pc}\,(x)\,(1_K + c) \doteq \mathrm{pc}\,(x) + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^0)$$

and (Corolary 3.76)

$$(\mathrm{pc}\,(w))^n\,(1_K + y_1)^n \doteq (\mathrm{pc}\,(w))^n\,(1_K + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^0)) \doteq (\mathrm{pc}\,(w))^n + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^0)\ ,$$

we get $\mathrm{pc}\,(x) = (\mathrm{pc}\,(w))^n$, which immediatly implies that $1_K + c = (1_K + y_1)^n$ and $y_1 = y$. □

For the scenario specified in the statement of Theorem 3.78, note that x and $\mathrm{pc}\,(x)$ have the same number of n-th roots in $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ and K, respectively.

Example 3.79. Since the ideal $\overset{\text{wo}}{\mathrm{P}}(\mathbb{Z})$ is an incremental full Rayner ideal on $\mathbb{Z}$ (Proposition 3.4, Item (b)), and since there are three cubic roots of unity in $\mathbb{C}$, we have that the element $1+\mathrm{X}$ of the Hahn field $\mathbb{C}[[\mathrm{X}^{\mathbb{Z}}]]$ has three cubic roots of unity x, x' and x'' in $\mathbb{C}[[\mathrm{X}^{\mathbb{Z}}]]$ (Theorem 3.78). Let us calculate by brute force the first coefficients of one of these roots. After a few simple calculations, we have

$$\begin{aligned} x^3 = x_0^3 + 3x_0^2x_1\mathrm{X} + (3x_0^2x_2 + 3x_0x_1^2)\,\mathrm{X}^2 + (3x_0^2x_3 + 6x_0x_1x_2 + x_1^3)\,\mathrm{X}^3 \\ + (3x_0^2x_4 + 6x_0x_1x_3 + 3x_0x_2^2 + 3x_1^2x_2)\,\mathrm{X}^4 + \cdots , \end{aligned}$$

and since $x^3 = 1+\mathrm{X}$, we obtain the system of equations

$$\begin{aligned} x_0^3 &= 1 & 3x_0^2x_3 + 6x_0x_1x_2 + x_1^3 &= 0 \\ 3x_0^2x_1 &= 1 & 3x_0^2x_4 + 6x_0x_1x_3 + 3x_0x_2^2 + 3x_1^2x_2 &= 0 \\ 3x_0^2x_2 + 3x_0x_1^2 &= 0 \end{aligned}$$

The equation $x_0^3 = 1$ has three solutions for x_0 in $\mathbb{C}$, namely 1, $\mathsf{e}^{2\pi i/3}$ and $\mathsf{e}^{4\pi i/3}$. For $x_0 = 1$, we can recursively obtain the following results for x_1, x_2, x_3 and x_4:

$$x_0 = 1\,, x_1 = \frac{1}{3x_0^2} = \frac{1}{3}\,, x_2 = \frac{-3x_0x_1^2}{3x_0^2} = -\frac{1}{9}\,, x_3 = \frac{-\,(6x_0x_1x_2 + x_1^3)}{3x_0^2} = \frac{5}{81}$$

$$x_4 = \frac{-\,(6x_0x_1x_3 + 3x_0x_2^2 + 3x_1^2x_2)}{3x_0^2} = -\frac{10}{243}\,.$$

Note that one could indefinitely go on calculating the coefficients $x_5x_6x_7x_8\ldots$ in the same fashion. Hence, one of the cubic roots of $1+\mathrm{X}$ in $\mathbb{C}[[\mathrm{X}^{\mathbb{Z}}]]$ is given by the power series

$$x = 1 + \frac{1}{3}\,\mathrm{X} - \frac{1}{9}\,\mathrm{X}^2 + \frac{5}{81}\,\mathrm{X}^3 - \frac{10}{243}\,\mathrm{X}^4 + \cdots = 1\cdot \mathrm{X}^{\overset{\mathcal{I}}{\mathrm{ms}}(1+\mathrm{X})/3}\,(1+y)\ ,$$

where $y \coloneqq \frac{1}{3}\,\mathrm{X} - \frac{1}{9}\,\mathrm{X}^2 + \frac{5}{81}\,\mathrm{X}^3 - \frac{10}{243}\,\mathrm{X}^4 + \cdots$, and the other two cubic roots of $1+\mathrm{X}$ in $\mathbb{C}[[\mathrm{X}^{\mathbb{Z}}]]$ are given by

$$x' = \mathsf{e}^{2\pi i/3}\mathrm{X}^{\overset{\mathcal{I}}{\mathrm{ms}}(1+\mathrm{X})/3}\,(1+y) = \mathsf{e}^{2\pi i/3} + \frac{\mathsf{e}^{2\pi i/3}}{3}\,\mathrm{X} - \frac{\mathsf{e}^{2\pi i/3}}{9}\,\mathrm{X}^2 + \frac{5\,\mathsf{e}^{2\pi i/3}}{81}\,\mathrm{X}^3 - \frac{10\,\mathsf{e}^{2\pi i/3}}{243}\,\mathrm{X}^4 + \cdots$$

and

$$x'' = \mathsf{e}^{4\pi i/3}\mathrm{X}^{\overset{\mathcal{I}}{\mathrm{ms}}(1+\mathrm{X})/3}\,(1+y) = \mathsf{e}^{4\pi i/3} + \frac{\mathsf{e}^{4\pi i/3}}{3}\,\mathrm{X} - \frac{\mathsf{e}^{4\pi i/3}}{9}\,\mathrm{X}^2 + \frac{5\,\mathsf{e}^{4\pi i/3}}{81}\,\mathrm{X}^3 - \frac{10\,\mathsf{e}^{4\pi i/3}}{243}\,\mathrm{X}^4 + \cdots .$$

Note that x is also a cubic root of $1+\mathrm{X}$ in the Hahn field $\mathbb{Q}[[\mathrm{X}^{\mathbb{Z}}]]$, but since the primary coefficient $\operatorname{pc}(1+\mathrm{X})=1$ has only one cubic root in $\mathbb{Q}$, we obtain that x is the only cubic root of $1+\mathrm{X}$ in $\mathbb{Q}[[\mathrm{X}^{\mathbb{Z}}]]$.

Corollary 3.80. *Let K be a field and let G be an ordered subgroup of $(\mathbb{R},+_{\mathbb{R}},<_{\mathbb{R}})$ and let $\mathcal{J}$ be an incremental arithmetic Rayner ideal on G. If every $\mathrm{X}^g \in \overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ for $g \in G$ has an n-th root in $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ for each $n \in \mathbb{N}$, then the group G is divisible and dense in $\mathbb{R}$.*

Proof. For each $g \in G$ and each $n \in \mathbb{N}$, we have $g/n \in G$, confirming the divisibility of G. Since every subgroup of $(\mathbb{R},+_{\mathbb{R}})$ of the form $r\mathbb{Z}$ for $r \in (0,\infty)_{\mathbb{R}}$ is not divisible, the group G is dense in $\mathbb{R}$ by Theorem 1.68.

□

Corollary 3.80 shows that we cannot remove the hypothesis that G is divisible in Theorem 3.72.

Example 3.81. The Hahn fields $\mathbb{Q}[[\mathrm{X}^{\mathbb{Z}}]]$, $\mathbb{R}[[\mathrm{X}^{\mathbb{Z}}]]$ and $\mathbb{C}[[\mathrm{X}^{\mathbb{Z}}]]$, the Levi-Civita fields $\overset{\mathrm{lf}}{\mathbb{Q}}[[\mathrm{X}^{\mathbb{Z}}]]$, $\overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Z}}]]$ and $\overset{\mathrm{lf}}{\mathbb{C}}[[\mathrm{X}^{\mathbb{Z}}]]$, and the Puiseux fields $\overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{D}_d}]]$, $\overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]$, $\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{D}_d}]]$, $\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]$, $\overset{\mathrm{bd}}{\mathbb{C}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{D}_d}]]$ and $\overset{\mathrm{bd}}{\mathbb{C}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]$, do not have n-th roots for some natural numbers $n \in \mathbb{N}$, since the ordered groups $\mathbb{Z}$, $\mathcal{D}_d$ and $\mathcal{P}$ are not divisible. In particular, those fields are not algebraically closed.

3.10 $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ as an ordered rng

Taking into account an ordered rng R, the Rayner rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ naturally inherits an order-dense ordered rng structure from the order of R, and the induced order on it places higher weight on the powers of X with low exponents in G, matching the notion of order of magnitude on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ resultant from the G-pseudovaluation $\overset{\mathcal{J}}{\mathrm{ms}}$ (Section 3.5). Furthermore, when R is an ordered ring, the Rayner ring $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is non-Archimedean. These facts are established in the following theorem:

Theorem 3.82. *Let R be an ordered rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G.*

(a) *The set*

$$P := \{x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \mid \operatorname{pc}(x) > 0_R\}$$

is a synthetic positive cone in the rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ (Definition 2.38).

Let $\overset{P}{<}$ be the order on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ that corresponds to the synthetic positive cone P in that rng (Theorem 2.39).

(b) *For all $x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, we have $x \overset{P}{<} y$ if, and only if, $x_{\mu_{x,y}} < y_{\mu_{x,y}}$, where $\mu_{x,y} \in \breve{G}$ is given by $\mu_{x,y} := \overset{\mathcal{J}}{\operatorname{ms}}(x - y)$;*

(c) *For all $r, s \in R$, the conditions $r \overset{P}{<} s$ and $r <_R s$ are equivalent;*

(d) *If $x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ are elements so that $0_R \overset{P}{\leqslant} x \overset{P}{\leqslant} y$ or $y \overset{P}{\leqslant} x \overset{P}{\leqslant} 0_R$, then $\overset{\mathcal{J}}{\operatorname{ms}}(x) \geqslant \overset{\mathcal{J}}{\operatorname{ms}}(y)$;*

(e) *If $x, y, z \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ are elements so that $0_R \overset{P}{\leqslant} x \overset{P}{\leqslant} y \overset{P}{\leqslant} z$ or $z \overset{P}{\leqslant} y \overset{P}{\leqslant} x \overset{P}{\leqslant} 0_R$, and if $\overset{\mathcal{J}}{\operatorname{ms}}(x) = \overset{\mathcal{J}}{\operatorname{ms}}(z)$, then $\overset{\mathcal{J}}{\operatorname{ms}}(x) = \overset{\mathcal{J}}{\operatorname{ms}}(y) = \overset{\mathcal{J}}{\operatorname{ms}}(z)$;*

(f) *If $x, y, z \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ are elements so that $x \overset{P}{\leqslant} y \overset{P}{\leqslant} z$, then the equality $\mu_{xz} = \overset{\breve{G}}{\min}\{\mu_{xy}, \mu_{yz}\}$ holds;*

(g) *For each $g \in G$, the sets $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ and $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ are $\overset{P}{<}$-order-convex;*

(h) *If $n \in \mathbb{N}$ and if $x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] - \{0_R\}$ are elements so that $x \overset{P}{\leqslant} y \overset{P}{\leqslant} nx$ or $nx \overset{P}{\leqslant} y \overset{P}{\leqslant} x$, then $\overset{\mathcal{J}}{\operatorname{ms}}(x) = \overset{\mathcal{J}}{\operatorname{ms}}(y)$ and $\operatorname{pc}(x) \overset{R}{\mathscr{A}} \operatorname{pc}(y)$;*

(i) *Each non-trivial Archimedean class of $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$ can be uniquely written in the form $A\mathrm{X}^g + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$, where A is a non-trivial Archimedean class of R and where $g \in G$. In particular, the ordered rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is non-Archimedean;*

(j) *The ordered rng $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$ is order-dense (Definition 1.24);*

(k) *If the ideal $\mathcal{J}$ is left-finite (Definition 3.9), then the set $R[\mathrm{X}^G]$ of generalised polynomials is strictly order-dense in the ordered Rayner rng $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$.*

Suppose R is an ordered ring and take any $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

(l) *If r and s are two infinitely close elements of R (Definition 2.42), then they are infinitely close in the ordered Rayner ring $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$;*

(m) *If $\overset{\mathcal{J}}{\mathrm{ms}}(x) > 0_G$, then the element x is infinitesimal in the ordered Rayner ring $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$;*

(n) *If $\overset{\mathcal{J}}{\mathrm{ms}}(x) < 0_G$, then the element x is infinite in the ordered Rayner ring $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$.*

Lastly, suppose R is an Archimedean ordered ring.

(o) *The converses of items (m) and (n) are valid;*

(p) *For each $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, we have $\overset{\mathcal{J}}{\mathrm{ms}}(x) = 0_G$ if, and only if, the element x is appreciable in the ordered Rayner ring $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$;*

(q) *Two elements r and s in R are infinitely close in the ordered Rayner ring $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$ if, and only if, they are equal.*

Proof. The proofs of items (b), (c) and (l) are immediate, item (e) follows from item (d), and item (i) follows from item (h). Item (p) is a direct consequence of items (m), (n) and (o). We shall prove the remaining items.

(a) We have $0_R \notin P$, and since $\mathrm{pc}(-x) = -\mathrm{pc}(x)$ $(\forall x \in R)$, the intersection $P \cap (-P)$ is empty. If x is a non-zero element of the rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, then $\mathrm{pc}(x) \in R - \{0_R\}$, and we either have $\mathrm{pc}(x) < 0_R$ or $\mathrm{pc}(x) > 0_R$, that is, we have $\mathrm{pc}(-x) = -\mathrm{pc}(x) > 0_R$ or $\mathrm{pc}(x) > 0_R$, implying that $x \in (-P) \cup P \cup \{0_R\}$ and proving that the sets $-P$, P and $\{0_R\}$ form a partition of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. One may easily check that P is closed under the addition of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. If x and y are elements of P, then $\mathrm{pc}(x), \mathrm{pc}(x) > 0_R$, and since every ordered rng has no zero divisors (Proposition 2.34, Item (b)), we obtain (Proposition 3.47, Item (a))

$$\mathrm{pc}(xy) = (xy)_{\overset{\mathcal{J}}{\mathrm{ms}}(xy)} = (xy)\,[\![\overset{\mathcal{J}}{\mathrm{ms}}(x) + \overset{\mathcal{J}}{\mathrm{ms}}(y)]\!] = \mathrm{pc}(x)\ \mathrm{pc}(y) > 0_R\,,$$

which gives us $xy \in P$ and proves that P is a synthetic positive cone in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

(d) It suffices to prove the assertion for the case $0_R \overset{P}{\leqslant} x \overset{P}{\leqslant} y$. If $x = 0_R$, then $\overset{\mathcal{J}}{\mathrm{ms}}(x) = \multimap \geqslant \overset{\mathcal{J}}{\mathrm{ms}}(y)$. Suppose $0_R \overset{P}{<} x$. Thus, we have $\mathrm{pc}(x), \mathrm{pc}(y) > 0_R$ and $\mathrm{pc}(y-x) \geqslant 0_R$. If $\overset{\mathcal{J}}{\mathrm{ms}}(x) < \overset{\mathcal{J}}{\mathrm{ms}}(y)$, then we have (Proposition 3.47, Item (a))

$$\mathrm{pc}(x) - y_{\overset{\mathcal{J}}{\mathrm{ms}}(x)} = \mathrm{pc}(x-y) = -\mathrm{pc}(y-x) \leqslant 0_R\,,$$

implying $\mathrm{pc}(x) \leqslant y_{\overset{\mathcal{J}}{\mathrm{ms}}(x)}$, $y_{\overset{\mathcal{J}}{\mathrm{ms}}(x)} \neq 0_R$, and $\overset{\mathcal{J}}{\mathrm{ms}}(y) \leqslant \overset{\mathcal{J}}{\mathrm{ms}}(x) < \overset{\mathcal{J}}{\mathrm{ms}}(y)$, which is absurd, proving the item.

(f) Since $x \overset{P}{\leqslant} y \overset{P}{\leqslant} z$, we get $x - z \overset{P}{\leqslant} y - z \overset{P}{\leqslant} 0_R$ and $0_R \overset{P}{\leqslant} y - x \overset{P}{\leqslant} z - x$, implying $\mu_{yz}, \mu_{xy} \geqslant \mu_{xz}$ by item (d). On the other hand, we have

$$\mu_{xz} = \overset{\mathcal{J}}{\mathrm{ms}}(z-x) \geqslant \overset{\breve{G}}{\min}\{\overset{\mathcal{J}}{\mathrm{ms}}(z-y), \overset{\mathcal{J}}{\mathrm{ms}}(y-x)\} = \overset{\breve{G}}{\min}\{\mu_{yz}, \mu_{x,y}\} \geqslant \mu_{xz}\,,$$

since $\overset{\mathcal{J}}{\mathrm{ms}}$ is a G-pseudovaluation on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

(g) Take $x, y, z \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ so that $x \overset{P}{\leqslant} y \overset{P}{\leqslant} z$. Thus, $0_R \overset{P}{\leqslant} y - x \overset{P}{\leqslant} z - x$, and by item (d) we get that $\overset{\mathcal{J}}{\mathrm{ms}}(y-x) \geqslant \overset{\mathcal{J}}{\mathrm{ms}}(z-x)$ and $y \doteq x + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(z-x)}\right)$. If $x, z \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ for some $g \in G$, then $z - x \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ and $\overset{\mathcal{J}}{\mathrm{ms}}(z-x) \geqslant g$, which implies

$$y \doteq x + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(z-x)}\right) \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g) + \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(z-x)}\right) \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)\,,$$

proving that the set $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ is $\overset{P}{<}$-order-convex. The proof that $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ is $\overset{P}{<}$-order-convex is analogous.

(h) It suffices to prove the assertion for the case $x \overset{P}{\leqslant} y \overset{P}{\leqslant} nx$. Note that that implies $0_R \overset{P}{\leqslant} x$. Since $\overset{\mathcal{J}}{\mathrm{ms}}(x) = \overset{\mathcal{J}}{\mathrm{ms}}(nx)$, we have $\overset{\mathcal{J}}{\mathrm{ms}}(x) = \overset{\mathcal{J}}{\mathrm{ms}}(y)$ by item (e). Furthermore, note that

$$-\mathrm{pc}(x) + \mathrm{pc}(y) = -\mathrm{pc}(x-y) = \mathrm{pc}(y-x) \geqslant 0_R$$

and

$$n\,\mathrm{pc}(x) - \mathrm{pc}(y) = \mathrm{pc}(nx-y) \geqslant 0_R\,,$$

resulting in $\mathrm{pc}(x) \leqslant \mathrm{pc}(y) \leqslant n\,\mathrm{pc}(x)$ and $\mathrm{pc}(x) \overset{R}{\mathscr{A}} \mathrm{pc}(y)$;

(j) and **(k)** Let x and y be two distinct elements of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ so that $x \overset{P}{<} y$, let r be a fixed positive element of R and let p be a fixed element of G so that $p > \overset{\mathcal{J}}{\mathrm{ms}}(y-x)$. Thus, we have

$$x \overset{P}{<} \Big(\sum_{g<p} x_g \mathrm{X}^g\Big) + (x_p + r)\,\mathrm{X}^p \overset{P}{<} y\,.$$

In particular, if $\mathcal{J}$ is left-finite, then the intersection $\mathrm{supp}\,(x) \cap (\leftarrow, p)_G$ is finite and we get

$$\Big(\sum_{g<p} x_g \mathrm{X}^g\Big) + (x_p + r)\,\mathrm{X}^p \in R\,[\mathrm{X}^G]\ .$$

(m) If $\overset{\mathcal{J}}{\mathrm{ms}}(x) > 0_G$, then we have

$$(\forall n \in \mathbb{N})\ \ \mathrm{pc}\,(1_R - n\,|x|) = 1_R > 0_R\,,$$

that is, $0_R \overset{P}{<} n\,|x| \overset{P}{<} 1_R$ $(\forall n \in \mathbb{N})$, implying that the element x is infinitesimal in $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$.

(n) If $\overset{\mathcal{J}}{\mathrm{ms}}(x) < 0_G$, then we have

$$(\forall n \in \mathbb{N})\ \ \mathrm{pc}\,(|x| - n_R) = |\mathrm{pc}\,(x)| > 0_R\,,$$

that is, $n1_R = n_R \overset{P}{<} |x|$ $(\forall n \in \mathbb{N})$, implying that the element x is infinite in $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$.

(o) Suppose x is a non-zero infinitesimal element in the ordered ring $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$, and suppose $\overset{\mathcal{J}}{\mathrm{ms}}(x) \leqslant 0_G$. We shall derive a contradiction from those suppositions. We have $n\,|x| \overset{P}{<} 1_R$ $(\forall n \in \mathbb{N})$, and, by item (n), we have $\overset{\mathcal{J}}{\mathrm{ms}}(x) = 0_G = \overset{\mathcal{J}}{\mathrm{ms}}(n\,|x|)$ $(\forall n \in \mathbb{N})$, which gives us

$$(\forall n \in \mathbb{N})\ \ \overset{\mathcal{J}}{\mathrm{ms}}(1_R - n\,|x|) \geqslant \overset{\breve{G}}{\min}\,\{\overset{\mathcal{J}}{\mathrm{ms}}(1_R)\,, \overset{\mathcal{J}}{\mathrm{ms}}(n\,|x|)\} = 0_G\,.$$

If there is an $n \in \mathbb{N}$ so that $(1_R - n\,|x|)_{0_G} = 0_R$, then $n\,|\mathrm{pc}\,(x)| = 1_R$ and

$$\begin{aligned}(1_R - (n+1)\,|x|)_{0_G} &= (1_R)_{0_G} - n\,|x|_{0_G} - |x|_{0_G}\\ &= 1_R - n\,|\mathrm{pc}\,(x)| - |\mathrm{pc}\,(x)|\\ &= -\,|\mathrm{pc}\,(x)| < 0_R\,,\end{aligned}$$

implying $\mathrm{pc}\,(1_R - (n+1)\,|x|) = -\,|\mathrm{pc}\,(x)| < 0_R$ and $1_R \overset{P}{<} (n+1)\,|x|$, which is absurd by the suppositions. Thus, we have

$$(\forall n \in \mathbb{N})\ \ (1_R - n\,|x|)_{0_G} \neq 0_R \quad \text{and} \quad \overset{\mathcal{J}}{\mathrm{ms}}\,(1_R - n\,|x|) = 0_G\,,$$

implying

$$(\forall n \in \mathbb{N})\ \ 1_R - n\,|\mathrm{pc}\,(x)| = (1_R - n\,|x|)_{0_G} = \mathrm{pc}\,(1_R - n\,|x|) > 0_R\,,$$

that is, $\mathrm{pc}\,(x)$ is non-zero and infinitesimal in R, contradicting the assumption that R is Archimedean. The proof of the converse of item (n) is analogous.

(q) If $r - s$ is infinitesimal in $(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{P}{<})$, then, by item (**o**), we have $\overset{\mathcal{J}}{\mathrm{ms}}\,(r-s) > 0_G$ and

$$0_R = (r-s)_{0_G} = r_{0_G} - s_{0_G} = r - s\,.$$

□

From now on, whenever R is an ordered rng, we shall assume that the Rayner rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is endowed with the order $\overset{P}{<}$ on it defined in Theorem 3.82, and we shall say that $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is an **ordered Rayner rng**. In addition, the order $\overset{P}{<}$ shall be denoted just by $<$, and, with the intention of evading the cumbersome notations

$$(x,y)_{\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]}\,,\ [x,y]_{\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]}\,,\ [x,y)_{\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]}\,,\text{etc.}$$

for intervals in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, these shall be plainly denoted by (x,y), $[x,y]$, $[x,y)$, etc.

Corollary 3.83. *Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. The rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is orderable if, and only if, the rng R is orderable.*

Example 3.84. All Rayner rings of the forms $\overset{\mathcal{J}}{\mathbb{C}}[[\mathrm{X}^G]]$ and $\overset{\mathcal{J}}{\mathbf{H}}[[\mathrm{X}^G]]$ are non-orderable, for the rings $\mathbb{C}$ and $\mathbf{H}$ are non-orderable.

3.11 The pure part function

An element of an ordered Rayner ring $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ may be infinitely close to an element r of R, and that element r may not be unique in general. This issue shall be addressed here. First, we need a few definitions:

Definition 3.85. Let R be an ordered ring, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. We have the following notations and terminology:

- ▷ An element x of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is **near-pure** if there is an $r \in R$ such that $x \sim r$, where $\sim$ is the relation of infinite proximity on the ordered ring $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ (Definition 2.42). Such r is unique if R is Archimedean (Proposition 3.82, Item (q)). The set of near-pure elements of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ shall be denoted by $\mathrm{NP}\,(R, G)$ or $\overset{\mathcal{J}}{\mathrm{NP}}\,(R, G)$ in this section;
- ▷ Suppose R is Archimedean. The **pure-part function** relative to $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is the function
$$\mathrm{pp} = \underset{R,G}{\overset{\mathcal{J}}{\mathrm{pp}}} \,:\, \mathrm{NP}\,(R, G) \to R$$
that associates each near-pure element x in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ to the unique element $\mathrm{pp}\,(x)$ of R such that $x \sim \mathrm{pp}\,(x)$.

The reader acquainted with the theory of Non-standard Analysis may notice that the pure-part function $\mathrm{pp} : \mathrm{NP}\,(R, G) \to R$ is an analogue of the **standard-part function** $\mathrm{st} : \mathrm{Fin}\,({}^*\mathbb{R}) \to \mathbb{R}$, where $\mathrm{Fin}\,({}^*\mathbb{R})$ is the set of finite hyperreals (Definition 2.42). In their own ways, each of these functions provides an essential connection between two distinct mathematical realms.

Example 3.86. The pure-part function $\mathrm{pp} : \mathrm{NP}\,(R, G) \to R$ cannot be defined when the ordered ring R is non-Archimedean. For instance, take $R := {}^*\mathbb{R}$, $G := \mathbb{Q}$ and take a positive infinitesimal hyperreal a. Thus, we have

$$(\forall n \in \mathbb{N}) \;\; |n\,(\mathrm{X} - 0)| = n\mathrm{X} < 1 \quad \text{and} \quad n\,|a - \mathrm{X}| = na - n\mathrm{X} < 1 \;(\forall n \in \mathbb{N}) \;,$$

which gives us $0 \sim \mathrm{X} \sim a$ in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, that is, X is infinitely close to two distinct elements of ${}^*\mathbb{R}$.

Proposition 3.87. *Let R be an ordered ring, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G.*

(a) $\mathrm{NP}(R,G) = \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{0_G})$; **(b)** $\mathrm{NP}(R,G)$ *is a subring of* $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

Suppose R is Archimedean.

(c) *The pure part function* $\mathrm{pp} : \mathrm{NP}(R,G) \to R$ *is a surjective, non-strictly increasing, unital homomorphism between ordered rings such that* $\mathrm{pp}(x) = x_{0_G}$ $(\forall x \in \mathrm{NP}(R,G))$;

(d) $\mathrm{Ker}(\mathrm{pp}) = \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G})$ *and* $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{0_G}) / \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G}) \overset{\mathbf{Ring}}{\cong} R$.

Proof. Item (b) is a consequence of item (a) and Subitem 9 of Item (b) of Theorem 3.47. Item (d) follows directly from item (c), Item (o) of Theorem 3.82 and the First Isomorphism Theorem (Appendix B, Theorem B.49). We shall prove the remaining items.

(a) If $x \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{0_G})$, then $x \doteq x_{0_G} + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G})$ and $x \sim x_{0_G}$ (Theorem 3.82, Item (m)). Suppose x is an element of $\mathrm{NP}(R,G)$ and take an r in R so that $x \sim r$. Thus, $x - r$ is infinitesimal in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, and, in particular, $x - r$ is not infinite in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, implying $\overset{\mathcal{J}}{\mathrm{ms}}(x - r) \geqslant 0_G$ (Theorem 3.82, Item (n)) and $x \doteq r + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{0_G}) \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{0_G})$.

(c) The function pp is surjective, given that $\mathrm{pp}(r) = r$ $(\forall r \in R)$. Take two elements x and y of $\mathrm{NP}(R,G) = \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{0_G})$. Thus, we have $x \doteq x_{0_G} + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G})$, $x \sim x_{0_G}$ (Theorem 3.82, Item (m)) and $\mathrm{pp}(x) = x_{0_G}$, which gives us $\mathrm{pp}(1_R) = (1_R)_{0_G} = 1_R$ in particular. Moreover, note that

$$\mathrm{pp}(x + y) = (x + y)_{0_G} = x_{0_G} + y_{0_G} = \mathrm{pp}(x) + \mathrm{pp}(y) ,$$

and

$$\begin{aligned} xy &\doteq (x_{0_G} + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G}))(y_{0_G} + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G})) \\ &\doteq x_{0_G} y_{0_G} + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G + 0_G}) + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G + 0_G}) + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G + 0_G}) \\ &\doteq x_{0_G} y_{0_G} + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G}) , \end{aligned}$$

which implies $\mathrm{pp}(xy) = x_{0_G} y_{0_G} = \mathrm{pp}(x)\,\mathrm{pp}(y)$ and proves that pp is a unital homomorphism. Lastly, if $\mathrm{pp}(y) < \mathrm{pp}(x)$, then

$$(x - y)_{0_G} = x_{0_G} - y_{0_G} = \mathrm{pp}(x) - \mathrm{pp}(y) > 0_R ,$$

and, since
$$\overset{\mathcal{J}}{\mathrm{ms}}(x-y) \geqslant \overset{\breve{G}}{\min}\{\overset{\mathcal{J}}{\mathrm{ms}}(x), \overset{\mathcal{J}}{\mathrm{ms}}(y)\} \geqslant 0_G,$$
we obtain $\overset{\mathcal{J}}{\mathrm{ms}}(x-y) = 0_G$ and $y < x$, proving that the function pp is non-strictly increasing.

□

3.12 $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ as a differential ring

We shall introduce a derivation of Rayner rings, and, although not much will result from that derivation in this work, the author felt the need to notify the reader about its existence, so that he or she may be inspired to draw their own mathematical conclusions out of it. In Computer Science, specifically on the area of Computational Differentiation, that derivation has been successfully utilised in order to provide a practical method for the computation of derivatives (21, 19, 207).

Proposition 3.88. *Let R be a commutative ring, let G be an ordered subgroup of $(R, +_R, <_R)$ so that*[10] $1_R \in G$, *let $\mathcal{J}$ be an arithmetic Rayner ideal on G and let $\partial = \underset{R,G}{\partial} : \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ be the function given by $(\partial x)_g := (g + 1_R)\, x_{g+1_R}$.*

(a) *The function ∂ is a derivation of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ (Definition 2.54);*

(b) $(\forall r \in R)\ \partial r = 0_R$;

(c) $(\forall g \in G)\ \partial\left(\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)\right) \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g-1_R})$ and $\partial(\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)) \doteq \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g-1_R})$;

(d) $\partial\left(\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{0_G})\right) \doteq \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{-1_R})$;

(e) $(\forall x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]])\ \overset{\mathcal{J}}{\mathrm{ms}}(\partial x) \neq -1_R$;

(f) *If R has no zero divisors and if $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is so that $\overset{\mathcal{J}}{\mathrm{ms}}(x) \neq 0_G = 0_R$, then we have $\overset{\mathcal{J}}{\mathrm{ms}}(\partial x) = \overset{\mathcal{J}}{\mathrm{ms}}(x) - 1_R$.*

Proof. Item (b) follows immediately from the definition of ∂. We shall prove the remaining items.

[10] Note that we also have $0_R = 0_G \in G$.

(**a**) Take x and y in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and take a g in G. Note that

$$\begin{aligned}(\partial(x+y))_g &= (g+1_R)(x+y)_{g+1_R} \\ &= (g+1_R)\,x_{g+1_R} + (g+1_R)\,y_{g+1_R} \\ &= (\partial(x))_g + (\partial(y))_g\ ,\end{aligned}$$

and since the ring R is commutative, we have

$$\begin{aligned}(\partial(xy))_g &= (g+1_R)(xy)_{g+1_R} \\ &= \sum_{\substack{p,q\in G\\ p+q=g+1_R}} (p+q)\,x_p y_q \\ &= \sum_{\substack{p,q\in G\\ (p-1_R)+q=g}} (px_p)\,y_q \ + \sum_{\substack{p,q\in G\\ p+(q-1_R)=g}} x_p\,(qy_q) \\ &= \sum_{\substack{s,q\in G\\ s+q=g}} ((s+1_R)\,x_{s+1_R})\,y_q \ + \sum_{\substack{p,t\in G\\ p+t=g}} x_p\,((t+1_R)\,y_{t+1_R}) \\ &= (\partial x\cdot y + x\cdot\partial y)_g\end{aligned}$$

proving that ∂ is a derivation of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

(**c**) If h is an element of G so that $h \leqslant g-1_R$, then $h+1_R \leqslant g$ and

$$[\partial(\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g))]_h \doteq (h+1_R)(\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g))_{h+1_R} \doteq (h+1_R)\cdot 0_R = 0_R\,,$$

which shows that $\partial(\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)) \doteq \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g-1})$. If x is an element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ so that $x \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$, then x is of the form $r\mathrm{X}^g + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ and we get

$$\partial x \doteq \partial(r\mathrm{X}^g) + \partial(\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)) \doteq gr\mathrm{X}^{g-1_R} + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g-1_R}) \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g-1_R})\ .$$

(**d**) If x is an element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ so that $x \doteq \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{0_G})$, then x is of the form $r\mathrm{X}^{0_G} + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G})$ and we get

$$\partial x \doteq \partial(r\mathrm{X}^{0_G}) + \partial(\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G})) \doteq 0_G\cdot r\mathrm{X}^{-1_R} + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{-1_R}) \doteq \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{-1_R})$$

by item (c).

(**e**) If x is an element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ so that $\overset{\mathcal{J}}{\mathrm{ms}}(\partial x) = -1_R$, then we get

$$0_R \neq (\partial x)_{-1_R} = (-1_R+1_R)\,x_{-1_R+1_R} = 0_R\cdot x_{0_R} = 0_R\,,$$

which is absurd.

(f) Suppose R has no zero divisors and suppose x is an element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ so that $\overset{\mathcal{J}}{\mathrm{ms}}(x) \neq 0_G = 0_R$. The desired result is certainly true in the case $x = 0_R$. Assume $x \neq 0_R$. Thus, we have

$$(\partial x)_{\overset{\mathcal{J}}{\mathrm{ms}}(x)-1_R} = \overset{\mathcal{J}}{\mathrm{ms}}(x)\, x_{\overset{\mathcal{J}}{\mathrm{ms}}(x)} = \overset{\mathcal{J}}{\mathrm{ms}}(x)\,\mathrm{pc}(x) \neq 0_R$$

given that $\overset{\mathcal{J}}{\mathrm{ms}}(x) \neq 0_G = 0_R \neq \mathrm{pc}(x)$, implying $\overset{\mathcal{J}}{\mathrm{ms}}(\partial x) \leqslant \overset{\mathcal{J}}{\mathrm{ms}}(x) - 1_R$ and $\overset{\mathcal{J}}{\mathrm{ms}}(\partial x) = \overset{\mathcal{J}}{\mathrm{ms}}(x) - 1_R$ by item (c).

□

Example 3.89. Let K be a field of characteristic zero, let G be an ordered subgroup of $(K, +_K, <_K)$ so that $1_K \in G$ and let $\mathcal{J}$ be an arithmetic Rayner ideal on G so that the subset $(\mathbb{N}_0)_K$ of K (Definition 2.14) is an element of $\mathcal{J}$. The derivation $\partial : \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ of Proposition 3.88 has the trivial fixed point $\partial\, 0_R = 0_R$, but it also has non-trivial ones. Consider the element

$$e := \sum_{n\in\mathbb{N}_0} \frac{1_K}{(n!)_K}\,\mathrm{X}^{n_K} = 1_K + \mathrm{X} + \frac{1_K}{2_K}\,\mathrm{X}^{2_K} + \frac{1_K}{6_K}\,\mathrm{X}^{3_K} + \cdots + \frac{1_K}{(n!)_K}\,\mathrm{X}^{n_K} + \cdots \in \overset{\mathcal{J}}{K}[[\mathrm{X}^G]]\ .$$

For all $g \in G - ([-1,\infty)_{\mathbb{Z}})_K$, we have that $g + 1_K \notin (\mathbb{N}_0)_K = \mathrm{supp}(e)$ and $(\partial e)_g = (g + 1_K)\, e_{g+1_K} = 0_K$, and for all $g \in ([-1,\infty)_{\mathbb{Z}})_K$ so that $g = n_K$ for some $n \in [-1,\infty)_{\mathbb{Z}}$, we have

$$e_{g+1_K} = e_{n_K+1_K} = e_{(n+1)_K} = \frac{1_K}{((n+1)!)_K} = \begin{cases} 1_K & \text{if } n = -1\,, \\ \frac{1_K}{(n_K+1_K)(n!)_K} & \text{if } n \geqslant 0\,, \end{cases}$$

implying

$$(\partial e)_g = (g + 1_K)\, e_{g+1_K} = (n_K + 1_K)\, e_{g+1_K} = \begin{cases} 0_K & \text{if } n = -1 \text{ (i.e. } g = -1_K)\ , \\ \frac{1_K}{(n!)_K} & \text{if } n \geqslant 0\,. \end{cases}$$

Therefore, we have $\mathrm{supp}(\partial e) = (\mathbb{N}_0)_K$ and $\partial e = \sum_{n\in\mathbb{N}_0} \frac{1_K}{(n!)_K}\,\mathrm{X}^{n_K} = e$. We leave to the reader the proof that every fixed point of ∂ is of the form re for $r \in K$.

