A categorial foundation for a representation theory of logics

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Resumo

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Neste trabalho estabelecemos uma base teórica para a construção de uma teoria de representação de lógicas proposicionais. Iniciamos identificando uma relação precisa entre a categoria das lógicas (Blok-Pigozzi) algebrizáveis e a categoria de suas classes de álgebras associadas. Assim obtemos codificações funtoriais para as equipolências e morfismos densos entre lógicas. Na tentativa de generalizar os resultados obtidos sobre a codificação dos morfismos entre lógicas algebrizáveis, introduzimos a noção de funtor filtro e sua lógica associada. Classificamos alguns tipos especiais de lógicas e um estudo da propriedade metalógica de interpolação de Craig via amalgamação em matrizes para lógicas não-protoalgebrizáveis, e estabelecemos a relação entre a categoria dos funtores filtros e a categoria de lógicas. Em seguida, empregamos noções da teoria das instituições para definir instituições para as lógicas proposicionais abstratas, para uma lógica algebrizável e para uma lógica Lindenbaum algebrizável. Sobre a instituição das lógicas algebrizáveis (lógicas Lindenbaum algebrizáveis), estabelecemos uma versão abstrata do Teorema de Glivenko e que é exatamente o tradicional teorema de Glivenko quando aplicado entre a lógica clássica e intuicionista. Por fim, influenciado pela teoria de representação para anéis, apresentamos os primeiros passos da teoria de representação de lógicas. Introduzimos as definições de diagramas modelos à esquerda para uma lógica, Morita equivalência e Morita equivalência estável para lógicas. Mostramos que quaisquer representações para lógica clássica são estavelmente Morita equivalentes, entretanto a lógica clássica e intuicionista não são estavelmente Morita equivalentes.

Palavras-chave: Lógicas algébricas abstratas, Lógicas algebrizáveis, Teoria das categorias.

Abstract

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In this work we provide a framework in order to build a representation theory of propositional logics. We begin identifying a precise relation between the category of (Blok-Pigozzi) algebraizable logic and the category of their classes of associated algebras. Then, we have a functorial codification for the equipollence and dense morphisms between logics. Attempting generalize the results found before about codification of morphisms among algebraizable logics, we introduce the notion of filter functor and its associated logic. We classify some special kinds of logics and a study of a meta-logical Craig interpolation property via matrices amalgamation for non-protoalgebraizable logics, and we establish a relation between the category of filter functors and the category of logics. In the sequel, we employ notions of institution theory to define the institutions for the abstract propositional logics, for an algebraizable logic and Lindenbaum algebraizable logic. On the institutions for algebraizable logics (Lindenbaum algebraizable logics), we introduce the abstract Glivenko's theorem and this notion is exactly the traditional Glivenko's theorem when applied between the classical logic and intuitionistic logic. At last, influenced by the representation theory of rings, we present the first steps on the representation theory of logics. We introduce the definition of left diagram model for a logic, Morita equivalence of logics and stably-Morita equivalence for logics. We have showed that any presentation for classical logic are stably-Morita equivalent, but the classical logic and intuitionistic logic are not stably-Morita equivalent.

Keywords: Abstract algebraic logic, Algebraizable logic, Category theory.

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Introduction

The aim of this thesis is establishing a concise foundation in order to build a "good" representation theory for propositional logics and then use that to unify properties among logics, solve many different open problems in abstract logic and universal logic, study translations of meta-logical properties between logics and classify the many kinds of logics. The idea to begin the construction of a representation theory of logic has been the well-known and useful representation theory of rings in algebra. We intend get results in logic through of the similar way of the results in algebra have emerged. The fundamental tools that we used here to get results to develop the representation theory of logic are the theory of *Abstract logics* and *Category Theory*. Influenced by the methods of combining logics, it has emerged the study of the category of abstract logics, or simply the category of logics. From that the researchers started to use the wide knowledge about abstract algebraic logic and the category theory to solve problems about logics.

The theory of *Abstract Algebraic Logics* (*AAL*) nowadays can be seen as a theory that studies the connections between logic and algebra. Those connections allow one to use the powerful tools of universal algebra to study meta-logical properties. *AAL* was born with the work of Boole, Pierce, De Morgan, Schröder, etc. on classical logic. Through of Hilbert's idea of metamathematics, the study of logics has been focused around the formal notions of assertion, i.e., logical validity and theoremhood, and logical inference. Thus, we have two approach to the subject, one centered on the notion of logical equivalence and the other centered on the notions of assertion and inference. On those two distinct approaches about logic began the attempts to connect them. Lindenbaum and Tarski were the first ones to describe a precise connection between those distinct approaches. On the Lindenbaum's idea of viewing the set of formulas as an algebra with operators induced by the logical connectives, Tarski gave the precise connection between classical propositional calculus and Boolean algebras. The logical equivalence is an equivalence relation on the formulas algebra, and the associated quotient algebra turns out to be a free Boolean algebra. This method to connect logic and algebra is the so-called $Lindenbaum - Tarski$ *method*. This method consists of interpret the logical equivalence of formulas φ and ψ in classical propositional logic as theoremhood of a suitable formula $(\varphi \leftrightarrow \psi)$ in the assertional system.

Others logics like intuitionistic logic or multiple-valued logic, can also be approached on the two point of view, the equivalent and the assertional. For instance, using the *Lindenbaum – Tarski method*, it is possible connect the Intuitionistic propositional calculus and the Heyting algebra. The developing of the theory of algebraic logic was made trying generalize the *Lindenbaum–Tarski method*. This theory consists of the investigation whether or not a logic can be connected with a class of algebras by means *Lindenbaum – Tarki method*. Generalizing those ideas, Blok and Pigozzi [BP89] introduced the notion of algebraizable logic. Superficially speaking, an algebraizable logic consists of a set of formulas in two variables such that interpret the logical equivalence between two formulas as theoremhood of this set of formulas. On this idea it is possible, for instance, to build the connection between classical propositional logic with signature containing just the implication as binary connective $\{\longrightarrow\}$, and the class of Boolean algebras using a set of formulas given by $\{\varphi \longrightarrow \psi,\ \psi \longrightarrow \varphi\}$. Due to Blok and Pigozzi's works, it has emerged the theory of *Abstract algebraic logic* (AAL). The theory of AAL is a powerful tool to investigate meta-logical properties, for instance, using the connection of an algebraizable logic with its class of algebras, it is possible to interpret some meta-logical properties through a well-known property on a "good" class of algebras. The converse is also possible to do, i.e., one can analysis if certain property on a class of algebras holds just verifying if the associated logic for the class of algebras satisfies a correspondent meta-logical property.

Another way to study meta-logical properties or intrinsic properties of a logic is looking at through its "alternative" or "complementary" logics. For instance of "alternative" logics of classical logic we have the intuitionistic logic, many-valued logic, paraconsistent logic. For "complementary" logics for classical logic we can consider modal logics such as temporal logic, epistemic logic, erotetic logic, doxastic logic, and so on. On this alternative to study a logic has born the methods of combining logics. The study of combining a logic appear in dual aspect: as a processes of decomposition or analysis of logics (e.g.,the "Possible Translation Semantics" of W. Carnielli, [Car90]) or as processes of composition or synthesis of logics (e.g., the "Fibrings" of D. Gabbay, [Gab96]). The methods of combining logics have been the main motivation to consider a category of logics. Below we will explain, with more details, how combining logics has motivated the categories of logics.

The consideration of category of logics give us a possibility to interpret a logical system (objects), have a notion of translations (morphisms), identify equivalences between logics (isomorphisms) and combine logics (limits and co-limits) using the well-known tools of category theory. The major concern in the study of categories of logics (CLE-UNICAMP-Brazil, IST-Lisboa-Portugal) is to describe condition for preservation, under the combination method, of meta-logical properties ([CCC+03], [ZSS01]). The initial steps on categories of logics are given in the sequence of papers [AFLM05], [AFLM06] and [AFLM07]. They present a category of logic such that the morphisms send n-ary connective to n-ary connective. More flexible notions of morphisms between logics are considered in [FC04], [BCC04], [BSCC06], [CG07]. The morphisms in this category send n-ary connective to a n-ary formula. Thus, we must consider a different way for composition of morphism, implying some categorial "defects". A "refinement" of those ideas is provided in [MM14]. Every category above has the same objects: the propositional finitary logics, i.e., a pair given by a signature and Tarskian consequence relation on its formula algebra; the morphisms considered are (some kind of) "logical translations", i.e. some functions that preserves consequence relations.

Overview of the thesis. In this thesis we begin giving a background of basic notion to develop the subsequent chapters. We start introducing the methods of combining logics and giving some examples. In this section, we give an explanation to choice of representing a logical system by a signature and a Tarskian consequence relation. After reading this first section, we believe that one can so understand why the theory of combination of logics has been the main motivation to study of categories of logics. In the sequel we introduce the different kinds of categories of signatures and then the categories of logics and some of their subcategories, namely the category of signatures and logics with strict and flexible morphisms, the category of congruential (selfextensional) logics and the quotient of these categories. We introduce the notion of algebraizable logic with examples and some important results we use in this work. Finally, we present the category of algebraizable logics, that one can see as a special subcategory of the former categories, the category of Lindenbaum algebraizable logics and the relationship among these categories.

The chapter 2 is dedicated to explain the relation between signatures and structure. Roughly speaking we present an anti-isomorphism between the category of signature with flexible morphisms and the category of structure which the morphisms are functors that "commute over Set", i.e., commute with the forgetful functors $U : \Sigma - Str \to Set$. Looking at the category of algebraizable logics, we have that the anti-isomorphism, established before, "restricts" to an anti-isomorphism between the category of algebraizable logics and a category such that the objects are pairs $(\Sigma - Str, a)$ where $QV(a) \subseteq \Sigma - Str$. The main results in this chapter are the codification of isomorphisms in the quotient category of algebraizable logics Q^c_f by means functors of structures such that restrict to isomorphisms of quasivarieties, and a codification of dense morphisms in the category from an arbitrary logic to an algebraizable logic by means functors such that are full, faithful, injective on objects and satisfy an additional property named "heredity".

Trying to construct a codification of morphisms of arbitrary logics in the same way provided in the chapter 2, we introduce in the chapter 3 the notion of a *filter pair* (G, i) and its associated logic. On this notion we have a classification of some special kinds of logics, namely the protoalgebraic logics, equivalential logics, truth-equational logics and algebraizable logics, just analysing the relation between the Leibniz operator and a specific filter pair. In the sequel we provide, by means filter functor, an analysis of meta-logical properties, more specifically, we have proved a relation between the amalgamation property in matrices and Craig entailment interpolation property in non-protoalgebaic logics. The last part of this chapter we introduce the category of filter functors and a "codification" of morphisms of logics, i.e., we have an adjunction between the category of logic and the category of filter functors.

In the chapter 4 we employ the notions of *Institutions* and π-*Institutions*. We have proved an adjunction between the category of institutions and the category of π-institutions. This adjunction is not completely new because there is a proof in [FS88] of the relation between the objects of those categories, but was not found a proof of the relation between their morphisms. We introduce an institution for the abstract propositional logics and an institution for each (Lindembaum) algebraizable logic, providing a new approach to the identity problem of logics ([B́05]). As an application of the results provided before, we have a generalization of Glivenko's theorem in algebraizable logics (Lindenbaum algebaizable logics). It is presented the notion of Glivenko's context in the institutions for algebraizable logics (Lindenbaum algebraizable logics) and then the main result in this chapter is the theorem 4.3.6 (4.3.12) stating that for a Glivenko's context there is an institution morphism associated. As a consequence of this theorem we have that given a Glivenko's context, there is a abstract Glivenko's theorem associated for algebraizable logics 4.3.7 (Lindenbaum algebraizable logics 4.3.13). The abstract Glivenko's theorem between the propositional classical logic and the propositional intuicionistic logic is exactly the classical Glivenko's theorem or a variation of it 4.3.8.

The chapter 5 is dedicated to introduce the first notions in order to define a precise representation theory of logics. We start presenting a notion of left diagram model for an arbitrary logic. Weakening the notion of isomorphisms between left diagram modules of logics, we propose a notion of Morita equivalence of logics and weakening even more we have the notion of left-stably-Morita equivalence. A result here that give us evidence that the definition of left-stably-Morita equivalence is working is the theorem 5.3.10 stating that the representations of classical logics are left-stably-Morita equivalence, but the classical logic and the intuitionistic logic are not left-stably-Morita equivalent (5.3.11).

A Conclusion chapter, with indications of future research, end our thesis.

Chapter 1

Preliminaries

We start this thesis presenting the main motivation to study the category of logics, namely the many methods of combining logics ([CC]). They appear in dual aspects: as processes of decomposition or analysis of logics (e.g., the "Possible Translation Semantics" of W. Carnielli, [Car90]) or as processes of composition or synthesis of logics (e.g., the "Fibrings" of D. Gabbay, [Gab96]). The combining of logics is still a young topic in contemporary logic. Besides the pure philosophical interest of define mixed logic systems in which distinct operators obey logical relations of different nature (syntactical and/or semantical), there also exist many pragmatical and methodological reasons to consider combined logics. We introduce these two ways to combine logics and we present some example of combining logics as $Algebraic\ fibring$ and $Possible - translation\ semantics$. Being the last one the main motivation to begin the study of representation of logic.

The initial steps on "global" approach to categories of logics are given in the sequence of papers [AFLM05], [AFLM06] and [AFLM07]: they present very simple but too strict notions of logical morphisms, having "good" categorial properties ([AR94]) but unsatisfactory treatment of the "identity problem" of logics ([B0́5]). More flexible notions of morphisms between logics are considered in [FC04], [BCC04], [BSCC06], [CG07]: this alternative notion allows better approach to the identity problem, however, has many categorial "defects". A "refinement" of those ideas is provided in [MM14]: are considered categories of logics satisfying *simultaneously* certain natural conditions: (i) represent the major part of logical systems; (ii) have good categorial properties; (iii) allow a natural notion of algebraizable logical system ([BP89], [Cze01]); (iv) allow satisfactory treatment of the "identity problem" of logics. Here we present these diferentes kids of categories of logics. Firstly we introduce the categories of signature with strict and flexile morphisms. On the categories of signature we have the categories of logics such that the morphisms are exactly strict and flexible.

Generalizing the ideas that describe a precise connection between Boolean algebra and classic propositional logic presented by $Lindenbaum - Tarski$, Blok and Pigozzi introduced in [BP89] the concept of *algebraizable logic*, for the first time, as a mathematical definition based on the notions of algebraizing pair and equivalent algebraic semantics. We present the precise definition given by Blok and Pigozzi, and some important results that we will use in the following sections of this thesis. Here, another relevant category of logics has, as objects, the algebraizable logics; the morphisms between them are the translations that preserve algebraizing pairs.

1.1 Combining Logics

The combining of logics is still a young topic in contemporary logic [CC]. Besides the pure philosophical interest on defining mixed logic systems in which distinct operators obey logics of different nature, as for instance combining epistemic and deontic logics, there also exist many pragmatical and methodological reasons for considering combined logics. In fact, the use of formal logic as a tool in Computer Science frequently requires the integration of several logic systems into a homogeneous environment.

The idea of looking at logic as an entirety avoiding fragmentation is not new. Philosophers and logicians from Ramón Lull (1235-1316), in *Air Magna* to Gottfried W. Leibniz (1646-1716), with *calculus ratiocinator*, have thought of building schemes where different logics could interact and cooperate.

Currents researches in logics have a strong trend to look for pluralism and compartmentalization. On one hand, we have alternative logics to the classical logic, such as multi-valued logics, intuitionistic logic, paraconsistent logic. On the other hand, we also have logics complementary to classical, such as modal logics, and, in particular, temporal logic, epistemic logic, doxastic logic, erotetic logic, deontic logic, and so on. Because of this, the study of combining of logics appear in dual aspects: as a processes of decomposition or analysis of logics or as processes of composition or synthesis of logics. In both cases, we seek to determine conditions which preserve the meta-logic properties as: Soundness, Completeness, Craig's Interpolation, Decidability, and so on.

1.1.1 Synthesis and analysis of logics

The methods for combining logics appear in dual aspects: Analysis and Synthesis of logics. Roughly speaking , one can decompose a given logic into factors of lower complexity, in order to facilitate the study the former one through of the simpler factors. This method is the decomposition or analysis logic.

$$\begin{tikzcd}
 & a \\
l \arrow[ur] \arrow[r] \arrow[dr] & a' \\
 & a''
\end{tikzcd}$$

The other method is to compose a new logical system over existing ones. In this process we intend to study properties of various logics into one.This method is called of composition or synthesis of logic.

$$\begin{tikzcd}
a \arrow[dr] & \\
a' \arrow[r] & l \\
a'' \arrow[ur] &
\end{tikzcd}$$

The combined logic should be minimal in some sense. In the case of synthesis of logics, it is expected that the logic assumes the role of infimum in a certain sense, i.e., if a logic l is obtained from two other a and a' then: 1) l extends a and a' and 2) l is the smaller extension of a and a'.

On the other hand in process of analysis of logics, a logic under analysis l assumes the role of supremum of a and a'

Before we give examples of combining of logics, an important question about the presentation of logics arises. Is it possible to combine logics defined in different ways? e.g., how could one combine a logic L_1, defined by natural deduction, with a logic L_2, defined by a Hilbert's axiomatic system? How should the resulting logic L be represented: as a natural deduction, as an axiomatic system or as a mixed proof system?

Consider now a logic L_1 described by semantical means (Valuations or Kripke models), whereas the logic L_2 is presented through a syntactical proof system, such as a sequent calculus or Hilbert-style axiomatization. which presentation fits better for the resulting (combined) logic: semantical or syntactical?

One way of solve this problem of combine logics of different presentations is consider something in common in most of logics: their consequence relations. Given two logics L_1 and L_2 represented by different forms, it is always possible to extract their consequence relations in order to combines them.

Definition 1.1.1. *A signature is a sequence of pairwise disjoint sets $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$. In what follows, $X = \{x_0, x_1, ..., x_n, ...\}$ will denote a fixed enumerable set (written in a fixed order).*

Denote $F(\Sigma)$ (or by Fm) (respectively $F(\Sigma)[n]$ (or $Fm[n]$)), the set of Σ-formulas over X (respec. the set of $\Sigma-$formulas ψ such that $var(\psi) = \{x_0, ..., x_{n-1}\}$).

Definition 1.1.2. *A Tarskian consequence relation is a relation $\vdash \subseteq \wp(F(\Sigma)) \times F(\Sigma)$, on a signature $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$, such that, for every set of formulas Γ, Δ and every formula φ, ψ of $F(\Sigma)$, it satisfies the following conditions:*

- **Reflexivity** :*If $\varphi \in \Gamma$, $\Gamma \vdash \varphi$*
- **Cut** :*If $\Gamma \vdash \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash \psi$, then $\Delta \vdash \varphi$*
- **Monotonicity** :*If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$*

  –The finitary-structural-Tarskian consequence relation is a Tarskian consequence relation such that the following conditions hold:

- **Finitarity** :*If $\Gamma \vdash \varphi$, then there is a finite subset Δ of Γ such that $\Delta \vdash \varphi$.*
- **Structurality** :*If $\Gamma \vdash \varphi$ and $\sigma : X \to F(\Sigma)$ is a substitution, then $\sigma[\Gamma] \vdash \sigma(\varphi)$*

In this thesis we consider a Tarskian relation as a finitary-structural-Tarskian relation.

The notion of logic consider in this work is given in the next:

Definition 1.1.3. *A Tarskian logic of type Σ, or a $\Sigma - logic$, or simply a logic of type Σ, is a pair $(\Sigma, \vdash)$ where Σ is a signature and $\vdash$ is a Tarskian consequence relation.*

The set of all consequence relations on a signature Σ, denoted by $Cons_\Sigma$, is endowed with the partial order: $\vdash_0 \leq \vdash_1$ iff for each $\Gamma \subseteq F(\Sigma)$, $\{\varphi \in F(\Sigma);\ \Gamma \vdash_0 \varphi\} = \overline{\Gamma}^0 \subseteq \overline{\Gamma}^1 = \{\varphi \in F(\Sigma);\ \Gamma \vdash_1 \varphi\}$.

Remark 1.1.4. *For each signature Σ, the poset $(Cons_\Sigma, \leq)$ is a complete lattice. It is in fact an algebraic lattice where the compact elements are the "finitely generated logics", i.e., the logics over Σ given by a finite set of axioms and a finite set of (finitary) inference rules.*

1.1.2 Examples of combining of logics

Category theory stands for a useful and important mathematical tool when we are interested on the relationship among objects. A possible approach to the process of combining of logics is given by a categorial treatment. The examples of combining of logics below use the category theory [Mac71].

Algebraic Fibring:

Let $\mathcal{L}$ to be a category of logics and $L_1, L_2 \in \mathcal{L}$, then the Fibring between L_1 and L_2 is the co-product $(L_1 \coprod L_2)$.

$co - product$

$$\begin{tikzcd} & L & \\ L_1 \arrow[ur] \arrow[r] & L_1 \coprod L_2 \arrow[u, dashed] & L_2 \arrow[ul] \arrow[l] \end{tikzcd}$$

$product$

$$\begin{tikzcd} & L \arrow[dl] \arrow[dr] \arrow[d, dashed] & \\ L_1 & L_1 \times L_2 \arrow[l] \arrow[r] & L_2 \end{tikzcd}$$

In this process, due to universal property of co-product, we have that the logic $(L_1 \coprod L_2)$, in a way plays the role of supremum. This process of combining of logics is example of composition or synthesis of logics.

The Algebraic Fibring was introduced by A. Sernadas, C. Sernadas and C. Caleiro in [SSC99] in order to overcome limitations on the fibring process proposed originally by D. Gabbay in [Gab96].

Another example of combining of logics is:

Possible-Translations Semantics:

A translation between two logics $L = (\Sigma, \vdash)$ and $L' = (\Sigma', \vdash')$ is a function $f : F(\Sigma) \to F(\Sigma')$ such that preserve derivability, i.e., if $\Gamma \vdash \varphi$ then $f[\Gamma] \vdash' f(\varphi)$. A pair $P = (\{L_i\}_{i\in I}, \{f_i\}_{i\in I})$ where $f_i : F(\Sigma) \to F(\Sigma_i)$ is called possible-translation frame to $L = (\Sigma, \vdash)$. P is a possible-translation semantics to L if for all $\Gamma \cup \{\varphi\} \subseteq F(\Sigma)$,

$$\Gamma \vdash \varphi \; iff \;\; for \; all \; i \in I, \;\; f_i[\Gamma] \vdash f_i(\varphi)$$

Thus possible to check up the derivability of the logic L through translations in logics L_i.

On the categorial point of view, we have that L is "faithfully encoded" in the "product" of objects $(L_i)_{i\in I}$.

This is an example of combining of logics in sense of composition or synthesis of logics. The possible-translations semantic was introduced by W. Carnielli in [Car90]. Actually, several paraconsistent logics which are not characterizable by finite matrices can be characterized by suitable combinations of many-valued logics [JBS].

Some other types of combining of logics (see [CC]) are.

- Product (introduced by K. Segerberg(1973) and independently by V. Sehtman(1978)), Fusion (introduced by R. Thomason (1984)), Fibring (introduced by D. Gabbay (1996))

  They all combining only modal logics.

- Temporalization, Parametrization

- Institutions (introduced by J. Goguen e R. Burstall, the institutions was introduced with use of categorical language making a type of abstract models theory applied in computer science).[1]

In the late 90s, the logic group of IST Lisboa-Portugal and the group theoretical and applied logic of CLE-UNICAMP considered systematically the categorical perspective of phenomenon of (some types of) combining of logics. On one hand the analytic process that occurs in possible-translation semantics produces a conservative translation between the logic in study and the product (or weak product) of simpler logics. On the other hand, the "unrestricted" fibring, i.e., without connective sharing of the logical constituents is described for dual categorical construction of product, coproduct and the notion of "restricted" fibring, i.e. with connective sharing of the logical constituents, is described for dual categorical construction of fibred product (or pullback) and amalgamated sum (or pushout) ([SSC99], [CCRS05]).

pullback

$$\begin{tikzcd}
& L' \arrow[d, dashed] \arrow[dl, bend right=10] \arrow[dr, bend left=10] & \\
& L \arrow[dl] \arrow[dr] & \\
L_1 \arrow[r] & L_2 & L_3 \arrow[l]
\end{tikzcd}$$

pushout

$$\begin{tikzcd}
& L' & \\
& L \arrow[u, dashed] & \\
L_1 \arrow[ur] \arrow[uur, bend left=10] & L_2 \arrow[l] \arrow[r] & L_3 \arrow[ul] \arrow[uul, bend right=10]
\end{tikzcd}$$

The categorical approach to the notion of fibring is relevant because, besides such approach requires that object of study and the relationship among them are totally accurate, the characterization of fibring as a universal construction (as coproduct or amalgamated sum) in a given choice of category of logics allows the *definition* through the same universal properties of the fibring notion of logics in other categories that capture other aspects of the logic systems. Thus, there were proposed new categories of logics that present treatment of two problems that occurred in certain fibring in the first categories of logics, *collapsing problem* and *anti-collapsing problem*:

(i) The *collapsing problem* in fibring occurs when the logic obtained by fibring presents more relations in the combined language than expected, according to the description of fibring made by D. Gabbay. The "solution" to this problem would be defining other categories of logic systems which still represent the original component logics, but such that the combined logics, obtained through the correspondent concept of fibring in these new categories, are accordingly weaker, in some sense, then the logic obtained through fibring in the original category. Two examples in this sense are the *modular fibring*: with syntactic character, obtained through restriction or control in the interactions among the components

[1] In the chapter 4 we present its precise definition.

logics ([SRC02]); and the *criptofibring*: with semantic character, obtained through relaxation of the relations between the model combined and the constituent models ([CR04]).

(ii) The *anti-collapsing* problem occurs when the logic obtained through fibring presents less relations in the combined language than expected, e.g., the absence of some natural meta-properties. The "solution" to this problem would be defining other categories of logic systems which still represent the original component logics, but such that the combined logics, obtained through correspondent concept of fibring in these new categories, are accordingly stronger, in some sense, than the logic obtained through fibring in the original category. A solution proposal, of syntactic character, is the notion of *meta-fibring* ([Con05]): in this case, the morphisms between logics are signature morphisms which induce, in the set of formulas, meta-translations of the source logic on the target logic.

1.2 Categories of logics

The appearance of several processes of combining of logics was the main motivation to the systematic study of categories of logics. The category theory is concerned to the relations among different mathematical objects. This is exactly the proposal in that we will apply this theory in logic. Here, the objects in those categories of logics are signature and consequence operator pairs, the morphisms are translations between logics. In the study of categories of logics, some problems relating the logic properties and categories arise. In view of this, different definitions of categories of logics appear, more precisely, different definitions of morphisms between logic systems.

1.2.1 Categories of signatures and logics with strict morphism

Initially we define the category of signature with "strict" morphism $\mathcal{S}_s$ according to [AFLM05], [AFLM06] and [AFLM07].

Definition 1.2.1. *The objects of the category $\mathcal{S}_s$ are signature. If Σ, Σ' are signature then a morphism $f : \Sigma \to \Sigma'$ is a sequence of functions $f = (f_n)_{n\in\mathbb{N}}$, where $f_n : \Sigma_n \to \Sigma'_n$. For each morphism $f : \Sigma \to \Sigma$ there is only one function $\hat{f} : F(\Sigma) \to F(\Sigma')$, called the extension of f, such that:*

- $\hat{f}(x) = x$ *if* $x \in X$ *(X is a fixed enumerable set)*
- $\hat{f}(c) = f_0(c)$ *if* $c \in \Sigma_0$
- $\hat{f}(c_n(\psi_0, ..., \psi_{n-1})) = f_n(c_n)(\hat{f}(\psi_0), ..., \hat{f}(\psi_{n-1}))$ *if* $c_n \in \Sigma_n,\ n > 0$

Then, by indution, $\hat{f}(\varphi(\psi_0, ..., \psi_{n-1})) = \hat{f}(\varphi)(\hat{f}(\psi_0), ..., \hat{f}(\psi_{n-1}))$

The categories $\mathcal{S}_s$ and $Set^{\mathbb{N}}$ are equivalent, thus we have that $\mathcal{S}_s$ has good categorial properties, namely $\mathcal{S}_s$ is a finitely locally presentable category and the finitely presentable (fp) signatures are the "finite support" signatures.

Remark 1.2.2. *(i) (**Sub**): For any substitution function $\sigma : X \to F(\Sigma)$, there is only one extension $\tilde{\sigma} : F(\Sigma) \to F(\Sigma)$ such that $\tilde{\sigma}$ is an homomorphism $\tilde{\sigma}(x) = \sigma(x)$, for all $x \in X$ and*

$$\tilde{\sigma}(c_n(\psi_0, ..., \psi_{n-1}) = c_n(\tilde{\sigma}(\psi_0), ..., \tilde{\sigma}(\psi_{n-1}))$$

for all $c_n \in \Sigma_n, n \in \mathbb{N}$. The identity substitution induces the identity homomorphism on the formula algebra; the composition substitution of the substitutions $\sigma, \sigma' : X \to F(\Sigma)$ is the substitution $\sigma'' : X \to F(\Sigma)$, $\sigma'' = \sigma \star \sigma' := \tilde{\sigma} \circ \sigma'$ and $\tilde{\sigma''} = (\sigma \star \sigma')^{\sim} = \tilde{\sigma} \circ \tilde{\sigma'}$.

Let $f : \Sigma \to \Sigma$ be a $\mathcal{S}_s$-morphism. Then for any substitution $\sigma : X \to F(\Sigma)$ there is another substitution σ' such that $\tilde{\sigma'} \circ \hat{f} = \hat{f} \circ \tilde{\sigma}$.

(ii) Let $f : \Sigma \to \Sigma'$ and $\theta \in F(\Sigma)$. If $var(\theta) \subseteq \{x_{i_0}, ..., x_{i_{n-1}}\}$, then

$$\hat{f}(\theta(\vec{x})[\vec{x}|\vec{\psi}]) = \hat{f}(\theta(\vec{x}))[\vec{x}|\hat{f}(\vec{\psi})].$$

Moreover $var(\hat{f}(\theta)) = var(\theta)$ and then $\hat{f}$ restricts to maps $\hat{f}_n : F(\Sigma)[n] \to F(\Sigma')[n]$

Now we give the definition of category of logics with "strict" morphism $\mathcal{L}_s$.

Definition 1.2.3. *The objects of $\mathcal{L}_s$ are $l = (\Sigma, \vdash)$, where Σ is a signature and $\vdash$ is a tarskian consequence operator. A $\mathcal{L}_s$-morphism, $f : l \to l'$ is a (strict) signature morphism $f \in \mathcal{S}_s(\Sigma, \Sigma')$ such that $\hat{f} : F(\Sigma) \to F(\Sigma')$ is a $(\vdash, \vdash')$-translation: $\Gamma \vdash \psi \Rightarrow \hat{f}(\Gamma) \vdash' \hat{f}(\psi)$*

$\mathcal{L}_s$ is a ω-locally presentable category and the fp logics are given by a finite set of "axioms" and "inference rules" over a fp signature.

Between the categories $\mathcal{L}_s$ and $\mathcal{S}_s$, there exists a forgetful functor such that forget the consequence relation.

The categories above mentioned have good categorial properties, but unsatisfactory treatment for the logic problems, e.g., the "identity problem" of logics [B0́5]. Two presentations of classic propositional logic with signatures $\{\neg, \to\}$ and $\{\neg, \vee\}$ do not admit strict morphism between them (because any such morphism must takes $\to$ to $\vee$ and they do not preserve $\vdash$) while it was expected that these presentations should be isomorphic.

1.2.2 Categories of signatures and logics with flexible morphism

At this moment, it is given a definition of category of logics essentially described in [JKE96] [FC04], [BCC04], [BSCC06] and [CG07]. This definition gives a more appropriated treatment for the "identity problem" of logics.

Similarly to the previous case, firstly we define the category of signature with "flexible" morphism $\mathcal{S}_f$. Before defining this category, it is introduced the following notation:

If $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ is a signature, then $T(\Sigma) := (F(\Sigma)[n])_{n\in\mathbb{N}}$ is a signature too.

A **flexible** morphism $f : \Sigma \to \Sigma'$ is a sequence of functions $f_n^\sharp : \Sigma_n \to F(\Sigma')[n], n \in \omega$.

For each signature Σ and $n \in \mathbb{N}$, consider the particular flexible morphism:

$$\begin{array}{rll} (j_\Sigma)_n : & \Sigma_n \to & F(\Sigma)[n] \\ & c_n \mapsto & c_n(x_0, ..., x_{n-1}) \end{array}$$

For each flexible morphism $f : \Sigma \to \Sigma'$, there is only one function $\check{f} : F(\Sigma) \to F(\Sigma')$, called the extension of f, such that:

(i) $\check{f}(x) = x$, if $x \in X$;

(ii) $\check{f}(c_n(\psi_0, ..., \psi_{n-1})) = f(c_n)(x_0, ..., x_{n-1})[x_0|\check{f}(\psi_0), ..., x_{n-1}|\check{f}(\psi_{n-1})]$, if $c_n \in \Sigma_n, n \in \mathbb{N}$.

We have the inverse bijections (just notations): $h \in \mathcal{S}_f(\Sigma, \Sigma') \leftrightarrow h^\sharp \in \mathcal{S}_s(\Sigma, T(\Sigma'))$; $f \in \mathcal{S}_s(\Sigma, T(\Sigma')) \leftrightarrow f^\flat \in \mathcal{S}_f(\Sigma, \Sigma')$.

Definition 1.2.4. *The category $\mathcal{S}_f$ is the category of signature and flexible morphism as above. The composition in $\mathcal{S}_f$ is given by $(f' \bullet f'')^\sharp := (\check{f}(\restriction) \circ f^\sharp)$. The identity id_Σ in $\mathcal{S}_f$ is given by $(id_\Sigma)^\sharp := ((j_\Sigma)_n)_{n\in\mathbb{N}}$*

As well as the category $\mathcal{S}_s$, we have that the category $\mathcal{S}_f$ satisfies the properties in 1.2.2

Definition 1.2.5. *The category $\mathcal{L}_f$ is the category of propositional logics and flexible translations as morphisms. This is a category "built above" the category $\mathcal{L}_f$, that is, there is an obvious forgetful functor $U_f : \mathcal{L}_f \to \mathcal{S}_f$.*

If $l = (\Sigma, \vdash), l' = (\Sigma', \vdash')$ are logics then a flexible translation morphism $f : l \to l'$ in $\mathcal{L}_f$ is a flexible signature morphism $f : \Sigma \to \Sigma'$ in $\mathcal{S}_f$ such that "preserves the consequence relation", that is, for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, if $\Gamma \vdash \psi$ then $\check{f}[\Gamma] \vdash' \check{f}(\psi)$. Composition and identities are similar to $\mathcal{S}_f$.

Remark 1.2.6. *Due to "flexibility", this category allows a better approach to the identity problem of logics. Consider the flexible morphisms $t : (\to, \neg) \longrightarrow (\vee', \neg')$ such that $t(\to) = \neg' x_0 \vee' x_1$ (formula in two variables), $t(\neg) = \neg' x_0$ and $t' : (\vee', \neg') \longrightarrow (\to, \neg)$ such that $t'(\vee') = \neg x_0 \to x_1$, $t'(\neg') = \neg x_0$. This pair of morphisms induce an* equipollence *between these presentations of classic logics [CG07].*

However this category does not has good categorical properties, because of the lack of many kinds of limits and colimits (which are useful for combining logics).

Remark 1.2.7. *It follows easily from the facts above that the forgetful functor* $U_f : \mathcal{L}_f \to \mathcal{S}_f : ((\Sigma, \vdash) \to ((\Sigma', \vdash')) \mapsto (\Sigma \to \Sigma')$ *has left and right adjoint functors: the left adjoint* $\perp_f$: $\mathcal{S}_f \to \mathcal{L}_f$ *and the right adjoint* $\top_f : \mathcal{S}_f \to \mathcal{L}_f$ *take a signature* Σ *to, respectively,* $\perp_f (\Sigma) = (\Sigma, \vdash_{min})$ *(the first element of* $Cons_\Sigma$*) and* $\top_f(\Sigma) = (\Sigma, \vdash_{max})$ *(the last element of* $Cons_\Sigma$*). Moreover,* $U_f \circ \perp_f = Id_{\mathcal{S}_f} = U_f \circ \top_f$ *and* U_f *preserves all limits and colimits that exists in* $\mathcal{L}_f$.

Remark 1.2.8. *It is known that* $\mathcal{L}_f$ *has weak products, coproducts and some pushouts, and in the Remark above we see that* U_f *preserves limits and colimits. As* U_f *also "lift" limits and colimits - the constructions in* $\mathcal{L}_f$ *are analogous to in* $\mathcal{L}_s$ *(in [AFLM07]) , just replace* $\hat{f}$ *by* $\check{f}$ *- then given a small category* $\mathcal{I}$, $\mathcal{L}_f$ *is* $\mathcal{I}$*-complete (respectively,* $\mathcal{I}$*-cocomplete) if and only if* $\mathcal{S}_f$ *is* $\mathcal{I}$*-complete (respectively,* $\mathcal{I}$*-cocomplete). As the category* $\mathcal{S}_f$ *has colimits for any (small) diagram entails that* $\mathcal{L}_f$ *has colimits for any (small) diagram "in* $\mathcal{L}_s$*", in particular, it has all unconstrained fibrings (= coproducts) and the constrained fibrings (= pushouts) "based in* $\mathcal{L}_s$*".*

1.2.3 Other categories of logics

Due to the drawbacks in the categories of logics mentioned above, other categories of logics that help the overcome these "defects" are presented.

- On the category $\mathcal{L}_f$ we take the quotient category $Q\mathcal{L}_f$: $f, g \in \mathcal{L}_f(l, l')$, $f \sim g \; iff \; \check{f}(\varphi) \dashv'\vdash \check{g}(\varphi)$. Thus two logics l, l' are equipollent if only if l and l' are $Q\mathcal{L}_f$−isomorphic [CG07]

- Still on the category $\mathcal{L}_f$ we have the "congruential"[2] logics $\mathcal{L}_f^c$. This category is a subcategory of $\mathcal{L}_f$ where the logics are congruential, i.e., logics that satisfies:

$$\varphi_0 \dashv\vdash \psi_0, ..., \varphi_{n-1} \dashv\vdash \psi_{n-1} \Rightarrow c_n(\varphi_0, ..., \varphi_{n-1}) \dashv\vdash c_n(\psi_0, ..., \psi_{n-1}).$$

  The inclusion functor $\mathcal{L}_f^c \hookrightarrow \mathcal{L}_f$ has a left adjoint given by congruential closure operator.

- In [MM14] we found the category $Q\mathcal{L}_f^c$ (or simply $\mathcal{Q}_f^c$). This category of logics satisfies *simultaneously* certain natural conditions:
  (i) it represents the major part of logical systems;
  (ii) it has a good categorial approach (e.g., they are complete, cocomplete and accessible categories);
  (iii) it allows a natural notion of algebraizable logical system ([BP89],[Cze01]);
  (iv) it allows satisfactory treatment of the "identity problem" of logics.

[2]In many references the authors are calling this logic as **selfextensional** logic and they say that a logic is **congruential** when it is a **fullyselfextensional** logic.

In [MM14] it is shown that the categories $\mathcal{S}_s$ and $\mathcal{S}_f$ are well related, more precisely, there is a pair of adjoint functors between them, namely $(+)_S : \mathcal{S}_s \to \mathcal{S}_f$ and $(-)_S : \mathcal{S}_f \to \mathcal{S}_s$. Moreover there is a monad or triple $\mathcal{T} = (T_S, \mu_S, \eta_S)$ on $\mathcal{S}_s$ canonically associated with this adjunction such that T preserves filtered colimits, reflects isomorphisms and, mainly, that $Kleisli(\mathcal{T}) = \mathcal{S}_f$ [Mac71], which derives some additional informations about the category $\mathcal{S}_f$: e.g., it has all coproducts.

This adjunction between $\mathcal{S}_s$ and $\mathcal{S}_f$ through forgetful functors U_s and U_f gives a pair of adjoint functors $(+)_L : \mathcal{L}_s \to \mathcal{L}_f$, $(-)_L : \mathcal{L}_f \to \mathcal{L}_s$. i.e.:

- $U_f \circ (+)_L = (+)_S \circ U_s$
- $U_s \circ (-)_L = (-)_S \circ U_f$
- $U_s \eta_L = \eta_S U_f$
- $U_f \varepsilon_L = \varepsilon_S U_f$

The signature monad $\mathcal{T}_S = (T_S, \mu_S, \eta_S)$ associated to the signature adjunction (η_S, ε_S) (i.e.,$\mu_S = (-)_S \varepsilon_S (+)_S$) "lifts" to a logic monad $\mathcal{T}_L = (T_L, \mu_L, \eta_L)$ associated to the signature adjunction (η_L, ε_L) (i.e.,$\mu_L = (-)_L \varepsilon_L (+)_L$) and is such that $Kleisli(\mathcal{T}_L) = \mathcal{L}_f$. Moreover, the functors $(+)_L$ and $(-)_L$ are precisely the canonical functors associated to the adjunction of the Kleisli category of a monad.

1.3 Algebraizable logics

The idea behind algebraizing a logic has emerged from the need to connect two independent approaches to logic. On one hand, there was the logic equivalence and on the other hand there was the assertion and inference. The attempts of connecting them beginning with the ideas of Hilbert.

Tarski described a precise connection between Boolean algebra and classical propositional logic, following the Lindenbaum idea of looking at the set of formulas as an algebra with the induced operators by logic connectives. This is so-called *Lindenbaum-Tarski* method.

Application of Lindenbaum-Tarski method to intuitionistic logic provides a connection between intuitionistic propositional calculus and Heyting algebras.

Traditionally algebraic logic has focused on the algebraic investigation of particular classes of algebras of logic, whether they could be connected to some known assertional system by means of the Lindenbaum-Tarski method or not. However, when such a connection could be established, there was interest on investigating the relationship among various meta-logical properties of the logical system and the algebraic properties of the associated class of algebras.

Now we describe the Lindenbaum-Tarski method.

Let $l = (\Sigma, \vdash)$ be a presentation of classical logic, e.g., $\Sigma = (\neg, \rightarrow)$. $T \subseteq F(\Sigma)$ is a theory of l iff $T \vdash \psi \Rightarrow \psi \in T$ for all $\psi \in F(\Sigma)$.

We define the following relation on $F(\Sigma)$:

$$\varphi \equiv_T \psi \quad iff \quad \varphi \rightarrow \psi \in T \text{ and } \psi \rightarrow \varphi \in T.$$

$\equiv_T$ is a congruence relation of the algebra of formulas. The quotient algebra $F(\Sigma)/\equiv_T$ is know as the Lindenbaum-Tarski algebra determined by T. $F(\Sigma)/\equiv_T$ is a Boolean algebra.

About the ideas of $Lindenbaum - Tarski$, Blok and Pigozzi in 1989 [BP89] gave the concept of algebraizable logics for the first time as a general mathematical definition instead of a particular construction. The idea behind the definition is the following: a logic is algebraizable if there exists a class of related algebras with the logic of the same way that class of Boolean algebras is related with the classical propositional logic.

Definition 1.3.1. *Let Σ be a signature. We will denote by $\Sigma - Str$ the category with objects given by all the structures (or algebras) on the signature Σ and morphisms Σ-homomorphisms between them. A fundamental example of Σ-structure is $F(\Sigma)$, the absolutely free Σ-algebra on the set of variables X.*

Definition 1.3.2. *Given a class of algebras $\boldsymbol{K}$ over the signature Σ, the equational consequence associated with $\boldsymbol{K}$ is the relation $\models_{\boldsymbol{K}}$ between a set of equations Γ and a single equation $\varphi \equiv \psi$ over Σ defined by:*

$$\Gamma \models_{\boldsymbol{K}} \varphi \equiv \psi \text{ iff } for\ every\ A \in \boldsymbol{K}\ and\ every\ \Sigma - homomorphism\ \ h : F(\Sigma) \rightarrow A,$$

$$if\ h(\eta) = h(\nu)\ for\ all\ \eta \equiv \nu \in \Gamma,\ then\ h(\varphi) = h(\psi).$$

Definition 1.3.3. *Let $l = (\Sigma, \vdash)$ be a logic and $\boldsymbol{K}$ be a class of $\Sigma-$algebra. $\boldsymbol{K}$ is a equivalent algebraic semantics for l if $\vdash$ can be faithfully interpreted in $\models_{\boldsymbol{K}}$ in the following sense:*

(1) there is a finite set $\tau(p) = \{(\delta_i(p), \epsilon_i(p)), i = 1, ..., n\}$ of equations in a single variable p such that for all $\Gamma \cup \{\varphi\} \subseteq F(\Sigma)$ and for $j < n$ has been:

$\Gamma \vdash \varphi \Leftrightarrow \{\tau(\gamma) : \gamma \in \Gamma\} \models_{\boldsymbol{K}} \tau(\varphi)$ *where* $\tau(\varphi) = \{(\delta_i(p)[p/\varphi], \epsilon_i(p)[p/\varphi]), i = 1, ..., n\}$.

(2) there is a finite system $\Delta_j(p, q), j = 1, ..., m$ of two variables formulas (formed by derived binary connectives) such that for all equation $\varphi \equiv \psi$,

$\varphi \equiv \psi =\!\!|_{\boldsymbol{K}}\!\!\models \tau(\varphi \Delta \psi)$

where $\varphi \Delta \psi = \Delta(\varphi, \psi)$, $\Delta(\varphi, \psi) = \{\Delta_j(\varphi, \psi), j = 1, ..., m\}$ *and* $\tau(\varphi \Delta \psi) = \{\delta_i(\Delta_j(\varphi, \psi)) \equiv \epsilon_i(\Delta_j(\varphi, \psi));\ i = 1, ..., n \text{ and } j = 1, ..., m\}$.

In this case we shall say that a logic l is algebraizable. The set $\langle \tau(p), \Delta(p, q) \rangle$ (or just $\langle \tau, \Delta \rangle$) is called an "algebraizing pair", with $\tau = (\delta, \epsilon)$ as the "defining equations" and Δ as the "equivalence formulas".

Proposition 1.3.4. *Let K an equivalent algebraic semantic for the algebrizable logic $a = (\Sigma, \vdash)$ with algebraizing pair $\langle \tau, \Delta \rangle$, then:*

1. *For all set of equations Γ and for all equation $\varphi \equiv \psi$, we have that*

$$\Gamma \models_K \varphi \equiv \psi \;\Leftrightarrow\; \{\xi \Delta \eta : \xi \equiv \eta \in \Gamma\} \vdash \varphi \Delta \psi$$

2. *For each $\psi \in F(\Sigma)$ we have that*

$$\psi \dashv\vdash \Delta(\tau(\psi)).$$

Conversely, if there is a logic $a = (\Sigma, \vdash)$ and formulas $\langle \Delta(p,q), \tau(p) \rangle$ that satisfy the conditions 1. and 2., then K is an equivalent algebraic semantics for a.

Remark 1.3.5. *By a direct application of the definition above, if $l = (\Sigma, \vdash)$ is an algebraizable logic and $\phi, \psi \in F(\Sigma)$, then $\phi, \phi \Delta \psi \vdash \psi$ (detachment property).*

As examples of algebraizable logics we have, in addition to CPC (Classic Propositional Calculus) and IPC (Intuitionistic Propositional Calculus), some modal logics, the Post and Lukasiewicz multi-valued logics, and many of several versions of quantum logic.

In case of CPC (IPC), a possible algebraizing pair $\langle \Delta(p,q), \tau(p) \rangle = \langle \Delta(p,q), (\epsilon(p), \delta(p)) \rangle$ is:

1. $\Delta(p,q) = \{p \leftrightarrow q\}$
2. $\epsilon(p) = p$
3. $\delta(p) = \top$

and K is the class of Boolean algebras (respectively the class of Heyting algebras).

Another class of algebras that is an[3] equivalent algebraic semantic for an algebrizable logic, but present in many branches of mathematics, is the class of all groups ([BP]). To the (equational) theory of groups over the signature $\Sigma = \{\cdot, ^{-1}, e\}$, it is associated the following propositional logic l_{Gr}, the "logic of groups" over the same signature Σ, that is

Axioms of l_{Gr}

G_1 $((p \cdot q) \cdot r) \cdot (p \cdot (q \cdot r))^{-1}$

G_2 $(p \cdot e) \cdot p^{-1}$

G_3 $(e \cdot p) \cdot p^{-1}$

[3]See Proposition 1.3.6.

G_4 $p \cdot p^{-1}$

G_5 $p^{-1} \cdot p$

Rules

R_1 $p \cdot q^{-1} \vdash q \cdot p^{-1}$

R_2 $p \cdot q^{-1} \vdash p^{-1} \cdot q^{-1^{-1}}$

R_3 $\{p \cdot q^{-1}, q \cdot r^{-1}\} \vdash p \cdot r^{-1}$

R_4 $\{p \cdot q^{-1}, r \cdot s^{-1}\} \vdash (p \cdot r) \cdot (q \cdot s)^{-1}$

R_5 $p \vdash p \cdot e^{-1}$

R_6 $p \cdot e^{-1} \vdash p$

The logic of groups theory has as algebrizing pair $\langle \Delta(p,q), \tau(p) = \langle \epsilon(p), \delta(p) \rangle \rangle$:

1. $\Delta(p,q) = p \cdot q^{-1}$
2. $\delta(p) = p$
3. $\epsilon(p) = e$

K, in this case, is the class of groups. Worth pointing out that the logic of groups, in some sense, does not admit Deduction Theorem.

Recall that a quasivariety is a class of algebras K such that it is axiomatizable by quasi-identities, i.e., formulas of the form

$$(p_1 \equiv q_1 \wedge ... \wedge p_n \equiv q_n) \rightarrow p \equiv q \ for \ n \geq 1$$

when $n = 0$ the quasi-identity is

$$\top \rightarrow p \equiv q.$$

Now we will recall a result about "uniqueness" of algebraizing pair and the quasivariety semantics of an algebraizable logic. For any class K of Σ-algebras let us denote $(K)^Q$ the Σ-quasivariety generated by K.

Proposition 1.3.6 (2.15-[BP89])**.** *Let a be an algebraizable logic.*

(a) Let $\langle (\delta_i(p), \varepsilon_i(p)), \Delta_i(p,q) \rangle$, an algebraizing pair for a, and K_i an equivalent algebraic semantic associated with a, for each $i \in \{0,1\}$. Then $(K_0)^Q, (K_1)^Q$ are equivalent algebraic semantics for a. Moreover, some uniqueness conditions hold:

- *on quasivariety semantics: $(K_0)^Q = (K_1)^Q$;*

- *on equivalence formulas:* $\Delta_0(p,q) \dashv\vdash \Delta_1(p,q)$*;*
- *on defining equations:* $(\delta_0(p) \equiv \varepsilon_0(p)) \ =\!|_K\!\models\ (\delta_1(p) \equiv \varepsilon_1(p))$ *(where* $K := (K_0)^Q = (K_1)^Q$*).*

(b) Let $\langle(\delta_i(p), \varepsilon_i(p)), \Delta_i(p,q)\rangle$*. Suppose that the following conditions holds:*

- $(\delta_0(p), \varepsilon_0(p)), \Delta_0(p,q)\rangle$ *is an algebraizing pair for a;*
- $\Delta_0(p,q) \dashv\vdash \Delta_1(p,q)$*;*
- $(\delta_0(p) \equiv \varepsilon_0(p)) \ =\!|_{(K_0)^Q}\!\models\ (\delta_1(p) \equiv \varepsilon_1(p))$*.*

Then $\langle(\delta_1(p), \varepsilon_1(p)), \Delta_1(p,q)\rangle$*is an algebraizing pair for a and* $(K_1)^Q = (K_0)^Q$*.*

If $a = (\Sigma, \vdash)$ is an algebraizable logic then, by the Proposition above, we can (and we will) denote by $QV(a)$ the unique quasivariety on the signature Σ that is an equivalent algebraic semantics for a.

