**A study in set-theoretic functional analysis:
Extensions of $c_0(I)$ -valued operators on linearly ordered compacta
and weaker forms of normality on Ψ -spaces**

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candidato Victor dos Santos Ronchim, tal como
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Resumo

RONCHIM, Victor S. **Um estudo em análise funcional conjuntista: Extensões de operadores valorados a $c_0(I)$ em compactos totalmente ordenados e enfraquecimentos de normalidade em Ψ -espaços.** 2021. 98 f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2010.

Na primeira parte deste trabalho nós investigamos generalizações do clássico teorema de Sobczyk, que afirma que todo operador limitado valorado a c_0 e definido em um subespaço fechado de um espaço de Banach separável admite uma extensão limitada. Em direção a este objetivo, nós exploramos a generalização deste problema quando o espaço c_0 é substituído pela sua versão não separável $c_0(I)$, abordando o problema de estender operadores limitados definidos em uma subálgebra de Banach unital de $C(K)$, onde K é um compacto totalmente ordenado em uma tentativa de generalizar os resultados de D.V. Tausk e C. Correa. Nós descrevemos uma classe de compactos totalmente ordenados, chamada de separavelmente determinada, onde os critérios para extensão de operadores valorados a c_0 e para operadores valorados a $c_0(I)$ coincidem. Na segunda parte, nós examinamos enfraquecimentos de normalidade em espaços de Mrówka-Isbell. Estudamos o conceito de semi-normalidade nestes espaços, provendo resultados estruturais que conectam normalidade, semi-normalidade e quase-normalidade. Nós definimos o conceito de separação, chamado fortemente $(\aleph_0, < \mathfrak{c})$ -separado, e provamos a existência genérica de famílias quase disjuntas, completamente separáveis e fortemente $(\aleph_0, < \mathfrak{c})$ -separadas sob a hipótese $\mathfrak{s} = \mathfrak{c}$ and $\mathfrak{b} = \mathfrak{c}$, respondendo uma questão proposta por P. Szeptycki e S. Garcia-Balan.

Palavras-chave: Análise Funcional, Teorema de Sobczyk, Extensão de operadores, Compactos totalmente ordenados, Topologia conjuntista.

Abstract

RONCHIM, Victor S. **A study in set-theoretic functional analysis: Extensions of $c_0(I)$ -valued operators on linearly ordered compacta and weaker forms of normality on Ψ -spaces.** 2021. 98 f. thesis (Phd) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2010.

In the first part of this work we investigate generalizations of the classical theorem of Sobczyk, which states that every c_0 -valued bounded operator defined on a closed subspace of a separable Banach space admits a bounded extension to the entire space. Towards this goal, we explore the generalization of this problem when c_0 is replaced by the non-separable space $c_0(I)$, addressing the problem of extending bounded operators defined on Banach unital subalgebras of $C(K)$, where K is a linear compact space in an attempt to extend the results of D.V. Tausk and C. Correa. We describe a class of linear compact spaces, called separably determined, where the criteria for extending c_0 -valued operators and the one for extending $c_0(I)$ -valued operators are the same. On the second part of this work we examine weaker forms of normality in Mrówka-Isbell spaces. We study the concept of semi-normality in spaces providing structural results connecting normality, semi-normality and almost-normality. We define the separation concept of strongly $(\aleph_0, < \mathfrak{c})$ -separated almost disjoint families and prove the generic existence of completely separable strongly $(\aleph_0, < \mathfrak{c})$ -separated almost disjoint families under the assumption $\mathfrak{s} = \mathfrak{c}$ and $\mathfrak{b} = \mathfrak{c}$, answering a question from P. Szeptycki and S. Garcia-Balan.

Keywords: Functional Analysis, Sobczyk's theorem, Operator extension, Linearly ordered compacta, Set-theoretic topology.

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List of Abbreviations

AD	Almost disjoint
c_0 -EP	c_0 -extension property
$c_0(I)$ -EP	$c_0(I)$ -extension property
CH	Continuous Hypothesis
c.i.s.	Continuous increasing surjection
DA	Double-arrow space
κ -SF	κ -separating family
LOTS	Linearly ordered topological spaces
MA	Martin's axiom
MAD	Maximal almost disjoint
NBV	Normalized bounded variation
Sad	Separable adjacency
SD	Separably determined
SF	separating family
SFIP	Strongly finite intersection property
WLOG	Without loss of generality

List of Symbols

$C(K)$	The Banach space of continuous functions defined on a compact Hausdorff space endowed with supremum norm
X^*	The dual of a Banach space
$X \equiv Y$	The spaces are isometric
X^+	The set of right-isolated points of a ordered space X
X^-	The set of left-isolated points of a ordered space X
$I \cup \{\infty\}$	The one-point compactification of the discrete set I
0_X	The minimum of a totally ordered set X
1_X	The maximum of a totally ordered set X
$\tau_{\leq Y}$	The order topology in Y generated by the restricted order of a superset to Y
$(\tau_{\leq}) _Y$	The topology of subspace for a subset Y of an ordered set
$y_n \searrow x$	Limit of a decreasing sequence to x
$y \searrow x$	Limit from above x
$y_n \nearrow x$	Limit of an increasing sequence to x
$y \nearrow x$	Limit from below x
$DA(K, X)$	The double-arrow space for $X \subseteq K$
$w(X)$	The weight of a topological space
$d(X)$	The density of a topological space
$L(X)$	The Lindelöf degree of a topological space
$c(X)$	The cellularity of a topological space
$h\theta(X)$	The hereditariness of a cardinal function θ of a topological space X
$\chi(x, X)$	The character of a point x in a topological space X
$\chi(X)$	The character of a topological space X
$\chi_L(x, K)$	The left-character of a point in a linear compact space
$\chi_R(x, K)$	The right-character of a point in a linear compact space
$\text{Clop}(K)$	The collection of all clopen sets of K
$M(K)$	The Banach space of all regular finite scalar-valued Borel measures on a compact Hausdorff K
$V(f, P)$	The variation of f in respect to the partition P
$V(f)$	The total variation of f
$\text{span}(A)$	The linear span of a set A
$BV(K)$	The Banach space of functions of bounded variation defined on K endowed with the total variation norm

$\text{NBV}(K)$	The Banach space of right-continuous functions of bounded variation defined on K endowed with the total variation norm
$S(f, F, P)$	The Riemann-Stieltjes sum of f in respect to F and P
φ^*	The composition operator
φ_*	The operator of push-forward measures
$[A]^{<\kappa}$	The collection of subsets of A with size less than κ
$[A]^\kappa$	The collection of subsets of A with size κ
$\mathcal{P}(A)$	The powerset of A
$C_{is}(K, L)$	The set of continuous increasing surjections between K and L
$Q(\varphi)$	The set $\{t \in \text{ran}(\varphi) : \varphi^{-1}(t) \geq 1\}$ for a c.i.s. map
$Q_0(\varphi)$	The set $\{t \in Q(\varphi) : t \text{ is an internal point of } \text{ran}(\varphi)\}$ for a c.i.s. map
$\text{Sad}(K)$	The set of all points in K with separable adjacency
$\kappa\text{-SF}$	Indicates that a family of sets is a κ -separating family
SF	Indicates that a family of sets is a 2-separating family
$f \equiv c$	The function f is constant equal c
$f =^* g$	f and g are equal outside a finite set
$s \frown t$	The finite function $s = \{(s , t)\}$
$(s, \mathbb{t}$	The function $f \in [0, 1]^\omega$ with $f _{ s } = s$ and $f _{[s , \omega[} \equiv t$
P_j^k	The projection of $[0, 1]^k$ onto $[0, 1]^j$
P_n	The projection of $[0, 1]^\omega$ onto $[0, 1]^n$
H_E	The separable closed set generated by a countable set E
$\hat{x}$	The set $\{x n : n \in \omega\}$
$\tilde{S}$	The points in $[0, 1]^\omega$ that extend infinitely many points of S
$\mathcal{A}$	An almost disjoint family
$\Psi(\mathcal{A})$	The Mrówka-Isbell space associated with $\mathcal{A}$
$\mathfrak{a}$	The almost disjointness number
$\mathfrak{p}$	The pseudointersection number
$\mathfrak{b}$	The bounding number
$\mathfrak{s}$	The splitting number
$\mathcal{J}^+(\mathcal{A})$	The positive sets of an almost disjoint family

Chapter 1

Introduction

1.1 On the operator extension problem

Banach spaces of the form $C(K)$, the space of continuous real-valued functions defined on a compact Hausdorff space, endowed with the supremum norm, play a central role in the Banach space theory development ([Kos07, Ros03]). For instance, several questions in Banach space theory are answered through the construction of examples and counter-examples in the class of the $C(K)$ spaces ([Kos04]).

In this thesis we study, on the class of spaces $C(K)$, classical problems of Banach space theory, in special the problem of extending bounded operators. Recall that the Sobczyk's Theorem ([Sob41]) states that the space c_0 is separably injective, i.e., every c_0 -valued operator T defined on a closed subspace Y of a separable Banach space X admits a bounded extension $\tilde{T}$ to X with the estimate $\|\tilde{T}\| \leq 2\|T\|$.

The search for generalizations of Sobczyk's Theorem is a research field where much work has been done in the last decades ([ACG⁺02, CT14, CT13, CT15, Mol91, Pat93, Sán06, SCY00]).

In [CT13, CT15] are studied pairs of Banach spaces $Y \subseteq X$ for which the conclusion of Sobczyk's Theorem hold, i.e., pairs of spaces satisfying the following definition: a closed subspace Y of a Banach space X has the c_0 -extension property (c_0 -EP) in X if every bounded operator $T : Y \rightarrow c_0$ admits a bounded extension to X . We simply say that X has the c_0 -EP when every closed subspace Y of X has the c_0 -EP in X .

Obviously, every complemented subspace of a Banach space X has the c_0 -EP in X . With the terminology just introduced, the Sobczyk's Theorem says that every separable Banach space has the c_0 -EP.

More generally, if the quotient X/Y is separable, then Y has the c_0 -EP in X ([CT13, Proposition 2.2, (a)]). Moreover, when a Banach space X has the c_0 -EP, then every isomorphic copy of c_0 in X is complemented; spaces with this property are frequently called *Sobczyk spaces*.

Evidently, every bounded operator $T : X \rightarrow \ell_\infty$ is characterized as a bounded sequence of functionals in X^* as follows:

$$T(x) = (\alpha_n(x))_{n \geq 1}, \quad (1.1.1)$$

where $\|T\| = \sup_{n \geq 1} \|\alpha_n\|$. In particular it follows that the range of such an operator is contained in c_0 if, and only if, $(\alpha_n)_{n \geq 1}$ is weak*-null. In this way, Y has the c_0 -EP in X when every weak*-null sequence in Y^* admits an extension to a weak*-null sequence in X^* .

Since Banach spaces of the form $C(K)$ has a structure of unital Banach algebra, an interesting class of closed subspaces of $C(K)$ are the unital Banach subalgebras and it follows from Stone–Weierstrass Theorem that such subalgebras are precisely the images of *composition operators*:

$$\varphi^* : C(L) \ni f \mapsto f \circ \varphi \in C(K), \quad (1.1.2)$$

induced by continuous surjective functions $\varphi : K \rightarrow L$, where L is a compact Hausdorff space.

It follows from the Riesz Representation Theorem that the dual space of $C(K)$ is isometrically

identified with $M(K)$, the space of signed σ -additive regular Borel measures in K , endowed with the total variation norm. It follows that the adjoint of the composition operator (1.1.2) is the pushforward of measures $\varphi_* : M(K) \rightarrow M(L)$ given by:

$$(\varphi_*\mu)(B) = \mu(\varphi^{-1}[B]).$$

It follows that $\varphi^*C(L)$ has the c_0 -EP in $C(K)$ iff every weak*-null sequence $(\mu_n)_{n \geq 1}$ in $M(L)$ admits a weak*-null sequence $(\tilde{\mu}_n)_{n \geq 1}$ in $M(K)$ such that $\varphi_*\tilde{\mu}_n = \mu_n$, for all $n \geq 1$.

In [CT15] the problem of characterizing maps $\varphi : K \rightarrow L$ such that $\varphi^*[C(L)]$ has the c_0 -EP in $C(K)$ was studied in the context of totally ordered spaces K and L , under the assumption that φ is increasing, surjective and continuous. The case where φ is not an increasing function is far more complicated to analyse and was considered in [CT13], only with L countable. Moreover, in [KK12] there is a characterization of the continuous increasing surjections $\varphi : K \rightarrow L$ with $\phi^*C(L)$ complemented in $C(K)$.

In an attempt to generalize the main results in [CT15], we replace c_0 by its non-separable version $c_0(I)$, where I is an arbitrary infinite set (frequently supposed to be uncountable):

$$c_0(I) = \left\{ (x_i)_{i \in I} \in \mathbb{R}^I : \forall \varepsilon > 0, \{i \in I : |x_i| \geq \varepsilon\} \text{ is finite} \right\}.$$

In other words, the space $c_0(I)$ is formed by families of real numbers $(x_i)_{i \in I}$ such that $x_i \rightarrow 0$ when $i \rightarrow \infty$ and $I \cup \{\infty\}$ is the Alexandroff compactification of the discrete set I . It is easy to see that bounded operators $T : X \rightarrow c_0(I)$ are characterized as a weak*-null families in X^* by:

$$T(x) = (\alpha_i(x))_{i \in I}, \quad (1.1.3)$$

where weak*-null means that $\lim_{i \rightarrow \infty} \alpha_i = 0$ in the weak* topology, or equivalently that $(\alpha_i(x))_{i \in I} \in c_0(I)$ for all $x \in X$.

By now the reader may find that it is natural to study generalizations of the Sobczyk's Theorem and c_0 -EP like results when we replace c_0 by $c_0(I)$. For instance, in [ACG⁺02] were studied conditions about a Banach space X and the cardinality of I that guarantee that every isomorphic copy of $c_0(I)$ is complemented.

We introduce the notion of $c_0(I)$ -EP in the same fashion as the c_0 -EP. Notice that when $|J| \leq |I|$ it follows that $c_0(J)$ has a complemented copy in $c_0(I)$ and it follows that if Y has the $c_0(I)$ -EP in X , then Y has the $c_0(J)$ -EP in X . In particular, Y has the c_0 -EP in X .

The converse holds when Y is separable because in this case, every bounded operator defined on Y has its image contained in a $c_0(J)$ with J countable. Although, the converse is not always true: in [ACG⁺02, Theorem 1.1, (b)] the authors construct a WCG space $C(K)$ that has a non-complemented copy of $c_0(I)$ with $|I| \geq \beth_\omega$ and it follows from ([CT13, Proposition 2.2, (b)]) that $C(K)$ has the c_0 -EP.

The following questions naturally arises from this discussion:

Question 1.1.1. *Let K and L be totally ordered compact spaces and $\varphi : K \rightarrow L$ be an increasing, continuous and surjective function. Given a bounded operator $T : C(L) \rightarrow c_0(I)$, under what conditions does T admit a bounded extension to $C(K)$? Under what conditions the subalgebra $\phi^*C(L)$ has the $c_0(I)$ -EP in $C(K)$?*

Our work revolves around answering, at least, partially these questions. One should keep in mind that when working on these problems, the answers may depend in a non-trivial way on the cardinality of I as well as other set-theoretical assumptions such as MA or CH : this already happens in [ACG⁺02] and in [CT15].

1.2 On the weaker forms of normality

Mrówka-Isbell spaces are the topological spaces associated to almost disjoint families and they form an important class of topological spaces, which provides plenty of examples and counter examples to a wide range of questions in general topology. By its nature, it often occurs that topological properties of such spaces are connected with the combinatorial properties of their associated almost disjoint families (e.g. a Mrówka-Isbell is pseudocompact iff its associated almost disjoint family is maximal). We cite the surveys [Hru14] and [HHH18] as references for this field of study.

Recall that if N is a countable infinite set such that $N \cap [N]^\omega = \emptyset$, an almost disjoint family (over N) is an infinite collection $\mathcal{A}$ of infinite subsets of N such that for all distinct $a, b \in \mathcal{A}$, $a \cap b$ is finite. A MAD family (maximal almost disjoint family) is an almost disjoint family which is not properly contained in any other almost disjoint family. It easily follows from Zorn's Lemma that every almost disjoint family can be extended to a MAD family. Moreover there exist almost disjoint families of size $\mathfrak{c}$, this is a folklore result that we present in Proposition 5.1.2 for the sake of completeness.

The Mrówka-Isbell space associated with an almost disjoint family $\mathcal{A}$ over N , denoted by $\Psi(\mathcal{A})$ and briefly called Ψ -space, is the topological space whose underlying set is $N \cup \mathcal{A}$ endowed with the topology generated by $\{\{n\} : n \in N\} \cup \{\{a\} \cup (a \setminus F) : a \in \mathcal{A}, F \in [N]^{<\omega}\}$. It is immediate that these spaces are Hausdorff, locally compact (therefore Tychonoff), not countably-compact, zero-dimensional and separable.

Although Ψ -spaces have so many nice topological properties, they are not always normal: if $\mathcal{A}$ is an almost disjoint family of size $\mathfrak{c}$, notice that then $\mathcal{A}$ is a closed discrete subspace of $\Psi(\mathcal{A})$, so it is not normal by Jones's Lemma. However, Mrówka-Isbell spaces can be normal, for instance, $\Psi(\mathcal{A})$ is metrizable iff $\mathcal{A}$ is countable. The existence of an uncountable normal Isbell-Mrówka space is independent of the Axioms of ZFC, and is equivalent to the existence of a normal separable non-metrizable Moore space [Tal77] [HHH05], [SS73].

In this work we study weaker forms of normality on such spaces. In the literature various weakenings of normality have been proposed and studied, such as quasi-normality [Zai68], almost-normality [SA70], mildly-normality [Shc72] and semi-normality [SA70], see [GBS20] and [AK20] for results regarding this area of research. Here we will focus on the relationship between normality, semi-normality and almost normality.

Given a topological space X , a regular closed set of X is a closed set F such that $F = \text{cl}(\text{int}(F))$, similarly an open set U is said to be regular open iff $U = \text{int}(\text{cl}(U))$. We say a topological space X is *almost-normal* iff whenever F is a closed set and K is a regular closed set disjoint from F , there exist disjoint open sets U, V such that $F \subseteq U$, $K \subseteq V$. A topological space X is said to be *semi-normal* iff for every closed set F and every open set U containing F there exists a regular open set V such that $F \subseteq V \subseteq U$. The following proposition connects the aforementioned concepts:

Proposition 1.2.1 ([SA70]). *A topological space is normal iff it is almost-normal and semi-normal.*

Recall that an almost disjoint family $\mathcal{A}$ is called of true cardinality $\mathfrak{c}$ if for every $B \subseteq \omega$, the set $\mathcal{A}|_B = \{a \in \mathcal{A} : |a \cap B| = \omega\}$ is either finite or it has cardinality $\mathfrak{c}$. In [GBS20], P. Szeptycki, S. Garcia-Balan provided, among several other examples, an example in ZFC, of an almost disjoint family of true cardinality $\mathfrak{c}$ which is not almost-normal but satisfies weaker separation properties, such as quasi-normality. They also showed that this example could be made MAD if there exists a MAD family of true cardinality $\mathfrak{c}$. They asked the following question (Question 4.2 of [GBS20]):

Question 1.2.2. *Is there an almost-normal not normal almost disjoint family?*

In [RR20] the authors provide a partial answer to Question 1.2.2 (consistently yes) and, recently in [Cor21] the author presents a construction of an almost-normal MAD family under CH .

In [GBS20], they also define the concept of strongly $\aleph_0$ -separated almost disjoint family, which is related to almost-normal almost disjoint families, as follows: an almost disjoint family $\mathcal{A}$ (over N) is said to be strongly $\aleph_0$ -separated iff for every two countable disjoint subsets $\mathcal{B}, \mathcal{C}$ of $\mathcal{A}$ there exists $X \subseteq N$ such that:

- (1) For every $a \in \mathcal{A}$, $a \subseteq^* X$ or $A \cap X =^* \emptyset$;
- (2) for every $a \in \mathcal{B}$, $a \subseteq^* X$;
- (3) for every $a \in \mathcal{C}$, $a \cap X =^* \emptyset$.

They showed that every almost-normal almost disjoint family is strongly $\aleph_0$ -separated and showed that, under CH , there exist MAD families which are strongly $\aleph_0$ -separated. In the last Chapter of this work we define a stronger concept we call strongly $(\aleph_0, < \mathfrak{c})$ -separated almost disjoint family and prove that $\mathfrak{b} = \mathfrak{c}$ plus $\mathfrak{s} = \mathfrak{c}$ implies the generic existence of $(\aleph_0, < \mathfrak{c})$ -separated completely separable MAD families.

1.3 Outline of the thesis

To read this thesis, the reader is required to have some knowledge on general topology, functional analysis and set theory. More specifically, we use classical results about separation axioms such as Urysohn's Lemma, few results of cardinal functions defined on topological spaces such as Jones's Lemma and the basic tools in functional analysis such as Riesz Representation Theorem for functionals and concepts related to regular measures on compact Hausdorff spaces. While writing this work I made an effort to, at some extent, make it self-contained: there is a brief introduction to most of prerequisites topics and references were given to each of them in an attempt to ease the burden on the reader and to cover any of the requisites aforementioned. We will now present an overview of the topics discussed in this thesis.

This document is divided in two parts, the first one that ranges from Chapter 2 to Chapter 4 concerns the functional analysis results regarding operator extension. While the second part, presented on Chapter 5 concerns the results obtained on set-theoretic topology.

We present in Chapter 2 the basic definitions and constructions results regarding totally ordered spaces. The beginning of this Chapter contains basic definitions and classical results such as Ostaszewski's classification theorem for separable linear compact spaces. Section 2.3 is devoted exclusively to introduce cardinal functions that will be used through this work. One could think that this section is also a warm-up for what is presented in Chapter 5, where set-theoretic topology plays a fundamental role. Later on Chapter 2 we present all the Riesz-like representation theorems for bounded functionals in $C(K)$, where K is a linear compact space as well as the tools needed to describe weak*-convergence through the representation theorems. It is in Section 2.5 that we define, in the view of all this machinery aforementioned, what it means to a bounded operator $T : C(L) \rightarrow c_0(I)$ to have an extension to the space $C(K)$ through a continuous increasing surjection $\varphi : K \rightarrow L$ between linear compact spaces. In Section 2.5 we also study some ways to describe weak* convergence and basic results regarding linearly dense subsets in Banach spaces.

In Chapter 3 we introduce a central tool to our study: the $c_0\ell_1$ decompositions. These decompositions characterize which $c_0(I)$ -valued bounded operators defined on $C(L)$ admit extension to $C(K)$ through a continuous increasing surjection $\varphi : K \rightarrow L$. Section 3.1 is where we develop most of the results concerning the relationship between $c_0\ell_1$ decompositions and operator extension theory. With some of these results we were able to provide in Section 3.2 another characterization of weak*-null families on linear compact spaces. Finally, in Section 3.3 we define an entire class of linear compact spaces, that we call Separably Determined, which are the ordered compacta that do not distinguish between c_0 -EP and $c_0(I)$ -EP, i.e., continuous increasing maps ranging in a separably determined linear compact space have the c_0 -EP iff they have the $c_0(I)$ -EP.

Chapter 4 is devoted to prove that there exists a linear compact space that is not separably determined, namely, our example is the space $[0, 1]^\omega$ endowed with the lexicographic order. We choose to divide this Chapter in two parts. In Section 4.1 we introduce a topological definition that is worth of study by itself: the κ -separating families. These families are straightforward connected to separation axioms and are related to σ -discreteness in metrizable spaces. After presenting some

basic relations, in the end of the section we build a bridge between $c_0\ell_1$ decompositions and ω_1 -separating families. Section 4.2 is devoted to prove that $[0, 1]^\omega$ is not separably determined. Through this section we made several reductions to tackle the problem and then we actually prove the result in Theorem 4.2.20. This concludes all the elements of functional analysis contained in this work.

In Chapter 5, as promised, we made a shift on the subject to study weaker forms of normality in Ψ -spaces. This entire Chapter is devoted to present part of the results published in [RR20]. We start Section 5.1 with an introduction to almost disjoint families and the topological spaces associated with them. There we discuss basic topological properties of these spaces and how the combinatorial properties of an almost disjoint family can reflect on topological properties of the Ψ -space. Section 5.2 is a catalogue of different weaker forms of normality and puts them on a hierarchy. This is a rather short section that serves both as an introduction to later sections and consulting material for the reader. On Section 5.3 we provide a combinatorial characterization of semi-normality and provide some structural results. Finally, on Section 5.4 we define a new separation property, called strongly $(\aleph_0, < \mathfrak{c})$ -separated almost disjoint, that generalizes the concept of strongly $\aleph_0$ -separated almost disjoint family, introduced in [GBS20]. We discuss the property we defined in the light of infinite combinatorics and small cardinals, giving some structural results such as Lemma 5.4.5 and Corollary 5.4.6. In the end of Section 5.4, we prove in Theorem 5.4.9 that completely separated MAD families having this property exists generically under the assumption that $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$. This theorem has both a weaker assumption and a stronger conclusion than the one proved in [GBS20].

1.4 The contribution of this work

We gladly can say that in both parts of this work we made concrete advances to each of the respective fields of research. Here we describe the most important contributions of each part of the thesis.

In the first part of the work, the one that consists of results in functional analysis, we were able to analyse the problem of extending $c_0(I)$ -bounded operators in unital Banach subalgebras $\varphi^*C(L)$ of $C(K)$, induced by a continuous increasing surjection $\varphi : K \rightarrow L$ between totally ordered compact spaces. To tackle this problem, we use the classical Riesz Representation Theorem for functionals and another representation theorem that describes functionals $\alpha \in C(K)^*$ as right-continuous functions of bounded variation $F \in \text{NBV}(K)$ when K is a linear compact space. Together, this representation results yields the diagram:

$$\begin{array}{ccc} M(K) & \xrightarrow{\varphi_*} & M(L) \\ \parallel & & \parallel \\ \text{NBV}(K) & \xrightarrow[\varphi_*]{\text{-----}} & \text{NBV}(L) \end{array}$$

that describes exactly what it means to an operator extend another in the aforementioned context. For reference, this diagram is in Section 2.5, preceding Definition 2.5.17, that is central to our study.

After we realize how wild the operator extension could behave in comparison to the criteria for c_0 -valued bounded operators, we provide a new tool that characterizes the extension of such operators: $c_0\ell_1$ -decompositions. We dedicate an entire section just to obtain (not so) basic results regarding these compositions and they were used throughout the entire work. The examples that we mentioned before are Example 3.0.4 and Example 3.1.22. Only in the end of Section 3.1, with Theorem 3.1.30, that we were able to provide a result of operator extension that resembles the classical Sobczyk's Theorem.

After realizing that the criteria to extend operators, with $\varphi : K \rightarrow L$ and L being a separable linear compact space, was really closed to the result for c_0 -valued operators, we were able to present the class of Separably determined compact spaces. We dedicate Section 3.3 to study these spaces with two goals in mind: the first one being to provide closure constructions on this class of spaces: these are Proposition 3.3.13 and Theorem 3.3.14 and the second goal is to characterize these linear compact spaces in terms of extensions as we did in Theorem 3.3.7 and Theorem 3.3.8.

Chapter 4 is devoted to prove that $[0, 1]^\omega$ is not a separably determined space. We made several reductions to tackle this problem and were able to define an operator associated with characteristic functions that could not be extended in Corollary 4.2.21. What is curious about this result is that we were able to provide such an example with $|I| = \omega_1$, much lower than we expected, due to a result in [ACG⁺02].

On the second part, we obtained a combinatorial characterization for the concept of semi-normality in Ψ -spaces. In the process we observed that this characterization was also equivalent to another property concerning these spaces: weakly separated families. Thus we were able to prove that two topological aspects studied in the literature of Mrówka-Isbell spaces were equivalent in Proposition 5.3.2. Among the structural results concerning the weaker forms of normality it is worth to mention Corollary 5.3.6 and, in contrast, Proposition 5.3.7 that put a restriction to the technique used to obtain the former result. In Section 5.4 we provide a stronger separation condition, namely the strongly $(\aleph_0, < \mathfrak{c})$ -separated almost disjoint family, in Definition 5.4.2 and using infinite combinatorics we were able to improve a result from S. Garcia and P. Szeptycki in Theorem 5.4.9. Although we just presented part of the results of [RR20] in this work, there the reader may find other interesting results that combine set-theory and topology, but they are not in the scope of this thesis.

Chapter 2

Linearly Ordered Topological Spaces

In this chapter we introduce the basic topological properties of linearly ordered topological spaces and the basic tools needed to study the structure of the bounded operators in Banach spaces of the form $C(K)$.

2.1 Linearly ordered spaces

Definition 2.1.1. Let X be a set. We say that a relation $\leq$ in X is a *total order* if the following properties hold:

- Reflexivity: $x \leq x$, for all $x \in X$;
- Antisymmetry: $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in X$;
- Transitivity: $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in X$;
- Comparability: For all $x, y \in X$, either $x \leq y$ or $y \leq x$.

In this case, we say that X is *totally ordered* by $\leq$.

Totally ordered sets have a natural topology that generalizes the usual topology of open intervals of $\mathbb{R}$ as we shall see in the next:

Definition 2.1.2. Let X be a totally ordered set. The *order topology* in X is the topology induced by the subbase of open rays:

$$]-\infty, a[= \{x \in X : x < a\} \quad \text{and} \quad]b, +\infty[= \{x \in X : b < x\},$$

for all $a, b \in X$.

By a *linearly ordered topological space* (**LOTS**), we mean a totally ordered set X endowed with the order topology.

The reader may notice that LOTS satisfies several topological separation properties, e.g., T_0 to T_4 axioms. For a detailed study of such spaces we refer [E⁺77], [SS78] and [Cat06]. In what follows we will focus our attention to other topological aspects of the LOTS, such as compactness, connectedness, continuity and the behaviour of the order topology when restricted for subsets. For that purpose, we need to introduce some notation:

Definition 2.1.3. Let X be a totally ordered set and $x, y \in X$.

- We denote the *closed interval* defined by x and y by $[x, y] = \{a \in X : x \leq a \leq y\}$;
- We say that $C \subset X$ is *convex* if for all $a, b \in C$, $[a, b] \subseteq C$;

- x is *right-isolated* if it admits a successor in X , i.e., there exists (a unique) $x^+ \in X$ such that $x^+ > x$ and $]x, x^+[= \emptyset$. When x is not right-isolated, we say that x is a *right-limit point*;
- x is *left-isolated* if it admits a predecessor in X , i.e., there exists (a unique) $x^- \in X$ such that $x^- < x$ and $]x^-, x[= \emptyset$. When x is not left-isolated, we say that x is a *left-limit point*;
- x is *isolated* if it is both right-isolated and left-isolated;
- x is an *internal point* if it is both right-limit and left-limit point;
- $X^+ = \{x \in X : x \text{ is right-isolated in } X\}$ and $X^- = \{x \in X : x \text{ is left-isolated in } X\}$;
- We say that $x, y \in X$ are *consecutive points* of X if $]x, y[= \emptyset$.

In some circumstances, it is easier to work with basic open sets of the order topology of the form $]x, y[= \{a \in X : x < a < y\}$. Nevertheless, when X is unbounded not every basic open set of X can be written this way. In this case one may consider $\overline{X} = X \cup \{-\infty, +\infty\}$ and define $-\infty < x < +\infty$ for all $x \in X$. Then, the basic open sets of X are all listed as $\{]a, b[: a, b \in \overline{X}\}$. Using this argument, the reader may notice that, if x is not right-isolated in X , then every open neighbourhood of x contains points strictly greater than x . On the other hand, if x is a right-isolated point of X , it is immediate that $] - \infty, x[$ is a clopen set.

Definition 2.1.4. We say that a totally ordered set X is *Dedekind-complete* iff for every non-empty subset $Y \subseteq X$ bounded above, there exists $\sup Y$.

Equivalently one may define Dedekind-completeness in terms of the existence of infimum for non-empty bounded below subsets of X . From the previous definitions, we have the following characterizations:

Theorem 2.1.5. *Let X be a LOTS. Then X is compact iff X is bounded and Dedekind-complete*

Proof. Suppose that X is compact. Suppose towards a contradiction that X is unbounded, then there exists a finite subset F of X such that $X = \bigcup_{x \in F}] - \infty, x[$ and because X is totally ordered it follows that $\max F = \max X$, a contradiction. Similarly one shows that X is lower bounded.

Now, suppose towards a contradiction that there exists a non-empty set $S \subseteq X$ without supremum. Then:

$$X = \bigcup_{x \in S}] - \infty, x[\cup \bigcup \{]y, +\infty[: y \text{ is an upper bound of } S \}$$

is an open cover of X . Thus, there exist finite sets $F \subseteq S$ and G a set of upper bounds of S such that:

$$X = \bigcup_{x \in F}] - \infty, x[\cup \bigcup_{y \in G}]y, +\infty[.$$

It follows that $\min G = \sup S$, a contradiction. Thus, X is Dedekind-complete.

Conversely, assume that X is bounded and Dedekind-complete. Given an open cover $\mathcal{U}$ of X , consider:

$$S = \{t \in X : [\min X, t] \text{ can be covered by a finite subcover of } \mathcal{U}\}.$$

Notice that $\min X \in S$ and $t' \in S$, provided that $t' \leq t$ with $t \in S$. Denote $s = \sup S$, it follows that $[\min X, s] \subseteq S$ and because $\mathcal{U}$ is an open cover, it follows that $[\min X, s] \subseteq S$. We claim that $s = \max X$.

Suppose towards a contradiction that $s < \max X$. Since s is right-isolated, It is clear that s is right-isolated and it follows that its immediate successor, s^+ belongs to S , contradicting the fact that s is an upper bound of S . $\square$

In view of the previous result, by a *linear compact space* (or a compact line), we mean a LOTS X that is compact in respect to the order topology. Since linear compact spaces are bounded, we denote $0_X = \min X$ and $1_X = \max X$.

Example 2.1.6. All of the following are linear compact spaces:

- The interval $[0, 1]$ with the usual order inherited from $\mathbb{R}$;
- $[0, \alpha] = \{\beta : \beta \text{ is an ordinal and } \beta \leq \alpha\}$;
- Closed subsets of linear compact spaces (e.g. the Cantor set).

More examples and constructions involving linear compact spaces will be provided throughout this chapter. For linear ordered spaces, there is also a well known characterization of connectedness:

Theorem 2.1.7. *Let X be a LOTS. Then X is connected iff X is Dedekind-complete and has no consecutive points.*

Proof. Similarly as we did in Theorem 2.1.5, this proof mimics the proof that a interval of real numbers is connected and we just provide a sketch of the proof.