4 The Strong and the Weak Topologies on Rayner Rngs

In this chapter, we shall systematically examine two highly remarkable topologies on Rayner rngs $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, the so-called **strong** and **weak topologies** on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, where the latter can only be defined when R has an ordered rng structure. We shall study these topologies with a high degree of generality, discussing their properties for all Rayner rngs and specifying how they interact with the algebraic structure of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

4.1 The strong topology on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$

We begin with a study of the so-called **strong topology** on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, which we shall define below. Strong topologies are key for the development of analytical theories on the classical Hahn, Puiseux and Levi-Civita fields, and the most consequential conclusions of that kind concern the latter fields (cf. Introduction). Such results are truly startling, since the strong topologies happen to be zero-dimensional and totally disconnected, which are properties that one would definitely not expect from a topology that is suited for analytical considerations.

Definition 4.1. Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. The **strong topology on** $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is the topology $\overset{\overset{\mathcal{J}}{\mathrm{ms}}}{\mathrm{Valt}}$ on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ induced by the G-pseudovaluation $\overset{\mathcal{J}}{\mathrm{ms}} : \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to \breve{G}$

(Definition 1.82), and it is denoted by $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$ or $\underset{R,G}{\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}}$. Thus, the strong topology is generated by the basis that is formed by the sets

$$x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) = \{y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \mid (\forall h \in (\leftarrow, g]_G)\ y_h = x_h\}$$

for $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and $g \in G$.

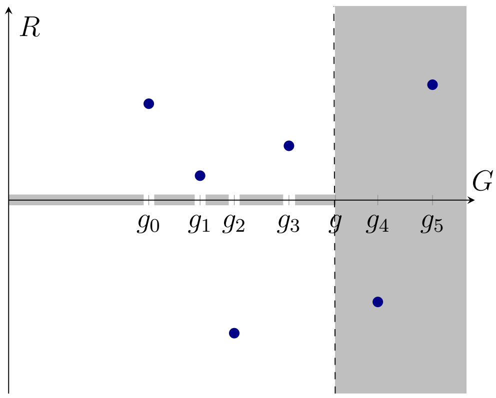

Figure (1): Absolute graphical representation of the basic $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-open set $x + \mathrm{o}\,(\mathrm{X}^g)$.

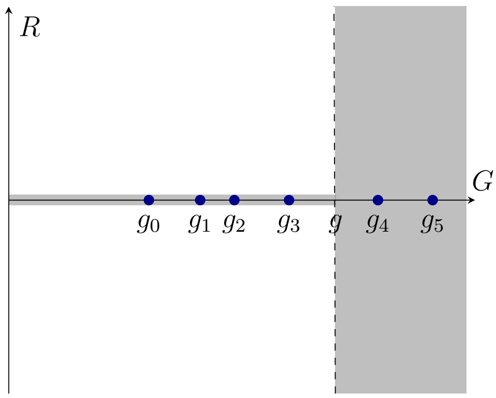

Figure (2): Relative graphical representation of the basic $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-open set $x + \mathrm{o}\,(\mathrm{X}^g)$.

Visual aids are often helpful when dealing with topological problems, and, taking that into account, we provide here two graphical representations of the basic $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-open set $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$, depicted in Figures (1) and (2), which may facilitate the reader's comprehension of what is transpiring as he or she follows the proofs of the theorems. These pictures surely do help the author considerably. We shall explain how they are meant to be understood in the following paragraphs.

Let the family $\{g_\alpha\}_{\alpha<\gamma}$ be the increasing ordinal sequence of elements of the support $\mathrm{supp}\,(x)$, and consider Figure (1). For each $h \in G$ and for each $y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, let us call the ordered pair (h, y_h) the **point of** y **with exponent** h **and with coefficient** y_h, and let us call the intersection point of the R-axis with the G-axis the **origin** of our graph. The G-axis embodies the ordering of the exponents of the points, that is, it depicts how the order $<_G$ of G arranges the elements of G relative to each other, where the exponent of the origin must be an arbitrary element $h_0 \in G$ that is less than $g_0 = \mathrm{ms}\,(x)$ and g. The particular choice of h_0 is unimportant for our purposes. The R-axis, which is orthogonal to the G-axis, measures the coefficients of the points, so that the distance between a point (h, y_h) and the G-axis is directly proportional to the absolute value $|y_h|$, and so that the points that lie above (resp. below, on) the G-axis have positive (resp. negative, zero) coefficients in R. The points (g_α, x_{g_α}) of x with exponents in $\mathrm{supp}\,(x)$ are drawn as dots in the graph[1], and the points $(h, x_h) = (h, 0_R)$ of x for $h \in (\leftarrow, g]_G - \mathrm{supp}\,(x)$ are represented by a bold ray drawn over the G-axis with starting point $(g, 0_R)$ that extends endlessly in the negative direction of the G-axis and that gets occasionally interrupted whenever it arrives at an element of $\mathrm{supp}\,(x)$.

Consider an arbitrary element y of the basic $\overset{\mathcal{J}}{\mathsf{St}}$-open set $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$. Since $y_h = x_h$ $(\forall h \in (\leftarrow, g]_G)$, all points of y must coincide with the points of x up to the exponent g, including g itself, and, therefore, there is no freedom for the points of y in that region of the graph. On the other hand, for exponents h greater than g, the points of y are completely free to reside anywhere within the shaded area drawn on the right hand side of the dotted, vertical line $h = g$. Thus, the combination of the bold ray, the dots and the shaded area represent all possible elements of the set $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$. We call this type of graph an **absolute graphical representation of** $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$. Note that one may add more basic $\overset{\mathcal{J}}{\mathsf{St}}$-open sets to that graph, provided that one appropriately discriminates the points belonging to different sets, and that is the reason why these representations are optimal for situations in which two or more basic $\overset{\mathcal{J}}{\mathsf{St}}$-open sets are considered simultaneously.

Whenever only one basic $\overset{\mathcal{J}}{\mathsf{St}}$-open set $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$ is considered[2], it is preferable to make use of the neater representation depicted in Figure (2), which we shall describe. For each $h \in G$ and each $y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, let us call the ordered

[1] The set $\mathrm{supp}\,(x)$ is infinite in general, but, of course, only a finite number of dots can actually be drawn in the graph. A handful of them is enough to convey the idea of the representation.

[2] Or at least only one at a time, not two or more simultaneously.

pair $(h, y_h - x_h)$ the **x-relative point of y with exponent h and with x-relative coefficient** $y_h - x_h$. As with Figure (1), the G-axis still embodies the ordering of the exponents of the points, and the exponent of the origin must still be any arbitrary element $h_0 \in G$ that is less than $g_0 = \mathrm{ms}(x)$ and g. But now, the R-axis measures the differences between the coefficients of the points of y and the coefficients of the points of x, so that the distance between an x-relative point $(h, y_h - x_h)$ and the G-axis is directly proportional to the absolute value $|y_h - x_h|$, and so that the points that lie above (resp. below, on) the G-axis have positive (resp. negative, zero) x-relative coefficients in R, that is, the points that lie above (resp. below, on) the G-axis have coefficients that are greater than (resp. lower than, equal to) the coefficients of the corresponding points of x. The x-relative points $(g_\alpha, x_{g_\alpha} - x_{g_\alpha}) = (g_\alpha, 0_R)$ of x with exponents in $\mathrm{supp}(x)$ are drawn as dots on the G-axis, and the x-relative points $(h, x_h - x_h) = (h, 0_R)$ of x for $h \in (\leftarrow, g]_G - \mathrm{supp}(x)$ are represented by a bold ray drawn over the G-axis with starting point $(g, 0_R)$ that extends endlessly in the negative direction of the G-axis and that gets occasionally interrupted whenever it arrives at an element of $\mathrm{supp}(x)$.

Taking into consideration the arbitrary element y of $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ once more, we have $y_h - x_h = 0_R$ $(\forall h \in (\leftarrow, g]_G)$, implying that all x-relative points of y must be on the G-axis up to the exponent g, including g itself, and, therefore, there is no freedom for the x-relative points of y in that region of the graph. On the other hand, for exponents h greater than g, the x-relative points of y are completely free to reside anywhere within the shaded area drawn on the right hand side of the dotted, vertical line $h = g$. Thus, the combination of the bold ray, the dots and the shaded area represent all possible elements of the set $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$, with the caveat that those abstractions are all centred at x in a sense. We call this type of graph a **relative graphical representation of** $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$.

In the following theorem, we shall cover the fundamental properties of the strong topology on Rayner rngs:

Theorem 4.2. *Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G.*

(**a**) *For each $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, the sets*

$$\left\{x + \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g) \;\middle|\; g \in G\right\} \quad \text{and} \quad \left\{x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g) \;\middle|\; g \in G\right\}$$

are local $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-bases of x which consist of $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-clopen sets;

(**b**) *If $\{{}_\lambda x\}_{\lambda\in\Lambda}$ is a net in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and if x is an element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ such that ${}_\lambda x \xrightarrow[\lambda\in\Lambda]{\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}} x$, then for each $g \in G$, the net $\{{}_\lambda x_g\}_{\lambda\in\Lambda}$ in R is eventually constant at the value x_g;*

(**c**) *If the ideal $\mathcal{J}$ is left-finite (Definition 3.1), then for each $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ of infinite support*[3]*, we have $\displaystyle\sum_{n=1}^{N} x_{g_n}\mathrm{X}^{g_n} \xrightarrow[N\to\infty]{\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}} x$, where $\{g_n\}$ is the increasing sequence of elements of $\operatorname{supp}(x)$. In particular, the set of generalised polynomials $R[\mathrm{X}^G]$ is $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-dense in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ in that case;*

(**d**) *$\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is a topological rng when endowed with the topology $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$;*

(**e**) *The topology $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$ is $T^{7/2}$, perfect, zero-dimensional and totally disconnected;*

(**f**) *The $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-subspace of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ induced by the subset R is discrete;*

(**g**) *If D is an $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-dense subset of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, then $|R|, |G| \leqslant |D|$. In particular, if $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$ is separable, then R and G are countable;*

(**h**) *If $\mathcal{J}$ is left-finite and if R and G are countable, then $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$ is separable;*

(**i**) *If the rng R has at least one positive element r_0 so that $\overset{+R}{\phi}(r_0) = \infty$ (Definition 1.5), then no inhabited $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-open subspace of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is countably compact. In that case, the topology $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$ is neither countably compact nor locally compact;*

(**j**) *If $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$ is complete (Definition 1.55), then $\mathcal{J}$ is incremental (Definition 3.1);*

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[3] Recall that the existence of an infinite element S of $\mathcal{J}$ implies $\mathrm{cf}(G) = \omega$ and $|S| = \omega$ (Proposition 3.2, Item (b)).

(**k**) *The following conditions are equivalent:*

- $\triangleright$ $\overset{\mathcal{J}}{\mathsf{St}}$ *is metrizable;*
- $\triangleright$ $\mathrm{cf}\,(G) = \omega\,.$
- $\triangleright$ $\overset{\mathcal{J}}{\mathsf{St}}$ *is ultrametrizable;*

If those conditions hold, then there is an invariant ultrametric ρ on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ such that $\overset{\mathcal{J}}{\mathsf{St}} = \mathsf{t}\,(\rho)$, and, furthermore, if $\mathcal{J}$ is incremental, then we may also assume that ρ is complete.

Proof. Item (e) is an immediate consequence of item (d), Proposition 1.61 and Item (d) of Proposition 1.83. We shall prove the remaining items.

(**a**) We know that for each $g \in G$, the sets $x + \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g)$ and $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$ are $\overset{\mathcal{J}}{\mathsf{St}}$-clopen (Propositions 1.81 and 1.83) and the set $\{x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) \mid g \in G\}$ is clearly a local $\overset{\mathcal{J}}{\mathsf{St}}$-basis of x. It remains to prove that the set $\big\{x + \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^g) \mid g \in G\big\}$ is a local $\overset{\mathcal{J}}{\mathsf{St}}$-system of neighbourhoods of x. Indeed, if $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$ is a basic open $\overset{\mathcal{J}}{\mathsf{St}}$-neighbourhood of x and if h is any element of the ordered group G so that $h > g$, then we have the inclusion $x + \overset{\mathcal{J}}{\mathrm{O}}\,(\mathrm{X}^h) \subset x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$, thus proving the item.

(**b**) If ${}_\lambda x \xrightarrow[\lambda\in\Lambda]{\overset{\mathcal{J}}{\mathsf{St}}} x$ and if g is an element of G, then there is a $\lambda_0 \in \Lambda$ such that we have ${}_\lambda x \doteq x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$ $(\forall \lambda \in [\lambda_0, \rightarrow)_\Lambda)$, and that implies the condition ${}_\lambda x_g = x_g$ $(\forall \lambda \in [\lambda_0, \rightarrow)_\Lambda)$.

(**c**) From the comments after Proposition 3.46, we have

$$(\forall N \in \mathbb{N}) \sum_{n=1}^{N} x_{g_n} \mathrm{X}^{g_n} \doteq x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^{g_N})\ ,$$

and the result follows from the fact that the sequence $\{g_n\}$ is cofinal in G (Proposition 3.10, Item (d));

(**d**) The proofs that the addition and the additive inversion functions on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ are continuous with respect to $\overset{\mathcal{J}}{\mathsf{St}}$ are straightforward consequences of Subitem 8 of Item (b) of Proposition 3.47, and they are left to the reader. We shall prove that the multiplication on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is continuous with respect to $\overset{\mathcal{J}}{\mathsf{St}}$. Take a basic $\overset{\mathcal{J}}{\mathsf{St}}$-neighbourhood $xy + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$ of a product xy, where $x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and $g \in G$, and let h be an element of G such that

$$h > \begin{cases} \overset{G}{\max}\left\{(-\overset{\mathcal{J}}{\mathrm{ms}}(x)) + g, g + (-\overset{\mathcal{J}}{\mathrm{ms}}(y)), |g|\right\} & \text{if } x \neq 0_R \neq y\,, \\ \overset{G}{\max}\left\{g + (-\overset{\mathcal{J}}{\mathrm{ms}}(y)), |g|\right\} & \text{if } x = 0_R \neq y\,, \\ \overset{G}{\max}\left\{(-\overset{\mathcal{J}}{\mathrm{ms}}(x)) + g, |g|\right\} & \text{if } x \neq 0_R = y\,, \\ |g| & \text{if } x = 0_R = y\,. \end{cases}$$

In any case, we have

$$\overset{\mathcal{J}}{\mathrm{ms}}(x) + h > g\,,\quad h + \overset{\mathcal{J}}{\mathrm{ms}}(y) > g \quad\text{and}\quad 2h > 2\,|g| > g\,.$$

Hence, if $x', y' \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ are so that $x' \doteq x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^h)$ and $y' \doteq y + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^h)$, then we get

$$\begin{aligned} x'y' &\doteq (x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^h))\,(y + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^h)) \\ &\doteq xy + \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{h+\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right) + \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)+h}\right) + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{2h}) \\ &\doteq xy + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)\,, \end{aligned}$$

in view of the calculations

$$\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^h)\,\overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right) \doteq \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{h+\overset{\mathcal{J}}{\mathrm{ms}}(y)}\right) \quad\text{and}\quad \overset{\mathcal{J}}{\mathrm{O}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)}\right)\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^h) \doteq \overset{\mathcal{J}}{\mathrm{o}}\left(\mathrm{X}^{\overset{\mathcal{J}}{\mathrm{ms}}(x)+h}\right)\,,$$

and that concludes the proof.

(f) Let g_0 be a fixed positive element of G and take an arbitrary $r \in R$. We shall show that $R \cap (r + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})) = \{r\}$. If $r' \in R$ is such that $r' \doteq r + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})$ and $r' \neq r$, then $0_G = \overset{\mathcal{J}}{\mathrm{ms}}(r' - r) > g_0$, which is absurd, and that proves the item.

(g) Let D be an $\overset{\mathcal{J}}{\mathsf{St}}$-dense subset of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, let r_0 be a fixed positive element of R, let g_0 be a fixed positive element of G and take the following families of $\overset{\mathcal{J}}{\mathsf{St}}$-open sets:

$$P \coloneqq \{r_0\mathrm{X}^g + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g+g_0})\}_{g\in G} \quad\text{and}\quad Q \coloneqq \{r + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})\}_{r\in R}\,.$$

If $g \in G$ and if $x \doteq r_0\mathrm{X}^g + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g+g_0})$, then $\overset{\mathcal{J}}{\mathrm{ms}}(x) = g$, showing that the family P is disjoint. Similarly, if $r \in R$ and if $x \doteq r + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})$, then $x_{0_G} = r$, showing that the family Q is disjoint. Thus, since any element of P or of Q contains at least one element of D, we have $|G| = |P| \leqslant |D|$ and $|R| = |Q| \leqslant |D|$.

(h) Since R and G are countable, the set of generalised polynomials $R\,[\mathrm{X}^G]$ is countable (Proposition 3.39), and since $\mathcal{J}$ is left-finite, the set $R\,[\mathrm{X}^G]$ is $\overset{\mathcal{J}}{\mathsf{St}}$-dense in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ by item (c).

(i) Suppose r_0 is a positive element of R so that $\overset{+_R}{\phi}(r_0) = \infty$, let U be an inhabited $\overset{\mathcal{J}}{\mathsf{St}}$-open subspace of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, let $x \in U$ be arbitrary and let $g \in G$ be so that $g > 0_G$ and $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g) \subset U$. Consider the sequence $\{x + nr_0\mathrm{X}^{2g}\}_{n\in\mathbb{N}}$ in $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ and suppose it has an $\overset{\mathcal{J}}{\mathsf{St}}$-cluster point y in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. Hence, there are infinitely many natural numbers n such that $x + nr_0\mathrm{X}^{2g} \doteq y + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{3g})$, implying

$$x_{2g} + nr_0 = (x + nr_0\mathrm{X}^{2g})_{2|g|} \doteq (y + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{3g}))_{2g} \doteq y_{2g}\,,$$

but, since $\overset{+_R}{\phi}(r_0) = \infty$, that cannot be true for two distinct values of n and we have a contradiction. Therefore, the sequence $\{x + nr_0\mathrm{X}^{2g}\}_{n\in\mathbb{N}}$ has no $\overset{\mathcal{J}}{\mathsf{St}}$-cluster points in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, entailing that U and $\mathrm{Cl}_{\overset{\mathcal{J}}{\mathsf{St}}}(U)$ are not countably compact. In particular, the set U is not $\overset{\mathcal{J}}{\mathsf{St}}$-relatively compact in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and the topology $\overset{\mathcal{J}}{\mathsf{St}}$ is not locally compact.

(j) Suppose S is a subset of G such that

$$(\forall h \in G)\;\; S_h \coloneqq S \cap (\leftarrow, h]_G \in \mathcal{J}\,,$$

let r be a fixed non-zero element of the rng R and consider the net $\{{}_hs\}_{h\in G} \coloneqq \left\{\sum_{g\in S_h} r\mathrm{X}^g\right\}_{h\in G}$ in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. If $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^p)$ is a basic $\overset{\mathcal{J}}{\mathsf{St}}$-open neighbourhood of 0_R, then for all $h, h' \in [p, \rightarrow)_G$ so that $h \geqslant h'$, we have $S_h - S_{h'} \subset (h', h]_G$ and

$${}_hs - {}_{h'}s = \sum_{g\in S_h} r\mathrm{X}^g - \sum_{g\in S_{h'}} r\mathrm{X}^g = \sum_{g\in S_h - S_{h'}} r\mathrm{X}^g \doteq \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{h'}) \doteq \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^p)\,,$$

proving that the net $\{{}_hs\}_{h\in G}$ is $\overset{\mathcal{J}}{\mathsf{St}}$-Cauchy. Thus, the completeness of $\overset{\mathcal{J}}{\mathsf{St}}$ implies that there is an $l \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ such that ${}_hs \xrightarrow[h\in G]{\overset{\mathcal{J}}{\mathsf{St}}} l$. Each net $\{{}_hs_g\}_{h\in G}$ for $g \in G - S$ is constant at the value 0_R and each net $\{{}_hs_g\}_{h\in G}$ for $g \in S$ is eventually constant at the value r. Therefore, by item (b) we have $S = \mathrm{supp}\,(l) \in \mathcal{J}$, proving that $\mathcal{J}$ is incremental.

(**k**) By Proposition 1.60 and Theorems 3.41 and 1.84, it suffices to prove that $\overset{\mathcal{J}}{\mathsf{St}}$ is complete when $\mathrm{cf}(G) = \omega$ and $\mathcal{J}$ is incremental. Let $\{g_n\}$ be an increasing, cofinal sequence in G and let $\{{}_n s\}$ be an $\overset{\mathcal{J}}{\mathsf{St}}$-Cauchy sequence in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. For each $h \in G$, let $\lfloor h \rfloor$ be the smallest natural number such that $h \leqslant g_{\lfloor h \rfloor}$, and for each $k \in \mathbb{N}$, let $\mathrm{c}(k)$ be a natural number such that

$$(\forall m, n \in [\mathrm{c}(k), \infty)_{\mathbb{N}}) \ \ {}_m s \doteq {}_n s + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_k}) .$$

In fact, the numbers $\mathrm{c}(k)$ shall be chosen so that the sequence $\{\mathrm{c}(k)\}_{k\in\mathbb{N}}$ is increasing. Let $l : G \to R$ be the function defined by $l_h := {}_{\mathrm{c}(\lfloor h \rfloor)} s_h$. Thus, if $h, h' \in G$ are so that $h \leqslant h'$, then $\lfloor h \rfloor \leqslant \lfloor h' \rfloor$, $\mathrm{c}(\lfloor h \rfloor) \leqslant \mathrm{c}(\lfloor h' \rfloor)$, ${}_{\mathrm{c}(\lfloor h \rfloor)} s \doteq {}_{\mathrm{c}(\lfloor h' \rfloor)} s + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_{\lfloor h \rfloor}})$, and, as $h \leqslant g_{\lfloor h \rfloor}$, we get

$$l_h = {}_{\mathrm{c}(\lfloor h \rfloor)} s_h = {}_{\mathrm{c}(\lfloor h' \rfloor)} s_h ,$$

leading up to

$$(\forall h' \in G) \ \ \mathrm{supp}(l) \cap (\leftarrow, h']_G \subset \mathrm{supp}\left({}_{\mathrm{c}(\lfloor h' \rfloor)} s\right) \in \mathcal{J} .$$

Therefore, since the ideal $\mathcal{J}$ is incremental, we have $l \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and $l \doteq {}_{\mathrm{c}(\lfloor h' \rfloor)} s + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{h'})$ $(\forall h' \in G)$, which implies that ${}_n s \overset{\overset{\mathcal{J}}{\mathsf{St}}}{\underset{n\to\infty}{\dashrightarrow}} l$ and ${}_n s \xrightarrow[n\to\infty]{\overset{\mathcal{J}}{\mathsf{St}}} l$ (Proposition 1.58, Item (b)).

□

Example 4.3. Let R be a rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. Every **polynomial function**

$$\begin{cases} p : \left(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{\mathcal{J}}{\mathsf{St}}\right) \to \left(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{\mathcal{J}}{\mathsf{St}}\right) \\ p(x) := {}_n a x^n + {}_{n-1} a x^{n-1} + \cdots + {}_1 a x + {}_0 a \end{cases}$$

is continuous (Theorem 4.2, Item (d)).

Example 4.4. The strong topologies (Example 1.64)

$$\overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{R},\mathbb{Z}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{C},\mathbb{Z}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{R},\mathbb{Q}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{C},\mathbb{Q}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{Z},\mathbb{R}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{Q},\mathbb{R}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{R},\mathbb{R}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{C},\mathbb{R}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{R},\mathsf{BS}_\ell} \quad \text{and} \quad \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{C},\mathsf{BS}_\ell}$$

are not separable, where $\mathcal{J}$ is any arithmetic Rayner ideal on each respective ordered group of exponents (Theorem 4.2, Item (g)). In particular, the strong topologies on the classical Hahn, Levi-Civita and Puiseux fields are not separable.

Example 4.5. The strong topologies on the Hahn rings $\mathbb{Z}[[X^{\mathbb{Z}}]]$ and $\mathbb{Q}[[X^{\mathbb{Z}}]]$, the strong topologies on the Levi-Civita rings $\overset{\mathrm{lf}}{\mathbb{Z}}[[X^{\mathbb{Q}}]]$, $\overset{\mathrm{lf}}{\mathbb{Q}}[[X^{\mathbb{Q}}]]$, $\overset{\mathrm{lf}}{\mathbb{Z}}[[X^{\mathsf{BS}_\ell}]]$ and $\overset{\mathrm{lf}}{\mathbb{Q}}[[X^{\mathsf{BS}_\ell}]]$ and the strong topologies on the Puiseux rings $\overset{\mathrm{bd}}{\mathbb{Z}}_{\mathbb{Z}}[[X^{\mathbb{Q}}]]$, $\overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[X^{\mathbb{Q}}]]$, $\overset{\mathrm{bd}}{\mathbb{Z}}_{\mathbb{Z}}[[X^{\mathcal{D}_d}]]$, $\overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[X^{\mathcal{D}_d}]]$, $\overset{\mathrm{bd}}{\mathbb{Z}}_{\mathbb{Z}}[[X^{\mathcal{P}}]]$ and $\overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[X^{\mathcal{P}}]]$ (Examples 3.20 and 3.21) are separable (Theorem 4.2, Item (h)).

Example 4.6. The strong topologies

$$\overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{Z},\mathbb{Z}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{Q},\mathbb{Z}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{R},\mathbb{Z}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{C},\mathbb{Z}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{Z},\mathbb{Q}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{Q},\mathbb{Q}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{R},\mathbb{Q}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{C},\mathbb{Q}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{Z},\mathbb{R}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{Q},\mathbb{R}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{R},\mathbb{R}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{C},\mathbb{R}}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{Z},\mathsf{BS}_\ell}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{Q},\mathsf{BS}_\ell}, \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{R},\mathsf{BS}_\ell} \text{ and } \overset{\mathcal{J}}{\mathsf{St}}_{\mathbb{C},\mathsf{BS}_\ell}$$

are neither countably compact nor locally compact, where $\mathcal{J}$ is any arithmetic Rayner ideal on each respective ordered group of exponents (Theorem 4.2, Item (i)).

Example 4.7. The strong topologies on a Rayner rng $R[X^G]$ of generalised polynomials and on a Puiseux rng of the form $\overset{\mathrm{bd}}{R}_{\mathbb{Z}}[[X^{\mathbb{Q}}]]$ are not complete, where R is any rng and G is any ordered group (Theorem 4.2, Item (j); See comment before Example 3.20).

Example 4.8. Since we have $\mathrm{cf}(\mathbb{Z}) = \mathrm{cf}(\mathbb{Q}) = \mathrm{cf}(\mathbb{R}) = \mathrm{cf}(\mathsf{BS}_\ell) = \omega$, the strong topologies on the Hahn rngs of the forms $R[[X^{\mathbb{Z}}]]$, $R[[X^{\mathbb{Q}}]]$, $R[[X^{\mathbb{R}}]]$ and $R[[X^{\mathsf{BS}_\ell}]]$, and the strong topologies on the Levi-Civita rngs of the forms $\overset{\mathrm{lf}}{R}[[X^{\mathbb{Z}}]]$, $\overset{\mathrm{lf}}{R}[[X^{\mathbb{Q}}]]$, $\overset{\mathrm{lf}}{R}[[X^{\mathbb{R}}]]$ and $\overset{\mathrm{lf}}{R}[[X^{\mathsf{BS}_\ell}]]$, are all completely ultrametrizable by invariant ultrametrics, where R is any rng (Theorem 4.2, Item (k)).

Example 4.9. Combining the results from the Examples 4.5 and 4.8, we get that the strong topologies on the Hahn rings $\mathbb{Z}[[X^{\mathbb{Z}}]]$ and $\mathbb{Q}[[X^{\mathbb{Z}}]]$, and the strong topologies on the Levi-Civita rings $\overset{\mathrm{lf}}{\mathbb{Z}}[[X^{\mathbb{Q}}]]$, $\overset{\mathrm{lf}}{\mathbb{Q}}[[X^{\mathbb{Q}}]]$, $\overset{\mathrm{lf}}{\mathbb{Z}}[[X^{\mathsf{BS}_\ell}]]$ and $\overset{\mathrm{lf}}{\mathbb{Q}}[[X^{\mathsf{BS}_\ell}]]$ are all **polish topologies**, that is, they are completely metrizable and separable. Polish topological spaces play important roles in Descriptive Set Theory (127), Measure Theory (30) and Probability Theory (108, 30).

The following theorem shows that the strong topology on an ordered Rayner rng coincides with the order topology on it:

Proposition 4.10. *Let R be an ordered rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G.*

(a) $\operatorname{Ordt}(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]) = \overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$;

(b) *For all $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and all positive $r_0 \in R$, the set $\{[x - r_0\mathrm{X}^g, x + r_0\mathrm{X}^g]\}_{g\in G}$ of $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-closed intervals in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is a local $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-system of neighbourhoods of x;*

(c) *Every non-trivial Archimedean class in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-open;*

(d) *No inhabited $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-open subspace of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is countably compact. In that case, the topology $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$ is not countably compact nor locally compact.*

Proof. Item (d) is a direct consequence of Item (i) of Theorem 4.2.

(a) We shall show that the identity function

$$\mathrm{id} : (\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \operatorname{Ordt}(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]])) \to \left(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}\right)$$

is a homeomorphism. Take an element x in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. If $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ is a basic $\overset{\mathcal{J}}{\check{\mathsf{S}}\mathsf{t}}$-neighbourhood of x, and if y is an element of the open interval $(x - r\mathrm{X}^{2|g|}, x + r\mathrm{X}^{2|g|})$, then we have that $-r\mathrm{X}^{2|g|} < y - x < r\mathrm{X}^{2|g|}$ and $\overset{\mathcal{J}}{\mathrm{ms}}(y - x) \geqslant 2|g| > g$ (Theorem 3.82, Item (g)). Thus, we have proved the inclusion $(x - r\mathrm{X}^{2|g|}, x + r\mathrm{X}^{2|g|}) \subset x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ for that case, and that proves the continuity of id.

Consider an open interval $(a, \rightarrow)$ containing x, which is a subbasic neighbourhood of x with respect to the order topology $\operatorname{Ordt}(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]])$, let $g_a \coloneqq \overset{\mathcal{J}}{\mathrm{ms}}(x - a)$ and take a $y \doteq x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_a})$. Hence, we have

$$\overset{\mathcal{J}}{\mathrm{ms}}(y - a) \geqslant \overset{\breve{G}}{\min}\{\overset{\mathcal{J}}{\mathrm{ms}}(y - x), \overset{\mathcal{J}}{\mathrm{ms}}(x - a)\} = g_a,$$

and, since $(y - a)_{g_a} = (x - a)_{g_a} > 0_R$, we get $\overset{\mathcal{J}}{\mathrm{ms}}(y - a) = g_a$ and $a < y$, proving the inclusion $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_a}) \subset (a, \rightarrow)$. Analogously, if (a, b) is an interval containing x, then one can show that $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{\max\{g_a, g_b\}}) \subset (a, b)$, where $g_b \coloneqq \overset{\mathcal{J}}{\mathrm{ms}}(b - x)$. Therefore, the inverse id^{-1} is continuous and the proof is complete.

(b) Firstly, we shall prove that each closed interval $[x - r_0\mathrm{X}^g, x + r_0\mathrm{X}^g]$ is an $\overset{\mathcal{J}}{\mathsf{St}}$-neighbourhood of x. In fact, if y is an element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ so that $y \doteq x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$, then

$$\overset{\mathcal{J}}{\mathrm{ms}}\,(y - x \pm r_0\mathrm{X}^g) \geqslant \min\{\overset{\mathcal{J}}{\mathrm{ms}}\,(y - x)\,, \overset{\mathcal{J}}{\mathrm{ms}}\,(r_0\mathrm{X}^g)\} = g$$

and

$$(y - x \pm r_0\mathrm{X}^g)_g \doteq (\overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) \pm r_0\mathrm{X}^g)_g \doteq \pm r_0 \neq 0_R\,,$$

implying $\overset{\mathcal{J}}{\mathrm{ms}}\,(y - x \pm r_0\mathrm{X}^g) = g$ and $y - x - r_0\mathrm{X}^g < 0_R < y - x + r_0\mathrm{X}^g$, that is, $-r_0\mathrm{X}^g < y - x < r_0\mathrm{X}^g$. Thus, we have proved that the inclusion $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) \subset [x - r_0\mathrm{X}^g, x + r_0\mathrm{X}^g]$ holds, and that proves that the interval $[x - r_0\mathrm{X}^g, x + r_0\mathrm{X}^g]$ is an $\overset{\mathcal{J}}{\mathsf{St}}$-neighbourhood of x.

Let $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$ be an $\overset{\mathcal{J}}{\mathsf{St}}$-basic neighbourhood of x and let r_0 be a fixed positive element of R. If y is an element of the closed interval $[x - r_0\mathrm{X}^{2|g|}, x + r_0\mathrm{X}^{2|g|}]$, then we have that $-r_0\mathrm{X}^{2|g|} \leqslant y - x \leqslant r_0\mathrm{X}^{2|g|}$ and $\overset{\mathcal{J}}{\mathrm{ms}}\,(y - x) \geqslant 2\,|g| > g$ (Theorem 3.82, Item (g)). Hence, we have just proved the inclusion $[x - r_0\mathrm{X}^{2|g|}, x + r_0\mathrm{X}^{2|g|}] \subset x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$, thus proving the item.

(c) Note that each non-trivial Archimedean class in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ can be uniquely written in the form

$$A\mathrm{X}^g + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) = \bigcup_{r \in A} (r\mathrm{X}^g + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g))\ ,$$

where A is a non-trivial Archimedean class in R and where g is an element of G (Theorem 3.82, Item (i)).

□

4.2 The weak topology on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$

As we have seen in Section 4.1, the strong topology $\overset{\mathcal{J}}{\mathsf{St}}$ is defined on all Rayner rngs. Interestingly, when $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is an ordered Rayner rng, a coarser (or weaker, smaller) topology on the set $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, called the **weak topology on** $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, is also fundamental for the development of analytical theories on structures of that kind, especially concerning the convergence of power series with coefficients in the real and complex Levi-Civita fields (cf. Introduction). We shall define that coarser topology for a general ordered Rayner rng and we shall examine its fundamental properties in this section.

Definition 4.11. Let R be an ordered rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. The **weak topology on** $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is the topology on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ generated by the subbasis consisting of the sets of the form

$$\mathsf{W}^g_r(x) := \{y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \mid (\forall h \in (\leftarrow, g]_G)\ \ |y_h - x_h| < r\ \}$$

for $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, $r \in (0_R, \rightarrow)_R$ and $g \in G$, and that topology shall be denoted by $\overset{\mathcal{J}}{\mathsf{Wt}}$ or $\underset{R,G}{\overset{\mathcal{J}}{\mathsf{Wt}}}$.

Just as we did in Section 4.1 with reference to the strong topology on $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, we shall provide here two graphical representations of the subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open set $\mathsf{W}^g_r(x)$, depicted in Figures (3) and (4), which may facilitate the reader's comprehension of what is transpiring as he or she follows the proofs of the theorems. We shall explain how they are meant to be understood in the following paragraphs.

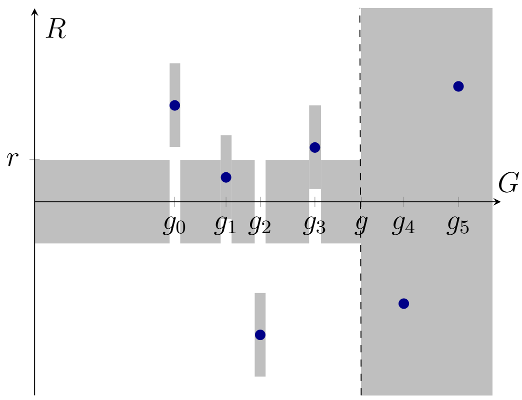


Figure (3): Absolute graphical representation of the subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open set $\mathsf{W}^g_r(x)$.

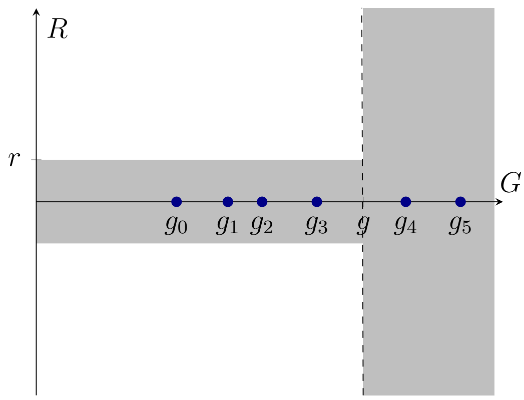


Figure (4): Relative graphical representation of the subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open set $\mathsf{W}^g_r(x)$.