Proposition 1.3.7 (2.17 [BP89]). *Let a be an algebraizable logic a and* $\langle(\delta, \epsilon), \Delta\rangle$ *be an algebraizing pair for a. Then the quasivariety* $QV(a)$ *is axiomatized by the set given by the 3 kinds of quasi-equations below:*

- $\delta(x_0 \Delta x_0) \equiv \epsilon(x_0 \Delta x_0)$*;*
- $\delta(x_0 \Delta x_1) \equiv \epsilon(x_0 \Delta x_1) \ \rightarrow\ x_0 \equiv x_1$*;*
- $(\bigwedge_{i<n} \delta(\psi_i) \equiv \epsilon(\psi_i)) \ \rightarrow\ \delta(\phi) \equiv \epsilon(\phi)$*, for each* $\{\phi, \psi_0, \cdots, \psi_{n-1}\} \subseteq F(\Sigma)$ *such that* $\{\psi_0, \cdots, \psi_{n-1}\} \vdash \phi$*, for* $n \geq 0$*.*

The proposition below give us a syntactic characterization algebraizable logics.

Proposition 1.3.8 (4.7-[BP89]). *Let* $a = (\Sigma, \vdash)$ *be a logic and* $\Delta \subseteq_{fin} F(\Sigma)[2]$*,* $(\delta \equiv \epsilon) \subseteq_{fin} (F(\Sigma)[1] \times F(\Sigma)[1])$ *such that the conditions below are satisfied*

(a) $\vdash \varphi \Delta \varphi$*, for all* $\varphi \in F(\Sigma)$*;*

(b) $\varphi \Delta \psi \vdash \psi \Delta \varphi$*, for all* $\varphi, \psi \in F(\Sigma)$*;*

(c) $\varphi \Delta \psi, \psi \Delta \vartheta \vdash \varphi \Delta \vartheta$*, for all* $\varphi, \psi, \vartheta \in F(\Sigma)$*;*

(d) $\varphi_0 \Delta \psi_0, ..., \varphi_{n-1} \Delta \psi_{n-1} \vdash c_n(\varphi_0, ..., \varphi_{n-1}) \Delta c_n(\psi_0, ..., \psi_{n-1})$*, for all* $c_n \in \Sigma_n$ *and all* $\varphi_0, \psi_0, ..., \varphi_{n-1}, \psi_{n-1} \in F(\Sigma)$*;*

(e) $\vartheta \dashv\vdash \Delta(\tau(\vartheta))$*, for all* $\vartheta \in F(\Sigma)$*.*

Then a is an algebraizable logic with Δ *as equivalence formulas and* τ *as defining equations.*

Conversely if $a = (\Sigma, \vdash)$ *is a algebrizable logics with algebraizing pair* $\langle\Delta(p,q), \tau(p)\rangle$*, then the conditions* (a) *to* (e) *are satisfied for these formulas.*

Remark 1.3.9. *It follows from the characterization above that, if* $\vdash_0, \vdash_1$ *are consequence operators over the same signature* Σ*, if* $l_0 = (\Sigma, \vdash_0)$ *is an algebraizable logic with algebraizing pair* $\langle\Delta(p,q), \tau(p)\rangle$ *and* $\vdash_0 \leq \vdash_1$ *(for any* $\Gamma \cup \{\varphi\}$*,if* $\Gamma \vdash_0 \varphi$ *then* $\Gamma \vdash_1 \varphi$*), then* $l_1 = (\Sigma, \vdash_1)$ *is an algebraizable logic and* $\langle\Delta(p,q), \tau(p)\rangle$ *is an algebraizing pair.*

Definition 1.3.10. *Let Σ be a signature, A be a Σ-algebra and $F \subseteq A$.*

(a) Let θ be a congruence in A. θ is said to be compatible with F if, for all $a, b \in A$, if $a \in F$ and $\langle a, b\rangle \in \theta$ then $b \in F$.

(b) We will denote by $\Omega^A(F)$ the largest congruence of A compatible with F. We say that the function Ω^A with domain the set of all subsets of A is called the Leibiniz operator on A.

Definition 1.3.11. *Let $l = (\Sigma, \vdash)$ be a logic and $A \in \Sigma - Str$. A subset F of A is a l-filter if for every $\Gamma \cup \{\varphi\} \subseteq F(\Sigma)$ such that $\Gamma \vdash \varphi$ and every valuation $v : F(\Sigma) \to A$, if $v[\Gamma] \subseteq F$ then $v(\varphi) \in F$. The pair $\langle M, F\rangle$ is then said to be a matrix model of l or just l-matrix. The set of all l-matrix is denoted by $Matr_l$.*

An l-matrix $\langle A, F\rangle$ is reduced (or is a reduced matrix) if its Leibniz congruence is the identity. Thus, in a reduced a matrix $\langle A, F\rangle$ the interval $[Id_A, \Omega^A(F)]$ is a unitary set. Given an l-matrix $\mathcal{M} = \langle A, F\rangle$, the quotient matrix

$$\mathcal{M}/\Omega(\mathcal{M}) = \langle A/\Omega^A(F), F/\Omega^A(F)\rangle$$

is called the reduction $\mathcal{M}$. We denote it by $\mathcal{M}^*$.

The class of all reduced l-matrix is denoted by $Matr_l^*$. We denote by Alg_l^* the class of all algebras A such that there is $F \subseteq A$ witch $\langle A, F\rangle \in Matr_l^*$.

Fact 1.3.12. *(a) [1.5-[BP89]] Let $a = (\Sigma, \vdash)$ be an algebraizable logic, $A \in \Sigma - Str$ and $F \subseteq A$ be a $l-filter$. Then $\Omega_A F = \{\langle a, b\rangle : \varphi^A(a, c_0, ..., c_{k-1}) \in F \;\Leftrightarrow \varphi^A(b, c_0, ..., c_{k-1}) \in F$, for all $\varphi \in Fm_L$ and $c_i \in A\}$*

(b) [5.2-[BP89]] Let $a = (\Sigma, \vdash)$ be an algebraizable logic over the language Σ, and let $\Delta(x_0, x_1)$ be a system of equivalence formulas. Then

$$\Omega_A F = \{\langle a, b\rangle : a\Delta^A b \in F\}$$

for every $A \in \Sigma - Str$ and every $l-$filter F of A.

Theorem 1.3.13. *(The Isomorphism Theorem, first version [BP89]).*

Let l be a logic and K a quasivariety. The following are equivalent.

1. *l is algebraizable with equivalent semantics K*

2. *For every algebra A the Leibiniz operator Ω^A is an isomorphism between $Fi_l(A)$ and $Co_K(A)$.*

Theorem 1.3.14. *(The Isomorphism Theorem, 2nd version [Fon16]).*

Let l be a logic and K be a quasivariety. The following conditions are equivalent:

1. *l is algebraizable with equivalent algebraic semantics the class K.*

2. *For every algebra A there is an isomorphism Φ^A between the lattices $Fi_l(A)$ and $Co_K(A)$ that commutes with endomorphisms, i.e., for every $F \in Fi_l(A)$ and every $h \in End(A)$, $\Phi^A h^{-1}(F) = h^{-1}\Phi^A F$.*

3. *There is an isomorphim Φ between the lattices $\mathcal{T}h(l)$ and $Co_K(Fm)$ that commutes with substitutions, i.e., for every $T \in \mathcal{T}h(l)$ and every $\sigma \in End(Fm)$, $\Phi\sigma^{-1}T = \sigma^{-1}\Phi T$*

Definition 1.3.15. 1. *Let L be a lattice. A element $a \in L$ is compact if for every directed subset $\{d_i\}$ of L we have $a \leq \bigvee_i d_i \Leftrightarrow \exists i(a \leq d_i)$. L is said algebraic if it is complete lattice such that every element is join of compact elements. We denote the category of algebraic lattice by AL.*

2. *Let $l = (\Sigma, \vdash)$ be a logic and $K \subseteq \Sigma - Str$, here $Fi_l : \Sigma - Str \to AL$ is the functor such that given a algebra $A \in \Sigma - Str$, $Fi_l(A)$ is the lattice of filters and given $f \in Mor_{\Sigma - Str}(A, B)$, $Fi_l(f) = f^{-1}$. The application $Co_K : \Sigma - Str \to AL$ is the functor such that for every $A \in \Sigma - Str$ $Co_K(A)$ is the lattice of relative congruence and given $f \in Hom_{\Sigma - Str}(A, B)$, $Co_K(f) = f^{-1}(f^{-1} \times f^{-1})$.*

Below we present a categorial version of the Isomorphism theorem. We put here the proof of this corollary because there is no a direct proof in the literatures.

Corollary 1.3.16. *(The Isomorphism Theorem, 3rd version)*

Let l be a logic and K be a generalized quasivariety. The following conditions are equivalent:

1. *l is an algebraizable logic with algebraic semantics the class K.*

2. *There is a natural isomorphism between the functors Fi_l and Co_K.*

Proof:

$''1 \Rightarrow 2''$ That l is an algebebraizable logic. By theorem 1.3.13 we have that for every $A \in \Sigma - Str$, the Leibiniz operator $\Omega^A : Fi_l(A) \to Co_K(A)$ is a isomorphism. Let $\Omega = (\Omega^A)_{A \in \Sigma - Str}$. We prove that Ω is a natural transformation. In other to do that, it is enough to prove that given $f \in Hom_{\Sigma - Str}(A, B)$, the following diagram commutes

$$\begin{array}{ccc} Fi_l(A) & \xrightarrow{\Omega^A} & Co_K(A) \\ {\scriptstyle f^{-1}}\uparrow & & \uparrow{\scriptstyle f^{-1}} \\ Fi_l(B) & \xrightarrow[\Omega^B]{} & Co_K(B) \end{array}$$

Let $F \in Fi_l(B)$. Firstly we prove that $f^{-1}(\Omega^B(F))$ is compatible with $f^{-1}(F)$. Let $(a, b) \in f^{-1}(\Omega^B(F))$ and suppose that $a \in f^{-1}(F)$. Then $(f(a), f(b)) \in \Omega^B(F)$ and $f(a) \in F$. Therefore $f(b) \in F$, thus $b \in f^{-1}(F)$.

Now let $(a,b) \in \Omega^A(f^{-1}(F))$, then by algebraizability of l, we have $\Delta^A(a,b) \in f^{-1}(F)$. Thus $\Delta^B(f(a),f(b)) = f(\Delta^A(a,b)) \subseteq F$. Therefore $(f(a),f(b)) \in \Omega^B(F)$ and finally $(a,b) \in f^{-1}(\Omega^B(F))$. Then $\Omega^A(f^{-1}(F)) = f^{-1}(\Omega^B(F))$. That proves the naturality of Ω.

$''2 \Rightarrow 1''$ Suppose that there is a natural isomorphism $\Phi : Fi_l \to Co_K$. In particular we have that for every $A \in \Sigma - Str$, $\Phi^A : Fi_l(A) \to Co_K(A)$ is a isomorphism and commutes with endomorphisms. By theorem 1.3.14 we have that l is an algebraizable logic. □

1.3.1 The category of algebrizable logics

With the definition of categories of logics given above, it is possible to define categories of algebraizable logics. Other categories of algebraizable logics can be found in [JKE96], [FC04].

- $\mathcal{A}_s$ is the category of algebraizable logics with morphisms in $\mathcal{L}_s$ which preserve algebraizing pairs. In the sequence of works, [AFLM06], [AFLM07] it is proved that the category $\mathcal{A}_s$ is a relatively complete ω-accessible category [AR94].

- $\mathcal{A}_f$ is the category of algebraizable logics with morphisms in $\mathcal{L}_f$ which preserve algebraizing pairs. $\mathcal{A}_f$ is a subcategory of $\mathcal{L}_f$, $\mathcal{A}_f \hookrightarrow \mathcal{L}_f$.

- Related to the category $\mathcal{A}_f$, we have the following subcategories: $\mathcal{A}_f^c, Q\mathcal{A}_f$ and $Q\mathcal{A}_f^c$.

- The "Lindenbaum algebraizable" logics are the algebraizable logics l such that given formulas $\varphi,\psi \in F(\Sigma)$, $\varphi \dashv\vdash \psi \Leftrightarrow \vdash \varphi\Delta\psi$. The Lindenbaum algebraizable logics gives a subcategory of the category of algebraizable logics ($j : Lind(\mathcal{A}_f) \hookrightarrow \mathcal{A}_f$). The inclusion functor $Lind(\mathcal{A}_f) \hookrightarrow \mathcal{A}_f$ has a left adjoint functor $L : \mathcal{A}_f \to Lind(\mathcal{A}_f)$ and $Lind(\mathcal{A}_f)$ is relevant in the representation theory of logics that we will present later.

Definition 1.3.17. *(a) Let $l' = (\Sigma',\vdash') \in \mathcal{L}_f$, $a = (\alpha,\vdash) \in \mathcal{A}_f$ and $f : l' \to a$ be a $\mathcal{L}_f$-morphism. Suppose $a \in \mathcal{A}_f$, then f is called $\Delta-$dense when, given $n \in \mathbb{N}$ and $\varphi \in F(\alpha)[n]$, there is a $\varphi' \in F(\Sigma)[n]$ such that $\vdash \check{f}(\varphi')\Delta\varphi$, for some equivalence set of formulas Δ of a. Obviously, if $a \in Lind(\mathcal{A}_f)$, then a morphism $f \in \mathcal{L}_f(l',a)$ is Δ-dense iff it is dense.*

(b) Let $l = (\Sigma,\vdash) \in \mathcal{L}_f$, $a' = (\alpha',\vdash') \in \mathcal{A}_f$. Define the binary relation $\approx_{a'}$ in the set $\mathcal{L}_f(l,a')$ by, $g_0,g_1 \in \mathcal{L}_f(l,a')$,

$$g_0 \approx_{a'} g_1 \; iff \; \forall\phi \in F(\Sigma)(X) \vdash' \check{g}_0(\phi)\Delta'\check{g}_1(\phi),$$

where Δ' is any equivalence set of formulas for a'. It follows from Fact 1.3.8 that this is an equivalence relation.

When $a = (\alpha',\vdash') \in \mathcal{A}_f$, we have an equivalence relation ${}_a\approx_{a'}$ in the set $\mathcal{A}_f(a,a')$. Moreover by the definition of morphisms in $\mathcal{A}_f$, the family $\{{}_a\approx_{a'}: a,a' \in \mathcal{A}_f\}$ defines a *congruence relation*[4] *on the category* $\mathcal{A}_f$ *(see [Mac71], Chapter II, Section 8). Denote* $\overline{\mathcal{A}_f}$ *the quotient category. It is clear that* $\overline{Lind(\mathcal{A}_f)} = Q(Lind(\mathcal{A}_f))$.

By Fact 1.3.8, clearly $Lind(\mathcal{A}_f) \subseteq \mathcal{A}_f^c$. In the sequel, we establish the equality between these categories . In particular, we obtain that the left adjoint functor $L : \mathcal{A}_f \to Lind(\mathcal{A}_f)$ of the inclusion $Lind(\mathcal{A}_f) \hookrightarrow \mathcal{A}_f$ is simply given by $l \in \mathcal{A}_f \mapsto l^{(c)} \in \mathcal{A}_f^c$ (see Remark 1.3.9).

The following proposition give us a characterization for Lindenbaum algebraizable logics.

Proposition 1.3.18. *Let l be a logic. Then l is Lindenbaum algebraizable iff it is an algebraizable and selfextensional logic.*

Proof:

"$\Rightarrow$" Suppose $l \in Lind(\mathcal{A}_f)$ By Fact 1.3.8, it follows that for every equivalence set of formulas Δ associated to l, the relation defined by $\vdash \Delta(\varphi, \psi)$ is a congruence relation. Therefore that the relation $\dashv\vdash$ is a congruence, thus $l \in \mathcal{A}_f^c$.

"$\Leftarrow$" Suppose $l \in \mathcal{A}_f^c$ and let $\varphi, \psi \in F(\Sigma)$. We only have to prove $\varphi \dashv\vdash \psi$ entails $\vdash \varphi\Delta\psi$ (see Remark 1.3.5).

Consider $T := \{\gamma \in F(\Sigma) : \vdash \gamma\}$ the set of all theorems of l. Let $\varphi, \psi \in F(\Sigma)$ be such that $\varphi \dashv\vdash \psi$. Then $\varphi \in T$ iff $\psi \in T$. Thus $\dashv\vdash$ is a Σ-congruence compatible with T. By definition of Ω^A 1.3.11, $\dashv\vdash \subseteq \Omega T$, thus $\langle \varphi, \psi \rangle \in \Omega T$.

It is straightforward that T is a filter in $F(\Sigma)$. By the Fact 1.3.12.(b) above, $\Omega T = \{\langle \sigma, \sigma' \rangle : \sigma\Delta\sigma' \in T\}$. Therefore $\varphi\Delta\psi \in T$, which means $\vdash \varphi\Delta\psi$. Therefore $l \in Lind(\mathcal{A}_f)$. $\square$

The following diagram represents the functors (and its adjoints) between the categories mentioned above:

$$\begin{tikzcd}
\mathcal{A}_f \arrow[r, "incl"] \arrow[d, shift left, "L"] & \mathcal{L}_f \arrow[r, "q"] \arrow[d, shift left, "c"] & \mathcal{Q}_f \arrow[d, shift left, "\bar{c}"] \\
Lind(\mathcal{A}_f) \arrow[r, "incl"'] \arrow[u, shift left, "j"] & \mathcal{L}_f^c \arrow[r, "q^c"'] \arrow[u, shift left, "i"] & \mathcal{Q}_f^c \arrow[u, shift left, "\bar{i}"]
\end{tikzcd}$$

1.3.2 Some generalizations of algebraizable logics

We introduce now some special kids of logics that are generalizations for algebraizable logics. In the chapter 3 we present sufficient conditions to get those logics via filter functor theory.

[4]If $f \in \mathcal{A}_f(b, a), f' \in \mathcal{A}_f(a', b')$, then $(f' \circ g_0 \circ f)_b \approx_{b'} (f' \circ g_1 \circ f)$.

[4]We thank prof. Ramon Jansana for suggesting this result.

Definition 1.3.19. *Let $l = (\Sigma, \vdash)$ a logic:*

- *l is Protoalgebraic logic if for any theory $T \in Th(l)$,*

$$if \langle \varphi, \psi \rangle \in \Omega(T) \ then \ T, \varphi \vdash \psi \ and \ T, \psi \vdash \varphi.$$

- *l is weakly algebraizable logic if it is protoalgebric and Ω is injective.*

- *l is Equivalential logic if there is a set of congruence formulas, i.e., a set of formulas $\Delta(q,p)$ in at most variables q and p such that for any $A \in \Sigma{-}Str$, any filter $F \in Fi_l(A)$ and any $a, b \in A$,*

$$\langle a, b \rangle \in \Omega^A(F) \quad iff \quad \Delta^A(a,b) \subseteq F.$$

- *A class of matrices M has its filters equationally definable by a set of equations $\tau(p)$ if for every matrix $\langle A, F \rangle \in M$, for every $a \in A$,*

$$a \in F \quad iff \quad \delta^A(a) = \varepsilon^A(a), \ for \ every \ \delta \approx \varepsilon \in \tau(p).$$

  l is Truth-equational logic if the class of reduced matrix $Matr^*(l)$ has its filters equationally definable.

Now we remind some characterizations of the logics defined above.

Theorem 1.3.20. *Let l be a logic:*

- *l is protoalgebraizable iff Ω is monotone on set of theories $Th(l)$.*

- *l is equivalential logic iff $(\Omega^A)_{A \in \Sigma - Str}$ commutes with homomorphism and Ω is monotone.*

- *l is truth-equational logic iff there exists a set of equations $\tau(p)$ such that for every algebra A and every $F \in Fi_l(A)$,*

$$F = \{a \in A; \ \tau^A(a) \subseteq \Omega^A(F)\}$$

The following diagrams represent the relations among those logics and the algebraizable logics, i.e., the first one represents the inclusions among those classes of logics. The second one represents how to "build" them by "intersection".

$$
\begin{tikzcd}
 & Algebraizable \arrow[dl] \arrow[dr] & & \\
Equivalential \arrow[dr] & & weakly\ algebrizable \arrow[dl] \arrow[dr] & \\
 & Protoalgebraizable & & Truth - equational
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & Protoalgebraizable \arrow[dr] & & Truth - equational \arrow[dl] \\
Equivalential \arrow[dr] & & weakly\ algebraizable \arrow[dl] & \\
 & Alebegraizable & &
\end{tikzcd}
$$

Chapter 2

Functorial encoding of algebraizable logics morphisms

In this chapter we start to provide tools to build the representation theory of logics. We establish some (categorial) relations between logics and their categories of structures, more precisely, given a morphism of algebraizable logics, there is an induced functor between the category of all structures over the underlying signatures such that it restricts to the quasivarieties that are their equivalent algebraic semantics. About this relation: (i) we provide an anti-isomorphism between the class of morphisms of signatures and some functors between the categories of associated structures; (ii) we prove that this anti-isomorphism restricts to an anti-isomorphism between morphisms of (Lindenbaum) algebraizable logics and some functors on its categories structures that restrict to its quasivarieties.

We verify some "translation" to properties about morphisms of logics through properties of functors between quasivarieties. In the following chapters we use this codification to obtain important results about studying of meta-logical properties, relations among others categories and to define (stably) Morita equivalence of logics. This chapter helps us understand better the local "behavior" of logics. We hope use the representation theory of logics that we start to develop in the chapter 5, to study the global "behavior" of logics.

In the three sections below, we present: (i) results on certain adjoint pairs of functors between quasivarieties; (ii) some results about functors between quasivarieties associated to morphisms of (Lindenbaum) algebraizable logics; (iii) a complete (functorial) codification of morphisms of signatures and morphisms of algebraizable logics.

In the sequel: (i) a quasivariety $\mathcal{K}$ on the signature Σ will be viewed as a full subcategory of the category of all structures on that given signature; (ii) for an algebraizable logic $a = (\alpha, \vdash)$, we will denote by $QV(a)$ the unique quasivariety semantics associated to a (see Fact 1.3.6).

2.1 Quasivarieties and signature functors

Here we analyze (adjoint pairs of) functors between quasivarieties associated to combination of two fronts: (i) inclusion functors: $\mathcal{K} \hookrightarrow \Sigma - Str$; (ii) "signature" functors i.e. each a $\mathcal{S}_f$-morphism, $h : \Sigma \longrightarrow \Sigma'$, induces a functor $h^\star : \Sigma' - Str \longrightarrow \Sigma - Str$. Natural transformations associated to the above mentioned adjunctions also play a significant role here and in the next subsections.

Recall that, by a classical result in universal algebra due to Mal'cev, a subclass $\mathcal{K} \subseteq \Sigma - Str$ is a quasivariety iff it is closed under isomorphisms, substructures, products and ultraproducts (or directed colimits).

The following Lemma was prove by Mal'cev, but we provide an alternative proof for it.

Lemma 2.1.1. *Let $\mathcal{K}$ be a quasivariety on the signature α. The inclusion functor has a left adjoint $(L, I) : \mathcal{K} \rightleftarrows \alpha - Str$: given by $M \mapsto M/\theta_M$ where θ_M is the least Σ-congruence in M such that $M/\theta_M \in \mathcal{K}$. Moreover, the unity of the adjunction (L, I) has components $(q_M)_{M \in \Sigma - Str}$, where $q_M : M \twoheadrightarrow M/\theta_M$ is the quotient homomorphism.*

Proof: Consider $\Gamma_M = \{\theta \subseteq |M| \times |M|$; is congruence relation and $M/\theta \in \mathcal{K}\}$. Γ is not empty, because $\theta = |M|\times|M|$ is a congruence relation and $M/\theta = \{\star\} \in \mathcal{K}$. Let $\theta_M = \bigcap \Gamma_M$. We will show first that $\theta \in \Gamma_M$: as θ_M is a Σ-congruence in M, it remains to check that $M/\theta_M \in \mathcal{K}$.

Consider the "diagonal" Σ-homomorphism:

$$\delta_M : M \to \prod_{\theta \in \Gamma_M} M/\theta;\ m \mapsto ([m]_\theta)_{\theta \in \Gamma_M}.$$

We will show that $Ker(\delta_M) = \theta_M$:

$(m, n) \in Ker(\delta_M) \Leftrightarrow ([m]_\theta)_{\theta \in \Gamma_M} = ([n]_\theta)_{\theta \in \Gamma_M} \Leftrightarrow [m]_\theta = [n]_\theta, \forall\ \theta \in \Gamma_M \Leftrightarrow m\theta n\ \forall\ \theta \in \Gamma_M \Leftrightarrow m\theta_M n$.

Thus, by the "theorem of homomorphism" on $\Sigma - Str$, there is a unique Σ-<u>mono</u>morphism $\bar{\delta}_M : M/\theta_M \rightarrowtail \prod_{\theta \in \Gamma_M} M/\theta$ such the diagram below commutes

$$\begin{tikzcd} M \arrow[r, "\delta_M"] \arrow[d, "q_M"'] & \prod_{\theta \in \Gamma_M} M/\theta \\ M/\theta_M \arrow[ur, "\bar{\delta}_M"'] & \end{tikzcd}$$

As $\mathcal{K}$ is closed under products, we have that $\prod_{\theta \in \Gamma_M} M/\theta \in \mathcal{K}$. We also have that $\mathcal{K}$ is closed under substructures and isomorphisms, then M/θ_M is $\mathcal{K}$.

Denote $L(M) := M/\theta_M$. We will show that $q_M : M \twoheadrightarrow I(L(M))$ satisfies the universal property relatively to Σ-homomorphisms $f : M \longrightarrow I(N)$, with $N \in \mathcal{K}$.

Thus we obtain a injective Σ-homomorphism $\bar{f} : M/Ker(f) \rightarrowtail I(N)$. As $\mathcal{K}$ is closed by substructures and isomorphisms, so we have that $M/Ker(f) \in \mathcal{K}$. Hence $Ker(f) \in \Gamma_M$

and $\theta_M \subseteq Ker(f)$. Then, again by the theorem of homomorphism, there is a unique homomorphism $\tilde{f} : M/\theta_M \longrightarrow N$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "q_M"] \arrow[dr, "f"'] & I(L(M)) \arrow[d, "I(\tilde{f})"] \\ & I(N) \end{tikzcd}$$

It follows from an well-known result on adjoint functors, see for instance [Mac71], Theorem 2 in page 81, that there is a unique way to obtain a functor $L : \Sigma - Str \to \mathcal{K}$ such that $(q_M)_{M \in \Sigma - Str}$ become the unity of an adjunction $(L, I) : \mathcal{K} \rightleftarrows \alpha - Str$. Given $g \in \Sigma - Str(M, P)$, then $q_P \circ g \in \Sigma - Str(M, I(P))$ and, as $(g \times g)^{-1}[\theta_P] = Ker(q_P \circ g)$, we have that $L(g) = \widetilde{q_P \circ g} : M/\theta_M \longrightarrow P/\theta_P : [m]_{\theta_M} \mapsto [g(m)]_{\theta_P}$. $\square$

Remark 2.1.2. *Let Σ be a signature and $\mathcal{K} \subseteq \Sigma - Str$ be a quasivariety.*

(a) The forgetful functor ($\Sigma - Str \overset{U}{\to} Set$) has the "absolutely free algebra" functor ($Set \overset{F}{\to} \Sigma - Str$), $Y \mapsto F(Y)$, as left adjoint. The unity of this adjunction has components the inclusion maps $\sigma_Y : Y \rightarrowtail U(F(Y))$, for each set Y.

(b)The (forgetful) functor ($\mathcal{K} \overset{I}{\to} \Sigma - Str \overset{U}{\to} Set$) has the (free) functor ($Set \overset{F}{\to} \Sigma - Str \overset{L}{\to} \mathcal{K}$), $Y \mapsto F(Y)/\theta_{F(Y)}$, as left adjoint. Moreover, if $\sigma_Y : Y \to U \circ F(Y)$ is the Y-component of the unity of the adjunction (F, U), then $(Y \overset{t_Y}{\to} UILF(Y)) := (Y \overset{\sigma_Y}{\to} UF(Y) \overset{U(q_{F(Y)})}{\to} UILF(Y))$ is the Y-component of the adjunction $(L \circ F, U \circ I)$.

Proposition 2.1.3. *Let $a = (\Sigma, \vdash)$ be an algebraizable and consider the binary relation on $F(X)$,*

$$\phi \sim_\Delta \psi \ iff \ \vdash \phi \Delta \psi,$$

where Δ is an equivalence formula for a. Then:
(a) $\sim_\Delta$ is a Σ-congruence on $F(X)$.
(b) $F(X)/\Delta := F(X)/\sim_\Delta \in QV(a)$.
(c) $\sim_\Delta = \theta_{F(X)}$ (see Lemma 2.1.1), thus $F(X)/\Delta = L(F(X))$ is the free $QV(a)$-object over the set $X = \{x_0, \ldots, x_n, \ldots\}$.
In particular, when a is a Lindenbaum algebraizable logic, $F(X)/\Delta = F(X)/(\dashv\vdash)$ is the free $QV(a)$-object over the set X.

Proof:

(a) By items (a)-(d) in Fact 1.3.8 is clear that $\sim_\Delta$ is a Σ-congruence on $F(X)$.

(b) By (a) above, thus $F(X)/\Delta := F(X)/\sim_\Delta$ is a Σ-structure. Thus, to obtain $F(X)/\Delta \in QV(a)$, it is enough to show that $F(\Sigma)/\Delta$ satisfies the conditions of Fact 1.3.7.

- Let $\varphi := x_0\Delta x_0$, then $\vdash \varphi$. As a is algebraizable logics, $\varphi \dashv\vdash \delta(\varphi)\Delta\varepsilon(\varphi)$. So $\vdash \delta(\varphi)\Delta\varepsilon(\varphi)$. Therefore $[\delta(\varphi)]_\Delta = [\varepsilon(\varphi)]_\Delta$. Hence $F(X)/\Delta \models \delta(\varphi) \equiv \varepsilon(\varphi)$.

- Suppose $F(X)/\Delta \models \delta(x_0\Delta x_1) \equiv \epsilon(x_0\Delta x_1)$. Then

$$[\delta(x_0\Delta x_1)]_\Delta = [\varepsilon(x_0\Delta x_1)]_\Delta$$

therefore $\vdash \delta(x_0\Delta x_1)\Delta\epsilon(x_0\Delta x_1) \rightarrow\ x_0 \equiv x_1$. As a is an algebraizable logic, $(x_0\Delta x_1) \dashv\vdash \delta(x_0\Delta x_1)\Delta\epsilon(x_0\Delta x_1) \rightarrow\ x_0 \equiv x_1$, we obtain $\vdash x_0\Delta x_1$, i.e. $[x_0]_\Delta = [x_1]_\Delta$. Hence $F(X)/\Delta \models (x_0 \equiv x_1)$ and $F(X)/\Delta \models \delta(x_0\Delta x_1) \equiv \epsilon(x_0\Delta x_1)\ \rightarrow\ x_0 \equiv x_1$.

- Given $\psi_0, ..., \psi_{n-1}, \varphi \in F(X)$ such that $\{\psi_0, ..., \psi_{n-1}\} \vdash \varphi$ and suppose $F(X)/\Delta \models \delta(\psi_0) \equiv \varepsilon(\psi_0) \wedge ... \wedge \delta(\psi_{n-1}) \equiv \varepsilon(\psi_{n-1})$. Then $[\delta(\psi_0)]_\Delta = [\varepsilon(\psi_0)]_\Delta, ..., [\delta(\psi_{n-1})]_\Delta = [\varepsilon(\psi_{n-1})]_\Delta$. Therefore

$$\vdash \delta(\psi_0)\Delta\varepsilon(\psi_0), ..., \vdash \delta(\psi_{n-1})\Delta\varepsilon(\psi_{n-1}).$$

As a is algebraizable logic, $\psi_i \dashv\vdash \psi_i$, $\forall i < n$, thus $\vdash \psi_0, ..., \vdash \psi_{n-1}$ and, by cut, we obtain $\vdash \varphi$. Again, as a is algebraizable, we obtain $\vdash \delta(\varphi)\Delta\varepsilon(\varphi)$. Hence $F(X)/\Delta \models \delta(\varphi) \equiv \varepsilon(\varphi)$ and $F(X)/\Delta \models (\bigwedge_{i<n} \delta(\psi_i) \equiv \varepsilon(\psi_i))\ \rightarrow\ \delta(\varphi) \equiv \varepsilon(\varphi)$.

(c) Let $M \in QV(a)$. The universal property of $\sigma_X : X \longrightarrow U(F(X))$ induces a bijection $\Sigma - Str(F(X), I(M)) \cong Set(X, U(I(M))$: for each function $v : X \longrightarrow U(I(M))$ there is an unique Σ-homomorphism $V : F(X) \longrightarrow I(M)$ such that $V \circ \sigma_X = v$. Establish the equality $\sim_\Delta = \theta_{F(X)}$ is equivalent to prove that $\sim_\Delta \subseteq Ker(V)$, for each function $v : X \longrightarrow U(I(M))$. Suppose $\phi \sim_\Delta \psi$, then $\vdash \phi\Delta\psi$. As a is an algebraizable logic we obtain, by Fact 1.3.4 $\models_{QV(a)} \phi \equiv \psi$, i.e. for each $M \in QV(a)$ and each Σ-homomorphism $H : F(X) \longrightarrow I(M)$, $H(\phi) = H(\psi)$. Thus $\sim_\Delta \subseteq Ker(V)$ for each function $v : X \longrightarrow U(I(M))$. □

Remark 2.1.4. *By reasoning analogous to the proof above we can establish that, for every $Y \subseteq X$, the binary relation on $F(Y)$ given by $(\sim_\Delta) \restriction := (\sim_\Delta) \cap (F(Y) \times F(Y))$ coincides with $\theta_{F(Y)}$, thus $F(Y)/\Delta \restriction := F(Y)/ \sim_\Delta\restriction$ is the free $QV(a)$-object over the set Y.*

2.1.5. Signature functors: *Given a morphism in $\mathcal{S}_f$, $\Sigma \xrightarrow{h} \Sigma'$, we associate a functor $\Sigma - Str \xleftarrow{h^\star} \Sigma' - Str$ in the following way*

- *For each $M' \in \Sigma' - Str$ denote $h^\star(M') = (M')^h$ the Σ-structure such that*

– *$|(M')^h| = |M'|$ (structures with same underlying set);*

– *Let $k \geq 0$ and $c_k \in \Sigma_k$, then $h(c_k) \in F(\Sigma')[k]$ is a* first-order k-ary term over Σ' *and its interpretation in the Σ'-structure M' is a certain k-ary operation on $|M'|$, $M'^{h(c_k)} : |M'|^k \rightarrow |M'|$; define $(c_k)^{(M')^h} := h(c_k)^{M'}$ (it is a k-ary operation on $|M'^h|$).*

If $\phi \in F(\Sigma)$ has exactly n variables, then it can be viewed as n-ary first-order Σ-term and its interpretation over $(M')^h$ is defined (by recursion on complexity); analogously the n-ary first-order Σ'-term $\check{h}(\phi)$ can be interpreted on M'. We can prove, by induction on the *complexity of ϕ, that the n-operations on the same set $|(M')^h| = |M'|$, $(\phi)^{(M'^h)}$, $(\check{h}(\phi))^{(M')}$, coincide.*

- *Let $g \in \Sigma - Str(M', N')$, we define $h^\star(M', g, N') = (M'^h, g, N'^h) \in \Sigma - Str(M'^h, N'^h)$: clearly, the function g determines a Σ-homomorphism from M'^h to N'^h).*

It is clear that $h^\star$ preserves identities and composition, thus it is a (covariant) functor.

By construction, the functor $h^\star : \Sigma' - Str \to \Sigma - Str$ "commutes over Set", i.e., $U \circ h^\star = U'$. It is straitforward that $h^\star$ preserves, strictly, *the following constructions: substructures, products, directed inductive limits, reduced products, congruences and quotients.*

Proposition 2.1.6. *Consider a signature morphism $h \in \mathcal{S}_f(\Sigma, \Sigma')$ and quasivarieties $I : \mathcal{K} \hookrightarrow \Sigma - Str$, $I' : \mathcal{K}' \hookrightarrow \Sigma' - Str$. Suppose that the induced functor $h^\star : \Sigma' - Str \longrightarrow \Sigma - Str$ restricts to a $h^\star\!\restriction : \mathcal{K}' \to \mathcal{K}$, i.e. there is a (unique) functor $h^\star\!\restriction$ such that $I \circ h^\star\!\restriction = h^\star \circ I'$, then*

(a) $h^\star\!\restriction : \mathcal{K}' \to \mathcal{K}$ has a left adjoint $G : \mathcal{K} \to \mathcal{K}'$.

(b) Suppose that $h^\star\!\restriction : \mathcal{K}' \to \mathcal{K}$ satisfies the following conditions:

(b1) $h^\star\!\restriction$ is faithful;

(b2) $h^\star\!\restriction$ is full;

(b3) $h^\star\!\restriction$ is injective on objects;

(b4) $h^\star\!\restriction$ is hereditary*, i.e., given $M \in \mathcal{K}$, $N' \in \mathcal{K}'$ such that there is an injective Σ-homomorphism $j : M \rightarrowtail h^\star\!\restriction(N')$, then there is $M' \in \mathcal{K}'$ such that $h^\star\!\restriction(M') = M$.*

Then the left adjoint G can be defined on objects $M \in \mathcal{K}$ as "a quotient" $G(M) \in \mathcal{K}'$, with $h^\star\restriction (G(M)) = M/\rho_M$, where ρ_M is the least Σ-congruence in M such that $M/\rho_M = h^\star(M')$, for some $M' \in \mathcal{K}'$ (that is automatically unique by (l3)); moreover the M-component of the unity of the adjunction is the quotient map $p_M : M \twoheadrightarrow M/\rho_M$.

Proof: (a) We will give here an indirect proof of the existence of the left adjoint G: we will prove that the hypothesis on "Freyd Left Adjoint Theorem" (see [Mac71], Theorem 2, page 117) are satisfied by $h^\star\!\restriction : \mathcal{K}' \to \mathcal{K}$.

- As $\mathcal{K} \subseteq \Sigma - Str$ and $\mathcal{K}' \subseteq \Sigma' - Str$ are closed under isomorphisms, substructures and products, $\mathcal{K}$ and $\mathcal{K}'$ are complete categories, i.e. they have all small limits. Moreover, as $h^\star : \Sigma' - Str \to \Sigma - Str$ (strictly) preserves: isomorphisms, substructures and products, then the same holds for $h^\star\!\restriction : \mathcal{K}' \to \mathcal{K}$. Thus $h^\star\!\restriction : \mathcal{K}' \to \mathcal{K}$ preserves all small limits.

- We show that the "solution set condition" holds for $h^\star\!\restriction$. Let $M \in \mathcal{K}$ and consider $\kappa := card(|M|)$ and consider the *class* $C_M := \{N' \in \mathcal{K}'$: such that N' has a $\mathcal{K}'$-generator subset of size $\leq \kappa\}$. It is clear that there is a *set* $S_M \subseteq C_M$ of representatives of C_M modulo isomorphism. We will show that $\bigcup_{S' \in S_M} \mathcal{K}(M, h^\star\!\restriction (S'))$ is a *set* that satisfies the solution set condition for M'.

Let $P' \in \mathcal{K}'$ and $f : M \to h^\star\!\restriction (P')$ be a $\mathcal{K}$-morphisms. Let $N' \subseteq P'$ be the Σ'-substructure of P' that is generated by $image(f)$. Then $N' \in C_M$ and we can take $S' \in S_M$ such that $S' \cong_{\mathcal{K}'} N'$. Consider a fixed $\mathcal{K}'$-isomorphism $t : S' \to N'$ and let $i : N' \hookrightarrow P'$ be the inclusion. Then we have shown that the homomorphism $f : M \to h^\star\restriction (P')$ factors through some member g of the set $\bigcup_{S' \in S_M} \mathcal{K}(M, h^\star\restriction (S'))$ (i.e. $f = h^\star(i \circ t) \circ g$).

(b) Let $M \in \mathcal{K}$, we will prove that the "quotient map" $p_M : M \to h^\star\restriction (G(M))$, $h^\star\restriction (G(M)) = M/\rho_M$, satisfies the universal property. Consider $\Omega_M := \{\theta \subseteq M \times M : \theta$ is a Σ-congruence in M and there is a (unique) $P'_\theta \in \mathcal{K}'$ such that $M/\theta = h^\star\restriction (P'_\theta)\}$. We show first that Ω_M has minimum by verifying that $\rho_M := \bigcap \Omega_M \in \Omega_M$. Indeed we have a *injective* Σ-homomorphism $j : M/\rho_M \rightarrowtail \prod_{\theta\in\Omega_M} M/\theta$, $[m]_{\rho_M} \mapsto ([m]_\theta)_{\theta\in\Omega_M}$. By definition of Ω_M, $\prod_{\theta\in\Omega_M} M/\theta = \prod_{\theta\in\Omega_M} h^\star\restriction (P'_\theta)$. As $h^\star\restriction$ preserves products we have the *injective* Σ-homomorphism $j : M/\theta_M \rightarrowtail h^\star\restriction (\prod_{\theta\in\Omega_M} P'_\theta)$. By conditions (b4) and (b3), $M/\rho_M = h^\star\restriction (M')$ for a unique $M' \in \mathcal{K}$. Thus $\rho_M = \bigcap \Omega_M \in \Omega_M$.

Let $N' \in \mathcal{K}'$ and $f : M \to h^\star\restriction (N')$ be a Σ-homomorphism: we will show that there is a unique Σ'-homomorphism $f' : M' \to N'$ such that:

$$(M \xrightarrow{f} h^\star\restriction (N')) \;=\; (M \overset{p_M}{\twoheadrightarrow} h^\star\restriction (M') \xrightarrow{h^\star\restriction(f')} h^\star\restriction (N'))$$

Then f factors through the quotient homomorphism $q_f : M \twoheadrightarrow M/Ker(f)$ by the *injective* Σ-homomorphism $\bar{f} : M/Ker(f) \rightarrowtail h^\star\restriction (N')$. Then, by conditions (b4) and (b3), $M/ker(f) = h^\star\restriction (P')$ for a unique $P' \in \mathcal{K}$. As $Ker(f) \in \Omega_M$, we have $\rho_M \subseteq Ker(f)$ and, by the theorem of homomorphism, there is a unique Σ-homomorphism $\bar{f} : M/\rho_M \to h^\star\restriction (N')$ such that $\bar{f} \circ p_M = f$. As $M/\rho_M = h^\star\restriction (M')$ for a unique $M' \in \mathcal{K}$, the conditions (b1) and (b2) ensures that there is a unique Σ'-homomorphism $f' : M' \to N'$ such that $h^\star\restriction (f') = \bar{f}$. Then f' is the unique Σ'-homomorphism such that $f = h^\star\restriction (f') \circ p_M$. □

Proposition 2.1.7. *Consider a signature morphism $h \in \mathcal{S}_f(\Sigma, \Sigma')$ and quasivarieties $I : \mathcal{K} \hookrightarrow \Sigma - Str$, $I' : \mathcal{K}' \hookrightarrow \Sigma' - Str$. Suppose that the induced functor $h^\star : \Sigma' - Str \to \Sigma - Str$ restricts to a (unique) functor $h^\star\restriction : \mathcal{K}' \to \mathcal{K}$, i.e. $I \circ h^\star\restriction = h^\star \circ I'$. Denote G and $\bar{G}$ the (unique up to natural isomorphism) left adjoint functors of, respectively , $h^\star$ and $h^\star\restriction$ (they exists by Proposition 2.1.6 above). Then:*

(a) $(G \circ F) \cong F'$ and $(\bar{G} \circ L) \cong (L' \circ G)$.

(b) There is a natural epimorphism $\tilde{h} : L \circ h^\star \twoheadrightarrow h^\star\restriction \circ L$, that restricts to $L \circ h^\star \circ I' = h^\star\restriction \circ L' \circ I'$.

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, shift left, "\bar{G}"] \arrow[d, shift right, "I"'] \arrow[ddr, bend right=50] \arrow[drr, phantom, "\nearrow"] & & \mathcal{K}' \arrow[ll, shift left, "h^\star\restriction"] \arrow[d, shift left, "I'"] \arrow[ddl, bend left=50] \\
\Sigma\text{--}Str \arrow[u, shift right, "L"'] \arrow[rr, shift right, "G"'] \arrow[dr, shift right, "U"'] & & \Sigma'\text{--}Str \arrow[u, shift left, "L'"] \arrow[ll, shift right, "h^\star"'] \arrow[dl, shift left, "U'"] \\
& Set \arrow[ul, shift right, "F"'] \arrow[ur, shift left, "F'"] \arrow[uul, bend left=70] \arrow[uur, bend right=70] &
\end{tikzcd}$$

Proof:

(a) The uniqueness up to isomorphism of left adjoints entails that $U \circ h^\star$ has a left adjoint isomorphic to $G \circ F$. As $U \circ h^\star = U'$ and F' is a left adjoint of U', again the uniqueness of left adjoints up to isomorphism ensures that $(G \circ F) \cong F'$. Analogously, from the equality $I \circ h^\star\restriction = h^\star \circ I'$, we obtain the natural isomorphism $(\bar{G} \circ L) \cong (L' \circ G)$.

(b) Let $M' \in \Sigma' - Str$ and consider the canonical arrow in $\Sigma' - Str$ $q'_{M'} : M' \twoheadrightarrow M'/\theta'_{M'} = I'(L'(M'))$. Applying $h^\star$, we obtain the (surjective) Σ-homomorphism $h^\star(q'_{M'}) : h^\star(M') \twoheadrightarrow h^\star(M'/\theta'_{M'})$ and the induced Σ-isomorphism

$$\overline{h^\star(q'_{M'})} : h^\star(M')/ker(h^\star(q'_{M'}) \xrightarrow{\cong} h^\star(M'/\theta'_{M'}).$$

As the functor $h^\star$ commutes over Set and (strictly) preserves substructures and products, then $h^\star(\theta'_{M'})$ is a Σ congruence over $h^\star(M')$ and $h^\star(M'/\theta'_{M'}) = h^\star(M')/h^\star(\theta'_{M'})$; thus $ker(h^\star(q'_{M'})) = h^\star(\theta'_{M'})$ and

$$\overline{h^\star(q_{M'})} = Id : h^\star(M')/h^\star(\theta'_{M'}) \longrightarrow h^\star(M'/\theta'_{M'}).$$

In particular, $h^\star(M')/h^\star(\theta'_{M'}) = h^\star(M'/\theta'_{M'}) \in \mathcal{K}$ and $\theta_{h^\star(M')} \subseteq h^\star(\theta'_{M'})$. Therefore, there is a *canonical surjective* Σ-homomorphism

$$\tilde{h}_{M'} : h^\star(M')/\theta_{h^\star(M')} \twoheadrightarrow h^\star(M')/h^\star(\theta'_{M'}) = h^\star(M'/\theta'_{M'}) :$$

this defines a $\mathcal{K}$-morphism $\tilde{h}_{M'} : L(h^\star(M')) \twoheadrightarrow h^\star \restriction (L'(M'))$.

When $M' \in \mathcal{K}'$, then $h^\star(M') \in \mathcal{K}$, $\theta'_{M'} = \Delta_{|M'|}$ and $\theta_{h^\star(M')} = \Delta_{|h^\star(M')|} = h^\star(\Delta_{|M'|}) = h^\star(\theta'_{M'})$. Thus, in this case, $\tilde{h}_{M'} = Id : L(h^\star(M')) \longrightarrow h^\star \restriction (L'(M'))$.

If $f' : M' \longrightarrow N'$ is a Σ'-homomorphism, then $h^\star(f') : h^\star(M') \longrightarrow h^\star(N')$ is a Σ-homomorphism. To show that the diagram below commutes

$$
\begin{tikzcd}
M' \arrow[d, "f'"'] & L \circ h^\star(M') \arrow[r, "\tilde{h}_{M'}"] \arrow[d, "L\circ h^\star(f')"'] & h^\star_\upharpoonright \circ L'(M') \arrow[d, "h^\star_\upharpoonright \circ L'(f')"] \\
N' & L \circ h^\star(N') \arrow[r, "\tilde{h}_{N'}"'] & h^\star_\upharpoonright \circ L'(N')
\end{tikzcd}
$$

it is enough to realize that

$$h^\star_\upharpoonright(L'(f')) \circ \tilde{h}_{M'} \circ q_{h^\star}(M') = \tilde{h}_{N'} \circ L(h^\star(f')) \circ q_{h^\star}(M'),$$

where $q_{h^\star}(M') : h^\star(M') \twoheadrightarrow h^\star(M')/\theta_{h^\star(M')}$ is the canonical *surjective* Σ- homomorphism, but this follows immediately from a diagram chase. Thus $\tilde{h} := (\tilde{h}(M')_{M' \in \Sigma' - Str})$ is a natural transformation. □

2.2 Algebraizable logics and functors

In this part of the work, we verify that the general results on the functors between quasivarieties presented in the previous subsection can be applied to functors induced by logical morphisms between algebraizable logics. Are established the first connections between properties of the logical morphisms and the properties of its induced functors.

Proposition 2.2.1. *Let $a = (\alpha, \vdash)$ and $a' = (\alpha', \vdash')$ be algebraizable logics and let $h \in \mathcal{A}_f(a, a')$. Then the induced functor $h^\star : \alpha' - Str \to \alpha - Str$ restricts to $h^\star\!\upharpoonright : QV(a') \to QV(a)$ (i.e. $I \circ h^\star\!\upharpoonright = h^\star \circ I'$).*

Proof: As $QV(a) \subseteq \alpha - Str$ and $QV(a') \subseteq \alpha' - Str$ are full subcategories, it is enough to show that: for each $M' \in QV(a')$ we have $h^\star(M') \in QV(a)$.

It follows from the description of a set of quasi-identities that determines the unique equivalent quasivariety semantics associated to algebraizable logic in Fact 1.3.7 it follows that, if $(\Delta, (\delta, \epsilon))$ is an algebraizable pair for $a = (\alpha, \vdash)$, then the set of quasi-identities $S_a = S_a^0 \cup S_a^1 \cup S_a^2$ axiomatizes $QV(a)$, where:
$S_a^0 = \{\delta(x_0 \Delta x_0) \equiv \epsilon(x_0 \Delta x_0)\}$;
$S_a^1 = \{\delta(x_0 \Delta x_1) \equiv \epsilon(x_0 \Delta x_1)) \ \to \ x_0 \equiv x_1\}$;
$S_a^2 = \{(\delta(\psi_0) \equiv \epsilon(\psi_0) \wedge ... \wedge \delta(\psi_{n-1}) \equiv \epsilon(\psi_{n-1})) \ \to \ \delta(\varphi) \equiv \epsilon(\varphi) \ : \{\psi_0, ..., \psi_{n-1}\} \vdash \varphi\}$.