Suppose that X is connected. Thus there are no consecutive points $x < y$ in X because otherwise we would have $X =]-\infty, x] \cup [y, +\infty[$ are clopen sets of X . Suppose towards a contradiction that there exists a non-empty upper bounded set $S \subseteq X$ that does not admit supremum. Then:

$$X = \underbrace{\left(\bigcup \{]-\infty, x[: x \in S \} \right)}_A \cup \underbrace{\left(\bigcup \{]y, +\infty[: y \text{ is an upper bound of } S \} \right)}_B.$$

It follows that A and B are disjoint, non-empty open sets, contradicting that X is connected.

Conversely, suppose that X is Dedekind-complete without consecutive points but X is not connected. Then there exist non-empty disjoint open sets such that $X = A \cup B$. Fix $x \in A$ and $y \in B$ and assume **WLOG** that $x < y$. Because A and B are disjoint and open, it follows that $]x, y[\cap A \neq \emptyset$. One shows that $c = \sup (]x, y[\cap A)$ is such that $c \notin A$ and $c \notin B$, a contradiction. $\square$

Given a subset Y of a LOTS X , one may ask if the topology induced by X coincides with the topology generated by the total order restricted to Y . However, it is not the case:

Example 2.1.8. Consider $X = \mathbb{R}$ endowed with the order topology and $Y = \{-1\} \cup \{\frac{1}{n} : n \geq 1\}$.

Then, in the topology induced by X , $\{-1\}$ is isolated in Y . On the other hand, if we consider the order topology generated by the total order $\leq_Y$ in Y , then $\{-1\}$ is an accumulation point.

For Y a subset of a LOTS X , we denote by $\leq_Y$, the order in Y induced by the order of X (i.e. $\leq_Y = \leq|_{Y \times Y}$). The order topology in Y that is induced by this order will be denoted as $\tau_{\leq_Y}$ and the topology of Y as a subspace of X will be denoted by $(\tau_{\leq})|_Y$. It is easy to see that $\tau_{\leq_Y} \subseteq (\tau_{\leq})|_Y$ for every $Y \subseteq X$. To avoid confusion with the standard notation of intervals, we will make it clear where an interval is being considered by using superscripts in the following way:

$$\text{For } a, b \in X \text{ and } Y \subseteq X : \quad [a, b]^Y = [a, b]^X \cap Y.$$

The reader may notice that the problem in the Example 2.1.8 is that Y is not a convex subset of X and due to this, some neighbourhoods of the order topology of Y are not neighbourhoods in the topology generated by the order restricted to Y . This yields the following result, that is left to the reader:

Proposition 2.1.9. *Let X be a LOTS and $Y \subseteq X$. If Y is convex in X , then $\tau_{\leq_Y} = (\tau_{\leq})|_Y$.*

To better understand when these two topologies coincide for subsets Y of a LOTS X , it is helpfull to characterize continuity of increasing functions between LOTS.

Definition 2.1.10. Given X, Y LOTS, a function $f : X \longrightarrow Y$ is *increasing* if:

$$x_1 \leq x_2 \implies f(x_1) \leq f(x_2), \quad \text{for all } x_1, x_2 \in X.$$

When f is bijective and the converse holds, we say that f is an *order-isomorphism*.

In the case when f is continuous, increasing and surjective, we denote $f : X \xrightarrow{\text{c.i.s.}} Y$.

Let X and Y be LOTS, $f : X \rightarrow Y$ be an arbitrary function and let $x \in X$. We consider the lower limit topology in X , the topology generated by X and the right-half open intervals $\{[a, +\infty[: a \in X\}$. When f is continuous in x in respect to the lower limit topology in X , we say that f is right-continuous in x . The reader may notice that when f is right-continuous, then $\lim_{y \searrow x} f(y) = f(x)$, where $y \searrow x$ denotes the limit from above in the order. Similarly, one may define the upper limit topology in X and left-continuity, where it holds $\lim_{y \nearrow x} g(y) = g(x)$, where $y \nearrow x$ denotes the limit from below.

It follows that f is continuous at x iff f is left-continuous in x and right-continuous in x . Since these concepts are symmetrical, it is enough to characterize right-continuity for increasing functions:

Proposition 2.1.11. *Let $f : X \rightarrow Y$ be an increasing function between LOTS and let $x \notin X^+$. Are equivalent:*

- (a) f is not right-continuous at x ;
- (b) there exists $y \in Y$, $y > f(x)$ such that $f[]x, +\infty[] \subseteq [y, +\infty[$

Since right-continuity is trivial for right isolated points, we have:

Corollary 2.1.12. *Let X and Y be LOTS, $f : X \rightarrow Y$ be an increasing function and let $x \in X$. Then f is right-continuous at x if, and only if, one of the following cases hold:*

- (a) x is right-isolated in X ;
- (b) x is not right-isolated in X and for each $y \in Y$, $y > f(x)$ there exists $a \in X$, $a > x$, such that $f(x) \leq f(a) < y$.

An analogous characterization holds for left-continuity at a given point. Now we are able to present another case in which the topology of subspace is well behaved in terms of the order:

Proposition 2.1.13. *Let X be a LOTS and $Y \subseteq X$ be a closed subspace. Then $\tau_{\leq_Y} = (\tau_{\leq})|_Y$.*

Proof. Since $\tau_{\leq_Y} \subseteq (\tau_{\leq})|_Y$ always holds, it suffices to prove that $i : (Y, \tau_{\leq_Y}) \rightarrow (X, \tau_{\leq_X})$ is continuous.

Suppose, towards a contradiction, that there exists $y \in Y$ such that i is not right-continuous at y . Then y is not right-isolated in Y and there exists $x \in X$, $x > y$ such that $i[]y, +\infty[^Y] \subseteq [x, +\infty[^X$. The set $[x, +\infty[^Y$ is non-empty because y is not right-isolated, then it follows that $y_0 = \inf[x, +\infty[^Y$ must be the successor of y , a contradiction. Similarly, i is left-continuous. $\square$

Recall that a topological space $\mathcal{X}$ is *zero-dimensional* iff $\mathcal{X}$ has a basis consisting of clopen sets. We present the following characterization:

Proposition 2.1.14. *Let K a linear compact space. Are equivalent:*

- (a) K is zero-dimensional;
- (b) Every non-empty $]a, b[$ contains a right-isolated point;
- (c) Every non-empty $]a, b[$ contains a left-isolated point;

Proof. (a) $\implies$ (b) : Given a non-empty open interval $I =]a, b[$, fix $x \in I$. Since K is zero-dimensional, there exists $C \in \text{Clop}(K)$ such that $C \subseteq I$. We claim that $y = \sup C \in C$ is right-isolated. Indeed, suppose towards a contradiction that y is not right-isolated. There exists an open neighbourhood $]z_1, z_2[\subseteq C$ of y , then $]y, z_2[\neq \emptyset$ and y is not an upper-bound of C , a contradiction.

(b) $\implies$ (c) : If $I =]a, b[\neq \emptyset$, there exists $x \in I$ right-isolated in K . If x^+ is the successor of x in K , then $x^+ \leq b$ is left-isolated since $]x, x^+[= \emptyset$.

(c) $\implies$ (a) : We will prove that $\text{Clop}(K)$ is a basis. Given $I =]a, b[\neq \emptyset$, fix $x \in I$. If x is an isolated point, then $\{x\}$ is a clopen set and we are done. Suppose that x is not isolated, then WLOG, x is not right-isolated. By our assumption, there exists $y \in]x, b[$ left-isolated, with predecessor $y^- > x$. Now we have two cases to consider.

If $]a, x[= \emptyset$, then $C =]a, y[= [x, y^-]$ is a clopen set containing x with $C \subseteq I$. On the other hand, if $]a, x[\neq \emptyset$, it contains a left isolated point z with predecessor $z^- > a$, thus $C =]z^-, y[= [z, y^-]$ is a clopen set containing x with $C \subseteq I$. $\square$

2.2 Lexicographic order and double-arrow spaces

In this section we introduce a new class of linearly ordered spaces. Such spaces are obtained by ordering the product of linear compact spaces and we shall see that, in a way, they describe all separable compact ordered spaces.

Definition 2.2.1. Let X be a linearly ordered space and let $(Y_x)_{x \in X}$ be a family of non-empty linearly ordered spaces. The *lexicographic order* in $\bigcup_{x \in X} (\{x\} \times Y_x)$ is defined as follows:

$$(x, y) \leq (x', y') \iff x < x' \quad \text{or} \quad (x = x' \text{ and } y < y')$$

It is straightforward to check that the lexicographic order is a total order in $\bigcup_{x \in X} (\{x\} \times Y_x)$ and we have the following:

Proposition 2.2.2. Let X be a linear compact space and let $(Y_x)_{x \in X}$ be a family of non-empty linear compact spaces, then $\bigcup_{x \in X} \{x\} \times Y_x$, endowed with the lexicographic order, is a linear compact space.

Proof. By Theorem 2.1.5, it suffices to show that $K = \bigcup_{x \in X} (\{x\} \times Y_x)$ is bounded and Dedekind-complete. Denote by $0_X = \min X$, and $1_X = \max X$.

Then K is bounded by $(0_X, \min Y_{0_X})$ and $(1_X, \max Y_{1_X})$. In addition, if $S \subseteq K$ is bounded above, then the points $x_0 = \sup\{x \in X : \exists y \in Y_x, (x, y) \in S\}$ and $y_0 = \sup\{y \in Y_{x_0} : (x_0, y) \in S\}$ are well defined and one can check that $(x_0, y_0) = \sup S$. $\square$

Proposition 2.2.3. Let K be a linear compact space, $X \subseteq K$, $(Y_x)_{x \in X}$ be a family of non-empty linear compact and $\widehat{K} = ((K \setminus X) \times \{\min K\}) \cup \bigcup_{x \in X} (\{x\} \times Y_x)$, endowed with the lexicographic order. Then the projection $\pi : \widehat{K} \longrightarrow K$ is increasing, surjective and continuous.

Proof. It is immediate that π is increasing and surjective. Let $(t, y) \in \widehat{K}$ be a point that is not right-isolated and $s \in K$ be such that $t < s$. Then there exists $(t', y') \in](t, y), (s, \min Y_s)[$. It follows that $t' < s$ and $\pi((t, y)) \leq \pi((t', y')) < s$.

By Corollary 2.1.12, π is right continuous. The argument for left-continuity is similar. $\square$

Particular cases of the previous construction are given by the following examples:

Example 2.2.4 (Double-arrow space). Let K be a linear compact space and $X \subseteq K$. The *double-arrow space* is defined by $\text{DA}(K; X) = (K \times \{0\}) \cup (X \times \{1\})$, endowed with the lexicographic order. In the case when $K = [0, 1]$, we denote $\text{DA}(X)$. Furthermore, when both $K = X = [0, 1]$, we denote the double-arrow space by DA .

Example 2.2.5 (Compact long line). Let α be an infinite ordinal. The *compact long line of length α* is the space $K = [0, \alpha] \times [0, 1]$ endowed with the lexicographic order.

In what follows, we investigate some topological properties of the double-arrow spaces of real numbers. For the next result, the definition of *weight* of a topological space can be found in the Section 2.3, where we define all cardinal functions related to topological properties.

Proposition 2.2.6. *Let $X \subseteq [0, 1]$. Then $\text{DA}(X)$ is separable and $w(\text{DA}(X)) = |X| + \omega$.*

Proof. It is easy to check that $(\mathbb{Q} \cap [0, 1]) \times \{0\}$ is dense in $\text{DA}(X)$.

For $t, s \in (\mathbb{Q} \cap [0, 1]) \cup X$ and $\varepsilon, \delta \in 2$, the family of the following sets:

- $] (t, \varepsilon), (s, \delta) [$;
- $] -\infty, (t, \varepsilon) [$;
- $] (t, \varepsilon), +\infty [$;
- $\text{DA}(X)$,

form a basis for the topology of $\text{DA}(X)$, hence $w(\text{DA}(X)) \leq |X| + \omega$.

Fix $\mathcal{B}$ a topological basis. Since $\text{DA}(X)$ is compact, $\text{Clop}(\text{DA}(X))$ is a family of compact open sets and we have that $|\mathcal{B}|^{<\omega} \twoheadrightarrow \text{Clop}(\text{DA}(X))$. Thus $|\text{Clop}(\text{DA}(X))| \leq |\mathcal{B}|$. In addition, $\{[\min \text{DA}(X)], (x, 0] : x \in X\} \subseteq \text{Clop}(\text{DA}(X))$ and it follows that $|X| + \omega \leq |\text{Clop}(\text{DA}(X))| \leq |\mathcal{B}|$. Since $\mathcal{B}$ is arbitrary, $|X| + \omega \leq w(\text{DA}(X))$. $\square$

Proposition 2.2.7. *Let $X \subseteq [0, 1]$. The following are equivalent:*

- (a) $\text{DA}(X)$ is zero-dimensional;
- (b) X is dense in $[0, 1]$.

Proof. (a) $\implies$ (b) Suppose that X is not dense in $[0, 1]$. Then, there exists a non-empty basic open set $]a, b[$ that is disjoint from X . It follows that $]a, b[\times \{0\}$ is a non-empty, connected open subset of $\text{DA}(X)$. Thus, $\text{Clop}(\text{DA}(X))$ does not form a basis for the order topology.

(b) $\implies$ (a) One can check that the sets of the form:

- $] (a, 0), (b, 1) [$;
- $] -\infty, (b, 1) [$;
- $] (a, 0), +\infty [$;
- $\text{DA}(X)$,

for all $a, b \in X$, form a basis for the order topology in $\text{DA}(X)$. $\square$

We already know that the Cantor set is a linear compact space, since it is a closed subset of $[0, 1]$. But it also may be described as a double-arrow space, as we show next:

As a topological space, the Cantor set is homeomorphic to 2^ω , endowed with the product topology, where each component of the product carries the discrete topology. Then we may consider $\varphi : 2^\omega \rightarrow [0, 1]$, given by:

$$\varphi((\varepsilon_n)_{n \in \omega}) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{2^{n+1}}.$$

The function φ is increasing and surjective, although it is not injective since some elements of $[0, 1]$ have more than one binary expansion. Let $X = \{\frac{k}{2^n} : n \geq 1 \text{ and } 1 \leq k < 2^n\}$ the set of dyadic rationals, it follows that, for every $x \in X$, $|\varphi^{-1}[x]| = 2$ and for every $y \in [0, 1] \setminus X$, $|\varphi^{-1}[y]| = 1$.

Thus, φ induces an order-isomorphism from 2^ω onto $\text{DA}(X)$ given by:

$$\widehat{\varphi}(s) = \begin{cases} (\varphi(s), 0), & \text{if } \varphi(s) \notin X; \\ (\varphi(s), 0), & \text{if } \varphi(s) \in X \text{ and } s = \min \varphi^{-1}[s] \text{ in the lexicographic order;} \\ (\varphi(s), 1), & \text{if } \varphi(s) \in X \text{ and } s = \max \varphi^{-1}[s] \text{ in the lexicographic order.} \end{cases}$$

What we have done here motivates a classical characterization theorem for separable linear compact spaces, due to Ostaszewski [Ost74], that we state next. We opt to omit the proof since the original paper is easy to read and the proof relatively long to include here.

Theorem 2.2.8 (Ostaszewski's Theorem). *K is a linear compact separable space iff there exists a closed set $F \subseteq [0, 1]$ and $Y \subseteq F$ such that K order-isomorphic to $\text{DA}(F, Y)$.*

We finish this section with two applications of the Ostaszewski's Theorem.

Proposition 2.2.9. *Let K be a linear compact space. If K is not zero-dimensional, then there exists a closed separable subset $H \subseteq K$ with $|H| = \mathfrak{c}$.*

Proof. Since K is compact Hausdorff space, it follows that K is not totally disconnected. Thus, there exists a connected component C of K with $|C| > 1$. Using Urysohn's Lemma and the connectness of C , there exists a continuous function $f : K \rightarrow [0, 1]$ such that $f[C] = [0, 1]$.

Take $S \subseteq C$ a countable set with $f[S] = \mathbb{Q} \cap [0, 1]$. Then $H = \text{cl}(S)$ is a closed separable subspace of K with $f[H] = [0, 1]$ and we have that $|H| \geq \mathfrak{c}$.

On the other hand, by Ostaszewski's Theorem H is order isomorphic to $\text{DA}(F, Y)$ where F is a closed set of $[0, 1]$ and $Y \subseteq F$ and we have that $|H| \leq \mathfrak{c}$. $\square$

Proposition 2.2.10. *Let K be an uncountable separable linear compact space. Then $|K \setminus K^+| = \mathfrak{c}$.*

Proof. By Ostaszewski's Theorem, K is order isomorphic to $\text{DA}(F, Y)$. Since K is uncountable, F is a closed uncountable subset of $[0, 1]$ and we have that $|F| = \mathfrak{c}$. Since F is second countable we have that $|F^+| \leq \omega$, thus $|F \setminus F^+| = \mathfrak{c}$. We consider $f : F \setminus F^+ \rightarrow K \setminus K^+$ given by:

$$f(t) = \begin{cases} (t, 0), & \text{if } t \notin Y; \\ (t, 1), & \text{if } t \in Y. \end{cases}$$

It follows that f is injective, thus $|K \setminus K^+| = \mathfrak{c}$. $\square$

2.3 Cardinal functions on linear compact spaces

This section is devoted to introduce some cardinal functions on linear compact spaces (or more generally in totally ordered sets) and to compare them with well known classical cardinals that are defined on arbitrary topological spaces.

The study of cardinal functions is of great importance for set-theoretic topology since they constitute an useful tool that extends topological aspects of a space, and allow one to formulate and prove results in an elegant manner. Moreover, cardinal functions are mostly used to obtain quantitative information about a topological space X through some of its topological properties, for example, if we suppose that X is a Hausdorff topological space with countable base, then 2^ω is an upper bound to both the cardinality of X and to the number of its open sets. To a more detailed reference on this subject, we refer [KV14].

Definition 2.3.1. For a topological space X we define the following:

(a) The *weight* of X is the cardinal given by:

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\} + \omega.$$

(b) The *density* of X is the cardinal given by:

$$d(X) = \min\{|S| : S \subseteq X, \overline{S} = X\} + \omega.$$

(c) The *Lindelöf degree* of X is the cardinal given by:

$$L(X) = \min\{\kappa : \text{every open cover of } X \text{ has a subcover of size } \leq \kappa\} + \omega$$

Obviously all of these cardinal functions generalize well known aspects of topological spaces. For instance, X is second countable iff $w(X) = \omega$, X is separable iff $d(X) = \omega$ and X is Lindelöf iff $L(X) = \omega$, and in case when X is metrizable all the three cardinals coincide.

Also, they can be used to give short arguments to well known facts regarding topological spaces. For instance, let X be a compact Hausdorff space, then $w(X) = d(C(X))$. The countable case of this equality, together with the Urysohn metrization theorem yields:

$$X \text{ is metrizable} \iff C(X) \text{ is separable}$$

Definition 2.3.2. Let X be a topological space. We say that a collection $\mathcal{C}$ of open sets is an *antichain* (or *cellular*) if the elements of $\mathcal{C}$ are non-empty and pairwise disjoint. We define the *cellularity* of X as follows:

$$c(X) = \sup\{|\mathcal{C}| : \mathcal{C} \text{ is an antichain in } X\} + \omega$$

This cardinal function generalizes the ccc (countable chain condition) for topological spaces and is straightforward to check that $c(X) \leq d(X)$. Some authors also use the name “Suslin Condition” to mean ccc due to the classical Suslin’s problem of characterizing totally ordered spaces isomorphic to $\mathbb{R}$. We would like to mention that under $MA + \neg CH$, ccc is preserved under products, on the other hand, a Suslin Line is ccc and the product of a Suslin line with itself is not ccc - and Suslin lines exist under the diamond principle. For more on that subject, we refer [Kun11, Cie97].

We say that a cardinal function θ is monotone when $\theta(Y) \leq \theta(X)$ for every subspace Y of X . For example, the weight is monotone and the Lindelöf degree is not. For a cardinal function θ that is not monotone we can introduce the cardinal of hereditariness of θ :

$$h\theta(X) = \sup\{\theta(Y) : Y \subseteq X\}$$

In the case when K is a separable linear compact space, then $hd(K) = hL(K) = \omega$, i.e. K is both hereditarily separable and hereditarily Lindelöf and $hL(K) = \omega$. The proofs of these are given in [CT15].

All of the previously defined cardinals are global in the sense of each of them gives information based on a topological property of the entire space. Now we want to talk about cardinals concerning local properties:

Definition 2.3.3. Let X be a topological space and $x \in X$. The *character* of x in X is defined as follows:

$$\chi(x, X) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a local base for } x\}.$$

The character of X is $\chi(X) = \sup\{\chi(x, X) : x \in X\}$.

Thus a topological space X is first countable iff $\chi(X) = \omega$. All of the cardinals defined so far can be used to determine bounds to the cardinality of X itself, such as the classical Arhangel’skii result for Hausdorff spaces:

$$|X| \leq 2^{L(X)\chi(X)}$$

For instance, the countable case of this result answers the question posed by Alexandroff and Urysohn about the existence of a compact, first countable space with size greater than the continuum. To see more on bound results, see [KV14]. Now we turn our attention to cardinal functions that can be defined on linear compact spaces:

Definition 2.3.4. Let K be a linear compact space and $t \in K$. We define the *right-character* of t in K as follows:

$$\chi_R(t, K) = \begin{cases} \min\{|A| : A \subseteq]t, \max K], \inf A = t\}, & \text{if } t \in K \setminus K^+; \\ \omega, & \text{if } t \in K^+. \end{cases}$$

Similarly, we define the *left-character of t in K* by:

$$\chi_L(t, K) = \begin{cases} \min\{|A| : A \subseteq [\min K, t[, \sup A = t\}, & \text{if } t \in K \setminus K^-; \\ \omega, & \text{if } t \in K^-. \end{cases}$$

Now that we have established two notions to analyse the behaviour of a local basis at a point, we need to check how they are connected.

Lemma 2.3.5. *Let K be a linear compact space and $t \in K \setminus K^+$ and denote $\chi_R(t, K) = \kappa$. There exists a continuous strictly decreasing function $\varphi : [0, \kappa] \rightarrow K$ with $\varphi(\kappa) = t$.*

Proof. Take $X \subset]t, \max K]$ with $\inf X = t$ and $|X| = \kappa$ and write $X = \{z_\alpha : \alpha < \kappa\}$. We define φ recursively as follows:

- $\varphi(0) = z_0$;
- $\varphi(\alpha + 1) \in]t, \min\{z_\alpha, \varphi(\alpha)\}[$;
- For a limit ordinal γ we set $\varphi(\gamma) = \sup]t, \inf\{\varphi(\alpha) : \alpha < \gamma\}]$.

In this way φ is well defined because in the successor case we always have $]t, \min\{z_\alpha, \varphi(\alpha)\}[\neq \emptyset$ due to $|X| = \kappa > \alpha$. In addition φ is clearly strictly decreasing and we prove that it is continuous:

Indeed, the continuity is trivial for successor ordinals because they are discrete points in $[0, \kappa]$ and for γ a limit ordinal, fix an open set $A \subseteq K$ with $\varphi(\gamma) \in A$, **WLOG** we can assume that either $A = [\min K, s[$ or $A =]s, \max K]$. The former case is trivial since $\varphi[[0, \gamma[\subseteq A$ and in the latter we take $\beta < \gamma$ such that $\varphi(\beta) \in]s, \varphi(\gamma)[$, then $\varphi[[\beta, \gamma[\subseteq A$. $\square$

Of course a similar statement is true about points $t \in K \setminus K^-$ and the existence of an increasing continuous function $\varphi : [0, \kappa] \xrightarrow{\text{c.i.s.}} K$ with $\varphi(\kappa) = t$, where $\chi_L(t, K) = \kappa$. As a consequence of this result we have:

Proposition 2.3.6. *Let K be a linear compact space and $t \in K$. Then:*

$$\chi(t, K) = \max\{\chi_L(t, K), \chi_R(t, K)\}$$

Proof. In case t is an isolated point, all of the cardinal functions are ω and we are done. Suppose that $t \in K \setminus (K^+ \cup K^-)$ and **WLOG** we can assume that $\max\{\chi_L(t, K), \chi_R(t, K)\} = \chi_R(t, K)$. By Lemma 2.3.5 we take the maps $\varphi : [0, \chi_L(t)] \rightarrow [\min K, t]$ continuous, strictly increasing and $\psi : [0, \chi_R(t)] \rightarrow [t, \max K]$ continuous and strictly decreasing. We set

$$\mathcal{V} = \left\{]\varphi(\alpha), \psi(\beta)[: \alpha < \chi_L(t) \text{ and } \beta < \chi_R(t) \right\}.$$

It follows that $\mathcal{V}$ is a local base for t and $|\mathcal{V}| = \max\{\chi_L(t, K), \chi_R(t, K)\}$, thus we have proved that

$$\chi(t, K) \leq \max\{\chi_L(t, K), \chi_R(t, K)\}$$

Now we prove the other inequality. Fix $\mathcal{V}$ an arbitrary local base for t , we will show that $|\mathcal{V}| \geq \max\{\chi_L(t, K), \chi_R(t, K)\}$. We can assume **WLOG** that the $\mathcal{V}$ is a local base of basic open sets. Suppose towards a contradiction that $|\mathcal{V}| < \max\{\chi_L(t, K), \chi_R(t, K)\}$ and write:

$$\mathcal{V} = \left\{]a_\alpha, b_\alpha[: \alpha \in |\mathcal{V}| \right\}.$$

Since we are assuming that $\chi_R(t, K) = \max\{\chi_L(t, K), \chi_R(t, K)\}$, the set $B = \{b_\alpha : \alpha \in |\mathcal{V}|\}$ is such that $B \subseteq [t, \max K[$ with $|B| = |\mathcal{V}| < \chi_L(t, K)$, thus $\inf B > t$, which contradicts the fact that $\mathcal{V}$ is a local base for t .

The cases where t is isolated only from one side are similar to the previous case and this concludes the proof. $\square$

The importance of Proposition 2.3.6 is to connect the character defined for arbitrary topological spaces and the character defined specifically looking at the structure of a total order gives to a space. Since in this work we are mostly concerning with linear compact spaces, we opt to choose the way that most resembles the structure of the space, i.e., we follow Definition 2.3.4 in the following sections.

2.4 Representations of $C(K)^*$ for linear compact spaces

In the previous sections of this chapter we introduced basic results and terminology that will be adopted throughout the text. Now we are ready to give a characterization of the space $C(K)^*$ when K is a linear compact space.

When K is a compact Hausdorff space, we denote by $M(K)$ the space of all regular finite scalar-valued Borel measures on K . It follows that $M(K)$, endowed with the bounded variation norm, is a Banach space. The following result is due to Riesz and is one of the most classical results regarding $C(K)$ spaces, we refer [Rud87] for details.

Theorem 2.4.1 (Riesz Representation Theorem). *Let K be a compact Hausdorff space. Then $C(K)^*$ is isometrically isomorphic to $M(K)$ via the operator:*

$$\Phi : M(K) \ni \mu \xrightarrow{\equiv} \alpha_\mu \in C(K),$$

where $\alpha_\mu(f) = \int_K f d\mu$, for all $f \in C(K)$.

We will provide an analogous theorem when K is a linear compact space. For that purpose, we need the following:

Definition 2.4.2. Let K be a linear compact space. We say that P is a *finite partition* of K if $P \in [K]^{<\omega}$. For a function $f : K \rightarrow \mathbb{R}$, we define the *variation of f in respect to the partition P* as follows:

$$V(f, P) = \sum_{i=0}^{k-1} |f(x_{i+1}) - f(x_i)|,$$

where $P = \{x_0, \dots, x_k\}$ with $x_0 < \dots < x_k$.

The *total variation of f* is defined by:

$$V(f) = \sup\{V(f, P) : P \in [K]^{<\omega}\} \in [0, +\infty],$$

and we say that f is a function of *bounded variation* if $V(f) < +\infty$.

We denote by $BV(K)$ the vector space of all functions of bounded variation. In addition, $BV(K)$ is a Banach space when it is endowed with the norm

$$\|f\| = |f(\min K)| + V(f).$$

The subspace $NBV(K) = \{f \in BV(K) : f \text{ is right-continuous}\}$ is a Banach space.

Remark. Notice that the set of all partitions of K , $[K]^{<\omega}$, is a directed set when ordered by inclusion and one can always consider that a fixed finite set $F \subseteq K$ is contained in an arbitrary partition.

Moreover, the previous definition can be adapted for a totally ordered set $\mathfrak{X}$ with the following modification on the bounded variation norm:

$$\|f\| = |\text{i.v.}(f)| + V(f),$$

where the initial value of $f \in BV(\mathfrak{X})$ is given by $\text{i.v.}(f) = f(\min \mathfrak{X})$ when there exists a minimum in $\mathfrak{X}$, or $\text{i.v.}(f) = \lim_{x \rightarrow -\infty} f(x)$ otherwise. The next result shows that this norm is well defined.

Proposition 2.4.3. *Let $\mathfrak{X}$ be a totally ordered set and $F \in \text{BV}(\mathfrak{X})$. Then $\lim_{x \rightarrow -\infty} F(x)$ exists. In particular, for every $y \in \mathfrak{X} \setminus \mathfrak{X}^+$, $\lim_{x \searrow y} F(x)$ exists.*

Proof. Fix $x_0 \in \mathfrak{X}$ and define an auxiliary function given by $\varphi(x) = V(F|_{[x, x_0]})$, for all $x \leq x_0$. Then φ is decreasing with $0 \leq \text{ran}(\varphi) \leq V(F)$ and it follows that $\lim_{x \rightarrow -\infty} \varphi(x) = \sup_{x \rightarrow -\infty} \varphi(x)$ exists.

Given $\varepsilon > 0$, there exists $x_1 \leq x_0$ such that $|\varphi(x) - \varphi(y)| < \varepsilon$ for all $x, y \leq x_1$ and we have that:

$$\forall x \leq y \leq x_1 : |F(x) - F(y)| \leq V(F|_{[x, y]}) = \varphi(x) - \varphi(y) = |\varphi(x) - \varphi(y)| < \varepsilon.$$

Hence $(F(x))_{x \rightarrow -\infty}$ is a Cauchy net in $\mathbb{R}$, therefore it is convergent. Observe that, if $\mathfrak{X}$ has minimum $0_{\mathfrak{X}}$, then $\varphi(0_{\mathfrak{X}}) = \sup_{x \rightarrow -\infty} \varphi(x) = \lim_{x \rightarrow -\infty} F(x)$.

For a fixed $y \in \mathfrak{X}$, consider the linear ordered space $\mathfrak{Z} = \{x \in \mathfrak{X} : x \geq y\}$. Since $F \in \text{BV}(\mathfrak{X})$ implies $F|_{\mathfrak{Z}} \in \text{BV}(\mathfrak{Z})$, $\lim_{x \searrow y} F(x) = \lim_{x \rightarrow -\infty} F|_{\mathfrak{Z}}(x)$ exists. $\square$

In the special case where the linear compact space is a closed interval $[a, b] \subseteq \mathbb{R}$, there is the classical representation theorem for bounded linear functionals on $C([a, b])$, published by F. Riesz in [Rie09]. Next we state this result, that also can be found in [Kre78, Theorem 4.4.1]:

Theorem 2.4.4. *Every functional f on $C([a, b])$ can be represented by a Riemann-Stieltjes integral*

$$f(x) = \int_a^b x(t)dw(t),$$

where $w \in \text{BV}([a, b])$ with $V(w) = \|f\|$.

We aim to generalize the previous result to an arbitrary linear compact space K in a way that one can represent bounded linear functionals on $C(K)$ as functions in $\text{NBV}(K)$. For that purpose, we will need some technical results:

Proposition 2.4.5. *Let $\mu \in M(K)$ and consider $F_\mu : K \rightarrow \mathbb{R}$ given by $F_\mu(x) = \mu([\min K, x])$ for all $x \in K$. Then F_μ is right-continuous. Furthermore, if $x \in K$ is not left-isolated, then $\lim_{y \nearrow x} F_\mu(y) = \mu([\min K, x])$ and, in particular, $\mu(\{x\}) = F_\mu(x) - \lim_{y \nearrow x} F_\mu(y)$.*

Proof. Let $x \in K \setminus K^+$ and $\varepsilon > 0$. Since μ is regular, there exists an open set $U \subseteq K$, with $[0_K, x] \subseteq U$ and $|\mu|(U \setminus [0_K, x]) < \varepsilon$. Because x is not right-isolated, there exists $y > x$ such that $[0_K, y] \subseteq U$, hence $|\mu|([x, y]) < \varepsilon$. Thus F_μ is right continuous, since for each $z \in [x, y]$, we have:

$$|F_\mu(z) - F_\mu(x)| \leq |\mu|([x, z]) \leq |\mu|([x, y]) < \varepsilon.$$

To prove that $\lim_{y \nearrow x} F_\mu(y) = \mu([\min K, x])$, let $x \in K \setminus K^-$ and $\varepsilon > 0$. There exists a closed set $F \subseteq [0_K, x]$ such that $|\mu|([0_K, x] \setminus F) < \varepsilon$. It follows that $y = \sup F < x$ and, since x is not left-isolated, $]y, x[\neq \emptyset$. For each $z \in]y, x[$ we have:

$$|\mu([0_K, x]) - F_\mu(z)| \leq |\mu|([z, x]) \leq |\mu|([0_K, x] \setminus F) < \varepsilon. \quad \square$$

Now we are able to define a map between $M(K)$ and $\text{NBV}(K)$ as follows:

$$M(K) \ni \mu \xrightarrow{T} F_\mu \in \text{NBV}(K).$$

It is straightforward to check that T is linear. Moreover, T is bounded because for an arbitrary partition P , we have:

$$V(F_\mu, P) = \sum_{i=0}^{k-1} |F_\mu(t_{i+1}) - F_\mu(t_i)| = \sum_{i=0}^{k-1} |\mu([t_i, t_{i+1}])| \leq \sum_{i=0}^{k-1} |\mu|([t_i, t_{i+1}]) = |\mu|(K \setminus \{0_K\}).$$

Thus, $V(F_\mu) \leq |\mu|(K \setminus \{0_K\})$, and it follows that

$$\|T(\mu)\| = \|F_\mu\| = |F_\mu(0_K)| + V(F_\mu) \leq |F_\mu(0_K)| + |\mu|(K \setminus \{0_K\}) = |\mu|(K) = \|\mu\|.$$

Lemma 2.4.6. *If $f \in \text{BV}(K)$ is continuous at 0_K , then $F : K \rightarrow \mathbb{R}$ given by $F(x) = V(f|_{[0_K, x]})$ is continuous at 0_K .*

Proof. Since F is increasing, if 0_K is not right-isolated, it follows that:

$$\lim_{x \searrow 0_K} F(x) = \inf\{F(x) : x > 0_K\} = L \geq 0.$$

Given $\varepsilon > 0$, let $z > 0_K$ be such that $L \leq F(z) < L + \varepsilon$ and $|f(x) - f(0_K)| < \varepsilon$ for all $x \in [0_K, z]$. For $P = \{0_K = t_0, \dots, t_k = z\}$ an arbitrary partition of $[0_K, z]$, we have that:

$$\begin{aligned} V(f|_{[0_K, z]}, P) &= |f(t_1) - f(0_K)| + \sum_{i=1}^{k-1} |f(t_{i+1}) - f(t_i)| \\ &\leq \varepsilon + V(f|_{[t_1, z]}) \\ &\leq \varepsilon + (F(z) - F(t_1)) \\ &< 2\varepsilon. \end{aligned}$$

Thus, $F(z) \leq 2\varepsilon$ and it follows that F is continuous at 0_K . $\square$

Corollary 2.4.7. *If $f \in \text{NBV}(K)$, then f can be written as difference of two increasing functions of $\text{NBV}(K)$.*

Proof. Since f is right-continuous, a small adaptation on the previous proof shows that $F : K \rightarrow \mathbb{R}$, given by $F(x) = V(f|_{[0_K, x]})$ is right-continuous. Moreover, $F \in \text{BV}(K)$ since it is increasing. Notice that, if $x \leq y$ in K :

$$F(y) - F(x) = V(f|_{[x, y]}) \geq |f(y) - f(x)| \geq f(y) - f(x).$$

Thus, $f = F - (F - f)$ is the difference of two increasing, right-continuous functions of bounded variation. $\square$

Lemma 2.4.8. *Let K be a linear compact space and $\mathcal{U}$ be an open cover of K . There exists a partition $P = \{0_K = x_0, \dots, x_n = 1_K\}$ such that for all $0 \leq i \leq n-1$, one of the cases are verified:*

- *There exists $U \in \mathcal{U}$ such that $[t_i, t_{i+1}] \subseteq U$;*
- *$]t_i, t_{i+1}[= \emptyset$.*

Proof. Consider $\overline{K} = K \cup \{-\infty, +\infty\}$. Since $\mathcal{U}$ is an open cover of K , it admits a refinement by basic open sets of $\overline{K}$:

$$K \subseteq \bigcup_{i \in I} \underbrace{]a_i, a_{i+1}[}_{A_i}, \quad \text{with }]a_i, a_{i+1}[\subseteq U_i \in \mathcal{U}.$$

By compactness, $K \subseteq \bigcup_{i=0}^k]a_i, a_{i+1}[$. If $A_i \neq \emptyset$, fix $a_i < b_i < a_{i+1}$, and consider the following partition of K :

$$P = \{0_K, 1_K\} \cup \{a_i : a_i \in K, 0 \leq i \leq k\} \cup \{b_i : A_i \neq \emptyset\}.$$

Arranging in increasing order, $P = \{0_K = x_0, \dots, x_n = 1_K\}$ is such that, for every $0 \leq j \leq n-1$, we have the following cases to consider:

Case 1: If $t_j \in \{b_i : A_i \neq \emptyset\}$, then $[t_j, t_{j+1}[= [b_i, a_{i+1}[\subseteq]a_i, a_{i+1}[\subseteq U_i \in \mathcal{U}$.