Let the family $\{g_\alpha\}_{\alpha<\gamma}$ be the increasing ordinal sequence of elements of the support $\operatorname{supp}(x)$, and consider Figure (3). The G-axis, the R-axis, the dots and the shaded area drawn on the right hand side of the dotted, vertical line $h = g$ here play the same roles they played in the absolute graphical representation of the basic $\overset{\mathcal{J}}{\mathsf{St}}$-open set $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ (Section 4.1, Figure (1)). The novelties here are twofold: first, for each $\alpha < \gamma$, the points (g_α, s) for $s \in (x_{g_\alpha} - r, x_{g_\alpha} + r)_R$ are represented by a vertical, bold line segment drawn with length $2r$ and with (g_α, x_{g_α}) as its midpoint; and second, the points (h, s) for $h \in (\leftarrow, g]_G - \operatorname{supp}(x)$ and $s \in (-r, r)_R$ are represented by a shaded, horizontally extending area that has vertical extension $2r$, is centred at the G-axis, and is drawn in the graph on the left hand side of the dotted, vertical line $h = g$, extending endlessly in the negative direction of the G-axis and occasionally getting interrupted whenever it arrives at points with exponents in $\operatorname{supp}(x)$.

Now, consider an arbitrary element y of the subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open set $\mathsf{W}_r^g(x)$. Since $|y_h - x_h| < r$ $(\forall h \in (\leftarrow, g]_G)$, we have

$$(\forall h \in (\leftarrow, g]_G - \operatorname{supp}(x)) \;\; y_h \in (-r, r)_R$$

and

$$(\forall h \in (\leftarrow, g]_G \cap \operatorname{supp}(x)) \;\; y_h \in (x_h - r, x_h + r)_R \; .$$

This precisely means that, up to the exponent g, including g itself, the points of y either lie in the shaded, horizontally extending area or on one of the vertical, bold line segments we have described above. Thus, the combination of the vertical, bold line segments, the horizontally extending area on the left hand side of the vertical line $h = g$ and the shaded area on the right hand side of that same line represent all possible elements of the set $\mathsf{W}_r^g(x)$, conveying that there is some restricted freedom for the coefficients of the points of y with exponents less or equal to g, and there is unhindered freedom for the coefficients of the points of y with exponents greater than g. We call this type of graph an **absolute graphical representation of** $\mathsf{W}_r^g(x)$. Note that one may add more subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open sets to that graph, provided that one appropriately discriminates the points belonging to different sets, and that is the reason why these representations are optimal for situations in which two or more subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open sets are considered simultaneously.

Whenever only one subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open set $\mathsf{W}_r^g(x)$ is considered[4], it is preferable to make use of the neater representation depicted in Figure (4), which we shall describe. The G-axis, the R-axis, the dots and the shaded area drawn on the right hand side of the dotted, vertical line $h = g$ here play the same roles they played in the relative graphical representation of the basic $\overset{\mathcal{J}}{\mathsf{St}}$-open set $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ (Section 4.1, Figure (2)). Additionally, here the points (h, s) for $h \in (\leftarrow, g]_G$ and $s \in (-r, r)_R$ are represented by a shaded, horizontally extending area that has vertical extension $2r$, is centred at the G-axis, and is drawn in the graph on the left hand side of the dotted, vertical line $h = g$, extending endlessly in the negative direction of the G-axis.

Taking into consideration the arbitrary element y of $\mathsf{W}_r^g(x)$ once more, since $|y_h - x_h| < r$ $(\forall h \in (\leftarrow, g]_G)$, all x-relative points of y up to the exponent g, including g itself, must reside in the shaded, horizontally extending area on the left hand side of the dotted, vertical line $h = g$, representing the fact that there is some restricted freedom for the coefficients of the points of y in this region of the graph, as long as they do not stray too far from the corresponding coefficients of x. On the other hand, the coefficients of the points of y with exponents greater than g are completely free to assume any values in R whatsoever. Thus, the combination of the horizontally extending area on the left hand side of the vertical line $h = g$ and the shaded area on the right hand side of that same line represent all possible elements of the set $\mathsf{W}_r^g(x)$, with the caveat that those abstractions are all centred at x in a sense. We call this type of graph a **relative graphical representation of** $\mathsf{W}_r^g(x)$.

If the reader properly understood how Figures (1)-(4) are meant to be taken, then, by comparing Figures (1) and (3) (resp. (2) and (4)), he or she should be able to notice that $x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g) \subset \mathsf{W}_r^g(x)$ straightaway, since all relevant points (resp. relevant x-relative points) of the former representation are contained in the latter.

Before proceeding with the properties of $\overset{\mathcal{J}}{\mathsf{Wt}}$, we must first consider a special class of functions which becomes relevant whenever the arithmetic Rayner

[4] Or at least only one at a time, not two or more simultaneously.

ideal $\mathcal{J}$ on G is left-finite. To start with, note that for each $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, the intersection $\operatorname{supp}(x) \cap (\leftarrow, g]_G$ is finite in that case, and, thus, the set $\{|x_h| \mid h \in (\leftarrow, g]_G\}$ is finite as well, implying that it has a maximum in the interval $[0_R, \rightarrow)_R$.

Definition 4.12. Let R be an ordered rng, let G be an ordered group and let $\mathcal{J}$ be a left-finite arithmetic Rayner ideal on G (Definition 3.9). The g-**amplitude function on** $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is the function $\mathsf{p}^g \colon \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \to [0_R, \rightarrow)_R$ that is given by $\mathsf{p}^g(x) := \overset{R}{\max}_{h \in (\leftarrow, g]_G} |x_h|$.

When $\mathcal{J}$ is left-finite, the subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open sets $\mathsf{W}^g_r(x)$ may be written in the form

$$\mathsf{W}^g_r(x) = \{y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \mid \mathsf{p}^g(y - x) < r\} = x + (\mathsf{p}^g)^{-1}\langle [0_R, r)_R \rangle \ .$$

Proposition 4.13. *Let R be an ordered rng, let G be an ordered group and let $\mathcal{J}$ be a left-finite arithmetic Rayner ideal on G.*

- ▷ *For all $g_1, g_2 \in G$ so that $g_1 \leqslant g_2$, we have $\mathsf{p}^{g_1} \leqslant \mathsf{p}^{g_2}$;*
- ▷ $(\forall x, y, z \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]])\,(\forall g \in G)\ \ \mathsf{p}^g(x + y) \leqslant \mathsf{p}^g(x) + \mathsf{p}^g(y)$;
- ▷ $(\forall x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]])\,(\forall r \in R)\,(\forall g \in G)\, \mathsf{p}^g(rx) = |r|\, \mathsf{p}^g(x)$ and $\mathsf{p}^g(xr) = \mathsf{p}^g(x)\,|r|$.

Proof. The proofs of these items are entirely straightforward and are left to the reader. □

Example 4.14. The reader familiar with the notion of a **seminorm** on a vector $\mathbb{R}$-space may have noticed that in the case $R = \mathbb{R}$ each g-amplitude function $\mathsf{p}^g \colon \overset{\mathcal{J}}{\mathbb{R}}[[\mathrm{X}^G]] \to [0, \infty)_{\mathbb{R}}$ is actually a seminorm on the vector $\mathbb{R}$-space $\overset{\mathcal{J}}{\mathbb{R}}[[\mathrm{X}^G]]$ (cf. Proposition 3.34, Item (d)).

In the two following theorems, we shall cover the fundamental properties of the weak topology on ordered Rayner rngs:

Theorem 4.15. *Let R be an ordered rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G.*

(a) *Let $\{{}_\lambda x\}_{\lambda\in\Lambda}$ be a net in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and let x be an element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.*

- ▷ *If ${}_\lambda x \xrightarrow[\lambda\in\Lambda]{\overset{\mathcal{J}}{\mathsf{Wt}}} x$, then ${}_\lambda x_g \xrightarrow[\lambda\in\Lambda]{\overset{R}{\mathrm{Ordt}}} x_g$ $(\forall g\in G)$;*
- ▷ *If ${}_\lambda x \overset{\overset{\mathcal{J}}{\mathsf{Wt}}}{\underset{\lambda\in\Lambda}{\dashrightarrow}} x$, then ${}_\lambda x_g \overset{\overset{R}{\mathrm{Ordt}}}{\underset{\lambda\in\Lambda}{\dashrightarrow}} x_g$ $(\forall g\in G)$.*

(b) *The converses of the subitems of item (a) hold true if the union $\bigcup\limits_{\lambda\in\Lambda}\operatorname{supp}({}_\lambda x)$ is finite;*

(c) *The topology $\overset{\mathcal{J}}{\mathsf{Wt}}$ is T2.*

(d) $\overset{\mathcal{J}}{\mathsf{Wt}} \subset \overset{\mathcal{J}}{\mathsf{St}}$;

(e) *The order topology $\overset{R}{\mathrm{Ordt}}$ is the subspace topology on R induced by $\overset{\mathcal{J}}{\mathsf{Wt}}$;*

(f) *R is $\overset{\mathcal{J}}{\mathsf{Wt}}$-closed;*

(g) *The additive inversion of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is a continuous function of type*

$$\left(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]],\overset{\mathcal{J}}{\mathsf{Wt}}\right)\to\left(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]],\overset{\mathcal{J}}{\mathsf{Wt}}\right);$$

(h) *If $\operatorname{ci}((0_R,\to)_R)\leqslant\omega$ and if $\operatorname{cf}(G)=\omega$, then $\overset{\mathcal{J}}{\mathsf{Wt}}$ is first-countable;*

(i) *If $\mathcal{J}$ is ω_1-dominated (Definition 3.1) and if $\overset{\mathcal{J}}{\mathsf{Wt}}$ is separable, then $\overset{R}{\mathrm{Ordt}}$ is separable and G is countable;*

(j) *The ordered rng R has a least positive element $r_{\min}$ if, and only if, we have $\overset{\mathcal{J}}{\mathsf{Wt}}=\overset{\mathcal{J}}{\mathsf{St}}$. In that case, we have $\mathsf{W}^g_{r_{\min}}(x)=x+\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ for each $x\in\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and each $g\in G$;*

(k) *Every countably compact* $\overset{\mathcal{J}}{\mathsf{Wt}}$*-subspace of* $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ *has empty* $\overset{\mathcal{J}}{\mathsf{Wt}}$*-interior. In particular, the weak topology* $\overset{\mathcal{J}}{\mathsf{Wt}}$ *is neither countably compact nor locally compact;*

(l) *Take the elements* $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, $r_1, r_2 \in (0_R, \rightarrow)_R$ *and* $g_1, g_2 \in G$. *We have* $\mathsf{W}_{r_1}^{g_1}(x) \subset \mathsf{W}_{r_2}^{g_2}(x)$ *if, and only if,* $g_1 \geqslant g_2$ *and* $r_1 \leqslant r_2$.

Suppose the ordered rng R *has no least positive element and the ideal* $\mathcal{J}$ *is left-finite.*

(m) *For each* $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, *the sets of the form* $\mathsf{W}_r^g(x)$ *for* $g \in G$ *and* $r \in (0_R, \rightarrow)_R$ *constitute a local* $\overset{\mathcal{J}}{\mathsf{Wt}}$*-basis of* x*;*

(n) *For all* $x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ *so that* $x < y$, *each of the following intervals*

$$(\leftarrow, y), (\leftarrow, y], (x, y), [x, y), (x, y], [x, y], (x, \rightarrow) \text{ and } [x, \rightarrow)$$

in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ *contains not one inhabited* $\overset{\mathcal{J}}{\mathsf{Wt}}$*-open set. In particular, these intervals are neither* $\overset{\mathcal{J}}{\mathsf{Wt}}$*-open nor* $\overset{\mathcal{J}}{\mathsf{Wt}}$*-closed.*

(o) *The addition operation of* $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ *is a continuous function of type*

$$\left(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{\mathcal{J}}{\mathsf{Wt}}\right) \times \left(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{\mathcal{J}}{\mathsf{Wt}}\right) \to \left(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{\mathcal{J}}{\mathsf{Wt}}\right) ;$$

(p) *The topology* $\overset{\mathcal{J}}{\mathsf{Wt}}$ *is* $T^{7/2}$*;*

(q) $\overset{\mathcal{J}}{\mathsf{Wt}}$ *is first-countable if, and only if, we have* $\mathrm{ci}\left((0_R, \rightarrow)_R\right) = \omega = \mathrm{cf}\,(G)$.

Proof. The items (c) and (f) are immediate consequences of item (a) and Proposition 1.51. Item (p) follows from Proposition 1.61 and items (c), (g) and (o).

(a) Take a fixed $g \in G$. If ${}_\lambda x \xrightarrow[\lambda \in \Lambda]{\overset{\mathcal{J}}{\mathsf{Wt}}} x$, then for each $r \in (0_R, \rightarrow)_R$, there is a $\lambda_0 \in \Lambda$ such that

$$(\forall \lambda \in [\lambda_0, \rightarrow)_\Lambda)\, (\forall h \in (\leftarrow, g]_G)\ \left|{}_\lambda x_h - x_h\right| < r\,,$$

which, in particular, implies $|{}_\lambda x_g - x_g| < r$ $(\forall \lambda \in [\lambda_0, \rightarrow)_\Lambda)$ and gives us ${}_\lambda x_g \xrightarrow[\lambda\in\Lambda]{\overset{R}{\mathrm{Ordt}}} x_g$. If ${}_\lambda x \overset{\overset{\mathcal{J}}{\mathsf{Wt}}}{\underset{\lambda\in\Lambda}{\dashrightarrow}} x$, then there is a subnet $\{{}_{\lambda_\mu} x\}_{\mu\in M}$ of $\{{}_\lambda x\}_{\lambda\in\Lambda}$ such that ${}_{\lambda_\mu} x \xrightarrow[\mu\in M]{\overset{\mathcal{J}}{\mathsf{Wt}}} x$, implying ${}_{\lambda_\mu} x_g \xrightarrow[\mu\in M]{\overset{R}{\mathrm{Ordt}}} x_g$ $(\forall g \in G)$ and ${}_\lambda x_g \overset{\overset{R}{\mathrm{Ordt}}}{\underset{\lambda\in\Lambda}{\dashrightarrow}} x_g$ $(\forall g \in G)$.

(b) Suppose ${}_\lambda x_g \xrightarrow[\lambda\in\Lambda]{\overset{R}{\mathrm{Ordt}}} x_g$ $(\forall g \in G)$, suppose the union $\bigcup\limits_{\lambda\in\Lambda} \operatorname{supp}({}_\lambda x)$ is finite, and let $g_1 g_2 \dots g_n$ be the increasing finite sequence of elements of that union. Note that for all $\lambda \in \Lambda$ and all $h \in G - \{g_1 \dots g_n\}$, we have ${}_\lambda x_h = x_h = 0_R$, which immediately gives us $|{}_\lambda x_h - x_h| = 0_R$. Let $r \in (0_R, \rightarrow)_R$ be arbitrary. For each index element $i \in [1, n]_{\mathbb{N}}$, there is an element $\lambda_i \in \Lambda$ such that $|{}_\lambda x_{g_i} - x_{g_i}| < r$ $(\forall \lambda \in [\lambda_i, \rightarrow)_\Lambda)$, and, since Λ is directed, there is a $\lambda_0 \in \Lambda$ such that $\lambda_0 \geqslant \lambda_i$ $(\forall i \in [1, n]_{\mathbb{N}})$. Thus, we have

$$(\forall \lambda \in [\lambda_0, \rightarrow)_\Lambda)\,(\forall h \in G)\;\; |{}_\lambda x_h - x_h| < r\,,$$

implying, in particular, that for every element g in G, we have ${}_\lambda x \in \mathsf{W}^g_r(x)$ $(\forall \lambda \in [\lambda_0, \rightarrow)_\Lambda)$, which gives us ${}_\lambda x \xrightarrow[\lambda\in\Lambda]{\overset{\mathcal{J}}{\mathsf{Wt}}} x$. The proof of the converse of the second subitem of item (a) is analogous.

(d) Let U be a $\overset{\mathcal{J}}{\mathsf{Wt}}$-open set and let x be an element of U. Thus, there are ${}_1y\,{}_2y \dots {}_ny \in \overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]]$, $r_1 r_2 \dots r_n \in (0_R, \rightarrow)_R$ and $g_1 g_2 \dots g_n \in G$ such that $x \in \bigcap\limits_{i=1}^{n} \mathsf{W}^{g_i}_{r_i}({}_iy) \subset U$. Let $g \coloneqq \max\limits_{i\in[1,n]_{\mathbb{N}}} g_i$ and let z be an element of $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)$. Since $z_h = x_h$ $(\forall h \in (\leftarrow, g]_G)$, for each $i \in [1, n]_{\mathbb{N}}$ we have

$$(\forall h \in (\leftarrow, g_i]_G)\;\; |{}_iy_h - z_h| = |{}_iy_h - x_h| < r_i\,,$$

that is, $z \in \bigcap\limits_{i=1}^{n} \mathsf{W}^{g_i}_{r_i}({}_iy)$, proving the inclusion $x + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g) \subset \bigcap\limits_{i=1}^{n} \mathsf{W}^{g_i}_{r_i}({}_iy) \subset U$ and proving that U is $\overset{\mathcal{J}}{\mathsf{St}}$-open.

A clever, alternative way of proving item (d) is to show (or just notice) that each subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open set $\mathsf{W}^g_r(x)$ may be written as the union

$$\mathsf{W}^g_r(x) = \bigcup_{y\in\mathsf{W}^g_r(x)} \left(y + \overset{\mathcal{J}}{\mathrm{o}}\,(\mathrm{X}^g)\right) \in \overset{\mathcal{J}}{\mathsf{St}}\,,$$

and that readily implies that every $\overset{\mathcal{J}}{\mathsf{Wt}}$-open set is $\overset{\mathcal{J}}{\mathsf{St}}$-open. The reader is encouraged to compare Figures (1) and (3) (or Figures (2) and (4)) in order to grasp the reason why the equation above holds true.

(e) Consider the identity function

$$\mathrm{id} : (R, \overset{R}{\mathrm{Ordt}}) \to \big(R, \overset{\mathcal{J}}{\mathsf{Wt}}\restriction_R\big)$$

and let $r \in R$ be fixed. The inverse function id^{-1} is continuous by item (a). If the intersection $R \cap \mathsf{W}_s^g(x)$ is a subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_R$-neighbourhood of r so that we have $g \geqslant 0_G$ (without loss of generality), then we get $|x_h| < s$ $(\forall h \in (\leftarrow, g]_G - \{0_G\})$ and $|r - x_{0_G}| < s$, and one can easily check[5] that $(x_{0_G} - s, x_{0_G} + s)_R \subset R \cap \mathsf{W}_s^g(x)$, proving that id is continuous.

(g) Let x be a fixed element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and let U be a $\overset{\mathcal{J}}{\mathsf{Wt}}$-neighbourhood of $-x$. Thus, there are ${}_1y\,{}_2y \ldots {}_ny \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, $r_1 r_2 \ldots r_n \in (0_R, \to)_R$ and $g_1 g_2 \ldots g_n \in G$ such that $-x \in \bigcap_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}({}_iy) \subset U$. That means that for every $i \in [1, n]_{\mathbb{N}}$, we have

$$(\forall h \in (\leftarrow, g_i])\ \ |x_h - (-{}_iy_h)| = |(-x)_h - {}_iy_h| < r_i\,,$$

which implies $x \in \bigcap_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}(-{}_iy)$. If z is another element of $\bigcap_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}(-{}_iy)$, then for every $i \in [1, n]_{\mathbb{N}}$, we have

$$(\forall h \in (\leftarrow, g_i])\ \ |(-z)_h - {}_iy_h| = |z_h - (-{}_iy_h)| < r_i\,,$$

that is, we have that $-z \in \bigcap_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}({}_iy)$, and we obtain the inclusions $-\bigcap_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}(-{}_iy) \subset \bigcap_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}({}_iy) \subset U$, thus proving the item.

(h) If $\{r_n\}$ is a coinitial sequence in $(0_R, \to)_R$, if $\{g_n\}$ is a cofinal sequence in G and if x is an element of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, then the sequence $\{\mathsf{W}_{r_n}^{g_n}(x)\}$ is clearly a countable local $\overset{\mathcal{J}}{\mathsf{Wt}}$-basis of x.

[5] In order to help the understanding of the situation, the reader is encouraged to sketch the absolute and relative graphical representations of the subbasic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open set $\mathsf{W}_s^g(x)$, just as we did in Figures (3) and (4).

(**i**) Let D be a countable $\overset{\mathcal{J}}{\mathsf{Wt}}$-dense subset of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, let $S \coloneqq \{x_{0_G} \mid x \in D\}$ and let $E \coloneqq \bigcup\limits_{x \in D} \operatorname{supp}(x)$. The set S is countable, and, since the ideal $\mathcal{J}$ is ω_1-dominated, we have

$$|E| = \Bigl|\bigcup_{x \in D} \operatorname{supp}(x)\Bigr| \leqslant |D|\,\omega \leqslant \omega\omega = \omega\,,$$

that is, E is countable. If r is an arbitrary element of R, then there is a sequence $\{{}_n x\}$ in D such that ${}_n x \xrightarrow[n\to\infty]{\overset{\mathcal{J}}{\mathsf{Wt}}} r\mathrm{X}^{0_G}$, implying ${}_n x_{0_G} \xrightarrow[n\to\infty]{\overset{R}{\mathrm{Ordt}}} r$ by item (a) and showing that S is $\overset{R}{\mathrm{Ordt}}$-dense in R. Likewise, if g is an element of G and if r_0 is a non-zero element of R, then there is a sequence $\{{}_n x\}$ in D such that ${}_n x \xrightarrow[n\to\infty]{\overset{\mathcal{J}}{\mathsf{Wt}}} r_0\mathrm{X}^g$, and that gives us ${}_n x_g \xrightarrow[n\to\infty]{\overset{R}{\mathrm{Ordt}}} r_0 \neq 0_R$ by item (a). In that case, there are infinitely many natural numbers n such that ${}_n x_g \neq 0_R$ and $g \in \operatorname{supp}({}_n x) \subset E$, proving that $G = E$ and that G is countable.

(**j**) Suppose that the ordered rng R has no least positive element. We shall prove that the basic $\overset{\mathcal{J}}{\mathsf{St}}$-open set $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G})$ is not $\overset{\mathcal{J}}{\mathsf{Wt}}$-open. Consider a basic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open neighbourhood $\bigcap\limits_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}({}_i x)$ of 0_R, let r be a positive element of R less than each r_i and let g be a negative element of G less than each g_i and each $\overset{\mathcal{J}}{\mathrm{ms}}({}_i x)$. Since $0_R \in \bigcap\limits_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}({}_i x)$, we have $|{}_i x_h| < r_i$ $(\forall h \in (\leftarrow, g_i])$ for each $i \in [1,n]_{\mathbb{N}}$. Thus, considering the element $r\mathrm{X}^g$ of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, one notices that $r\mathrm{X}^g \in \bigcap\limits_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}({}_i x) - \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G})$, proving that $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{0_G})$ is not $\overset{\mathcal{J}}{\mathsf{Wt}}$-open.

Conversely, suppose R has a least positive element $r_{\min}$, take an element x of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and let g be a fixed element of G. Note that $\mathsf{W}_{r_{\min}}^{g}(x) \supset x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$. If y is an element of $\mathsf{W}_{r_{\min}}^{g}(x)$, then we get $|y_h - x_h| < r_{\min}$ $(\forall h \in (\leftarrow, g]_G)$, that is, $y_h = x_h$ $(\forall h \in (\leftarrow, g]_G)$, implying $y \doteq x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ and proving that the equation $\mathsf{W}_{r_{\min}}^{g}(x) = x + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ holds true. Therefore, since x and g are arbitrary, we have $\overset{\mathcal{J}}{\mathsf{Wt}} \supset \overset{\mathcal{J}}{\mathsf{St}}$ and $\overset{\mathcal{J}}{\mathsf{Wt}} = \overset{\mathcal{J}}{\mathsf{St}}$ by item (d).

(k) Suppose A is a $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspace of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ whose $\overset{\mathcal{J}}{\mathsf{Wt}}$-interior has an element x, let $U = \bigcap_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}({}_ix)$ be a basic $\overset{\mathcal{J}}{\mathsf{Wt}}$-neighbourhood of x contained in A, let r be a fixed non-zero element of R and let $g_{\max} := \overset{G}{\max}_{i\in[1,n]_{\mathbb{N}}} g_i$. If the sequence

$$\left\{\left(\sum_{g\leqslant g_{\max}} x_g \mathrm{X}^g\right) + nr\mathrm{X}^{2|g_{\max}|}\right\}_{n\in\mathbb{N}} \subset U$$

has a $\overset{\mathcal{J}}{\mathsf{Wt}}$-cluster point y in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, then we get $nr \underset{n\to\infty}{\overset{\overset{R}{\mathrm{Ordt}}}{\dashrightarrow}} y_{2|g_{\max}|}$ by item (a), which is absurd (Proposition 1.65, Item (b)). Hence, the $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspaces A and $\mathrm{Cl}_{\overset{\mathcal{J}}{\mathsf{Wt}}}(A)$ are not countably compact.

(l) One can easily see that the conditions $g_1 \geqslant g_2$ and $r_1 \leqslant r_2$ imply $\mathsf{W}_{r_1}^{g_1}(x) \subset \mathsf{W}_{r_2}^{g_2}(x)$. Suppose this last inclusion holds. If $g_1 < g_2$, then we get

$$\left(\sum_{g<g_2} x_g \mathrm{X}^g\right) + (x_{g_2} + r_2)\,\mathrm{X}^{g_2} \in \mathsf{W}_{r_1}^{g_1}(x) - \mathsf{W}_{r_2}^{g_2}(x)\ ,$$

which is absurd, proving that $g_1 \geqslant g_2$. Likewise, if $r_1 > r_2$, then we get

$$\left(\sum_{g<g_2} x_g \mathrm{X}^g\right) + (x_{g_2} + r_2)\,\mathrm{X}^{g_2} + \left(\sum_{g_2<g\leqslant g_1} x_g \mathrm{X}^g\right) \in \mathsf{W}_{r_1}^{g_1}(x) - \mathsf{W}_{r_2}^{g_2}(x)\ ,$$

which is absurd, proving that $r_1 \leqslant r_2$.

(m) Take a basic $\overset{\mathcal{J}}{\mathsf{Wt}}$-neighbourhood $\bigcap_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}({}_ix)$ of x, let $g := \overset{G}{\max}_{i\in[1,n]_{\mathbb{N}}} g_i$ and let r be a positive element of R so that $r < \overset{R}{\min}_{i\in[1,n]_{\mathbb{N}}} (r_i - \mathsf{p}^{g_i}(x - {}_ix))$. If y is an element of $\mathsf{W}_r^g(x)$, then for each $i \in [1,n]_{\mathbb{N}}$, we have

$$(\forall h \in (\leftarrow, g_i]_G)\ \ |y_h - {}_ix_h| \leqslant |y_h - x_h| + |x_h - {}_ix_h| < r + \mathsf{p}^{g_i}(x - {}_ix) < r_i\,,$$

which gives us $\mathsf{W}_r^g(x) \subset \bigcap_{i=1}^{n} \mathsf{W}_{r_i}^{g_i}({}_ix)$, thus proving the item.

(n) We shall prove the item only for the intervals whose left endpoints are equal to x. Let I be one of these intervals and take arbitrary $z \in I$, $r \in (0_R, \rightarrow)_R$ and $g \in G$. We are to show that $\mathsf{W}_r^g(z) \not\subset I$, which implies the desired result by item item (m). Let r' be a positive element of R less than r and let g' be an element of G less than g, $\overset{\mathcal{J}}{\mathrm{ms}}(x)$ and $\overset{\mathcal{J}}{\mathrm{ms}}(z)$. Consider the element

$$w := -r'\mathrm{X}^{g'} + \sum_{\substack{h\in\mathrm{supp}(z)\\ h\leqslant g}} (r' + z_h)\,\mathrm{X}^h\,.$$

Thus, we get $\mathsf{p}^g(w-z) = r' < r$ and $w \in \mathsf{W}^g_r(z)$, while on the other hand we have $\overset{\mathcal{J}}{\mathrm{ms}}(w-x) = g'$ and

$$\mathrm{pc}\,(w-x) = w_{g'} - x_{g'} = -r' < 0_R\,,$$

that is, $w < x$, implying $w \notin I$. Therefore, we obtain $\mathsf{W}^g_r(z) \not\subset I$, proving the item.

(o) Let x and y be two fixed elements of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and consider a basic $\overset{\mathcal{J}}{\mathsf{Wt}}$-neighbourhood $\mathsf{W}^g_r(x+y)$ of $x+y$ (item (m)). Take a positive element r' of R so that $2r' < r$ (Proposition 1.77). If u and v are elements of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ so that $u \in \mathsf{W}^g_{r'}(x)$ and $v \in \mathsf{W}^g_{r'}(y)$, then we have

$$(\forall h \in (\leftarrow, g]_G)\ \ |(u+v)_h - (x+y)_h| \leqslant |u_h - x_h| + |v_h - y_h| < 2r' < r\,,$$

implying $\mathsf{W}^g_{r'}(x) + \mathsf{W}^g_{r'}(y) \subset \mathsf{W}^g_r(x+y)$ and proving the item.

(q) The sufficient condition follows from item (h). Suppose that $\overset{\mathcal{J}}{\mathsf{Wt}}$ is first-countable and let $\{U_n\}$ be a countable $\overset{\mathcal{J}}{\mathsf{Wt}}$-system of neighbourhoods of 0_R. By item (m), for each $n \in \mathbb{N}$ there are $g_n \in G$ and $r_n \in (0_R, \rightarrow)_R$ such that $\mathsf{W}^{g_n}_{r_n}(0_R) \subset U_n$. Moreover, for all $g \in G$ and $r \in (0_R, \rightarrow)_R$, there is an $n \in \mathbb{N}$ so that $\mathsf{W}^{g_n}_{r_n}(0_R) \subset U_n \subset \mathsf{W}^g_r(0_R)$, leading up to $g_n \geqslant g$ and $r_n \leqslant r$ by item (l). Therefore, the sequence $\{g_n\}$ is cofinal in G and the sequence $\{r_n\}$ is coinitial in $(0_R, \rightarrow)_R$, proving that $\mathrm{ci}\,((0_R, \rightarrow)_R) = \omega = \mathrm{cf}\,(G)$.

□

Example 4.16. The weak topologies

$$\underset{\mathbb{Z},\mathbb{Z}}{\overset{\mathcal{J}}{\mathsf{Wt}}}\,,\ \underset{\mathbb{Q},\mathbb{Z}}{\overset{\mathcal{J}}{\mathsf{Wt}}}\,,\ \underset{\mathbb{R},\mathbb{Z}}{\overset{\mathcal{J}}{\mathsf{Wt}}}\,,\ \underset{\mathbb{Z},\mathbb{Q}}{\overset{\mathcal{J}}{\mathsf{Wt}}}\,,\ \underset{\mathbb{Q},\mathbb{Q}}{\overset{\mathcal{J}}{\mathsf{Wt}}}\,,\ \underset{\mathbb{R},\mathbb{Q}}{\overset{\mathcal{J}}{\mathsf{Wt}}}\,,\ \underset{\mathbb{Z},\mathbb{R}}{\overset{\mathcal{J}}{\mathsf{Wt}}}\,,\ \underset{\mathbb{Q},\mathbb{R}}{\overset{\mathcal{J}}{\mathsf{Wt}}}\,,\ \underset{\mathbb{R},\mathbb{R}}{\overset{\mathcal{J}}{\mathsf{Wt}}}\,,\ \underset{\mathbb{Z},\mathsf{BS}_\ell}{\overset{\mathcal{J}}{\mathsf{Wt}}}\,,\ \underset{\mathbb{Q},\mathsf{BS}_\ell}{\overset{\mathcal{J}}{\mathsf{Wt}}}\ \text{ and }\ \underset{\mathbb{R},\mathsf{BS}_\ell}{\overset{\mathcal{J}}{\mathsf{Wt}}}$$

are all first-countable, where $\mathcal{J}$ is any arithmetic Rayner ideal on each respective ordered group of exponents (Theorem 4.15, Item (h)).

Example 4.17. Since the ordered field of hyperreal numbers ${}^*\mathbb{R}$ (Example 2.43) is not separable with respect to its order topology, since every arithmetic Rayner ideal on an ordered subgroup of $(\mathbb{R}, +_\mathbb{R}, <_\mathbb{R})$ is ω_1-dominated (Example 3.7), and since the set $\mathbb{R}$ is uncountable, we have that the weak topologies on the ordered Rayner rngs of the forms

$$ {}^*\mathbb{R}\,[[\mathrm{X}^G]]\ ,\ {}^*\overset{\mathrm{lf}}{\mathbb{R}}\,[[\mathrm{X}^G]]\ ,\ {}^*\overset{\mathrm{bd}}{\mathbb{R}}_\mathbb{Z}[[\mathrm{X}^G]]\ ,\ \overset{\mathcal{J}}{R}\,[[\mathrm{X}^\mathbb{R}]] $$

are not separable, where G is any ordered subgroup[6] of $(\mathbb{R}, +_\mathbb{R}, <_\mathbb{R})$, R is any ordered rng and $\mathcal{J}$ is any arithmetic Rayner ideal on $\mathbb{R}$ (Theorem 4.15, Item (i)).

Example 4.18. Since the ordered ring $\mathbb{Z}$ has the number 1 as its least positive element, the weak topology coincides with the strong topology on every ordered Rayner ring of the form $\overset{\mathcal{J}}{\mathbb{Z}}\,[[\mathrm{X}^G]]$, where G is any ordered group and $\mathcal{J}$ is any arithmetic Rayner ideal on G (Theorem 4.15, Item (j)).

As it happens, some ordered Rayner rings become topological vector spaces when endowed their canonical left action (cf. Proposition 3.34, Item (d)), as we shall demonstrate in the following theorem. Our proof of that fact is nothing but a generalisation of Shamseddine's proof that the real Levi-Civita field is a topological vector $\mathbb{R}$-space (203), written in different notations.

Theorem 4.19. *Let K be an ordered division ring, let G be a cofinal ordered subgroup of $(K, +_K, <_K)$ and let $\mathcal{J}$ be a left-finite arithmetic Rayner ideal on G.*

(**a**) *The sets of the form*

$$ \mathsf{S}^g(x) := \{y \in \overset{\mathcal{J}}{K}\ [[\mathrm{X}^G]] \mid \mathsf{p}^g(y - x) < g^{-1}\} $$

for $x \in \overset{\mathcal{J}}{K}\ [[\mathrm{X}^G]]$ and $g \in (0_K, \rightarrow)_G$ constitute a local $\overset{\mathcal{J}}{\mathsf{Wt}}$-basis of x;

(**b**) *Take the elements $x \in \overset{\mathcal{J}}{K}\ [[\mathrm{X}^G]]$ and $g, h \in G$. We have $\mathsf{S}^g(x) \subsetneq \mathsf{S}^h(x)$ if, and only if, $g > h$;*

[6] Of course, we also assume that $\mathbb{Z}$ is a Puiseux ordered subgroup of G in the case of the Puiseux rng ${}^*\overset{\mathrm{bd}}{\mathbb{R}}_\mathbb{Z}[[\mathrm{X}^G]]$.

(c) $\left(\overset{\mathcal{J}}{K}[[\mathrm{X}^G]], \overset{\mathcal{J}}{\mathsf{Wt}}\right)$ *is a topological vector K-space when endowed with the left action*

$$\odot : (K, \overset{K}{\mathsf{Ordt}}) \times \left(\overset{\mathcal{J}}{K}[[\mathrm{X}^G]], \overset{\mathcal{J}}{\mathsf{Wt}}\right) \to \left(\overset{\mathcal{J}}{K}[[\mathrm{X}^G]], \overset{\mathcal{J}}{\mathsf{Wt}}\right)$$

given by $(\odot(r,x))_g = (rx)_g \coloneqq rx_g$.

Proof.

(a) Note that each set of the form $\mathsf{S}^g(x) = \mathsf{W}^g_{g^{-1}}(x)$ is $\overset{\mathcal{J}}{\mathsf{Wt}}$-open. Take a basic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open neighbourhood $\mathsf{W}^g_r(x)$ of x (Theorem 4.15, Item (m)). Since G is cofinal in K, there is a $g_0 \in G$ such that $\overset{K}{\max}\{g, r^{-1}\} \leqslant g_0$, and if z is an element of $\mathsf{S}^{g_0}(x)$, then we get

$$\mathsf{p}^g(z-x) = \overset{K}{\max_{h\in(\leftarrow,g]}} |z_h - x_h| \leqslant \overset{K}{\max_{h\in(\leftarrow,g_0]}} |z_h - x_h| < g_0^{-1} \leqslant r\,,$$

which gives us $z \in \mathsf{W}^g_r(x)$. Hence, we have $\mathsf{S}^{g_0}(x) \subset \mathsf{W}^g_r(x)$, thus proving the item.

(b) The condition $g \geqslant h$ clearly implies $\mathsf{S}^g(x) \subset \mathsf{S}^h(x)$. If $\mathsf{S}^g(x) \subsetneq \mathsf{S}^h(x)$ and $g \leqslant h$, then $\mathsf{S}^g(x) \subsetneq \mathsf{S}^h(x) \subset \mathsf{S}^g(x)$, which is absurd, proving the necessary condition of the item. On the other hand, if $g > h$, then $g^{-1} < h^{-1}$ and note that

$$\left(\sum_{g'<h} x_{g'}\mathrm{X}^{g'}\right) + (x_h + g^{-1})\,\mathrm{X}^h \in \mathsf{S}^h(x) - \mathsf{S}^g(x)\,,$$

which implies $\mathsf{S}^g(x) \neq \mathsf{S}^h(x)$, proving the item.

