Analysis of the Ricci flow on compact manifolds

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Dedicated to E. Lages Lima and M. do Carmo

Resumo

PETZET RUDNIK, A. **Análise do fluxo de Ricci em variedades compactas**. 2019. 120 f. Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2019.

Um problema clássico em geometria é o de procurar métricas especiais numa variedade. Dentre os espaços mais simples encontram-se os de curvatura (seccional) constante, a saber, os espaços modelo: $\mathbb{S}^N$, $\mathbb{H}^N$ e $\mathbb{R}^N$ que são únicos, num certo contexto, a menos de isometrias. Em dimensão 2, nós temos o Teorema da Uniformização que afirma que toda superfície fechada admite uma métrica de curvatura constante -1, 0 ou 1, de acordo com seu gênero. Então, uma pergunta natural que surge é se esta conjectura pode ser estendida para dimensões maiores. E aqui o fluxo de Ricci entra em cena para tentar responder a esta pergunta que é conhecida como Conjectura de Geometrização de Thurston. O fluxo de Ricci é um fluxo geométrico, que significa que é definido **independente de coordenadas**, no qual um começa com uma variedade Riemanniana suave (M, g_0) e evolui a sua métrica pela equação $\frac{\partial}{\partial t}g = -2Rc(g)$, onde $Rc(g)$ denota o tensor de Ricci da métrica g . O fluxo de Ricci foi introduzido por Hamilton no seu artigo amplamente conhecido de 1982, "*Three manifolds with positive Ricci curvature*", e pode ser visto como uma equação do calor na métrica Riemanniana.

Palavras-chave: fluxo de Ricci, uniformização, solitons de Ricci, variedades compactas, fluxo geométrico.

Abstract

PETZET RUDNIK, A. **Analysis of the Ricci flow on compact manifolds**. 2019. 120 f. Dissertação (Mestrado)
- Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2019.

A classical problem in geometry is to seek special metrics on a manifold. Among the most simple spaces are those of constant (sectional) curvature, namely, the model spaces: $\mathbb{S}^N$, $\mathbb{H}^N$ and $\mathbb{R}^N$, which are unique, in a sense, up to isometries. In dimension 2, we have the Uniformization Theorem which states that every closed surface admits a metric of constant curvature -1, 0 or 1, according to its genus. So a natural question that arises is if this conjecture may be extended to higher dimensions. And here the Ricci flow enters the stage to try to answer this question which is known as Thurston's Geometrization Conjecture. The Ricci flow is a geometric flow, meaning that it is defined in a **coordinate-independent** manner, in which one starts with a differentiable Riemannian manifold (M, g_0) and evolve its metric by the equation $\frac{\partial}{\partial t}g = -2Rc(g)$, where $Rc(g)$ denotes the Ricci tensor of the metric g . The Ricci flow was introduced in Hamilton's seminal 1982 paper, "Three manifolds with positive Ricci curvature", and it can be seen as a heat equation for the Riemannian metric on a manifold.

Keywords: Ricci flow, uniformization, Ricci solitons, compact manifolds, geometric flow.

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Notations and Abbreviations

LHS Left Hand Side.

RHS Right Hand Side.

PDE Partial Differential Equation.

ODE Ordinary Differential Equation.

$\Gamma(E) = C^\infty(E)$ The set of differentiable sections over a bundle $\pi : E \rightarrow M$.

$\mathfrak{X}(M)$ The space of vector fields over M .

$\sigma[L](\xi)$ the principal symbol of a differential operator L in the direction of ξ .

G the matrix of the metric in some system of coordinates.

$\Gamma(S_2^+ T^* M)$ the space of all Riemannian metrics.

$\text{Diff}(M)$ The Set of Diffeomorphisms (C^∞) of M .

$dV_g = dV(g)$ the volume form of the metric g on N -dimensional spaces.

$d\mu(g) = d\mu$ The volume form of 2-manifolds.

Γ_{ij}^k the Christoffel symbols in coordinates.

$\text{tr}, \det, \text{div}, \text{trace}, \text{determinant and divergence.}$

$\text{grad}(\cdot) = g^{ij}(\partial_i \cdot) \partial_j$ the gradient.

$\doteq$ defined to be equal to.

$\mathcal{L}$ Lie derivative.

∇ covariant derivative.

$X^\flat$ dual 1-form to the vector field X .

$\alpha^\sharp$ dual vector field to the 1-form α .

$R, R_c, \mathcal{R} R_m$ scalar, Ricci, $\binom{3}{1}$ and $\binom{4}{0}$ Riemann curvature tensors.

$\langle \cdot, \cdot \rangle$ inner-product on some vector bundle.

♣ the end of a proof or an example, depending on the context.

$\int$ integral over the manifold from the context.

N dimension of the manifold from the context.

Δ tensorial Laplace or Laplace, depending on the context.

$g(X, Y) = \langle X, Y \rangle$ metric.

$\{\partial_k\}$ coordinate vector fields or frame, depending on the context.

T^*M cotangent bundle.

$T_l^k M$ the bundle of k -covariant and l -contravariant tensors over M .

(x^i) local coordinates.

f^*E Pullback bundle of E by f .

${}^f\nabla$ Pullback connection on f^*E .

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Introduction

A classical problem in geometry is to seek special metrics on a manifold. Among the most simple spaces are those of constant (sectional) curvature, namely, the model spaces: $\mathbb{S}^N$, $\mathbb{H}^N$ and $\mathbb{R}^N$, which are unique, in a sense, up to isometries. In dimension 2, we have the Uniformization Theorem which states that every closed surface admits a metric of constant curvature -1, 0 or 1, according to its genus. So a natural question that arises is to ask if this conjecture may be extended to higher dimensions. The 3-dimensional version of the uniformization theorem is known as Thurston's geometrization conjecture, and to try to approach it the Ricci flow enters the stage. The Ricci flow is a geometric flow, meaning that it is defined in a **coordinate-independent** manner, in which one starts with a differentiable Riemannian manifold (M, g_0) and evolve its metric by the equation $\frac{\partial}{\partial t} g = -2Rc(g)$, where $Rc(g)$ denotes the Ricci tensor of the metric g . The Ricci flow was introduced by Hamilton in the early 1980s on [Ham82]. Inspired by Eels and Sampson's work on harmonic map heat flow in 1964, Hamilton conjectured that it should be possible to take other geometric objects and try to 'improve it' by means of a heat-type equation, for example, the metric g_{ij} . The properties of the Ricci flow equation are therefore connected with properties of heat equations on manifolds. For example, maximum principles are fundamental tools for the understanding of long time existence and the development of singularities. There is also an independent subject which has strengthened its importance after Hamilton's paper, namely Differential Harnack Inequalities (see [MS06]). Hamilton's Ricci flow has led to many applications in Differential Geometry. Most notably is Perelman's work, [Per02], which is believed to have led to prove Poincaré's conjecture, which says that every simply connected, closed 3-manifold is homeomorphic to the 3-sphere. Another very interesting application of the Ricci flow is the differentiable sphere theorem, which was proved by Simon Brendle on, for example, [Bre10] and which has a nice exposition in [Bre10].

0.1 Contributions

The main contributions of this work are the following:

- i. To expose a simple and accessible language of tensors, its derivatives and others differential operators in such a manner that people outside geometry can follow the text.
- ii. To explore the language of tensors and its expressions in the context of the Ricci flow, and to provide full understanding of facts that specialists take, in general, for granted. For example, when defining the Harmonic Map Flow, we give a complete exposition of the expression of the maps involved, the pullback connection and other quantities.

0.2 Overview

In chapter 1 we give a brief introduction to the basic facts from Riemannian geometry that we are going to need throughout the text. In general we omit proofs of facts one can encounter in basic Riemannian geometry books.

In chapter 2 We introduce the Ricci flow equation. We also explore (differentiable) one parameter families of Riemannian metrics and its evolution equation and the evolution of all the tensors a metric generates. For that we see that although the connection is not a tensor, its derivative with respect to time is, that is, $\partial_t \nabla$ is a tensor, which simplifies some calculations. For further understanding of the Ricci tensor seen as a differential operator on the space of metrics we define principal symbols of operators on vector bundles and investigate the fact that the linearized Ricci tensor has a nontrivial kernel and that it is related to the nonparabolicity of the Ricci flow equation. Then, assuming existence of the Ricci DeTurck's flow, we go into existence theory which employs the Ricci DeTurck's diffeomorphisms, which is a family of diffeomorphisms that pulls the Ricci DeTurck's flow back to the Ricci flow. To prove uniqueness, we use the Harmonic Map flow introduced by Eels and Sampson (1964).

In chapter 3 we explore more evolution equations now in the direction of the Ricci flow. We also investigate maximum principles and see some applications. Then, we see in which circumstances we may extend the Ricci flow and that it is closely related to the blow-up of the $\binom{4}{0}$ -tensor field of curvature.

In chapter 4 we investigate the Ricci flow in 2-dimensional Riemannian manifolds. In fact, we see that it is easier to work with the normalized Ricci flow, since it enjoys long time existence given any initial data (see [Ham88]). We then explore some properties of Ricci solitons and gradient Ricci solitons and see that the tensor M defined as the trace free part of the Hessian of the Ricci potential being zero means that the metric is a gradient Ricci soliton.

Chapter 1

Preliminaries from Riemannian geometry

1.1 Differentiable manifolds and tangent spaces

Definition 1.1. A differentiable manifold of dimension N is a Hausdorff and second countable topological space $\mathbb{M}$ together with a family of homeomorphisms $\chi_\alpha : U_\alpha \subset \mathbb{M} \rightarrow \mathbb{R}^N$ of open sets U_α of $\mathbb{M}$ into $\mathbb{R}^N$ such that

- $\bigcup_\alpha (U_\alpha) = \mathbb{M}$.
- for any pair α, β with $U_\alpha \cap U_\beta = W \neq \emptyset$ the mappings $\chi_\alpha \circ \chi_\beta^{-1}$ are differentiable as maps in the euclidean sense.
- The family $\{(U_\alpha, \varphi_\alpha)\}$ is maximal relatively to the previous two conditions.

The pair (U_α, χ_α) is called a chart on $\mathbb{M}$ or a system of coordinates. We will denote it by $(U, (x^i))$, or simply (x^i) , or even $(U, \chi = (x^i))$, when we want to explicit both the map χ and its coordinates. We will call each x^i a coordinate function of the chart (U, χ) . Finally, the term differentiable means smooth, that is, infinitely differentiable. The manifolds we are working with will be always of class C^∞ . Whenever we use the term differentiable, it is implied that we are meaning C^∞ .

Definition 1.2. Let $\mathbb{M}$ be a differentiable manifold. We say that a function $f : \mathbb{M} \rightarrow \mathbb{R}$ is differentiable provided

$$f \circ \chi^{-1} : \chi(U) \subset \mathbb{R}^N \rightarrow \mathbb{R}$$

is differentiable, where (U, χ) is any chart. A curve $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{M}$ is said to be differentiable if its coordinate representation is differentiable, that is, if

$$\alpha(t) = \chi^{-1}(\alpha_1(t), \dots, \alpha_N(t)) = \chi^{-1}((x^1 \circ \alpha)(t), \dots, (x^N \circ \alpha)(t)),$$

in some system of coordinates (U, χ) such that $U \cap \alpha(I) \neq \emptyset$, $\chi = (x^1, \dots, x^N)$, then $\alpha_i(t)$ is differentiable, for $i = 1, \dots, N$. More generally, given manifolds $\mathbb{M}$ and P , of dimensions N and K respectively, we say that a map $f : \mathbb{M} \rightarrow P$ is differentiable if

$$\psi \circ f \circ \chi^{-1} : \chi(U) \subset \mathbb{R}^N \rightarrow \mathbb{R}^K$$

is differentiable in the euclidean sense, for some choice of differentiable charts (U, χ) on $\mathbb{M}$ and (V, ψ) on P , provided $f(U) \subset V$.

It is clear that Definition 1.2 does not depend on choice of charts. In particular, for any chart $(U, (x^i))$, its coordinate functions $x^i : U \rightarrow \mathbb{R}$ are differentiable. We will denote the space of all differentiable functions on $\mathbb{M}$ by $C^\infty(\mathbb{M})$.

Definition 1.3. Let $\mathbb{M}$ be a differentiable manifold and let p be a point in $\mathbb{M}$. A linear map $v : C^\infty(\mathbb{M}) \rightarrow \mathbb{R}$ is called a **derivation at p** if it satisfies

$$v(fg) = f(p)v(g) + g(p)v(f), \text{ for all } f, g \in C^\infty(\mathbb{M}).$$

The set of all derivations of $C^\infty(\mathbb{M})$ at p , denoted by $T_p\mathbb{M}$, is a vector space called the **tangent space to $\mathbb{M}$ at p** . An element of $T_p\mathbb{M}$ is called a **tangent vector at p** .

Definition 1.4. Given a differentiable manifold $\mathbb{M}$, a point $p \in \mathbb{M}$ and a differentiable curve $\alpha : (-\delta, \delta) \rightarrow \mathbb{M}$, with $\alpha(0) = p$ the tangent vector to the curve α at $t = 0$, is a function that takes differentiable functions over $\mathbb{M}$ into real numbers

$$\alpha'(0) : C^\infty(\mathbb{M}) \rightarrow \mathbb{R} \text{ defined by } \alpha'(0)(f) = \left. \frac{d}{dt} \right|_{t=0} (f \circ \alpha)(t).$$

Any tangent vector at p is the tangent vector at $t = 0$ of some curve $\alpha : (-\delta, \delta) \rightarrow \mathbb{M}$, with $\alpha(0) = p$.

Definition 1.5. Let $\mathbb{M}$ and P be differentiable manifolds and $F : \mathbb{M} \rightarrow P$ a differentiable map. Then, for each $p \in \mathbb{M}$ we define a map

$$dF_p : T_p\mathbb{M} \rightarrow T_{F(p)}P$$

called the **differential of F at p** , as follows. Given $v \in T_p\mathbb{M}$ we let $dF_p v$ be the derivation at $F(p)$ that acts on $f \in C^\infty(P)$ by the rule

$$dF_p v(f) = v(f \circ F).$$

Before continuing, let's introduce a manifold which we will be working with the text: the **tangent bundle $T\mathbb{M}$** . Let $\mathbb{M}$ be a differentiable N -dimensional manifold and define

$$T\mathbb{M} = \{(p, v); p \in \mathbb{M}, v \in T_p\mathbb{M}\}.$$

The tangent bundle comes equipped with a natural topology and differentiable structure. With this natural structure and topology the tangent bundle is a $2N$ -dimensional manifold. This is the natural space to work with positions, velocities and vector fields, which are sections of the tangent bundle and will be explained hereafter.

1.2 Vector Fields

Definition 1.6. A **vector field** X on a manifold $\mathbb{M}$ is a correspondence that associates to each point $p \in \mathbb{M}$ a tangent vector $X(p) = X|_p \in T_p\mathbb{M}$ on the tangent space of $\mathbb{M}$ at p . Thus, X is a map from $\mathbb{M}$ into the tangent bundle $T\mathbb{M}$ and is also called a section of the tangent bundle

$$X : \mathbb{M} \rightarrow T\mathbb{M}.$$

The vector field X is said to be differentiable if the map $X : \mathbb{M} \rightarrow T\mathbb{M}$ is differentiable. The set of all vector fields over a manifold $\mathbb{M}$ is denoted by $\mathfrak{X}(\mathbb{M})$.

Coordinate vector fields are an example of local vector fields and they are obtained as follows. Given a chart $(U, (x^i))$ and writing $\chi = (x^1, \dots, x^N)$, then its coordinate vector fields are defined by

$$\left. \frac{\partial}{\partial x^i} \right|_p = d(\chi^{-1})_{\chi(p)} e_i|_{\chi(p)}, \quad \text{for } i = 1, \dots, N,$$

where $\{e_i\}$ is the standard basis of $\mathbb{R}^N$ at the point $\chi(p)$. We will denote these local vector fields by $\frac{\partial}{\partial x^i} = \partial_i$. We may express any vector field X in local coordinates as $X = X^k \partial_k$, where the Einstein summation convention is used, and the functions $X^k : U \subset \mathbb{M} \rightarrow \mathbb{R}$ depend on the system of coordinates chosen and are differentiable if X is.

Now, with the notion of coordinate vector fields in hands we may revisit Definition 1.5 and see how differentials of maps looks like in local coordinates.

Under the same hypothesis of Definition 1.5, let $(U, (x^i))$ be a chart on $\mathbb{M}$ and let $(V, (y^j))$ be a chart on P with $F(U) \cap V \neq \emptyset$, and consider $v = \left. \frac{\partial}{\partial x^k} \right|_p$. Writing $\psi = (y^1, \dots, y^K)$, with K the dimension of P , we have

$$dF_p \left(\left. \frac{\partial}{\partial x^k} \right|_p \right) (f) = \left. \frac{\partial}{\partial x^k} \right|_p (f \circ \psi^{-1} \circ \psi \circ F) = \left. \frac{\partial(y^j \circ F)}{\partial x^k} \right|_p \left. \frac{\partial}{\partial y^j} \right|_{F(p)} (f).$$

Thus,

$$dF_p \left(\left. \frac{\partial}{\partial x^k} \right|_p \right) = \left. \frac{\partial(y^j \circ F)}{\partial x^k} \right|_p \cdot \left. \frac{\partial}{\partial y^j} \right|_{F(p)}, \quad F(p) \in V.$$

We call $dF_p v$ the pushforward of $v \in T_p\mathbb{M}$ by F and denote it by $dF_* v$.

Occasionally we regard vector fields as mappings acting on the space of functions into the space of functions, that is, $X : C^\infty(\mathbb{M}) \rightarrow C^\infty(\mathbb{M})$. To describe this situation it is useful to look at coordinate vector fields. Given a chart (U, χ) on $\mathbb{M}$, we define the action of its coordinate vector fields by

$$\partial_k|_p(f) = \frac{\partial(f \circ \chi^{-1})}{\partial x^k}(\chi(p))$$

Hence, for $X = X^k \partial_k$, we have $X|_p(f) = X^k(p) \partial_k|_p(f)$. There is another way of looking at vector fields regarding them as a functional acting on the space of 1-forms. We will describe this approach later, when we have the notion of cotangent bundle.

If $\Phi : \mathbb{M} \rightarrow P$ is a diffeomorphism between manifolds and $X \in \mathfrak{X}(\mathbb{M})$ is a vector field, we are able to pushforward X to a vector field on P in the following way. For each $p \in \mathbb{M}$ define $Y_{\Phi(p)} = d\Phi_p X_p$. Since Φ is a diffeomorphism this is a well defined vector field in P . We will denote the vector field Y by $\Phi_* X = Y$ when Y is obtained this way and say that Y is the pushforward of X via of Φ . We can describe this using local coordinates. Let $(U, \chi = (x^k))$ be a system of coordinates on $\mathbb{M}$ and $(V, \psi = (y^\alpha))$ on P and consider their coordinates vectors respectively denoted $\frac{\partial}{\partial x^k}$ and $\frac{\partial}{\partial y^\alpha}$. Then, we have

$$d\Phi_p \frac{\partial}{\partial x^k} \Big|_p = d(\Phi \circ \chi^{-1})_{\chi(p)} e_k \Big|_{\chi(p)} = \frac{\partial \Phi^\alpha}{\partial x^k}(\chi(p)) \frac{\partial}{\partial y^\alpha} \Big|_{(\psi \circ \Phi)(p)}$$

where $\frac{\partial \Phi^\alpha}{\partial x^k}(\chi(p)) = \frac{\partial (y^\alpha \circ \Phi \circ \chi^{-1})}{\partial x^k}(\chi(p))$. For an arbitrary vector field $X \in \mathfrak{X}(\mathbb{M})$, we write X in these coordinates as $X = X^k \frac{\partial}{\partial x^k}$ and use the linearity of $d(\Phi)$ to obtain a similar formula, now omitting the point,

$$d\Phi X = X^k \frac{\partial \Phi^\alpha}{\partial x^k} \frac{\partial}{\partial y^\alpha}$$

The interpretation of vector fields as operators acting on the space of functions allows us to consider compositions of these operators. For instance, if X and Y belong to $\mathfrak{X}(\mathbb{M})$, and f belongs to the space of differentiable functions, $f \in C^\infty(\mathbb{M})$, then (Xf) belongs to $C^\infty(\mathbb{M})$ and again $Y(Xf)$ is a differentiable function. Then, $Y(Xf) = (Y \circ X)(f)$. Unfortunately, this composition does not define a vector field in general. But there is a way of obtaining a new vector field from the given ones:

$$[X, Y] = X \circ Y - Y \circ X \tag{1.1}$$

The bracket operation is then defined $[\cdot, \cdot] : \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \rightarrow \mathfrak{X}(\mathbb{M})$ by the equation (1.1) above. In general, we will omit the composition among vector fields and simply write XY instead of $X \circ Y$. We just point out that this operation enjoys anticommutativity, linearity and also the Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

There are several other properties involving the brackets some of which will be discussed opportunely.

1.3 Tensors on Vector Spaces

We're going now to establish some nomenclature for tensors that will appear throughout the text. The reader already acquainted with the subject may skip this part.

Let W be a finite-dimensional real vector space. Given an integer $k \in \mathbb{N}$, a covariant k -tensor on W is an element of the k -fold tensor product $W^* \otimes \dots \otimes W^*$, that is, a real-valued multilinear function of k elements of W :

$$\beta : W \times \dots \times W \rightarrow \mathbb{R}$$

We denote the vector space of all covariant k -tensors on W by the notation $T^k(W^*) = W^* \otimes \dots \otimes W^*$.

On any finite-dimensional real vector space W , we can define the notion of a contravariant l -tensor on W , which is an element of the l -fold tensor product $W^* \otimes \dots \otimes W^*$, that is, a real-valued multilinear function of l elements of W^* ,

$$\beta : W^* \times \dots \times W^* \rightarrow \mathbb{R}.$$

We denote the vector space of all contravariant l -tensors on W^* by the notation $T_l(W) = W^* \otimes \dots \otimes W^*$.

More generally, given a finite-dimensional real vector space W , and integers $k, l \in \mathbb{N}$, we define the space of mixed tensors on W of type $\binom{k}{l}$ as

$$T_l^k(W) = W^* \otimes \dots \otimes W^* \otimes W \otimes \dots \otimes W,$$

where we consider k copies of the tensor fold product of W^* and l copies of the tensor fold product of W . We are going to refer to these functions as $\binom{k}{l}$ -tensors on W and it is implicit that the tensor has k arguments from the vector space W , and l arguments from W^* . Be aware, however, that the notation $T_l^k(W)$ is not universal.

It is fundamental for this work to know how tensors interact under composition, namely tensor product. Given any $F \in T_l^k(W)$ and $G \in T_q^p(W)$, we denote the tensor product of F by G as $F \otimes G$ and we define it by

$$F \otimes G(X_1, \dots, X_{k+p}, \eta^1, \dots, \eta^{l+q}) = F(X_1, \dots, X_k, \eta^1, \dots, \eta^l)G(X_{k+1}, \dots, X_{k+p}, \eta^{l+1}, \dots, \eta^{l+q})$$

It is straightforward that, given a N -dimensional vector space W , considering any basis $\{E_1, \dots, E_N\}$ for W , and $\{\varepsilon^1, \dots, \varepsilon^N\}$ its dual basis, then given any $\beta \in T_l^k(W)$, we may write

$$\beta = \beta_{i_1 \dots i_k}^{j_1 \dots j_l} \varepsilon^{j_1} \otimes \dots \otimes \varepsilon^{j_l} \otimes E_{i_1} \otimes \dots \otimes E_{i_k},$$

where $\beta_{i_1 \dots i_k}^{j_1 \dots j_l} = \beta(E_{i_1}, \dots, E_{i_k}, \varepsilon^{j_1}, \dots, \varepsilon^{j_l})$. We are going to refer for these functions as $\binom{k}{l}$ -tensors on W and it is implicit that the tensor has k arguments from the vector space W , and l arguments from W^* . Notice also that the first arguments of tensors of mixed type are vectors, and their last arguments are covectors.

Trace. We define an operation called trace or contraction on the space of tensors which lowers the rank of a tensor by two. We define $\text{tr} : T_{l+1}^{k+1}(W) \rightarrow T_l^k(W)$ by letting $\text{tr}F(X_1, \dots, X_k, \eta^1, \dots, \eta^l)$ be the trace of the endomorphism $F(X_1, \dots, X_k, \cdot, \eta^1, \dots, \eta^l, \cdot) \in T_1^1(W)$. More generally, we can define the trace of a given tensor on any pair of indices such that one is covariant and one is contravariant. For example, given a $\binom{3}{1}$ -tensor on W written in some basis as

$$\beta = \beta_{ijk}^p \varepsilon^i \otimes \varepsilon^j \otimes \varepsilon^k \otimes E_p,$$

then the trace of β on its third and fourth arguments gives the $\binom{2}{0}$ -tensor $\text{tr}\beta = \beta_{ijp}^p \varepsilon^i \otimes \varepsilon^j$.

Alternating and Symmetric Tensors. Let W be a finite-dimensional real vector space. A covariant k -tensor α on W is said to be **symmetric** if its value is unchanged by interchanging any pair of arguments:

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k),$$

for all $1 \leq i \leq j \leq k$ and all vectors $v_1, \dots, v_k \in W$. For example, the inner product is a symmetric 2-tensor. A k -tensor β on W is said to be **alternating** (or **antisymmetric**) if it changes sign whenever two of its arguments are interchanged

$$\beta(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = \beta(v_1, \dots, v_j, \dots, v_i, \dots, v_k),$$

for all pair of indices i, j , and all vector $v_1, \dots, v_k \in W$.

1.4 Vector Bundles and Tensor Fields on Manifolds

Definition 1.7. Let $\mathbb{M}$ be a differentiable manifold. A real vector bundle of rank k over $\mathbb{M}$ (the base space) is a differentiable manifold E (the total space), together with a differentiable surjective map $\pi : \mathbb{M} \rightarrow E$, (the projection) satisfying the following conditions:

- for each $p \in \mathbb{M}$, the set $E_p = \pi^{-1}(p)$ (the fibre of E over p) is endowed with a structure of a k -dimensional real vector space.
- for each $p \in \mathbb{M}$, there exists a neighborhood U of p and a diffeomorphism $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ (called a local trivialization of E over U) such that for each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k$
- $\pi_U \circ \Phi = \pi$, where $\pi_U : U \times \mathbb{R}^k \rightarrow U$ is the projections and Φ and U are as in the previous condition.

For example, the tangent bundle $T\mathbb{M}$ of a given manifold $\mathbb{M}$ is a vector bundle of rank N , whereas its dimension as a manifold is $2N$. A differentiable vector field X over a differentiable manifold $\mathbb{M}$ can now be interpreted as a section, that is, if $\pi : T\mathbb{M} \rightarrow \mathbb{M}$ is the natural projection, then $\pi \circ X = Id_{\mathbb{M}}$. The space of sections of the tangent bundle will be denoted by $\Gamma(T\mathbb{M})$, so that $\mathfrak{X}(\mathbb{M}) = \Gamma(T\mathbb{M})$.

Definition 1.8. Given a vector bundle $\pi : \mathbb{M} \rightarrow E$, a section on E is a map $\tau : \mathbb{M} \rightarrow E$ such that $\pi \circ \tau = Id_{\mathbb{M}}$ is the identity map of $\mathbb{M}$. The set of differentiable sections over a vector bundle E is going to be denoted by $\Gamma(E)$.

Example 1.9. [The pullback bundle]

Let $\mathbb{M}$ and P be differentiable manifolds and let $\pi : E \rightarrow \mathbb{M}$ be a differentiable vector bundle. If $f : \mathbb{M} \rightarrow P$ is a differentiable map, then we define the **pullback bundle** f^*E **over** P as the set

$$f^*E \doteq \{(p, v) \in P \times E : f(p) = \pi(v)\},$$

whose fiber at $x \in P$ is $(f^*E)_x = E_{f(x)}$ and whose local trivializations are induced from the local trivializations of E (where one uses the map $\Gamma(E) \rightarrow \Gamma(f^*E)$, $s \mapsto f^*s \doteq s \circ f$). ♣

Example 1.10. Next, we look at the operation that associates to two vector spaces V and W , the vector space $Hom(V, W)$, consisting of all linear maps from V to W . Given now two (differentiable) vector bundles E and F over $\mathbb{M}$, we form the new vector bundle $Hom(E, F)$ over $\mathbb{M}$ with fibers

$$Hom(E, F)_x = Hom(E_x, F_x).$$

We see that local frames of E and F induce a local frame of $\text{Hom}(E, F)$, and then we obtain a canonical differentiable structure on this new vector bundle. A section of $\text{Hom}(E, F)$ is a differentiable map $u : E \rightarrow F$ (of vector bundles over $\mathbb{M}$) with the property that, for each $x \in \mathbb{M}$, u sends E_x to F_x and the map

$$u_x \doteq u|_{E_x} : E_x \rightarrow F_x,$$

is linear. We can identify sections of $\text{Hom}(E, F)$ with $C^\infty(\mathbb{M})$ -linear maps $\Gamma(E) \rightarrow \Gamma(F)$, but we will not discuss this here. For more details, you can consult [Lee03, Chapter 10]. ♣

Now let $\mathbb{M}$ be a differentiable manifold. Given any point $p \in \mathbb{M}$, we consider the tangent space to $\mathbb{M}$ at p and we can then perform the same algebraic constructions over $T_p\mathbb{M}$ and its dual space, $(T_p\mathbb{M})^*$, tensor product, traces, etc., producing tensors at p . For example, a $\binom{k}{l}$ -tensor at p is an element of $T_l^k(T_p\mathbb{M})$. Tensors defined over $\mathbb{M}$ will be called tensor fields. A $\binom{k}{l}$ -tensor field over $\mathbb{M}$ is a section of the vector bundle $\pi : T_l^k(\mathbb{M}) \rightarrow \mathbb{M}$,

$$F : \mathbb{M} \rightarrow T_l^k(\mathbb{M}), \quad p \mapsto F|_p \in T_l^k(T_p\mathbb{M}),$$

meaning that

$$\pi \circ F = \text{Id}_{\mathbb{M}}, \quad \text{where } \pi : \mathbb{M} \rightarrow T_l^k(\mathbb{M})$$

is the vector bundle of mixed tensors of type $\binom{k}{l}$. Sometimes, we describe the bundle itself by writing it as a union of the fibres at each point,

$$T_l^k(\mathbb{M}) = \bigcup_{p \in \mathbb{M}} T_l^k(T_p\mathbb{M}).$$

Whenever speaking about tensors, we will use the vertical line notation which is useful to distinguish the tensor from its coefficients functions, when represented in a system of coordinates. To explain what the vertical line notation means we give some examples.

Example 1.11. [Covariant 1-tensor fields]

Consider the cotangent bundle $T^*(\mathbb{M})$, and an arbitrary section $\tau \in C^\infty(T^*\mathbb{M})$, i.e., a (differential) 1-form. Consider then a system of coordinates $(U, (x^i))$ and its coordinate covector fields $dx^j : U \rightarrow T^*\mathbb{M}$, and write $\tau = a_i dx^i$ as linear combination of these covector fields, where $a_j : U \rightarrow \mathbb{R}$ are the coefficients functions of τ in the coordinates (x^i) . We have,

$$\tau : \mathbb{M} \rightarrow T^*\mathbb{M}, \quad p \in \mathbb{M} \mapsto \tau|_p = a_i(p) dx^i|_p. \clubsuit$$

Example 1.12. [Contravariant 1-tensor fields]

Given a system of coordinates (U, x^i) we have the coordinate vector fields

$$\partial_j = \frac{\partial}{\partial x^j} : U \rightarrow TU, \quad p \in \mathbb{M} \mapsto \partial_j|_p \in (T_p\mathbb{M})^{**},$$

which form a basis for the tangent space at any point $p \in U$. Therefore, given any $X \in \mathfrak{X}(\mathbb{M})$, we write it locally as $X = X^i \partial_i$, where $X^j : U \rightarrow \mathbb{R}$ are the coefficient functions of X in the system (x^i) and point out that it acts on covector fields according to the formula

$$X(dx^k) = \delta_k^i X^i = X^k$$

Thus, vector fields can be seen as sections acting on the space of (differential) 1-forms. ♣

From now on and throughout this work we will write any tensor field in some system of coordinates as in the previous two examples and we will make no reference for its coefficient functions, which should

be understood from the context. Also, observe that we will always be using the Einstein summation convention for expressions with repeat indices.

Example 1.13. [*Covariant k -tensor fields*]

Consider the vector bundle $\pi : T^k\mathbb{M} \rightarrow \mathbb{M}$. Given a section $\tau \in C^\infty(T^k\mathbb{M})$, also called a $\binom{k}{0}$ -tensor field, and a system of coordinates (U, x^i) , we will write

$$\tau : \mathbb{M} \rightarrow T^k\mathbb{M}, \quad p \in \mathbb{M} \mapsto \tau|_p = a_{i_1 \dots i_k}(p)(dx^{i_1} \otimes \dots \otimes dx^{i_k})|_p$$

In this case, we will use parentheses for the tensor product of these kind of tensor fields. ♣

Example 1.14. [*Alternate covariant k -tensor fields*]

Consider now the bundle $\Lambda^k\mathbb{M}$ of k -forms over $\mathbb{M}$. We will use the more traditional notation $\Omega^k(\mathbb{M})$ for the space of sections of this bundle, instead of $C^\infty(\Lambda^k\mathbb{M})$, and for a (differentiable) k -form $\alpha \in \Omega^k(\mathbb{M})$, we will write, in a given system of coordinates (U, x^i) ,

$$\alpha : \mathbb{M} \rightarrow \Lambda^k\mathbb{M}, \quad p \in \mathbb{M} \mapsto \alpha|_p = a_{i_1 \dots i_k}(p)(dx^{i_1} \wedge \dots \wedge dx^{i_k})|_p$$

Remembering that in the summation for alternating tensors we only consider the indices $i_1 < \dots < i_k$, where $i_j = 1, \dots, N$, where $j \in \{1, \dots, k\}$ and N is, as always, the dimension of the manifold $\mathbb{M}$. ♣

Example 1.15. [*k -covariant, l -contravariant tensor fields*]

Finally, we consider the bundle $T_l^k\mathbb{M}$ of $\binom{k}{l}$ -tensor fields over $\mathbb{M}$, whose space of sections will be denoted throughout the text by the more convenient notation $\Gamma(T_l^k\mathbb{M})$. We have, $\Gamma(T_l^k\mathbb{M}) = C^\infty(T_l^k\mathbb{M})$. Given any system of coordinates (U, x^i) , we know that the coordinate vector fields $\{\partial_i|_p\} \subset T_p\mathbb{M}$ and the coordinate covector fields $\{dx^j|_p\} \subset (T_p\mathbb{M})^*$ form a basis for these vector spaces at each point $p \in U$. Then, given any tensor field $F \in \Gamma(T_l^k\mathbb{M})$, we may write

$$F = F_{j_1 \dots j_l}^{i_1 \dots i_k} dx^{j_1} \otimes \dots \otimes dx^{j_l} \otimes \partial_{i_1} \otimes \dots \otimes \partial_{i_l}$$

for certain coordinate functions $F_{j_1 \dots j_l}^{i_1 \dots i_k} \in C^\infty(U)$, for $i_n, j_m = 1, \dots, N$, where $n \in \{1, \dots, k\}$ and $m \in \{1, \dots, l\}$. ♣

Convention 1.16. When writing any tensor field of mixed type we write the tensor's covariant part in front of the contravariant part. In other words, the first arguments of any mixed tensor field will be vector fields, and its last arguments will be covector fields. In the above example, we have first $dx^{j_1} \otimes \dots \otimes dx^{j_l}$, which we call the covariant part, and after we have $\partial_{i_1} \otimes \dots \otimes \partial_{i_l}$ which is the contravariant part. Notice also that we always choose indices' positions so that covector fields have upper indices, while the component functions of covectors fields have lower indices. On the other hand, vector fields have lower indices and their component functions have upper indices. We are going to refer for these functions as $\binom{k}{l}$ -tensor fields on $\mathbb{M}$ and it is implicit that the tensor has k arguments from the tangent bundle $T\mathbb{M}$ and l arguments from the cotangent bundle $T^*\mathbb{M}$.

1.5 Lie Derivative

In order to define a reasonable derivative of vector fields and ultimately arbitrary tensor fields, we must further investigate objects related to the manifold itself in order to find an intrinsic way to define a derivative. There is no way of copying directional derivatives in $\mathbb{R}^N$, since manifolds lack vector space structure, in general.

Given a Manifold $\mathbb{M}$, and a vector field X , we consider the family of diffeomorphisms generated by X , $\psi_t : \mathbb{M} \rightarrow \mathbb{M}$, for $t \in (-\epsilon, \epsilon)$, given by the solution of the equation

$$\frac{\partial}{\partial t} \psi = X, \quad \psi_0 = Id.$$

This ψ is called the local flow of X . The existence of flows on manifolds follows from basic theory of differential equations. For existence of flows on manifolds, we refer the reader [Lee03, Chapter 9]. Following this reference, we define the **Lie derivative of W with respect to X** by

$$(\mathcal{L}_X W)_p = \lim_{t \rightarrow 0} \frac{d(\psi_{-t})_{\psi_t(p)} W_{\psi_t(p)} - W_p}{t},$$

provided the limit exists. We are interested in defining the Lie derivative of more general tensor fields. Let us define first the pull of tensors under local diffeomorphisms. Let F be a $\binom{k}{l}$ -tensor field over $\mathbb{M}$, and consider a local diffeomorphism $\Phi : \mathbb{M} \rightarrow \mathbb{M}$. We define the $\binom{k}{l}$ -tensor field $\Phi^* F$ on $\mathbb{M}$ called the **pullback of F by Φ** by

$$\Phi^* F_{(p)}(X_1, \dots, X_k, \eta^1, \dots, \eta^l) = F|_{\Phi(p)}(d(\Phi)_p X_1|_p, \dots, d(\Phi)_p X_k|_p, (\Phi^{-1})^* \eta^1|_p, \dots, (\Phi^{-1})^* \eta^l|_p),$$

for all $p \in \mathbb{M}$, and vector fields $X_1, \dots, X_k \in \mathfrak{X}(\mathbb{M})$, and covector fields $\eta^1, \dots, \eta^l \in \Gamma(T^*\mathbb{M})$. Since Φ is a local diffeomorphism, Φ^{-1} is well defined on a neighborhood of $\Phi(p)$ and the symbol $(\Phi^{-1})^* \eta^j$ is the pullback of the 1-form η^j restricted to the neighborhood where Φ is a diffeomorphism. Thus, $(\Phi^{-1})^* \eta^j$ is a well defined local 1-form, for $j = 1, \dots, l$. Now we define the Lie derivative of general tensor fields. Given a $\binom{k}{l}$ -tensor field F and a vector field $X \in \mathfrak{X}(\mathbb{M})$ whose flow is ψ_t , we define the **Lie derivative of F with respect to X** at $p \in \mathbb{M}$ by

$$(\mathcal{L}_X F)_{(p)} = \left. \frac{d}{dt} \right|_{t=0} (\psi_t^* F_{(p)}).$$

There are several properties from the Lie Derivative we're going to use throughout the text. For future reference, we list them here.

Proposition 1.17. Let $\mathbb{M}$ be a differentiable manifold and let $X \in \mathfrak{X}(\mathbb{M})$ be any $\binom{0}{1}$ -tensor field on $\mathbb{M}$. Suppose f is a differentiable function on $\mathbb{M}$ and let F and G be differentiable covariant tensor fields on $\mathbb{M}$.

1. $\mathcal{L}_X f = X(f)$
2. $\mathcal{L}_X(fF) = (\mathcal{L}_X f)F + f\mathcal{L}_X F$
3. $\mathcal{L}_X(F \otimes G) = (\mathcal{L}_X F) \otimes G + F \otimes (\mathcal{L}_X G)$
4. If $Y \in \mathfrak{X}(\mathbb{M})$ is another vector field on $\mathbb{M}$ then $(\mathcal{L}_X Y) = [X, Y]$

5. $\mathcal{L}_X$ commutes with all traces, i.e.,

$$(\mathcal{L}_X \text{tr} F) = \text{tr}(\mathcal{L}_X F)$$

for all traces (over any indices) of the tensor F (after raising one index).

Proof. See [Lee03]. ♣

Among the most important features of the Lie derivative, we highlight the following

Lemma 1.18. *Suppose $\mathbb{M}$ is a differentiable manifold with or without boundary and let $X \in \mathfrak{X}(\mathbb{M})$. If the boundary of $\mathbb{M}$, $\partial\mathbb{M}$, is non-empty assume further that X is tangent to $\partial\mathbb{M}$. Let ψ_t be the flow of X . Then, for any differentiable $\binom{k}{0}$ -tensor field F , at any point in the domain of the flow, we have*

$$\left. \frac{d}{dt} \right|_{t=t_0} (\psi_t)^* F_p = (\psi_{t_0}^* (\mathcal{L}_X F))_p$$

Proof. See [Lee03, Chapter 12]. ♣

1.6 Riemannian Metrics

Definition 1.19. A **Riemannian metric** on a manifold $\mathbb{M}$ is a map $g : \mathbb{M} \rightarrow S_2^+ T^* \mathbb{M}$ which associates to each point $p \in \mathbb{M}$ an inner product $\langle \cdot, \cdot \rangle_p$ on the tangent space $T_p \mathbb{M}$ (that is, a *symmetric, bilinear* and *positive definite* form), which varies differentiably in the following sense: if $(U, (x^i))$ is a system of coordinates around p , then

$$g_{ij} \doteq \langle \partial_i, \partial_j \rangle : U \rightarrow \mathbb{R}, \quad x \in U \mapsto g_{ij}(x) = \left\langle \left. \frac{\partial}{\partial x^i} \right|_x, \left. \frac{\partial}{\partial x^j} \right|_x \right\rangle_x,$$

is a differentiable function on the open set $U \subset \mathbb{M}$. (Notice that we are using the letter x to denote both the variable belonging to a neighborhood of $p \in \mathbb{M}$ and to denote the coordinate functions $x^i : U \rightarrow \mathbb{R}$, $x \mapsto x^i(x)$ of some chart $(x^1, \dots, x^N) : U \rightarrow \mathbb{R}^N$).

It is clear that Definition (1.19) does not depend on the choice of the system of coordinates. Indeed, if $(V, (y^k))$ is another system of coordinates around $p \in U \cap V$ then denoting by

$$h_{lk} = \left\langle \left. \frac{\partial}{\partial y^l} \right|_p, \left. \frac{\partial}{\partial y^k} \right|_p \right\rangle, \quad \text{and by } g_{ij} = \left\langle \left. \frac{\partial}{\partial x^i} \right|_p, \left. \frac{\partial}{\partial x^j} \right|_p \right\rangle, \quad (1.2)$$

the functions obtained by applying the metric $g = \langle \cdot, \cdot \rangle$ on the coordinate vector fields obtained from the corresponding charts and recalling that $\left. \frac{\partial}{\partial y^k} \right|_p = \frac{\partial x^i(p)}{\partial y^k} \left. \frac{\partial}{\partial x^i} \right|_p$ for all $p \in U \cap V$, we have

$$h_{lk} = \left\langle \left. \frac{\partial}{\partial y^l} \right|_p, \left. \frac{\partial}{\partial y^k} \right|_p \right\rangle = \frac{\partial x^i}{\partial y^l} \frac{\partial x^j}{\partial y^k} g_{ij} \quad (1.3)$$

This proves the independence claimed. A Riemannian metric g on a differentiable manifold $\mathbb{M}$ may be alternatively characterized by saying that g is a $\binom{2}{0}$ -tensor field, $g \in \Gamma(T^2 \mathbb{M})$, that is symmetric (i.e.,

$g(X, Y) = g(Y, X)$, for all $X, Y \in \mathfrak{X}(\mathbb{M})$ and positive definite (i.e., $g(X, X) > 0$, for all $0 \neq X \in \mathfrak{X}(\mathbb{M})$). We thus observe that, a Riemannian metric determines an inner product on each tangent space $T_p\mathbb{M}$.

Given any coordinates (x^i) on the open set $U \subset \mathbb{M}$, we may also write the Riemannian metric g in the tensorial form

$$g = g_{ij}dx^i \otimes dx^j : U \rightarrow S_2T^*\mathbb{M}, \quad x \in \mathbb{M} \mapsto g(x) \in S_2^+T^*\mathbb{M}_x \subset S_2T^*\mathbb{M}_x \cong \mathbb{R}^{\frac{N(N+1)}{2}}.$$

We claim that the space of all symmetric and positive definite $\binom{2}{0}$ -tensor fields (which is not a vector space) is open in the vector space of all symmetric $\binom{2}{0}$ -tensor fields, if $\mathbb{M}$ is compact. Let's prove this now. Let $g \in \Gamma(S_2^+T^*\mathbb{M})$ be any Riemannian metric, and take any nonzero $h \in \Gamma(S_2^+T^*\mathbb{M})$. The **unit tangent bundle** is the set

$$UT(\mathbb{M}) = \{(x, v) \in T\mathbb{M} : g_x(v, v) = 1\} \subset T\mathbb{M}.$$

When $\mathbb{M}$ is compact, one may prove that $UT(\mathbb{M})$ is compact and, in general, $UT(\mathbb{M})$ an embedded submanifold (not only when $\mathbb{M}$ is compact) in $T\mathbb{M}$, the tangent bundle. The functions

$$\varrho : UT(\mathbb{M}) \rightarrow \mathbb{R}, \quad (x, v) \mapsto g_x(v, v), \quad \text{and} \quad \tau : UT(\mathbb{M}) \rightarrow \mathbb{R}, \quad (x, v) \mapsto h_x(v, v),$$

are continuous. Define the numbers $\rho \doteq \inf_{p \in UT(\mathbb{M})} \varrho(p)$, $m_1 \doteq \inf_{p \in UT(\mathbb{M})} \tau(p)$ and $m_0 \doteq \sup_{p \in UT(\mathbb{M})} \tau(p)$. There exists $\delta > 0$ such that $\rho + tm_0 > 0$, for all $t \in (-\delta, \delta)$. Choose δ to be $\min\{|\rho/2m_0|, |\rho/2m_1|\}$. The function,

$$\zeta : UT(\mathbb{M}) \times [-\delta, \delta] \rightarrow \mathbb{R}, \quad \text{defined by} \quad (x, v, t) \mapsto g_x(v, v) + th_x(v, v),$$

satisfies $\zeta(p) > 0$, for all $p \in UT(\mathbb{M}) \times (-\delta, \delta)$, and therefore $g + th$ is positive definite for all $t \in (-\delta, \delta)$. This shows that the space of Riemannian metrics

$$\Gamma(S_2^+T^*\mathbb{M}) \subset \Gamma(S_2T^*\mathbb{M})$$

is open in $\Gamma(S_2T^*\mathbb{M})$, if $\mathbb{M}$ is compact. In particular, the tangent space to each fiber of $S_2^+T^*\mathbb{M}$, have the same dimension of each fiber of $S_2T^*\mathbb{M}$. A similar argument, also using the unit tangent bundle, shows that any two Riemannian metrics are uniformly equivalent, if $\mathbb{M}$ is compact. (See A.17)

Considering a system of coordinates $(U, (x^i))$, it is also clear that $g_{ij} = g_{ji}$ and these N^2 function defines a square symmetric matrix which is often regarded as a map denoted by

$$G = (g_{ij}) : U \rightarrow GL_N(\mathbb{R}) \cap \text{Sym}(N),$$

where $\text{Sym}(N)$ is the vector space of all symmetric matrices with real entries. Then, if G denotes the matrix (g_{ij}) and H denotes the matrix (h_{lk}) as given in equations (1.2) then we may rewrite equation (1.3) in the matrix form

$$H = [D(\chi \circ \psi^{-1})^T \circ G \circ D(\chi \circ \psi^{-1})] \quad (1.4)$$

where $D(\chi \circ \psi^{-1})$ is the differential of the change of coordinates, considering $\chi = (x^1, \dots, x^N)$ and $\psi = (y^1, \dots, y^N)$ and this provides another way of looking at the independence of coordinates claimed in terms of the matrices G and H .

Since $G(p)$ is an invertible matrix at each $p \in U$, we may use a well known formula to compute its inverse matrix $G(p)^{-1}$ (see [HK71, Chapter 5, Section 5.4]), which will be very useful throughout. Using the standard notation A_{ij} to denote the entry of a matrix A localized in the i th row and j th column and denoting by $\text{adj } A$ the transpose of the matrix of cofactors of A , the mentioned formula is

$$[G(p)^{-1}]_{ij} = (-1)^{i+j} \frac{1}{\det G(p)} \det G_{[ji]}(p), \quad \left[G^{-1} = \frac{1}{\det G} \text{adj } G \right]$$

where $G_{[ji]}(p)$ is the $(N-1)^2$ square matrix obtained from $G(p)$ by eliminating its j th row and its i th column. We will also denote the inverse matrix by $[G]^{-1} = (g^{ij})$ in any system of coordinates $(U, (x^i))$. In fact, these N^2 real valued functions g^{ij} are the coefficients of the symmetric and positive definite $\binom{0}{2}$ -tensor field denoted by g^{-1} , which is the pointwise inverse of the (Riemannian) metric tensor and which is written locally as

$$g^{-1} = g^{ij} \partial_i \otimes \partial_j.$$

where these coefficients are given by $g^{ij} = g^{-1}(dx^i, dx^j) = [G^{-1}]_{ij}$. (By pointwise inverse of the metric tensor we mean in terms of matrices, that is, at each $x \in U \subset \mathbb{M}$, $(g(x)^{ij})$ is the inverse matrix of the matrix $g(x)_{ij}$, where the coordinates are fixed.)

The metric g on a differentiable manifold also gives rise to metrics on various vector bundles. The usual inner product on the tangent bundle is as follows. Given vector fields X and Y in $\mathfrak{X}(\mathbb{M})$, which are given in local coordinates $(U, (x^i))$ by $X = X^i \partial_i$ and $Y = Y^k \partial_k$, then

$$\langle X, Y \rangle = g_{ik} X^i Y^k.$$

Given covector fields $\xi = \xi_i dx^i, \zeta = \zeta_k dx^k \in \Gamma(T^*\mathbb{M})$ we define the inner product on the cotangent bundle by (what else?)

$$\langle \xi, \zeta \rangle = g^{ik} \xi_i \zeta_k.$$

Notice we're using the same symbol $\langle \cdot, \cdot \rangle$ to denote inner products on different vector bundles. We will do the same for different vector bundles and the space where this inner product is taking place should always be clear from context. Before continuing, let's illustrate a situation that should clarify what is being done. Let us recall a basic fact from euclidean spaces. Consider the vector space of square matrices, $M_N(\mathbb{R})$. Consider then the bilinear function

$$(\cdot, \cdot) : M_N(\mathbb{R}) \times M_N(\mathbb{R}) \rightarrow \mathbb{R}, \quad (A, B) \mapsto (A, B) = \text{tr} A \cdot B^*,$$

where $\text{tr} A = \sum_{i=1}^N a_{ii}$ is the ordinary trace of a matrix in euclidean spaces. (The reason why we are using parentheses to denote this inner product on $M_N(\mathbb{R})$ instead of $\langle \cdot, \cdot \rangle$ is to differentiate the inner product in $\mathbb{R}^{N^2}$, which has a canonical structure, from inner products in vector bundles over Riemannian manifolds.) Then, $(\cdot, \cdot)$ defines an inner product, whose induced norm is

$$\|A\|^2 = \sum_{i,j=1}^N a_{ij}^2, \quad \text{where } A = (a_{ij})_{ij}.$$

In this case, the Cauchy-Schwarz inequality reads

$$(\text{tr} A \cdot B^*)^2 \leq (\text{tr} A \cdot A^*)(\text{tr} B \cdot B^*).$$

In particular, if $B = Id$ is the identity then, for any $A \in M_N(\mathbb{R})$, we have

$$(\text{tr} A \cdot Id)^2 \leq (\text{tr} A \cdot A^*)(\text{tr} Id) = N \|A\|^2, \quad \text{which implies, } \|A\|^2 \geq \frac{1}{N} (\text{tr} A)^2$$

Now, let T and S be two $\binom{2}{0}$ -tensor fields over the Riemannian manifold $\mathbb{M}$, given in local coordinates $(U, (x^s))$ by $T = T_{ij} dx^i \otimes dx^j$ and $S = S_{kl} dx^k \otimes dx^l$. On each fiber of the vector bundle $T^2\mathbb{M}$, the map

$$\langle \cdot, \cdot \rangle : \Gamma(T^2\mathbb{M}) \times \Gamma(T^2\mathbb{M}) \rightarrow C^\infty(\mathbb{M}), \quad (T, S) \mapsto \langle S, T \rangle \doteq \text{tr}(ST^*) = g^{ik} g^{jl} T_{ij} S_{kl}$$

is known as **Hilbert-Schmidt inner product**. The Hilbert-Schmidt inner product also enjoys the property that its induced norm (the norm of a $\binom{2}{0}$ -tensor) dominates trace in the following sense.