Case 2: If $t_{j+1} \in \{b_i : A_i \neq \emptyset\}$, it is analogous to Case 1.

Case 3: If $\{t_j, t_{j+1}\} \cap \{b_i : A_i \neq \emptyset\} = \emptyset$, then $t_j = a_i$ and $t_{j+1} = a_{i+1}$ for some $0 \leq i \leq k$ and $]t_j, t_{j+1}[= \emptyset$. $\square$

Corollary 2.4.9. *Let K be a linear compact space, $f \in C(K)$ and $\varepsilon > 0$. There exists a partition $P = \{0_K = t_0, \dots, t_n = 1_K\}$ such that, for all $0 \leq i \leq n-1$:*

$$]t_i, t_{i+1}[= \emptyset \quad \text{or} \quad \text{diam}\left(f\left[[t_i, t_{i+1}]\right]\right) < \varepsilon$$

Proof. Consider the open over $\mathcal{U} = \{f^{-1}\left[B(x, \frac{\varepsilon}{2})\right]\}$ and apply Lemma 2.4.8 □

Definition 2.4.10. Consider K a linear compact space, $F \in \text{NBV}(K)$, $P = \{0_K = x_0, \dots, x_k = 1_K\}$ and $f \in C(K)$, we define the *Riemann-Stieltjes sum of f in respect to F and P* as follows:

$$S(f, F, P) = f(0_K)F(0_K) + \sum_{i=0}^{k-1} f(x_{i+1})\left(F(t_{i+1}) - F(t_i)\right)$$

Theorem 2.4.11. *Let K be a linear compact space. Then $M(K)$ is isometrically isomorphic to $\text{NBV}(K)$.*

Proof. Fix $F \in \text{NBV}(K)$ and consider $\alpha : C(K) \rightarrow \mathbb{R}$ given by $\alpha(f) = \lim\{S(f, F, P) : P \in [K]^{<\omega}\}$. We say that α defined in this way corresponds to F via Riemann-Stieltjes.

Claim 1: α is well defined, i.e., for every $f \in C(K)$, $(S(f, F, P))_P$ is a convergent net in $\mathbb{R}$.

Given $\varepsilon > 0$, by Corollary 2.4.9, there exists a partition $P_0 : 0_K = t_0 < \dots < t_k = 1_K$ such that:

$$]t_i, t_{i+1}[= \emptyset \quad \text{or} \quad \text{diam}\left(f\left[[t_i, t_{i+1}]\right]\right) < \frac{\varepsilon}{2V(F)}$$

If Q is a refinement of P_0 , we can write $Q = \{s_{ij} : 0 \leq i \leq k-1, 0 \leq j \leq p_i-1\} \cup \{t_k\}$ such that:

$$t_0 = s_{0,0} < \dots < s_{0,p_0-1} < t_1 = s_{1,0} < \dots < t_i = \underbrace{s_{i,0} < \dots < s_{i,p_i-1}}_{\text{points of } Q \text{ between } t_i \text{ and } t_{i+1}} < t_{i+1} < \dots < t_k$$

This arranging of Q yields the following:

$$f(t_{i+1})(F(t_{i+1}) - F(t_i)) = \sum_{j=0}^{p_i-1} f(t_{i+1})(F(s_{i,j+1}) - F(s_{i,j})), \quad \text{for all } 0 \leq i \leq k-1$$

Thus:

$$\begin{aligned} |S(f, F, P_0) - S(f, F, Q)| &= \left| \sum_{i=0}^{k-1} \sum_{j=0}^{p_i-1} (f(t_{i+1}) - f(s_{i,j+1}))(F(s_{i,j+1}) - F(s_{i,j})) \right| \\ &\leq \sum_{i=0}^{k-1} \sum_{j=0}^{p_i-1} \underbrace{|(f(t_{i+1}) - f(s_{i,j+1}))|}_{< \frac{\varepsilon}{2V(F)} \text{ or }]t_i, t_{i+1}[= \emptyset} |F(s_{i,j+1}) - F(s_{i,j})| \\ &\leq \frac{\varepsilon}{2V(F)} \sum_{i=0}^{k-1} \sum_{j=0}^{p_i-1} |F(s_{i,j+1}) - F(s_{i,j})| \\ &\leq \frac{\varepsilon}{2}. \end{aligned}$$

Claim 2: α is linear and bounded.

It is straightforward to check linearity. For boundedness, observe that $\|\alpha\| \leq \|F\|$, since for

every $f \in C(K)$ with $\|f\| = 1$, we have:

$$|S(f, F, P)| = \left| f(0_K)F(0_K) + \sum_{i=0}^{k-1} f(t_{i+1})(F(t_{i+1}) - F(t_i)) \right| \leq |F(0_K)| + V(F) = \|F\|.$$

Claim 3: If $\mu \in M(K)$ represents α as in the Riesz Representation Theorem 2.4.1, then $F = F_\mu$. Suppose that $F \in \text{NBV}(K)$ is increasing. Since μ represents α , we have:

$$\forall f \in C(K) : \quad \alpha(f) = \int_K f d\mu.$$

If $x \in K$ is right-isolated, then $\chi_{[0_K, x]}$ is continuous and it follows:

$$F(x) = \lim\{S(\chi_{[0_K, x]}, F, P) : P \in [K]^{<\omega}\} = \int_K \chi_{[0_K, x]} d\mu = \mu([0_K, x]) = F_\mu(x).$$

If $x \in K$ is not right-isolated, given $\varepsilon > 0$, there exists an open set U with $[0_K, x] \subseteq U$ and $|\mu|(U \setminus [0_K, x]) < \varepsilon$. Since x is not right-isolated and F is increasing and right-continuous, there exists $y > x$ such that $[x, y] \subseteq U$ and for all $z \in [x, y]$:

- $|\mu|([x, z]) = |\mu|([0_K, z] \setminus [0_K, x]) \leq |\mu|(U \setminus [0_K, x]) < \varepsilon$;
- $|F(z) - F(x)| = F(z) - F(x) < \varepsilon$.

By Urysohn's lemma, take $g : K \rightarrow [0, 1]$ continuous such that $g|_{[0_K, x]} \equiv 1$ and $g|_{[y, 1_K]} \equiv 0$. It follows that:

$$\alpha(g) = \int_K g d\mu = \int_{[0_K, x]} d\mu + \int_{]x, z[} g d\mu \leq \mu([0_K, x]) + |\mu|([x, z]) < F_\mu(x) + \varepsilon.$$

On the other hand, if $P = \{0_K = t_0, \dots, t_k = 1_K\}$ is a partition containing $x = t_m$ and $y = t_n$ such that $|S(g, F, P) - \alpha(g)| < \varepsilon$, we have that:

$$\begin{aligned} S(g, F, P) &= g(0_K)F(0_K) + \sum_{i=0}^{k-1} g(t_{i+1})(F(t_{i+1}) - F(t_i)) \\ &= F(0_K) + \sum_{i=0}^{m-1} (F(t_{i+1}) - F(t_i)) + \sum_{i=m}^{n-2} g(t_{i+1})(F(t_{i+1}) - F(t_i)) + \\ &\quad + \sum_{i=n-1}^{k-1} g(t_{i+1}) \overset{0}{\rightarrow} (F(t_{i+1}) - F(t_i)) \\ &= F(x) + \underbrace{\sum_{i=m}^{n-2} g(t_{i+1})(F(t_{i+1}) - F(t_i))}_{< \varepsilon} \\ &\leq \alpha(g) \\ &< F_\mu(x) + \varepsilon. \end{aligned}$$

Using the reverse triangle inequality with $\alpha(g)$ and $S(g, F, P)$ yields $|F(x) - \alpha(g)| < 2\varepsilon$. Since $F_\mu(x) \leq \alpha(g) < F_\mu(x) + \varepsilon$, we have that $|\alpha(g) - F_\mu(x)| < \varepsilon$. Thus $|F_\mu(x) - F(x)| < 3\varepsilon$ and, since ε is arbitrary, we conclude that $F(x) = F_\mu(x)$.

By Corollary 2.4.7, every $F \in \text{NBV}(K)$ is the difference of two increasing functions. Then, if $G \in \text{NBV}(K)$, $G = F - H$ and we proved above that $F = F_\mu$ and $H = H_\nu$. Therefore $G = G_{\mu-\nu}$.

Claim 4: For $\mu \in M(K)$, let $F_\mu \in \text{NBV}(K)$ and α be the linear functional that corresponds to F_μ via Riemann-Stieltjes. Then α is the linear functional that corresponds to μ in the Riesz Representation Theorem 2.4.1.

Suppose that $\nu \in M(K)$ is the measure representing α in Theorem 2.4.1. We need to show that, for every $f \in C(K)$:

$$\int_K f d\mu = \int_K f d\nu = \alpha(f) = \lim\{S(f, F_\mu, P) : P \in [K]^\omega\}.$$

Given $\varepsilon > 0$, there exists a partition P_0 such that $|S(f, F_\mu, P) - \alpha(f)| < \frac{\varepsilon}{2}$, whenever $P \geq P_0$. Moreover, by Corollary 2.4.9, we can assume that a partition $P = \{0_K = t_0, \dots, x_k = 1_K\} \geq P_0$ satisfies:

$$\forall i < k : \quad]t_i, t_{i+1}[\neq \emptyset \implies \text{diam}\left(f\left[]t_i, t_{i+1}[\right]\right) < \frac{\varepsilon}{2\|\mu\|}.$$

It follows that:

$$S(F, F_\mu, P) = f(0_K)\mu(\{0_K\}) + \sum_{i=0}^{k-1} f(x_{i+1})\mu(]t_i, t_{i+1}[).$$

On the other hand, since $K = \{0_K\} \cup \bigcup_{i < k}]t_i, t_{i+1}[$, it follows that:

$$\int_K f d\mu = f(0_K)\mu(\{0_K\}) + \sum_{i=0}^{k-1} \int_{]t_i, t_{i+1}[} f d\mu.$$

Now we are able to estimate the following:

$$\begin{aligned} |S(F, F_\mu, P) - \int_K f d\mu| &= \left| \sum_{i=0}^{k-1} \underbrace{\left(f(x_{i+1})\mu(]t_i, t_{i+1}[) - \int_{]t_i, t_{i+1}[} f d\mu \right)}_{=0 \text{ if }]t_i, t_{i+1}[= \emptyset} \right| \\ &\leq \sum_{i=0}^{k-1} \frac{\varepsilon}{2\|\mu\|} |\mu|(]t_i, t_{i+1}[) \\ &\leq \frac{\varepsilon\|\mu\|}{2\|\mu\|} < \varepsilon \end{aligned}$$

Thus:

$$\left| \int_K f d\mu - \alpha(f) \right| \leq \left| \int_K f d\mu - S(F, F_\mu, P) \right| + |S(F, F_\mu, P) - \alpha(f)| < \varepsilon.$$

Claim 5: $\text{NBV}(K)$ and $M(K)$ are isometrically isomorphic.

The previous claims yields the following diagram:

$$\begin{array}{ccccc} \text{NBV}(K) & \xrightarrow{T} & C^*(K) & \xrightarrow{S} & M(K) \\ & & \nwarrow Z & \nearrow & \\ & & & & \end{array}$$

where S is the isometry in the Riesz Representation Theorem 2.4.1, $Z(\mu) = F_\mu$ with $\|Z(\mu)\| \leq \|\mu\|$ and $T(F) = \alpha$ is defined in the Claims 1 and 2, with $\|T(F)\| \leq \|\alpha\|$. From the Claims 3 and 4, it follows

$$\text{id}_{\text{NBV}(K)} = Z \circ S \circ T \quad \text{and} \quad T \circ Z = S^{-1}.$$

Thus, T and Z are bijective. Moreover, T and Z are isometries, since

$$\forall F \in \text{NBV}(K) : \quad \|F\| = \|F_\mu\| \leq \|\mu\| \leq \|\alpha\| \leq \|F\|. \quad \square$$

Theorem 2.4.12. *Let K be a zero dimensional linear compact space. Then $\text{NBV}(K)$ is isometrically isomorphic to $\text{BV}(K^+)$.*

Proof. We will prove that the following map is an isometry:

$$\text{NBV}(K) \ni F \xrightarrow{T} F|_{K^+} \in \text{BV}(K^+).$$

It is clear that T is a bounded linear operator with $\|T(F)\| \leq \|F\|$.

Claim 1: T is an isometry.

Let $\varepsilon > 0$ and $P = \{x_0, \dots, x_k = 1_K\}$ be a partition in K . Consider the partition $\tilde{P} = \{s_0, \dots, s_k\}$ defined as follows. If $t_j \in K^+$, take $s_j = t_j$. If $t_j \in K \setminus K^+$, since F is right-continuous, by Proposition 2.1.14, there exists $t_j < s_j < t_{j+1}$ such that $s_j \in K^+$ and $|F(t_j) - F(s_j)| < \frac{\varepsilon}{2k}$. It follows that:

$$|F(t_{i+1}) - F(t_i)| \leq |F(t_{i+1}) - F(s_{i+1})| + |F(s_{i+1}) - F(s_i)| + |F(s_i) - F(t_i)| < |F(s_{i+1}) - F(s_i)| + \frac{\varepsilon}{k}.$$

Thus,

$$\begin{aligned} V(F, P) &= \sum_{i=0}^{k-1} |F(t_{i+1}) - F(t_i)| \\ &\leq \sum_{i=0}^{k-1} \left(\underbrace{|F(t_{i+1}) - F(s_{i+1})|}_{< \frac{\varepsilon}{2k}} + |F(s_{i+1}) - F(s_i)| + \underbrace{|F(s_i) - F(t_i)|}_{< \frac{\varepsilon}{2k}} \right) \\ &< \sum_{i=0}^{k-1} \left(|F(s_{i+1}) - F(s_i)| + \frac{\varepsilon}{k} \right) \\ &= V(F|_{K^+}, \tilde{P}) + \varepsilon \\ &\leq V(F|_{K^+}) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $V(F, P) = V(F|_{K^+})$. Moreover, since F is right-continuous and K is zero-dimensional,

$$F(0_K) = \lim_{x \searrow 0_K} F(x) = \lim_{\substack{x \in K^+ \\ x \rightarrow -\infty}} F(x) = \lim_{\substack{x \in K^+ \\ x \rightarrow -\infty}} F|_{K^+}(x),$$

and we have that T is an isometry:

$$\|F\| = |F(0_K)| + V(F) = \left| \lim_{\substack{x \in K^+ \\ x \rightarrow -\infty}} F|_{K^+}(x) \right| + V(F|_{K^+}) = \|F|_{K^+}\|.$$

Claim 2: T is surjective (therefore bijective):

Given $G \in \text{BV}(K^+)$, notice that by Proposition 2.4.3, there exists $\lim_{x \searrow y} G(x)$ for all $y \in K \setminus K^+$.

Consider $F : K \rightarrow \mathbb{R}$, given by:

$$F(x) = \begin{cases} G(x), & \text{if } x \in K^+ \\ \lim_{t \searrow x} G(t), & \text{if } x \in K \setminus K^+. \end{cases}$$

It is clear that $F|_{K^+} = G$. We will prove that F is right-continuous. Given $x \in K \setminus K^+$ and $\varepsilon > 0$, by the definition of $F(x)$, there exists $x_0 \in K^+$ such that:

$$t \in [x, x_0] \cap K^+ \implies |F(x) - F(t)| = |F(x) - G(t)| < \frac{\varepsilon}{2}.$$

Similarly, given $y \in]x, x_0[\setminus K^+$, there exists $t_y \in K^+$ with $y < t_y < x_0$ and $|F(y) - F(t_y)| < \frac{\varepsilon}{2}$. We have that $|F(x) - F(y)| \leq |F(x) - F(t_y)| + |F(t_y) - F(y)| < \varepsilon$. Therefore $|F(x) - F(z)| < \varepsilon$ for all $z \in [x, x_0]$ and F is right-continuous.

The same argument used to prove that T is an isometry shows that F is of bounded variation,

with $V(F) = F(F|_{K^+}) = V(G)$. Thus, T is an isometry. $\square$

2.5 The weak* topology induced by isometries

The weak and the weak* topologies rise naturally in the study of Banach Space theory due to the fact that the norm topology of such spaces is too strong in the sense that it has few compact subsets. For example, in a normed space X , to ensure that the closed unit ball B_X is a compact subset of X , it is necessary and sufficient that X is finite dimensional. It turns out that these topologies play a central role in a wide range of topics of the Banach space theory. As a reference to the study of weak, weak* topologies and nets in arbitrary Banach spaces, we refer [Meg12] and [Wil12].

The remaining of this section is devoted to use the previous representations of $C(K)^*$, Theorem 2.4.11 and Theorem 2.4.12, to carry the weak* topology defined in $C(K)^*$ to $\text{NBV}(K)$, $M(K)$ and $\text{BV}(K^+)$, respectively. Afterwards we characterize $c_0(I)$ -valued operators in a way that interacts with the respective carried weak* topology.

Our notation is most standard. If X is a Banach space and $J : X \longrightarrow X^{**}$ is the canonical isometric embedding, the weak* topology in X^* is the topology induced by the family of bounded linear functionals $\{J(x) : x \in X\}$.

It is well known that in a set Z , one can use a suitable notion of net convergence to describe a topology in Z and it is, indeed, very common in analysis to use topologies this way. This can be done because nets in topological spaces satisfies the following:

- (a) if $z_i = x$ for all $i \in I$, then $z_i \longrightarrow x$;
- (b) if $z_i \longrightarrow z$, then every subnet of $(z_i)_{i \in I}$ converges to z ;
- (c) if every subnet of $(z_i)_{i \in I}$ has a subnet converging to z , then $(z_i)_{i \in I}$ converges to z ;
- (d) (Diagonal Principle) if $z_i \longrightarrow z$ and, for each $i \in I$, a net $(z_j^i)_{j \in J_i}$ converges to x_i , then the net $(z_j^i)_{\substack{i \in I \\ j \in J_i}} \longrightarrow z$, where $\bigcup_{i \in I} \{i\} \times J_i$ is lexicographically ordered, has a subnet converging to z .

Furthermore, if a notion of net convergence in a set Z satisfies the previous conditions, when defining the closure of $E \subseteq Z$ by $\overline{E} = \{z \in Z : z_i \longrightarrow z \text{ for some } (z_i)_{i \in I} \subseteq E\}$ the result is a topological space where the notion of net convergence is as the one originally specified. From now on, we will treat topologies and their net convergence descriptions interchangeably.

Definition 2.5.1. Let X, Y be Banach spaces and $T : X^* \longrightarrow Y$ be an isometry. The weak* topology in Y is the topology characterized by the following notion of net convergence:

$$y_i \xrightarrow{w^*} y \iff T^{-1}(y_i) \xrightarrow{w^*} T^{-1}(y)$$

In some cases it is possible to give a better characterization of the weak topology for bounded nets, for instance, in [Hög77] it is proved that in $\text{NBV}([0, 1])$, a bounded net $F_i \xrightarrow{w^*} F$ iff $F_i(1) \longrightarrow F(1)$ and $F_i \xrightarrow{L_1([0, 1], \mathfrak{m})} F$, where $\mathfrak{m}$ denotes the Lebesgue measure. In what follows we will state more results about the convergence of bounded nets.

We say that a subset D of a Banach space X is linearly dense in X if $\text{cl}(\text{span}(D)) = X$. The reader may notice that the weak* convergence of bounded nets in X^* is characterized by the convergence of a given linearly dense set D , but for the sake of completeness we give a proof of this fact.

Example 2.5.2. [Kub07, Proposition 3.2] Let K be a linear compact space. Then $\{f \in C(K) : f \text{ is increasing}\}$ is a linearly dense set of $C(K)$.

Lemma 2.5.3. If X is a Banach space, $(\alpha_i)_{i \in I}$ is a bounded net in X^* and $\alpha \in X^*$, then $\Lambda = \{x \in X : \alpha_i(x) \longrightarrow \alpha(x)\}$ is a closed subspace of X .

Proof. It is straightforward to prove that Λ is a subspace. Take $y \in \text{cl}(\Lambda)$ and $\varepsilon > 0$. There exists $x \in \Lambda$ such that $\|x - y\| \leq \frac{\varepsilon}{4 \max\{\sup \|\alpha_i\|, \|\alpha\|\}}$. Since $x \in \Lambda$, there exists $i_0 \in I$ such that $|\alpha_i(x) - \alpha(x)| < \frac{\varepsilon}{4}$, for all $i \geq i_0$. It follows that, for all $i \geq i_0$:

$$\begin{aligned} |\alpha_i(y) - \alpha(y)| &\leq |\alpha_i(y) - \alpha_i(x)| + |\alpha_i(x) - \alpha(x)| + |\alpha(x) - \alpha(y)| \\ &\leq \|\alpha_i\| \|x - y\| + \frac{\varepsilon}{4} + \|\alpha\| \|x - y\| \\ &\leq \frac{3\varepsilon}{4}, \end{aligned}$$

thus, $y \in \Lambda$. □

Proposition 2.5.4. *Let X be a Banach space, $(\alpha_i)_{i \in I}$ be a bounded net in X^* and $D \subseteq X$ a linearly dense set. Then $\alpha_i \xrightarrow{w^*} \alpha$ iff $\alpha_i(x) \rightarrow \alpha(x)$, for all $x \in D$.*

Proof. Suppose that $\alpha_i(x) \rightarrow \alpha(x)$ for all $x \in D$. By Lemma 2.5.3 it follows that $\text{span}(D) \subseteq \Lambda$ and since $\text{span}(D)$ is dense in X , we have that $X = \Lambda$. □

Example 2.5.5. Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $\ell_p^* = \ell_q$ and the canonical Schauder basis $(e_n)_{n \in \omega}$ is a linearly dense set in ℓ_p . It follows that the weak* topology in $B_{\ell_p^*}$ (or any bounded region of ℓ_p^*) coincides with the product topology of $\ell_q \subseteq \mathbb{R}^\omega$:

$$\begin{aligned} \alpha_i \xrightarrow{w^*} \alpha &\iff \alpha_i(e_n) \rightarrow \alpha(e_n), \quad \forall n \in \omega \\ &\iff \alpha_i^n \rightarrow \alpha^n, \quad \forall n \in \omega \\ &\iff \pi_n(\alpha_i) \rightarrow \pi_n(\alpha), \quad \forall n \in \omega. \end{aligned}$$

Proposition 2.5.6. *Let K be a compact, Hausdorff, zero-dimensional space. Then $\{f \in C(K) : \text{ran}(f) \text{ is finite}\}$ is a dense subspace of $C(K)$. In particular, $\{\chi_C \in C(K) : C \in \text{Clop}(K)\}$ is linearly dense in $C(K)$.*

Proof. Given $f \in C(K)$ and $\varepsilon > 0$, we will define a function g with finite range and $\|f - g\| < \varepsilon$. For each $p \in K$, let $U_p = f^{-1}\left[B(f(p), \frac{\varepsilon}{4})\right]$. Since K is zero-dimensional, for every $p \in K$, there exists a clopen neighborhood of p such that $C_p \subseteq U_p$. Then $K = \bigcup_{i=0}^n C_{p_i}$ and, **WLOG**, the C_{p_i} are all disjoint from each other.

Thus, we have that $K = \bigcup_{i=0}^n C_{p_i}$, with $p_i \in C_{p_i} \in \text{Clop}(K)$ and $\text{diam}(f[C_{p_i}]) < \varepsilon$ for all $i \leq n$.

Consider the simple function $g = \sum_{i=0}^n f(p_i)\chi_{C_{p_i}} \in C(K)$, since for every $x \in K$, there exists $i \leq n$ such that $x \in C_i$, it follows that:

$$\|f - g\| = \sup_{x \in K} |f(x) - g(x)| \leq \sup_{i \leq n} \text{diam}(f[C_{p_i}]) < \varepsilon. \quad \square$$

Corollary 2.5.7. *If K is a compact, Hausdorff, zero-dimensional space, then the convergence of bounded nets in $M(K) \equiv C(K)^*$ is characterized by:*

$$\mu_i \xrightarrow{w^*} \mu \iff \mu_i(C) \rightarrow \mu(C), \quad \text{for all } C \in \text{Clop}(K)$$

Proof. Since the weak topology on $M(K)$ is given by Theorem 2.4.1, it follows that:

$$\begin{aligned} \mu_i \xrightarrow{w^*} \mu &\iff \alpha_{\mu_i} \xrightarrow{w^*} \alpha_\mu \\ &\iff \alpha_{\mu_i}(\chi_C) \xrightarrow{w^*} \alpha_\mu(\chi_C), \quad \text{for all } C \in \text{Clop}(K) \\ &\iff \mu_i(C) \rightarrow \mu(C), \quad \text{for all } C \in \text{Clop}(K). \end{aligned} \quad \square$$

To address the problem of characterization of weak topologies in a zero-dimensional linear compact space, we first need the following:

Definition 2.5.8. Let K be a zero-dimensional linear compact space. We say that $S \subseteq \text{Clop}(K)$ is an *additive class* if the following conditions are satisfied:

- $\emptyset \in S$;
- (disjoint union) for all $A, B \in S$, with $A \cap B = \emptyset$, then $A \cup B \in S$;
- (proper difference) for all $A, B \in S$ with $B \subseteq A$, then $A \setminus B \in S$.

For a set $Y \subseteq \text{Clop}(K)$, we say that $\text{Clop}(K)$ is generated by Y as an additive class iff every element of $\text{Clop}(K)$ can be written as a finite (disjoint) union of elements Y and proper differences of elements of Y .

Proposition 2.5.9. Let K be a zero-dimensional linear compact space. If $\text{Clop}(K)$ is generated by S as an additive class, then the weak topology in $M(K)$ for bounded nets is determined by the convergence on S .

Proof. Let $(\mu_i)_{i \in I}$ be a bounded net in $M(K)$ such that $\mu_i \xrightarrow{w^*} \mu$. By our assumption, for every $C \in \text{Clop}(K)$, we can write $C = \bigcup_{j=0}^{n_C} X_j$, where each X_j is an element of:

$$S = \{A \setminus B : A, B \in S \text{ and } B \subseteq A\} \cup \{A \cup B : A, B \in S \text{ e } A \cap B = \emptyset\}.$$

Then, by Corollary 2.5.7, we have the following:

$$\begin{aligned} \mu_i \xrightarrow{w^*} \mu &\iff \mu_i(C) \longrightarrow \mu(C), \quad \text{for all } C \in \text{Clop}(K) \\ &\iff \sum_{j=0}^{n_C} \mu_i(X_j) \longrightarrow \sum_{j=0}^{n_C} \mu(X_j), \quad \text{for all } C \in \text{Clop}(K) \\ &\iff \mu_i(A) \longrightarrow \mu(A), \quad \text{for all } A \in S. \end{aligned} \quad \square$$

In the view of the previous result, we need to find a family of clopen sets in K that generates $\text{Clop}(K)$ as an additive class to characterize the weak convergence of bounded nets. For this purpose we need some topological results regarding linear compact spaces.

Definition 2.5.10. Let K be a linear compact space and $S \subseteq K$. We define a equivalence relation on S as follows:

$$\forall x, y \in S : \quad x \sim y \iff [\min\{x, y\}, \max\{x, y\}] \subseteq S$$

The equivalence class $[x]_{\sim}$ are called convex components of S .

Remark. It is easy to prove that the convex components of S are closed in the topology of subspace. Moreover, when S is an open set of K , the convex components of S are open sets of K .

In particular, if $S \in \text{Clop}(K)$, then it follows from Proposition 2.1.13 that S is written as a finite (disjoint) union of its convex components (that are also clopen sets of K).

Before we give more sophisticated arguments involving convex components, we provide two basic results regarding convex components that will be used in the following chapters of this work:

Proposition 2.5.11. Let K be a linear compact space, $F \in \text{BV}(K)$ and $0 < \varepsilon_2 < \varepsilon_1$. Then the number of convex components of the set

$$A = \{t \in K : F(t) \geq \varepsilon_2\}$$

that intersects the set

$$B = \{t \in K : F(t) \geq \varepsilon_1\}$$

is finite and less than $1 + \frac{V(F)}{2(\varepsilon_1 - \varepsilon_2)}$. In particular, the set B can be covered by finitely many convex sets contained in A .

Proof. We first show that there exists only finitely many convex components of A that intersects B . Suppose towards a contradiction that this is not the case, then there exists a infinite countable number of distinct convex components $(C_n)_{n \in \omega}$. For each n , take $t_n \in C_n \cap B$. We can assuming, reordering if necessary, that $(t_n)_{n \in \omega}$ is an increasing sequence. For each $n \in \omega$, there exists s_n with $t_n < s_n < t_{n+1}$ with $|F(s_n)| < \varepsilon_2$ because t_n and t_{n+1} are not in the same convex component of A . We have the following estimate for the partition $P_n = \{t_0, s_0, t_1, s_1, \dots, t_{n-1}, s_{n-1}, s_n, t_n\}$:

$$\sum_{i=0}^{n-1} (|F(s_i) - F(t_{i+1})| + |F(t_i) - F(s_i)|) \leq V(F)$$

On the other hand, we have that both $|F(t_i) - F(s_i)| > \varepsilon_1 - \varepsilon_2$ and $|F(s_i) - F(t_{i+1})| > \varepsilon_1 - \varepsilon_2$, so:

$$\sup\{V(F, P_n) : n \in \omega\} \geq \sup\{(n-1)(\varepsilon_1 - \varepsilon_2) : n \in \omega\} = \infty,$$

a contradiction to F being of bounded variation. This shows us that we have only finitely many convex components of A that intersects B as well as the number N of convex components satisfies:

$$2(N-1)(\varepsilon_1 - \varepsilon_2) < V(F).$$

To see that B can be covered by finitely many convex components contained in A , just notice that $B \subseteq A$. $\square$

Proposition 2.5.12. *Let K be a zero-dimensional linear compact space. Then $\text{Clop}(K)$ is generated as an additive class by $\mathcal{C} = \{[0_K, x] : x \in K^+\}$.*

Proof. Consider $\mathcal{S} = \mathcal{C} \cup \{]x, y] : x, y \in K^+, x < y\} \subseteq \text{Clop}(K)$.

Claim 1 $\mathcal{S}$ is a basis of open sets of K :

Let U be a non-empty open set in K and $x \in U$. Since K is zero-dimensional, there exists a clopen set C such that $x \in C \subseteq U$. Let $C = \bigcup_{i=0}^n I_n$ be the finite union of its convex components, which are also clopen sets of K . Take $j \leq n$ such that $x \in I_j$. Since $\inf I_j \in K^-$ and $\sup I_j \in K^+$, it follows that $x \in](\inf I_j)^-, \sup I_j] \subseteq U$.

Claim 2 $\mathcal{S}$ contains all proper differences of $\mathcal{C}$. Moreover, if $A, B \in \mathcal{S}$, then $A \setminus B$ is the finite disjoint union of (at most two) elements of $\mathcal{S}$.

The set $\mathcal{S}$ is defined to contain all the proper differences of $\mathcal{C}$ and it is straightforward to check the second part of Claim 2 for every pair of elements of $\mathcal{S}$.

Claim 3 Every finite union of elements of $\mathcal{S}$ can be written as a finite disjoint union of elements of $\mathcal{S}$:

Consider $\tilde{\mathcal{S}}$ the collection of all finite disjoint unions of elements of $\mathcal{S}$. Then $\mathcal{S} \subseteq \tilde{\mathcal{S}}$ and:

- (a) $A, B \in \mathcal{S} \implies A \setminus B \in \tilde{\mathcal{S}}$;
- (b) $A \in \tilde{\mathcal{S}}, B \in \mathcal{S} \implies A \setminus B \in \tilde{\mathcal{S}}$;
- (c) $A, B \in \tilde{\mathcal{S}} \implies A \setminus B \in \tilde{\mathcal{S}}$.

Given $A_0, \dots, A_n \in \mathcal{S}$, define $B_j = A_j \setminus \bigcup_{i < j} A_i$ for $j \leq n$. It follows that, for all $j \leq n$:

$$B_j = (\dots (\underbrace{(A_j \setminus A_1) \setminus A_2}_{\in \tilde{\mathcal{S}}} \dots) \setminus A_{j-1}) \in \tilde{\mathcal{S}}.$$

Then $\bigcup_{i=0}^n A_i = \bigcup_{i=0}^n B_i \in \tilde{\mathcal{S}}$.