(c) We know that $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ is a vector K-space when endowed with the action $\odot$ (Proposition 3.34, Item (d)). We shall prove that $\odot$ is continuous. Let $r \in K$ and $x \in \overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ be fixed elements and let $\mathsf{S}^g(rx)$ be a basic $\overset{\mathcal{J}}{\mathsf{Wt}}$-neighbourhood of rx (item (a)). Note that we have $2_K \neq 0_K \neq 4_K$ (Proposition 2.17) and also $2_K, 2_K^{-1}, 4_K, 4_K^{-1} \in \mathrm{Z}(K)$ (Proposition 2.17).

The proof shall be divided in three cases:

- ▷ Suppose that $r = 0_K = \mathsf{p}^g(x)$. Thus, if $r' \in (-1_K, 1_K)_K$ and if $x' \in \mathsf{S}^g(x)$, then we get (Proposition 4.13)

  $$\mathsf{p}^g(r'x') = |r'|\,\mathsf{p}^g(x') < \mathsf{p}^g(x') \leqslant \mathsf{p}^g(x'-x) + \mathsf{p}^g(x) = \mathsf{p}^g(x'-x) < g^{-1}\,,$$

  implying $r'x' \in \mathsf{S}^g(0_K) = \mathsf{S}^g(rx)$ and proving that

  $$\odot\,\langle (-1_K, 1_K)_K \times \mathsf{S}^g(x)\rangle \subset \mathsf{S}^g(rx)\ .$$

- ▷ Suppose that $r = 0_K \neq \mathsf{p}^g(x)$ and let r_0 be the positive element of K given by

  $$r_0 := \overset{K}{\min}\left\{ \frac{1_K}{2_K}\,,\, \frac{g^{-1}(\mathsf{p}^g(x))^{-1}}{2_K} \right\}\ .$$

  If $r' \in (-r_0, r_0)_K$ and if $x' \in \mathsf{S}^g(x)$, then

  $$\begin{aligned} \mathsf{p}^g(r'x') &\leqslant \mathsf{p}^g(r'\,(x'-x)) + \mathsf{p}^g(r'x) \\ &= |r'|\,\mathsf{p}^g(x'-x) + |r'|\,\mathsf{p}^g(x) \\ &< \frac{1_K}{2_K}\,g^{-1} + \frac{g^{-1}(\mathsf{p}^g(x))^{-1}}{2_K}\,\mathsf{p}^g(x) = g^{-1}\,, \end{aligned}$$

  implying $r'x' \in \mathsf{S}^g(0_K) = \mathsf{S}^g(rx)$ and proving that

  $$\odot\,\langle (-r_0, r_0)_K \times \mathsf{S}^g(x)\rangle \subset \mathsf{S}^g(rx)\ .$$

- ▷ Suppose that $r \neq 0_K$, let g_0 be a positive element of G greater than $2g$ and $2g\,|r|$, and let r_0 be the positive element of K given by

  $$r_0 := \begin{cases} \frac{1_K}{2_K} & \text{if } \mathsf{p}^g(x) = 0_K, \\ \overset{K}{\min}\left\{ \frac{1_K}{2_K}\,,\, \frac{g^{-1}(\mathsf{p}^{g_0}(x))^{-1}}{4_K} \right\} & \text{otherwise.} \end{cases}$$

  Note that $g_0 > 2g > g$. Finally, if we take any $r' \in (r - r_0, r + r_0)_K$ and $x' \in \mathsf{S}^{g_0}(x)$, then we get

  $$|r'-r|\,\mathsf{p}^{g_0}(x) < \begin{cases} r_0 \cdot 0_K = 0_K < \frac{g^{-1}}{4_K} & \text{if } \mathsf{p}^g\,(x) = 0_K, \\ r_0\,\mathsf{p}^{g_0}(x) \leqslant \frac{g^{-1}(\mathsf{p}^{g_0}(x))^{-1}}{4_K}\ \mathsf{p}^{g_0}(x) = \frac{g^{-1}}{4_K} & \text{otherwise.} \end{cases}$$

and

$$\begin{aligned}\mathsf{p}^{g}(r'x'-rx) &= \mathsf{p}^{g}((r'-r)\,(x'-x)+(r'-r)\,x+r\,(x'-x))\\ &\leqslant |r'-r|\,\mathsf{p}^{g_0}(x'-x)+|r'-r|\,\mathsf{p}^{g_0}(x)+|r|\,\mathsf{p}^{g_0}(x'-x)\\ &< \frac{1_K}{2_K}\,g_0^{-1}+\frac{g^{-1}}{4_K}+|r|\,g_0^{-1}\\ &< \frac{1_K}{2_K}\,\frac{g^{-1}}{2_K}+\frac{g^{-1}}{4_K}+|r|\,\frac{|r|^{-1}\,g^{-1}}{2_K}=g^{-1}\,,\end{aligned}$$

implying $r'x'\in\mathsf{S}^{g}(rx)$ and proving that

$$\odot\,\langle (r-r_0,r+r_0)_K\times\mathsf{S}^{g_0}(x)\rangle\subset\mathsf{S}^{g}(rx)\ .$$

$\square$

Example 4.20. The underlying additive groups of the ordered Rayner fields

$$\mathbb{Q}[[\mathrm{X}^{\mathbb{Z}}]],\ \mathbb{R}[[\mathrm{X}^{\mathbb{Z}}]],\ \overset{\mathrm{lf}}{\mathbb{Q}}\,[[\mathrm{X}^{\mathbb{Q}}]],\ \overset{\mathrm{lf}}{\mathbb{R}}\,[[\mathrm{X}^{\mathbb{Q}}]],\ \overset{\mathrm{lf}}{\mathbb{R}}\,[[\mathrm{X}^{\mathbb{R}}]],\ \overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}\,[[\mathrm{X}^{\mathbb{Q}}]],\ \overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}\,[[\mathrm{X}^{\mathbb{Q}}]],$$
$$\overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}\,[[\mathrm{X}^{\mathcal{D}_d}]],\ \overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}\,[[\mathrm{X}^{\mathcal{D}_d}]],\ \overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}\,[[\mathrm{X}^{\mathcal{P}}]]\ \text{ and }\ \overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}\,[[\mathrm{X}^{\mathcal{P}}]]$$

are all topological vector spaces when endowed with their respective weak topologies and their respective canonical left actions (Theorem 4.19, Item (c)).

4.3 Conditions for $\overset{\mathcal{J}}{\mathsf{Wt}}$ to be metrizable and separable

In this section, we shall prove a theorem that provides equivalent conditions for the weak topology $\overset{\mathcal{J}}{\mathsf{Wt}}$ to be metrizable and separable in the case in which the arithmetic Rayner ideal $\mathcal{J}$ on G is left-finite.

Theorem 4.21. *Let R be an ordered rng that has no least positive element, let G be an ordered group and let $\mathcal{J}$ be a left-finite arithmetic Rayner ideal on G. The following conditions are equivalent:*

(a) $\overset{\mathcal{J}}{\mathsf{Wt}}$ *is metrizable and separable;*

(b) $\overset{\mathcal{J}}{\mathsf{Wt}}$ *is second-countable;*

(c) $\overset{\mathcal{J}}{\mathsf{Wt}}$ *is separable;*

(d) $\overset{R}{\mathsf{Ordt}}$ *is separable and* $|G|=\omega$.

Proof. The result (a)⇒(b) follows from Proposition 1.47, we have (b)⇒(c) by Proposition 1.39 and we have (c)⇒(d) by Item (d) of Proposition 3.10 and Item (i) of Theorem 4.15.

(d)⇒**(a)**: The underlying idea of the proof is to show that the topology $\overset{\mathcal{J}}{\mathsf{Wt}}$ is second-countable and then apply Item (p) of Theorem 4.15, Theorem 1.48 and Proposition 1.47 to conclude that $\overset{\mathcal{J}}{\mathsf{Wt}}$ is metrizable and separable. Let S be a countable $\overset{R}{\mathrm{Ordt}}$-dense subset of R containing 0_R. We shall show that the sets $\mathsf{W}^g_r(q)$ for $q \in S\,[\mathrm{X}^G]$, $g \in G$ and $r \in (0_R, \rightarrow)_S$ form a countable basis of $\overset{\mathcal{J}}{\mathsf{Wt}}$. Firstly, there is only a countable number of these sets, given that the sets $S\,[\mathrm{X}^G]$, G and $(0_R, \rightarrow)_S$ are countable (Proposition 3.39). Take a basic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open neighbourhood $\mathsf{W}^g_r(x)$ of an arbitrary element $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. Let r_0 be a positive element of R so that $2r_0 < r$ (Proposition 1.77), let r_1 be an element of the inhabited intersection $S \cap (0_R, r_0)_R$, and, for each element $h \in \operatorname{supp}(x) \cap (\leftarrow, g]_G$, let q_h be an element of the inhabited intersection $S \cap (x_h - r_1, x_h + r_1)_R$. Lastly, let $q \coloneqq \sum_{\substack{h \in \operatorname{supp}(x) \\ h \leqslant g}} q_h \mathrm{X}^h \in S\,[\mathrm{X}^G]$. By construction, we have $x \in \mathsf{W}^g_{r_1}(q)$, and for each $y \in \mathsf{W}^g_{r_1}(q)$ and each $h \in (\leftarrow, g]_G$, we get

$$|y_h - x_h| \leqslant |y_h - q_h| + |q_h - x_h| < r_1 + r_1 < r_0 + r_0 < r\,,$$

that is, $y \in \mathsf{W}^g_r(x)$, implying the inclusion $\mathsf{W}^g_{r_1}(q) \subset \mathsf{W}^g_r(x)$ and proving that $\overset{\mathcal{J}}{\mathsf{Wt}}$ is second-countable (Theorem 4.15, Item (m)).

□

Example 4.22. Since the ordered fields $\mathbb{Q}$ and $\mathbb{R}$ have no least positive element and are separable with respect to their order topologies, and since the sets $\mathbb{Z}$, $\mathbb{Q}$, BS_ℓ, $\mathcal{D}_d$ and $\mathcal{P}$ are countable, we have that the weak topologies on the ordered Rayner rings

$$\mathbb{Q}\,[[\mathrm{X}^{\mathbb{Z}}]]\,,\ \mathbb{R}\,[[\mathrm{X}^{\mathbb{Z}}]]\,,\ \overset{\mathrm{lf}}{\mathbb{Q}}\,[[\mathrm{X}^{\mathbb{Q}}]]\,,\ \overset{\mathrm{lf}}{\mathbb{R}}\,[[\mathrm{X}^{\mathbb{Q}}]]\,,\ \overset{\mathrm{lf}}{\mathbb{Q}}\,[[\mathrm{X}^{\mathsf{BS}_\ell}]]\,,\ \overset{\mathrm{lf}}{\mathbb{R}}\,[[\mathrm{X}^{\mathsf{BS}_\ell}]]\,,$$
$$\overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]\,,\ \overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]\,,\ \overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{D}_d}]]\,,\ \overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{D}_d}]]\,,\ \overset{\mathrm{bd}}{\mathbb{Q}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]] \text{ and } \overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]$$

are all metrizable and separable.

4.4 Conditions for $\overset{\mathcal{J}}{\mathsf{Wt}}$ to be connected; An Intermediate Value Theorem

In this section, we shall consider the property of connectedness in relation to the weak topology $\overset{\mathcal{J}}{\mathsf{Wt}}$ on an ordered Rayner rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, and such considerations shall naturally lead us to a new version of the Intermediate Value Theorem for $\overset{\mathcal{J}}{\mathsf{Wt}}$ that is valid when R are $\mathcal{J}$ satisfy certain conditions.

Theorem 4.23. *Let R be an ordered rng, let G be an ordered group and let $\mathcal{J}$ be an arithmetic Rayner ideal on G. If $\overset{\mathcal{J}}{\mathsf{Wt}}$ is connected, then $\overset{R}{\mathrm{Ord}}\mathsf{t}$ is connected. Conversely, if $\overset{R}{\mathrm{Ord}}\mathsf{t}$ is connected and $\mathcal{J}$ is left-finite, then we have:*

(**a**) *The $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspaces of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ of the forms $x+\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ and $x+\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)$ for $x \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ and $g \in G$ are connected. In particular, the topology $\overset{\mathcal{J}}{\mathsf{Wt}}$ is connected;*

(**b**) *The basic $\overset{\mathcal{J}}{\mathsf{Wt}}$-open subspaces $\mathsf{W}_r^g(x)$ are connected;*

(**c**) *For all $x, y \in \overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ so that $x<y$, the $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspaces of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ given by the intervals*

$$(\leftarrow, y),(\leftarrow, y],(x, y),[x, y),(x, y],[x, y],(x, \rightarrow) \text{ and } [x, \rightarrow)$$

are connected.

Proof. Since the canonical 0_G-projection $\mathrm{pr}_{0_G}:\left(\overset{\mathcal{J}}{R}[[\mathrm{X}^G]], \overset{\mathcal{J}}{\mathsf{Wt}}\right) \rightarrow(R, \overset{R}{\mathrm{Ord}}\mathsf{t})$ given by $\mathrm{pr}_{0_G}(x) \coloneqq x_{0_G}$ is continuous (Theorem 4.15, Item (a)), if $\overset{\mathcal{J}}{\mathsf{Wt}}$ is connected, then $\overset{R}{\mathrm{Ord}}\mathsf{t}$ is connected (Proposition 1.42). Assume $\overset{R}{\mathrm{Ord}}\mathsf{t}$ is connected and the Rayner ideal $\mathcal{J}$ is left-finite in the following items.

(**a**) and (**b**): It suffices to prove the items for $x=0_R$ (Theorem 1.57, Item (a)). Consider the two $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspaces $S \coloneqq \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^g)$ and $W \coloneqq \mathsf{W}_r^g(0_R)$ of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, where $g \in G$ and $r \in(0_R, \rightarrow)_R$, let

$${}_1x, {}_2x \in R[\mathrm{X}^G] \cap S \quad \text{and} \quad {}_1y, {}_2y \in R[\mathrm{X}^G] \cap W$$

be generalised polynomials, let

$$g_1 g_2 \dots g_m \in \operatorname{supp}({}_1x) \cup \operatorname{supp}({}_2x) \quad \text{and} \quad h_1 h_2 \dots h_n \in \operatorname{supp}({}_1y) \cup \operatorname{supp}({}_2y)$$

be the respective increasing finite sequences of elements of those unions and let k be the greatest number in the interval $[1,n]_{\mathbb{N}}$ such that $h_k \leqslant g$. Consider the functions

$$f : \left(R, \overset{R}{\mathsf{Ordt}}\right)^{\overset{m}{\times}} \to \left(S, \overset{\mathcal{J}}{\mathsf{Wt}}\restriction_S\right)$$

and

$$g : \left((-r,r)_R, \overset{R}{\mathsf{Ordt}}\restriction_{(-r,r)_R}\right)^{\overset{k}{\times}} \times \left(R, \overset{R}{\mathsf{Ordt}}\right)^{\overset{n-k}{\times}} \to \left(W, \overset{\mathcal{J}}{\mathsf{Wt}}\restriction_W\right)$$

given by

$$f(r_1 \dots r_m) \coloneqq \sum_{i=1}^{m} r_i \mathrm{X}^{g_i} \quad \text{and} \quad g(r_1 \dots r_n) \coloneqq \sum_{i=1}^{n} r_i \mathrm{X}^{h_i}.$$

Those functions are continuous (Theorem 4.15, Item (b)) and their domains are connected (Propositions 1.44 and 1.54), implying that the image $\operatorname{Im}(f)$ forms a connected $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_S$-subspace of S and the image $\operatorname{Im}(g)$ forms a connected $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_W$-subspace of W (Proposition 1.42). Since ${}_1x, {}_2x \in \operatorname{Im}(f)$ and ${}_1y, {}_2y \in \operatorname{Im}(g)$, and since ${}_1x$, ${}_2x$, ${}_1y$, ${}_2y$ were arbitrarily taken, we have

$$R[\mathrm{X}^G] \cap S \subset C_1 \quad \text{and} \quad R[\mathrm{X}^G] \cap W \subset C_2,$$

where C_1 is the $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_S$-connected component that extends $\operatorname{Im}(f)$ and where C_2 is the $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_W$-connected component that extends $\operatorname{Im}(g)$. Since C_1 is $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_S$-closed and C_2 is $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_W$-closed (Proposition 1.40), and since $\mathcal{J}$ is left-finite, we get (Theorem 4.2, Item (c))

$$S = \operatorname{Cl}_{\overset{\mathcal{J}}{\mathsf{St}}\restriction_S}\left(R[\mathrm{X}^G] \cap S\right) \subset \operatorname{Cl}_{\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_S}\left(R[\mathrm{X}^G] \cap S\right) \subset C_1$$

and

$$W = \operatorname{Cl}_{\overset{\mathcal{J}}{\mathsf{St}}\restriction_W}\left(R[\mathrm{X}^G] \cap W\right) \subset \operatorname{Cl}_{\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_W}\left(R[\mathrm{X}^G] \cap W\right) \subset C_2,$$

leading up to $C_1 = S$ and $C_2 = W$ and proving that the topologies $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_S$ and $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_W$ are connected. The proof that the topology $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^g)}$ is connected is analogous. Lastly, since $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{s_1}) \cap \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{s_2}) \neq \emptyset$ $(\forall s_1, s_2 \in G)$ and $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]] = \bigcup_{s \in G} \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^s)$, the topology $\overset{\mathcal{J}}{\mathsf{Wt}}$ is connected (Proposition 1.41).

(c) Firstly, we shall prove that the interval $(\leftarrow, y)$ forms a connected $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspace of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$. Let us denote $\mu_{uv} \coloneqq \overset{\mathcal{J}}{\mathrm{ms}}(v-u)$ for all $u, v \in R[\mathrm{X}^G]$, and let $a, b \in R[\mathrm{X}^G] \cap (\leftarrow, y)$ be two fixed generalised polynomials so that $a < b$. Thus, we have the inequality $a < b < y$ and $\mu_{ay} = \overset{\smile}{\min}_{G} \{\mu_{ab}, \mu_{by}\}$ (Theorem 3.82, Item (f)). Let $g_1 g_2 \dots g_m$ and $h_1 h_2 \dots h_n$ be the increasing finite sequences of elements of the finite sets $(\operatorname{supp}(a) \cup \operatorname{supp}(b)) \cap (\mu_{ay}, \rightarrow)_G$ and $(\operatorname{supp}(a) \cup \operatorname{supp}(b)) \cap \left(\overset{\smile}{\max}_{G} \{\mu_{ab}, \mu_{by}\}, \rightarrow\right)_G$, respectively. We define a function f in three possible ways depending on how the elements μ_{ay}, μ_{ab} and μ_{by} are arranged in G:

Case 1. If $\mu_{ay} = \mu_{ab} = \mu_{by}$, then let

$$f : (\leftarrow, y_{\mu_{ay}})_R \times R^{n}_{\times} \longrightarrow \left((\leftarrow, y), \overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow, y)}\right)$$

be the function given by

$$f(r, t_1 \dots t_n) := \underbrace{\Big(\sum_{g < \mu_{ay}} y_g \mathrm{X}^g\Big)}_{= \sum_{g < \mu_{ay}} a_g \mathrm{X}^g = \sum_{g < \mu_{ay}} b_g \mathrm{X}^g} + r\mathrm{X}^{\mu_{ay}} + \Big(\sum_{i=1}^{n} t_i \mathrm{X}^{h_i}\Big).$$

Case 2. If $\mu_{ay} = \mu_{by} < \mu_{ab}$, then let

$$f : R \times R^{n}_{\times} \longrightarrow \left((\leftarrow, y), \overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow, y)}\right)$$

be the function given by

$$f(r, t_1 \dots t_n) := \underbrace{\Big(\sum_{g < \mu_{ab}} a_g \mathrm{X}^g\Big)}_{= \sum_{g < \mu_{ab}} b_g \mathrm{X}^g} + r\mathrm{X}^{\mu_{ab}} + \Big(\sum_{i=1}^{n} t_i \mathrm{X}^{h_i}\Big).$$

Case 3. If $\mu_{ay} = \mu_{ab} < \mu_{by}$, then let

$$f : (\leftarrow, y_{\mu_{ay}})_R \times R^{m}_{\times} \longrightarrow \left((\leftarrow, y), \overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow, y)}\right)$$

be the function given by

$$f(r, s_1 \dots s_m) := \underbrace{\Big(\sum_{g < \mu_{ay}} y_g \mathrm{X}^g\Big)}_{= \sum_{g < \mu_{ay}} a_g \mathrm{X}^g = \sum_{g < \mu_{ay}} b_g \mathrm{X}^g} + r\mathrm{X}^{\mu_{ay}} + \Big(\sum_{i=1}^{m} s_i \mathrm{X}^{g_i}\Big).$$

In any case, the function f is continuous (Theorem 4.15, Item (b)) and its domain is connected (Propositions 1.44 and 1.54), implying that both the image $\mathrm{Im}\,(f)$ and its $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow,y)}$-closure $\mathrm{Cl}_{\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow,y)}}(\mathrm{Im}\,(f))$ form connected $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow,y)}$-subspaces of $(\leftarrow, y)$ (Proposition 1.42 and 1.40).

In Case 1 and Case 2, we clearly have $a, b \in \mathrm{Im}\,(f)$, and we have $a \in \mathrm{Im}\,(f)$ in Case 3. We shall show that $b \in \mathrm{Cl}_{\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow,y)}}(\mathrm{Im}\,(f))$ in Case 3. In consideration of the assumption that the order topology $\overset{R}{\mathsf{Ordt}}$ is connected, it is clear that R has no least positive element. If $\mathsf{W}^g_r(b)$ is a basic $\overset{\mathcal{J}}{\mathsf{Wt}}$-neighbourhood of b so that $g \geqslant \mu_{ay}$ (without loss of generality), and if r' is a positive element of R so that $r' < r$, then we have

$$(\forall h \in G - \{\mu_{ay}\}) \;\; \left(f\left(b_{\mu_{ay}} - r', b_{g_1} \ldots b_{g_m}\right)\right)_h = b_h$$

and

$$\left|\left(f\left(b_{\mu_{ay}} - r', b_{g_1} \ldots b_{g_m}\right)\right)_{\mu_{ay}} - b_{\mu_{ay}}\right| = \left|\left(b_{\mu_{ay}} - r'\right) - b_{\mu_{ay}}\right| = r' < r\,,$$

which gives us $f\left(b_{\mu_{ay}} - r', b_{g_1} \ldots b_{g_m}\right) \in \mathsf{W}^g_r(b)$ and proves that the condition $b \in \mathrm{Cl}_{\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow,y)}}(\mathrm{Im}\,(f))$ holds in Case 3.

Since a and b were arbitrarily taken, we have $R\,[\mathrm{X}^G] \cap (\leftarrow, y) \subset C$, where C is the $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow,y)}$-connected component that extends $\mathrm{Im}\,(f)$. Since C is $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow,y)}$-closed, and since $\mathcal{J}$ is left-finite, we get (Theorem 4.2, Item (c))

$$(\leftarrow, y) = \mathrm{Cl}_{\overset{\mathcal{J}}{\mathsf{St}}\restriction_{(\leftarrow,y)}}\left(R\,[\mathrm{X}^G] \cap (\leftarrow, y)\right) \subset \mathrm{Cl}_{\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow,y)}}\left(R\,[\mathrm{X}^G] \cap (\leftarrow, y)\right) \subset C\,,$$

leading up to $C = (\leftarrow, y)$ and proving that the topology $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(\leftarrow,y)}$ is connected. The proof that $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{(x,\rightarrow)}$ is connected is analogous.

Note that $y \in \mathrm{Cl}_{\overset{\mathcal{J}}{\mathsf{St}}}((\leftarrow, y)) \subset \mathrm{Cl}_{\overset{\mathcal{J}}{\mathsf{Wt}}}((\leftarrow, y))$ (Theorem 4.10, Item (a)). Thus, the interval $(\leftarrow, y]$ is a connected $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspace of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, and so is the interval $[x, \rightarrow)$ by the same token. Lastly, the four intervals $(x, y) = (x, \rightarrow) \cap (\leftarrow, y)$, $[x, y) = [x, \rightarrow) \cap (\leftarrow, y)$, $(x, y] = (x, \rightarrow) \cap (\leftarrow, y]$ and $[x, y] = [x, \rightarrow) \cap (\leftarrow, y]$ are all connected $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspaces of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$.

$\square$

Example 4.24. Since the ordered field $\mathbb{R}$ is connected with respect to its order topology, the weak topologies on the ordered Rayner rings

$$\mathbb{R}\,[[\mathrm{X}^{\mathbb{Z}}]],\ \overset{\mathrm{lf}}{\mathbb{R}}\,[[\mathrm{X}^{\mathbb{Q}}]]\,,\ \overset{\mathrm{lf}}{\mathbb{R}}\,[[\mathrm{X}^{\mathbb{R}}]]\,,\ \overset{\mathrm{lf}}{\mathbb{R}}\,[[\mathrm{X}^{\mathsf{BS}_\ell}]]\,,\ \overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]\,,\ \overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{D}_d}]] \text{ and } \overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]$$

are connected (Theorem 4.23, Item (a)).

The standard proof of the classical Intermediate Value Theorem on $\mathbb{R}$ relies on two crucial facts: every interval $[a,b]_{\mathbb{R}}$ in $\mathbb{R}$ is a connected topological subspace of $\mathbb{R}$, and, in turn, every connected subspace of $\mathbb{R}$ is order-convex in $\mathbb{R}$ (Definition 1.24). The argument goes as follows: taking a continuous function $f : [a,b]_{\mathbb{R}} \to \mathbb{R}$ so that $f(a) < f(b)$ and taking an intermediate element $y \in (f(a), f(b))_{\mathbb{R}}$, we have that the image $f\langle [a,b]_{\mathbb{R}}\rangle$ is a connected subspace of $\mathbb{R}$ (Proposition 1.42), implying that it is also order-convex in $\mathbb{R}$, and, hence, we finally obtain $y \in f\langle [a,b]_{\mathbb{R}}\rangle$, that is, there is a $c \in (a,b)_{\mathbb{R}}$ so that[7] $y = f(c)$.

It is reasonable to project that the same strategy may work to derive a version of the Intermediate Value Theorem for the weak topology $\overset{\mathcal{J}}{\mathsf{Wt}}$. We already know that every interval in $\overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]]$ is a connected $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspace of $\overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]]$ when the order topology $\overset{R}{\mathsf{Ordt}}$ on R is connected and the Rayner ideal $\mathcal{J}$ is left-finite (Theorem 4.23, Item (c)). With that, now we would need only to show that every connected $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspace of $\overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]]$ is order-convex in $\overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]]$ in that case. Unfortunately, that statement is false, as the following example shows:

Example 4.25. Consider the real Levi-Civita field $\mathcal{R} = \overset{\mathrm{lf}}{\mathbb{R}}\,[[\mathrm{X}^{\mathbb{Q}}]]$. The basic $\overset{\overset{\mathrm{lf}}{\mathrm{P}(\mathbb{Q})}}{\underset{\mathbb{R},\mathbb{Q}}{\mathsf{Wt}}}$-open subspace $\mathsf{W}^1_2(0)$ is connected (Theorem 4.23, Item (b)), we have $0, 1 \in \mathsf{W}^1_2(0)$, but the positive infinitesimal $2\mathrm{X}$ is not in $\mathsf{W}^1_2(0)$. Thus, the set $\mathsf{W}^1_2(0)$ is not order-convex in $\mathcal{R}$.

It turns out that, by controlling the order of magnitude of the elements of a connected $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspace C and by relaxing the notion of order-convexity for subsets of $\overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]]$, we shall be able to prove that C is order-convex in a sense.

[7] Note that $c \notin \{a,b\}$, for $f(c) = y \notin \{f(a), f(b)\}$.

Definition 4.26. Let R be an ordered rng, let G be an ordered group, let $\mathcal{J}$ be an arithmetic Rayner ideal on G and let g_0 be a fixed element of G. A subset S of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ is **order-convex (in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$) modulo** $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})$ if for all $x, y \in S$ so that $x < y$ and for every $z \in (x, y)$, there is a $z' \in S$ such that $z' \doteq z + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})$.

One may intuitively think that the notion of order-convexity modulo $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})$ for the set S essentially means that any intermediate element z between two elements $x < y$ of S is "fairly close" to an element z' of S, so that the order of magnitude of the difference $z' - z$ is strictly lower than the order of magnitude of an element of the form $r\mathrm{X}^{g_0}$. For instance, if R is an ordered ring and if $g_0 \geqslant 0_G$, then z is infinitely close to z' in $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ (Definition 2.42).

Proposition 4.27. *Let R be an ordered rng that has no least positive element, let G be an ordered group, let $\mathcal{J}$ be an arithmetic Rayner ideal on G and let g_0 be a fixed element of G. If C is a connected $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspace of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ contained in $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$, then C is order-convex modulo $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})$.*

Proof. Take $x, y \in C$ so that $x < y$ and suppose there is a $z \in (x, y)$ such that $(z + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})) \cap C = \emptyset$. Since $x < z < y$, we have $z \in \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$ (Theorem 3.82, Items (d) and (e)), and since $C \subset \overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$, we get $w_{g_0} \neq z_{g_0}$ $(\forall w \in C)$. For each $w \in C \cap (\leftarrow, z)$ (resp. $w \in C \cap (z, \rightarrow)$), let $r(w)$ be a positive element of R such that $r(w) < z_{g_0} - w_{g_0}$ (resp. $r(w) < w_{g_0} - z_{g_0}$), and consider the $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_C$-open sets U and V given by

$$U := C \cap \bigcup_{w \in C \cap (\leftarrow, z)} \mathsf{W}^{g_0}_{r(w)}(w) \quad \text{and} \quad V := C \cap \bigcup_{w \in C \cap (z, \rightarrow)} \mathsf{W}^{g_0}_{r(w)}(w)\ .$$

Note that $C = U \cup V$, $x \in U$ and $y \in V$. If $u \in U \cap V$, then there are ${}_1w \in C \cap (\leftarrow, z)$ and ${}_2w \in C \cap (z, \rightarrow)$ such that $u \in \mathsf{W}^{g_0}_{r({}_1w)}({}_1w) \cap \mathsf{W}^{g_0}_{r({}_2w)}({}_2w)$, which implies

$$u_{g_0} - {}_1w_{g_0} \leqslant |u_{g_0} - {}_1w_{g_0}| < r({}_1w)\ , \quad {}_2w_{g_0} - u_{g_0} \leqslant |u_{g_0} - {}_2w_{g_0}| < r({}_2w)$$

and

$$u_{g_0} < {}_1w_{g_0} + r({}_1w) < z_{g_0} < {}_2w_{g_0} - r({}_2w) < u_{g_0}\ ,$$

which is absurd. Therefore, U and V are disjoint, inhabited, $\overset{\mathcal{J}}{\mathsf{Wt}}\restriction_C$-open sets that cover C, contradicting the connectedness of C.

□

Theorem 4.28 (An Intermediate Value Theorem for $\overset{\mathcal{J}}{\mathsf{Wt}}$). *Let R be an ordered rng that has no least positive element, let G be an ordered group, let $\mathcal{J}$ be an arithmetic Rayner ideal on G and let g_0 be a fixed element of G. If C is a connected $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspace of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, if*

$$f : \left(C, \overset{\mathcal{J}}{\mathsf{Wt}}\restriction_C\right) \to \left(\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0}), \overset{\mathcal{J}}{\mathsf{Wt}}\restriction_{\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})}\right)$$

is a continuous function, and if a and b are elements of C so that $f(a) < f(b)$, then for every $y \in (f(a), f(b))$ there is a $c \in C$ such that $f(c) \doteq y + \overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})$.

Proof. The image $f\langle C\rangle$ is a connected $\overset{\mathcal{J}}{\mathsf{Wt}}$-subspace of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ contained in $\overset{\mathcal{J}}{\mathrm{O}}(\mathrm{X}^{g_0})$ (Proposition 1.42), and therefore it is order-convex modulo $\overset{\mathcal{J}}{\mathrm{o}}(\mathrm{X}^{g_0})$ (Proposition 4.27).

□

Example 4.29. Consider the real Levi-Civita field $\mathcal{R} = \overset{\mathrm{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]$ and, omitting the ideal $\overset{\mathrm{lf}}{\mathrm{P}}(\mathbb{Q})$ in the big-O and little-O notations, consider the function

$$f : \left(\mathrm{O}(\mathrm{X}^0), \mathsf{Wt}\restriction_{\mathrm{O}(\mathrm{X}^0)}\right) \to \left(\mathrm{O}(\mathrm{X}^0), \mathsf{Wt}\restriction_{\mathrm{O}(\mathrm{X}^0)}\right)$$

given by $f(x) := x_0$, where Wt is the weak topology on $\mathcal{R}$. Note that f is continuous, given that for all $x \in \mathrm{O}(\mathrm{X}^0)$ we have

$$(\forall g \in [0,\infty)_{\mathbb{Q}})\,(\forall r \in (0,\infty)_{\mathbb{R}})\;\; f\langle \mathsf{W}^g_r(x)\rangle = (x_0 - r, x_0 + r)_R \subset \mathsf{W}^g_r(x_0) = \mathsf{W}^g_r(f(x)) \,.$$

Thus, since $\mathbb{R}$ has no least positive element, since $\mathrm{O}(\mathrm{X}^0)$ is a connected Wt-subspace of $\mathcal{R}$ (Theorem 4.23, Item (a)), and since $\mathrm{Im}(f) \subset \mathrm{O}(\mathrm{X}^0)$, the Intermediate Value Theorem we have just proved holds for f. In particular, we have $f(0) = 0 < 2 = f(2)$, and the element $1 + \mathrm{X}$, which clearly is not in the image $\mathrm{Im}(f)$, is in the interval $(0,2)_{\mathcal{R}}$. Hence, according to the theorem, there must be a $c \in \mathrm{O}(\mathrm{X}^0)$ so that $f(c) \doteq 1 + \mathrm{X} + \mathrm{o}(\mathrm{X}^0)$. Indeed, any c in the set $1 + \mathrm{o}(\mathrm{X}^0)$ will do.

The example above is certainly rather eccentric, and it may be taken as a peripheral curiosity. Unfortunately, other non-trivial examples of applications of Theorem 4.28 are hard to find at this stage of the discussion. Other versions of the Intermediate Value Theorem that are much more applicable to ordinary situations were obtained for the classical Hahn, Levi-Civita and Puiseux fields (209, 210, 208, 16), but these theorems are not directly related to the weak topologies on these ordered Rayner fields, and they are strict, in the sense that the functions f of which they concern assume every value in the interval $(f(a), f(b))$.

Final Considerations

In the current research, we have generalised Rayner's work on power series fields in order to obtain a general class of power series rngs, and we have determined several direct connections between the properties of the Rayner rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$, the rng R, the ordered group G and the Rayner ideal $\mathcal{J}$ on G. Our work builds on the works of many mathematicians over the span of almost two centuries, too many to mention in full in these final considerations, which include Puiseux, Levi-Civita, Hahn, Mac Lane, Rayner, Berz, Shamseddine, Krapp, Kuhlmann and Serra. Their studies provide much information on the mathematical systems of formal power series considered in this thesis, and we have sought not only to address some of the gaps they had left in their considerations, but also to expand the domain of applicability of the theory.

The need to generalise Rayner's construction emanates from the realisation that infinitely many Rayner rngs that are not fields have rich mathematical structures, and many of such structures had not been previously considered in the literature. Our approach consists in providing generalisations to many basic definitions and results concerning the Theory of Rayner Ideals and the Theory of Levi-Civita Fields, and it also employs new simplifying notations that bring out the arithmetic and valuation-theoretic elements of the discussion. The scarcity of existing research on Rayner fields was the sole limiting factor in the development of this thesis, but thankfully Krapp, Kuhlmann and Serra's work was detailed and thorough enough to provide a solid basis for investigations on the subject.

We shall conclude with some suggestions for future research. Several ideas presented below were kindly hinted by Prof. Hugo Luiz Mariano.