Lemma 1.20. [Norm of a 2-covariant tensor dominates trace]
 Let T be any $\binom{2}{0}$ -tensor field over $\mathbb{M}$. Then, we have

$$|T|^2 \geq \frac{1}{N} (g^{jk} T_{jk})^2. \quad (1.5)$$

Proof.

Just choose coordinates where $g_{ij} = \delta_{ij}$. ♣

(Coordinates in which $g_{ij}(p) = \delta_{ij}$ are called normal coordinates based at $p \in \mathbb{M}$ and in Lemma 1.40 we give its properties. Since T is a tensor, equation (1.5) holds in any system of coordinates and at any point.)

More generally, suppose we're given two $\binom{k}{0}$ -tensor fields κ and σ written in local coordinates as

$$\kappa = \kappa_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k} \quad \text{and} \quad \sigma = \sigma_{j_1 \dots j_k} dx^{j_1} \otimes \dots \otimes dx^{j_k},$$

then, we define the inner product on the fibers of $T^k \mathbb{M}$ by

$$\langle \kappa, \sigma \rangle \doteq g^{i_1 j_1} \dots g^{i_k j_k} \kappa_{i_1 \dots i_k} \sigma_{j_1 \dots j_k}. \quad (1.6)$$

Yet more generally, suppose we're given $\binom{k}{l}$ -tensor fields Λ and Ξ written in local coordinates as

$$\begin{aligned} \Lambda &= \lambda_{i_1 \dots i_k}^{j_1 \dots j_l} dx^{i_1} \otimes \dots \otimes dx^{i_k} \otimes \partial_{j_1} \otimes \dots \otimes \partial_{j_l}, \quad \text{and} \\ \Xi &= \xi_{p_1 \dots p_k}^{q_1 \dots q_l} dx^{p_1} \otimes \dots \otimes dx^{p_k} \otimes \partial_{q_1} \otimes \dots \otimes \partial_{q_l} \end{aligned}$$

then define the inner product on each fiber of $T_l^k \mathbb{M}$ by

$$\langle \Lambda, \Xi \rangle \doteq g_{j_1 q_1} \dots g_{j_l q_l} g^{i_1 p_1} \dots g^{i_k p_k} \lambda_{i_1 \dots i_k}^{j_1 \dots j_l} \xi_{p_1 \dots p_k}^{q_1 \dots q_l}. \quad (1.7)$$

Of course, the latter definition for $\binom{k}{l}$ -tensor fields is general enough to include the previous case. The reason why we separated these definitions is because the inner product on the vector bundle $T^k \mathbb{M}$ of $\binom{k}{0}$ -tensors is going to be very common in this work.

1.6.1 Raising and Lowering Indices

Given a metric g on a differentiable manifold $\mathbb{M}$, define a map called **flat** from $T\mathbb{M}$ to $T^*\mathbb{M}$ by sending a vector field $X = X^k \partial_k \in \mathfrak{X}(\mathbb{M})$ to the 1-form $X^\flat$

$$X^\flat \doteq g_{ik} X^i dx^k = X_k dx^k.$$

We say that $X^\flat$ is obtained from X by **lowering an index**. The action of the 1-form $X^\flat$ on vector fields is determined by the rule

$$X^\flat(Y) = \langle X, Y \rangle.$$

Similarly, we may define an operator called **sharp** from $T^*\mathbb{M}$ to $T\mathbb{M}$, which sends 1-forms to vector fields. Given a 1-form $\omega = \omega_k dx^k \in \Gamma(T^*\mathbb{M})$, we define

$$\omega^\sharp \doteq g^{ik} \omega_i \partial_k = \omega^k \partial_k,$$

where, by definition, g^{ik} are the components of the inverse matrix $G^{-1} = (g_{ik})^{-1}$. We say that $\omega^\sharp$ is obtained from ω by **raising an index**. These operations can be applied to any tensors of any rank, in any index position to transform tensors from covariant to contravariant or vice versa. For example, if $F \in \Gamma(T_1^3 \mathbb{M})$ is given in local coordinates $(U, (x^i))$ as

$$F = F_{ij}{}^k{}_l dx^i \otimes dx^j \otimes \partial_k \otimes dx^l \quad (1.8)$$

then, we can apply the sharp operator to raise the second index, to get

$$\begin{aligned} F^\sharp &= g^{jp} F_{ip}{}^k{}_l dx^i \otimes \partial_j \otimes \partial_k \otimes dx^l \\ &= F_i{}^{jk}{}_l dx^i \otimes \partial_j \otimes \partial_k \otimes dx^l. \end{aligned} \quad (1.9)$$

Notice that, in the case of the $\binom{3}{1}$ -tensor field F , as we can raise three indices the notation $F^\sharp$ could be ambiguous, at first. When there is possibility of confusion, we just describe in words each time it arises. In the case of the tensor F , we wrote that by $F^\sharp$ we meant the tensor F with its second index raised. Notice also that the tensor F is not in accordance with convention (1.16), which produces no problem. A very important example of raising an index is applied to differential of functions. Let $f \in C^\infty(\mathbb{M})$. Then, we know the expression of its differential in coordinates, $df = \partial_k f dx^k$ (which is exactly the same for $\mathbb{R}^N$). Then, applying the sharp to the 1-form df , we get

$$df^\sharp = g^{ik} \partial_i f \partial_k$$

This agrees with the definition of gradient:

$$\langle \text{grad} f(p), X \rangle \doteq df(p)X, \quad \text{for all } X \in T_p \mathbb{M},$$

so that, $\text{grad} f = g^{ik} \partial_i f \partial_k$.

Trace of Tensor Fields

Another application of raising and lowering indices is to take trace of tensors. This is possible as long as they have an upper and one lower index as we did in section (1.3). For example, we may take the trace of the $\binom{3}{1}$ -tensor field F given by (1.8) in its third and fourth indices, obtaining

$$\text{tr} F = F_{ij}{}^k{}_k dx^i \otimes dx^j \in \Gamma(T_1^3 \mathbb{M}). \quad (1.10)$$

Notice also that we could take the trace of F in its second and third arguments. We could suppress this problem by writing

$$\text{tr}_{(3,4)} F = F_{ij}{}^k{}_k dx^i \otimes dx^j,$$

so it is known in which indices we are taking trace. But the form in (1.10) is preferable to avoid troublesome notation. Since we explain in words in which indices we are taking trace, it also avoids ambiguity. Another example is as follows. Consider the $\binom{2}{2}$ -tensor field $F^\sharp$ defined in local coordinates by (1.9) and take the trace in its first and second indices, obtaining

$$\text{tr}(F^\sharp) = F_i^{ik} \partial_k \otimes dx^i = g^{ij} F_{ij}^k \partial_k \otimes dx^i.$$

This illustrates a situation in which we raise an index before taking trace. This is possible for tensors whose rank is bigger than or equal to 2. There is a situation of particular interest for taking traces, namely $\binom{2}{0}$ -tensor fields.

The trace of 2-covariant tensors. Let $h = h_{ij} dx^i \otimes dx^j$ be any $\binom{2}{0}$ -tensor field, and raise its first index obtaining the $\binom{1}{1}$ -tensor field

$$h^\sharp = g^{ik} h_{kj} \partial_i \otimes dx^j = h^i_j \partial_i \otimes dx^j. \quad (1.11)$$

We define the trace of the tensor h , in these coordinates, by

$$\text{tr} h \doteq h^j_j = g^{ij} h_{ij}.$$

Notice that if we had raised the second index, then we would have obtained $h^\sharp = g^{jk} h_{ik} dx^i \otimes \partial_j$ (this $h^\sharp$ is not the same as in equation (1.11)!) which would give $\text{tr} h^\sharp = h_i^i = g^{ik} h_{ik}$. Moreover, although the tensors $h^\sharp$ defined above with the first and second index raised are $\binom{1}{1}$ -tensor fields, they have not the same nature, but still their traces are the same, because of the symmetry of g^{-1} .

1.6.2 Length and Riemannian Volume Form

Definition 1.21. Given a Riemannian metric g on a differentiable manifold $\mathbb{M}$, we define the **length** of a piecewise C^1 curve

$$\gamma : [a, b] \rightarrow \mathbb{M}, \text{ by } l(\gamma) = \int_a^b |\gamma'(t)|_g dt$$

where $|u|_g = \sqrt{\langle u, u \rangle}$ is the norm induced by g on the tangent space.

When the manifold is connected, the Riemannian metric g induces a distance function as follows.

Definition 1.22. The **distance** from two arbitrary points $p, q \in \mathbb{M}$ is defined by

$$d(p, q) = \inf\{l(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve in } \mathbb{M} \text{ starting at } p \text{ and ending at } q\}$$

A metrics g also allow us to construct a unique structure, called the **Riemannian volume form**, which we now describe.

Definition 1.23. We shall say that a differentiable manifold $\mathbb{M}$ is **orientable** if it is possible to define a C^∞ N -form Ω on $\mathbb{M}$ which is not zero at any point, in which case $\mathbb{M}$ is said to be oriented by the choice of Ω .

We say that a N -form Ω as in definition (1.23) is an **orientation form** for $\mathbb{M}$. A basis $(f_1, \dots, f_N)$ for $T_p \mathbb{M}$ is said to be (positively) oriented if given a choice of orientation Ω , we have

$$\Omega(f_1, \dots, f_N) > 0.$$

A local frame $(F_1, \dots, F_N)$ for $T\mathbb{M}$ defined on an open set $U \subset \mathbb{M}$ (i.e., $F_i : U \rightarrow T\mathbb{M}$, $i = 1, \dots, N$ and $(F_1|_p, \dots, F_N|_p)$ is a basis for $T_p\mathbb{M}$) is said to be a (local) oriented frame if $\Omega(F_1|_p, \dots, F_N|_p) > 0$, for all $p \in U$.

Proposition 1.24. Suppose $(\mathbb{M}, g)$ is an oriented N -dimensional Riemannian manifold with or without boundary, and $N \geq 1$. Then, there is a unique differentiable orientation form $dV_g \in \Omega^N(\mathbb{M})$, called the Riemannian volume form, satisfying

$$dV_g(E_1, \dots, E_N) = 1$$

for every local oriented orthonormal frame (E_i) for $\mathbb{M}$. In particular, in local coordinates $(U, (x^i))$, the Riemannian volume form has the coordinate expression

$$dV_g = \sqrt{\det G} \, dx^1 \wedge \dots \wedge dx^N. \quad (1.12)$$

Proof. See [Lee03]. ♣

In particular, if the manifold $\mathbb{M}$ is compact, we define its volume by

$$V(g) = \int_{\mathbb{M}} dV_g.$$

This N -form also allows us to compute integrals of functions over manifolds. When the manifold is compact, we define the integral of a function $f \in C^0(\mathbb{M})$ by $\int_{\mathbb{M}} f dV_g$.

A manifold together with a Riemannian metric is called a **Riemannian manifold** and this pair is denoted by $(\mathbb{M}, g)$. It is also well known that every differentiable manifold has a Riemannian metric. Actually, the space of Riemannian metrics over a manifold is infinite dimensional.

Definition 1.25. A differentiable one parameter family of metrics is a curve

$$g : I \subset \mathbb{R} \rightarrow \Gamma(S_2^+ T^* \mathbb{M}), \quad t \in I \mapsto g(t)$$

defined in some nondegenerate interval I taking values in the space of Riemannian metrics, which is also differentiable (C^∞) with respect to t .

For a one parameter family of metrics $g(t)$, given a system of coordinates $(U, (x^i))$, the component functions of the metric tensor are characterized by

$$g_{ij} : U \times I \rightarrow \mathbb{R}, \quad (p, t) \mapsto g_{ij}(p, t)$$

which is differentiable on the product manifold $U \times I$. Sometimes we will write $g_{ij}[p, t]$ instead of $g_{ij}(p, t)$. The derivative of the metric with respect to t will give rise to a symmetric $\binom{2}{0}$ -tensor field which will be denoted by

$$\partial_t g = \frac{\partial}{\partial t} g = \frac{\partial g_{ij}}{\partial t} dx^i \otimes dx^j,$$

and which, in general, fails to be positive definite.

1.7 The Covariant Derivative

Definition 1.26. Let $\mathbb{M}$ be a differentiable manifold and let $\pi : E \rightarrow \mathbb{M}$ be a (real) vector bundle. A connection on E is a map

$$\nabla : \Gamma(T\mathbb{M}) \times \Gamma(E) \rightarrow \Gamma(E), \quad (X, Y) \mapsto \nabla(X, Y) = \nabla_X Y$$

satisfying the following properties:

- a. $\nabla_X Y$ is linear in X over $C^\infty(\mathbb{M})$, i.e., $\nabla_{fX+gZ} Y = f\nabla_X Y + g\nabla_Z Y$,
- b. $\nabla_X Y$ is linear in Y over $\mathbb{R}$, i.e., $\nabla_X(aY + bY^\bullet) = a\nabla_X Y + b\nabla_X Y^\bullet$,
- c. ∇ satisfies a product rule over C^∞ in its second variable, i.e., $\nabla_X fY = f\nabla_X Y + X(f)Y$,

where $X, Z \in \Gamma(T\mathbb{M})$ and $Y, Y^\bullet \in \Gamma(E)$ are sections, $f, g \in C^\infty(\mathbb{M})$ are differentiable functions and $a, b \in \mathbb{R}$.

We say that $\nabla_X Y$ is the covariant derivative of Y in the direction of (the vector field) X . In particular, if $\Gamma(E) = \Gamma(T\mathbb{M})$, that is, if the vector bundle E is the tangent bundle $T\mathbb{M}$, then the connection ∇ is an affine (or simply a linear) connection. We are going to denote by ∇ all kind of derivatives on arbitrary vector bundles although, strictly speaking, they are not the same.

Lemma 1.27. Let $\mathbb{M}$ be a differentiable manifold equipped with an affine connection ∇ . Then, there exists a unique connection in each tensor bundle $T_l^k(\mathbb{M})$, also denoted by ∇ , and satisfying the following properties:

- a. On $T\mathbb{M}$, ∇ agrees with the given connection.
- b. On $C^\infty(\mathbb{M})$, ∇ is given by the usual differentiation of functions, that is, $\nabla_X f = X(f)$, $f \in C^\infty(\mathbb{M})$.
- c. $\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$ for any tensors F and G
- d. ∇_X commutes with all traces, that is, $\nabla_X \text{tr} Y = \text{tr} \nabla_X Y$, for all traces (over any indices) of the tensor Y .

This connection satisfies the following additional property:

- i. For any $\binom{k}{l}$ -tensor field $F \in \Gamma(T_l^k \mathbb{M})$, vector fields Y_i and 1-forms ω^j ,

$$\begin{aligned} (\nabla_X F)(Y_1, \dots, Y_k, \omega^1, \dots, \omega^l) &= X(F(Y_1, \dots, Y_k, \omega^1, \dots, \omega^l)) \\ &\quad - \sum_{i=1}^k F(Y_1, \dots, \nabla_X Y_i, \dots, Y_k, \omega^1, \dots, \omega^l) \\ &\quad - \sum_{j=1}^l F(Y_1, \dots, Y_k, \omega^1, \dots, \nabla_X \omega^j, \dots, \omega^l). \end{aligned}$$

There is a way to describe a connection locally by giving N^3 functions called the Christoffel symbols whose definition is made relative to a (given) coordinate frame (and, therefore, to all frames). Let $(U, (x^i))$ be a system of coordinates on a manifold $\mathbb{M}$ equipped of a linear connection ∇ . Denote by $(\partial_1, \dots, \partial_N)$ the coordinate vector fields from (x^i) . Then $\nabla_{\partial_i} \partial_j$ is a linear combination of (∂_k) at each point which gives raise to N^3 functions called the **Christoffel symbols** of the connection ∇ with respect to the given coordinates and are defined by

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k.$$

The Christoffel symbols completely describe a connection on U (see [Tu17, Chapter 3, Section 13.3]).

Lemma 1.28. *Let $\mathbb{M}$ be a differentiable manifold equipped with an affine connection ∇ and let $F \in \Gamma(T_l^k \mathbb{M})$. The map $\nabla F : \mathfrak{X}(\mathbb{M}) \times \dots \times \mathfrak{X}(\mathbb{M}) \times \Gamma(T^* \mathbb{M}) \times \dots \times \Gamma(T^* \mathbb{M})$ defined by*

$$\nabla F(X, Y_1, \dots, Y_k, \omega^1, \dots, \omega^l) = (\nabla_X F)(Y_1, \dots, Y_k, \omega^1, \dots, \omega^l)$$

*defines a $\binom{k+1}{l}$ -tensor field over $\mathbb{M}$, called the **total covariant derivative** of F .*

Proof.

See [Lee06, Chapter 4]. ♣

The covariant derivative may then be applied to any tensor (of any rank) to produce another tensor, whose rank is increased by one. Let us give now some examples to fix the ideas and the notation we're using on this text.

Example 1.29. [Covariant derivative of 1-forms]

Let $(U, (x^i))$ be a system of coordinates and $\omega \in \Gamma(T^* \mathbb{M})$ be any 1-form, which is written in those coordinates as $\omega = \omega_j dx^j$. Then, its total covariant derivative $\nabla \omega$ is a $\binom{2}{0}$ -tensor field which is expressed in components as

$$\nabla \omega = (\partial_i \omega_j - \Gamma_{ij}^k \omega_k) dx^i \otimes dx^j,$$

where $\{\Gamma_{ij}^k\}$ are the Christoffel of the given connection on $T\mathbb{M}$ in these coordinates. To verify this, we compute $\nabla \omega$ at coordinate vector fields to obtain

$$\nabla \omega(\partial_i, \partial_j) = \partial_i(\omega(\partial_j)) - \omega(\Gamma_{ij}^k \partial_k) = \partial_i \omega_j - \Gamma_{ij}^k \omega_k.$$

When writing the components of the total covariant derivative of a 1-form, we will use the shorthand notation $\nabla_{\partial_i} \omega_j = \nabla_i \omega_j$ so that

$$\nabla \omega = (\nabla_i \omega_j) dx^i \otimes dx^j, \text{ where } \nabla_i \omega_j = \partial_i \omega_j - \Gamma_{ij}^k \omega_k.$$

Example 1.30. [Covariant derivative of vector fields]

Let (U, x^i) be a system of coordinates and $Y \in \mathfrak{X}(\mathbb{M}) = \Gamma(T\mathbb{M})$ be any vector field which is written in the coordinates (x^i) as $Y = Y^j \partial_j$. Then, the total covariant derivative of Y is a $\binom{1}{1}$ -tensor field which may be expressed locally as

$$\nabla Y = (\partial_i Y^j + \Gamma_{ik}^j Y^k) dx^i \otimes \partial_j = (\nabla_i Y^j) dx^i \otimes \partial_j,$$

where $\{\Gamma_{ij}^k\}$ are the Christoffel symbols of the given connection ∇ on $T\mathbb{M}$ in the coordinates (x^i) . Indeed, using the formula for the covariant derivative of a $\binom{0}{1}$ -tensor field, we have

$$\nabla Y(\partial_i, dx^j) = \partial_i(Y(dx^j)) - Y(\nabla_{\partial_i} dx^j) = \partial_i Y^j - Y(-\Gamma_{ik}^j dx^k) = \partial_i Y^j + \Gamma_{ik}^j Y^k. \clubsuit$$

When we write the components of a total covariant derivative of a vector field, we will use ∇ with a lower index to separate indices resulting from differentiation from the preceding indices. Thus, in our previous example, the components of ∇Y are written $\nabla_{\partial_i} Y^j = \nabla_i Y^j$, so that

$$\nabla_i Y^j = \partial_i Y^j + \Gamma_{ik}^j Y^k.$$

Example 1.31. [Covariant derivative of a k -covariant, l -contravariant tensor fields]

More generally, we consider an arbitrary mixed tensor $F \in \Gamma(T_l^k \mathbb{M})$ given in local coordinates (U, x^i) as

$$F = F_{j_1 \dots j_k}^{i_1 \dots i_l} dx^{j_1} \otimes \dots \otimes dx^{j_k} \otimes \partial_{i_1} \otimes \dots \otimes \partial_{i_l}. \quad (1.13)$$

We have, $\nabla F \in \Gamma(T_l^{k+1} \mathbb{M})$ whose components are given by

$$\nabla_p F_{j_1 \dots j_k}^{i_1 \dots i_l} = \partial_p F_{j_1 \dots j_k}^{i_1 \dots i_l} - \sum_{m=1}^k F_{j_1 \dots q \dots j_k}^{i_1 \dots i_l} \Gamma_{pj_m}^q + \sum_{m=1}^l F_{j_1 \dots j_k}^{i_1 \dots m \dots i_l} \Gamma_{pm}^{i_m},$$

so that,

$$\nabla F = (\nabla_p F_{j_1 \dots j_k}^{i_1 \dots i_l}) dx^p \otimes dx^{j_1} \otimes \dots \otimes dx^{j_k} \otimes \partial_{i_1} \otimes \dots \otimes \partial_{i_l}. \clubsuit$$

1.7.1 Riemannian Connection

As is known from Riemannian geometry, there are plenty affine connections in the tangent bundle, but there is only one that has more geometric significance when $\mathbb{M}$ is endowed with a Riemannian metric. We now describe this connection.

Definition 1.32. Let $\mathbb{M}$ be a differentiable manifold equipped with a affine connection ∇ and define a map

$$\tau : \mathfrak{X}(\mathbb{M}) \times \mathfrak{X} \rightarrow \mathfrak{X}(\mathbb{M}), \quad \text{by} \quad \tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

We call τ the **torsion tensor** of ∇ and we say that the affine connection ∇ is symmetric if its torsion vanishes identically. Suppose, moreover, that the manifold $\mathbb{M}$ is endowed with a Riemannian metric $g = \langle \cdot, \cdot \rangle$. Then, we say that the affine connection ∇ is **compatible with** the metric g if and only if

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle, \quad \text{for all } X, Y, Z \in \mathfrak{X}(\mathbb{M}).$$

The torsion τ of an affine connection ∇ is a $\binom{2}{1}$ -tensor field over $\mathbb{M}$, whereas a connection ∇ is not $\binom{2}{1}$ -tensor field, since $\nabla_X Y$ is not linear over $C^\infty(\mathbb{M})$ in Y , but the difference of any two affine connections on $\mathbb{M}$, say ∇^0 and ∇^1 , does define a tensor, that is, if $A : \mathfrak{X}(\mathbb{M}) \times \mathfrak{X} \rightarrow \mathfrak{X}(\mathbb{M})$, is defined by $A(X, Y) = \nabla_X^1 Y - \nabla_X^0 Y$ then A is a $\binom{2}{1}$ -tensor field over $\mathbb{M}$.

Now we are able to state the fundamental lemma of Riemannian Geometry.

Lemma 1.33. Let $(\mathbb{M}, g)$ be a differentiable Riemannian manifold. Then, there exists a unique affine connection ∇ over $\mathbb{M}$ that is symmetric and compatible with the metric g .

Proof.

See [FdC13, Chapter 2]. ♣

This unique affine connection is known as the **Riemannian connection** or **Levi-Civita connection** of the metric g . Its Christoffel symbols are given in local coordinates as

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \quad (1.14)$$

From now on, $\mathbb{M}$ will denote a Riemannian manifold and its metric will be clear from context. When talking about its connection, we're always referring to its Riemannian connection. The construction of geodesics, and its derivative along vector fields may be done using only affine (linear) connections, but we're exclusively interested in the Riemannian case.

1.7.2 Notation for higher derivatives

When considering higher order derivatives a notational issue arises. Let us clarify here the notations we are following when considering these operations.

Example 1.34. [*Derivative of 2-covariant tensor fields*]

Let $h \in \Gamma(T^2\mathbb{M})$ be any $\binom{2}{0}$ -tensor field. The second covariant derivative of h is a $\binom{4}{0}$ -tensor field $\nabla\nabla h = \nabla^2 h$ which, in local coordinates $(U, (x^i))$, has the form

$$\nabla\nabla h = \nabla_i \nabla_j h_{kl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

where

$$\nabla_i \nabla_j h_{kl} = (\nabla_{\partial_i} \nabla_{\partial_j} h)(\partial_k, \partial_l) = (\nabla\nabla h)(\partial_i, \partial_j, \partial_k, \partial_l). \clubsuit$$

Yet in the previous example, using the same notation, we have

$$\nabla^2 : \Gamma(T^2\mathbb{M}) \rightarrow \Gamma(T^4\mathbb{M}),$$

and if $X, Y \in \mathfrak{X}(\mathbb{M})$ then $\nabla_X \nabla_Y h$ is the unique $\binom{2}{0}$ -tensor field such that

$$(\nabla_X \nabla_Y h)(Z, W) = \nabla^2 h(X, Y, Z, W), \text{ for all } Z, W \in \mathfrak{X}(\mathbb{M}).$$

Note that,

$$(\nabla^2 h)(X, Y) \doteq \nabla_X \nabla_Y h \neq \nabla_X (\nabla_Y h).$$

More generally, for any tensor field $A \in \Gamma(T_q^p \mathbb{M})$ and $X_1, \dots, X_n \in \mathfrak{X}(\mathbb{M})$, we adopt the convention (not universally followed) that

$$\nabla_{X_1} \nabla_{X_2} \dots \nabla_{X_n} A$$

is the unique $\binom{p}{q}$ -tensor field such that

$$\nabla_{X_1} \nabla_{X_2} \dots \nabla_{X_n} A(Y_1, \dots, Y_p, \omega^1, \dots, \omega^q) = (\nabla^n A)(X_1, \dots, X_n, Y_1, \dots, Y_p, \omega^1, \dots, \omega^q),$$

for all vector fields $Y_1, \dots, Y_p$, and all covector fields $\omega^1, \dots, \omega^q$. For example, in our convention,

$$\nabla_X(\nabla_Y A) = \nabla_X \nabla_Y A + \nabla_{\nabla_X Y} A = (\nabla^2 A)(X, Y) + (\nabla A)(\nabla_X Y). \quad (1.15)$$

1.7.3 The Lie derivative revisited

Now that we have the notion of a Riemannian connection we may see how the Lie derivative relates with connections. In this section we are going to be more informal when stating facts about the Lie derivative.

Under the same hypotheses of Lemma 1.17 (see item 4), the Lie derivative is defined on vector fields $X, Y \in \mathfrak{X}(\mathbb{M})$ so that

$$\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X.$$

If $\omega \in \Gamma(T^*\mathbb{M})$ is any (differentiable) 1-form and X, Y are vector fields, the Lie derivative of ω is given by

$$(\mathcal{L}_X \omega)Y = X(\omega(Y)) - \omega([X, Y]).$$

For the proof of this fact, see [Lee03, Chapter 12]. More generally, the Lie derivative of a $\binom{k}{l}$ -tensor field F on $\mathbb{M}$ satisfies

$$\begin{aligned} (\mathcal{L}_X F)(Y_1, \dots, Y_k; \omega^1, \dots, \omega^l) &= X(F(Y_1, \dots, Y_k; \omega^1, \dots, \omega^l)) \\ &\quad - \sum_{i=1}^k F(Y_1, \dots, [X, Y_i], \dots, Y_k; \omega^1, \dots, \omega^l) \\ &\quad - \sum_{j=1}^l F(Y_1, \dots, Y_k; \omega^1, \dots, \mathcal{L}_X \omega^j, \dots, \omega^l). \end{aligned}$$

For the proof of this fact with F replaced by a $\binom{k}{0}$ -tensor field, see [Lee03, Chapter 12]. For the general case, see [CKKS04]. In particular, on a Riemannian manifold $(\mathbb{M}, g)$ with connection ∇ , we have

$$(\mathcal{L}_X g)(\partial_i, \partial_j) = \nabla_i X_j + \nabla_j X_i,$$

for any vector field X .

1.7.4 Some Differential Operators

In this section we will give some examples of partial differential operators without giving a formal definition of this object. For our purposes, it is interesting to know some of them, and their expression in

local coordinates. For this section, we give the following references: [JJ08, Chapter 2], [Gil95, Chapter 1] and [CvdB09].

A first order differential operator of particular interest is the divergence. From basic Riemannian geometry, the **divergence** of a vector field $X \in \mathfrak{X}(\mathbb{M})$ is given by

$$\operatorname{div} : \mathfrak{X}(\mathbb{M}) \rightarrow C^\infty(\mathbb{M}), \quad \operatorname{div} X(p) = \operatorname{trace}(\xi \mapsto \nabla_\xi X),$$

where ξ ranges over $T_p\mathbb{M}$. The divergence is linear over $\mathbb{R}$ and satisfies

$$\operatorname{div}(fX) = \langle \operatorname{grad} f, X \rangle + f \operatorname{div}(X), \quad \text{for all } f \in C^\infty(\mathbb{M}), \text{ and } X \in \mathfrak{X}(\mathbb{M}).$$

Let $(U, (x^i))$ be a system of coordinates. Writing $X = X^i \partial_i$ in these coordinates, we have

$$\nabla_j X = \nabla_{\partial_j} X = (\partial_j X^k + X^l \Gamma_{lj}^k) \partial_k, \quad \text{which implies } \operatorname{div}(X) = \nabla_j X^j = \partial_j X^j + X^k \Gamma_{kj}^j.$$

Differentiating the components of the metric with respect to ∂_k and multiplying it by its inverse, we get

$$g^{ji} \partial_k g_{ij} = g^{ji} (\Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il}) = 2\Gamma_{kj}^j.$$

Then,

$$\begin{aligned} \operatorname{div} X &= \partial_j X^j + X^k \frac{1}{2} (g^{ji} \partial_k g_{ij}) \\ &= \partial_j X^j + \frac{X^k}{2} \frac{1}{\det G} ((-1)^{i+j} \det G_{[ij]} \partial_k g_{ij}) \\ &= \partial_j X^j + \frac{X^k}{2} \partial_k (\log \det G) \\ &= \frac{1}{\sqrt{\det G}} \partial_k (X^k \sqrt{\det G}), \end{aligned} \tag{1.16}$$

where $\det G$ is the determinant of the matrix (g_{ij}) in the given coordinates. This gives another way of interpreting the divergence on vector fields (see [CRD84]). We now generalize the notion of divergence to arbitrary $\binom{k}{0}$ -tensor fields, which are the tensors we are going to work with in this text.

Definition 1.35. Let $(\mathbb{M}, g)$ be a Riemannian manifold. The **divergence** is the family of operators

$$\operatorname{div} : \Gamma(T^k \mathbb{M}) \rightarrow \Gamma(T^{k-1} \mathbb{M}), \quad \text{defined by } \operatorname{div} \alpha = \operatorname{tr}(\nabla \alpha)^\sharp,$$

where $\sharp$ is with respect to the first index of $\nabla \alpha$ and the trace is taken with respect to the first and second indices of $(\nabla \alpha)^\sharp$.

This definition may be given with respect to an arbitrary orthonormal frame. Here we compute it in coordinates. Let $\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$ be the coordinate expression of $\alpha \in \Gamma(T^k \mathbb{M})$. Then, apply the covariant derivative to α to get the coordinate expression

$$\nabla \alpha = \nabla_p \alpha_{i_1 \dots i_k} dx^p \otimes dx^{i_1} \otimes \dots \otimes dx^{i_k}.$$

Raising the first index of $\nabla \alpha$ we get $(\nabla \alpha)^\sharp = g^{pq} \nabla_q \alpha_{i_1 \dots i_k} \partial_p \otimes dx^{i_1} \otimes \dots \otimes dx^{i_k}$, whose trace in the first and second indices is

$$\operatorname{tr}(\nabla \alpha)^\sharp = g^{pq} \nabla_q \alpha_{p i_2 \dots i_k} dx^{i_2} \otimes \dots \otimes dx^{i_k}.$$

Thus, the divergence of any $\binom{k}{0}$ -tensor field α is a $\binom{k-1}{0}$ -tensor field whose components in any system of coordinates are

$$\operatorname{div}(\alpha)_{i_1 \dots i_{k-1}} = \operatorname{div}(\alpha)(\partial_{i_1}, \dots, \partial_{i_{k-1}}) = \nabla^p \alpha_{p i_1 \dots i_{k-1}}.$$

In particular, if $h = h_{ij} dx^i \otimes dx^j$ is a $\binom{2}{0}$ -tensor field, then $\operatorname{div}(h) = g^{pq} \nabla_q h_{pk} dx^k$. Yet another example, if $X = X^i \partial_i$ is a vector field, then $X^b = X_j dx^j$, $X_j = g_{ij} X^i$ and

$$\operatorname{div} X^b = g^{pq} \nabla_q X_p = \nabla_p X^p$$

so that the divergence on 1-forms coincide with the corresponding g -dual vector fields, as shown above.

Definition 1.36. The **rough Laplacian** is the family of operators

$$\Delta : \Gamma(T_l^k \mathbb{M}) \rightarrow \Gamma(T_l^k \mathbb{M}), \text{ defined by } \Delta F \doteq g^{pq} \nabla_p \nabla_q F,$$

where $(\mathbb{M}, g)$ is a Riemannian manifold.

The coordinate expression of the laplacian applied to functions which we will frequently use is

$$\Delta f = g^{pq} (\partial_p \partial_q - \Gamma_{pq}^k \partial_k) f.$$

Also, recalling the formula for the gradient of a function f , $\operatorname{grad} f = g^{ij} \partial_i f \partial_j$, and applying the formula obtained at (1.16) to the vector field $\operatorname{grad} f$, we get

$$\Delta f = \operatorname{div}(\operatorname{grad} f) = \frac{1}{\sqrt{\det G}} \partial_i (g^{ij} \sqrt{\det G} \partial_j f)$$

1.8 Geodesics and Normal Coordinates

Before defining geodesics we need to address the issue of differentiating vector field along curves. Since this tool is not central to the development of this work, we will just point out its properties. Let $\gamma : I \rightarrow \mathbb{M}$ be a curve in a manifold $\mathbb{M}$. Then the connection ∇ determines a unique operator $D_t = \frac{D}{dt} : \Gamma(\gamma) \rightarrow \Gamma(\gamma)$, where $\Gamma(\gamma)$ is the set of vector fields along γ . This operator D_t satisfies the following rules. D_t is linear over $\mathbb{R}$ and it satisfies the product rule for functions over $C^\infty(I)$. Furthermore, given $V \in \Gamma(\gamma)$ if V is extendible, that is, V admits an extension $\tilde{V}$ defined on an open set containing $\gamma(I)$ and which satisfies $\tilde{V}(\gamma(t)) = V(t)$ then $D_t V(t) = \nabla_{\gamma'(t)} \tilde{V}$. For any $V \in \Gamma(\gamma)$, we say that $D_t V$ is the covariant derivative of V along γ . For clarity, let us just remind what is the covariant derivative of a vector field along a curve γ . Suppose γ is given in local coordinates $(U, \chi = (x^i))$ as

$$\gamma(t) = \chi^{-1}(\gamma_1(t), \dots, \gamma_N(t))$$

Then,

$$\gamma'(t) = \gamma'_i(t) \partial_i \big|_{\gamma(t)}$$

Let $V : I \rightarrow T\mathbb{M}$ be a differentiable vector field along γ , that is, $V(t) \in T_{\gamma(t)}\mathbb{M}$, for all $t \in I$, and write V in these coordinates as $V = V^j \partial_j$, where $V^j = V^j(t) : I \rightarrow \mathbb{R}$ are differentiable functions. Then, by the properties of D_t , since ∂_j are extendible, we have

$$\begin{aligned} D_t V &= \left(\frac{d}{dt} V^j \right) \partial_j + V^j \nabla_{\gamma'} \partial_j \\ &= \left(\frac{d}{dt} V^k + V^j \gamma'_i (\Gamma_{ij}^k \circ \gamma) \right) \partial_k. \end{aligned}$$

Definition 1.37. Let $\mathbb{M}$ be a Riemannian manifold with its connection ∇ and let $\gamma : I \rightarrow \mathbb{M}$ be a differentiable curve in $\mathbb{M}$. The **acceleration** of γ is the vector field $D_t \gamma'$ along γ . A curve γ is called a **geodesic** with respect to ∇ if its acceleration is zero: $D_t \gamma' \equiv 0$.

Theorem 1.38. [Existence and uniqueness of geodesics]

Let $(\mathbb{M}, g)$ be a Riemannian manifold with a Riemannian connection ∇ . For any $p \in \mathbb{M}$ and any $V \in T_p \mathbb{M}$ and any $t_0 \in \mathbb{R}$, there exists an open interval $I \subset \mathbb{R}$ containing t_0 and a geodesic $\gamma : I \rightarrow \mathbb{M}$ satisfying $\gamma(t_0) = p$ and $\gamma'(t_0) = V$.

Proof.

See [Lee06, Chapter 4]. ♣

Remark 1.39. All these concepts about geodesics can be defined using only linear connections but we will not need this extension here.

Our purpose now is to construct a special system of coordinates, named normal coordinates. To this end, we need the exponential map, which is vital to Riemannian geometry, but which has little importance for our work. So, we briefly pass through these notions and leave [Mor98], [FdC13] as references.

Let $p \in \mathbb{M}$ and $V \in T_p \mathbb{M}$ be given. Then, there exists a unique maximal geodesic $\gamma : I \rightarrow \mathbb{M}$ with $\gamma(0) = p$ and $\gamma'(0) = V$, defined on some open interval I . This maximal geodesic is denoted by γ_V . Now, define a subset $\mathcal{E}$ of $T\mathbb{M}$ the **domain of the exponential map** by

$$\mathcal{E} \doteq \{V \in T\mathbb{M} : \gamma_V \text{ is defined on an interval containing } [0, 1]\} \quad (1.17)$$

and then define the exponential map at $p \in \mathbb{M}$, and $V \in T_p \mathbb{M}$ by

$$\exp_p(V) \doteq \gamma_V(1).$$

The exponential map is differentiable and for any $p \in \mathbb{M}$ there is a star shaped neighborhood V of the origin of $T_p \mathbb{M}$ and a neighborhood U of p in $\mathbb{M}$ such that $\exp_p : V \rightarrow U$ is a diffeomorphism. Such a neighborhood U of $p \in \mathbb{M}$ is called a **normal neighborhood** of p . An orthonormal basis $\{E_1, \dots, E_N\}$ of $T_p \mathbb{M}$ gives rise to an isomorphism $E : \mathbb{R}^N \rightarrow T_p \mathbb{M}$ defined by $E(x^1, \dots, x^N) = x^i E_i$ and if $U \subset \mathbb{M}$ is a normal neighborhood of p in $\mathbb{M}$, then we may compose this isomorphism with the exponential map to produce a system of coordinates

$$\chi = E^{-1} \circ \exp_p^{-1} : U \rightarrow \mathbb{R}^N.$$

Any such coordinates are called **(Riemannian) normal coordinates** centered at $p \in \mathbb{M}$.

Lemma 1.40. [Properties of normal coordinates]

Let $(U, (x^i))$ be any normal coordinates centered at $p \in \mathbb{M}$.

1. The coordinates of p are $(0, \dots, 0)$, that is, $x^i(p) = 0$, for $i = 1, \dots, N$.
2. The components of the metric at p are $g_{ij} = \delta_{ij}$, with respect to the basis $(\partial_k|_p) \subset T_p \mathbb{M}$, generated by the coordinate functions.

3. The Christoffel symbols and the first partial derivatives of g_{ij} vanish at p :

$$(\nabla_{\partial_i} \partial_j)(p) = 0, \text{ and } \partial_k|_p g_{ij} = 0.$$

4. Given any mixed tensor $F \in \Gamma(T_l^k \mathbb{M})$, written in these coordinates as in (1.13), then we have, at p

$$\nabla_p F_{j_1 \dots j_k}^{i_1 \dots i_l} = \partial_p F_{j_1 \dots j_k}^{i_1 \dots i_l} = \frac{\partial}{\partial x^p} F_{j_1 \dots j_k}^{i_1 \dots i_l}.$$

Proof.

See [Lee06, Chapter 5]. ♣

1.9 Isometries and Connections

Definition 1.41. Let $(\mathbb{M}, g)$ and (P, h) be Riemannian manifolds of the same dimension. A diffeomorphism $f : \mathbb{M} \rightarrow P$ is called an **isometry** if

$$\langle u, v \rangle_{g(p)} = \langle df_p u, df_p v \rangle_{h(f(p))}$$

for every $u, v \in T_p \mathbb{M}$, and $p \in \mathbb{M}$ or, equivalently, $f^* h = g$, where f^* denotes the pull back via f .

There is a remarkable property relating Riemannian connections and isometries.

Lemma 1.42. Suppose $f : (\mathbb{M}, g) \rightarrow (P, \tilde{g})$ is a local isometry. Then, f takes the Riemannian connection ∇ of g to the Riemannian connection $\tilde{\nabla}$ of $\tilde{g}$, in the sense that

$$f_* \nabla_Y X = \tilde{\nabla}_{f_* Y} f_* X, \quad Y, X \in \mathfrak{X}(\mathbb{M})$$

Proof.

Take an open set $U \subset \mathbb{M}$, where $f|_U : U \rightarrow \tilde{U}$ is an isometry and apply [Lee06, Chapter 5, Proposition 5.6].

♣

Even more remarkable is a formula in coordinates interlinking these Riemannian connections via the isometry f . Let us first consider a slightly more general context. Let f be a diffeomorphism between differentiable manifolds:

$$f : \mathbb{M} \rightarrow P.$$

It is clear that if h is a Riemannian metric for P then $f^* h$ is a Riemannian metric for $\mathbb{M}$ which turns the diffeomorphism above into an isometry. Analogously, if g is a Riemannian metric for $\mathbb{M}$ then $(f^{-1})^* g$ is a Riemannian metric for P , which turns the above diffeomorphism into an isometry. Let us now see how the connections generated by (Riemannian) metrics relate via isometries.

Suppose $f : \mathbb{M} \rightarrow P$ is an isometry and let $\bar{g}$ be any (Riemannian) metric for $\mathbb{M}$. Let $(U, (x^i))$ and $(V, (y^\alpha))$ be system of coordinates for $\mathbb{M}$ and P respectively and denote their coordinate vector fields by the shorthand notation

$$\frac{\partial}{\partial x^i} = \partial x^i, \quad \text{and} \quad \frac{\partial}{\partial y^\alpha} = \partial y^\alpha.$$

Let us denote the Riemannian connection of $\bar{g}$ by $\bar{\Gamma}$ and denote the Riemannian connection of $(f^{-1})^*\bar{g}$ by Γ . Objects related to the connection ∇ are written without the tilde above. Using the equation from Lemma 1.42, we have

$$\begin{aligned} \bar{\Gamma}_{ij}^k \partial x^k &= \left[(f^{-1})_* \circ f_* \right] \bar{\nabla}_{\partial x^i} \partial x^j \\ &= (f^{-1})_* \nabla_{(f_* \partial x^i)} (f_* \partial x^j) \\ &= (f^{-1})_* \nabla_{f_* \partial x^i} \left(\frac{\partial f^\beta}{\partial x^j} \partial y^\beta \right) \\ &= (f^{-1})_* \left\{ \left(f_* \partial x^i \left(\frac{\partial f^\beta}{\partial x^j} \right) \right) \partial y^\beta + \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \nabla_{\partial y^\alpha} \partial y^\beta \right\} \\ &= (f^{-1})_* \left\{ \frac{\partial^2 f^\beta}{\partial x^j \partial x^i} \partial y^\beta + \Gamma_{\alpha\beta}^\gamma \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \partial y^\gamma \right\} \\ &= \left\{ \frac{\partial^2 f^\beta}{\partial x^j \partial x^i} \frac{\partial (f^{-1})^k}{\partial y^\beta} + \Gamma_{\alpha\beta}^\gamma \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \frac{\partial (f^{-1})^k}{\partial y^\gamma} \right\} \partial x^k. \end{aligned}$$

which yields, for each $k = 1, \dots, N$, and $i, j \in \{1, \dots, N\}$ fixed,

$$\bar{\Gamma}_{ij}^k = \frac{\partial^2 f^\beta}{\partial x^j \partial x^i} \frac{\partial (f^{-1})^k}{\partial y^\beta} + \Gamma_{\alpha\beta}^\gamma \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \frac{\partial (f^{-1})^k}{\partial y^\gamma}.$$

Multiplying now by $\frac{\partial f^\nu}{\partial x^k}$, for each k , summing over k and using the formula $\frac{\partial f^\nu}{\partial x^k} \frac{\partial (f^{-1})^k}{\partial y^\beta} = \delta_\beta^\nu$, we get

$$\bar{\Gamma}_{ij}^k \frac{\partial f^\nu}{\partial x^k} = \frac{\partial^2 f^\nu}{\partial x^j \partial x^i} + \Gamma_{\alpha\beta}^\nu \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}.$$

We finally multiply the equation above by $\bar{g}^{ij}$, for each fixed i and j , then sum over these indexes to get the desired formula:

$$((f^{-1})^* \bar{g})^{\alpha\beta} \Gamma_{\alpha\beta}^\nu = \bar{g}^{ij} \left[- \frac{\partial^2 f^\nu}{\partial x^j \partial x^i} + \bar{\Gamma}_{ij}^k \frac{\partial f^\nu}{\partial x^k} \right].$$

For later reference, we enunciate what we just did.

Lemma 1.43. [Relation between connections under isometry]

Suppose $f : (\mathbb{M}, \bar{g}) \rightarrow (P, g)$ is an isometry between Riemannian manifolds, where $g = (f^{-1})^* \bar{g}$. Let us denote by $\bar{\Gamma}$ the Riemannian connection $\bar{g}$ and denote by Γ the Riemannian connection of g . Then,

$$g^{\alpha\beta} \Gamma_{\alpha\beta}^\nu = \bar{g}^{ij} \left[- \frac{\partial^2 f^\nu}{\partial x^j \partial x^i} + \bar{\Gamma}_{ij}^k \frac{\partial f^\nu}{\partial x^k} \right].$$

1.10 Curvature

This section aims at defining the basic tensors of curvature, which are of utmost importance for this work.

Definition 1.44. The (Riemann) $\binom{3}{1}$ -tensor field of curvature on a Riemannian manifold $\mathbb{M}$ is a map

$$\mathcal{R} : \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \rightarrow \mathfrak{X}(\mathbb{M}), \text{ defined by } \mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z. \quad (1.18)$$

According to the discussion at Section 1.7.2, the $\binom{3}{1}$ -tensor field of curvature is written as

$$\mathcal{R}(X, Y)Z = (\nabla^2 Z)(X, Y) - (\nabla^2 Z)(Y, X),$$

for all vector fields $X, Y, Z \in \mathfrak{X}(\mathbb{M})$. By equation (1.15), one may also write (1.18) in the more familiar way

$$\mathcal{R}(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z.$$

In local coordinates $(U, (x^i))$ the brackets $[\partial_i, \partial_j]$ vanish identically, so that, one has

$$\mathcal{R}(\partial_i, \partial_j)\partial_k = [\partial_i, \partial_j](\partial_k) \equiv (\nabla_i \nabla_j - \nabla_j \nabla_i)\partial_k.$$

Moreover, in any coordinates, the $\binom{3}{1}$ -tensor field may be written as

$$\mathcal{R} = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l, \text{ where } \mathcal{R}(\partial_i, \partial_j)\partial_k = R_{ijk}{}^l \partial_l.$$

It is straightforward to check that its coefficients are given by

$$R_{ijk}{}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^p \Gamma_{pi}^l - \Gamma_{ik}^p \Gamma_{pj}^l, \quad (1.19)$$

so that, in normal coordinates based at $p \in \mathbb{M}$, equation (1.19) becomes

$$R_{ijk}{}^l(p) = (\partial_i|_p) \Gamma_{jk}^l - (\partial_j|_p) \Gamma_{ik}^l.$$

Definition 1.45. The (Riemann) $\binom{4}{0}$ -tensor field of curvature on a Riemannian manifold $\mathbb{M}$ is a map

$$Rm : \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \rightarrow C^\infty(\mathbb{M}) \text{ defined by } Rm(X, Y, Z, W) = \langle \mathcal{R}(X, Y)Z, W \rangle$$

The $\binom{4}{0}$ -tensor field is obtained from the $\binom{3}{1}$ -tensor field of curvature $\mathcal{R}$ by lowering its last index and, in coordinates $(U, (x^i))$, it is written as

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l, \text{ where } R_{ijkl} = g_{lm} R_{ijk}{}^m.$$

Lemma 1.46. [Local isometry invariants]

The Riemann $\binom{3}{1}$ -tensor field of curvature $\mathcal{R}$ and the Riemann $\binom{4}{0}$ -tensor field of curvature Rm are local isometry invariants. More precisely, if $\phi : (\mathbb{M}, g) \rightarrow (\tilde{\mathbb{M}}, \tilde{g})$ is a local isometry, then

$$\phi^* \tilde{Rm} = Rm \text{ and } \tilde{\mathcal{R}}(\phi_* X, \phi_* Y)\phi_* Z = \phi_*(\mathcal{R}(X, Y)Z)$$

Proof.

It follows from Lemma 1.42. ♣

Lemma 1.47. [Symmetries of the curvature tensor]

The $\binom{4}{0}$ -tensor field of curvature satisfies, for any $X, Y, Z, W, V \in \mathfrak{X}(\mathbb{M})$, the following identities

a. Rm is skew-symmetric in its first two and last two entries:

$$Rm(X, Y, Z, W) = -Rm(Y, X, Z, W) = -Rm(X, Y, W, Z).$$

b. Rm is symmetric between the first two and last two entries:

$$Rm(X, Y, Z, W) = Rm(Z, W, Y, X, Y).$$

c. Rm satisfies a cyclic permutation property known as Bianchi's first identity:

$$Rm(X, Y, Z, W) + Rm(Y, Z, X, W) + Rm(Z, X, Y, W) = 0.$$

d. ∇Rm satisfies a cyclic permutation property known as Bianchi's second identity:

$$\nabla Rm(X, Y, Z, W, V) + \nabla Rm(X, Y, W, V, Z) + \nabla Rm(X, Y, V, Z, W) = 0.$$

Proof. See [Pet06]. ♣

The third identity is an immediate consequence of the well known Bianchi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

The last identity (fourth identity) is also known as the differential Bianchi identity. Computing it at an arbitrary point $p \in \mathbb{M}$ in normal coordinates based at $p \in \mathbb{M}$, gives the result.

The two first properties of symmetry from the $\binom{4}{0}$ -tensor field Rm allows us to regard it as a section of the bundle

$$\Lambda^2 T^* \mathbb{M} \otimes_S \Lambda^2 T^* \mathbb{M},$$

of symmetric bilinear forms. This affords us the following. If we denote by $T \doteq \text{tr}_{1,4} \mathcal{R}$ the trace of the $\binom{3}{1}$ -tensor field of curvature in its first and last indices then, in any system of coordinates, if $\mathcal{R} = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l$, we have

$$T = R_{ijk}{}^i dx^j \otimes dx^k = g^{im} R_{ikm} dx^j \otimes dx^k.$$

Then, using (1.47), we have

$$T_{jk} = g^{im} R_{ikm} = g^{im} R_{kmi} = g^{mi} R_{mkji} = T_{kj},$$

therefore, the tensor T is symmetric. It is so important that it deserves a name.