Denote $\mathfrak{h}$ the extension of h to first-order formulas, instead $\check{h}$ that is the extension of h for propositional α- formulas (= first-order terms). For instance, $\mathfrak{h}((\delta(\psi_0) \equiv \epsilon(\psi_0) \wedge \ldots \wedge \delta(\psi_{n-1}) \equiv \epsilon(\psi_{n-1})) \ \to \ \delta(\varphi) \equiv \epsilon(\varphi)) = (\check{h}\delta(\check{h}\psi_0) \equiv \check{h}\epsilon(\check{h}\psi_0) \wedge \ldots \wedge \check{h}\delta(\check{h}\psi_{n-1}) \equiv \check{h}\epsilon(\check{h}\psi_{n-1})) \ \to \ \check{h}\delta(\check{h}\varphi) \equiv \check{h}\epsilon(\check{h}\varphi)$.

As $h \in \mathcal{A}_f(a, a')$ then:

- $((\check{h}(\delta), \check{h}(\epsilon)), \check{h}(\Delta))$ is an algebraizable pair for a'.

- If $\{\psi_0, ..., \psi_{n-1}\} \vdash \varphi$, then $\{\check{h}\psi_0, ..., \check{h}\psi_{n-1}\} \vdash' \check{h}\varphi$.

From these, it follows that: $\mathfrak{h}[S^0_a] = S^0_{a'}$, $\mathfrak{h}[S^1_a] = S^1_{a'}$ and $\mathfrak{h}[S^2_a] \subseteq S^2_{a'}$. Thus, for each quasi-equation $\Omega \in S^0_a \cup S^1_a \cup S^2_a$, we have $M' \vDash_{\alpha'} \mathfrak{h}(\Omega)$. On the other hand, for each first-order formula Θ holds the following equivalence:

$$M' \vDash_{\alpha'} \mathfrak{h}(\Theta) \ \Leftrightarrow\ h^\star(M') \vDash_\alpha \Theta.$$

Thus $h^\star(M') \in QV(a)$, as we wish. □

Proposition 2.2.2. *Let $l = (\Sigma, \vdash) \in \mathcal{L}_f$ and $a, a' \in \mathcal{A}_f$. Keeping the notation in the definition 1.3.17, we have:*

(a) Let $g_0, g_1 : l \to a'$ be $\mathcal{L}_f$-morphisms. Then

$$g_0 \approx_{a'} g_1 \ \Leftrightarrow\ g_0^\star\restriction = g_1^\star\restriction : QV(a') \to \Sigma - Str.$$

(b) Let $g_0, g_1 : a \to a'$ be $\mathcal{A}_f$-morphisms. Then

$$[g_0]_\approx = [g_1]_\approx \in \overline{\mathcal{A}_f} \ \Leftrightarrow\ g_0^\star\restriction = g_1^\star\restriction : QV(a') \to QV(a).$$

Proof: Item (b) follows from item (a), since a quasivariety on signature α determines a full subcategory of $\alpha - Str$.

(a)" $\Rightarrow$ "

Let $M' \in QV(a')$ and $c_n \in \Sigma_n$. As $g_0 \approx_{a'} g_1$, we have that

$$\vdash_{a'} \check{g}_0(c_n)(x_0, ..., x_{n-1}) \Delta \check{g}_1(c_n)(x_0, ..., x_{n-1}).$$

Thus, by Fact 1.3.4, $\vDash_{QV(a')} \check{g}_0(c_n)(x_0, ..., x_{n-1}) \equiv \check{g}_1(c_n)(x_0, ..., x_{n-1})$. Therefore:

$$c_n^{M'^{g_0}} = (g_0(c_n))^{M'} = (g_1(c_n))^{M'} = c_n^{M'^{g_1}}$$

Thus $g^\star_{0\restriction}(M') = g^\star_{1\restriction}(M')$ and, as $g^\star_{0\restriction}, g^\star_{1\restriction}$ commute over Set, they coincide also on the arrow level. Therefore $g^\star_{0\restriction} = g^\star_{1\restriction}$.

" $\Leftarrow$ "

Suppose that $g^\star_{0\restriction} = g^\star_{1\restriction}$. Let $\varphi \in F(\Sigma)$, hence $\varphi^{M'^{g_0}} = \varphi^{M'^{g_1}}$ for all $M' \in QV(a')$. So $\vDash_{QV(a')} \check{g}_0(\varphi) \equiv \check{g}_1(\varphi)$. Due to a' to be algebraizable, by Fact 1.3.4, $\vdash_{a'} \check{g}_0(\varphi)\Delta\check{g}_1(\varphi)$. Therefore $g_0 \approx_{a'} g_1$. □

Corollary 2.2.3. *Let $l = (\Sigma, \vdash) \in \mathcal{L}_f$ and $a, a' \in \mathcal{A}^c_f$.*

(a) Let $g_0, g_1 : l \to a'$ be $\mathcal{L}_f$-morphisms. Then

$$[g_0]_{\dashv\vdash} = [g_1]_{\dashv\vdash} \in \mathcal{Q}_f \ \Leftrightarrow\ g_0^\star\restriction = g_1^\star\restriction : QV(a') \to \Sigma - Str.$$

(b) Let $g_0, g_1 : a \to a'$ be $\mathcal{A}^c_f$-morphisms. Then

$$[g_0]_{\dashv\vdash} = [g_1]_{\dashv\vdash} \in Q\mathcal{A}^c_f \ \Leftrightarrow\ g_0^\star\restriction = g_1^\star\restriction : QV(a') \to QV(a).$$

Proposition 2.2.4. *Let a and a' be algebraizable logics and $a \underset{h}{\overset{h'}{\rightleftarrows}} a'$ be a pair of $\mathcal{A}_f$-morphisms. Then $a \underset{[h]_\approx}{\overset{[h']_\approx}{\rightleftarrows}} a'$ is a pair of inverse $\overline{\mathcal{A}_f}$-isomorphisms iff $QV(a) \underset{h^\star\upharpoonright}{\overset{h'^\star\upharpoonright}{\leftrightarrows}} QV(a')$ is an isomorphism of categories.*

Proof: The induced $\overline{\mathcal{A}_f}$-morphisms $a \underset{[h]_\approx}{\overset{[h']_\approx}{\rightleftarrows}} a'$ is a pair of inverse $\overline{\mathcal{A}_f}$-isomorphisms iff

$$[id_a]_\approx = [h']_\approx \circ [h]_\approx = [h' \bullet h]_\approx \textit{ and}$$

$$[id_{a'}]_\approx = [h]_\approx \circ [h']_\approx = [h \bullet h']_\approx$$

$$iff \text{ (by Corollary 2.2.2.(b))}$$

$$id_a^\star\upharpoonright = (h' \bullet h)^\star\upharpoonright = (h^\star \circ h'^\star)\upharpoonright = h^\star\upharpoonright \circ h'^\star\upharpoonright \textit{ and } id_{a'}^\star\upharpoonright = (h \bullet h')^\star\upharpoonright = (h'^\star \circ h^\star)\upharpoonright = h'^\star\upharpoonright \circ h^\star\upharpoonright$$

$$iff$$

the pair of functors $QV(a) \underset{h^\star\upharpoonright}{\overset{h'^\star\upharpoonright}{\leftrightarrows}} QV(a')$ is a pair of inverse isomorphism of categories. $\square$

Restricting the above result to the setting of Lindenbaum algebarizable logics, we obtain the

Corollary 2.2.5. *Let a and a' be Lindenbaum algebraizable logics and $a \underset{h}{\overset{h'}{\rightleftarrows}} a'$ be a pair of $\mathcal{A}_f^c$-morphisms. Then $a \underset{[h]_{\dashv\vdash}}{\overset{[h']_{\dashv\vdash}}{\rightleftarrows}} a'$ is a pair of inverse $Q(\mathcal{A}_f^c)$-isomorphisms[1] iff $QV(a) \underset{h^\star\upharpoonright}{\overset{h'^\star\upharpoonright}{\leftrightarrows}} QV(a')$ is an isomorphism of categories.*

Proposition 2.2.6. *Let $l = (\Sigma, \vdash) \in \mathcal{L}_f$ and $a, a' \in \mathcal{A}_f$.*

(a) Let $h : l \to a'$ be a $\mathcal{L}_f$-morphism. Consider the conditions:

(a1) h is a Δ-dense $\mathcal{L}_f$-morphism.

(a2) The functor $h^\star\upharpoonright: QV(a') \to \Sigma - Str$ is full, faithful, injective on objects and satisfies the heredity condition (see 2.1.6.(b4)).

(b) Let $h : a \to a'$ be a $\mathcal{A}_f$-morphism. Consider the conditions:

(b1) h is a Δ-dense $\mathcal{A}_f$-morphism.

(b2) The functor $h^\star\upharpoonright: QV(a') \to QV(a)$ is full, faithful, injective on objects and satisfies the heredity condition.

Then (a1) $\Rightarrow$ (a2) and (b1) $\Rightarrow$ (b2)

Proof: The implication [(b1) $\Rightarrow$ (b2)] follows from [(a1) $\Rightarrow$ (a2)] and Proposition 2.2.1, since the inclusion functor $I : QV(a) \hookrightarrow \alpha - Str$ is clearly full, faithful, injective on objects and satisfies the heredity condition.

We will prove [(a1) $\Rightarrow$ (a2)]

---

[1] Remember that $Q(\mathcal{A}_f^c) = Q(Lind(\mathcal{A}_f)) = \overline{Lind(\mathcal{A}_f)}$.

Full: Let $M', N' \in QV(a')$ and $f : h^\star_\restriction(M') \to h^\star_\restriction(N')$ be a Σ-homomorphism. As $h^\star_\restriction$ commutes over *Set*, we have $U(f) : |M'| \to |N'|$ is a function. We will prove that $f : M' \to N'$ is a α'-homomorphism.

By the hypothesis (h is Δ-dense), for each $c'_n \in \alpha'_n$ there is $\varphi_n \in F(\Sigma)[n]$ such that $\vdash' \check{h}(\varphi_n(x_0, ..., x_{n-1}))\Delta' c'_n(x_0, ..., x_{n-1})$. Thus, as a' is an algebraizable logic, then

$$\models_{QV(a)} \check{h}(\varphi_n(x_0, ..., x_{n-1})) \equiv c'_n(x_0, ..., x_{n-1}).$$

Let $v : X \to |M'|$ be a function. Consider $m_0 = v(x_0), ..., m_{n-1} = v(x_{n-1})$. So

$$\check{h}(\varphi_n(x_0, ..., x_{n-1})))^{M'}[\overrightarrow{x}/\overrightarrow{m}] = (c'_n(x_0, ..., x_{n-1}))^{M'}[\overrightarrow{x}/\overrightarrow{m}].$$

$$\begin{aligned} f((c'_n(x_0, ..., x_{n-1}))^{M'}[\overrightarrow{x}/\overrightarrow{m}]) &= f(\check{h}(\varphi_n(x_0, ..., x_{n-1})))^{M'}[\overrightarrow{x}/\overrightarrow{m}]) \\ &= f((\varphi_n(x_0, ..., x_{n-1}))^{M'^h}[\overrightarrow{x}/\overrightarrow{m}]) \\ &= (\varphi_n(x_0, ..., x_{n-1}))^{N'^h}[\overrightarrow{x}/f(\overrightarrow{m})] \\ &= (c'_n(x_0, ..., x_{n-1}))^{N'}[\overrightarrow{x}/f(\overrightarrow{m})] \end{aligned}$$

Therefore f is a $QV(a)$-morphism.

Faithful: Let $f_1, f_2 \in QV(a')(M', N')$. As $h^\star_\restriction(M', f, N') = (M'^h, f, N'^h)$, if $h^\star_\restriction(f_1) = h^\star_\restriction(f_2) \in \Sigma - Str(M'^h, N'^h)$ then $f_1 = f_2$.

Injective on objects: Let $M', N' \in QV(a')$ such that $h^\star_\restriction(M') = h^\star_\restriction(N')$, so $|M'| = |N'|$. Given $c'_n \in \alpha_n$ there is $\varphi_n(x_0, ..., x_{n-1}) \in F(\Sigma)[n]$ such that

$$\vdash' \check{h}(\varphi_n(x_0, ..., x_{n-1}))\Delta' c'_n(x_0, ..., x_{n-1}) \;\Rightarrow$$

$$\models_{QV(a)} \check{h}(\varphi_n(x_0, ..., x_{n-1})) \equiv c'_n(x_0, ..., x_{n-1})$$

Hence, given $m_0, ..., m_{n-1} \in |M'|$

$$\begin{aligned} c'^{M'}_n(m_0, ..., m_{n-1}) &= \check{h}(\varphi_n)^{M'}(m_0, ..., m_{n-1}) \\ &= \varphi_n^{M'^h}(m_0, ..., m_{n-1}) \\ &= \varphi_n^{N'^h}(m_0, ..., m_{n-1}) \\ &= \check{h}(\varphi_n)^{N'}(m_0, ..., m_{n-1}) \\ &= c'^{N'}_n(m_0, ..., m_{n-1}) \end{aligned}$$

Therefore $M' = N'$

Heredity:

Let $M \in \Sigma - Str$, $N' \in QV(a')$ be such that there is an injective Σ-homomorphism $j : M \rightarrowtail h^\star\restriction (N')$. We must show that there is $M' \in QV(a')$ such that $h^\star\restriction (M') = M$. Remark that:

(i) as $h^\star \restriction$ is injective on objects, then M' is unique;

(ii) as $h^\star \restriction$ is full and faithful, then $j : M' \rightarrowtail N'$ is an injective α'-homomorphism.

Thus, as $N' \in QV(a')$, it is enough construct an α'-structure M' such that $h^\star(M') = M$, because then $M' \in QV(a')$ automatically.

As h is Δ-dense, given $c'_n \in \alpha_n$ *select a formula* $\langle c'_n \rangle(x_0, ..., x_{n-1}) \in F(\Sigma)[n]$ such that

$$\vdash' \check{h}(\langle c'_n \rangle(x_0, ..., x_{n-1}))\Delta' c'_n(x_0, ..., x_{n-1}) \;\Rightarrow$$

$$\models_{QV(a')} \check{h}(\langle c'_n \rangle(x_0, ..., x_{n-1})) \equiv c'_n(x_0, ..., x_{n-1})$$

Define $|M'| := |M|$. Let $m_0, ..., m_{n-1} \in |M'|$, define $c'^{M'}_n(m_0, ..., m_{n-1}) := \langle c'_n \rangle^M(m_0, ..., m_{n-1})$. Then:

$$j(c'^{M'}_n(m_0, ..., m_{n-1})) = j(\langle c'_n \rangle^M(m_0, ..., m_{n-1})) =$$

$$= \langle c'_n \rangle^{N'^h}(j(m_0), ..., j(m_{n-1})) = (\check{h}(\langle c'_n \rangle))^{N'}(j(m_0), ..., j(m_{n-1})) =$$

$$= c'^{N'}_n(j(m_0), ..., j(m_{n-1})).$$

Thus $j : M' \rightarrowtail N'$ is an injective α'-homomorphism. In particular, $M' \in QV(a')$.

Now let $c_k \in \Sigma_k$ and $a_0, \cdots, a_{k-1} \in |M'|$. Then

$$j(c^{M'^h}_k(a_0, \cdots, a_{k-1})) = j((\check{h}(c_k))^{M'}(a_0, \cdots, a_{k-1})) =$$

$$= (\check{h}(c_k))^{N'}(j(a_0), \cdots, j(a_{k-1})) = c^{N'^h}_k(j(a_0), \cdots, j(a_{k-1})) =$$

$$= j(c^M_k(a_0, \cdots, a_{k-1})).$$

As j is injective, then $c^{M'^h}_k(a_0, \cdots, a_{k-1}) = c^M_k(a_0, \cdots, a_{k-1})$. Hence $h^\star\restriction (M') = M$. $\square$

As density and Δ-density of morphisms coincide on Lindenbaum algebraizable logics, we immediately obtain the

Corollary 2.2.7. *Let* $l = (\Sigma, \vdash) \in \mathcal{L}_f$ *and* $a, a' \in \mathcal{A}^c_f$.

(a) Let $h : l \to a'$ *be a* $\mathcal{L}_f$*-morphism. Consider the conditions:*

(a1) h is a dense $\mathcal{L}_f$*-morphism.*

(a2) The functor $h^\star\restriction: QV(a') \to \Sigma-Str$ *is full, faithful, injective on objects and satisfies the heredity condition .*

(b) Let $h : a \to a'$ *be a* $\mathcal{A}^c_f$*-morphism. Consider the conditions:*

(b1) h is a dense $\mathcal{A}^c_f$*-morphism.*

(b2) The functor $h^\star\restriction: QV(a') \to QV(a)$ *is full, faithful, injective on objects and satisfies the heredity condition.*

Then (a1) $\Rightarrow$ *(a2) and (b1)* $\Rightarrow$ *(b2)*

In the next subsection we will be able to prove that the implications presented in Proposition 2.2.6 and Corollary 2.2.7 are, in fact, equivalences.

Proposition 2.2.8. *Let* $a \xrightarrow{h} a' \in \mathcal{A}^c_f$*, then:*

(a) $h^\star_\restriction : QV(a') \to QV(a)$ *has a left adjoint* $G : QV(a) \to QV(a')$.

(b) In case that h is a dense morphism, then the left adjoint G can be defined on objects $M \in QV(a)$ as "a quotient" $G(M) \in QV(a')$, with $h^\star \restriction (G(M)) = M/\rho_M$, where ρ_M is the least Σ-congruence in M such that $M/\rho_M = h^\star(M')$, for some (and unique) $M' \in \mathcal{K}'$ (that is automatically unique by (l3)); moreover the M-component of the unity of the adjunction, $M \to h^\star(G(M))$, is the quotient map $p_M : M \twoheadrightarrow M/\rho_M$.

Proof: Item (a) follows from Propositions 2.2.1 and 2.1.6.(a). Item (b) is a direct consequence of Proposition 2.1.6.(b) and Corollary 2.2.7.(b). □

Remark 2.2.9. *Let $a = IPC$ and $a' = CPC$ both Lindenbaum algebraizable logics with the same signature. We have the inclusion morphism $h : IPC \to CPC$. So $h^\star_\restriction : BA \to HA$ has left a adjoint functor $G : HA \to BA$. Observe that $h^\star_\restriction$ is the inclusion functor. Hence given $H \in HA$, $G(H) = H/F_H$, where F_H is the filter in H generated by the subset $\{a \leftrightarrow \neg\neg a : a \in H\}$. Its possible to proof that $G(H) \cong H_{\neg\neg}$, where $H_{\neg\neg}$ denote the poset of regular elements of H, that is, those elements $x \in H$ such that $\neg\neg x = x$.*

This fact motivate us to investigate the relation of Gödel translation with the left adjoint functor G.

As an application of some of the general results in the present work, we derive in [MP] a generalized "Glinvenko's Theorem" related to an $\mathcal{A}^c_f$-morphism $h : a \to a'$, whenever holds a simple condition of the unity of the adjunction $(G, h^\star_\restriction) : QV(a') \leftrightarrows QV(a)$.

2.3 Functorial encoding of logical morphisms

Here we apply the previous results to determine a faithful representation of algebraizable logic morphisms as certain functors. We start presenting a functorial encoding of signature morphisms.

Lemma 2.3.1. *Let $\Sigma, \Sigma' \in Obj(\mathcal{S}_f)$. Consider $H : \Sigma' - Str \to \Sigma - Str$ a functor that "commutes over Set" (i.e. $U \circ H = U'$) and, for each set Y, let $\eta_H(Y) : F(Y) \to H(F'(Y))$ be the unique Σ-morphism such that $(Y \overset{\sigma_X}{\to} UF(Y) \overset{U(\eta_Y)}{\to} UHF'(Y)) = (Y \overset{\sigma'_X}{\to} U'F'(Y))$ (by the universal property of σ_Y). Then:*
(a) $(\eta_H(Y))_{Y \in Set}$ is a natural transformation $\eta_H : F \to H \circ F'$.
(b) For each set Y and each $\psi \in F(Y)$, $Var(\eta_H(Y)(\psi)) \subseteq Var(\psi)$.
When $\forall \psi \in F(Y)$, $Var(\eta_H(Y)(\psi)) = Var(\psi)$, we will say that $\eta_H(Y)$ "preserves variables".
(c) For each $n \in \mathbb{N}$, let $X_n := \{x_0, \cdots, x_{n-1}\} \subseteq X$, if $\eta_H(X_n)$ preserves variables, then the mapping $c_n \in \Sigma_n \mapsto \eta_H(X_n)(c_n(x_0, \cdots, x_{n-1})) \in F'(X_n)$ determines a unique $\mathcal{S}_f$-morphism $m_H : \Sigma \to \Sigma'$ such that $\check{m}_H = \eta_H(X)$.

Proof:

(a) Let a function $f : Y \to Z$. Consider the diagram:

$$\begin{tikzcd}
Y \arrow[r, "\sigma_Y"] \arrow[d, "f"'] \arrow[rr, bend left, "\sigma'_Y"] & UF(Y) \arrow[r, "U(\eta_H(Y))"] \arrow[d, "UF(f)"'] & UHF'(Y) \arrow[d, "UHF'(f)"] \\
Z \arrow[r, "\sigma_Z"'] \arrow[rr, bend right, "\sigma'_Z"'] & UF(Z) \arrow[r, "U(\eta_H(Z))"'] & UHF'(Z)
\end{tikzcd}$$

The left square commutes because σ is the unit of adjunction between U and F and the external diagram commutes because σ' is the unit of adjunction between $U' = U \circ H$ and F'. We also have that $U(\eta_H(Y)) \circ \sigma_Y = \sigma'_Y$; the same is valid when we change Y for Z. Thus, a diagram chase entails ensures that

$$UHF'(f) \circ U(\eta_H(Y)) \circ \sigma_Y = U(\eta_H(Z)) \circ UF(f) \circ \sigma_Y.$$

As U is a functor, the universal property of σ_Y give us

$$U(HF'(f) \circ \eta_H(Y)) = U(\eta_Z \circ F(f)).$$

As U is faithful, we obatin

$$HF'(f) \circ \eta_H(Y) = \eta_Z \circ F(f).$$

Thus $\eta_H : F \Rightarrow H \circ F'$ is a natural transformation.

(b) Let Y a set and $\psi \in F(Y)$. Consider $Z = Var(\psi)$. and denote $i : Z \hookrightarrow Y$ the inclusion function. As η_H is a natural transformation, we have the follow commutative diagram:

$$\begin{tikzcd}
F(Y) \arrow[r, "\eta_H(Y)"] & HF'(Y) \\
F(Z) \arrow[u, "F(i)=incl"] \arrow[r, "\eta_H(Z)"'] & HF'(Z) \arrow[u, "HF'(i)=incl"']
\end{tikzcd}$$

As $\psi \in F(Z)$, we have $\eta_H(Z)(\psi) = \eta_H(Y)(\psi)$. Therefore $Var(\eta_H(Y)(\psi)) \subseteq Z = Var(\psi)$.

(c) Follows directly from the definition of flexible morphism of signatures. $\square$

Note that any functor $H : \Sigma' - Str \to \Sigma - Str$ that "commutes over Set" ($U \circ H = U'$) is automatically faithful, since U' is a faithfull functor. Another simple but useful result is given by the following

Fact 2.3.2. *Keeping the notation above, are equivalent:*

(a) $\eta_H(Y)$ preserves variables, for each set Y.

(b) $\eta_H(Y)$ preserves variables, for each set $Y \subseteq X$.

(c) $\eta_H(X_n)$ preserves variables, for each $n \in \mathbb{N}$.

Proof: We only have to show that $(c) \Rightarrow (a)$.

Let Y be an arbitrary set and let $\psi \in F(Y)$. Let $\{y_0, \cdots, y_{n-1}\} \subseteq Y$ be a (bijective) enumeration of $Var(\psi)$ and consider the injection $x_i \overset{f}{\mapsto} y_i$, $i < n$, from X_n into Y. As $F(f)$ is injective, denote $\tilde{\psi} \in F(X_n)$ the unique member such that $F(f)(\tilde{\psi}) = \psi \in F(Y)$. By hypothesis $Var(\tilde{\psi}) = X_n = Var(\eta_H(X_n)(\tilde{\psi}))$. As $F(f)$ and $H(F'(f))$ are injective and η_H is a natural transformation, then a diagram chase entails

$$Var(\eta_H(Y)(\psi)) = Var(\psi).$$

□

When η_H satisfies the equivalent conditions above, we say that η_H "preserves variables". As we will see in the sequence, this is a fundamental concept in this work, leading us to the following

Definition 2.3.3. *Let $\Sigma, \Sigma' \in Obj(\mathcal{S}_f)$ and $H : \Sigma' - Str \to \Sigma - Str$ be functor. We will say that H is a "signature" functor if it satisfies the conditions below:*
$(s1)$ H commutes over Set (i.e. $U \circ H = U'$);
$(s2)$ η_H preserves variables.

Proposition 2.3.4. *(a) Let $\Sigma - Str \overset{id}{\to} \Sigma - Str$. Then $\eta_{id_{\Sigma-Str}} = id_F$ and $id_{\Sigma-Str}$ is a signature functor; moreover $m_{id_{\Sigma-Str}} = id_\Sigma \in \mathcal{S}_f(\Sigma, \Sigma)$.*

(b) Let $(\Sigma - Str \overset{H}{\leftarrow} \Sigma' - Str \overset{H'}{\leftarrow} \Sigma'' - Str)$ be functors that commutes over Set. Then $\eta_{H\circ H'} = H(\eta_{H'}) \circ \eta_H$. If H and H' are signature functors, then $H \circ H'$ is a signature functor and, moreover, in this case, $m_{H\circ H'} = m_{H'} \bullet m_H \in \mathcal{S}_f(\Sigma, \Sigma'')$.

Proof: (a) It is clear that $id_{\Sigma-Str}$ commutes over Set. For each set Y, notice that the function $id_{F(Y)} : F(Y) \to id_{\Sigma-Str}(F(Y))$ satisfies $\sigma_Y \circ U(id_{F(Y)}) = \sigma_Y$ where $\sigma_Y : Y \to UF(Y)$ is the unit of the adjunction $F \dashv U$. Then, the universal property of σ_Y entails $\eta_{id_{\Sigma-Str}}(Y) = id_{F(Y)}$. Thus, in particular, $\eta_{id_{\Sigma-Str}}$ preserves variables, i.e., $id_{\Sigma-Str}$ is a signature functor. Moreover, $\check{m}_{id_{\Sigma-Str}} = \eta_{id_{\Sigma-Str}}(X) = id_{F(X)} : F(X) \to F(X)$, thus $m_{id_{\Sigma-Str}} = id_\Sigma \in \mathcal{S}_f(\Sigma, \Sigma)$.

(b) As H' and H commute over Set, we have that $H' \circ H$ also commutes over Set. We have that the following commutative diagrams:

$$\begin{tikzcd} Y \arrow[r, "\sigma_Y"] \arrow[dr, "\sigma'_Y"'] & UF(Y) \arrow[d, "U(\eta_H(Y))"] \\ & U'F'(Y) \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\sigma'_Y"] \arrow[dr, "\sigma''_Y"'] & U'F'(Y) \arrow[d, "U'(\eta'_H(Y))"] \\ & U''F''(Y) \end{tikzcd}$$

As $U \circ H = U'$, we obtain

$$U(\eta_{H\circ H'}) \circ \sigma_Y = \sigma''_Y =$$
$$= UH(\eta_{H'}(Y)) \circ U(\eta_H(Y)) \circ \sigma_Y = U(H(\eta_{H'}(Y)) \circ \eta_H(Y)) \circ \sigma_Y.$$

By the universal property of σ_Y, for each set Y, we obtain $\eta_{H\circ H'} = H(\eta_{H'}) \circ \eta_H$.

For each $n \in \mathbb{N}$, $\eta_{H\circ H'}(X_n) = H(\eta_{H'}(X_n)) \circ \eta_H(X_n)$. Now suppose that H and H' are signature functors. As $\eta_H(X_n)$ and $\eta_{H'}(X_n)$ preserve variables and H commutes over Set, then $\eta_{H\circ H'}(X_n)$ preserves variables. Thus $H \circ H'$ is a signature functor. Moreover, in this case, $\check{m}_{H\circ H'} = \eta_{H\circ H'}(X) = H(\eta_{H'}(X)) \circ \eta_H(X) = H(\check{m}_{H'}) \circ \check{m}_H$: this means that $m_{H\circ H'} = m_{H'} \bullet m_H$. □

In the sequence, we will see that, among the functors $H : \Sigma' - Str \to \Sigma - Str$ that commutes over Set, there are two kinds of functors that also preserves variables: the isomorphisms $\Sigma' - Str \to \Sigma - Str$ and the functors $h^\star$, induced by $\mathcal{S}_f$-morphisms $h : \Sigma \to \Sigma'$.

Proposition 2.3.5. *Let $H : \Sigma' - Str \to \Sigma - Str$ be an isomorphism of categories such that $U \circ H = U'$. Then H is a signature functor.*

Proof: As $H : \Sigma' - Str \to \Sigma - Str$ is an isomorphism of categories such that $U \circ H = U'$, then $H^{-1} : \Sigma - Str \to \Sigma' - Str$ is an isomorphism of categories and obviously $U' \circ H^{-1} = U$. Let Y be a set and consider $\psi \in F(Y)$. By the Lemma 2.3.1.(b), $Var(\eta_H(Y)(\psi)) \subseteq Var(\psi))$. On the other hand, by the Proposition 2.3.4 $id_{F(Y)} = \eta_{H\circ H^{-1}}(Y) = H(\eta_{H^{-1}}(Y)) \circ \eta_H(Y)$, thus $Var(\psi) = Var(H(\eta_{H^{-1}}(Y))(\eta_H(Y)(\psi))) \subseteq Var(\eta_H(Y)(\psi))$, since we can apply Lemma 2.3.1.(b) to H^{-1} and H commutes over Set . □

Proposition 2.3.6. *Let $h \in \mathcal{S}_f(\Sigma, \Sigma')$, then for all $Y \subseteq X$, $\eta_{h^\star}(Y) = \check{h}_{\restriction Y} : F(Y) \to F'(Y)^h$. In particular, $\eta_{h^\star}$ preserves variables and $h^\star$ is a signature functor according the Definition 2.3.3.*

Proof: Firstly observe that the function $\check{h} : U(F(X)) \to U'(F'(X))$ (see subsection 2.2) is such that $Var(\check{h}(\phi)) = Var(\phi)$, for each $\phi \in U(F(X))$. Thus, for each $Y \subseteq X$, it restricts to $\check{h}_{\restriction Y} : U(F(Y)) \to U'(F'(Y))$ and for each $\varphi \in U(F(Y))$, $Var(\check{h}_{\restriction Y}(\varphi)) = Var(\varphi)$.

Now, remark that $\check{h}_{\restriction Y}$ determines a Σ-homomorphism.

$$\begin{array}{rll} \check{h}_{\restriction Y} : & F(Y) & \to h^\star(F'(Y)) \\ & \varphi & \mapsto \check{h}(\varphi) \end{array}$$

Clearly, the diagram below commutes:

$$\begin{tikzcd} Y \arrow[r, "\sigma_Y"] \arrow[dr, "\sigma'_Y"'] & UF(Y) \arrow[d, "U(\check{h}_{\restriction Y})"] \\ & Uh^\star F'(Y) \end{tikzcd}$$

Due to the universal property of σ_Y, we have $\eta_{h^\star}(Y) = \check{h}_{\restriction Y}$, for each $Y \subseteq X$. Thus, by Fact 2.3.2, $\eta_{h^\star}$ preserves variables and, as $h^\star$ commutes over Set, then $h^\star$ is a signature functor according the Definition 2.3.3. □

The family of functors $h^\star$, induced by $\mathcal{S}_f$-morphisms h, have a nice categorial behavior:

Proposition 2.3.7. *(a) Let $\Sigma \overset{id_\Sigma}{\to} \Sigma$ be the identity $\mathcal{S}_f$-morphism on the signature Σ. Then $id_\Sigma^\star = id_{\Sigma-Str} \in Cat(\Sigma - Str, \Sigma - Str)$.*

(b) Let $(\Sigma \overset{h}{\to} \Sigma' \overset{h'}{\to} \Sigma'')$ be $\mathcal{S}_f$-morphisms. Then $(h' \bullet h)^\star = h^\star \circ h'^\star \in Cat(\Sigma'' - Str, \Sigma - Str)$.

Proof: Since the functors induced by signature morphisms are faithful and commute over Set, we only have to verify the equalities of functors in (a) and (b) at level of the objects.

It is clear that, for each $M \in Obj(\Sigma - Str)$, $M = id_{\Sigma-Str}(M) = id_\Sigma^\star(M)$, establishing item (a).

To prove item (b), note first that, for each $M'' \in Obj(\Sigma'' - Str)$,

$$U((h' \bullet h)^\star(M'')) = U''(M'') = U'(h'^\star(M'')) =$$

$$= (U \circ h^\star)(h'^\star(M'')) = U((h^\star \circ h'^\star)(M'')),$$

Thus, the Σ-structures $(h' \bullet h)^\star(M'')$ and $(h^\star \circ h'^\star)(M'')$ shares the same underlying set. It remains verify that, for each $n \in \mathbb{N}$ and each $c_n \in \Sigma_n$,

$$(I): \ (c_n)^{(h'\bullet h)^\star(M'')} = (c_n)^{(h^\star \circ h'^\star)(M'')}.$$

Developing the left hand side of (I) we obtain

$$(L): \ (c_n)^{(h'\bullet h)^\star(M'')} = ((h' \bullet h)(c_n))^{M''} = ((h' \bullet h)(c_n))^{M''} =$$

$$= ((\check{h}' \circ h)(c_n))^{M''} = ((\check{h}'(h(c_n)))^{M''} = (h(c_n))^{M''h'}.$$

Developing the right hand side of (I) we obtain

$$(R): \ (c_n)^{(h^\star \circ h'^\star)(M'')} = (c_n)^{(h^\star(h'^\star(M''))} =$$

$$= (c_n)^{(h^\star((M'')^{h'})} = (h(c_n))^{M''h'}.$$

Summing up, we obtain $(h' \bullet h)^\star(M'') = (h^\star \circ h'^\star)(M'')$. Thus $(h' \bullet h)^\star = (h^\star \circ h'^\star)$. □

At this point, is natural consider the following

Definition 2.3.8. *Let $\mathcal{S}_f^\dagger$ denote the (non-full) subcategory of the category of all the (large) categories[2] given by the categories $\Sigma - Str$, for each signature Σ, and with the signature functors as morphisms between them.*

[2] I.e., the category whose objects are large categories and the arrows are functors between categories.

Theorem 2.3.9. *The categories $\mathcal{S}_f$ and $\mathcal{S}_f^\dagger$ are anti-isomorphic. More precisely:*
(a) The mapping $\Sigma \in Obj(\mathcal{S}_f) \mapsto \Sigma - Str \in Obj(\mathcal{S}_f^\dagger)$ is bijective;
(b) Given $\Sigma, \Sigma' \in \mathcal{S}_f$, the mappings $h \in \mathcal{S}_f(\Sigma, \Sigma') \mapsto h^\star \in \mathcal{S}_f^\dagger(\Sigma' - Str, \Sigma - Str)$ and $H \in \mathcal{S}_f^\dagger(\Sigma' - Str, \Sigma - Str) \mapsto m_H \in \mathcal{S}_f(\Sigma, \Sigma')$ are (well defined) inverse bijections.
(c) $id_\Sigma^\star = id_{\Sigma - Str}$ and $(h' \bullet h)^\star = h^\star \circ h'\star$;
$m_{id_{\Sigma-Str}} = id_\Sigma$ and $m_{H \circ H'} = m_{H'} \bullet m_H$.

Proof: The equalities in item (c) were established in Propositions 2.3.4 and 2.3.7.

(a) The mapping $\Sigma \in Obj(\mathcal{S}_f) \mapsto \Sigma - Str \in Obj(\mathcal{S}_f^\dagger)$ is surjective, by definitions of $\mathcal{S}_f^\dagger$. Note that $\Sigma \neq \Sigma' \Rightarrow \Sigma - Str \neq \Sigma' - Str$ (in fact, $\Sigma \neq \Sigma' \Rightarrow Obj(\Sigma - Str) \cap Obj(\Sigma' - Str) = \emptyset$).

(b) The mappings $H \mapsto m_H$ and $h \mapsto h^\star$ are well defined by, respectively, Lemma 2.3.1.(c) and Proposition 2.3.6. Moreover, these results ensures that $\check{m}_{h^\star} = \eta_{h^\star}(X) = \check{h}$. Therefore $m_{h^\star} = h$. It remains only to prove that, for each signature functor $H : \Sigma' - Str \to \Sigma - Str$, $(m_H)^\star = H$.

It is enough to prove that $H(M') = (m_H)^\star(M')$ for each Σ'-structure M', because, as $U \circ H = U' = U \circ (m_H)^\star$, then for each Σ'-homomorphism $(M' \xrightarrow{g} N')$ we will have

$$H(M' \xrightarrow{g} N') = (m_H)^\star(M' \xrightarrow{g} N').$$

Claim H and $(m_H)^\star$ coincide on free Σ'-structures:
Indeed, consider a set Y and the diagram below:

$$\begin{tikzcd} Y \arrow[r, "\sigma_Y"] \arrow[dr, "\sigma'_Y"'] & UFY \arrow[d, "U\eta_H(Y) = U\eta_{m_H^\star}(Y)"] \\ & U'F'Y \end{tikzcd}$$

As $U \circ H = U' = U \circ (m_H)^\star$ and due to the universal property of σ_Y, them

$$(FY \xrightarrow{\eta_H(Y)} HF'Y) \;=\; (FY \xrightarrow{\eta_{m_H^\star}(Y)} m_H^\star F'Y)$$

as morphisms of $\Sigma - Str$, hence

$$(+)\ H(F'Y) = m_H^\star(F'Y).$$

Now we will prove the general case: $H(M') = (m_H)^\star(M')$, for each $M' \in \Sigma' - Str$. Note that $UH(M') = U'(M') = U(m_H)^\star(M')$, thus the Σ-structures $H(M')$ and $(m_H)^\star(M')$ shares the same underlying set. We must show the the interpretation of all Σ-symbols in $H(M')$ and $(m_H)^\star(M')$ coincide.

Let $\varepsilon' : F'U' \Rightarrow Id_{\Sigma' - Str}$ be the natural transformation that is the co-unit of the adjunction between F' and U'. It is clear that, for each $M' \in \Sigma - Str$, $\varepsilon'_{M'} : F'U'(M') \twoheadrightarrow M'$ is a surjective Σ'-homomorphism, thus the Isomorphism Theorem gives the following commutative diagram:

$$\begin{tikzcd} F'U'M' \arrow[r, two heads, "\varepsilon'_{M'}"] \arrow[dr, "q_{M'}"'] & M' \\ & \frac{F'U'M'}{ker(\varepsilon_{M'})} \arrow[u, "\cong\ \bar{q}_{M'}"'] \end{tikzcd}$$

In particular, the Σ'-structure M', on the underlying set $U'(M')$, is completely determined by the surjective Σ'-homomorphims $\varepsilon'_{M'} : F'U'(M') \twoheadrightarrow M'$.

Applying H and $m^\star_H$ to $\varepsilon'_{M'} : F'U'(M') \twoheadrightarrow M'$ we obtain the *surjective* Σ-homomorphisms

$$\begin{tikzcd} HF'U'(M') \arrow[r, two heads, "H(\varepsilon'_{M'})"] \arrow[dr, "H(q_{M'})"'] & H(M') \\ & H(\frac{F'U'M'}{ker(\varepsilon_{M'})}) \arrow[u, "\cong\ H(\bar{q}_{M'})"'] \end{tikzcd} \qquad \begin{tikzcd} m^\star_H F'U'(M') \arrow[r, two heads, "m^\star_H(\varepsilon'_{M'})"] \arrow[dr, "m^\star_H(q_{M'})"'] & m^\star_H(M') \\ & m^\star_H(\frac{F'U'M'}{ker(\varepsilon_{M'})}) \arrow[u, "\cong\ m^\star_H(\bar{q}_{M'})"'] \end{tikzcd}$$

By (+) above, we have $H(F'(U'(M'))) = m^\star_H(F'(U'(M')))$, as Σ-structures. Now, as $U \circ H = U' = U \circ m^\star_H$, we have

$$(UHF'U'(M') \overset{UH(\varepsilon'_{M'})}{\twoheadrightarrow} UH(M')) \;=\; (U'F'U'(M') \overset{U'(\varepsilon'_{M'})}{\twoheadrightarrow} U'(M')) \;=$$
$$= (Um^\star_H F'U'(M') \overset{Um^\star_H(\varepsilon'_{M'})}{\twoheadrightarrow} Um^\star_H(M')).$$

Thus the Σ-structures $H(M')$ and $m^\star_H(M')$ on the same underlying set coincide, since they are determined by the same surjective Σ-homomorphism.

□

We will denote the inverse (contravariant) functors in the Theorem above by:

$$E_S : \mathcal{S}_f \longrightarrow \mathcal{S}^\dagger_f \qquad\qquad E^\dagger_S : \mathcal{S}^\dagger_f \longrightarrow \mathcal{S}_f$$

$$\begin{tikzcd} \Sigma \arrow[d, "h"'] & \Sigma - Str \\ \Sigma' & \Sigma' - Str \arrow[u, "h^\star"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} \Sigma - Str & \Sigma \arrow[d, "m_H"] \\ \Sigma' - Str \arrow[u, "H"'] & \Sigma' \end{tikzcd}$$

The characterization Theorem 2.3.9 provides some interesting

Corollary 2.3.10. *Let $H : \Sigma' - Str \to \Sigma - Str$ be a signature functor. Then:*

(a) H preserves, strictly, *the following constructions: substructures, products, directed inductive limits, reduced products, congruences and quotients.*

(b) H has a left adjoint $G : \Sigma - Str \to \Sigma' - Str$ with unity of the adjunction $\lambda : id_{\Sigma - str} \Rightarrow H \circ G$. Moreover G and λ can be chosen such that $G \circ F = F'$ and $\lambda_{F(Y)} = \eta_H(Y) : F(Y) \to$

$H(F'(Y))$, for each set Y and, in particular, from Proposition 2.3.6, for each $Y \subseteq X$, $\eta_H(Y) = (\check{m}_H)_{\restriction Y} : F(Y) \to F'(Y)^{m_H}$.

Proof: (a) This follows from 2.1.5 and characterization Theorem above.

(b) By characterization Theorem above and Proposition 2.1.6.(a), the functor H has a left adjoint G and, by Proposition 2.1.7.(a) $G \circ F \cong F'$. Now we will analyze the additional conditions. As adjoint functors are determined up to natural isomorphism by the choice of universal arrows, it is enough to show that, for each set Y, the Σ-homomorphism $\eta_H(Y) : F(Y) \to H(F'(Y))$ is such that for each $M' \in Obj(\Sigma' - Str)$ and each Σ-homomorphism $f : F(Y) \to H(M')$, there is an unique Σ'-homomorphism $f' : F'(Y) \to M'$ such that $H(f') \circ \eta_H(Y) = f$. I.e., we must show that, for each $M' \in Obj(\Sigma' - Str)$, the mapping $f' \in \Sigma' - Str(F'(Y), M') \stackrel{t}{\mapsto} H(f') \circ \eta_H(Y) \in \Sigma - Str(F(Y), H(M'))$ is a bijection. Consider the bijections given by the pairs of adjoint functors (F, U) and (F', U'):

$$f \in \Sigma - Str(F(Y), H(M')) \stackrel{j}{\mapsto} U(f) \circ \sigma_Y \in Set(Y, U(H(M')))$$

$$f' \in \Sigma' - Str(F'(Y), M') \stackrel{j'}{\mapsto} U'(f') \circ \sigma'_Y \in Set(Y, U'(M'))$$

As $Set(Y, U'(M')) = Set(Y, U(H(M')))$ and $U(\eta_H(Y)) \circ \sigma_Y = \sigma'_Y$ we conclude that $j \circ t = j'$, i.e., the diagram below commutes

$$\begin{tikzcd}
Set(Y, U'(M')) \arrow[r, "="] & Set(Y, U(H(M')) \\
\Sigma' - Str(F'(Y), M') \arrow[u, "\cong\ j'"] \arrow[r, "t"'] & \Sigma - str(F(Y), H(M')) \arrow[u, "j\ \cong"']
\end{tikzcd}$$

Thus, as j and j' are bijections, then t is a bijection. This entails the additional results.

□

Now, having a detailed functorial encoding of (flexible) signature morphisms, we can proceed to a functorial description of logical morphisms between algebraizable logics.

Lemma 2.3.11. *Let $I : \mathcal{K} \hookrightarrow \Sigma - Str$ and $I' : \mathcal{K}' \hookrightarrow \Sigma' - Str$ full inclusions, where $\mathcal{K}$ and $\mathcal{K}'$ are quasivarieties. Let $H : \Sigma' - Str \to \Sigma - Str$ be a signature functor such that it restricts (uniquely) to a functor $H\restriction : \mathcal{K}' \to \mathcal{K}$ (thus $I \circ H\restriction = H \circ I'$). Keeping the notation in Remark 2.1.2, for each set Y, let (by the universal property of t_Y) $\bar{\eta}_H(Y) : LF(Y) \to H\restriction (L'F'(Y))$ be the unique $\mathcal{K}$-morphism such that $(Y \stackrel{t_Y}{\to} UILF(Y) \stackrel{UI(\bar{\eta}_Y)}{\to} UIH\restriction L'F'(Y)) =$ $(Y \stackrel{t'_Y}{\to} U'I'L'F'(Y))$. Then:*

(a) $(\bar{\eta}_H(Y))_{Y \in Set}$ is a natural transformation $\bar{\eta}_H : L \circ F \to H\restriction \circ L' \circ F'$.

(b) Both the diagrams below commute

$$\begin{tikzcd}
Y \arrow[r, "\sigma_Y"] \arrow[rr, bend left, "t_Y"] \arrow[d, "id_Y"'] & UF(Y) \arrow[r, "U(q_{F(Y)})"] \arrow[d, "U(\eta_H(Y))"'] & UILF(Y) \arrow[d, "UI(\bar{\eta}_H(Y))"] \\
Y \arrow[r, "\sigma'_Y"'] \arrow[rr, bend right, "t'_Y"'] & UHF'(Y) \arrow[r, "UH(q'_{F'(Y)})"'] & UIH\restriction L'F'(Y)
\end{tikzcd}$$

$$\begin{tikzcd}
F(Y) \arrow[r, "q_{F(Y)}"] \arrow[d, "\eta_H(Y)"'] & ILF(Y) \arrow[d, "I(\bar{\eta}_H(Y))"] \\
HF'(Y) \arrow[r, "H(q'_{F'(Y)})"'] & IH\restriction L'F'(Y)
\end{tikzcd}$$

(c) H and $H\restriction$ have left adjoints, respectively $G : \Sigma - Str \to \Sigma' - Str$ and $\bar{G} : \mathcal{K} \to \mathcal{K}'$, the respective unities of the adjunctions $\lambda : id_{\Sigma-str} \Rightarrow H \circ G$ and $\bar{\lambda} : id_{\mathcal{K}} \Rightarrow H \restriction \circ \bar{G}$. Moreover $G, \bar{G}$ and $\lambda, \bar{\lambda}$ can be chosen such that:

- *$G \circ F = F'$ and $\bar{G} \circ L \circ F = L' \circ F' = L' \circ G \circ F$;*
- *$\lambda_{F(Y)} = \eta_H(Y) : F(Y) \to H(F'(Y))$ and $\bar{\lambda}_{LF(Y)} = \bar{\eta}_H(Y) : LF(Y) \to$
$H\restriction (L'F'(Y))$, for each set Y.*

Proof: Item (a) follows in an analogous fashion to the proof of Lemma 2.3.1.(a): by analyzing the commutativity of the diagram below from the universal property of t_Y, for each function $f : Y \to Z$.

$$\begin{tikzcd}
Y \arrow[r, "t_Y"] \arrow[rr, bend left, "t'_Y"] \arrow[d, "f"'] & UILF(Y) \arrow[r, "UI(\bar{\eta}_H(Y))"] \arrow[d, "UILF(f)"'] & UIH \restriction L'F'(Y) \arrow[d, "UIH\restriction L'F'(f)"] \\
Z \arrow[r, "t_Z"'] \arrow[rr, bend right, "t'_Z"'] & UILF(Z) \arrow[r, "UI(\bar{\eta}_H(Z))"'] & UIH \restriction L'F'(Z)
\end{tikzcd}$$

Item (b) follows in an analogous fashion to the proof of Lemma 2.3.1.(a): the top diagram commutes, by analyzing the commutativity of the diagram below from the universal property of σ_Y; the bottom diagram commutes since the functor U is faithful and the inner right square in the top diagram commutes.

Item (c) follows in an analogous fashion to the proof of Corollary 2.3.10.(b): first, by applying Proposition 2.1.7, and then, by a diagram chase to shows that, for each $M' \in \mathcal{K}'$, the mapping $f' \in \mathcal{K}'(L'F'(Y), M') \mapsto H \restriction (f') \circ \bar{\eta}_H(Y) \in \mathcal{K}(LF(Y), H \restriction (M'))$ is a bijection.

□

Proposition 2.3.12. *Let $l = (\Sigma, \vdash) \in \mathcal{L}_f$ and $a, a' \in \mathcal{A}_f$.*

(a) Let $h : l \to a'$ be a $\mathcal{L}_f$-morphism. Then are equivalent:

(a1) h is a Δ-dense $\mathcal{L}_f$-morphism.

(a2) The functor $h^\star\restriction : QV(a') \to \Sigma - Str$ is full, faithful, injective on objects and satisfies the heredity condition (see 2.1.6.(b4)).

(b) Let $h : a \to a'$ be a $\mathcal{A}_f$-morphism. Then are equivalent:

(b1) h is a Δ-dense $\mathcal{A}_f$-morphism.

(b2) The functor $h^\star\restriction : QV(a') \to QV(a)$ is full, faithful, injective on objects and satisfies the heredity condition.

Proof: The implications (a1) $\Rightarrow$ (a2) and (b1) $\Rightarrow$ (b2) were established in Proposition 2.2.6.

(a1) $\Rightarrow$ (a2): by Theorem 2.3.9, Lemma 2.3.11.(b), Remark 2.1.4 and Corollary 2.3.10.(b), the following diagram commutes, for each $Y \subseteq X$.

$$\begin{tikzcd}
F(Y) \arrow[r, two heads, "id_{F(Y)}"] \arrow[d, "\check{h}\restriction_Y"'] & F(Y) \arrow[d, "I(\bar{\eta}_{h^\star}(Y))"] \\
h^\star(F'(Y)) \arrow[r, two heads, "h^\star(q'_{F'(Y)})"'] & I'(h^\star \restriction (F'(Y)/\Delta' \restriction))
\end{tikzcd}$$

By hypothesis (a1), Lemma 2.3.11.(c) and Proposition 2.1.6.(b), the Σ'- homomorphism $\bar{\eta}_H(Y) : F(Y) \to I'(F'(Y)/\Delta' \restriction)$ is *surjective*. Thus a diagram chase shows that for each $\phi' \in F'(Y')$ there is $\phi \in F(Y)$ such that $\vdash' \check{h}(\phi)\Delta'\phi'$. Therefore, the $\mathcal{L}_f$-morphism $h : l \to a$ is Δ-dense.