- ▷ Let R be a rng, let $G_1 G_2 \ldots G_n$ be a finite sequence of ordered groups, and let $\mathcal{J}_1 \mathcal{J}_2 \ldots \mathcal{J}_n$ be a finite sequence so that each $\mathcal{J}_i$ is an arithmetic Rayner ideal on G_i. The product group $G_1 \times G_2 \times \cdots \times G_n$ is a partially ordered group when endowed with the partial order $<$ such that the condition $(g_1 g_2 \dots g_n) < (h_1 h_2 \dots h_n)$ is equivalent to the conjunction of $g_i \leqslant h_i$ $(\forall i)$ and $g_i < h_i$ $(\exists i)$. Let $\mathcal{J} = \mathcal{J}_{\mathcal{J}_1 \mathcal{J}_2 \dots \mathcal{J}_n}$ be the set of subsets S of the product $G_1 \times G_2 \times \cdots \times G_n$ such that for all $g_1 \in G_1$, $g_2 \in G_2$, ... and $g_n \in G_n$, we have

$$(\forall i \in [1,n]_{\mathbb{N}}) \ \ \{g \in G_i \mid (g_1 \dots g_{i-1}, g, g_{i+1} \dots g_n) \in S\} \in \mathcal{J}_i \,.$$

Consider the iteratively constructed Rayner rng

$$\overset{\mathcal{J}}{R}[[\mathrm{X}^{G_1 \times G_2 \times \cdots \times G_n}]] \coloneqq \Big(\cdots \Big(\big(\overset{\mathcal{J}_1}{R}[[\mathrm{X}_1^{G_1}]] \big) \overset{\mathcal{J}_2}{[[\mathrm{X}_2^{G_2}]]} \Big) \overset{\cdot^{\cdot^{\cdot^{\mathcal{J}_n}}}}{\cdots} \Big) [[\mathrm{X}_n^{G_n}]] \,.$$

An element of this rng is a function of type $G_n \to {}^{G_{n-1}}\big({}^{G_{n-2}} \cdots ({}^{G_1}R)\big)$, which, by iteratively applying $n-1$ times the Currying natural isomorphism ${}^{A}\left({}^{B}C\right) \overset{\mathbf{Set}}{\cong} {}^{A \times B}C$, may be regarded as a family $x = \{x_{g_1 g_2 \dots g_n}\}_{g_i \in G_i (\forall i)}$ in R so that its support

$$\operatorname{supp}(x) \coloneqq \{(g_1 g_2 \dots g_n) \in G_1 \times G_2 \times \cdots \times G_n \mid x_{g_1 g_2 \dots g_n} \neq 0_R\}$$

belongs to the set $\mathcal{J}$. Hence, the element x may be represented as a **multivariate Rayner series**

$$x = \sum_{g_i \in G_i (\forall i)} x_{g_1 g_2 \dots g_n} \mathrm{X}_1^{g_1} \mathrm{X}_2^{g_2} \cdots \mathrm{X}_n^{g_n} \,,$$

the set $\mathcal{J}$ may be called an **arithmetic Rayner ideal** on the product $G_1 \times G_2 \times \cdots \times G_n$, and the Rayner rng $\overset{\mathcal{J}}{R}[[\mathrm{X}^{G_1 \times G_2 \times \cdots \times G_n}]]$ may be called a **multivariate Rayner rng**. We may define the **Levi-Civita ideal**

$$\overset{\mathrm{lf}}{\mathrm{P}}(G_1 \times G_2 \times \cdots \times G_n) \coloneqq \mathcal{J}_{\overset{\mathrm{lf}}{\mathrm{P}}(G_1), \overset{\mathrm{lf}}{\mathrm{P}}(G_2), \dots, \overset{\mathrm{lf}}{\mathrm{P}}(G_n)}$$

and the **Hahn ideal**

$$\overset{\mathrm{wo}}{\mathrm{P}}(G_1 \times G_2 \times \cdots \times G_n) \coloneqq \mathcal{J}_{\overset{\mathrm{wo}}{\mathrm{P}}(G_1), \overset{\mathrm{wo}}{\mathrm{P}}(G_2), \dots, \overset{\mathrm{wo}}{\mathrm{P}}(G_n)}$$

on $G_1 \times G_2 \times \cdots \times G_n$. Note that the theory presented in Section 3.1 does not provide an axiomatisation for $\mathcal{J}$, since the product $G_1 \times G_2 \times \cdots \times G_n$ is not totally ordered.

A generalisation of the Hahn rngs was obtained by Ribenboim in 1990 for the case in which the exponents of the formal power series lie in a partially ordered monoid M (187, 188). In his approach, a so-called **generalised formal power series** is a family $x = \{x_m\}_{m\in M}$ whose support is **Artinian and narrow in** M, meaning that every sequence in $\operatorname{supp}(x)$ has a non-strictly increasing subsequence (cf. Lemma 1.32). Denoting by $\overset{\text{An}}{\mathrm{P}}(M)$ the set of Artinian and narrow subsets of M, it is easy to check that $\mathcal{J} \subset \overset{\text{An}}{\mathrm{P}}(G_1 \times G_2 \times \cdots \times G_n)$ and

$$\overset{\text{lf}}{\mathrm{P}}(G_1 \times G_2 \times \cdots \times G_n) \subset \overset{\text{wo}}{\mathrm{P}}(G_1 \times G_2 \times \cdots \times G_n) = \overset{\text{An}}{\mathrm{P}}(G_1 \times G_2 \times \cdots \times G_n) .$$

All these constructions suggest that it might be possible to obtain an axiomatisation of the Theory of Rayner Ideals that accounts for ideals on partially ordered groups (or perhaps even monoids). One immediate possibility would be to say that an **arithmetic Rayner ideal** on a partially ordered group G is an ideal $\mathcal{J}$ on G that is a subideal of $\overset{\text{An}}{\mathrm{P}}(G)$ and is such that $A + B \in \mathcal{J}$ $(\forall A, B \in \mathcal{J})$. If all details lined up perfectly, then such development would lead to a significant expansion of the scope of the Theory of Rayner Rngs.

▷ Let K be a field and let G be a divisible ordered group. One may prove that $K[[\mathrm{X}^G]](i) = (K(i))[[\mathrm{X}^G]]$, where $K(i)$ is the splitting field of the polynomial $\mathrm{X}^2 + 1$ over K. As Alling noted in 1962, a corollary of that fact is that if K is real-closed, then the Hahn field $K[[\mathrm{X}^G]]$ is real-closed (3). Similar results are likely to hold true for at least some Rayner fields other than the Hahn fields.

▷ We have not dealt with the notion of **Euclidean fields** in the main body of this work. These are the ordered fields K such that every positive element of K has a square root in K (Example 2.47). By Theorem 3.78, if K is an Euclidean field, if G is a 2-divisible ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ and if $\mathcal{J}$ is an incremental full Rayner ideal on G, then the ordered field $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ is Euclidean. This conclusion also holds for some full Rayner ideals $\mathcal{J}$ that are not incremental. For instance, if $x = \sum\limits_{n\in[2n_0,\infty)_{\mathbb{Z}}} x_{n/d}\mathrm{X}^{n/d}$ is a positive element of the ordered Puiseux field $\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]$, where the denominator d has been chosen so that the index n begins at an even number, and if $y = \sum_{n\in[n_0,\infty)_{\mathbb{Z}}} y_{n/d}\mathrm{X}^{n/d}$ is a positive square root of x in $\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]$, then we obtain

$$\sum_{n\in[2n_0,\infty)_{\mathbb{Z}}} x_{n/d}\mathrm{X}^{n/d} = x = y^2 = \sum_{n\in[2n_0,\infty)_{\mathbb{Z}}} \Bigg(\sum_{\substack{p+q=n\\ p,q\geqslant n_0}} y_{p/d}\, y_{q/d} \Bigg) \mathrm{X}^{n/d} ,$$

implying

$$(\forall n \in [2n_0,\infty)_{\mathbb{Z}})\ \ x_{n/d} = \sum_{\substack{p+q=n\\ p,q\geqslant n_0}} y_{p/d}\, y_{q/d} = \sum_{p=n_0}^{n-n_0} y_{p/d}\, y_{(n-p)/d}$$

and that the coefficients of y are recursively given by

$$y_{(n-n_0)/d} = \begin{cases} \sqrt{x_{2n_0/d}} & \text{if } n = 2n_0 \,, \\ \dfrac{1}{2\sqrt{x_{2n_0/d}}} \Bigg(x_{n/d} - \displaystyle\sum_{p=n_0+1}^{n-n_0-1} y_{p/d}\, y_{(n-p)/d} \Bigg) & \text{if } n \in (2n_0,\infty)_{\mathbb{Z}} \,. \end{cases}$$

Thus, the positive square root y of x indeed exists and belongs to $\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]$, proving that the ordered Puiseux field $\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathbb{Q}}]]$ is Euclidean. On the other hand, the positive square root $\mathrm{X}^{1/4}$ of the positive element $\mathrm{X}^{1/2}$ of the ordered Puiseux field $\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]$ (Example 3.21) does not belong to $\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]$, showing that $\overset{\mathrm{bd}}{\mathbb{R}}_{\mathbb{Z}}[[\mathrm{X}^{\mathcal{P}}]]$ is not Euclidean. Therefore, perhaps the assumption that $\mathcal{J}$ is incremental in the statement of Theorem 3.78 could be replaced by a weaker hypothesis on $\mathcal{J}$, one that is fulfilled by $\overset{\mathrm{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathbb{Q})$ and not by $\overset{\mathrm{bd}}{\mathrm{P}}_{\mathbb{Z}}(\mathcal{P})$. Furthermore, since Theorem 3.78 is a direct consequence of the Fixed Point Theorem (Theorem 3.61), it is possible that $\mathcal{J}$ does not have to be incremental there too.

▷ An **ultraproduct** is a model-theoretic construction that takes an indexed family of models of a mathematical theory, along with an ultrafilter on the set of indices of that family, and gives as a result another structure of the same kind. The first-order properties of an ultraproduct are the ones that hold for "most" members of the family, in a sense, while its higher-order properties can be quite different from those of its generating structures. The most renowned example of an ultraproduct is the ordered field of hyperreal numbers $^*\mathbb{R}$ (192), which is a non-Archimedean ordered field that has all the same first-order properties as $\mathbb{R}$. Reasonably, ultraproducts of families $\left\{\overset{\mathcal{J}_i}{R_i}[[\mathrm{X}^{G_i}]]\right\}_{i\in I}$ of Rayner rngs with respect to ultrafilters $\mathcal{U}$ on I can potentially give rise to rngs with interesting features, and such rngs might be related to the ultraproducts of the families $\{R_i\}_{i\in I}$, $\{G_i\}_{i\in I}$ and $\{\mathcal{J}_i\}_{i\in I}$ with respect to $\mathcal{U}$.

Other model-theoretic explorations could be conducted in order to obtain more information on the structure of the $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ construction. For instance, one could wonder if the first-order properties of $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]]$ depend on the higher-order properties of R, G or $\mathcal{J}$. Strictly speaking, consider two rngs R and R', two groups G and G', and two arithmetic Rayner ideals $\mathcal{J}, \mathcal{J}'$ on G, G', respectively, so that $R \equiv R'$, $G \equiv G'$ and $\mathcal{J} \equiv \mathcal{J}'$. Here, the symbol '$\equiv$' represents the relation of **elementary equivalence**, which treats as equivalent any two structures of the same kind that have precisely the same first-order properties. On this footing, a pressing question would be to determine whether the equivalence $\overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \equiv \overset{\mathcal{J}'}{R'}[[\mathrm{X}^{G'}]]$ holds, and if it does not in general, then it could be that it holds when additional hypotheses are assumed. Last but not least, we point out that the Hahn fields have been singled out as special cases in several recent studies on the Henselian valued fields (55, 185, 68, 110, 116), and such studies are likely to provide a good starting point for several model-theoretic considerations concerning the Rayner rngs.

▷ In Theorem 3.56, we learned that the functor $\mathcal{R} : \mathbf{Rng} \to \mathbf{Rng}$ given by

$$\begin{cases} \mathcal{R}\,(R) \coloneqq \overset{\mathcal{J}}{R}[[\mathrm{X}^G]] \\ \left(\mathcal{R}\,(\phi : R \xrightarrow{\mathbf{Rng}} S)\,(x)\right)_g \coloneqq \phi\,(x_g) \end{cases}$$

preserves object-finite limits and quotients modulo ideals in **Rng**, where G is an ordered group and $\mathcal{J}$ is an arithmetic Rayner ideal on G, and that implies that $\mathcal{R}$ preserves at least one kind of colimit, the initial object $\mathbf{0} = \{0\}$ in **Rng**, since $\mathbf{0}$ also happens to be the terminal object in **Rng**. Let us briefly discuss the reason why the preservation of colimits of non-null functors is more involved. Given a limit cone $\chi = \{\chi_i : \mathcal{F}\,(i) \xrightarrow{\mathbf{Rng}} L\}_{i\in \boldsymbol{I}_0}$ under a (possibly finite or object-finite) non-null functor $\mathcal{F} : \boldsymbol{I} \to \mathbf{Rng}$, and given a cone $\lambda = \left\{\lambda_i : \overset{\mathcal{J}}{\mathcal{F}}(i)\,[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\longrightarrow} V\right\}_{i\in \boldsymbol{I}_0}$ under $\mathcal{R}\circ\mathcal{F}$, then, in order to show that $\mathcal{R}$ preserves $\operatorname{colim}(\mathcal{F})$, one would need to show that there is a unique morphism $\overline{\lambda} : \overset{\mathcal{J}}{L}\,[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\longrightarrow} V$ such that the digraph

$$\begin{tikzcd}
\overset{\mathcal{J}}{\mathcal{F}}(i)\,[[\mathrm{X}^G]] \arrow[r, "\lambda_i"] \arrow[d, "\mathcal{R}(\chi_i)"'] & V \\
\overset{\mathcal{J}}{L}\,[[\mathrm{X}^G]] \arrow[ur, "\overline{\lambda}"'] &
\end{tikzcd}$$

in **Rng** commutes for all $i \in \boldsymbol{I}_0$. The sole piece of information available at this point is that χ is a limit cone under $\mathcal{F}$, that is, one can perform colimit lowerings of cones under $\mathcal{F}$ along χ. If one were to follow the procedure adopted in the proof of Item (b) of Theorem 3.56, then one would follow the given steps:

(**1**) For each $i \in \boldsymbol{I}_0$, define a family $\{\lambda_{i,j} : \mathcal{F}(i) \overset{\mathbf{Mon}}{\longrightarrow} V\}_{j\in J}$ that captures all the information contained in the morphism $\lambda_i : \overset{\mathcal{J}}{\mathcal{F}}(i)\,[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\longrightarrow} V$;

(**2**) For each $j \in J$, obtain the colimit lowering $\overline{\lambda_j} : L \overset{\mathbf{Mon}}{\longrightarrow} V$ of the cone $\lambda_j := \{\lambda_{i,j} : \mathcal{F}(i) \overset{\mathbf{Mon}}{\longrightarrow} V\}_{i\in \boldsymbol{I}_0}$ along χ;

(**3**) Define a function $\overline{\lambda} : \overset{\mathcal{J}}{L}\,[[\mathrm{X}^G]] \to V$ that captures all the information contained in the family $\{\overline{\lambda_j} : L \overset{\mathbf{Mon}}{\longrightarrow} V\}_{j\in J}$;

(**4**) Prove that $\overline{\lambda}$ is a homomorphism between rngs and that it satisfies the universal property of colimits for λ and $\mathcal{R}(\chi)$.

Step (2) is trivial, and Step (4) is attainable depending on how the other steps go. The aggravating issues lie in Steps (1) and (3), for there seems to be no direct means of storing all the data contained in a homomorphism of the form $\phi : \overset{\mathcal{J}}{R}\,[[\mathrm{X}^G]] \overset{\mathbf{Rng}}{\longrightarrow} S$ into a family of homomorphisms of the form $\psi := \{\psi_k : R \overset{\mathbf{Mon}}{\longrightarrow} S\}_{k\in K}$. The values of ϕ at the generalised polynomials in $R\,[\mathrm{X}^G]$ can be easily stored in ψ, for instance by taking $K := G$ and $\psi_g(x) := \phi\,(x\mathrm{X}^g)$ for each $g \in G$, but the values of ϕ at the formal power series with infinite support always seem to be left out of the conversion process. Therefore, if $\mathcal{R}$ preserves some sort of colimit other than the initial object in **Rng**, then proving this fact would probably require the use of a different proof strategy. Future examinations of "simple" specific examples of colimits in **Rng** might reveal new insights on the matter, such as determining whether the rngs (cf. Example B.39)

$$\left((\mathbb{Z}/2\mathbb{Z}) \overset{\mathrm{lf}}{\sqcup} (\mathbb{Z}/2\mathbb{Z})\right)[[\mathrm{X}^{\mathbb{Q}}]] \quad \text{and} \quad (\mathbb{Z}\overset{\mathrm{lf}}{/}2\mathbb{Z})[[\mathrm{X}^{\mathbb{Q}}]] \sqcup (\mathbb{Z}\overset{\mathrm{lf}}{/}2\mathbb{Z})[[\mathrm{X}^{\mathbb{Q}}]]$$

are isomorphic in **Rng**.

▷ As we have seen in the Introduction, a comprehensive theory of Differential and Integral Calculus on the real Levi-Civita field $\mathcal{R}$ has gradually been developed since the late 19th century. Several theorems of this thesis are generalisations of preliminary results of that theory, and it is presumable that many other results of the Analysis on $\mathcal{R}$ could be generalised in like manner, revealing to what extent the analytical properties of $\mathcal{R}$ hold for other ordered Rayner fields. So far, with the results of this work, we can conjecture that an ordered Rayner field $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$ is most likely to be "suitable for Analysis" if the following conditions are met for all formal power series $x = \sum_{g \in G} x_g \mathrm{X}^g$ in $\overset{\mathcal{J}}{K}[[\mathrm{X}^G]]$:

(**C1**) The coefficients x_g lie in an ordered field K (Theorems 3.67 and 3.82) such that every positive element of K has an n-th root for each natural number n (Theorem 3.78);

(**C2**) The exponents g lie in a divisible (Theorem 3.78) ordered subgroup G of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ (Theorem 3.61) that is also a cofinal ordered subgroup of $(K, +_K, <_K)$ (Theorem 4.19, Item (c));

(**C3**) The supports $\mathrm{supp}\,(x)$ lie in an incremental (Theorems 3.61 and 3.78; Theorem 4.2, Item (j); Theorem 4.19, Item (c)), left-finite (Theorem 4.2, Item (c); Theorem 4.15, Item (o)) full Rayner ideal $\mathcal{J}$ on G (Theorem 3.67).

We leave to the reader the proof that Condition (C2) implies that the ordered field K is Archimedean, and, therefore, the ordered group $(K, +_K, <_K)$ must be an ordered subgroup of $(\mathbb{R}, +_{\mathbb{R}}, <_{\mathbb{R}})$ in that case (Theorem 1.72). In addition, note that the Levi-Civita ideals $\overset{\mathrm{lf}}{\mathrm{P}}(G)$ are the only ones that are guaranteed to satisfy Condition (C3) among the main kinds of Rayner ideals that we have considered.

As expected, the Conditions (C1), (C2) and (C3) hold for the ordered field $\mathcal{R} = \overset{\text{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{Q}}]]$, and it is not hard to come up with other examples of ordered Levi-Civita fields that are also in accordance with those requirements. For instance, denoting by A the set of algebraic real numbers (37), the ordered fields

$$\overset{\text{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathbb{R}}]]\,,\ \overset{\text{lf}}{\mathbb{R}}[[\mathrm{X}^{\mathrm{A}}]]\,,\ \overset{\text{lf}}{\mathrm{A}}[[\mathrm{X}^{\mathbb{Q}}]] \ \text{ and } \ \overset{\text{lf}}{\mathrm{A}}[[\mathrm{X}^{\mathrm{A}}]]$$

have a number of interesting analytical properties which are worthy of further inspection.

Appendix

A Set Theory, Mathematical Structures and the TG System

The notions of **set**, **membership** and **inclusion** have been employed implicitly and explicitly by philosophers and mathematicians in every period of history, being occasionally deployed as intuitive dialectical tools for delineating complex arguments. For millennia, deliberations regarding finite sets never gave rise to controversies, being considered to be trivial and self-evident, but the situation was quite different for infinite sets, for they seemed to violate a few laws that many thinkers took for granted. For instance, the principle that states that "the whole is greater than the part" was widely accepted, but many set-theoretic counter-examples to that statement could easily be found, including proper subsets of $\mathbb{N}$ that are equipotent to $\mathbb{N}$, and that led many authors to distrust the use of actual infinities in their works, such as Aristotle, Galois, Pasch and Hilbert (40). These philosophical difficulties raised no serious concerns up until the late 19th century, for one could always phrase the arguments in ways that avoid mentioning infinite sets, sometimes going through great lengths to do so, and also because these issues did not affect the main body of mathematical results that had been established so far.

The first spark of the coming revolution on the nature of infinity occurred in 1874 when the German mathematician Georg Cantor provided a rigorous proof that the infinite set of algebraic real numbers has less elements than the set $\mathbb{R}$ in a sense (47). Shortly after, he showed that there are infinitely many sizes of infinity which can be well-ordered in a natural manner (50). Since these results defied thousands of years of philosophical tradition, they initially faced fierce opposition from many influential intellectuals of the 19th century, above all Kronecker (46), but a few others showed swift appreciation for Cantor's discoveries, including Dedekind and Weierstraß. By the beginning of the 20th century, the bulk of the mathematical community had already been convinced that the properties of sets are worthy of extensive investigation, and that realisation gave rise to a new branch of Mathematics called **Set Theory**.

However, Cantor himself knew that there were some rather strange predicaments resulting from his theory. In an unpublished letter to Hilbert sent in 1896, he remarked that there cannot be a set of all ordinals, since that set would be well-ordered and would be order-isomorphic to a proper initial segment of itself, which is absurd (40), and then in 1899 he wrote to Dedekind detailing the contradictions that arise when the set of all cardinals and the set of all sets are considered (50). These findings led him to distinguish the regular sets (*Mengen*) from what he called "multiplicities" (*Vielheiten*), the latter being sets that are "too large" to be thought of as single objects (51). Other prominent figures noticed similar irregularities on Set Theory around that period, such as Burali-Forti (45), Zermelo (180) and Russell (198), and even some inconsistencies involving finite sets were found, such as Richard's Paradox[1] (184). Soon it became unanimously clear that the foundations of Mathematics required an urgent revision in order to eliminate set-theoretic paradoxes. "*Where else is there to be found certainty and truth when even mathematical thinking fails?*", as Hilbert put the general sentiment amongst many researchers around that time (98).

We shall not retrace here the many philosophies, doctrines and programs proposed in the 20th century to address that matter, and we shall not address the enormous repercussions of Gödel's Incompleteness Theorems (86) on the discussions regarding the limitations of the Axiomatic Method. For our purposes, it suffices to describe the approach that the mathematical community eventually embraced as their preferred solution to all foundational issues:

[1] A simplified version of Richard's Paradox due to Berry and Russell (198) is as follows: the set of integers which can be named in less than nineteen syllables is finite, but the integer that is 'the least integer not nameable in fewer than nineteen syllables' is named is eighteen syllables, which is absurd.

- ▷ Up until the late 19th century, all mathematical assertions were articulated in natural human languages, which are non-designed languages that have evolved naturally in human societies through millennia of unceasing use and repetition, and sentences in these communication systems may have multiple alternative interpretations, often causing confusion in the assessment of texts. That hindrance was eradicated by the introduction of a completely formalised language with fixed syntactic rules in which every assertion can be expressed with no possible ambiguity, and, in that framework, the notion of property is reduced to well-formed assemblies of symbols which have no abstract meaning *per se*. Yet it must be noted that researchers do not usually enunciate their conclusions in such formalised language, not even logicians, model-theorists or set-theorists, still routinely making use of natural languages throughout, but there is a general consensus that all results should be appropriately formalizable. This constraint is not satisfied for all paradoxes of Set Theory generated by inventive plays upon words, such as Richard's Paradox;

- ▷ All logical deductions are to be carried out according to a completely formalised version of the Axiomatic Method that is greatly inspired by Hilbert's thorough and profound *Grundlagen der Geometrie* (99). Mathematicians are free to choose the axioms that support their conclusions, and, once these are chosen and fixed, all valid results of the theory must be provable from the axioms according to a finite set of formal, strictly syntactic, inference rules. In order to check the validity of a proof, one can, in principle, check in finitely many stages if the inference rules were applied correctly, so that one does not need to take the individual psychological reactions of the proof's creator into account. Thus, one may regard each such axiomatic system as a one-player game with a rigorous finite body of regulations. The assemblage of the formalised language employed to convey sentences and the inference rules in which logical deductions are performed is called **First-order Logic**, and it is the crown jewel of modern Mathematical Logic;

▷ A highly influential and widely accepted system of axioms was introduced for Set Theory, the so-called **Zermelo-Fraenkel-Choice system** (or **ZFC**), which is mostly due to the works of Zermelo (243, 245, 246), Skolem (218), von Neumann (146) and Fraenkel (77, 78, 79). In that system, the word '**set**' and the **membership** conditions $x \in y$ are left undefined, being taken as primitive notions of the theory, and all conditions are said to be L_ϵ**-formulas**.

Most axioms of ZFC are mere formal translations of the most basic human intuitions regarding the notion of set. In addition, there is the **Axiom of Foundation** (*Fundierungsaxiom*) introduced in order to discard some pathological sets from the theory, such as sets that belong to themselves, and there is an **Axiom Schema of Separation** (*Aussonderungsschema*) which is cautiously designed to avoid the appearance of Cantor's *Vielheiten,* stating that the set of all objects x satisfying an L_ϵ-formula $P(x, y_1 \ldots y_n)$ **exists** (**in ZFC**) if P implies a condition of the form $x \in A$, where $y_1 \ldots y_n$ and A are sets that have already been shown to exist. When a collection defined by an L_ϵ-formula P does not exist in that sense, one may still consider its "intrinsic nature" as a meta-object by *defining* that collection to be P itself, and such meta-formula can be encoded as an object of ZFC [2]. And then there is the **Axiom of Choice** (or **AC**) (*Auswahlaxiom*), which stirred a great deal of controversy in the early 20th century despite having a fairly intuitive and plain statement. That controversy emerged over the fact that the AC establishes the existence of mathematical objects without explicitly defining them, thus breaking with a few philosophical principles regarding the notion of constructibility in Mathematics. Since many well-established theorems require it for their proofs, resistance to the AC has gradually vanished through time, so that that axiom has been securely incorporated into standard Set Theory.

[2] One way of achieving that is by defining the language of First-order Logic within a countably generated free monoid (148), where the sets $y_1 \ldots y_n$ are added to the alphabet of the language.

Although Gödel's Second Incompleteness Theorem has crushed all hopes of proving that ZFC is consistent, the vast majority of mathematicians have high confidence in the consistency of that system, and that is due to the fact that no contradiction has been found over roughly a century of intense, exhaustive testing of the axioms (63);

- ▷ The earliest alternative to ZFC within the framework of First-order Logic was introduced in 1925 by von Neumann (146), and shortly later it was reformulated and simplified by R. Robinson (193), Bernays (13) and Gödel (87), eventually being denominated the **Neumann-Bernays-Gödel system** (or **NBG**). This system takes the **membership** conditions $x \in y$ and the words '**class**' and '**set**' as its primitive notions[3], and, unlike ZFC, it has a finite number of axioms[4]. Cantor's *Vielheiten* are formalised within NBG precisely as the classes A such that $A \notin B$ for every class B, and such classes, which at some past time caused so much trouble for many thinkers, are called **proper classes**, and this name stuck for the general concept even in contexts that are not related to NBG. It has been proved that NBG is a **conservative extension** of ZFC, meaning that a sentence written in the language of ZFC is a theorem of ZFC if, and only if, it is a theorem of NBG, and that immediately implies that the consistency of NBG is equivalent to that of ZFC. Many other alternatives to ZFC were proposed, most notably the Morse-Kelley system (232, 111, 143), but none of them has achieved the same degree of acceptance and dissemination as ZFC and NBG. We refer the reader to (127, 71, 194, 106, 159, 224, 199, 100) for details on the prevailing formalisations of Set Theory within the framework of First-order Logic;

- ▷ Mathematics has been largely unified under Set Theory, so that a strong foundation for the latter provides a strong foundation for the former.

[3] A formalisation of NBG that takes only the notions of **class** and **membership** as primitives was introduced by Mendelson in 1964. We refer the reader to (159) for details, and we call his or her attention to the fact that Mendelson assumes ZFC's Axiom of Choice for his system, while von Neumann, Bernays and Gödel assumed the stronger **Axiom of Global Choice**. Many authors refer to Mendelson's axiomatic system as **Mendelson's NBG**.

[4] ZFC has two axiom schemas which generate countably infinitely many axioms for the theory, while NBG has no axiom schema.

The desire to unify the "queen of sciences"[5] into a single conceptual framework can be traced back to the Pythagorean maxim "All things are numbers", and that feeling has remained throughout history taking many forms. A major step was taken toward that goal in the 17th century when Descartes and Fermat unravelled a deep connection between Algebra and Geometry which would culminate in the development of Calculus. At the beginning of the 19th century, the scenario was so that the natural numbers were seen as "exclusive products of our intellect" (85), the continuous quantities were viewed as lengths, areas and volumes of geometric constructions, and the rational numbers were intrinsically connected to the notion of subdivision of a magnitude into equal parts. In the span of a century, these conceptions were inherently altered due to the works of many experts in varied fields, including Graßmann, Hankel, Weierstraß, Cantor, Dedekind and Méray, who showed that the integers, the rationals, the reals and the complex numbers could all be fully described in set-theoretic terms (40), and gradually all other areas of Mathematics met the same fate, so that Set Theory became the *lingua franca* for the development of mathematical ideas. Deep results in Topos Theory and first-order axiomatisations of the category of sets and the category of categories have led many theorists to make intriguing claims that the notion of category will eventually be prevailingly viewed as the most fundamental notion of Mathematics (133, 134, 154, 88), but so far there is no indication that the community is reaching an agreement in that direction;

▷ Caught in the atmosphere of set-theoretic unification, in 1935 the French group Nicolas Bourbaki, headed by Cartan, Chevalley, Delsarte, Dieudonné and Weil, set out a monumental attempt to encapsulate nearly all of the Mathematics of its time into a modern, rigorous and comprehensive series of textbooks called *Éléments de mathématique*[6]. The series was intended to be completely self-contained: references to other works were explicitly

[5] Phrase coined by Gauss (231).

[6] The use of the unusual, archaic French word *'mathématique'* in place of the customary *'mathématiques'* is deliberate (35). "*The absence of the 's' was of course quite intentional, one way for Bourbaki to signal its belief in the unity of mathematics*", as the Bourbaki member Armand Borel explained (31).

forbidden within the main body of the exposition, and all results employed in the text were to be proved in the series itself as theorems of an axiomatic system essentially equivalent to ZFC. The group, which is still operational today with a whole new personnel, presented their main founding ambition in one of their thought-provoking articles:

> As every one knows, all mathematical theories can be considered as extensions of the general theory of sets [...] On these foundations, I state that I can build up the whole of the Mathematics of the present day; and, if there is anything original in my procedure, it lies solely in the fact that, instead of being content with such a statement, I proceed to prove it in the same way as Diogenes proved the existence of motion; and my proof will become more and more complete as my treatise grows. (36)

That stance led some authors to claim that Bourbaki did not understand the profundity of Gödel's First Incompleteness Theorem, to which the Bourbaki member Alain Connes responded that "*[Gödel's theorem] states that, for every finite or recursively defined set of axioms, there are always questions that cannot be answered. [...] But if a question is not decidable, [...] we can give it an answer and continue to reason. This means that [...] each undecidable question creates a bifurcation and imposes a choice.*" (54). Thus, it seems that at least a few members of the group used to believe that mathematicians are destined to work on an increasing sequence of axiomatisations of Set Theory that tends toward completeness without ever reaching it. However, the group never voiced its definitive choices for many well-known problems that have been proved to be independent of the ZFC system, such as the Generalised Continuum Hypothesis (48, 87, 60) and Martin's Axiom (155, 212), and to this day most mathematicians have shown great reluctance to do so, since there is no acknowledged basis on which those decisions are to be made. The posture that has been adopted by logicians and model-theorists is so that all reasonable variations of the axiomatisations of Set Theory must be carefully studied and the logical connections between these systems must be disclosed.

In the remainder of this appendix, we shall examine how mathematical structures are defined in formal Set Theory, and then we shall give special consideration to structures whose universes are "too large", advocating for the use of an extension of ZFC due to Tarski and Grothendieck that is appropriate to overcome most technical limitations regarding such constructs.

A.1 Structures in modern Mathematics

A **mathematical structure** is an abstract encapsulation of the component elements, inherent configuration and operational data of a multipart mathematical idea that satisfies a predetermined set of axioms. For example, consider the set $\mathbb{C}$ of complex numbers, the usual addition and multiplication operations $+_{\mathbb{C}}\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ and $\times_{\mathbb{C}}\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}$, the usual topology $\tau_{\mathbb{C}}$ on $\mathbb{C}$ (Example 1.46), and bear in mind that the following conditions hold true:

(P1$_{\mathbb{C}}$) $+_{\mathbb{C}}$ is a commutative and associative operation on $\mathbb{C}$ that has an identity element 0 in $\mathbb{C}$, and every element of $\mathbb{C}$ is $+_{\mathbb{C}}$-invertible (Definitions 1.1, 1.2 and 1.5);

(P2$_{\mathbb{C}}$) $\times_{\mathbb{C}}$ is a commutative and associative operation on $\mathbb{C}$ that has an identity element 1 in $\mathbb{C}$, and every element of $\mathbb{C}-\{0\}$ is $\times_{\mathbb{C}}$-invertible;

(P3$_{\mathbb{C}}$) $0\neq 1$;

(P4$_{\mathbb{C}}$) $(\forall x,y,z\in\mathbb{C})\ \ (x+y)\,z = xz+yz \quad\text{and}\quad x\,(y+z)=xy+xz\,;$

(P5$_{\mathbb{C}}$) $\tau_{\mathbb{C}}$ is a topology on $\mathbb{C}$ such that the functions

$$+_{\mathbb{C}}\colon (\mathbb{C},\tau_{\mathbb{C}})\times(\mathbb{C},\tau_{\mathbb{C}})\to(\mathbb{C},\tau_{\mathbb{C}})\ \ ,\quad \times_{\mathbb{C}}\colon (\mathbb{C},\tau_{\mathbb{C}})\times(\mathbb{C},\tau_{\mathbb{C}})\to(\mathbb{C},\tau_{\mathbb{C}})$$
$$\text{and}\quad \overset{\times_{\mathbb{C}}}{\mathrm{Inv}}\colon (\mathbb{C}-\{0\}\,,\tau_{\mathbb{C}}\restriction(\mathbb{C}-\{0\}))\to(\mathbb{C},\tau_{\mathbb{C}})$$

are continuous (Definitions 1.36), where $\overset{\times_{\mathbb{C}}}{\mathrm{Inv}}:\mathbb{C}-\{0\}\to\mathbb{C}$ is the function given by $\overset{\times_G}{\mathrm{Inv}}(z):=z^{-1}$ (Definition 1.7).

All these considerations are called upon at once when one asserts that the set $\mathbb{C}$ of complex numbers has a **topological field structure** when **endowed** with the operations $+_{\mathbb{C}}$, $\times_{\mathbb{C}}$ and the topology $\tau_{\mathbb{C}}$, or, more briefly, that the ordered 4-tuple $(\mathbb{C}, +_{\mathbb{C}}, \times_{\mathbb{C}}, \tau_{\mathbb{C}})$ is a **topological field** (Definition 2.30; Example 2.31).

That observation may seem pedantic since it simply reflects the definition of the phrase 'topological field' and since the properties (P1$_{\mathbb{C}}$)-(P5$_{\mathbb{C}}$) are historically inspired upon the arithmetic and topological properties of number systems. However, the recurring use of such structural, axiomatic abbreviations has revolutionised the way in which mathematicians organise the ideas and results of their discipline for the past two hundred years. Note that the phrase 'topological field' stands for any 4-tuple of the form

$$(K, +_K : K \times K \to K, \times_K : K \times K \to K, \tau_K \in \mathrm{P}^2(K))$$

that satisfies the **axioms** (A1$_K$)-(A5$_K$), where each (An_K) is the condition (P$n_{\mathbb{C}}$) with the set $\mathbb{C}$ replaced by K. All results that one may derive from these axioms for *any* such 4-tuple must hold true for *every* particular topological field, and that saves one the effort of having to prove these results again everytime one identifies a topological field. Clearly, that is valid for every kind of mathematical structure, as Bourbaki explained:

> Its most salient feature [of the axiomatic method], [...], is to achieve considerable *economy of thought*. "Structures" are *tools* for the mathematician; once he has discerned [...] the relations satisfying the axioms of a structure of a known type, he immediately has at his disposal the whole arsenal of general theorems relating to structures of that type, when before he had to laboriously forge for himself means of attack, whose powers depended on his personal talents and which were often encumbered with the unnecessarily restrictive hypotheses arising from the particularities of the problem under study. (35)

The reader can find in the literature several definitions of the notion of mathematical structure (38, 1, 101, 177), some broader than others, some clumsier than others. In essence, all definitions (very roughly) agree that a **structure $\mathfrak{S}$ with universe** (**set**) X is an ordered pair of sets $\mathfrak{S} := (X, R)$, where R is any element of some set S_X obtained from X by applying a finite number of operations, such that $\mathfrak{S}$ must satisfy some set of axioms. For instance, a semigroup can be defined to be an ordered pair $(M, \times_M)$ so that $\times_M$ is any element of the set of functions $S_M := {}^{M \times M}M$ and such that the associativity axiom $x\,(yz) = (xy)\,z$ $(\forall x, y, z \in M)$ is satisfied (Definition 1.2). Since ordered pairs can be easily encoded as sets, such as in Wiener's definition $(a, b) := \{\{\{a\}, \emptyset\}, \{\{b\}\}\}$ (236) or as in Kuratowski's now-accepted definition $(a, b) := \{\{a\}, \{a, b\}\}$ (125), that entails that all the data of a mathematical structure can be effectively enclosed within a unified entity in Set Theory. However, such definition of the notion of structure is inappropriate for specific applications within the prevailing axiomatisations of Set Theory, as we shall address in the next section.