1.10.1 Ricci and Scalar Curvatures

Definition 1.48. The **Ricci curvature** is the symmetric $\binom{2}{0}$ -tensor field defined as the trace of the $\binom{3}{1}$ -tensor field of curvature in its first and last indices and it is denoted by Rc . Its components are denoted by R_{ij} , so that, in coordinates $(U, (x^i))$, one has

$$Rc = R_{ij}dx^i \otimes dx^j, \text{ where } R_{ij} \doteq R_{kij}{}^k = g^{km}R_{kijm}.$$

From (1.19), one readily obtains

$$\begin{aligned} R_{ij} = R_{kij}{}^k &= \partial_k \Gamma_{ij}^k - \partial_i \Gamma_{kj}^k + \Gamma_{ij}^p \Gamma_{pk}^k - \Gamma_{kj}^p \Gamma_{pi}^k \\ &= \partial_k \left(\frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \right) - \partial_i \left(\frac{1}{2} g^{kl} (\partial_k g_{jl} + \partial_j g_{kl} - \partial_l g_{jk}) \right) + \Gamma_{ij}^p \Gamma_{pk}^k - \Gamma_{kj}^p \Gamma_{pi}^k \\ &= \frac{1}{2} g^{kl} \left(\partial_k \partial_j g_{il} - \partial_k \partial_l g_{ij} - \partial_i \partial_j g_{kl} + \partial_i \partial_l g_{jk} \right) \\ &\quad + \frac{1}{2} (\partial_k g^{kl}) (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) - \frac{1}{2} (\partial_i g^{kl}) (\partial_k g_{jl} + \partial_j g_{kl} - \partial_l g_{jk}) \\ &\quad + \Gamma_{ij}^p \Gamma_{pk}^k - \Gamma_{kj}^p \Gamma_{pi}^k. \end{aligned} \tag{1.20}$$

Definition 1.49. The **scalar curvature** R is defined as the trace of the Ricci tensor

$$R = g^{ij} R_{ij}.$$

Lemma 1.50. [Contracted Bianchi identity]

The covariant derivatives of the Ricci tensor and scalar curvature satisfy the following identity

$$\operatorname{div} Rc = \frac{1}{2} \nabla R, \text{ in components, this is: } \nabla^j R_{mj} = \frac{1}{2} \nabla_m R$$

where div is the divergence operator defined in 1.35.

Proof.

Consider the Differential Bianchi identity in coordinates:

$$\nabla_m R_{ijkl} + \nabla_k R_{ijlm} + \nabla_l R_{ijmk} = 0. \tag{1.21}$$

Then, multiply equation (1.21) by the function g^{il} and sum over i and l . Multiply the equation obtained by g^{jk} and sum over j and k , obtaining

$$\begin{aligned} 0 &= g^{jk} \nabla_m R_{jk} - g^{jk} \nabla_k R_{jm} - g^{il} \nabla_l R_{im} \\ &= \nabla_m (g^{jk} R_{jk}) - 2 \nabla^j R_{jm}, \end{aligned}$$

with m being the only fixed index. In particular, $2 \operatorname{div} Rc = 2(g^{jk} \nabla_j R_{km}) dx^m = (\partial_m R) dx^m$. ♣

1.10.2 Sectional curvature

Proposition 1.51. Let $\mathbb{M}$ be a Riemannian manifold. Let $p \in \mathbb{M}$, let $\sigma \subset T_p\mathbb{M}$ be any two dimensional vector subspace of the tangent space and let $x, y \in \sigma$ be two linearly independent vectors. Then,

$$K(x, y) = \frac{Rm(x, y, x, y)}{|x|^2|y|^2 - \langle x, y \rangle^2}, \quad (1.22)$$

does not depend on the choice of vectors $x, y \in \sigma$.

Proof. See [FdC13, Chapter 4]. ♣

Definition 1.52. Given a point $p \in \mathbb{M}$, and a two dimensional subspace $\sigma \subset T_p\mathbb{M}$, the real number $K(x, y) = K(\sigma)$, where $\{x, y\} \subset \sigma$ is any basis of σ , is called the sectional curvature of σ at p .

On two dimensional manifolds, Proposition 1.51 is known as "**Gauss's Theorema Egregium**" (see [Lee06, pp.143]). Furthermore, in two dimensional spaces, the sectional curvature $K = K(\sigma)$ is the Gaussian curvature. In fact, in two dimensional manifolds, we have stronger claims, as the next result shows.

Proposition 1.53. The Gaussian curvature of a 2-dimensional Riemannian manifold is related to the $\binom{4}{0}$ curvature tensor, Ricci tensor, and scalar curvature by the formulas:

$$Rm(X, Y, Z, W) = K(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle)$$

$$Rc(X, Y) = K \langle X, Y \rangle$$

$$R = 2K$$

Additionally, one may conclude (again) that K is independent of frames, and completely determines the curvature tensor.

Proof. See [Lee06]. ♣

For future reference, we point out that Proposition (1.53) reads, in local coordinates $(U, (x^i))$,

$$R_{ijkl} = K(g_{il}g_{jk} - g_{ik}g_{jl}) \quad \text{and} \quad R_{ij} = \frac{R}{2}g_{ij} \quad (1.23)$$

1.11 Some results in comparison geometry

Definition 1.54. Let $(\mathbb{M}, g)$ be a Riemannian manifold. The **injectivity radius** $\text{inj}(p)$ at a point $p \in \mathbb{M}$ is defined by

$$\text{inj}(p) \doteq \sup\{r > 0, \text{ such that, } \exp_p : B_r(p) \subset T_p\mathbb{M} \rightarrow \mathbb{M} \text{ is injective}\}.$$

The injectivity radius of a Riemannian manifold $\mathbb{M}$ with metric g is defined by

$$\text{inj}(\mathbb{M}, g) \doteq \inf\{\text{inj}(p) : p \in \mathbb{M}\}.$$

Figure 1.1: The injectivity radius: $\text{inj}(A)$ is small, $\text{inj}(B)$ is large.

In what follows, $\mathbb{M}$ will be a differentiable manifold with fixed Riemannian metric g and we will denote the sectional curvature of $\mathbb{M}$ by $K_{\mathbb{M}}$.

Definition 1.55. We let $\mathbb{M}_{\delta}$ denote the **space form of constant sectional curvature** δ , namely,

- i. the N -sphere of constant sectional curvature δ , when $\delta > 0$.
- ii. $\mathbb{R}^N$, when $\delta = 0$.
- iii. the hyperbolic space of constant sectional curvature δ , when $\delta < 0$.

Lemma 1.56. [Klingenberg's lemma]

If $K \geq K_{\mathbb{M}} \geq H > 0$, then

$$\text{inj}(\mathbb{M}) \geq \min\left\{\frac{\pi}{\sqrt{K}}, \text{half the length of the shortest smooth closed geodesic in } \mathbb{M}\right\}$$

Proof.

See [CE75, Chapter 5]. ♣

Let $\gamma : [0, \infty) \rightarrow \mathbb{M}$ be a geodesic and suppose that γ is parametrized by arc-length and that $\gamma(0) = p$. For sufficiently small $t > 0$ we know that $\text{dist}(\gamma(t), \gamma(0)) = t$, since $\exp_p$ is injective on a sufficiently small ball in $T_p\mathbb{M}$, where the distance function was defined in 1.22. On the other hand if $\text{dist}(\gamma(t_0), \gamma(0)) < t_0$, then for any $\alpha > 0$,

$$\text{dist}(\gamma(t_0 + \alpha), \gamma(0)) \leq \text{dist}(\gamma(t_0 + \alpha), \gamma(t_0)) + \text{dist}(\gamma(t_0), \gamma(0)) < t_0 + \alpha.$$

It follows that the set of t for which $\text{dist}(\gamma(t), \gamma(0)) = t$ is either $[0, \infty)$ or $[0, t_0)$ for some $t_0 > 0$. In the latter case, the $\gamma(t_0)$ is called the **cut point** of γ with respect to p . The union of all cut points is called the **cut locus of p** and is denoted by $C(p)$.

Definition 1.57. Let $\gamma : [0, a] \rightarrow \mathbb{M}$ be a geodesic and put $\gamma(0) = p$. The point $q = \gamma(t_0)$, $t_0 \in (0, a]$, is conjugate to p along γ if $v_0 = t_0 \gamma'(0) \in T_p \mathbb{M}$ is a critical point of $\exp_p$.

Theorem 1.58. [Klingenberg]

Let $\mathbb{M}$ be an even-dimensional, orientable manifold with $K \geq K_{\mathbb{M}} > 0$. If $p, q \in \mathbb{M}$ are such that $q \in C(p)$ and $\text{dist}(p, q) = \text{inj}(\mathbb{M})$ then q is conjugate to p . In this case, we have

$$\text{inj}(\mathbb{M}) \geq \frac{\pi}{\sqrt{K}}.$$

Proof.

[CE75, Chapter 5].

Notation. Given $x \in \mathbb{M}$, we let $V(x; r)$ denote the volume of $B(x; r)$, which is the ball centered at x , with radius r measured with respect to the Riemannian metric g of $\mathbb{M}$, that is

$$V(x; r) = \int_{B(x; r)} dV_g.$$

We denote by $V_\delta(r)$ the volume of the ball with radius $r > 0$ in the space form $\mathbb{M}_\delta$ and centered at any point.

Theorem 1.59. [Volume comparison theorem]

Assume that the sectional curvatures of $\mathbb{M}$ are all less than or equal to δ . Then, for every $x \in \mathbb{M}$, we have

$$V(x; r) \geq V_\delta(r),$$

for all $r \leq \min\{\text{inj}(x), \frac{\pi}{\sqrt{\delta}}\}$, with equality for some fixed r if and only if $B(x; r)$ is isometric to the disk of radius r in the constant curvature space form $\mathbb{M}_\delta$.

Proof.

See [Cha95, Chapter 3]. ♣

Now we compute the area of the ball of radius $0 < r < \delta$ centered at the north pole of the sphere of radius δ , and use the theorem above to compare the volumes of balls in 2-dimensional manifolds whose (sectional) curvature is less than or equal to δ with the volume of the ball in the sphere.

Example 1.60. [Area comparison theorem for the sphere]

Let $\mathbb{M}$ be any differentiable compact surface with Riemannian metric g for which the scalar curvature is positive, $R(g) > 0$. Then, the sectional curvature $K_{\mathbb{M}}$ is bounded above by $\frac{R_{\max}}{2} \doteq \delta$. According to Theorem 1.59, we have

$$V_g(x; r) \geq V_\delta(r),$$

whenever r is less than or equal to $\min\{\text{inj}(x), \frac{\pi}{\sqrt{\delta}}\}$. Let $\mathbb{S}^2(\delta)$ be the 2-sphere of radius δ . Let's choose a parametrization for this sphere.

Let $\alpha : V \subset \mathbb{S}^2(\delta) \rightarrow U = (-\pi/2, \pi/2) \times (0, 2\pi)$ be a system of coordinates such that its inverse is given by

$$\alpha^{-1}(\theta, \varphi) = (\delta \cos \theta \cos \varphi, \delta \cos \theta \sin \varphi, \delta \sin \theta).$$

It is a straightforward calculation to get

$$\left| \left(\frac{\partial}{\partial \theta} \alpha^{-1} \right) \wedge \left(\frac{\partial}{\partial \varphi} \alpha^{-1} \right) \right| = \delta^2 \cos \theta.$$

Figure 1.2: A cap of radius r centered at the north pole, embedded in the sphere of radius δ

Then, if we restrict the angle θ to range from $(\pi/2 - r, \pi/2)$ we almost cover the ball centered at the north pole of radius r , which we denote by $B_r(N)$. It is a direct computation to get

$$\text{Area}(B_r(N)) = \int_{(\frac{\pi}{2}-r, \frac{\pi}{2}) \times (0, 2\pi)} \delta^2 \cos \theta d\theta \wedge d\varphi = 2\pi\delta^2(1 - \cos r).$$

Hence, theorem 1.59, in this case, reads

$$\int_{B_g(x, r)} d\mu \geq 2\pi\delta^2(1 - \cos r). \clubsuit$$

Chapter 2

Existence and uniqueness

This chapter is almost entirely based on the book [CKKS04], except for an example and some material from [MTTS07], [AH10] and from [TT06]. In this chapter we study some properties of the Ricci flow and various evolution equations of the familiar tensors from Riemannian geometry. This chapter aims at presenting a detailed proof of the existence and uniqueness of the Ricci flow using standard results on systems of parabolic PDEs and assuming existence and uniqueness for the Harmonic Map flow problem, which is also a system of parabolic PDEs. Whenever is convenient we give a reference. Besides, in order to establish the character of the Ricci flow equation we define the symbol of a partial differential operator on vector bundles, so we make a brief discussion about it, which is based on [TT06].

2.1 Properties of The Ricci Flow Equation

We begin by defining the Ricci flow and after that we analyse its behaviour under a special system of coordinates, called harmonic.

Definition 2.1. The Ricci flow equation is the following evolution equation for a Riemannian metric

$$\frac{\partial}{\partial t}g(t) = -2Rc(g(t)). \quad (2.1)$$

A solution to this equation (or a Ricci flow) is a differentiable one-parameter family of differentiable Riemannian metrics $g(t)$ parameterized by t on some non-degenerate interval I on a differentiable manifold $\mathbb{M}$. If I contains an initial point t_0 then $(\mathbb{M}, g(t_0))$ is called an initial condition for the Ricci flow equation. If I is an interval of the type $[t_0, T)$ then we will assume the solution $g(t)$ to be differentiable (C^∞) in time on (t_0, T) and continuous on I .

2.1.1 Diffeomorphism Invariance

The Ricci flow equation is invariant under the full group of diffeomorphisms in the following sense. As a result of Lemma 1.46, the Ricci tensor is also a (local) isometry invariant. In particular, let $\varphi : \bar{\mathbb{M}} \rightarrow (\mathbb{M}, g(t))$ be an arbitrary time-independent diffeomorphism where $g(t)$ is a one-parameter family of metrics satisfying the Ricci flow equation. Define the family of metrics in the domain manifold $\bar{\mathbb{M}}$ by $\bar{g}(t) = \varphi^*(g(t))$. Then, we have

$$\begin{aligned} \frac{\partial}{\partial t} \bar{g}(t) &= \frac{\partial}{\partial t} (\varphi^*(g(t))) \\ &= \varphi^* \left(\frac{\partial}{\partial t} g(t) \right) \\ &= \varphi^* (-2\text{Rc}(g(t))) \\ &= -2\text{Rc}(\varphi^* g(t)) \\ &= -2\text{Rc}(\bar{g}(t)) \end{aligned}$$

where the fourth equality follows from Lemma 1.46. Therefore, the family $\bar{g}(t)$ also satisfies equation (2.1) and we conclude that the Ricci flow equation is invariant under the full group of diffeomorphisms of $\mathbb{M}$. Note that this observation is almost tautological, since the Ricci flow involves only local isometry invariants. Moreover, if φ is a **time-dependent** diffeomorphism then an additional Lie derivative is introduced into the equation (see equation (2.26)).

2.1.2 Ricci Flow in Harmonic Coordinates

Definition 2.2. Let $(\mathbb{M}, g)$ be a Riemannian manifold. A local system of coordinates $(U, (x^i))$ is called **harmonic coordinates** if each coordinate function x^i is harmonic, that is,

$$\Delta x^i = g^{jk} (\partial_j \partial_k - \Gamma_{jk}^l \partial_l) x^i = -g^{jk} \Gamma_{jk}^i = 0.$$

Suppose we have a Ricci flow $g = g(t)$ defined on the interval I , $g : I \rightarrow \Gamma(S_2^+ T^* \mathbb{M})$. For a given instant $t_0 \in I$, we fix the metric $g(t_0)$ and consider a harmonic system of coordinates $(U, (x^i))$ according to the fixed metric, that is, $\Delta_{g(t_0)} x^j = 0$. We cannot expect these coordinates to be harmonic for different times $t \neq t_0$. In general we will omit the parameter t and that should cause no confusion.

Lemma 2.3. [Ricci flow in harmonic coordinates]

Let Δ be the Laplace Beltrami operator acting on the space of functions. In harmonic coordinates with respect to $g(t)$, the Ricci flow assumes the form of a heat equation at instant t in the components of the metric $g(t)$, that is, at instant t the following equation holds

$$\frac{\partial}{\partial t} g_{jk} = \Delta g_{jk} + Q_{jk}(g, g^{-1})$$

where Q_{jk} denotes a sum of terms which contains components of the inverse metric g^{-1} and, at most, first order (spatial) partial derivatives in the components of the metric g .

Remark 2.4. Before going into details, let's present a useful formula. First notice that, in harmonic coordinates with respect to $g(t)$, we have

$$g^{pq} \partial_k (\Gamma_{pq}^n g_{nj}) = \partial_k (g^{pq} \Gamma_{pq}^n g_{nj}) - (\partial_k g^{pq}) (\Gamma_{pq}^n g_{nj}) = -(\partial_k g^{pq}) (\Gamma_{pq}^n g_{nj})$$

and this equation means that the term on the left has at most first order partial derivatives in the components of the metric g , where indexes p, q and n are free. But also, $\partial_k (\Gamma_{pq}^n g_{nj})$ has terms involving second order derivatives. Thus, we compute only the terms involving second order partial derivatives in g and call S what remains:

$$g^{pq} \partial_k (\Gamma_{pq}^n g_{nj}) = [g^{pq} g_{nj}] \frac{1}{2} g^{ni} (\partial_k \partial_p g_{iq} + \partial_k \partial_q g_{ip} - \partial_k \partial_i g_{qp}) + S_{jk},$$

where $S_{jk} = g^{pq} (\frac{1}{2} \partial_k g^{ni}) (\partial_p g_{iq} + \partial_q g_{ip} - \partial_i g_{qp}) + g^{pq} \Gamma_{pq}^n (\partial_k g_{nj})$, which has clearly only lower order terms. Thus,

$$\begin{aligned} g^{pq} \partial_k (\Gamma_{pq}^n g_{nj}) + g^{pq} \partial_j (\Gamma_{pq}^n g_{nk}) &= \frac{1}{2} g^{pq} \{ \partial_k \partial_p g_{jq} + \partial_k \partial_q g_{jp} - \partial_k \partial_j g_{pq} \} + \\ &+ \frac{1}{2} [g^{pq}] (\partial_j \partial_p g_{kq} + \partial_j \partial_q g_{kp} - \partial_j \partial_k g_{qp}) + S_{jk} + S_{kj} \\ &= g^{pq} \{ \partial_k \partial_p g_{jq} \} + g^{pq} \{ \partial_j \partial_p g_{kq} \} - g^{pq} \{ \partial_j \partial_k g_{pq} \} + S_{jk} + S_{kj}. \end{aligned}$$

Proof. [of Lemma 2.3]

By (1.20), we have

$$\begin{aligned} -2R_{jk} &= -2\{R_{pjk}{}^p\} \\ &= -2\{\partial_p \Gamma_{jk}^p - \partial_j \Gamma_{pk}^p + \Gamma_{jk}^q \Gamma_{pq}^p - \Gamma_{pk}^q \Gamma_{jq}^p\} \\ &= -\partial_p \{g^{pq} (\partial_j g_{kq} + \partial_k g_{jq} - \partial_q g_{jk})\} + \partial_j \{g^{pq} (\partial_p g_{kq} + \partial_k g_{pq} - \partial_q g_{kp})\} + \{-2\Gamma_{jk}^q \Gamma_{pq}^p + 2\Gamma_{pk}^q \Gamma_{jq}^p\} \\ &= -g^{pq} \{\partial_p \partial_j g_{kq} + \partial_p \partial_k g_{jq} - \partial_p \partial_q g_{jk}\} + g^{pq} \{\partial_j \partial_p g_{kq} + \partial_j \partial_k g_{pq} - \partial_j \partial_q g_{kp}\} + T_{jk} \\ &= \Delta g_{jk} + g^{pq} \{\partial_j \partial_k g_{pq}\} - g^{pq} \{\partial_p \partial_k g_{jq}\} - g^{pq} \{\partial_j \partial_p g_{kq}\} + T_{jk} \\ &= \Delta g_{jk} - g^{pq} \partial_k (\Gamma_{pq}^n g_{nj}) - g^{pq} \partial_j (\Gamma_{pq}^n g_{nk}) + S_{jk} + S_{kj} + T_{jk} \\ &= \Delta g_{jk} + Q_{jk}(\partial g, g^{-1}) \end{aligned}$$

where $T_{jk} = -(\partial_p g^{pq}) \{\partial_j g_{kq} + \partial_k g_{jq} - \partial_q g_{jk}\} + (\partial_j g^{pq}) (\partial_p g_{kq} + \partial_k g_{pq} - \partial_q g_{kp}) + \{-2\Gamma_{jk}^q \Gamma_{pq}^p + 2\Gamma_{pk}^q \Gamma_{jq}^p\}$ has clearly only lower order terms. ♣

We refer to [DK81] for the proof of the existence of harmonic coordinates and some basic material in these coordinates.

2.1.3 Normalized Ricci Flow

We will frequently employ the normalized Ricci-flow equation, which is obtained from the Ricci-flow equation by rescaling the metric by a function of time and a reparametrization of time. The normalized Ricci-flow equation is the following PDE in the metric:

$$\frac{\partial}{\partial t} g = -2Rc + \frac{2}{N} rg$$

where N is, as usual, the dimension of the manifold $\mathbb{M}$ and $r = \frac{\int R dV}{\int dV}$ is the average scalar curvature. One of the advantages of working with the normalized Ricci flow is that in the direction of this flow, the volume of the manifold is preserved. Suppose that $g(t)$ is a Ricci flow. We would like to find a function ψ of time such that the volume $\tilde{V}$ of $\mathbb{M}$ according to the new metric $\tilde{g} = \psi g$ is constant and equal to $\mathbb{1}$. By (1.12), if $(U, (x^i))$ is a system of coordinates, then the unique Riemannian volume form $d\tilde{V}$ of $\tilde{g}$ satisfies

$$d\tilde{V} = \sqrt{\det \tilde{G}} \, dx^1 \wedge \dots \wedge dx^N = \psi^{\frac{N}{2}} \sqrt{\det G} \, dx^1 \wedge \dots \wedge dx^N = \psi^{\frac{N}{2}} dV,$$

and, as we wish

$$1 = \tilde{V}(t) = \int d\tilde{V} = \psi^{\frac{N}{2}} \int dV = \psi^{\frac{N}{2}} V(t)$$

we define $\psi(t) = V(t)^{-\frac{2}{N}}$. We need to compute quantities related to $\tilde{g}$, like curvature and the volume form, in order to show that $\tilde{g}$ satisfies the normalized Ricci flow equation with respect to some parametrization of time. Define

$$\tilde{t}(t) = \int_0^t \psi(\tau) d\tau$$

and observe $\tilde{t}' = \psi > 0 \Rightarrow \tilde{t}$ is invertible and denote its inverse by t . Then, $t'(\tilde{t}(t))\tilde{t}'(t) = 1$. Objects with the tilde above refers to quantities related to $\tilde{g}$. Then,

$$\begin{aligned} \tilde{\Gamma}_{ij}^k &= \frac{1}{2} \tilde{g}^{lk} \left(\partial_i \tilde{g}_{jl} + \partial_j \tilde{g}_{il} - \partial_l \tilde{g}_{ij} \right) \\ &= \frac{1}{2} \frac{1}{\psi} g^{kl} \left(\psi (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \right) = \Gamma_{ij}^k \end{aligned}$$

Therefore, it is straightforward that $\tilde{R}_{ij} = R_{ij}$ and $\tilde{R} = \frac{1}{\psi} R$, by the formula of the Riemannian $\binom{3}{1}$ -tensor field of curvature and because $R_{ij} = R_{kij}{}^k$, and by the formula $\tilde{R} = \tilde{g}^{ij} \tilde{R}_{ij}$, respectively. Next, we use the evolution equation from Lemma 2.22 (which will be presented further in this document), to obtain

$$\partial_t dV = \frac{1}{2} g^{ij} (\partial_t g_{ij}) dV = -R dV, \text{ which implies } \frac{V'}{V} = -\psi \tilde{r},$$

and then

$$\begin{aligned}
\frac{\partial}{\partial \tilde{t}} \tilde{g} &= \left(\frac{\partial t}{\partial \tilde{t}} \frac{\partial}{\partial t} \right) \{ \psi g \} \\
&= (\ln \psi)' g + \frac{1}{\psi} \psi \left(\frac{\partial}{\partial t} g \right) \\
&= \left(\ln V(t)^{-\frac{2}{N}} \right)' g + \left(-2Rc(g) \right) \\
&= -2\tilde{R}c - \frac{2}{N} \frac{V'}{V} g \\
&= -2\tilde{R}c + \frac{2}{N} \tilde{r} \tilde{g}.
\end{aligned}$$

Remark 2.5. The differentiability of ψ follows from the observation that if (U, χ) is any system of coordinates then

$$\sqrt{\cdot} \circ \det \circ G : U \times [0, T) \rightarrow \mathbb{R}_+^*, \quad (\det G > 0, \text{ on } U \times [0, T))$$

is a composition of differentiable functions, where $G(x, t) \in GL_N(\mathbb{R})$ is as usual the matrix of $g(t)$ at these coordinates, at $x \in \mathbb{M}$. After a moment's thought, one concludes that ψ is differentiable.

For later reference let's establish what we have done so far concerning this change of flows.

Lemma 2.6. [The Ricci flow in relation to the normalized Ricci flow]

Suppose that $(\mathbb{M}, g(t))$ satisfies the Ricci flow equation on a compact differentiable manifold of dimension N . Then, for any differentiable function of time $\psi(t)$ such that $\tilde{g}(t) = \psi(t)g(t)$ and $\psi > 0$, we have

- The Riemannian connection of $\tilde{g}$ and g are related by $\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k$.
- The $\binom{3}{1}$ -tensor field of curvature of $\tilde{g}$ satisfies $\tilde{R}_{ijk}{}^l = R_{ijk}{}^l$.
- The Ricci tensor of $\tilde{g}$ satisfies $\tilde{R}_{ij} = R_{ij}$.
- The scalar curvature of $\tilde{g}$ satisfies $\tilde{R} = \frac{1}{\psi} R$.
- The $\binom{4}{0}$ -tensor field of curvature $\tilde{R}m$ of $\tilde{g}$ satisfies $\tilde{R}_{ijkl} = \psi R_{ijkl}$.
- The Riemannian volume form of $\tilde{g}$ satisfies $d\tilde{V} = \psi^{N/2} dV$.

In particular, if $\psi(t) = V(t)^{-2/N}$, where $V = V(t)$ is the volume of $\mathbb{M}$ with respect to the metric g then $\tilde{g} = \tilde{g}(\tilde{t})$, where $\tilde{t}(t) = \int_0^t \psi(\tau) d\tau$, satisfies the normalized Ricci flow equation.

The Figure 2.1 presents the properties of the Ricci flow equation studied along this section.

Figure 2.1: Scheme of the properties of the Ricci flow equation

2.2 Second-order partial differential equations

We begin by defining the principal symbol of a linear scalar partial differential operator over the space of functions and then generalize for vector bundles.

Let $\mathbb{M}$ be a closed manifold and consider the PDE for $u \in C^\infty(\mathbb{M})$ given by

$$L(u) = \frac{\partial}{\partial t}(u), \quad (2.2)$$

where $L : C^\infty(\mathbb{M}) \rightarrow C^\infty(\mathbb{M})$ can be written with respect to local coordinates $(U, (x^i))$ as

$$L(u) = a_{ij} \partial_i \partial_j u + b_i \partial_i u + cu, \quad (2.3)$$

where all the coefficients a_{ij}, b_i, c are differentiable (C^∞), locally defined functions.

Definition 2.7. The equation (2.2) is said to be **parabolic** if a_{ij} is positive definite for all $x \in \mathbb{M}$, that is, if

$$a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2, \text{ for any covector } \xi = \xi_i dx^i \in (T_x \mathbb{M})^*.$$

We notice that this definition does not depend on the choice of the coordinate system.

Example 2.8. Let $f_0 \in C^\infty(\mathbb{M})$ be any differentiable function, and consider the PDE, written in coordinate $(U, (x^i))$ as

$$\frac{\partial}{\partial t} f = \frac{g^{ij}}{e_f} \partial_i \partial_j f + v f, \quad f(0) = f_0, \quad \text{and } v \in C^\infty(U).$$

This equation is obviously parabolic and may be rewritten, in these coordinates, as

$$\frac{\partial}{\partial t} f = \frac{1}{e_f} \Delta f + u, \quad \text{where } u = \frac{g^{ij}}{e_f} \Gamma_{ij}^k \partial_k f + v f. \clubsuit$$

Definition 2.9. Given a second order partial differential operator $L : C^\infty(\mathbb{M}) \rightarrow C^\infty(\mathbb{M})$, written as in equation (2.3), we define its principal $\sigma[L]$ in coordinates as the operator

$$\sigma[L] : T^*\mathbb{M} \rightarrow \mathbb{R}, \text{ given by } \sigma[L](x, \xi) = a_{ij}(x)\xi_i\xi_j \quad (2.4)$$

where $\xi = \xi_k dx^k \in T_x^*\mathbb{M}$ is an arbitrary covector. We say that the operator L is elliptic if the matrix (a_{ij}) is positive definite at every $x \in U$, i.e., $a_{ij}(x)\xi_i\xi_j > 0$, for all x .

Example 2.10. The Laplacian $\Delta : C^\infty(\mathbb{M}) \rightarrow C^\infty(\mathbb{M})$ given by

$$\Delta f = \frac{1}{\sqrt{\det G}} \partial_i \left(g^{ij} \sqrt{\det G} \partial_j f \right) = g^{ij} \partial_i \partial_j f - g^{ij} \Gamma_{ij}^k \partial_k f,$$

in some coordinates $(U, (x^i))$, where G is the matrix (g_{ij}) in these coordinates, is clearly an elliptic operator on all of $\mathbb{M}$ since

$$\sigma[\Delta](x, \xi) = g^{ij}(x)\xi_i\xi_j = \langle \xi, \xi \rangle = |\xi|^2 > 0,$$

for any $\xi \in T_x^*(\mathbb{M})$, with $\xi \neq 0$, and the (inner) product above is considered at the cotangent bundle at $x \in U$. ♣

2.3 Extension of PDEs to Vector Bundles

Let E be a differentiable vector bundle over a closed differentiable manifold $\mathbb{M}$, with differentiable canonical projection $\pi_E : E \rightarrow \mathbb{M}$, and consider a partial differentiable operator of second order

$$L : \Gamma(E) \rightarrow \Gamma(E),$$

on the space of sections of E . Using local trivializations, it is possible to find a local frame field for E over an open set $U \subset \mathbb{M}$, i.e., a set of sections $(f^\alpha)_\alpha$ such that $(f^\alpha(q))_\alpha \subset E_q$ is a basis for this vector space, and then we may write, for any $v \in \Gamma(E)$,

$$v = v_\alpha f^\alpha, \text{ where } v_\alpha : U \rightarrow \mathbb{R},$$

are (local) differentiable functions. Suppose, shrinking U if necessary, that there is a system of coordinates $(U, (x^i))$ and consider the equation

$$L(v) = \partial_t v, \quad (2.5)$$

where L may be written as

$$L(v) = [a_{\alpha\beta}^{ij} \partial_i \partial_j v_\alpha + b_{\alpha\beta}^i \partial_i v_\alpha + c_{\alpha\beta} v_\alpha] f^\beta \quad (2.6)$$

where the coefficients $a_{\alpha\beta}^{ij}$, $b_{\alpha\beta}^i$ and $c_{\alpha\beta}$ are real, differentiable functions defined over U . In order to define the principal symbol of L , let us make some preliminary algebraic considerations. Let π^*E be the pullback bundle of E over $T^*\mathbb{M}$ (see Definition 1.9) and denote by

$$\Pi : \pi^*E \rightarrow T^*\mathbb{M}, \quad (x, \xi, w) \mapsto (x, \xi),$$

its canonical projection, so that the fibers of π^*E are $(\pi^*E)_{\xi_x} = E_x$.

$$\begin{array}{ccc} \pi^*E & \dashrightarrow & E \\ \Pi \downarrow & & \downarrow \pi_E \\ T^*\mathbb{M} & \xrightarrow{\pi} & \mathbb{M} \end{array}$$

We are now ready to define the principal symbol of these operators.

Definition 2.11. Let $L : \Gamma(E) \rightarrow \Gamma(E)$ be a second order partial differential operator as above, given in local coordinates as in equation (2.6). We define its **principal symbol** $\sigma[L] : \pi^*E \rightarrow \pi^*E$ as a vector bundle homomorphism

$$(T_x^*\mathbb{M}) \ni \xi_x \mapsto \sigma[L](\xi_x) \in \text{Hom}(E_x)$$

in the direction of $\xi \in \Gamma(T^*\mathbb{M})$ obtained by replacing the directional derivatives ∂_i by the covector ξ_i at the second order terms, i.e.,

$$\sigma[L](\xi_x)(v) = [a_{\alpha\beta}^{ij}(x)\xi_i(x)\xi_j(x)v_\alpha(x)]f^\beta(x).$$

Definition 2.12. We say that a linear partial differential operator $L : \Gamma(E) \rightarrow \Gamma(E)$ is **elliptic** if its principal symbol

$$\sigma[L](\xi)(x, \cdot) : E_x \rightarrow E_x, \quad \xi \in \Gamma(T^*\mathbb{M}),$$

is an isomorphism whenever $\xi(x) \neq 0$. We say that a nonlinear partial differential operator is elliptic if its linearization is elliptic.

(See the discussion on linearisation at Section A.4 of the appendix.)

Definition 2.13. We say that the equation $L(v) = \partial_t v$ (2.5), is **parabolic** if there exists $\lambda > 0$ such that

$$\langle \sigma[L](\xi_x)(v), v \rangle_{E_x} \geq \lambda |\xi|^2 |v|^2 \quad (2.7)$$

for all $\xi \in \Gamma(T^*\mathbb{M})$ and $v \in \Gamma(E)$.

Remark 2.14. The equation (2.5) is parabolic if the inequality (2.7) holds for any fibre metric, that is, for any metric (inner product) defined on the vector space E_x .

More generally, let $\pi : E \rightarrow \mathbb{M}$ and $\pi^\bullet : F \rightarrow \mathbb{M}$ be real differentiable vector bundles over a differentiable manifold $\mathbb{M}$. Let $v \in \Gamma(E)$ be any section on E written as $v = v_\alpha f^\alpha$ on a local frame $(f^\alpha)_\alpha$, where $v_\alpha : U \rightarrow \mathbb{R}$ are differentiable functions and $U \subset \mathbb{M}$ is an open set where there is a system of coordinates

(x^i) . Shrinking U if necessary, suppose further that $L(v|_p) \in (\pi^\bullet)^{-1}(V)$, for all $p \in U$, where $V \subset \mathbb{M}$ is an open set where there is a local frame $(g^\beta)_\beta$ for F , that is, $(g^\beta(q))_\beta$ is a basis for F_q , for all $q \in V$. Now, let L be a second order partial differential operator which may be written, in those frames, as

$$L : \Gamma(E) \rightarrow \Gamma(F), \quad L(v) = [a_{\alpha\beta}^{ij} \partial_i \partial_j v_\alpha + b_{\alpha\beta}^i \partial_i v_\alpha + c_{\alpha\beta} v_\alpha] g^\beta. \quad (2.8)$$

Finally, consider the equation $L(v) = \frac{\partial}{\partial t}(v)$. The definition of parabolicity of this equation is done similarly as well as the principal symbol of these operators.

$$\sigma[L](\xi_x) : E_x \rightarrow F_x, \quad \sigma[L](\xi)(v) = [a_{\alpha\beta}^{ij} \xi_i \xi_j v_\alpha] g^\beta.$$

Similarly, one could define the symbol of a higher order partial differential operator. For a slightly different discussion on the matter, we refer to [CKKS04, Chapter 3]. There is, however, an important property we will need from this theory.

Proposition 2.15. [The principal symbol of the composition of differential operators]

Let E, F and G be real vector bundles over $\mathbb{M}$ and let

$$L : \Gamma(E) \rightarrow \Gamma(F), \quad \text{and} \quad X : \Gamma(G) \rightarrow \Gamma(E)$$

be linear partial differential operators. Let $\xi \in \Gamma(T^*\mathbb{M})$ be any covector field. Then,

$$\sigma[L \circ X](\xi) = \sigma[L](\xi) \circ \sigma[X](\xi). \quad (2.9)$$

Proof. See [CvdB09, Lemma 1.2.4]. ♣

2.4 Evolution Equations

Now, we will analyze some properties of the Ricci tensor regarding its principal symbol. For this purpose, we will look at the evolution of quantities such as metrics, connections and Ricci tensor under a family of metrics. The idea is the following. If one has a metric g on a manifold $\mathbb{M}$ then one also has a uniquely determined Riemannian connection ∇ on $\mathbb{M}$. Now, if we have a one parameter family of metrics $g(t)$ defined in an interval, say $I = [0, T)$, then we also have a one parameter family of Riemannian connections:

$$\nabla = \nabla(t) : [0, T) \times \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \rightarrow \mathfrak{X}(\mathbb{M}),$$

and we would like to compute its time derivative to understand how it evolves under the change of the metrics. We then observe that $\nabla_X Y$ is not a tensor on the Y variable, since $\nabla_X fY = f\nabla_X Y$ does not hold for functions $f \in C^\infty(\mathbb{M})$ in general. But,

$$\Theta : \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \rightarrow \mathfrak{X}(\mathbb{M}), \quad \text{defined by} \quad \Theta(X, Y) = \frac{\partial}{\partial t} \nabla_X Y$$

does define a $\binom{2}{1}$ -tensor field. This means that the evolution of a Riemannian connections will define a tensor field and therefore it is easier to calculate its local evolution equation, since tensors are pointwise

objects. Thus, the analysis of the evolution of Riemannian metrics, connections and curvatures will lead us to local expressions of derivatives of Christoffel symbols with respect to t .

The Figure 2.2 is an illustration of two one parameter families of metrics in different directions under the same initial condition g_0 . The idea is that these families of metrics deform the initial manifold, and this evolution may be more or less interesting. We would like the reader to bear in mind that any differentiable curve of metrics will generate differentiable curves of each tensor field generated by the metric. The figure spotlight some of the main objects we will consider.

Figure 2.2: The growth of two different curves of Riemannian metrics with the same initial metric

We warn the reader that each one of these tensors depend on the metric $g(t)$, and do not depend on the parameter t directly. For instance, the volume form element dV depends only on the Riemannian metric chosen. Then, rigorously speaking, the one parameter family of these N-forms, may be thought as

$$[0, T] \rightarrow \Omega^N(\mathbb{M}), \quad t \mapsto dV_{g(t)}.$$

Now we are going to turn the attention to the evolution equations of these tensors under the flow of any differentiable one parameter family of metrics. We start with the (directional) derivative of the inverse metric.

Lemma 2.16. [Spatial derivative of the inverse metric g^{-1}]

Let $(\mathbb{M}, g)$ be a Riemannian Manifold. In any system of coordinates $(U, (x^i))$ we have

$$\frac{\partial}{\partial x^k} g^{ij} = -g^{ip} g^{jq} \frac{\partial}{\partial x^k} g_{pq}. \quad (2.10)$$

Proof

Multiplying line i of the matrix $G^{-1} = (g^{ij})$ by column q of $G = (g_{ij})$, summing over p , and taking derivatives, we have

$$\frac{\partial}{\partial x^k} (g^{ip} g_{pq}) = \frac{\partial}{\partial x^k} (\delta_i^q) = 0.$$

Hence, $0 = (\frac{\partial}{\partial x^k} g^{ip}) g_{pq} + g^{ip} (\frac{\partial}{\partial x^k} g_{pq})$. Multiplying now by g^{jq} with q and j fixed and then summing over q , we get

$$0 = \left(\frac{\partial}{\partial x^k} g^{ip}\right) g_{pq} g^{jq} + g^{ip} \left(\frac{\partial}{\partial x^k} g_{pq}\right) g^{jq} = \left(\frac{\partial}{\partial x^k} g^{ip}\right) \delta_p^j + g^{ip} g^{jq} \left(\frac{\partial}{\partial x^k} g_{pq}\right) = \frac{\partial}{\partial x^k} g^{ij} + g^{ip} g^{jq} \left(\frac{\partial}{\partial x^k} g_{pq}\right),$$

where p and q are free indexes going from 1 to $N = \dim(\mathbb{M})$. ♣

A similar argument gives us the

Lemma 2.17. [Evolution equation for the inverse metric g^{-1}]

If $g(t)$ is a one parameter differentiable family of differentiable Riemannian metrics defined in some interval, then the evolution equation for its inverse metric g^{-1} is given by, in local coordinates,

$$\frac{\partial}{\partial t} g^{ij} = -g^{ip} g^{jq} \frac{\partial}{\partial t} g_{pq}. \quad (2.11)$$

Proof.

Exactly as the previous. ♣

Lemma 2.18. [Evolution of the Riemannian connection]

The evolution of the Riemannian connection $\Gamma = \Gamma(g(t))$ in any system of coordinates is given by

$$\frac{\partial}{\partial t} \Gamma_{ij}^k = \frac{1}{2} g^{kl} \{ \nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \} \quad (2.12)$$

where the $\binom{2}{0}$ -tensor field h is the time derivative of g , $\partial_t g = h$.

Proof.

Let $(U, (x^i))$ be any system of coordinates. Recall the formula for the Christoffel symbols in these coordinates,

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \{ \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \} \quad (2.13)$$

Writing ∂_t for partial derivatives with respect to t and using Lemma 2.17, we get

$$\begin{aligned} \partial_t \Gamma_{ij}^k &= -\frac{1}{2} g^{kp} g^{lq} (\partial_t g_{pq}) \{ \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \} + \frac{1}{2} g^{kl} \{ \partial_i \partial_t g_{jl} + \partial_j \partial_t g_{il} - \partial_l \partial_t g_{ij} \} \\ &= -g^{kp} h_{pq} \left[\frac{1}{2} g^{lq} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \right] + \\ &\quad + \frac{1}{2} g^{kl} \{ (\nabla_i h_{jl} + \Gamma_{ij}^q h_{ql} + \Gamma_{il}^q h_{jq}) + (\nabla_j h_{il} + \Gamma_{ji}^q h_{ql} + \Gamma_{jl}^q h_{iq}) - (\nabla_l h_{ij} + \Gamma_{li}^q h_{qj} + \Gamma_{lj}^q h_{iq}) \} \\ &= -g^{kp} h_{pq} \Gamma_{ij}^q + \frac{1}{2} g^{kl} \{ \nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \} + \frac{1}{2} g^{kp} \{ 2\Gamma_{ij}^q h_{pq} \} \\ &= \frac{1}{2} g^{kl} \{ \nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \}, \end{aligned}$$

where, in the penultimate equality, we substitute the free index l for p and in the second equality we used the ordinary formula for the covariant derivative of a $\binom{2}{0}$ -tensor field. ♣

Next, we obtain the evolution equation for the (Riemann) curvature endomorphism. Recall first that this tensor is given in local coordinates by $\mathcal{R} = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l$, where these components are easily obtained from the definition of this tensor in terms of the Christoffel symbols.

Lemma 2.19. [Evolution of the Riemann curvature endomorphism]

The evolution of the $\binom{3}{1}$ -tensor field of curvature $\mathcal{R}$, is given by

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijk}{}^l = \frac{1}{2} g^{lp} \left\{ \nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} \right. \\ \left. - \nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik} \right\} \end{aligned} \quad (2.14)$$

Proof.

We first establish a formula for the first term on the RHS of the equation above. Let $(U, (x^i))$ be any system of coordinates. Using equation (2.12) and equation (2.10), we obtain

$$\begin{aligned} \partial_t \partial_i \Gamma_{jk}^l &= \partial_i \partial_t \Gamma_{jk}^l \\ &= \partial_i \left\{ \frac{1}{2} g^{lp} (\nabla_j h_{kp} + \nabla_k h_{jp} - \nabla_p h_{jk}) \right\} \\ &= -\frac{1}{2} g^{lq} g^{ps} (\partial_i g_{qs}) (\nabla_j h_{kp} + \nabla_k h_{jp} - \nabla_p h_{jk}) + \frac{1}{2} g^{lp} \{ \partial_i (\nabla_j h_{kp} + \nabla_k h_{jp} - \nabla_p h_{jk}) \} \\ &= -g^{lq} (\partial_i g_{qs}) \left\{ \frac{1}{2} g^{ps} (\nabla_j h_{kp} + \nabla_k h_{jp} - \nabla_p h_{jk}) \right\} + \frac{1}{2} g^{lp} \{ \nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} \} + \\ &\quad + \Gamma_{ik}^m \left\{ \frac{1}{2} g^{lp} (\nabla_j h_{mp} + \nabla_m h_{jp} - \nabla_p h_{jm}) \right\} + \Gamma_{ip}^m \left\{ \frac{1}{2} g^{lp} (\nabla_j h_{km} + \nabla_k h_{jm} - \nabla_m h_{jk}) \right\} + \\ &\quad + \frac{1}{2} g^{lp} \{ \Gamma_{ij}^m \nabla_m h_{kp} + \Gamma_{ij}^m \nabla_k h_{mp} - \Gamma_{ij}^m \nabla_p h_{mk} \} \\ &= -g^{lq} g_{ms} \Gamma_{iq}^m (\partial_t \Gamma_{jk}^s) - \Gamma_{is}^m (\partial_t \Gamma_{jk}^s) \delta_m^l + \frac{1}{2} g^{lp} \{ \nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} \} + \\ &\quad + \Gamma_{ik}^m \{ \partial_t \Gamma_{jm}^l \} + \Gamma_{ip}^m \left\{ \frac{1}{2} g^{lp} (\nabla_j h_{km} + \nabla_k h_{jm} - \nabla_m h_{jk}) \right\} \\ &\quad + \frac{1}{2} g^{lp} \{ \Gamma_{ij}^m \nabla_m h_{kp} \} + \frac{1}{2} g^{lp} \{ \Gamma_{ij}^m \nabla_k h_{mp} \} - \frac{1}{2} g^{lp} \{ \Gamma_{ij}^m \nabla_p h_{mk} \}. \end{aligned}$$

But, we also have

$$\begin{aligned} \Gamma_{ip}^m \left\{ \frac{1}{2} g^{lp} (\nabla_j h_{km} + \nabla_k h_{jm} - \nabla_m h_{jk}) \right\} &= \frac{1}{2} [\partial_i g_{ps} + \partial_p g_{is} - \partial_s g_{ip}] g^{lp} \left\{ \frac{1}{2} g^{ms} (\nabla_j h_{km} + \nabla_k h_{jm} - \nabla_m h_{jk}) \right\} \\ &= \frac{1}{2} [2\Gamma_{pi}^m g_{ms}] g^{lp} (\partial_t \Gamma_{jk}^s). \end{aligned}$$

Finally, taking derivative the with respect to t of equation (1.19) and renaming indexes we obtain the desired formula. ♣

We would like to mention that this evolution equation could be more easily obtained using normal coordinates based at any point $p \in \mathbb{M}$. Since the Riemann curvature endomorphism is a tensor field, it is a pointwise object and therefore one may obtain any formula concerning this tensor using any system of coordinates, in particular, using normal coordinates. However, we wanted to do it on arbitrary coordinates to show an alternative way.

We would like now to obtain a formula for the evolution of the Ricci tensor in components. We regard this tensor as a second order partial differential operator,

$$Rc : \Gamma(S_2^+ T^* \mathbb{M}) \rightarrow \Gamma(S_2 T^* \mathbb{M})$$

where $\Gamma(S_2^+ T^* \mathbb{M})$ denotes the space of symmetric and positive definite $\binom{2}{0}$ -tensor fields, that is, Riemannian metrics, and $\Gamma(S_2 T^* \mathbb{M})$ denotes the space of symmetric $\binom{2}{0}$ -tensor fields. That is, we look at the Ricci tensor as a "function" of metrics. Recall first that, in local coordinates, the Ricci tensor is given by $Rc = R_{jk} dx^j \otimes dx^k$, where $R_{jk} = R_{ijk}{}^i$.

Lemma 2.20. *[Evolution of the Ricci tensor]*

The evolution of the Ricci tensor is given, in local coordinates, by

$$\frac{\partial}{\partial t} R_{jk} = \frac{1}{2} g^{pq} \{ \nabla_q \nabla_j h_{pk} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{pq} \} \quad (2.15)$$

Proof.

Take partial derivatives of $R_{jk} = R_{ijk}{}^i$ with respect to time and apply the previous lemma noticing that $g^{ip} \nabla_j \nabla_i h_{kp} = g^{ip} \nabla_j \nabla_p h_{ik}$ once i and p are free indexes. ♣

The equation above can be interpreted in the following manner.

$$\partial t R_{jk} = \partial t [Rc(g)(\partial j, \partial k)] = DRc(g)(\partial t g)(\partial j, \partial k) = DRc(g)(h)(\partial j, \partial k).$$

Hence, we may also write

$$DRc(g)h = \frac{1}{2} g^{pq} \left[\nabla_q \nabla_j h_{pk} + \nabla_q \nabla_k h_{pj} + \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{pq} \right] dx^j \otimes dx^k \quad (2.16)$$

where, in this case, $\partial t g = h$ and we say that $DRc(g)$ is the linearization of the Ricci tensor (see equation (A.4) and the discussion on linearization presented there).

Let us compute now the evolution of the scalar curvature. Recall that, in local coordinates $(U, (x^i))$, the scalar curvature is given by, $R = g^{jk} R_{jk}$.

Lemma 2.21. *[Evolution of the scalar curvature]*

The evolution of the scalar curvature is given in local coordinates by

$$\partial t R = -\Delta H + \nabla^p \nabla^q h_{pq} - \langle Rc, h \rangle, \quad (2.17)$$

where H is the trace of the $\binom{2}{0}$ -tensor field $\partial t g = h$, i.e., $H = g^{pq} h_{pq}$ and $\nabla^q h_{qj} = (\operatorname{div} h)_j$ (see Definition 1.35), and $\langle Rc, h \rangle = \langle Rc, h \rangle_{T^2 \mathbb{M}}$ is the inner product on the bundle of $\binom{2}{0}$ -tensors, see equation (1.6).

Proof.

If $v, k \in \Gamma(T^2 \mathbb{M})$ are two $\binom{2}{0}$ -tensor fields, written in coordinates as

$$v = v_{ij} dx^i \otimes dx^j, \quad \text{and} \quad k = k_{lm} dx^l \otimes dx^m,$$

then their inner-product according to equation (1.6), is

$$\langle v, k \rangle = g^{il} g^{jm} v_{ij} k_{lm}.$$

Using equations (2.11) and (2.15), we get

$$\begin{aligned} \partial t R &= (\partial t g^{ij}) R_{ij} + g^{ij} (\partial t R_{ij}) \\ &= -g^{ip} g^{jq} (\partial t g_{pq}) R_{ij} + \frac{1}{2} g^{pq} \{ g^{ij} \nabla_q \nabla_i h_{pj} + g^{ij} \nabla_q \nabla_j h_{pi} \} \\ &\quad - \frac{1}{2} g^{pq} \{ g^{ij} \nabla_q \nabla_p h_{ij} \} - \frac{1}{2} g^{ij} \{ g^{pq} \nabla_i \nabla_j h_{pq} \} \\ &= -\langle h, Rc \rangle + \frac{1}{2} g^{pq} \{ \nabla_q \nabla^j h_{pj} + \nabla_q \nabla^i h_{pi} \} \\ &\quad - \frac{1}{2} g^{pq} \nabla_q \nabla_p (g^{ij} h_{ij}) - \frac{1}{2} g^{ij} \nabla_i \nabla_j (g^{pq} h_{pq}) \\ &= -\langle h, Rc \rangle + \nabla^p \nabla^i h_{pi} - \Delta H. \quad \clubsuit \end{aligned}$$

Next, let us compute evolution of the Riemannian volume form.