(b1) $\Rightarrow$ (b2): is proved in an analogous way, by a chase on the commutative diagram below

$$\begin{tikzcd}
F(Y) \arrow[r, two heads, "q_{F(Y)}"] \arrow[d, "\check{h}\restriction_Y"'] & I(F(Y)/\Delta \restriction) \arrow[d, "I(\bar{\eta}_{h^\star}(Y))"] \\
h^\star(F'(Y)) \arrow[r, two heads, "h^\star(q'_{F'(Y)})"'] & I'(h^\star \restriction (F'(Y)/\Delta' \restriction))
\end{tikzcd}$$

□

As density and Δ-density of morphisms coincide on Lindenbaum algebraizable logics, we immediately obtain the

Corollary 2.3.13. *Let $l = (\Sigma, \vdash) \in \mathcal{L}_f$ and $a, a' \in \mathcal{A}_f^c$.*

(a) Let $h : l \to a'$ be a $\mathcal{L}_f$-morphism. Then are equivalent:

(a1) h is a dense $\mathcal{L}_f$-morphism.

(a2) The functor $h^\star\restriction: QV(a') \to \Sigma - Str$ is full, faithful, injective on objects and satisfies the heredity condition .

(b) Let $h : a \to a'$ be a $\mathcal{A}_f^c$-morphism. Then are equivalent:

(b1) h is a dense $\mathcal{A}_f^c$-morphism.

(b2) The functor $h^\star\restriction: QV(a') \to QV(a)$ is full, faithful, injective on objects and satisfies the heredity condition.

Having in mind the Definitions 2.3.3 and 2.3.8, it is natural to consider the following

Definition 2.3.14. *(a) Let $a = (\Sigma, \vdash), a' = (\Sigma', \vdash')$ be algebraizable logics. A functor $H : \Sigma' - Str \to \Sigma - Str$ will be called a "BP-functor", H is a signature functor also satisfying $(l1), (l2), (l3)$:*

$(l1)$ H has a (unique) restriction to the associated quasivarieties $H\restriction: QV(a') \to QV(a)$;
There are algebraizing pairs $(\Delta, (\delta, \varepsilon))$ and $(\Delta', (\delta', \varepsilon'))$ of, respectively, a and a' such that:
$(l2)$ $\check{m}_H(\Delta) \dashv'\vdash \Delta'$;
$(l3)$ $\check{m}_H(\delta) \equiv \check{m}_H(\varepsilon) = |_{QV(a')}| = \delta' \equiv \varepsilon'$.

It is straightforward that:

- *$id_{\Sigma - Str} : \Sigma - Str \to \Sigma - Str$ is a BP-functor;*
- *If $(\Sigma - Str \overset{H}{\leftarrow} \Sigma' - Str \overset{H'}{\leftarrow} \Sigma'' - Str)$ are BP-functors, then $H \circ H' : \Sigma'' - Str \to \Sigma - str$ is a BP-functor.*[3]

(b) Denote $\mathcal{A}_f^\dagger$ the category with:

- Objects: *are pairs $(\Sigma - Str, a)$ where $a = (\Sigma, \vdash)$ is an algebraizable logic;*
- Arrows: *are BP-functors $(\Sigma' - Str, a') \overset{H}{\to} (\Sigma - Str, a)$;*
- identities and composition: *as (BP-)functors.*

(c) Denote $Lind(\mathcal{A}_f)^\dagger$ the full subcategory of $\mathcal{A}_f^\dagger$ with objects, the pairs $(\Sigma - Str, a)$ where $a = (\Sigma, \vdash)$ is a Lindenbaum algebraizable logic.

Below we present the results that encompass most part of the present work

Theorem 2.3.15. *The pair of inverse anti-isomorphisms of categories $\mathcal{S}_f \underset{E_S^\dagger}{\overset{E_S}{\rightleftarrows}} \mathcal{S}_f^\dagger$ in Theorem 2.3.9 "restricts", via the forgetful functors $\mathcal{A}_f \to \mathcal{S}_f$ and $\mathcal{A}_f^\dagger \to \mathcal{S}_f^\dagger$, to a pair of inverse anti-isomorphisms of categories $\mathcal{A}_f \underset{E_A^\dagger}{\overset{E_A}{\rightleftarrows}} \mathcal{A}_f^\dagger$.*

[3] Note that, for each $M'' \in QV(a'')$, $(M'')^{m_{H'}} = H'(M'') \in QV(a')$ and $(M'')^{m_{H'}} \vDash_{\Sigma'} \check{m}_H(\delta) \equiv \check{m}_H(\varepsilon) \leftrightarrow \delta' \equiv \varepsilon'$ iff $M'' \vDash_{\Sigma''} \check{m}_{H'}(\check{m}_H(\delta)) \equiv \check{m}_{H'}(\check{m}_H(\varepsilon)) \leftrightarrow \check{m}_{H'}(\delta') \equiv \check{m}_{H'}(\varepsilon')$.

$$
\begin{array}{ccc}
E_A : \mathcal{A}_f \longrightarrow \mathcal{A}_f^\dagger & \qquad & E_A^\dagger : \mathcal{A}_f^\dagger \longrightarrow \mathcal{A}_f
\end{array}
$$

$$
\begin{tikzcd}
a = (\Sigma, \vdash) \arrow[d, "h"'] & (\Sigma - Str, a) & (\Sigma - Str, a) & a \arrow[d, "m_H"] \\
a' = (\Sigma', \vdash') & (\Sigma' - Str, a') \arrow[u, "h^\star"'] & (\Sigma' - Str, a') \arrow[u, "H"'] & a'
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{A}_f \arrow[r, shift left, "E_A^\dagger"', leftarrow] \arrow[r, shift right, "E_A"'] \arrow[d, "Forget"'] & \mathcal{A}_f^\dagger \arrow[d, "Forget"] \\
\mathcal{S}_f \arrow[r, shift left, "E_S^\dagger"', leftarrow] \arrow[r, shift right, "E_S"'] & \mathcal{S}_f^\dagger
\end{tikzcd}
$$

Moreover, if $h \in \mathcal{A}_f(a, a')$ *and* $H \in \mathcal{A}_f^\dagger((\Sigma' - Str, a'), (\Sigma - Str, a))$ *are in correspondence, then the pair of inverse anti-isomorphisms* $(E_A, E_A^\dagger)$ *is such that:*

(a) It establishes a correspondence between the equivalence class $\{h' \in \mathcal{A}_f(a, a') : [h]_\approx = [h']_\approx \in \overline{\mathcal{A}_f}(a, a')\}$ *and the equivalence class* $\{H' \in \mathcal{A}_f^\dagger((\Sigma' - Str, a'), (\Sigma - Str, a)) : H' \restriction = H \restriction\}$*.*

(b) $[h]_\approx$ *is a* $\overline{\mathcal{A}_f}$*-isomorphism* $\Leftrightarrow$ $H\restriction$ *is an isomorphism between quasivarieties.*

(c) h *is a* Δ*-dense morphism* $\Leftrightarrow$ $H\restriction$ *is full, faitful, injective on object and heredity.*

Proof: After the pair of ("restricted") inverse anti-isomorphisms $(E_A, E_A^\dagger)$ were established, then: item (a) follows from Proposition 2.2.2.(b); item (b) follows from Proposition 2.2.4; item (c) follows from Proposition 2.3.12.(b).

It follows from directly from Theorem 2.3.9 and the definitions of the object part of the functors $(E_A, E_A^\dagger)$ that they establishes an well defined pair of inverse bijections between the classes of objects $Obj(\mathcal{A}_f)$ and $Obj(\mathcal{A}_f^\dagger)$.

If we establish that the (arrow) mappings below are well defined:
$h \in \mathcal{A}_f(a, a') \overset{E_A}{\mapsto} h^\star \in \mathcal{A}_f^\dagger((\Sigma' - Str, a'), (\Sigma - Str, a))$;
$H \in \mathcal{A}_f^\dagger((\Sigma' - Str, a'), (\Sigma - Str, a)) \overset{E_A^\dagger}{\mapsto} m_H \in \mathcal{A}_f(a, a')$,
then it will follow from Theorem 2.3.9 that the pair of inverse anti-isomorphisms $\mathcal{S}_f \underset{E_S^\dagger}{\overset{E_S}{\rightleftarrows}} \mathcal{S}_f^\dagger$ in Theorem 2.3.9 "restricts" to a pair of inverse anti-isomorphisms $\mathcal{A}_f \underset{E_A^\dagger}{\overset{E_A}{\rightleftarrows}} \mathcal{A}_f^\dagger$.

Let $h \in \mathcal{A}_f(a, a')$. By Proposition 2.3.6, $h^\star : \Sigma' - str \to \Sigma - Str$ is a signature functor and, by Proposition 2.2.1, it restricts (uniquely) to a functor $h^\star\restriction : QV(a') \to QV(a)$: thus condition ($l1$) is fulfilled. By Theorem 2.3.9, $m_{h^\star} = h$; as h preserves algebraizable pairs, then Fact 1.3.6.(a) ensures that the conditions ($l2$) and ($l3$) are satisfied. Therefore E_A is an well defined functor.

Let $H \in \mathcal{A}_f^\dagger((\Sigma' - Str, a'), (\Sigma - Str, a))$. Lemma 2.3.1.(c) entails that $m_H : \Sigma \to \Sigma'$ is a $\mathcal{S}_f$-morphism. Conditions ($l2$) and ($l3$) and Fact 1.3.6.(b) ensures that m_H preserves algebraizing pairs. It remains to show that m_H is a $\mathcal{L}_f$-morphism, i.e. given $\Gamma \cup \{\varphi\} \subseteq F(X)$, we must have

$$\Gamma \vdash \varphi \;\Rightarrow\; \check{m}_H[\Gamma] \vdash' \check{m}_H(\varphi)$$

But, as a and a' are algebraizable logics, it is enough to prove that

$$\{\varepsilon(\psi) \equiv \delta(\psi);\ \psi \in \Gamma\} \models_{QV(a)} \varepsilon(\varphi) \equiv \delta(\varphi) \;\Rightarrow$$

$$\{\varepsilon'(\check{m}_H(\psi)) \equiv \delta'(\check{m}_H(\psi));\ \psi \in \Gamma\} \models_{QV(a')} \varepsilon'(\check{m}_H(\varphi)) \equiv \delta'(\check{m}_H(\varphi)).$$

Let $M' \in QV(a')$ and suppose that $M' \models_{\Sigma'} \varepsilon'(\check{m}_H(\psi)) \equiv \delta'(\check{m}_H(\psi))$ for each $\psi \in \Gamma$. As m_H satisfies condition ($l3$), then holds, for each $\psi \in \Gamma$,

$$M' \models_{\Sigma'} \check{m}_H(\varepsilon)(\check{m}_H(\psi)) \equiv \check{m}_H(\delta)(\check{m}_H(\psi))$$

I.e.:

$$M' \models_{\Sigma'} \check{m}_H(\varepsilon(\psi)) \equiv \check{m}_H(\delta(\psi))$$

By Theorem 2.3.9, $H = (m_H)^\star$, thus we get

$$H(M') \models_{\Sigma} \varepsilon(\psi) \equiv \delta(\psi)$$

From the hypothesis, $H(M') \in QV(a)$, and as $\{\varepsilon(\psi) \equiv \delta(\psi);\ \psi \in \Gamma\} \models_{QV(a)} \varepsilon(\varphi) \equiv \delta(\varphi)$, we obtain

$$H(M') \models_{\Sigma} \varepsilon(\varphi) \equiv \delta(\varphi)$$

Therefore, as above,

$$M' \models_{\Sigma'} \check{m}_H(\varepsilon(\varphi)) \equiv \check{m}_H(\delta(\varphi))$$

and

$$M' \models_{\Sigma'} \varepsilon'(\check{m}_H(\varphi)) \equiv \delta'(\check{m}_H(\varphi)).$$

As $M' \in QV(a')$ was taken arbitrarily, then $\{\varepsilon'(\check{m}_H(\psi)) \equiv \delta'(\check{m}_H(\psi));\ \psi \in \Gamma\} \models_{QV(a')} \varepsilon'(\check{m}_H(\varphi)) \equiv \delta'(\check{m}_H(\varphi))$.

Summing up, m_H is a logical morphism that preserves algebraizable pairs. Therefore $E_A^\dagger$ is an well defined functor. This finishes the proof. $\square$

Restricting the result above to the setting of Lindenbaum algebraizable logics, we obtain the

Corollary 2.3.16. *The pair of inverse anti-isomorphisms of categories* $\mathcal{A}_f \underset{E_A^\dagger}{\overset{E_A}{\rightleftarrows}} \mathcal{A}_f^\dagger$ *in Theorem 2.3.15 "restricts", via the (full) inclusion functors* $Lind(\mathcal{A}_f) \hookrightarrow \mathcal{A}_f$ *and* $Lind(\mathcal{A}_f)^\dagger \hookrightarrow \mathcal{A}_f^\dagger$*, to a pair of inverse anti-isomorphisms of categories* $Lind(\mathcal{A}_f) \underset{E_L^\dagger}{\overset{E_L}{\rightleftarrows}} Lind(\mathcal{A}_f)^\dagger$.

$$\begin{tikzcd}
Lind(\mathcal{A}_f) \arrow[r, leftrightarrow, "E_L^\dagger", "E_L"'] \arrow[d, "Incl"'] & Lind(\mathcal{A}_f)^\dagger \arrow[d, "Incl"] \\
\mathcal{A}_f \arrow[r, leftrightarrow, "E_A^\dagger", "E_A"'] & \mathcal{A}_f^\dagger
\end{tikzcd}$$

Moreover, if $h \in Lind(\mathcal{A}_f)(a, a')$ *and* $H \in Lind(\mathcal{A}_f)^\dagger((\Sigma' - Str, a'), (\Sigma - Str, a))$ *are in correspondence, then the pair of inverse anti-isomorphisms* $(E_L, E_L^\dagger)$ *is such that:*

(a) It establishes a correspondence between the equivalence class

$$\{h' \in Lind(\mathcal{A}_f)(a, a') : [h]_{\dashv\vdash} = [h']_{\dashv\vdash} \in QLind(\mathcal{A}_f)(a, a')\}$$

and the equivalence class

$$\{H' \in Lind(\mathcal{A}_f)^\dagger((\Sigma' - Str, a'), (\Sigma - Str, a)) : H' \restriction = H \restriction\}.$$

(b) $[h]_{\dashv\vdash}$ *is a* $QLind(\mathcal{A}_f)$*-isomorphism* $\Leftrightarrow$ $H\restriction$ *is an isomorphism between quasivarieties.*

(c) h *is a dense morphism* $\Leftrightarrow$ $H\restriction$ *is full, faitful, injective on object and heredity.*

Proof: It is clear that $(E_A, E_A^\dagger)$ establishes a bijective correspondence between the subclasses $Obj(Lind(\mathcal{A}_f))$ and $Obj(Lind(\mathcal{A}_f)^\dagger)$. As $Lind(\mathcal{A}_f) \hookrightarrow \mathcal{A}_f$ and $Lind(\mathcal{A}_f)^\dagger \hookrightarrow \mathcal{A}_f^\dagger$ are full subcategories, then $(E_A, E_A^\dagger)$ restricts to a pair of inverse anti-isomorphisms $Lind(\mathcal{A}_f) \underset{E_L^\dagger}{\overset{E_L}{\rightleftarrows}} Lind(\mathcal{A}_f)^\dagger$.

On the additional results: item (a) follows from Corollary 2.2.3.(b); item (b) follows from Corollary 2.2.5; item (c) follows from Corollary 2.3.13.(b). $\square$

Chapter 3

Filter functors in logic and application

We have seen that for any algebraizable logic a there is a quasivariety $QV(a)$ associated. This quasivariety $QV(a)$ keeps the semantic information of a. Unfortunately, for an arbitrary Tarskian logic there is no a class of algebra endowed of its semantic information. To an arbitrary Tarskian logic $l = (\Sigma, \vdash)$, the set of filters $Fi_l(M)$ for an arbitrary algebra M of Σ-Str, in a certain way, has the semantic information of the logic l. That was the main motivation to start studying the notion of filter pairs and its associated logics.

It is well-known that every Tarskian logic gives rise to an algebraic lattice contained in the powerset $\wp(Fm_\Sigma(X))$, namely the lattice of theories. This lattice is closed under arbitrary intersections and filtered unions.

Conversely an algebraic lattice $L \subseteq \wp(Fm_\Sigma(X))$ that is closed under arbitrary intersections and unions of increasing chains gives rise to a finitary closure operator (assigning to a subset $A \subseteq Fm_\Sigma(X)$ the intersection of all members of L containing A). This closure operator need not be structural — this is an extra requirement.

We observe that the structurality of the logic just defined is equivalent to the *naturality* (in the sense of category theory) of the inclusion of the algebraic lattice into the power set of formulas with respect to endomorphisms of the formula algebra: Structurality means that the preimage under a substitution of a theory is a theory again or, equivalently, that the following diagram commutes:

$$\begin{tikzcd} Fm_\Sigma(X) \arrow[d, "\sigma"'] & L \arrow[r, hook, "i"] & \wp(Fm_\Sigma(X)) \\ Fm_\Sigma(X) & L \arrow[u, "\sigma^{-1}|_L"] \arrow[r, hook, "i"] & \wp(Fm_\Sigma(X)) \arrow[u, "\sigma^{-1}"'] \end{tikzcd}$$

Further, it is equivalent to demand this naturality for all Σ-algebras and homomorphisms instead of just the formula algebra.

We thus arrive at the definition of *filter pair*, Def. 3.1.1: A filter pair for the signature Σ is a contravariant functor G from Σ-algebras to algebraic lattices together with a natural transformation $i\colon G \to \wp(-)$ from G to the functor taking an algebra to the power set of its underlying set, which preserves arbitrary infima and directed suprema.

The logic associated to a filter pair (G, i) is simply the logic associated (in the above fashion) to the algebraic lattice given by the image $i(G(Fm_\Sigma(X))) \subseteq \wp(Fm_\Sigma(X))$.

In particular it is clear that different filter pairs can give rise to the same logic, indeed this will happen precisely if the images of i for the formula algebra are the same. A filter pair can thus be seen as a *presentation* of a logic, and there can of course be different presentations of the same logic. We could have removed a bit of this ambiguity by requiring that i be an inclusion, but it is one of the insights of this chapter that it is beneficial not to do this. Indeed this will give us greater flexibility for the choice of the functor G, and injectivity of i can become a meaningful extra feature. Thus, for example, if G is the functor associating to a Σ-structure the lattice of relative congruences to some quasivariety K, then by Prop. 3.1.9 the injectivity of i means that the associated logic is algebraizable.

In this section we show how to recognize classes of logics through their presentations by filter functors and how these presentations permit to use algebraic methods even outside the realm of protoalgebraic logics.

A second aim of this chapter is to continue the work of the last chapter: Remembering, we establish a correspondence of certain functors between categories of Σ-structures and translations between algebraizable logics. Here we introduce a notion of morphism of filter functors and it is shown that it encodes translations between their associated logics. This encoding will play a role in the long-term project of studying arbitrary logics through their translations into algebraizable logics and their associated categories of matrices.

3.1 Filter Functors

In the following sections we present: firstly the definition of filter pair and a study of the functors Co and Co_K where K is a class of algebras. In the sequel we dedicated to study of the Craig entailment interpolation property end its correspondence with amalgamation property in filters. We introduce the category of Filters $\mathcal{F}i$ and the relationship between this category and the category of logic.

Now we start studying some general aspects present on the functor filter for a given logic and then we collect these ideas in order to begin the correspondence between logics and its "algebraic" counterpart.

Observe that given $M \in \Sigma - str$, the inclusion $i_M : Fi_l(M) \to (\mathcal{P}(M); \subseteq)$ preserves order and satisfies the following condition. Given $A \subseteq M$, there is a $F \in Fi_l(M)$ such that $A \subseteq F$ (just take $F := M$).

Moreover, given a morphism $h : M \to N$ we have the following diagram commuting:

$$\begin{tikzcd} M \arrow[d, "h"'] & Fi_l(M) \arrow[r, "i_M"] & (\mathcal{P}(M);\subseteq) \\ N & Fi_l(N) \arrow[u, "Fi_l(h)"] \arrow[r, "i_N"'] & (\mathcal{P}(N);\subseteq) \arrow[u, "h^{-1}"'] \end{tikzcd}$$

Definition 3.1.1. *Now let Σ be a structure. A* **filter pair** (G, i^G) *consists of a contravariant functor $G : \Sigma - str \to AL$ and $i^G = (i^G_M)_{M\in\Sigma-Str}$ such that for any $M \in \Sigma - str$ there is a function preserving order $i^G_M : G(M) \to (\mathcal{P}(M);\subseteq)$ (inside of the category of poset) with the following properties:*

1. *For any $M \in \Sigma - str$, $i^G_M(\top) = M$ and i^G_M preserves inf and directed sup.*

2. *Given a morphism $h : M \to N$ the following diagram commutes:*

$$\begin{tikzcd} M \arrow[d, "h"'] & G(M) \arrow[r, "i^G_M"] & (\mathcal{P}(M);\subseteq) \\ N & G(N) \arrow[u, "G(h)"] \arrow[r, "i^G_N"'] & (\mathcal{P}(N);\subseteq) \arrow[u, "h^{-1}"'] \end{tikzcd}$$

Remark 3.1.2. *1. Condition **2** says that i^G is a natural transformation from G to the functor $\wp$: $\Sigma - str^{op} \to AL$ sending a Σ-structure to the power set of its underlying set and a homomorphism of Σ-structures to its associated inverse image function.*

2. Notice that given a Tarskian logic $l = (\Sigma, \vdash)$, we have a filter pair (Fi_l, i^{Fi_l}) where Fi_l is a functor (see 1.3.15) and i^{Fi_l} is the inclusion as above.

Proposition 3.1.3. *Let (G, i^G) be a filter pair, then there is a logic $l_G = (\Sigma, \vdash_G)$ as follows: Given $\Gamma \cup \{\varphi\} \subseteq F(X)$.*

$$\Gamma \vdash_G \varphi \;\; \textit{iff} \;\; \textit{for any } a \in G(F(X)),\ \textit{if } \Gamma \subseteq i_{F(X)}(a) \textit{ then } \varphi \in i_{F(X)}(a).$$

Proof:

It is easy to see that $\vdash_G$ satisfies reflexivity, cut and monotonicity.

The structurality is a consequence of condition **2** (naturality). Indeed, let $\sigma \in hom(F(X), F(X))$ and $\Gamma \cup \{\varphi\} \subseteq F(X)$ such that $\Gamma \vdash_G \varphi$. Consider $a \in G(F(X))$ such that $\sigma[\Gamma] \subseteq i^G_{F(X)}(a)$. This implies $\Gamma \subseteq \sigma^{-1}(i^G_{F(X)}(a))$. By naturality we have $\sigma^{-1}(i^G_{F(X)}(a)) = i^G_{F(X)}(G(\sigma)(a))$. Therefore $\varphi \in i^G_{F(X)}(G(\sigma)(a)) = \sigma^{-1}(i^G_{F(X)}(a))$ and finally $\sigma(\varphi) \in i^G_{F(X)}(a)$.

Now we are going to prove the finitarity. Let $\Gamma \cup \{\varphi\} \subseteq F(X)$. Consider the set $S = \{\Gamma' \subseteq F(X);\ \Gamma' \subseteq_{fin} \Gamma\}$. Notice that S is a directed set. Suppose that for any $\Gamma' \in S$, $\Gamma' \nvdash_G \varphi$, hence there is $a \in G(F(X))$ such that $\Gamma' \subseteq i_{F(X)}(a)$ and $\varphi \notin i_{F(X)}(a)$. Denote by $a_{\Gamma'} = \wedge\{a \in G(F(X));\ \Gamma' \subseteq i_{F(X)}(a)\}$. $i_{F(X)}$ preserves inf, thus $\Gamma' \subseteq i_{F(X)}(a_{\Gamma'})$ and $\varphi \notin i_{F(X)}(a_{\Gamma'})$. We obtain that the set $s = \{a_{\Gamma'};\ \Gamma' \in S\}$ is a directed set.

By the assumption $i_{F(X)}$ preserves directed sup, hence

$$\Gamma = \cup S \subseteq \bigcup_{\Gamma' \in S} i_{F(X)}(a_{\Gamma'}) = i_{F(X)}(\vee s).$$

On the other hand $\varphi \notin \bigcup_{\Gamma' \in S'} i_{F(X)}(a_{\Gamma'}) = i_{F(X)}(\vee s)$. Therefore $\Gamma \nvdash_G \varphi$. □

Remark 3.1.4. *1. One can define a logic l^G as follows: $\Gamma \vdash^G \varphi$ iff for any algebra M, for any $a \in G(M)$ and any valuation $v : F(X) \to M$, if $v[\Gamma] \subseteq i_M(a)$ then $v(\varphi) \in i_M(a)$. It is easy to see that both logics are the same $l_G = l^G$.*

2. Notice that for any $M \in \Sigma - str$ and $a \in G(M)$, $\langle M; i_M(a) \rangle$ is a matrix of l_G. Indeed, just apply the naturality of i. This shows us that for every $M \in \Sigma - Str$, $i^G_M[G(M)] \subseteq Fi_{l_G}(M)$, then we can consider the natural transformation $i^G : G \Rightarrow Fi_{l_G}$. We denote $Matr^G = \{\langle M, i^G_M(a) \rangle;\ a \in G(M) \text{ and } M \in \Sigma - Str\}$. Thus $Matr^G \subseteq Matr_{l_G}$, we have that

$$\vdash_{Matr^G} = \vdash^G = \vdash_G = \vdash_{Matr_{l_G}}.$$

Notice that for every set X we can define a logic over a filter pair. Here, given a function $f : X \to Y$, we will denote by the same $f : F_\Sigma(X) \to F_\Sigma(Y)$. We also denote $i^G_Z = i^G_{F_\Sigma(Z)}$ just to simplify.

Proposition 3.1.5. *Let X, Y sets and (G, i^G) a filter pair on Σ.*

1. For any function $f : X \to Y$ and $\Gamma \cup \{\varphi\} \subseteq F_\Sigma(X)$:

$$\Gamma \vdash^X \varphi \ \Rightarrow\ f[\Gamma] \vdash^Y f(\varphi).$$

2. For any injective function $f : X \rightarrowtail Y$ and $\Gamma \cup \{\varphi\} \subseteq F_\Sigma(X)$:

$$\Gamma \vdash^X \varphi \ \Leftrightarrow\ f[\Gamma] \vdash^Y f(\varphi).$$

3. $\Gamma \vdash^X \varphi$ iff there is a finite sets $X' \subseteq_f X$ and $\Gamma' \subseteq_f \Gamma$ such that $var(\Gamma' \cup \{\varphi\}) \subseteq X'$ and $\Gamma' \vdash^{X'} \varphi$.

Proof: *1.* Let $a \in G(F_\Sigma(Y))$ such that $f[\Gamma] \subseteq i^G_Y(a)$. Then $\Gamma \subseteq f^{-1}(i^G_Y(a)) = i^G_X(G(f)(a))$. Since $\Gamma \vdash^X \varphi$, we have that $\varphi \in i^G_X(G(f)(a))$. Therefore $f(\varphi) \in i^G_Y(a)$. As a was arbitrary we have $f[\Gamma] \vdash^Y f(\varphi)$.

2. Let $f : X \to Y$ injective. By *1* we have that $\Gamma \vdash^X \varphi \ \Rightarrow\ f[\Gamma] \vdash^Y f(\varphi)$. Remains to prove the converse. Let $a \in G(F_\Sigma(X))$ such that $\Gamma \subseteq i^G_X(a)$. Since f is injective there is $g : Y \to X$ such that $g \circ f = Id_X$. Hence $g \circ f[\Gamma] = \Gamma$. Then $f[\Gamma] \subseteq g^{-1}(i^G_X(a)) = i^G_Y(G(g)(a))$.

Since $f[\Gamma] \vdash^Y f(\varphi)$, then $f(\varphi) \in i^G_Y(G(g)(a)) = g^{-1}(i^G_X(a))$. Therefore $\varphi = g(f(\varphi)) \in i^G_X(a)$.

3. " $\Rightarrow$ " Since $\vdash^Z$ is finitary, there is a finite set $\Gamma' \subseteq_f \Gamma$ such that $\Gamma' \vdash^Z \varphi$. Consider $Z' = var(\Gamma') \cup var(\varphi)$. Let $a \in G(F_\Sigma(Z'))$. Suppose $\Gamma' \subseteq i^G_{Z'}(a)$. We have the inclusion function $j : Z' \hookrightarrow Z$ such that $j[\Gamma'] = \Gamma$. Due to *2* we have $\Gamma' \vdash^{Z'} \varphi$.

" $\Leftarrow$ " Let $a \in G(F_\Sigma(Z))$ such that $\Gamma \subseteq i^G_Z(a)$. By assumption we have that there are $\Gamma' \subseteq_f \Gamma$ and $Z' \subseteq_f Z$ such that $var(\Gamma' \cup \{\varphi\}) \subseteq Z'$ and $\Gamma' \vdash^{Z'} \varphi$. Consider the inclusion function $j : Z' \hookrightarrow Z$. Notice that $j[\Gamma'] = \Gamma'$. By item *2* we have that $\Gamma' \vdash^Z \varphi$, thus $\Gamma \vdash^Z \varphi$. $\square$

Proposition 3.1.6. *Let (G, i^G) a filter pair on Σ. For any set Z, if $F \in Fi_{l_G}(F_\Sigma(Z))$ then there is $a \in G(F_\Sigma(Z))$ such that $i^G_Z(a) = F$.*

Proof: Consider the set $S = \{a \in G(F_\Sigma(Z));\ F \subseteq i^G_Z(a)\}$. Denote $a_F = \wedge S$. Notice that $F \subseteq i^G_Z(a_F)$. Suppose that there is $\varphi \in i^G_Z(a_F)$ such that $\varphi \notin F$. We consider two cases: $|Z| \leq |X|$ and $|X| \leq |Z|$ where X is the set which l_G is defined.

($|Z| \leq |X|$): In this case there is a injective function $f : Z \to X$. By 3.1.5 we have $F \vdash^Z \varphi$ iff $f[F] \vdash f(\varphi)$. Suppose that $f[F] \vdash f(\varphi)$. Then, since $F \in Fi_{l_G}(F_\Sigma(Z))$, we have that for any evaluation $v : X \to F_\Sigma(Z)$ if $v(f[F]) \subseteq F$ then $v(f(\varphi)) \in F$. Consider $g : X \to Z$ such that $g \circ f = Id_Z$, then g can be seen as a evaluation with $g \circ f[F] = F$ and $g \circ f(\varphi) = \varphi$. Then $\varphi \in F$ which is a contradiction. Therefore $f[F] \nvdash f(\varphi)$. Thus there is $a \in G(F_\Sigma(X))$ such that $f[F] \subseteq i^G_X(a) = i^G_Z(G(f)(a))$. Hence $G(f)(a) \in S$. Thus $a_F \leq G(f)(a)$, since i^G_Z preserves inf, $i^G_Z(a_F) \subseteq i^G_Z(G(f)(a))$. Hence $\varphi \in i^G_Z(G(f)(a))$, and this implies a contradiction. Then $F = i^G_F(a_F)$.

($|X| \leq |Z|$): Observe that for any finite set $F' \subseteq_f F$ one can define a injective function $f_{F'} : X \rightarrowtail Z$ such that there is a set $X' \subseteq X$ which $f_{F'}[X'] = var(F') \cup var(\varphi)$. Moreover $f_{F'\restriction_{X'}}$ is a bijection.

Suppose that $F \vdash^Z \varphi$. Then there is a finite set $F' \subseteq_f F$ such that $F' \vdash^Z \varphi$. Consider $f_{F'}$ as above. So there is a retraction $g : Z \to X$ such that $g_{\restriction_{f_{F'}[X']}}$ is the inverse of $f_{F'\restriction_{X'}}$. Due to 3.1.5 we have $F' \vdash^Z \varphi \;\Leftrightarrow\; f_{F'} \circ g[F'] \vdash^Z f_{F'} \circ g[\varphi] \;\Leftrightarrow\; g[F'] \vdash g(\varphi)$. Thus, since $F \in Fi_{l_G}(F(Z))$, for any evaluation $v : X \to F(Z)$ we have that if $v(g[F']) \subseteq F$ then $v(g(\varphi)) \in F$. Note that $f_{F'}$ can be seen as evaluation and $f_{F'}(g[F']) = F' \subseteq F$. Then $\varphi = f_{F'}(g(\varphi)) \in F$. This implies a contradiction. Hence $F \nvdash^Z \varphi$. Therefore there is $a \in G(F_\Sigma(Z))$ such that $F \subseteq i^G_Z(a)$ and $\varphi \notin i^G_Z(a)$. Thus $a \in S$. So $a_F \leq a$ and then $i^G_Z(a_F) \subseteq i^G_Z(a)$. Hence $\varphi \in i^G_Z(a)$ which is a contradiction. Finally $F = i^G_Z(a_F)$. $\square$

3.1.1 Filter pairs over Co and Co_K, and a classification of those filter logics

In this section we present a analysis of two special filter pairs, more precisely, the filter pairs over the functors $Co : \Sigma - Str \to AL$ and $Co_K : \Sigma - Str \to AL$ where $K \subseteq \Sigma - Str$ is a class of algebras.

Suppose that there is a natural transformation $i^{Co} : Co \Rightarrow (\mathcal{P}(\ ), \subseteq)$ such that (Co, i^{Co}) is a filter pair. We consider its associated logic l_{Co}. Hence $A \in \Sigma - Str$, $Im(i^{Co}_A) \subseteq Fi_{l_{Co}}$. We have $i^{Co}_A : Co(A) \to Fi_{l_{Co}}$.

We are going give now a study of variants of algebraizable logics (introduced in 1.3.19) via this specific filter functors.

Proposition 3.1.7. *If $\Omega^A(i^{Co}_A(\theta)) = \theta$, i.e., Ω^A is a retraction to i^{Co}_A, then l_{Co} is a protoalgebraic logic.*

Proof:

Due to 3.1.6 we have that for any $T \in Th(l_{Co})$, there is $\theta \in Co(Fm)$ such that $i^{Co}_{Fm}(\theta) = T$. Let $T, T' \in Th(l_{Co})$ such that $T \subseteq T'$, then $i^{Co}_{Fm}(\theta \cap \theta') = i^{Co}_{Fm}(\theta) \cap i^{Co}_{Fm}(\theta') = T \cap T' = T = i^{Co}_{Fm}(\theta)$. Observe that i^{Co}_A is injective. Indeed, let $\theta, \theta' \in Co(A)$ such that $i^{Co}_A(\theta) = i^{Co}_A(\theta')$. Applying Ω^A we have $\theta = \Omega^A(i^{Co}_A(\theta)) = \Omega^A(i^{Co}_A(\theta')) = \theta'$. Since i^{Co}_A is injective we have that $\theta \cap \theta' = \theta$, thus $\theta \subseteq \theta'$. Therefore $\Omega(T) = \Omega(i^{Co}_{Fm}(\theta)) = \theta \subseteq \theta' = \Omega(i^{Co}_{Fm}(\theta')) = \Omega(T')$. So Ω is monotonic. By theorem 1.3.20 we have that l_{Co} is protoalgebraic logic. $\square$

Proposition 3.1.8. *Let $K \subseteq \Sigma - Str$ be a class of algebras and let $i^K : Co_K \Rightarrow (\mathcal{P}(\ ), \subseteq)$ such that $\langle Co_K, i^K \rangle$ is a filter pair. We denote by l_K the logic associated with $\langle Co_K, i^K \rangle$. If Ω^A is a retraction to i^K_A for any $A \in \Sigma - Str$ and $(\Omega^A)_{A \in \Sigma - Str}$ is a natural transformation, then:*

1. *l_K is an equivalential locic and $K \subseteq Alg^* l_K$*
2. *If K is closed under isomorphism then $K = Alg^* l_K$*

Proof:

1) We have seen in 3.1.7 that Ω is monotone. Since $(\Omega^A)_{A \in \Sigma - Str}$ is a natural transformation we have by 1.3.20 that l_K is an equivalential logic. Consider $i^K_A(Id_A)$. Then $\Omega^A(i^K_A(Id_A)) = Id_A$, hence $A/\Omega(i^K_A(Id_A)) = A$, thus $\langle A, i^K_A(Id_A) \rangle \in Matr^*(l_K)$. Therefore $A \in Alg^*(l_{Co})$

2) Let $A \in Alg^* l_K$ Let X be a set of the cardinality of A. Then consider a surjective morphism $f : F(X) \twoheadrightarrow A$. Since $A \in Alg^* l_K$ there is $F \in Fi_{l_K}(A)$ such that $\langle A, F \rangle \in Matr^* l_K$. Let $T = f^{-1}(F)$. By 3.1.6, we have that there is $\theta \in Co_K(F(X))$ such that

$i^K_{F(X)}(\theta) = T$, thus

$$\begin{aligned}
\theta &= \Omega^{F(X)}(i^K_{F(X)}(\theta)) \\
&= \Omega^{F(X)}(T) \\
&= \Omega^{F(X)(f^{-1}(F))} \\
&= f^{-1}(\Omega^A(F)) \\
&= f^{-1}(Id_A) \\
&= ker(f)
\end{aligned}$$

Therefore $A \cong F(X)/ker(f) = F(X)/\theta \in K$. Since K is closed under isomorphisms, $A \in K$. □

Proposition 3.1.9. *Let Σ be a signature and $K \subseteq \Sigma - str$ a quasivariety. If there exist some injective natural transformation $i^K : Co_k \Rightarrow (\mathcal{P}(\ ), \subseteq)$ such that (Co_K, i^K) is a filter pair, then the logic l_K associated with it is an algebraizable logic.*

Proof:

It is known that $i^K_{Fm}[Co_K(Fm)] = Th(l_K)$, As i^K_{Fm} is injective, we have that i^K_{Fm} is bijective. Then i^K_{Fm} is an isomorphism. Now let $\sigma \in hom(Fm, Fm)$. As i^K is a natural transformation we have the following diagram commuting:

$$\begin{array}{ccccc}
Fm & & Co_K(Fm) & \xrightarrow{i^K_{Fm}} & (\mathcal{P}(Fm); \subseteq) \\
\big\downarrow{\scriptstyle\sigma} & & \big\uparrow{\scriptstyle Co_K(\sigma)} & & \big\uparrow{\scriptstyle\sigma^{-1}} \\
Fm & & Co_K(Fm) & \xrightarrow[i^K_{Fm}]{} & (\mathcal{P}(Fm); \subseteq)
\end{array}$$

Notice that $\sigma^{-1}(T) \in Th(l_K)$ for any $T \in Th(l_k)$. Therefore i^K_{Fm} is a isomorphism such that commutes with substitution. By isomorphism theorem 1.3.14, l_K is an algebraizable logic. □

Lemma 3.1.10. *Let Σ be a signature, $K \subseteq \Sigma - str$ a quasivariety and τ a set of equations in at most one variable. The map $i^K = (i^K_M)_{M \in \Sigma - Str}$ where:*

$$\begin{aligned}
i^K_M : Co_K(M) &\rightarrow (\mathcal{P}(M), \subseteq) \\
\theta &\mapsto \{m \in M;\ \tau^M(m) \subseteq \theta\}
\end{aligned}$$

is a natural transformation and for any $M \in \Sigma - Str$, i^K_M preserves inf and sup directed, i.e., (Co_K, i^K) is a filter pair.

Proof:

Let $f \in hom(M, N)$.

Denote here $f(\tau^M(m)) = \{\langle f(\varepsilon^M(m)), f(\delta^M(m))\rangle;\ \langle \varepsilon, \delta\rangle \in \tau\}$.

$$
\begin{array}{ccc}
M & \quad & \begin{tikzcd} Co_K(M) \arrow[r, "i^K_M"] & (\mathcal{P}(M);\subseteq) \\ Co_K(N) \arrow[u, "Co_K(f)"] \arrow[r, "i^K_N"'] & (\mathcal{P}(N);\subseteq) \arrow[u, "f^{-1}"'] \end{tikzcd} \\
\Big\downarrow f & & \\
N & &
\end{array}
$$

Given $\theta \in Co_K(N)$ then

$$
\begin{aligned}
f^{-1}(i^K_N(\theta)) &= f^{-1}(\{n \in N;\ \tau^N(n) \subseteq \theta\}) \\
&= \{m \in M;\ \tau^N(f(m)) \subseteq \theta\} \\
&= \{m \in M;\ f(\tau^M(m)) \subseteq \theta\} \\
&= \{m \in M;\ \tau^M(m) \subseteq Co_K(f)(\theta)\} \\
&= i^K_M(f^{-1}(\theta))
\end{aligned}
$$

Let $\theta,\ \theta' \in Co_K(M)$.

$$
\begin{aligned}
i^K_M(\theta \cap \theta') &= \{m \in M;\ \tau^M(m) \subseteq \theta \cap \theta'\} \\
&= \{m \in M;\ \tau^M(m) \in \theta \textit{ and } \tau^M(m) \subseteq \theta'\} \\
&= \{m \in M;\ \tau^M(m) \subseteq \theta\} \cap \{m \in M;\ \tau^M(m) \subseteq \theta'\}
\end{aligned}
$$

Then i^K_M preserves inf.

Now let $U = \{\theta_i;\ i \in I\}$ be an up-directed set.

$$
\begin{aligned}
i^K_M(\bigvee U) &= \{m \in M;\ \tau^M(m) \subseteq \bigvee U\} \\
&= \{m \in M;\ \tau^M(m) \subseteq \bigcup_{i\in I} \theta_i\} \\
&= \bigcup_{i\in I}\{m \in M;\ \tau^M(m) \subseteq \theta_i\}
\end{aligned}
$$

□

Corollary 3.1.11. *Let Σ be a signature, $K \subseteq \Sigma - str$ a quasivariety and τ a set of equations. If i^K, defined as above, is injective then the logic l_K of the filter pair (Co_K, i^K) is an algebraizable logic.*

Remark 3.1.12. *This corollary give us an alternative proof to theorem 5.2 [BR]. The condition of injectivity assumed here is exactly the condition put there to get algebraizability.*

Corollary 3.1.13. *If $i^K_A(\Omega^A(F)) = F$, i.e., Ω^A is a section to i^K_A, then l_K is a truth-equational logic.*

Proof:

As $i^K_A(\Omega^A(F)) = F$ then $F = \{a \in A;\ \tau^A(a) \subseteq \Omega^A(F)\}$. By Theorem 1.3.20 we have that l_K is a truth-equational logic. □

Lemma 3.1.14. *Let K be a pointed quasivariety and consider the set of equations $\tau = \{\langle x, 0\rangle\}$. Then Ω^A is a section to i^K_A for any $A \in \Sigma - Str$.*

Proof:

Notice that in the logic l_K we have that $\vdash_K 0$, $x, \varphi(x,\bar{z}) \vdash_K \varphi(0,\bar{z})$ and $x, \varphi(0,\bar{z}) \vdash_K \varphi(x,\bar{z})$ for any $\varphi(x,\bar{z}) \in Fm$. Indeed, let $\theta \in Co_K(Fm)$, then $\langle 0, 0\rangle \in \theta$, thus $0 \in i^K_{Fm}(\theta)$

and then $\vdash_K 0$. Now let $\theta \in Co_K(Fm)$ and suppose that $x, \varphi(x,\bar{z}) \in i^K_{Fm}(\theta)$, then $\langle x, 0\rangle \in \theta$ and $\langle \varphi(x,\bar{z}), 0\rangle \in \theta$. Since θ is a congruence, we have that $\langle \varphi(x,\bar{z}), \varphi(0,\bar{z})\rangle \in \theta$. Therefore $\langle \varphi(0,\bar{z}), 0\rangle \in \theta$, so $\varphi(0,\bar{z}) \in i^K_{Fm}(\theta)$. Hence $x, \varphi(x,\bar{z}) \vdash_K \varphi(0,\bar{z})$. The same proof can be used to prove that $x, \varphi(0,\bar{z}) \vdash_K \varphi(x,\bar{z})$.

Now we are able to prove that for any $A \in \Sigma - Str$ and $F \in Fi_{l_K}(A)$, $F = i^K_A(\Omega^A(F))$. Let $a \in F$ and $\varphi(x,\bar{z}) \in Fm$. Let $\bar{c} \in A$ and suppose that $\varphi^A(a,\bar{c}) \in F$. Since $x, \varphi(x,\bar{z}) \vdash_K \varphi(0,\bar{z})$, we have that $\varphi^A(0,\bar{c}) \in F$. Analogously we have that if $\varphi^A(0,\bar{c}) \in F$, $\varphi^A(a,\bar{c}) \in F$. Hence $\langle a, 0\rangle \in \Omega^A(F)$. By definition of i^K_A we have $a \in i^K_A(\Omega^A(F))$. Thus $F \subseteq i^K_A(\Omega^A(F))$. Let $a \in i^K_A(\Omega^A(F))$, then $\langle a, 0\rangle \in \Omega^A(F)$. Since $\vdash_K 0$, then $0 \in F$, therefore $a \in F$. Hence $i^K_A(\Omega^A(F)) \subseteq F$.

Due to Corollary 3.1.13 l_K is a truth-equational logic. $\square$

Corollary 3.1.15. *Let $K \subseteq \Sigma - Str$ a pointed quasivariety. Consider $\tau = \{\langle x, 0\rangle\}$ and the logic l_K obtained as in the Lemma 3.1.10). Then*

- *l_K is truth-equational.*
- *If Ω^A is a section for i^K_A for any $A \in \Sigma - Str$, then l_K is algebraizable.*

Example 3.1.16. *Let Σ be the signature of group theory, i.e. $\Sigma = (\Sigma_n)_{n\in\omega}$ where $\Sigma_0 = \{e\}$, $\Sigma_1 = \{\ ^{-1}\}$, $\Sigma_2 = \{\cdot\}$ and $\Sigma_n = \emptyset$ for any $n > 2$. Consider $\tau = \{\langle x, e\rangle\}$. Let $\mathbf{K}$ be the variety of group theory. By 3.1.10 we have that (Co_K, i^K), as above, is a filter pair and then there is a logic l_K associated. It is easy to see that i^K_M is injective for all $M \in \Sigma - Str$. Thus by 3.1.11 l_K is an algebraizable logic. From that we have that Ω^M is a retraction of i^K_M for any $M \in \Sigma - Str$. Since $\mathbf{K}$ is closed by isomorphism we have by 3.1.8 $\mathbf{K} = Alg^* l_K$, then the variety of group theory is an equivalent algebraic semantic for l_K. $l_K = l_{Gr}$ defined in 1.3.*

Now we will give a characterization for selfextensional logics.

Definition 3.1.17. *Let $\langle G, i^G\rangle$ be a filter pair. We say that $l_G = (\Sigma, \vdash)$ is compatible with $\langle G, i^G\rangle$ if for every $\varphi, \psi \in Fm$ such that $\varphi \dashv\vdash \psi$, and every $a \in G(Fm)$ and $\rho(p,\vec{z}) \in Fm$*

$$\rho(\varphi,\vec{\sigma}) \in i^G_{Fm}(a) \ \ iff \ \ \rho(\psi,\vec{\sigma}) \in i^G_{Fm}(a), \ for\ \vec{\sigma} \subseteq Fm$$

Proposition 3.1.18. *Let $\langle G, i^G\rangle$ be a filter pair. Then l_G is a selfextensional logic if, and only if, it is compatible with $\langle G, i^G\rangle$*

Proof:

" $\Leftarrow$ " Suppose that $\varphi_0 \dashv\vdash \psi_0, ..., \varphi_{n-1} \dashv\vdash \psi_{n-1}$. Let $c_n \in \Sigma_n$ and $a \in G(Fm)$. By compatibility,

$$\begin{aligned} c_n(\varphi_0, ..., \varphi_{n-1}) \in i^G_{Fm}(a) \ \ & iff \ \ c_n(\varphi_0, ..., \psi_{n-1}) \in i^G_{Fm}(a) \\ & iff \ \ c_n(\psi_0, ..., \psi_{n-2}, \psi_{n-1}) \in i^G_{Fm}(a) \end{aligned}$$

Hence $c_n(\varphi_0, ..., \varphi_{n-1}) \dashv\vdash c_n(\psi_0, ..., \psi_{n-1})$. Therefore $\dashv\vdash$ is congruence relation.

" $\Rightarrow$ " suppose $\varphi \dashv\vdash \psi$. Let $\rho(p, \vec{z}) \in Fm$. We know that for every variable x, $x \dashv\vdash x$. Then by congruentiality, given $a \in G(Fm)$, $\rho(\varphi, \bar{x}) \in i^G_{Fm}(a)$ *iff* $\rho(\psi, \bar{x}) \in i^G_{Fm}(a)$. Hence l_G is compatible with $\langle G, i^G \rangle$. □

Corollary 3.1.19. *Let $K \subseteq \Sigma - str$ be a quasivariety. If $\langle Co_K, i^K \rangle$ is a filter pair, then l_K is compatible with $\langle Co_K, i^K \rangle$ iff l_K is a Lindenbaum algebraizable logic.*

Proof:

Since $\langle Co_K, i^K \rangle$ is a filter pair and due to 3.1.9, l_K is an algebraizable logic. By 3.1.18, l_K is congruential. As any congruential logic is Lindenbaum, thus l_K is Lindenbaum algebraizable logic.

Every Lindenbaum algebraizable logic is selfextensional logic, then by 3.1.18 l_K is compatible with respect to $\langle Co_K, i^K \rangle$. □

3.2 Craig entailment interpolation property and filter functors

In this section we present a correspondence between Craig entailment interpolation property on a logic given by a filter pair on Co_K and the matrix amalgamation property in the class of matrix of this logic.