A.2 Troubles with large structures in ZFC and NBG

In foundational areas of Mathematics, such as Model Theory, Category Theory and Sheaf Theory, it is often convenient to deal with structures whose universes are "too large" to be regarded as single objects, which are sometimes called **large structures**. We shall consider how these are treated in ZFC and NBG.

As previously mentioned, there is a trick that allows one to circumvent, without risk of contradiction, the fact that proper classes are not formal objects in ZFC, and it consists in *defining* **classes** as being the L_ϵ-formulas $P(v)$ with one free variable v and with set parameters[7] (106). With that definition, the elements of a class $P(v)$ are taken to be the objects x such that $P(x)$ holds in ZFC, so that the sentence $P(x)$ is denoted by $x \in P$ by abuse of language, and each set A can be identified with the class $v \in A$. Furthermore, a **proper class** is a class $P(v)$ such that the existence of the set $\{x \mid x \in P\}$ implies a contradiction, and a **class-structure** may be (loosely) defined as an ordered pair $(P(v), R(v))$ that satisfies some set of axioms, where $P(v)$ is a class and $R(v)$ is a class whose

[7] We shall omit the parameters here.

elements belong to some class S_P obtained from P by applying a finite number of operations. A **large structure**, in that vein, is a class-structure whose universe is a proper class, and the regular structures are called **small structures**. As an example, we have that $\mathsf{On} \coloneqq$ 'v is an ordinal' is a proper class that forms a "large well-ordered class" when endowed with the "class-well-order" given by

$$<_{\mathsf{On}} \coloneqq (\exists\alpha)\,(\exists\beta)\,(v = (\alpha,\beta) \wedge (\text{`}\alpha \text{ and } \beta \text{ are ordinals'}) \wedge \alpha < \beta)\ ,$$

whose elements (α, β) belong to the class

$$S_{\mathsf{On}} \coloneqq \text{“}\mathsf{On} \times \mathsf{On}\text{”} = (\exists\alpha)\,(\exists\beta)\,(v = (\alpha,\beta) \wedge (\text{`}\alpha \text{ and } \beta \text{ are ordinals'}))\ .$$

Unfortunately, that approach has a significant downside: a result that is true for all small structures of a particular kind may not hold true for some large structures of the same type. For instance, we know that every well-ordered set is order-isomorphic to some ordinal, but the large structure $(\mathsf{On}, <_{\mathsf{On}})$ from our previous example is clearly not "class-order-isomorphic" to any ordinal, and even if we had defined the class On to be an ordinal as a special case, that would force us to redefine the class On, leading up to a long, tedious series of uninteresting changes in many definitions and results of Set Theory just to accommodate that new notion of ordinal. Thus, for each kind of structure, mathematicians would need to keep track of which theorems hold only for the small representatives of such structures. Such nuisance can be tolerated, and in fact it has been by many theorists, but perhaps a better strategy for dealing with large structures would be less troublesome, as we shall indicate in a short while.

In NBG, classes are taken as primitive objects of the theory, and proper classes are precisely the classes that cannot be elements of any class. However, given a proper class P and a class R, one is not able to formally construct the (Kuratowski's) ordered pair $(P, R) = \{\{P\}, \{P, R\}\}$ within the theory, since P would appear as an element of $\{P\}$ in that case. Hence, although NBG is ontologically richer than ZFC, incorporating the notion of (proper) class into the mix, that is not of use for defining large structures as *unified* objects in NBG. On this account, many authors routinely make reference to large structures only by abuse of language, as they actually mean to implement the defining parts of such structures *separately*, and these abuses often build up quickly as more and more constructions are perfomed using such structures. Even if one were to cope with these linguistic fictions, one would still face the same difficulty in NBG as that described for ZFC, that is, one would still have results that are true for all small structures of a particular kind but that are false for some large structures of the same type.

Therefore, the ways in which large structures are treated in ZFC and NBG are rather problematic. In the next section, we shall discuss an elegant alternative to these axiomatic systems that allows the user to operate with large structures more freely and straightforwardly.

A.3 From the inaccessibles to the TG system

The first paper to present a broad formulation of the general Theory of Totally Ordered Sets was Hausdorff's *Grundzüge einer Theorie der geordneten Mengen*, published in 1908, in which the author systematically developed the Theory of (Total) Order-types with careful attention to the special cases of the ordinal and cardinal numbers (94). As Cantor had previously remarked, some cardinals are equal to their own cofinality (Definition 1.27), such as $0, \omega, \omega_1, \omega_2$, etc., and some are not the immediate successor of any cardinal, such as $0, \omega, \omega_\omega, \omega_{\omega_\omega}$, etc., the former numbers being called **regular cardinals** and the latter **initial cardinals**. Contrarily, a **singular cardinal** is one that is not regular, and a **successor cardinal** is one that is not an initial cardinal. Hausdorff effortlessly noticed that 0 and ω are regular initial cardinals, and he wondered if there are other cardinals of that kind. Referring to cardinals simply as numbers, and taking for granted the fact that $\omega_\alpha \geqslant \alpha$ for every ordinal α, he reflected:

> The question of whether [...] there are also regular initial numbers with a limit index must remain undecided here, but the following can be remarked on this subject. An initial number [ω_α] with a limit index [α] is singular if its index is smaller than itself; accordingly, a regular initial number with a limit index can only be included among those initial numbers which are equal to their own indices [that is, $\omega_\alpha = \alpha$] [...]. The existence of such a number appears at least problematic hereafter, but must be considered as a possibility [...]. (94)

That contemplation echoed in the works of several mathematicians in the early 20th century, including Kuratowski (126), Zermelo (246), Sierpiński (214) and Tarski (221), and it was widely believed that a verdict on the matter could potentially provide some useful insight onto Cantor's Generalised Continuum Hypothesis. The regular initial cardinals were eventually called **weakly inaccessible cardinals**, and it was revealed that such numbers are precisely the regular cardinals κ such that $\alpha^+ < \kappa$ $(\forall \alpha < \kappa)$. Zermelo, who referred to the inaccessible cardinals as "boundary numbers" (*Grenzzahlen*), hinted in 1930 that the following axiom should be considered for Set Theory:

Axiom of (Weak) Inaccessibles (AWI). (246) For every cardinal α, there is a weakly inaccessible cardinal κ such that $\alpha < \kappa$.

Moreover, in that same year Tarski and Sierpiński discovered that, assuming the validity of the Generalised Continuum Hypothesis, every weakly inaccessible cardinal κ is such that $\alpha^\beta < \kappa$ $(\forall \alpha, \beta < \kappa)$ (213), and the weakly inaccessible cardinals that satisfy this extra condition were eventually called **strongly inaccessible cardinals**. Accordingly, the analogous version of axiom AWI for the strong inaccessible cardinals was considered, and soon the name 'Axiom of Inaccessibles' became tied only to this new version of the axiom, which we shall denote by the acronym **AI**. The status of both notions of inaccessibles within the mathematical community in 1938 was concisely summarised by Tarski:

> The concept of inaccessible cardinal number has long been encountered in set-theoretic investigations [...]. Initially, however, the unattainable figures were viewed more as a curiosity; [...] Hausdorff expresses the opinion that these numbers are of such an exorbitant size that they will hardly ever come into consideration for the usual purposes of set theory. It was only later that the meaning of the term under consideration for the basic questions was recognised, and in recent years it has turned out that the inaccessible numbers are not without meaning for certain factual problems of set theory, indeed they even play an essential role in some investigations. For these reasons, it seems worthwhile today to subject the concept of inaccessible cardinal number to a closer examination. (222)

Tarski was the first to realise that, in the presence of the axioms of ZFC except the Axiom of Choice, the existence of arbitrarily large strongly inaccessible cardinals is implied by the existence of arbitrarily large "inaccessible sets", that is, it is implied by the following statement:

(Tarski's) Axiom of Inaccessible Sets (TA). (222, 223) *For every set N, there is a set M such that:*

- ▷ $N \in M$;
- ▷ *If* $X \in M$, *then* $\mathrm{P}(X) \in M$;
- ▷ *If* $X \in M$ *and* $Y \subset X$, *then* $Y \in M$;
- ▷ *If* $X \subset M$ *is so that* $|X| < |M|$, *then* $X \in M$.

Hence, "*If one wants to ensure the existence of arbitrarily large cardinal numbers that are strongly inaccessible, one must enrich the Zermelo-Fraenkel system with a new principle, that is, formally speaking, with a new axiom*", he argued (222). He also noticed that axiom TA implies the Axiom of Infinity, the Axiom of Power Set and the Axiom of Choice, so that these may be removed from the axiomatisation of the system ZFC+TA, which is why Tarki's Axiom was regarded as being "too strong" by many set-theorists, being treated as a suspicious eccentricity in the following decades.

Some mathematicians saw Tarski's "inaccessible sets" as a potential opportunity to avoid proper classes in foundational areas of Mathematics, especially in the booming area of Category Theory. In 1959, Alexander Grothendieck, a member of the Bourbaki group at the time, presented his Theory of Universes in the group's internal newsletter *La Tribu* (118). That theory consisted in an adaptation of Tarski's notion of "inaccessible set" and the introduction of an axiom for Set Theory, as we shall now describe.

In most applications of the notion of proper class, one does not actually need these classes to contain *all* their elements. It is sufficient that such collections contain the objects that are relevant for the specific context under consideration, and that they are closed under all basic set-theoretic operations. That idea is formalised in the following definition:

Definition. (81, 5) A (**Grothendieck**) **Universe** is an inhabited set $\mathbf{U}$ such that the following conditions hold:

- ▷ If $x \in \mathbf{U}$ and if $y \in x$, then $y \in \mathbf{U}$;
- ▷ If $x, y \in \mathbf{U}$, then $\{x, y\} \in \mathbf{U}$;
- ▷ If $x \in \mathbf{U}$, then $\mathrm{P}(x) \in \mathbf{U}$;
- ▷ If $\{x_i\}_{i \in I}$ is a family in $\mathbf{U}$ so that $I \in \mathbf{U}$, then $\bigcup_{i \in I} x_i \in \mathbf{U}$.

The four items above which axiomatise the notion of universe are **closure properties** in essence, that is, they are statements that specify that if some objects belong to the set $\mathbf{U}$, then the products of the applications of some mathematical operations to these objects must remain in $\mathbf{U}$. Thus, one can deal with sets within $\mathbf{U}$ without worrying too much about which set-theoretic operations one may legitimately apply to these entities, and, in that regard, the definition of a universe is meant to provide a set in which all Mathematics can be done. It turns out that, within ZFC, the universes are precisely the sets of the form V_κ, where $\{\mathsf{V}_\alpha\}_{\alpha \in \mathsf{On}}$ is Zermelo's cumulative hierarchy of sets and where κ is a non-zero strongly inaccessible cardinal (242), and that every uncountable universe forms a transitive model of ZFC (120, 242, 142). In particular, it follows from Gödel's Second Incompleteness Theorem that the ZFC system cannot prove the existence of an uncountable universe, and, in fact, the countable set V_ω of hereditarily finite sets is the only universe that can be constructed in ZFC.

Following the pattern of Tarski's Axiom, Grothendieck proposed that the following axiom should be added to Set Theory:

(Grothendieck's) Axiom of Universes (AU). (81, 90) For every set x, there is a universe $\mathbf{U}$ such that $x \in \mathbf{U}$.

It is a standard exercise to prove that ZFC+AI, ZFC+TA and ZFC+AU are equivalent, and these axiomatic systems are conjointly called the **Tarski-Grothendieck system** (or **TG**). The following proposition reveals the current status of the consistency of TG relative to that of ZFC:

Proposition. *(5, 238)*

(**a**) *If* ZFC *is consistent, then* $\mathrm{ZFC} + \neg\mathrm{AU}$ *is consistent;*

(**b**) *One cannot prove within* ZFC *that the consistency of* ZFC *implies the consistency of* TG. *In standard logical notation, we have*

$$\mathrm{ZFC} \nvdash \mathrm{Con}\,(\mathrm{ZFC}) \to \mathrm{Con}\,(\mathrm{TG})\ .$$

Proof.

(**a**) Suppose $\mathcal{M} = (M, E)$ is a model of ZFC. If $\mathcal{M}$ has no uncountable universe as an element, then $\mathcal{M}$ is a model of ZFC+¬AU. If $\mathcal{M}$ has an uncountable universe $\mathbf{U}$ as an element, then $\mathbf{U}$ is of the form V_κ, where κ is an uncountable strongly inaccessible cardinal in $\mathcal{M}$, and taking κ' to be the least cardinal of that kind in $\mathcal{M}$, then $\mathsf{V}_{\kappa'}$ is a model of ZFC+¬AU.

(**b**) If we assume otherwise, then, since the consistency of ZFC is a theorem of TG, then the consistency of TG is a theorem of TG (by *modus ponens*), which is absurd by Gödel's Second Incompleteness Theorem.

□

Therefore, by item (b), any possible proof of relative consistency of TG with respect to ZFC would require the use of tools outside of the ZFC system. Even so, since TG has been tested, sometimes inadvertently, by many set theorists and logicians for nearly as much time as ZFC, and since no contradiction has been found in that long appraisal, the TG system is largely believed to be consistent (120). Besides, most proposed axiomatisations of Set Theory featuring large cardinals imply the existence of strongly inaccessible cardinals, so that axiom AI is regarded as one of the tamest in a long list of large-cardinal axioms (239).

Grothendieck consistently employed axiom AU in order to avoid the use of proper classes in his categorical treatment of Algebraic Geometry, and that allowed him to define large categories as standard structures in a Bourbaki-like fashion (118). He described such technique as follows, in the context of Category Theory:

> To avoid certain logical difficulties, we will here admit the notion of *Universe*, which is a set "big enough" so that we do not get out of it by the usual operations of set theory; an "*Axiom of Universes*" guarantees that any object is in a Universe [...]. Thus, the acronym **Ens** designates not the category of all sets (a notion which has no meaning), but the category of sets that are found in a given Universe [8] (which we will not specify here in the notation). Similarly, **Cat** will designate the category of categories that are found in the Universe in question [...]. (90)

Along these lines, the source of the troubles that mathematicians have faced when dealing with large structures (Section A.2) can be entirely eliminated, since universes are plain, well-behaved sets, and since mathematical structures, small or large, can be defined as standard ordered pairs (X, R) in TG (Section A.1), with X and R being elements of some universe $\mathbf{U}$ (219). Even if an operation leads a mathematical object outside of a given universe, the Axiom of Universes guarantees that there is a bigger universe that contains that object. There is a caveat to that approach: some universal constructs (limits, colimits, adjoints, Kan extensions, etc.) are universe-dependent, as some concrete examples have shown (235, 142). Luckily, most of these issues have been overcome by the development of clever techniques (5, 152, 82, 142).

It is worth noting here that most Bourbaki members did not cope well with Grothendieck's innovative foundational proposals, mainly due to the fact that the group had already committed itself to a rigid course of incremental presentation with several already-published volumes (10). Additionally, the lack

[8] We denote this category by **Set** in this thesis, as usual in English-language texts (Example B.17).

of a proof of relative consistency of AU with respect to ZFC caused some members to worry that the introduction of AU would cause the loss of "empirical certainty" of the axiomatic system (118). After many heated, shouted discussions[9] at the *École Normale Supérieure* in 1959, mostly between Grothendieck and Cartan, it was decided that Grothendieck would resign from the group, and after that it took half a century for Bourbaki to finally introduce the notion of category in the *Éléments de mathématique* series. Nevertheless, the group agreed to release some of its internal papers to be published as an appendix entitled *Univers* for the notes of the *Séminaire de géométrie algébrique du Bois Marie*, directed by Grothendieck (5). Bourbaki's concerns toward the Theory of Universes have echoed and disseminated in the mathematical community to this day, but, by contrast, Grothendieck's approach to Algebraic Geometry has been widely acclaimed.

The best argument for TG is based on its usefulness. Let us recall that the Axiom of Choice was not well-received by many prominent constructivists following its introduction by Zermelo in 1908 (244), but it eventually became almost unanimously accepted in the mathematical community for pragmatic reasons. As Martin-Löf put it:

> Despite the strength of the initial opposition against it, Zermelo's axiom of choice gradually came to be accepted mainly because it was needed at an early stage in the development of several branches of mathematics, not only set theory, but also topology, algebra and functional analysis, for example. Towards the end of the thirties, it had become firmly established and was made part of the standard mathematical curriculum in the form of Zorn's lemma. (156)

Hence, it was neither some major philosophical breakthrough on the nature of "empirical certainty" nor some convincing heuristic evidence that provoked that mood shift in favour of the AC, but rather a gradual acknowledgement that this axiom vastly enriches Mathematics. As it turns out, the study of large cardinals has been very fruitful in a number of areas of Mathematics (109), not only in foundational, model-theoretic studies on Set Theory but also in many seemingly

[9] Such intense discussions between Bourbaki members were quite common, and, curiously, they very rarely culminated in someone's resignation.

unrelated areas such as Descriptive Set Theory, Measure Theory, Determinacy Theory and Combinatorics (138). Furthermore, the notion of universe has proved to be a valuable supporting tool in several important applications, for instance in Wiles' proof of the Taniyama-Shimura conjecture, in Deligne's proof of the Weil conjectures and in Faltings' proof of the Mordell conjecture (158). Therefore, the rationale for endorsing the AC applies for accepting TG as a valid mathematical framework, at least to a certain degree.

B Concepts of Category Theory

This appendix serves as a reference to a number of basic definitions of Category Theory that are employed in Chapters 1, 2 and 3, most crucially in Section 3.6. Following the approach of Bourbaki (41), we shall first introduce the notion of digraph and its associated basic concepts in Section B.1, and then we shall define categories as digraphs with additional structure in Section B.2. The remaining sections are devoted to three fundamental universal constructs in Category Theory, namely the concepts of limit, colimit and quotient. We refer the reader to (7) for a thorough account of Digraph Theory, and the works (2, 6, 215, 220) are easy-to-read introductory texts on Category Theory in which the reader can find further explanations on the concepts included here.

B.1 Digraphs

A **digraph** is an abstract structure that represents a set of **vertices** together with the connections existing between these vertices, so that each such connection, called an **arrow**, is expressly oriented from a vertex to another vertex.

Definition B.1. A **directed graph** or a **digraph**[1] is a set $\boldsymbol{G}$ endowed with a set V and two functions $\mathrm{i}, \mathrm{t} : \boldsymbol{G} \to V$. We have the following notations and terminology:

- ▷ We shall denote digraphs by boldface uppercase latin letters, such as $\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{C}, \boldsymbol{D} \ldots$;

[1] Many authors call this structure a **directed multigraph** or a **multidigraph**, reserving the names 'directed graph' and 'digraph' to a more restricted notion.

- ▷ An **arrow in $\boldsymbol{G}$** is an element of the set $\boldsymbol{G}$. Arrows are denoted by lowercase latin letters, such as $f, g, h \ldots$;

- ▷ A **vertex in $\boldsymbol{G}$** is an element of the set V. Vertices are denoted by uppercase latin letters, such as $A, B, U, V, \ldots$. Whenever no particular notation is attributed to the set of vertices V of $\boldsymbol{G}$, it shall be denoted by $\boldsymbol{G}_0$;

- ▷ The **initial vertex function of $\boldsymbol{G}$** is the function $\mathrm{i} : \boldsymbol{G} \to V$ and the **terminal vertex function of $\boldsymbol{G}$** is the function $\mathrm{t} : \boldsymbol{G} \to V$. Whenever no particular notation is attributed to these functions, they shall be denoted by $\mathrm{i}_{\boldsymbol{G}}$ and $\mathrm{t}_{\boldsymbol{G}}$, respectively. For each arrow f in $\boldsymbol{G}$, the **initial vertex of f** is the image $\mathrm{i}(f)$ and the **terminal vertex of f** is the image $\mathrm{t}(f)$;

- ▷ For each $A, B \in V$, the set of arrows in $\boldsymbol{G}$ with initial vertex A and with terminal vertex B is denoted by $\overset{\boldsymbol{G}}{\mathrm{Arr}}(A, B)$, and the condition $f \in \overset{\boldsymbol{G}}{\mathrm{Arr}}(A, B)$ is denoted by $f : A \xrightarrow{\boldsymbol{G}} B$;

- ▷ A digraph $\boldsymbol{G}$ is **finite** (resp. **vertex-finite**, **arrow-finite**) if it has a finite number of vertices and arrows (resp. a finite number of vertices, a finite number of arrows). Otherwise, it is **infinite** (resp. **vertex-infinite**, **arrow-infinite**);

- ▷ A **subdigraph of $\boldsymbol{G}$** is a digraph $\boldsymbol{S}$ such that $\boldsymbol{S} \subset \boldsymbol{G}$, $\boldsymbol{S}_0 \subset \boldsymbol{G}_0$, $\mathrm{i}_{\boldsymbol{S}} \subset \mathrm{i}_{\boldsymbol{G}}$ and $\mathrm{t}_{\boldsymbol{S}} \subset \mathrm{t}_{\boldsymbol{G}}$. In that case, we also say that **$\boldsymbol{G}$ contains $\boldsymbol{S}$**;

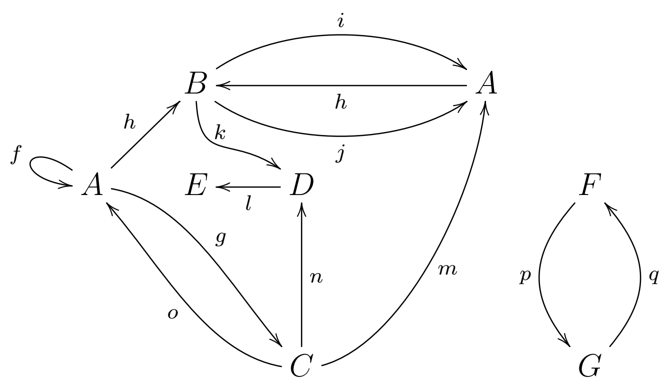
- ▷ A subdigraph $\boldsymbol{S}$ of $\boldsymbol{G}$ is **full in $\boldsymbol{G}$** if every arrow in $\boldsymbol{G}$ whose initial and terminal vertices belong to $\boldsymbol{S}_0$ is an arrow in $\boldsymbol{S}$.

Our definition of digraph places greater emphasis on the set of arrows and less on the set of vertices of the digraph. Such definitional choice is due to the fact that we shall define the notion of category by use of the notion of digraph (Definition B.15), bearing in mind the fact that the results of Category Theory characteristically "*subjugate the role of the mathematical object to the role of its network of relationships*" (157).

Example B.2. Let V be a set. The **empty digraph on** V is the set $\emptyset$ endowed with the set V and the functions $\mathrm{i}, \mathrm{t} : \emptyset \to V$, and it is denoted by Null_V. In particular, if $V = \emptyset$, then the empty digraph $\mathrm{Null}_V = \mathrm{Null}_\emptyset$ is called the **null digraph**, and if V is a singleton $\{X\}$, then the empty digraph $\mathrm{Null}_V = \mathrm{Null}_{\{X\}}$ is called the **trivial digraph with vertex** X.

Some digraphs may be fully described by graphical representations:

Example B.3. The graphical representation

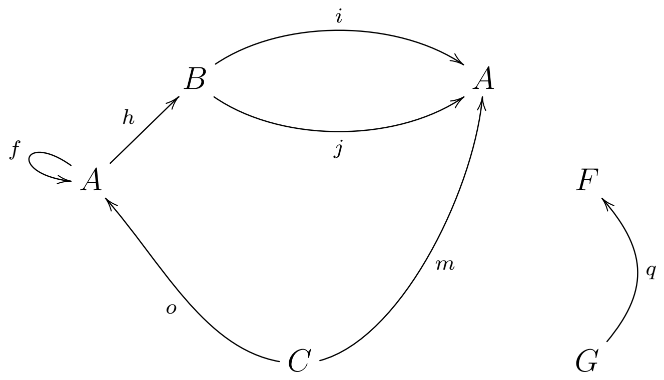


depicts the finite digraph $\boldsymbol{G} \coloneqq \{f, g, h, i, j, k, l, m, n, o, p, q\}$ with set of vertices given by $\boldsymbol{G}_0 \coloneqq \{A, B, C, D, E, F, G\}$ and with

$$\begin{cases} \mathrm{i}_{\boldsymbol{G}} \coloneqq \{(f, A), (g, A), (h, A), (i, B), (j, B), (k, B), (l, D), (m, C), (n, C), (o, C), (p, F), (q, G)\} \ , \\ \mathrm{t}_{\boldsymbol{G}} \coloneqq \{(f, A), (g, C), (h, B), (i, A), (j, A), (k, D), (l, E), (m, A), (n, D), (o, A), (p, G), (q, F)\} \ . \end{cases}$$

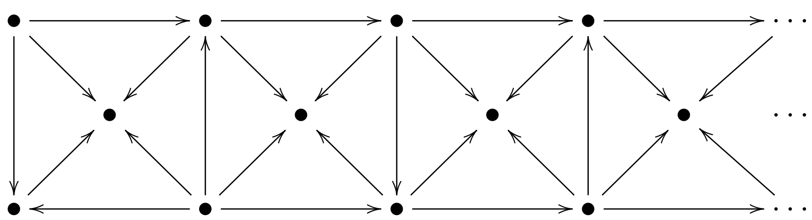
That representation is by no means unique. As a matter of fact, a vertex and an arrow may appear in different locations in a graphical representation of a digraph, with the provision that repeated arrows must appear with the same initial and terminal vertices assigned to them to keep consistency. Also, some parts of a digraph may appear completely disconnected from the rest of the digraph. Accordingly, a digraph may have infinitely many different graphical representations.

By arbitrarily deleting a few vertices and a few arrows, we obtain the following subdigraph $\boldsymbol{S}$ of $\boldsymbol{G}$:



By abuse of language, we often say that a graphical depiction of a digraph *is* the digraph itself.

Example B.4. Sometimes we are mostly interested in the *shape* of the digraph and we omit the labels of the vertices and arrows. For example, we may speak of the infinite digraph



When such label omission is performed, there can be no repeated vertices or repeated arrows, though, meaning that any two drawn dots represent different vertices of the digraph and any two drawn arrows represent different arrows of the digraph.

One may straightforwardly obtain new digraphs from known ones, as the following definition shows:

Definition B.5. Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be two digraphs.

▷ The **opposite digraph of $\boldsymbol{G}$** is the digraph $(\boldsymbol{G}, \boldsymbol{G}_0, \mathrm{i}^{\mathrm{op}}, \mathrm{t}^{\mathrm{op}})$, where i^{op} and t^{op} are the functions of type $\boldsymbol{G} \to \boldsymbol{G}_0$ given by

$$\mathrm{i}^{\mathrm{op}}(f) \coloneqq \mathrm{t}_{\boldsymbol{G}}(f) \quad \text{and} \quad \mathrm{t}^{\mathrm{op}}(f) \coloneqq \mathrm{i}_{\boldsymbol{G}}(f) \ ,$$

and it is denoted by $\boldsymbol{G}^{\mathrm{op}}$;

▷ The **product digraph of $\boldsymbol{G}$ and $\boldsymbol{H}$** is the digraph

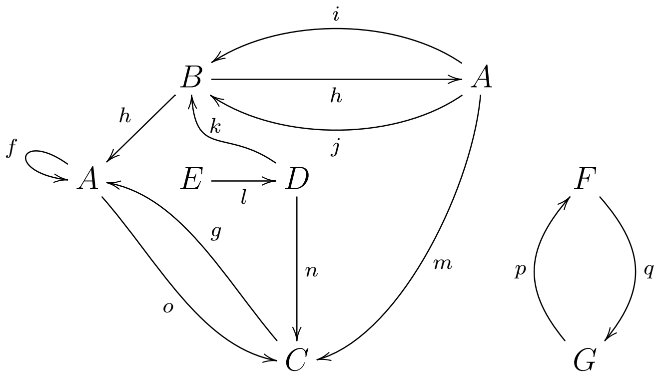
$$(\boldsymbol{G} \times \boldsymbol{H}, \boldsymbol{G}_0 \times \boldsymbol{H}_0, \mathrm{i}_{\boldsymbol{G}\times\boldsymbol{H}}, \mathrm{t}_{\boldsymbol{G}\times\boldsymbol{H}}) \ ,$$

where $\mathrm{i}_{\boldsymbol{G}\times\boldsymbol{H}}, \mathrm{t}_{\boldsymbol{G}\times\boldsymbol{H}} \colon \boldsymbol{G} \times \boldsymbol{H} \to \boldsymbol{G}_0 \times \boldsymbol{H}_0$ are the functions given by

$$\mathrm{i}_{\boldsymbol{G}\times\boldsymbol{H}}(f, g) \coloneqq (\mathrm{i}_{\boldsymbol{G}}(f)\,, \mathrm{i}_{\boldsymbol{H}}(g)) \quad \text{and} \quad \mathrm{t}_{\boldsymbol{G}\times\boldsymbol{H}}(f, g) \coloneqq (\mathrm{t}_{\boldsymbol{G}}(f)\,, \mathrm{t}_{\boldsymbol{H}}(g)) \ ,$$

and it is denoted by $\boldsymbol{G} \times \boldsymbol{H}$, by abuse of language.

Example B.6. The opposite digraph $\boldsymbol{G}^{\mathrm{op}}$ of the finite digraph $\boldsymbol{G}$ described in Example B.3 is given by



The primary tools for comparing digraphs are given in the following definition:

Definition B.7. Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be two digraphs.

▷ A **morphism** $\phi : \boldsymbol{G} \to \boldsymbol{H}$ is a 4-tuple

$$\phi = (\phi_0 \colon \boldsymbol{G}_0 \to \boldsymbol{H}_0, \phi_{\mathrm{Arr}} \colon \boldsymbol{G} \to \boldsymbol{H}, \boldsymbol{G}, \boldsymbol{H})$$

such that $\mathrm{i}_{\boldsymbol{H}} \circ \phi_{\mathrm{Arr}} = \phi_0 \circ \mathrm{i}_{\boldsymbol{G}}$ and $\mathrm{t}_{\boldsymbol{H}} \circ \phi_{\mathrm{Arr}} = \phi_0 \circ \mathrm{t}_{\boldsymbol{G}}$, that is, such that for every arrow $f : A \xrightarrow{\boldsymbol{G}} B$, we have $\phi_{\mathrm{Arr}}(f) : \phi_0(A) \xrightarrow{\boldsymbol{H}} \phi_0(B)$;

- ▷ By abuse of language, for each $A \in \boldsymbol{G}_0$ and each $f \in \boldsymbol{G}$, the vertex $\phi_0(A)$ and the arrow $\phi_{\mathrm{Arr}}(f)$ are denoted simply by $\phi(A)$ and $\phi(f)$. It is customary to state the definition of a morphism $\phi : \boldsymbol{G} \to \boldsymbol{H}$ as if it were a single function, specifying its values at every object and morphism in $\boldsymbol{G}$ and omitting the mention of the functions ϕ_0 and ϕ_{Arr};
- ▷ The **shape** of a morphism $\phi : \boldsymbol{G} \to \boldsymbol{H}$ is the digraph $\boldsymbol{G}$;
- ▷ A morphism $\phi : \boldsymbol{G} \to \boldsymbol{H}$ is **finite** (resp. **vertex-finite**, **arrow-finite**) if its shape is finite (resp. vertex-finite, arrow-finite);
- ▷ A morphism $\phi : \boldsymbol{G} \to \boldsymbol{H}$ is **surjective** if both functions ϕ_0 and ϕ_{Arr} are surjective;
- ▷ An **isomorphism** $\phi : \boldsymbol{G} \to \boldsymbol{H}$ is a morphism ϕ of that type such that the functions ϕ_0 and ϕ_{Arr} are bijective. The digraphs $\boldsymbol{G}$ and $\boldsymbol{H}$ are **isomorphic** if there is an isomorphism $\phi : \boldsymbol{G} \to \boldsymbol{H}$.

Example B.8. Let $\boldsymbol{G}$ be a digraph and let $\boldsymbol{S}$ be a subdigraph of $\boldsymbol{G}$. The **canonical immersion on $\boldsymbol{G}$ induced by $\boldsymbol{S}$** is the morphism $\phi : \boldsymbol{S} \to \boldsymbol{G}$ given by $\phi(f) \coloneqq f$ $(\forall f \in \boldsymbol{S})$ and $\phi(A) \coloneqq A$ $(\forall A \in \boldsymbol{S}_0)$. Hence, every subdigraph of $\boldsymbol{G}$ gives rise to a morphism which is inherently associated to it.

Example B.9. Let $\boldsymbol{G}$ be the finite digraph described in Example B.3, let $\boldsymbol{H}$ be a digraph and let $\phi : \boldsymbol{G} \to \boldsymbol{H}$ be a morphism. The digraph

$$
\begin{tikzcd}
 & \phi(B) \arrow[rr, bend left, "\phi(i)"] \arrow[rr, bend right, "\phi(j)"'] \arrow[dr, bend right, "\phi(k)"] & & \phi(A) \arrow[ll, "\phi(h)"] & \\
\phi(A) \arrow[loop left, "\phi(f)"] \arrow[ur, "\phi(h)"] \arrow[drr, bend left, "\phi(g)"] & \phi(E) & \phi(D) \arrow[l, "\phi(l)"] & & \phi(F) \arrow[d, bend right, "\phi(p)"'] \\
 & & \phi(C) \arrow[ull, bend left, "\phi(o)"] \arrow[u, "\phi(n)"] \arrow[uur, bend right, "\phi(m)"'] & & \phi(G) \arrow[u, bend right, "\phi(q)"']
\end{tikzcd}
$$

is a subdigraph of $\boldsymbol{H}$. As a particular example, we might define $\boldsymbol{H}$ to be the digraph

$$\begin{tikzcd}
 & L \arrow[dl, bend left, "v"] \arrow[dr, "u"] & & N \arrow[loop right, "y"] \\
K \arrow[loop left, "r"] \arrow[ur, bend left, "s"] \arrow[rr, bend left, "t"] & & M \arrow[ll, bend left, "w"] \arrow[ur, "x"] &
\end{tikzcd}$$

and we might define ϕ to be as follows:

$$\begin{cases} \phi\left(A\right), \phi\left(D\right), \phi\left(F\right), \phi\left(G\right) \coloneqq K \\ \phi\left(B\right), \phi\left(E\right) \coloneqq L \\ \phi\left(C\right) \coloneqq M \end{cases} \quad \text{and} \quad \begin{cases} \phi\left(f\right), \phi\left(p\right), \phi\left(q\right) \coloneqq r \\ \phi\left(h\right), \phi\left(l\right) \coloneqq s \\ \phi\left(g\right) \coloneqq t \\ \phi\left(i\right), \phi\left(j\right), \phi\left(k\right) \coloneqq v \\ \phi\left(m\right), \phi\left(n\right), \phi\left(o\right) \coloneqq w \end{cases}$$

Note that the vertex N and the arrows u, x and y in $\boldsymbol{H}$ are not images of ϕ.

A morphism between digraphs may be described graphically, as the following example shows:

Example B.10. The graphical representation

$$\begin{tikzcd}
 & B \arrow[dd, "i"] \\
A \arrow[ur, "f"] \arrow[dr, bend left, "h"] \arrow[dr, bend right, "g"'] & \\
 & C
\end{tikzcd}
\overset{\phi}{\Longrightarrow}
\begin{tikzcd}
 & Q \arrow[dd, "t"] \\
P \arrow[ur, "r"] \arrow[dr, bend left, "s"] \arrow[dr, bend right, "s"'] & \\
 & R
\end{tikzcd}$$

depicts a morphism $\phi : \boldsymbol{G} \to \boldsymbol{H}$, where $\boldsymbol{G}$ is the digraph

$$\begin{tikzcd}
 & B \arrow[dd, "i"] \\
A \arrow[ur, "f"] \arrow[dr, bend left, "h"] \arrow[dr, bend right, "g"'] & \\
 & C
\end{tikzcd}$$

and $\boldsymbol{H}$ is a digraph that contains the digraph

$$\begin{tikzcd} & Q \arrow[d, "t"] \\ P \arrow[ur, "r"] \arrow[r, bend left, "s"] \arrow[r, bend right, "s"'] & R \end{tikzcd}$$

Such morphism ϕ is given by

$$\begin{cases} \phi\,(A) \coloneqq P\,, \\ \phi\,(B) \coloneqq Q\,, \\ \phi\,(C) \coloneqq R \end{cases} \quad \text{and} \quad \begin{cases} \phi\,(f) \coloneqq r\,, \\ \phi\,(g)\,, \phi\,(h) \coloneqq s\,, \\ \phi\,(i) \coloneqq t\,. \end{cases}$$

That description of ϕ is still not complete, since the representation does not provide the codomain $\boldsymbol{H}$ of ϕ. When that codomain is not specified in the context, we assume ϕ is surjective, that is, we assume that the codomain is entirely shown in the right hand side of the drawing.

In many applications, it is necessary to keep track of ordered pairs of arrows (f, g) such that the second arrow ends where the first begins:

Definition B.11. Let $\boldsymbol{G}$ be a digraph. We call the binary relation

$$\mathrm{Link}_{\boldsymbol{G}} \coloneqq \{(f, g) \in \boldsymbol{G} \times \boldsymbol{G} \mid \mathrm{t}_{\boldsymbol{G}}\,(g) = \mathrm{i}_{\boldsymbol{G}}\,(f)\}$$

on $\boldsymbol{G}$ the **linking relation on $\boldsymbol{G}$**. We have the following terminology:

- ▷ Two arrows f and g in $\boldsymbol{G}$ are **linked in** G if $(f, g) \in \mathrm{Link}_{\boldsymbol{G}}$ or $(g, f) \in \mathrm{Link}_{\boldsymbol{G}}$;
- ▷ The pair of arrows $(f, g) \in \boldsymbol{G} \times \boldsymbol{G}$ is **composable in $\boldsymbol{G}$** if $(f, g) \in \mathrm{Link}_{\boldsymbol{G}}$.