Lemma 2.22. *[Evolution of the Riemannian volume form]*

The evolution of the Riemannian volume form is given in local coordinates by

$$\partial_t dV = \frac{1}{2} H dV, \quad \text{where } H = g^{pq} \left(\frac{\partial g_{pq}}{\partial t} \right). \quad (2.18)$$

Proof.

In local coordinates $(U, (x^i))$, we have $dV = \sqrt{\det G} dx^1 \wedge \dots \wedge dx^N$ (see equation (1.12)). Therefore,

$$\begin{aligned} \partial_t dV &= \frac{1}{2} \frac{1}{\sqrt{\det G}} \left(\frac{\partial \det G}{\partial t} \right) dx^1 \wedge \dots \wedge dx^N \\ &= \frac{1}{2} \frac{1}{\det G} \left(\frac{\partial \det G}{\partial g_{ij}} \right) (\partial_t g_{ij}) \sqrt{\det G} dx^1 \wedge \dots \wedge dx^N \\ &= \frac{1}{2} \frac{(-1)^{i+j} \det G_{[ij]}}{(\det G)} (\partial_t g_{ij}) dV \\ &= \frac{1}{2} (g^{ji} \partial_t g_{ij}) dV. \clubsuit \end{aligned}$$

It is worth establishing now the evolution of the total curvature. First, recall that we have, from the divergence theorem,

$$\int \Delta H dV = 0, \quad \text{and} \quad \int \operatorname{div}(\operatorname{div} h) dV = 0,$$

since $\mathbb{M}$ is closed. Thus, we have the following

Corollary 2.23. *[Evolution of the total scalar curvature]*

The evolution of the total scalar curvature $\int R dV$ is given by,

$$\partial_t \int R dV = \int [-\langle Rc, h \rangle + \frac{1}{2} RH] dV \quad (2.19)$$

Proof.

By differentiating under the integral sign, we have:

$$\begin{aligned} \partial_t \int R dV &= \int (\partial_t R) dV + \int R \partial_t dV \\ &= \int [-\Delta H + \nabla^p \nabla^q h - \langle Rc, h \rangle + \frac{1}{2} RH] dV \\ &= \int [-\langle Rc, h \rangle + \frac{1}{2} RH] dV. \clubsuit \end{aligned}$$

Suppose now that the family of metrics $g(t)$ satisfy the normalized Ricci flow equation $\partial_t g = -2Rc + \frac{2}{N}rg$. Then, in the direction of this flow, the volume of the manifold remains constant as g changes, that is,

$$\partial_t \int dV = - \int R dV + r \int dV = 0,$$

since the trace of $h = \partial_t g$ is given by $g^{ij} \partial_t g_{ij} = g^{ij} (-2R_{ij} + \frac{2}{N} r g_{ij}) = -2R + 2r$, in any system of coordinates.

In particular, one may now compute the evolution of the average scalar curvature $r = \frac{\int R dV}{\int dV}$ in the direction

of the normalized Ricci flow as a consequence of equation (2.19), which becomes, in this context, the following evolution equation

$$\partial_t \int R dV = \int \left(2|Rc|^2 - R^2 + \frac{rR}{N}(N+2) \right) \quad (2.20)$$

What would happen to the evolution of the average scalar curvature under the normalized Ricci flow if the manifold was two dimensional, compact and oriented?

2.5 The principal symbol of the Ricci operator

We are now going to see that the nonlinear differential operator Rc fails to be an elliptic operator in the metric g , and that this is due to the fact that the kernel of its linearization is nontrivial.

To begin with, let's write the principal symbol of the linearized operator $DRc(g)$ in the direction of some covector field $\xi \in \Gamma(T^*\mathbb{M})$. According to Definition 2.11, to compute the principal symbol of DRc , we just replace the covariant derivatives by the components of the covector taken in some local chart. Let $\xi \in \Gamma(T^*\mathbb{M})$ be any covector field. By equation (2.16) and Definition 2.11, we have

$$\sigma[DRc](\xi)(h) = \frac{1}{2}g^{pq} \left[\xi_q \xi_j h_{pk} + \xi_q \xi_k h_{pj} - \xi_q \xi_p h_{jk} - \xi_j \xi_k h_{pq} \right] dx^j \otimes dx^k. \quad (2.21)$$

Observe that $S_2^+T^*\mathbb{M}$ is not a vector bundle. But, $S_2^+T^*\mathbb{M}$ is open in $S_2T^*\mathbb{M}$ so that its tangent bundle has the same dimension of $S_2T^*\mathbb{M}$. By Definition 2.12, the principal symbol of the linearisation of the Ricci tensor acts fiberwise: for each $x \in \mathbb{M}$, we have

$$\sigma[DRc](\xi)(x, \cdot) : T_x(S_2^+T^*\mathbb{M}) \cong (S_2T^*\mathbb{M})_x \rightarrow (S_2T^*\mathbb{M})_x.$$

We are going to show that this map is not an isomorphism. In order to compute the dimension of the kernel of DRc , we are going to compose Rc with first order differential operators, and see that its kernel contains a N -dimensional vector space on each fiber. For this purpose, recall the first order differential operator

$$\text{div} : \Gamma(S_2T^*\mathbb{M}) \rightarrow \Gamma(T^*\mathbb{M}), \quad \text{div}(h) = g^{pq}(\nabla_q h_{pk})dx^k = (\nabla^p h_{pk})dx^k.$$

Then, consider the formal adjoint of $\delta_g = -\text{div}$

$$\delta_g^* : \Gamma(T^*\mathbb{M}) \rightarrow \Gamma(S_2T^*\mathbb{M}), \quad \delta_g^*X = \frac{1}{2}(\nabla_j X_k + \nabla_k X_j)dx^j \otimes dx^k,$$

considered with respect to the inner product

$$(\cdot, \cdot) : \mathfrak{X}(\mathbb{M}) \times \mathfrak{X}(\mathbb{M}) \rightarrow \mathbb{R}, \quad (X, Y) \doteq \int \langle X, Y \rangle dV.$$

We consider now the composition of $DRc(g)$ with δ_g^* , which is a third order differential operator

$$DRc(g) \circ \delta_g^* : \Gamma(T^*\mathbb{M}) \rightarrow \Gamma(S_2T^*\mathbb{M}).$$

We have,

$$\begin{aligned}
[DRc \circ \delta^* X]_{jk} &= \frac{1}{2} g^{pq} \{ \nabla_q \nabla_j (\delta^* X)_{pk} + \nabla_q \nabla_k (\delta^* X)_{pj} - \nabla_q \nabla_p (\delta^* X)_{jk} - \nabla_j \nabla_k (\delta^* X)_{pq} \} \\
&= + \frac{1}{4} \nabla^p \{ \nabla_j \nabla_p - \nabla_p \nabla_j \} X_k + \frac{1}{4} \nabla^p \{ \nabla_k \nabla_p - \nabla_p \nabla_k \} X_j + \\
&\quad \frac{1}{4} g^{pq} \{ \nabla_q \nabla_j \nabla_k - \nabla_j \nabla_k \nabla_q \} X_p + \frac{1}{4} g^{pq} \{ \nabla_q \nabla_k \nabla_j - \nabla_j \nabla_k \nabla_q \} X_p \\
&= - \frac{1}{4} [\nabla^q R_{jqk}{}^l] X_l - \frac{1}{4} [\nabla^q R_{kqj}{}^l] X_l - \frac{1}{4} g^{pq} [R_{jqk}{}^l \nabla_p X_l + R_{kqj}{}^l \nabla_p X_l] + \\
&\quad \frac{1}{4} g^{pq} \{ \nabla_j R_{kp} + \nabla_k R_{jp} \} X_q + \frac{1}{2} g^{pq} \{ R_{kp} \nabla_j X_q + R_{jp} \nabla_k X_q \} + \\
&\quad [\frac{1}{2} R_{jqk}{}^m \nabla_m X^q + \frac{1}{4} g^{pq} R_{jkp}{}^l \nabla_q X_l] \\
&= - \frac{1}{4} X^p \{ \nabla^q R_{jqkp} + \nabla^q R_{kqip} \} + \frac{1}{4} X^p \{ \nabla_j R_{kp} + \nabla_k R_{jp} \} + \\
&\quad \frac{1}{2} \{ R_k^p \nabla_j X_p + R_j^p \nabla_k X_p \} \\
&= \frac{1}{2} [L_{X^\#} Rc]_{jk}.
\end{aligned}$$

Therefore, the composition of these operators is actually a first-order differential operator. So, its (degree three) principal symbol

$$\sigma[DRc \circ \delta_g^*](\xi) : T^*\mathbb{M} \rightarrow S_2 T^*\mathbb{M}$$

is the zero map and, according to Proposition 2.15, we have

$$0 = [DRc \circ \delta_g^*](\xi) = \sigma[DRc](\xi) \circ \sigma[\delta_g^*](\xi),$$

and hence $\text{Im}(\sigma[\delta_g^*](\xi)) \subseteq \text{Ker}(\sigma[DRc](\xi))$, which means that the kernel of the linearized Ricci operator contains an N -dimensional space in each fiber, since the adjoint of the divergence δ_g^* is injective.

On the other hand, consider the partial differential operator given in local coordinates $(U, (x^i))$ by

$$B_g : \Gamma(S_2 T^*\mathbb{M}) \rightarrow \Gamma(T^*\mathbb{M}), \quad B_g h = g^{pq} (\nabla_q h_{pk} - \frac{1}{2} \nabla_k h_{pq}) dx^k \quad (2.22)$$

And notice, in particular, the contracted second Bianchi identity implies $B_g Rc(g) = 0$. In order to compute the dimension of the kernel of $\sigma[DRc](\xi)$, for $0 \neq \xi \in \Gamma(T^*\mathbb{M})$ we are going to show that, restricted so some subspace of $S_2 T^*\mathbb{M}$ of dimension $\frac{N(N-1)}{2}$, the principal symbol $\sigma[DRc](\xi)$ is an isomorphism. We begin by providing the orthogonal complement of the mentioned subspace.

Consider the space

$$A_\xi = \{ \xi \otimes W + W \otimes \xi - \langle \xi, W \rangle g : W \in T^*\mathbb{M} \} \subseteq S_2 T^*\mathbb{M}$$

where $\xi \in \Gamma(T^*\mathbb{M})$ is a nonzero $\binom{1}{0}$ -tensor field over $\mathbb{M}$. We would like to describe the relationship between the space A_ξ and the kernel $\text{Ker}(\sigma[B_g](\xi)) \doteq K_\xi$. By (2.22), for all $X \in T^*\mathbb{M}$, we have

$$\begin{aligned}
\langle \sigma[B_g](\xi)(h), X \rangle &= \langle g^{ij} \xi_j h_{ik} dx^k, X_l dx^l \rangle - \frac{1}{2} \langle g^{ik} \xi_j h_{ik} dx^j, X_l dx^l \rangle \\
&= g^{ij} g^{kl} \xi_j X_l h_{ik} - \frac{1}{2} g^{ik} g^{jl} \xi_j X_l h_{ik},
\end{aligned}$$

which yields

$$\begin{aligned}
\langle \xi \otimes X + X \otimes \xi - \langle \xi, X \rangle g, h \rangle &= \langle \xi_j X_l dx^j \otimes dx^l, h_{ik} dx^i \otimes dx^k \rangle + \langle \xi_j X_l dx^l \otimes dx^j, h_{ki} dx^k \otimes dx^i \rangle - \langle \xi, X \rangle g^{ik} h_{ik} \\
&= 2\{g^{ij} g^{kl} \xi_j X_l h_{ik} - \frac{1}{2} g^{jl} g^{ik} \xi_j X_l h_{ik}\} \\
&= 2\langle \sigma[B_g](\xi)(h), X \rangle.
\end{aligned}$$

Hence $A_\xi^\perp = K_\xi$. Then, given any $h \in K_\xi$, we have

$$0 = \sigma[B_g](\xi)(h)_k = g^{pq} \xi_q h_{pk} - \frac{1}{2} g^{pq} \xi_k h_{pq},$$

and therefore

$$\begin{aligned}
\sigma[DRc](\xi)(h)(\partial_j, \partial_k) &= \frac{1}{2} [\xi_j (g^{pq} \xi_q h_{pk}) + \xi_k (g^{pq} \xi_q h_{pj})] - \frac{1}{2} [\xi_j \xi_k g^{pq} h_{pq}] - \frac{1}{2} [(g^{pq} \xi_p \xi_q) h_{jk}] \\
&= -\frac{1}{2} |\xi|^2 h(\partial_j, \partial_k),
\end{aligned}$$

hence that $\sigma[DRc](\xi)|_{K_\xi} : K_\xi \rightarrow K_\xi$ is an automorphism given by

$$\sigma[DRc](\xi)|_{K_\xi}(h) = -\frac{1}{2} |\xi|^2 h.$$

Suppose we knew that $\dim A_\xi = N$. Recalling that $A_\xi^\perp = K_\xi$, we have $\dim K_\xi = \frac{N(N-1)}{2}$, because $\dim S_2 T^* \mathbb{M} = \frac{N(N+1)}{2}$. These calculations leads to

$$\frac{N(N+1)}{2} = \dim \text{Ker}(\sigma[DRc](\xi)) + \dim \text{Im}(\sigma[DRc](\xi)) \geq \dim \text{Ker} \sigma[DRc](\xi) + \frac{N(N-1)}{2} \quad (2.23)$$

according to the rank-nullity theorem, and equation (2.23) implies that $\dim \text{Ker}(\sigma[DRc](\xi)) \leq N$. Thus, since we already know that it is $\geq N$, we have

$$\dim \text{Ker}(\sigma[DRc](\xi)) = N.$$

It remains only to show that $\dim A_\xi$ is equal to N . For this, let us define the following operator. Fix any covector field $\xi \in \Gamma(T^* \mathbb{M})$ such that $\xi|_p \neq 0$, for all $p \in \mathbb{M}$ and set

$$T : T^* \mathbb{M} \rightarrow S_2 T^* \mathbb{M}, \quad T(W) = \xi \otimes W + W \otimes \xi - \langle \xi, W \rangle g, \quad \text{for } W \in T^* \mathbb{M}.$$

We show that $\ker T = \{0\}$. Take any nonzero $W \in T^* \mathbb{M}$ and regard it as a functional $W : T_p \mathbb{M} \rightarrow \mathbb{R}$. Certainly, there is some $0 \neq u \in T_p \mathbb{M}$ which belongs to $\text{Ker} W$ if $N \geq 2$. In all cases it is certainly true that either $\langle \xi, W \rangle \neq 0$ or $\langle \xi, W \rangle = 0$. Let's consider both possibilities.

Suppose first $\langle \xi, W \rangle \neq 0$. Then, by the reasoning above, there's some $u \in T_p \mathbb{M}$, $u \neq 0$, such that

$$T(W)(u, u) = -\langle \xi, W \rangle |u|^2 \neq 0.$$

Suppose now that $\langle \xi, W \rangle = 0$. Then, consider any system of coordinates (U, χ) around p and write in these coordinates $W = W_i dx^i|_p$ and $\xi = \xi_j dx^j$. Consider the $\binom{0}{1}$ -tensor fields over U defined by $X = W^i \partial_i$ and $Y = \xi^j \partial_j$ and compute, at p ,

$$(\xi \otimes W)(X|_p, Y|_p) = (\xi_j(p) W_j)^2 > 0, \quad (W \otimes \xi)(X|_p, Y|_p) = (W_j)^2 (\xi_k(p))^2 > 0,$$

hence that $T(W)(X|_p, Y|_p) > 0$. This shows that T is injective, and since the dimension of each fiber in $T^* \mathbb{M}$ is N , the dimension of A_ξ follows.

2.6 DeTurck's trick and existence

Hamilton's original paper relied on the Nash-Moser Inverse Function Theorem. He showed that given a closed 3-manifold X with a C^∞ metric g_0 , then there exists a unique solution $g(t)$ to the Ricci flow equation defined on $[0, \epsilon)$, for some $\epsilon > 0$, which satisfies $g(0) = g_0$. Later on, Dennis DeTurck proposed a modified flow to which the standard theory of strictly parabolic systems of PDEs can be applied, and is equivalent to the Ricci flow in a sense to be defined shortly (see Definition 2.24).

We would like to obtain a strictly parabolic system of partial differential equations from the Ricci flow equation. As we saw in the previous section, the kernel of the linearization of the Ricci operator is nontrivial. We begin by defining an operator such that its sum with $-2Rc$, is an elliptic partial differential operator. Define the operator

$$P : \Gamma(S_2^+ T^* \mathbb{M}) \rightarrow \Gamma(S_2 T^* \mathbb{M}), \quad \text{by } P(g) = L_W g$$

where $W : \mathbb{M} \rightarrow T\mathbb{M}$ is a $\binom{0}{1}$ -tensor field defined, in local coordinates $(U, (x^i))$, by

$$W^k = g^{pq}(\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k), \quad (2.24)$$

where $\tilde{\Gamma}$ is a Riemannian connection obtained from a fixed background Riemannian metric $\tilde{h}$ and $L_W g$ is the Lie derivative of the $\binom{2}{0}$ -tensor field g in the direction of W . By (A.12) the vector field W defined in (2.24) is independent of local coordinates and is therefore well defined. It is a straightforward computation that in local coordinates, we have

$$P(g)(\partial_i, \partial_j) = \nabla_i W_j + \nabla_j W_i$$

where $W_j = g_{jk} W^k$ are the components of the $\binom{1}{0}$ -tensor field W^b . To understand the linearization of the operator P and ultimately its principal symbol, we first take its derivative with respect to time along some flow (that is, along some differentiable one parameter family of metrics) and care only about the second order derivatives of $h = \partial_t g$. To understand the calculations to come, be aware that we will systematically replace lower order derivatives of h by components of $\binom{2}{0}$ -tensor fields to be described opportunistically. Let (U, χ) be normal coordinates centered at $p \in U$. We have, at p ,

$$\begin{aligned} \partial_t \nabla_i W_j &= \partial_t \partial_i W_j - \partial_t (\Gamma_{ij}^l W_l) \\ &= \partial_i (g_{jk} \partial_t W^k) + S'_{ij} \\ &= g_{jk} \partial_i [g^{pq} (\partial_t \Gamma_{pq}^k)] + S'_{ij} \\ &= g_{jk} g^{pq} (\partial_i \partial_t \Gamma_{pq}^k) + S'_{ij} \\ &= g_{jk} g^{pq} \left\{ \frac{1}{2} g^{kl} [\partial_i (\nabla_p h_{ql} + \nabla_q h_{pl} - \nabla_l h_{pq})] \right\} + S'_{ij} \\ &= \frac{1}{2} g^{pq} (\nabla_i \nabla_p h_{qj} + \nabla_i \nabla_q h_{pj} - \nabla_i \nabla_j h_{pq}) + S'_{ij}, \end{aligned}$$

which yields

$$\partial_t P_{ij} = g^{pq} \nabla_i \nabla_q h_{pj} + g^{pq} \nabla_j \nabla_q h_{pi} - \nabla_i \nabla_j g^{pq} h_{pq} + S_{ij},$$

where S is a $\binom{2}{0}$ -tensor field which contains only lower order derivatives of h , $S_{ij} = S'_{ij} + S''_{ji}$ and the tensors S' and S'' have components, as you can check, $S'_{ij} = \partial_i [(\partial_t g_{jk}) W^k] - \partial_t (\Gamma_{ij}^l W_l)$ and $S''_{ij} = S'_{ij} + g_{jk} \partial_i [(\partial_t g^{pq}) (\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k)]$. And, since both sides of this equation are components of tensors, the equation holds at every point and at any system of coordinates. Now, define a $\binom{1}{0}$ -tensor field $V \in \Gamma(T^* \mathbb{M})$ by

$$V = (B_g h)_j dx^j, \quad h = \partial t g,$$

where B_g is the operator given locally by $(B_g h)_j = g^{pq}(\nabla_q h_{pj} - \frac{1}{2} \nabla_j h_{pq})$. It is directly obtained then that

$$\partial t P_{ij} = \nabla_i V_j + \nabla_j V_i + S_{ij} \quad (\star)$$

Recall that the trace of h is given in coordinates by $H = g^{pq} h_{pq}$. By equation (2.15), we have

$$\begin{aligned} -2\partial t R_{jk} &= g^{pq} [\nabla_q \nabla_p h_{jk} + \nabla_j \nabla_k h_{pq} - \nabla_q \nabla_j h_{pk} - \nabla_q \nabla_k h_{pj}] \\ &= \Delta h_{jk} + \nabla_j \nabla_k H - g^{pq} \nabla_j \nabla_q h_{pk} - g^{pq} \nabla_k \nabla_q h_{pj} \\ &\quad - g^{pq} \{ \nabla_q \nabla_j - \nabla_j \nabla_q \} h_{pk} - g^{pq} \{ \nabla_q \nabla_k - \nabla_k \nabla_q \} h_{pj} \\ &= \Delta h_{jk} - \nabla_j V_k - \nabla_k V_j + T_{jk}, \end{aligned}$$

where T is a $\binom{2}{0}$ -tensor field which contains only lower order derivatives of h and whose components are given by T_{jk} . Because of its importance, we rewrite the linearization of $-2Rc$ as

$$-2\partial t R_{jk} = \Delta h_{jk} - \nabla_j V_k - \nabla_k V_j + T_{jk}. \quad (\star\star)$$

Now, consider the operator

$$Q : \Gamma(S_2^+ T^* \mathbb{M}) \rightarrow \Gamma(S_2 T^* \mathbb{M}), \quad Q(g) = (-2Rc + P)(g).$$

By equations $(\star)$ and $(\star\star)$, the evolution of the operator Q is

$$\partial t Q_{ij} = \Delta h_{ij} + S_{ij} + T_{ij}$$

and then, by Definition 2.11, its principal symbol in the direction of $\xi \in \Gamma(T^* \mathbb{M})$ is

$$\sigma[DQ(g)](\xi)(h)_{ij} = |\xi|^2 h_{ij}.$$

Therefore, Q is an elliptic operator and according to standard theory of PDEs (see [LSU68] and [Aub98, Theorem 4.51]) there exists a unique solution $g(t)$ to the initial value problem (2.25) below, that is, since the linearization of Q is elliptic, there exists a unique family of metrics $g(t)$ defined on $[0, \epsilon)$ for some $\epsilon > 0$ satisfying the Ricci-DeTurck flow equation, that is

$$\begin{cases} \frac{\partial g(t)}{\partial t} = -2Rc(g(t)) + \mathcal{L}_{W(t)} g(t), & \left[\text{in components: } \frac{\partial g_{ij}}{\partial t} = -2R_{ij} + (\mathcal{L}_W g)_{ij} \right] \\ g(0) = g_0. \end{cases} \quad (2.25)$$

Now we define what it means for two families of metrics to be equivalent.

Definition 2.24. Given two one parameter families of Riemannian metrics $\tilde{g}(t)$ and $\hat{g}(t)$ defined in the same time interval I over a differentiable manifold $\mathbb{M}$, we say that $\tilde{g}(t)$ is equivalent to $\hat{g}(t)$ if there exists a one parameter family of diffeomorphisms $\psi_t : \mathbb{M} \rightarrow \mathbb{M}$, for $t \in I$ such that

$$(\psi_t)^* \hat{g}(t) = \tilde{g}(t).$$

The key point in DeTurck's proof of the existence of the Ricci flow is to show that there is a one parameter family of diffeomorphisms that push the Ricci-DeTurck flow back to the Ricci flow, and the family of such diffeomorphisms is given by the solutions of the following system of ODEs:

$$\partial t \psi(x, t) = -W(\psi(x, t), t), \quad \psi(\cdot, 0) = Id_{\mathbb{M}},$$

where W is the vector field locally defined by

$$W^k = g^{pq}(t)(\Gamma(t)_{pq}^k - \tilde{\Gamma}_{pq}^k),$$

where $g(t)$ is the solution of the Ricci-DeTurck flow equation and W exists for as long as the Ricci-DeTurck flow exists, namely for $[0, \epsilon)$. If the family of maps $\psi = \psi(x, t) : \mathbb{M} \times [0, \epsilon) \rightarrow \mathbb{M}$ exists for as long as W is defined, then we employ Lemma 1.18 to show that the pull-back of $g(t)$ via these diffeomorphisms defines a Ricci flow.

Since

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{\psi_{t+s}^* g(t+s) - \psi_t^* g(t)}{s} &= \lim_{s \rightarrow 0} \frac{\psi_{t+s}^* [g(t+s) - g(t)]}{s} + \lim_{s \rightarrow 0} \frac{[\psi_{t+s}^* - \psi_t^*] g(t)}{s} \\ &= \psi_t^* \left[\frac{\partial}{\partial t} g(t) \right] + \frac{\partial}{\partial s} \Big|_{s=0} [\psi_{t+s}^* g(t)] \end{aligned}$$

we have,

$$\begin{aligned} \frac{\partial}{\partial t} [\psi_t^* g(t)] &= \frac{\partial}{\partial s} \Big|_{s=0} [\psi_{t+s}^* g(t+s)] \\ &= \psi_t^* \left[\frac{\partial}{\partial t} g(t) \right] + \frac{\partial}{\partial s} \Big|_{s=0} [\psi_{t+s}^* g(t)] \\ &= \psi_t^* \left[-2\text{Rc}(g(t)) + \mathcal{L}_{W(t)} g(t) \right] - \psi_t^* [\mathcal{L}_{W(t)} g(t)] \\ &= \psi_t^* \left[-2\text{Rc}(g(t)) \right] \\ &= -2\text{Rc}[\psi_t^* g(t)], \end{aligned} \tag{2.26}$$

where $\frac{\partial}{\partial s} \Big|_{s=0} [\psi_{t+s}^* g(t)] = -\psi_t^* [\mathcal{L}_{W(t)} g(t)]$ follows from Lemma 1.18 with slight modifications, and the last line follows from the invariance of the Ricci tensor under diffeomorphisms (see Section 2.1.1). Therefore, $\bar{g}(t) = \psi_t^* g(t)$ is indeed a Ricci flow. In particular, the family of metrics $g(t)$ satisfying the Ricci-DeTurck flow equation (2.25) is equivalent to the family of metrics $\bar{g}(t)$, according to equation (2.26). Hence, the Ricci-DeTurck flow is equivalent to the Ricci flow, according to Definition 2.24. This proves the existence claimed. It remains only to show the existence of the Deturck's diffeomorphisms. We will not prove the existence of these diffeomorphisms here and we refer to [CKKS04, Chapter 3] for the proof. For clarity of exposition, we will state the result used.

Theorem 2.25. *Let $X = X(t) : [0, T) \rightarrow \mathfrak{X}(\mathbb{M})$ be a continuous time-dependent family of vector fields on a compact N -dimensional manifold $\mathbb{M}$, defined for $0 \leq t < T \leq \infty$. Then, there exists a one-parameter family of diffeomorphisms $\psi = \psi(t) : \mathbb{M} \rightarrow \mathbb{M}$ defined on the same time interval such that*

$$\partial_t \psi(x, t) = -X(\psi(x, t), t), \quad \psi(\cdot, 0) = \text{Id}_{\mathbb{M}}, \tag{2.27}$$

for all $x \in \mathbb{M}$, for all $t \in [0, T)$.

2.7 Harmonic Map Heat Flow

This section aims at giving the background for proving the uniqueness of the Ricci flow. The main ingredient we need is the harmonic map flow, which we will consider only for diffeomorphisms, for which we will admit existence and uniqueness.

2.7.1 The pullback connection

Let $f : \mathbb{M} \rightarrow P$ be a differentiable map between differentiable manifolds. We want to interpret the differential of f as a section in some vector bundle. For each $p \in \mathbb{M}$, we have the linear map $f_*(p) : T_p\mathbb{M} \rightarrow T_{f(p)}P$, that is $f_*(p) \in T_p^*\mathbb{M} \otimes (f^*TP)_p$, so f_* is a differentiable section of $T^*\mathbb{M} \otimes (f^*TP)$. Given a differentiable vector field $X \in \mathfrak{X}(\mathbb{M})$, the pushforward of X is a section $f_*X \in \Gamma(f^*TP)$, obtained by applying f_* to X . Let us express this in local coordinates. Let $(V, \psi = (y^\alpha))$ be a chart on P and let $(U, \chi = (x^i))$ be a chart on $\mathbb{M}$ such that $f(U) \cap V \neq \emptyset$. Denoting by $\frac{\partial}{\partial x^i}$ the coordinate vector fields from (x^i) , we have

$$df\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial(\psi^{-1} \circ \psi \circ f \circ \chi^{-1})}{\partial x^i} = \frac{\partial f^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha},$$

where $f^\alpha = y^\alpha \circ f \circ \chi^{-1}$. Then, it is clear that $\left[df\left(\frac{\partial}{\partial x^i}\right) \right] dy^\beta = \frac{\partial f^\beta}{\partial x^i}$ and therefore, we may write

$$df = \frac{\partial f^\beta}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^\beta}.$$

Notice that df is a sort of $\binom{1}{1}$ -tensor field whose first argument is a vector field on $\mathbb{M}$ and whose second argument is a covector field over P , which is in accordance with Convention 1.16.

Theorem 2.26. *[Existence and uniqueness of the pullback connection]*

let ${}^P\nabla$ be a connection on TP over P and $f : \mathbb{M} \rightarrow P$ a differentiable map. There is a unique connection ${}^f\nabla$ on f^*TP , referred to as the pullback connection, such that

$${}^f\nabla_v(Y_f) = {}^P\nabla_{f_*v}Y, \text{ for all } v \in T\mathbb{M} \text{ and } Y \in \Gamma(TP) = \mathfrak{X}(P).$$

Proof

See [AH10, pp. 39]. ♣

Remark 2.27. The theorem above may be generalized to arbitrary vector bundles over P , equipped with any connection.

Let us determine the action of the pullback connection on any section of f^*TP . Let $Y \in \Gamma(f^*TP)$ be any section, recalling that, $(f^*Y)(p) = Y(f(p)) \in T_{f(p)}P$, for any $p \in \mathbb{M}$. Fix an arbitrary $p \in \mathbb{M}$ and choose a local frame $\{\partial_\alpha\}$ about $f(p)$ for TP . Then, we may write $Y = Y^\alpha(\partial_\alpha)_f$, where each Y^α is a differentiable function defined near p . Let $X \in \mathfrak{X}(\mathbb{M})$ be any vector field on $\mathbb{M}$. So the rules for a connection together with the pullback connection condition give

$$\begin{aligned}
f^* \nabla_X Y &= f^* \nabla_X (Y^\alpha (\partial_\alpha)_f) \\
&= (Y^\alpha)^f \nabla_X (\partial_\alpha)_f + (\partial_\alpha)_f (f^* \nabla_X Y^\alpha) \\
&= (Y^\alpha)^P \nabla_{f_* X} (\partial_\alpha)_f + (X(Y^\alpha)) (\partial_\alpha)_f \\
&= (Y^\alpha)^P \nabla_{\xi^\beta (\partial_\beta)_f} (\partial_\alpha)_f + X(Y^\alpha) (\partial_\alpha)_f \\
&= ((Y^\alpha \xi^\beta) ({}^P \Gamma \circ f)_{\beta\alpha}^\gamma + X(Y^\gamma)) (\partial_\gamma)_f
\end{aligned}$$

where $f_* X = \xi^\beta (\partial_\beta)_f$ is the pushforward of X written in the basis $\{(\partial_\alpha)_f\}$ and $({}^P \Gamma \circ f)_{\alpha\beta}^\gamma$ are the components of the connection in the referred basis. This serves to give us a sense of how the connection will act on section of the bundle $T^* \mathbb{M} \otimes (f^* TP)$.

2.7.2 The harmonic map laplacian

Suppose now that $f : (\mathbb{M}, g) \rightarrow (P, h)$ is a diffeomorphism between Riemannian manifolds. Given a system of coordinates $(U, (x^i))$ in $\mathbb{M}$, and a system of coordinates $(V, (y^\alpha))$ in P , let us denote the Riemannian connection of g , in these coordinates, by $\Gamma(g)_{jk}^i$ and by $\tilde{\Gamma}(h)_{\beta\gamma}^\alpha$ the Riemannian connection of h in the coordinates (y^α) . Let us assume that the induced connection on $\Gamma(T^* \mathbb{M} \otimes f^* TP)$, is determined by

$$\nabla : \Gamma(T^* \mathbb{M} \otimes f^* TP) \rightarrow \Gamma(T^* \mathbb{M} \otimes T^* \mathbb{M} \otimes f^* TP), \quad (f^* \Gamma)_{i\beta}^\alpha = \frac{\partial f^\gamma}{\partial x^i} (\tilde{\Gamma}(h) \circ f)_{\gamma\beta}^\alpha$$

Hence,

$$\nabla(df) = \left(\nabla_j \frac{\partial f^\beta}{\partial x^i} \right) dx^j \otimes dx^i \otimes \frac{\partial}{\partial y^\beta} \quad (2.28)$$

where

$$\begin{aligned}
\nabla_j \frac{\partial f^\beta}{\partial x^i} &= \partial_j \left\{ \frac{\partial f^\beta}{\partial x^i} \right\} - \Gamma(g)_{ji}^k \frac{\partial f^\beta}{\partial x^k} + (f^* \Gamma(h))_{j\alpha}^\beta \frac{\partial f^\alpha}{\partial x^i} \\
&= \frac{\partial^2 f^\beta}{\partial x^j \partial x^i} - \Gamma(g)_{ij}^k \frac{\partial f^\beta}{\partial x^k} + \frac{\partial f^\gamma}{\partial x^j} \frac{\partial f^\alpha}{\partial x^i} (\Gamma(h) \circ f)_{\alpha\gamma}^\beta.
\end{aligned}$$

Notice now that, by looking at equation (2.28), we may raise the first index of the $\binom{2}{1}$ -tensor field $\nabla(df)$ in order to get the trace of the resulting tensor. Applying then the sharp operator to raise the first index, we get

$$\nabla(df)^\sharp = \left(\nabla^j \frac{\partial f^\beta}{\partial x^i} \right) \frac{\partial}{\partial x^j} \otimes dx^i \otimes \frac{\partial}{\partial y^\beta}$$

and define the harmonic map laplacian for diffeomorphisms with domain metric g and codomain metric h by the formula

$$\Delta_{g,h}f \doteq \text{tr}(\nabla(df)^\sharp) \in \Gamma(f^*TP)$$

which, in local coordinates, have the form

$$(\Delta_{g,h}f)dy^\alpha = g^{ij} \left(\nabla_j \frac{\partial f^\alpha}{\partial x^i} \right).$$

Theorem 2.28. [*Harmonic map laplacian for diffeomorphisms*]

Suppose $f : (\mathbb{M}, \bar{g}) \rightarrow (P, h)$ is a diffeomorphism between Riemannian manifolds. Then, the harmonic map laplacian may be written, in any system of coordinates, as

$$(\Delta_{\bar{g},h}f)dy^\nu = [(f^{-1})^*(\bar{g})]^{\alpha\beta} \left\{ -\Gamma((f^{-1})^*\bar{g})_{\alpha\beta}^\nu + \tilde{\Gamma}(h)_{\alpha\beta}^\nu \right\}.$$

Proof

Let's denote by $g = (f^{-1})^*\bar{g}$ the metric induced by f on the codomain manifold. It is a straightforward calculation to get the formula once we employ the equation (1.43), obtained at chapter one,

$$g^{\alpha\beta} \Gamma_{\alpha\beta}^\nu = \bar{g}^{ij} \left[-\frac{\partial^2 f^\nu}{\partial x^j \partial x^i} + \bar{\Gamma}_{ij}^k \frac{\partial f^\nu}{\partial x^k} \right]$$

from the definition of the harmonic map laplacian in coordinates, and recalling that

$$\bar{g}^{ij} \frac{\partial f^\zeta}{\partial x^j} \frac{\partial f^\xi}{\partial x^i} = g^{\zeta\xi} \clubsuit$$

2.7.3 The harmonic map laplacian in relation to the DeTurck's diffeomorphisms

As an application of the Theorem 2.28, we now state the most important theorem of this section

Theorem 2.29. [*The harmonic map laplacian in relation to the DeTurck's diffeomorphisms*]

Let $g(t)$ be a Ricci DeTurck flow and consider the one-parameter family of vector fields

$$W : [0, \epsilon) \rightarrow \mathfrak{X}(\mathbb{M}), \quad W = g(t)^{pq} \{ \Gamma(t)_{pq}^k - \tilde{\Gamma}(h)_{pq}^k \} \partial_k,$$

where $\Gamma = \Gamma(t)$ is the Riemannian connection generated by the metric $g(t)$. Consider also the family of diffeomorphisms $\psi_t : (\mathbb{M}, \bar{g}(t)) \rightarrow (\mathbb{M}, h)$ obtained from the vector fields above through the system of EDOs

$$\frac{\partial}{\partial t} \psi = -W, \quad \psi(0) = Id_{\mathbb{M}},$$

where $\bar{g}(t)$ satisfies the Ricci flow equation. Then,

$$\Delta_{\bar{g},h} \psi_t = \frac{\partial}{\partial t} \psi_t.$$

Proof.

Since ψ_t pushes the Ricci DeTurck flow back to the Ricci flow, i.e., $\psi_t^* g(t) = \bar{g}(t)$, it also certainly satisfies

$$(\psi_t^{-1})^* \bar{g}(t) = g(t),$$

and recall that W exists as long as $g(t)$ exists. From the equation obtained for the harmonic map laplacian for diffeomorphisms (Theorem 2.28), we have

$$\begin{aligned} (\Delta_{\bar{g},h}\psi_t)dx^k &= [(\psi_t^{-1})^*\bar{g}]^{pq}\{-\Gamma((\psi_t^{-1})^*\bar{g})_{pq}^k + \tilde{\Gamma}(h)_{pq}^k\} \\ &= -g^{pq}\{\Gamma_{pq}^k - \tilde{\Gamma}(h)_{pq}^k\} \\ &= -W^k. \end{aligned}$$

The result then follows from the independence of coordinates. ♣

The Harmonic Map Flow. Given a diffeomorphism $f_0 : \mathbb{M} \rightarrow P$, the harmonic map flow is the following parabolic system of PDEs

$$\frac{\partial}{\partial t}f = \Delta f, \quad f(0) = f_0$$

which have a unique solution for a short time. (See [Paz83] or [Aub98]).

2.8 Uniqueness of the Ricci flow

In this section we prove uniqueness of the Ricci flow since we have all tools needed to do it. Let's go straight to the proof.

Suppose $\hat{g}(t)$ and $\bar{g}(t)$ are two solutions of the Ricci flow equation with the same initial condition:

$$\hat{g}(0) = \bar{g}_0, \quad \text{and} \quad \bar{g}(0) = \bar{g}_0.$$

We fix a background metric h and then we solve two harmonic map heat flows

$$\Delta_{\hat{g}(t),h}\hat{\psi} = \frac{\partial}{\partial t}\hat{\psi}, \quad \Delta_{\bar{g}(t),h}\bar{\psi} = \frac{\partial}{\partial t}\bar{\psi}, \quad \hat{\psi}(0) = Id_{\mathbb{M}}, \quad \text{and} \quad \bar{\psi}(0) = Id_{\mathbb{M}}$$

for diffeomorphisms $\hat{\psi}_t, \bar{\psi}_t : \mathbb{M} \rightarrow \mathbb{M}$, where $\Delta_{\bar{g},h}$ represents the map laplacian with respect to the domain metric $\bar{g}$ and codomain metric h , whose connection is still denoted by $\tilde{\Gamma}$.

Now we may compute that both metrics $\hat{h}(t) = (\hat{\psi}_t^{-1})^*\hat{g}(t)$ and $\bar{h}(t) = (\bar{\psi}_t^{-1})^*\bar{g}(t)$ are solutions of the Ricci DeTurck flow equation, using exactly the same method in equation (2.26). The metrics $\hat{h}$ and $\bar{h}$ are seen to satisfy a strictly parabolic system of PDEs. Recalling the operator Q , which now is seen to depend on a second variable: a vector field,

$$Q[g, W] = -2Rc(g) + \mathcal{L}_W g = -2Rc(g) + P(g, W),$$

this means that,

$$\begin{cases} \frac{\partial \hat{h}}{\partial t} = Q[\hat{h}, \hat{W}], & \text{where } \hat{W}^k = \hat{h}^{pq}\{\hat{\Gamma}_{pq}^k - \tilde{\Gamma}_{pq}^k\} \\ \frac{\partial \bar{h}}{\partial t} = Q[\bar{h}, \bar{W}], & \text{where } \bar{W}^k = \bar{h}^{pq}\{\bar{\Gamma}_{pq}^k - \tilde{\Gamma}_{pq}^k\}, \end{cases}$$

Since $\hat{h}(0) = \bar{h}(0)$ one concludes that $\hat{h}(t) = \bar{h}(t)$, from existing theory.

On the other hand, given these maps $\bar{h}$ and $\hat{h}$, we can recover the family of diffeomorphisms $\hat{\psi}$ and $\bar{\psi}$ through the ODE

$$\frac{\partial}{\partial t} \hat{\psi} = \hat{W}[\hat{h}, h](\hat{\psi}, t), \text{ and } \frac{\partial}{\partial t} \bar{\psi} = \bar{W}[\bar{h}, h](\bar{\psi}, t),$$

and this means that both $\hat{\psi}$ and $\bar{\psi}$ are solutions of the same ODE generated by the same vector field $W = \bar{W} = \hat{W}$, since them both are defined by the same family of metrics. Then, from existing theory of ODE, we have $\hat{\psi}_t = \bar{\psi}_t$. Finally, we conclude that the initial family of metrics is the same

$$\hat{g}(t) = (\hat{\psi}_t)^* \hat{h}(t) = (\bar{\psi}_t)^* \bar{h}(t) = \bar{g}(t). \clubsuit$$

For later reference, we state the main theorem we have achieved so far.

Theorem 2.30. *[Existence and uniqueness of the Ricci flow]*

Let g_0 be a differentiable Riemannian metric on a differentiable closed manifold $\mathbb{M}$. Then, there exists a unique solution to the Ricci flow equation $g(t)$ defined on the positive time interval $[0, \epsilon)$ such that $g(0) = g_0$.

We finish this section with a simple instance of Ricci flow.

Example 2.31. *[Ricci flow on Einstein manifolds]*

Let $\mathbb{M}$ be a differentiable manifold. We say that a Riemannian metric g_0 on $\mathbb{M}$ is Einstein if its Ricci tensor is a multiple of g_0 by a constant, that is, $Rc(g_0) = \mu g_0$, for some $\mu \in \mathbb{R}$. Suppose that g_0 is Einstein. We showed that (see Lemma 2.6) any Riemannian metric g and any of its multiple μg generate the same Riemannian connection, and therefore their Ricci tensors are also the same. In particular

$$Rc(\psi(t)g_0) = Rc(g_0) = \mu g_0 = \frac{\mu}{\psi(t)}g(t).$$

for any differentiable positive function of time $\psi : [0, \infty) \rightarrow]0, \infty)$, setting $g(t) = \psi(t)g_0$.

Using this we may produce solutions to the Ricci flow equation as follows. Suppose there is a solution of the Ricci flow equation of the type $g(t) = \psi(t)g_0$. Then,

$$\psi'(t)g_0 = g'(t) = -2Rc(g(t)) = -2\mu g_0$$

Hence, $\psi'(t) = -2\mu$ and therefore $\psi(t) = -2\mu t + 1$. On the other hand, if $\psi(t) = -2\mu t + 1$ then $g(t) = \psi(t)g_0$ is a solution to the Ricci flow equation, and therefore the unique solution defined for $t \geq 0$, satisfying $g(0) = g_0$. If $\mu > 0$, then the solution is defined for $0 \leq t < 1/2\mu$. If $\mu < 0$ then the solution is defined for all $t \geq 0$.

In this example, let's take a look at what happens with the curvature and the volume of the manifold when the metric evolves. We have, in local coordinates,

$$R(g(t)) = g^{ij}(t)R_{ij}(g(t)) = \frac{1}{\psi(t)}g_0^{ij}R_{ij}(g_0) = \frac{N\mu}{1 - 2\mu t}$$

If $\mu > 0$ then the scalar curvature R converges to $+\infty$ when $t \rightarrow \left(\frac{1}{2\mu}\right)^-$. If $\mu < 0$ then the Ricci flow $g(t)$ is defined for all $t \geq 0$ and the scalar curvature converges to zero as $t \rightarrow +\infty$. Now we analyze what happens with the volume. Since, in local coordinates, the Riemannian volume form is given by $dV(g(t)) = \sqrt{\det G(t)} dx^1 \wedge \dots \wedge dx^N$, we have $dV(g(t)) = \psi^{\frac{N}{2}}(t)dV(g_0)$. The volume, in the direction of $g(t) = \psi(t)g_0$ is, then, given by

$$V(t) = \int dV(g(t)) = \psi^{\frac{N}{2}} \int dV(g_0) = \psi^{\frac{N}{2}} V(0).$$

If $\mu < 0$ then $1 - 2\mu t \longrightarrow +\infty$, when $t \longrightarrow \infty$ and therefore

$$V(t) \xrightarrow{t \longrightarrow \infty} +\infty.$$

In particular, while the curvature of the manifold converges to zero, its volume grows without bounds. On the other hand, if $\mu > 0$, then

$$V(t) = (1 - 2\mu t)^{\frac{N}{2}} V(0) \xrightarrow{t \longrightarrow \frac{1}{2\mu}^-} 0.$$

In particular, while the curvature grows without bounds, the volume converges to zero.

Chapter 3

Long Time Existence

This chapter aims at showing that the obstacle to long-time existence of the Ricci flow is the ($\binom{4}{0}$ -tensor field of) curvature becoming unbounded. For this purpose, in summary, we will need estimates for both the metric and the Ricci tensor. Another prerequisite is to study evolution equations under Ricci flow. In this chapter we will also present maximum principles and show that all manifolds that start with a positive (scalar) curvature will develop a singularity in finite time. This chapter is mainly based on [CLN] and [CKKS04]. We also took some ideas from [CS12].

3.1 Evolution equations under Ricci flow

In this section we review the evolution equations obtained in Chapter 2, at Section 2.4, but now we would like to understand their behavior in the direction of the Ricci flow. In order to summarize the main evolution equations, let us briefly recall their expressions in the direction of any family of metrics and after that we apply them to the Ricci flow.

Let $\mathbb{M}$ be a differentiable manifold and let $g(t)$ be any differentiable one parameter family of Riemannian metrics. The following equations hold in any system of coordinates $(U, (x^i))$.

1. The evolution of the inverse metric g^{-1} is given by

$$\frac{\partial}{\partial t} g^{ij} = -g^{ip} g^{iq} \frac{\partial}{\partial t} g_{pq}.$$

2. The Riemannian connection $\Gamma(g(t))$ evolves by

$$\frac{\partial}{\partial t} \Gamma_{ij}^k = \frac{1}{2} g^{kl} \{ \nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \}, \quad \text{where } h \doteq \partial_t g.$$

3. The $\binom{3}{1}$ -tensor field of curvature $\mathcal{R}(g(t))$ evolves by

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijk}{}^l = \frac{1}{2} g^{lp} \bigg\{ & \nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} \\ & - \nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik} \bigg\}. \end{aligned}$$

4. The symmetric $\binom{2}{0}$ -tensor field $Rc(g(t))$ evolves according to the law

$$\frac{\partial}{\partial t} R_{jk} = \frac{1}{2} g^{pq} \{ \nabla_q \nabla_j h_{pk} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{pq} \}.$$

5. The $\binom{0}{0}$ -tensor field of curvature, the scalar curvature, $R(g(t))$ evolves by

$$\partial_t R = -\Delta H + \nabla^p \nabla^q h_{pq} - \langle Rc, h \rangle, \text{ where } H \doteq \text{tr} h.$$

6. The Riemannian volume form $dV_{g(t)}$ evolves by

$$\partial_t dV = \frac{1}{2} H dV.$$

7. The evolution of the total scalar curvature $\int R dV$ is given by,

$$\partial_t \int R dV = \int \left[-\langle Rc, h \rangle + \frac{1}{2} R H \right] dV.$$

In the next lemma, we use all these evolutions.

Lemma 3.1. [*Evolution equations under the Ricci flow*]
Let $\mathbb{M}$ be a differentiable manifold and let $g(t)$ be a Ricci flow,

$$\frac{\partial}{\partial t} g = -2Rc,$$

and fix some system of coordinates $(U, (x^i))$. Then, the following equations holds on $U \subset \mathbb{M}$ for as long as $g(t)$ exists.

1. The evolution of the inverse metric g^{-1} is given by

$$\frac{\partial}{\partial t} g^{jk} = 2R^{jk}.$$

2. The Riemannian connection $\Gamma(g(t))$ evolves by

$$\frac{\partial}{\partial t} \Gamma_{ij}^k = g^{kl} \{ -\nabla_i R_{jl} - \nabla_j R_{il} + \nabla_l R_{ij} \}.$$

3. The $\binom{3}{1}$ -tensor field of curvature $\mathcal{R}(g(t))$ evolves by

$$\frac{\partial}{\partial t} R_{ijk}{}^l = R_{ijk}{}^p R_p{}^l + R_{ijp}{}^l R_k{}^p + \left\{ -\nabla_i \nabla_k R_j{}^l + \nabla_i \nabla^l R_{jk} + \nabla_j \nabla_k R_i{}^l - \nabla_j \nabla^l R_{ik} \right\}.$$

4. The symmetric $\binom{2}{0}$ -tensor field $Rc(g(t))$ evolves by

$$\frac{\partial}{\partial t} R_{jk} = \Delta R_{jk} - 2R_j^p R_{kp} + 2R_{pjkq} R^{pq}.$$

5. The $\binom{0}{0}$ -tensor field of curvature, scalar curvature, $R(g(t))$ evolves by

$$\partial_t R = \Delta R + 2|Rc|^2.$$

6. The Riemannian volume form $dV_{g(t)}$ evolves by

$$\partial_t dV = -R dV.$$

7. The evolution of the total scalar curvature $\int R dV$ is given by,

$$\partial_t \int R dV = \int [2|Rc|^2 - R^2] dV$$

Proof.