By the previous claims, every $C \in \text{Clop}(K)$ can be written as a finite (disjoint) union of elements of $\mathcal{S}$. Then $\text{Clop}(K)$ is generated as an additive class by $\mathcal{C}$, because $\mathcal{S}$ contains all the proper differences of $\mathcal{C}$. $\square$

Corollary 2.5.13. *Let K be a zero-dimensional linear compact space. Then for bounded nets in $M(K) \equiv \text{BV}(K^+)$, are equivalent:*

- (a) $\mu_i \xrightarrow{w^*} \mu$;
- (b) $\mu_i([0_K, x]) \xrightarrow{w^*} \mu([0_K, x])$, for all $x \in K^+$;
- (c) $F_i \xrightarrow{\text{pointwise}} F$.

Proof. Use the isometry given in Theorem 2.4.12 and Proposition 2.5.12. $\square$

When I is an infinite set (not necessarily a directed set) and X is a Banach space, we say that $(\alpha_i)_{i \in I} \subseteq X^*$ is weak*-null family if $(\alpha_i(x))_{i \in I} \in c_0(I)$ for all $x \in X$. When it is clear in the context, may abbreviate that to $\alpha_i \xrightarrow{w^*} 0$, even though I has no partial order structure because the definition of weak*-null family is equivalent to say that $w^*\text{-}\lim_{i \rightarrow \infty} \alpha_i = 0$, where $I \cup \{\infty\}$ is the one-point compactification of the discrete set I . We finish this section establishing a relation between bounded $c_0(I)$ -valued operators, the weak* topology and how this will be used in the following parts of our work.

Proposition 2.5.14. *Let X be a Banach space and I be an infinite set. The following are equivalent:*

- (a) $T : X \longrightarrow c_0(I)$ is a bounded operator;
- (b) There exists a weak*-null family $(\alpha_i)_{i \in I} \subseteq X^*$ such that $\|T\| = \sup_{i \in I} \|\alpha_i\|$.

Proof. For $i \in I$, denote $\pi_i : c_0(I) \longrightarrow \mathbb{R}$ the canonical projections.

- (a) $\implies$ (b) Define $\alpha_i = \pi_i \circ T$, for every $i \in I$. Then $(\alpha_i)_{i \in I}$ is a weak*-null family and $\|T\| = \sup_{i \in I} \|\alpha_i\|$.
- (b) $\implies$ (a) Define $T : X \longrightarrow c_0(I)$ by $T(x) = (\alpha_i(x))_{i \in I}$. Then T is a well defined operator and $\|T\| = \sup_{i \in I} \|\alpha_i\|$. $\square$

It is usual then, when working on the Banach space $C(K)$, where K is a linear compact space, to associate bounded $c_0(I)$ -valued operators with the suitable weak*-converging family given by the previous result and one of the Riesz's Representations theorems in this section.

Definition 2.5.15. Given K, L compact Hausdorff spaces and a continuous function $\varphi : K \longrightarrow L$, the composition operator induced by φ is $\varphi^* : C(L) \longrightarrow C(K)$ is defined by $\varphi^*(f) = f \circ \varphi$, for all $f \in C(L)$.

Proposition 2.5.16. *Given K, L compact Hausdorff spaces and a continuous function $\varphi : K \longrightarrow L$, then:*

- (a) φ is surjective $\iff \varphi^*$ is an isometric embedding;
- (b) φ is injective $\iff \varphi^*$ is surjective.

Proof. If φ is surjective, then for all $f \in C(L)$:

$$\|f\| = \sup_{y \in L} |f(y)| = \sup_{x \in K} |f \circ \varphi(x)| = \|\varphi^*(f)\|.$$

For the converse, suppose that φ is not surjective. Then, by Urysohn's lemma, take $y \in L \setminus \text{ran}(\varphi)$ and a continuous function $g : L \rightarrow [0, 1]$ such that $g(y) = 1$ and $g|_{\text{ran}(\varphi)} \equiv 0$. Then $\|g\| > 0$ and $\varphi^*(g) \equiv 0$ and we conclude that φ^* is not an isometric embedding.

If φ is injective, it follows from Tietze's extension theorem that φ^* is surjective. On the other hand, suppose towards a contradiction that φ^* is surjective and there exist $x_1 \neq x_2$ such that $\varphi(x_1) = \varphi(x_2)$. Consider $g : K \rightarrow [0, 1]$ such that $g(x_1) = 0$ and $g(x_2) = 1$, given by the Urysohn's lemma. By our assumption, there exists $f \in C(L)$ with $\varphi^*(f) = g$, that leads to the following contradiction:

$$0 = g(x_1) = f \circ \varphi(x_1) = f \circ \varphi(x_2) = g(x_2) = 1. \quad \square$$

By the previous result, given $\varphi : K \xrightarrow{\text{c.i.s.}} L$ between linear compact spaces, it follows that $C(L)$ is identified with the subspace $\varphi^*C(K) \subseteq C(K)$ and one can ask whether a $c_0(I)$ -valued operator defined on $C(L)$ admits an extension to $C(K)$. Since these operators are associated with weak*-null families of linear functionals, this lead us to the following diagrams:

$$\begin{array}{ccc} M(K) & \xrightarrow{\varphi_*} & M(L) \\ \parallel & & \parallel \\ \text{NBV}(K) & \xrightarrow{\varphi_*} & \text{NBV}(L) \end{array} \quad \begin{array}{ccc} M(K) & \xrightarrow{\varphi_*} & M(L) \\ \parallel & & \parallel \\ \text{BV}(K^+) & \xrightarrow{\varphi_*} & \text{BV}(L^+) \end{array}$$

The vertical arrows on the left diagram are given by Theorem 2.4.11 and, in the right diagram both K and L are considered to be zero-dimensional and the vertical arrows are given by Theorem 2.4.12. In both cases, the map $\varphi_* : M(K) \rightarrow M(L)$ is the usual pushforward of measures, given by $\varphi_*(\mu)(A) = \mu(\varphi^{-1}[A])$ and the maps $\varphi_* : \text{NBV}(K) \rightarrow \text{NBV}(L)$ and $\varphi_* : \text{BV}(K^+) \rightarrow \text{BV}(L^+)$ are the maps induced by these identifications, that we describe as follows. For every $t \in L$, denote $\varphi^{-1}[t] = [a_t, b_t]$, then for every $F \in \text{NBV}(K)$ and $G \in \text{BV}(K^+)$:

$$\varphi_*(F)(t) = F(b_t), \text{ for all } t \in L \quad \text{and} \quad \varphi_*(G)(s) = G(b_s), \text{ for all } s \in L^+.$$

This motivates the following definition:

Definition 2.5.17. Let $\varphi : K \xrightarrow{\text{c.i.s.}} L$. We say that a bounded operator $T : C(L) \rightarrow c_0(I)$ extends through φ to $C(K)$ iff one of the (equivalent) statements are satisfied:

- (a) There exists a bounded operator $\tilde{T} : C(K) \rightarrow c_0(I)$ such that $\tilde{T} \circ \varphi^* = T$;
- (b) There exists a weak*-null family $(\tilde{\mu}_i)_{i \in I} \subseteq M(K)$ such that $T \equiv (\varphi_*(\tilde{\mu}_i))_{i \in I}$;
- (c) There exists a weak*-null family $(\tilde{F}_i)_{i \in I} \subseteq \text{NBV}(K)$ such that $T \equiv (\varphi_*(\tilde{F}_i))_{i \in I}$;
- (d) Provided both K and L are zero-dimensional, there exists a weak*-null family $(\tilde{F}_i)_{i \in I} \subseteq \text{BV}(K^+)$ such that $T \equiv (\varphi_*(\tilde{F}_i))_{i \in I}$;

Lemma 2.5.18. Let $\varphi : K \xrightarrow{\text{c.i.s.}} L$, $G \in \text{BV}(K)$ and $F = \varphi_*(G) \in \text{BV}(L)$. Then:

$$\max \left\{ \|F\|_{\text{BV}}, \sum_{t \in L} V(G|_{\varphi^{-1}[t]}) \right\} \leq \|G\|_{\text{BV}} \leq \|F\|_{\text{BV}} + 2 \sum_{t \in L} V(G|_{\varphi^{-1}[t]}).$$

Moreover, $G \in \text{NBV}(K)$ if and only if $F \in \text{NBV}(L)$.

Proof. It is straightforward to check that $\|F\|_{\text{BV}} \leq \|G\|_{\text{BV}}$ and $\sum_{t \in L} V(G|_{\varphi^{-1}[t]}) \leq \|G\|_{\text{BV}}$.

On the other hand, if $P : \min K = x_0 < \dots < x_{n+1} = \max K$ is an arbitrary partition, denote $\varphi[P] = \{t_0 < \dots < t_{k+1}\}$. **WLOG** we can consider that $a_{t_i}, b_{t_i} \in P$ for all $i \leq k+1$ by replacing

P , if necessary, by $P \cup \{a_{t_i} : i \leq k+1\} \cup \{b_{t_i} : i \leq k+1\}$. It follows that

$$\begin{aligned}
 V(G, P) &= \sum_{i=0}^{k+1} V(G, P \cap [a_{t_i}, b_{t_i}]) + \sum_{i=0}^k |G(a_{t_{i+1}}) - G(b_{t_i})| \\
 &\leq \sum_{i=0}^{k+1} V(G|_{\varphi^{-1}[t_i]}) + \sum_{i=0}^k \left(|G(b_{t_i}) - G(b_{t_{i+1}})| + |G(b_{t_{i+1}}) - G(a_{t_{i+1}})| \right) \\
 &\leq \sum_{i=0}^{k+1} V(G|_{\varphi^{-1}[t_i]}) + \sum_{i=0}^k |F(t_i) - F(t_{i+1})| + \sum_{i=1}^{k+1} |G(b_{t_i}) - G(a_{t_i})| \\
 &\leq \|F\|_{BV} + 2 \sum_{t \in L} V(G|_{\varphi^{-1}[t]}). \quad \square
 \end{aligned}$$

Remark. The reader may notice that the preceding result holds for arbitrary functions $G : K \rightarrow \mathbb{R}$ provided that $G(b_t) = F(t)$ for all $t \in L$. In this case, the estimates holds when $G \in BV(K)$. In addition, G is right continuous if, and only if, F is right-continuous and $G|_{\varphi^{-1}[t]}$ is right-continuous for all $t \in L$.

We finish this section with characterizations of weak* convergence when K is not considered zero-dimensional.

Definition 2.5.19. Let K be a linear compact space and $[a, b] \subseteq K$. The *canonical retraction of K over the closed interval $[a, b]$* is the function $R : K \rightarrow [a, b]$ given by $R|_{[\min K, a]} \equiv a$, $R|_{[a, b]} \equiv id|_{[a, b]}$ and $R|_{[b, \max K]} \equiv b$.

Notice that a canonical retraction $R : K \rightarrow [a, b]$ is a c.i.s. function, hence the composition map $R^* : C[a, b] \rightarrow C(K)$ is an isometric embedding. It is straightforward to check that the elements of $R^*(C[a, b])$ are the functions $f \in C(K)$ that are constant on the closed intervals $[\min K, a]$ and $[b, \max K]$.

Proposition 2.5.20. Let $\varphi : K \xrightarrow{c.i.s.} L$. For each $t \in L$, denote $R_t : K \rightarrow \varphi^{-1}[t]$ the canonical retraction. Then the set

$$\varphi^*C(L) \cup \bigcup_{t \in L} R_t^*C(\varphi^{-1}[t])$$

is a linearly dense in $C(K)$.

Proof. See [CT13, Lemma 2.2] and notice that $R_{t_0} \equiv t_0$, provided $|\varphi^{-1}[t_0]| = 1$. $\square$

Corollary 2.5.21. Let $\varphi : K \xrightarrow{c.i.s.} L$, $\mu \in M(K)$ and $(\mu_i)_{i \in I}$ be a bounded family in $M(K)$. The following are equivalent:

- (a) $\mu_i \xrightarrow{w^*} \mu$;
- (b) $\varphi_*(\mu_i) \xrightarrow{w^*} \varphi_*(\mu)$ and $(R_t)_*(\mu_i) \xrightarrow{w^*} (R_t)_*(\mu)$, for all $t \in L$.

Proof. See [CT13, Corollary 2.3]. $\square$

Chapter 3

Extension of bounded operators in $C(K)$

The problem involving the extension of bounded operators is a classical and well investigated theme in functional analysis. For instance, the Hahn-Banach theorem ([Hah27, Ban29]) states that every bounded linear functional can be extended from a subspace of a normed vector space. On the other hand, R.S. Phillips proved in [Phi40] that c_0 is not a complemented subspace of ℓ_∞ , showing that there exist bounded operators between Banach spaces that cannot be extended.

Another classical result regarding this area of study is given by the classical Sobczyk's theorem ([Sob41, Vee71, SCY00]), which states that every c_0 -valued bounded operator defined on a closed subspace of a separable Banach space admits a bounded extension to the entire space. The classical Sobczyk's Theorem holds when the space c_0 is replaced by its non-separable version, $c_0(I)$:

Theorem 3.0.1. *Let X be a Banach space and $Y \subseteq X$ be a closed subspace. If the quotient space X/Y is separable, then every bounded operator $T : Y \rightarrow c_0(I)$ admits an extension $\tilde{T} : X \rightarrow c_0(I)$ with $\|\tilde{T}\| \leq 2\|T\|$.*

Towards the goal of investigating generalizations of the classical theorem of Sobczyk, in [CT13], the following definition is given: a closed subspace Y of a Banach space X has the c_0 -extension property (c_0 -EP) in X if every bounded operator $T : Y \rightarrow c_0$ admits a bounded extension to X . In [CT15], the authors investigate the c_0 -EP for unital Banach subalgebras of $C(K)$, with K a linearly ordered compact space. More precisely, it is given a characterization of the $\phi : K \xrightarrow{\text{c.i.s.}} L$ such that the corresponding subalgebra $\phi^*C(L) \subseteq C(K)$ has the c_0 -EP. To state this characterization, we will need to introduce some notation.

Given K, L linear compact spaces and $\varphi : K \xrightarrow{\text{c.i.s.}} L$, we define the sets $Q(\varphi) = \{t \in L : |\varphi^{-1}(t)| > 1\}$ and $Q_0(\varphi) = \{t \in Q(\varphi) : t \text{ is an internal point of } L\}$ and we say that φ has the c_0 -EP when the range of the composition operator, $\varphi^*C(L) \subseteq C(K)$ has the c_0 -EP in $C(K)$. We are now able to state the main results regarding operator extension in [CT15]:

Proposition 3.0.2. *Let $T : C(L) \rightarrow c_0$ be a bounded operator associated with a weak*-null sequence $(F_n)_{n \geq 1} \subseteq \text{NBV}(L)$. The following conditions are equivalent:*

- (a) *there exists a bounded operator $\tilde{T} : C(K) \rightarrow c_0$ with $\tilde{T} \circ \varphi^* = T$;*
- (b) *the set of points $t \in Q(\varphi)$ such that $F_n(t) \not\rightarrow 0$ is countable;*
- (c) *the set of points $t \in Q_0(\varphi)$ such that $F_n(t) \not\rightarrow 0$ is countable;*
- (d) *there exists a bounded operator $\tilde{T} : C(K) \rightarrow c_0$ with $\tilde{T} \circ \varphi^* = T$ and $\|\tilde{T}\| \leq 2\|T\|$.*

Theorem 3.0.3. *Let K, L be linear compact spaces and $\varphi : K \xrightarrow{\text{c.i.s.}} L$. Are equivalent:*

- (a) *φ has the c_0 -EP;*
- (b) *For each separable G_δ set $A \subseteq L$ such that every point $x \in A$ is an internal point of A , then $A \cap Q(\varphi) = A \cap Q_0(\varphi)$ is countable.*

The following construction shows that in order to replace c_0 for its non separable version, $c_0(I)$, we cannot obtain analogous equivalences of Proposition 3.0.2:

Example 3.0.4. A weak*-null family $(F_i)_{i \in I} \subseteq \text{NBV}(L)$ that extends through $\varphi : K \xrightarrow{\text{c.i.s.}} L$ but the set $\{t \in Q(\varphi) : F_i(t) \not\rightarrow 0\}$ is uncountable:

Consider K, L linear compact spaces, a function $\varphi : K \xrightarrow{\text{c.i.s.}} L$ and a bounded operator $T : C(L) \rightarrow c_0$ that extends through φ to $C(K)$. By Definition 2.5.17, it follows that $T \equiv (F_n)_{n \in \omega} \subseteq \text{NBV}(L)$ and there exists a weak*-null sequence $(G_n)_{n \in \omega} \subseteq \text{NBV}(K)$ such that for each $n \in \omega$, $\varphi_*(G_n) = F_n$.

For an ordinal γ , we consider the linear compact spaces $\mathbb{R}_\gamma^L = [0, \gamma] \times L$ and $\mathbb{R}_\gamma^K = [0, \gamma] \times K$, both endowed with the lexicographic order and $\Phi : \mathbb{R}_\gamma^K \xrightarrow{\text{c.i.s.}} \mathbb{R}_\gamma^L$ given by $\Phi(\alpha, x) = (\alpha, \varphi(x))$.

For each $\beta \leq \gamma$ and each $n \in \omega$, we define $F_{\beta,n} : \mathbb{R}_\gamma^L \rightarrow \mathbb{R}$ and $G_{\beta,n} : \mathbb{R}_\gamma^K \rightarrow \mathbb{R}$ as follows:

$$F_{\beta,n}(\alpha, t) = \chi_{\{\beta\}}(\alpha)F_n(t) \quad \text{and} \quad G_{\beta,n}(\alpha, s) = \chi_{\{\beta\}}(\alpha)G_n(s),$$

for each $\alpha \leq \gamma$, $s \in K$ and $t \in L$.

It is clear that $(F_{\beta,n})_{\beta < \gamma, n \in \omega} \subseteq \text{NBV}(\mathbb{R}_\gamma^L)$, $(G_{\beta,n})_{\beta < \gamma, n \in \omega} \subseteq \text{NBV}(\mathbb{R}_\gamma^K)$.

Claim 1: For each $\beta \leq \gamma$ and each $n \in \omega$, $\Phi_*(G_{\beta,n}) = F_{\beta,n}$.

Indeed, for every $(\alpha, t) \in \mathbb{R}_\gamma^L$, denote $[a_t, b_t] = \varphi^{-1}[t]$. It follows that $\Phi^{-1}[(\alpha, t)] = [(\alpha, a_t), (\alpha, b_t)]$ and we have

$$F_{\beta,n}(\alpha, t) = \chi_{\{\beta\}}(\alpha)F_n(t) = \chi_{\{\beta\}}(\alpha)G_n(b_t) = G_{\beta,n}(\alpha, b_t) = \Phi_*(G_{\beta,n})(\alpha, t),$$

and the claim is proved.

For each $\beta \leq \gamma$ we denote $L_\beta = \{\beta\} \times L \subseteq \mathbb{R}_\gamma^L$ and we consider $i_\beta^L : L \rightarrow \mathbb{R}_\gamma^L$ and $\pi_\beta^L : L_\beta \rightarrow L$ given by $i_\beta^L(t) = (\beta, t)$ and $\pi_\beta^L(\beta, t) = t$, for each $t \in L$. Then, for every $\mu \in M(L)$, $(i_\beta^L)_*(\mu) \in M(\mathbb{R}_\gamma^L)$ is given by:

$$(i_\beta^L)_*(\mu)(B) = \mu\left(\pi_\beta^L[B \cap L_\beta]\right).$$

Claim 2: If $(\mu_n)_{n \in \omega} \subseteq M(L)$ is such that $F_n \equiv \mu_n$ for all $n \in \omega$, it follows that

$$F_{\beta,n} \equiv \mu_{\beta,n}, \text{ where } \mu_{\beta,n} = \begin{cases} (i_\beta^L)_*(\mu_n) - F_n(\max L) \cdot \delta_{(\beta+1, \min L)}, & \text{if } \beta < \gamma \\ (i_\gamma^L)_*(\mu_n), & \text{if } \beta = \gamma \end{cases}$$

where $\delta_{(\beta+1, \min L)}$ denotes the Dirac measure.

We will prove the claim when $\beta < \gamma$ since when $\beta = \gamma$ the proof is similar. For a fixed $(\alpha, t) \in \mathbb{R}_\gamma^L$, we have three cases to consider:

$\alpha < \beta$: Then $\delta_{(\beta+1, \min L)}\left([(0, \min L), (\alpha, t)]\right) = 0$. In addition:

$$(i_\beta^L)_*(\mu_n)\left([(0, \min L), (\alpha, t)]\right) = \mu_n\left((i_\beta^L)^{-1}\left([(0, \min L), (\alpha, t)]\right)\right) = \mu_n(\emptyset) = 0.$$

Thus $F_{\beta,n} \equiv \mu_{\beta,n}$ trivially.

$\alpha = \beta$: We still have $\delta_{(\beta+1, \min L)}\left([(0, \min L), (\beta, t)]\right) = 0$. And it follows that:

$$(i_\beta^L)_*(\mu_n)\left([(0, \min L), (\beta, t)]\right) = \mu_n\left([\min L, t]\right) = F_n(t) = F_{\beta,n}(\beta, t).$$

$\alpha > \beta$: In this case $\delta_{(\beta+1, \min L)}\left([(0, \min L), (\alpha, t)]\right) = 1$ and $(i_\beta^L)^{-1}\left([(0, \min L), (\alpha, t)]\right) = L$. It follows that:

$$\left((i_\beta^L)_*(\mu_n) - F_n(\max L) \cdot \delta_{(\beta+1, \min L)}\right)\left([(0, \min L), (\alpha, t)]\right) = \mu_n(L) - F_n(\max L) = 0 = F_{\beta,n}(\alpha, t).$$

Thus $F_{\beta,n} \equiv \mu_{\beta,n}$ and the claim is done.

We need to show that the family $(F_{\beta,n}) \xrightarrow{w^*} 0$. For each $\beta \leq \gamma$, we consider

$$\mathcal{F}_\beta = \{f \in C(\mathbb{R}_\gamma^L) : \exists(\beta, s) < (\beta, t), f|_{[(0, \min L), (\beta, s)]} \equiv 1, f|_{[(\beta, t), (\gamma, \max L)]} \equiv 0\}.$$

Claim 3: The family $\mathcal{F} = \bigcup_{\beta \leq \gamma} \mathcal{F}_\beta \cup \{\chi_{[(0, \min L), (\beta, \max L)]} : \beta \leq \gamma\}$ is linearly dense in $C(\mathbb{R}_\gamma^L)$.

Indeed, it follows from Urysohn's lemma that $\mathcal{F}_\beta \neq \emptyset$ for every $\beta \leq \gamma$. It is straightforward to check that $\mathcal{F}$ separate points of $\mathbb{R}_\gamma^L$, is closed under products and $1 \in \mathcal{F}$. Then $\text{span}(\mathcal{F})$ is the subalgebra generated by $\mathcal{F}$ and it is dense in $C(\mathbb{R}_\gamma^L)$ by the Stone-Weierstrass theorem.

Claim 4: $F_{\beta,n} \xrightarrow{w^*} 0$.

Since $F_{\beta,n} \equiv \mu_{\beta,n}$ by the Claim 2 and $\mathcal{F}$ is linearly dense in $C(\mathbb{R}_\gamma^L)$, by Proposition 2.5.4 it suffices to show that $\mu_{\beta,n}(f) \rightarrow 0$ for every $f \in \mathcal{F}$. We will prove this claim when $\beta < \gamma$ since the case where $\beta = \gamma$ is similar. For each $f \in \mathcal{F}$ we have:

$$\begin{aligned} \mu_{\beta,n}(f) &= \int_{\mathbb{R}_\gamma^L} f d\left((i_\beta^L)_*(\mu_n) - F_n(\max L) \cdot \delta_{(\beta+1, \min L)}\right) \\ &= \int_{\mathbb{R}_\gamma^L} f d\left((i_\beta^L)_*(\mu_n)\right) - \int_{\mathbb{R}_\gamma^L} f d\left(F_n(\max L) \cdot \delta_{(\beta+1, \min L)}\right) \\ &= \int_L f \circ i_\beta^L d\mu_n - F_n(\max L) f(\beta + 1, \min L). \end{aligned}$$

In the case when $f = \chi_{[(0, \min L), (\alpha, \max L)]}$, it follows that:

$$\mu_{\beta,n}(f) = \begin{cases} \mu_n(L), & \text{if } \alpha = \beta; \\ 0, & \text{otherwise.} \end{cases}$$

Since by our assumption $\mu_n(L) \rightarrow 0$, it follows that $\mu_{\beta,n}(f) \rightarrow 0$. For the case when $f \in \mathcal{F}_\alpha$ is such that $f|_{[(0, \min L), (\beta, t)]} \equiv 0$, $f|_{[(\beta, t), (\gamma, \max L)]} \equiv 1$, consider $\tilde{f} \in C(L)$ given by $\tilde{f}(x) = f \circ i_\beta^L(x)$. Then:

$$\mu_{\beta,n}(f) = \begin{cases} \mu_n(\tilde{f}), & \text{if } \alpha = \beta; \\ 0, & \text{otherwise.} \end{cases}$$

Again we have that $\mu_n(\tilde{f}) \rightarrow 0$ by our assumption and it follows that $\mu_{\beta,n}(f) \rightarrow 0$ and the claim is done.

Under similar arguments of Claims (2), (3) and (4), one can show that $G_{\beta,n} \xrightarrow{w^*} 0$. Then by Definition 2.5.17 it follows that $T \equiv (F_{\beta,n})_{\beta \leq \gamma, n \in \omega}$ extends through Φ to $C(\mathbb{R}_\gamma^L)$.

Claim 5: If $\{t \in Q(\varphi) : F_n(t) \not\rightarrow 0\} \neq \emptyset$ and γ is uncountable, then $\{(\alpha, t) \in Q(\Phi) : F_{\beta,n}(t) \not\rightarrow 0\}$ is uncountable.

This follows from the fact that if $t \in Q(\varphi)$ and $F_n(t) \not\rightarrow 0$, then for every $\alpha \leq \gamma$, $F_{\beta,n}(\alpha, t) \not\rightarrow 0$.

Even without being able to obtain analogous equivalences of Proposition 3.0.2 when considering $c_0(I)$ -valued operators, there is much work to be done. After this brief introduction to the problem, we will devote the following sections of this chapter to investigate when it is possible to extend operators under these circumstances.

3.1 Families of type $c_0\ell_1$ and operator extensions

In this section we provide a tool that gives a criterion for the existence of an extension for a given $c_0(I)$ -valued operators.

Definition 3.1.1. Let Q be a set and $(F_i)_{i \in I}$ be a $\|\cdot\|_\infty$ -bounded family of real-valued maps

defined on Q . We say that $(F_i)_{i \in I}$ is of *type* $c_0\ell_1$ if it admits a decomposition:

$$F_i(t) = a_{i,t} + b_{i,t}$$

where the families of real numbers, $(a_{i,t})_{i \in I, t \in Q}$ and $(b_{i,t})_{i \in I, t \in Q}$ satisfies the following:

(a) for each $t \in Q$, $\lim_{i \in I} a_{i,t} = 0$;

(b) $\sup_{i \in I} \sum_{t \in Q} |b_{i,t}| < +\infty$.

In this case, we say that $(a_{i,t})_{i \in I, t \in Q}$ and $(b_{i,t})_{i \in I, t \in Q}$ form a $c_0\ell_1$ decomposition of $(F_i)_{i \in I}$. In addition, given a family $(k_i)_{i \in I}$ of non-negative real numbers, we say that $(F_i)_{i \in I}$ is of *type* $c_0\ell_1$ *bounded by* $(k_i)_{i \in I}$ *over* Q when the families $(a_{i,t})_{i \in I, t \in Q}$ and $(b_{i,t})_{i \in I, t \in Q}$ satisfying (a) and (b) also satisfies:

(c) For all $i \in I$: $\sum_{t \in Q} |b_{i,t}| \leq k_i$.

To make the notation easier, given a family $(F_i)_{i \in I}$ of real valued functions defined on Q and a subset $S \subseteq Q$, we say that $(F_i)_{i \in I}$ satisfies a property **P** over S when the family of restrictions $(F_i|_S)_{i \in I}$ satisfies the property **P**.

Proposition 3.1.2. *Let Q be a set and $(F_i)_{i \in I}$ be a $\|\cdot\|_\infty$ -bounded family of real-valued maps defined on Q . The collection $\mathcal{I} = \{S \subseteq Q : (F_i)_{i \in I} \text{ is of type } c_0\ell_1 \text{ over } S\}$ is an ideal of $\mathcal{P}(Q)$, containing $[Q]^{<\omega}$.*

Proof. It is immediate that $\mathcal{I}$ is downward closed. Since the family $(F_i)_{i \in I}$ is $\|\cdot\|_\infty$ -bounded, it follows that $[Q]^{<\omega} \subseteq \mathcal{I}$. Suppose that $A, B \in \mathcal{I}$. Then there exists $c_0\ell_1$ decompositions:

$$F_i(t) = a_{i,t} + b_{i,t} \quad \text{and} \quad F_i(s) = c_{i,s} + d_{i,s},$$

for all $t \in A$ and $s \in B$. We consider

$$a'_{i,x} = \begin{cases} a_{i,x}, & \text{if } x \in A; \\ c_{i,x}, & \text{if } x \in B \setminus A. \end{cases} \quad \text{and} \quad b'_{i,x} = \begin{cases} b_{i,x}, & \text{if } x \in A; \\ d_{i,x}, & \text{if } x \in B \setminus A. \end{cases}$$

Then it follows that $(a'_{i,t})_{i \in I, t \in A \cup B}$ and $(b'_{i,t})_{i \in I, t \in A \cup B}$ form a $c_0\ell_1$ decomposition of $(F_i)_{i \in I}$, i.e., $A \cup B \in \mathcal{I}$. $\square$

It follows directly from the definition that any $\|\cdot\|_\infty$ -bounded family $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over the set $Z = \{t \in Q : ((F_i(t)) \in c_0(I))\}$. The next result provides an upper bound of the cardinality of $Q \setminus Z$, provided that $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q .

Proposition 3.1.3. *Let $(F_i)_{i \in I}$ be a family of type $c_0\ell_1$ over Q . Then:*

$$|\{t \in Q : (F_i(t))_{i \in I} \notin c_0(I)\}| \leq |I|.$$

Proof. Let $(a_{i,t})_{i \in I, t \in Q}$ and $(b_{i,t})_{i \in I, t \in Q}$ be a $c_0\ell_1$ decomposition of $(F_i)_{i \in I}$. For each $i \in I$ the set $Z_i = \{t \in Q : b_{i,t} \neq 0\}$ is countable because $(b_{i,t})_{t \in Q} \in \ell_1(Q)$. Thus:

$$|\{t \in Q : (F_i(t))_{i \in I} \notin c_0(I)\}| \leq \left| \bigcup_{i \in I} Z_i \right| \leq |I|. \quad \square$$

Next theorem shows the relationship between families of type $c_0\ell_1$ and the operator extension problem.

Theorem 3.1.4. *Let $\varphi : K \xrightarrow{c.i.s.} L$ and a weak*-null family $(F_i)_{i \in I} \subseteq \text{NBV}(L)$. Are equivalent:*

- (a) $(F_i)_{i \in I}$ extends through φ ;
- (b) $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q(\varphi)$.

Moreover, if $(F_i)_{i \in I}$ is of type $c_0\ell_1$ bounded by $(k_i)_{i \in I}$ over $Q(\varphi)$, then one can choose an extension $(G_i)_{i \in I} \subseteq \text{NBV}(K)$ with $\varphi_*(G_i) = F_i$ and $\|G_i\| \leq \|F_i\| + 2k_i$ for all $i \in I$.

Proof. (a) $\implies$ (b): Let $(G_i)_{i \in I} \subseteq \text{NBV}(K)$ be a weak*-null family such that $\varphi_*(G_i) = F_i$ for all $i \in I$ and denote $\mu_i \in M(K)$ the measure that corresponds to G_i in Theorem 2.4.11. We will show that $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q(\varphi)$.

For each $t \in Q(\varphi)$ we denote $\varphi^{-1}[t] = [a_t, b_t]$ and, by the Urysohn's lemma, let $f_t \in C(K)$ such that $f_t|_{[\min K, a_t]} \equiv 1$ and $f_t|_{[b_t, \max K]} \equiv 0$. For each $t \in Q(\varphi)$ and $i \in I$, consider:

$$a_{i,t} = \int_K f_t d\mu_i \quad \text{and} \quad b_{i,t} = F_i(t) - a_{i,t}.$$

Since $\mu_i \xrightarrow{w^*} 0$, it follows that for each $t \in Q(\varphi)$, $a_{i,t} \xrightarrow{w^*} 0$. Furthermore, we have:

$$\begin{aligned} |F_i(t) - a_{i,t}| &= \left| \int_K \chi_{[\min K, b_t]} - f_t d\mu_i \right| \\ &\leq \int_K |\chi_{[\min K, b_t]} - f_t| d|\mu_i| \\ &\leq \int_K \chi_{[a_t, b_t]} d|\mu_i| \\ &= |\mu_i|([a_t, b_t]). \end{aligned}$$

This estimate yields the following:

$$\sum_{t \in Q(\varphi)} |b_{i,t}| \leq \sum_{t \in Q(\varphi)} |\mu_i|([a_t, b_t]) \leq \|\mu_i\|.$$

Taking the supremum over I , it follows that $(a_{i,t})_{i \in I, t \in Q(\varphi)}$ and $(b_{i,t})_{i \in I, t \in Q(\varphi)}$ form a $c_0\ell_1$ decomposition of $(F_i)_{i \in I}$.

(b) $\implies$ (a): Suppose that $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q(\varphi)$ and let $(a_{i,t})_{i \in I, t \in Q(\varphi)}$ and $(b_{i,t})_{i \in I, t \in Q(\varphi)}$ be a $c_0\ell_1$ decomposition.

For each $i \in I$, define $G_i : K \rightarrow \mathbb{R}$ as follows:

$$\forall t \in L : \quad G_i|_{[a_t, b_t]}(x) = \begin{cases} a_{i,t}, & \text{if } a_t \leq x < b_t; \\ F_i(t), & \text{if } x = b_t. \end{cases}$$

It follows from Lemma 2.5.18 that

$$\begin{aligned} \|G_i\|_{BV} &\leq \|F_i\|_{BV} + 2 \sum_{t \in L} V(G_i|_{\varphi^{-1}[t]}) \\ &\leq \|F_i\|_{BV} + 2 \sum_{t \in Q(\varphi)} V(G_i|_{\varphi^{-1}[t]}) \\ &\leq \|F_i\|_{BV} + 2 \sum_{t \in Q(\varphi)} |F_i(t) - a_{i,t}| \\ &\leq \|F_i\|_{BV} + 2 \sum_{t \in Q(\varphi)} |b_{i,t}|. \end{aligned}$$

Then $(G_i)_{i \in I}$ is $\|\cdot\|_{BV}$ -bounded and each G_i is right-continuous. To show that $G_i \xrightarrow{w^*} 0$, let $\mu_i \equiv G_i$ for each $i \in I$.