Definition 3.2.1. *Let l be a logic of type Σ.*

- *Given $M, N \in \Sigma$, $F \in Fi_l(M)$ and $F' \in Fi_l(N)$, a function $f : \langle M, F \rangle \to \langle N, F' \rangle$ is a matrix-embedding if $f : M \to N$ is a embedding such that $f[F] \subseteq F'$ and $f[M \setminus F] \subseteq N \setminus F'$.*

- *l has the Craig entailment interpolation property if for every set of formulas Γ and every formula φ, if $\Gamma \vdash \varphi$ then there is a set of formulas Γ' with the variables in $var(\Gamma) \cap var(\varphi)$ such that $\Gamma \vdash \Gamma'$ and $\Gamma' \vdash \varphi$.*

- *l has the theory amalgamation property if for every two no disjoint sets of variables X and Y, and every $l-filter$ T of the formulas algebra $F(X)$ there is an $l-filter$ R of the formula algebra $F(X \cup Y)$ such that $R \cap F(X) = T$ and $R \cap F(Y) = Fi_l[T \cap F(X \cap Y)] = \bigcap\{T' \in Th(F(Y));\ T \cap F(X \cap Y) \subseteq T'\}$*

Definition 3.2.2. *Let $K \subseteq \Sigma - Str$ be a class of algebras, $Co_K : \Sigma - Str \to AL$ and a natural transformation $i^K : Co_K \Rightarrow (\mathcal{P}(\ ), \subseteq)$ such that (Co_K, i^K) is a filter pair.*

- *The logic l_K is filter-weak-equivalential if given $Y \subseteq X$ and $\theta \in Co_K(F(Y))$, if $\langle\varphi, \psi\rangle \in \theta$ then $\varphi \in i^K_{F(Y)}(\theta)$ iff $\psi \in i^K_{F(Y)}(\theta)$.*

- *We shall say that K has the i^K-matrix-amalgamation property if given $A_1, A_2, A_3 \in K$ and $F_{A_i} \in Fi_{l_K}(A_i)$ for all $1 \leq i \leq 3$ and a matrix-embedding $i_j : \langle A_3, F_3\rangle \to \langle A_j, F_j\rangle$ where $j \in \{1, 2\}$, there exists a matrix $\langle A_4, F_4\rangle$ with $A_4 \in K$ and embeddings $e_j : \langle A_j, F_j\rangle \to \langle A_4, F_4\rangle$ such that $e_1 \circ i_1 = e_2 \circ i_2$.*

Remark 3.2.3. *under the conditions above, if $\Omega^A(i^K_A(\theta)) = \theta$ for any $A \in \Sigma - Str$ and $\theta \in Co_K(\theta)$, we have that Ω^A is surjective. In this case l_K is a filter-weak-equivalential logic. Indeed, let $Y \subseteq X$, $\varphi, \psi \in F(Y)$ and $\theta \in Co_K(F(Y))$ such that $\langle\varphi, \psi\rangle \in \theta$. Suppose that $\varphi \in i^K_{F(Y)}(\theta)$. Since $\Omega^{F(Y)}$ is surjective, there is $T \in Fi_{l_K}(F(Y))$ such that $\Omega^{F(Y)}(T) = \theta$, namely, $T = i^K_{F(Y)}(\theta)$. By proposition 3.1.7 we have that l_K is protoalgebraic logic, then $\langle\varphi, \psi\rangle \in \Omega^{F(Y)}(T)$, hence $T, \varphi \vdash^Y \psi$ and $T, \psi \vdash^Y \varphi$. Since $\varphi \in i^K_{F(Y)}(\theta) = T$, we have that $\psi \in i^K_{F(Y)}(\theta)$. Analogously we can prove that if $\psi \in i^K_{F(Y)}(\theta)$ then $\varphi \in i^K_{F(Y)}(\theta)$. In this way we have that l_K is filter-weak-equivalential logic.*

Lemma 3.2.4. *([CP99]) If a logic l has the theory amalgamation property, then it has the Craig entailment interpolation property.*

Theorem 3.2.5. *Let Σ be a signature, $K \subseteq \Sigma - Str$ a class of algebras and (Co_K, i^K) a filter pair. If l_K is a filter-weak-equivalential logic and K has the i^K-matrix-amalgamation property restricted to reduced filters, then l_K has the Craig entailment property.*

Proof:

In order to prove that l_K has the Craig entailment interpolation property, we will prove first that l_K has the theory amalgamation property. Let X, Y be non disjoint sets and $T \in Fi_{l_k}(F(X))$. Denote by $Z = X \cap Y$ and $W = X \cup Y$. Consider $T' = Fi^Y_{l_K}(T \cap F(Z))$. So $T' \cap F(Z) = T \cap F(Z)(= T'')$. Indeed, it is clear that $T \cap F(Z) \subseteq T' \cap F(Z)$. Suppose that there is $\varphi \in T' \cap F(Z)$ such that $\varphi \notin T \cap F(Z)$, hence $T \cap F(Z) \nvdash_{l_K} \varphi$. Notice that $Z \subseteq Y$, due to lemma 3.1.5 we have $T \cap F(Z) \nvdash^Y \varphi$, then $\varphi \notin Fi^Y_{l_K}(T \cap F(Z)) = T'$, a contradiction.

By proposition 3.1.6 consider now $\theta_T \in Co_K(F(X))$, $\theta_{T'} \in Co_K(F(Y))$ and $\theta_{T''} \in Co_K(F(Z))$ such that $T = i^K_{F(X)}(\theta_T)$, $T' = i^K_{F(Y)}(\theta_{T'})$ and $T'' = i^K_{F(Z)}(\theta_{T''})$. Let $A_1 = F(X)/\theta_T$, $A_2 = F(Y)/\theta_{T'}$ and $A_3 = F(Z)/\theta_{T''}$. Define $i_j : A_3 \to A_j$ by $i_j([\varphi]_{\theta_{T''}}) = [\varphi]_{\theta_T}$ for $j \in \{1, 2\}$. It is easy to see that i_j is embedding for $j \in \{1, 2\}$. If $[\varphi]_{\theta_{T''}} \in T''/\theta''$, then there is $\psi \in T''$ such that $\langle\varphi, \psi\rangle \in \theta_{T''}$. Since l_K is filter-weak-equivalential, we have that $\varphi \in T'' = T \cap F(Z)$. Hence $[\varphi]_{\theta_{T''}} \in T''/\theta_{T''}$ implies $[\varphi]_{\theta_T} \in T/\theta_T$. Now let $[\varphi]_{\theta_{T''}} \in A_3 \setminus (T''/\theta_{T''})$. Suppose that $[\varphi]_{\theta_T} \in T/\theta_T$, therefore $\varphi \in T$. Since $\varphi \in F(Z)$, so $\varphi \in T \cap F(Z)$, and then $\varphi \in T''$. Thus $[\varphi]_{\theta_{T''}} \in T''/\theta_{T''}$. Implying a contradiction. Hence

$[\varphi]_{\theta_T} \in A_1 \setminus (T/\theta_T)$. Due to that we have that i_1 is a matrix-embedding. Analogously one can prove that i_2 is a matrix-embedding as well.

By i^K-matrix-amalgamation property restricted to reduced filters, there is $A_4 \in K$ and $F_4 \in Fi_{l_K}(A_4)$ such that $\langle A_4, F_4\rangle \in Matr^* l_K$ with $e_1 : \langle A_1, T/\theta_T\rangle \to \langle A_4, F_4\rangle$ and $e_2 : \langle A_2, T'/\theta_{T'}\rangle \to \langle A_4, F_4\rangle$ such that $e_1 \circ i_1 = e_2 \circ i_2$.

Define $f : W \to A_4$ such that $f(x) = e_1([x]_{\theta_T})$ if $x \in X$ and $f(x) = e_2([x]_{\theta_{T'}})$ if $x \in Y \setminus Z$. Notice that for every $z \in Z$, $e_1([z]_{\theta_T}) = e_1(i_1([z]_{\theta_{T''}})) = e_2(i_2([z]_{\theta_{T''}})) = e_2([z]_{\theta_{T'}})$. Therefore $e_1([z]_{\theta_T}) = f(z) = e_2([z]_{\theta_{T'}})$. Hence we can consider $f : F(W) \to A_4$ and then if $\varphi \in F(X)$, $f(\varphi) = e_1([\varphi]_{\theta_T})$ and if $\varphi \in F(Y)$, $f(\varphi) = e_2([\varphi]_{\theta_{T'}})$.

Let $R = f^{-1}(F_4)$, then

$$\begin{aligned}
\varphi \in R \cap F(X) \quad & iff \quad f(\varphi) \in F_4 \\
& iff \quad e_1([\varphi]_{\theta_T}) \in F_4 \\
& iff \quad [\varphi]_{\theta_T} \in T/\theta_T \ (i^K - matrix - amalg.) \\
& iff \quad \varphi \in T \ (filter - weak - equiv.)
\end{aligned}$$

Hence $R \cap F(X) = T$. Analogously we have that $R \cap F(Y) = T'$. Due to Lemma 3.2.4, l_K has Craig entailment interpolation property. $\square$

Lemma 3.2.6. *Let $K \subseteq \Sigma - Str$ be a quasivariety. Suppose that there is $i^K : Co_K \to \langle \mathcal{P}(\ ); \subseteq\rangle$ such that $\langle Co_K, i^K\rangle$ is a filter pair and l_K a truth-equational logic. K has the i^K-matrix-amalgamation property then K has the amalgamation property.*

Proof:

Let $A_3 \overset{f_i}{\hookrightarrow} A_i$ embeddings for i=1,2. Since l_K is a truth-equational logic, there is a set of equation τ such that defines the filters in $Matr^* l$. Then consider the filters defined by $F_i = \{a \in A_i;\ A_i \models \tau(a)\}$ for any i=1,2,3. We prove that $f_i : \langle A_3, F_3\rangle \to \langle A_i, F_i\rangle$ is a matrix-embedding for any i=1,2. Indeed, let $a \in F_3$, for any $\langle \delta, \varepsilon\rangle \in \tau$, we have that $\delta^{A_i}(f_i(a)) = f_i(\delta^{A_3}(a))$ and $\varepsilon^{A_i}(f_i(a)) = f_i(\varepsilon^{A_3}(a))$. Since $a \in F_3$, we have $\delta^{A_3}(a) = \varepsilon^{A_3}(a)$, hence $\delta^{A_i}(f_i(a)) = \varepsilon^{A_i}(f_i(a))$, proving $f_i[F_3] \subseteq F_i$. Now suppose that $f_i(a) \in F_i$. Then $\delta^{Ai}(f_i(a)) = \varepsilon^{A_i}(f_i(a))$ for all $\langle \delta, \varepsilon\rangle \in \tau$. Thus $f_i(\delta^{A_3}(a)) = f_i(\varepsilon^{A_3}(a))$. Since f_i is an embedding, we have that $\delta^{A_3}(a) = \varepsilon^{A_3}(a)$, therefore $a \in \{b \in A_3;\ A_3 \models \tau(b)\} = F_3$. Thus $f_i[A_3 \setminus F_3] \subseteq A_i \setminus F_i$.

By l_K-matrix-amalgamation, we have that there is a matrix $\langle A_4, F_4\rangle$ together matrix-embeddings $\langle A_i, F_i\rangle \overset{g_i}{\hookrightarrow} \langle A_4, F_4\rangle$, for i=1,2, such that the following diagram commutes:

$$\begin{tikzcd} & \langle A_1, F_1\rangle \arrow[dr, "g_1"] & \\ \langle A_3, F_3\rangle \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & \langle A_4, F_4\rangle \\ & \langle A_2, F_2\rangle \arrow[ur, "g_2"'] & \end{tikzcd}$$

In particular we have the following diagram commuting:

$$\begin{tikzcd} & A_1 \arrow[dr, "g_1"] & \\ A_3 \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & A_4 \\ & A_2 \arrow[ur, "g_2"'] & \end{tikzcd}$$

□

Definition 3.2.7. *A logic l has a Deduction-Detachment Theorem (DDT) if there is a set of formulas $\Delta(p, q)$ such that for every set of formulas Γ and every formulas α and β*

$$\Gamma, \alpha \vdash \beta \quad iff \quad \Gamma \vdash \Delta(\alpha, \beta).$$

The set Δ is called deduction-detachment set for l or shortly DD-set.

Lemma 3.2.8. *([CP99]) If l has the Craig entailment interpolation property then for any set of variables X, every set $\Gamma \subseteq Fm(X)$ and every $\varphi \in Fm(X)$, if $\varphi \in Fi_l^{Fm(X)}(\Gamma)$ then there is a set $\Gamma' \subseteq \Gamma$ such that $var(\Gamma') \subseteq var(\Gamma) \cap var(\varphi)$ and $\varphi \in Fi_l^{Fm(X)}(\Gamma')$.*

Theorem 3.2.9. *Let l be a equivalential logic with DDT. If l has Craig entailment property then Alg*l has the l-matrix-amalgamation restrict to reduced filters.*

Proof:

Let $A_i \in Alg^*l$ and $F_i \in Fi_l(A_i)$ reduced filter for $i = 1, 2, 3$. Suppose without lost of generality that A_3 is a subalgebra of A_1 and A_2 and $F_j \cap A3 = F_3$ for $j = 1, 2$. Consider X, Y set of variables such that X is of the cardinality of A_1, Y is of the cardinality of A_2 and $X \cap Y$ is of the cardinality of A_3. Denote by $Z = X \cap Y$ and $W = X \cup Y$. Consider the homomorphisms $h : F(X) \twoheadrightarrow A_1$ and $g : F(Y) \twoheadrightarrow A_2$, such that

- $h \restriction X$ is a bijection between X and A_1.
- $g \restriction Y$ is a bijection between Y and A_2.
- $h \restriction Z = g \restriction Z$.

- $g \restriction Z$ is a bijection between Z and A_3.

Let $\Gamma_1 = h^{-1}[F_1]$ and $\Gamma_2 = g^{-1}[F_2]$. Consider T the l-filter of $F(W)$ generated by $\Gamma_1 \cup \Gamma_2$.

Claim 1: $T \cap F(X) = \Gamma_1$. The inclusion $\Gamma_1 \subseteq T \cap F(X)$ is clear. Let $\varphi \in T \cap F(X)$, then $\varphi \in T = Fi_l^{F(W)}(\Gamma_1 \cup \Gamma_2)$. l is finitary, then there are $\varphi_1, ..., \varphi_n \in \Gamma_1$ such that $\varphi \in Fi_l^{F(W)}(\Gamma_2, \varphi_1, ..., \varphi_n)$. By DDT there is a set of formulas $\Delta^*(p_1, ..., p_n, q)$ such that

$$\Delta^* = \Delta^*(\varphi_1, ..., \varphi_n, \varphi) \subseteq Fi_l^{F(W)}(\Gamma_2).$$

By Craig entailment interpolation property, using the Lemma 3.2.8, for each $\psi \in \Delta*$, there is a set Γ_ψ such that $var(\Gamma_\psi) \subseteq var(\Gamma_2) \cap var(\psi)$, $\Gamma_\psi \subseteq Fi_l^{F(W)}(\Gamma_2)$ and $\psi \in Fi_l^{F(W)}(\Gamma_\psi)$. Hence $\bigcup_{\psi \in \Delta^*} \Gamma_\psi \subseteq Fi_l^{F(W)}(\Gamma_2)$ and $\psi \in Fi_l^{F(W)}(\Gamma_\psi)$.

Notice that $var(\Gamma) \subseteq Y$ and for any $\psi \in \Delta^*$, $var(\psi) \subseteq X$ then $var(\bigcup_{\psi \in \Delta*} \Gamma_\psi) \subseteq Z$. Using detachment we have that $\varphi \in Fi_l^{F(W)}(\bigcup_{\psi \in \Delta^*} \Gamma_\psi, \varphi_1, ..., \varphi_n)$.

Notice that the variables in the formulas $\varphi_1, ..., \varphi_n$ and in the formulas $\bigcup_{\psi \in \Delta^*} \Gamma_\psi$ are all in X. We prove that $\varphi \in \Gamma_1$. In order to do that, we prove that $\bigcup_{\psi \in \Delta^*} \Gamma_\psi \subseteq \Gamma_1$. Extending the homomorphism $g : F(Y) \twoheadrightarrow A_2$ to morphism $g' : F(W) \twoheadrightarrow A_2$, we have $g' \restriction F(Y) = g$. $\Gamma_2 \subseteq g^{-1}[F_2]$. So $\bigcup_{\psi \in \Delta^*} \Gamma_\psi \subseteq Fi_l^{F(W)}(\Gamma_2) \subseteq g'^{-1}[F_2]$. Since $\bigcup_{\psi \in \Delta^*} \Gamma_\psi \subseteq F(Z) \subseteq F(Y)$, we have $g[\bigcup_{\psi \in \Delta^*} \Gamma_\psi] \subseteq g[g'^{-1}[F_2]] \subseteq F_2$. Therefore $h[\bigcup_{\psi \in \Delta^*} \Gamma_\psi] = g[\bigcup_{\psi \in \Delta^*} \Gamma_\psi] \subseteq F_2 \cap A_3 = F_1 \cap A_3 \subseteq F_1$. Thus $\bigcup_{\psi \in \Delta^*} \Gamma_\psi \subseteq \Gamma_1$, then $\varphi \in Fi_l^{F(W)}(\Gamma_1)$. By Lemma 3.1.5 $\varphi \in Fi_l^{F(X)}(\Gamma_1)$. Then $\varphi \in \Gamma_1$.

Claim 2: $T \cap F(Y) = \Gamma_2$. The same proof of Claim 1.

Notice that $\Omega^X(\Gamma_1) = ker(h)$. Indeed,

$$\begin{aligned}
\langle \varphi, \psi \rangle \in \Omega^X(\Gamma_1) \quad &\Leftrightarrow \quad \langle \varphi, \psi \rangle \in \Omega^X(h^{-1}(F_1)) \\
&\Leftrightarrow \quad \langle \varphi, \psi \rangle \in h^{-1}(\Omega^A(F_1))(l \textit{ is equiv. logic}) \\
&\Leftrightarrow \quad \langle h(\varphi), h(\psi) \rangle \in \Omega^A(F_1)(F_1 \textit{ is reduced filter}) \\
&\Leftrightarrow \quad h(\varphi) = h(\psi)
\end{aligned}$$

Analogously we have $\Omega^Y(\Gamma_2) = ker(g)$. Then $F(X)/\Omega^X(\Gamma_1) \cong A_1$, $F(Y)/\Omega^Y(\Gamma_2) \cong A_2$ and $F(Z)/\Omega^Z(\Gamma_1 \cap F(Z)) \cong A_3$. Observe that these isomorphisms send $\Gamma_i/\Omega(\Gamma_i)$ to F_i for $i = 1, 2, 3$ and $\Gamma_3 = \Gamma_1 \cap F(Z)$. Consider $D = F(W)/\Omega^W(T)$ and $F_D = T/\Omega^W(T)$. With that we have $\langle D, F_D \rangle \in Matr^*(l)$ and $D \in Alg^*l$.

Define $e_1 : F(X)/\Omega^X(\Gamma_1) \to D$ where $e_1([\varphi]_{\Omega^W(\Gamma_1)}) = [\varphi]_{\Omega^W(T)}$. In the same way to $e_2 : F(Y)/\Omega^Y(\Gamma_2) \to D$. Observe that $\Omega^X(\Gamma_1) = \Omega^W(T) \cap F(X)^2$. Indeed, we can see $T \cap F(X) = j^{-1}[T]$ and $\Omega^Y(T) \cap F(X)^2 = j^{-1}(\Omega^W(T))$ where $j : F(X) \hookrightarrow F(W)$ is the inclusion morphism. Since l is equivalential logic we have

$$\begin{aligned}\Omega^X(\Gamma_1) &= \Omega^X(T \cap F(X)) \\ &= \Omega^X(j^{-1}(T)) \\ &= j^{-1}(\Omega^W(T)) \\ &= \Omega^W(T) \cap F(X)^2\end{aligned}$$

With this equality we have that e_1 is well defined and injective. Analogously to e_2. It is easy to see that the following diagram commutes

$$\begin{tikzcd} & F(X)/\Omega(\Gamma_1) \cong A_1 \arrow[dr] & \\ A_3 \cong F(Z)/\Omega(\Gamma_1 \cap F(Z)) \arrow[ur] \arrow[dr] & & D \\ & F(Y)/\Omega(\Gamma_2) \cong A_2 \arrow[ur] & \end{tikzcd}$$

It clear the $e_i[\Gamma_i/\Omega(\Gamma_1)] \subseteq T/\Omega(T)$. Let $\varphi \in F(X)$ such that $[\varphi]_{\Omega(T)} \in T/\Omega(T)$, then $\varphi \in T$ due to Leibniz operator. Thus by Claim 1 $\varphi \in T \cap F(X) = \Gamma_1$, then $[\varphi]_{\Omega(\Gamma_1)} \in \Gamma/\Omega(\Gamma_1)$. Analogously for e_2. Therefore e_i is a matrix-embedding for $i = 1, 2$. □

Corollary 3.2.10. *Let $K \subseteq \Sigma - Str$ closed by isomorphism. Suppose that there is i^K : $Co_K \Rightarrow (\mathcal{P}(\ ), \subseteq)$ such that (Co_K, i^K) is a filter pair, $(\Omega^A)_{A\in\Sigma-Str}$ is natural transformation such that is a retraction for i^K. If l_K has DDT then K has i^K-matrix-amalgamation property restrict to reduced filters if, and only if l_K has Craig entailment interpolation property.*

Proof:

" $\Rightarrow$ " : Due to 3.2.3 we have that l_K is filter-weak-equivalential logic, then applying Theorem 3.2.5 it is done.

" $\Leftarrow$ " : By Proposition 3.1.8 we have l_K is equivalential logic and $K = Alg^*l_K$. So it is just apply 3.2.9. □

Corollary 3.2.11. *Let $K \subseteq \Sigma - Str$ be a quasivariety and τ be a set of equations in at most one variable. By 3.1.10 we have that we can define a filter par (Co_K, i^K) and consequently a logic l_K. Suppose that Ω^A is a section to i^K_A for any $A \in \Sigma - Str$. If K has i^K-matrix-amalgamation property then K has amalgamation property.*

Proof:

Using 3.1.13 we have that l_K is truth-equational logic. Then apply 3.2.6 □

3.3 The category of filters $\mathcal{F}i$

In this section we present the definition of the category of filter functors and establish the correspondence between it and the category of logics with flexible morphisms.

The category $\mathcal{F}i$ is composed by:

• **Objects**: Filters pairs (G, i^G).

• **Morphisms**: Pairs (H, j^H) such that $H : \Sigma' - str \to \Sigma - str$ is a structure functor, i.e., it commutes over set and the natural transformation $\eta^H : F \Rightarrow H \circ F'$ preserves variables, where F, F' are free functors from the category Set to structures and $j^H : G' \Rightarrow G \circ H$ is a natural transformation such that given $M' \in \Sigma' - str$, $i^G_{H(M')} \circ j^H_{M'} = i^{G'}_{M'}$ where (G, i^G) and $(G', i^{G'})$ are filters pairs.

$$\begin{tikzcd} \Sigma' - str \arrow[rr, "H"] \arrow[dr, bend right, "\mathcal{P}"'] \arrow[dr, "G^{L'}"] & & \Sigma - str \arrow[dl, bend left, "\mathcal{P}"] \arrow[dl, "G^{L}"'] \\ & AL & \end{tikzcd}$$

-Composition: Given $(H, j^H), (H', j^{H'}) \in Ob\mathcal{F}i$.

$$(H, j^H) \bullet (H', j^{H'}) = (H \circ H', j^H \bullet j^{H'})$$

where $(j^H \bullet j^{H'})_{M''} := j^H_{H'(M'')} \circ j^{H'}_{M''}$. Observe that

$$i^G_{H \circ H'(M'')} \circ ((j^H \bullet j^{H'})_{M''}) = i^{G''}_{M''}.$$

Indeed

$$\begin{aligned} i^G_{H \circ H'(M'')} \circ ((j^H \bullet j^{H'})_{M''}) &= i^G_{H \circ H'(M'')} \circ (j^H_{H'(M'')} \circ j^{H'}_{M''}) \\ &= (i^G_{H \circ H'(M'')} \circ j^H_{H'(M'')}) \circ j^{H'}_{M''} \\ &= i^{G'}_{H'(M'')} \circ j^{H'}_{M''} \\ &= i^{G''}_{M''}. \end{aligned}$$

In the former sections we have seen a correspondence between the objects of $\mathcal{L}_f$ and $\mathcal{F}i$. Now we show the correspondence between morphisms of $\mathcal{L}_f$ and of $\mathcal{F}i$.

In chapter 2 we have that given a functor $H : \Sigma' - Str \to \Sigma - Str$ such that it is a structure functor, then there is a signature morphism $m_H : \Sigma' \to \Sigma$, such that $m_H(c_n) = \eta_H(X)(c_n(x_0, ..., x_{n-1}))$. We can consider the functor

$$\begin{array}{cccc} \mathcal{N} : & \mathcal{F}i & \to & \mathcal{L}_f \\ & (G', i^{G'}) & & l_{G'} \\ & \downarrow & \mapsto & \uparrow m_H \\ & (G, i^G) & & l_G \end{array}$$

We must prove that m_H is a translation.

Let $\Gamma \cup \{\varphi\} \subseteq F(X)$ such that $\Gamma \vdash_G \varphi$. Let $a \in G'(F'(X))$. Suppose that $\check{m}_H(\Gamma) \subseteq i^{G'}_{F'(X)}(a) = i^G_{H(F(X))}(j^H_{F'(X)}(a))$. We can see $\check{m}_H$ as an evaluation. Since the pair

$$\langle H(F'(X)), i^G_{H(F'(X))}(j^H_{F'(X)}(a))\rangle$$

is a matrix model of l_G, we have $\check{m_H}(\varphi) \in i^G_{H(F'(X))}(j^H_{F'(X)}(a)) = i^{G'}_{F(X)}(a)$. As a has been taken arbitrary, we conclude that $\check{m_H}[\Gamma] \vdash_{G'} \check{m_H}(\varphi)$.

Now consider the functor (see 1.2.2)

$$\begin{array}{rccl} \mathcal{N}': & \mathcal{L}_f & \to & \mathcal{F}i \\ & l & & (Fi_l, i^l) \\ & h\downarrow & \mapsto & h^\star\uparrow \\ & l' & & (Fi_{l'}, i^{l'}) \end{array}$$

Observe that given $l \in \mathcal{L}_f$, $\mathcal{N} \circ \mathcal{N}'(l) = \mathcal{N}((Fi_l, i^l)) = l_{Fi_l} = l$. Let $h \in hom_{\mathcal{L}_f}(l, l')$, then $\mathcal{N} \circ \mathcal{N}'(h) = \mathcal{N}((h^\star, j^\star))$, where $j^\star : Fi_{l'} \Rightarrow Fi_l \circ h^\star$ given by the inclusion, i.e., let $M \in \Sigma' - str$ and $F \in Fi_{l'}(M)$, $j^\star(F) = F$. Indeed $j^\star$ is well defined, let $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash_l \varphi$ and $v : X \to h^\star(M)$ where $F(X) = Fm$ such that $\bar{v}[\Gamma] \subseteq F$.

$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[rr, bend left, "\sigma'_X"] \arrow[dr, "v"'] & Fm \arrow[r, "\check{h}"] \arrow[d, "\bar{v}"] & h^\star Fm' \arrow[dl, "\bar{v}'"] \\ & h^\star M' & \end{tikzcd}$$

Consider σ_X and σ'_X the respective unit of adjunction between the free functor and forgetful functor over $\Sigma - str$ and $\Sigma - str$. Consider $\bar{v}' : F'm \to M$ the unique morphism, given by the universal property such that $\bar{v}' \circ \sigma'_X = v$. As $h^\star(F'm)$ has the same universe of $F'm$, we can see $\bar{v}' : h^\star(F'm) \to h^\star(M)$ as a morphism, i.e., $\bar{v}' = h^\star(\bar{v}')$. It holds that $\sigma'_X = \check{h} \circ \sigma_X$. Therefore, $\bar{v}' \circ \check{h} \circ \sigma_x = v$. Notice that $\bar{v}$ is the unique morphism such that $\bar{v} \circ \sigma_X = v$. Hence $\bar{v}' \circ \check{h} = \bar{v}$. Therefore, $\bar{v}' \circ \check{h}(\Gamma) \subseteq F$. As $F \in Fi_{l'}(M)$ and $\check{\Gamma} \vdash_{l'} \check{h}(\varphi)$, then $\bar{v}' \circ \check{h}(\varphi) \subseteq F$, hence $\bar{v}(\varphi) \in F$. Since v has been taken arbitrary, $F \in Fi_l(h^\star M)$.

Applying $\mathcal{N}((h^\star, j^\star)) = m_{h^\star} = h$. Then $\mathcal{N} \circ \mathcal{N}' = Id_{\mathcal{L}_f}$.

On the other hand $\mathcal{N}' \circ \mathcal{N}((G, i^G)) \neq (G, i^G)$,

On the class of filter pairs one can define the following relation:

$$(G, i^G) \sim (G', i^{G'}) \quad iff \quad G, G' : \Sigma - str \to AL \; and \; i^G_{Fm} = i^{G'}_{Fm}$$

The relation $\sim$ is a equivalence relation.

This relation means that if two filter pairs are in correspondence, then they define the same logic.

So we have that $\mathcal{N}' \circ \mathcal{N}((G, i^G))$ and (G, i^G) are in the same class with respect to $\sim$.

The fuctors $\mathcal{N}$ and $\mathcal{N}'$ give us, in a similar way as in chapter 2, a "codification" for morphisms in $\mathcal{L}_f$.

Moreover, these functors establish a adjunction $\mathcal{N}' \dashv \mathcal{N}$. Indeed, let (G, i^G) be a filter pair. We have seen that $\mathcal{N}' \circ \mathcal{N}((G, i^G)) = \mathcal{N}'(l_G) = (Fi_{l_G}, i)$. Given l a logic and (H, j) :

$(G, i^G) \to \mathcal{N}'(l)$, we have that $m_H : l \to l_G$ is a morphism in $\mathcal{L}_f$ such that the following diagram commutes:

$$\begin{tikzcd}
(G, i^G) \arrow[r, "{(Id,i^G)}"] \arrow[dr, "{(H,j)}"'] & (Fi_{l_G}, i) \arrow[d, "{(m_H^* = H,i)}"] \\
& (Fi_l, i)
\end{tikzcd}$$

So we have $\mathcal{N}' \dashv \mathcal{N}$. Moreover we can so see $\mathcal{L}_f$ as a reflexive subcategory of $\mathcal{F}i$ and then, by Proposition 5.3.1 in [Bor94] we have that there exist the category of fractions such that is equivalente to $\mathcal{N}'$ where Σ (a class of morphisms such that the category of fractions exists) is all morphism $f \in \mathcal{L}_f$ such that $\mathcal{N}'(f)$ is a isomorphism.

Chapter 4

Institutions for propositional logics and an abstract approach to Glinvenko's theorem

The notion of *Institution* was introduced for the first time by Goguen and Burstall in [GB92]. This concept formalizes the informal notion of logical system into a mathematical object. The main (model-theoretical) characteristic is that an institution contains a satisfaction relation between models and sentences that are coherent under change of notation: That motivated us to consider an institution of a logic, i.e., an institution for a propositional logic l represents all logic l' such that is *equipollent* with l ([CG07]). A variation of the formalism of institutions, the notion of π-*Institution*, were defined by Fiadeiro and Sernadas in [FS88] providing an alternative (proof-theoretical) approach to deductive system. In [FS88] and [Vou02] was showed a way to relate institutions with π-institutions. On the best of our knowledge, there is no literature on categorial connections between the category of institutions and the category of π-institutions. Here, we provide a categorial relationship using the well-known relation between objects of those categories, more precisely, in section 1 we determine a pair of adjoint functors between those categories.

Connecting those abstract logical settings with the notions presented in the previous chapters of the present thesis, we introduce, in the subsequent sections, institutions for abstract logics, algebraizable logics and Lindenbaum algebraizable logics. Concerning the latter, we present the definition of a *Glivenko's context* between two algebraizable logics. Recalling the classical Glivenko's theorem, proved by Valery Glivenko in 1929 that says one can translate the classical logic into intuitionistic logic by means double-negation of classical formulas, we prove in 4.3.6 (4.3.12) that for each Glivenko's context relating two algebraizable logics (respectively, Lindenbaum algebraizable logics), can be associated a institutions morphism between the corresponding logical institutions . Moreover, in 4.3.7 (4.3.13) we have that a Glivenko's context between institutions of algebraizable logics (Lindenbaum algebraizable logics) provides an abstract Glivenko's theorem between those logics, generalizing the results presented in [Tor08]. In particular, considering the institutions of classical logic and of intuitionistic logic 4.3.8, we build a Glivenko's context and thus an abstract Glivenko's theorem such that is exactly the traditional Glivenko's theorem.

4.1 Categorial relationship between Institution and Π-Institution

In this part of the work we establish an adjunction between the category of Institutions $Inst$ and the category of π-Institution $\pi - Inst$. We are going to present now the correspondence between the objects in those categories, but this relation is not original. One can find it in [Vou02], [FS88]. The new result here is the relationship between their morphisms.

We start giving the definition of institution and π-institution with their respective notions of morphisms (and comorphisms), and consequently their categories.

Definition 4.1.1. *An Institution* $I = (\mathbb{S}ig, Sen, Mod, \models)$ *consists of*

$$\begin{tikzcd} & \mathbb{S}ig \arrow[dl, "Mod"'] \arrow[dr, "Sen"] & \\ (\mathbb{C}at)^{op} & \models & \mathbb{S}et \end{tikzcd}$$

1. *a category* $\mathbb{S}ig$*, whose the objects are called* signature*,*
2. *a functor* $Sen : \mathbb{S}ig \to \mathbb{S}et$*, for each signature a set whose elements are called* sentence *over the signature*
3. *a functor* $Mod : (\mathbb{S}ig)^{op} \to \mathbb{C}at$*, for each signature a category whose the objects are called* model*,*
4. *a relation* $\models_\Sigma \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$ *for each* $\Sigma \in |\mathbb{S}ig|$*, called* Σ-satisfaction*, such that for each morphism* $h : \Sigma \to \Sigma'$*, the compatibility condition*

$$M' \models_{\Sigma'} Sen(h)(\phi) \ if \ and \ only \ if \ Mod(h)(M') \models_\Sigma \phi$$

holds for each $M' \in |Mod(\Sigma')|$ *and* $\phi \in Sen(\Sigma)$

Example 4.1.2. *For each pair of cardinals* $\aleph_0 \leq \kappa, \lambda \leq \infty$*, the category of languages* $L = (C, F, R)$[1] *and language morphisms, endowed with the usual notion of* $L_{\kappa,\lambda}$*-sentences (=* $L_{\kappa,\lambda}$*-formulas with no free variable), with the usual association of category of structures and with the usual (tarskian) notion of satisfaction, gives rise to an institution* $I(\kappa, \lambda)$*.*

[1] Where C is a set of symbols of constants, F is a set of symbols of (finitary) function symbols and R is a set of symbols of (finitary) relation symbols.

Definition 4.1.3. *Let I and I' be institutions.*

(a) An Institution **morphism** *$h = (\Phi, \alpha, \beta) : I \to I'$ consists of*

$$\begin{tikzcd}
 & \mathbb{S}ig \arrow[dl, bend right, "Sen"'] \arrow[d, "\Phi"] \arrow[dr, bend left, "(Mod)^{op}"] \arrow[dl, bend left] \arrow[dr, bend right] & \\
\mathbb{S}et & \mathbb{S}ig' \arrow[l, "Sen'"] \arrow[r, "Mod'^{op}"'] & \mathbb{C}at^{op}
\end{tikzcd}$$

1. *a functor $\Phi : \mathbb{S}ig \to \mathbb{S}ig'$*

2. *a natural transformation $\alpha : Sen' \circ \Phi \Rightarrow Sen$*

3. *a natural transformation $\beta : Mod \Rightarrow Mod' \circ \Phi^{op}$*

such that the following compatibility condition *holds:*

$$m \models_\Sigma \alpha_\Sigma(\varphi') \ \ iff \ \ \beta_\Sigma(m) \models'_{\Phi(\Sigma)} \varphi'$$

For any $\Sigma \in \mathbb{S}ig$, any Σ-model m and any $\Phi(\Sigma)$-sentence φ'.

(b) A triple $f = \langle \phi, \alpha, \beta \rangle : I \to I'$ is a **comorphism** *between the given institutions if the following conditions hold:*

- *$\phi : \mathbb{S}ig \to \mathbb{S}ig'$ is a functor.*
- *$\alpha : Sen \Rightarrow Sen' \circ \phi$ and $\beta : Mod' \circ \phi^{op} \Rightarrow Mod$ are natural transformations such that satisfy:*

$$m' \models'_{\phi(\Sigma)} \alpha_\Sigma(\varphi) \ iff \ \beta_\Sigma(m') \models_\Sigma \varphi$$

For any $\Sigma \in \mathbb{S}ig$, $m' \in Mod'(\phi(\Sigma))$ and $\varphi \in Sen(\Sigma)$.

Example 4.1.4. *Given two pairs of cardinals (κ_i, λ_i), with $\aleph_0 \leq \kappa_i, \lambda_i \leq \infty$, $i = 0, 1$, such that κ_0, κ_1 and $\lambda_0 \leq \lambda_1$, then it is induced a morphism and a comorphism of institutions $(\Phi, \alpha, \beta) : I(\kappa_0, \lambda_0) \to I(\kappa_1, \lambda_1)$, given by the* same data*: $\mathbb{S}ig_0 = Lang = \mathbb{S}ig_1$, $Mod_0 = Mod_1 : (Lang)^{op} \to \mathbb{C}at$, $\Phi = Id_{Lang} : \mathbb{S}ig_0 \to \mathbb{S}ig_1$, $\beta := Id : Mod_i \Rightarrow Mod_{1-i}$, $\alpha := inclusion : Sen_0 \Rightarrow Sen_1$.*

Given $f : I \to I'$ and $f' : I \to I''$ comorphisms of institutions, then $f' \bullet f := \langle \phi' \circ \phi, \alpha' \bullet \alpha, \beta' \bullet \beta \rangle$ defines a comorphism $f' \bullet f : I \to I''$, where $(\alpha' \bullet \alpha)_\Sigma = \alpha'_{\phi(\Sigma)} \circ \alpha_\Sigma$ and $(\beta' \bullet \beta)_\Sigma = \beta_\Sigma \circ \beta'_{\phi(\Sigma)}$. Let $Id_I := \langle Id_{\mathbb{S}ig}, Id, Id \rangle : I \to I$. It is straitforward to check that these data determines a category[2]. We will denote by **Inst** this category of institutions

[2]As usual in category theory, the set theoretical size issues on such global constructions of categories can be addressed by the use of, at least, two Grothendieck's universes.

where the arrows are **comorphisms** of institutions. Of course, it can also be formed a category whose objects are institutions and the arrows are **morphisms** of institutions, but that will be less important here.

Definition 4.1.5. *A π-Institution $J = \langle \mathbb{S}ig, Sen, \{C_\Sigma\}_{\Sigma\in|\mathbb{S}ig|}\rangle$ is a triple with its first two components exactly the same as the first two components of an institution and, for every $\Sigma \in |\mathbb{S}ig|$, a closure operator $C_\Sigma : \mathcal{P}(Sen(\Sigma)) \to \mathcal{P}(Sen(\Sigma))$, such that the following coherence conditions holds, for every $f : \Sigma_1 \to \Sigma_2 \in Mor(\mathbb{S}ig)$:*

$$Sen(f)(C_{\Sigma_1}(\Gamma)) \subseteq C_{\Sigma_2}(Sen(f)(\Gamma)),\ \textit{for all } \Gamma \subseteq Sen(\Sigma_1).$$

Definition 4.1.6. *Let J and J' be π-institutions, $g = \langle \phi, \alpha\rangle : J \to J'$ is a* ***comorphism*** *between π-institution when the following conditions hold:*

- *$\phi : \mathbb{S}ig \to \mathbb{S}ig'$ is a functor*
- *$\alpha : Sen \Rightarrow Sen' \circ \phi$ is a natural transformation such that satisfies the compatibility condition:*

$$\varphi \in C_\Sigma(\Gamma) \Rightarrow \alpha_\Sigma(\varphi) \in C_{\phi(\Sigma)}(\alpha_\Sigma(\Gamma))\ \textit{for all } \Gamma \cup \{\varphi\} \subseteq \mathbb{S}ig(\Sigma)$$

Let $g : J \to J'$ and $g' : J' \to J''$ be morphisms of π-institutions. $g' \bullet g$ is defined as the two first components of composition of comorphisms of institutions. The identity morphism is given as the two first components of the comorphism identity of institution. We will denote by π-**Inst** the category of π-institutions and with arrows its comorphisms.

Example 4.1.7. *(a) The each of categories of propositional $\mathcal{L}_f$ and $\mathcal{L}_s$ is associated an π-institution J_f (respectively, J_s) in the following way:*
• $\mathbb{S}ig_f := \mathcal{L}_f$;
• $Sen_f : \mathbb{S}ig_f \to \mathbb{S}et$, given by $(f : (\Sigma, \vdash) \to (\Sigma', \vdash)) \mapsto (\check{f} : F_\Sigma(X) \to F_{\Sigma'}(X))$;
• For each $l = (\Sigma, \vdash) \in |\mathbb{S}ig_f|$, $C_l : P(F_\Sigma(X)) \to P(F_\Sigma(X))$ is given by $C_l(\Gamma) := \{\phi \in F_\Sigma(X) : \Gamma \vdash_l \phi\}$, for each $\Gamma \subseteq F_\Sigma(X)$.

(b) The "inclusion" functor $(+)_L : \mathcal{L}_s \to \mathcal{L}_f$, mentioned in the section 2 of Chapter 1, induces a comorphism (and also a morphism!) of the associated π-institutions $(+) := ((+)_L, \alpha^+) : J_s \to J_f$, where, for each $l = (\Sigma, \vdash) \in \mathbb{S}ig_s = \mathcal{L}_s$, $\alpha^+(l) = Id_{F_\Sigma(X)} : F_\Sigma(X) \to F_\Sigma(X)$.[3]

[3] A lateral question in this chapter, that is interesting by its own, is understand the role of the adjoint functor $(-)_L : \mathcal{L}_f \to \mathcal{L}_s$ in the π-institutional level J_f, J_s.

In order to establish the adjunction between $Inst$ and $\pi - Inst$ we introduce the following: Let $I = \langle \mathbb{S}ig, Sen, Mod, \models \rangle$ be an institution. Given $\Sigma \in |\mathbb{S}ig|$, consider

$$\Gamma^\star = \{m \in Mod(\Sigma);\ m \models_\Sigma \varphi \ for\ all\ \varphi \in \Gamma\} \ \ and$$

$$M^\star = \{\varphi \in Sen(\Sigma);\ m \models_\Sigma \varphi \ for\ all\ m \in M\}$$

for any $\Gamma \subseteq Sen(\Sigma)$ and $M \subseteq Mod(\Sigma)$. Clearly, these mappings establishes a Galois connection. Thus $C^I_\Sigma(\Gamma) := \Gamma^{\star\star}$, defines a closure operator for any $\Sigma \in |\mathbb{S}ig|$ ([Vou02]).

The following lemma describes the behavior of these Galois connections through institutions comorphisms.

Lemma 4.1.8. *Let $f = \langle \phi, \alpha, \beta \rangle : I \to I'$ an arrow in* **Inst**. *Then given $\Gamma \subseteq Sen(\Sigma)$ and $M \subseteq |Mod(\phi(\Sigma))|$, the following conditions holds:*

1) $\beta_\Sigma[(\alpha_\Sigma[\Gamma])^\star] \subseteq \Gamma^\star$

2) $\alpha_\Sigma[(\beta_\Sigma[M])^\star] \subseteq M^\star$

Proof: *1)* Let $m \in \beta_\Sigma[(\alpha_\Sigma[\Gamma])^\star]$. So there is $m' \in \alpha_\Sigma[\Gamma]^\star$ such that $\beta_\Sigma(m') = m$. As $m' \in \alpha_\Sigma[\Gamma]^\star$, hence $m' \models'_{\phi(\Sigma)} \alpha_\Sigma[\Gamma] \ \Leftrightarrow\ \beta_\Sigma(m') \models_\Sigma \Gamma \ \Leftrightarrow\ m \models_\Sigma \Gamma$. Then $m \in \Gamma^\star$.

2) Let $\varphi \in \alpha_\Sigma[(\beta_\Sigma[M])^\star]$. So there is $\psi \in \beta_\Sigma[M]^\star$such that $\alpha_\Sigma(\psi) = \varphi$. Since $\psi \in (\beta_\Sigma[M])^\star$, hence $\beta_\Sigma[m] \models_\Sigma \psi \ \Leftrightarrow\ m \models_{\phi(\Sigma)} \alpha_\Sigma(\psi) \ \Leftrightarrow\ m \models_{\phi(\Sigma)} \varphi$ for any $m \in M$. Therefore $\varphi \in M^\star$. □

Define the following application:

$$\begin{array}{rcl} F: \ \mathbf{Inst} & \longrightarrow & \pi - \mathbf{Inst} \\ I & \longmapsto & F(I) = \langle \mathbb{S}ig, Sen, \{C^I_\Sigma\}_{\Sigma \in |\mathbb{S}ig|} \rangle \end{array}$$

In order to establish that F is well defined, it is enough to prove the compatibility condition for $\{C^I_\Sigma\}_{\Sigma \in |\mathbb{S}ig|}$, i.e., given $f : \Sigma_1 \to \Sigma_2$ and $\Gamma \subseteq Sen(\Sigma_1)$, then $Sen(f)(C^I_{\Sigma_1}(\Gamma)) \subseteq C^I_{\Sigma_2}(Sen(f)(\Gamma))$. Let $\varphi_2 \in Sen(f)(C^I_{\Sigma_1}(\Gamma))$, then there is $\varphi_1 \in \Gamma^{**}$ such that $Sen(f)(\varphi_1) = \varphi_2$. Let $m \in (Sen(f)(\Gamma))^*$. So $m \models_{\Sigma_2} Sen(f)(\Gamma)$. By compatibility condition in institutions we have that $Mod(f)(m) \models_{\Sigma_1} \Gamma$, thus $Mod(f)(m) \in \Gamma^*$. Since $\varphi_1 \in \Gamma^{**}$ we have that $Mod(f)(m) \models_{\Sigma_1} \varphi_1$, hence $m \models_{\Sigma_2} Sen(f)(\varphi_1) = \varphi_2$. Therefore $\varphi_2 \in (Sen(f)(\Gamma))^{**} = C^I_{\Sigma_2}(Sen(f)(\Gamma))$.

Now let $f = \langle \phi, \alpha, \beta \rangle : I \to I'$ be a comorphism of institutions. Then consider $F(f) = \langle \phi, \alpha \rangle$. Notice that $F(f)$ is a comorphism between $F(I)$ and $F(I')$. Indeed, in order to prove that, it is enough to prove that $F(f)$ satisfies the compatibility condition. Let $\Gamma \cup \{\varphi\} \subseteq Sen(\Sigma)$ for some $\Sigma \in |\mathbb{S}ig|$. Suppose that $\alpha_\Sigma(\varphi) \notin C^I_{\phi(\Sigma)}(\alpha_\Sigma[\Gamma])$. Hence $\alpha_\Sigma(\varphi) \notin \alpha_\Sigma[\Gamma]^{\star\star}$. Therefore $\alpha_\Sigma[\Gamma]^\star \not\models'_{\phi(\Sigma)} \alpha_\Sigma(\alpha)$. Thus there is $m \in \alpha_\Sigma[\Gamma]^\star$ such that $m \not\models'_{\phi(\Sigma)} \alpha_\Sigma(\varphi)$. Hence $\beta_\Sigma(m) \not\models_\Sigma \varphi$. Due to 4.1.8 1) we have that $\beta_\Sigma(m) \in \Gamma^{\star\star}$. Therefore $\varphi \notin \Gamma^{\star\star} = C^I_\Sigma(\Gamma)$.

Now let $f : I \to I'$ and $f' : I' \to I''$ comorphism of institutions. $F(f' \bullet f) = \langle \phi' \circ \phi, \alpha' \bullet \alpha\rangle = F(f') \bullet F(f)$ and $F(Id_I) = Id_{F(I)}$. Then F is a functor.

Consider now the application:

$$\begin{array}{rccl} G: & \pi-\mathbf{Inst} & \longrightarrow & \mathbf{Inst} \\ & J & \longmapsto & G(J) = \langle \mathbb{S}ig, Sen, Mod^J, \models^J \rangle \end{array} \quad \text{Where:}$$

• The two first components of the $\pi-$institution are preserved.

• $Mod^J : \mathbb{S}ig \to \mathbb{C}at^{op}$.

$Mod^J(\Sigma) := \{C_\Sigma(\Gamma);\ \Gamma \subseteq Sen(\Sigma)\} \subseteq P(Sen(\Sigma))$ is viewed as a poset category and, given $f : \Sigma \to \Sigma'$, $Mod^J(f) = Sen(f)^{-1}$.

$Mod^J(f)$ is well defined. Indeed: Let $\Gamma \subseteq Sen(\Sigma')$ and $\varphi \in C_\Sigma(Sen(f)^{-1}(C_{\Sigma'}(\Gamma)))$.

$$\begin{aligned} Sen(f)(\varphi) \in Sen(f)[C_\Sigma(Sen(f)^{-1}(C_{\Sigma'}[\Gamma]))] &\subseteq C_\Sigma[Sen(f)(Sen(f)^{-1}(C_\Sigma[\Gamma]))] \\ &\subseteq C_{\Sigma'}(C_{\Sigma'}[\Gamma]) = C_{\Sigma'}[\Gamma] \end{aligned}$$

Therefore $\varphi \in Sen(f)^{-1}(C_\Sigma[\Gamma])$. It is easy to see that Mod^J is a contravariant functor.

• Define $\models^J \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$ as a relation such that given $m \in Mod(\Sigma)$ and $\varphi \in Sen(\Sigma)$, $m \models^J_\Sigma \varphi$ if and only if $\varphi \in m$. Let $f : \Sigma \to \Sigma'$, $\varphi \in Sen(\Sigma)$ and $m' \in |Mod(\Sigma')|$.

$$\begin{aligned} Mod^J(f)(m') \models^J_\Sigma \varphi &\Leftrightarrow Sen(f)^{-1}(m') \models^J_\Sigma \varphi \\ &\Leftrightarrow \varphi \in Sen(f)^{-1}(m') \\ &\Leftrightarrow Sen(f)(\varphi) \in m' \\ &\Leftrightarrow m' \models^J_{\Sigma'} Sen(f)(\varphi) \end{aligned}$$

Therefore the compatibility condition is satisfied and then we have that $G(J)$ is a institution.

Now let $h = \langle \phi, \alpha\rangle : J \to J'$ be a comorphism of π-institution. Define for any $\Sigma \in |\mathbb{S}ig|$ $\beta_\Sigma : Mod^{J'} \circ \phi(\Sigma) \to Mod^J(\Sigma)$ where $\beta_\Sigma(m) = \alpha_\Sigma^{-1}(m)$. We prove that β_Σ is well defined, i.e., $\alpha_\Sigma^{-1}(m) \in Mod^J(\Sigma)$. Let $\varphi \in C_\Sigma(\alpha_\Sigma^{-1}(m))$. h is a morphism of π-institution, then $\alpha_\Sigma(\varphi) \in C_{\phi(\Sigma)}(\alpha_\Sigma(\alpha_\Sigma^{-1}(m))) \subseteq C_{\phi(\Sigma)}(m) = m$. Therefore $\varphi \in \alpha_\Sigma^{-1}(m)$.

Now we are going to prove that β is a natural transformation. Let $f : \Sigma_1 \to \Sigma_2$. Since α is a natural transformation, observe that the following diagram commutes:

$$\begin{tikzcd} P(Sen(\Sigma_1)) & P(Sen'(\phi(\Sigma_1))) \arrow[l, "\alpha_{\Sigma_1}^{-1}"'] \\ P(Sen(\Sigma_2)) \arrow[u, "Sen(f)^{-1}"] & P(Sen'(\phi(\Sigma_2))) \arrow[l, "\alpha_{\Sigma_2}^{-1}"] \arrow[u, "Sen'(\phi(f))^{-1}"'] \end{tikzcd}$$

Using this commutative diagram we are able to prove that the following diagram commutes:

$$\begin{tikzcd} Mod^{J'} \circ \phi(\Sigma_1) \arrow[r, "\beta_{\Sigma_1}"] & Mod^J(\Sigma_1) \\ Mod^{J'} \circ \phi(\Sigma_2) \arrow[u, "Mod^{J'}(\phi(f))"] \arrow[r, "\beta_{\Sigma_2}"'] & Mod^J(\Sigma_2) \arrow[u, "Mod^J(f)"'] \end{tikzcd}$$

Let $m \in Mod^{J'} \circ \phi(\Sigma_2)$.

$$\begin{aligned} Mod^J(f) \circ \beta_{\Sigma_2}(m) &= Mod^J(f)(\alpha_{\Sigma_2}^{-1}(m)) \\ &= Sen(f)^{-1}(\alpha_{\Sigma_2}^{-1}(m)) \\ &= \alpha_{\Sigma_1}^{-1}(Sen(\phi(f))^{-1}(m)) \\ &= \beta_{\Sigma_1}(Sen(\phi(f))^{-1}(m)) \\ &= \beta_{\Sigma_1} \circ Mod^{J'}(\phi(f))(m) \end{aligned}$$

$G(h) = \langle \phi, \alpha, \beta \rangle$ is a comorphism of institution. Indeed, it is enough to prove the compatibility condition. Let $m \in Mod^{J'}(\phi(\Sigma))$ and $\varphi \in Sen(\Sigma)$.

$$\begin{aligned} m \models^{J'}_{\phi(\Sigma)} \varphi\alpha_\Sigma(\varphi) &\Leftrightarrow \alpha_\Sigma(\varphi) \in m \\ &\Leftrightarrow \varphi \in \alpha_\Sigma^{-1}(m) \\ &\Leftrightarrow \varphi \in \beta_\Sigma(m) \\ &\Leftrightarrow \beta_\Sigma(m) \models^J_\Sigma (m)\varphi \end{aligned}$$

It is easy to see that G is a functor.