Most of the proofs are straightforward. For example, the evolution of the inverse metric is

$$\partial_t g^{jk} = -g^{ji} g^{kl} \partial_t g_{il} = 2g^{ji} R_i^k = 2R^{jk}.$$

The evolution of the $\binom{3}{1}$ -tensor field of curvature requires only to substitute the tensor field $h = (\partial_t g)$ by $-2Rc$ and to apply the commutator formula for a $\binom{2}{0}$ -tensor field (see Lemma A.6). We have,

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijk}{}^l &= g^{lp} \left\{ \nabla_j \nabla_i - \nabla_i \nabla_j \right\} R_{kp} + g^{lp} \left\{ -\nabla_i \nabla_k R_{jp} + \nabla_i \nabla_p R_{jk} + \nabla_j \nabla_k R_{ip} - \nabla_j \nabla_p R_{ik} \right\} \\ &= R_{ijk}{}^p R_p^l + R_{ijp}{}^l R_k^p + \left\{ -\nabla_i \nabla_k R_j^l + \nabla_i \nabla^l R_{jk} + \nabla_j \nabla_k R_i^l - \nabla_j \nabla^l R_{ik} \right\}. \end{aligned}$$

For a not so straightforward example, take the evolution of the scalar curvature. Recall first the contracted Bianchi identity (see Lemma 1.50):

$$g^{jk} \nabla_j R_{kl} = \frac{1}{2} \nabla_l R.$$

Applying this identity to the evolution of the curvature, we get

$$\begin{aligned} \partial_t R &= 2\Delta R - 2\nabla^p \{ \nabla^q R_{pq} \} - \langle Rc, -2Rc \rangle \\ &= \Delta R + 2|Rc|^2 \end{aligned} \tag{3.1}$$

Now we compute the evolution of the Ricci tensor, by applying the same contracted Bianchi identity and the commutator formulas for $\binom{2}{0}$ -tensor field (see Lemma A.6). We have

$$\begin{aligned} \frac{\partial}{\partial t} R_{jk} &= \Delta R_{jk} - g^{pq} \{ \nabla_q \nabla_j - \nabla_j \nabla_q \} R_{pk} - g^{pq} \{ \nabla_q \nabla_k - \nabla_k \nabla_q \} R_{pj} \\ &\quad - \nabla_j g^{pq} \nabla_q R_{pk} - \nabla_k g^{pq} \nabla_q R_{pj} + \nabla_k \nabla_j R \\ &= \Delta R_{jk} - 2R_j^p R_{kp} + 2R_{pjkq} R^{pq}. \end{aligned}$$

3.2 Maximum principles for scalar equations

We will initially discuss some relevant results concerning maximum principles which have a fundamental role in the study of the Ricci flow. For our purposes, there are only two main results worth stating.

In this section we will consider a closed manifold $\mathbb{M}$, by which we mean that $\mathbb{M}$ is compact and has empty boundary, $\partial\mathbb{M} = \emptyset$. In general, we will consider an arbitrary differentiable one-parameter family of Riemannian metrics $g : [0, T) \rightarrow \Gamma(S_2^+ T^* \mathbb{M})$, and the heat equation

$$\frac{\partial}{\partial t} u = \Delta_{g(t)} u.$$

Lemma 3.2. [*Scalar maximum principle, first version*]

Let $u : \mathbb{M} \times [0, T) \rightarrow \mathbb{R}$ be a differentiable function on a closed manifold $\mathbb{M}$ satisfying

$$\frac{\partial}{\partial t} u \geq \Delta_{g(t)} u, \quad \text{on } \mathbb{M} \times [0, T).$$

If $u \geq c$, at $t=0$, for $c \in \mathbb{R}$, then $u \geq c$, for all times $t \in [0, T)$.

Proof.

Let $\epsilon > 0$ and define

$$u_\epsilon : \mathbb{M} \times [0, T) \rightarrow \mathbb{R}, \quad \text{by } u_\epsilon(x, t) = u(x, t) + \epsilon(1 + t).$$

It is clear that $u_\epsilon(x, 0) > c$, for all $x \in \mathbb{M}$, by hypothesis. Suppose, by contradiction, that for some point $(x_0, t_0) \in \mathbb{M} \times [0, T)$ we have $u_\epsilon(x_0, t_0) \leq c$. Then, consider the set

$$\mathcal{A} \doteq \{t \in [0, T) : u_\epsilon(x, t) \leq c, \text{ for some } x \in \mathbb{M}\}.$$

$\mathcal{A}$ is bounded below by 0. Take $t_1 \doteq \inf \mathcal{A}$. Therefore, there exists some point $(x_1, t_1) \in \mathbb{M} \times [0, T)$, such that

$$u_\epsilon(x_1, t_1) = c \quad \text{and} \quad u_\epsilon(x, t) > c, \quad \text{for all } t \geq 0, \text{ satisfying } t < t_1.$$

It is then clear that $u_\epsilon(x_1, t_1)$ is the minimum of the function

$$u_\epsilon(\cdot, t_1) : \mathbb{M} \rightarrow \mathbb{R}, \quad x \in \mathbb{M} \mapsto u_\epsilon(x, t_1).$$

Hence, $u(x_1, t_1) \leq u(x, t_1)$, for all $x \in \mathbb{M}$. Therefore, we have $\nabla u(x_1, t_1) = 0$ and also $\Delta u(x_1, t_1) \geq 0$, which is essentially the second derivative test in calculus. Finally,

$$0 \geq \frac{\partial u_\epsilon}{\partial t}(x_1, t_1) = \frac{\partial}{\partial t} (u(x, t) + \epsilon(1 + t))(x_1, t_1) \geq \Delta u(x_1, t_1) + \epsilon > 0,$$

which is an absurd. Thus, we conclude that $u_\epsilon > c$ at all points on $\mathbb{M} \times [0, T)$ and

$$u(x, t) > c - \epsilon(1 + t), \quad \text{for all } x \in \mathbb{M} \text{ and } t \in [0, T).$$

The theorem follows once ϵ was arbitrary. ♣

We will now consider a slightly different equation in which we consider a gradient term together with an arbitrary differentiable family of vector fields defined in the same time interval on which the family of

metrics is defined:

$$\frac{\partial}{\partial t}v = \Delta v + \langle X(t), \text{grad}v \rangle \quad (3.2)$$

We say that a differentiable function $u : \mathbb{M} \times [0, T) \rightarrow \mathbb{R}$ is a supersolution to equation (3.2) at $(x, t) \in \mathbb{M} \times [0, T)$ if

$$\frac{\partial u}{\partial t}(x, t) \geq \Delta u(x, t) + \langle X, \text{grad}u \rangle(x, t)$$

and we say that it is a subsolution to equation (3.2) if

$$\frac{\partial u}{\partial t}(x, t) \leq \Delta u(x, t) + \langle X, \text{grad}u \rangle(x, t)$$

In the next theorem, we show that subsolutions of (3.2) preserve upper bounds. The proof that follows is eminently similar to the proof of the first version, so the fast reader may skip it.

Theorem 3.3. [*Scalar maximum principle, second version*]

Let $g(t)$ and $X(t)$ be a family of metrics and of vector fields, respectively, defined on the same time interval $[0, T)$, on a closed manifold $\mathbb{M}$. Let $u : \mathbb{M} \times [0, T) \rightarrow \mathbb{R}$ be a differentiable function such that u is a subsolution for the heat equation (3.2) at all points (x, t) such that $u(x, t) > c$, for some $c \in \mathbb{R}$. Thus, if $u(x, 0) \leq c$, for all $x \in \mathbb{M}$, then $u(x, t) \leq c$ for all $x \in \mathbb{M}$ and $t \in [0, T)$.

Proof.

Let $\epsilon > 0$ be any positive real number and define the function

$$h : \mathbb{M} \times [0, T) \rightarrow \mathbb{R}, \quad \text{by } h = [u - c] - \epsilon t - \epsilon.$$

By hypothesis, $h(x, 0) < 0$. We are going to prove that $h(x, t) < 0$, for all $x \in \mathbb{M}$ and all $t \in [0, T)$. Suppose, by contradiction, that for some (x_0, t_0) , we have $h(x_0, t_0) \geq 0$. Then, there is a first moment t_1 in which h annihilates for some $x_1 \in \mathbb{M}$, that is, there exists $t_1 \in [0, T)$ such that

$$h(x_1, t_1) = 0, \quad \text{and } 0 \leq t < t_1 \implies h(x, t) < 0, \quad \text{for all } x \in \mathbb{M}$$

This is true since $h(x, 0) < 0$ and because $\mathbb{M}$ is compact. Certainly, x_1 is a local maximum of

$$h(\cdot, t_1) : \mathbb{M} \times [0, T) \rightarrow \mathbb{R},$$

which yields $\nabla h(x_1, t_1) = 0$ and $\Delta h(x_1, t_1) \leq 0$. Furthermore, at (x_1, t_1)

$$0 \leq \frac{\partial}{\partial t}h = \frac{\partial}{\partial t}u - \epsilon \leq \Delta u - \epsilon = \Delta h - \epsilon < 0,$$

which is a contradiction. Then, $h(x, t) < 0$ at all points and the result follows. ♣

Now, let $X = X(t)$ be a differentiable one parameter family of differentiable vector fields defined on the same time interval as $g(t)$, and consider the semilinear heat equation

$$\frac{\partial}{\partial t}v = \Delta_{g(t)}v + \langle X(t), \text{grad}v \rangle + F(v) \quad (3.3)$$

where $F : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function, and $\Delta_{g(t)}$ is the laplacian varying with time, which we will denote simply by Δ . We will say that a function

$$u : \mathbb{M} \times [0, T) \rightarrow \mathbb{R}$$

is a supersolution for the semilinear equation (3.3), if

$$\frac{\partial}{\partial t}u \geq \Delta_{g(t)}u + \langle X(t), \text{grad}u \rangle + F(u),$$

and a subsolution if

$$\frac{\partial}{\partial t} u \leq \Delta_{g(t)} u + \langle X(t), \text{gradu} \rangle + F(u).$$

Theorem 3.4. [Scalar maximum principle, third version]

Let $\mathbb{M}$ be a closed manifold and $g(t)$ be a differentiable family of metrics and suppose $u : \mathbb{M} \times [0, T) \rightarrow \mathbb{R}$ is a differentiable function which is also a supersolution to equation (3.3). Suppose there exists a constant $c_1 \in \mathbb{R}$ such that

$$u(x, 0) \geq c_1, \text{ for all } x \in \mathbb{M}.$$

Furthermore, suppose that ζ_1 is the solution to the associated ODE

$$\frac{d}{dt} \zeta_1 = F(\zeta_1), \text{ satisfying } \zeta_1(0) = c_1.$$

Then, $u(x, t) \geq \zeta_1(t)$ for all $x \in \mathbb{M}$ and for all $t \in [0, T)$ such that $\zeta_1(t)$ exists.

Analogously, suppose $u : \mathbb{M} \times [0, T) \rightarrow \mathbb{R}$ is a differentiable subsolution to equation (3.3), and suppose $u(\cdot, 0) \leq c_2$ for some constant c_2 . Suppose, furthermore, that ζ_2 is a solution to the associated ODE

$$\frac{d}{dt} \zeta_2 = F(\zeta_2), \text{ satisfying } \zeta_2(0) = c_2.$$

Then, $u(x, t) \leq \zeta_2(t)$ for all $x \in \mathbb{M}$ and for all $t \in [0, T)$ such that $\zeta_2(t)$ exists.

Remark 3.5. The associated ODE from the PDE (3.3) is thought of as just neglecting the terms involving direction derivatives, namely the gradient and the Laplacian terms.

Proof.

Let us prove the case of subsolutions. We claim that $(u(x, t) - \zeta_2(t)) \leq 0$, at all points $x \in \mathbb{M}$ and times $t \in [0, T)$. Fix any $\tau \in [0, T)$ and let's prove the claim on the compact set $\mathbb{M} \times [0, \tau]$. For simplicity, set $\zeta_2 = \zeta$. We have,

$$\frac{\partial}{\partial t} (u - \zeta) \leq \Delta u + \langle X, \nabla u \rangle + F(u) - F(\zeta) = \Delta(u - \zeta) + \langle X, \nabla(u - \zeta) \rangle + F(u) - F(\zeta).$$

Now, since $\mathbb{M}$ is a compact space, there is a positive constant $K_\tau \in \mathbb{R}$ such that $|u|$ and $|\zeta|$ are bounded above K_τ on $\mathbb{M} \times [0, \tau]$, which means that both $u(x, t)$ and $\zeta(t)$ lies on the set $[-K_\tau, K_\tau]$, for $x \in \mathbb{M}$, and $t \in [0, \tau]$. Since F is locally Lipschitz, there is another constant L_τ which satisfies

$$|F \circ u(x, t) - F \circ \zeta(t)| \leq L_\tau |u(x, t) - \zeta(t)|, \text{ for all } x \in \mathbb{M}, \text{ and } t \in [0, \tau].$$

Therefore,

$$\frac{\partial}{\partial t} (u - \zeta) \leq \Delta(u - \zeta) + \langle X, \nabla(u - \zeta) \rangle + L_\tau |u - \zeta|.$$

Consider then the quantity $v = e^{-L_\tau t} (u - \zeta)$ and take its partial derivative with respect to time to get

$$\begin{aligned} \frac{\partial}{\partial t} v &\leq -L_\tau e^{-L_\tau t} (u - \zeta) + e^{-L_\tau t} \left\{ \Delta(u - \zeta) + \langle X, \nabla(u - \zeta) \rangle + L_\tau |u - \zeta| \right\} \\ &= \Delta v + \langle X, \nabla v \rangle + L_\tau (|v| - v) \end{aligned}$$

Therefore, for all $(x, t) \in \mathbb{M} \times [0, \tau]$ such that $v(x, t) \geq 0$, we have

$$\frac{\partial}{\partial t} v \leq \Delta v + \langle X, \nabla v \rangle.$$

Then, since $u(x, 0) - \zeta(0) \leq 0$, for all $x \in \mathbb{M}$, we have $v(x, 0) \leq 0$, for all $x \in \mathbb{M}$. Then we may apply the previous theorem to get $v(x, t) \leq 0$ on $\mathbb{M} \times [0, \tau]$. ♣

3.3 Long-time existence of the flow

In this section we show that the only obstacle for the long-time existence of the flow is the $\binom{4}{0}$ -tensor field of curvature becoming unbounded. For this purpose, we need to assume the following important result.

Theorem 3.6. [*Metric and Ricci derivative estimates*]

Let $g(t)$ be a solution of the Ricci flow equation on a compact N -dimensional manifold $\mathbb{M}$. Suppose that the domain of definition of the flow is the half-open time interval $[0, T)$. Let $(U, (x^i))$ be any system of coordinates. If there exists $K > 0$ such that

$$|Rm(x, t)|_{g(t)} \leq K, \quad \text{on } \mathbb{M} \times [0, T),$$

then, for each $k \in \mathbb{N}$, there exist constants c_k and c'_k depending on the chosen coordinates, k, N, K, T and $g(0)$ such that

$$|\partial^k g(x, t)| \leq c_k \quad \text{and} \quad |\partial^k Rc(x, t)| \leq c'_k$$

holds for all $(x, t) \in U \times [0, T)$, where the norms are taken with respect to the Euclidean metric in the coordinates (x^i) .

Remark 3.7. This theorem can be read in the following manner. Given an arbitrary Ricci flow, if the curvature is uniformly bounded along the flow then any derivative of the metric and of the Ricci tensor become uniformly bounded with respect to a fixed metric and corresponding connection.

Remark 3.8. Let us explain the nomenclature used in the theorem. Recall that by Euclidean metric we mean the metric g defined on U whose components in the chosen system of coordinates $(U, (x^i))$ are $g_{ij} = \delta_{ij}$. It is just the pull-back of the standard metric in $\mathbb{R}^N$ via the chart. By $\partial^k g$ we mean the $\binom{k+2}{0}$ -tensor field also defined only on U , whose components in the coordinates (x^i) are

$$\partial^k g = \partial_{i_1} \dots \partial_{i_k} g_{jl} dx^{i_1} \otimes \dots \otimes dx^{i_k} \otimes dx^j \otimes dx^l.$$

For the proof of the theorem and a more detailed exposition of these remarks we refer [CKKS04, Chapter 6].

Let us now prove an independent result which is interesting in its own. Recall that given any symmetric $\binom{2}{0}$ -tensor field k on a differentiable manifold $\mathbb{M}$, we write $k \geq 0$ to indicate that $k|_p(x, x) \geq 0$ for all $p \in M$ and every $x \in T_p \mathbb{M}$. Similarly, we write $k \geq \nu$ for symmetric covariant 2-tensor fields if $k - \nu \geq 0$.

Lemma 3.9. *[Equivalence of metrics through integral estimates]*

Suppose that, for $0 \leq t < T \leq \infty$, $g(t)$ is any differentiable one parameter family of differentiable Riemannian metrics on a closed N -dimensional manifold M . If there exists $c \in \mathbb{R}$ such that

$$\int_0^T \left| \frac{\partial}{\partial t} g[x, t] \right|_{g(t)} dt \leq c, \quad \text{for all } x \in \mathbb{M}$$

then

$$e^{-c} g[x, 0] \leq g[x, t] \leq e^c g[x, 0], \quad \text{for all } x \in \mathbb{M} \text{ and } t \in [0, T).$$

Additionally, as $t \rightarrow T^-$ the metrics $g(t)$ converge uniformly to a continuous metric $g(T)$ such that

$$e^{-c} g[x, 0] \leq g[x, T] \leq e^c g[x, 0], \quad \text{for all } x \in \mathbb{M}.$$

Remark. As an immediate consequence of Lemma 3.9, we also have

$$\int_0^T \left| \frac{\partial}{\partial t} g[x, t] \right|_{g(0)} dt \leq c', \quad \text{for all } x \in \mathbb{M} \text{ and some constant } c'.$$

In fact, fixing any $x \in \mathbb{M}$, given $u, v \in T_x \mathbb{M}$, with $|u|_0, |v|_0 = 1$, we have

$$|u|_t \leq e^{\frac{c}{2}} |u|_0 \quad \text{implies} \quad \left| \frac{\partial}{\partial t} g[x, t](u, v) \right| \leq \left| e^c \frac{\partial}{\partial t} g[x, t] \right|_{g(t)},$$

where $|u|_s$ means the norm of u with respect to $g[x, s]$. Therefore,

$$\int_0^T \left| \frac{\partial}{\partial t} g[x, t] \right|_{g(0)} dt \leq e^c \int_0^T \left| \frac{\partial}{\partial t} g[x, t] \right|_{g(t)} dt \leq ce^c,$$

and so the assertion follows with $c' = ce^c$.

Proof.. *[of Lemma 3.9]*

Choose $x \in \mathbb{M}$, $t_0 \in [0, T)$ and $0 \neq w \in T_x \mathbb{M}$ arbitrarily. Then,

$$\begin{aligned} \left| \log \left(\frac{g[x, t_0](w, w)}{g[x, 0](w, w)} \right) \right| &= \left| \int_0^{t_0} \frac{\partial}{\partial t} [\log g[x, t](w, w)] dt \right| \\ &= \left| \int_0^{t_0} \frac{\frac{\partial}{\partial t} g[x, t](w, w)}{g[x, t](w, w)} dt \right| \\ &\leq \int_0^{t_0} \left| \frac{\partial}{\partial t} g[x, t] \left(\frac{w}{|w|_t}, \frac{w}{|w|_t} \right) \right| dt \\ &\leq \int_0^{t_0} \left| \frac{\partial}{\partial t} g[x, t] \right|_{g(t)} dt \\ &\leq c \end{aligned}$$

exponentiating this inequality gives us

$$e^{-c} g[x, 0](w, w) \leq g[x, t_0](w, w) \leq e^c g[x, 0](w, w), \quad \text{for all } w \in T_x \mathbb{M},$$

and hence, all metrics are uniformly equivalent at all points $x \in \mathbb{M}$:

$$e^{-c} g[x, 0] \leq g[x, t_0] \leq e^c g[x, 0], \quad t_0 \in [0, T), \quad \text{for all } x \in \mathbb{M}$$

Now, define

$$g[x, T] = g[x, 0] + \int_0^T \frac{\partial}{\partial t} g[x, t] dt$$

The RHS is well defined since the integrand is absolutely integrable with respect to the norm induced by $g[x, s]$, for any fixed $s \in [0, T]$, by exactly the same argument from the remark preceding the proof.

We have,

$$|g[x, T] - g[x, t]|_{g(0)} \leq \int_t^T \left| \frac{\partial}{\partial t} g[x, \xi] \right|_{g(0)} d\xi \quad (3.4)$$

and the RHS of equation above goes to zero as t approaches T . Indeed, let $F : \mathbb{M} \times [0, T] \rightarrow \mathbb{R}$ be given by $F(x, t) = \int_t^T \left| \frac{\partial}{\partial t} g[x, \xi] \right|_{g(0)} d\xi$. Fix any $x_0 \in \mathbb{M}$. Then, the function $t \in [0, T] \mapsto F(x_0, t)$ is bounded above by $c' < \infty$ and below by 0. Moreover, $F(x_0, \cdot)$ is monotone nonincreasing and therefore the limit $\lim_{t \rightarrow T^-} F(x_0, t)$ exists and is equal to zero. Now, let us show that $F(x, t)$ converges uniformly on $\mathbb{M}$ to zero. Let $\epsilon > 0$. For each point $x_0 \in \mathbb{M}$ there exists some $t_{x_0} = t_{x_0}(\epsilon)$ such that $F(x_0, t) < \epsilon$, if $t \geq t_{x_0}$. Now, the function $x \in \mathbb{M} \mapsto F(x, t_{x_0})$ is continuous, whence the preimage of open sets by $F(\cdot, t_{x_0})$ are open, so that

$$[F(\cdot, t_{x_0})]^{-1}((-\epsilon, \epsilon)) = \{x \in \mathbb{M} : -\epsilon < F(x, t_{x_0}) < \epsilon\},$$

is an open neighborhood of $x_0 \in \mathbb{M}$. Then, we have an open cover for $\mathbb{M}$:

$$\mathbb{M} = \bigcup_{x \in \mathbb{M}} [F(\cdot, t_x)]^{-1}((-\epsilon, \epsilon)).$$

Since $\mathbb{M}$ is compact, there exists a finite number of points $x_1, \dots, x_K \in \mathbb{M}$ such that $\mathbb{M} = [F(\cdot, t_{x_1})]^{-1}((-\epsilon, \epsilon)) \cup \dots \cup [F(\cdot, t_{x_K})]^{-1}((-\epsilon, \epsilon))$, with $x_k \in [F(\cdot, t_{x_k})]^{-1}((-\epsilon, \epsilon))$ for each $k = 1, \dots, K$. Now, take $t_S \doteq \max_{1 \leq k \leq K} t_{x_k}$, and observe that

$$\mathbb{M} = [F(\cdot, t_S)]^{-1}((-\epsilon, \epsilon)).$$

We have shown that, for each $\epsilon > 0$ there exists a time $t_S < T$ such that $F(x, t_S) < \epsilon$ independently of $x \in \mathbb{M}$, hence $F(x, t)$ converges uniformly on $\mathbb{M}$ to zero. From equation (3.4), we have

$$|g[x, T] - g[x, t]|_{g(0)} \leq \int_t^T \left| \frac{\partial}{\partial t} g[x, \xi] \right|_{g(0)} d\xi = F(x, t) \rightarrow 0,$$

so that $g(t)$ converges to $g(T)$ uniformly on $\mathbb{M}$, therefore $g(T)$ is continuous. Now, we proved

$$e^{-c} g_{[x, 0]} \leq g_{[x, t]} \leq e^c g_{[x, 0]}, \quad \text{holds for all } t \in [0, T],$$

and then taking the limit we have the desired formula, which proves that $g(T)$ is positive definite and hence a continuous metric. Note that the symmetry follows from the definition of $g(T)$. ♣

Corollary 3.10. *[Equivalence of metrics under Ricci flow]*

Let $g(t)$ be a solution of the Ricci flow equation on a compact N -dimensional manifold $\mathbb{M}$. Suppose that the domain of definition of the flow is the half-open finite time interval $[0, T)$. If the Ricci tensor is bounded by some constant K , that is, if

$$|Rc(x, t)|_{g(t)} \leq K, \quad \text{holds for all } (x, t) \in \mathbb{M} \times [0, T),$$

then the metris $g(t)$, $0 \leq t < T$, are uniformly equivalent and they converge to a continuous metric $g(T)$ as $t \rightarrow T^-$.

Proof.

According to the Ricci flow equation, $\left| \frac{\partial}{\partial t} g[x, t] \right|_{g(t)}$ is uniformly bounded on $\mathbb{M} \times [0, T)$, hence its integral on $[0, T)$ is as well and therefore Lemma 3.9 applies. In particular,

$$e^{-2KT} g[x, 0] \leq g[x, t] \leq e^{2KT} g[x, 0], \quad \text{holds for all } (x, t) \in \mathbb{M} \times [0, T]. \clubsuit$$

Theorem 3.11. [Differentiability of the limit metric under Ricci flow]

Let $g(t)$ be a solution of the Ricci flow on a compact N -dimensional manifold $\mathbb{M}$. Suppose that the domain of definition of the flow is the half-open finite time interval $[0, T)$. If the $\binom{4}{0}$ -tensor field of curvature Rm is uniformly bounded by some constant K , that is, if

$$|Rm(x, t)|_{g(t)} \leq K, \quad \text{holds for all } (x, t) \in \mathbb{M} \times [0, T),$$

then the metris $g(t)$ are uniformly equivalent and converge to a differentiable (C^∞) Riemannian metric $g(T)$ as $t \rightarrow T^-$.

Proof.

Notice that any uniform bound on Rm implies a uniform bound on the Ricci tensor. Therefore, this family of metrics is uniformly equivalent and converge to a continuous metric $g(T)$ which we now show to be differentiable (C^∞). In order to do this, we should take partial derivatives of the components of $g(T)$ with respect to some system of coordinates which we choose arbitrarily. Let $(U, (x^i))$ be any local coordinates. From the Ricci flow equation, we get

$$g_{jk}[x, T] = g_{jk}[x, t] - 2 \int_t^T R_{jk}[x, \xi] d\xi, \quad \text{for all } t \in [0, T).$$

If we show that the RHS of the equation above is differentiable then we are done. But, in order to do this, we must differentiate under the integral sign. For, given any multi-index $\alpha \in \mathbb{N}^N$, with $|\alpha| = k$, from Theorem 3.6, $\partial^{|\alpha|} R_{jk}$ is uniformly bounded on $U \times [0, T)$, and therefore we may apply differentiation under integral sign to get

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} \left\{ g_{jk}[x, t] - 2 \int_t^T R_{jk}[x, \xi] d\xi \right\} = \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{jk} \right)[x, t] - 2 \int_t^T \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} R_{jk} \right)[x, \xi] d\xi$$

This shows that the function $g_{jk}(T)$ is differentiable on U and else: the formula

$$\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{jk} \right)[x, T] = \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{jk} \right)[x, t] - 2 \int_t^T \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} R_{jk} \right)[x, \xi] d\xi,$$

holds at every $x \in U$, for any multi-index α and system of coordinates. Thus, $g(T)$ is differentiable.

Now we show that $g(t)$ converges uniformly to $g(T)$ in any C^k norm as t approaches T , in the following sense: since M is compact, we cover it with a finite number of domains of local coordinates such that given any multi-index α , with $|\alpha| = k$, and any $\epsilon > 0$ there exists $\delta > 0$ satisfying

$$\left| \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{jk} \right)[x, t] - \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{jk} \right)[x, T] \right|_{g(0)} < \epsilon, \quad \text{for all } T - \delta \leq t < T, \quad \text{and } x \in \mathbb{M}. \quad (3.5)$$

Notice that we could replace $g(0)$ by any metric $g(s)$ from the family because they are uniformly equivalent. Actually, we could employ theorem A.17 and replace the metrics from the family by any Riemannian metric, since $\mathbb{M}$ is compact.

Let α be any multi-index. We then cover $\mathbb{M}$ with a finite number of such coordinate domains which we choose to be the image, under the corresponding parametrization, of the closed unit balls centered at the origin in $\mathbb{R}^N$. On each of these domains, which are compact, we choose the Euclidean metric. Because

these coordinate domains are compact and there are only a finite number of them, these corresponding metrics are uniformly equivalent to $g(0)$. (In fact, they are uniformly equivalent to all $g(s)$). Thus, it is enough to show the convergence 3.5 with respect to the norm induced by the Euclidean metric in each corresponding coordinate domain. According to theorem 3.6 and the Remarks thereafter, for each given $k \in \mathbb{N}$ and fixed coordinate domain U there exists c'_k such that $|\partial^k Rc| \leq c'_k$ holds on U . Since there are only a finite number of these coordinate domains, we choose c_k to be the largest of them, i.e., c_k satisfies $|\partial^k Rc| \leq c_k$, for every domain chosen. Then,

$$\begin{aligned} \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{jk}[x, t] - \frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{jk}[x, T] \right| &= \left| 2 \int_t^T \frac{\partial^{|\alpha|}}{\partial x^\alpha} R_{jk}[x, \xi] d\xi \right| \\ &\leq 2 \int_t^T \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} R_{jk}[x, \xi] \right| d\xi \\ &\leq 2 \int_t^T |\partial^k Rc[x, \xi]| d\xi \\ &\leq 2c_k(T - t). \end{aligned}$$

Thus, $g(t)$ converges uniformly on M to $g(T)$ in any C^k norm. ♣

Theorem 3.12. *Let g_0 be a differentiable Riemannian metric on a differentiable compact manifold $\mathbb{M}$. The unique Ricci flow with $g(0) = g_0$ has a solution defined on a maximal time interval $[0, T)$, where $T \leq \infty$. Furthermore, if $T < \infty$, then*

$$\lim_{t \rightarrow T^-} \left(\sup_{x \in \mathbb{M}} |Rm(x, t)| \right) = \infty.$$

Proof.

Let us suppose by contradiction that $T < \infty$ and that the $\binom{4}{0}$ -tensor field of curvature Rm is bounded above by some constant $K > 0$. It follows then by the previous theorem, theorem 3.11, that $g(t)$ converges uniformly to a differentiable metric $g(T)$ in every C^k norm. Then, consider the differentiable manifold $\mathbb{M}$ together with the differentiable Riemannian metric $g(T)$. According to theorem 2.30, there exists $\epsilon > 0$ and a one parameter family of metrics $\bar{g} : [0, \epsilon) \rightarrow \Gamma(S_2^+ T^* \mathbb{M})$ which is the unique solution of the Ricci flow equation starting at $g(T)$, that is,

$$\begin{cases} \frac{\partial}{\partial t} \bar{g} = -2Rc(\bar{g}), \\ \bar{g}(0) = g(T), \end{cases}$$

Now we try to "glue" the solutions obtained to get another solution defined past T . The only difficulty that arises is to prove that the solution obtained is differentiable (C^∞) in time at $t = T$. Consider then the new family of metrics defined on the interval $[0, T + \epsilon)$ by

$$\tilde{g}(t) = \begin{cases} g(t), & \text{if } t \in [0, T) \\ \bar{g}(t - T), & \text{if } t \in [T, T + \epsilon). \end{cases}$$

Then, $\tilde{g}(0) = g_0$ and if $\tilde{g}$ is differentiable at $t = T$ then it is an extension of the initial maximal solution, since it is a solution to the Ricci flow equation on each interval separately. To show that $\tilde{g}$ is differentiable at $t = T$ we start observing that it is continuous on $[0, T + \epsilon)$ since $g(t)$ converges uniformly to $g(T) = \bar{g}(0)$. Also, $\tilde{g}$ is obviously differentiable on $(0, T) \cup (T, T + \epsilon)$ so we show it is of class C^1 at $t = T$, just comparing lateral limits and using continuity of the Ricci tensor. We have,

$$\begin{aligned} \lim_{t \rightarrow T^-} \frac{\partial \tilde{g}}{\partial t}(t) &= \lim_{t \rightarrow T^-} \frac{\partial g}{\partial t}(t) = \lim_{t \rightarrow T^-} Rc(g(t)) = Rc(g(T)) \\ &= Rc(\bar{g}(0)) = \lim_{t \rightarrow T^+} Rc(\bar{g}(t - T)) = \lim_{t \rightarrow T^+} \frac{\partial \bar{g}}{\partial t}(t - T) = \lim_{t \rightarrow T^+} \frac{\partial \tilde{g}}{\partial t}(t) \end{aligned}$$

This shows that $\tilde{g}$ is of class C^1 at $t = T$ and also

$$\lim_{t \rightarrow T} \frac{\partial \tilde{g}}{\partial t}(t) = Rc(\tilde{g}(T))$$

Now, using the chain rule it becomes clear that $\tilde{g}$ is also of class C^2 at $t = T$, because

$$\begin{aligned} \lim_{t \rightarrow T^-} \frac{\partial^2 \tilde{g}}{\partial t^2}(t) &= \lim_{t \rightarrow T^-} \frac{\partial}{\partial t} \{Rc(\tilde{g}(t))\} = \lim_{t \rightarrow T^-} DRc(\tilde{g}(t)) \left(\frac{\partial \tilde{g}}{\partial t}(t) \right) \\ &= DRc(\tilde{g}(T)) \left(\frac{\partial \tilde{g}}{\partial t}(T) \right) = \lim_{t \rightarrow T^+} DRc(\tilde{g}(t)) \left(\frac{\partial \tilde{g}}{\partial t}(t) \right) = \lim_{t \rightarrow T^+} \frac{\partial^2 \tilde{g}}{\partial t^2}(t) \end{aligned}$$

Thus, one concludes that $\tilde{g}$ is differentiable at $t = T$ by recursive application of the chain rule and observing that the Ricci flow equation allows us to write time derivatives of the metric in terms of quantities related to the metric and their derivatives which are supposed to be continuous. Therefore, the solution $\tilde{g}$ is an extension past $t = T$ of the initial solution which is a contradiction. So, the initial assumption that the norm of Rm must be uniformly bounded must be incorrect. This completes the proof of the result. ♣

3.4 Further Evolution Equations

Following the idea of the beginning of this chapter, we would like to put some evolution equations in a more convenient form to apply maximum principles and therefore deduce the behavior of the scalar curvature along the Ricci flow. We are going to evolve components of tensors in order to show that they follow a heat-type equation.

We begin by establishing the evolution equation for the $\binom{3}{1}$ -tensor field of curvature.

Theorem 3.13. [*Evolution of the Riemann curvature endomorphism*]
The components of the $\binom{3}{1}$ -tensor field of curvature evolve by

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijk}{}^l &= \Delta R_{ijk}{}^l + g^{pq} \left\{ R_{ijp}{}^m R_{mqk}{}^l - 2R_{pik}{}^m R_{jqm}{}^l + 2R_{pim}{}^l R_{jqk}{}^m \right\} \\ &\quad - R_i^m R_{mjk}{}^l - R_j^m R_{imk}{}^l - R_k^m R_{ijm}{}^l + R_m^l R_{ijk}{}^m \end{aligned}$$

Proof.

First, observe that the differential Bianchi identity (see Lemma 1.47, item *d*) implies

$$g^{ls} \nabla_q R_{ijks} = -\nabla_i R_{jqk}{}^l - \nabla_j R_{qik}{}^l,$$

then recall the commutator formula for the $\binom{3}{1}$ -tensor field of curvature (see equation A.6):

$$\{\nabla_p \nabla_i - \nabla_i \nabla_p\} R_{jqk}{}^l = -R_{pij}{}^m R_{mqk}{}^l - R_{piq}{}^m R_{jmk}{}^l - R_{pik}{}^m R_{jqm}{}^l + R_{pim}{}^l R_{jqk}{}^m.$$

Using again the second Bianchi identity in the form

$$-g^{pq}\nabla_p R_{jqk}{}^l = -g^{pq}g^{lm}\nabla_p R_{kmjq} = \nabla^l R_{kj} - \nabla_k R_j^l,$$

we now systematically apply the previous three formulas to get a suitable equation for the Laplacian applied to each component of $\mathcal{R}$ in a system of coordinates:

$$\begin{aligned} \Delta R_{ijk}{}^l &= g^{pq}g^{ls}\nabla_p\nabla_q R_{ijks} \\ &= g^{pq}\left\{-\nabla_p\nabla_i R_{jqk}{}^l - \nabla_p\nabla_j R_{qik}{}^l\right\} \\ &= -g^{pq}\left\{\nabla_p\nabla_i - \nabla_i\nabla_p\right\}R_{jqk}{}^l + g^{pq}\left\{\nabla_p\nabla_j - \nabla_j\nabla_p\right\}R_{iqk}{}^l \\ &\quad - g^{pq}\nabla_i\nabla_p R_{jqk}{}^l + g^{pq}\nabla_j\nabla_p R_{iqk}{}^l \\ &= -g^{pq}\left\{-R_{pij}{}^m R_{mqk}{}^l - R_{piq}{}^m R_{jmk}{}^l - R_{pik}{}^m R_{jqm}{}^l + R_{pim}{}^l R_{jqk}{}^m\right\} \\ &\quad + g^{pq}\left\{-R_{pji}{}^m R_{mqk}{}^l - R_{pjq}{}^m R_{imk}{}^l - R_{pjk}{}^m R_{iqm}{}^l + R_{pjm}{}^l R_{iqk}{}^m\right\} \\ &\quad + \nabla_i\left\{\nabla^l R_{kj} - \nabla_k R_j^l\right\} - \nabla_j\left\{\nabla^l R_{ki} - \nabla_k R_i^l\right\} \\ &= -g^{pq}\left\{R_{ijp}{}^m R_{mqk}{}^l - R_{pik}{}^m R_{jqm}{}^l + R_{pim}{}^l R_{jqk}{}^m + R_{pjk}{}^m R_{iqm}{}^l - R_{pjm}{}^l R_{iqk}{}^m\right\} \\ &\quad - R_i^m R_{jmk}{}^l + R_j^m R_{imk}{}^l + \left\{-\nabla_i\nabla_k R_j^l + \nabla_i\nabla^l R_{jk} + \nabla_j\nabla_k R_i^l - \nabla_j\nabla^l R_{ik}\right\} \\ &= -g^{pq}\left\{R_{ijp}{}^m R_{mqk}{}^l - 2R_{pik}{}^m R_{jqm}{}^l + 2R_{pim}{}^l R_{jqk}{}^m\right\} \\ &\quad - R_i^m R_{mjk}{}^l - R_j^m R_{imk}{}^l - \left\{R_m^l R_{ijk}{}^m + R_k^m R_{ijm}{}^l\right\} \\ &\quad + R_m^l R_{ijk}{}^m + R_k^m R_{ijm}{}^l + \left\{-\nabla_i\nabla_k R_j^l + \nabla_i\nabla^l R_{jk} + \nabla_j\nabla_k R_i^l - \nabla_j\nabla^l R_{ik}\right\} \end{aligned} \tag{3.6}$$

where the penultimate line follows from the simplest of the Bianchi identities, which implies

$$g^{pq}R_{pij}{}^m R_{mqk}{}^l - g^{pq}R_{pji}{}^m R_{mqk}{}^l = -g^{pq}R_{ijp}{}^m R_{mqk}{}^l,$$

and from rearranging the terms like, for example,

$$g^{pq}R_{piq}{}^m R_{jmk}{}^l = -g^{pq}R_{ipq}{}^m R_{jmk}{}^l = R_i^m R_{jmk}{}^l.$$

The last line follows by observing that it is possible to rearrange terms, by interchanging the roles of p and q . Then, recalling the evolution equation for this tensor under the Ricci flow

$$\frac{\partial}{\partial t}R_{ijk}{}^l = R_p^l R_{ijk}{}^p + R_k^p R_{ijp}{}^l + \left\{-\nabla_i\nabla_k R_j^l + \nabla_i\nabla^l R_{jk} + \nabla_j\nabla_k R_i^l - \nabla_j\nabla^l R_{ik}\right\},$$

the desired equation follows from observing that the last line of equation (3.6) is the RHS of the equation above. ♣

For the next evolution equation, recall that the $\binom{4}{0}$ -tensor field of (Riemann) curvature is given, in local coordinates $(U, (x^i))$, by $Rm = R_{ijkl}dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ and that the components of this tensor may be written as $R_{ijkl} = g_{pl}R_{ijk}{}^p$ according to the procedure of raising and lowering indices.

Theorem 3.14. [Evolution of the covariant 4-tensor field of curvature under Ricci flow]
The evolution of the $\binom{4}{0}$ -tensor field of curvature under the Ricci flow is given, in local coordinates, by

$$\begin{aligned} \frac{\partial}{\partial t}R_{ijkl} &= \Delta R_{ijkl} + g^{pq}\left\{R_{ijp}{}^m R_{mqkl} - 2R_{pik}{}^m R_{jqml} + 2R_{jqk}{}^m R_{piml}\right\} \\ &\quad - R_i^m R_{mjkl} - R_j^m R_{imkl} - R_k^m R_{ijml} - R_l^m R_{ijkm} \end{aligned}$$

Proof.

Using the previous evolution, we have

$$\begin{aligned}
\frac{\partial}{\partial t} R_{ijkl} &= \left(\frac{\partial}{\partial t} g_{sl} \right) R_{ijk}{}^s + g_{sl} \left(\frac{\partial}{\partial t} R_{ijk}{}^s \right) \\
&= (-2R_{sl}) R_{ijk}{}^s + g_{sl} \Delta R_{ijk}{}^s + g_{sl} g^{pq} \left\{ R_{ijp}{}^m R_{mqk}{}^s - 2R_{pik}{}^m R_{jqm}{}^s + 2R_{pim}{}^s R_{jqk}{}^m \right\} \\
&\quad + g_{sl} \left(-R_i^m R_{mjk}{}^l - R_j^m R_{imk}{}^l - R_k^m R_{ijm}{}^l + R_m^l R_{ijk}{}^m \right) \\
&= \Delta R_{ijkl} + g^{pq} \left\{ R_{ijp}{}^m R_{mqkl} - 2R_{pik}{}^m R_{jqml} + 2R_{jqk}{}^m R_{piml} \right\} \\
&\quad - R_i^m R_{mjk}{}^l - R_j^m R_{imk}{}^l - R_k^m R_{ijm}{}^l - R_l^m R_{ijk}{}^m. \clubsuit
\end{aligned}$$

Let's see now the evolution of the laplacian acting on the space of smooth functions and possibly also depending on time.

Theorem 3.15. [Evolution of the Laplace Beltrami operator]

Let $g(t)$ be any differentiable one parameter family of Riemannian metrics on a differentiable manifold $\mathbb{M}$. Denoting the Laplace by

$$\Delta_{g(t)} : C^\infty(\mathbb{M} \times [0, T)) \rightarrow C^\infty(\mathbb{M} \times [0, T))$$

acting on the space of functions, its evolution is given by

$$\frac{\partial}{\partial t} \circ \Delta_{g(t)} = - \left(\frac{\partial}{\partial t} g_{ij} \right) \nabla^i \nabla^j - g^{kl} (g^{pq} \nabla_p h_{ql} - \frac{1}{2} \nabla_l H) \nabla_k + \Delta \circ \frac{\partial}{\partial t}$$

Proof.

First, according to the evolution equation of the Riemannian connection (2.12), we have

$$\begin{aligned}
g^{pq} \frac{\partial}{\partial t} \Gamma_{pq}^k &= g^{pq} \left\{ \frac{1}{2} g^{kl} (\nabla_p h_{ql} + \nabla_q h_{pl} - \nabla_l h_{pq}) \right\} \\
&= g^{kl} (g^{pq} \nabla_p h_{ql} - \frac{1}{2} \nabla_l H),
\end{aligned}$$

where again we're denoting $h = \partial_t g$. Then, the evolution of the Laplacian becomes

$$\begin{aligned}
\frac{\partial}{\partial t} \Delta u &= \frac{\partial}{\partial t} \left\{ g^{pq} (\partial_p \partial_q - \Gamma_{pq}^k \partial_k) u \right\} \\
&= -g^{pi} g^{qj} \left(\frac{\partial}{\partial t} g_{ij} \right) \nabla_p \nabla_q u - g^{pq} \left(\frac{\partial}{\partial t} \Gamma_{pq}^k \right) \partial_k u + \Delta \partial_t u \\
&= - \left(\frac{\partial}{\partial t} g_{ij} \right) \nabla^i \nabla^j u - g^{kl} (g^{pq} \nabla_p h_{ql} - \frac{1}{2} \nabla_l H) \partial_k u + \Delta \partial_t u
\end{aligned}$$

Corollary 3.16. [Evolution of The Laplacian Under Conformal Deformations]

Let $g(t)$ be a differentiable one parameter family of Riemannian metrics that evolve according to $\partial_t g = fg$ for some differentiable function that depends on both space and time

$$f : M \times I \rightarrow \mathbb{R}$$

where I is the interval which $g(\cdot)$ is defined. Then, the evolution of the Laplace is

$$\frac{\partial}{\partial t} \circ \Delta = -f \Delta + \left(\frac{N}{2} - 1 \right) \langle \nabla f, \nabla(\cdot) \rangle + \Delta \circ \frac{\partial}{\partial t} \quad (3.7)$$

Proof.

One just applies the previous formula to the evolution of the metric to get

$$\begin{aligned}\partial_t \Delta u &= -f g_{ij} \nabla^i \nabla^j u - g^{kl} \left(\delta_q^l \nabla_p f - \frac{1}{2} N \nabla_l f \right) \partial_k u + \Delta \partial_t u \\ &= -f \Delta u + \left(\frac{N}{2} - 1 \right) \langle \nabla f, \nabla u \rangle + \Delta \partial_t u. \clubsuit\end{aligned}$$

3.4.1 Estimate to positive scalar curvature

Let us see that if the (scalar) curvature is strictly positive according to some metric g_0 then the Ricci flow will encounter a singularity in finite time. We consider the N -dimensional evolution of the scalar curvature (5) under Ricci flow:

$$\partial_t R = \Delta R + 2|Rc|^2$$

Now, according the Lemma 1.20, we have $|Rc|^2 \geq \frac{1}{N} R^2$. Thus,

$$\partial_t R \geq \Delta R + \frac{2}{N} R^2.$$

This means that R is a supersolution for the heat equation. Now, according to the third version of maximum principle, Theorem 3.4, R is bounded below by the solution of the ordinary differential equation

$$\zeta'(t) = \frac{2}{N} \zeta^2(t), \quad \zeta(0) = \zeta_0, \quad (3.8)$$

where $\zeta_0 \doteq \inf_{x \in M} R(x, 0) > 0$, since R is supposed to be positive according to g_0 . Now, the solution to the problem (3.8) is $\zeta(t) = \frac{N}{N\zeta_0^{-1} - 2t}$ which is defined for $[0, T)$, where $T = \frac{N\zeta_0^{-1}}{2}$. We have, $R(x, t) \geq \zeta(t)$, for all $x \in M$, for all $t \in [0, T)$ which shows that positive scalar curvature will develop a singularity in finite time.

Chapter 4

Ricci Flow on Surfaces

In the two dimensional case, the Ricci tensor is a multiple of the metric, i.e., $Rc = \frac{Rg}{2}$, where R is the scalar curvature of g , and then the Ricci flow equation may be written as

$$\frac{\partial}{\partial t}g = -Rg.$$

We are going to consider the normalized Ricci flow which, in dimension two, is written as

$$\frac{\partial}{\partial t}g = (r - R)g, \quad \text{where } r \doteq \frac{\int R dV}{\int dV} \quad (4.1)$$

is the average scalar curvature. We denote the volume element, in this context, by $d\mu$ instead of dV . In particular, according to the Gauss-Bonnet formula, $\partial_t \int R d\mu = 0$ and since $\partial_t \int d\mu = 0$ holds in the direction of the normalized Ricci flow (in any dimension), we have $\partial_t(r) = 0$. Thus, r is independent of time.

The aim of this chapter is to prove the following theorem.

Theorem 4.1. [Convergence to a constant curvature metric]

Let $\mathbb{M}$ be a closed 2-dimensional manifold with Riemannian metric g_0 . If $r > 0$ suppose, in addition, that $R[g_0] > 0$. Then, there exists a unique solution to the normalised Ricci flow equation (4.1), with $g(0) = g_0$. The solution exists for all time and the flow $g(t)$ converges uniformly in any C^n -norm to a constant curvature Riemannian metric g_∞ , as $t \rightarrow \infty$.

In this chapter we are going to need the concept of conformal classes. Let us define what it is.

Definition 4.2. Two Riemannian metrics $g, \tilde{g} \in \Gamma(S_2^+ T^* \mathbb{M})$ on the differentiable manifold $\mathbb{M}$ of dimension $N \geq 2$ are said to be **conformally equivalent**, if there exists a function $\psi \in C^\infty(\mathbb{M})$ such that

$$\tilde{g} = e^{2\psi} g.$$

By a conformal class or conformal structure we understand a class of conformally equivalent Riemannian metrics. If C is a conformal structure on $\mathbb{M}$, then the pair $(\mathbb{M}, C)$ is called a conformal manifold.

4.1 Evolution equations in dimension two

In this section we are going to evolve quantities using the normalized Ricci flow. Some of these evolutions will only be applied when dealing with convergence issues, so the reader may skip them at first and return to them when needed.

Lemma 4.3. [Evolution of the connection in the direction of the normalized Ricci flow]

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a surface $\mathbb{M}$. Then, the connection evolves according to

$$\partial_t \Gamma_{ij}^k = -\frac{1}{2} \left\{ \delta_j^k \nabla_i R + \delta_i^k \nabla_j R - g_{ij} \nabla^k R \right\} \quad (4.2)$$

Proof.

It is an application of formula (2.12) with $\partial_t g_{ij} = (r - R)g_{ij}$:

$$\begin{aligned} \partial_t \Gamma_{ij}^k &= \frac{1}{2} g^{kl} \left\{ \nabla_i (\partial_t g_{jl}) + \nabla_j (\partial_t g_{il}) - \nabla_l (\partial_t g_{ij}) \right\} \\ &= \frac{1}{2} g^{kl} \left\{ -g_{jl} \nabla_i R - g_{il} \nabla_j R + g_{ij} \nabla_l R \right\} \\ &= -\frac{1}{2} \left\{ \delta_j^k \nabla_i R + \delta_i^k \nabla_j R - g_{ij} g^{kl} \nabla_l R \right\}. \clubsuit \end{aligned}$$

Lemma 4.4. [Evolution of the curvature in the direction of the normalized Ricci flow]

Let $g(t)$ be any solution of the normalized Ricci flow equation (4.1) on a surface $\mathbb{M}$. Then, the scalar curvature evolves by

$$\partial_t R = \Delta R + R(R - r) \quad (4.3)$$

Proof.

Using equation (3.1), we see that, in the direction of the normalized Ricci flow and in two dimensional spaces, $N=2$, we have

$$\begin{aligned} \partial_t R &= 2\Delta R - 2\nabla^p \left[g^{qj} \nabla_j \frac{R}{2} g_{pq} \right] - \left\langle \frac{R}{2} g, (r - R)g \right\rangle \\ &= 2\Delta R - \delta_{jp} \nabla^p \nabla_j R + (R - r)R \frac{|g|^2}{2} \\ &= \Delta R + R(R - r). \clubsuit \end{aligned}$$

Next, we compute the evolution of the Laplacian under the flow (4.1).

Lemma 4.5. [Evolution of the Laplacian under normalized Ricci flow]

Let $g(t)$ be a solution to the normalized Ricci flow equation on a surface $\mathbb{M}$. Then, the evolution of the Laplace operator is given by

$$\frac{\partial}{\partial t} \Delta = (R - r)\Delta. \quad (4.4)$$

Proof.

One just applies equation 3.7 in the conformal class (see Definition 4.2) the metric is evolving, namely, the normalized Ricci flow direction. ♣

The next evolution equations of this section concerns derivatives of the curvature and their square norms.

Lemma 4.6. *[Evolution of the covariant derivative of the curvature]*

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a surface $\mathbb{M}$. Then, ∇R evolves by

$$\partial_t \nabla R = \Delta \nabla R + \frac{3}{2} R \nabla R - r \nabla R. \quad (4.5)$$

Proof.

It follows by an application of the Ricci identity for a surface $\nabla \Delta - \Delta \nabla = -\frac{1}{2} R \nabla$ and the evolution of the scalar curvature (4.3)

$$\begin{aligned} \partial_t \nabla R &= \nabla(\partial_t R) = \nabla(\Delta R + R(R - r)) \\ &= \nabla \Delta R + (R - r) \nabla R + R \nabla R \\ &= \Delta \nabla R + \frac{3}{2} R \nabla R - r \nabla R. \clubsuit \end{aligned}$$

Lemma 4.7. *[Evolution of the square norm of the gradient of the curvature]*

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a surface $\mathbb{M}$. Then, $|\nabla R|^2$ evolves by

$$\partial_t |\nabla R|^2 = \Delta |\nabla R|^2 - 2 |\nabla \nabla R|^2 + (4R - 3r) |\nabla R|^2. \quad (4.6)$$

Proof.