Note that $\mathrm{Link}_{\boldsymbol{G}^{\mathrm{op}}} = (\mathrm{Link}_{\boldsymbol{G}})^{-1}$.

Example B.12. Let $\boldsymbol{G}$ be the finite digraph described in Example B.3. The linking relation on $\boldsymbol{G}$ is given by

$$\mathrm{Link}_{\boldsymbol{G}} = \{(f,f)\,,(g,f)\,,(h,f)\,,(m,g)\,,(n,g)\,,(o,g)\,,(i,h)\,,(j,h)\,,(k,h)\,,(h,i)\,,\\ (h,j)\,,(l,k)\,,(h,m)\,,(l,n)\,,(f,o)\,,(g,o)\,,(h,o)\,,(q,p)\,,(p,q)\}\ .$$

A simple yet extremely important concept in Digraph Theory is given in the following definition:

Definition B.13. Let $\boldsymbol{G}$ be a digraph, let A and B be two vertices in $\boldsymbol{G}$ and let $n \in \mathbb{N}_0$. A **walk from** A **to** B **in** $\boldsymbol{G}$ (**of length** n) is an ordered $(n+2)$-tuple $w = (A, f_1 \dots f_n, B)$, where $f_1 \dots f_n$ is a finite sequence of arrows in $\boldsymbol{G}$ such that:

- ▷ either $n = 0$ and $A = B$
- ▷ or $n \geqslant 1$, $\mathrm{i}_{\boldsymbol{G}}(f_1) = A$, $(f_{i+1}, f_i) \in \mathrm{Link}_{\boldsymbol{G}}$ $(\forall i \in [1,n)_{\mathbb{N}})$ and $\mathrm{t}_{\boldsymbol{G}}(f_n) = B$.

In that case, we denote $w : A \xrightarrow[\text{walk}]{\boldsymbol{G}} B$ and we define $\mathrm{i}_{\boldsymbol{G}}(w) \coloneqq A$ and $\mathrm{t}_{\boldsymbol{G}}(w) \coloneqq B$. A **parallel pair of walks** in $\boldsymbol{G}$ is a pair of walks $w_1, w_2 : A \xrightarrow{\boldsymbol{G}} B$ which have the same initial and terminal vertices.

Example B.14. Let $\boldsymbol{G}$ be the finite digraph described in Example B.3. The following tuples are walks in $\boldsymbol{G}$:

- ▷ $c_1 \coloneqq (E, \emptyset, E)$
  (the **trivial cycle around** E);
- ▷ $c_2 \coloneqq (A, f, f, f, A)$;
- ▷ $c_3 \coloneqq (C, n, D)$;
- ▷ $c_3 \coloneqq (A, h, j, h, k, l, E)$;
- ▷ $c_4 \coloneqq (C, o, f, g, m, h, k, D)$.

B.2 Categories

Perhaps the most influential and impactful mathematical theory since the mid-20th century has been Eilenberg and Mac Lane's **Category Theory** (130). Introduced in 1945 in order to study the notion of natural isomorphism, and *ab initio* showing applications to Algebraic Topology and Homological Algebra (70), this field has provided a powerful conceptual framework under which many frequently recurrent patterns in a variety of areas of Mathematics could be identified and described. Its range of applications has flourished since its introduction, reaching fields of study as diverse as Algebraic Geometry, Mathematical Physics, Computer Science and the Foundations of Mathematics (130, 119). In this section, we shall present a compendium of basic definitions of Category Theory for reference purposes.

Definition B.15. A **category** is a digraph $\boldsymbol{C}$ endowed with a function $\circ : \mathrm{Link}_{\boldsymbol{C}} \to \boldsymbol{C}$ (Definition B.11) such that:

- ▷ If $f : A \xrightarrow{\boldsymbol{C}} B$ and $g : B \xrightarrow{\boldsymbol{C}} C$ are arrows, then $g \circ f : A \xrightarrow{\boldsymbol{C}} C$;
- ▷ If $f : A \xrightarrow{\boldsymbol{C}} B$, $g : B \xrightarrow{\boldsymbol{C}} C$ and $h : C \xrightarrow{\boldsymbol{C}} D$ are arrows, then we have $h \circ (g \circ f) = (h \circ g) \circ f$;
- ▷ For each vertex A in $\boldsymbol{C}$, there is an arrow $\mathrm{id}_A : A \xrightarrow{\boldsymbol{C}} A$ such that $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$ for all arrows $f, g \in \boldsymbol{C}$ so that $\mathrm{t}_{\boldsymbol{C}}(f) = \mathrm{i}_{\boldsymbol{C}}(g) = A$. One can easily verify that the arrow id_A is unique.

We have the following notations and terminology:

- ▷ The vertices in $\boldsymbol{C}$ are called **objects in $\boldsymbol{C}$**;
- ▷ The arrows in $\boldsymbol{C}$ are called **morphisms in $\boldsymbol{C}$**;
- ▷ The **composition operation on $\boldsymbol{C}$** is the function $\circ : \mathrm{Link}_{\boldsymbol{C}} \to \boldsymbol{C}$. Whenever no particular notation is attributed to that function, it shall be denoted by $\circ_{\boldsymbol{C}}$;

- ▷ The sets of the form $\overset{\boldsymbol{C}}{\mathrm{Arr}}(A, B)$ (Definition B.1) shall be denoted by $\overset{\boldsymbol{C}}{\mathrm{Hom}}(A, B)$, where A and B are two objects in the category $\boldsymbol{C}$;
- ▷ A category $\boldsymbol{C}$ is **finite** (resp. **object-finite**) if its underlying digraph is finite (resp. vertex-finite). Otherwise, it is **infinite** (resp. **object-infinite**);
- ▷ A **subcategory of** $\boldsymbol{C}$ is a subdigraph $\boldsymbol{S}$ of the underlying digraph of $\boldsymbol{C}$ such that $\boldsymbol{S}$ is a category when endowed with the restriction $\circ_{\boldsymbol{C}} \restriction_{\mathrm{Link}(\boldsymbol{S})}$. In that case, we also say that $\boldsymbol{C}$ **contains** $\boldsymbol{S}$;
- ▷ A **digraph in** $\boldsymbol{C}$ is a subdigraph $\boldsymbol{G}$ of the underlying digraph of $\boldsymbol{C}$. The **subcategory of** $\boldsymbol{C}$ **generated by** $\boldsymbol{G}$ is the smallest subcategory of $\boldsymbol{C}$ containing all the vertices and arrows of $\boldsymbol{G}$, and it shall be denoted by $\mathrm{Cat}_{\boldsymbol{C}}(\boldsymbol{G})$. We have $(\mathrm{Cat}_{\boldsymbol{C}}(\boldsymbol{G}))_0 = \boldsymbol{G}_0$, that is, the category $\mathrm{Cat}_{\boldsymbol{C}}(\boldsymbol{G})$ is obtained from $\boldsymbol{G}$ only by introducing arrows to $\boldsymbol{G}$;
- ▷ The ($\circ_{\boldsymbol{C}}$-)**composition** of a walk $(A, f_1 \ldots f_m, B)$ in $\boldsymbol{C}$ of length $m \geqslant 2$ (Definition B.13) is the composition $f_m \circ \cdots \circ f_1 : A \xrightarrow{\boldsymbol{C}} B$;
- ▷ A digraph $\boldsymbol{G}$ in $\boldsymbol{C}$ **commutes** or is **commutative** (**in** $\boldsymbol{C}$) if any two parallel walks $w_1, w_2 : A \xrightarrow[\text{walk}]{\boldsymbol{G}} B$ of lengths $\geqslant 2$ have the same composition in $\boldsymbol{C}$.

The notion of category is a generalisation of the notion of monoid. Recall that a monoid is a set M endowed with an associative operation $\times_M$ on M that has an identity element 1_M in M (Definition 1.5). In a similar way, as we have stated above, a category is a digraph $\boldsymbol{C}$ endowed with an associative (partial) operation $\circ_{\boldsymbol{C}}$ on $\boldsymbol{C}$ that has an identity element id_A for each object A in $\boldsymbol{C}$. As a matter of fact, every monoid M may be seen as a category with only one object whose (endo)morphisms are the elements of M and whose composition operation is the operation of M. Thus, a category is an assemblage of interconnected monoids, in a sense.

Example B.16. Note that $\mathrm{Link}_{\mathrm{Null}_\emptyset} = \emptyset$ (Example B.2; Definition B.11). The **null category** is the null digraph $\mathrm{Null}_\emptyset$ endowed with the empty function $\circ : \emptyset \to \emptyset$, and it shall be denoted by $\mathrm{Null}_\emptyset$ by abuse of language.

Example B.17. A **large category** is a category such that the existence of its set of morphisms implies a contradiction in ZFC. Some examples of large categories are the following:

- ▷ The **category of sets** is the category whose objects are the sets, whose morphisms are the functions between sets and whose composition operation is the usual functional composition. This category is denoted by **Set**;
- ▷ The **category of semigroups**, **SGrp** (Definition 1.2);
- ▷ The **category of monoids**, **Mon** (Definition 1.5);
- ▷ The **category of groups**, **Grp** (Definition 1.7);
- ▷ The **category of ordered sets**, **SetOrd** (Definition 1.26);
- ▷ The **category of topological spaces**, **Top** (Definition 1.36);
- ▷ The **category of rngs**, **Rng** (Definition 2.1);
- ▷ The **category of rings**, **Ring** (Definition 2.5).

The category **Grp** is a subcategory of **Mon**, which in turn is a subcategory of **SGrp**. The category **Ring** is a subcategory of **Rng**. In Sections A.2 and A.3 of Appendix A, we discuss how large structures are usually dealt with in the ZFC, NBG and TG axiomatic frameworks.

Example B.18. Let $I = \{i, j, k, \dots\}$ be a set. The **discrete category on** I is the digraph

$$\begin{tikzcd} i \arrow[loop left, "\mathrm{id}_i"] & j \arrow[loop left, "\mathrm{id}_j"] & k \arrow[loop left, "\mathrm{id}_k"] & l \arrow[loop left, "\mathrm{id}_l"] & \cdots \end{tikzcd}$$

endowed with the composition operation given by $\mathrm{id}_x \circ \mathrm{id}_x \coloneqq \mathrm{id}_x$ $(\forall x \in I)$, and we shall denote it by $\mathrm{Disc}\,(I)$. A category $\boldsymbol{C}$ is **discrete** if there is a set I such that $\boldsymbol{C} = \mathrm{Disc}\,(I)$.

One may prove that for any digraph $\boldsymbol{G}$, there is a smallest[2] category $\boldsymbol{C}$ such that $\boldsymbol{G}$ is a digraph in $\boldsymbol{C}$ and that no morphism in $\boldsymbol{G}$ is the $\circ_{\boldsymbol{C}}$-composition

[2] Up to isomorphism (Definition B.25).

of a walk in $\boldsymbol{G}$ of length $\geqslant 2$. With that, we say that each graphical representation of $\boldsymbol{G}$ (Section B.1) is a **graphical representation of $\boldsymbol{C}$**. Note that no $\circ_{\boldsymbol{C}}$-identity morphism is in $\boldsymbol{G}$.

Example B.19. The graphical representation

$$\begin{tikzcd} C \arrow[r, "f"] & D \arrow[r, "g"] & E \end{tikzcd}$$

depicts the finite category $\boldsymbol{C}$ whose underlying digraph is

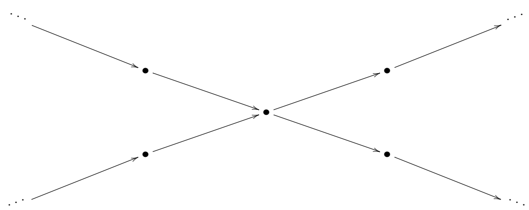
$$\begin{tikzcd} C \arrow[loop above, "\mathrm{id}_C"] \arrow[r, "f"] \arrow[rr, bend right, "h"'] & D \arrow[loop above, "\mathrm{id}_D"] \arrow[r, "g"] & E \arrow[loop above, "\mathrm{id}_E"] \end{tikzcd}$$

and whose composition operation is given by

- ▷ $g \circ f := h$;
- ▷ $\mathrm{id}_C \circ \mathrm{id}_C := \mathrm{id}_C$;
- ▷ $f \circ \mathrm{id}_C := f$;
- ▷ $h \circ \mathrm{id}_C := h$;
- ▷ $\mathrm{id}_D \circ \mathrm{id}_D := \mathrm{id}_D$;
- ▷ $\mathrm{id}_D \circ f := f$;
- ▷ $g \circ \mathrm{id}_D := g$;
- ▷ $\mathrm{id}_E \circ \mathrm{id}_E := \mathrm{id}_E$;
- ▷ $\mathrm{id}_E \circ g := g$;
- ▷ $\mathrm{id}_E \circ h := h$.

By abuse of language, we often say that a graphical depiction of a category *is* the category itself.

Example B.20. Sometimes we are mostly interested in the *shape* of a category and we omit the labels of the vertices and arrows. For example, we may speak of the infinite category



When such label omission is performed, there can be no repeated vertices or repeated arrows, though, meaning that any two drawn dots represent different objects of the category and any two drawn arrows represent different morphisms of the category.

One may straightforwardly obtain new categories from known ones, as the following definition shows:

Definition B.21. Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two categories.

- ▷ The **opposite category of $\boldsymbol{C}$** is the digraph $\boldsymbol{C}^{\mathrm{op}}$ (Definition B.5) endowed with the **opposite operation induced by $\circ_{\boldsymbol{C}}$**, which is the function
$$\circ^{\mathrm{op}}_{\boldsymbol{C}} : \mathrm{Link}_{\boldsymbol{C}^{\mathrm{op}}} = (\mathrm{Link}_{\boldsymbol{C}})^{-1} \longrightarrow \boldsymbol{C}$$
given by $f \circ^{\mathrm{op}}_{\boldsymbol{C}} g \coloneqq g \circ_{\boldsymbol{C}} f$. We always assume that the opposite digraph $\boldsymbol{C}^{\mathrm{op}}$ is endowed with the operation $\circ^{\mathrm{op}}_{\boldsymbol{C}}$, turning $\boldsymbol{C}^{\mathrm{op}}$ into a category;
- ▷ The **product category of $\boldsymbol{C}$ and $\boldsymbol{D}$** is the product digraph $\boldsymbol{C} \times \boldsymbol{D}$ (Definition B.5) endowed with the **product operation induced by $\circ_{\boldsymbol{C}}$ and $\circ_{\boldsymbol{D}}$**, which is the function
$$\underset{\boldsymbol{C}\times\boldsymbol{D}}{\circ} : \mathrm{Link}_{\boldsymbol{C}\times\boldsymbol{D}} \longrightarrow \boldsymbol{C} \times \boldsymbol{D}$$
given by $(f, g) \underset{\boldsymbol{C}\times\boldsymbol{D}}{\circ} (f', g') \coloneqq (f \circ_{\boldsymbol{C}} f', g \circ_{\boldsymbol{D}} g')$. We always assume that the product digraph $\boldsymbol{C} \times \boldsymbol{D}$ is endowed with the operation $\underset{\boldsymbol{C}\times\boldsymbol{D}}{\circ}$, turning $\boldsymbol{C} \times \boldsymbol{D}$ into a category.

Some morphisms in a category $\boldsymbol{C}$ deserve special attention:

Definition B.22. Let $\boldsymbol{C}$ be a category.

- ▷ A morphism $f : A \xrightarrow{\boldsymbol{C}} B$ is an **endomorphism (in $\boldsymbol{C}$)** if $A = B$. In that case, we say that $f : A \xrightarrow{\boldsymbol{C}} A$ is an **endomorphism of** A;
- ▷ A morphism $f : A \xrightarrow{\boldsymbol{C}} B$ is **mono** or is a **monomorphism (in $\boldsymbol{C}$)** if for all objects C in $\boldsymbol{C}$ and all morphisms $g, h : C \xrightarrow{\boldsymbol{C}} A$, the equation $f \circ g = f \circ h$ implies $g = h$;

- ▷ A morphism $f : A \xrightarrow{\boldsymbol{C}} B$ is **epi** or is an **epimorphism** (**in** $\boldsymbol{C}$) if for all objects C in $\boldsymbol{C}$ and all morphisms $g, h : B \xrightarrow{\boldsymbol{C}} C$, the equation $g \circ f = h \circ f$ implies $g = h$;
- ▷ A morphism $f : A \xrightarrow{\boldsymbol{C}} B$ is **iso** or is an **isomorphism** (**in** $\boldsymbol{C}$) if there is a morphism $g : B \xrightarrow{\boldsymbol{C}} A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. In that case, the morphism g is unique, it is called the **inverse of** f (**in** $\boldsymbol{C}$) and it is denoted by f^{-1};
- ▷ An endomorphism $f : A \xrightarrow{\boldsymbol{C}} A$ is an **automorphism of** A (**in** $\boldsymbol{C}$) if it is iso;
- ▷ Two objects A and B in $\boldsymbol{C}$ are **isomorphic** (**in** $\boldsymbol{C}$), or simbolically $A \overset{\boldsymbol{C}}{\cong} B$, if there is an isomorphism $f : A \xrightarrow{\boldsymbol{C}} B$. One can easily verify that the binary relation
$$\overset{\boldsymbol{C}}{\cong} := \{(A, B) \in \boldsymbol{C}_0 \times \boldsymbol{C}_0 \mid A \text{ is isomorphic to } B \text{ in } \boldsymbol{C}\}$$
is an equivalence relation on $\boldsymbol{C}_0$.

A morphism $f : A \xrightarrow{\boldsymbol{C}} B$ is mono (resp. epi, iso) if, and only if, it is epi (resp. mono, iso) in $\boldsymbol{C}^{\mathrm{op}}$.

Example B.23. (2) In **Set**, the monomorphisms are the injective functions, the epimorphisms are the surjective functions, and the isomorphisms are the bijective functions.

Example B.24. (2) In **Ring**, the monomorphisms are the injective homomorphisms and the isomorphisms are the bijective homomorphisms. Every surjective homomorphism between rings is epi, but not all epimorphisms are surjective. For instance, since any homomorphism of type $\mathbb{Q} \xrightarrow{\mathbf{Ring}} R$ is entirely determined by its values on the integers, where R is any ring, note that the canonical inclusion of type $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in **Ring** that is not surjective.

Definition B.25. Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two categories. A **functor** $\mathcal{F}:\boldsymbol{C}\to\boldsymbol{D}$ is a morphism between the underlying digraphs (Definition B.7) such that:

- ▷ $\mathcal{F}(g\circ f)=\mathcal{F}(g)\circ\mathcal{F}(f)$ for every pair (f,g) of composable morphisms in $\boldsymbol{C}$;
- ▷ $(\forall C\in\boldsymbol{C}_0)\ \mathcal{F}(\mathrm{id}_C)=\mathrm{id}_{\mathcal{F}(C)}$.

We have the following notations and terminology:

- ▷ We denote the function $\mathcal{F}_{\mathrm{Arr}}\colon\boldsymbol{C}\to\boldsymbol{D}$ (Definition B.7) by $\mathcal{F}_{\mathrm{Hom}}$;
- ▷ A functor $\mathcal{F}:\boldsymbol{C}\to\boldsymbol{D}$ is **faithful** if for all $X,Y\in\boldsymbol{C}_0$, the restriction
$$\mathcal{F}_{\mathrm{Hom}}\restriction\left(\overset{\boldsymbol{C}}{\mathrm{Hom}}(X,Y)\right):\overset{\boldsymbol{C}}{\mathrm{Hom}}(X,Y)\to\overset{\boldsymbol{D}}{\mathrm{Hom}}(\mathcal{F}(X),\mathcal{F}(Y))$$
is injective;
- ▷ The **shape** of a functor $\mathcal{F}:\boldsymbol{C}\to\boldsymbol{D}$ is the category $\boldsymbol{C}$;
- ▷ A functor is **finite** (resp. **object-finite**) if its shape category is finite (resp. object-finite);
- ▷ A functor $\mathcal{F}:\boldsymbol{C}\to\boldsymbol{D}$ is **surjective** if both functions $\mathcal{F}_0\colon\boldsymbol{C}_0\to\boldsymbol{D}_0$ and $\mathcal{F}_{\mathrm{Hom}}\colon\boldsymbol{C}\to\boldsymbol{D}$ are surjective;
- ▷ An **isomorphism** $\mathcal{F}:\boldsymbol{C}\to\boldsymbol{D}$ is a functor $\mathcal{F}$ of that type such that the functions $\mathcal{F}_0\colon\boldsymbol{C}_0\to\boldsymbol{D}_0$ and $\mathcal{F}_{\mathrm{Hom}}\colon\boldsymbol{C}\to\boldsymbol{D}$ are bijective. The categories $\boldsymbol{C}$ and $\boldsymbol{D}$ are **isomorphic** if there is an isomorphism $\mathcal{F}:\boldsymbol{C}\to\boldsymbol{D}$.

Note that the equation $\mathcal{F}(g\circ f)=\mathcal{F}(g)\circ\mathcal{F}(f)$ is only valid when it is known beforehand that the pair (f,g) is composable. Case in point, there may be a composable pair $(\mathcal{F}(f),\mathcal{F}(g))$ in $\boldsymbol{D}$ such that the pair (f,g) is not composable in $\boldsymbol{C}$, and, in that case, the term $\mathcal{F}(g\circ f)$ is meaningless. Three examples of functors are discussed at length in Section 3.6.

Example B.26. Let $\boldsymbol{C}$ be a category. A functor of type $\mathrm{Null}_{\emptyset}\to\boldsymbol{C}$ (Example B.16) is said to be a **null functor** of such type.

Example B.27. Let $\boldsymbol{C}$ be a category and let $\boldsymbol{G}$ be a digraph in $\boldsymbol{C}$. The **canonical immersion on $\boldsymbol{C}$ induced by** $\mathrm{Cat}_{\boldsymbol{C}}(\boldsymbol{G})$ (Definition B.15) is the functor $\mathcal{F} : \mathrm{Cat}_{\boldsymbol{C}}(\boldsymbol{G}) \to \boldsymbol{C}$ given by $\mathcal{F}(f) \coloneqq f$ $(\forall f \in \mathrm{Cat}_{\boldsymbol{C}}(\boldsymbol{G}))$ and $\mathcal{F}(C) \coloneqq C$ $(\forall C \in \boldsymbol{G}_0)$.

Example B.28. Let $\boldsymbol{C}$ be the finite category described in Example B.19, let $\boldsymbol{D}$ be a category and let $\mathcal{F} : \boldsymbol{C} \to \boldsymbol{D}$ be a functor. The category

$$\mathcal{F}(C) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(D) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(E)$$

is a subcategory of $\boldsymbol{D}$. As a particular example, we might define $\boldsymbol{D} \coloneqq \mathbf{Set}$ (Example B.17) and $\mathcal{F}$ to be the functor given by

$$\begin{cases} \mathcal{F}(C), \mathcal{F}(E) \coloneqq \mathbb{R}, \\ \mathcal{F}(D) \coloneqq \mathbb{R} \times \mathbb{R} \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{F}(f)(x) \coloneqq (x, 0), \\ \mathcal{F}(g)(x, y) \coloneqq y. \end{cases}$$

Since $\mathcal{F}(h) = \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$, the image $\mathcal{F}(h) : \mathbb{R} \to \mathbb{R}$ is the constant function with value 0.

A functor between categories may be described graphically, as the following example shows:

Example B.29. The graphical representation

$$\begin{tikzcd}
& A \arrow[rr, "f_4"] \arrow[dl, "f_1"'] \arrow[dd, "g_1"] & & D \arrow[dd, "g_4"] \\
B \arrow[rr, "f_2"] \arrow[dd, "g_2"'] & & C \arrow[ur, "f_3"'] \arrow[dd, "g_3"] & \\
& E \arrow[rr, "h_4"'] & & H \arrow[dl, "h_3"] \\
F \arrow[ur, "h_1"] \arrow[rr, "h_2"'] & & G &
\end{tikzcd}
\overset{\mathcal{F}}{\Longrightarrow}
\begin{tikzcd}
& X \arrow[rr, "q"] \arrow[dl, "p"'] \arrow[dd, "r"] & & Y \arrow[dd, "\mathrm{id}_Y"] \\
Y \arrow[rr, "\mathrm{id}_Y"] \arrow[dd, "\mathrm{id}_Y"'] & & Y \arrow[ur, "\mathrm{id}_Y"'] \arrow[dd, "u"] & \\
& Y \arrow[rr, "\mathrm{id}_Y"] & & Y \arrow[dl, "t"] \\
Y \arrow[ur, "\mathrm{id}_Y"] \arrow[rr, "s"'] & & Z &
\end{tikzcd}$$

depicts a functor $\mathcal{F} : \boldsymbol{C} \to \boldsymbol{D}$, where $\boldsymbol{C}$ is the finite category

$$\begin{tikzcd}
& A \arrow[rr, "f_4"] \arrow[dl, "f_1"'] \arrow[dd, "g_1"] & & D \arrow[dd, "g_4"] \\
B \arrow[rr, "f_2"] \arrow[dd, "g_2"'] & & C \arrow[ur, "f_3"'] \arrow[dd, "g_3"] & \\
& E \arrow[rr, "h_4"'] & & H \arrow[dl, "h_3"] \\
F \arrow[ur, "h_1"] \arrow[rr, "h_2"'] & & G &
\end{tikzcd}$$

and $\boldsymbol{D}$ is a category that contains the category

$$\begin{tikzcd}
& X \arrow[rr, "q"] \arrow[dl, "p"'] \arrow[dd, "r"] & & Y \arrow[dd, "\mathrm{id}_Y"] \\
Y \arrow[rr, "\mathrm{id}_Y"] \arrow[dd, "\mathrm{id}_Y"'] & & Y \arrow[ur, "\mathrm{id}_Y"'] \arrow[dd, "u"] & \\
& Y \arrow[rr, "\mathrm{id}_Y"] & & Y \arrow[dl, "t"] \\
Y \arrow[ur, "\mathrm{id}_Y"] \arrow[rr, "s"'] & & Z &
\end{tikzcd}$$

Such functor $\mathcal{F}$ is given by

$$\begin{cases} \mathcal{F}(A) \coloneqq X\,, \\ \mathcal{F}(B)\,, \mathcal{F}(C)\,, \mathcal{F}(D)\,, \mathcal{F}(E)\,, \mathcal{F}(F)\,, \mathcal{F}(H) \coloneqq Y\,, \\ \mathcal{F}(G) \coloneqq Z \end{cases}$$

and

$$\begin{cases} \mathcal{F}(f_1) \coloneqq p\,, \mathcal{F}(f_4) \coloneqq q\,, \mathcal{F}(g_1) \coloneqq r\,, \\ \mathcal{F}(f_2)\,, \mathcal{F}(f_3)\,, \mathcal{F}(g_2)\,, \mathcal{F}(g_4)\,, \mathcal{F}(h_1)\,, \mathcal{F}(h_4) \coloneqq s\,, \\ \mathcal{F}(h_2) \coloneqq s\,, \mathcal{F}(h_3) \coloneqq t\,, \mathcal{F}(g_3) \coloneqq u\,. \end{cases}$$

That description of $\mathcal{F}$ is still not complete, since the representation does not provide the codomain $\boldsymbol{D}$ of $\mathcal{F}$. When that codomain is not specified in the context, we assume $\mathcal{F}$ is surjective, that is, we assume that the codomain is entirely shown in the right hand side of the drawing.

The following proposition shows that functors preserve isomorphisms:

Proposition B.30. *(2) Let $\mathcal{F} : \boldsymbol{C} \to \boldsymbol{D}$ be a functor between categories. If $f : C \xrightarrow{\boldsymbol{C}} D$ is an isomorphism, then $\mathcal{F}(f) : \mathcal{F}(C) \xrightarrow{\boldsymbol{D}} \mathcal{F}(D)$ is an isomorphism.*

Proof. Since f is an isomorphism in $\boldsymbol{C}$, there is a morphism $g : C \xrightarrow{\boldsymbol{C}} D$ so that $g \circ f = \mathrm{id}_C$ and $f \circ g = \mathrm{id}_D$, and that gives us

$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(g \circ f) = \mathcal{F}(\mathrm{id}_C) = \mathrm{id}_{\mathcal{F}(C)}$$

and

$$\mathcal{F}(f) \circ \mathcal{F}(g) = \mathcal{F}(f \circ g) = \mathcal{F}(\mathrm{id}_D) = \mathrm{id}_{\mathcal{F}(D)},$$

proving that $\mathcal{F}(f)$ is an isomorphism in $\boldsymbol{D}$.

$\square$

B.3 Limits and colimits

Many mathematical constructions are defined by **universal properties**, loosely meaning that they are specified, up to isomorphism, by their relationship toward all related structures. In this section, we shall present two dual notions of that kind, the **limit** and the **colimit** of a functor, which are considerably relevant almost everywhere in Mathematics. Truth be told that only the limits play a meaningful role in the main body of this work, but the colimits were also included here since these two concepts are almost always presented together in literature.

Definition B.31. Let $\boldsymbol{I}$ and $\boldsymbol{C}$ be two categories, let $\mathcal{F} : \boldsymbol{I} \to \boldsymbol{C}$ be a functor and let V be an object in $\boldsymbol{C}$. A **cone over** (resp. **under**) $\mathcal{F}$ **with vertex** V is a family

$$\lambda = \{\lambda_i : V \xrightarrow{\boldsymbol{C}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0} \qquad (\text{resp. } \lambda = \{\lambda_i : \mathcal{F}(i) \xrightarrow{\boldsymbol{C}} V\}_{i \in \boldsymbol{I}_0})$$

such that for each morphism $k : i \xrightarrow{\boldsymbol{I}} i'$, the digraph

$$\begin{tikzcd} & V \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_{i'}"] & \\ \mathcal{F}(i) \arrow[rr, "\mathcal{F}(k)"'] & & \mathcal{F}(i') \end{tikzcd} \qquad \left(\text{resp.} \quad \begin{tikzcd} \mathcal{F}(i) \arrow[rr, "\mathcal{F}(k)"] \arrow[dr, "\lambda_i"'] & & \mathcal{F}(i') \arrow[dl, "\lambda_{i'}"] \\ & V & \end{tikzcd}\right)$$

in $\boldsymbol{C}$ is commutative. A **limit cone over** (resp. **under**) $\mathcal{F}$ is a cone

$$\chi = \{\chi_i : L \xrightarrow{\boldsymbol{C}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0} \qquad (\text{resp. } \chi = \{\chi_i : \mathcal{F}(i) \xrightarrow{\boldsymbol{C}} L\}_{i \in \boldsymbol{I}_0})$$

over (resp. under) $\mathcal{F}$ with a vertex L that satisfies the **universal property of limits** (resp. **colimits**) **in** $\boldsymbol{C}$, that is, it is such that for each cone λ over (resp. under) $\mathcal{F}$ with vertex V, there is a unique morphism $\overline{\lambda} : V \xrightarrow{\boldsymbol{C}} L$ (resp. $\overline{\lambda} : L \xrightarrow{\boldsymbol{C}} V$) such that the digraph

$$\begin{tikzcd} & L \arrow[d, "\chi_i"] \\ V \arrow[ur, "\overline{\lambda}"] \arrow[r, "\lambda_i"'] & \mathcal{F}(i) \end{tikzcd} \qquad \left(\text{resp.} \quad \begin{tikzcd} \mathcal{F}(i) \arrow[r, "\lambda_i"] \arrow[d, "\chi_i"'] & V \\ L \arrow[ur, "\overline{\lambda}"'] & \end{tikzcd}\right)$$

in $\boldsymbol{C}$ is commutative for all $i \in \boldsymbol{I}_0$. In that case, the morphism $\overline{\lambda}$ is called the **limit lifting** (resp. **colimit lowering**) **of the cone** λ **along** χ.

The term 'limit lifting' (resp. 'colimit lowering') comes from a physical analogy, as we shall describe. In the second diagram shown in Definition B.31, one may imagine that the morphism χ_i of the limit cone χ represents the direction of the gravitational field in the surroundings. Thus, in order to reorientate the morphism λ_i of the cone λ into the direction of the morphism $\overline{\lambda}$, one would need to lift (resp. lower) the terminal (resp. initial) end of the morphism λ_i from the object $\mathcal{F}(i)$ to the object L.

When a functor $\mathcal{F} : \boldsymbol{I} \to \boldsymbol{C}$ has a limit cone over (resp. under) it with vertex L, then L is unique up to a unique isomorphism. More precisely, if

$$\chi = \{\chi_i : L \xrightarrow{\boldsymbol{C}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0} \quad \text{and} \quad \chi' = \{\chi'_i : L' \xrightarrow{\boldsymbol{C}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$$

$$(\text{resp. } \chi = \{\chi_i : \mathcal{F}(i) \xrightarrow{\boldsymbol{C}} L\}_{i \in \boldsymbol{I}_0} \quad \text{and} \quad \chi' = \{\chi'_i : \mathcal{F}(i) \xrightarrow{\boldsymbol{C}} L'\}_{i \in \boldsymbol{I}_0})$$

are limit cones over (resp. under) $\mathcal{F}$, then there is a unique isomorphism $\alpha : L \xrightarrow{\boldsymbol{C}} L'$ such that the digraph

$$\begin{tikzcd} L \arrow[d, "\alpha"'] \arrow[dr, "\chi_i"] & \\ L' \arrow[r, "\chi'_i"'] & \mathcal{F}(i) \end{tikzcd} \qquad \left(\text{resp.} \quad \begin{tikzcd} & L \arrow[d, "\alpha"] \\ \mathcal{F}(i) \arrow[ur, "\chi_i"] \arrow[r, "\chi'_i"'] & L' \end{tikzcd} \right)$$

in $\boldsymbol{C}$ is commutative for all $i \in I$. With that in mind, we say that the object L (or any object in $\boldsymbol{C}$ isomorphic to L) is the **limit** (resp. **colimit**) **of** $\mathcal{F}$ (**in** $\boldsymbol{C}$) and we denote it by $\overset{\boldsymbol{C}}{\mathrm{Lim}}(\mathcal{F})$ (resp. $\overset{\boldsymbol{C}}{\mathrm{Colim}}(\mathcal{F})$). We shall denote the limit cone χ over (resp. under) $\mathcal{F}$ by $\overset{\boldsymbol{C}}{\mathrm{lim}}(\mathcal{F})$ (resp. $\overset{\boldsymbol{C}}{\mathrm{colim}}(\mathcal{F})$).

Remark B.32. It has been observed that limits are unique up to unique isomorphism. That fact has a sort of converse: if $\chi = \{\chi_i : L \xrightarrow{\boldsymbol{C}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ is a limit cone over $\mathcal{F}$ with vertex L and if $f : L' \xrightarrow{\boldsymbol{C}} L$ is an isomorphism, then the family $\{\chi_i \circ f : L' \xrightarrow{\boldsymbol{C}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ is a limit cone over $\mathcal{F}$ with vertex L'.

The reader may have noticed that only the "shape" of the category $\boldsymbol{I}$ plays an important role in Definition B.31, that is, the labels of the objects and the morphisms of $\boldsymbol{I}$ are irrelevant to the essence of the limit and the colimit of a functor $\mathcal{F} : \boldsymbol{I} \to \boldsymbol{C}$. That is the reason why the shape $\boldsymbol{I}$ of $\mathcal{F}$ is usually called the **index category of** $\mathcal{F}$ in this context. Thus, graphical representations of $\boldsymbol{I}$ usually omit all labels as shown in Example B.19, but whenever it is necessary to assign labels to the objects of $\boldsymbol{I}$, they are often denoted by lowercase latin letters, such as $i, j, k \ldots$, just as one would often denote the elements of the index set of a family.

Definition B.33.

- $\triangleright$ A category $\boldsymbol{C}$ is **complete** (resp. **cocomplete**) if every[3] functor $\mathcal{F}$ with codomain $\boldsymbol{C}$ has a limit (resp. colimit) in $\boldsymbol{C}$. A category is **bicomplete** if it is complete and cocomplete;
- $\triangleright$ A category $\boldsymbol{C}$ is **finitely complete** (resp. **finitely cocomplete**) if every finite functor $\mathcal{F}$ with codomain $\boldsymbol{C}$ has a limit (resp. colimit) in $\boldsymbol{C}$. A category is **finitely bicomplete** if it is finitely complete and finitely cocomplete.