Theorem 4.1.9. *The above defined functors $F : \mathbf{Inst} \to \pi-\mathbf{Inst}$ and $G : \pi-\mathbf{Inst} \to \mathbf{Inst}$, establish an adjunction $G \dashv F$ between the categories $\mathbf{Inst}$ and $\pi-\mathbf{Inst}$.*

Proof:

Define the application $\eta_J = \langle Id_{\mathbb{S}ig}, Id_{Sen} \rangle : J \to F(G(J))$ for each π-Institution $J = \langle \mathbb{S}ig, Sen, \{C_\Sigma\}_{\Sigma \in |\mathbb{S}ig|} \rangle$. This application is well define. Indeed, we prove that $C_\Sigma = C_\Sigma^{G(I)}$ for any $\Sigma \in |\mathbb{S}ig|$. By definition of the functor G, notice that given $\Sigma \in |\mathbb{S}ig|$ and $\Gamma \subseteq Sen(\Sigma)$, $C_\Sigma(\Gamma) \in \Gamma^\star = \{m \in Mod(\Sigma);\ m \models^J_\Sigma \Gamma\}$. Moreover $C_\Sigma(\Gamma) \subseteq m$ for every $m \in \Gamma^\star$. Then for any $\varphi \in Sen(\Sigma)$

$$\begin{aligned} \varphi \in C_\Sigma(\Gamma) &\Leftrightarrow \varphi \in m \ for\ all\ m \in \Gamma^\star \\ &\Leftrightarrow m \models^J_\Sigma \varphi \ for\ all\ m \in \Gamma^\star \\ &\Leftrightarrow \varphi \in \Gamma^{\star\star} = \{\psi \in Sen(\Sigma);\ \Gamma^\star \models^J_\Sigma \psi\} \\ &\Leftrightarrow \varphi \in C_\Sigma^{G(J)}(\Gamma). \end{aligned}$$

It is clear that $(\eta_J)_{J \in |\pi-\mathbf{Inst}|}$ is a natural transformation. Remains to prove that η_J satisfies the universal property for any $J \in |\pi - \mathbf{Inst}|$.

Let $h = \langle \phi, \alpha \rangle : J \to F(I)$ where $J = \langle \mathbb{S}ig, Sen, \{C_\Sigma\}_{\Sigma \in |\mathbb{S}ig|} \rangle$ is a $\pi-$institution, $I = \langle \mathbb{S}ig', Sen', Mod', \models' \rangle$ an institution and h a morphism of $\pi-$institution. Define $\bar{h} = \langle \phi, \alpha, \beta \rangle : G(J) \to I$ where the first two components are the same of h and given $\Sigma \in |\mathbb{S}ig|$, $\beta_\Sigma : Mod' \circ \phi(\Sigma) \to Mod^J(\Sigma)$ such that $\beta_\Sigma(m) = \alpha_\Sigma^{-1}[m^\star]$. β_Σ is well defined. Indeed, notice that $m^\star = m^{\star\star\star}$ for any $m \in Mod'(\phi(\Sigma))$. Since $C^I_\Sigma(\Gamma) = \Gamma^{\star\star}$, therefore $m^\star = C^I_\Sigma(m^\star)$. We have shown that as h is a morphism of $\pi-$institution, $\alpha_\Sigma^{-1}(m^\star) = \alpha_\Sigma^{-1}(C^I_\Sigma(m^\star)) \in Mod^J$.

Now we prove that $(\beta_\Sigma)_{\Sigma\in|\mathbb{S}ig|}$ is a natural transformation. Let $f : \Sigma_1 \to \Sigma_2$. Then given $m \in Mod' \circ \phi(\Sigma_2)$

$$\begin{tikzcd}
Mod' \circ \phi(\Sigma_1) \arrow[r, "\beta_{\Sigma_1}"] & Mod^J(\Sigma_1) \\
Mod' \circ \phi(\Sigma_2) \arrow[r, "\beta_{\Sigma_2}"'] \arrow[u, "Mod'(\phi(f))"] & Mod^J(\Sigma_2) \arrow[u, "Mod^J(f)"']
\end{tikzcd}$$

$$\begin{aligned}
Mod^J(f)(\beta_{\Sigma_2}(m)) &= Sen(f)^{-1}(\alpha_{\Sigma_2}^{-1}(m^\star)) \\
&= \alpha_{\Sigma_1}^{-1}(Sen(\phi(f)^{-1})(m^\star)) \\
&=_\dagger \alpha_{\Sigma_1}^{-1}((Mod(\phi(f))(m^\star))^\star) \\
&= \beta_{\Sigma_1}(Mod(\phi(f))(m)).
\end{aligned}$$

The justification of the equality ($\dagger$) is:

$$\begin{aligned}
\varphi \in Sen(\phi(f))^{-1}(m^\star) &\Leftrightarrow Sen(\phi(f))(\varphi) \in m^\star \\
&\Leftrightarrow m \models_{\phi(\Sigma_2)} Sen(\phi(f))(\varphi) \\
&\Leftrightarrow Mod(\phi(f))(m) \models \Sigma_2\varphi \\
&\Leftrightarrow \varphi \in (Mod(\phi(f))(m))^\star
\end{aligned}$$

Hence β is a natural transformation. Therefore $\bar{h}$ is a comorphism between $G(I)$ and I. Observe that $F(\bar{h}) = \langle \phi, \alpha \rangle = h$. Then we have the following diagram commuting:

$$\begin{tikzcd}
J \arrow[r, "\eta_J"] \arrow[dr, "h"'] & F(G(J)) \arrow[d, "F(\bar{h})"] \\
& F(I)
\end{tikzcd}$$

Moreover, clearly $\bar{h}$ is the unique arrow such that the diagram above commutes. Hence $G \dashv F$. □

4.2 Institutions for abstract propositional logics

The "proof-theoretical" Example 4.1.7.(a), that provides a *π-institution* for a category of propositional logics, lead us to search an analogous "model-theoretical" version of it that is different from the canonical one (i.e., that obtained by applying the functor $G : \pi - \mathbf{Inst} \to \mathbf{Inst}$): In the first subsection of this section, we provide (another) *institution* for a category of propositional logics. That is naturally interesting because the theory of institutions was firstly used by computer scientist for first order logic.

However, the main motivation for the use of institution theory in this work is because it relates the sentences and models of a logic *independently of its presentations*, retaining only its "essence". More precisely, in the second subsection, we are going to define institutions for each (equivalence class of) algebraizable logic and Lindenbaum algebraizable logic: this will enable us to apply notions and results from institutions to study meta-logic properties of a (equivalence class of) well-behaved logic, as we will exemplify in the next section.

4.2.1 An institution for the abstract propositional logics

From to the category of logics $\mathcal{L}_f$, we define:

- $\mathbb{S}ig := \mathcal{L}_f$, the category of propositional logics $l = (\Sigma, \vdash)$ and flexible morphisms.

- $Sen : \mathbb{S}ig \to \mathbb{S}et$ where $Sen(l) = \mathcal{P}(F(\Sigma)) \times F(\Sigma)$ and given $f \in Mor_{\mathbb{S}ig}(l_1, l_2)$ then $Sen(f) : Sen(l_1) \to Sen(l_2)$ is such that $Sen(f)(\langle \Gamma, \varphi \rangle) = \langle \check{f}[\Gamma], \check{f}(\varphi) \rangle$. It is easy to see that Sen is a functor.

- $Mod : \mathbb{S}ig \to \mathbb{C}at^{op}$ where $Mod(l) = Matr_l$ and given $f \in Mor_{\mathbb{S}ig}(l_1, l_2)$, $Mod(f) : Matr_{l_2} \to Matr_{l_1}$ such that $Mod(f)(\langle M, F \rangle) = \langle f^\star(M), F \rangle$. $Mod(f)$ is well defined, indeed:

It is enough to prove that given $\langle M, F \rangle \in Matr_{l_2}$, then F is a l_1-filter in $f^\star(M)$. Let $\Gamma \cup \{\varphi\} \subseteq F(\Sigma_1)$ such that $\Gamma \vdash_1 \varphi$. Let $v : F(\Sigma_1) \to f^\star(M)$ and suppose that $v[\Gamma] \subseteq F$. We define $\bar{v} : F(\Sigma_2) \to M$ where $\bar{v}(x) = v(x)$ for all variable x and $\bar{v}(c_n(\psi_0, ..., \psi_{n-1})) = c_n^M(\bar{v}(\psi_0), ..., \bar{v}(\psi_{n-1}))$ for all formula $\varphi = c_n(\psi_0, ..., \psi_{n-1})$ where c_n is a n-ary connective. As we saw in the Chapter 2, the function $\check{f} : F(\Sigma_1) \to f^\star(F(\Sigma_2))$ is a morphism in $\Sigma_1 - Str$. Therefore the following diagram commutes

$$\begin{tikzcd} F(\Sigma_1) \arrow[r, "v"] \arrow[d, "\check{f}"'] & f^\star(M) \\ f^\star(F(\Sigma_2)) \arrow[ur, "\bar{v}"'] & \end{tikzcd}$$

This follows directly from Proposition 2.3.6, since $\check{f} = \eta_{f^\star}(X) : F(\Sigma_1)(X) \to f^\star(F(\Sigma_2)(X))$ is the unity of the adjunction between $\Sigma_1 - Str$ and $\Sigma_2 - Str$, described in Chapter 2. Anyway, we provide here a more explicit proof: For any variable x we have that $\bar{v} \circ \check{f}(x) = v(x)$. Now suppose that for a formula $c_n(\psi_0, ..., \psi_{n-1})$ we have $\bar{v} \circ \check{f}(\psi_i) = v(\psi_i)$ with $i \in \{0, ..., n-1\}$ then

$$\begin{aligned} \bar{v} \circ \check{f}(c_n(\psi_0, ..., \psi_{n-1})) &= \bar{v}(f(c_n)(\check{f}(\psi_0), ..., \check{f}(\psi_{n-1}))) \\ &= f(c_n)^M(\bar{v} \circ \check{f}(\psi_0), ..., \bar{v} \circ \check{f}(\psi_{n-1})) \\ &= c_n^{f^\star(M)}(\psi_0), ..., \bar{v} \circ \check{f}(\psi_{n-1})) \\ &= c_n^{f^\star(M)}(v(\psi_0), ..., v(\psi_{n-1})) \\ &= v(c_n(\psi_0, ..., \psi_{n-1})) \end{aligned}$$

Since $v[\Gamma] \subseteq F$ we have $\bar{v} \circ \check{f}[\Gamma] \subseteq F$. f is a morphism between logics, so $\check{f}[\Gamma] \vdash_2 \check{f}(\varphi)$. Since $\langle M, F \rangle \in Matr_2$ therefore $\bar{v} \circ \check{f}(\varphi) \in F$. Hence F is a filter of l_1.

- Given $l \in \mathbb{S}ig$ We define a relation $\models \subseteq |Mod(l)| \times Matr_l$ as:

Given $\langle M, F\rangle \in Mod(l)$ and $\langle \Gamma, \varphi\rangle \in Sen(l)$,

$$\langle M, F\rangle \models_l \langle \Gamma, \varphi\rangle \text{ } iff \text{ } for \text{ } all \text{ } v : F(\Sigma_l) \to M, \text{ } if \text{ } v[\Gamma] \subseteq F, \text{ } then \text{ } v(\varphi) \in F.$$

Now we prove that $\models$ satisfies the compatibility condition. Let $f : l \to l'$ be a morphism in $\mathbb{S}ig$, $\langle M', F'\rangle \in Mod(l')$ and $\langle \Gamma, \varphi\rangle \in Sen(l)$.

The universal property of $\check{f}$ defines a bijection:

$$v' \in \Sigma' - Str(F(\Sigma')(X), M') \leftrightsquigarrow v \in \Sigma - Str(F(\Sigma)(X), f^\star(M'))$$

such that the diagram of functions below commutes

$$\begin{tikzcd} f^\star(F(\Sigma')) \arrow[r, "v'"] & f^\star(M') \\ F(\Sigma) \arrow[u, "\check{f}"] \arrow[ur, "v"'] & \end{tikzcd}$$

Thus

$$\begin{aligned} \langle f^\star(M'), F'\rangle \models_l \langle \Gamma, \varphi\rangle \quad & iff \quad for\ all\ v : F(\Sigma) \to f^\star(M'),\ if\ v[\Gamma] \subseteq F',\ then\ v(\varphi) \in F' \\ & iff \quad for\ all\ v' : F(\Sigma') \to M',\ if\ v'[\check{f}[\Gamma]] \subseteq F',\ then\ v'(\check{f}(\varphi)) \in F' \\ & iff \quad \langle M', F'\rangle \models_{l'} \langle \check{f}[\Gamma], \check{f}(\varphi)\rangle \end{aligned}$$

Definition 4.2.1. *We denote by* $I_f = \langle \mathbb{S}ig, Sen, Mod, \models\rangle$ *the above defined institution of abstract propositional logics associated with* $\mathcal{L}_f$.

4.2.2 (Lindenbaum) algebraizable logics as institutions

In this section we define institutions for each (equivalence class of) algebraizable logic and Lindenbaum algebraizable logic: this will enable us to apply notions and results from institutions to study meta-logic properties of a (equivalence class of) well-behaved logic, as we will exemplify in the next section.

The institution of an algebraizable logic

Let $a = (\Sigma, \vdash)$ any algebraizable logic and Δ any of its a set of equivalence formulas. Given $\varphi \in F(\Sigma)$, consider φ/Δ the class of formulas ψ of a such that $\vdash \varphi\Delta\psi$ (this does not depend on the particular choice of Δ). If $\Gamma \subseteq F(\Sigma)$, still denote $\Gamma/\Delta := \{\varphi/\Delta;\ \varphi \in \Gamma\}$. Recall that $\overline{\mathcal{A}}_f$ denotes the quotient category of $\mathcal{A}_f$ by the congruence relation given by $f, f' : a_1 \to a_2$, $f \equiv f'$ iff for each $\varphi_1 \in F(\Sigma_1)$, $\vdash_2 \check{f}(\varphi_1)\Delta_2\check{f}'(\varphi_1)$, where Δ_2 is any equivalence formulas for a_2 (see Chapter 1, section 3).

Now fix a an algebraizable logic. Consider:

• $\mathbb{S}ig_a$ is the category whose objects are the algebraizable logics isomorphic to a in $\overline{\mathcal{A}}_f$ and the morphisms in $\mathbb{S}ig_a$ are the isomorphisms in $\overline{\mathcal{A}}_f$ (i.e., the equivalence class of$\mathcal{A}_f$-morphisms $f : a_1 \to a_2$ is such that there exists a $\mathcal{A}_f$-morphism $g : a_2 \to a_1$ such that $\vdash_1 \check{g} \circ \check{f}(\varphi_1)\Delta_2\varphi_1$ and $\vdash_2 \check{f} \circ \check{g}(\psi_2)\Delta_2\psi_2$, for each $\varphi_1 \in F(\Sigma_1), \psi_2 \in F(\Sigma_2)$).

• $Sen_a : \mathbb{S}ig_a \to \mathbb{S}et$ such that $Sen_a(a_1) = \mathcal{P}_{fin}(F(\Sigma_1)/\Delta_1) \times F(\Sigma_1)/\Delta_1$ and given $[h] : a_1 \to a_2$, $sen_a([h])(\langle\Gamma/\Delta_1, \varphi/\Delta_1\rangle) = \langle\check{h}[\Gamma]/\Delta_2, \check{h}(\varphi)/\Delta_2\rangle$. This is well defined because, if $h \equiv h'$, then for any $\varphi, \varphi' \in F(\Sigma_1)$, if $\varphi/\Delta = \varphi'/\Delta$ then $\vdash_1 \varphi\Delta_1\varphi'$ and since h, h' are represent the same morphism in $\overline{\mathcal{A}}_f$ we have that $\vdash_2 \check{h}(\varphi)\Delta_2\check{h}'(\varphi')$.

• $Mod_a : \mathbb{S}ig_a \to \mathbb{C}at^{op}$ is such that $Mod_a(a') := Matr^*_{a'}$ and given $[f] : a_1 \to a_2$ we define $Mod_a([f]) : Matr^*_{a_2} \to Matr^*_{a_1}$ where $Mod_a([f])(\langle M, F\rangle) := \langle f^\star M, F\rangle$, by Proposition 2.2.2, this does not depend on the particular representation of $[f]$. We must prove that Mod is well defined, i.e. that $\langle f^\star M, F\rangle$ is a reduced matrix. We saw in the previous subsection that F is a a_1-filter for $f^\star M$ thus, firstly, we prove that $\Omega^{f^\star M}(F)$ is a congruence in M. Let $(a_i, b_i) \in \Omega^{f^\star M}(F)$ such that $0 \leq i \leq n-1$ and c_n a n-ary connective in a_2; denote $c_n(\vec{x}) := c_n(x_0, \cdots, x_{n-1})$. As $[f]$ is a morphism in $\mathbb{S}ig_a$, then there exists $g : a_2 \to a_1 \in \mathcal{A}_f$ such that $\vdash_2 \check{f} \circ \check{g}(c_n(\vec{x}))\Delta_2 c_n(\vec{x})$. Since a_2 is algebraizable logic, we have that $\models_{QV(a_2)} \check{f} \circ g(c_n) \approx c_n(\vec{x})$. As $\langle M, F\rangle \in Matr^*_{a_2}$, then $M \in QV(a_2)$. Hence $g(c_n)^{f^\star M} = \check{f}(g(c_n))^M = c_n^M$. We know that $\Omega^{f^\star M}(F)$ is a congruence in $f^\star M$, thus $(c_n^M(a_0, ..., a_{n-1}), c_n^M(b_0, ..., b_{n-1})) = (g(c_n)^{f^\star M}(a_0, ..., a_{n-1}), g(c_n)^{f^\star M}(b_0, ..., b_{n-1})) \in \Omega^{f^\star M}(F)$. Therefore $\Omega^{f^\star M}(F)$ is a congruence on M. Moreover, it is compatible with F. Hence $\Omega^{f^\star M}(F) \subseteq \Omega^M(F) = Id_{|M| \times |M|}$. Then $\Omega^{f^\star M}(F) = Id_{|f^\star M| \times |f^\star M|}$, so $\langle f^\star M, F\rangle$ is a reduced matrix.

• To $\models$ we use here a similar definition as in the subsection above, namely given $\langle M, F\rangle \in Matr{*}_{a_1}$ and $\langle\Gamma/\Delta, \varphi/\Delta\rangle \in Sen_a(a_1)$ then $\langle M, F\rangle \models \langle\Gamma/\Delta, \varphi/\Delta\rangle$ iff for any valuation $v : F(\Sigma_1)(X) \to M$, if $v[\Gamma] \subseteq F$ then $v(\varphi) \in F$. As $M \in Qv(a_1)$, this is well-defined, i.e., if $\vdash \theta\Delta\theta'$ then $v(\theta) = v(\theta')$, since v factors uniquely through the quotient morphism $F(\Sigma_1)(X) \twoheadrightarrow F(\Sigma_1)(X)/\Delta$. The proof of the compatibility follows from the same way as in the subsection above.

Definition 4.2.2. *We denote by $InsAL_a = \langle\mathbb{S}ig_a, Sen_a, Mod_a, \models\rangle$ the above defined institution. This will be called the algebraizable institution of a.*

The institution of a Lindenbaum algebraizable logics

Before define the Institution of Lindenbaum algebraizable logics, we define a notion of satisfiability of class of formulas:

Definition 4.2.3. *Let a be algebraizable logic. Given $M \in QV(a)$, $M \models_{QV(a)} [\varphi] \approx [\psi]$ iff* *for every valuation* $v : F(\Sigma_a)(X) \to M$,

$$v(\varphi') = v(\psi') \ \ \textit{such that} \ \ \varphi' \dashv\vdash \varphi \ \ \textit{and} \ \ \psi' \dashv\vdash \psi$$

Remark 4.2.4. *If* $a \in Lind(\mathcal{A}_f)$ *then, since* $F(\Sigma_a)(X)/\dashv\vdash\, = F(\Sigma_a)(X)/\Delta$ *is the free* $QV(a)$*-structure on* X *(see Remark 2.1.4), then* $\models_{QV(a)} [\varphi] \approx [\psi] \ \Leftrightarrow \models_{QV(a)} \varphi \approx \psi$.

Given $a \in Lind(\mathcal{A}_f)$. Consider the following maps:

• $\mathbb{S}ig'_a$ is the category whose the objects are $a_1 = (\Sigma_1, \vdash_1) \in Lind(\mathcal{A}_f)$, that are isomorphic to a in the quotient category $QLind(\mathcal{A}_f) = Q(\mathcal{A}^c_f)$ and the morphisms are only the isomorphisms in $QLind(\mathcal{A}_f)$.

• $Mod'_a : \mathbb{S}ig'^{op}_a \to \mathbb{C}at$ such that $Mod'_a(a_1) = QV(a_1)$ for all $a_1 \in |\mathbb{S}ig'_a|$ and $Mod'_a(a_1 \stackrel{[h]}{\to} a_2) = (QV(a_2) \stackrel{h^\star}{\to} QV(a_1))$ (see Corollary 2.2.3).

• We define now the functor $Sen'_a : \mathbb{S}ig'_a \to \mathbb{S}et$.

Let $a_1 \in |\mathbb{S}ig'_a|$. The idea here is to describe a convenient set of tuples that represents quasi-equations in Σ_1 (i.e., $Eq_0 \wedge ... \wedge Eq_{n-1} \to Eq$).

For each $s = ([\varphi_0], \cdots, [\varphi_{n-1}], [\psi])$, a non-empty finite sequence in $F(\Sigma_1)/\dashv\vdash$ (the free $QV(a_1)$-structure on the set X) and each (τ, Δ), an algebraizable pair of a_1, where $\tau = \{(\varepsilon^j, \delta^j);\ j = 1, ..., m \ for \ some \ m \in \omega\}$, let

$$q(s, (\Delta, \tau)) := (([\varepsilon(\varphi_0)], [\delta(\varphi_0)]), \cdots, ([\varepsilon(\varphi_{n-1})], [\delta(\varphi_{n-1})]), ([\varepsilon(\psi)], [\delta(\psi)]))$$

where the *notation* $([\varepsilon(\theta)], [\delta(\theta)])$ abbreviates the pair of finite sequence of equivalence class of formulas: $([\varepsilon^j(\theta), [\delta^j(\theta)])_j$ *with* $j = 1, \cdots, m$.. Note that, as a_1 is a *congruential algebraizable logic*, then:

(*) If $[\theta] = [\theta']$ (i.e., $\theta \dashv\vdash \theta'$), then $\delta(\theta) \dashv\vdash \delta(\theta')$ and $\varepsilon(\theta) \dashv\vdash \varepsilon(\theta')$. Thus we have an well defined mapping $\varphi/\Delta \stackrel{t}{\mapsto} (\varepsilon(\varphi)/\Delta, \delta(\varphi)/\Delta)$ and $q(s, (\Delta, \tau))$ is well-defined;

(**) conversely, as $\varphi \dashv\vdash \Delta(\epsilon(\varphi), \delta(\varphi))$, then we have and well defined map $(\varepsilon(\varphi)/\Delta, \delta(\varphi)/\Delta) \stackrel{r}{\mapsto} \varphi/\Delta$ and $r \circ t = id$.

Define $q_s := \{q(s, (\tau, \Delta)) : (\tau, \Delta)$ is an algebraizable pair of $a_1\}$ and then take $Sen'_a(a_1) := \{q_s : s$ is a non-empty finite sequence in $F(\Sigma_1)/\Delta_1\}$. Note that, by the above remark, the mapping $s \stackrel{t}{\mapsto} q_s$ determine a *bijection* between the set of non-empty finite sequences in $F(\Sigma_1)/\Delta$ and $Sen'a(a_1)$

Let $[f] : a_1 \to a_2$ be an isomorphism in $QLind(\mathcal{A}_f)$, in particular $\check{f}/\dashv\vdash : F(\Sigma_1)/\dashv_1\vdash \to F(\Sigma_2)/\dashv_2\vdash$ is a bijection. Let $s = ([\varphi_0], \cdots, [\varphi_{n-1}], [\psi])$ be a non-empty finite sequence in $F(\Sigma_1)/\dashv_1\vdash$ and $((\varepsilon, \delta), \Delta)$ be an algebraizable pair of a_1. Then $f{*}s := ([\check{f}(\varphi_0)], \cdots, [\check{f}(\varphi_{n-1})], [\check{f}(\psi)])$ is a non-empty finite sequence in $F(\Sigma_2)/\dashv_2\vdash$ and the mapping

$$q(s, ((\varepsilon, \delta), \Delta)) \ \mapsto \ q(f * s, ((\check{f}(\varepsilon), \check{f}(\delta)), \check{f}(\Delta)))$$

determines a bijection: $f^+ : q_s \stackrel{\cong}{\to} q_{f*s}$.

Then $Sen'_a([f]) : Sen_a(a_1) \to Sen'_a(a_2)$ is given by $Sen'_a([f])(q_s) := q_{f*s}$ (this map is well defined). It is straitforward check that $Sen'_a : \mathbb{S}ig'_a \to \mathbb{S}et$ is a functor.

Just to simplify notation, from now on we will denote the any element of the set q_s by $(([\alpha_0], [\beta_0]), \cdots, ([\alpha_{n-1}], [\beta_{n-1}]), ([\alpha], [\beta])) = (([\varepsilon(\varphi_0)], [\delta(\varphi_0)]), ..., ([\varepsilon(\varphi_{n-1})]), ([\varepsilon(\psi)], [\delta(\psi)]))$.

• Given $a' \in \mathbb{S}ig_a$, $M' \in QV(a')$ and $q' \in Sen_a(a')$, we say that $M' \models^a q'$ when, for any (and thus for all!) element $(([\alpha'_0], [\beta'_0]), \cdots, ([\alpha'_{n-1}], [\beta'_{n-1}]), ([\alpha'], [\beta']))$ of q', if

$$M' \models_{QV(a')} [\alpha'_i] \approx [\beta'_i] \ \forall\ i = 0, ..., n-1$$

then

$$M' \models_{QV(a)} [\alpha'] \approx [\beta']$$

Let $[f] : a_1 \to a_2 \in \mathbb{S}ig'_a$, $M_2 \in QV(a_2)$ and $q \in Sen'_a(a_1)$. Then, as $[f] : a_1 \to a_2$ which is a isomorphism in $QLind(\mathcal{A}_f)$, then it is easy to see that

$$M_2 \models^a Sen(f)(q) \ \Leftrightarrow\ Mod(f)(M_2) \models^a q$$

Definition 4.2.5. *Then we have that $InsLAL_a = \langle \mathbb{S}ig'_a, Sen'_a, Mod'_a, \models' \rangle$ is a institution called the Lindenbaum institution of a.*

Remark 4.2.6. *As can be easily checked, each Lindenbaum algebraizable logic a, determines the following* comorphism of institutions*: $h^a = (\Phi^a, \alpha^a, \beta^a) : InsLAL_a \to InsAL_a$, where:*

• *$\Phi^a : \mathbb{S}ig'_a \to \mathbb{S}ig_a$ consists of inclusion of categories: $\Phi^a(a_1 \overset{[h_1]}{\to} a_2) = a_1 \overset{[h_1]}{\to} a_2$;*

• *$\beta^a : Mod_a \circ (\Phi^a)^{op} \Rightarrow Mod'_a$, given by, for each $a_1 \in |\mathbb{S}ig'_a|$, $\beta^a(a_1) : Matr^*_a \to QV(a_1)$ is the forgetful functor;*

• *$\alpha^a : Sen'_a \Rightarrow Sen_a \circ \Phi^a$, given by, for each $a_1 \in |\mathbb{S}ig'_a|$, for each $q \in Sen'_a(a_1)$, let $s = ([\varphi_0], \cdots, [\varphi_{n-1}], [\psi])$ be the* unique *non-empty finite sequence in $F(\Sigma_1)/ \dashv\vdash$ such that $q = q_s$, then $\alpha^a(a_1)(q) := (\{[\varphi_0], \cdots, [\varphi_{n-1}]\}, [\psi]) \in \mathcal{P}_{fin}(F(\Sigma_1)/\Delta_) \times F(\Sigma_1)/\Delta = Sen_a(a_1)$.*

• *It holds the compatiblitity condition: for each $a_1 \in |\mathbb{S}ig'_a|$, each $(M, F) \in |Mod_a(\Phi^a(a_1)| = Matr^*(a_1)$ and each $q_s \in Sen'_a(a_1)$*

$$(M, F) \models^{I_a} \alpha^a(q_s) \ iff\ M \models^{I'_a} q_s$$

And this follows from:

(+) For each $v : X \to M$ and $\varphi \in F(\Sigma_1)$:

$$v(\varphi) \in F \ iff\ v(\varepsilon(\varphi)) = v(\delta(\varphi))^4$$

Remark 4.2.7. *One can ask "why do use different notion of institution of a Lindenbaum algebraizable logic instead of the restrict the notion of institution of algebraizable logic to the*

[4] Indeed, as $\varphi \dashv\vdash \Delta(\epsilon(\varphi), \delta(\varphi))$, *then* $v(\varphi) \in F\ iff\ v(\Delta(\epsilon(\varphi), \delta(\varphi))) \in F\ iff\ (v(\epsilon(\varphi)), v(\delta(\varphi))) \in \Omega^M(F) = id$.

class of Lindenbaum algebraizable logic?" The answer to this question is that those institutions seem not be isomorphic, but there are notions of abstract Glivenko's theorem for both of them. This means that we have two different approaches to abstract Glivenko's theorem as follow in the next section. We believe that those two different approaches for the abstract Glivenko's theorem can be applied for special classes of logics, for instance we can use the idea behind of the institution for an algebraizable logic as 4.2.2 to provide an institution for an equivalential logic. On the other hand, we can use the idea behind of the institution for a Lindenbaum algebraizable logic as 4.2.2 to provide an institution for a truth-equational logic.

4.3 The abstract Glivenko's theorem

The Glivenko's theorem allows one translate the classical logic into the intuitionistic logic by means double negation. More precisely, if Σ be a commom signature for expressing presentations of classical propositional logic (CPC) and intuitionistic propositional logic (IPC) – for instance, $\Sigma = \{\neg, \rightarrow, \wedge, \vee\}$– and $\Gamma \cup \{\varphi\} \subseteq F(\Sigma)$, then $\Gamma \vdash_{CPC} \varphi$ iff $\neg\neg\Gamma \vdash_{IPC} \neg\neg\varphi$. Here we generalize the Glivenko's theorem between arbitrary algebraizable logics (Lindenbaum algebraizable logics) using the ideas and notions of the Institution Theory applied to the former defined institutions for algebraizable logics (Lindenbaum algebraizable logics).

Remark 4.3.1. *(a) Recalling the Remark 2.2.9:*
Let $a = IPC$ and $a' = CPC$ both Lindenbaum algebraizable logics with the same signature. We have the "inclusion" morphism $h : IPC \to CPC$. Denote BA and HA, the quasivarieties of Boolean algebras and of Heyting algebras on that commom signature. So $h^\star_\restriction = incl : BA \to HA$ has left a adjoint functor $G : HA \to BA$. Observe that $h^\star_\restriction$ is the inclusion functor. Hence given $H \in HA$, $G(H) = H/F_H$, where F_H is the filter in H generated by the subset $\{a \leftrightarrow \neg\neg a : a \in H\}$, and the quotient HA-homomorphism $q_H : H \twoheadrightarrow incl(G(H))$ is the H-component of the unity of this adjunction. It is possible to proof that $G(H) \cong H_{\neg\neg}$, where $H_{\neg\neg}$ denote the (boolean algebra) of regular elements of H, that is, those elements $x \in H$ such that $\neg\neg x = x$. Moreover, the surjective HA-homomorphism $x \in H \mapsto \neg\neg x \in H_{\neg\neg}$ has HA-section $H_{\neg\neg} \mapsto \neg\neg y \in H$.

(b) Let $h : a \to a' \in \mathcal{A}_f$. Then $h^\star$ and $h^\star\restriction$ have respective left adjoints L_h and $\bar{L_h}$. Consider $\partial : Id \Rightarrow h^\star \circ L_h$ and $\bar{\partial} : Id \Rightarrow h^\star\restriction \circ\bar{L_h}$ the units of the adjunctions between $h^\star, L_h$ and $h^\star\restriction, \bar{L_h}$ respectively. Given $X \in \mathbb{S}et$ the following diagram commute: (Here $\partial_X = \partial_{FX} = \check{h}$. The same for $\bar{\partial}$)

$$\begin{tikzcd} FX \arrow[r, "\partial_X"] \arrow[d] & F'X \arrow[d] \\ FX/\Delta \arrow[r, "\bar{\partial}_X"'] & F'X/\Delta' \end{tikzcd}$$

Due to Proposition 2.3.6, $\partial_X = \check{h}$. *Moreover, observe that* $\bar{\partial}_X$ *and* $[\check{h}] : FX/\Delta \to h^{\star}\restriction (F'X/\Delta')$ *both satisfies the universal property, so there exist an isomorphism between* $\bar{L}_h(FX/\Delta)$ *and* $F'X/\Delta$. *With this we can consider* $\bar{\partial}_X : FX/\Delta \to h^{\star}\restriction (F'X/\Delta')$

Now we are ready to propose the following

Definition 4.3.2. *A* ***Glivenko's context*** *is a pair* $\mathbb{G} = (h : a \to a', \bar{\rho})$ *where* $h \in \mathcal{A}_f(a, a')$ *and* $\bar{\rho} : h^{\star}\restriction \circ L_h \Rightarrow Id$ *is a natural transformation that is a* section *of the unit* $\bar{\partial} : Id \Rightarrow h^{\star}\restriction \circ L_h$).

Remark 4.3.3. *Let* $\mathbb{G} = (h : a \to a', \bar{\rho})$ *is a* ***Glivenko's context*** *then:*

(a) $[\check{h} = \bar{\partial}_X : FX/\Delta \to h^{\star}\restriction (F'X/\Delta')$ *is a surjective homomorphism thus* h *is a* Δ*-dense morphism (see also Propositions and 2.1.6 2.3.12). For each* $Y \subseteq X$, *can be chosen (non naturally) a "lifting"* $\rho_Y : F'Y \to FY$, *for each of the natural sections* $\bar{\rho}_Y : F'Y/\Delta_Y \to F'Y/\Delta_Y$*:*

$$\begin{tikzcd} FY \arrow[d] & F'Y \arrow[l, "\rho_Y"'] \arrow[d] \\ FY/\Delta & F'Y/\Delta' \arrow[l, "\bar{\rho}_Y"] \end{tikzcd}$$

$\bar{\partial}_X[\theta] = [\check{h}(\theta)]$, *for all* $\theta \in FX$.

(b) On the other hand, the condition of being a Δ*-dense on a* $\mathcal{A}_f$*-morphism* h *is not sufficient to ensure that* h *is part of a Glivenko's context: Consider a the "logic of abelian groups" and* a' *the "logic of groups" (see Chapter 1, section 3): both are algebraizable logics; then* $QV(a) = Ab, QV(a') = Gr$ *and, for each group* G, *the unity of this adjunction at* G *is the quotient homomorphism* $q_G : G \twoheadrightarrow incl(G/[G,G])$; *taking* $G = F(x, y)$, *the free group in 2 generators, then* $G/[G, G] \cong \mathbb{Z} \oplus \mathbb{Z}$ *is the free abelian group in 2 generators and is straitforward* $q_G : G \twoheadrightarrow incl(G/[G, G])$ *does not have a section! It will be interesting determine additional condition on a* Δ*-dense morphism, that ensures it be a part of a Glivenko's context.*

(c) Observe that for any $M' \in QV(a')$ *there is* $M \in QV(a)$ *such that* $L_h(M) \cong M'$*: indeed, as* $h : a \to a'$ *is a* Δ*-dense morphism, thus combination of the results 2.2.6 (or 2.2.7) and 2.1.6, ensures that* $h^{\star}\restriction: QV(a') \to QV(a)$ *is a full and faithfull functor with a left adjoint and a well known result on adjunctions, entails that the* co-unity *of the adjunction* κ *must be an isomorphism, thus* $\kappa_{M'} : L_h(h^{\star}(M')) \overset{\cong}{\to} M'$, *for each* $M' \in QV(a')$.

Remark 4.3.4. *If $\mathbb{G} = (h : a \to a', \bar{\rho})$ is a* ***Glivenko's context*** *then, taking $Y = \{x_0\} \subseteq X$, then $E_Y(x_0) \in F(Y)$ is a Σ'-formula in at most one variable x_0 such that $[x_0] = [\check{h}(\rho_Y(x_0))] \in F'(Y)/\Delta'$ and thus $[\rho_Y(x_0)] = [\rho_Y(\check{h}(\rho_Y(x_0)))] \in F(Y)/\Delta$.*

(Note that the formula $\neg\neg(x)$ appears as a "fixed formulas" in CPC and as an "idempotent formula" in IPC.) Conversely, give a "fixed formula" seems to be also a sufficient condition for exists a Glivenko's context, i.e. give a Σ_a-formula in at most variable x_0, $\theta(x_0)$, such that $\vdash'_a x_0\Delta'(\check{h}(\theta(x_0))$. Further investigation is needed to establish (and explore) a precise relation between fixed/idempotent formulas and Glivenko's contexts.

4.3.5. *Let $\mathbb{G} = (h : a \to a', \bar{\rho})$ be a Glivenko's context and suppose that a_1 is an algebraizable logic and $[e_1] : a \to a_1$ is an isomorphism in the quotient category $\overline{\mathcal{A}_f}$. Let $[h_1] : a_1 \to a'$ be the unique $\overline{\mathcal{A}_f}$ such that the diagram below commutes*

$$\begin{tikzcd} a \arrow[r, "{[h]}"] \arrow[d, "{[e_1]}"'] & a' \arrow[d, "{[id'_a]}"] \\ a_1 \arrow[r, "{[h_1]}"'] & a' \end{tikzcd}$$

Then $h_1 : a_1 \to a$ is a Δ-dense morphism in $\mathcal{A}_f$.

From the choice of left adjoints of functors between quasivarieties induced by Δ-dense morphisms (see Chapter 2), we have the strict *equalities $L_{h_1} \circ L_{e_1} = L_{h_1 \circ e_1} = L_h$ and then also the diagram below commutes (L_{e_1} is the inverse isomorphism of $e_1^\star$)*

$$\begin{tikzcd} QV(a) \arrow[r, "L_h"] & QV(a') \\ QV(a_1) \arrow[r, "L_{h_1}"'] \arrow[u, "e_1^\star"] & QV(a') \arrow[u, "id_{QV(a')}"'] \end{tikzcd}$$

Thus, the (natural) section, $\bar{\rho}$, of the unity of the adjunction $L_h \dashv h$ induces uniquely a (natural) section, $\bar{\rho}^{a_1}$, of the unity of the adjunction $L_{h_1} \dashv h_1$.

In more details: if $M_1 \in QV(a_1)$ and $\partial^{a_1}_{M_1} : M_1 \twoheadrightarrow h_1^\star(L_{h_1}(M_1))$ is the (canonical) unity of $L_{h_1} \dashv h_1$ (remember that h_1 is Δ-dense, since h is Δ-dense and $[e_1]$ is an isomorphism), then

$$e_1^\star(\partial^{a_1}_{M_1}) : e_1^\star(M_1) \twoheadrightarrow e_1^\star(h_1^\star(L_{h_1}(M_1))) \ =$$

$$\partial^a_{e_1^\star(M_1)} : e_1^\star(M_1) \twoheadrightarrow h^\star(L_h(e_1^\star(M_1)))$$

Thus take $\bar{\rho}^{a_1}_{M_1} := L_{e_1}(\bar{\rho}_{e_1^\star(M_1)})$

4.3.1 The abstract Glivenko's theorem in *InsAL*

On the category *InsAL* we are going to present the abstract Glivenko's theorem through morphisms in this category.

Theorem 4.3.6. *Let a, a' be algebraizable logics, then each $\mathbb{G} = (h : a \to a', \rho)$* ***Glivenko's context*** *induces a institutions morphism $InsAL_a \to InsAL_{a'}$. More precisely, fixing a choice of isomorphisms $\varepsilon : Obj(\mathbb{S}ig_a) \to Mor(\mathbb{S}ig_a)$, $a_1 \mapsto \varepsilon(a_1) = [e_1] : a \overset{\cong}{\to} a_1$, we define a institution morphism $N_{(G,\varepsilon_a)} : InsAL_a \to InsAL_{a'}$*[5]

Proof:

By simplicity, we will write (G, ε) for (G, ε_a). We will define

$$N_{(G,\varepsilon)} = \langle \Phi^{(G,\varepsilon)}, \alpha^{(G,\varepsilon)}, \beta^{(G,\varepsilon)} \rangle$$

(this will depend only on the choice of isomorphisms in the *domain* institution $InsAL_a$):

- $\Phi^{(G,\varepsilon)} : \mathbb{S}ig_a \to \mathbb{S}ig_{a'}$

The object part of $\Phi^{(G,\varepsilon)}$ is easy do define: for $a_1 \in |\mathbb{S}ig_a|$, set $\Phi^{(G,\varepsilon)}(a_1) := a'$.

It follows from adaptations of results in [AFLM07] and [MM14] that $\overline{\mathcal{A}_f}$ is a finitely accessible category that has all colimits (except initial object) and is relatively complete (i.e, has limits for all diagrams that admits a cone). In particular $\overline{\mathcal{A}_f}$ has pushouts, and for each $\overline{\mathcal{A}_f}$-isomorphism $[f] : a \to a$, we consider the following pushout

$$\begin{tikzcd} a \arrow[r, "{[h]}"] \arrow[d, "{[f]}"'] & a' \arrow[d, "{[f'_1]}"] \\ a \arrow[r, "{[h_1]}"'] & a'_1 \end{tikzcd}$$

As a pushout of an iso is an iso and a pushout of an epi is an epi (recall that h is a Δ-dense morphisms, i.e., $[h]$ is an epi), we may suppose that the vertex of the pushout is a', $[f^h] : a' \to a'$ is an isomorphism and the diagram below commutes[6]

$$\begin{tikzcd} a \arrow[r, "{[h]}"] \arrow[d, "{[f]}"'] & a' \arrow[d, "{[f^h]}"] \\ a \arrow[r, "{[h]}"'] & a' \end{tikzcd}$$

Note that, as $[h]$ is an epi, then $[f^h]$ is uniquely determined.

Now let $a_1, a_2 \in \mathbb{S}ig_a$ and $[g] : a_1 \to a_2$ be an arrow in $\mathbb{S}ig_a$ (i.e., $[g]$ is a $\overline{\mathcal{A}_f}$-isomorphism). Then, as $e_i : a \to a_i$ is an isomorphism, $i = 1, 2$, then there is a unique isomorphism $[g_\varepsilon] : a \to a'$ such that left diagram below commutes.

$$\begin{tikzcd} a_1 \arrow[d, "{[g]}"'] & a \arrow[l, "{[e_1]}"'] \arrow[r, "{[h]}"] \arrow[d, "{[g_\varepsilon]}"] & a' \arrow[d, "{[g^h_\varepsilon]}"] \\ a_2 & a \arrow[l, "{[e_2]}"] \arrow[r, "{[h]}"'] & a' \end{tikzcd}$$

[5]Such induced morphisms are "isomorphic", for different choices of isomorphisms $\varepsilon^0, \varepsilon^1$.

[6]In this case, this is a necessary and sufficient condition to be a pushout.

Then define $\Phi^{(G,\varepsilon)}([g] : a_1 \to a_2) := [g^h_\varepsilon] : a' \to a'$. As $[g_\varepsilon]$ and $[g^h_\varepsilon]$ are *uniquely determined* by g, it follows that $\Phi^{(G,\varepsilon)}$ preserves identities and composition of arrows in $\mathbb{S}ig_a$, thus being a functor.

- $\alpha^{(G,\varepsilon)} : Sen_{a'} \circ \Phi^{(G,\varepsilon)} \Rightarrow Sen_a$ where, for a we have that $\alpha^{(G,\varepsilon)}(a) : Sen_{a'} \circ \Phi^{(G,\varepsilon)}(a) = Sen_{a'}(a') \to Sen_a(a)$ such that $\alpha^{(G,\varepsilon)}(a)(\langle \Gamma'/\Delta', \varphi'/\Delta' \rangle) = \langle \rho_X[\Gamma']/\Delta, \rho_X[\varphi']/\Delta \rangle$. Now for $a_1 \in \mathbb{S}ig_a$, let $[e_1] : a \to a_1$ the isomorphism corresponding by the choice ε at a_1 then, by 4.3.5, $\alpha^{(G,\varepsilon)}(a_1) : Sen_{a'} \circ \Phi^{(G,\varepsilon)}(a_1) \to Sen_a(a_1)$ such that for $\langle \Gamma'/\Delta', \varphi'/\Delta' \rangle \in Sen_{a'}(a')$, $\alpha^{(G,\varepsilon)}(a_1)(\langle \Gamma'/\Delta', \varphi'/\Delta' \rangle) = \langle \rho^{a_1}_X[\Gamma']/\Delta_1, \rho^{a_1}_X(\varphi')/\Delta_1 \rangle = \langle \check{e_1} \circ \rho_X[\Gamma']/\Delta_1, \check{e_1} \circ \rho_X(\varphi')/\Delta_1 \rangle$. If $[g] : a_1 \to a_2$ is an isomorphism in $\mathbb{S}ig_a$, then for each $\theta' \in F'X$, $\vdash_2 \check{g}(\rho^{a_1}_X(\theta'))\Delta_2 \rho^{a_2}_X(\check{g}^h_\varepsilon(\theta'))$, thus $\alpha^{(G,\varepsilon)}$ is a natural transformation.

- $\beta^{(G,\varepsilon)} : Mod_a \Rightarrow Mod_{a'} \circ (\Phi^{(G,\varepsilon)})^{op}$ where for a we have $\beta^{(G,\varepsilon)}(a) : Mod_a(a) = Matr^*_a \to Mod_{a'}(\Phi^{(G,\varepsilon)}(a)) = Matr^*_{a'}$ such that $\beta_h(a)(\langle M, F_M \rangle) = \langle L_h(M), F_{L_h(M)} \rangle$, where $F_{L_h(M)} := \bar{\partial}_M[F_M]$ (note that $L_h(M) \in QV(a)$)[7]. Now for $a_1 \in \mathbb{S}ig_a$, $\beta^{(G,\varepsilon)}(a_1) : Mod_a(a_1) = Matr^*_{a_1} \to Mod_{a'}(\Phi^{(G,\varepsilon)}(a_1)) = Matr^*_{a'}$ such that $\beta^{(G,\varepsilon)}(a_1)(\langle M, F_M \rangle) = \langle L_h(e^\star_1(M')), F_{L_h(e^\star_1(M'))} \rangle$. Similarly of above we have the well definition of $\beta^{(G,\varepsilon)}$. The naturality is proved using the functorial encoding of equipollence that we have proved in Chapter 2.

- The proof the compatibility condition will be splited in two parts:

(I) The first part consist of the compatibility on the logic a:

Claim. Given $\langle M, F_M \rangle \in Mod_a(a) = Matr^*_a$ and $\langle \Gamma'/\Delta', \varphi'/\Delta' \rangle \in Sen_{a'}$ then

$$\beta_h(a)(\langle M, F_M \rangle) \models' \langle \Gamma'/\Delta', \varphi'/\Delta' \rangle \ \ \textit{iff} \ \ \langle M, F_M \rangle \models \alpha_h(a)(\langle \Gamma'/\Delta', \varphi'/\Delta' \rangle)$$

In other notation

$$\langle L_h(M), F_{L_h(M)} \rangle \models' \langle \Gamma'/\Delta', \varphi'/\Delta' \rangle \ \ \textit{iff} \ \ \langle M, F_M \rangle \models \langle \rho_X[\Gamma']/\Delta, \rho_X(\varphi')/\Delta \rangle$$

Proof of the Claim.

" $\Rightarrow$ ": Let $v : X \to M$ be an evaluation such that $v[\rho_X[\Gamma']] \subseteq F$. We can consider $\bar{w} = h^* \circ L_h(\bar{v}) : (F'(X)/\Delta')^h \to h^*(L_h(M))$ and then the following diagram commutes:

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] \arrow[ddl, bend right, "v"'] \arrow[ddr, bend left, "w"] & \\
\frac{FX}{\Delta} \arrow[rr, shift left, "\bar{\partial}_X"] \arrow[d, "\bar{v}"] & & (\frac{F'X}{\Delta'})^h \arrow[ll, shift left, "\bar{\rho}_X"] \arrow[d, "\bar{w}"] \\
M \arrow[rr, shift left, "\bar{\partial}_M"] & & h^\star\!\restriction L_h M \arrow[ll, shift left, "\bar{\rho}_M"]
\end{tikzcd}$$

[7]That $\langle L_h(M), F_{L_h(M)} \rangle \in Matr^*_{a'}$, follows from an argument analogous to the proof of compatibility condition.