By (4.5), we get

$$\begin{aligned} \partial_t |\nabla R|^2 &= \partial_t (g^{ij} \nabla_i R \nabla_j R) \\ &= -g^{ik} g^{jl} (r - R) g_{kl} \nabla_i R \nabla_j R + 2g^{ij} (\partial_t \nabla_i R) \nabla_j R \\ &= -(r - R) g^{ik} \delta_{jk} \nabla_i R \nabla_j R + 2g^{ij} \Delta \nabla_i R \nabla_j R + 3R g^{ij} \nabla_i R \nabla_j R - 2r g^{ij} \nabla_i R \nabla_j R \\ &= (R - r) |\nabla R|^2 + 2 \langle \Delta \nabla R, \nabla R \rangle + 3R |\nabla R|^2 - 2r |\nabla R|^2 \\ &= 2 \langle \Delta \nabla R, \nabla R \rangle + (4R - 3r) |\nabla R|^2 \end{aligned}$$

Now, using the following identity for the scalar curvature (see equation (A.2))

$$\Delta |\nabla R|^2 = 2 \langle \Delta \nabla R, \nabla R \rangle + 2 |\nabla \nabla R|^2,$$

the desired result follows. ♣

Lemma 4.8. *[Evolution of the double covariant derivative of the curvature]*

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a surface $\mathbb{M}$. Then, $\nabla \nabla R$ evolves by

$$\partial_t \nabla \nabla R = \Delta \nabla \nabla R + 2 \nabla R \otimes \nabla R + R(\Delta R)g - r \nabla \nabla R. \quad (4.7)$$

Proof.

According to (4.2) and (4.3), we have

$$\begin{aligned}
\partial_t \nabla_i \nabla_j R &= \partial_t (\partial_i \nabla_j R - \Gamma_{ij}^k \partial_k R) \\
&= \partial_i \partial_j \partial_t R - (\partial_t \Gamma_{ij}^k) \partial_k R - \Gamma_{ij}^k \partial_k \partial_t R \\
&= \nabla_i \nabla_j (\Delta R + R(R - r)) + \nabla_i R \nabla_j R - \frac{1}{2} g_{ij} |\nabla R|^2.
\end{aligned}$$

Now, consider the commutator formula $[\Delta, \nabla \nabla]$ applied to the scalar curvature (see A.7):

$$\nabla_i \nabla_j \Delta R - \Delta \nabla_i \nabla_j R = -\nabla_i R \nabla_j R - 2R \nabla_i \nabla_j R + R(\Delta R) g_{ij} + \frac{1}{2} |\nabla R|^2 g_{ij}.$$

Applying this to the previous equation, we get

$$\begin{aligned}
\partial_t \nabla_i \nabla_j R &= \nabla_i \nabla_j \Delta R + 2R \nabla_i \nabla_j R + 3\nabla_i R \nabla_j R - r \nabla_i \nabla_j R - \frac{1}{2} g_{ij} |\nabla R|^2 \\
&= \Delta \nabla_i \nabla_j R + 2\nabla_i R \nabla_j R + R(\Delta R) g_{ij} - r \nabla_i \nabla_j R. \clubsuit
\end{aligned}$$

Lemma 4.9. [Evolution of the square norm of the double covariant derivative of the curvature]
Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a surface $\mathbb{M}$. Then, $|\nabla \nabla R|^2$ evolves by

$$\partial_t |\nabla \nabla R|^2 = \Delta |\nabla \nabla R|^2 - 2|\nabla \nabla \nabla R| + 4\langle \nabla R \otimes \nabla R, \nabla \nabla R \rangle + 2R(\Delta R)^2 + 2R|\nabla \nabla R|^2 - 4r|\nabla \nabla R|^2. \quad (4.8)$$

Proof.

$$\begin{aligned}
\partial_t |\nabla \nabla R|^2 &= \partial_t (g^{ik} g^{jl} \nabla_i \nabla_j R \nabla_k \nabla_l R) \\
&= -2(r - R)(g^{ik} g^{jl} \nabla_i \nabla_j R \nabla_k \nabla_l R) + 2\{g^{ik} g^{jl} (\partial_t \nabla_i \nabla_j R) \nabla_k \nabla_l R\} \\
&= -2(r - R)|\nabla \nabla R|^2 + 2g^{ik} g^{jl} \{\Delta \nabla_i \nabla_j R + 2\nabla_i R \nabla_j R + R(\Delta R) g_{ij} - r \nabla_i \nabla_j R\} \nabla_k \nabla_l R \\
&= 2(R - r)|\nabla \nabla R|^2 + 2\langle \Delta \nabla \nabla R, \nabla \nabla R \rangle + 4\langle \nabla R \otimes \nabla R, \nabla \nabla R \rangle + 2R(\Delta R)(\Delta R) - 2r|\nabla \nabla R|^2
\end{aligned}$$

Next, we use the fact that $\Delta|Q| = 2\langle \Delta Q, Q \rangle + 2|\nabla Q|^2$, for $Q \in \Gamma(T^2 M)$, (see A.2) obtaining

$$\partial_t |\nabla \nabla R|^2 = \Delta |\nabla \nabla R|^2 - 2|\nabla \nabla \nabla R| + 4\langle \nabla R \otimes \nabla R, \nabla \nabla R \rangle + 2R(\Delta R)^2 + 2R|\nabla \nabla R|^2 - 4r|\nabla \nabla R|^2. \clubsuit$$

Lemma 4.10. [Evolution of the third covariant derivative of the curvature]
Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a surface $\mathbb{M}$. Then, $\nabla \nabla \nabla R$ evolves by

$$\begin{aligned}
\partial_t \nabla_i \nabla_j \nabla_k R &= \Delta \nabla_i \nabla_j \nabla_k R + R g_{ij} (\Delta \nabla_k R) + R g_{ik} (\Delta \nabla_j R) \\
&\quad - R \nabla_j \nabla_i \nabla_k R - R \nabla_k \nabla_i \nabla_j R - \frac{R}{2} \nabla_i \nabla_j \nabla_k R \\
&\quad + 2\{(\nabla_j R)(\nabla_i \nabla_k R) + (\nabla_k R)(\nabla_i \nabla_j R)\} + (\Delta R)(\nabla_i R) g_{jk} + R g_{jk} (\nabla_i \Delta R) - r \nabla_i \nabla_j \nabla_k R
\end{aligned} \quad (4.9)$$

Proof.

We have,

$$\begin{aligned}
\partial_t \nabla_i \nabla_j \nabla_k R &= \partial_t \left(\partial_i \nabla_j \nabla_k R - \Gamma_{ij}^l \nabla_l \nabla_k R - \Gamma_{ik}^l \nabla_j \nabla_l R \right) \\
&= \partial_i \left(\partial_t \nabla_j \nabla_k R \right) - \Gamma_{ij}^l \left(\partial_t \nabla_l \nabla_k R \right) - \Gamma_{ik}^l \left(\partial_t \nabla_j \nabla_l R \right) \\
&\quad - \left(\partial_t \Gamma_{ij}^l \right) \nabla_l \nabla_k R - \left(\partial_t \Gamma_{ik}^l \right) \nabla_j \nabla_l R \\
&= \nabla_i \left\{ \partial_t \nabla_j \nabla_k R \right\} + (\nabla_i R) (\nabla_j \nabla_k R) \\
&\quad + \frac{1}{2} \left\{ (\nabla_j R) (\nabla_i \nabla_k R) + (\nabla_k R) (\nabla_i \nabla_j R) \right\} - \frac{1}{2} \left\{ g_{ij} (\nabla^l R) (\nabla_k \nabla_l R) + g_{ik} (\nabla^l R) (\nabla_j \nabla_l R) \right\}
\end{aligned}$$

Now, by (4.7), we have

$$\begin{aligned}
\partial_t \nabla_i \nabla_j \nabla_k R &= \nabla_i \Delta \nabla_j \nabla_k R + \frac{5}{2} \left\{ (\nabla_j R) (\nabla_i \nabla_k R) + (\nabla_k R) (\nabla_i \nabla_j R) \right\} + (\Delta R) (\nabla_i R) g_{jk} \\
&\quad + R (\nabla_i \Delta R) g_{jk} - r \nabla_i \nabla_j \nabla_k R - \frac{1}{2} \left\{ g_{ij} (\nabla^l R) (\nabla_k \nabla_l R) + g_{ik} (\nabla^l R) (\nabla_j \nabla_l R) \right\}
\end{aligned}$$

Substituting formula (A.8) with f replaced by the curvature, that is,

$$\begin{aligned}
\nabla_i \Delta \nabla_j \nabla_k R &= \Delta \nabla_i \nabla_j \nabla_k R + R g_{ij} (\Delta \nabla_k R) + R g_{ik} (\Delta \nabla_j R) \\
&\quad - R \nabla_j \nabla_i \nabla_k R - R \nabla_k \nabla_i \nabla_j R - \frac{R}{2} \nabla_i \nabla_j \nabla_k R \\
&\quad - \frac{1}{2} (\nabla_j R) (\nabla_i \nabla_k R) - \frac{1}{2} (\nabla_k R) (\nabla_i \nabla_j R) + \frac{1}{2} g_{ij} (\nabla^q R) (\nabla_q \nabla_k R) + \frac{1}{2} g_{ik} (\nabla^q R) (\nabla_j \nabla_q R)
\end{aligned}$$

we get (4.9). ♣

The reader is invited to compute the evolution for square norm of the third covariant derivative of the curvature. Now we generalize these evolution equations to covariant derivatives of the scalar curvature of any order.

Lemma 4.11. [Evolution of the n th covariant derivative of the curvature and its square norm]

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a surface $\mathbb{M}$. Then, $\nabla^n R$ and $|\nabla^n R|^2$ evolves, respectively, by

$$\partial_t (\nabla^n R) = \Delta \nabla^n R + \sum_{j=0}^{\lfloor n/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-j} R) - r \nabla^n R \quad (4.10)$$

and,

$$\begin{aligned}
\partial_t |\nabla^n R|^2 &= \Delta |\nabla^n R|^2 - 2 |\nabla^{n+1} R|^2 - (n+2) r |\nabla^n R|^2 \\
&\quad + (\nabla^n R) \otimes_{g(t)} \left[\sum_{j=0}^{\lfloor n/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-j} R) \right], \quad (4.11)
\end{aligned}$$

where $X \otimes_{g(t)} Y$ denotes a finite linear combination of contractions (see section 1.6.1) of the tensors X and Y with respect to the metric $g(t)$.

Proof.

Recalling the inner-product on the bundle $T^k \mathbb{M}$ of $\binom{k}{0}$ -tensors given by (1.6), applying the formulas (A.10), (A.11), (A.12) and (A.13), we obtain the lemma. ♣

4.2 Ricci solitons on surfaces

Recall that a map $f : U \subset \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined on an open set $U \subset \mathbb{R}^N$ is said to be **conformal** if the (nonoriented) angles of intersecting curves are preserved, that is, if the (nonoriented) angles of any two vectors v_1 and v_2 at $p \in U$ are equal to the angle formed by $df_p v_1$ with $df_p v_2$. More generally, if $\mathbb{M}$ is a Riemannian manifold, with metric tensor g , a transformation $\varphi : \mathbb{M} \rightarrow \mathbb{M}$ is said to be conformal if $\varphi^* g = \rho g$, where $\rho : \mathbb{M} \rightarrow \mathbb{R}$ is a positive function on $\mathbb{M}$. For instance, if ρ is a constant function then φ is a homothetic transformation. If ρ is identically equal to one then φ is an isometry. (For more information on conformal transformations in Riemannian geometry see [KN96]).

Definition 4.12. Let $g(t)$ be a solution of the normalized Ricci flow (4.1) on a two dimensional manifold $\mathbb{M}$. We say that $g(t)$ is a **self-similar solution** if there exists a one parameter family $\psi(t)$ of conformal diffeomorphisms such that

$$g(t) = [\psi(t)]^* g(0) \quad (4.12)$$

Differentiating equation (4.12) with respect to time (see Lemma 1.18), we get

$$\frac{\partial}{\partial t} g = (\mathcal{L}_X g),$$

where $X = X(t)$ is the one parameter family of vector fields generated by $\psi(t)$. In local coordinates, we have

$$(r - R)g_{ij} = \nabla_i X_j + \nabla_j X_i, \quad (4.13)$$

using the fact that $g(t)$ is a solution to the normalized Ricci flow equation, where $X_i = g_{ik} X^k$ is the 1-form obtained from X . We shall refer to equation (4.13) as the **Ricci soliton equation**. If the vector field X is the gradient of some function $f \in C^\infty(\mathbb{M} \times [0, T))$, i.e., $X = -\text{grad} f$ at all point and times then we may rewrite equation (4.13) as

$$(R - r)g_{ij} = 2\nabla_i \nabla_j f. \quad (4.14)$$

We shall refer to equation (4.14) as the **gradient Ricci soliton equation**.

Definition 4.13. We say that a Riemannian metric $g \in \Gamma(S_2^+ T^* \mathbb{M})$ on a surface $\mathbb{M}$ is a **Ricci soliton** if it satisfies equation (4.13). We say that g is a **gradient Ricci soliton** if it satisfies equation (4.14).

Taking traces of both sides on equation (4.14) we get the equation

$$\Delta f = R - r, \quad (4.15)$$

which always has a solution $f \in C^\infty(\mathbb{M} \times [0, T))$. The function f satisfying (4.15) is known as a **Ricci potential** or as a **potential of the curvature**. In particular, given two Ricci potentials f and h , we have $\Delta(f - h) = 0$. Then, according to Corollary A.16, in the appendix, we have $f = h + \lambda(t)$ for some function $\lambda(t)$ of time alone, which shows that two Ricci potentials differ only by a function of time. Now, equation (4.15) is always solvable on compact manifolds since

$$\int (r - R) d\mu = 0.$$

Any solution to the associated homogeneous equation $\Delta u = 0$ on $\mathbb{M} \times [0, T)$, depends only on time (see A.16). For any $c = c(t) : [0, T) \rightarrow \mathbb{R}$, considering the inner product space $L^2(\mathbb{M}, d\mu)$, we have

$$(R - r, c(t))_{L^2} = \int c(t)(R - r)d\mu = c(t) \int (R - r)d\mu = 0,$$

that is, $R - r$ is orthogonal to any solution of the homogeneous equation $\Delta u = 0$.

Lemma 4.14. [The relation between Ricci potentials and gradient Ricci solitons]

The trace free part of the Hessian of any Ricci potential annihilates at gradient Ricci solitons, and vice-versa. That is, the tensor $M(g) \doteq \nabla \nabla f - \frac{1}{2} \Delta f g$ vanishes if, and only if, g is a gradient Ricci soliton (see Definition 4.13). Moreover, the gradient Ricci soliton equation (4.14) implies that

$$\nabla R + R \nabla f = 0. \quad (4.16)$$

In particular, $\log R + f$ is constant in space if the curvature is positive.

Proof.

The first part is immediate. For the second part suppose that the Ricci potential f satisfies the gradient Ricci soliton equation. Recall the Ricci identity for surfaces, $\nabla \Delta - \Delta \nabla = -\frac{1}{2} R \nabla$, and apply the divergence for the $\binom{2}{0}$ -tensor field M to get

$$\begin{aligned} (\operatorname{div} M)(\partial_i) &= g^{jk} \{\nabla_k M_{ij}\} = \Delta \nabla_i f + \frac{1}{2} g^{jk} g_{ij} \nabla_k \Delta f \\ &= \nabla_i \Delta f + \frac{1}{2} R \nabla_i f - \frac{1}{2} \nabla_i \Delta f \\ &= \frac{1}{2} (\nabla_i \Delta f + R \nabla_i f). \end{aligned}$$

In particular, if $R > 0$ then $\nabla(\log R + f) = \frac{1}{R}(\nabla R + R \nabla f) = 0$, and $\log R + f$ is constant in space. ♣

In the next lemma we show the existence of a Ricci potential which satisfies the evolution equation $\partial_t f = \Delta f + r f$. We will soon see that, among the Ricci potentials, this is a preferred one.

Lemma 4.15. Let $g(t)$ be a solution to the normalized Ricci flow equation on a closed surface $\mathbb{M}$ and let f_0 be an arbitrary Ricci potential. Then, there exists a function $c = c(t)$ of time such that the potential function

$$f = f_0 + c, \quad \text{satisfies} \quad \frac{\partial}{\partial t} f = \Delta f + r f. \quad (4.17)$$

Proof.

According to the evolution equation of the Laplace operator (4.4)

$$\frac{\partial}{\partial t} \circ \Delta = -(r - R) \Delta + \Delta \circ \frac{\partial}{\partial t},$$

and taking the time derivative on the left of the Ricci potential equation $\Delta f_0 = R - r$, we get

$$\partial_t \Delta f_0 = -(r - R) \Delta f_0 + \Delta \partial_t f_0 = (R - r)^2 + \Delta \partial_t f_0.$$

On the other hand, recalling the evolution of the curvature (4.3), we have

$$\partial_t R = \Delta \circ \Delta f_0 + r(R - r) + (R - r)^2 = \Delta \circ \Delta f_0 + r \Delta f + (R - r)^2$$

Therefore,

$$\Delta(\partial_t f_0 - \Delta f_0 - r f_0) = 0.$$

Since $\mathbb{M}$ is closed, by applying again Corollary A.16, we conclude that there exists a function of time alone $\gamma(t)$ such that

$$\partial_t f_0 - \Delta f_0 - r f_0 = \gamma(t)$$

Now, if $c(t) = -e^{rt} \int_0^t e^{-r\tau} \gamma(\tau) d\tau$ then $c' = rc - \gamma$, which implies

$$\partial_t(f_0 + c) = \Delta(f_0 + c) + r(f_0 + c). \clubsuit$$

Next, we present a corollary of the maximum principle applied to the Ricci potential f obtained in the previous lemma.

Corollary 4.16. Let $g(t)$ be any family of metrics defined on the time interval $[0, T)$ on a compact differentiable manifold $\mathbb{M}$. Let $f : \mathbb{M} \times [0, T) \rightarrow \mathbb{R}$ be any differentiable function that satisfies the evolution equation $\partial_t f = \Delta f + rf$. Then, there exists $c > 0$ such that

$$|f| \leq ce^{rt}, \text{ on } M \times [0, T). \quad (4.18)$$

Proof.

Since $\mathbb{M}$ is compact, there exists $c > 0$ such that $|f(x, 0)| \leq c$, for all $x \in M$. The corollary now follows from observing that $\partial_t(fe^{-rt}) = \Delta(fe^{-rt})$ and the maximum principle applied to the function fe^{-rt} . $\clubsuit$

Now, let $g(t)$ be a normalized Ricci flow on a closed surface $\mathbb{M}$ and let f be a Ricci potential such that its evolution equation is $\partial_t f = \Delta f + rf$ and define the quantity

$$H = R - r + |\nabla f|^2. \quad (4.19)$$

The evolution of H determines nice bounds for the scalar curvature R . Let us compute the evolution of $|\nabla f|^2$ first.

Lemma 4.17. [The evolution of the square norm of the Ricci potential f]

Under the normalized Ricci flow $(\mathbb{M}, g(t))$ on a closed surface, the quantity $|\nabla f|^2$ evolves by

$$\frac{\partial}{\partial t} |\nabla f|^2 = \Delta |\nabla f|^2 - 2|\nabla \nabla f|^2 + r|\nabla f|^2. \quad (4.20)$$

Proof.

Using the Ricci identity $\Delta \nabla_i = \nabla_i \Delta + \frac{R}{2} \nabla_i$ for a surface, we have

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla f|^2 &= \left(\frac{\partial}{\partial t} g^{ij} \right) \nabla_i f \nabla_j f + 2g^{ij} \left(\frac{\partial}{\partial t} \nabla_i f \right) \nabla_j f \\ &= -g^{ik} g^{jl} \left(\frac{\partial}{\partial t} g_{kl} \right) \nabla_i f \nabla_j f + 2g^{ij} (\nabla_i \Delta f) \nabla_j f + 2g^{ij} (\nabla_i r f) \nabla_j f \\ &= (R - r) |\nabla f|^2 + 2g^{ij} (\Delta \nabla_i f) \nabla_j f - R |\nabla f|^2 + 2r |\nabla f|^2 \\ &= \Delta |\nabla f|^2 - 2|\nabla \nabla f|^2 + r |\nabla f|^2. \clubsuit \end{aligned}$$

Lemma 4.18. [The evolution of H]

Under the normalized Ricci flow $(\mathbb{M}, g(t))$ on a closed surface the quantity H defined by equation (4.19) evolves according to the rule

$$\frac{\partial}{\partial t} H = \Delta H - 2|M|^2 + rH \quad (4.21)$$

where M is the trace free part of $\nabla^2 f$, that is,

$$M = \nabla \nabla f - \frac{1}{2} \Delta f g \quad (4.22)$$

Proof.

It is clear that

$$|M|^2 = |\nabla \nabla f|^2 - \Delta f g^{ij} \nabla_i \nabla_j f + \frac{1}{4} (\Delta f)^2 |g|^2 = |\nabla \nabla f|^2 - \frac{1}{2} (\Delta f)^2,$$

and

$$0 = (\Delta f)^2 - (r - R)^2.$$

Now, by (4.3) and (4.20), we get

$$\begin{aligned} \partial_t H &= \Delta R + R(R - r) + \Delta |\nabla f|^2 - 2 |\nabla \nabla f|^2 + r |\nabla f|^2 \\ &= \Delta (R - r + |\nabla f|^2) - 2 |\nabla \nabla f|^2 + (\Delta f)^2 - (r - R)^2 + R(R - r) + r |\nabla f|^2 \\ &= \Delta H - 2 |M|^2 + r H. \clubsuit \end{aligned}$$

Theorem 4.19. [Uniform equivalence of metrics under normalised Ricci flow]

Let $g(t)$ be a solution to the normalised Ricci flow equation on a compact surface $\mathbb{M}$ with $r \leq 0$ defined on the half-open time interval $[0, T)$. Then, there exists some constant $c \geq 1$ such that these metrics are uniformly equivalent, that is,

$$\frac{1}{c} g[x, 0] \leq g[x, t] \leq c g[x, 0], \quad \text{for all } 0 \leq t < T, \quad \text{and all } x \in \mathbb{M}.$$

Proof.

Let f be a Ricci potential such that $\partial_t f = \Delta f + r f$, whose existence is guaranteed by Lemma (4.15). We have

$$\frac{\partial}{\partial t} g = (r - R)g = \Delta f g = \left(\frac{\partial}{\partial t} f - r f \right) g.$$

Integrating this equation with respect to time from 0 to t yields

$$g[x, t] = \exp \left[f(x, t) - f(x, 0) - r \int_0^t f(x, \tau) d\tau \right] g[x, 0], \quad x \in \mathbb{M}. \quad (4.23)$$

By Corollary 4.16, by the triangle inequality and because $r \leq 0$, we have

$$\left| f(x, t) - f(x, 0) - r \int_0^t f(x, \tau) d\tau \right| \leq c e^{rt} + c - r \int_0^t c e^{r\tau} d\tau = c(e^{rt} + 1) + c(1 - e^{rt}) = 2c.$$

Then, for any $0 \neq w \in T_x \mathbb{M}$, dividing equation (4.23) by $|w|_0^2$, where $|\cdot|_0$ is the norm with respect to $g[x, 0]$, applying the logarithm to the resulting equation and using the above estimate gives us

$$\left| \log \left(\frac{g_{[x, t]}(w, w)}{g_{[x, 0]}(w, w)} \right) \right| \leq 2c$$

Exponentiating this inequality implies

$$e^{-2c} g_{[x, 0]}(w, w) \leq g_{[x, t]}(w, w) \leq e^{2c} g_{[x, 0]}(w, w), \quad \text{for all } w \in T_x \mathbb{M}, \quad \text{and all } x \in \mathbb{M}. \clubsuit$$

4.2.1 Kazdan-Warner Identity and Gradient Ricci solitons

Here we state without proof one of the main results of this chapter, namely the Kazdan-Warner identity, which will be used to prove that gradient Ricci solitons (see Definition 4.13) are constant curvature metrics, in the case of surfaces of positive Euler characteristic. The Kazdan-Warner identity uses the uniformization theorem and therefore this work does not provide a proof of existence of metrics of constant curvature using only Ricci flow theory. It is possible, however, to obtain such a proof relying only on Ricci flow (see [CLT05]).

Definition 4.20. Let $(\mathbb{M}, g)$ be an N -dimensional Riemannian manifold. A vector field $X \in \mathfrak{X}(\mathbb{M})$ is said to be a conformal Killing vector field if

$$\mathcal{L}_X g = \frac{2}{N}(\operatorname{div} X)g, \quad [\text{in coordinates: } \nabla_i X_j + \nabla_j X_i = \frac{2}{N}(\operatorname{div} X)g_{ij}].$$

A vector field $Y \in \mathfrak{X}(\mathbb{M})$ is said to be a conformal vector field if

$$\mathcal{L}_Y g = (\operatorname{div} Y)g, \quad [\text{in coordinates: } \nabla_i Y_j + \nabla_j Y_i = (\operatorname{div} Y)g_{ij}].$$

Theorem 4.21. [*Kazdan-Warner identity*]

For any conformal vector field Y on a Riemannian 2-sphere (S^2, g) , we have

$$\int_{S^2} \langle \nabla_g R, Y \rangle_g d\mu_g = 0.$$

Proof.

See [Gao11] or [CKKS04, Chapter 5].

The theorem above employs the Uniformization Theorem arguing that the metric g belongs to the conformal class (see Definition 4.2) of some constant curvature metric, say $\bar{g}$, thus allowing one to write $g = e^{2u}\bar{g}$, for some function $u \in C^\infty(S^2)$. The fact that any gradient Ricci soliton is a constant curvature metric then follows from Theorem 4.21.

Corollary 4.22. [*Ricci solitons are constant curvature metrics*]

If (S^2, g, X) is a Ricci soliton on a Riemannian 2-sphere then g has constant curvature.

Proof.

Since g is a Ricci soliton, it satisfies $(r - R)g_{ij} = \nabla_i X_j + \nabla_j X_i$, where $X^b = X_i dx^i$ is the 1-form obtained from X . Taking the trace of this equation, we get

$$2(r - R) = 2g^{ij}\nabla_i X_j = 2\nabla_k X^k = 2\operatorname{div} X.$$

Therefore, X is a conformal vector field. According to the Kazdan-Warner identity (4.21), and according to the divergence theorem applied to the vector field RX , we have

$$0 = \int_{S^2} \langle \nabla R, X \rangle d\mu = - \int_{S^2} R \operatorname{div} X d\mu.$$

Now, the integral of $(r - R)$ over S^2 is zero whence

$$0 = - \int_{S^2} R \operatorname{div} X d\mu = - \int_{S^2} R(r - R) d\mu + \int_{S^2} r(r - R) d\mu = \int_{S^2} (r - R)^2 d\mu,$$

and $(r - R) \equiv 0$. ♣

Corollary 4.23. *[Ricci solitons are constant curvature metrics]*

If $(\mathbb{M}, g, X)$ is a Ricci soliton on a closed Riemannian surface then g is a constant curvature metric.

Proof.

This is an application of Corollary (4.22). Since $\mathbb{M}$ has $r > 0$, is compact and has no boundary we observe that it is diffeomorphic to a (topological) sphere. If $f : S^2 \rightarrow \mathbb{M}$ is a diffeomorphism then we push the Ricci soliton g back to S^2 so that these two manifolds become geometrically the same, and apply the referred corollary to conclude that g is a constant curvature metric. ♣

4.3 First Estimates on the Curvature

We shall consider three cases regarding the values of the average scalar curvature in proving convergence of the normalized Ricci flow to a metric of constant curvature: $r > 0$, $r = 0$ and $r < 0$. We will estimate bounds for the scalar curvature, based on maximum principles and then deal with convergence issues. Let us first make a sketch about some of those bounds for the scalar curvature, under the normalized Ricci flow assuming already global existence. More specifically, in all estimates that are about to come we will assume the existence of a differentiable family of metrics $g(t)$ defined for all $t \geq 0$ which satisfies the normalized Ricci flow equation with bounded curvature on a closed surface.

Case $r < 0$.

In this case, we are going to obtain the estimate

$$|R - r| \leq ke^{rt}, \quad \text{for all } t \geq 0, \quad (4.24)$$

where k is a constant to be determined. The associated ODE to equation (4.3) is

$$\frac{\partial}{\partial t} \zeta = F(\zeta), \quad \text{with } \zeta(0) = R_{\min}(0) \leq r, \quad (4.25)$$

where $F : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $F(y) = y^2 - yr$. The solution to this ODE is $\zeta(t) = \frac{r}{1 - (1 - \frac{r}{\zeta(0)})e^{rt}}$. Then, according to the maximum principle (see Theorem (3.4)), we have

$$R(x, t) \geq \zeta(t), \quad \text{for all } t \geq 0, \quad \text{and } x \in M. \quad (4.26)$$

Consider the function $u : \mathbb{M} \times [0, \infty) \rightarrow \mathbb{R}$ defined by $u(x, t) = (\zeta(t) - r)e^{-rt}$ which is a supersolution to the heat equation $\frac{\partial}{\partial t} v = \Delta v$. We have $\zeta(t) - r \geq (R_{\min}(0) - r)e^{rt}$, since supersolutions maintain lower bounds and $u(x, 0) = \zeta(0) - r$. These inequalities together with (4.26) imply

$$R(x, t) - r \geq \zeta(t) - r \geq (R_{\min}(0) - r)e^{rt}, \quad \text{for all } t \geq 0, \quad \text{and } x \in \mathbb{M}. \quad (4.27)$$

Now, consider the quantity $H = R - r + |\nabla f|^2$ defined in (4.19). By (4.21), we have

$$\partial_t (He^{-rt}) = (\Delta H)e^{-rt} - 2|M|^2 e^{-rt} \leq \Delta (He^{-rt}), \quad (4.28)$$

and therefore He^{-rt} is a subsolution to the heat equation and then it preserves upper bounds (see Theorem (3.3)). Since $\mathbb{M}$ is compact, we have $H \leq c$, for some $c > 0$ which implies

$$R - r \leq H \leq ce^{rt}, \text{ for all } t \geq 0 \text{ and } x \in \mathbb{M}. \quad (4.29)$$

The estimates (4.27) and (4.29) together prove (4.24).

Case $r=0$.

In this case, we are going to show that the scalar curvature satisfies

$$-\frac{1}{t} \leq R \leq c, \text{ for all } t \geq 0, \text{ and } x \in \mathbb{M}. \quad (4.30)$$

If $r = 0$, then the evolution of the scalar curvature (4.3), becomes

$$\partial_t R = \Delta R + R^2,$$

whose associated ODE is

$$\frac{d}{dt} \zeta = \zeta^2, \text{ with initial condition } \zeta_0 = R_{\min}(0) \leq 0.$$

The solution to this ODE is $\zeta(t) = \frac{\zeta_0}{1 - \zeta_0 t}$. According to Theorem (3.4), we have

$$R(x, t) \geq \zeta(t) = \frac{\zeta(0)}{1 - \zeta(0)t}, \text{ for all } t \geq 0, \text{ and } x \in \mathbb{M}. \quad (4.31)$$

Now, let f be a Ricci potential and consider the quantity $H = R + |\nabla f|^2$ defined in (4.19). By (4.21), H is a subsolution to the heat equation $\partial_t H \leq \Delta H$. By (3.3), there exists some $c > 0$ such that

$$R \leq R + |\nabla f|^2 = H \leq c, \text{ for all } t \geq 0, \text{ and } x \in \mathbb{M}. \quad (4.32)$$

The estimates (4.31) and (4.32) together prove (4.30).

Case $r>0$.

Suppose $R_{\min}(0) \leq 0$. We are going to obtain the following estimate

$$ke^{rt} \geq R \geq R_{\min}(0)e^{-rt} \quad (4.33)$$

The ODE associated with the evolution of the curvature (4.3) is (4.25), whose solution is $\zeta = \frac{r}{1 - \left(1 - \frac{r}{R_{\min}(0)}\right)e^{rt}}$.

Thus, by Theorem (3.4), we have

$$R(x, t) \geq \zeta(t), \text{ for all } t \geq 0, \text{ and } x \in \mathbb{M}. \quad (4.34)$$

Now, the quantity $(\zeta(t)e^{rt})$ satisfies $\partial_t(\zeta(t)e^{rt}) = \zeta(t)^2 e^{rt} \geq \Delta(\zeta(t)e^{rt})$. By (3.3), we conclude that

$$(\zeta(t)e^{rt}) \geq R_{\min}(0), \text{ for all } t \geq 0.$$

This inequality together with (4.34) proves that

$$R(x, t) \geq R_{\min}(0)e^{-rt}, \text{ for all } t \geq 0, \text{ and } x \in \mathbb{M}. \quad (4.35)$$

Finally, the quantity He^{-rt} is known to satisfy the evolution equation (4.28) and therefore it maintains

upper bounds wherever it is defined, namely

$$R - r \leq H \leq ce^{rt}, \text{ for all } x \in M, \text{ for all } t \geq 0. \quad (4.36)$$

The inequalities (4.36) and (4.35) prove (4.33).

To summarize, we establish what we have done so far concerning these estimates for the scalar curvature.

Theorem 4.24. [First estimates on the curvature under normalised Ricci flow]

Let $g(t)$ be a solution to the normalized Ricci flow equation (4.1) on a compact surface $\mathbb{M}$ and defined for all $t \geq 0$. Then, there exists a constant $k > 0$ such that

$$|R - r| \leq ke^{rt}, \text{ for all } t \geq 0. \quad (4.37)$$

Proof. ♣

4.4 The flow evolving inside a conformal class

In this section we show that if the Ricci flow flows at all then it flows inside a conformal class (see Definition 4.2).

Lemma 4.25. [The effect of a conformal change of metrics]

If g and h are metrics on a surface $\mathbb{M}$ conformally related by

$$g = e^{2u} h, \quad u \in C^\infty(\mathbb{M}),$$

then, their scalar curvatures are related by

$$R(g) = e^{-2u} (-2\Delta_h u + R(h)). \quad (4.38)$$

Proof. See [CKKS04, Chapter 5] or [CS12]. ♣

Lemma 4.26. [The normalised Ricci flow by conformal classes]

Let $(\mathbb{M}, g_0)$ be a Riemannian manifold of dimension 2. A given family of metrics $g(t) = u(\cdot, t)g_0$, defined on some interval $[0, T)$, is a solution to the normalised Ricci flow equation (4.1) if, and only if, u satisfies the evolution equation

$$\partial_t u = \Delta_{g_0}(\log u) + ru - R(g_0), \quad (4.39)$$

where, $u(\cdot, t) \in C^\infty(\mathbb{M})$ is a function on $\mathbb{M}$, for each $t \in [0, T)$.

Proof.

Set $f \doteq \frac{1}{2} \log u$ and write $g(t) = e^{2f} g_0$. According to (4.38), we have

$$R(g) = e^{-2f} (-2\Delta_{g_0} f + R(g_0)) = \frac{1}{u} (-\Delta_{g_0}(\log u) + R(g_0)).$$

Therefore, if $g(t)$ satisfies (4.1) we have

$$\begin{aligned} (\partial_t u)g_0 &= \partial_t g = [r - R(g)]g \\ &= [\Delta_{g_0}(\log u) - R(g_0) + ru]g_0. \end{aligned}$$

If u satisfies the equation (4.39) then it is clear that $g(t)$ satisfies (4.1). ♣

The next theorem shows that on 2-dimensional Riemannian manifolds, the normalized Ricci flow evolves inside a conformal class. This is due to the parabolicity of the flow in dimension 2.

Theorem 4.27. [The normalised Ricci flow evolves under conformal classes]

Let (M, g_0) be a Riemannian manifold of dimension 2. Then, there exists a unique function $u : \mathbb{M} \times [0, \epsilon) \rightarrow \mathbb{R}$ satisfying the evolution equation (4.39) and therefore the family of metrics defined by $g(t) \doteq u(t)g_0$ is the unique solution to (4.1) such that $g(0) = g_0$.

Proof.

We have

$$\Delta_{g_0} \log u = g_0^{ij} (\partial_i \partial_j - \Gamma_{ij}^k \partial_k) \log u = \frac{1}{u} \Delta_{g_0} u - \frac{1}{u^2} |\nabla_{g_0} u|_{g_0}^2.$$

Thus, the equation (4.39) with $u(0) \equiv 1$ is equivalent to the problem

$$\partial_t u = \frac{1}{u} g_0^{ij} \nabla_i \nabla_j u - \frac{1}{u^2} |\nabla_{g_0} u|_{g_0}^2 - R(g_0) + ru, \quad \text{with } u(0) = 1,$$

which is a parabolic equation (see Definition 2.7 and Example 2.8) and hence a solution exists for a short time (see [Aub98, Theorem 4.51]). Moreover, by (2.30) we have uniqueness of the Ricci flow which implies the uniqueness claimed. ♣

4.5 Long time existence

From Lemma 3.9 we see that under a priori estimates of the derivative of a family of metrics, we get convergence to a continuous metric. In the 2-dimensional case, since the flow evolves under conformal classes, the estimates obtained in equation (4.37), are sufficient to obtain a similar result.

Lemma 4.28. [Equivalence of metrics under normalised Ricci flow]

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a closed surface $\mathbb{M}$ and suppose $g(t)$ is defined on the time interval $0 \leq t < T \leq \infty$. By Theorem 4.27, write $g = e^u g_0$. If there exists $k > 0$ such that

$$\int_0^T \left| \frac{\partial}{\partial t} u(x, t) \right| dt \leq k, \quad \text{for all } x \in \mathbb{M}$$

then,

$$e^{-k}g[x, 0] \leq g[x, t] \leq e^k g[x, 0], \text{ for all } x \in \mathbb{M}, \text{ and } t \in [0, T].$$

Additionally, as $t \rightarrow T^-$ the family $g(t)$ converges uniformly to a continuous metric $g(T)$, such that

$$e^{-K}g[x, 0] \leq g[x, T] \leq e^K g[x, 0], \text{ for all } x \in \mathbb{M},$$

for some constant K . Note that this means that $g(T)$ is uniformly equivalent to g_0 .

Proof.

Define

$$u(x, T) = u(x, 0) + \int_0^T \partial_t u(x, t) dt.$$

This integral exists as the integrand is absolutely integrable with respect to the norm on $\mathbb{R}$. This equation implies

$$|u(x, T) - u(x, \tau)| \leq \int_\tau^T |\partial_t u(x, t)| dt \rightarrow 0,$$

as $\tau \rightarrow T$, for each $x \in M$. Since M is compact, the convergence is uniform on $\mathbb{M}$ (see the argument given in Lemma 3.9). Hence $u(x, t) \rightarrow u(x, T)$, uniformly so that $u(x, T)$ is continuous. Therefore,

$$|u(x, \tau)| \leq |u(x, T)| + \int_\tau^T |\partial_t u(x, t)| dt \leq K, \text{ for all } \tau \in [0, T].$$

Exponentiating, we get $e^{-K} \leq e^{u(x, t)} \leq e^K$ for all $t \in [0, T]$ which implies

$$e^{-K}g[x, 0] \leq g[x, t] \leq e^K g[x, 0], \text{ for all } x \in \mathbb{M}, \text{ and all } t \in [0, T]. \quad (4.40)$$

One defines $g(T) = e^{u(\cdot, T)}g_0$, which is seen to be a continuous metric. Taking the limit of equation (4.40), we see that these metrics are uniformly equivalent. ♣

The next step for proving global existence is to show that the limit metric is differentiable (C^∞). This is another consequence of the BBS (Bernstein-Bando-Shi) estimates, discussed at Chapter 3. Now, we use these estimates again for the scalar curvature which, in two dimensional manifolds, contains the whole information of the curvature tensors. More precisely, Theorem 3.6 implies the following: if we have uniform bounds on the scalar curvature on $\mathbb{M} \times [0, T]$, then we can get uniform bounds on the scalar curvature and the metric and all its derivatives. But, by Theorem 4.37, we already have finite time uniform bounds on the scalar curvature. In addition to this assumption, we are going to assume that bounds on the covariant derivative implies bounds on coordinate derivatives, which is done locally without great effort. More precisely, suppose we have

$$|\nabla^n R| \leq K_n, \text{ on } \mathbb{M} \times [0, T], \quad (4.41)$$

Then, since M is compact, we can choose a finite set of coordinate charts such that the closed unit balls of the coordinate charts cover M . Let $(U, (x^i))$, be any of these charts. Then, by our assumption that uniform bounds on covariant derivatives of the curvature imply uniform bounds on coordinate derivatives of the curvature and the metric, for any multi-index α such that $|\alpha| = n$, we also have

$$\left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} R \right| \leq P_n \text{ and } \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} g \right| \leq Q_n \quad (4.42)$$

for constants P_n and Q_n , so that these quantities are uniformly bounded on $\mathbb{M} \times [0, T]$. Moreover, if we write the normalised Ricci flow as $g(t) = e^u g_0$, then we can get uniform bounds for all derivatives of the function u . For instance,

$$\partial_k(g(t)_{ij}) = (\partial_k u)e^u g(0)_{ij} + e^u(\partial_k g(0)_{ij}),$$

shows that the first derivatives of u are bounded (on that coordinate domain). By induction, one obtains uniform bounds for all derivatives of u .

Corollary 4.29. (of Lemma 4.28). Suppose in addition to the hypotheses of Lemma 4.28, that all derivatives of the scalar curvature are uniformly bounded as in equation (4.41). Suppose moreover that

$$\int_0^T |\nabla^n R(x, t)| dt < \infty, \text{ for all } x \in \mathbb{M},$$

for all $n \geq 1$. Then, the metric $g(T)$ of Lemma 4.28 is differentiable (C^∞) and the metrics $g(t)$ converge uniformly in every C^n norm to $g(T)$ as $t \rightarrow T$.

Proof.

The function $u(\cdot, T)$ obtained at Lemma 4.28 can be written as

$$u(x, T) = u(x, t) + \int_t^T \partial_t u(x, \tau) d\tau = u(x, t) + \int_t^T (r - R(x, \tau)) d\tau.$$

Using a finite number of coordinate charts (whose domain are compact sets, say $\mathbb{B}_1, \dots, \mathbb{B}_{N_0} \subset \mathbb{M}$ and $\cup_{i=1}^{N_0} \mathbb{B}_i = \mathbb{M}$), we argue that both $\frac{\partial^{|\alpha|}}{\partial x^\alpha} R$ and $\frac{\partial^{|\alpha|}}{\partial x^\alpha} u$ are uniformly bounded on the corresponding coordinate domain times the interval of time, where α is an arbitrary fixed multi-index. Hence, we can take the derivative under the integral sign:

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} u(x, T) = \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} u \right)(x, t) - \int_t^T \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} R \right)(x, \tau) d\tau.$$

In particular, the LHS of this equation exists for all α , so that $u(\cdot, T)$ is differentiable. Now, we write these N_0 coordinate domains as $(\mathbb{B}_1, (x_1^i)), \dots, (\mathbb{B}_{N_0}, (x_{N_0}^i))$ and then we take P_{N_0} to be the greatest number among the numbers P_n which satisfy the equation $\left| \frac{\partial^{|\alpha|}}{\partial x_k^\alpha} R \right| \leq P_n$ on $\mathbb{B}_k \times [0, T)$ for all multi-indices $|\alpha| = n$ (as in equation 4.42), so that $\left| \frac{\partial^{|\alpha|}}{\partial x_k^\alpha} R \right| \leq P_{N_0}$ holds on $\mathbb{B}_k \times [0, T)$, for all $k = 1, \dots, N_0$ and therefore on all of $\mathbb{M} \times [0, T)$ and for all multi-indices $|\alpha| = n$. Therefore,

$$\left| \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} u \right)(x, T) - \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} u \right)(x, t) \right| \leq \int_t^T \left| \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} R \right)(x, \tau) \right| d\tau \leq \int_t^T P_{N_0} d\tau.$$

The RHS of the inequality above converges to zero as $t \rightarrow T$ uniformly on $\mathbb{M}$. (Notice that this is a particular version of the argument given in Theorem 3.11, in a simpler context.) Therefore, the metric $g(x, T) = e^{u(x, T)} g(x, 0)$ is differentiable and uniformly equivalent to all metrics from the family $g(t)$. The convergence claimed then follows. ♣

Next, we present the 2-dimensional version of Theorem 3.12.

Theorem 4.30. [Long time existence of the normalised Ricci flow]

Let g_0 be a differentiable Riemannian metric on a differentiable compact manifold $\mathbb{M}$ of dimension 2. The unique solution to the normalised Ricci flow equation (4.1) with $g(0) = g_0$ is defined for all times $t \geq 0$.

Proof.

Suppose that the flow is defined for all t such that $0 \leq t \leq T < \infty$. The estimate from equation (4.37) readily gives all estimates to apply Corollary (4.29) and get a Riemannian metric $g(T)$ so that we can restart the flow obtaining a contradiction. ♣

In Sections 4.5.1 and 4.5.2 we will derive bounds for all derivatives of the curvature on surfaces of nonpositive Euler characteristic $\chi(\mathbb{M}) \leq 0$, therefore allowing a direct proof of long-time existence in those cases. The special case $R[g_0] \geq 0$, will be treated separately and we will rely on Theorem 4.30 to assure long time-existence of the flow in this case.

4.5.1 Convergence when $r < 0$.

In this section, the main goal is to establish the following theorem.

Theorem 4.31. [Convergence to a constant curvature metric in the case $r < 0$]

Let $\mathbb{M}$ be a closed 2-dimensional manifold with Riemannian metric g_0 and average scalar curvature $r < 0$. Let $g(t)$ be the unique solution to the normalised Ricci flow equation (4.1) with $g(0) = g_0$. Then, the flow $g(t)$ converges exponentially in any C^n -norm to a constant curvature Riemannian metric g_∞ , as $t \rightarrow \infty$.

Proof.

By Theorem 4.30, the flow is defined for all times $0 \leq t < \infty$. By Theorem 4.19 the metrics are all uniformly equivalent. By Theorem 4.37, the curvature is exponentially approaching its average. By Corollary 4.29, the result will be accomplished if we show that all derivatives of the curvature are dying exponentially. Now, we show this by induction.

Lemma 4.32. [Estimate to the first derivative of the curvature when $r < 0$]

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a closed surface $\mathbb{M}$ with $r < 0$. Then, there exists $C_1 > 0$ such that

$$|\nabla R|^2 \leq C_1 e^{\frac{r}{2}t} \quad (4.43)$$

Proof.

By (4.6) and (4.24), we have

$$\begin{aligned} \partial_t |\nabla R|^2 &= \Delta |\nabla R|^2 - 2 |\nabla \nabla R|^2 + 4(R - r) |\nabla R|^2 + r |\nabla R|^2 \\ &\leq \Delta |\nabla R|^2 - 2 |\nabla \nabla R|^2 + (4ke^{rt} + r) |\nabla R|^2 \\ &\leq \Delta |\nabla R|^2 + \frac{r}{2} |\nabla R|^2, \end{aligned}$$

for $t \geq t_0$, for some t_0 big enough. But then

$$\frac{\partial}{\partial t} \left(|\nabla R|^2 e^{-\frac{r}{2}t} \right) \leq \Delta \left(|\nabla R|^2 e^{-\frac{r}{2}t} \right), \quad \text{for } t \geq t_0$$

Choose some $C_1 > 0$ such that $|\nabla R|^2 e^{-\frac{r}{2}t} \leq C_1$ holds at t_0 . By the fact that $|\nabla R|^2 e^{-\frac{r}{2}t}$ is a subsolution to the heat equation and by the maximum principle, the thesis follows. ♣

Lemma 4.33. [Preliminary lemma]

Let $\alpha \geq 1$, $r < 0$ and $A > 0$. Consider the solution $\phi(t)$ to the differential equation

$$\frac{\partial \phi}{\partial t} = \alpha r \phi + K e^{rt/2}. \quad (4.44)$$

There exists a constant $D > 0$ such that $\phi \leq D e^{rt/2}$.

Proof.

The solution to (4.44) is given by

$$\phi(t) = e^{\alpha r t} \left\{ \phi(0) + \frac{K}{(\frac{1}{2} - \alpha)r} (e^{(\frac{1}{2} - \alpha)r t} - 1) \right\}.$$

We have,

$$\phi(t) e^{-rt/2} - \frac{K}{(\frac{1}{2} - \alpha)r} = \left(\phi(0) - \frac{K}{(\frac{1}{2} - \alpha)r} \right) e^{rt(\alpha - \frac{1}{2})}.$$

Since the RHS of the equation above is converging to zero, when $t \rightarrow \infty$, $\phi(t) e^{-rt/2}$ converges to $\frac{A}{(\frac{1}{2} - \alpha)r}$.

The lemma follows. ♣

Lemma 4.34. [Estimate to the n th derivative of the curvature]

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a closed surface $\mathbb{M}$ with $r < 0$. Then, for each natural $n \in \mathbb{N}$ there exists a constant $C_n > 0$ such that

$$|\nabla^n R|^2 \leq C_n e^{\frac{r}{2}t} \quad (4.45)$$

Proof.

We suppose by induction that the lemma holds for all $0 \leq i \leq n-1$. Using (4.11), the induction hypotheses implies that

$$\begin{aligned} \partial_t |\nabla^{n-1} R|^2 &= \Delta |\nabla^{n-1} R|^2 - 2 |\nabla^n R|^2 - (n+1)r |\nabla^{n-1} R|^2 \\ &\quad + (\nabla^{n-1} R) \otimes_{g(t)} \left[\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-1-j} R) \right] \\ &\leq \Delta |\nabla^{n-1} R|^2 - 2 |\nabla^n R|^2 + C e^{rt}. \end{aligned}$$

The induction hypotheses also allows us to get the estimate

$$\begin{aligned} \partial_t |\nabla^n R|^2 &= \Delta |\nabla^n R|^2 - 2 |\nabla^{n+1} R|^2 - (n+2)r |\nabla^n R|^2 \\ &\quad + (\nabla^n R) \otimes_{g(t)} \left[\sum_{j=0}^{\lfloor n/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-j} R) \right] \\ &\leq \Delta |\nabla^n R|^2 - (n+2)r |\nabla^n R|^2 + C e^{rt/2} (1 + |\nabla^n R|). \end{aligned}$$

Define $\Phi : \mathbb{M} \times [0, \infty) \rightarrow \mathbb{R}$, by

$$\Phi \doteq |\nabla^n R|^2 - (n+1)r |\nabla^{n-1} R|^2.$$

One readily obtains

$$\begin{aligned}
\partial_t \Phi &\leq \Delta \Phi + nr |\nabla^n R|^2 + ke^{rt/2} (1 + |\nabla^n R|) + k' e^{rt/2} \\
&\leq \Delta \Phi + nr |\nabla^n R|^2 + ke^{rt/2} (|\nabla^n R|^2 + 1) + k'' e^{rt/2} \\
&\leq \Delta \Phi + (nr + ke^{rt/2}) \Phi + (nr + ke^{rt/2})(n+1)r |\nabla^{n-1} R|^2 + k'' e^{rt/2}.
\end{aligned}$$

for some positive constants k'' and k' . Then, there exists a constant k''' such that

$$\partial_t \Phi \leq \Delta \Phi + \frac{nr}{2} \Phi + k''' e^{rt/2}.$$

One sets $k''' = K$ and applies Lemma 4.33 with Theorem 3.4 to get the following estimate on the curvature

$$|\nabla^n R|^2 \leq \Phi \leq De^{rt/2},$$

for some $D > 0$, where D is any constant that satisfies $\phi \leq De^{rt}$, where ϕ is a solution to (4.44) with $\alpha = \frac{n}{2}$.

♣

4.5.2 Convergence when $r=0$.

In this section we are going to establish the following theorem.