If $f \in \varphi^*C(L)$, then there exists $g \in C(L)$ such that $f = g \circ \varphi$. Since $F_i = \varphi_*(G_i)$, we have that $F_i \equiv \varphi_*(\mu_i)$. By our assumption, $F_i \xrightarrow{w^*} 0$, hence:

$$\mu_i(f) = \int_K (g \circ \varphi) d\mu_i = \int_L g d\varphi_*(\mu_i) \xrightarrow{i} 0.$$

Moreover, for each $t \in Q(\varphi)$, for the canonical retractions (see Definition 2.5.19):

$$(R_t)_*(\mu_i) = a_{i,t}\delta_{a_t} + (F_i(\max L) - a_{i,t})\delta_{b_t}.$$

Since $a_{i,t} \xrightarrow{i} 0$ and $F_i(\max L) \xrightarrow{i} 0$, for each $f \in (R_t)^*C([a_t, b_t])$, $f = g \circ R_t$ and:

$$\mu_i(f) = \mu_i(g \circ R_t) = (R_t)_*(\mu_i)(g) = a_{i,t}g(a_t) + (F_i(\max L) - a_{i,t})g(b_t) \xrightarrow{i} 0.$$

By Proposition 2.5.4 and Proposition 2.5.20, it follows that $G_i \xrightarrow{w^*} 0$. □

In what follows we shall see that some subsets of a linear compact space K can be ignored when checking that a weak*-null family is of type $c_0\ell_1$. We have the following basic result:

Proposition 3.1.5. *Let K be a linear compact space and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$. Then:*

- (a) $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over K^+ ;
- (b) $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over K^- ;

Proof. It is immediate that for every $t \in K^+$, it follows that:

$$F_i(t) = \int_K \chi_{[\min K, t]} d\mu_i \xrightarrow{i} 0,$$

where $M(K) \ni \mu_i \equiv F_i$ for all $i \in I$. Since the family $(F_i)_{i \in I}$ is bounded, it follows that taking $b_{i,t} = 0$ already gives us a $c_0\ell_1$ decomposition over K^+ .

Furthermore, if $s \in K^-$, it follows that its immediate predecessor s^- is right-isolated. Then define $a_{i,s} = F_i(s^-)$ for all $s \in K^-$ gives us a $c_0\ell_1$ decomposition over K^- :

$$F_i(s) = \underbrace{F_i(s^-)}_{a_{i,s}} + \underbrace{F_i(s) - F_i(s^-)}_{b_{i,s}},$$

with:

$$\sup_{i \in I} \sum_{s \in Q^-} |b_{i,s}| \leq \sup_{i \in I} \|F_i\|_{BV} < +\infty. \quad \square$$

By the preceding result tell us that in order to determine if a weak*-null family in $\text{NBV}(K)$ is of type $c_0\ell_1$, we only need to check if it is of type $c_0\ell_1$ over the internal points of K . More precisely:

Corollary 3.1.6. *Let $\varphi : K \xrightarrow{c.i.s.} L$ and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$. Are equivalent:*

- (a) $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q(\varphi)$;
- (b) $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q_0(\varphi)$.

Proof. Notice that $Q(\varphi) = Q_0(\varphi) \cup (Q(\varphi) \cap (L^+ \cup L^-))$. Then equivalence follows from Proposition 3.1.2 and Proposition 3.1.5. □

The Cantor-Bendixson derivative is classical construction used to uniquely decompose any topological space in the union of a perfect set with a scattered set. In what follows we define an analogous construction for linearly ordered sets (not necessarily compact):

Definition 3.1.7. Let X be a linearly ordered set and $A \subseteq X$. The *two-sided derivative of A in X* is defined as follows:

$$\text{Der}(A, X) = \{x \in X : x \text{ is an internal point of } A \text{ relative to } X\}.$$

For $n \in \omega$ we define recursively the n^{th} two-sided derivative of A by:

- $\text{Der}_0(A, X) = A$;
- $\text{Der}_{n+1}(A, X) = \text{Der}(\text{Der}_n(A, X), X) = A$, for $n \in \omega$.

We say that A has *finite two-sided height in X* if there exists a $n \in \omega$ such that $\text{Der}_n(A, X) = \emptyset$, in this case the least such n is called the two-sided height of A in X .

The same problem that was discussed when the topology inherited by a subset A of X coincides with the topology induced by the restricted order may occur here:

Example 3.1.8. Consider $X = \{0\} \cup \{\pm \frac{1}{n} : n \geq 1\}$, endowed with the induced order from the real numbers, and $A = \{-\frac{1}{2}\} \cup \{\frac{1}{n} : n \geq 1\}$. Then $0 \in \text{Der}(A, X) \setminus \text{Der}(A, A)$.

We would like to observe that the two-sided height of A in X that we defined here is equal to the internal order of A plus 1. The recursive definition of internal order was firstly presented in [KK12] by Kubiś and Kalenda and the authors proved that the family of sets all subsets of X of finite internal order form an ideal of $\mathcal{P}(X)$. Moreover, they are related to the image of composition operators being a complemented subspace.

Proposition 3.1.9. Let K, L be linear compact spaces and $\varphi : K \xrightarrow{\text{c.i.s.}} L$. Are equivalent:

- (a) $Q_0(\varphi)$ has finite internal order;
- (b) $\varphi^*C(L)$ is complemented in $C(K)$.

Proof. See assertion (2) in [KK12, Lemma 2.7]. □

Proposition 3.1.10. Let K be a linear compact space and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$. Then $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over every set $A \subseteq K$ of finite two-sided height in K .

Proof. Consider the first projection $\pi_1 : \text{DA}(K, A) \xrightarrow{\text{c.i.s.}} K$. Since $A = Q_0(\pi_1)$ has finite two-sided height in K , it follows from Proposition 3.1.9 that $\pi_1^*C(K)$ is complemented subspace of $C(\text{DA}(K, A))$. Thus every weak*-null family $(F_i)_{i \in I} \subseteq \text{NBV}(K)$ extends through π_1 . By Theorem 3.1.4, $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $A = Q_0(\pi_1)$. □

We will do a small digression to give another interesting consequence of Proposition 3.1.9 and later we will continue the study of $c_0\ell_1$ decompositions. In Chapter 2 we established an order isomorphism between 2^ω and $\text{DA}(X)$, where $X = \{\frac{k}{2^n} : n \geq 1 \text{ and } 1 \leq k < 2^n\}$. Consider the first projection $\pi_1 : \text{DA}(X) \xrightarrow{\text{c.i.s.}} [0, 1]$, then $Q_0(\pi_1) = X$ and $\text{Der}_n(X, [0, 1]) = X$ for every $n \in \omega$. Thus $\pi_1^*C([0, 1])$ is not a complemented subspace of $C(\text{DA}(X))$. As a consequence of this, there exists a short exact sequence starting at $C([0, 1])$ that does not split. Next we show how to obtain such a sequence.

The space $\pi_1^*C([0, 1])$ consists of all functions $g \in C(\text{DA}(X))$ that are constant over $\pi_1^{-1}[x]$, for every $x \in X$. Thus, it is not difficult to see that the following is a short exact sequence, that not splits:

$$0 \longrightarrow C([0, 1]) \xrightarrow{\pi_1^*} \underbrace{C(\text{DA}(X))}_{\cong C(2^\omega)} \xrightarrow{T} c_0(X) \longrightarrow 0,$$

where $T : C(\text{DA}(X)) \longrightarrow c_0(X)$ is defined by $T(g) = \left(g(x, 0) - g(x, 1) \right)_{x \in X}$.

Now we will continue the study of $c_0\ell_1$ decompositions. For this purpose, we need the following:

Definition 3.1.11. Given two families of real numbers $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$, we say:

$$x_i \leq y_i \pmod{c_0(I)}$$

if, and only if, there exists $(z_i)_{i \in I} \in c_0(I)$ such that $x_i \leq y_i + z_i$ for all $i \in I$.

Remark. The reader may notice that $x_i \leq y_i \pmod{c_0(I)}$ is equivalent to:

$$\forall \varepsilon > 0: \quad \{i \in I : x_i \geq y_i + \varepsilon\} \text{ is finite.}$$

When a family $(F_i)_{i \in I}$ admits a $c_0\ell_1$ decomposition over a set Q that is bounded by $(k_i)_{i \in I}$ and $S \subseteq Q$ is finite, it follows that:

$$\sum_{t \in S} |F_i(t)| \leq \sum_{t \in S} |a_{i,t}| + \sum_{t \in S} |b_{i,t}| \leq k_i \pmod{c_0(I)}.$$

Next result shows us that the converse of this fact holds when Q is a countable set, provided we have the following lemma:

Lemma 3.1.12. Let $c_1, \dots, c_k$ real numbers, let $c \geq 0$ and $\varepsilon > 0$ be such that $\sum_{n=1}^k c_n \leq c + \varepsilon$. Then there exists $a_1, \dots, a_k$ and $b_1, \dots, b_k$ real numbers satisfying:

$$(a) \quad c_n = a_n + b_n, \text{ for each } 1 \leq n \leq k;$$

$$(b) \quad \sum_{n=1}^k |b_n| \leq c;$$

$$(c) \quad \max\{a_1, \dots, a_k\} \leq \varepsilon.$$

Proof. If $|c_n| < \varepsilon$ for all $1 \leq n \leq k$ the result is trivial defining $a_n = c_n$ and $b_n = 0$ for every n . Otherwise, if there exists m such that $|c_m| \geq \varepsilon$, then we can assume that $c_m > 0$ (since the case $c_m < 0$ is analogous) and we define:

$$b_n = \begin{cases} c_n & \text{if } n \neq m, \\ c_n - \varepsilon & \text{if } n = m. \end{cases} \quad \text{and} \quad a_n = \begin{cases} 0 & \text{if } n \neq m, \\ \varepsilon & \text{if } n = m. \end{cases} \quad \square$$

Proposition 3.1.13. Let $(F_i)_{i \in I}$ be a family of real valued functions defined on Q and $(k_i)_{i \in I} \subseteq \mathbb{R}$ be a bounded family of non-negative numbers such that:

$$\forall S \in [Q]^{<\omega} : \quad \sum_{t \in S} |F_i(t)| \leq k_i \pmod{c_0(I)}.$$

If Q is a countable set, then $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q , bounded by $(k_i)_{i \in I}$.

Proof. Consider an enumeration $Q = \{t_n : n \in \omega\}$ and for each $n \geq 1$ we denote $Q_n = \{t_0, \dots, t_n\}$ for each $n \geq 1$. By our assumption, we can find an increasing sequence $(I_n)_{n \geq 1}$ of finite subsets of I such that

$$\forall n \geq 1, \forall i \in I \setminus I_n : \quad \sum_{t \in Q_n} |F_i(t)| < k_i + \frac{1}{n}. \quad (3.1.1)$$

We will define $(a_{i,t})_{i \in I, t \in Q}$ and $(b_{i,t})_{i \in I, t \in Q}$ in respect to the family of $(I_n)_{n \geq 1}$ as follows:

- If $i \in I \setminus \bigcup_{n \geq 1} I_n$, we define $a_{i,t} = 0$, for all $t \in Q$ and $b_{i,t} = F_i(t)$;
- If $i \in I_1$, we define $a_{i,t} = F_i(t)$ and $b_{i,t} = 0$, for all $t \in Q$;
- If $n \geq 1$, $i \in I_{n+1} \setminus I_n$ and $t \in Q \setminus Q_n$, we define $a_{i,t} = F_i(t)$ and $b_{i,t} = 0$;
- If $n \geq 1$, $i \in I_{n+1} \setminus I_n$ and $t \in Q_n$, we apply Lemma 3.1.12 with $F_i(t_0), \dots, F_i(t_n), k_i, \frac{1}{n}$ to obtain a finite decomposition $a_{i,t}, b_{i,t}$ satisfying:

- (a) $F_i(t) = a_{i,t} + b_{i,t}$, for all $t \in Q_n$;
- (b) $\max\{|a_{i,t}| : t \in Q_n\} \leq \frac{1}{k}$;
- (c) $\sum_{t \in Q_n} |b_{i,t}| \leq k_i$.

We shall prove that this is a $c_0\ell_1$ decomposition for $(F_i)_{i \in I}$ over Q . For a fixed $t_m \in Q$ and $\varepsilon > 0$, take $k > m$ such that $\frac{1}{k} < \varepsilon$. By our definition of the $(a_{i,t_m})_{i \in I}$ we have that $|a_{i,t_m}| < \frac{1}{k} < \varepsilon$ for all $i \in I \setminus I_k$. Thus $\lim_{i \in I} a_{i,t_m} = 0$.

Now, for a fixed $i \in I$ we have the following cases to consider:

Case 1 : If $i \in I \setminus \bigcup_{n \in \omega} I_n$, then we have the following estimate:

$$\sum_{t \in Q} |b_{i,t}| = \sum_{t \in Q} |F_i(t)| \leq k_i,$$

where the last inequality holds because every finite set $S \subseteq Q$ is contained in a Q_m and (3.1.1) holds for every $n \geq 1$. Taking the supremum over I yields:

$$\sup_{i \in I} \sum_{t \in Q} |b_{i,t}| \leq \sup_{i \in I} k_i < +\infty.$$

Case 2 : If $i \in \bigcup_{n \geq 1} I_n$. In case $i \in I_1$, all $b_{i,t} = 0$ and we are done. Otherwise, let $m \in \omega$ such that $i \in I_{m+1} \setminus I_m$. Then:

$$\sum_{t \in Q} |b_{i,t}| = \sum_{t \in Q_m} |b_{i,t}| \leq k_i$$

Taking the supremum over I , it follows that $\sup_{i \in I} \sum_{t \in Q} |b_{i,t}| \leq \sup_{i \in I} k_i < +\infty$. $\square$

Proposition 3.1.14. *Let K be a linear compact space, $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$ such that each F_i is associated to $\mu_i \in M(K)$. For all $t, s \in K$, $t < s$ we have that:*

- (a) $|F_i(t)| \leq |\mu_i|([t, s[) \mod c_0(I)$;
- (b) $|F_i(s)| \leq |\mu_i|([t, s]) \mod c_0(I)$;
- (c) $|F_i(t)| + |F_i(s)| \leq |\mu_i|([t, s]) \mod c_0(I)$

Proof. (a) : Using Urysohn's Lemma, let $f : K \rightarrow [0, 1]$ be a continuous functions such that $f|_{[\min K, t]} \equiv 1$ and $f|_{[s, \max K]} \equiv 0$. Then, for each $i \in I$, it follows that

$$\left| F_i(t) - \int_K f d\mu_i \right| = \left| \int_K \chi_{[\min K, t]} - f d\mu_i \right| \leq \int_K \chi_{[t, s[} d|\mu_i|.$$

Thus,

$$|F_i(t)| \leq \left| F_i(t) - \int_K f d\mu_i \right| + \left| \int_K f d\mu_i \right|,$$

and the first part is proved, since $\lim_{i \in I} \mu_i(f) = 0$.

The proof of (b) uses the same technique and is left as an exercise. To prove (c), first suppose that $]t, s[= \emptyset$. Then t is right-isolated and $w^*\text{-}\lim_{i \in I} F_i(t) = 0$, and the assertion is proved. On the other hand, if $]t, s[\neq \emptyset$ we fix $u \in]t, s[$ and combine the items (a) with $t < u$ and (b) with $u < s$. $\square$

Example 3.1.15. It is not true that $|F_n(s)| \leq |\mu_n|([t, s[) \mod c_0$.

Indeed, suppose that there is a decreasing convergent sequence $s_n \searrow s$ in K and let $F_n = \chi_{[s, s_n[}$. Then $\delta_{s_n} \xrightarrow{w^*} \delta_s$ and since $F_n \equiv \delta_{s_n} - \delta_s$, we have that $F_n \xrightarrow{w^*} 0$. On the other hand, $|F_n(s)| = 1$ for every $n \in \omega$ and $|\delta_{s_n} - \delta_s|([t, s[) = 0$.

Corollary 3.1.16. Let K be a linear compact space, $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$ associated with $(\mu_i)_{i \in I}$ in $M(K)$. Given a non-empty convex open subset $J \subseteq K$ and $t \in J$, we have:

$$(a) \quad |F_i(t)| \leq |\mu_i|([\min K, t] \cap J) \mod c_0(I);$$

$$(b) \quad |F_i(t)| \leq |\mu_i|([t, \max K] \cap J) \mod c_0(I).$$

Proof. In case $t = \min J$, since J is a convex open set it follows that t is left-isolated in K . Then either $t = \min K$, in which case $F_i(t) = \mu_i(\{t\})$, or t admits an immediate predecessor t^- right-isolated and $F_i(t) = F_i(t^-) + \mu_i(\{t\})$. In both cases (a) follows. In case $t = \max J$, then t is right-isolated in K and (b) holds trivially.

If t is not the minimum of J , to obtain (a) we take $s \in J, s < t$ and applying Proposition 3.1.14 yields:

$$|F_i(t)| \leq |\mu_i|([s, t]) \mod c_0(I) \leq |\mu_i|(J \cap [\min K, t]) \mod c_0(I).$$

With a similar reasoning we conclude (b) assuming that t is not maximum of J . $\square$

Proposition 3.1.17. Let K be a linear compact space, $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$ associated with $(\mu_i)_{i \in I} \subseteq M(K)$, $J \subseteq K$ be a non-empty convex set and $S \subseteq J$ finite. Then the following are satisfied:

$$(a) \quad \text{If } |J| > 1, \text{ then } \sum_{t \in S} |F_i(t)| \leq |\mu_i|(J) \mod c_0(I);$$

$$(b) \quad \text{If } J \text{ is open, then } \sum_{t \in S} |F_i(t)| \leq \frac{1}{2} |\mu_i|(J) \mod c_0(I).$$

Proof. (a): When J is not a singleton we can also suppose that S has more than one point. Write $S = \{t_0 < \dots < t_k\}$ with $k \geq 1$. Due to Proposition 3.1.14, we have for all $0 \leq j \leq k-1$:

$$|F_i(t_j)| + |F_i(t_{j+1})| \leq |\mu_i|([t_j, t_{j+1}]) \mod c_0(I).$$

Adding all these inequalities yields:

$$|F_i(t_0)| + |F_i(t_k)| + 2 \sum_{j=1}^{k-1} |F_i(t_j)| \leq \sum_{j=0}^k |\mu_i|([t_j, t_{j+1}]) \mod c_0(I). \quad (3.1.2)$$

Thus, it follows that

$$\sum_{j=0}^k |F_i(t_j)| \leq |\mu_i|(J) \mod c_0(I).$$

(b): In the case when J is an open singleton, then this point is right-isolated and the result follows trivially from Corollary 3.1.16. When J is an open convex set and $S = \{t_0 < \dots < t_k\}$, it

follows from Corollary 3.1.16 that

$$\begin{aligned} |F_i(t_0)| &\leq |\mu_i| \left([\min K, t_0] \cap J \right) \mod c_0(I) \\ |F_i(t_k)| &\leq |\mu_i| \left([t_k, \max K] \cap J \right) \mod c_0(I) \end{aligned}$$

Adding these inequalities with (3.1.2) yields

$$2 \left(\sum_{j=0}^k |F_i(t_j)| \right) \leq |\mu_i|(J) \mod c_0(I),$$

which finishes the part (b). $\square$

Corollary 3.1.18. *Let K be a linear compact space and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$. Given $\varepsilon > 0$ and an infinite subset A of K , the following set is finite:*

$$\{i \in I : |F_i(t)| \geq \varepsilon \text{ for all } t \in A\}.$$

Proof. Denote by $k = \sup_{i \in I} \|F_i\|_{BV}$ and let $n \geq 1$ such that $n\varepsilon > \frac{k}{2}$ and take a subset S of A with n elements. Applying Proposition 3.1.17 with $J = K$, we have:

$$\sum_{t \in S} |F_i(t)| \leq \frac{k}{2} \|F_i\|_{BV} \mod c_0(I).$$

Thus:

$$\left\{ i \in I : \sum_{t \in S} |F_i(t)| \geq \frac{k}{2} + (n\varepsilon - \frac{k}{2}) \right\} = \left\{ i \in I : \sum_{t \in S} |F_i(t)| \geq n\varepsilon \right\}$$

is finite and contains $\{i \in I : |F_i(t)| \geq \varepsilon \text{ for all } t \in A\}$. $\square$

Corollary 3.1.19. *Let K be a linear compact space and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$. Given $\varepsilon > 0$ and a convex subset J of K having more than one point, the following set is finite:*

$$\{i \in I : |F_i(t)| \geq \varepsilon \text{ for all } t \in A\}.$$

Proof. If J is infinite, the result follows from Corollary 3.1.18. Otherwise, $t_0 = \min J$ is right-isolated and $(F_i(t_0))_{i \in I} \in c_0(I)$, thus $\{i \in I : |F_i(t_0)| \geq \varepsilon\}$ is finite and contains $\{i \in I : |F_i(t)| \geq \varepsilon \text{ for all } t \in A\}$. $\square$

Corollary 3.1.20. *Let K be a linear compact space and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$. Then $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over every countable subset of K .*

Proof. Combine Proposition 3.1.17 and Proposition 3.1.13. $\square$

The previous result has an interesting consequence when combined with the criterion given by Theorem 3.1.4. More specifically, given K, L linear compact spaces and $\varphi : K \xrightarrow{\text{c.i.s.}} L$ our criterion only depends of φ in terms of the set $Q(\varphi)$. Indeed, if $\psi : K \xrightarrow{\text{c.i.s.}} L$ is such that $Q(\varphi) = Q(\psi)$, then the following conditions are equivalent:

- (a) $(F_i)_{i \in I}$ extends through φ ;
- (b) $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q(\varphi) (= Q(\psi))$;
- (c) $(F_i)_{i \in I}$ extends through ψ .

Thus, it is natural to consider the set:

$$C_{is}(K, L) = \{\varphi : K \longrightarrow L : \varphi \text{ is a continuous increasing surjection}\},$$

endowed with the equivalence relation $\varphi \sim \psi$ iff $Q(\varphi) \triangle Q(\psi)$ is countable. This equivalence relation has the property that whenever we are able to extend a $c_0(I)$ -valued operator through φ , then we are able to extend the operator through all $\psi \in [\varphi]_{\sim}$. In addition, we have the following criterion to determine whether a family $(F_i)_{i \in I}$ is of type $c_0\ell_1$:

Proposition 3.1.21. *Given $\varphi : K \xrightarrow{\text{c.i.s.}} L$ and a weak* null family $(F_i)_{i \in I}$ in $\text{NBV}(L)$. If the set $\{t \in Q(\varphi) : (F_i(t))_{i \in I} \notin c_0(I)\}$ is countable, then $(F_i)_{i \in I}$ extends through φ .*

Proof. By Theorem 3.1.4, we just need to check that $(F_i)_{i \in I}$ is $c_0\ell_1$ over $Q(\varphi)$ and this is satisfied since

$$Q(\varphi) = \{t \in Q(\varphi) : (F_i(t))_{i \in I} \in c_0(I)\} \cup \{t \in Q(\varphi) : (F_i(t))_{i \in I} \notin c_0(I)\},$$

where $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over these two sets. $\square$

The previous result recovers the extension criterion given in Proposition 3.0.2 (and presented in [CT15]), i.e., for a fixed countable set I , extending a bounded operator associated with a weak*-null sequence $(F_i)_{i \in I} \subseteq \text{NBV}(L)$ through a map $\varphi : K \xrightarrow{\text{c.i.s.}} L$ is equivalent to say that $Q(\varphi)$ is countable. On the other hand, Example 3.0.4 shows that this does not hold for arbitrary sets on indexes. In fact, more can be achieved, because the next example shows that we can construct an operator that extends through any c.i.s. map:

Example 3.1.22. Let γ be an uncountable ordinal, $L = [0, \gamma] \times [0, 1]$ be lexicographically ordered and $I = [0, \gamma] \times \omega$. For each $i = (\alpha, n) \in I$ we consider

$$F_{(\alpha, n)} = \chi_{[(\alpha, 0), (\alpha, \frac{1}{n+1})[\cdot}$$

It follows that $(F_{(\alpha, n)})_{(\alpha, n) \in I}$ is a bounded family in $\text{NBV}(L)$. We will prove that $(F_{(\alpha, n)})_{(\alpha, n) \in I}$ is weak*-null.

For $(\alpha, n) \in I$ and $\beta \in [0, \gamma]$, we have

$$\forall \beta \in [0, \gamma] : \quad (\pi_1)_*(F_{(\alpha, n)})(\beta) = F_{(\alpha, n)}(\max \pi_1^{-1}[\beta]) = F_{(\alpha, n)}((\beta, 1)) = 0.$$

Furthermore, for $\delta \in [0, \gamma]$ and the canonical retraction $R_\delta : L \xrightarrow{\text{c.i.s.}} \{\delta\} \times [0, 1]$, for each $t \in [0, 1]$:

$$\begin{aligned} (R_\delta)_*(F_{(\alpha, n)})(\delta, t) &= F_{(\alpha, n)}(\max R_\delta^{-1}[(\delta, t)]) \\ &= \begin{cases} F_{(\alpha, n)}(\delta, t) & \text{if } t < 1 \\ F_{(\alpha, n)}(\gamma, 1) & \text{if } t = 1 \end{cases} \end{aligned}$$

Thus, by Corollary 2.5.21 with $\pi_1 : L \xrightarrow{\text{c.i.s.}} [0, \gamma]$, it follows that $(F_{(\alpha, n)})_{(\alpha, n) \in I}$ is weak*-null.

Now consider an arbitrary $\varphi : K \xrightarrow{\text{c.i.s.}} L$, we will show that $(F_{(\alpha, n)})_{(\alpha, n) \in I}$ is of type $c_0\ell_1$ over L . Indeed, it follows that

$$\begin{aligned} L &= \left\{t \in L : (F_{(\alpha, n)}(t))_{(\alpha, n) \in I} \notin c_0(I)\right\} \cup \left\{t \in L : (F_{(\alpha, n)}(t))_{(\alpha, n) \in I} \in c_0(I)\right\} \\ &= ([0, \gamma] \times \{0\}) \cup \left\{t \in L : (F_{(\alpha, n)}(t))_{(\alpha, n) \in I} \in c_0(I)\right\}. \end{aligned}$$

Since $\text{Der}([0, \gamma] \times \{0\}, L) = \emptyset$, it follows from Proposition 3.1.10 that $(F_{(\alpha, n)})_{(\alpha, n) \in I}$ is of type $c_0\ell_1$ over L . In particular, $(F_{(\alpha, n)})_{(\alpha, n) \in I}$ extends through φ by Theorem 3.1.4.

In the remaining of this section we will prove that the extension criterion presented in Theorem 3.1.4 also satisfies the same norm condition of the Sobczyk's theorem, i.e., whenever one

can extend a bounded operator T through a c.i.s. map φ , there exist an extension $\tilde{T}$ such that $\|\tilde{T}\| \leq 2\|T\|$.

Lemma 3.1.23. *Let K be a linear compact space, $t \in K$ and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$. If $\chi_R(t, K) > \omega$, then $F_i(t) \xrightarrow{i} 0$.*

Proof. Assume towards a contradiction that $\lim_{i \in I} F_i(t) \neq 0$. Then, there exist $\varepsilon > 0$ and a countable set $J \subseteq I$ such that $|F_j(t)| \geq \varepsilon$ for all $j \in J$.

For every $j \in J$, by the regularity of $\mu_j \in M(K)$ associated to F_j , there exists an open set $[\min K, t] \subseteq W \subseteq K$ such that $|\mu_j|(W \setminus [\min K, t]) < \frac{\varepsilon}{2}$. Since t is not right-isolated, there exists $t_j \in W$, $t_j > t$ and it follows that $|\mu_j|([t, t_j]) < \frac{\varepsilon}{2}$. Since $\chi_R(t, K) > \omega$, we have that $s = \inf\{t_j : j \in J\} > t$. Using Urysohn's Lemma, take a continuous function $f : K \rightarrow [0, 1]$ such that $f|_{[\min K, t]} \equiv 1$ and $f|_{[s, \max K]} \equiv 0$, it follows that for every $j \in J$:

$$\begin{aligned} \left| \int_K f d\mu_j \right| &= \left| F_j(t) + \int_{[t, s]} f d\mu_j \right| \\ &\geq |F_j(t)| - \left| \int_{[t, s]} f d\mu_j \right| \\ &\geq |F_j(t)| - |\mu_j|([t, t_j]) \\ &> \frac{\varepsilon}{2}, \end{aligned}$$

contradicting the fact that $(\mu_i)_{i \in I}$ is weak*-null in $M(K)$. $\square$

Proposition 3.1.24. *If K is a linear compact space and $(F_i)_{i \in I}$ is a weak*-null family in $\text{NBV}(K)$ associated to $(\mu_i)_{i \in I} \subseteq M(K)$, then for every $t \in K$, the set $\{i \in I : F_i(t) \neq 0\}$ is countable.*

Proof. When $\chi_R(t, K) > \omega$, the results follow from Lemma 3.1.23. Suppose that $\chi_R(t, K) = \omega$. In case t is a right-isolated point of K the result is trivial, therefore we can assume that $t \in K \setminus K^+$.

In this case, there exists a strictly decreasing sequence $(t_n)_{n \in \omega} \subseteq]t, \max K]$ with $t_n \searrow t$. Using Urysohn's Lemma, we take for every $n \in \omega$ a continuous function $f_n : K \rightarrow [0, 1]$ such that $f_n|_{[\min K, t]} \equiv 1$ and $f_n|_{[t_n, \max K]} \equiv 0$. Since $(\mu_i)_{i \in I}$ is a weak*-null family in $M(K)$, for every $n \in \omega$ the set $\{i \in I : \int_K f_n d\mu_i \neq 0\}$ is countable. Thus the following set is also countable:

$$I' \doteq \bigcup_{n \in \omega} \left\{ i \in I : \int_K f_n d\mu_i \neq 0 \right\}.$$

Notice that $(f_n)_{n \in \omega}$ is pointwise convergent to $\chi_{[\min K, t]}$ and $|f_n| \leq 1$ for all $n \in \omega$. In addition, if $i \in I \setminus I'$, it follows from the Dominated Convergence Theorem:

$$0 = \int_K f_n d\mu_i \longrightarrow \int_K \chi_{[\min K, t]} d\mu_i = F_i(t),$$

that concludes our proof. $\square$

For the next result we first need the following:

Definition 3.1.25. We say that a family of subsets $(A_j)_{j \in J} \subseteq X$ is *point-countable* if for every $x \in X$, the set $\{j \in J : x \in A_j\}$ is countable.

Lemma 3.1.26. *Suppose that $(A_j)_{j \in J} \subseteq X$ is a point-countable family of countable sets. Then there exists a partition $J = \bigcup_{\lambda \in \Lambda} J_\lambda$ satisfying the following:*

- (a) For every $\lambda \in \Lambda$, $|J_\lambda| \leq \omega$;

(b) $A_{j_1} \cap A_{j_2} = \emptyset$ whenever $j_1 \in J_\lambda$, $j_2 \in J_\mu$ with $\lambda \neq \mu$.

Proof. Define on J the following relation:

$$j \sim j' \stackrel{\text{def}}{\iff} \exists j_0, \dots, j_n \in J, j_0 = j, j_n = j' : A_{j_i} \cap A_{j_{i+1}} \neq \emptyset \quad (0 \leq i \leq n-1).$$

It is straightforward to check that this is an equivalence relation on J . We claim that the equivalence classes are countable.

Indeed, for a fixed $j_0 \in J$, it follows that $[j_0]_\sim = \bigcup_{n \in \omega} C_n$ where the C_n 's are recursively defined as follows:

- $C_0 = \{j_0\}$;
- $C_{n+1} = \{j \in J : \exists j' \in C_n, A_j \cap A_{j'} \neq \emptyset\}$.

We prove by induction that C_n is countable for all $n \in \omega$. Assume that C_n is countable, then:

$$C_{n+1} = \bigcup_{j' \in C_n} \bigcup_{p \in A_{j'}} \{j \in J : p \in A_j\},$$

thus C_{n+1} is countable.

Since every equivalence class is countable, $J = \bigcup_{j \in J} [j]_\sim$ is a partition satisfying (a) and (b). $\square$

Proposition 3.1.27. *Let Q be a set, $(k_i)_{i \in I}$ be a family of non-negative real numbers and $(F_i)_{i \in I}$ be a family of real valued maps defined on Q . Suppose that the following are satisfied:*

- (1) *For every $S \subseteq Q$ finite: $\sum_{t \in S} |F_i(t)| \leq k_i \pmod{c_0(I)}$;*
- (2) *For all $t \in Q$, the set $\{i \in I : F_i(t) \neq 0\}$ is countable.*

Then the following are equivalent:

- (a) $(F_i)_{i \in I}$ is of type $c_0 \ell_1$;
- (b) *There exists a partition $I \times Q = A \cup B$ such that:*
 - (b-i) *For all $t \in Q$: $\lim_{i \in I} F_i(t) \chi_A(i, t) = 0$;*
 - (b-ii) *For all $i \in I$, the set $B_i = \{t \in Q : (i, t) \in B\}$ is countable.*
- (c) $(F_i)_{i \in I}$ is of type $c_0 \ell_1$ bounded by $(k_i)_{i \in I}$.

Proof. (a) $\implies$ (b): By our assumption, we can decompose $F_i(t) = a_{i,t} + b_{i,t}$ for all $i \in I$ and $t \in Q$ with $\lim_{i \in I} a_{i,t} = 0$ and $\sup_{i \in I} \sum_{t \in Q} |b_{i,t}| < +\infty$ and consider the sets:

$$B = \{(i, t) \in I \times Q : b_{i,t} \neq 0\} \quad \text{and} \quad A = (I \times Q) \setminus B.$$

Then it follows that B_i is countable for every $i \in I$. Moreover for each $t \in Q$, we have:

$$F_i(t) \chi_A(i, t) = \begin{cases} 0, & \text{if } (i, t) \in B; \\ a_{i,t}, & \text{if } (i, t) \in A. \end{cases}$$

Thus $\lim_{i \in I} F_i(t) \chi_A(i, t) = 0$, proving (b).