Since $v \circ \rho_X[\Gamma'] \subseteq F_M$ we have that $\Gamma' \subseteq \rho_X^{-1} \circ v^{-1}[F_M]$. Consider (Δ', τ') a algebraizable pair for a'. Then we have that for all $\psi \in \Gamma'$ and $(\varepsilon'^j, \delta'^j) \in \tau'$, $(\varepsilon'^j(\psi), \delta'^j(\psi)) \in \Omega^{F'(X)^h}(\rho_X^{-1} \circ v^{-1}(F_M)) = \rho_X^{-1} \circ v^{-1}(\Omega^M(F_M))$. Therefore $(v \circ \rho_X(\varepsilon'^j(\psi)), v \circ \rho_X(\delta'^j(\psi))) \in \Omega^M(F_M)$. Since $\langle M, F_M \rangle$ is a reduced matrix, we have for all $\psi \in \Gamma'$

$$\begin{array}{rcl}
v \circ \rho_X(\varepsilon'^j(\psi)) & = & v \circ \rho_X(\delta'^j(\psi)) \\
\bar{v} \circ \bar{\rho_X}(\varepsilon'^j(\psi)/\Delta') & = & \bar{v} \circ \bar{\rho_X}(\delta'^j(\psi)/\Delta') \\
\bar{\rho_M} \circ \bar{w}(\varepsilon'^j(\psi)/\Delta') & = & \bar{\rho_M} \circ \bar{w}(\delta'^j(\psi)/\Delta') \\
\bar{\partial}_M \circ \bar{\rho_M} \circ \bar{w}(\varepsilon'^j(\psi)/\Delta') & = & \bar{\partial}_M \circ \bar{\rho_M} \circ \bar{w}(\delta'^j(\psi)/\Delta') \\
\bar{w}(\varepsilon'^j(\psi)/\Delta') & = & \bar{w}(\delta'^j(\psi)/\Delta') \\
w(\varepsilon'^j(\psi)) & = & w(\delta'^j(\psi))
\end{array}$$

Then $(w(\varepsilon'^j(\psi)), w(\delta'^j(\psi))) \in \Omega^{L_h M}(F_{L_h M})$. Thus $w(\psi) \in F_{L_h M}$ for all $\psi \in \Gamma'$, by assumption $w(\varphi') \in F_{L_h M}$. So $(w(\varepsilon'^j(\varphi')), w(\delta'^j(\varphi'))) \in \Omega^{L_h M}(F_{L_h M})$. Therefore

$$\begin{array}{rcl}
w(\varepsilon'^j(\varphi')) & = & w(\delta'^j(\varphi')) \\
\bar{w}(\varepsilon'^j(\varphi')/\Delta') & = & \bar{w}(\delta'^j(\varphi')/\Delta') \\
\bar{\rho_M} \circ \bar{w}(\varepsilon'^j(\varphi')/\Delta') & = & \bar{\rho_M} \circ \bar{w}(\delta'^j(\varphi')/\Delta') \\
\bar{v} \circ \bar{\rho_X}(\varepsilon'^j(\varphi')/\Delta') & = & \bar{v} \circ \bar{\rho_X}(\delta'^j(\varphi')/\Delta') \\
v \circ \rho_X(\varepsilon'^j(\varphi')) & = & v \circ \rho_X(\delta'^j(\varphi'))
\end{array}$$

Then $(v \circ \rho_X(\varepsilon'^j(\varphi')), v \circ \rho_X(\delta'^j(\varphi'))) \in \Omega^M(F_M)$. Therefore $v \circ \rho_X(\varphi') \in F_M$.

" $\Leftarrow$ ": Let $w : X \to L_h M$ a valuation such that $w[\Gamma'] \subseteq F_{L_h M}$. Consider $\bar{w} : F'(X)/\Delta' \to L_h(M)$ given by w such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "w"'] & F'(X) \arrow[r, "q"] \arrow[d, "w"] & F'(X)/\Delta \arrow[dl, "\bar{w}"] \\
 & M &
\end{tikzcd}$$

Let $\bar{v} = \bar{\rho}_M \circ \bar{w} \circ \bar{\partial}_X$, then $\bar{\partial}_M \circ \bar{v} = \bar{\partial}_M \circ \bar{\rho} \circ \bar{w}\bar{\partial}_X = \bar{w} \circ \bar{\partial}_X$.

Since $w[\Gamma'] \subseteq F_{L_h M}$, we have that $(w(\varepsilon'(\psi)), w(\delta'(\psi))) \in \Omega^{L_h M}(F_{L_h M})$ for all $\psi \in \Gamma$ and $(\varepsilon', \delta') \in \tau'$. Since $\langle L_h M, F_{L_h M} \rangle$ is a reduced matrix, we have that

$$\begin{array}{rcl}
w(\varepsilon'(\psi)) & = & w(\delta'(\psi)) \\
\bar{w}(\varepsilon'(\psi)) & = & \bar{w}(\delta'(\psi)) \\
\bar{\rho_M} \circ \bar{w}(\varepsilon'(\psi)) & = & \bar{\rho_M} \circ \bar{w}(\delta'(\psi)) \\
\bar{v} \circ \bar{\rho_X}(\varepsilon'(\psi)) & = & \bar{v} \circ \bar{\rho_X}(\delta'(\psi))
\end{array}$$

From $\bar{\rho}_X$ there is ρ_X such that the square in the bellow diagram commutes, and then the diagram commutes:

$$\begin{tikzcd} F'(X) \arrow[r, "\rho_X"] \arrow[d] & F(X) \arrow[r, "v"] \arrow[d] & M \\ F'(X)/\Delta' \arrow[r, "\bar{\rho}_X"'] & F(X)/\Delta \arrow[ur, "\bar{v}"'] & \end{tikzcd}$$

With that we have $v\circ\rho_X(\varepsilon(\psi)) = v\circ\rho_X(\delta(\psi))$, so $(v\circ\rho_X(\varepsilon(\psi)), v\circ\rho_X(\delta(\psi))) \in \Omega^M(F_M)$ for all $\psi \in \Gamma'$. By algebraizability we have $v \circ \rho_X(\psi) \in F_M$ for all $\psi \in F_M$. By assumption $v \circ \rho_X(\varphi) \in F_M$. Thus $(v \circ \rho_X(\varepsilon(\varphi)), v \circ \rho_X(\delta(\varphi))) \in \Omega^M(F_M)$. Since $\langle M, F_M\rangle$ is a reduced matrix, we have that

$$\begin{aligned} v \circ \rho_X(\varepsilon'(\varphi)) &= v \circ \rho_X(\delta'(\varphi)) \\ \bar{v} \circ \bar{\rho}_X(\varepsilon'(\varphi)/\Delta') &= \bar{v} \circ \bar{\rho}_X(\delta'(\varphi)/\Delta') \\ \bar{\rho}_M \circ \bar{w}(\varepsilon'(\varphi)/\Delta') &= \bar{\rho}_M \circ \bar{w}(\delta'(\varphi)/\Delta') \\ \bar{\partial}_M \circ \bar{\rho}_M \circ \bar{w}(\varepsilon'(\varphi)/\Delta') &= \bar{\partial}_M \circ \bar{\rho}_M \circ \bar{w}(\delta'(\varphi)/\Delta') \\ \bar{w}(\varepsilon'(\varphi)/\Delta') &= \bar{w}(\delta'(\varphi)/\Delta') \\ w(\varepsilon'(\varphi)) &= w(\delta'(\varphi)) \end{aligned}$$

With that we have $(w(\varepsilon'(\varphi)), w(\delta'(\varphi))) \in \Omega^{L_hM}(F_{L_hM})$. Therefore $w(\varphi) \in F_{L_hM}$.

(II) One can use similar argument to prove the second part, i.e., given $a_1 \in \mathbb{S}ig(a)$, $\langle M_1, F_{M_1}\rangle \in Mod_a(a_1) = Matr^*_{a_1}$ and $\langle \Gamma'/\Delta', \varphi'/\Delta'\rangle \in Sen_{a'}(a')$ then:

$$\langle L_h(e_1^*(M_1)), F_{L_h(e_1^*(M_1))}\rangle \models' \langle \Gamma'/\Delta', \varphi'/\Delta'\rangle \;\; \textit{iff} \;\; \langle M_1, F_{M_1}\rangle \models_1 \langle \check{e}_1\rho_X[\Gamma']/\Delta_1, \check{f}\rho_X(\varphi')/\Delta_1\rangle$$

□

As a consequence of this theorem we have the abstract Glivenko's theorem between algebraizable logics.

Corollary 4.3.7. *For each Glivenko's context $\mathbb{G} = (h : a \to a', \rho)$, is associated an abstract Glivenko's theorem between a and a' i.e; given $\Gamma' \cup \{\varphi'\} \subseteq F'(X)$ then*

$$\rho_X[\Gamma'] \vdash \rho_X(\varphi') \;\Leftrightarrow\; \Gamma' \vdash' \varphi'$$

Proof:

We know that for any algebraizable logic a, $\vdash_a = \vdash_{Matr^*_a}$. For any reduced matrix in a' $\langle M', F_{M'}\rangle$ we have that $M' \in QV(a')$ and then there is $M \in QV(a)$ such that $L_hM \cong M'$ (see Remark 4.3.3.(c)) , moreover $\langle L_hM, F_{L_hM}\rangle \cong \langle M', F_{M'}\rangle$. With that it is enough to prove that

$$\rho_X[\Gamma'] \vdash_{Matr^*_a} \rho_X(\varphi') \;\Leftrightarrow\; \Gamma' \vdash_{Matr^*_{a'}} \varphi'$$

And that is equivalent to prove that for any $\langle M, F_M\rangle \in Matr^*_a$,

$$\langle M, F_M\rangle \models \langle \rho_X[\Gamma'], \rho_X(\varphi')\rangle \ \ \textit{iff} \ \ \langle L_h M, F_{L_h M}\rangle \models' \langle \Gamma', \varphi'\rangle$$

Or even,

$$\langle M, F_M\rangle \models \langle \rho_X[\Gamma']/\Delta, \rho_X(\varphi')/\Delta\rangle \ \ \textit{iff} \ \ \langle L_h M, F_{L_h M}\rangle \models' \langle \Gamma'/\Delta', \varphi'/\Delta'\rangle$$

But this last one follows from the previous theorem. □

Now we present that the abstract Glivenko's theorem restricts to the classical Glivenko's theorem.

Example 4.3.8. *Let $\Sigma = (\Sigma_n)_{n\in\omega}$ such that $\Sigma_0 = \emptyset,\ \Sigma_1 = \{\neg\},\ \Sigma_2 = \{\longrightarrow\}$ and $\Sigma_n = \emptyset$ for all $n > 2$. Let the map $h : IPC \to CPC$ such that IPC and CPC both are defined with the signature Σ, $h(\neg) = \neg$ and $h(\longrightarrow) = \longrightarrow$, i.e., h is the inclusion map from the intuitionistic propositional logic to the classical propositional logic. IPC and CPC are (Lindenbaum) algebraizable logics and h is a morphism in $\mathcal{A}_f$. Notice that h^* is the identity functor and its restriction $h^*\restriction: Bool \hookrightarrow Heyt$ has a left adjoint given by $L_h : Heyt \to Bool$ such that for any $A \in Heyt$, $L_h(A) = A_{\neg\neg}$ where is the boolean algebra of regular element, i.e., $a \in A$ such that $\neg\neg a = a$. The unit of this adjunction is $\partial_A : A \to A_{\neg\neg}$ such that $\partial_A(a) = \neg\neg a$ for all $A \in \Sigma{-}Str$. It is easy to see that this map define a natural transformation, moreover it has a natural transformation such that is a section given by $\rho_A : A_{\neg\neg} \to A$ where $\rho_A(a) = \neg\neg a = a$. Then we have that $(h : IPC \to CPC, \rho)$ is a Glivenko's context.*

We know that $\psi \dashv_{CPC}\vdash \neg\neg\psi$ and then we have that $\psi/\Delta = \neg\neg\psi/\Delta$ where $\Delta = \{x \to y,\ y \to x\}$. Using the abstract Glivenko's theorem we have that given $\Gamma\cup\{\varphi\}$ set of formulas, then to prove that $\neg\neg\Gamma \vdash_{IPC} \neg\neg\varphi \ \Leftrightarrow\ \Gamma \vdash_{CPC} \varphi$ is enough to prove that for all matrix $\langle M, F_M\rangle \in Matr^*_{ICP}$,

$$\langle M_{\neg\neg}, F_{M_{\neg\neg}}\rangle \models_{CPC} \langle \Gamma/\Delta, \varphi/\Delta\rangle \ \ \textit{iff} \ \ \langle M, F_M\rangle \models_{IPC} \langle \neg\neg\Gamma/\Delta, \neg\neg\varphi/\Delta\rangle$$

That is exactly the same to prove that

$$\langle L_h M, F_{L_h M}\rangle \models_{CPC} \langle \Gamma/\Delta, \varphi/\Delta\rangle \ \ \textit{iff} \ \ \langle M, F_M\rangle \models_{CPC} \langle \rho_X[\Gamma]/\Delta, \rho_X(\varphi)/\Delta\rangle.$$

This last follows from the previous corollary.

Remark 4.3.9. *We believe that the notion of abstract Glivenko's theorem provided here, partially generalizes the approach that has been developed in [Tor08] In that paper, the author consider abstract Glivenko's theorem in the algebraizable logic setting (and also in some variants) but just relating logics defined over the* same signature *by means of an essentially idempotent formula with a free variable.*

4.3.2 The abstract Glivenko's theorem in *InsLAL*

We also have that a Glivenko's context induces an abstract Glivenko's theorem for $InsLAL$ and we present now.

In this subsection we consider fixed: a, a' Lindenbaum algebraizable logics, $G = (h : a \to a', \rho)$ a Glivenko's context and a choice of isomorphisms

$$\varepsilon_a : Obj(\mathbb{S}ig'_a) \to \bigcup_{a_1 \in Obj(\mathbb{S}ig'_a)} Hom_{\mathbb{S}ig_a}(a, a_1)$$

given by $a_1 \mapsto ([e_1] : a \to a_1)$

4.3.10. *For each $s = ([\varphi_0], \cdots, [\varphi_{n-1}], [\psi])$, a non-empty finite sequence in $F(\Sigma)/ \dashv\vdash$ (the free $QV(a)$-structure on the set X) and each (τ, Δ), an algebraizable pair of a, where $\tau = \{(\varepsilon^j, \delta^j);\ j = 1, ..., m\ for\ some\ m \in \omega\}$, let*

$$q(s, (\Delta, \tau)) := (([\varepsilon(\varphi_0)], [\delta(\varphi_0)]), \cdots, ([\varepsilon(\varphi_{n-1})], [\delta(\varphi_{n-1})]), ([\varepsilon(\psi)], [\delta(\psi)]))$$

where the notation $([\varepsilon(\theta)], [\delta(\theta)])$ *abbreviates the pair of finite sequence of equivalence class of formulas: $([\varepsilon^j(\theta), [\delta^j(\theta)])_j$ with $j = 1, \cdots, m$.. Note that, as a is a* congruential algebraizable logic, *then:*
(*) If $[\theta] = [\theta']$ (i.e., $\theta \dashv\vdash \theta'$), then $\delta(\theta) \dashv\vdash \delta(\theta')$ and $\varepsilon(\theta) \dashv\vdash \varepsilon(\theta')$.Thus we have an well defined mapping $\varphi/\Delta \overset{t}{\mapsto} (\varepsilon(\varphi)/\Delta, \delta(\varphi)/\Delta)$ and $q(s, (\Delta, \tau))$ is well-defined;
(**) conversely, as $\varphi \dashv\vdash \Delta(\epsilon(\varphi), \delta(\varphi))$, then we have and well defined map $(\varepsilon(\varphi)/\Delta, \delta(\varphi)/\Delta) \overset{r}{\mapsto} \varphi/\Delta$ and $r \circ t = id$.

Recall that $q_s := \{q(s, (\tau, \Delta)) : (\tau, \Delta)$ is an algebraizable pair of $a_1\}$ and $Sen'_a(a) := \{q_s : s$ is a non-empty finite sequence in $F(\Sigma)\}$. Note that, by the above remark, the mapping $s \overset{t}{\mapsto} q_s$ determine a bijection *between the set of non-empty finite sequences in $F(\Sigma)/\Delta$ and $Sen'a(a)$*

Then, in particular $\check{h}/ \dashv\vdash: F(\Sigma)/ \dashv\vdash \to F(\Sigma')/ \dashv'\vdash$ has a section $\bar{\rho}_X : F(\Sigma')/ \dashv'\vdash \to F(\Sigma)/ \dashv\vdash$. Let $s' = ([\varphi'_0], \cdots, [\varphi'_{n-1}], [\psi'])$ be a non-empty finite sequence in $F(\Sigma')/ \dashv'\vdash$ and $((\varepsilon', \delta'), \Delta')$ be an algebraizable pair of a'. Then $\rho_* s' := ([\rho_X(\varphi'_0)], \cdots, [\rho_X(\varphi'_{n-1})], [\rho_X(\psi')])$ is a non-empty finite sequence in $F(\Sigma)/ \dashv\vdash$ and the mapping $q'_{s'} \in Sen'_{a'}(a') \ \mapsto\ q_{\rho_* s'} \in Sen'_a(a)$ is a section of the map on non-empty finite sequences induced by $\check{h}/ \dashv\vdash: F(\Sigma)/ \dashv\vdash \to F(\Sigma')/ \dashv'\vdash$.

Now, we start providing the following

Proposition 4.3.11. *Let $L_h : QV(a) \to QV(a')$ be the left adjoint of $h^\star\restriction: QV(a') \to QV(a)$ as defined in Chapter 2 (see Proposition 2.1.6), then for each $M \in QV(a)$ and $q' \in Sent'_a(a')$, the following compatibility relation holds:*

$$M \models^a \bar{\rho} q' \ \Leftrightarrow\ L_h M \models^{a'} q'$$

Proof:

" $\Rightarrow$ " Let $q' \in Sen'(a')$. Suppose that given $M \in QV(a)$, $M \models^a \bar{\rho}q'$. Given $w : X \to L_h(M)$ ($|L_hM| = |h^\star L_hM|$, we can consider $w : X \to h^\star L_h(M)$) such that

$$\bar{w}[\varepsilon'(\varphi_i')] = \bar{w}[\delta'(\varphi_i')],\ i = 0, ..., n-1$$

Look to diagram below:

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] \arrow[ddl, bend right, "v"'] \arrow[ddr, bend left, "w"] & \\
\frac{FX}{\Delta} \arrow[rr, shift left, "\bar{\partial}_X"] \arrow[d, "\bar{v}"] & & (\frac{F'X}{\Delta'})^h \arrow[ll, shift left, "\bar{\rho}_X"] \arrow[d, "\bar{w}"] \\
M \arrow[rr, shift left, "\bar{\partial}_M"] & & h^\star\!\upharpoonright L_hM \arrow[ll, shift left, "\bar{\rho}_M"]
\end{tikzcd}$$

Consider $\bar{v} = \bar{\rho}_M\bar{w}\bar{\partial}_X$ (there is $v : X \to M$ such that to be corresponding with $\bar{v}$). Hence $\bar{v}\bar{\rho}_X = \bar{\rho}_M\bar{w}\bar{\partial}_X\bar{\rho}_X = \bar{\rho}_M\bar{w}$

$$\begin{aligned}
\bar{\rho}_M\bar{w}[\varepsilon'(\varphi_i')] &= \bar{\rho}_M\bar{w}[\delta'(\varphi_i')]\ (i = 0, ..., n-1) \\
\bar{v}\bar{\rho}_X[\varepsilon'(\varphi_i')] &= \bar{v}\bar{\rho}_X[\delta'(\varphi_i')]\ (i = 0, ..., n-1) \\
\bar{v}\bar{\rho}_X[\varepsilon'(\varphi')] &= \bar{v}\bar{\rho}_X[\delta'(\varphi')]\ Hypo. \\
\bar{\partial}_M\bar{v}\bar{\rho}_X[\varepsilon'(\varphi')] &= \bar{\partial}_M\bar{v}\bar{\rho}_X[\delta'(\varphi')] \\
\bar{\partial}_M\bar{\rho}_M\bar{w}[\varepsilon'(\varphi')] &= \bar{\partial}_M\bar{\rho}_M\bar{w}[\delta'(\varphi')] \\
\bar{w}[\varepsilon'(\varphi')] &= \bar{w}[\delta'(\varphi')]
\end{aligned}$$

w was taken arbitrary, so $L_hM \models^{a'} q$

" $\Leftarrow$ " Suppose that $L_hM \models^{a'} q'$. Let $v : X \to M$ such that $\bar{v}\bar{\rho}_X[\varepsilon'(\varphi_i')] = \bar{v}\bar{\rho}_X[\delta'(\varphi_i')]\ \forall\, i = 0, ..., n-1$.

Consider $\bar{w} = h^\star\!\upharpoonright L_h(\bar{v})$(exist $w : X \to h^\star\!\upharpoonright L_hM$ extends to $\bar{w}$). So $\bar{\rho}_M\bar{w} = \bar{v}\bar{\rho}_X$ and $\bar{w}\bar{\partial}_X = \bar{\partial}_M\bar{v}$. Therefore

$$\begin{aligned}
\bar{\rho}_M\bar{w}[\varepsilon'(\varphi_i')] &= \bar{\rho}_M\bar{w}[\delta'(\varphi_i')]\ i = 0, ..., n-1 \\
\bar{\partial}_M\bar{\rho}_M\bar{w}[\varepsilon'(\varphi_i')] &= \bar{\partial}_M\bar{\rho}_M\bar{w}[\delta'(\varphi_i')]\ i = 0, ..., n-1 \\
\bar{w}[\varepsilon'(\varphi_i')] &= \bar{w}[\delta'(\varphi_i')]\ i = 0, ..., n-1 \\
\bar{w}[\varepsilon'(\varphi')] &= \bar{w}[\delta'(\varphi')]\ Hypo.
\end{aligned}$$

Hence $\bar{v}\bar{\rho}_X[\varepsilon'(\varphi')] = \bar{\rho}_M\bar{w}[\varepsilon'(\varphi')] = \bar{\rho}_M\bar{w}[\delta'(\varphi')] = \bar{v}\bar{\rho}_X[\delta'(\varphi')]$.

Then $M \models^a \bar{\rho}_X q'$ □

We also have that a Glivenko's context induces an abstract Glivenko's theorem for $InsLAL$ and we present now and Proposition 4.3.11 above is part of it.

Theorem 4.3.12. *Let a, a' be* Lindenbaum *algebraizable logics, then each $\mathbb{G} = (h : a \to a', \rho)$* ***Glivenko's context*** *induces a institutions morphism $InsLAL_a \to InsLAL_{a'}$. More* *precisely, fixing a choice of isomorphisms* $\varepsilon : Obj(\mathbb{S}ig'_a) \to Mor(\mathbb{S}ig'_a)$, $a_1 \mapsto \varepsilon(a_1) = [e_1] : a \overset{\cong}{\to} a_1$, *we define a institution morphism* $M_{(G,\varepsilon_a)} : InsLAL_a \to InsLAL_{a'}$[8]

Proof:

By simplicity, we will write (G, ε) for (G, ε_a). We will define

$$M_{(G,\varepsilon)} = \langle \Phi'^{(G,\varepsilon)}, \alpha'^{(G,\varepsilon)}, \beta'^{(G,\varepsilon)} \rangle$$

(this will depend only on the choice of isomorphisms in the *domain* institution $InsLAL_a$):

$\Phi'^{(G,\varepsilon)} : \mathbb{S}ig'_a \to \mathbb{S}ig'_{a'}$: it is defined in the same way as $\Phi^{(G,\varepsilon)} : \mathbb{S}ig_a \to \mathbb{S}ig_{a'}$was defined in 4.3.1.

Now the definition of $\alpha'^{(G,\varepsilon)}$.

Firstly for a we have $\alpha'^{(G,\varepsilon)}(a) : Sen_{a'} \circ \Phi'^{(G,\varepsilon)}(a) = Sen_{a'}(a') \to Sen_a(a)$ is the mapping $q'_{s'} \in Sen'_{a'}(a') \mapsto q_{\rho_* s'} \in Sen'_a(a)$, as defined in 4.3.10.

For an arbitrary $a_1 \in \mathbb{S}ig'_a$ we define $\alpha'^{(G,\varepsilon)}(a_1) : Sen_{a'}(a') \to Sen_a(a_1)$ by for $q' \in Sen_a(a')$,

$$\alpha'^{(G,\varepsilon)}(a')(q') = \bar{\rho}_X^{a_1}(q')$$

such that for each component of $\bar{\rho}_X^{a_1} q'$ is $([\check{e}_1 \rho_X(\varepsilon'(\varphi_k))], [\check{e}_1 \rho_X(\delta'(\varphi_k))])$ for $k = 1, ..., n-1$ and the last component is $([\check{e}_1 \rho_X(\varepsilon'(\varphi))], [\check{e}_1 \rho_X(\delta'(\varphi))])$. This defines a natural transformation. Indeed, first observe that the diagram below commutes:

$$\begin{tikzcd}
F(\Sigma')/\Delta' \arrow[r, "\bar{\rho}_X"] \arrow[d, "{[\check{g}^g_\varepsilon]}"'] & F(\Sigma)/\Delta \arrow[r, "{[\check{e}_1]}"] \arrow[d, "{[\check{g}_\varepsilon]}"] & F(\Sigma_1)/\Delta_1 \arrow[d, "{[\check{g}]}"] \\
F(\Sigma')/\Delta' \arrow[r, "\bar{\rho}_X"'] & F(\Sigma)/\Delta \arrow[r, "{[\check{e}_2]}"'] & F(\Sigma_2)/\Delta_2
\end{tikzcd}$$

then we have the following diagram commuting:

$$\begin{tikzcd}
Sen_{a'}(\Phi'(a_1)) \arrow[r, "\alpha'^{(G,\varepsilon)}(a_1)"] \arrow[d, "{[\check{g}^h_\varepsilon]}"'] & Sen_a(a_1) \arrow[d, "{[\check{g}]}"] \\
Sen_{a'}(\Phi'(a_2)) \arrow[r, "\alpha'^{(G,\varepsilon)}(a_2)"'] & Sen_a(a_2)
\end{tikzcd}$$

Let now to define $\beta'^{(G,\varepsilon)}$. For a we define $\beta'^{(G,\varepsilon)} : Mod'_a \Rightarrow Mod'_{a'} \circ (\Phi'^{(G,\varepsilon)})^{op}$ is define as: $\beta'^{(G,\varepsilon)}(a) = L_h : QV(a) = Mod'_a(a) \to Mod'_{a'}(\Phi'^{(G,\varepsilon)}(a)) = QV(a')$

The corresponding definition works for an arbitrary $a_1 \in \mathbb{S}ig_a$ because since a and a_1 are Q^c_f-isomorphic, we have by 2.2.4 that $QV(a)$ and $QV(a_1)$ are isomorphic. I.e., $\beta'^{(G,\varepsilon)}(a_1) = L_{h_1} : QV(a_1) = Mod'_a(a_1) \to Mod'_{a'}(\Phi'^{(G,\varepsilon)}(a_1)) = QV(a')$, where $(a \overset{[h]}{\to} a') = (a \overset{[e_1]}{\to} a_1 \overset{[h_1]}{\to} a')$. This defines a natural transformation. Indeed, notice that the following diagram commutes:

[8]Such induced morphisms are "isomorphic", for different choices of isomorphisms $\varepsilon^0, \varepsilon^1$.

$$\begin{array}{ccc} QV(a_1) & \xleftarrow{h_1^\star\upharpoonright} & QV(a') \\ {\scriptstyle g^\star\upharpoonright}\uparrow & & \uparrow{\scriptstyle (g_\varepsilon^h)^\star\upharpoonright} \\ QV(a_2) & \xleftarrow[h_2^\star\upharpoonright]{} & QV(a') \end{array}$$

And then we have the following diagram commuting:

$$\begin{array}{ccc} QV(a_1) & \xrightarrow{\beta_{a_1}} & QV(a') \\ {\scriptstyle g^\star\upharpoonright}\uparrow & & \uparrow{\scriptstyle (g_\varepsilon^h)^\star\upharpoonright} \\ QV(a_2) & \xrightarrow[\beta_{a_2}]{} & QV(a') \end{array}$$

On the compatibility condition. First for the logic a we must guarantee that $M \models^a \bar{\rho}q' \;\Leftrightarrow\; L_h M \models^{a'} q'$: this is the content of Proposition 4.3.11.

For an arbitrary logic $a_1 \in \mathbb{S}ig'_a$ we must to prove that for any $M_1 \in QV(a_1)$ and $q_{s'} \in Sen(a')$:

$$\beta'^{(G,\varepsilon)}(a_1)(M_1) \models^{a'} q_{s'} \;\Leftrightarrow\; M_1 \models^{a_1} \alpha'(G,\varepsilon)(a_1)(q_{s'})$$

in other notation

$$L_{h_1}(M_1) \models^{a'} q_{s'} \;\Leftrightarrow\; M_1 \models^{a_1} \bar{\rho}_X^{a_1}(q_{s'}).$$

In fact, since $[\Phi(e_1)]$ is an isomorphism, we have that $\Phi(e_1)^\star\upharpoonright$ is an isomorphism. Therefore:

$$\begin{aligned} L_{h_1}(M_1) \models^{a'} q_{s'} &\Leftrightarrow \Phi(e_1)^\star L_{h_1}(M_1) \models q_{(\Phi(e_1)^+)^{-1}(s_1)} \\ &\Leftrightarrow L_h(e_1^\star(M_1)) \models q_{(\Phi(e_1)^+)^{-1}(s_1)} \\ &\Leftrightarrow e_1^\star(M_1) \models \alpha'^{(G,\varepsilon)}(a)(q_{(\Phi(e_1)^+)^{-1}(s_1)}) \\ &\Leftrightarrow M_1 \models^{a_1} e_1^+\alpha'^{(G,\varepsilon)}(a)(q_{(\Phi(e_1)^+)^{-1}(s_1)}) \\ &\Leftrightarrow M_1 \models^{a_1} \alpha_{a_1}(q_{s'}). \end{aligned}$$

$\square$

Corollary 4.3.13. *For each Glivenko's context* $\mathbb{G} = (h : a \to a', \bar{\rho})$*, is associated an abstract Glivenko's theorem between* a *and* a' *i.e; given* $\Gamma' \cup \{\varphi'\} \subseteq F'(X)$ *then*

$$\rho_X[\Gamma'] \vdash \rho_X(\varphi') \;\Leftrightarrow\; \Gamma' \vdash' \varphi'$$

Proof:

Firstly, remark that it is enough consider Γ finite. Because a and a' are algebraizable logics, and h preserves algebraizing pairs, it is enough to show that

$$\begin{array}{ccc} \{\varepsilon(\rho_X(\psi')) \approx \delta(\rho_X(\psi')), \psi' \in \Gamma'\} & \models_{QV(a)} & \varepsilon(\rho_X(\varphi')) \approx \delta(\rho_X(\varphi')) \\ & \Updownarrow & \\ \{\varepsilon'(\psi') \approx \delta'(\psi'), \psi' \in \Gamma'\} & \models_{QV(a')} & \varepsilon'(\varphi') \approx \delta'(\varphi') \end{array}$$

Consider $\Gamma = \{\psi_0, ..., \psi_{n-1}\}$, $s' = (\psi'_0/\Delta', ..., \psi'_{n-1}/\Delta', \varphi'/\Delta')$. Then:
(i) $q' = q'_{s'}$ is determined by any of its elements

$$(([\varepsilon'(\psi'_0)], [\delta'(\psi'_0)]), ..., ([\varepsilon'(\psi'_{n-1})], [\delta'(\psi'_{n-1})])([\varepsilon'(\varphi')], [\delta'(\varphi)]));$$

(ii) $\alpha(a)(q') = q_{\rho_* s'}$ is determined by any of its elements

$$(([\varepsilon(\rho_X(\psi'_0))], [\delta(\rho_X(\psi'_0))]), ..., ([\varepsilon(\rho_X(\psi'_{n-1}))], [\delta(\rho_X(\psi'_{n-1}))])([\varepsilon(\rho_X(\varphi'))], [\delta(\rho_X(\varphi))]))$$

Thus we have to show:

$$(\forall M \in QV(a), M \models^a \bar{\rho}_X q') \ \Leftrightarrow \ (\forall M' \in QV(a')M' \models^{a'} q')$$

By Remark 4.3.3.(c) given $M' \in QV(a')$ there is $M \in QV(a)$ such that $L_h(M) \cong M'$. With this, it is enough to show that for every $M \in QV(a)$,

$$M \models^a \bar{\rho}_X q' \Leftrightarrow L_h M \models^{a'} q'$$

And this last equivalence is established the Proposition 4.3.11 above.

□

Remark 4.3.14. *Since the CPC and IPC are Lindenbaum algebraizable logic, one can see that the example 4.3.8 follows a consequence of the abstract Glivenko's theorem fo $InsLAL$ as well as the abstract Glivenko's theorem for $InsAL$.*

Remark 4.3.15. *A simple analysis of the derivations of "logical" forms of Glivenko's Theorem (Corolaries 4.3.7 and 4.3.13) from the corresponding "instituitional" form of Glivenko's Theorem (Theorems 4.3.6 and 4.3.12), i.e. the existence of certain (induced) morphisms of institutions make clear that the latter form is stronger than the former one. We can interpret this as another evidence[9] of the (virtually unexplored) relevance of institution theory in propositional logic.*

4.4 Category of algebraizable logics with Glivenko's morphisms

In this section we present that the definition of Glivenko's context given in 4.3.2 offer more information about the relationship of logics, it give us a category of algebraizable logics such that the morphisms are Glivenko's contexts, i.e., the objects are the same of in $\mathcal{A}_f$ and given a and a' algebraizable logics, a Glivenko's morphism is a Glivenko's context $(h : a \to a', \rho)$. Denote by $G\mathcal{A}_f$ this category.

[9]Beside the nice approach of the identity problem for (algebraizable) propositional logics: "a logic is an institution, thus manifested through many signatures".

Theorem 4.4.1. *$G\mathcal{A}_f$ is a category*

Proof:

In this category the composition is the usual, i.e., given $G = (h : a \to a', \rho)$ and $G' = (h' : a' \to a'', \rho')$, we have that $G' \circ G = (h' \circ h : a \to a'', \rho' \bullet \rho)$ where $(\rho' \bullet \rho)_M = \rho_M \circ \rho'_{L_h M}$ (this is natural in $M \in QV(a)$). In order to prove that the composition is well defined, we must to prove that $\rho' \circ \rho$ a section for the unit of the adjunction $L_{h'\circ h} \dashv (h' \circ h)^*$. The composition of adjunctions is a adjunction and $\partial' \circ \partial$ is its the unit. Remember that $L_{h' \circ h} = L_{h'} \circ L_h$ (an strict equality, with the choice of adjoints given in Chapter 2, as quotients) and $(h' \circ h)^* = h^* \circ h'^*$. Then we have that $M \overset{(\partial' \circ \partial)_M}{\longrightarrow} h^* h'^* L_{h'} L_h M = M \overset{h^*(\partial'_{L_h M}) \circ \partial_M}{\longrightarrow} h^* h'^* L_{h'} L_h M = M \overset{\partial'_{L_h M} \circ \partial_M}{\longrightarrow} h^* h'^* L_{h'} L_h M$. Then we have

$$\begin{aligned}
(\partial' \circ \partial)_M \circ (\rho' \bullet \rho)_M &= (\partial'_{L_h M} \circ \partial_M) \circ (\rho_M \circ \rho'_{L_h M}) \\
&= \partial'_{L_h M} \circ (\partial_M \circ \rho_M) \circ \rho'_{L_h M} \\
&= \partial'_{L_h M} \circ \rho'_{L_h M} \\
&= Id_{L_h M}.
\end{aligned}$$

Thus $(\rho' \bullet \rho)_M$ is a section for $(\partial' \circ \partial)_M$ for all $M \in \Sigma - Str$. Clearly there is the identity Glivenko's context for an algebraizable logic a given by $(Id_a : a \to a, \rho = (Id_M)_{M \in \Sigma - Str})$. To prove the associativity let $G = (h : a \to a', \rho)$, $G' = (h' : a' \to a'', \rho')$ and $G'' = (h'' : a'' \to a''', \rho'')$ be Glivenko's morphisms (Glivenko's context). Since $\mathcal{A}_f$ is a category we have that $h'' \circ (h' \circ h) = (h'' \circ h') \circ h$. Remains to prove that $\rho'' \bullet (\rho' \bullet \rho) = (\rho'' \bullet \rho') \bullet \rho$. Let $M \in \Sigma - Str$, then

$$\begin{aligned}
(\rho'' \bullet (\rho' \bullet \rho))_M &= (\rho' \bullet \rho)_M \circ \rho''_{L_{h' \circ h} M} \\
&= (\rho_M \circ \rho'_{L_h M}) \circ \rho''_{L_{h' \circ h} M} \\
&= (\rho_M \circ \rho'_{L_h M}) \circ \rho''_{L_{h'} \circ L_h M} \\
&= \rho_M \circ (\rho'_{L_h M} \circ \rho''_{L_{h'} L_h} M) \\
&= \rho_M \circ (\rho'' \bullet \rho')_{L_h M} \\
&= ((\rho'' \bullet \rho') \bullet \rho)_M
\end{aligned}$$

Therefore $G\mathcal{A}_f$ is a category □

The theorems 4.3.6 and 4.3.12 say that for any Glivenko's context there is a institution morphism associated, more precisely, given a Glivenko's context $(h : a \to a', \rho)$ and a choice of isomorphisms $\varepsilon_a : Obj(\mathbb{S}ig_a) \to \bigcup_{a_1 \in Obj(\mathbb{S}ig_a)} Hom_{\mathbb{S}ig_a}(a, a_1)$, we have a institution morphism $\langle \Phi_{G,\varepsilon}, \alpha_{G,\varepsilon}, \beta_{G,\varepsilon} \rangle$. Notice that there are more than one possible choice for the family $(\varepsilon_a)_{a \in |\mathcal{A}_f|}$, but the application below still define a functor.

$$\begin{array}{rrcl}
\mathcal{G}_\varepsilon : & G\mathcal{A}_f & \rightarrow & \mathbf{Inst} \\
& a & & InsAL_a \\
& (h,\rho)\downarrow & \mapsto & \downarrow \langle \Phi_{(G,\varepsilon)}, \alpha_{(G,\varepsilon)}, \beta_{(G,\varepsilon)} \rangle \\
& a' & & InsAL_{a'}
\end{array}$$

Another natural functor that arise is $\mathcal{U} : G\mathcal{A}_f \to \mathcal{A}_f$ such that $\mathcal{U}((h : a \to a', \rho)) = (h : a \to a')$ for any Glivenko's context $(h : a \to a', \rho)$.

Naturally, we can defined in analogous way a (full) subcategory $G\mathcal{A}_f^c \subseteq G\mathcal{A}_f$, with objects being the Lindenbaum algebraizable logics and, for each choice of isomorphisms $(\varepsilon_a)_{a \in |\mathbb{S}ig'_a|}$, we get a functor:

$$\begin{array}{rrcl}
\mathcal{G}^c_\varepsilon : & G\mathcal{A}^c_f & \rightarrow & \mathbf{Inst} \\
& a & & InsLAL_a \\
& (h,\rho)\downarrow & \mapsto & \downarrow \langle \Phi'_{(G,\varepsilon)}, \alpha'_{(G,\varepsilon)}, \beta'_{(G,\varepsilon)} \rangle \\
& a' & & InsLAL_{a'}
\end{array}$$

Once established those relations we have the following diagram that represents the relation among the categories studied in this thesis.

$$\begin{tikzcd}
\pi - Inst \arrow[r, bend right] & Inst \arrow[l, bend right] & \mathcal{F}i \arrow[dd, bend left] \\
G\mathcal{A}^c_f \arrow[ur] \arrow[r] \arrow[d] & G\mathcal{A}_f \arrow[u] \arrow[d] & \\
\mathcal{A}^c_f \arrow[r, tail] & \mathcal{A}_f \arrow[r, tail] & \mathcal{L}_f \arrow[uu, bend left]
\end{tikzcd}$$

On the other hand,we saw that the categories $\mathcal{L}_s$ and $\mathcal{L}_f$ determines institutions and π-institutions. Having in mind the adjunctions $\mathcal{L}_s \rightleftarrows \mathcal{L}_f \rightleftarrows \mathcal{F}_i$, we believe that is possible establish a (extended) direct relation from $\mathcal{F}i$ to $\mathbf{Inst}$ and $\Pi - \mathbf{Inst}$. This is part of the future works on the thesis.

Chapter 5

First steps on the Representation Theory of Logic

In the representation theory of rings, the category of rings is functorially encoded into the category of categories: a ring R is encoded by the category of (left/right) linear representation of R (respectiv. $R - Mod$, $Mod - R$) or some convenient essentially small subcategories given by finitely generated modules (respect. $R - Mod$, $Mod - R$). It is proposed, based on the results in chapter 2, an encoding of a general propositional logic by a diagram of categories and functors given by the quasivarieties canonically associated to the algebraizable logics (in the sense of [BP89]) connected with the given propositional logic.

In this setting, we start the study of left "Morita equivalence" of logics and variants. We introduce the concepts of left-stably-Morita-equivalent logics and show that the presentations of classical logics are stably-Morita-equivalent, but classical logics and intuitionistic logics are not stably-Morita-equivalent: they are only stably-Morita-adjointly related. We start the development mainly of *left* representation theory of logics –related to *analysis* process of combination of logics– because, differently from representation theory of rings, the right representation theory of logics –related to *synthesis* process of combination of logics– is technically more involved than the "left" case (it requires tools from the theory of 2-categories). Fragments of this approach to representation theory of propositional logics can be found in [AFLM05].

We will denote the forgetful functors: $U : \mathcal{A}_f \to \mathcal{L}_f$, $U^c : \mathcal{A}_f^c \to \mathcal{L}_f^c$, $\bar{U} : \overline{\mathcal{A}_f} \to Q\mathcal{L}_f$ and $\bar{U}^c : Q\mathcal{A}_f^c \to Q\mathcal{L}_f^c$.

5.1 General Logics and Categories

We start this chapter presenting some important results about the categories $l \downarrow \bar{U}$ and $l \downarrow \bar{U}^c$. The following theorem give us an way to build a co-product in the category $l \downarrow \bar{U}^c$.

Proposition 5.1.1. *Let $l \in \mathcal{L}_f$, $a \in \mathcal{A}_f$ and $f : l \to a$ be a Δ-dense morphism in $\mathcal{L}_f$. Then:*

(i) There exists a logic a' over the signature of l such that there is a conservative translation to a, it is algebraizable and $l \leq a'$. In particular, $[f] : a' \overset{\cong}{\to} a$ in the quotient category $\overline{\mathcal{A}_f}$.

(ii) $f^\star[QV(a)] \subseteq \Sigma - Str$ is a quasivariety and it is an equivalent algebraic semantic to a'.

Proof:

(i) Consider $l = (\Sigma, \vdash)$ and $a = (\alpha, \vdash_a)$. We define $\vdash' \subseteq \mathcal{P}(F(\Sigma)) \times F(\Sigma)$ by definition 2.9 of [AFLM07], that is, $\Gamma \vdash' \varphi$ iff $\check{f}(\Gamma) \vdash_a \check{f}(\varphi)$. $\vdash'$ is a Tarskian consequence, indeed, the reflexivity, cut, monotonicity and finitarity are easy to prove. We will prove the structurality. Suppose $\Gamma \vdash' \varphi$. Let $s : X \to F(\Sigma)$. Define $s' = \check{f} \circ s$ ($s' : X \to F(\alpha)$). Consider $s : F(\Sigma) \to F(\Sigma)$ the extension of s and $s' : F(\alpha) \to F(\alpha)$ the extension of s'. The following diagram commute:

$$\begin{tikzcd}
F(\Sigma) \arrow[r, "s"] \arrow[d, "\check{f}"'] & F(\Sigma) \arrow[d, "\check{f}"] \\
F(\alpha) \arrow[r, "s'"'] & F(\alpha)
\end{tikzcd}$$

Indeed, just apply induction on complexity. Let $x \in X$, $\check{f} \circ s(x) = s'(x) = s' \circ \check{f}(x)$ ($\check{f}$ carries variable to variable). Now let $\varphi = c_n(\psi_0, ..., \psi_{n-1})$ and suppose that $\check{f} \circ s(\psi_i) = s' \circ \check{f}(\psi_i)$ such that $i = 0, ..., n-1$.

$$\begin{aligned}
\check{f} \circ s(\varphi) &= \check{f} \circ s(c_n(\psi_0, ..., \psi_{n-1})) \\
&= \check{f}(c_n(s(\psi_0)), ..., s(\psi_{n-1})) \\
&= f(c_n)(\check{f}(s(\psi_0)), ..., \check{f}(s(\psi_{n-1}))) \\
&= f(c_n)(s' \circ \check{f}(\psi_0), ..., s' \circ \check{f}(\psi_{n-1})) \\
&= s'(f(c_n)(\check{f}(\psi_0), ..., \check{f}(\psi_{n-1}))) \\
&= s' \circ \check{f}(c_n(\psi_0, ..., \psi_{n-1})) \\
&= s' \circ \check{f}(\varphi).
\end{aligned}$$

By definition $\Gamma \vdash' \varphi$ iff $\check{f}(\Gamma) \vdash_a \check{f}(\varphi)$. s' is a substitution and $\vdash_a$ is structural, so $s'(\check{f}[\Gamma]) \vdash_a s'(\check{f}(\varphi)) \Leftrightarrow \check{f} \circ s[\Gamma] \vdash_a \check{f} \circ s(\varphi) \Leftrightarrow s[\Gamma] \vdash' s(\varphi)$. Therefore $\vdash'$ is a Tarskian consequence relation. consider $a' = (\Sigma, \vdash')$. By definition of $\vdash'$ we have that f can be seen as an application from a' to a and it is a Δ-dense conservative translate.

Now we prove that a' is algebraizable logic. Fix (Δ, τ) any algebraizing pair of a. For each $\sigma_i \in \Delta$, $i \leq m$, choose, by Δ-density of f, $\sigma'_i \in F(\Sigma)[2]$ such that $\vdash_a \check{f}(\sigma'_i)\Delta\sigma_i$. Define

$$\Delta' = \{\sigma'_i \in F(\Sigma)[2] : i \leq m\}.$$

Then Δ' is a finite set of Σ-formulas in the variables x_0, x_1. Similarly, for each $\langle t^1_j, t^2_j \rangle \in \tau$, $j \leq n$, choose $t'^l_j \in F(\Sigma)[1]$ such that $\vdash_a \check{f}(t'^l_j)\Delta t^l_j$, $l = 1, 2$. Define

$$\tau' = \{\langle t'^1_j, t'^2_j \rangle : j \leq n\}.$$

Then τ' is a finite set of pairs Σ-formulas in the variable x_0.

We are going to show that (Δ', τ) is an algebraizable pair to a'. For this we use the Theorem 1.3.8.

(a) Let $\varphi \in F(\Sigma)$. Δ is a equivalence set of formulas to a, then $\vdash_a \Delta(x, x)$, i.e., for each $i \leq m$, $\vdash_a \sigma_i(x, x)$. By definition of Δ', for each $\sigma'_i \in \Delta'$ there is $\sigma_i \in \Delta$ such that $\vdash_a \check{f}(\sigma'_i)\Delta\sigma_i$. Therefore, as Δ satisfies the transitivity property, $\vdash_a \check{f}(\sigma'_i(x, x))$, i.e $\vdash_a \check{f}(\Delta'(x, x))$, thus $\vdash_{a'} \Delta'(x, x)$. and then, as $\vdash_{a'}$ is structural, then $\vdash_{a'} \Delta'(\psi, \psi)$, for each $\psi \in F(\Sigma)$.

The proofs of conditions **(b)**, **(c)** and **(d)** are analogous to the item **(a)**. For the item **(d)**, we use the Δ-density to choose, for each $c_n \in \alpha_n$, a formula $\psi_n \in F(\Sigma)[n]$ such that $\vdash_a \check{f}(\psi_n)\Delta c_n(x_0, \cdots, x_{n-1})$.

(e) Since $\vdash_{a'}$ is structural, it is enough to prove that $x \dashv_{a'}\vdash \Delta'(\tau'(x))$. Let $\langle t'_1, t'_2\rangle \in \tau'$, then $\vdash_a \Delta(\check{f}(t'_1), t_1)$ and $\vdash_a \Delta(\check{f}(t'_2), t_2)$ for some $\langle t_1, t_2\rangle \in \tau$. Thus $\check{f}(t'_1) \dashv_a\vdash t_1$ and $\check{f}(t'_2) \dashv_a\vdash t_2$. As Δ satisfies the transitivity and symmetric properties we have that $\Delta(\check{f}(t'_1), \check{f}(t'_2)) \dashv_a\vdash \Delta(t_1, t_2)$. Therefore $x \dashv_a\vdash \Delta(\check{f}(t'_1), \check{f}(t'_2))$. By definition of Δ' we have that $x \dashv_a\vdash \check{f}(\Delta')(\check{f}(t'_1), \check{f}(t'_2)) = \check{f}(\Delta'(t'_1, t'_2))$. Hence $x \dashv_{a'}\vdash \Delta'(t'_1(x), t'_2(x))$. $\langle t'_1, t'_2\rangle$ was taken arbitrary, then $x \dashv_{a'}\vdash \Delta'(\tau'(x))$

Finally, let $\Gamma \cup \{\varphi\} \subseteq F(\Sigma)$. Suppose that $\Gamma \vdash \varphi$, so $\check{f}[\Gamma] \vdash_a \check{f}(\varphi)$ and then $\Gamma \vdash' \varphi$. Therefore $\vdash \leq \vdash'$. Moreover, as $f : a' \to a$ is a Δ-dense morphism that is a conservative translation and that also preserves algebraizing pair, then $[f] : a' \to a$ is an $\mathcal{A}_f$-isomorphism.

(ii) As $[f] : a' \to a$ is an $\mathcal{A}_f$-isomorphism, then $f^\star : QV(a) \overset{\cong}{\to} QV(a')$ (see proposition 2.2.4), thus $QV(a') = f^\star[QV(a)]$.

We provide also a direct proof. By Proposition 2.1.6 we have that $f^\star[QV(a)]$ is closed under substructure. As $f^\star\restriction$ preserves product, so $f^\star[QV(a)]$ is closed under product. Let $M \in \Sigma - Str$ such that $I : M \cong f^\star(A)$ where $A \in QV(a)$. It is easy to see that this isomorphism establishes a isomorphism between A and A' where $|A'| = |M|$ and given $c'_n \in \alpha_n$, $c_n'^{A'}(m_0, ..., m_{n-1}) = I^{-1}(c_n'^{A}(I(m_0), ..., I(m_{n-1})))$. Therefore $f^\star[QV(a)]$ is closed under isomorphism. Remains to show that it is closed under ultraproducts.

Let $I \in Set$. Given U ultrafilter in I, θ_U is a congruence on $\prod_{i\in I} A_i \in f^\star[QV(a)]$. We know that $f^\star$ preserves strict products, then there is $A'_i \in QV(a)$ for each $i \in I$ such that $\prod_{i\in I} f^\star(A'_i) = \prod_{i\in I} A_i$. Remember that the definition of θ_U is $\langle a, b\rangle \in \theta_U \; iff \; \{i \in I;\ a(i) = b(i)\} \in U$. So θ_U is a congruence in $\prod_{i\in I} A'_i$. Then we have the morphisms $q' : \prod_{i\in I} f^\star(A'_i) \to f^\star(\prod_{i\in I} A_i/U)$ and $q : \prod_{i\in I} f^\star(A'_i) \to \prod_{i\in I} f^\star(A'_i)/U$. Observe that $ker(q') = \{\langle a, b\rangle \in \prod_{i\in I} f^\star(A'_i);\ a/\theta_U = b/\theta_U\} = \theta_U$. By isomorphism theorem $f^\star(\prod_{i\in I} A'_i/U) \cong \prod_{i\in I} f^\star(A'_i)/U$. We proved that $f^\star[QV(a)]$ is closed under isomorphism, hence $\prod_{i\in I} f^\star(A'_i)/U \in f^\star[QV(a)]$. Therefore $f^\star[QV(a)]$ is closed under ultraproduct. With this $f^\star[QV(a)]$ is a quasivariety.