Theorem 4.35. [Convergence to a constant curvature metric in the case $r=0$]

Let $\mathbb{M}$ be a closed 2-dimensional manifold with Riemannian metric g_0 and average scalar curvature $r = 0$. Let $g(t)$ be the unique solution to the normalised Ricci flow equation (4.1) with $g(0) = g_0$. Then, the flow $g(t)$ converges uniformly in any C^n -norm to a constant curvature Riemannian metric g_∞ , as $t \rightarrow \infty$.

Proof.

By Theorem 4.30, the flow is defined for all times $0 \leq t < \infty$. By Theorem 4.19 the metrics are all uniformly equivalent. To prove the result we need to show that the scalar curvature and all its derivatives are uniformly converging to zero as $t \rightarrow \infty$. Let us show this now.

Lemma 4.36. [Estimate to the gradient of the potential of the curvature]

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a closed surface $\mathbb{M}$ with $r = 0$. Then, there exists $K > 0$ such that, the Ricci potential satisfy

$$|\nabla f|^2 \leq \frac{K}{1+t}, \text{ for all } t \geq 0, \text{ for all } x \in \mathbb{M}. \quad (4.46)$$

Proof.

By (4.20), we have

$$\partial_t |\nabla f|^2 = \Delta |\nabla f|^2 - 2 |\nabla \nabla f|^2 \leq \Delta |\nabla f|^2,$$

so that $|\nabla f|^2$ is a subsolution to the heat equation. Theorem 3.3 then implies

$$|\nabla f|^2 \leq K_0, \text{ for all } t \geq 0, \text{ and all } x \in \mathbb{M}. \quad (4.47)$$

Furthermore, the two equations

$$\partial_t(t|\nabla f|^2) \leq \Delta(t|\nabla f|^2) + |\nabla f|^2, \quad \text{and} \quad \Delta f^2 = g^{ij}(\partial_i \partial_j - \Gamma_{ij}^k \partial_k)f^2 = 2f\Delta f + 2|\nabla f|^2,$$

together imply

$$\partial_t f^2 = 2f\Delta f = \Delta f^2 - 2|\nabla f|^2,$$

since $\partial_t f = \Delta f + rf$. Putting these equations together, we get

$$\partial_t(t|\nabla f|^2 + f^2) \leq \Delta(t|\nabla f|^2 + f^2).$$

By Theorem 3.3, we conclude that there exists some $K_1 > 0$ such that $t|\nabla f|^2 + f^2 \leq \frac{1}{K_1}$, which together with equation (4.47), implies (4.46). ♣

Lemma 4.37. [Estimate to the curvature when $r=0$]

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a closed surface $\mathbb{M}$ with $r = 0$. Then, there exists $K > 0$ such that for all $t \geq 0$, we have

$$|R| + |\nabla f|^2 \leq \frac{K}{1+t}, \quad \text{for all } t \geq 0, \quad \text{and } x \in \mathbb{M}. \quad (4.48)$$

where f is a Ricci potential.

Proof.

By (4.20) and (4.3), we compute that

$$\partial_t(R + 2|\nabla f|^2) = \Delta(R + 2|\nabla f|^2) + R^2 - 4|\nabla f|^2.$$

Taking the norm of (4.22) yields $|\nabla \nabla f|^2 - \frac{1}{2}(\Delta f)^2 \geq 0$, which implies $-4|\nabla \nabla f|^2 \leq -2(\Delta f)^2 = -2R^2$, so that

$$\partial_t(R + 2|\nabla f|^2) \leq \Delta(R + 2|\nabla f|^2) - R^2.$$

Thus,

$$\begin{aligned} \partial_t \left[t(R + 2|\nabla f|^2) \right] &\leq t\Delta \left[(R + 2|\nabla f|^2) \right] + R + 2|\nabla f|^2 \\ &\quad - tR^2 - 2tR|\nabla f|^2 - 2t|\nabla f|^4 + 2tR|\nabla f|^2 - 2t|\nabla f|^4 \\ &= \Delta \left[t(R + 2|\nabla f|^2) \right] + R + 2|\nabla f|^2 \\ &\quad - \frac{t}{2}R^2 - \frac{t}{2}(R + 2|\nabla f|^2) + 2t|\nabla f|^2(R + |\nabla f|^2). \end{aligned}$$

By Lemma 4.36, there is K_1 such that $t|\nabla f|^2 \leq K_1$. If $R < 0$ then $tR + 2t|\nabla f|^2 < 2K_1$ and $tR + 2t|\nabla f|^2$ is uniformly bounded on its domain. Now, if there is K' large enough so that $t(R + 2|\nabla f|^2) \geq K'$, implies $R \geq 0$ and therefore

$$\begin{aligned}
\partial_t \left[t(R + 2|\nabla f|^2) \right] &\leq \Delta \left[t(R + 2|\nabla f|^2) \right] - \frac{t}{2}R^2 - \frac{t}{2}(R + 2|\nabla f|^2) \\
&\quad + R(1 + 2t|\nabla f|^2) + 2|\nabla f|^2(1 + t|\nabla f|^2) \\
&\leq \Delta \left[t(R + 2|\nabla f|^2) \right] - \frac{1}{2t} \left[t(R + 2|\nabla f|) \right]^2 \\
&\quad - \left(\frac{t}{2}R^2 - R(1 + 2K_1) + \frac{1}{2t}(1 + 2K_1)^2 \right) \\
&\quad + \frac{1}{2t}(1 + 2K_1)^2 + \frac{2K_1}{t}(1 + K_1) \\
&\leq \Delta \left[t(R + 2|\nabla f|^2) \right] - \frac{1}{2t} \left[t(R + 2|\nabla f|) \right]^2 \\
&\quad - \left(\sqrt{\frac{t}{2}}R - \sqrt{\frac{1}{2t}}(1 + 2K_1) \right)^2 + \frac{K_2}{t} \\
&\leq \Delta \left[t(R + 2|\nabla f|^2) \right] - \frac{1}{2t} \left\{ \left[t(R + 2|\nabla f|) \right]^2 - 2K_2 \right\}
\end{aligned}$$

for some K_2 large enough. Then, there is some $K > 0$ large enough that $t(R + 2|\nabla f|^2) \geq K$ implies

$$\partial_t \left[t(R + 2|\nabla f|^2) \right] \leq \Delta \left[t(R + 2|\nabla f|^2) \right],$$

so that $t(R + 2|\nabla f|^2)$ is a subsolution to the heat equation and by Theorem 3.3 it maintains upper bounds. Since Theorem 4.37 uniformly bounds the curvature and Lemma 4.36 uniformly bound $|\nabla f|^2$, the lemma follows. ♣

Lemma 4.38. *[Estimate to the gradient of the curvature when $r=0$]*

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a closed surface $\mathbb{M}$ with $r = 0$. Then, there exists $K > 0$ such that for all $t \geq 0$, we have

$$|\nabla R|^2 \leq \frac{K}{(1+t)^3}, \text{ for all } t \geq 0, \text{ for all } x \in \mathbb{M}. \quad (4.49)$$

Proof.

When $r = 0$, equation (4.6) takes the form

$$\partial_t |\nabla R|^2 = \Delta |\nabla R|^2 - 2|\nabla \nabla R|^2 + 4R|\nabla R|^2 \leq \Delta |\nabla R|^2 + 4R|\nabla R|^2.$$

Let α be a constant to be determined later and define the quantity

$$\varphi : \mathbb{M} \times [0, \infty) \rightarrow \mathbb{R}, \text{ by } \varphi = t^4 |\nabla R|^2 + \alpha t^3 R^2.$$

By (4.3), we have

$$\partial_t (\alpha t^3 R^2) = \Delta (\alpha t^3 R^2) - 2\alpha t^3 |\nabla R|^2 + 2\alpha t^3 R^3 + 3\alpha t^2 R^2$$

and

$$\partial_t (t^4 |\nabla R|^2) \leq 4t^3 |\nabla R|^2 + t^4 (\Delta |\nabla R|^2 + 4R|\nabla R|^2) \Delta (t^4 |\nabla R|^2) + 4t^3 (tR + 1) |\nabla R|^2.$$

Then, we get

$$\partial_t \varphi \leq \Delta \varphi + \alpha(2t^3 R^3 + 3t^2 R^2) + 2t^3(2tR + 2 - \alpha)|\nabla R|^2.$$

By Lemma 4.37, we can choose $2 < \alpha < \infty$ such that $2tR + 2 \leq \alpha$. Therefore

$$\partial_t \varphi \leq \Delta \varphi + K, \quad \text{where } K = \alpha\left(\frac{1}{4}\alpha^3 + \frac{3}{4}\alpha^2\right).$$

By Theorem 3.4, we obtain

$$t^4 |\nabla R|^2 + \alpha t^3 R^2 \leq Kt. \clubsuit$$

Lemma 4.39. [Estimate to the n th derivative of the curvature when $r=0$]

Let $g(t)$ be any solution to the normalized Ricci flow equation (4.1) on a closed surface $\mathbb{M}$ with $r = 0$. Then, for each positive integer n there exists $C_n > 0$ such that

$$|\nabla^n R|^2 \leq \frac{C_n}{(1+t)^{n+2}}, \quad \text{for all } t \geq 0, \quad \text{for all } x \in \mathbb{M}. \quad (4.50)$$

Proof.

We suppose by induction that the lemma holds for all $0 \leq i \leq n-1$. Let N be a constant to be determined later and consider the quantity

$$\Phi : \mathbb{M} \times [0, \infty) \rightarrow \mathbb{R}, \quad \text{defined by } \Phi = t^{n+3} |\nabla^n R|^2 + N t^{n+2} |\nabla^{n-1} R|^2.$$

First, let us estimate each of these terms separately. By (4.11), on one hand, we have

$$\begin{aligned} \partial_t \left(t^{n+3} |\nabla^n R|^2 \right) &= \Delta \left(t^{n+3} |\nabla^n R|^2 \right) - 2t^{n+3} |\nabla^{n+1} R|^2 \\ &\quad + t^{n+3} (\nabla^n R) \otimes_{g(t)} \left[\sum_{j=0}^{\lfloor n/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-j} R) \right] + (n+3)t^{n+2} |\nabla^n R|^2 \\ &\leq \Delta \left(t^{n+3} |\nabla^n R|^2 \right) + t^{\frac{n+2}{2}} (\nabla^n R) \otimes_{g(t)} \left[t^{\frac{n+4}{2}} \sum_{j=0}^{\lfloor n/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-j} R) \right] + (n+3)t^{n+2} |\nabla^n R|^2. \end{aligned}$$

By the induction hypotheses, there are constants a, b_j and B sufficiently large such that

$$\begin{aligned} t^{\frac{n+2}{2}} (\nabla^n R) \otimes_{g(t)} \left[t^{\frac{n+4}{2}} \sum_{j=0}^{\lfloor n/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-j} R) \right] &\leq at^{n+2} (tR) |\nabla^n R|^2 \\ &\quad + \sqrt{t^{n+2} |\nabla^n R|^2} \cdot \sum_{j=1}^{\lfloor n/2 \rfloor} b_j \sqrt{t^{j+2} |\nabla^j R|^2} \sqrt{t^{n-j+2} |\nabla^{n-j} R|^2} \\ &\leq at^{n+2} (tR) |\nabla^n R|^2 + B \sqrt{t^{n+2} |\nabla^n R|^2}. \end{aligned}$$

On the other hand, by (4.11), we have

$$\begin{aligned} \partial_t \left(t^{n+2} |\nabla^{n-1} R|^2 \right) &= \Delta \left(t^{n+2} |\nabla^{n-1} R|^2 \right) - 2t^{n+2} |\nabla^n R|^2 \\ &\quad + t^{\frac{n+1}{2}} (\nabla^{n-1} R) \otimes_{g(t)} \left[t^{\frac{n+3}{2}} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-j-1} R) \right] + (n+2)t^{n+1} |\nabla^{n-1} R|^2 \end{aligned}$$

The induction hypotheses imply that there are constants c_j and C large enough so that

$$t^{\frac{n+1}{2}} (\nabla^{n-1} R) \otimes_{g(t)} \left[t^{\frac{n+3}{2}} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-j-1} R) \right] \leq \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} c_N \sqrt{t^{n+1} |\nabla^{n-1} R|^2 t^{j+2} |\nabla^j R|^2 t^{n-j+1} |\nabla^{n-j-1} R|^2} \leq C.$$

By induction, there exists large E such that

$$\partial_t \Phi \leq \Delta \Phi + t^{n+2} |\nabla^n R|^2 \left\{ atR + (n+3-2N) \right\} + B + Bt^{n+2} |\nabla^n R|^2 + CN + E$$

By our estimate on the curvature from Lemma 4.37, choosing sufficiently large N , it become clear that

$$\partial_t \Phi \leq \Delta \Phi + D.$$

From the maximum principle, Theorem 3.4, we have $\Phi(t) \leq Dt$, for all times, which implies $|\nabla^n R|^2 \leq \frac{D}{t^{n+2}}$. Since $\mathbb{M}$ is compact, we estimate the quantity $|\nabla^n R|^2$ on $\mathbb{M} \times [0, 1]$ and combine these estimates to get the result. ♣

4.6 The case $R(0) > 0$.

In this section we will work the case $R[g_0] > 0$, that is, the scalar curvature at an initial metric g_0 is strictly positive. The more general case $r > 0$ is not within the scope of this text and can be found in [CKKS04].

In this section we will consider a modified Ricci flow

$$\partial_t g = (r - R)g + \mathcal{L}_{\text{grad} f} g = 2M, \quad (4.51)$$

where f is a Ricci potential and the quantity M is defined in equation (4.22). Now we show that the modified Ricci flow is equivalent to the normalised Ricci flow and that the tensor M is invariant under diffeomorphisms.

Lemma 4.40. [The tensor M is invariant by diffeomorphisms]

The trace-free part of the Hessian of the Ricci potential is invariant under diffeomorphisms, that is, if $\psi : \mathbb{M} \rightarrow \mathbb{M}$ is any diffeomorphism then $\psi^* M(g) = M(\psi^* g)$.

Proof.

Notice that if g is any Riemannian metric on $\mathbb{M}$ then $\psi^* g$ is a metric that turn the diffeomorphism ψ into an isometry. Then, the lemma becomes a consequence of Lemma 1.46. ♣

Next, consider the family of vector fields $X(t) = \text{grad} f(\cdot, t) \in \mathfrak{X}(\mathbb{M})$ generated by the Ricci potential, and consider the problem

$$\partial_t \psi_t(p) = -X[\psi_t(p), t] = -\text{grad} f(\psi_t(p), t), \quad \text{with } \psi_0 = Id_{\mathbb{M}}. \quad (4.52)$$

Lemma 4.41. [The solution to the modified flow is equivalent to the normalized Ricci flow]

The solution $g(t)$ to the modified flow (4.51) is equivalent to the normalized Ricci flow in the sense of Definition 2.24, that is, there exists a one-parameter family of diffeomorphisms $\psi(t)$ indexed in time and defined in the same time interval as $g(t)$ such that

$$\bar{g}(t) = \psi(t)^* g(t)$$

is a solution to the normalized Ricci flow.

Proof.

By Theorem 2.25, the solution to the problem (4.52) exists and remains a family of diffeomorphisms for as long as the Ricci potential is defined. The family $\psi(t)$ obtained satisfy (see equation (2.26))

$$\begin{aligned} \partial_t \bar{g}(t) &= \frac{\partial}{\partial s} \Big|_{s=0} \psi_{t+s}^* g(t+s) \\ &= \psi_t^* (\partial_t g) + \frac{\partial}{\partial s} \Big|_{s=0} \psi_{t+s}^* g(t) \\ &= \psi_t^* ((r-R)g + \mathcal{L}_{\text{grad} f} g) - \psi_t^* \mathcal{L}_{\text{grad} f} g \\ &= (r + \bar{R})\bar{g}, \end{aligned}$$

so that, $\bar{g}$ is a solution to the normalized Ricci flow. ♣

We would like to show that this new flow converges to a metric of constant positive scalar curvature. So, if this is the case then eventually the curvature must become greater than a positive constant independent of time, that is, there shall exist $c > 0$ such that $R \geq c$ holds after a while. In order to understand which implications the fact $R \geq c > 0$ has, we must further understand the $\binom{2}{0}$ -tensor field M . Thereby, let us first study the evolution equation satisfied by M and its square norm.

Lemma 4.42. [Evolution of the Tensor M under normalised Ricci flow]

Let $g(t)$ be a solution to the normalized Ricci flow on a surface $\mathbb{M}$. Then, the tensor M evolves by

$$\partial_t M = \Delta M + (r - 2R)M.$$

Proof.

We have,

$$\begin{aligned} \partial_t M_{ij} &= \partial_t \left((\partial_i \partial_j f - \Gamma_{ij}^k \partial_k f) - \frac{1}{2}(R-r)g_{ij} \right) \\ &= \left(\partial_i \partial_j \partial_t f - \Gamma_{ij}^k \partial_k \partial_t f \right) - (\partial_t \Gamma_{ij}^k)(\partial_k f) - \frac{1}{2}(\partial_t R)g_{ij} - \frac{1}{2}(R-r)(\partial_t g_{ij}) \\ &= \left(\nabla_i \nabla_j \partial_t f \right) - (\partial_t \Gamma_{ij}^k)(\partial_k f) - \frac{1}{2}(\Delta R + R(R-r))g_{ij} - \frac{1}{2}(R-r)(r-R)g_{ij} \\ &= \nabla_i \nabla_j \{ \Delta f + r f \} - \frac{1}{2}(\Delta R)g_{ij} - (\partial_t \Gamma_{ij}^k)(\partial_k f) - \frac{1}{2}(R-r)(R+r-R)g_{ij} \\ &= \nabla_i \nabla_j \Delta f - \frac{1}{2}(\Delta(R-r))g_{ij} + r \nabla_i \nabla_j f - r \frac{1}{2} \Delta f g_{ij} - (\partial_t \Gamma_{ij}^k) \partial_k f \\ &= \nabla_i \nabla_j \Delta f - \frac{1}{2}(\Delta \Delta f)g_{ij} + r M_{ij} + \frac{1}{2}(\nabla_i R \nabla_j f + \nabla_j R \nabla_i f - \langle \nabla R, \nabla f \rangle g_{ij}) \end{aligned}$$

where the last line follows from the evolution (4.2), observing that

$$\begin{aligned}
(\partial_t \Gamma_{ij}^k) \partial_k f &= -\frac{1}{2} \left\{ \delta_j^k \nabla_i R + \delta_i^k \nabla_j R - g_{ij} g^{kl} \nabla_l R \right\} \nabla_k f \\
&= -\frac{1}{2} \left\{ \nabla_i R \nabla_j f + \nabla_j R \nabla_i f - \langle \nabla R, \nabla f \rangle g_{ij} \right\}
\end{aligned}$$

Recall the commutator formula $[\Delta, \nabla \nabla](f)$ for functions, obtained at (A.7),

$$\nabla_i \nabla_j \Delta f - \Delta \nabla_i \nabla_j f = -\frac{1}{2} (\nabla_i R \nabla_j f + \nabla_j R \nabla_i f - \langle \nabla R, \nabla f \rangle g_{ij}) - 2R \left(\nabla_i \nabla_j f - \frac{1}{2} \Delta f g_{ij} \right) \quad (4.53)$$

to get

$$\begin{aligned}
\partial_t M_{ij} &= \Delta \left\{ \nabla_i \nabla_j f - \frac{1}{2} (\Delta f) g_{ij} \right\} - 2R \left(\nabla_i \nabla_j f - \frac{1}{2} \Delta f g_{ij} \right) + r M_{ij} \\
&= \Delta M_{ij} - 2R M_{ij} + r M_{ij}. \clubsuit
\end{aligned}$$

Lemma 4.43. [Evolution of the square norm of M]

Let $g(t)$ be a solution to the normalized Ricci flow on a surface $\mathbb{M}$. Then, the tensor M evolves by

$$\partial_t |M|^2 = \Delta |M|^2 - 2|\nabla M|^2 - 2R|M|^2. \quad (4.54)$$

Proof.

It is a straightforward computation:

$$\begin{aligned}
\partial_t |M|^2 &= \partial_t (g^{ik} g^{jl} M_{ij} M_{kl}) \\
&= 2(\partial_t g^{ik}) g^{jl} M_{ij} M_{kl} + 2g^{ik} g^{jl} (\partial_t M_{ij}) M_{kl} \\
&= -2(g^{ip} g^{kq} (\partial_t g_{pq}) g^{jl} M_{ij} M_{kl}) + 2(g^{ik} g^{jl} (\partial_t M_{ij}) M_{kl}) \\
&= -2(r - R)|M|^2 + 2\langle \Delta M, M \rangle + 2(r - 2R)|M|^2 \\
&= \Delta |M|^2 - 2|\nabla M|^2 - 2R|M|^2,
\end{aligned}$$

where in the penultimate equality we have used Lemma A.2. $\clubsuit$

Now, equation 4.54 readily implies that if eventually the scalar curvature becomes greater than a constant $c > 0$, then we can obtain an estimate of the sort $|M| \leq C e^{-ct}$. Indeed, suppose that $R \geq c > 0$. We have

$$\partial_t \left(|M|^2 e^{2ct} \right) = \Delta \left(|M|^2 e^{2ct} \right) - 2|\nabla M|^2 e^{2ct} + 2(c - R)|M|^2 e^{2ct} \leq \left(\Delta |M|^2 e^{2ct} \right),$$

and then, according to the maximum principle, there exists some constant $C > 0$ such that

$$|M| \leq C e^{-ct}. \quad (4.55)$$

This inequality is the key point to study convergence to a metric of constant positive curvature. In fact, we will show that if all derivatives of the tensor M are dying exponentially fast then the modified Ricci flow is converging to a constant positive curvature metric g_∞ . Moreover, if all derivatives of the Hessian of f are vanishing exponentially then the curvature is as well. It will follow that the curvature of the normalised Ricci flow is also converging to a positive constant curvature metric when $t \rightarrow \infty$, since the tensor M is invariant under diffeomorphism. The next step is to show that $R \geq c > 0$. Before this, observe that if the

Ricci flow starts with a metric of positive curvature then it remains so for all times.

Lemma 4.44. [The scalar curvature maintains positive]

Let $g(t)$ be a solution to the normalized Ricci flow equation on a closed surface $\mathbb{M}$ with $r > 0$. Then, if $R(\cdot, 0) > 0$, we have $R(\cdot, t) > 0$, for all times where $g(t)$ is defined, namely $0 \leq t < \infty$.

Proof.

If $s_0 = R_{\min}(0) > 0$, then the fact that $r/s_0 \geq 1$ implies $1 - (1 - r/s_0)e^{rt} > 0$, for all times, and therefore $\frac{r}{1 - (1 - r/s_0)e^{rt}} > 0$, for all $t \in [0, \infty)$. We use this to prove that the curvature is bounded below by the solution of the ODE

$$s' = s^2 - sr, \quad \text{with } s(0) = R_{\min}(0). \quad (4.56)$$

But the curvature evolves by $\partial_t R = \Delta R + R(R - r)$, whose associated ODE is (4.56). Then, the result follows from Theorem 3.4. ♣

4.6.1 Estimates on the surface entropy

We define the **surface entropy** as the quantity

$$N(g(t)) = \int R \log R d\mu. \quad (4.57)$$

The surface entropy is well defined along the normalized Ricci flow, when $R(\cdot, 0) > 0$, according to Lemma 4.44. Next, we compute the surface entropy's evolution equation.

Lemma 4.45. [Evolution of surface entropy]

Let $g(t)$ be a solution to the normalized Ricci flow on a closed surface M such that $R(\cdot, 0) > 0$. Then, the surface entropy evolves by

$$\frac{d}{dt} N = + \int (R - r)^2 d\mu - \int \frac{|\nabla R|^2}{R} d\mu.$$

Proof.

Observe first that the

$$\partial_t(Rd\mu) = (\Delta R + R(R - r))d\mu + R(r - R)d\mu = \Delta R d\mu$$

Next, applying the divergence formula for the vector field $\log R(\text{grad} R)$, we get

$$0 = \int \langle \nabla(\log R), \nabla R \rangle d\mu + \int \log R(\Delta R) d\mu = \int \frac{|\nabla R|^2}{R} d\mu.$$

Then, finding the entropy's evolution becomes straightforward

$$\begin{aligned}
\frac{d}{dt}N &= \int \log R \partial_t(R d\mu) + \int (\partial_t \log R)(R d\mu) \\
&= \int \log R (\Delta R) d\mu + \int \frac{1}{R} (\Delta R + R(R-r)) R d\mu \\
&= \int \Delta R (\log R + 1) d\mu + \int R(R-r) d\mu \\
&= - \int \frac{|\nabla R|^2}{R} d\mu + \int R(R-r) d\mu. \clubsuit
\end{aligned}$$

Lemma 4.46. [The entropy is a decreasing function of time]

Let $g(t)$ be a solution to the normalized Ricci flow on a closed surface $\mathbb{M}$ such that $R(\cdot, 0) > 0$. Then, the surface entropy is decreasing with time, that is

$$\frac{d}{dt}N \leq 0.$$

Proof.

Using the following identities for surfaces

$$\nabla \Delta = \Delta \nabla - \frac{1}{2} R \nabla \quad \text{and} \quad \Delta |\nabla f|^2 = 2 \langle \Delta \nabla f, \nabla f \rangle + 2 |\nabla \nabla f|^2,$$

we have

$$\begin{aligned}
\int \langle \nabla f, \nabla \Delta f \rangle d\mu &= \int \langle \nabla f, \Delta \nabla f \rangle d\mu - \frac{1}{2} \int R |\nabla f|^2 d\mu \\
&= \frac{1}{2} \int \Delta |\nabla f|^2 d\mu - \int |\nabla \nabla f|^2 d\mu - \frac{1}{2} \int R |\nabla f|^2 d\mu \\
&= - \int |\nabla \nabla f|^2 d\mu - \frac{1}{2} \int R |\nabla f|^2 d\mu
\end{aligned}$$

where the last equality follows from the divergence theorem. Next, applying the first Green identity to the function $(\Delta f)^2$, we get

$$\int (R-r)^2 d\mu = \int (\Delta f)^2 d\mu = - \int \langle \nabla f, \nabla \Delta f \rangle d\mu = \int |\nabla \nabla f|^2 d\mu + \frac{1}{2} \int R |\nabla f|^2 d\mu.$$

Now, recall that $|M|^2 = |\nabla \nabla f|^2 - \frac{1}{2} (\Delta f)^2$ to obtain

$$\begin{aligned}
-2 \int |M|^2 d\mu &= \int (\Delta f)^2 d\mu - 2 \int |\nabla \nabla f|^2 d\mu \\
&= - \int |\nabla \nabla f|^2 d\mu + \frac{1}{2} \int R |\nabla f|^2 d\mu \\
&= \int R |\nabla f|^2 d\mu - \int (R-r)^2 d\mu.
\end{aligned}$$

Finally, applying the Green identity to the middle term of the RHS of the equality

$$\frac{|\nabla R + R \nabla f|^2}{R} = \frac{|\nabla R|^2}{R} + 2 \langle \nabla R, \nabla f \rangle + \frac{R^2 |\nabla f|^2}{R},$$

gives us

$$\int \frac{|\nabla R + R \nabla f|^2}{R} d\mu = \int \frac{|\nabla R|^2}{R} d\mu + 2 \int R \Delta f d\mu + \int R |\nabla f|^2 d\mu,$$

and therefore

$$\begin{aligned}
\frac{d}{dt}N &= \int (R-r)^2 d\mu - \int \frac{|\nabla R|^2}{R} d\mu \\
&= - \int (R-r)^2 d\mu - \int \left(\frac{|\nabla R|^2}{R} - 2(R-r)^2 + R|\nabla f|^2 \right) d\mu + \int R|\nabla f|^2 d\mu \\
&= - \int \frac{|\nabla R + R\nabla f|^2}{R} d\mu - 2 \int |M|^2 d\mu \leq 0. \clubsuit
\end{aligned}$$

Corollary 4.47. [Bounds on the entropy]

Let $g(t)$ be a solution to the normalized Ricci flow equation on a closed surface $\mathbb{M}$ such that $R(\cdot, 0) > 0$. Then, there is a constant $C > 0$ depending only on $g(0)$ such that

$$N(t) \leq C, \quad (4.58)$$

for all times where $g(t)$ is defined. In particular, given any $x \in \mathbb{M}$ then, denoting by $B_{g(t)}(x, \epsilon)$ the ball centered at x with radius ϵ measured with respect to the metric $g(t)$, we have

$$C \geq \int_{B_{g(t)}(x, \epsilon)} R \log R d\mu - \frac{1}{e} \int d\mu,$$

where e is the Euler's constant.

Proof.

The inequality 4.58 follows directly from the fact that $\frac{d}{dt}N \leq 0$. For the second part, observe that $\inf_{y>0} (y \log y) = -\frac{1}{e}$. Therefore,

$$\begin{aligned}
C &\geq N(t) = \int R \log R d\mu = \int \left(R \log R + \frac{1}{e} - \frac{1}{e} \right) d\mu \\
&\geq \int_{B_{g(t)}(x, \epsilon)} \left(R \log R + \frac{1}{e} \right) d\mu - \frac{1}{e} \int d\mu \\
&\geq \int_{B_{g(t)}(x, \epsilon)} R \log R d\mu - \frac{1}{e} \int d\mu. \clubsuit
\end{aligned}$$

4.6.2 Bounds for the metric on a surface with $R(0)>0$

Lemma 4.48. [Doubling-time estimate]

Let $g(t)$ be a solution to the normalized Ricci flow equation on a closed surface $\mathbb{M}$ with $R(\cdot, 0) > 0$. Then, for any $t_0 \in [0, \infty)$, we have

$$R_{\max}(t) \leq 2R_{\max}(t_0), \text{ for all } t \in \left[t_0, t_0 + \frac{1}{2R_{\max}(t_0)} \right].$$

Proof.

By Lemma 4.44, $R > 0$ along the normalized Ricci flow. Using equation (4.3), we get

$$\partial_t R = \Delta R + R(R - r) \leq \Delta R + R^2, \text{ for all } x \in \mathbb{M}, \text{ and } t \geq 0.$$

Fixing any $t_0 > 0$ and considering the associated ordinary differential equation

$$\frac{d}{dt} \rho = \rho^2, \text{ with } \rho(t_0) = R_{\max}(t_0) \doteq \rho_0,$$

we get an upper bound for the curvature by the maximum principle, that is

$$R \leq \frac{1}{\rho_0^{-1} + t_0 - t} = \rho(t), \text{ for } t_0 \leq t < \rho_0^{-1} + t_0.$$

Now, shrinking the interval of definition of ρ by half, that is, $t_0 \leq t \leq t_0 + \frac{1}{2}\rho_0^{-1}$, we get

$$\frac{1}{\frac{1}{2}\rho_0^{-1} + t_0 - t} > \frac{1}{\rho_0^{-1} + t_0 - t} = \rho(t) \geq R_{\max}(t).$$

The lemma follows. ♣

We are assuming global existence of the normalized Ricci flow $g(t)$. This means, in particular, that the scalar curvature $R = R(x, g(t))$ is defined for all metrics along the flow. In general, we regard the curvature as a function depending on the point of application and the metric. Thus, along the flow, it doesn't depend on time directly. However, by composing the curvature with a certain curve, we may change this point of view. We may write

$$\tilde{R}(x, t) = R(x, g(t)).$$

So, now we may regard the curvature as a function depending directly on time. In addition, since these functions are the same, we will write $R(x, t)$ instead of $R(x, g(t))$. This affords the following reasoning. Given $x \in \mathbb{M}$, we have

$$g(x, t) = e^{\int_{t_0}^t (r - R(x, \tau)) d\tau} g(x, t_0). \quad (4.59)$$

From now on, we will often write

$$g(\tau) \leq g(s), \text{ to mean that } g(x, \tau)(u, u) \leq g(x, s)(u, u) \text{ for all } x \in \mathbb{M}, \text{ and all } u \in T_x \mathbb{M}.$$

Lemma 4.49. [Bounds on the metric along the normalized Ricci flow]

Let $g(t)$ be a solution to the normalized Ricci flow equation on a closed surface $\mathbb{M}$ with $R(\cdot, 0) > 0$. Then, given any $t_0 \in [0, \infty)$, we have

$$e^{r(t-t_0)} g(t_0) \geq g(t) \geq \frac{1}{e} g(t_0), \text{ for all } t \in \left[t_0, t_0 + \frac{1}{2R_{\max}(t_0)} \right], \quad (4.60)$$

where the first inequality holds for all times $t \geq t_0$.

Suppose, furthermore, that the curvature is uniformly bounded on $\mathbb{M} \times [0, \infty)$, that is, $R \leq C$, for some $C > 0$. Then, we have

$$e^{-C} g(t-1) \leq g(\tau) \leq e^r g(t-1), \text{ for all times } t \geq 1, \text{ and } \tau \in [t-1, t]. \quad (4.61)$$

Proof.

First we prove the lower bound for the metric in (4.60). Given $x \in \mathbb{M}$, by Lemma 4.48, we have

$$\begin{aligned} \int_{t_0}^t (r - R(x, \tau)) d\tau &\geq - \int_{t_0}^t R(x, \tau) d\tau \geq -2 \int_{t_0}^t R_{\max}(t_0) d\tau \\ &\geq -2R_{\max}(t_0) \int_{t_0}^{t_0 + \frac{1}{2R_{\max}(t_0)}} d\tau = -1, \end{aligned}$$

for $t_0 \leq t \leq t_0 + \frac{1}{2R_{\max}(t_0)}$. The lower bound follows from 4.59. For the second inequality we use the fact that $R(t) > 0$, along the normalized Ricci flow (see 4.44). Fixing an arbitrary $x \in \mathbb{M}$, by (4.59), we have

$$g(x, t) = e^{\int_{t_0}^t (r - R(x, \tau)) d\tau} g(x, t_0) \leq e^{\int_{t_0}^t r d\tau} g(x, t_0) = e^{r(t-t_0)} g(x, t_0).$$

In particular, for $0 \leq t-1 \leq \tau \leq t$, we have

$$g(x, \tau) = e^{\int_{t-1}^{\tau} (r - R(x, s)) ds} g(x, t-1) < e^{\int_{t-1}^{\tau} r ds} g(x, t-1) = e^r g(x, t-1).$$

For the last part, suppose that the curvature is uniformly bounded by C . Again, using (4.59), we have

$$g(x, \tau) \geq e^{\int_{t-1}^{\tau} (r - C) ds} g(x, t-1) \geq e^{-C \int_{t-1}^{\tau} ds} g(x, t-1) = e^{-C} g(x, t-1).$$

These inequalities together imply that, for $x \in \mathbb{M}$,

$$e^{-C} g(x, t-1) \leq g(x, \tau) < e^r g(x, t-1), \quad \text{for all } t \geq 1, \text{ and all } \tau \in [t-1, t]. \clubsuit$$

4.6.3 Estimates on the curvature

Lemma 4.50. [Bounds on the scalar curvature]

Let $g(t)$ be any solution to the normalized Ricci flow equation on a compact surface $\mathbb{M}$ with $r > 0$. Suppose that $|R(\cdot, 0)| \leq \kappa$, for some $\kappa > 0$. Then, we have the following estimate

$$|R(x, t)| \leq \frac{\kappa}{1 - \kappa t}, \quad \text{for all } x \in \mathbb{M} \text{ and } t \in [0, 1/\kappa) .$$

In particular, $|R| \leq 2\kappa$ and $4t|R| \leq 1$, on the time interval $[0, 1/(8\kappa))$.

Proof.

According to the evolution of the curvature, equation (4.3), at a maximum of R in space, we have $R_{\max}(t) > 0$, since $r > 0$, and

$$\partial_t R \leq R^2.$$

Considering the equation $\partial_t \rho = \rho^2$, with $\rho(0) = R_{\max}(0) > 0$, whose solution is $\rho(t) = \frac{1}{\rho_0^{-1} - t}$, where $\rho_0 = \rho(0)$, we have $R \leq \rho$, according to maximum principles and since R is a subsolution of the heat equation $\partial_t u = \Delta u + u^2$. Since $R_{\max}(0) \leq \kappa$, we have $\frac{1}{R_{\max}(0)} - t \geq \frac{1}{\kappa} - t$, which implies

$$R \leq \frac{1}{\frac{1}{\kappa} - t} = \frac{\kappa}{1 - \kappa t}, \quad \text{for } t \in [0, 1/\kappa). \quad (4.62)$$

On the other hand, at a minimum of R , we have $\partial_t R \geq R(R - r)$ and $R_{\min}(0) \geq -\kappa$. Considering the equation $\frac{d}{dt} s = s^2 - sr$, with $s(0) = R_{\min}(0)$, whose solution is $s(t) = \frac{r}{1 - (1-r/s_0)e^{rt}}$, where $s_0 = s(0)$, we have

$R \geq s$, since R is a supersolution to the heat equation $\partial_t u = \Delta u + u(u - r)$, and using maximum principles. Then, $0 \neq s_0 \geq -\kappa$ implies $\frac{e^{rt}}{s_0} \leq -\frac{1}{\kappa}$, and since $e^{rt} \geq rt + 1$, we get $\frac{1}{r} - \frac{e^{rt}}{r} \leq -t < t$. Summing these inequalities, we get $\frac{1}{r} - \frac{e^{rt}}{r} + \frac{e^{rt}}{s_0} \leq -\frac{1}{\kappa} + t$, and therefore

$$R(x, t) \geq s(t) \geq \frac{\kappa}{\kappa t - 1}, \text{ for } t \in [0, 1/k]. \quad (4.63)$$

Combining inequalities (4.63) and (4.62) gives $|R| \leq \frac{\kappa}{1-t\kappa}$ and since $-\kappa t \geq 1/2$ on $[0, 1/(2\kappa)]$, we have $\frac{1}{1-\kappa t} \leq 2$, on the same time interval. Therefore,

$$|R| \leq 2\kappa, \text{ and } 4t|R| \leq 1, \text{ on } 0 \leq t \leq 1/(8\kappa). \clubsuit$$

Lemma 4.51. [Bounds on the gradient of the scalar curvature]

Let $g(t)$ be any solution to the normalized Ricci flow on a compact surface $\mathbb{M}$ with $r > 0$. There exists a constant $c_\kappa < \infty$ such that if $|R(\cdot, 0)| \leq \kappa$, for some $\kappa > 0$ then, we have the following estimate

$$|\nabla R(x, t)| \leq \frac{c_\kappa}{\sqrt{t}}, \text{ for all } \mathbb{M} \in M \text{ and } t \in [0, 1/k].$$

Proof.

Let's compute the evolution equation of the quantity $G \doteq t|\nabla R|^2 + R^2$. It follows from equation 4.6 that

$$\partial_t(t|\nabla R|^2) = t(\Delta|\nabla R|^2 - 2|\nabla \nabla R|^2 + (4R - 3r)|\nabla R|^2) + |\nabla R|^2.$$

The fact that $\Delta R^2 = R\Delta R + \langle \nabla R, \nabla R \rangle + R\Delta R$, implies

$$\partial_t R^2 = 2R[\Delta R + R(R - r)] = \Delta R^2 - |\nabla R|^2 + 2R^2(R - r).$$

Therefore,

$$\begin{aligned} \partial_t G &= \Delta G + 4t|\nabla R|^2 + |\nabla R|^2 + 2R^3 \\ &\quad - (2t|\nabla \nabla R|^2 + 2|\nabla R|^2 + 2R^2 r) \\ &\leq \Delta G + (4tR - 1)|\nabla R|^2 + 2R^3 \end{aligned}$$

Now, applying Lemma 4.50, we have

$$\partial_t G \leq \Delta G + 16\kappa^3, \text{ on } 0 \leq t \leq 1/(8\kappa)$$

since both $(4tR - 1) \leq 0$ and $|R| \leq 2\kappa$ holds on $0 \leq t \leq 1/(8\kappa)$. Then, we see that G is a subsolution to the PDE $\partial_t u = \Delta u + 16\kappa^3$. Since $G(0) = R^2(0) \leq \kappa^2$, G is bounded above by the solution of $\xi'(t) = 16\kappa^3$, with $\xi(0) = \kappa^2$, according to the maximum principle, that is,

$$G \leq \kappa^2 + 16\kappa^3 t \leq 3\kappa^2, \text{ on } 0 \leq t \leq 1/(8\kappa).$$

In particular, $|\nabla R| \leq \sqrt{\frac{3}{t}}\kappa$, for $0 \leq t \leq 1/(8\kappa)$. $\clubsuit$

In the previous Lemma, if we had that the curvature were bounded above by κ on the time interval $[0, 1/(4\kappa)]$, i.e., if $R(\cdot, t) \leq \kappa$, on that interval then we would obtain $4tR \leq 1$ on the same interval, what would lead to

$$\partial_t G \leq \Delta G + (4tR - 1)|\nabla R|^2 + 2R^3 \leq \Delta G + 4\kappa^3, \text{ on } [0, 1/(4\kappa)].$$

Therefore, by the maximum principle, $t|\nabla R|^2 \leq G \leq 2\kappa^2$ and $|\nabla R| \leq \frac{\sqrt{2}\kappa}{\sqrt{t}}$, both on $[0, 1/4\kappa]$. This leads to the next result.

Corollary 4.52. *[Bounds on the gradient of the scalar curvature]*

Let $g(t)$ be any solution to the normalized Ricci flow equation on a compact surface $\mathbb{M}$ with $R(\cdot, 0) > 0$. Then, if the curvature satisfies $R(\cdot, t) \leq \kappa$ on $[t_0, t_0 + 1/(4\kappa)]$, we have

$$|\nabla R(x, t)| \leq \frac{\sqrt{2\kappa}}{\sqrt{t - t_0}}, \quad \text{on } [t_0, t_0 + 1/(4\kappa)].$$

Proof.

Define $G(x, t) \doteq (t - t_0)|\nabla R(x, t)|^2 + R^2(x, t)$ on $[t_0, t_0 + \frac{1}{4\kappa}]$ and observe that its evolution follows from the previous evolution, and it satisfies

$$\partial_t G \leq \Delta G + ((t - t_0)4R - 1)|\nabla R|^2 + 2R^3.$$

Since both inequalities $((t - t_0)4R - 1)|\nabla R|^2 \leq 0$ and $R \leq \kappa$ holds on the time interval $[t_0, t_0 + 1/(4\kappa)]$, we may apply the same technique as before to get an upper bound for G , that is,

$$(t - t_0)|\nabla R|^2 \leq G \leq 4\kappa^3 t + \kappa^2 \leq 2\kappa^2, \quad \text{on } [t_0, t_0 + 1/(4\kappa)].$$

The corollary follows. ♣

In the next result, use the fact that on a compact Riemannian manifold any two points can be connected by a curve of shortest length, and this curve is a geodesic (see [JJ08, Chapter 1]).

Lemma 4.53. *[Uniform bounds on the scalar curvature]*

Let $g(t)$ be a solution to the normalized Ricci flow equation on a closed surface $\mathbb{M}$ with $R[g_0] > 0$. Then, there exists a constant $1 \leq C < \infty$ depending only on $g(0)$ such that

$$\sup_{\mathbb{M} \times [0, \infty)} R \leq C.$$

Proof.

Since we have time-dependent bounds on the curvature, we may set

$$\kappa_1 = \max_{\mathbb{M} \times [0, 1]} R.$$

If, for all times $T > 1$, we have $\max_{\mathbb{M} \times [0, T]} R \leq \kappa_1$, then take $C = \kappa_1$. Suppose this is not the case. Let $T > 1$ be arbitrary and set

$$\kappa(T) \doteq \max_{\mathbb{M} \times [0, T]} (R) + 2. \tag{4.64}$$

We are going to show that $\kappa = \kappa(T)$ is bounded independently of T . There is some $(x_1, t_1) \in \mathbb{M} \times]1, T]$ such that $R(x_1, t_1) + 2 = \kappa$. Let $t_0 = t_1 - \frac{1}{4\kappa} > 0$, and observe that $[t_0, t_1] \subset [0, T]$ is contained in an interval on which $R \leq \kappa$. Applying Corollary 4.52, we get

$$|\nabla R(x, t)| \leq \frac{2\kappa}{\sqrt{t - t_0}}, \quad \text{on } [t_0, t_0 + 1/(4\kappa)].$$

In particular, at $t = t_1$, we have $|\nabla R(x, t_1)| \leq 4\kappa^{3/2}$, for all $x \in \mathbb{M}$. Denote by $B_{g(t_1)}(x, \epsilon)$ the ball centered at $x \in \mathbb{M}$ of radius $\epsilon > 0$ measured with respect to the metric $g(t_1)$. Let

$$y \in B_{g(t_1)}(x_1, r), \quad \text{where } r \doteq \frac{1}{\sqrt{64\kappa}}$$

and consider a unit speed minimal geodesic $\gamma : [0, s_0] \doteq I \rightarrow M$, such that $\gamma(s_0) = x_1$ and $\gamma(0) = y$. We have, by the fundamental theorem of calculus,

$$\begin{aligned}
R(x_1, t_1) - R(y, t_1) &= \int_I \frac{\partial}{\partial s} [R(\gamma(s), t_1)] ds \\
&= \int_I \langle \nabla R(\gamma(s), t_1), \gamma'(s) \rangle ds \\
&\leq \int_I |R(\gamma(s), t_1)| ds \\
&\leq \frac{4\kappa^{3/2}}{8\kappa^{1/2}} = \kappa/2,
\end{aligned}$$

for all $y \in B_{g(t_1)}(x_1, r)$. Now we are going to apply Klingenberg's Theorem. Observe that the sectional curvature is bounded above by $R_{\max}(t_1)/2$. Therefore, by Theorem 1.58, we have

$$\text{inj}_{g(t_1)}(\mathbb{M}) \geq \frac{\pi}{\sqrt{\frac{R_{\max}(t_1)}{2}}} > \frac{\pi}{\sqrt{\kappa}}$$

where the last inequality follows from equation (4.64). According to Corollary 4.47, there is some $C > 0$ such that

$$C \geq \int_{B_{g(t_1)}(x, r)} (R \log R) d\mu(g(t_1)) - \frac{1}{e} \int d\mu(g(t_1)).$$

Yet, note that

$$\text{inj}_{g(t_1)}(x_1) \geq \text{inj}_{g(t_1)}(\mathbb{M}) > \frac{8\pi}{8\sqrt{\kappa}} > \frac{1}{\sqrt{64\kappa}} = r.$$

Thanks to the fundamental limit $\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$, the function $\kappa^\bullet \in [0, \infty) \mapsto 8\sqrt{\kappa^\bullet} \cdot \sin\left(\frac{1}{8\sqrt{\kappa^\bullet}}\right)$ is bounded from below by some constant k_0 . Therefore, by the Volume Comparison Theorem 1.59, we have

$$\begin{aligned}
\int_{B_{g(t_1)}(x, r)} (R \log R) d\mu(g(t_1)) &\geq \left(\frac{\kappa}{2} \log \frac{\kappa}{2}\right) V_{g(t_1)}(x_1, r) \\
&\geq \left(\frac{\kappa}{2} \log \frac{\kappa}{2}\right) V_\delta(r) = \left(\frac{\kappa}{2} \log \frac{\kappa}{2}\right) 2\pi\delta^2(1 - \cos r) \\
&= \kappa\pi \left(\log \frac{\kappa}{2}\right) \frac{\kappa^2}{4} \frac{1 - \cos^2 r}{1 + \cos r} \geq \frac{\kappa^2\pi}{8} \left(\log \frac{\kappa}{2}\right) \frac{1}{64} \left(8\sqrt{\kappa} \cdot \sin \frac{1}{8\sqrt{\kappa}}\right)^2 \\
&\geq \frac{\kappa^2\pi}{8} \left(\log \frac{\kappa}{2}\right) \frac{1}{64} k_0^2 \doteq C_0 \left(\log \frac{\kappa}{2}\right).
\end{aligned}$$

In particular,

$$C \geq C_0 \left(\log \frac{\kappa}{2}\right) - \frac{1}{e} V(\mathbb{M}, g(t_1)),$$

which implies $\kappa \leq 2e^{C_1}$, where C_1 is a constant which depends only on $g(0)$, by Lemma 4.49. ♣

We now set uniform bounds on the diameter of the manifold. Recall that given a metric g on a manifold $\mathbb{M}$ then the diameter of $\mathbb{M}$ measured with respect to g is given by

$$\text{diam}_g(\mathbb{M}) \doteq \sup_{p, q \in \mathbb{M}} \text{dist}_g(p, q) \in \mathbb{R}_+ \cup \{\infty\}.$$

Before going into details, observe that although we don't have uniform lower bounds on the curvature, we have lower bounds on the time function $R_{\max} : [0, \infty) \rightarrow \mathbb{R}$. Indeed, since the average scalar curvature r is independent of time, we have

$$r = \frac{\int R d\mu}{\int d\mu} \leq R_{\max}.$$

Thus, $R_{\max}$ can not approach zero. In particular, there exists $c > 0$ such that $c \leq R_{\max}(t)$ holds for all times $t \geq 0$.

Corollary 4.54. [*Uniform bounds on the diameter of the manifold*]

Let $g(t)$ be any solution to the normalized Ricci flow equation on a closed surface $\mathbb{M}$ with $R[g_0] > 0$. Then, there exists a constant $C_1 > 0$ depending only on g_0 such that the diameter of the manifold remains bounded along the flow, that is,

$$\text{diam}(\mathbb{M}, g(t)) \leq C_1 \quad (4.65)$$

Proof.

Fix an arbitrary moment $t > 0$. If, for every two points $p, q \in \mathbb{M}$, we had

$$\text{dist}_{g(t)}(p, q) \leq \frac{2\pi}{\sqrt{R_{\max}(t)}},$$

then the diameter would be bounded by the quantity $2\pi/c$ for all times, where c^2 is any lower bound for $R_{\max}$. Let us suppose that this is not the case. Fix $t > 0$. Suppose that there are N points $p_1, \dots, p_N \in \mathbb{M}$ such that the distance between them is greater than $2\pi/\sqrt{R_{\max}(t)}$, that is,

$$\text{dist}_{g(t)}(p_i, p_j) \geq \frac{2\pi}{\sqrt{R_{\max}(t)}}, \quad \text{for all } i \neq j, \quad i, j \in \{1, \dots, N\}.$$

Now, the sectional curvature of $\mathbb{M}$ at t satisfies $K(t) = R(t)/2 \leq R_{\max}(t)/2$, so that, according to Klingenberg's Lemma 1.56, we have

$$\text{inj}_{g(t)}(\mathbb{M}) \geq \frac{\pi}{\sqrt{R_{\max}(t)/2}}.$$

Fixing $r \doteq \frac{\pi}{\sqrt{R_{\max}(t)/2}}$ and $\delta \doteq R_{\max}(t)/2$ we see that the balls $B_{g(t)}(p_i, r)$ are embedded and pairwise disjoint. Also, $\text{inj}_{g(t)}(p_i) \geq \text{inj}_{g(t)}(\mathbb{M}) \geq \frac{\pi}{\sqrt{\delta}} = r$. Thus, we may apply Theorem 1.59 to get

$$V_{g(t)}(p_i, r) \geq V_{\delta}(r) > 0$$

and the number $V_{\delta}(r) = 4\pi\delta^2(1 - \cos(r))$ cannot approach zero since the curvature $R = R(t)$ is bounded, by Lemma 4.53 (see Example 1.60). Therefore, Theorem 1.59 implies that there exists $\epsilon > 0$ such that

$$\int d\mu(g(t)) \geq \sum_{i=1}^N V_{g(t)}(p_i, r) \geq N\epsilon.$$

Finally, recall that the volume of any manifold is constant in the direction of the normalized Ricci flow and conclude that at any moment the number of disjoint open balls of radius $\geq \frac{\pi}{\sqrt{R_{\max}/2}}$ is bounded, because

$$N \leq \frac{1}{\epsilon} \int d\mu(g(t)). \clubsuit$$

4.6.4 Differential Harnack estimates

Let $L = \log R$. Since $R > 0$ for all times where the normalized Ricci flow is defined, L is well defined. Observe that, in this case, the gradient Ricci soliton equation (4.14) implies (see Lemma 4.14)

$$\nabla R + R\nabla f = 0, \quad (4.66)$$

and that $L + f$ is constant in space. Then, define the **differential Harnack quantity** by

$$Q \doteq \Delta L + R - r. \quad (4.67)$$

Next, we compute the evolution of these quantities.