(b) $\implies$ (c): By moving all points (i, t) with $F_i(t) = 0$ from B to A , we can assume **WLOG** that $F_i(t) \neq 0$ for all $(i, t) \in B$. Then, for every $t \in Q$:

$$B^t = \{i \in I : (i, t) \in B\} \subseteq \underbrace{\{i \in I : F_i(t) \neq 0\}}_{\text{countable}}.$$

In addition, $(B^t)_{t \in Q}$ is a point-countable family because, for each $i \in I$, we have

$$\{t \in Q : i \in B^t\} = \{t \in Q : t \in B_i\} \subseteq B_i.$$

By Lemma 3.1.26, there exists a partition of countable sets $Q = \bigcup_{\lambda \in \Lambda} Q_\lambda$ with $B^t \cap B^s = \emptyset$, provided that $s \in B_\lambda$, $t \in B_\mu$ and $\lambda \neq \mu$.

By Proposition 3.1.13, $(F_i)_{i \in I}$ is of type $c_0\ell_1$ bounded by $(k_i)_{i \in I}$ over Q_λ , for all $\lambda \in \Lambda$. We will prove that gluing all these $c_0\ell_1$ decompositions over the sets Q_λ forms a $c_0\ell_1$ decomposition over Q that is also bounded by $(k_i)_{i \in I}$.

Since $Q = \bigcup_{\lambda \in \Lambda} Q_\lambda$, we denote all these decompositions as families of real numbers $(a_{i,t})_{i \in I, t \in Q}$ and $(b_{i,t})_{i \in I, t \in Q}$ with:

- $F_i(t) = a_{i,t} + b_{i,t}$, for all $i \in I$, $t \in Q$;
- $\lim_{i \in I} a_{i,t} = 0$, for all $t \in Q$;
- $\sum_{t \in Q_\lambda} |b_{i,t}| \leq k_i$, for all $i \in I$, $\lambda \in \Lambda$.

We define the gluing of these decompositions as follows:

$$\forall i \in I, t \in Q : F_i(t) = a'_{i,t} + b'_{i,t}$$

where $a'_{i,t} = F_i(t)\chi_A(i, t) + a_{i,t}\chi_B(i, t)$ and $b'_{i,t} = b_{i,t}\chi_B(i, t)$.

For a fixed $t \in Q$ and a given $\varepsilon > 0$, the sets $\{i \in I : |a_{i,t}| \geq \varepsilon\}$ and $\{i \in I : |F_i(t)\chi_A(i, t)| \geq \varepsilon\}$ are finite, thus $\lim_{i \in I} a'_{i,t} = 0$. In addition, for a fixed $i \in I$ we have two cases to consider:

If $(i, t) \notin B$, for all $t \in Q$, then $b'_{i,t} \equiv 0$ and we are done. Suppose that there exists $t \in Q$ such that $(i, t) \in B$. Then there exists a unique $\lambda \in \Lambda$ such that $t \in Q_\lambda$. We claim that $B_i \subseteq Q_\lambda$. Indeed, suppose towards a contradiction that $s \neq t$ is such that $s \in Q_\mu$ with $\mu \neq \lambda$, then by the construction of the partition we have that $i \in B^t \cap B^s = \emptyset$, a contradiction. Thus, we have the following estimate:

$$\sum_{t \in Q} |b'_{i,t}| = \sum_{t \in B_i} |b_{i,t}| \leq \sum_{t \in Q_\lambda} |b_{i,t}| \leq k_i.$$

Then $(F_i)_{i \in I}$ is of type $c_0\ell_1$ bounded by $(k_i)_{i \in I}$, as required. $\square$

Corollary 3.1.28. *Let K a linear compact space, a convex subset $J \subseteq K$ with $|J| > 1$, $Q \subseteq J$ and $(F_i)_{i \in I}$ a weak*-null family in $\text{NBV}(K)$ associated with $(\mu_i)_{i \in I} \subseteq M(K)$. If $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q , then $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q bounded by $(|\mu_i|(J))_{i \in I}$. In addition, if J is open, then $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q bounded by $(\frac{1}{2}|\mu_i|(J))_{i \in I}$.*

Proof. It follows from Proposition 3.1.17 and Proposition 3.1.27. $\square$

Corollary 3.1.29. *Let K be a linear compact space, $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$ associated with $(\mu_i)_{i \in I} \subseteq M(K)$ and $(J_\lambda)_{\lambda \in \Lambda}$ a family of pairwise disjoint convex subsets of K having more than one point. If $Q \subseteq \bigcup_{\lambda \in \Lambda} J_\lambda$ and $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q \cap J_\lambda$ for all $\lambda \in \Lambda$, then $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q .*

Proof. It follows from Corollary 3.1.28 that for each $\lambda \in \Lambda$, $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q \cap J_\lambda$ bounded by $(|\mu_i|(J_\lambda))_{i \in I}$.

Since the sets $Q \cap J_\lambda$ with $\lambda \in \Lambda$ form a partition of Q , we can glue together all these partitions and write

$$\forall i \in I, t \in Q : F_i(t) = a_{i,t} + b_{i,t},$$

with $\lim_{i \in I} a_{i,t} = 0$ for all $t \in Q$ and $\sum_{t \in Q \cap J_\lambda} |b_{i,t}| \leq |\mu_i|(J_\lambda)$, for all $i \in I$. For all $i \in I$, we have the following:

$$\sum_{t \in Q} |b_{i,t}| = \sum_{t \in \bigcup_{\lambda \in \Lambda} Q \cap J_\lambda} |b_{i,t}| \leq |\mu_i| \left(\bigcup_{\lambda \in \Lambda} J_\lambda \right) \leq |\mu_i|(K) = \|F_i\|_{BV}.$$

Since $(F_i)_{i \in I}$ is weak*-null, it is bounded and the conclusion follows. $\square$

Theorem 3.1.30. *Consider $\varphi : K \xrightarrow{c.i.s.} L$ and a bounded operator $T : C(L) \rightarrow c_0(I)$. If T extends through φ , then there exists an extension $\tilde{T} : C(K) \rightarrow c_0(I)$ such that $\tilde{T} \circ \varphi^* = T$ and $\|\tilde{T}\| \leq 2\|T\|$.*

Proof. Let $T \equiv (F_i)_{i \in I} \subseteq \text{NBV}(L)$. By our assumption, Theorem 3.1.4 implies that $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q(\varphi)$. Using Corollary 3.1.28 with $J = L$, $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q(\varphi)$ bounded by $(\frac{1}{2}\|F_i\|)_{i \in I}$.

Theorem 3.1.4 yields a weak*-null family $(G_i)_{i \in I} \subseteq \text{NBV}(K)$ such that, for all $i \in I$, $\varphi_*(G_i) = F_i$ and $\|G_i\| \leq 2\|F_i\|$. The conclusion follows by considering $\tilde{T} \equiv (G_i)_{i \in I}$. $\square$

3.2 A characterization of weak*-null families

In this section we will use the results that were discussed and proved in Section 3.1 to provide a characterization of weak*-null families. We already briefly analysed cases of weak*-convergence in Section 2.5, mostly for zero-dimensional linear compact spaces. In what follows we take the discussion further and to present a characterization of weak*-null families in $\text{NBV}(K)$ for arbitrary linear compact spaces.

Lemma 3.2.1. *Let K be a linear compact space and $(F_i)_{i \in I}$ be a family in $\text{NBV}(K)$. The following are equivalent:*

- (a) $(F_i)_{i \in I}$ is weak*-null;
- (b) For every closed set Z of $[0, 1]$ and every $\varphi : K \xrightarrow{c.i.s.} Z$, $(\varphi_*(F_i))_{i \in I}$ is weak*-null in $\text{NBV}(Z)$.

Proof. (a) $\implies$ (b) is immediate.

(b) $\implies$ (a) It suffices to show that for every $f \in C(K)$, there exists a closed subset Z of $[0, 1]$ and $\varphi_Z : K \xrightarrow{c.i.s.} Z$ such that $f \in \varphi^*[C(Z)]$. Indeed, let $(\mu_i)_{i \in I}$ associated with $(F_i)_{i \in I}$. If $f = \varphi_Z^*(g) = g \circ \varphi_Z$, we have:

$$\mu_i(f) = \mu_i(g \circ \varphi_Z) = \int_K g \circ \varphi_Z d\mu_i = \int_Z g d(\varphi_Z)_* (\mu_i) \xrightarrow{i} 0.$$

Fix $f \in C(K)$, by Example 2.5.2 there exists a sequence $(f_n)_{n \in \omega}$ of continuous increasing functions $f_n : K \rightarrow [0, 1]$ such that $f \in \text{span}(\{f_n : n \geq 1\})$. Define

$$\varphi = \sum_{n \geq 1} \frac{f_n}{2^n} \quad \text{and} \quad Z = \text{ran}(\varphi).$$

It is straightforward to check that $\varphi : K \rightarrow Z$ is continuous, increasing and surjective. Moreover, to see that $f \in \text{ran}(\varphi^*)$ notice that for all $s, t \in K$, if $\varphi(s) = \varphi(t)$, then $f_n(s) = f_n(t)$ for all $n \geq 1$, thus $f(t) = f(s)$. $\square$

Theorem 3.2.2. *Let K be a linear compact space. A family $(F_i)_{i \in I}$ in $\text{NBV}(K)$ is weak*-null if, and only if, the following conditions are satisfied:*

- (a) $(F_i)_{i \in I}$ is bounded in $\text{NBV}(K)$;
- (b) $\lim_{i \in I} F_i(t) = 0$ for all $t \in K^+$;

(c) *There exists $k \geq 0$ such that $\sum_{t \in S} |F_i(t)| \leq k \pmod{c_0(I)}$ for every finite subset S of K .*

Proof. If $(F_i)_{i \in I}$ is weak*-null, then (a) and (b) follows easily and (c) follows from Proposition 3.1.17. Conversely, assume (a), (b) and (c). We will divide the proof in two parts:

Case 1: Assume that K is separable.

Let A be a countable dense set in K and consider $\pi_1 : \text{DA}(K, A) \xrightarrow{\text{c.i.s.}} K$. Notice that since A is dense in K , $\text{DA}(K, A)$ is zero-dimensional. We will construct a weak*-null family $(G_i)_{i \in I}$ in $\text{NBV}(\text{DA}(K, A))$ such that $(\pi_1)_*(G_i) = F_i$ for all $i \in I$.

By (c) and Proposition 3.1.13, $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over A . Let $F_i(t) = a_{i,t} + b_{i,t}$ be a $c_0\ell_1$ decomposition. For each $i \in I$ consider $G_i : \text{DA}(K, A) \rightarrow \mathbb{R}$ defined as follows:

$$G_i(t, n) = \begin{cases} a_{i,t}, & \text{if } t \in A \text{ and } n = 0; \\ F_i(t), & \text{if } t \in A \text{ and } n = 1; \\ F_i(t), & \text{if } t \in K \setminus A. \end{cases}$$

It follows that $(\pi_1)_*(G_i) = F_i$ for all $i \in I$. By (a) and Lemma 2.5.18 it follows that $(G_i)_{i \in I}$ is bounded in $\text{NBV}(\text{DA}(K, A))$.

Now, if $s \in \text{DA}(K, A)^+$, either $s = (t, 0)$ with $t \in A$, in which case $\lim_{i \in I} G_i(s) = \lim_{i \in I} a_{i,t} = 0$, or $s = (t, 1)$ with $t \in K^+$, and (b) implies that $\lim_{i \in I} G_i(s) = \lim_{i \in I} F_i(t) = 0$. Because $\text{DA}(K, A)$ is zero dimensional, this suffices to prove that $(G_i)_{i \in I}$ is weak*-null and we conclude the proof for the separable case.

Case 2: The general case.

We will apply Lemma 3.2.1 as follows. If $(F_i)_{i \in I}$ satisfies (a), (b) and (c) and $\varphi : K \xrightarrow{\text{c.i.s.}} Z$, with Z a closed subset of $[0, 1]$, then the family $(\varphi_*(F_i))_{i \in I}$ is bounded in $\text{NBV}(Z)$ and for every $t \in K^+$, $\lim_{i \in I} \varphi_*(F_i)(t) = 0$ and

$$\sum_{t \in S} |\varphi_*(F_i)(t)| \leq k \pmod{c_0(I)},$$

for each finite subset S of Z . Since Z is separable, it follows from the previous case that $(\varphi_*(F_i))_{i \in I}$ is weak*-null in $\text{NBV}(Z)$, which concludes our proof. $\square$

Corollary 3.2.3. *Let K be a linear compact space and $(F_i)_{i \in I}$ be a bounded family in $\text{NBV}(K)$. Suppose that $\{t \in K : \lim_{i \in I} F_i(t) \neq 0\}$ is countable and $\lim_{i \in I} F_i(t) = 0$ for all $t \in K^+$, then $(F_i)_{i \in I}$ is weak*-null.*

Proof. We prove that item (c) in Theorem 3.2.2 is verified.

Let $S = \{t_0, \dots, t_n\}$ be a finite subset of K . We have two cases to consider.

Case 1: Suppose that $[t_j, t_{j+1}]$ is uncountable.

Then there exists $x \in [t_j, t_{j+1}[$ with $\lim_{i \in I} F_i(x) = 0$. Then:

$$|F_i(t_{j+1})| \leq |F_i(t_{j+1}) - F_i(x)| + |F_i(x)| \leq |\mu_i|([t_j, t_{j+1})] + |F_i(x)|.$$

Thus, $|F_i(t_{j+1})| \leq |\mu_i|([t_j, t_{j+1}]) \pmod{c_0(I)}$.

Case 2: Suppose that $[t_j, t_{j+1}]$ is countable.

Then $[t_j, t_{j+1}]$ is a countable compact Hausdorff space, whence it is zero-dimensional. In particular, if $x \in [t_j, t_{j+1}[$ is right-isolated in $[t_j, t_{j+1}]$, then $x \in K^+$ and it follows that $\lim_{i \in I} F_i(x) = 0$ and with the same reasoning of the Case 1, we conclude that $|F_i(t_{j+1})| \leq |\mu_i|([t_j, t_{j+1}]) \pmod{c_0(I)}$.

Now observe that we have:

$$\sum_{j=0}^n |F_i(t_j)| \leq \sup_{i \in I} \|F_i\|_{BV} \quad \text{mod } c_0(I),$$

which concludes our proof. $\square$

3.3 Separably determined linear compact spaces and the $c_0(I)$ -EP

In the previous section, we have presented a criterion to extend bounded $c_0(I)$ -valued operators through a fixed $\varphi : K \xrightarrow{\text{c.i.s.}} L$. Although we were able to establish an extension result like Sobczyk's Theorem in this context, Examples 3.0.4 and 3.1.22 show us how the behavior of $c_0\ell_1$ decompositions can vary in some circumstances. So it is natural to ask under which conditions these extremely opposite behaviours are avoided. More specifically, in what kind of linear compact spaces and we have a criterion of $c_0(I)$ -valued operators that is similar to the one introduced in [CT15].

For that purpose, we generalize the definition of c_0 -EP as by following [CT15]. We say that φ has the $c_0(I)$ -extension property ($c_0(I)$ -EP) when every bounded linear operator $T : C(L) \rightarrow c_0(I)$ extends through φ . And we say that a closed subspace Y of a Banach space X has the $c_0(I)$ -EP in X iff every bounded $c_0(I)$ -valued operator defined on Y admits an extension to X . In this section we will describe a family of linear compact spaces so that the $c_0(I)$ -EP and the c_0 -EP coincide. We begin with the following result to motivate our analysis:

Theorem 3.3.1. *Let K, L be linear compact space, with L separable, and $\varphi : K \xrightarrow{\text{c.i.s.}} L$. If $T : C(L) \rightarrow c_0(I)$ is a bounded operator associated with $(F_i)_{i \in I} \subseteq \text{NBV}(L)$ that extends through φ , then:*

$$\left\{ t \in Q_0(\varphi) : \lim_{i \in I} F_i(t) \neq 0 \right\} \text{ is countable.}$$

Proof. By Ostaszewski's Theorem 2.2.8, there exist a closed set $F \subseteq [0, 1]$ and $X \subseteq F$ such that L is order isomorphic to $\text{DA}(F, X)$. Consider $\pi_1 : L \xrightarrow{\text{c.i.s.}} F$ and denote $\psi = \pi_1 \circ \varphi$. We have the following diagram:

$$\begin{array}{ccc} & C(K) & \\ \psi^* \uparrow \varphi^* & \nearrow T \equiv (F_i)_{i \in I} & \\ C(L) & \xrightarrow{\quad} & c_0(I) \\ \pi_1^* \uparrow & \nwarrow T|_{C(F)} \equiv (\pi_{1*}(F_i))_{i \in I} & \\ & C(F) & \end{array}$$

$$\forall s \in F : \quad \pi_{1*}(F_i)(s) = \begin{cases} F_i(s, 0), & \text{if } s \notin X; \\ F_i(s, 1), & \text{if } s \in X. \end{cases}$$

By our assumption, T extends through φ to $C(K)$, then $T|_{C(F)}$ extends to $C(K)$ through ψ . In addition, since F is metrizable, $C(F)$ is separable. Thus there exists $J \subseteq I$ countable such that $\pi_{1*}(F_i) \equiv 0$ for all $i \in I \setminus J$. By Proposition 3.0.2 it follows that:

$$\left\{ s \in Q_0(\psi) : \lim_{i \in I} \pi_{1*}(F_i)(s) \neq 0 \right\} \text{ is countable.}$$

Now notice that if $t \in L$ is an internal point, then $t = (s, 0)$ with $s \in F \setminus X$ and s is an internal point of F , in addition we have that $F_i(t) = \pi_{1*}F_i(s)$. This yields:

$$\left\{ t \in Q_0(\varphi) : \lim_{i \in I} F_i(t) \neq 0 \right\} \subseteq \left\{ s \in Q_0(\psi) : \lim_{i \in I} \pi_{1*}(F_i)(s) \neq 0 \right\} \quad \square$$

The previous result tells us that whenever the range of a map φ lies in a separable linear compact space, then the criteria for extending c_0 -valued operators from $C(L)$ through φ is the same for extend bounded $c_0(I)$ -valued operators. More precisely, for L separable and $\varphi : K \xrightarrow{\text{c.i.s.}} L$:

$$\varphi \text{ has the } c_0\text{-EP} \iff \varphi \text{ has the } c_0(I)\text{-EP}.$$

Lemma 3.3.2. *Let K be a linear compact space, H be a closed separable subset of K and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$. There exists a countable subset J of I such that:*

$$\text{for all } i \in I \setminus J : \quad F_i|_{H \setminus H^+} \equiv 0.$$

Proof. Let $(\mu_i)_{i \in I} \subseteq M(K)$ a weak*-null family associated with $(F_i)_{i \in I}$. Fix D be a countable dense set of H , for all $t, s \in D$ with $t < s$ using Urysohn's Lemma, pick a continuous function $f_{t,s} : C(K) \rightarrow [0, 1]$ such that $f_{t,s}|_{[\min K, t]} \equiv 1$ and $f_{t,s}|_{[s, \max K]} \equiv 0$ and define:

$$J = \bigcup_{\substack{t, s \in D \\ t < s}} \left\{ i \in I : \int_K f_{t,s} d\mu_i \neq 0 \right\}$$

First notice that J is countable set. Indeed, since $(\mu_i)_{i \in I}$ is weak*-null, we have:

$$\left\{ i \in I : \int_K f_{t,s} d\mu_i \neq 0 \right\} = \bigcup_{n \geq 1} \underbrace{\left\{ i \in I : \left| \int_K f_{t,s} d\mu_i \right| > \frac{1}{n} \right\}}_{\text{finite}}.$$

In what follows, we will prove that for $x \in H \setminus H^+$ and $i \in I \setminus J$, $F_i(x) = 0$. For a fixed $\varepsilon > 0$, since μ_i is regular and $[\min K, x]$ is a borel set, there exists an open set $U \subseteq K$ such that $|\mu_i|(U \setminus [\min K, x]) < \varepsilon$. Since $x \notin H^+$, there exists $y \in H \cap U$ with $y > x$ and we have that $|\mu_i|([x, y]) < \varepsilon$. Again, since x is not right-isolated on H , we can find $t, s \in H$ such that $x < t < s < y$ and we have the following estimate:

$$|F_i(x)| = \left| F_i(x) - \underbrace{\int_K f_{t,s} d\mu_i}_{=0} \right| = \left| \int_K \chi_{[\min K, x]} - f_{t,s} d\mu_i \right| \leq \int_K |\chi_{[x, s]}| d|\mu_i| = |\mu_i|([x, s]) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this finishes the proof. $\square$

Theorem 3.3.3. *Let K be a linear compact space, $Q \subseteq K$ and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$. If $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q , then for every closed separable subset $H \subseteq K$:*

$$\left\{ t \in Q \cap (H \setminus H^+) : \lim_{i \in I} F_i(t) \neq 0 \right\} \text{ is countable.}$$

In addition, when H is convex, the set

$$\left\{ t \in Q \cap H : \lim_{i \in I} F_i(t) \neq 0 \right\} \text{ is countable.}$$

Proof. Let H a closed separable subset of K and let $J \subseteq I$ a countable subset given by Lemma 3.3.2. Since $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q , by Proposition 3.1.2, $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q \cap (H \setminus H^+)$, thus if $(a_{i,t}), (b_{i,t})$ is a $c_0\ell_1$ decomposition, we have:

$$\begin{aligned} \left\{ t \in Q \cap (H \setminus H^+) : \lim_{i \in I} F_i(t) \neq 0 \right\} &\subseteq \bigcup_{i \in I} \{ t \in Q \cap (H \setminus H^+) : b_{i,t} \neq 0 \} \\ &= \bigcup_{i \in J} \underbrace{\{ t \in Q \cap (H \setminus H^+) : b_{i,t} \neq 0 \}}_{\text{countable}}, \end{aligned}$$

which concludes the first part.

For the second part, just notice that when H is convex we have that $H^+ \setminus K^+ \subseteq \{\max H\}$, thus:

$$\begin{aligned} \left\{ t \in Q \cap H : \lim_{i \in I} F_i(t) \neq 0 \right\} &= \left\{ t \in Q \cap (H \setminus H^+) : \lim_{i \in I} F_i(t) \neq 0 \right\} \cup \left\{ t \in Q \cap H^+ : \lim_{i \in I} F_i(t) \neq 0 \right\} \\ &\subseteq \left\{ t \in Q \cap (H \setminus H^+) : \lim_{i \in I} F_i(t) \neq 0 \right\} \cup \{\max H\}. \quad \square \end{aligned}$$

We will now study the class of compact linear spaces where the converse of the first part of Theorem 3.3.3 holds. This class of linear compact spaces will be the central part of this section.

Definition 3.3.4. A linear compact space K is *separably determined* if for every weak*-null family $(F_i)_{i \in I} \subseteq \text{NBV}(K)$ and every $Q \subseteq K$, the following is satisfied:

(SD) If for every closed separable set $H \subseteq K$ the set $\{t \in Q \cap (H \setminus H^+) : \lim_{i \in I} F_i(t) \neq 0\}$ is countable, then $(F_i)_{i \in I}$ is of type $c_0 \ell_1$ over Q .

Example 3.3.5. If L is a separable linear compact space, then L is separably determined due to Theorem 3.3.1

The next proof could be left as an exercise to the reader since it only relies on standard arguments of functional analysis. Nevertheless we opt to give these arguments in detail for the sake of completeness.

Proposition 3.3.6. If $\varphi : K \xrightarrow{c.i.s.} L$ has the $c_0(I)$ -EP, then φ has the c_0 -EP

Proof. Let $\theta : \omega \hookrightarrow I$ an injective function. This function produces a complemented isometric copy of c_0 in $c_0(I)$ as we describe next. Consider $\hat{\theta} : c_0(\omega) \rightarrow c_0(I)$ given by:

$$\hat{\theta}(x)(i) = \begin{cases} 0, & \text{if } i \in I \setminus \text{ran}(\theta); \\ x(\theta^{-1}[i]), & \text{otherwise.} \end{cases}$$

For every $\varepsilon > 0$, $\{i \in I : |\hat{\theta}(x)(i)| \geq \varepsilon\} \subseteq \{n \in \omega : |x(n)| \geq \varepsilon\}$ is finite, so $\hat{\theta}$ is well defined. Moreover, $\hat{\theta}$ is linear and it is an isometry, since for all $x \in c_0$:

$$\|\hat{\theta}(x)\|_{c_0(I)} = \sup_{i \in I} |\hat{\theta}(x)(i)| = \sup_{n \in \omega} |x(n)| = \|x\|_{c_0}.$$

Consider $\Pi : c_0(I) \rightarrow c_0$ defined by $\Pi(p)(n) = p(\theta(n))$, for all $n \in \omega$. It follows that Π is well defined since for every $\varepsilon > 0$:

$$\{n \in \omega : |\Pi(p)(n)| \geq \varepsilon\} \subseteq \{i \in I : |p(i)| \geq \varepsilon\}.$$

Also, Π is linear, surjective and for all $p \in c_0(I)$, we have:

$$\|\Pi(p)\|_{c_0} = \sup_{n \in \omega} |\Pi(p)(n)| \leq \sup_{i \in I} |p(i)| = \|p\|_{c_0(I)}.$$

By the way that we have defined $\hat{\theta}$ and Π , it follows that $\Pi \circ \hat{\theta} = id_{c_0}$:

$$\forall n \in \omega, x \in c_0 : \quad \Pi \circ \hat{\theta}(x) = \hat{\theta}(x)(\theta(n)) = x(\theta^{-1}[\theta(n)]) = x(n).$$

If φ has the $c_0(I)$ -EP, for every $T : C(L) \longrightarrow c_0$, we have the following diagram:

$$\begin{array}{ccccc} & & C(K) & \xrightarrow{\quad \exists S \quad} & \\ \varphi^* \uparrow & & & \searrow & \\ C(L) & \xrightarrow{T} & c_0 & \xrightarrow{\hat{\theta}} & c_0(I) \end{array}$$

where S is a bounded operator that extends $\hat{\theta} \circ T$ through φ^* . Then $\Pi \circ S$ is an extension of T as we required. $\square$

Next result shows the importance of the class of separably determined linear compact spaces. A particular case of this statement was already pointed-out for separable linear compact spaces in the discussion we made after Theorem 3.3.1.

Theorem 3.3.7. *Let L be a separably determined linear compact space and let $\varphi : K \xrightarrow{c.i.s.} L$. Then:*

$$\varphi \text{ has the } c_0\text{-EP} \iff \varphi \text{ has the } c_0(I)\text{-EP}$$

Proof. It suffices to show that if φ has the c_0 -EP, then φ has the $c_0(I)$ -EP. By Theorem 3.1.4 we only need to prove that every weak*-null family $(F_i)_{i \in I} \subseteq \text{NBV}(L)$ is of type $c_0\ell_1$ over $Q(\varphi)$.

Because L is separably determined, we have to check that for every separable closed set $H \subseteq L$ the set $\{t \in Q(\varphi) \cap (H \setminus H^+) : \lim_{i \in I} F_i(t) \neq 0\}$ is countable.

Fix $(F_i)_{i \in I}$ a weak*-null family of $\text{NBV}(L)$ and H a separable closed subset of L . Using Lemma 3.3.2, take a countable subset $J_H \subseteq I$ such that $F_i|_{H \setminus H^+} \equiv 0$ for all $i \in I \setminus J_H$. Then we have:

$$\left\{ t \in Q(\varphi) \cap (H \setminus H^+) : \lim_{i \in I} F_i(t) \neq 0 \right\} = \left\{ t \in Q(\varphi) \cap (H \setminus H^+) : \lim_{i \in J_H} F_i(t) \neq 0 \right\} = B$$

Since φ has the c_0 -EP, the countable weak*-null family $(F_i)_{i \in J_H}$ is of type $c_0\ell_1$ over $Q(\varphi)$. By Theorem 3.3.3, B is countable. $\square$

We also have that the separably determined linear compact spaces are exactly the linear compact spaces where the statement of the previous result holds, more precisely:

Theorem 3.3.8. *Let L be a linear compact space such that for all $\varphi : K \xrightarrow{c.i.s.} L$ and for all I :*

$$\varphi \text{ has the } c_0\text{-EP} \iff \varphi \text{ has the } c_0(I)\text{-EP}$$

Then L is separably determined.

Proof. Let $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(L)$ and $Q \subseteq L$ satisfying (SD). To prove that $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q it suffices to show that it is of type $c_0\ell_1$ over the set:

$$Q' = \left\{ t \in Q : \lim_{i \in I} F_i(t) \neq 0 \right\}.$$

Consider $K = \text{DA}(L, Q')$ and $\varphi = \pi_1 : K \xrightarrow{c.i.s.} L$. If φ has the c_0 -EP, by our assumption φ has the $c_0(I)$ -EP, thus $(F_i)_{i \in I}$ will be of type $c_0\ell_1$ over Q' and we are done. In what follows we prove that φ has the c_0 -EP.

Let $(G_n)_{n \in \omega}$ be a weak*-null family in $\text{NBV}(L)$ associated with $(\mu_n)_{n \in \omega} \subseteq M(L)$. For each $n \in \omega$, $\text{supp}(\mu_n)$ is a closed separable subset of L (see [Cor14, Prop 2.4.32]), then the following set is a closed separable subset of L :

$$H = \text{cl} \left(\bigcup_{n \in \omega} \text{supp}(\mu_n) \right).$$

By our hypothesis, $Q' \cap (H \setminus H^+) = \{t \in Q \cap (H \setminus H^+) : \lim_{i \in I} F_i(t) \neq 0\}$ is countable. Moreover, we have:

$$\begin{aligned} \left\{t \in Q' : \lim_{n \in \omega} G_n(t) \neq 0\right\} &= Q' \cap \left\{t \in L : \lim_{n \in \omega} G_n(t) \neq 0\right\} \\ &= Q' \cap \left\{t \in L \setminus L^+ : \lim_{n \in \omega} G_n(t) \neq 0\right\} \\ &\subseteq Q' \cap \left\{t \in L \setminus H^+ : \lim_{n \in \omega} G_n(t) \neq 0\right\} \\ &\subseteq Q' \cap (H \setminus H^+). \end{aligned}$$

The last inclusion holds because for all points $t \in L \setminus H$, $\lim_{n \in \omega} G_n(t) = 0$. By Proposition 3.0.2, $(G_n)_{n \in \omega}$ extends through φ and since $(G_n)_{n \in \omega}$ is arbitrary, φ has the c_0 -EP. $\square$

Definition 3.3.9. Let K be a linear compact space. We say that $t \in K$ have a *separable adjacency* in K if t belongs to a separable convex subset of K with more than one point. We denote by $\text{Sad}(K)$ the set of all points in K with separable adjacency in K .

Remark. The reader may notice that if t has separable adjacency if, and only if, one of the following is satisfied:

- (a) there exists $s \in]t, \max K]$ such that $[t, s]$ is separable;
- (b) there exists $s \in [\min K, t[$ such that $[s, t]$ is separable.

In addition, we could define a equivalence relation on K as follows:

$$x \sim y \stackrel{\text{def}}{\iff} [\min\{x, y\}, \max\{x, y\}] \text{ is separable}$$

This relation is similar to the ones that appear on the proof of Ostaszewski Theorem, in [Ost74], and have the following properties:

- The equivalence classes of $\sim$ are convex;
- $t \in \text{Sad}(K)$ if, and only if, $[t]_\sim$ has more than one point;
- If J is an equivalence class with more than one point, then J can be written as disjoint union of separable convex subsets of K , each having more than one point.

Proposition 3.3.10. Let K be a linear compact space, $Q \subseteq K$ and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$. Suppose that for every closed separable convex subset $H \subseteq K$, the following set is countable

$$\left\{t \in Q \cap H : \lim_{i \in I} F_i(t) \neq 0\right\}.$$

Then $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q \cap \text{Sad}(K)$.

Proof. Notice that $\text{Sad}(K) = \bigcup_{\lambda \in \Lambda} J_\lambda$, where each J_λ is a separable convex subset of K having more than one point. By our assumptions, the set $\{t \in Q \cap \text{cl}(J_\lambda) : \lim_{i \in I} F_i(t) \neq 0\}$ is countable for each $\lambda \in \Lambda$. By Corollary 3.1.20, $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q \cap J_\lambda$ for all $\lambda \in \Lambda$. By Corollary 3.1.29, the conclusion follows. $\square$

Theorem 3.3.11. Let K be a linear compact space. If the set $\{t \in K \setminus \text{Sad}(K) : \chi_R(t, K) = \omega\}$ can be written as the union of a countable set with a set of finite two-sided height, then K is separably determined.

Proof. By our assumption, we can write:

$$\{t \in K \setminus \text{Sad}(K) : \chi_R(t, K) = \omega\} = E \cup H,$$

where E is a countable set and H with finite two-sided height. To check that K is separably determined, let $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$ and $Q \subseteq K$ satisfying (SD). To show that $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q , it suffices to show that it is of type $c_0\ell_1$ over $Q' = \{t \in Q : \lim_{i \in I} F_i(t) \neq 0\}$. We have

$$Q' = \left(Q' \cap \text{Sad}(K) \right) \cup \underbrace{\left(Q' \setminus \text{Sad}(K) \right)}_{\subseteq E \cup H},$$

where the inclusion holds due to Lemma 3.1.23. By Proposition 3.3.10, $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q' \cap \text{Sad}(K)$, and combining Corollary 3.1.29 and Proposition 3.1.10, we have that $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $E \cup H$. $\square$

Now we will devote our attention to some closure properties of the class of separably determined linear compact spaces. To show that being separably determined is hereditary, we need the following:

Lemma 3.3.12. *If K is a separable linear compact space and $H \subseteq K$ is closed, then $H^+ \setminus K^+$ is countable.*

Proof. For each $t \in H^+ \setminus \{\max H\}$ denote by t_H^+ the immediate successor of t in H . The intervals $]t, t_H^+[$ with $t \in H^+ \setminus (K^+ \cup \{\max H\})$ are non-empty open sets of K and pairwise disjoint. Since K is separable, $H^+ \setminus K^+$ is countable. $\square$

Proposition 3.3.13. *If K is a separably determined linear compact space and L is a closed subset of K , then L is separably determined.*

Proof. Let $Q \subseteq L$ and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(L)$ satisfying (SD). Consider the inclusion map $i : L \hookrightarrow K$ and define $G_i = i_*(F_i)$ for all $i \in I$ according to the diagram:

$$\begin{array}{ccc} M(L) & \xhookrightarrow{i_*} & M(K) \\ \parallel & & \parallel \\ \text{NBV}(L) & \xhookrightarrow[i_*]{} & \text{NBV}(K) \end{array}$$

Since the map i_* is the adjoint of the restrictor map $R : C(K) \rightarrow C(L)$, we have that $(G_i)_{i \in I}$ is a weak*-null family in $\text{NBV}(K)$, in addition, $G_i|_L = F_i$ for all $i \in I$. By our hypothesis, K is separably determined, then it suffices to show that for a closed separable subspace H of K , the set $\{t \in Q \cap (H \setminus H^+) : \lim_{i \in I} G_i(t) \neq 0\}$ is countable.

Claim: For all $t \in K \setminus L$, $\lim_{i \in I} G_i(t) = 0$.