The most fundamental example of a bicomplete category is the following:

Example B.34. (190) The category of sets, **Set**, is bicomplete. In fact, given a functor $\mathcal{F} : \boldsymbol{I} \to \mathbf{Set}$, we have:

(**a**) Let L be the set of cones $\lambda = \{\lambda_i : \{\emptyset\} \xrightarrow{\mathbf{Set}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ over $\mathcal{F}$, all with the common vertex $\{\emptyset\}$, and let $\sigma = \{\sigma_i : L \xrightarrow{\mathbf{Set}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ be the family of morphisms in **Set** given by $\sigma_i(\lambda) := \lambda_i(\emptyset)$. The family σ is a limit cone over $\mathcal{F}$ with vertex L;

(**b**) Let $\sim$ be the smallest equivalence relation on the disjoint union $\bigsqcup_{i \in \boldsymbol{I}_0} \mathcal{F}(i)$ such that

$$(\forall i, j \in \boldsymbol{I}_0)\,(\forall x \in \mathcal{F}(i))\,(\forall f : i \xrightarrow{\boldsymbol{I}} j)\ \ (x, i) \sim (\mathcal{F}(f)(x), j)\ ,$$

let $L := \left(\bigsqcup_{i \in \boldsymbol{I}_0} F(i)\right) / \sim$ and let $\sigma = \{\sigma_i : \mathcal{F}(i) \xrightarrow{\mathbf{Set}} L\}_{i \in \boldsymbol{I}_0}$ be the family of morphisms in **Set** given by $\sigma_i(x) := (x, i) / \sim$. The family σ is a limit cone under $\mathcal{F}$ with vertex L.

---

[3] If the category $\boldsymbol{C}$ was defined in view of a universe $\mathcal{U}$ (Appendix A, Definition A.3) so that $\boldsymbol{C}, \boldsymbol{C}_0 \subset \mathcal{U}$, then we restrict the functors $\mathcal{F}$ considered in this definition, requiring the domain $\boldsymbol{I}$ of $\mathcal{F}$ to be an element of $\mathcal{U}$. The reader shall see that this assumption is not at all relevant for our discussions.

Example B.35. (2, 151) Other large categories mentioned in Example B.17 also turn out to be bicomplete, namely the categories **Mon**, **Top**, **Grp**, **Ring**, and **Rng**.

Some functors do not change the status of certain kinds of limits or colimits:

Definition B.36. Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two categories and let $\mathcal{K} : \boldsymbol{C} \to \boldsymbol{D}$ be a functor.

- ▷ The functor $\mathcal{K}$ **sends limits** (resp. **finite limits**, **object-finite limits**) **in $\boldsymbol{C}$ to limits in $\boldsymbol{D}$** if for every index category $\boldsymbol{I}$, every functor (resp. every finite functor, every object-finite functor) $\mathcal{F} : \boldsymbol{I} \to \boldsymbol{C}$ and every limit cone $\chi = \{\chi_i : V \xrightarrow{\boldsymbol{C}} \mathcal{F}(i)\}_{i \in \boldsymbol{I}_0}$ over $\mathcal{F}$, the cone
$$\mathcal{K}(\chi) = \{\mathcal{K}(\chi_i) : \mathcal{K}(V) \xrightarrow{\boldsymbol{D}} \mathcal{K}(\mathcal{F}(i))\}_{i \in \boldsymbol{I}_0}$$
over the composition $\mathcal{K} \circ \mathcal{F} : \boldsymbol{I} \to \boldsymbol{D}$ is a limit cone over $\mathcal{K} \circ \mathcal{F}$. If $\boldsymbol{C} = \boldsymbol{D}$, we say simply that $\mathcal{K} : \boldsymbol{C} \to \boldsymbol{C}$ **preserves limits** (resp. **finite limits**, **object-finite limits**) **in $\boldsymbol{C}$**;

- ▷ The functor $\mathcal{K}$ **sends colimits** (resp. **finite colimits**, **object-finite colimits**) **in $\boldsymbol{C}$ to colimits in $\boldsymbol{D}$** if for every index category $\boldsymbol{I}$, every functor (resp. every finite functor, every object-finite functor) $\mathcal{F} : \boldsymbol{I} \to \boldsymbol{C}$ and every limit cone $\chi = \{\chi_i : \mathcal{F}(i) \xrightarrow{\boldsymbol{C}} V\}_{i \in \boldsymbol{I}_0}$ under the functor $\mathcal{F}$, the cone
$$\mathcal{K}(\chi) = \{\mathcal{K}(\chi_i) : \mathcal{K}(\mathcal{F}(i)) \xrightarrow{\boldsymbol{D}} \mathcal{K}(V)\}_{i \in \boldsymbol{I}_0}$$
under the composition $\mathcal{K} \circ \mathcal{F} : \boldsymbol{I} \to \boldsymbol{D}$ is a limit cone under $\mathcal{K} \circ \mathcal{F}$. If $\boldsymbol{C} = \boldsymbol{D}$, we say simply that $\mathcal{K} : \boldsymbol{C} \to \boldsymbol{C}$ **preserves colimits** (resp. **finite colimits**, **object-finite colimits**) **in $\boldsymbol{C}$**;

- ▷ The functor $\mathcal{K} : \boldsymbol{C} \to \boldsymbol{D}$ is **left-exact** (resp. **right-exact**) if $\boldsymbol{C}$ is finitely complete and if $\mathcal{K}$ sends finite limits (resp. finite colimits) in $\boldsymbol{C}$ to limits (resp. colimits) in $\boldsymbol{D}$.

In Theorems 3.54 and 3.56, we show that two important functors in the Theory of Rayner Rngs preserve object-finite limits, and these functors also happen to be left-exact.

B.4 Notable limits and colimits

The simplest types of limits and colimits arise when null functors $\mathcal{F} : \mathrm{Null}_\emptyset \to \boldsymbol{C}$ are considered:

Definition B.37. Let $\boldsymbol{C}$ be a category. A **terminal** (resp. **initial**) **object in $\boldsymbol{C}$** is a limit (resp. colimit) of the null functor of type $\mathrm{Null}_\emptyset \to \boldsymbol{C}$ (Example B.26). When it exists, this terminal (resp. initial) object is denoted by $\mathbf{1}$ (resp. $\mathbf{0}$), its corresponding limit cone is empty, and it satisfies the **universal property of terminal** (resp. **initial**) **objects in $\boldsymbol{C}$**, that is, it is such that for every object V in $\boldsymbol{C}$, there is a unique morphism of type $V \xrightarrow{\boldsymbol{C}} \mathbf{1}$ (resp. $\mathbf{0} \xrightarrow{\boldsymbol{C}} V$).

Functors with discrete shapes (Example B.18) give rise to limits and colimits of the utmost importance in Mathematics, which extract the essence of the notions of Cartesian product and disjoint union of sets:

Definition B.38. Let $\boldsymbol{C}$ be a category, let $I = \{i', j', k', \dots\}$ be a set and let $\{X_i\}_{i \in I}$ be a family of objects in $\boldsymbol{C}$. A **product** (resp. **coproduct**) **of $\{X_i\}_{i\in I}$ in $\boldsymbol{C}$** is a limit (resp. colimit) of the functor $\mathcal{F} : \mathrm{Disc}\,(I) \to \boldsymbol{C}$ (Example B.18) given by

$$\begin{tikzcd} i' \arrow[loop above, "\mathrm{id}_{i'}"] & j' \arrow[loop above, "\mathrm{id}_{j'}"] & k' \arrow[loop above, "\mathrm{id}_{k'}"] & \cdots & {} \arrow[r, Rightarrow, "\mathcal{F}"] & {} & X_{i'} \arrow[loop above, "\mathrm{id}_{X_{i'}}"] & X_{j'} \arrow[loop above, "\mathrm{id}_{X_{j'}}"] & X_{k'} \arrow[loop above, "\mathrm{id}_{X_{k'}}"] & \cdots \end{tikzcd}$$

When it exists, this product (resp. coproduct) is denoted by $\prod\limits_{i\in I}^{\boldsymbol{C}} X_i$ (resp. $\coprod\limits_{i\in I}^{\boldsymbol{C}} X_i$), and its corresponding limit cone consists of a **family of projections** (resp. **injections**) that satisfies the **universal property of products** (resp. **coproducts**) **in $\boldsymbol{C}$**, that is, it consists of a family

$$\chi = \left\{ \chi_i : \prod_{i\in I}^{\boldsymbol{C}} X_i \xrightarrow{\boldsymbol{C}} X_i \right\}_{i\in I} \quad \left(\text{resp. } \chi = \left\{ \chi_i : X_i \xrightarrow{\boldsymbol{C}} \coprod_{i\in I}^{\boldsymbol{C}} X_i \right\}_{i\in I} \right)$$

such that for each family of morphisms $\lambda = \{\lambda_i : V \xrightarrow{\boldsymbol{C}} X_i\}_{i\in I}$ (resp. $\lambda = \{\lambda_i : X_i \xrightarrow{\boldsymbol{C}} V\}_{i\in I}$) with vertex V, there is a unique **product lifting** $\overline{\lambda} : V \xrightarrow{\boldsymbol{C}} \prod\limits_{i\in I}^{\boldsymbol{C}} X_i$ (resp. **coproduct lowering** $\overline{\lambda} : \coprod\limits_{i\in I}^{\boldsymbol{C}} X_i \xrightarrow{\boldsymbol{C}} V$) such that the digraph

$$\begin{tikzcd} & \prod\limits_{i\in I}^{\boldsymbol{C}} X_i \arrow[d, "\chi_i"] \\ V \arrow[ur, "\overline{\lambda}"] \arrow[r, "\lambda_i"'] & X_i \end{tikzcd} \qquad \left(\text{resp. } \begin{tikzcd} X_i \arrow[r, "\lambda_i"] \arrow[d, "\chi_i"'] & V \\ \coprod\limits_{i\in I}^{\boldsymbol{C}} X_i \arrow[ur, "\overline{\lambda}"'] & \end{tikzcd} \right)$$

in $\boldsymbol{C}$ is commutative for all $i \in I$.

If $I = \{i'_1, i'_2, \ldots, i'_n\}$ is a finite set, then the product $\prod\limits_{i\in I}^{\boldsymbol{C}} X_i$ (resp. the coproduct $\coprod\limits_{i\in I}^{\boldsymbol{C}} X_i$) is also denoted by

$$X_{i'_1} \times X_{i'_2} \times \cdots \times X_{i'_n} \ \ (\text{resp. } X_{i'_1} \sqcup X_{i'_2} \sqcup \cdots \sqcup X_{i'_n}) \ .$$

Example B.39. (61, 43) Let R and S be two rngs. We shall provide a construction[4] for the colimit $R \sqcup S$ in **Rng**. We may assume, without loss of generality, that the rngs R and S are disjoint, since the rng R (resp. S) is isomorphic to the Cartesian product $R \times \{1\}$ (resp. $S \times \{2\}$) endowed with the operations

$$(r, 1) + (r', 1) \coloneqq (r + r', 1) \ \ (\text{resp. } (s, 2) + (s', 2) \coloneqq (s + s', 2))$$

and

$$(r, 1)\,(r', 1) \coloneqq (rr', 1) \ \ (\text{resp. } (s, 2)\,(s', 2) \coloneqq (ss', 2)) \ ,$$

[4] The essence of this example is usually described in literature within the context of the Theory of R-Algebras. Our description of the coproducts in **Rng** is a simple adaptation of the construction of the so-called **free products of R-algebras**.

and since the rngs $R\times\{1\}$ and $S\times\{2\}$ are disjoint. Let $(R\cup S)^*$ be the set of finite sequences $p := p_1p_2\ldots p_n$ in $R\cup S$, not including the empty sequence, such that the conditions $p_i\in R$ and $p_{i+1}\in S$ are equivalent for all $i\in[1,n-1]_{\mathbb{N}}$, and let $R\sqcup S$ be the set of commutative, formal sums of elements of $(R\cup S)^*$:

$$R\sqcup S := \left\{\mathcal{O},\ \sum_{i=1}^{k} c_i\, p^i \ \middle|\ k\in\mathbb{N},\ c_i\in\mathbb{Z}-\{0\}\ (\forall i)\,,\ p^i\in(R\cup S)^*\ (\forall i)\right\}\,,$$

where $\sum_{i=1}^{k} c_i\, p^i$ is the formal sum notation for the function $f:(R\cup S)^*\to\mathbb{Z}$ with finite support $\{p^i \mid i\in[1,k]_{\mathbb{N}}\}$ such that $f(p^i) := c_i\ (\forall i)$, and where $\mathcal{O}:(R\cup S)^*\to\mathbb{Z}$ is the constant function with value 0. To end the construction, endow $R\sqcup S$ with the pointwise addition operation $(f+g)(x) := f(x)+g(x)$ and with the multiplication operation given by

$$\left(\sum_{i=1}^{k} c_i\, p_1^i\ldots p_{m_i}^i\right)\cdot\left(\sum_{j=1}^{l} d_j\, q_1^j\ldots q_{n_j}^j\right) := \sum_{\substack{i\in[1,k]_{\mathbb{N}}\\ j\in[1,l]_{\mathbb{N}}}} c_i d_j\, r_{ij}\,,$$

where

$$r_{ij} := \begin{cases} p_1^i\ldots p_{m_i}^i, q_1^j\ldots q_{n_j}^j & \text{if } p_{m_i}^i, q_1^j\notin R \text{ and } p_{m_i}^i, q_1^j\notin S\,,\\ p_1^i\ldots p_{m_i-1}^i, \left(p_{m_i}^i q_1^j\right), q_2^j\ldots q_{n_j}^j & \text{otherwise.}\end{cases}$$

and let $\chi_1 : R\to R\sqcup S$ and $\chi_2 : S\to R\sqcup S$ be the canonical homomorphisms given by

$$\chi_1(r)(x) := \begin{cases} 1 & \text{if } x = r\,,\\ 0 & \text{if } x\in(R\cup S)^*-\{r\}\,,\end{cases} \quad\text{and}\quad \chi_2(s)(x) := \begin{cases} 1 & \text{if } x = s\,,\\ 0 & \text{if } x\in(R\cup S)^*-\{s\}\,.\end{cases}$$

With these settings, we have that $R\sqcup S$ is a rng, and it is a coproduct of R and S in **Rng** with injections χ_1 and χ_2.

Functors whose shapes are generated by parallel pairs of morphisms give rise to important limits and colimits:

Definition B.40. Let $\boldsymbol{C}$ be a category and let $f, g : C \xrightarrow{\boldsymbol{C}} D$ be two morphisms. An **equaliser** (resp. **coequaliser**) **of** f **and** g **in** $\boldsymbol{C}$ is a limit (resp. colimit) of the functor $\mathcal{F}$ given by

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd} \xRightarrow{\mathcal{F}} \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & D \end{tikzcd}$$

whose codomain is $\boldsymbol{C}$. When it exists, this equaliser (resp. coequaliser) is denoted by $\overset{\boldsymbol{C}}{\mathrm{Eq}}(f,g)$ (resp. $\overset{\boldsymbol{C}}{\mathrm{Coeq}}(f,g)$), and its corresponding limit cone consists of an **equaliser morphism** (resp. **coequaliser morphism**) $\overset{\boldsymbol{C}}{\mathrm{eq}}(f,g)$ (resp. $\overset{\boldsymbol{C}}{\mathrm{coeq}}(f,g)$) that satisfies the **universal property of equalisers** (resp. **coequalisers**) **in** $\boldsymbol{C}$, that is, it consists of a morphism

$$\overset{\boldsymbol{C}}{\mathrm{eq}}(f,g) : \overset{\boldsymbol{C}}{\mathrm{Eq}}(f,g) \xrightarrow{\boldsymbol{C}} C \qquad \left(\text{resp. } \overset{\boldsymbol{C}}{\mathrm{coeq}}(f,g) : D \xrightarrow{\boldsymbol{C}} \overset{\boldsymbol{C}}{\mathrm{Coeq}}(f,g)\right)$$

such that the digraph

$$\begin{tikzcd} \overset{\boldsymbol{C}}{\mathrm{Eq}}(f,g) \arrow[r, "\overset{\boldsymbol{C}}{\mathrm{eq}}(f,g)"] & C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & D \end{tikzcd} \qquad \left(\text{resp. } \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & D \arrow[r, "\overset{\boldsymbol{C}}{\mathrm{coeq}}(f,g)"] & \overset{\boldsymbol{C}}{\mathrm{Coeq}}(f,g) \end{tikzcd}\right)$$

in $\boldsymbol{C}$ commutes and such that for each morphism $\lambda : V \xrightarrow{\boldsymbol{C}} C$ (resp. $\lambda : D \xrightarrow{\boldsymbol{C}} V$) making the digraph

$$\begin{tikzcd} V \arrow[r, "\lambda"] & C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & D \end{tikzcd} \qquad \left(\text{resp. } \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & D \arrow[r, "\lambda"] & V \end{tikzcd}\right)$$

in $\boldsymbol{C}$ commute, there is a unique **equaliser lifting** $\overline{\lambda} : V \xrightarrow{\boldsymbol{C}} \overset{\boldsymbol{C}}{\mathrm{Eq}}(f,g)$ (resp. **coequaliser lowering** $\overline{\lambda} : \overset{\boldsymbol{C}}{\mathrm{Coeq}}(f,g) \xrightarrow{\boldsymbol{C}} V$) such that the digraph

$$\begin{tikzcd} & \overset{\boldsymbol{C}}{\mathrm{Eq}}(f,g) \arrow[d, "\overset{\boldsymbol{C}}{\mathrm{eq}}(f,g)"] \\ V \arrow[ur, "\overline{\lambda}"] \arrow[r, "\lambda"'] & C \end{tikzcd} \qquad \left(\text{resp. } \begin{tikzcd} D \arrow[r, "\lambda"] \arrow[d, "\overset{\boldsymbol{C}}{\mathrm{coeq}}(f,g)"'] & V \\ \overset{\boldsymbol{C}}{\mathrm{Coeq}}(f,g) \arrow[ur, "\overline{\lambda}"'] & \end{tikzcd}\right)$$

in $\boldsymbol{C}$ commutes.

The fourth important type of limit-colimit pair that we shall define in this appendix may be seen as a generalisation of the product-coproduct conceptual pair:

Definition B.41. Let $\boldsymbol{C}$ be a category and let $f : C_1 \xrightarrow{\boldsymbol{C}} P$ and $g : C_2 \xrightarrow{\boldsymbol{C}} P$ (resp. $f : P \xrightarrow{\boldsymbol{C}} C_1$ and $g : P \xrightarrow{\boldsymbol{C}} C_2$) be two morphisms. A **pullback** (resp. **pushout**) **of** f **and** g **in** $\boldsymbol{C}$ is a limit (resp. colimit) of the functor $\mathcal{F}$ given by

$$
\begin{tikzcd}
 & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\quad \stackrel{\mathcal{F}}{\Longrightarrow} \quad
\begin{tikzcd}
 & C_2 \arrow[d, "g"] \\
C_1 \arrow[r, "f"] & P
\end{tikzcd}
$$

$$
\left(\text{resp.}\quad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \\
\bullet &
\end{tikzcd}
\quad \stackrel{\mathcal{F}}{\Longrightarrow} \quad
\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "g"'] & C_1 \\
C_2 &
\end{tikzcd}
\right)
$$

whose codomain is $\boldsymbol{C}$. When it exists, this pullback (resp. pushout) is denoted by $C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\times}} C_2$ (resp. $C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\sqcup}} C_2$) and it is also called the **fibred product** (resp. **fibred coproduct**[5]) **of** C_1 **and** C_2 **with respect to** f **and** g (**in** $\boldsymbol{C}$). Its corresponding limit cone consists of a pair of morphisms

$$\overline{f} : C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\times}} C_2 \xrightarrow{\boldsymbol{C}} C_2 \quad \text{and} \quad \overline{g} : C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\times}} C_2 \xrightarrow{\boldsymbol{C}} C_1$$
$$\left(\text{resp. } \overline{f} : C_2 \xrightarrow{\boldsymbol{C}} C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\sqcup}} C_2 \quad \text{and} \quad \overline{g} : C_1 \xrightarrow{\boldsymbol{C}} C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\sqcup}} C_2\right)$$

that satisfies the **universal property of pullbacks** (resp. **pushouts**) **in** $\boldsymbol{C}$, that is, it is such that the **pullback square** (resp. **pushout square**)

$$
\begin{tikzcd}
C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\times}} C_2 \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & C_2 \arrow[d, "g"] \\
C_1 \arrow[r, "f"] & P
\end{tikzcd}
\qquad
\left(\text{resp.}\quad
\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "g"'] & C_1 \arrow[d, "\overline{g}"] \\
C_2 \arrow[r, "\overline{f}"'] & C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\sqcup}} C_2
\end{tikzcd}
\right)
$$

[5] Some authors call the fibred coproduct $C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\sqcup}} C_2$ the **fibred sum** or the **amalgamated sum of** C_1 **and** C_2 **with respect to** f **and** g.

in $\boldsymbol{C}$ commutes and such that for each pair of morphisms $\lambda_i : V \xrightarrow{\boldsymbol{C}} C_i \ \ (i = 1, 2)$ (resp. $\lambda_i : C_i \xrightarrow{\boldsymbol{C}} V \ \ (i = 1, 2)$) making the digraph

$$
\begin{tikzcd}
V \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] & C_2 \arrow[d, "g"] \\
C_1 \arrow[r, "f"] & P
\end{tikzcd}
\qquad
\left(\text{resp.} \quad
\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "g"'] & C_1 \arrow[d, "\lambda_1"] \\
C_2 \arrow[r, "\lambda_2"'] & V
\end{tikzcd}
\right)
$$

in $\boldsymbol{C}$ commute, there is a unique **pullback lifting** $\overline{(\lambda_1, \lambda_2)} : V \xrightarrow{\boldsymbol{C}} C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\times}} C_2$ (resp. **pushout lowering** $\overline{(\lambda_1, \lambda_2)} : C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\sqcup}} C_2 \xrightarrow{\boldsymbol{C}} V$) such that the digraph

$$
\begin{tikzcd}
& C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\times}} C_2 \arrow[d, bend right, "\overline{f}"'] \arrow[dr, bend left, "\overline{g}"] & \\
& C_2 & C_1 \\
V \arrow[uur, bend left, "\overline{(\lambda_1,\lambda_2)}"] \arrow[ur, bend left, "\lambda_2"] \arrow[urr, bend right, "\lambda_1"'] & &
\end{tikzcd}
\qquad
\left(\text{resp.} \quad
\begin{tikzcd}
& & V \\
C_1 \arrow[urr, bend left, "\lambda_1"] \arrow[dr, bend right, "\overline{g}"'] & C_2 \arrow[ur, bend right, "\lambda_2"'] \arrow[d, bend left, "\overline{f}"] & \\
& C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\sqcup}} C_2 \arrow[uur, bend right, "\overline{(\lambda_1,\lambda_2)}"'] &
\end{tikzcd}
\right)
$$

in $\boldsymbol{C}$ commutes. The morphism $\overline{f}$ is called the **pullback** (resp. **pushout**) **of** f **along** g and the morphism $\overline{g}$ is called the **pullback** (resp. **pushout**) **of** g **along** f.

If the object P is a terminal (resp. initial) object in $\boldsymbol{C}$, then the morphisms $f : C_1 \xrightarrow{\boldsymbol{C}} P$ and $g : C_2 \xrightarrow{\boldsymbol{C}} P$ (resp. $f : P \xrightarrow{\boldsymbol{C}} C_1$ and $g : P \xrightarrow{\boldsymbol{C}} C_2$) are uniquely determined, and the fibred product $C_1 \underset{f,g}{\times} C_2$ (resp. fibred coproduct $C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\sqcup}} C_2$) is a product (resp. coproduct) of C_1 and C_2 in $\boldsymbol{C}$, that is, we have $C_1 \underset{f,g}{\times} C_2 \overset{\boldsymbol{C}}{\cong} C_1 \times C_2$ (resp. $C_1 \overset{\boldsymbol{C}}{\underset{f,g}{\sqcup}} C_2 \overset{\boldsymbol{C}}{\cong} C_1 \sqcup C_2$).

Several examples of the notable limits and colimits introduced in this section are shown in Sections 1.3 and 2.3, mostly instances of limits.

B.5 $\boldsymbol{A}$-Concrete categories; Quotients

Some categories are better understood when its objects and morphisms are viewed as those of another category.

Definition B.42. Let $\boldsymbol{A}$ be a category. An **$\boldsymbol{A}$-concrete category** is a category $\boldsymbol{C}$ endowed with a faithful functor $\mathcal{F} : \boldsymbol{C} \to \boldsymbol{A}$. Such functor is called the **forgetful functor of $\boldsymbol{C}$**, and whenever no particular notation is attributed to that functor, it shall be denoted by $\overset{\boldsymbol{C}}{\mathrm{U}} : \boldsymbol{C} \to \boldsymbol{A}$, while its images $\overset{\boldsymbol{C}}{\mathrm{U}}(X)$ are denoted by U^X for objects X in $\boldsymbol{C}$ that are not indicated by large expressions.

The case $\boldsymbol{A} = \mathbf{Set}$ is by far the most important in applications, but occasionally other choices of $\boldsymbol{A}$ are relevant in some areas of study.

Example B.43. All large categories shown in Example B.17 may be canonically regarded as **Set**-concrete categories. Of greater relevance for this thesis is the fact that the category **Rng** may also be canonically regarded as a **Mon**-concrete category, so that each rng R is associated to its underlying additive monoid $(R, +_R)$ and each homomorphism between rngs is associated to itself (cf. Section 2.2). For the record, note that the underlying additive monoid $(R, +_R)$ of a rng R is a (commutative) group, so that the category **Rng** may be seen as a **Grp**-concrete category. We chose to work with **Mon** instead of **Grp** in such contexts in order to achieve higher generality in Proposition 3.53 and Theorem 3.54.

Definition B.44. Let $\boldsymbol{A}$ be a category and let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two $\boldsymbol{A}$-concrete categories. An **$\boldsymbol{A}$-functor** $\mathcal{F} : \boldsymbol{C} \to \boldsymbol{D}$ is a functor between the underlying categories such that $\overset{\boldsymbol{C}}{\mathrm{U}} = \overset{\boldsymbol{D}}{\mathrm{U}} \circ \mathcal{F}$. Thus, one can easily verify that the $\boldsymbol{A}$-concrete categories and the $\boldsymbol{A}$-functors form a category.

Expressly, the equation $\overset{\boldsymbol{C}}{\mathrm{U}} = \overset{\boldsymbol{D}}{\mathrm{U}} \circ \mathcal{F}$ ensures that for each object X in $\boldsymbol{C}$ and for each morphism $f : X \xrightarrow{\boldsymbol{C}} X'$, the objects $X \in \boldsymbol{C}_0$ and $\mathcal{F}(X) \in \boldsymbol{D}_0$ are interpreted by the same object in $\boldsymbol{A}$ while the morphisms $f \in \boldsymbol{C}$ and $\mathcal{F}(f) \in \boldsymbol{D}$ are interpreted by the same morphism in $\boldsymbol{A}$.

Definition B.45. Let $f : X \to Y$ be a function between sets. The **equivalence relation on** X **induced by** f is the binary relation $\equiv$ on X such that for all $x, y \in X$, the condition $x \equiv y$ is equivalent to $f(x) = f(y)$. It is easy to check that that binary relation is an equivalence relation on X, and we shall denote it by $\overset{f}{\mathrm{eq}}$.

Given a mathematical structure C of a certain kind and given an equivalence relation $\equiv$ on the set of elements of C, it is common in Mathematics that the set of equivalence classes under $\equiv$ forms a structure Q of the same kind as that of C, and the structures C and Q often have similar features. This is made precise in the following definition:

Definition B.46. Let $\boldsymbol{C}$ be a **Set**-concrete category, let C be an object in $\boldsymbol{C}$ and let $\equiv$ be an equivalence relation on U^C. A (**universal**) **quotient of** C **modulo** $\equiv$ (**in** $\boldsymbol{C}$) is an object Q in $\boldsymbol{C}$ that satisfies the **universal property of quotients** (**in** $\boldsymbol{C}$), that is, it is such that there is a **quotient morphism** $\iota : C \xrightarrow{\boldsymbol{C}} Q$ such that $\equiv \subset \overset{\overset{\boldsymbol{C}}{\mathrm{U}}(\iota)}{\mathrm{eq}}$ and for each object D in $\boldsymbol{C}$ and each morphism $f : C \xrightarrow{\boldsymbol{C}} D$ so that $\equiv \subset \overset{\overset{\boldsymbol{C}}{\mathrm{U}}(f)}{\mathrm{eq}}$, there is a unique morphism $\overline{f} : Q \xrightarrow{\boldsymbol{C}} Y$ so that the digraph

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\iota"'] & D \\ Q \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

in $\boldsymbol{C}$ is commutative. In that case, the morphism $\overline{f} : Q \xrightarrow{\boldsymbol{C}} D$ is called the **quotient lowering of** f **modulo** $\equiv$ (**along** ι).

If C has a quotient Q modulo $\equiv$ in $\boldsymbol{C}$, then Q is unique up to unique isomorphism. More precisely, if Q and Q' are two quotients of C modulo $\equiv$ in $\boldsymbol{C}$

and if

$$\iota : C \xrightarrow{\boldsymbol{C}} Q \quad \text{and} \quad \iota' : C \xrightarrow{\boldsymbol{C}} Q'$$

are two morphisms that satisfy the conditions set out in Definition B.46, then there is a unique isomorphism $\alpha : Q \xrightarrow{\boldsymbol{C}} Q'$ such that the digraph

$$\begin{tikzcd} & Q \arrow[dd, "\alpha"] \\ C \arrow[ur, "\iota"] \arrow[dr, "\iota'"'] & \\ & Q' \end{tikzcd}$$

in $\boldsymbol{C}$ is commutative for all $i \in I$. With that in mind, we denote the quotient Q (or any object in $\boldsymbol{C}$ isomorphic to Q) by $C \overset{\boldsymbol{C}}{/} \equiv$.

Remark B.47. It has been observed that quotients are unique up to unique isomorphism. That fact has a sort of converse: if Q is a quotient of C modulo $\equiv$ in $\boldsymbol{C}$ with quotient morphism $\iota : C \xrightarrow{\boldsymbol{C}} Q$, and if $f : Q \xrightarrow{\boldsymbol{C}} Q'$ is an isomorphism, then the object Q' is a quotient of C modulo $\equiv$ in $\boldsymbol{C}$ with quotient morphism $f \circ \iota : C \xrightarrow{\boldsymbol{C}} Q'$.

Example B.48. Let A be a set, let $\equiv$ be an equivalence relation on A and let $A/\equiv$ be the canonical quotient set of A modulo $\equiv$, which is given by

$$A/\equiv \;\coloneqq \left\{ x/\equiv \;\vdots\; x \in A \right\}$$

where $x/\equiv \;\coloneqq\; \equiv\langle\{x\}\rangle$. It is easy to check that $A/\equiv$ is a quotient of A modulo $\equiv$ in **Set** with the (canonical) quotient morphism $\sigma : A \to A/\equiv$ given by $\sigma(x) \coloneqq x/\equiv$.

A couple of other examples of the notion of quotient are given in Example 3.58.

The following theorem is an elementary categorical generalisation of many standard results in Mathematics concerning quotients, most recurrently in algebraic settings:

Theorem B.49 (First Isomorphism Theorem)**.** *Let* $\boldsymbol{C}$ *be a* **Set**-*concrete category, let* $f : C \xrightarrow{\boldsymbol{C}} D$ *be a morphism, let* $\equiv$ *be an equivalence relation on* U^C *so that* $\equiv \subset \overset{\overset{\boldsymbol{C}}{\mathrm{U}}(f)}{\mathrm{eq}}$*, and suppose that* C *has a quotient* $C \overset{\boldsymbol{C}}{/} \equiv$ *in* $\boldsymbol{C}$ *modulo* $\equiv$ *with canonical morphism* $\iota : C \xrightarrow{\boldsymbol{C}} C \overset{\boldsymbol{C}}{/} \equiv$*.*

(**a**) $\mathrm{Im}\,(\overset{\boldsymbol{C}}{\mathrm{U}}\,(f)) \subset \mathrm{Im}\,(\overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f}))$*, where* $\overline{f} : C \overset{\boldsymbol{C}}{/} \equiv \xrightarrow{\boldsymbol{C}} D$ *is the quotient lowering of* f *modulo* $\equiv$ *along* ι*;*

(**b**) *If the function* $\overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota) : \mathrm{U}^C \to \overset{\boldsymbol{C}}{\mathrm{U}}\left(C \overset{\boldsymbol{C}}{/} \equiv\right)$ *is surjective, then we have the equality* $\mathrm{Im}\,(\overset{\boldsymbol{C}}{\mathrm{U}}\,(f)) = \mathrm{Im}\,(\overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f}))$*, and if, in addition, we have* $\equiv\, = \overset{\overset{\boldsymbol{C}}{\mathrm{U}}(f)}{\mathrm{eq}}$*, then* $\overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f})$ *is a bijection of type* $\overset{\boldsymbol{C}}{\mathrm{U}}\left(C \overset{\boldsymbol{C}}{/} \equiv\right) \to \mathrm{Im}\,(\overset{\boldsymbol{C}}{\mathrm{U}}\,(f))$*.*

Proof. Item (a) is a direct consequence of the equation $\overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f}) \circ \overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota) = \overset{\boldsymbol{C}}{\mathrm{U}}\,(f)$. We shall prove item (b).

(**b**) Since $\overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f}) \circ \overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota) = \overset{\boldsymbol{C}}{\mathrm{U}}\,(f)$ and since the function $\overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota)$ is surjective, we have $\mathrm{Im}\,(\overset{\boldsymbol{C}}{\mathrm{U}}\,(f)) = \mathrm{Im}\,(\overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f}))$. Suppose $\equiv\, = \overset{\overset{\boldsymbol{C}}{\mathrm{U}}(f)}{\mathrm{eq}}$. If $a, b \in \overset{\boldsymbol{C}}{\mathrm{U}}\left(C \overset{\boldsymbol{C}}{/} \equiv\right)$ are so that $\overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f})\,(a) = \overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f})\,(b)$, then there are $x, y \in \mathrm{U}^C$ such that $a = \overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota)\,(x)$ and $b = \overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota)\,(y)$, implying

$$\overset{\boldsymbol{C}}{\mathrm{U}}\,(f)\,(x) = \overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f})\left(\overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota)\,(x)\right) = \overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f})\,(a) = \overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f})\,(b) = \overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f})\left(\overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota)\,(y)\right) = \overset{\boldsymbol{C}}{\mathrm{U}}\,(f)\,(y)\,.$$

That being so, on account of $\equiv\, = \overset{\overset{\boldsymbol{C}}{\mathrm{U}}(f)}{\mathrm{eq}}$ we have $x \equiv y$, and, since $\equiv \subset \overset{\overset{\boldsymbol{C}}{\mathrm{U}}(\iota)}{\mathrm{eq}}$, we obtain

$$a = \overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota)\,(x) = \overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota)\,(y) = b\,,$$

proving that the function $\overset{\boldsymbol{C}}{\mathrm{U}}\,(\overline{f})$ is injective. □

In many instances of **Set**-concrete categories $\boldsymbol{C}$, mostly categories whose objects are algebraic structures (cf. Example B.24), a morphism $h : A \xrightarrow{\boldsymbol{C}} B$ is an isomorphism if, and only if, the function $\overset{\boldsymbol{C}}{\mathrm{U}}\,(h) : \mathrm{U}^A \to \mathrm{U}^B$ is bijective. Thus, in such categories, the quotient lowering $\overline{f} : C \overset{\boldsymbol{C}}{/} \equiv \xrightarrow{\boldsymbol{C}} D$ in Theorem B.49 is an isomorphism whenever $\overset{\boldsymbol{C}}{\mathrm{U}}\,(\iota)$ is surjective and $\equiv\, = \overset{\overset{\boldsymbol{C}}{\mathrm{U}}(f)}{\mathrm{eq}}$. This is precisely the reason why many results resembling Theorem B.49 in mathematical literature are said to be **isomorphism theorems**.

Last of all, some functors do not change the status of quotients in **Set**-concrete categories:

Definition B.50. Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two **Set**-concrete categories, let E be a set of equivalence relations $\equiv$ on sets of the form U^C for $C \in \boldsymbol{C}_0$ and let F be a set of equivalence relations $\approx$ on sets of the form U^D for $D \in \boldsymbol{D}_0$. A functor $\mathcal{K} : \boldsymbol{C} \to \boldsymbol{D}$ between the underlying categories **sends quotients modulo** E **in** $\boldsymbol{C}$ **to quotients modulo** F **in** $\boldsymbol{D}$ if for every object C in $\boldsymbol{C}$, for every equivalence relation $\equiv \in E$ on U^C and for every quotient Q of C modulo $\equiv$ in $\boldsymbol{C}$ with quotient morphism $\iota : C \xrightarrow{\boldsymbol{C}} Q$ so that $\equiv \subset \overset{\boldsymbol{C}}{\underset{}{\mathrm{eq}}}^{\mathrm{U}(\iota)}$, there is an equivalence relation $\approx \in F$ on $\mathrm{U}^{\mathcal{K}(C)}$ so that $\approx \subset \overset{\boldsymbol{D}}{\mathrm{eq}}^{\mathrm{U}(\mathcal{K}(\iota))}$ and such that $\mathcal{K}(Q)$ is a quotient of $\mathcal{K}(C)$ modulo $\approx$ in $\boldsymbol{D}$ with quotient morphism $\mathcal{K}(\iota) : \mathcal{K}(C) \xrightarrow{\boldsymbol{D}} \mathcal{K}(Q)$. If $\boldsymbol{C} = \boldsymbol{D}$ and $E = F$, we simply say that $\mathcal{K} : \boldsymbol{C} \to \boldsymbol{C}$ **preserves quotients modulo** E **in** $\boldsymbol{C}$.

In Theorems 3.54 and 3.56, we show that two important functors in the Theory of Rayner Rngs preserve quotients.

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