Lemma 4.55. [Evolution equation for the logarithm of the curvature]

Let $g(t)$ be any solution to the normalized Ricci flow equation on a surface $\mathbb{M}$ with $R_{\min}(0) > 0$. Then, the quantity $L = \log R$ evolves by

$$\partial_t L = \Delta L + |\nabla L|^2 + R - r. \quad (4.68)$$

Proof.

Since

$$\Delta L = g^{ij} \left(\frac{1}{R} \partial_i \partial_j R - \frac{1}{R} \Gamma_{ij}^k \partial_k R - \frac{1}{R^2} \partial_i R \partial_j R \right) = \frac{1}{R} \Delta R - \frac{1}{R^2} |\nabla R|^2,$$

by (4.3), we have

$$\partial_t L = \frac{1}{R} (\Delta R + R(R - r)) = \Delta L + |\nabla L|^2 + (R - r). \clubsuit$$

Lemma 4.56. [Evolution equation of the differential Harnack quantity]

Let $g(t)$ be any solution to the normalized Ricci flow equation on a surface $\mathbb{M}$ with $R_{\min}(0) > 0$. Then, the differential Harnack quantity Q evolves by

$$\partial_t Q = \Delta Q + 2\langle \nabla Q, \nabla L \rangle + 2 \left| \nabla \nabla L + \frac{1}{2}(R - r)g \right|^2 + rQ.$$

Proof.

Using the Ricci identity for surfaces $\Delta \nabla - \nabla \Delta = \frac{1}{2} R \nabla$ and equation (A.2), we get

$$\begin{aligned} \Delta |\nabla L|^2 &= 2\langle \Delta \nabla L, \nabla L \rangle + 2|\nabla \nabla L|^2 \\ &= 2\langle \nabla \Delta L, \nabla L \rangle + R|\nabla L|^2 + 2|\nabla \nabla L|^2 \\ &= 2\langle \nabla Q, \nabla L \rangle - 2\langle \nabla R, \nabla L \rangle + R|\nabla L|^2 + 2|\nabla \nabla L|^2 \\ &= 2\langle \nabla Q, \nabla L \rangle + 2|\nabla \nabla L|^2 - R|\nabla L|^2. \end{aligned}$$

By (4.68), we have

$$\Delta(\partial_t L) = \Delta Q + 2\langle \nabla Q, \nabla L \rangle + 2|\nabla \nabla L|^2 - R|\nabla L|^2$$

Therefore, using the evolution of the Laplacian, (4.4) and (4.3), we get

$$\begin{aligned}
\partial_t Q &= \partial_t \Delta L + \partial_t R \\
&= (R-r)\Delta L + \Delta \partial_t L + \Delta R + R(R-r) \\
&= \Delta Q + 2\langle \nabla Q, \nabla L \rangle - R|\nabla L|^2 - (R-r)\Delta L - (R-r)^2 + R(R-r)\Delta R \\
&\quad + 2|\nabla \nabla L|^2 + 2(R-r)\Delta L + (R-r)^2 \\
&= \Delta Q + 2\langle \nabla Q, \nabla L \rangle - \frac{1}{R}|\nabla R|^2 - (R-r)\{\Delta L + R-r\} + R(R-r) + \Delta R \\
&\quad + 2\left|\nabla \nabla L + \frac{1}{2}(R-r)g\right|^2.
\end{aligned}$$

The result follows from

$$-\frac{1}{R}|\nabla R|^2 - R\{\Delta L + R-r\} + R(R-r) + \Delta R = 0. \clubsuit$$

Corollary 4.57. [*Estimates for the differential Harnack quantity*]

Let $g(t)$ be any solution to the normalized Ricci flow equation on a compact surface M with $R_{\min}(0) > 0$. Then, the differential Harnack quantity Q satisfies

$$\partial_t Q \geq \Delta Q + 2\langle \nabla Q, \nabla L \rangle + Q^2 + rQ. \quad (4.69)$$

Proof.

Set $S = \nabla \nabla L + \frac{1}{2}(R-r)g$ and use the fact that $|T|^2 \geq \frac{1}{N}(\text{tr} T)^2$, for any $\binom{2}{0}$ -tensor field on an N -dimensional manifold (see Lemma 1.20). We have, $\text{tr} S = \Delta L + (R-r) = Q$. Therefore, $2|S|^2 \geq Q^2$, and this accomplishes Corollary 4.69. $\clubsuit$

Lemma 4.58. [*Bounds for the differential Harnack quantity*]

Let $g(t)$ be a complete solution to the normalized Ricci flow equation on a closed surface $\mathbb{M}$ with $R_{\min}(0) > 0$ and bounded scalar curvature. Then, there exists some $C > 1$ depending only on $g(0)$ such that

$$\partial_t \log R - |\nabla \log R|^2 = Q \geq -\frac{C e^{rt}}{C e^{rt} - 1}. \quad (4.70)$$

Proof.

Choose some $q_0 < -r < 0$ that bounds the continuous function Q from below at the initial moment, $Q(0) \geq q_0$. Then, since Q is a supersolution to the heat type equation $\partial_t u = \Delta u + 2\langle \nabla u, X(t) \rangle + u^2 + ru$, it satisfies, by the maximum principle, $Q(\cdot, t) \geq q(t)$, where q is the solution of the initial value problem:

$$q'(t) = q^2(t) + rq(t), \quad \text{with } q(0) = q_0,$$

namely, $q(t) = -\frac{rq_0 e^{rt}}{q_0 e^{rt} - q_0 - r}$. On the other hand, by 4.68, we have

$$\partial_t L - |\nabla L|^2 = \Delta L + R - r = Q,$$

which proves the equality. $\clubsuit$

Notice that the function $q(t)$ satisfies $q(0) < 0$ and q is decreasing, since $q' = q(q+r) < 0$.

Before continuing, let us make a brief digression. Given a differentiable curve $\gamma : I \rightarrow \mathbb{M}$, $\gamma = \gamma(t)$, then we may compose the scalar curvature R with γ and regard it as only a function of t , for appropriate time interval I , that is, $t \in I \mapsto R(\gamma(t), t)$. Then, taking the ordinary derivative of this composition, we get

$$\frac{d}{dt}(R(\gamma(t), t)) = \left\langle (\text{grad} R)(\gamma(t), t), \frac{d\gamma}{dt}(t) \right\rangle + \left(\frac{\partial}{\partial t} R \right)(\gamma(t), t), \quad (4.71)$$

where $\partial_t R = \frac{\partial}{\partial t} R$ denotes, as usual, the evolution of the curvature.

Lemma 4.59. [Bounding the quotient of curvatures]

Let $g(t)$ be a complete solution to the normalized Ricci flow equation with bounded and positive scalar curvature on a closed surface $\mathbb{M}$. Then, there exists constants C_1 and C_2 depending only on $g(0)$ such that, for all $x_1, x_2 \in \mathbb{M}$ and times $0 \leq t_1 < t_2$, we have

$$\frac{R(x_2, t_2)}{R(x_1, t_1)} \geq e^{-A/4} \frac{C_1 e^{rt_1} - 1}{C_1 e^{rt_2} - 1} \geq e^{-A/4 - C_2(t_2 - t_1)}, \quad (4.72)$$

where

$$A = A(x_1, t_1, x_2, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} \left| \frac{d\gamma}{dt} \right|^2 dt,$$

and the infimum is taken over all C^1 curves joining x_1 to x_2 and defined on $[t_1, t_2]$.

Proof.

Let $\gamma : [t_1, t_2] \rightarrow \mathbb{M}$ be any C^1 curve joining x_1 to x_2 , where x_1, x_2, t_1 , and t_2 are fixed. By the same reasoning of 4.71, using the estimate 4.70 and regarding the curvature as only a function of time, $t \mapsto (\log R)(\gamma(t), t)$, we have

$$\begin{aligned} \frac{d}{dt}(\log R(\gamma(t), t)) &= \left(\frac{\partial}{\partial t} \log R \right)(\gamma(t), t) + \left\langle (\nabla \log R)(\gamma(t), t), \frac{d\gamma}{dt}(t) \right\rangle \\ &\geq |\nabla \log R(\gamma(t), t)|^2 - \frac{C r e^{rt}}{C e^{rt} - 1} + \left\langle (\nabla \log R)(\gamma(t), t), \frac{d\gamma}{dt}(t) \right\rangle \\ &= \left| \nabla \log R(\gamma(t), t) + \frac{1}{2} \frac{d\gamma}{dt}(t) \right|^2 - \frac{1}{4} \left| \frac{d\gamma}{dt}(t) \right|^2 - \frac{C r e^{rt}}{C e^{rt} - 1} \\ &\geq -\frac{1}{4} \left| \frac{d\gamma}{dt}(t) \right|^2 - \frac{C r e^{rt}}{C e^{rt} - 1} \end{aligned}$$

Now, by the fundamental theorem of calculus, we get

$$\begin{aligned} \log \frac{R(x_2, t_2)}{R(x_1, t_1)} &= \int_{t_1}^{t_2} \frac{d}{dt}(\log R) dt \\ &\geq \int_{t_1}^{t_2} \left(-\frac{1}{4} \left| \frac{d\gamma}{dt} \right|^2 - \frac{C r e^{rt}}{C e^{rt} - 1} \right) dt \\ &= -\frac{1}{4} \int_{t_1}^{t_2} \left| \frac{d\gamma}{dt} \right|^2 dt - \log(C e^{rt_2} - 1) + \log(e^{C r t_1} - 1). \end{aligned}$$

From these inequalities, we get

$$\log \frac{R(x_2, t_2)}{R(x_1, t_1)} \geq -\frac{1}{4} A - \log(C e^{rt_2} - 1) + \log(e^{C r t_1} - 1),$$

from which the Lemma follows from exponentiating. ♣

Lemma 4.60. [Uniform positive lower bound on the curvature]

Let $g(t)$ be a complete solution to the normalized Ricci flow equation with bounded scalar curvature on a closed surface $\mathbb{M}$ with $R[g_0] > 0$. Then, there exist a constant c depending only on $g(0)$ such that,

$$R(x, t) \geq c > 0, \text{ for all } x \in \mathbb{M}, \text{ and } t \geq 0. \quad (4.73)$$

Proof.

In order to get estimates on the curvature, let us study the quantity $A = A(x_1, t_1, x_2, t_2)$ and try to get upper bounds for A for convenient variables $x_1, x_2 \in M$, and $t_1, t_2 \in [0, \infty)$. Fix any point and time (x, t) such that $t \geq 1$ and consider the curvature at $t - 1$. Since r is the average curvature, there must be some $x_1 \in M$ such that $R(x_1, t - 1) \geq r$. Let $\gamma : [t - 1, t] \rightarrow \mathbb{M}$ be a geodesic joining x_1 and x of constant speed with respect to $g(t)$ such that the distance from x_1 to x agrees with its length, i.e.,

$$\int_{t-1}^t |\gamma'(\tau)|_{g(t)} d\tau = |\gamma'(\tau)|_{g(t)} = \text{dist}_{g(t)}(x_1, x).$$

By (4.61), we have

$$|\gamma'(\tau)|_{g(\tau)}^2 \leq e^r \langle \gamma'(\tau), \gamma'(\tau) \rangle_{g(t-1)} \leq e^{r+C} \langle \gamma'(\tau), \gamma'(\tau) \rangle_{g(t)} = e^{r+C} |\gamma'(\tau)|_{g(t)}^2,$$

$\tau \in [t - 1, t]$, where C is the uniform bound on the curvature from Lemma 4.50.

$$\begin{aligned} A(x_1, t - 1, x, t) &= \inf_{\sigma} \int_{t-1}^t \left| \frac{d\sigma}{dt} \right|_{g(t)}^2 dt \leq \int_{t-1}^t \left| \frac{d\gamma}{d\tau} \right|_{g(\tau)}^2 d\tau \\ &\leq e^{r+C} \int_{t-1}^t \left| \frac{d\gamma}{d\tau} \right|_{g(t)}^2 d\tau \leq e^{r+C} [\text{dist}_{g(t)}(x_1, x)]^2 \end{aligned}$$

Using that the diameter of the manifold is bounded above by some constant C_1 along the flow, we have

$$A(x_1, t - 1, x, t) \leq e^{r+C} [\text{dist}_{g(t)}(x_1, x)]^2 \leq e^{r+C} C_1^2 \doteq k_1,$$

where the RHS of the equation above is independent of time and the point x . Now, applying Lemma 4.72, we get

$$R(x, t) \geq e^{-A/4-C_2} R(x_1, t - 1) \geq r e^{-A/4} e^{-C_2} \geq r e^{-k_1/4} e^{-C_2}.$$

Observe that for the interval $[0, 1]$, we have $ce^{-rt} \leq R$. These estimates on the curvature together prove the claim. ♣

4.6.5 Convergence when $R > 0$

In this section, the main goal is to establish the following theorem.

Theorem 4.61. [Convergence to a constant curvature metric]

Let $\mathbb{M}$ be a closed 2-dimensional manifold with Riemannian metric g_0 and positive scalar curvature $R[g_0] > 0$. Let $g(t)$ be the unique solution to the normalised Ricci flow equation (4.1) with $g(0) = g_0$. Then, the family $g(t)$ converges exponentially in any C^n -norm to a constant curvature Riemannian metric g_∞ , as $t \rightarrow \infty$.

For the proof of Theorem 4.61, we shall assume the following theorem.

Theorem 4.62. [All derivatives of M are dying exponentially]

Let $g(t)$ be any solution to the modified Ricci flow (4.51) defined for $0 \leq t < \infty$ on a closed surface $\mathbb{M}$ with $r > 0$. Then, for every $n \in \mathbb{N}$ there are (positive) constants K_n and k_n such that

$$|\nabla^n M| \leq K_n e^{-k_n t} \quad \text{and} \quad |\nabla^n \nabla^2 f| \leq K_n e^{-k_n t}. \quad (4.74)$$

Theorem 4.62 is the key point to obtain a constant curvature metric in the case $R[g_0] > 0$. Suppose it is valid. Then, the modified Ricci flow converges exponentially fast to a constant curvature metric when $t \rightarrow \infty$. To guide the reader's understanding, we summarize the relevant arguments here.

STEP 1. We apply Lemma 3.9 to conclude that all Riemannian metrics $g(t)$ are uniformly equivalent and that they are converging uniformly on $\mathbb{M}$ to a continuous, bilinear, symmetric and positive definite metric g_∞ when $t \rightarrow \infty$. Since we have $|M|^2 \leq C e^{-ct}$, by (4.55), this is well established.

STEP 2. We apply the same reasoning as in Theorem 3.11 to conclude that the limit metric g_∞ is differentiable (C^∞). In this step we require that both the tensor M and the family $g(t)$ to be uniformly bounded, so we can apply the derivative under the integral sign. Notice that we did a similar argument at Section 4.5. More precisely, in Corollary 4.29 we established differentiability of the limit metric under estimates to all derivatives of the curvature.

STEP 3. From the previous two steps we conclude that the tensor M is converging exponentially fast in all C^n -norms to a Riemannian metric g_∞ such that $M(g_\infty) \equiv 0$. By Lemma 4.14, g_∞ is a gradient Ricci soliton. By Corollary 4.22 g_∞ is a constant curvature metric on the surface $\mathbb{M}$.

STEP 4. The estimates (4.74) on all derivatives of the tensor M imply that all derivatives of the curvature are dying exponentially fast. Since the normalised Ricci flow (4.1) and the modified Ricci flow (4.51) are equivalent by Lemma 4.41 we obtain that all derivatives of the curvature from the normalised Ricci flow are dying exponentially fast and therefore converging to a constant curvature metric.

Steps 1,2 and 3 are clear. We show step 4. The curvature from the modified Ricci flow satisfies

$$|R - r| \leq |\nabla \nabla f - \frac{1}{2}(R - r)g| + |\nabla \nabla f| \leq K'_0 e^{-k'_0 t}.$$

Next, $\nabla M = \nabla^3 f - \frac{1}{2} \nabla R \otimes g$, so that

$$|\nabla R| = \frac{1}{\sqrt{2}} |\nabla R \otimes g| \leq \sqrt{2} |\nabla^3 f - \frac{1}{2} \nabla R \otimes g| + \sqrt{2} |\nabla^3 f| \leq K'_1 e^{-k'_1 t},$$

since $|\nabla R \otimes g|^2 = 2|\nabla R|^2$. In general, the n th (covariant) derivative of M is $\nabla^n M = \nabla^n \nabla^2 f - \frac{1}{2} \nabla^n R \otimes g$ and the same estimate follows readily. From these remarks, we obtain

$$|\nabla^n R| \leq K'_n e^{-k'_n t}.$$

Now, recall the one-parameter family of diffeomorphisms ψ_t obtained from (4.52). The one-parameter family of metrics $\bar{g}(t) = \psi(t)^* g(t)$ is a solution to the normalised Ricci flow equation by Lemma 4.41 and then, by diffeomorphism invariance it follows that the same estimates are obtained to the curvature of the unnormalised Ricci flow, that is, if $\bar{R}$ is the curvature from $\bar{g}$ then

$$|\bar{\nabla}^n \bar{R}| \leq K'_n e^{-k'_n t}.$$

(The scalar curvature is a local isometry invariant, see Lemma 1.46). This proves Theorem 4.61.

Appendix A

Miscellanea

A.1 Ricci calculus

A.1.1 Some formulas

Lemma A.1. [Bochner Identity]

Let $g = \langle \cdot, \cdot \rangle$ be any Riemannian metric on an N -dimensional manifold $\mathbb{M}$. Then, we have

$$\Delta|\nabla f|^2 = 2\langle \Delta \nabla f, \nabla f \rangle + 2|\nabla \nabla f|^2 + 2Rc(\nabla f, \nabla f), \quad (\text{A.1})$$

for functions $f \in C^\infty(\mathbb{M})$

Proof.

Let $f \in C^\infty(\mathbb{M})$ be any function. Let $(U, (x^i))$ be any system of coordinates on $\mathbb{M}$. We have

$$\begin{aligned} \frac{1}{2}\Delta(|\nabla f|^2) &= \frac{1}{2}g^{kl}\nabla_k\nabla_l(g^{ij}\partial_i f\partial_j f) \\ &= g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f\nabla_j f + g^{kl}g^{ij}\nabla_i\nabla_l\nabla_k f\nabla_j f \\ &= g^{kl}g^{ij}\left\{\{\nabla_k\nabla_l - \nabla_l\nabla_k\}\nabla_i f\right\}\nabla_j f + g^{ij}\nabla_i\Delta f\nabla_j f + |\nabla \nabla f|^2 \\ &= g^{kl}g^{ij}\left\{-R_{kil}{}^s\nabla_s f\right\}\nabla_j f + \langle \nabla \Delta f, \nabla f \rangle + |\nabla \nabla f|^2 \\ &= +g^{ij}g^{kl}R_{ik}{}^s\nabla_j f\nabla_l f + \langle \nabla \Delta f, \nabla f \rangle + |\nabla \nabla f|^2 \\ &= Rc(\nabla f, \nabla f) + \langle \nabla \Delta f, \nabla f \rangle + |\nabla \nabla f|^2. \end{aligned}$$

In particular, applying the formula $\Delta \nabla_i f = \nabla_i \Delta f + g^{kl}R_{il}{}^k \nabla_k f$ to the second term in the last equality above, we get

$$\langle \nabla \Delta f, \nabla f \rangle = \langle \Delta \nabla f, \nabla f \rangle - Rc(\nabla f, \nabla f),$$

which implies the useful formula

$$\frac{1}{2}\Delta|\nabla f|^2 = \langle \Delta \nabla f, \nabla f \rangle + |\nabla \nabla f|^2. \clubsuit \quad (\text{A.2})$$

Let's see now a slightly different result concerning the laplace applied to square norm of tensors.

Lemma A.2. *[Laplace applied to the square norm of 2-covariant tensors]*

Let $Q \in \Gamma(T^2\mathbb{M})$ be any $\binom{2}{0}$ -tensor field on an N -dimensional Riemannian manifold $\mathbb{M}$. Then, we have

$$\Delta|Q|^2 = 2\langle \Delta Q, Q \rangle + 2|\nabla Q|^2.$$

Proof.

Let $(U, (x^i))$ be any system of coordinates, and write $Q = Q_{ij}dx^i \otimes dx^j$ in these coordinates. We have

$$\begin{aligned} \frac{1}{2}\Delta|Q|^2 &= \frac{1}{2}g^{pq}\nabla_p\nabla_q(g^{ik}g^{jl}Q_{ij}Q_{kl}) \\ &= g^{pq}\nabla_p(g^{ik}g^{jl}(\nabla_q Q_{ij})Q_{kl}) \\ &= g^{pq}g^{ik}g^{jl}(\nabla_p\nabla_q Q_{ij})Q_{kl} + g^{pq}g^{ik}g^{jl}(\nabla_q Q_{ij})(\nabla_p Q_{kl}) \\ &= g^{ik}g^{jl}(\Delta Q_{ij})Q_{kl} + \langle \nabla Q, \nabla Q \rangle \\ &= \langle \Delta Q, Q \rangle + \langle \nabla Q, \nabla Q \rangle. \clubsuit \end{aligned}$$

Lemma A.3. *[Laplace applied to the square norm of 3-covariant tensors]*

Let $T \in \Gamma(T^3\mathbb{M})$ be any $\binom{3}{0}$ -tensor field on an N -dimensional Riemannian manifold $\mathbb{M}$. Then, we have

$$\Delta|T|^3 = 2\langle \Delta T, T \rangle + 2|\nabla T|^2.$$

Proof.

Let $(U, (x^i))$ be any system of coordinates, and write $T = T_{ijk}dx^i \otimes dx^j \otimes dx^k$ in these coordinates. We have

$$\begin{aligned} \frac{1}{2}\Delta|T|^2 &= \frac{1}{2}g^{pq}\nabla_p\nabla_q(g^{il}g^{jm}g^{kn}T_{ijk}T_{lmn}) \\ &= g^{pq}\nabla_p(g^{il}g^{jm}g^{kn}(\nabla_q T_{ijk})T_{lmn}) \\ &= g^{pq}g^{il}g^{jm}g^{kn}(\nabla_p\nabla_q T_{ijk})T_{lmn} + g^{pq}g^{il}g^{jm}g^{kn}(\nabla_q T_{ijk})(\nabla_p T_{lmn}) \\ &= g^{il}g^{jm}g^{kn}(\Delta T_{ijk})T_{lmn} + \langle \nabla T, \nabla T \rangle \\ &= \langle \Delta T, T \rangle + \langle \nabla T, \nabla T \rangle. \clubsuit \end{aligned}$$

In particular, if M is a $\binom{2}{0}$ -tensor field over $\mathbb{M}$ then $\nabla M \in \Gamma(T^3M)$ and the formula (A.3) above reads

$$\Delta|\nabla M|^3 = 2\langle \Delta \nabla M, \nabla M \rangle + 2|\nabla \nabla M|^2.$$

This formula may be extended to covariant tensors of any rank, as the next lemma shows.

Lemma A.4. *[Laplace applied to the square norm of k -covariant tensors]*

Let $S \in \Gamma(T^k\mathbb{M})$ be any $\binom{k}{0}$ -tensor field on an N -dimensional Riemannian manifold $\mathbb{M}$. Then, we have

$$\Delta|S|^2 = 2\langle \Delta S, S \rangle + 2|\nabla S|^2.$$

Proof.

One just arranges the indices appropriately and apply the same reasoning as above. $\clubsuit$

A.1.2 Ricci Identities

The commutator formulas for tensors are among the most useful tool in the Ricci calculus. We start showing the commutator for $\binom{1}{0}$ -tensors.

Lemma A.5. [Commutator formula for 1-forms]

Let $g = \langle , \rangle$ be any Riemannian metric on an N -dimensional manifold $\mathbb{M}$. Then, we have

$$\{\nabla_i \nabla_j - \nabla_j \nabla_i\} \alpha_k = -R_{ijk}{}^m \alpha_m. \quad (\text{A.3})$$

for any $\binom{1}{0}$ -tensor field α .

Proof.

It is sufficient to compute in normal coordinates (U, x^i) based at $p \in \mathbb{M}$. Let $\alpha = \alpha_k dx^k$ be the coordinate representation of the $\binom{1}{0}$ -tensor field α in those coordinates. We have, at p

$$\begin{aligned} \{\nabla_i \nabla_j - \nabla_j \nabla_i\} \alpha_k &= \partial_i (\partial_j \alpha_k - \Gamma_{jk}^m \alpha_m) - \partial_j (\partial_i \alpha_k - \Gamma_{ik}^m \alpha_m) \\ &= -(\partial_i \Gamma_{jk}^m) \alpha_m + (\partial_j \Gamma_{ik}^m) \alpha_m \\ &= R_{jik}{}^m \alpha_m. \clubsuit \end{aligned}$$

Lemma A.6. [Commutator formula for 2-covariant tensors]

Let $g = \langle , \rangle$ be any Riemannian metric on an N -dimensional manifold $\mathbb{M}$. Then, we have

$$\{\nabla_i \nabla_j - \nabla_j \nabla_i\} \alpha_{kl} = -R_{ijk}{}^m \alpha_{ml} - R_{ijl}{}^m \alpha_{km}, \quad (\text{A.4})$$

for any $\binom{2}{0}$ -tensor field α .

Proof.

Let $(U, (x^i))$ be normal coordinates based at some point $p \in \mathbb{M}$. We have, at p

$$\begin{aligned} \{\nabla_i \nabla_j - \nabla_j \nabla_i\} \alpha_{kl} &= \partial_i \partial_j \alpha_{kl} - (\partial_i \Gamma_{jk}^m) \alpha_{ml} - (\partial_i \Gamma_{jl}^m) \alpha_{km} \\ &\quad - \partial_j \partial_i \alpha_{kl} + (\partial_j \Gamma_{ik}^m) \alpha_{ml} + (\partial_j \Gamma_{il}^m) \alpha_{km} \\ &= -\left((\partial_i \Gamma_{jk}^m) \alpha_{ml} - (\partial_j \Gamma_{ik}^m) \alpha_{ml}\right) - \left((\partial_i \Gamma_{jl}^m) \alpha_{km} - (\partial_j \Gamma_{il}^m) \alpha_{km}\right) \\ &= -R_{ijk}{}^m \alpha_{ml} - R_{ijl}{}^m \alpha_{km}. \clubsuit \end{aligned}$$

This formula holds for tensors of any rank. Next, we enunciate the general formula for covariant tensors.

Lemma A.7. [Commutator formula for s -covariant tensors]

Let $g = \langle , \rangle$ be any Riemannian metric on an N -dimensional manifold $\mathbb{M}$. Then, we have

$$\{\nabla_i \nabla_j - \nabla_j \nabla_i\} \alpha_{k_1 k_2 \dots k_s} = - \sum_{l=1}^s R_{ijk_l}{}^m \alpha_{k_1 k_2 \dots k_{l-1} m k_{l+1} \dots k_s} \quad (\text{A.5})$$

for any $\binom{s}{0}$ -tensor field α .

Proof. ♣¹

More generally, given a $\binom{p}{q}$ -tensor field A , one has the commutator

$$\begin{aligned} [\nabla_i, \nabla_j] A_{k_1 \dots k_p}^{l_1 \dots l_q} &\doteq \nabla_i \nabla_j A_{k_1 \dots k_p}^{l_1 \dots l_q} - \nabla_j \nabla_i A_{k_1 \dots k_p}^{l_1 \dots l_q} \\ &= \sum_{r=1}^q R_{ijm}{}^{l_r} A_{k_1 \dots k_p}^{l_1 \dots l_{r-1} m \dots l_q} - \sum_{s=1}^p R_{ijk_s}{}^m A_{k_1 \dots k_{s-1} m \dots k_p}^{l_1 \dots l_q}. \end{aligned} \quad (\text{A.6})$$

Now, let us see some applications of these commutators. Let us start with the case of the bracket $[\Delta, \nabla^k]$ applied to functions $f \in C^\infty(\mathbb{M})$ on 2-dimensional manifolds. The N-dimensional case is obtained with little modifications.

Lemma A.8. *[A commutator formula for 2-Riemannian manifolds]*

Consider a function $f \in C^\infty(\mathbb{M})$, on a 2-Riemannian manifold $\mathbb{M}$, then the following formula holds

$$\nabla_i \nabla_j \Delta f - \Delta \nabla_i \nabla_j f = -\frac{1}{2}(\nabla_i R \nabla_j f + \nabla_j R \nabla_i f - \langle \nabla R, \nabla f \rangle g_{ij}) - 2R \left(\nabla_i \nabla_j f - \frac{1}{2} \Delta f g_{ij} \right) \quad (\text{A.7})$$

Proof.

We apply the commutator formulas (A.4) and (A.3) after summing and subtracting convenient terms:

$$\begin{aligned} \nabla_i \nabla_j \Delta f &= g^{kl} \nabla_i \{ \nabla_j \nabla_k - \nabla_k \nabla_j \} \nabla_l f + g^{kl} \nabla_i \nabla_k \nabla_j \nabla_l f \\ &= g^{kl} \nabla_i \{ -R_{jkl}{}^p \nabla_p f \} + g^{kl} \{ \nabla_i \nabla_k - \nabla_k \nabla_i \} \nabla_j \nabla_l f + g^{kl} \nabla_k \nabla_i \nabla_j \nabla_l f \\ &= -\frac{1}{2} \nabla_i R \nabla_j f - \frac{R}{2} \nabla_i \nabla_j f + g^{kl} \{ -R_{ikj}{}^p \nabla_p \nabla_l f - R_{ikl}{}^p \nabla_j \nabla_p f \} + g^{kl} \nabla_k \nabla_i \nabla_j \nabla_l f \\ &= -\frac{1}{2} \nabla_i R \nabla_j f - \frac{R}{2} \nabla_i \nabla_j f + \frac{R}{2} g_{ij} \Delta f - R \nabla_i \nabla_j f + g^{kl} \nabla_k \{ \nabla_i \nabla_l - \nabla_l \nabla_i \} \nabla_j f + \Delta \nabla_i \nabla_j f \\ &= \Delta \nabla_i \nabla_j f - \frac{1}{2} (\nabla_i R \nabla_j f + \nabla_j R \nabla_i f - \langle \nabla R, \nabla f \rangle g_{ij}) - 2R \nabla_i \nabla_j f + R \Delta f g_{ij}. \clubsuit \end{aligned}$$

Another application of the Ricci identities is as follows.

Theorem A.9. *[Commutator formula for 2-Riemannian manifolds]*

Consider a function $f \in C^\infty(\mathbb{M})$, on a 2-Riemannian manifold $\mathbb{M}$, then the following formula holds

$$\begin{aligned} \nabla_i \Delta \nabla_j \nabla_k f &= \Delta \nabla_i \nabla_j \nabla_k f + R g_{ij} (\Delta \nabla_k f) + R g_{ik} (\Delta \nabla_j f) \\ &\quad - R \nabla_j \nabla_i \nabla_k f - R \nabla_k \nabla_i \nabla_j f - \frac{R}{2} \nabla_i \nabla_j \nabla_k f \\ &\quad - \frac{1}{2} (\nabla_j R) (\nabla_i \nabla_k f) - \frac{1}{2} (\nabla_k R) (\nabla_i \nabla_j f) + \frac{1}{2} g_{ij} (\nabla^q R) (\nabla_q \nabla_k f) + \frac{1}{2} g_{ik} (\nabla^q R) (\nabla_j \nabla_q f) \end{aligned} \quad (\text{A.8})$$

Proof.

Let us organize the expression into simpler terms:

$$\begin{aligned} \nabla_i \Delta \nabla_j \nabla_k f &= g^{pq} \nabla_i \nabla_p \nabla_q \nabla_j \nabla_k f \\ &= g^{pq} \{ \nabla_i \nabla_p - \nabla_p \nabla_i \} \nabla_q \nabla_j \nabla_k f \\ &\quad + g^{pq} \nabla_p \{ \nabla_i \nabla_q - \nabla_q \nabla_i \} \nabla_j \nabla_k f + \Delta \nabla_i \nabla_j \nabla_k f \end{aligned} \quad (\text{A.9})$$

¹The proof is more an issue of organizing the indices than technical difficulty. After arranging the terms, one applies the reasoning as in the other proofs.

We calculate each of these terms separately. On one hand, we have

$$\begin{aligned} g^{pq} \{ \nabla_i \nabla_p - \nabla_p \nabla_i \} \nabla_q \nabla_j \nabla_k f &= g^{pq} \{ -R_{ipq}{}^s \nabla_s \nabla_j \nabla_k f - R_{ipj}{}^s \nabla_q \nabla_s \nabla_k f - R_{ipk}{}^s \nabla_q \nabla_j \nabla_s f \} \\ &= -\frac{R}{2} \nabla_i \nabla_j \nabla_k f - \frac{R}{2} \nabla_j \nabla_i \nabla_k f - \frac{R}{2} \nabla_k \nabla_j \nabla_i f \\ &\quad + g_{ij} \frac{R}{2} \Delta \nabla_k f + g_{ik} \frac{R}{2} \Delta \nabla_j f. \end{aligned}$$

On the other hand,

$$\begin{aligned} g^{pq} \nabla_p \{ \nabla_i \nabla_q - \nabla_q \nabla_i \} \nabla_j \nabla_k f &= g^{pq} \nabla_p \{ -R_{iqj}{}^s \nabla_s \nabla_k f - R_{iqk}{}^s \nabla_j \nabla_s f \} \\ &= -\frac{1}{2} (\nabla_j R) (\nabla_i \nabla_k f) - \frac{1}{2} (\nabla_k R) (\nabla_i \nabla_j f) - \frac{1}{2} R \nabla_j \nabla_i \nabla_k f - \frac{1}{2} R \nabla_k \nabla_j \nabla_i f \\ &\quad + \frac{1}{2} g_{ij} (\nabla^q R) (\nabla_q \nabla_k f) + \frac{1}{2} g_{ij} R (\Delta \nabla_k f) + \frac{1}{2} g_{ik} (\nabla^q R) (\nabla_j \nabla_q f) + \frac{1}{2} g_{ik} R (\Delta \nabla_j f) \end{aligned}$$

Substituting these terms into the organizing formula (A.9), we get (A.8). ♣

A.1.3 Higher order derivatives of the curvature

We would like now to generalize the formula (A.7) for the scalar curvature, that is, we are looking for a formula of the kind

$$[\Delta, \nabla^n]R = \Delta \nabla^n R - \nabla^n \Delta R.$$

Let us use induction on n to obtain the following.

Lemma A.10. [Commutator formula for the scalar curvature on 2-manifolds]

If $\mathbb{M}$ is a 2-dimensional Riemannian manifold in which there is a differentiable family of Riemannian metrics $g(t)$ then,

$$\nabla^n \Delta R - \Delta \nabla^n R = \sum_{j=0}^{\lfloor n/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-j} R), \quad (\text{A.10})$$

where $X \otimes_{g(t)} Y$ denotes a finite linear combination of contractions of the tensors X and Y with respect to the metric $g(t)$.

Proof.

We show the formula A.10 by induction on n . For $n=1$, the formula obviously holds. For $n=2$, it is equation A.7, which becomes, for f equals the scalar curvature,

$$\nabla_i \nabla_j \Delta R - \Delta \nabla_i \nabla_j R = -\nabla_i R \nabla_j R - \frac{R}{2} \nabla_i \nabla_j R + \frac{1}{2} (|\nabla R|^2 - R \Delta R) g_{ij}$$

Suppose the formula holds for some n . We are going to prove that it does for $n+1$. We have,

$$\nabla^{n+1} \Delta R - \Delta \nabla^{n+1} R = \nabla \{ \nabla^n \Delta R - \Delta \nabla^n R \} + \nabla \Delta \nabla^n R - \Delta \nabla^{n+1} R$$

On the first term on the RHS of the above equation we use induction to conclude that the covariant derivative of equation (A.10) is again a linear combination of the same sort. For the second term we use identity (A.5) and observe that, in coordinates, it becomes

$$\begin{aligned}
\nabla_i \Delta \nabla_{k_1} \dots \nabla_{k_n} R - \Delta \nabla_i \nabla_{k_1} \dots \nabla_{k_n} R &= \\
&= g^{pq} \{ \nabla_i \nabla_p - \nabla_p \nabla_i \} \nabla_q \nabla_{k_1} \dots \nabla_{k_n} R + g^{pq} \nabla_p \{ \nabla_i \nabla_q - \nabla_q \nabla_i \} \nabla_{k_1} \dots \nabla_{k_n} R \\
&= g^{pq} \left(-R_{ipq} {}^m \nabla_m \nabla_{k_1} \dots \nabla_{k_n} R - \sum_{l=1}^n R_{ipk_l} {}^m \nabla_q \nabla_{k_1} \dots \nabla_{k_{l-1}} \nabla_m \dots \nabla_{k_n} R \right) + \\
&\quad g^{pq} \nabla_p \left\{ - \sum_{l=1}^n R_{iqk_l} {}^m \nabla_{k_1} \dots \nabla_{k_{l-1}} \nabla_m \dots \nabla_{k_n} R \right\}
\end{aligned}$$

We now analyse some terms in particular so we may see that these contributions are also linear combinations of $\nabla^j R$ and $\nabla^{k-j} R$ for certain values of j . For instance, the following two terms appear among the calculations. First,

$$g^{pq} R_{ipq} {}^m = g^{mj} R_{ij} = g^{mj} g_{ij} R / 2 = \delta_i^m R / 2.$$

Second,

$$g^{pq} R_{ipk_l} {}^m = g^{pq} g^{mj} R_{ipk_l j} = g^{pq} g^{mj} (R/2) \{ g_{ij} g_{pk_l} - g_{ik_l} g_{pj} \} = \delta_i^m \delta_{k_l}^q (R/2) - \delta_p^m g^{pq} g_{ik_l} (R/2)$$

After a moment's thought one concludes that the terms remaining are of the desired kind. ♣

Lemma A.11. [Higher order derivatives of the scalar curvature]

Suppose $\mathbb{M}$ is a 2-dimensional manifold in which there is a family of Riemannian metrics $g(t)$ satisfying the normalized Ricci flow equation

$$\partial_t g = (r - R)g.$$

Then, we have the following formulas

$$\nabla^n R^2 = \sum_{j=0}^{\lfloor n/2 \rfloor} (\nabla^j R) \otimes_{g(t)} (\nabla^{n-j} R), \quad (\text{A.11})$$

$$\left(\frac{\partial}{\partial t} \Gamma \right) \otimes_{g(t)} (\nabla^j R) = (\nabla R) \otimes_{g(t)} (\nabla^j R), \quad (\text{A.12})$$

and,

$$\frac{\partial}{\partial t} \left(\nabla_{k_1} \nabla_{k_2} \dots \nabla_{k_n} R \right) = \nabla_{k_1} \{ \partial_t \nabla_{k_2} \dots \nabla_{k_n} R \} - \sum_{l=2}^n \left(\partial_t \Gamma_{k_1 k_l}^m \right) \nabla_{k_2} \dots \nabla_{k_{l-1}} \nabla_m \dots \nabla_{k_n} R. \quad (\text{A.13})$$

Proof.

All these formulas are straightforward. The first is achieved by induction. The second is an application of the evolution of the connection under the normalized Ricci flow obtained at Lemma (4.3). The third formula follows from

$$\begin{aligned}
 \partial_t \left(\nabla_{k_1} \nabla_{k_2} \dots \nabla_{k_n} R \right) &= \partial_t \left(\partial_{k_1} \nabla_{k_2} \dots \nabla_{k_n} R - \sum_{l=2}^n \Gamma_{k_1 k_l}^m \nabla_{k_2} \dots \nabla_{k_{l-1}} \nabla_m \dots \nabla_{k_n} R \right) \\
 &= \partial_{k_1} \partial_t \nabla_{k_2} \dots \nabla_{k_n} R - \sum_{l=2}^n (\Gamma_{k_1 k_l}^m) \partial_t \nabla_{k_2} \dots \nabla_{k_{l-1}} \nabla_m \dots \nabla_{k_n} R \\
 &\quad - \sum_{l=2}^n (\partial_t \Gamma_{k_1 k_l}^m) \left(\nabla_{k_2} \dots \nabla_{k_{l-1}} \nabla_m \dots \nabla_{k_n} R \right) \\
 &= \nabla_{k_1} \{ \partial_t \nabla_{k_2} \dots \nabla_{k_n} R \} - \sum_{l=2}^n (\partial_t \Gamma_{k_1 k_l}^m) \left(\nabla_{k_2} \dots \nabla_{k_{l-1}} \nabla_m \dots \nabla_{k_n} R \right). \clubsuit
 \end{aligned}$$

A.2 DeTurck's vector field W and independence of coordinates

Let $(U, (x^i))$ be any system of coordinates and recall the vector field W defined locally in (2.24) by

$$W = W^k \partial_k, \text{ where } W^k = g^{pq} (\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k), \quad (\text{A.14})$$

where the Γ is the Riemannian connection with respect to the metric g and $\tilde{\Gamma}$ is a Riemannian connection obtained from a Riemannian metric $\tilde{g}$. We are going to refer to this vector field as **DeTurck's vector field W**.

Theorem A.12. *[The DeTurck's vector field W is globally defined]
The vector field W defined in (2.24) is independent of coordinates.*

Proof.

Consider a system of coordinates (U, χ) and write its coordinate functions as $\chi = (x^1, \dots, x^N)$. Let (V, ψ) be another system of coordinates, write $\psi = (y^1, \dots, y^N)$ and denote the Christoffel symbols from ψ as Φ_{pq}^k . Let us show the following

$$W^k = h^{pq} (\Phi_{pq}^k - \tilde{\Phi}_{pq}^k)$$

where $H = (h_{ij})$ is the matrix obtained from applying the metric g to the coordinate vector fields $\frac{\partial}{\partial y^s}$, and (h^{ij}) is its inverse. Recall that we are considering the Riemannian connection ∇ generated by the metric g and a fixed background connection $\tilde{\nabla}$ generated by a metric $\tilde{g}$. This is a very long computation, so we highlight the main passages of the proof. Before doing so, we establish some formulas we're using along. Recall the formula 1.4 for the change of coordinates involving involving matrices

$$H = [D(\chi \circ \psi^{-1})^T \circ G \circ D(\chi \circ \psi^{-1})],$$

which implies

$$H^{-1} = [D(\psi \circ \chi^{-1}) \circ G^{-1} \circ D(\psi \circ \chi^{-1})^T]$$

and therefore

$$h^{\xi\zeta} = \frac{\partial y^\xi}{\partial x^l} \frac{\partial y^\zeta}{\partial x^k} g^{lk}, \quad \text{or also,} \quad g^{pq} = \frac{\partial x^p}{\partial y^\eta} \frac{\partial x^q}{\partial y^\nu} h^{\eta\nu}.$$

Let us now write the Christoffel symbols of Γ in terms of these changes. Recall first that

$$\Gamma_{pq}^k = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x^p} g_{ql} + \frac{\partial}{\partial x^q} g_{pl} - \frac{\partial}{\partial x^l} g_{pq} \right)$$

then rewrite the terms involving elements from G in terms of elements from H to get

$$\left(\frac{\partial}{\partial x^p} g_{ql} + \frac{\partial}{\partial x^q} g_{pl} - \frac{\partial}{\partial x^l} g_{pq} \right) = 2 \frac{\partial^2 y^\xi}{\partial x^p \partial x^q} \frac{\partial y^\zeta}{\partial x^l} h_{\xi\zeta} + \frac{\partial y^\xi}{\partial x^q} \frac{\partial y^\zeta}{\partial x^l} \frac{\partial y^\tau}{\partial x^p} \frac{\partial}{\partial y^\tau} h_{\xi\zeta} + \frac{\partial y^\xi}{\partial x^p} \frac{\partial y^\zeta}{\partial x^l} \frac{\partial y^\tau}{\partial x^q} \frac{\partial}{\partial y^\tau} h_{\xi\zeta} - \frac{\partial y^\xi}{\partial x^p} \frac{\partial y^\zeta}{\partial x^q} \frac{\partial y^\tau}{\partial x^l} \frac{\partial}{\partial y^\tau} h_{\xi\zeta}.$$

Then, we compute the difference of the symbols $\Gamma - \tilde{\Gamma}$ to get

$$\begin{aligned} \Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k &= \frac{1}{2} h^{\eta\nu} \left\{ \frac{\partial y^\xi}{\partial x^q} \frac{\partial y^\tau}{\partial x^p} \frac{\partial}{\partial y^\tau} h_{\xi\nu} + \frac{\partial y^\xi}{\partial x^p} \frac{\partial y^\tau}{\partial x^q} \frac{\partial}{\partial y^\tau} h_{\xi\nu} - \frac{\partial y^\xi}{\partial x^p} \frac{\partial y^\zeta}{\partial x^q} \frac{\partial}{\partial y^\nu} h_{\xi\zeta} \right\} \frac{\partial x^k}{\partial y^\eta} \\ &\quad - \frac{1}{2} \tilde{h}^{\eta\nu} \left\{ \frac{\partial y^\xi}{\partial x^q} \frac{\partial y^\tau}{\partial x^p} \frac{\partial}{\partial y^\tau} \tilde{h}_{\xi\nu} + \frac{\partial y^\xi}{\partial x^p} \frac{\partial y^\tau}{\partial x^q} \frac{\partial}{\partial y^\tau} \tilde{h}_{\xi\nu} - \frac{\partial y^\xi}{\partial x^p} \frac{\partial y^\zeta}{\partial x^q} \frac{\partial}{\partial y^\nu} \tilde{h}_{\xi\zeta} \right\} \frac{\partial x^k}{\partial y^\eta}. \end{aligned}$$

We are now ready to compute the components of the vector field W in this new system of coordinates:

$$\begin{aligned} g^{pq}(\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k) &= \frac{\partial x^p}{\partial y^\beta} \frac{\partial x^q}{\partial y^\gamma} h^{\beta\gamma} (\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k) \\ &= h^{\beta\gamma} \left\{ \frac{1}{2} h^{\eta\nu} \left(\frac{\partial}{\partial y^\beta} h_{\gamma\nu} + \frac{\partial}{\partial y^\gamma} h_{\beta\nu} - \frac{\partial}{\partial y^\nu} h_{\beta\gamma} \right) - \frac{1}{2} \tilde{h}^{\eta\nu} \left(\frac{\partial}{\partial y^\beta} \tilde{h}_{\gamma\nu} + \frac{\partial}{\partial y^\gamma} \tilde{h}_{\beta\nu} - \frac{\partial}{\partial y^\nu} \tilde{h}_{\beta\gamma} \right) \right\} \frac{\partial x^k}{\partial y^\eta} \\ &= h^{\beta\gamma} \{ \Phi_{\beta\gamma}^\eta - \tilde{\Phi}_{\beta\gamma}^\eta \} \frac{\partial x^k}{\partial y^\eta}, \end{aligned}$$

which means that

$$h^{\beta\gamma} \{ \Phi_{\beta\gamma}^\eta - \tilde{\Phi}_{\beta\gamma}^\eta \} \frac{\partial x^k}{\partial y^\eta} = W^k = W(dx^k).$$

Remembering the formula for the change of coordinates of 1-forms $dy^\alpha = \frac{\partial y^\alpha}{\partial x^k} dx^k$, we have

$$W(dy^\alpha) = h^{\beta\gamma} \{ \Phi_{\beta\gamma}^\alpha - \tilde{\Phi}_{\beta\gamma}^\alpha \}$$

This proves the independence claimed. ♣

Remark A.13. As we have mentioned in the preliminaries, given any two affine connections, say, ∇^0 and ∇^1 , we may define a $\binom{2}{1}$ -tensor field over $\mathbb{M}$ as the difference of these two connections, that is, if $A : \mathfrak{X}(\mathbb{M}) \times \mathfrak{X} \rightarrow \mathfrak{X}(\mathbb{M})$, is defined by $A(X, Y) = \nabla_X^1 Y - \nabla_X^0 Y$ then A is a tensor field over $\mathbb{M}$ (see the exercises in [Lee06, Chapter 4]). Now, if one notices that the coefficients of W (see equation (2.24)) is given as the difference of two connections, then one may also conclude that W is indeed a $\binom{0}{1}$ -tensor field (globally defined) over $\mathbb{M}$.

A.3 The Divergence Theorem

Theorem A.14. [The Divergence Theorem]

Let $(\mathbb{M}, g)$ be an oriented Riemannian manifold with or without boundary. For any compactly supported differentiable vector field $X \in \mathfrak{X}(\mathbb{M})$,

$$\int_{\mathbb{M}} (\operatorname{div} X) dV_g = \int_{\partial \mathbb{M}} \langle X, N \rangle_g dV_{\tilde{g}},$$

where N is the outward-pointing unit normal vector field along $\partial \mathbb{M}$ and $\tilde{g}$ is the induced Riemannian metric on $\partial \mathbb{M}$.

Proof. See [Lee03, Theorem 16.32]. ♣

Lemma A.15. [The only harmonic functions on compact Riemannian manifolds are constants]

Let $(\mathbb{M}, g)$ be a Riemannian manifold without boundary. A function $f \in C^\infty(\mathbb{M})$ is said to be harmonic if $\Delta f = 0$. If $\mathbb{M}$ is connected and compact then the only harmonic functions on $\mathbb{M}$ are constants.

Proof.

Let f be any harmonic function over $\mathbb{M}$. Since $\mathbb{M}$ has no boundary, according to the divergence theorem, we have

$$0 = \int \operatorname{div}(f \operatorname{grad} f) dV = \int |\operatorname{grad} f|^2 dV + \int f \Delta f dV = \int |\operatorname{grad} f|^2 dV.$$

Since $|\operatorname{grad} f|^2 \geq 0$, we have $|\operatorname{grad} f|^2 = 0$, and f is constant on connected components of $\mathbb{M}$. ♣

Corollary A.16. If $(\mathbb{M}, g(t))$ is a family of Riemannian metrics defined on the time interval $[0, T)$ then the only harmonic functions $f \in C^\infty(\mathbb{M} \times [0, T))$ are functions of time alone, $f(x, t) = \psi(t) : [0, T) \rightarrow \mathbb{R}$.

Proposition A.17. [Any two Riemannian metrics are equivalent on compact manifolds]

Suppose $\mathbb{M}$ is a differentiable manifold. Let $\Gamma(S_2^+ T^* \mathbb{M})$ be the space of Riemannian metrics over $\mathbb{M}$. If $\mathbb{M}$ is compact, then any two metrics are uniformly equivalent. In other words, given $g, h \in \Gamma(S_2^+ T^* \mathbb{M})$ there is a positive constant $k \in \mathbb{R}$ such that

$$\frac{1}{k} g(x) \leq h(x) \leq k g(x), \quad \text{for all } x \in \mathbb{M}.$$

A.4 Linearization of partial differential operators

Let us consider a non-linear partial differential operator L , and a vector bundle $\pi : E \rightarrow \mathbb{M}$

$$L : \Gamma(E) \rightarrow \Gamma(E), \quad (\text{A.15})$$

Let us define the linearisation of L . Given an arbitrary section $w \in \Gamma(E)$ and a section $v \in \Gamma(E)$ which may be thought of as a tangent vector to $\Gamma(E)$ at w then we define the linearisation of L at w as being

$$DL(w) : \Gamma(E) \rightarrow \Gamma(E), \quad DL(w)v = \left. \frac{d}{dt} \right|_{t=0} (L \circ \gamma)(t) \quad (\text{A.16})$$

where $\gamma : (-\delta, \delta) \rightarrow \Gamma(E)$ is any curve on the space of sections such that

$$\begin{cases} \gamma(0) = w \\ \gamma'(0) = v. \end{cases}$$

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