Indeed, denote by $\mu_i \in M(L)$ the measure associated with F_i . Then, if $t \in K \setminus L$, we have:

$$\begin{aligned} G_i(t) &= i_*(\mu_i)(t) \\ &= \mu_i(i^{-1}[\text{[min } K, t]]) \\ &= \mu_i([\text{min } K, t] \cap L) \\ &= \begin{cases} 0, & \text{if } [\text{min } K, t] \cap L = \emptyset; \\ F_i(s), & \text{where } s = \max([\text{min } K, t] \cap L) \in L^+. \end{cases} \end{aligned}$$

Since linear compact spaces are hereditarily separable, $H \cap L$ is a separable closed set of L and by (SD), we have that

$$\left\{ t \in Q \cap ((L \cap H) \setminus (L \cap H)^+) : \lim_{i \in I} F_i(t) \neq 0 \right\} \text{ is countable.}$$

Our previous claim yields:

$$\left\{ t \in Q \cap (H \setminus H^+) : \lim_{i \in I} G_i(t) \neq 0 \right\} \subseteq \left\{ t \in Q \cap L \cap (H \setminus H^+) : \lim_{i \in I} F_i(t) \neq 0 \right\} = (*)$$

Applying Lemma 3.3.12 for H and $L \cap H$ we have that $(L \cap H)^+ \setminus H^+$ is countable and decomposing $L \cap H = (L \cap H)^+ \cup ((L \cap H) \setminus (L \cap H)^+)$ we have:

$$(*) \subseteq ((L \cap H)^+ \setminus H^+) \cup \left\{ t \in Q \cap ((L \cap H) \setminus (L \cap H)^+) : \lim_{i \in I} F_i(t) \neq 0 \right\},$$

which completes the proof. $\square$

Theorem 3.3.14. *Let $\varphi : K \xrightarrow{\text{c.i.s.}} L$. If L is separably determined and $\varphi^{-1}[t]$ is separably determined for all $t \in L$, then K is separably determined.*

Proof. Let Q and $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$ satisfying (SD). We will show that $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q .

For each $t \in L$, let $R_t : K \xrightarrow{\text{c.i.s.}} \varphi^{-1}[t]$ be the canonical retraction and set $F_i^t = (R_t)_*(F_i)$ for all $i \in I$, so that $(F_i^t)_{i \in I}$ is weak*-null on $\text{NBV}(\varphi^{-1}[t])$ for all $t \in L$. For a fixed $t \in L$, since F_i and F_i^t are equal on $\varphi^{-1}[t] \setminus \{\max \varphi^{-1}[t]\}$, for all closed separable $H \subseteq \varphi^{-1}[t]$ we have:

$$\left\{ s \in Q \cap \varphi^{-1}[t] \cap (H \setminus H^+) : \lim_{i \in I} F_i^t(s) \neq 0 \right\} \text{ is countable.}$$

By our assumption on $\varphi^{-1}[t]$ being separably determined, we have that both $(F_i^t)_{i \in I}$ and $(F_i)_{i \in I}$ are of type $c_0\ell_1$ over $Q \cap \varphi^{-1}[t]$ for all $t \in L$. By Corollary 3.1.29 it follows that $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over $Q \cap \bigcup_{t \in Q(\varphi)} \varphi^{-1}[t]$.

Now we need to prove that $(F_i)_{i \in I}$ is also of type $c_0\ell_1$ over the set

$$Q' = Q \setminus \bigcup_{t \in Q(\varphi)} \varphi^{-1}[t]$$

For each $i \in I$, consider $G_i = \varphi_*(F_i)$, so that $(G_i)_{i \in I}$ is a weak*-null family in $\text{NBV}(L)$. In addition, $G_i(\varphi(s)) = F_i(s)$ for every $s \in Q'$. Thus, to show that $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q' , it suffices to prove that $(G_i)_{i \in I}$ is of type $c_0\ell_1$ over $\varphi[Q']$.

Because L is separably determined, we only need to prove that for each closed separable subset $H \subseteq L$, the set

$$A \doteq \left\{ t \in \varphi[Q'] \cap (H \setminus H^+) : \lim_{i \in I} G_i(t) \neq 0 \right\}$$

is countable.

For a fixed $H \subseteq L$ closed separable subset, let $\tilde{H}$ be a closed separable subset of K such that $\varphi[\tilde{H}] = H$. Such $\tilde{H}$ exists because for a countable dense $D \subseteq H$, we choose a countable set $\tilde{D}$ of K with $\varphi[\tilde{D}] = D$ and put $\tilde{H} = \text{cl}(\tilde{D})$. Since $\tilde{H}$ is closed and separable:

$$B \doteq \left\{ s \in Q \cap (\tilde{H} \setminus \tilde{H}^+) : \lim_{i \in I} F_i(s) \neq 0 \right\} \text{ is countable.}$$

We have that $\varphi[\tilde{H}] = H$. Moreover, given $t \in \varphi[Q'] \cap H$, $\varphi^{-1}[t] = \{s\}$, so the following are satisfied:

- $G_i(t) = F_i(s)$ for all $i \in I$;
- $t \in H^+$ if, and only if $s \in \tilde{H}^+$.

Thus $A \subseteq \varphi[B]$ is countable, what finishes the proof. $\square$

Corollary 3.3.15. *If K and L are separably determined linear compact spaces, then the product $K \times L$ endowed with the lexicographic order is a separably determined linear compact space.*

Proof. Apply Theorem 3.3.14 with $\pi_1 : K \times L \xrightarrow{\text{c.i.s.}} K$. $\square$

Chapter 4

A linear compact space that is not separably determined

In the previous chapter we studied the criterion to extend $c_0(I)$ -valued operators in terms of $c_0\ell_1$ families of real numbers, as well as several conditions to achieve this decomposition. Moreover, we were able to describe a family of linear compact spaces where the c_0 -EP and the $c_0(I)$ -EP coincide. One could ask if every linear compact space has this property, however, this is not true.

This chapter is devoted to the construction of an example of linear compact space that is not separably determined. The construction of such a space will be guided by Theorems 3.3.7 and 3.3.8, that characterize the separably determined linear compact spaces. More specifically, we want to determine what are the conditions needed to create a c.i.s. function to enforce that a $c_0(I)$ -valued operator cannot be extended and, at the same time, we do not lose the (SD) property.

4.1 Separating families in topological spaces

In this section we will introduce a topological concept, namely κ -separating family, that will help us to construct a linear compact space that is not separably determined in the following sections. Since this concept is well connected with metric spaces, we tried to make clear the distinction between the structure of metric spaces, where the metric is explicitly given and the structure of metrizable topological spaces, where none of the metrics that generate the topology are specified. To avoid confusion, our notation is the pair (M, d) for the former and the topological space X for the latter.

Definition 4.1.1. Let X be a topological space, $S \subseteq X$ and κ be a cardinal. A κ -separating family for S in X is a family $(V_x)_{x \in S}$ such that V_x is a neighbourhood of x for all $x \in S$ satisfying the following condition:

$$(\kappa\text{-SF}) \quad \text{If } A \subseteq S \text{ and } x \in V_y \text{ for all } x, y \in A, \text{ then } |A| < \kappa.$$

By a κ -separating family for X we mean a κ -separating family for X in itself. When $\kappa = 2$, we will simply call it a separating family for S in X (or a separating family for X).

It follows directly from the previous definition that S admits a κ -separating family in X if, and only if, S admits a κ -separating family when endowed with the subspace topology. Thus, if X admits a κ -separating family, every subspace of X also admits a κ -separating family.

The definition of separating family $(V_x)_{x \in S}$ for S can be expressed as follows:

$$(\text{SF}) \quad \text{For all } x, y \in S \text{ distinct, either } x \notin V_y \text{ or } y \notin V_x.$$

The reader may notice that a separating family is a family $(V_x)_{x \in S}$ of neighborhoods such that for every $x, y \in S$, V_x and V_y satisfy the T_0 -separation axiom.

It is straightforward to check that any refinement by neighborhoods of a κ -separating family is also a κ -separating family. In particular, for a metric space (M, d) we can assume that each V_x is

an open ball centered at x . Thus, M admits a separating family if, and only if, there exists a family of positive real numbers $(r_x)_{x \in M}$ such that:

$$\forall x, y \in M, x \neq y : d(x, y) \geq \min\{r_x, r_y\} \quad (4.1.1)$$

Proposition 4.1.2. *Every scattered topological space admits a separating family.*

Proof. Let X be a scattered topological space. For an ordinal α , denote by $X^{(\alpha)}$ the α -th Cantor-Bendixon derivative.

For each $x \in X$ there exists a (unique) ordinal α such that $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$. Thus, x is isolated in $X^{(\alpha)}$ and we can pick a neighborhood V_x of x in X such that $V_x \cap X^{(\alpha)} = \{x\}$. We claim that $(V_x)_{x \in X}$ is a separating family for X .

Indeed, let $x, y \in X$ be distinct and let α, β be such that $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$ and $y \in X^{(\beta)} \setminus X^{(\beta+1)}$. If $\alpha \leq \beta$, then $y \notin V_x$ and if $\beta \leq \alpha$, $x \notin V_y$. $\square$

Definition 4.1.3. A topological space X is called *σ -discrete* if it is a countable union of discrete subspaces.

Lemma 4.1.4. *If an infinite metrizable topological space X is σ -discrete, then $|X| = d(X)$.*

Proof. Notice that for metrizable spaces, if $Y \subseteq X$, $d(Y) \leq d(X)$. The conclusion follows from the fact that for an infinite discrete space Y , $|Y| = d(Y)$. $\square$

Corollary 4.1.5. *If X is metrizable and σ -discrete, every separable subspace of X is countable.*

Definition 4.1.6. A metric space (M, d) is called *strongly discrete* if there exists $\varepsilon > 0$ such that $d(x, y) \geq \varepsilon$ for all $x, y \in X$ distinct.

We say that M is *σ -strongly discrete* if it is a countable union of strongly discrete subspaces.

Notice that a strongly discrete metric space is discrete and complete, therefore it is closed in any larger metric space in which is isometrically embedded. We have the following characterization:

Proposition 4.1.7. *For a metric space (M, d) , the following are equivalent:*

- (a) M admits a separating family;
- (b) M is σ -strongly discrete;
- (c) M is σ -discrete.

Proof. (a) $\implies$ (b): Let $(r_x)_{x \in M}$ be a family of positive real numbers satisfying (4.1.1). For each $n \geq 1$, the set $M_n = \{x \in M : r_x \geq \frac{1}{n}\}$ is strongly discrete and $M = \bigcup_{n \geq 1} M_n$.

(c) $\implies$ (a): Since M is σ -discrete, we can write $M = \bigcup_{n \in \omega} M_n$ where each M_n is discrete and closed (see [PW15, Lemma 1]). We can assume **WLOG** that this is an increasing family of closed discrete subsets of M . For every $x \in M$, let $n_x = \min\{n \in \omega : x \in M_n\}$ and let $r_x > 0$ be such that $B(x, r_x) \cap M_{n_x} = \{x\}$. We claim that $(B(x, r_x))_{x \in M}$ is a separating family for M .

Indeed, let $x, y \in M$ distinct, the case where $n_x = n_y$ is trivial. Otherwise, we can assume that $n_x < n_y$ and it follows that $x \notin B(y, r_y)$. $\square$

Before we proceed with our analysis between the interplay of σ -discreteness, metric spaces and κ -separating families, we would like to refer [PW15, PW16] to the reader that wants to know more about σ -discrete metric spaces. These spaces have some nice structural properties, such as they either are scattered or contain an homeomorphic copy of $\mathbb{Q}$. We proceed with our study.

Example 4.1.8. The closed interval $[0, 1]$ does not admit a separating family.

Indeed, $[0, 1]$ is not σ -discrete by Corollary 4.1.5 and the result follows from Proposition 4.1.7.

Proposition 4.1.9. *Let (M, d) a σ -discrete metric space. Then, for all $A \subseteq M$, $|\text{cl}(A)| = |A|$.*

Proof. By Lemma 4.1.4, since $\text{cl}(A)$ is σ -discrete, we have:

$$|\text{cl}(A)| = d(\text{cl}(A)) = |A|. \quad \square$$

Proposition 4.1.10. *If (M, d) is a metric space satisfying the following:*

$$\forall A \subseteq M : |A| \leq \omega \implies |\text{cl}(A)| \leq \omega.$$

Then, for all $S \subseteq M$ with $|S| \leq \omega_\omega$, we have $|\text{cl}(S)| \leq \omega_\omega$.

Proof. We prove by induction on the cardinality. Let that $S = \{s_\alpha : \alpha < \omega_1\} \subseteq M$. If $x \in \text{cl}(S)$, there exists a countable subset A of S such that $x \in \text{cl}(A)$, in particular, $A \subseteq \{s_\alpha : \alpha < \beta\}$ for some $\beta < \omega_1$, thus:

$$\text{cl}(S) \subseteq \bigcup_{\beta < \omega_1} \text{cl}(\{s_\alpha : \alpha < \beta\}),$$

and we have that $|\text{cl}(S)| \leq \omega_1$.

Suppose that $|S| = \omega_n$ and for every set $A \subseteq M$ with $|A| \leq \omega_{n-1}$, it follows that $|\text{cl}(A)| \leq \omega_{n-1}$. Enumerate $S = \{s_\alpha : \alpha < \omega_n\}$, since $\text{cof}(\omega_n) = \omega_n > \omega$, every countable subset of S is bounded, that yields:

$$\text{cl}(S) \subseteq \bigcup_{\beta < \omega_n} \text{cl}(\{s_\alpha : \alpha < \beta\}).$$

Thus, $|\text{cl}(S)| \leq \omega_n$. $\square$

Lemma 4.1.11. *If X is a first countable topological space without isolated points, then every infinite subset $S \subseteq X$ is contained in a subspace of X without isolated points having the same cardinality of S .*

Proof. Fix S an infinite subset of X and recursively construct an increasing sequence $(S_n)_{n \in \omega}$ of subsets of X as follows:

- $S_0 = S$;
- Assuming that S_n is constructed, for each $x \in S_n$ pick a sequence in $X \setminus \{x\}$ converging to x and denote by A_x the range of such a sequence. We define

$$S_{n+1} = S_n \cup \bigcup_{x \in S_n} A_x.$$

It follows that $Y = \bigcup_{n \in \omega} S_n$ is the desired subspace because all $y \in Y$ is limit of converging sequence in Y and for all $n \in \omega$, $|S_{n+1}| = |S_n|$. $\square$

If we assume that X is completely metrizable (i.e., its topology is induced by some complete metric), we have another characterization:

Proposition 4.1.12. *Let X be a completely metrizable topological space. The following conditions are equivalent:*

- (a) X is scattered;
- (b) X is σ -discrete;
- (c) Every separable subspace of X is countable.

Proof. The fact that (a) implies (b) and that (b) implies (c) follows from Proposition 4.1.2 and Corollary 4.1.5, respectively.

(c) $\implies$ (a): Suppose towards a contradiction that X is not scattered, then there exists an infinite subset without isolated points. By Lemma 4.1.11, we can find a countable subspace Y of X without isolated points. Hence, $F = \text{cl}(Y)$ has no isolated points and by our assumption, F is countable. On the other hand, Baire's theorem implies that F has an isolated point, which is a contradiction. $\square$

Proposition 4.1.13. *A separable metric space that admits a ω_1 -separating family is countable.*

Proof. Let M a separable metric space and $(r_x)_{x \in M}$ be such that $(B(x, r_x))_{x \in M}$ forms a ω_1 -separating family for X . We consider for every $n \geq 1$:

$$M_n = \{x \in M : r_x \geq \frac{1}{n}\}$$

Notice that every set $A \subseteq M_n$ with $\text{diam}(A) < \frac{1}{n}$ is countable because of $(\omega_1\text{-SF})$. Since M_n is separable (and Lindelöf), we can cover M_n with countably many open sets having diameter less than $\frac{1}{n}$, thus each M_n is countable which concludes the proof. $\square$

Corollary 4.1.14. *If a metric space M admits a ω_1 -separating family, then every separable subspace of M is countable.*

We finish this section with an important relation between $c_0\ell_1$ decompositions and ω_1 -separating families:

Theorem 4.1.15. *Let K be a compact line, $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$ and Q be a subset of K . If Q admits an ω_1 -separating family, then $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over Q .*

Proof. Our strategy is to prove that condition (b) in Proposition 3.1.27 is satisfied, we will argue later that the conditions required are indeed satisfied. For each $n \geq 1$ and each $i \in I$, Proposition 2.5.11 yields a finite collection $\mathcal{J}_i(n, +)$ of pairwise disjoint convex subsets of K such that:

$$\{t \in K : F_i(t) \geq \frac{1}{n}\} \subset \bigcup \mathcal{J}_i(n, +) \subset \{t \in K : F_i(t) \geq \frac{1}{2n}\}. \quad (4.1.2)$$

With the same reasoning we apply Proposition 2.5.11 with $-F_i$ to obtain for each $i \in I$ and each $n \geq 1$ a finite collection $\mathcal{J}_i(n, -)$ of pairwise disjoint convex subsets of K such that:

$$\{t \in K : F_i(t) \leq -\frac{1}{n}\} \subset \bigcup \mathcal{J}_i(n, -) \subset \{t \in K : F_i(t) \leq -\frac{1}{2n}\}.$$

Let $(V_t)_{t \in Q}$ be an ω_1 -separating family for Q in K . For each $n \geq 1$, we define the sets:

$$\begin{aligned} B(n, +) &= \{(i, t) \in I \times Q : \exists J \in \mathcal{J}_i(n, +) \text{ and } t \in J \subseteq V_t\}; \\ B(n, -) &= \{(i, t) \in I \times Q : \exists J \in \mathcal{J}_i(n, -) \text{ and } t \in J \subseteq V_t\}. \end{aligned}$$

Claim: For each $i \in I$, the sets $B_i(n, +)$ and $B_i(n, -)$ are countable.

Suppose towards a contradiction that is uncountable. For each $t \in B_i(n, +)$, since $(i, t) \in B(n, +)$ we can pick $J_t \in \mathcal{J}_i(n, +)$ with $t \in J_t \subseteq V_t$. By a counting argument we can find an uncountable subset C of $B_i(n, +)$ such that the sets J_t are all equal for $t \in C$ and we denote $J = J_t$ for all $t \in C$. The fact that C is uncountable contradicts our assumption that $(V_t)_{t \in Q}$ is a ω_1 separating family because for all $s, t \in C$, $s \in J_t = J \subseteq V_t$ and we should have that $|C| < \omega_1$. Similarly one shows that $B_i(n, -)$ is countable and the claim is done.

Now set $A(n, +) = (I \times Q) \setminus B(n, +)$ for each $n \geq 1$ and let us show that for every $t \in Q$ the set

$$\{i \in I : F_i(t) \geq \frac{1}{n} \text{ and } (i, t) \in A(n, +)\} \quad (4.1.3)$$

is finite.

For each i in (4.1.3), we can find $J_i \in \mathcal{J}_i(n, +)$ with $t \in J_i$ by (4.1.2) and it follows that $J_i \not\subseteq V_t$ by the definition of $A(n, +)$. Assume towards a contradiction that the set (4.1.3) is infinite.

There are two cases to consider: either J_i intersects $]t, \max K] \setminus V_t$ for infinitely many i in (4.1.3) or J_i intersects $[\min K, t[\setminus V_t$ for infinitely many i in (4.1.3).

Let us start with the first case. Notice that if (4.1.3) is infinite then $(F_i(t))_{i \in I}$ is not in $c_0(I)$ and thus t cannot be right isolated and we can find $s > t$ with $[t, s] \subset V_t$. Then, if J_i intersects $]t, \max K] \setminus V_t$ then J_i contains an element greater than s and by the convexity of J_i , we conclude that $[t, s] \subset J_i$. Thus $[t, s] \subset J_i$ for infinitely many i in (4.1.3) and from (4.1.2) we obtain that $F_i(u) \geq \frac{1}{2n}$ for all $u \in [t, s]$ and infinitely many i in I . This contradicts Corollary 3.1.19.

Now assume that J_i intersects $[\min K, t[\setminus V_t$ for infinitely many i in (4.1.3). If t is not left isolated, we can find $s < t$ with $[s, t] \subset V_t$ and then as before we obtain that $F_i(u) \geq \frac{1}{2n}$ for all $u \in [s, t]$ and infinitely many i in I , also contradicting Corollary 3.1.19. If t is left isolated then t cannot be the minimum of K and its immediate predecessor s is in J_i , for infinitely many i in (4.1.3). Thus $F_i(s) \geq \frac{1}{2n}$ for infinitely many $i \in I$ with s being a right-isolated point, which yields a contradiction. This concludes the proof that (4.1.3) is finite.

For each positive integer n we set $A(n, -) = (I \times Q) \setminus B(n, -)$ and as before one shows with similar arguments that for every $t \in Q$ the set

$$\{i \in I : F_i(t) \leq -\frac{1}{n} \text{ and } (i, t) \in A(n, -)\} \quad (4.1.4)$$

is finite.

We now define the subsets A and B of $I \times Q$ by setting

$$B = \bigcup_{n=1}^{\infty} (B(n, +) \cup B(n, -))$$

and

$$A = (I \times Q) \setminus B = \bigcap_{n=1}^{\infty} (A(n, +) \cap A(n, -)).$$

For each $i \in I$, the set $B_i = \{t \in Q : (i, t) \in B\}$ is equal to

$$\bigcup_{n=1}^{\infty} (B_i(n, +) \cup B_i(n, -))$$

and therefore it is countable. To conclude the proof it remains to check that for every $t \in Q$ and $n \geq 1$ the sets

$$\{i \in I : F_i(t) \geq \frac{1}{n} \text{ and } (i, t) \in A\}, \quad \{i \in I : F_i(t) \leq -\frac{1}{n} \text{ and } (i, t) \in A\}$$

are finite. This follows from the fact that such sets are contained in the finite sets (4.1.3) and (4.1.4), respectively.

We want to observe that the conditions required in Proposition 3.1.27 are satisfied because taking $J = K$ in Proposition 3.1.17 yields (1) and we have shown:

$$\{i \in I : F_i(t) \neq 0\} = \bigcup_{n \geq 1} \left\{ i \in I : |F_i(t)| \geq \frac{1}{n} \right\} \subseteq \bigcup_{n \geq 1} B_i(n, +) \cup B_i(n, -)$$

is countable, as required in (2). □

Corollary 4.1.16. *Let K be a compact line, $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}(K)$ and T be a subset of K . If the set*

$$\{t \in T : (F_i(t))_{i \in I} \notin c_0(I)\}$$

admits an ω_1 -separating family, then $(F_i)_{i \in I}$ is of type $c_0\ell_1$ over T .

We already know that for a linear compact space K , the set of one-sided isolated points of K admits a $c_0\ell_1$ decomposition (see Proposition 3.1.5) and we can prove that both K^+ and K^- admit a separating family, as we show next:

Corollary 4.1.17. *If K is a compact line then the sets K^+ and K^- admit a separating family.*

Proof. For each $t \in K^-$, we have that $V_t = [t, \max K]$ is a neighborhood of t and $(V_t)_{t \in K^-}$ is a separating family for K^- ; namely, if $t, s \in K^-$ and $t < s$ then $t \notin V_s$. Similarly, one proves that K^+ admits a separating family by setting $V_t = [\min K, t]$ for all $t \in K^+$. $\square$

4.2 The space $[0, 1]^\omega$

We have proved in Section 3.3 that the lexicographic product of separably determined linear compact spaces is also separably determined. Thus, for all $n \in \omega$, the space $[0, 1]^n$ is separably determined. The natural question that arises from this reasoning is: what happens when we consider $[0, 1]^\omega$ ordered lexicographically? This is the first candidate that can be non-separably determined.

This section is devoted to prove that in $[0, 1]^\omega$ we are able to construct a c.i.s. map that distinguishes the c_0 -EP and the $c_0(I)$ -EP, answering the question that was left open in the beginning of this chapter. Before we proceed, we need several results concerning the structure of $[0, 1]^\omega$, only then we will be able to give the construction.

First of all, the order of $[0, 1]^\omega$ is formally defined as follows:

$$f \leq g \iff f = g \text{ or } \left(\exists n \in \omega : f|_n = g|_n \text{ and } f(n) < g(n) \right)$$

It is straightforward to check that this is a total order in $[0, 1]^\omega$. We adopt the following convention: given $t \in [0, 1]$, $s \in [0, 1]^{<\omega}$ we denote by $(s, \mathbb{t})$ the almost constant function $f \in [0, 1]^\omega$ such that $f|_{|s|} = s$ and $f|_{[|s|, \omega]} \equiv \mathbb{t}$, when s is empty, we simply denote the constant function by $\mathbb{t}$. The reader may notice that $f \in [0, 1]^\omega$ is almost constant iff there exists $t \in [0, 1]$ such that $f =^* \mathbb{t}$, where the relation $=^*$ is defined by:

$$f =^* g \iff \{n \in \omega : f(n) \neq g(n)\} \text{ is finite.}$$

Proposition 4.2.1. *$[0, 1]^\omega$ is a linear compact space.*

Proof. It is trivial to check that $\mathbb{0}$ and $\mathbb{1}$ are the minimum and the maximum of $[0, 1]^\omega$, respectively.

Let $S \subseteq [0, 1]^\omega$ be a non-empty set. We will prove that there exists $\sup S$. We will define $g \in [0, 1]^\omega$ as follows:

$$g(0) = \sup\{f(0) : f \in S\}.$$

Consider $S_0 = \{f \in S : f(0) = g(0)\}$. If $S_0 = \emptyset$ we set $g|_{[1, \omega]} \equiv 0$. Otherwise we define:

$$g(1) = \sup\{f(1) : f \in S_0\} \quad \text{and} \quad S_1 = \{f \in S_0 : f(1) = g(1)\}$$

Suppose that are defined $S_0 \supseteq \dots \supseteq S_{n-1}$ and $g(0), \dots, g(n)$. We consider $S_n = \{f \in S_{n-1} : f(n) = g(n)\}$ and define $g|_{[n+1, \omega]} \equiv 0$ if $S_n = \emptyset$, otherwise we define $g(n+1) = \sup\{f(n+1) : f \in S_n\}$.

Notice that we recursively construct a decreasing sequence $(S_n)_{n \in \omega}$ of subsets of S with the following properties:

- (a) For each $n \geq 1$, if $f \in S_n$, then $f(n) = g(n) = \sup\{h(n) : h \in S_{n-1}\}$;
- (b) For each $n \in \omega$, if $f \in S_n \setminus S_{n+1}$ and $h \in S_{n+1}$, then $f \leq h$;
- (c) If $f \in S_n$, then $f|_{n+1} = g|_{n+1}$.

Notice that g is an upper bound of S because for a fixed $f \in S$ we have three cases to consider. If $f \in S \setminus S_0$, then $f(0) < g(0)$ and $f < g$. If $f \in S_k \setminus S_{k+1}$, then $f|_{k+1} = g|_{k+1}$ and $f(k+1) < g(k+1)$, that yields $f < g$. If $f \in \bigcap_{n \in \omega} S_n$, then $f = g$. Now we argue that g is the least upper bound of S .

Case 1: There exists $n \in \omega$ such that $S_n = \emptyset$.

Suppose towards a contradiction that there exists an upper bound h of S with $h < g$. Then there exists $k \in \omega$ such that $h|_k = g|_k$ and $h(k) < g(k)$. If $k > n$ then $g(n) = 0$ and inductively one proves that $h = g$, a contradiction. If $k \leq n$, then $h(n) < g(n) = \sup\{f(n) : f \in S_{n-1}\}$ and this contradicts the fact that h is an upper bound of S .

Case 2: For all $n \in \omega$, $S_n \neq \emptyset$.

There exists $n \in \omega$ such that $h|_{n+1} = g|_{n+1}$ and $h(n) < g(n)$ and there exists $f \in S_{n-1}$ such that $h(n) < f(n)$, that contradicts the fact that h is an upper bound of S . $\square$

The reader may notice that Proposition 3.3.13 together with Corollary 3.3.15 tell us that for every $n \in \omega$ the lexicographic product $[0, 1]^n$ is a separably determined linear compact space, thus it is natural to ask whether $[0, 1]^\omega$ is separably determined as well. We proceed with some comparison between these structures. The following result is somewhat trivial and we just provide a sketch of the proof for the sake of completeness.

Proposition 4.2.2. *Let $n \geq 1$ and consider $P_n : [0, 1]^\omega \rightarrow [0, 1]^n$ the map given by $P_n(f) = f|_n$. Then P_n is a c.i.s. map.*

Proof. It is straightforward to check that P_n is surjective and increasing, where it is possible to occur $f < g$ and $P_n(f) \leq P_n(g)$.

For the continuity, notice that for $r < s$ in $[0, 1]^n$, we have $P_n^{-1}[]r, s[] =](r, \mathbb{1}), (s, \mathbb{0})[$. $\square$

Given $n \geq 1$ we also can consider the map $P_n^{n+1} : [0, 1]^{n+1} \rightarrow [0, 1]^n$, where $P_n^{n+1}(f) = f|_n$. This is also a c.i.s. map and we have the commutative diagrams:

$$\begin{array}{ccc}
 [0, 1]^\omega & & C([0, 1]^\omega) \\
 \downarrow P_{n+1} & & \uparrow (P_{n+1})^* \\
 [0, 1]^{n+1} & & C([0, 1]^{n+1}) \\
 \downarrow P_n^{n+1} & & \uparrow (P_n^{n+1})^* \\
 [0, 1]^n & & C([0, 1]^n)
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \curvearrowright \\
 & & (P_n)^* \\
 & & \curvearrowleft
 \end{array}$$

Thus, it follows that:

$$P_1^*C([0, 1]) \subseteq P_2^*C([0, 1]^2) \subseteq \cdots \subseteq P_n^*C([0, 1]^n) \subseteq \cdots \subseteq C([0, 1]^\omega).$$

A simple application of the Stone-Weierstrass for algebras shows us that $\bigcup_{n \geq 1} P_n^*C([0, 1]^n)$ is a (linearly) dense subset of $C([0, 1]^\omega)$. This yields the following:

Theorem 4.2.3. *Let $(F_i)_{i \in I}$ be a bounded family in $\text{NBV } [0, 1]^\omega$. Are equivalent:*

- (a) $(F_i)_{i \in I}$ is weak*-null in $[0, 1]^\omega$;
- (b) for all $n \geq 1$, $((P_n)_*F_i)_{i \in I}$ is weak*-null in $[0, 1]^n$;
- (c) $\lim_{i \in I} F_i(\mathbb{1}) = 0$ and for all $n \geq 1, x \in [0, 1]^{n-1}$, the set $\{t \in [0, 1] : \lim_{i \in I} F_i(x \smallfrown t, \mathbb{1}) \neq 0\}$ is countable.

Proof. We already established the equivalence between (a) and (b).

(a) $\implies$ (c): Assume that $(F_i)_{i \in I}$ is weak*-null in $\text{NBV } ([0, 1]^\omega)$. Then for every $n \geq 1$, $((P_n)_*F_i)_{i \in I}$ is weak*-null in $\text{NBV } ([0, 1]^n)$ and extends through P_n^* . We have the following diagram:

$$\begin{array}{ccc}
C([0, 1]^\omega) & \xrightarrow{(F_i)_{i \in I}} & c_0(I) \\
P_n^* \uparrow & \nearrow & \\
C([0, 1]^n) & & ((P_n)_* F_i)
\end{array}$$

For every $x \in [0, 1]^{n-1}$, the set $\{x\} \times [0, 1]$ is a convex, closed and separable subset of $[0, 1]^n$ and $Q(P_n) = [0, 1]^n$. By Theorem 3.3.3 it follows that the set

$$\left\{ y \in Q(P_n) \cap H : \lim_{i \in I} F_i(y) \neq 0 \right\} = \left\{ t \in [0, 1] : \lim_{i \in I} F_i(x \smallfrown t, \mathbb{1}) \neq 0 \right\}$$

is countable.

(c) $\implies$ (b): We will prove by induction.

For $n = 1$, by our assumption we have that $\{t \in [0, 1] : \lim_{i \in I} (P_1)_*(F_i)(t) \neq 0\}$ is countable and $\lim_{i \in I} (P_1)_*(F_i)(1) = \lim_{i \in I} F_i(\mathbb{1}) = 0$. Because $[0, 1]^+ = \{1\}$, by Corollary 3.2.3 it follows that $((P_1)_*(F_i))_{i \in I}$ is weak*-null in NBV $([0, 1])$.

Now suppose that $((P_n)_*(F_i))_{i \in I}$ is weak*-null in NBV $([0, 1]^n)$. In what follows, we will prove that $((P_{n+1})_*(F_i))_{i \in I}$ is weak*-null in NBV $([0, 1]^{n+1})$.

Consider the map $P_n^{n+1} : [0, 1]^{n+1} \xrightarrow{\text{c.i.s.}} [0, 1]^n$. By Proposition 2.5.20 it suffices to show that:

- (1) $\lim_{i \in I} (P_n^{n+1})_* \circ (P_{n+1})_*(F_i) = 0$;
- (2) For all $x \in [0, 1]^n : \lim_{i \in I} (R_x)_* \circ (P_{n+1})_*(F_i) = 0$.

Notice that (1) follows because $(P_n^{n+1})_* \circ (P_{n+1})_*(F_i) = (P_n)_*(F_i)$. Now fix $x \in [0, 1]^n$. Notice that $R_x : [0, 1]^{n+1} \longrightarrow \{x\} \times [0, 1]$ and that $(\{x\} \times [0, 1])^+ = \{x \smallfrown 1\}$. We have:

$$\begin{aligned}
(R_x)_* \circ (P_{n+1})_*(F_i)(x \smallfrown 1) &= (P_{n+1})_*(F_i)(1, \dots, 1) \\
&= F_i(\mathbb{1}).
\end{aligned}$$

Thus, $\lim_{i \in I} (R_x)_* \circ (P_{n+1})_*(F_i)(x \smallfrown 1) = 0$.

Now we argue that the set $A = \left\{ p \in \{x\} \times [0, 1] : \lim_{i \in I} (R_x)_* \circ (P_{n+1})_*(F_i)(p) \neq 0 \right\}$ is countable.

Indeed, if $p \in A$, then $p = x \smallfrown t$ with $t < 1$ and we have that:

$$(R_x)_* \circ (P_{n+1})_*(F_i)(p) = (P_{n+1})_*(F_i)(p) = F_i(x \smallfrown t, \mathbb{1})$$

By our assumption, $A \subseteq \{t \in [0, 1] : \lim_{i \in I} F_i(x \smallfrown t, \mathbb{1}) \neq 0\}$ is countable. By Corollary 3.2.3, part (2) follows. $\square$

In order to prove that $[0, 1]^\omega$ is not separably defined, we need to construct a bounded operator in $C([0, 1]^\omega)$ that not satisfy (SD). We will try to find an operator that is associated with a family of characteristic functions on NBV $([0, 1]^\omega)$, because for this type of family, we will not have to worry whether it is weak*-null. We have the following:

Proposition 4.2.4. *Consider $x \in [0, 1]^\omega$, $I = [0, 1]^{<\omega}$ and define $F_i = \chi_{[(i, \emptyset), (i, \mathbb{1})[}$ for each $i \in I$. Then:*

$$\lim_{i \in I} F_i(x) = 0 \iff x =^* \mathbb{1}.$$

Furthermore, $(F_i)_{i \in I}$ is weak*-null in NBV $([0, 1]^\omega)$.