Remains to show that $f^\star[QV(a)]$ is an equivalent algebraic semantic to l. It is enough to prove that $f^\star[QV(a)]$ is axiomatize for

$S^0_{a'} = \{\delta'(\psi'_0) \approx \varepsilon'(\psi'_0) \wedge ... \wedge \delta'(\psi'_{n-1}) \approx \varepsilon'(\psi'_{n-1}) \to \delta'(\varphi') \approx \varepsilon'(\varphi');$ $\{\psi'_0, ..., \psi'_{n-1}\} \vdash_{a'} \varphi'\}$

$S^1_{a'} = \{\delta'(x_0\Delta' x_1) \approx \varepsilon'(x_0\Delta' x_1) \to x_0 \approx x_1\}$

$S^2_{a'} = \{\delta'(x_0\Delta' x_0) \approx \varepsilon'(x_0\Delta' x_0)\}$

Consider $\{\psi'_0, ..., \psi'_{n-1}\} \vdash_{a'} \varphi'$. So $\{\check{f}(\psi'_0), ..., \check{f}(\psi'_{n-1})\} \vdash_a \check{f}(\varphi)$. Let $M' \in f^\star[QV(a)]$. Suppose that $M' \models \delta'(\psi'_i) \approx \varepsilon'(\psi'_i)$ for all $i = 0, ..., n-1$. Let $v : X \to M'$. Since $M' = f^\star(M)$ and $|f^\star(M)| = |M|$ then there are the extensions $\bar{v} : F(\Sigma) \to f^\star(M)$ and $\bar{v'} : F(\alpha) \to M$ such that $\bar{v'} \circ \check{f} = \bar{v}$ and $\bar{v}(\delta'(\psi'_i)) = \bar{v}(\varepsilon'(\psi'_i))$ for all $i = 0, ..., n-1$. Hence $\bar{v'} \circ \check{f}(\delta'(\psi'_i)) = \bar{v'} \circ \check{f}(\varepsilon(\psi'_i))$ for all $i = 0, ..., n-1$. Therefore $\bar{v'} \circ \check{f}(\delta'(\varphi')) = \bar{v'} \circ \check{f}(\varepsilon'(\varphi'))$ and then $\bar{v}(\delta'(\varphi')) = \bar{v}(\varepsilon'(\varphi'))$. v was taken arbitrary, so $M' \models \delta'(\varphi') \approx \varepsilon'(\varphi')$. With this $f^\star[QV(a)]$ satisfies $S^0_{a'}$. Analogously we have that $f^\star[QV(a)]$ satisfies the conditions $S^1_{a'}$ and $S^2_{a'}$. □

Now we recall the

Definition 5.1.2. *A Hilbert-style calculus H of type Σ is a set H of g-sequents of type Σ with the following property: for any substitution σ and any g-sequent $\langle\Gamma, \varphi\rangle$ of the calculus, the sequent $\langle\sigma[\Gamma], \sigma(\varphi)\rangle$ is also a g-sequent of the calculus; that is, it is a set of g-sequents closed under substitution instances. The g-sequents with an empty set of premises are called the axioms or axiom rules of H and the g-sequents with a non-empty set of premises are called the rules of inference of H.*

A finitary Hilbert-style calculus is a calculus all of whose g-sequents have a finite set of premises.

Given a finitary Hilbert-style calculus H and a set of formulas Γ, a deduction in H from Γ is a well-ordered sequence $\langle\varphi_0, ..., \varphi_m\rangle$ such that for any $q < m$, φ_q is (the conclusion of) an axiom of H or an element of Γ or is obtained by previous formulas by an inference rule of H, that is, there is a g-sequent $\langle\Delta, \varphi\rangle \in H$ such that $\Delta \subseteq \{\varphi_0, ..., \varphi_q\}$ and $\varphi_q = \varphi$.

Given a finitary Hilbert-style calculus H we say that a formula φ is deducible in H from a set of formulas Γ, if there is a deduction in H from Γ well-ordered and whose last element is φ.

The following result is well known:

Fact 5.1.3. *([JJLo58]) Every Tarskian logic is axiomatizable by a finitary Hilbert-style calculus.*

Now we are ready to present

Theorem 5.1.4. *Let $l = (\Sigma, \vdash)$ be a logic, $a_1 = (\alpha_1, \vdash_{a_1})$ and $a_2 = (\alpha_2, \vdash_{a_2})$ be Lindenbaum algebraizable logics (equivalently congruential algebraizable logics). If there are $f_1 \in hom_{\mathcal{L}_f}(l, a_1)$ and $f_2 \in hom_{\mathcal{L}_f}(l, a_2)$ then there is an Lindenbaum algebraizable logic $a_3 = (\alpha_3, \vdash_{a_3})$ and inclusion morphisms of signature $i_1 : \alpha_1 \hookrightarrow \alpha_3$ and $i_2 : \alpha_2 \hookrightarrow \alpha_3$ such that $i_1 \circ f_1 \dashv_{a_3}\vdash i_2 \circ f_2$. Moreover, a_3 is the co-product in the category $l \downarrow U$ where $\bar{U} : Q\mathcal{A}_f^c \to Q\mathcal{L}_f$*

Proof:

Firstly consider $\alpha_3 = \alpha_1 \dot{\cup} \alpha_2$. Define $\vdash_3$ the relation given by the following inference rules:

• $\Gamma \vdash_i \varphi$ such that $\Gamma \cup \{\varphi\} \subseteq Fm_i$ and $i \in \{1, 2\}$;

• $\check{f}_1(\varphi) \dashv\vdash \check{f}_2(\varphi)$ for all $\varphi \in Fm_l$.

• $\Delta_1 \dashv\vdash \Delta_2$ for some sets of equivalent formulas Δ_i for a_i such that $i = 1, 2$ (or equivalently $\Delta_1 \vdash \theta_2$ for all $\theta_2 \in \Delta_2$, and $\Delta_2 \vdash \theta_1$ for all $\theta_2 \in \Delta_2$).

With this rules is easy to see that the logic $a'_3 = (\alpha_3, \vdash_3)$ is an algebraizable logic. Just apply the Theorem 1.3.8 for every algebraizable pair (Δ, τ) for any logic a_i, $i \in \{1, 2\}$.

Consider a_3 the least Lindenbaum algebraizable logic such that extend a'_3.

Observe that we have the inclusions $j_i : a_i \to a_3$ for $i \in \{1, 2\}$ and $\check{j}_1 \circ \check{f}_1(\varphi) \dashv\vdash \check{j}_2 \circ \check{f}_2(\varphi)$ for every $\varphi \in Fm_l$.

Now consider for each $i \in \{1, 2\}$, $g_i : a_i \to a$ in $Lind(\mathcal{A}_f)$ such that $\check{g}_1 \circ \check{f}_1 \dashv_a\vdash \check{g}_2 \circ \check{f}_2$. Define $k : a_3 \to a$ which for any $c_n \in \alpha_3$, $k(c_n) = g_i(c_n)$ if $c_n \in \alpha_i$. Notice that with that definition we have that $k \circ j_i = g_i$ for $i \in \{1, 2\}$. Now we are going to prove that k is a translation.

Let $\Gamma \vdash_3 \varphi$. As $\vdash_3$ is finitary we have that there is $\{\psi_0, ..., \psi_n\} \subseteq \Gamma$ such that $\psi_0, ..., \psi_n \vdash_3 \varphi$. By Theorem 5.1.4 there is a Hilbert-style calculus H that axiomatize a_3. So there is a deduction in H from Γ $\langle \delta_0, ..., \delta_m \rangle$ to φ such that $\delta_m = \varphi$ and for any $q < m$, δ_q is a axiom of H or a element of Γ or there is a g-sequent $\langle \Lambda, \psi \rangle$ such that $\Lambda \subseteq \{\delta_0, ..., \delta_q\}$ and $\delta_q = \psi$. Now we prove that $\langle \check{k}(\delta_0), ..., \check{k}(\delta_m) \rangle$ is a deduction from $\check{k}[\Gamma]$ to $\check{k}(\varphi)$. Since $\delta_m = \varphi$, we have that $\check{k}(\delta_m) = \check{k}(\varphi)$.

If $\delta_q = \psi_i$ for every $i \in \{0, ..., n\}$ then just consider $\check{k}(\delta_q) = \check{k}(\psi_i)$ and then $\check{k}(\delta_q) \in \check{k}[\Gamma]$.

If δ_q is a theorem in a_3, then $\delta_q = \theta(\bar{\sigma})$ where $\theta \in Fm_i$ for some $i \in \{1, 2\}$ and $\bar{\sigma} = \{\sigma_0, ..., \sigma_d\} \subseteq Fm_3$. Thus $\vdash_i \theta(\bar{x})$, then $\vdash_a \check{g}(\theta(\bar{x}))$. Considering the substitution such that $\bar{x} \mapsto \check{k}(\bar{\sigma})$. Therefore $\vdash_a \check{g}_i(\theta(\bar{x}))[\bar{x}/\check{k}(\bar{\sigma})]$. So $\vdash_a \check{k}(\theta(\bar{(\sigma)}))$, hence $\vdash_a \check{k}(\delta_0)$. Or as the logic a_3 is a Lindenbaum algebraizable logic we can consider the theorems $\vdash_3 \check{f}_1(\varphi)\Delta\check{f}_2(\varphi)$, for some Δ an equivalence formula to a_i, instead of the rules $\check{f}_1(\varphi) \dashv_3\vdash \check{f}_2(\varphi)$. In this case we have that if $\delta_q = \check{f}_1(\varphi)\Delta\check{f}_2(\varphi)$ then $\check{k}(\delta_q) = \check{g}_1(\check{f}_1(\varphi))\check{g}_i(\Delta)\check{g}_2(\check{f}_2(\varphi))$. As g_i preserves algebraizable pair for any $i \in \{1, 2\}$ and $\check{g}_1 \circ \check{f}_1 \dashv_a\vdash \check{g}_2 \circ \check{f}_2$ then $\vdash_a \check{k}(\delta_q)$.

Suppose that there is $\Lambda \subseteq \{\delta_0, ..., \delta_q\}$ such that $\langle \Lambda, \delta_q \rangle$ is a inference rule. Then we have that

• there are $\theta_0(\bar{x}), ..., \theta_n(\bar{x}) \in Fm_i$ and a substitution σ such that $\{\theta_j(\bar{x})[\bar{x}/\bar{\sigma}];\ j = 0, ..., n-1\} = \Lambda$, $\theta_n(\bar{x})[\bar{x}/\bar{\sigma}] = \delta_q$ and $\Lambda(\bar{x}) := \{\theta_j(\bar{x});\ j = 0, ..., n-1\} \vdash_i \theta_n(\bar{x})$. Then $\check{g}_i[\Lambda(\bar{x})] \vdash_a \check{g}_i(\theta_n(\bar{x}))$. Considering a substitution $\bar{x} \mapsto \check{k}(\bar{\sigma})$ we have that

$$\check{k}[\Lambda] = \check{k}[\Lambda(\bar{x})][\bar{x}/\check{k}(\bar{\sigma})] \vdash_a \check{k}(\theta_n(\bar{x}))[\bar{x}/\check{k}(\bar{\sigma})] = \check{\delta q}$$

then $\check{k}(\delta_q)$ is obtained by $\check{k}[\Lambda]$.

Or

• there are a set $\Lambda(\bar{x}) = \Delta_i$ and $\theta(\bar{x}) \in \Delta_j$ with $i \neq j$ and $i, j \in \{1, 2\}$, and a substitution σ such that $\Lambda(\bar{x})[\bar{x}/\bar{\sigma}] = \Lambda$ and $\theta(\bar{x})[\bar{x}/\bar{\sigma}] = \delta_q$. Since g_p preserves algebraizable pair for $p = 1, 2$ we have that $\check{g}_1[\Delta_1] \dashv_a\vdash \check{g}_2[\Delta_2]$, thus $\check{g}_1[\Lambda(\bar{x})] \vdash_a \check{g}_2(\theta(\bar{x}))$. Therefore $\check{k}[\Lambda] = \check{g}_1[\Lambda(\bar{x})][\bar{x}/\check{k}(\bar{\sigma})] \vdash_a \check{g}_2(\theta(\bar{x}))[\bar{x}/\check{k}(\bar{\sigma})] = \check{k}(\delta_q)$.

With that we have that the g-sequent $\langle \check{k}(\delta_0), ..., \check{k}(\delta_m)\rangle$ is a deduction from $\check{k}[\Gamma]$ to $\check{k}(\varphi)$ and them $\check{k}[\Gamma] \vdash_a \check{k}(\varphi)$.

It is easy to see that k is the unique morphism such that $\check{k} \circ \check{j}_i \dashv_a\vdash \check{g}_i$ for $i \in \{1, 2\}$. □

We finish this short section with a direct application of the following

Fact 5.1.5. *([AR94]) The category $F_1 \downarrow F_2$ is accessible for arbitrary accessible functors $F_i : K_i \to L$ $(i = 1, 2)$.*

Since $Q\mathcal{L}_f^c$ and $Q\mathcal{A}_f^c$ are (finitely) accessible categories, and the functor $\bar{U}^c : Q\mathcal{A}_f^c \to Q\mathcal{L}_f^c$ is a accessible functor (see [MM14]) and due to the result above we have that:

Corollary 5.1.6. *(i) For any congruential logic k, category $k \downarrow \bar{U}^c$ is accessible.*
(ii) For any logic l, let $k = l^{(c)}$ the its congruential closure. Then, since $(l \downarrow \bar{U}^c) \cong (l^{(c)} \downarrow \bar{U}^c)$, the category $l \downarrow \bar{U}^c$ is accessible.

Moreover, from an adaption of Proposition 3.11 in [AFLM07] and an well-known result on limits in comma categories, it follows that $(l \downarrow \bar{U}^c)$ has products of all "bounded families", in particular, it has finite products.

Our intention is use "good" categorical properties of comma categories as $l \downarrow \bar{U}^c$ to apply in the study of meta-logical properties.

5.2 Diagram model of a logic

We begin providing notions in order to define the (left and right) diagram models of a logic. Notice that the definition of (left and right) diagram models use the encoding established in the chapter 2.

From now on, we will use frequently the notion of "concrete"' category, i.e. a pair $(\mathcal{C}, U)$, where $\mathcal{C}$ is a category and $U : \mathcal{C} \to Set$ is a faithful *functor. We denote by $concretCAT$, the category whose objects are concrete categories (on a smaller Grothendieck universe) and whose arrows are the concrete functor (= functors that "commute over Set").*

To each logic $l = (\Sigma, \vdash)$ is associated two pairs (left and right) of data:

(I) two comma categories (over $\mathcal{A}_f$):

- $(l \to U)$, the "left algebrizable spectrum of l";
- $(U \to l)$, the "right algebrizable spectrum of l".

One can see that the left spectrum is naturally associated with that "analysis processes" of combing logics. On the other hand the right spectrum is naturally associated with the "synthesis processes" of combing logics.

$$(l \to U) \qquad\qquad (U \to l)$$

$$\begin{tikzcd} a_0 \arrow[rr, "h"] & & a_1 \\ & l \arrow[ul, "f_0"] \arrow[ur, "f_1"'] & \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[rr, "h"] \arrow[dr, "f_0"'] & & a_1 \arrow[dl, "f_1"] \\ & l & \end{tikzcd}$$

(II) two diagrams (left and right "representation diagram"):

- l-$Mod \leftrightsquigarrow (l \to U, I)$;
- Mod-$l \leftrightsquigarrow (U \to l, L)$.

$$\begin{array}{llll} l - Mod: & (l \to U)^{op} & \longrightarrow & (\Sigma - str \leftarrow CAT) \\ & (a_0, f_0) & \mapsto & (\Sigma - str \overset{f_0^\star I_0}{\leftarrow} QV(a_0)) \\ & (a_0, f_0) \overset{h}{\to} (a_1, f_1) & \mapsto & ((QV(a_1), f_1^\star I_1) \overset{h^\star\upharpoonright}{\to} (QV(a_0), f_0^\star I_0)) \end{array}$$

$$\begin{tikzcd} QVa_0 \arrow[d, "I_0"] \arrow[dd, bend right=60] & QVa_1 \arrow[l, "h^\star\upharpoonright"'] \arrow[d, "I_1"'] \arrow[dd, bend left=60] \\ \alpha_0\text{--}str \arrow[d, "f_0^\star"'] & \alpha_1\text{--}str \arrow[l, "h^\star"] \arrow[d, "f_1^\star"] \\ \Sigma\text{--}str & \end{tikzcd} \qquad l - Mod$$

$$\begin{array}{llll} Mod - l: & (U \to l) & \longrightarrow & (2 - CAT \leftarrow \Sigma - str) \\ & (a_0, f_0) & \mapsto & (QV(a_0) \overset{L_0 f_0^\star}{\leftarrow} \Sigma - str) \\ & (a_0, f_0) \overset{h}{\to} (a_1, f_1) & \mapsto & ((QV(a_0), L_0 f_0^\star) \overset{(h^\star\upharpoonright, \tilde{h})}{\to} (QV(a_1), L_1 f_1^\star)) \end{array}$$

$$\begin{tikzcd}
QVa_0 & & QVa_1 \arrow[ll, "h^\star\restriction"'] \\
\alpha_0\text{–}str \arrow[u, "L_0"] & & \alpha_1\text{–}str \arrow[ll, "h^\star"] \arrow[u, "L_1"'] \arrow[ull, phantom, "\tilde{h}\nearrow"] \\
& \Sigma\text{–}str \arrow[ul, "f_0^\star"] \arrow[ur, "f_1^\star"'] \arrow[uul, bend left=60] \arrow[uur, bend right=60] &
\end{tikzcd}
\qquad Mod - l$$

$L_0 f_0^\star \overset{\tilde{h}f_1^\star}{\to} h^\star\!\restriction L_1 f_1^\star$ where $L_0 h^\star \overset{\tilde{h}}{\to} h^\star\!\restriction L_1$

Since: $(L_0 h^\star g^\star \overset{\tilde{h}g^\star}{\to} h^\star \restriction L_1 g^\star \overset{h^\star\restriction\tilde{g}}{\to} h^\star\!\restriction g^\star\!\restriction L_2) = L_0(gh)^\star \overset{\widetilde{gh}}{\to} (gh)^\star\!\restriction L_2$

then: $(L_0 f_0^\star \overset{\tilde{h}f_1^\star}{\to} h^\star\!\restriction L_1 f_1^\star \overset{h^\star\restriction(\tilde{g}f_2^\star)}{\to} h^\star\!\restriction g^\star\!\restriction L_2 f_2^\star) = L_0(gh)^\star \overset{\widetilde{gh}f_2^\star}{\to} (gh)^\star\!\restriction L_2 f_2^\star$

All functors and commutativity in both diagrams above is justified by Proposition 2.1.7 in Chapter 2, as well as the diagram below.

To each morphism between logics $t : l \to l'$ we have two pairs (left and right) of data:

(I) a left/right "spectral" functors, given by composition/precomposition with t:
$(l \to U) \overset{-\circ t}{\leftarrow} (l \to U)$;
$(U \to l) \overset{t\circ -}{\to} (U \to l')$.

(II) a left/right "representation diagram" morphism , given by precomposition/composition with $t^\star$:
$(l\text{-}Mod) \overset{t^\star\circ -}{\leftarrow} (l'\text{-}Mod)$;
$(Mod\text{-}l) \overset{-\circ t^\star}{\to} (Mod\text{-}l')$.

From now on, we will concern only on develop a "left" representation theory for propositional logics.

$$\begin{tikzcd}
QVa_0 \arrow[dr, "f_0^\star I_0"] \arrow[ddr, bend right=10] & & QVa_1 \arrow[ll, "h^\star\restriction"'] \arrow[dl, "f_1^\star I_1"'] \arrow[ddl, bend left=10] \\
& \Sigma'\text{–}str \arrow[d, "t^\star"] & \\
& \Sigma\text{–}str &
\end{tikzcd}
\qquad (l\text{-}Mod) \overset{t^\star\circ -}{\leftarrow} (l'\text{-}Mod)$$

The category of left diagram models: LM

Generalizing the information given above we define the category of left diagram model for a logic by:

objects: a left diagram model for a logic l, i.e. the functor

$$left(l) : (l \to U)^{op} \xrightarrow{l-Mod} (\Sigma - str \leftarrow concretCAT)_{Set} \hookrightarrow (CAT \leftarrow CAT)$$

I.e., $(\Sigma - str \leftarrow_{Set} concretCAT)_{Set}$ is a subcategory of the comma category, with objects and arrows being functors that "commute over Set".

arrows: a pair $(B, \tau) : left(l') \to left(l)$, where $B : (l' \to U) \to (l \to U)$ is a "change of bases" functor, and $\tau : left(l') \Rightarrow left(l) \circ B$ is a natural transformation. Notice that given $(a', f') \in (l \to U)$, $\tau_{a',f'} : left(l')(a', f') \to left(l)(B(a', f'))$ where $left(l')(a', f') \in (\Sigma' - Str \leftarrow concretCAT)_{Set}$ and $left(l)(B(a', f')) \in (\Sigma - Str \leftarrow concretCAT)_{Set}$.

Then we have that $\tau_{a',f'}$ has two components, namely a functor that commutes over Set, $Proj_1(\tau_{(a',f')}) : QV(cod(f')) \to QV(cod(B(f')))$, and a functor $Proj_2(\tau_{(a',f')}) : \Sigma' - Str \to \Sigma - Str$, that commutes over Set, such that the following diagram commutes (and commutes over Set):

$$\begin{tikzcd}
QV cod(f') \arrow[r] \arrow[d, "f'^{\star}"'] \arrow[dr, phantom, "\overset{\Rightarrow}{\tau_{(a',f')}}"] & QV cod(B(f')) \arrow[d, "B(f')^{\star}"] \\
\Sigma'\text{–}str \arrow[r] & \Sigma\text{–}str
\end{tikzcd}$$

And satisfies the following compatibility condition:

For each $(a'_0, f'_0), (a'_1, f'_1) \in (l' \to U)$:
$Proj_2(\tau_{(a'_0,f'_0)}) = Proj_2(\tau_{(a'_1,f'_1)}) : \Sigma' - str \to \Sigma - str$

$$\begin{tikzcd}
QV cod(f'_1) \arrow[r] \arrow[d, "f_1'^{\star}"'] \arrow[dr, phantom, "\overset{\Rightarrow}{\tau_{(a'_1,f'_1)}}"] & QV cod(B(f'_1)) \arrow[d, "B(f'_1)^{\star}"] \\
\Sigma'\text{–}str \arrow[r] & \Sigma\text{–}str \\
QV cod(f'_0) \arrow[r] \arrow[u, "f_0'^{\star}"] \arrow[ur, phantom, "\underset{\Rightarrow}{\tau_{(a'_0,f'_0)}}"] & QV cod(B(f'_0)) \arrow[u, "B(f'_0)^{\star}"']
\end{tikzcd}$$

Notice that given a logic morphism $t : l \to l'$ we have, by construction above, a morphism between diagram models $left(l)$ and $left(l')$.

One can define the left diagram model for logics over the functor U^c. We denote the left diagram model of a logic l over U^c by $left^c(l)$ and the category of left diagram model over U^c we denote by $\mathbf{LM}^c$.

As a first test of this definition, we present the following:

Proposition 5.2.1. *(a) Given a isomorphism $t : l \overset{\cong}{\to} l' \in \mathcal{L}_f$ we have an isomorphism*

$$(- \circ t, (id, t^\star)) : left(l') \overset{\cong}{\to} left(l) \in \ \mathbf{LM}.$$

(b) Consider $can_l : l \to l^{(c)}$ the morphism in $\mathcal{L}_f$ such that l^c is the congruential closure of l. It induces a $\mathbf{LM}^c$-isomorphism[1]

$$(- \circ can_l, (id, can_l^\star)) : left(l^{(c)}) \overset{\cong}{\to} left(l)$$

Proof:

(a): We have seen above that $(- \circ t) : (l' \longrightarrow U) \to (l \longrightarrow U)$ define a functor (the left "spectral" functor). Since t is an isomorphism, we have that $- \circ t^{-1} : (l \longrightarrow U) \to (l' \longrightarrow U)$ also is a functor, where t^{-1} is the inverse of t. Moreover, $(- \circ t^{-1})$ is the inverse functor of $(- \circ t)$. In order to prove the isomorphism between $left(l)$ and $left(l')$, remains to prove that $(id, t^\star)$ is a natural isomorphism. Notice that given $f' : l' \to a \in (l' \longrightarrow U)$ $(a = (\alpha, \vdash))$, $left(l')(f') = f'^\star \circ I$ and $left(l) \circ (- \circ t)(f') = (f' \circ t)^\star \circ I = t^\star \circ f'^\star \circ I$, where $I : QV(a) \to \alpha - Str$. Observe that $cod((- \circ t)(f')) = a$. Then we have the following diagram commuting:

$$\begin{tikzcd}
QV(a) \arrow[r, "id"] \arrow[d, "f'^\star"'] & QV(a) \arrow[d, "t^\star \circ f'^\star"] \\
\Sigma' - Str \arrow[r, "t^\star"'] & \Sigma - Str
\end{tikzcd}$$

This diagram represents $(id, t^\star)_{f'}$. Since t is an isomorphism we have that $(id, t^\star)$ is also an isomorphism.

Let $h : a \to a'$ such that $h \circ f_1' = f_2'$ for $f_i' : l' \to a_i$ for $i = 1, 2$. $(id, t^\star)$ is a natural transformation due to the following commutative diagram:

$$\begin{tikzcd}
QV(a_2) \arrow[rrr, "id_{QV(a_2)}"] \arrow[dd, "h^\star\restriction"'] \arrow[dr, "f_2'^\star"] & & & QV(a_2) \arrow[dl, "t^\star \circ f_2'^\star"'] \arrow[dd, "h^\star\restriction"] \\
& \Sigma'{-}str \arrow[r, "t^\star"] & \Sigma{-}str & \\
QV(a_1) \arrow[rrr, "id_{QV(a_1)}"'] \arrow[ur, "f_1'^\star"'] & & & QV(a_1) \arrow[ul, "t^\star \circ f_1'^\star"]
\end{tikzcd}$$

(b): In this case we have that $can_l : l \to l^{(c)}$ induces an isomorphism of categories given by the left "spectral" functor: $(- \circ can_l) : (l^{(c)} \longrightarrow U^c) \to (l \longrightarrow U^c)$. Moreover, as $can_l = id_\Sigma$, then $can_l^\star = id_{\Sigma - str} : \Sigma - str \to \Sigma - str$. Thus $(- \circ can_l, (id, can_l^\star))$ is a $\mathbf{LM}^c$-isomorphism following on the similar way of item (a). $\square$

[1]On the other hand, note that $l \cong l^{(c)}$ in $\mathcal{L}_f$ iff l is a congruential logic, iff $l = l^{(c)}$.

5.3 Morita equivalence of logics and variants

In this section we propose a definition of left Morita equivalence of logics and left-stably-Morita equivalence of logics. The definition of left-Morita equivalence is an weakening of the notion of isomorphism of left diagram models.

Definition 5.3.1. *(a) Let S be a subcategory of $(l \to U)$. S is called* generic *if $S \hookrightarrow (l \to U)$ is a* initial functor[2]

(b) The logics l and l' are left Morita equivalent when there are:

- *generic subcategories $S \hookrightarrow (l \to U)$ and $S' \hookrightarrow (l' \to U)$;*
- *equivalence quasi-inverse functors $B : S' \to S$ and $B' : S \to S'$;*
- *"natural comparations": (T, τ) and (T', τ') such that:*

$*$ $T : \Sigma' - str \to \Sigma - str$ is a functor that commutes over Set and for each $(a', f') \in S'$, $\tau_{(a',f')} : QVcod(f') \overset{\cong}{\to} QVcod(B(f'))$ is a concrete isomorphism of categories such that $B(f')^\star \circ \tau_{f'} = T \circ f'^\star$, i.e., the diagram below commutes (and commutes over Set):

$$\begin{tikzcd}
\Sigma'\text{–}str \arrow[r, "T"] & \Sigma\text{–}str \\
QVcod(f') \arrow[u, "f'^\star"] \arrow[r, "\tau_{(a',f')}"'] & QVcod(B(f')) \arrow[u, "B(f')^\star"']
\end{tikzcd}$$

and for each $(a'_0, f'_0) \overset{g}{\to} (a'_1, f'_1) \in S'$ we have the following diagram commuting (and commuting over Set):

$$\begin{tikzcd}
QVcod(f'_1) \arrow[rrr, "\tau_{(a'_1,f'_1)}", "\cong"'] \arrow[dr, "f'^\star_1"] \arrow[dd, "g'^\star"'] & & & QVcod(B(f'_1)) \arrow[dl, "B(f'_1)^\star"'] \arrow[dd, "B(g')^\star"] \\
& \Sigma'\text{–}str \arrow[r, "T"] & \Sigma\text{–}str & \\
QVcod(f'_0) \arrow[ur, "f'^\star_0"'] \arrow[rrr, "\cong", "\tau_{(a'_0,f'_0)}"'] & & & QVcod(B(f'_0)) \arrow[ul, "B(f'_0)^\star"]
\end{tikzcd}$$

$*$ analogous conditions for (T', τ') hold.

In the same vein we can define the notion of left Morita equivalence over the category $l \to U^c$. We call left^c Morita equivalence. Clearly, both notions are equivalence relations on the class of logics.

As a first test of these definitions, we present the following:

Proposition 5.3.2. *If $left(l) \cong left(l')$ then l and l' are left Morita equivalent. In particular:*

[2]Thus $S^{op} \hookrightarrow (l \to U)^{op} \overset{l-Mod}{\to} (concretCAT \to \Sigma - str)_{Set}$ is "relatively coinicial".

(a) If $l \cong l'$, then l and l' are left Morita equivalent.

(b) l and $l^{(c)}$ are left^c Morita equivalent.

Proof:

Since $left(l)$ and $left(l')$ are isomorphic, we have that there are pairs (B, τ), (B', τ') that establish the isomorphism. The proof that l and l' are left Morita equivalent follows from considering $S = (l \longrightarrow U)$ and $S' = (l' \longrightarrow U)$ as generic subcategories, $B : S' \to S$ and $B' = B^{-1} : S \to S'$, and τ and $\tau' = \tau^{-1}$ are the "natural comparisons".

The items (a) and (b) follow from the Theorem 5.2.1. $\square$

Proposition 5.3.3. *Let $a = (\Sigma, \vdash)$ and $a' = (\Sigma' \vdash')$ (respect. Lindenbaum) algebraizable logics. If a and a' are Δ-equipollent then they are left (respect. left^c) Morita equivalent.*

Proof:

In this case just consider $S := \{(a \overset{id_a}{\to} a)\}$ and $S' := \{(a' \overset{id_{a'}}{\to} a')\}$. S and S' are in fact generic subcategories. We define $B : S' \to S$ as $B(id_{a'}) = id_a$ and $B'(id_a) = id_{a'}$. In this way we have that B and B' are inverses functors. Since a and a' are Δ-equipollent, then they are $\overline{\mathcal{A}_f}$-isomrphic, i.e. there are $\mathcal{A}_f$-morphisms $f : a \to a'$ and $f' : a' \to a$ such that for any $\varphi' \in F(\Sigma')$, $\vdash' \check{f} \circ \check{f}'(\varphi')\Delta'\varphi'$ and for any $\varphi \in F(\Sigma)$, $\vdash \check{f}' \circ \check{f}(\varphi)\Delta\varphi$. Thus $f^\star : \Sigma' - Str \to \Sigma - Str$, $f'^\star : \Sigma - Str \to \Sigma' - Str$, $f^\star \restriction : QV(a') \overset{\cong}{\to} QV(a)$ and $f'^\star \restriction : QV(a) \overset{\cong}{\to} QV(a')$ (see Chapter 2) making the following diagrams commute:

$$\begin{tikzcd}
\Sigma'\text{–}str \arrow[r, "f^\star"] & \Sigma\text{–}str \\
QV(a') \arrow[u, "id^\star_{a'}\restriction"] \arrow[r, "f^\star\restriction"'] & QV(a) \arrow[u, "id^\star_a\restriction"']
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma\text{–}str \arrow[r, "f'^\star"] & \Sigma'\text{–}str \\
QV(a) \arrow[u, "id^\star_{a}\restriction"] \arrow[r, "f'^\star\restriction"'] & QV(a') \arrow[u, "id^\star_{a'}\restriction"']
\end{tikzcd}$$

Considering the pairs $(f^\star, f^\star\restriction)$ and $(f'^\star, f'^\star\restriction)$ the natural comparisons, we have proved that a and a' are left Morita equivalent. $\square$

Having in mind the result above, we define a new equivalence notion: of left Morita *equipollence.*

Definition 5.3.4. *Given $l = (\Sigma, \vdash)$ and $l' = (\Sigma', \vdash')$ logics, we shall say that they are left Morita* equipollent *if:*

• *There are categories Q and Q' and* initial *functors $K : (l \to U) \to Q$, $K' : (l' \to U) \to Q'$, such that $l - Mod : (l \to U)^{op} \to (concretCAT \to \Sigma - str)_{Set}$ factors through K^{op} and $l' - Mod : (l' \to U)^{op} \to (concretCAT \to \Sigma' - str)_{Set}$ factors through K'^{op}.*

• *There are quasi-inverse equivalence functors $B : Q' \to Q$ and $B' : Q \to Q'$.*

• *There are "natural comparisons" (T, τ) and (T', τ') as well as in the definition 5.3.1 such that obey the same properties described there.*

A natural test for this definition is the following

Proposition 5.3.5. *(a) If $left(l) \cong left(l')$ then l and l' are left Morita equivalent.*

(b) Let $l = (\Sigma, \vdash)$ and $l' = (\Sigma', \vdash')$ be logics. If l and l' are equipollent then they are left Morita equipollent.

Proof: The proof of item (a) works exactly as the proof of Proposition 5.3.2.

(b) Since l and l' are equipollent we have that there are $Q\mathcal{L}_f$-isomorphisms $[h] : l \to l'$ and $[h'] : l' \to l$. Consider the categories $Q = l \downarrow \bar{U}$, $Q' = l' \downarrow \bar{U}$, then we have that $B = ([h] \circ -)$ and $B' = ([h'] \circ -)$ establish pair of inverses isomorphisms between Q and Q'. Denoting the "projection functors" $K : (l \to U) \to (l \to \bar{U})$, $K' : (l' \to U) \to (l' \to \bar{U})$, by the results in Chapter 2, $l - mod$ factors through K^{op} and $l' - mod$ factors through K'^{op}. The natural comparisons here are given by $(h^\star, id)$, $(h'^\star, id)$.

Now we will provide another (diagrammatic) weak notion of equivalence of logics:

Definition 5.3.6. *The logics l and l' are left-stably Morita equivalent when:*

• *there are concrete functors: $\Sigma' - str \underset{F'}{\overset{F}{\rightleftarrows}} \Sigma - str$;*

• *there are concrete functors: $colim_{f' \in (l' \to U)}\ QVcod(f') \underset{E'}{\overset{E}{\rightleftarrows}} colim_{f \in (l \to U)}\ QVcod(f)$;*

such that:

• *E and E' are quasi-inverse equivalence functors;*

• *the diagram below commutes (and commutes over Set):*

$$\begin{tikzcd}
\Sigma\text{–}str \arrow[d, shift left, "F'"] & & colim\ QV(cod(f)) \arrow[ll, "can"'] \arrow[d, shift right, "E'"'] \\
\Sigma'\text{–}str \arrow[u, shift left, "F"] & & colim\ QV(cod(f')) \arrow[ll, "can'"] \arrow[u, shift right, "E"'] \arrow[u, phantom, "\simeq"]
\end{tikzcd}$$

The result below is parallel to Proposition 5.3.2:

Proposition 5.3.7. *Let l, l' be logics. If $left(l) \cong left(l')$, then l and l' are left-stably Morita equivalent.*

Proof:

Since $left(l)$ and $left(l')$ are isomorphic then we have that there are $B : (l' \longrightarrow U) \overset{\cong}{\to} (l \longrightarrow U)$ isomorphism and $\tau : left(l') \overset{\cong}{\Rightarrow} left(l) \circ B$ natural isomorphism. Remember that τ has two components, $Proj_1(\tau_{f'}) : QV(cod(f')) \overset{\cong}{\to} QV(cod(B(f')))$ and $Proj_2(\tau_{f'}) : \Sigma' - Str \overset{\cong}{\to} \Sigma - Str$ for any $f' \in (l' \longrightarrow U)$ such that $Proj_2(\tau_{f'_1}) = Proj_2(\tau_{f'_2})$ for all $f'_1, f'_2 \in (l' \longrightarrow U)$ that we denote just by $Proj_2(\tau)$. Since B is a isomorphism we have the concrete isomorphism of categories:

$$colim_{f \in (l \longrightarrow U)} QV(cod(f)) \cong colim_{f' \in (l' \longrightarrow U)} QV(B(f')).$$

Therefore there are

$$colim_{f' \in (l' \to U)}\ QV cod(f') \underset{E'}{\overset{E}{\rightleftarrows}} colim_{f \in (l \to U)}\ QV cod(f)$$

such that we have the following diagram commuting:

$$\begin{tikzcd}
\Sigma\text{–}str \arrow[d, shift right, "Proj_2(\tau^{-1})"'] & & colim\ QV(cod(f)) \arrow[ll, "can"'] \arrow[d, shift right, "E'"'] \\
\Sigma'\text{–}str \arrow[u, shift right, "Proj_2(\tau)"'] & & colim\ QV(cod(f')) \arrow[ll, "can'"] \arrow[u, shift right, "E"'] \arrow[u, phantom, "\simeq"]
\end{tikzcd}$$

□

The Proposition above can be derived from the more general result below:

Proposition 5.3.8. **(a)** *If l and l' are left Morita equivalent, then l and l' are left-stably Morita equivalent.*

(b) *If l and l' are left Morita equipollent, then l and l' are left-Stably Morita equivalent.*

We will register the following test of these notion on "well-behaved logics":

Proposition 5.3.9. *Let a_0 and a_1 be Lindenbaum algebrazaible logics such that $a_0 \cong a_1 \in |Q\mathcal{A}^c_f|$. Then a and a' are left^c stably Morita equivalent.*

Proof:

Let $a_0 \underset{h}{\overset{h'}{\rightleftarrows}} a_1$ be a pair of $\mathcal{A}_f$-morphisms that are inverse in the quotient category. By 2.2.4 we have that $QV(a_0) \underset{h^\star}{\overset{h'^\star}{\leftrightarrows}} QV(a_1)$ is an isomorphism of categories. Since a_i, $i = 0, 1$ are Lindenbaum algebraizable logics, the comma categories $(a_i \to \bar{U}^c)$ have an initial object $id_{a_i} : a_i \to a_i$, then $colimQV(a_i) \cong QV(a_i)$ and the canonical arrow can be identified with the inclusion $J_i : QV(a_i) \hookrightarrow \alpha_i - Str$, then we have the following diagram commuting:

$$
\begin{tikzcd}
\alpha\text{–}str \arrow[d, shift left, "h'^{\star}"] & & QV(a_0) \arrow[ll, "I_0"'] \arrow[d, shift right=2, "h'^{\star}\restriction"'] \arrow[d, phantom, "\cong"] \\
\alpha'\text{–}str \arrow[u, shift left, "h^{\star}"] & & QV(a_1) \arrow[ll, "I_1"] \arrow[u, shift right=2, "h^{\star}\restriction"']
\end{tikzcd}
$$

Therefore a and a' are stably-Morita-equivalent. □

As a consequence of this proposition, we have a solution for the "identity problem" for classical propositional logics.

Corollary 5.3.10. *The presentations of classical logics are left stably Morita equivalent.*

$$
\begin{tikzcd}
(\neg,\rightarrow)\text{–}str \arrow[d, shift left, "t'^{\star}"] & & BA(\neg,\rightarrow) \cong colim\ QV(cod(f)) \arrow[ll, "incl"'] \arrow[d, shift right=2, "t'^{\star}\restriction"'] \arrow[d, phantom, "\cong"] \\
(\neg',\vee')\text{–}str \arrow[u, shift left, "t^{\star}"] & & BA(\neg',\vee') \cong colim\ QV(cod(f')) \arrow[ll, "incl'"] \arrow[u, shift right=2, "t^{\star}\restriction"']
\end{tikzcd}
$$

The next proposition give us the first step on the way to measure distinction degrees between logic, i.e., this propositions tell us that the intuitionistic logic and classical logic are not left stably-Morita-equivalent, but they are left stably-Morita-adjoint (on the left and on the right).

Proposition 5.3.11. *Concerning the relations between Classical logics and Intuitionist logics:*

(a) They are not stably-Morita-equivalent.

(b) But they are (only) stably-Morita-adjointly related on the right and on the left:

I.e. there is a commutative diagram (that commutes over Set)

$$
\begin{tikzcd}
(\neg,\vee,\wedge,\rightarrow)\text{–}str \arrow[d, shift left, "id"] & & HA \cong colim\ QV(cod(f)) \arrow[ll, "incl"'] \\
(\neg,\vee,\wedge,\rightarrow)\text{–}str \arrow[u, shift left, "id"] & & BA \cong colim\ QV(cod(f')) \arrow[ll, "incl'"] \arrow[u, "{}_{L}J_{R}"']
\end{tikzcd}
$$

where the diagram with J commutes (and commutes over Set) and there are functors $L, R : HA \to BA$ such that $L \dashv J \dashv R$

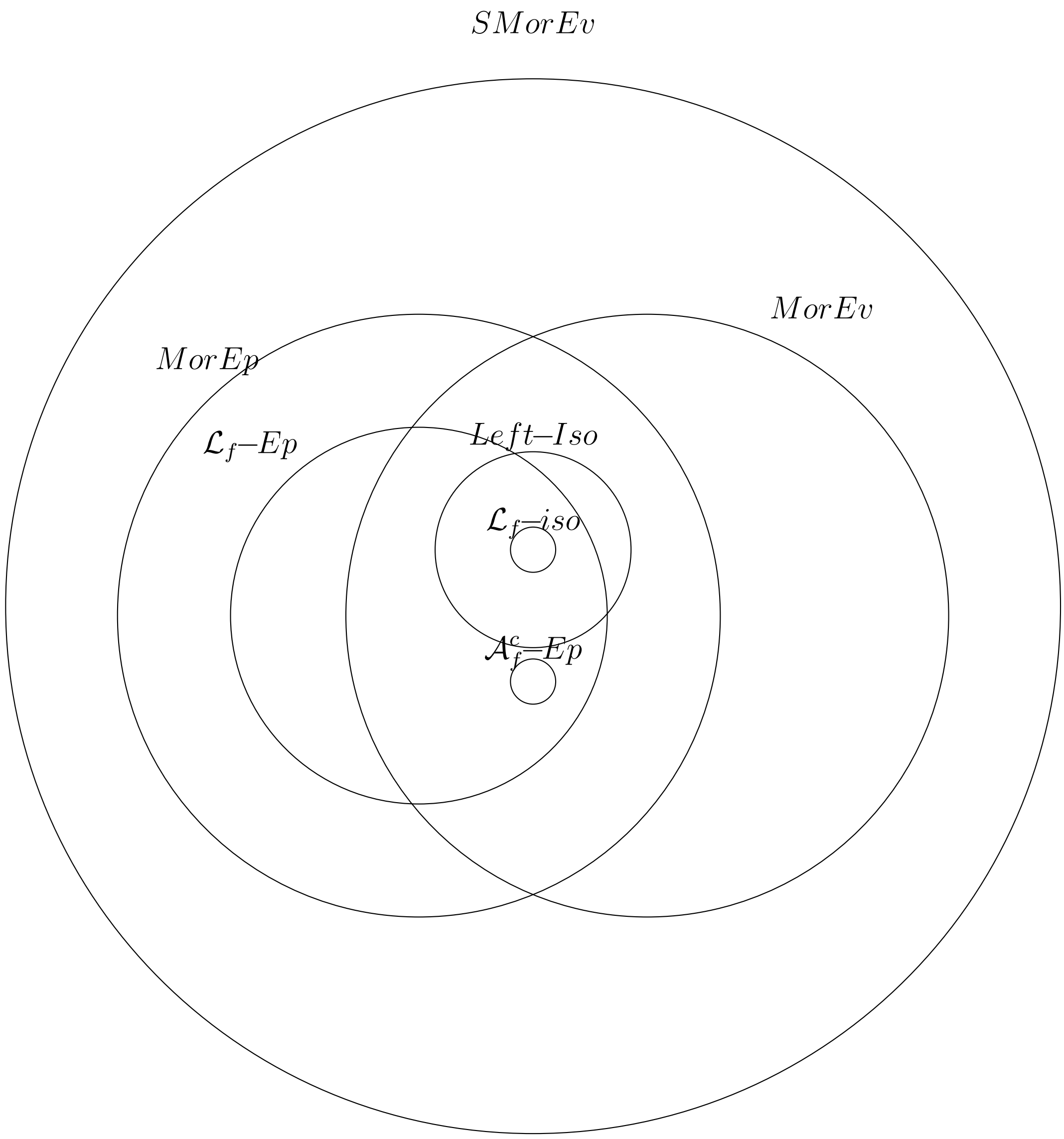
Proof: (a) Recall that the category BA of Boolean algebras satisfies the the famous "Stone representation Theorem": for any $B \in BA$ there is a set X and an BA-monomorphism into the X-power of $2 = \{0, 1\}$, $B \rightarrowtail 2^X$; this representation result is preserved under equivalence of categories, however HA, the category os Heyting algebras, has no object with the same role as 2 in BA.

(b) For instance consider c and i presentations of classical and intuitionist logics in the signature $\{\neg, \vee, \wedge, \rightarrow\}$ and BA and HA the categories of Boolean algebras and Heyting algebras in this signatures. Consider the inclusion functor $J : BA \hookrightarrow HA$. Then J clearly satisfies the diagrammatic commutative conditions an, moreover J admits left and right adjoints:

Let $L : HA \rightarrow BA : H \mapsto H/F_H$, where $F_H = \langle\{a \leftrightarrow \neg\neg a : a \in H\}\rangle$ is the filter generated by the above subset of H. As we recall in chapter 4, the quotient homomorphism $q_H : H \twoheadrightarrow J(L(H))$ has the universal property. Thus $L \dashv J$. Alternatively, if $Reg(H) = \{a \in H : \neg\neg a = a\}$ denotes the sub-boolean algebra of regular elements of H, then the map $H \twoheadrightarrow J(Reg(H))$, $b \mapsto \neg\neg b$ is an homomorphism and satisfies the universal property (thus $Reg(H) \cong L(H)$).

On the other hand, let $R(H) = \{b \in H : b \vee \neg b = 1\}$, the subalgebra of complemented elements of H: it is a boolean subalgebra. Then the inclusion homomorphism $i_H : J(R(H)) \hookrightarrow H$ has the following co-universal property: for each boolean algebra B and each HA-homomorphism $f : J(B) \rightarrow H$, it has a unique factorization $\tilde{f} : B \rightarrow R(H)$ through i_H. Thus $J \dashv R$. □

We finish this chapter, with a diagram that summarizes the order relation among the notions of identities of logics. We abbreviate: $\mathcal{L}_f$-isomorphism ($\mathcal{L}_f - Iso$), $\mathcal{L}_f$-equipollence ($\mathcal{L}_f - Ep$), Lindenbaum algebraizable equipollence ($\mathcal{A}_f^c - Ep$), left-diagram-models isomorphism ($Left - Iso$), left Morita equivalence ($MorEv$), left Motita equipollence ($MorEp$) and left-stably Morita equivalence ($SMorEv$).


$SMorEv$
$MorEv$
$MorEp$
$\mathcal{L}_f$–Ep
$Left$–Iso
$\mathcal{L}_f$–iso
$\mathcal{A}_f^c$–Ep

Chapter 6

Conclusion

We have provided a basic background in order to establish a representation theory of logic. Firstly, we gave preliminaries notions to develop this thesis and the main motivation to study the categories of logics. A survey on that subject was published in *São Paulo Journal of Mathematical science* [Pin15]. We have developed a functorial encoding for morphisms of algebraizable logics and characterization for two special kinds of morphisms, namely the dense morphisms and $Q(\mathcal{A}_f^c)$-isomorphisms. This part of the thesis was submitted in December 2015 to *Journal of the IGPL*, but one can find it in ArXiv [MP15]. We have introduced the notion of filter functors and its associated logic as a consequence of the attempting to generalize the codification presented before. Studying a special kind of filter functor, we have classified the protoalgebraizable logics, equivalential logics, truth-equational logics and the algebraizable logics. We also have applied the previous results about filter functors in order to study the meta-logical Craig entailment interpolation property via amalgamation in matrices for non-protoalgebraizable logics. At the end of the chapter 3, we have defined the category of filter functors and its relation with the category of logic $\mathcal{L}_f$. This material was developed in joint work with Prof. Peter Arndt, Prof. Dr. Ramon Jansana and Prof. Dr. Hugo Mariano and it is in the final process of typing for a submission [AJMP].

In the sequel we have established the categorial connection between institutions and π-institutions. It was introduced the institutions for abstract logics, algebraizable logics and Lindenbaum algebraizable logics. On the institutions for algebraizable logics we have defined the Glivenko's context and then the abstract Glivenko's theorem that restricts to the traditional Glivenko's theorem. There is an intermediate preprint about this chapter that we intend finish and submit within few months [MP].

In the chapter 5 we have presented just first steps toward establish a representation theory for propositional logics. It was introduced the notion of left diagram model of logics and the notions of left (stably-)Morita equivalence. This left-side approach is related to the processes of analysis of logics; the right-hand side is mathematically more involved and it is related to synthesis processes of logics. As a sample of that the definition of left (staby-

)Morita equivalence may be useful, we have proved that any presentation of classical logic are stably-Morita equivalent, but the classical and intuitionistic logics are not left stably-Morita equivalent. On the way to get a “algebraic topology” for logics, i.e., providing mathematical objects to distinguish logics, we have proved that the classical and intuitionistic logic are not stably-Morita equivalent, but they are stably-Morita adjoint. The first steps on this chapter one can find in [MP14].

Future works. Regarding the subject developed in chapter 2, we believe that is possible get characterizations for others kinds of algebraizable logic morphisms. Another attempt is establishing a functorial encoding for morphisms in some special logics like protoalgebraizable logics, equivalential logics and truth-equational logics.

In the direction of expanding the work of the chapter 3, we will try to develop the filter functor theory for non-finitary logics, i.e., replace the definition of filter functor for algebraic lattices to κ-algebraic lattices, addressing a question posed in [CN14]. Another theme is studying different meta-logical properties in this framework like Beth property or other notions of interpolation properties. Still on, classify others special kinds of logics like weakly-algebraizable logics and implicative logics. Study more about the category of filter functor and its connection with the category of logic in order to characterize some morphisms of logics as well as in the chapter 2. Realize what happens when the connection between category of filter functor and logics is restricted for protoalgebraizable logics, equivalential logics and truth-equational logics.

We believe we can establish a categorial relation of the categories of institution and π-institutions with different notions of morphisms we have considered in the chapter 4. Using the ideas behind of the institutions for algebraizable logics and Lindenbaum algebraizable logics, we intend define the institutions for special logics, i.e., protoalgebraizable logics, equivalential lolgics and truth-equational logics. Thus, generalize more the abstract Glivenko's theorem. Connecting the chapter 3 and 4 we intend understand the precise categorical relation between the category of filter functor and categories of institutions and π-institutions.

In order to develop the representation theory of logic we just have started in the chapter 5, we intend describe necessary/sufficient conditions for Morita equivalence of logics (and variants), analyze categories of fractions of categories of logics and define an “algebraic topology” for logics, i.e., define a general theory of “mathematical invariants” to measure the degree of distinction among arbitrary propositional logics and develop general methods of calculation of invariants. One of the goals of developing a representation theory is provide and work with new notions of “identities” between logics. Finally, we intend present the precise definition of the right diagram model of a logic that allows one get basic results analogous to the “left side”: This case is more involved because it needs a 2-categorial approach.

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