Proof. ($\implies$) Suppose that $x \neq^* \mathbb{1}$. Then for each $n \in \omega$, there exists $k_n \in \omega$, $k_n > n$ such that $x_{k_n} \neq 1$. For each $n \in \omega$ we consider $i_n = x|_{k_n}$, then:

$$\forall n \in \omega : F_{i_n}(x) = \chi_{[(x|_{k_n}, \emptyset), ((x|_{k_n}, \mathbb{1})[} = 1.$$

Thus, $\lim_{i \in I} F_i(x) \neq 0$.

($\Leftarrow$) Let $k = \min\{n \in \omega : x(n) \neq 1\}$. Then, for $i \in I$ we have:

$$F_i(x) = 1 \iff i \in \{x|_n : n \leq k\}$$

Thus, $\lim_{i \in I} F_i(x) = 0$. □

In what follows we will analyse the structure of the space $[0, 1]^\omega$. We will provide a fundamental system of neighbourhoods and describe the closed separable subspaces of $[0, 1]^\omega$.

Let $x \in [0, 1]^\omega$. We have three cases to consider:

Case 1: $x \neq^* 0$ and $x \neq^* 1$:

For each $n \in \omega$, we consider the set $V_x^n = \{y \in [0, 1]^\omega : y|_n = x|_n\}$ and we claim that $\mathcal{V}_x = \{V_x^n : n \in \omega\}$ is a local base for x .

Indeed, V_x^n is a neighbourhood because $(x|_n, 0), (x|_n, 1) \subseteq V_x^n$. If $f, g \in [0, 1]^\omega$ and $x \in]f, g[$, then:

$$\begin{aligned} \exists m \in \omega : f|_m = x|_m \text{ and } f(m) < x(m); \\ \exists n \in \omega : x|_n = g|_n \text{ and } x(n) < g(n). \end{aligned}$$

Taking $k = \max\{m, n\}$ it follows that $V_x^k \subseteq]f, g[$, which concludes the first case.

Notice that it is important that $x \neq^* 1$ because every open interval $]f, g[$ containing $(x|_n, 1)$ escapes from V_x^n . This also shows that V_x^n are not open.

Case 2: $x =^* 1$:

We write $x = (x|_{k+1}, 1)$, where $x(k) < 1$. Given $\varepsilon > 0$ and $n \in \omega$ we consider the following sets:

$$\begin{aligned} V_{x,\varepsilon}^1 &= \left\{ y \in [0, 1]^\omega : y|_k = x|_k \text{ and } y(k) \in]x(k), x(k) + \varepsilon[\right\}; \\ V_{x,n}^1 &= \left\{ y \in [0, 1]^\omega : y|_{k+1} = x|_{k+1} \text{ and } y(j) = 1, k+1 \leq j \leq k+n \right\}. \end{aligned}$$

We claim that $\mathcal{V}_x = \{V_{x,\varepsilon}^1 \cup V_{x,n}^1 : \varepsilon > 0, n \in \omega\}$ is a local base for x .

It is clear that $V_{x,\varepsilon}^1 \cup V_{x,n}^1$ is a neighbourhood of x because taking $f = (x|_{k+n+1}, 0)$ and $g = (x|_k \widehat{} t, 1)$ with $t \in]x(k), x(k) + \varepsilon[$ implies that $]f, x[\subseteq V_{x,n}^1$ and $[x, g[\subseteq V_{x,\varepsilon}^1$.

Suppose that $x \in]f, g[$. Then:

$$\begin{aligned} \exists n_1 \in \omega : f|_{n_1} = x|_{n_1} \text{ and } f(n_1) < x(n_1); \\ \exists n_2 \in \omega : x|_{n_2} = g|_{n_2} \text{ and } x(n_2) < g(n_2). \end{aligned}$$

If $n_1 < k$, then $V_{x,0}^1 \subseteq]f, x[$, otherwise $V_{x,n_1-k+1}^1 \subseteq]f, x[$. Because $x =^* 1$, $n_2 \leq k$. If $n_2 = k$ take $0 < \varepsilon < g(k) - x(k)$, then $V_{x,\varepsilon}^1 \subseteq]x, g[$. Otherwise, we have that $V_{x,\varepsilon}^1 \subseteq]x, g[$ for any ε .

Case 3: $x =^* 0$:

Similarly as we did in the previous case, we write $x = (x|_{k+1}, 0)$ with $x(k) > 0$ and for $\varepsilon > 0$ and $n \in \omega$ we define the sets:

$$\begin{aligned} V_{x,\varepsilon}^0 &= \left\{ y \in [0, 1]^\omega : y|_k = x|_k \text{ and } y(k) \in]x(k) - \varepsilon, x(k)[\right\}; \\ V_{x,n}^0 &= \left\{ y \in [0, 1]^\omega : y|_{k+1} = x|_{k+1} \text{ and } y(j) = 0, k+1 \leq j \leq k+n \right\}. \end{aligned}$$

Then $\mathcal{V}_x = \{V_{x,\varepsilon}^0 \cup V_{x,n}^0 : \varepsilon > 0, n \in \omega\}$ is a local base for x .

Now that we established a fundamental system of neighbourhoods we can describe the closed

separable subspaces of $[0, 1]^\omega$.

Definition 4.2.5. Let $E \subseteq [0, 1]$ be a countable set. We define:

$$\widehat{E} = \left\{ x \in [0, 1]^\omega : \exists k \in \omega, x|_k \in E^k \text{ and } (x|_{[k+1, \omega[} \equiv 0 \text{ or } x|_{[k+1, \omega[} \equiv 1) \right\}.$$

Proposition 4.2.6. For every countable subset E of $[0, 1]$, $H_E = E^\omega \cup \widehat{E}$ is a separable closed subset of $[0, 1]^\omega$.

Proof. First we show that H_E is closed. Fix $x \in \text{cl}(H_E)$ and we consider each of the following cases:

Case 1: $x \neq^* 0$ and $x \neq^* 1$:

We have that $x \notin \widehat{E}$, we argue that $x \in E^\omega$. Suppose towards a contradiction that $x \notin E^\omega$ and let $n = \min\{k \in \omega : x(k) \notin E\}$. Notice that for $j \geq n + 2$ and $y \in V_x^j \cap H_E$ we necessarily have that $y \in \widehat{E}$ because $x|_j = y|_j$ and $x(n) \notin E$. By the definition of $\widehat{E}$:

$$\exists k \leq n : y|_k \in E^k \text{ and } y|_{\omega \setminus k+1} \text{ is constant either equal to 0 or 1.}$$

Assume **WLOG** that $y|_{\omega \setminus k+1} \equiv 0$. We have that $x|_{[k, j[} = y|_{[k, j[} \equiv 0$, and ranging $j \geq n + 2$ one prove inductively that $x =^* 0$, a contradiction.

Case 2: $x =^* 1$:

We write $x = (x|_{k+1}, 1)$ with $x(k) < 1$. To show that $x \in H_E$, it suffices to show that $x|_k \in E^k$. Let $n = 1$ and $\varepsilon = \frac{1-x(k)}{2}$. Then there exists $y \in (V_{x, \varepsilon}^1 \cup V_{x, 1}^1) \cap H_E$. In case $y \in E^\omega$ we have that $x|_k = y|_k \in E^k$ and we are done.

If $y \in \widehat{E}$, then there exists $n \in \omega$ such that $y|_n \in E^n$ and $y|_{[n+1, \omega[}$ is constant equal either to 0 or 1. If $n \geq k$ then $x|_k \in E^k$ and we are done. Now, will argue that $n < k$ cannot happen:

Suppose that $n < k$. If $y \in V_{x, \varepsilon}^1$, then $y(k) \in]x(k), x(k) + \varepsilon[$ and this interval that is disjoint from $\{0, 1\}$, a contradiction. If $y \in V_{x, 1}^1$, observe that we must have $y|_{[n+1, \omega[} \equiv 1$, but $y(k) = x(k) < 1$, a contradiction.

Case 3: $x =^* 0$:

It is similar to the previous case.

Now we need to prove that H_E is separable. For each $k \in \omega$, we consider:

$$D_k = \left\{ y \in [0, 1]^\omega : y|_k \in E^k, y(k) \in [0, 1] \cap \mathbb{Q} \text{ and } (y|_{[k+1, \omega[} \equiv 0 \text{ or } y|_{[k+1, \omega[} \equiv 1) \right\}.$$

We claim that $D = \bigcup_{k \in \omega} D_k$ is dense in H_E . Notice that D is countable because each D_k is countable. Let $x \in H_E$ and V a basic neighbourhood of x , we argue that $V \cap D \cap H_E \neq \emptyset$:

Case 1: $x \neq^* 0$ and $x \neq^* 1$:

In this case we have that $x \in E^\omega$ and there exists $n \in \omega$ such that $V = V_x^n$. It follows that $(x|_n, 0) \in V \cap H_E \cap D$.

Case 2: $x =^* 1$:

We write $x = (x|_{k+1}, 1)$ with $x(k) < 1$. There exist $\varepsilon > 0$ and $n \in \omega$ such that $V = V_{x, \varepsilon}^1 \cup V_{x, n}^1$.

If $x \in E^\omega$, it follows that $(x|_{k+1}, 1) \in D \cap H_E \cap V_{x, n}^1$. If $x \in \widehat{E}$, there exists $j \in \omega$ such that $x|_j \in E^j$ and $x|_{[j+1, \omega[} \equiv 1$. We have the cases:

$j < k$: cannot happen because $x(k) < 1$;

$j = k$: choose $t \in]x(k), x(k) + \varepsilon[$, then $(x|_k \widehat{} t, 1) \in V_{x, \varepsilon}^1 \cap H_E \cap D$;

$j > k$: then $x \in D$.

Case 3: $x =^* 0$:

It is similar to the previous case. $\square$

Corollary 4.2.7. *Every closed separable subset of $[0, 1]^\omega$ is contained in some H_E , for a countable subset E of $[0, 1]$.*

Proof. Let F be a closed separable subset of $[0, 1]^\omega$ and let D be a countable dense set of F . Define $E = (\mathbb{Q} \cap [0, 1]) \cup \bigcup_{n \in \omega} \pi_n[D]$, where $\pi_n : [0, 1]^\omega \rightarrow [0, 1]$ is given by $\pi_n(f) = f(n)$. It follows that E is countable and that $F \subseteq H_E$. $\square$

Proposition 4.2.8. *Let $E \subseteq [0, 1]$ be countable with $0, 1 \in E$. Then:*

$$H_E^+ = \{1\} \cup \{x \in [0, 1]^\omega : \exists n \in \omega, x|_n \in E^n, x_n \notin E \text{ and } x|_{[n+1, \omega[} \equiv 0\}.$$

Proof. We just provide the steps of the proof. Firstly, observe that if x is a right-isolated point, then $x \notin E^\omega$. Then $x \in \widehat{E}$ and we must have that $x = (x|_n, x(n), 0)$.

To check every step, one must consider the three cases accordingly to the fundamental system we provided for the points of $[0, 1]^\omega$. $\square$

Proposition 4.2.9. *Let $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}([0, 1]^\omega)$, $Q \subseteq [0, 1]^\omega$ and $E \subseteq [0, 1]$ be countable with $0, 1 \in E$. The following are equivalent:*

- (a) $\{t \in Q \cap (H_E \setminus H_E^+) : \lim_{i \in I} F_i(t) \neq 0\}$ is countable;
- (b) $\{t \in E^\omega \cap Q : \lim_{i \in I} F_i(t) \neq 0\}$ is countable.

Proof. Notice that, since $0, 1 \in E$, it follows from Proposition 4.2.8:

$$H_E \setminus H_E^+ = (E^\omega \setminus \{1\}) \cup \{x \in \widehat{E} : \exists k \in \omega, x|_k \in E^k, x(k) \notin E \text{ and } x|_{[k+1, \omega[} \equiv 1\}.$$

Thus, (a) $\implies$ (b) is trivial.

(b) $\implies$ (a) By Theorem 4.2.3 we have that $\lim_{i \in I} F_i(1) = 0$ and for all $n \geq 1, x \in [0, 1]^{n-1}$, the set $\{t \in [0, 1] : \lim_{i \in I} F_i(x \smallfrown t, 1) \neq 0\}$ is countable. We consider the set

$$X = \left\{ x \in \widehat{E} : \exists k \in \omega, x|_k \in E^k, x(k) \notin E, x|_{[k+1, \omega[} \equiv 1 \text{ and } \lim_{i \in I} F_i(x) \neq 0 \right\}.$$

Notice that X is countable because we can write

$$X = \bigcup_{k \in \omega} \bigcup_{y \in E^k} \underbrace{\left\{ (y \smallfrown t, 1) : t \notin E \text{ and } \lim_{i \in I} F_i(y \smallfrown t, 1) \neq 0 \right\}}_{\text{is countable}}.$$

This, together with our assumption implies (a). $\square$

Remember that when H_1, H_2 are closed separable subsets of linear compact space K with $H_1 \subseteq H_2$, we always have that $(H_1 \setminus H_1^+) \subseteq (H_2 \setminus H_2^+)$. This yields the following characterization:

Corollary 4.2.10. *Let $(F_i)_{i \in I}$ be a weak*-null family in $\text{NBV}([0, 1])$ and let $Q \subseteq [0, 1]^\omega$. Then:*

$$\begin{array}{ccc} \text{For all separable closed } H \subseteq [0, 1]^\omega & & \text{For all countable set } E \subseteq [0, 1] \\ |\{x \in Q \cap (H \setminus H^+) : \lim_{i \in I} F_i(x) \neq 0\}| \leq \omega & \iff & |\{x \in Q \cap E^\omega : \lim_{i \in I} F_i(x) \neq 0\}| \leq \omega \end{array}$$

From what we have done until now, if we consider $[0, 1]^\omega$ ordered lexicographically, $I = [0, 1]^{<\omega}$ and let $F_i = \chi_{[(i, 0), (i, 1)[}$, by Proposition 4.2.4, $(F_i)_{i \in I}$ is a weak*-null family in $\text{NBV}([0, 1]^\omega)$ and

$\lim_{i \in I} F_i(x) = 0$ iff $x =^* \mathbb{1}$. Also, by Proposition 3.1.27 with $Q = [0, 1]^\omega$, $(F_i)_{i \in I}$ is of type $c_0 \ell_1$ over $[0, 1]^\omega$ iff $I \times [0, 1]^\omega = A \cup B$ such that

$$\lim_{i \in I} F_i(x) \chi_A(i, x) = 0 \quad \text{and} \quad B_i = \{x \in [0, 1]^\omega : (i, x) \in B\} \text{ is countable,}$$

for all $x \in [0, 1]^\omega$ and for all $i \in I$.

Now if we define $B'_i = \pi_2[B] \cap [(i, 0), (i, 1)[$ for all $i \in I$ it follows that $x \in B'_i$ iff $(i, x) \in B_i$ and $F_i(x) = 1$. On the other hand, if $i \in I$ is such that $x \in [(i, 0), (i, 1)[\setminus B'_i$, then $(i, x) \in A$. Thus:

$$\sup_{i \in I} |B'_i| = \sup_{i \in I} \sum_{x \in [0, 1]^\omega} |F_i(x)| < +\infty;$$

for all $x \in [0, 1]^\omega$, $\{i \in I : x \in [(i, 0), (i, 1)[\setminus B'_i\}$ is finite.

And if one consider a set $Q \subsetneq [0, 1]^\omega$, then $B'_i \subseteq [(i, 0), (i, 1)[\cap Q$ for all $i \in I$. What we have done is summarized in the following:

Proposition 4.2.11. *If $I = [0, 1]^{<\omega}$ and $F_i = \chi_{[(i, 0), (i, 1)[} \in \text{NBV}([0, 1]^\omega)$, then:*

$$(F_i)_{i \in I} \text{ is of type } c_0 \ell_1 \text{ over } Q \iff \begin{cases} \text{There exists a family of sets } (B_i)_{i \in I} \text{ such that:} \\ (1) \text{ For all } i \in I, B_i \subseteq [(i, 0), (i, 1)[\cap Q; \\ (2) \sup_{i \in I} |B_i| < +\infty; \\ (3) \text{ For all } x \in Q, |\{i \in I : x \in [(i, 0), (i, 1)[\setminus B_i\}| < \omega. \end{cases}$$

The reader may notice that combining Corollary 4.2.10 and Proposition 4.2.11, we are making reductions to tackle the problem of showing that $[0, 1]^\omega$ is not separably determined. We proceed with our study: we shall discuss what happens when we pass to an infinite subset J of $I = [0, 1]^{<\omega}$.

Proposition 4.2.12. *Let J be an infinite subset of $I = [0, 1]^{<\omega}$, $Q \subseteq [0, 1]^\omega$ and $F_i = \chi_{[(i, 0), (i, 1)[} \in \text{NBV}([0, 1]^\omega)$. Then we have the following characterizations for the weak*-null family $(F_i)_{i \in J}$:*

$$\begin{aligned} &\text{For all countable set } E \subseteq [0, 1] \\ &|\{x \in Q \cap E^\omega : \lim_{i \in J} F_i(x) \neq 0\}| \leq \omega \iff \begin{aligned} &\text{For all countable set } E \subseteq [0, 1] \\ &|\{x \in Q \cap E^\omega : |\{i \in J : x \supseteq i\}| \geq \omega\}| \leq \omega. \end{aligned} \end{aligned}$$

And:

$$(F_i)_{i \in J} \text{ is of type } c_0 \ell_1 \text{ over } Q \iff \begin{cases} \text{There exists a family of sets } (B_i)_{i \in J} \text{ such that:} \\ (1) \text{ For all } i \in J, B_i \setminus \{(i, 1)\} \subseteq [(i, 0), (i, 1)[\cap Q; \\ (2) \sup_{i \in J} |B_i| < +\infty; \\ (3) \text{ For all } x \in Q, |\{i \in J : x \supsetneq i \text{ and } x \notin B_i\}| < \omega. \end{cases}$$

Proof. For each $i \in J$ define $G_i = \chi_{[(i, 0), (i, 1)]}$ and consider the sets:

$$X = \{x \in Q \cap [0, 1]^\omega : \lim_{i \in J} F_i(x) \neq 0\} \quad \text{and} \quad Y = \{x \in Q \cap [0, 1]^\omega : \lim_{i \in J} G_i(x) \neq 0\}.$$

By the definition of $(G_i)_{i \in J}$, $\lim_{i \in J} G_i(x) \neq 0$ iff $\{i \in J : x \supseteq i\}$ is infinite. In addition, $X \subseteq Y$, moreover:

$$Y \setminus X = \{y \in Y : y =^* \mathbb{1}\},$$

because for $y \neq^* \mathbb{1}$ it follows that $F_i(y) = 1$ whenever $G_i(y) = 1$. Since E is a countable set, we have that $E^\omega \cap \{x \in [0, 1]^\omega : x =^* \mathbb{1}\}$ is countable, thus, by Corollary 4.2.10 we finished the first part of the proof.

Suppose that $(F_i)_{i \in J}$ is of type $c_0 \ell_1$. By Proposition 4.2.11 we obtain a family of sets $(B_i)_{i \in J}$ where $B_j \subseteq [(i, 0), (i, 1)[\cap Q$, $\sup_{i \in J} |B_i| < +\infty$ and $\{i \in J : x \in [(i, 0), (i, 1)[\setminus B_i\}$ is finite for all

$x \in Q$. Define for all $i \in J$, $B'_i = B_i \cup \{(i, 1)\}$ and notice that for each $x \in Q$:

$$\{i \in J : x \not\supseteq i \text{ and } x \notin B'_i\} \subseteq \{i \in J : x \in [(i, 0), (i, 1)[\setminus B_i\}.$$

Thus the family $(B'_i)_{i \in J}$ satisfies the required conditions.

Conversely, assume that there exists a family of sets $(B_i)_{i \in J}$ satisfying (1) through (3). We set $B'_i = B_i \setminus \{(i, 1)\}$ and by Proposition 4.2.11, $(F_i)_{i \in J}$ is of type $c_0\ell_1$ over $[0, 1]^\omega$. $\square$

To find a suitable set Q in $[0, 1]^\omega$ we can take advantage of the fact that with the set $[0, 1]^\omega$ we have the tree $[0, 1]^{<\omega}$, that until now we are using to index the family of characteristic functions in NBV $([0, 1])$. We need to introduce some notation:

Definition 4.2.13. Given $x \in [0, 1]^\omega$ and $s \in [0, 1]^{<\omega}$ we define the sets:

$$\hat{x} = \{x|_n : n \in \omega\} \quad \text{and} \quad C_s = \{x \in [0, 1]^\omega : s \subseteq x\}.$$

The reader may notice that $\hat{x}$ represents the path in the tree $[0, 1]^{<\omega}$ that leads to $x \in [0, 1]^\omega$. On the other hand, C_s represents all points $x \in [0, 1]^\omega$ with $x|_{|s|} = s$. Moreover, $C_s = [(s, 0), (s, 1)]$.

Let τ be the the product topology where each $[0, 1]$ is discrete in $[0, 1]^\omega$. It follows that $\{C_s : s \in [0, 1]^{<\omega}\}$ is a base of clopen sets for $([0, 1]^\omega, \tau)$ and we have that $([0, 1]^\omega, \tau)$ is a metrizable, zero-dimensional space with weight $\mathfrak{c}$.

Definition 4.2.14. For a subset S of $[0, 1]^{<\omega}$ we define:

$$\tilde{S} = \{x \in [0, 1]^\omega : |\hat{x} \cap S| = \omega\}.$$

Observe that the $\tilde{S}$ is the set of points that extends infinitely many elements of S , thus the points in $\tilde{S}$ are precisely the points $x \in [0, 1]^\omega$ where $\lim_{s \in S} \chi_{C_s}(x) \neq 0$. Moreover, for the points $x \in \tilde{S}$, the collection $\{C_s : s \in \hat{x} \cap S\}$ forms a local base for x in the product topology $[0, 1]^\omega$ with $[0, 1]$ discrete.

We can rewrite the conditions in Proposition 4.2.12 using the language that we introduced as follows, the first one being:

$$\text{For all countable set } E \subseteq [0, 1] : |\tilde{S} \cap Q \cap E^\omega| \leq \omega. \quad (4.2.1)$$

That is equivalent to $|\{x \in Q \cap (H \setminus H^+) : \lim_{s \in S} F_s(x) \neq 0\}| \leq \omega$ for each separable closed set $H \subseteq [0, 1]^\omega$.

And the second one:

$$(F_s)_{s \in S} \text{ is of type } c_0\ell_1 \text{ over } Q \iff \begin{cases} \text{There exists a family of sets } (B_s)_{s \in S} \text{ and } k \in \omega \text{ such that:} \\ (1) \text{ For all } s \in S, B_s \subseteq C_s \cap Q; \\ (2) \sup_{s \in S} |B_s| \leq k; \\ (3) \text{ For all } x \in \tilde{S} \cap Q, \{s \in S \cap \hat{x} : x \notin B_s\} \text{ is finite.} \end{cases} \quad (4.2.2)$$

Observe that in (4.2.2), if we define $s(x)$ as the biggest element of $\{s \in \tilde{S} \cap \hat{x} : x \notin B_s\}$, this leads to:

$$(F_s)_{s \in S} \text{ is of type } c_0\ell_1 \text{ over } Q \iff \begin{cases} \text{There exists } k \in \omega \text{ and there exists } (s(x))_{x \in \tilde{S} \cap Q} \text{ such that:} \\ (1) \text{ For all } x \in \tilde{S}, s(x) \in \hat{x} \cap S; \\ (2) \text{ For all } s \in S, |\{x \in \tilde{S} \cap Q : s \subseteq x \text{ and } s(x) \subseteq s\}| \leq k. \end{cases} \quad (4.2.3)$$

In the view of the last equivalence, we have:

Proposition 4.2.15. Let $S \subseteq [0, 1]^{<\omega}$ and $(F_s)_{s \in S}$ be the weak*-null family of characteristic functions $F_s = \chi_{[(s, 0), (s, 1)[}$. Are equivalent:

- (a) $(F_s)_{s \in S}$ is of type $c_0\ell_1$ over Q with $k \geq 1$ satisfying (4.2.3);
- (b) $\tilde{S} \cap Q$ admits a $(k+1)$ -separating family as subspace of $[0, 1]^\omega$ endowed with the product topology, where each $[0, 1]$ is discrete.

Proof. (a) $\implies$ (b) Let $k \geq 1$ and $(s(x))_{x \in \tilde{S} \cap Q}$ satisfying (4.2.3). We claim that $(C_{s(x)} \cap \tilde{S} \cap Q)_{x \in \tilde{S} \cap Q}$ is a $(k+1)$ -separating family for $\tilde{S} \cap Q$. Indeed, suppose towards a contradiction that there exists $A \subseteq \tilde{S} \cap Q$ with $|A| = k+1$ and for every distinct $x, y \in A$, $x \in C_{s(y)}$. We write $A = \{x_j : j \in k+1\}$ we have that the points $s(x_j)$ are compatible. For $s = \max\{s(x_j) : j \in k+1\}$, both $s \subseteq x_j$ and $s(x_j) \subseteq s$ for $j \in k+1$, which contradicts the fact that $\{x \in \tilde{S} \cap Q : s \subseteq x \text{ and } s(x) \subseteq s\}$ has size at most k .

(b) $\implies$ (a) Let $(U_x \cap \tilde{S} \cap Q)_{x \in \tilde{S} \cap Q}$ be a $(k+1)$ -separating family for $\tilde{S} \cap Q$. For every $x \in \tilde{S} \cap Q$, $\{C_s \cap \tilde{S} \cap Q : s \in \hat{x} \cap S\}$ is a local base for x , then there exists $s(x) \in \hat{x} \cap S$ such that $C_{s(x)} \subseteq U_x$. It follows that $(C_{s(x)} \cap \tilde{S} \cap Q)_{x \in \tilde{S} \cap Q}$ is also a $(k+1)$ -separating family because it is a refinement of the one we started with. We claim that $(s(x))_{x \in \tilde{S} \cap Q}$ satisfies (4.2.3):

Fix $s \in S$, notice that if $A = \{x \in \tilde{S} \cap Q : s \subseteq x \text{ and } s(x) \subseteq s\}$, then for all $x, y \in A$ we have that $x \in C_{s(y)} \cap \tilde{S} \cap Q$, thus $|A| \leq k$. $\square$

Definition 4.2.16. Let A be a subset of a limit ordinal γ . We say that:

- (a) A is *unbounded* iff $\sup A = \gamma$;
- (b) A is *closed* iff for all limit $\nu < \gamma$, $\nu \in A$ whenever $A \cap \nu$ is unbounded in ν ;
- (c) A is *club* in γ iff A is closed and unbounded.

It follows that a subset A of γ is closed if, and only if, it is closed in the order topology. Moreover, the club sets of γ form a filter base. We have:

Definition 4.2.17. For a limit ordinal γ with $\text{cof}(\gamma) > \omega$ the *club filter* is defined as follows:

$$\text{club}_\gamma = \{X \subseteq \gamma : \exists C \subseteq \gamma, C \text{ is club}, C \subseteq X\}$$

We say that a set $X \subseteq \gamma$ is *stationary* if, and only if, X meets every club subset of γ .

We need just two results regarding the relationship between clubs and stationary sets, that we just refer here:

Proposition 4.2.18. If $\kappa > \omega$ is regular, T is a stationary subset of κ and $f : T \longrightarrow E$ is a function where $|E| < \kappa$. Then some $f^{-1}\{e\}$ is stationary.

Proof. See [Kun11, Lemma III.6.7]. $\square$

Lemma 4.2.19 (Pressing Down Lemma). Let $\kappa > \omega$ be an uncountable regular cardinal, $S \subseteq \kappa$ be a stationary set and $f : S \longrightarrow \kappa$ be such that $f(\beta) < \beta$, for all $\beta \in S$. Then there exists $\alpha < \kappa$ such that $f^{-1}[\alpha]$ is stationary.

Proof. See [Kun11, Lemma III.6.14]. $\square$

Theorem 4.2.20. There exist $S \subseteq [0, 1]^{<\omega}$ and $Q \subseteq [0, 1]^\omega$ such that:

- (a) For all countable set E of $[0, 1]$, $|\tilde{S} \cap Q \cap E^\omega| \leq \omega$;
- (b) $\tilde{S} \cap Q$ viewed as a subspace of $[0, 1]^\omega$ with the product topology of discrete $[0, 1]$, does not admit k -separating family for any $k \in \omega$.

Proof. Fix an injection $\theta : \omega_1 \hookrightarrow [0, 1]$ and consider the induced injection $\psi : \omega_1^\omega \hookrightarrow [0, 1]^\omega$ given by:

$$\text{For all } n \in \omega : \quad \psi(f)(n) = \theta \circ f(n).$$

Let $S = (\text{ran}(\theta))^{<\omega}$. For every $\beta < \omega_1$, choose a surjective function $x_\beta : \omega \rightarrow \beta$ and define $Q = \{\psi(x_\beta) : \beta < \omega_1\}$.

Notice that $Q \subseteq \tilde{S}$ because $\psi(\hat{x}_\beta) = \{\{\theta(x_\beta(0)), \dots, \theta(x_\beta(n-1))\} : n \in \omega\}$, thus $\tilde{S} \cap Q = Q$.

Claim 1: For all $\alpha \in \omega_1$, $Q \cap \alpha^\omega$ is countable.

Notice that $\beta > \alpha$, then $\text{ran}(x_\beta) \not\subseteq \text{ran}(x_\alpha) = \alpha$. Thus $x_\beta \notin \alpha^\omega$. It follows that

$$\{x_\beta : \beta \in \omega_1\} \cap \alpha^\omega = \{x_\beta : \beta < \alpha\}$$

is a countable set.

Claim 2: For all countable set E of $[0, 1]$, $|Q \cap E^\omega| \leq \omega$.

If $\text{ran}(\theta) \cap E = \emptyset$, then $\text{ran}(\psi) \cap E^\omega = \emptyset$, in particular $Q \cap E^\omega = \emptyset$. Otherwise, we have that $\text{ran}(\theta) \cap E$ is countable. Then there exists $\alpha \in \omega_1$ such that $\theta^{-1}[E] \subseteq \alpha$. It follows that

$$|Q \cap E^\omega| \leq |Q \cap \alpha^\omega| \leq \omega,$$

where the last inequality holds prior to claim 1.

Claim 3: Q does not admit a k -separating family.

Suppose towards a contradiction that $(V_x \cap Q)_{x \in Q}$ is a k -separating family. We can suppose that V_x are all basic open sets of $[0, 1]^\omega$, and we can define a function $f : \omega_1 \rightarrow \omega$ such that $(C_{\psi(x_\beta)|_{f(\beta)}} \cap Q)_{\beta \in \omega_1}$ that refines $(V_x \cap Q)_{x \in Q}$, thus it is also a k -separating family for Q .

We have that $\omega_1 = \bigcup_{m \in \omega} f^{-1}[m]$, then there exists $n \in \omega$ such that $S = f^{-1}[n]$ is a stationary set. Notice that $(C_{\psi(x_\beta)|_n} \cap Q)_{\beta \in S}$ is a k -separating family for $\{\psi(x_\beta) : \beta \in S\} \subseteq Q$.

For $j \in n$, consider the function $f_j : S \rightarrow \omega_1$ given by

$$f_j(\beta) = x_\beta(j).$$

By the Pressing Down Lemma 4.2.19, there exists a stationary set $S_0 \subseteq S$ such that $f_0|_{S_0} \equiv \lambda_0$. Inductively we can find a sequence of stationary sets $S \supseteq S_0 \supseteq \dots \supseteq S_n$ such that the $f_j|_{S_j} \equiv n_j$. Thus, the function $g : S_n \rightarrow \omega_1^n$ given by

$$g(\beta) = (x_\beta(0), \dots, x_\beta(n-1)) = x_\beta|_n$$

is constant equal to $(\lambda_0, \dots, \lambda_{n-1})$.

We have that $(C_{\psi(x_\beta)|_n} \cap Q)_{\beta \in S_n}$ is a k -separating family for $\{\psi(x_\beta) : \beta \in S_n\}$. But it follows from our construction of S_n that, for all $\beta \in S_n$,

$$C_{\psi(x_\beta)|_n} = C_{(\theta(\lambda_0), \dots, \theta(\lambda_{n-1}))},$$

which leads us to a contradiction. □

Remember that in [CT15] the authors characterized c_0 -EP for $\varphi : K \xrightarrow{\text{c.i.s.}} L$ as follows:

$$\begin{array}{ccc} \varphi \text{ has the } c_0\text{-EP} & \iff & \begin{array}{l} \text{For each separable } G_\delta \text{ set } A \subseteq L \text{ such that} \\ \text{every point } x \in A \text{ is an internal point of } A, \\ \text{then } A \cap Q(\varphi) = A \cap Q_0(\varphi) \text{ is countable.} \end{array} \end{array}$$

This characterization, together with all the work we done here yields the following:

Corollary 4.2.21. *There exist a linear compact space $\tilde{K}$ and $\varphi : \tilde{K} \xrightarrow{\text{c.i.s.}} [0, 1]^\omega$ such that φ has the c_0 -EP and there exist a set of indexes I with $|I| = \omega_1$ and a $c_0(I)$ -valued operator defined on $C([0, 1]^\omega)$ that does not extend through φ . In particular, $[0, 1]^\omega$ is not separably determined.*

Proof. Notice that in Theorem 4.2.20, we can fix a injective function $\theta : \omega_1 \rightarrow]0, 1[$. Then for $S = (\text{ran } \theta)^{<\omega}$ and $Q = \{x_\beta : \beta < \omega_1\} \subseteq \tilde{S}$ it follows that the family $(F_s)_{s \in S}$ is not of type $c_0\ell_1$ over Q . By Proposition 4.2.15, the family $(F_s)_{s \in S}$ does not extend through

$$\pi_1 : \text{DA}([0, 1]^\omega, Q) \xrightarrow{\text{c.i.s.}} [0, 1]^\omega.$$

This yields the following diagram:

$$\begin{array}{ccc} C(\text{DA}([0, 1]^\omega, Q)) & & \\ \uparrow \pi_1^* & \searrow \nexists (G_s)_{s \in S} & \\ C([0, 1]^\omega) & \xrightarrow{(F_s)_{s \in S}} & c_0(S) \end{array}$$

We claim that π_1 has the c_0 -EP. Let A be a separable G_δ of $[0, 1]^\omega$ such that every $x \in A$ is a internal point of A , we will argue that $A \cap Q(\pi_1) = A \cap Q$ is countable.

Indeed, let $H = \text{cl}(A)$. Then H is a closed separable subset of $[0, 1]^\omega$. Moreover, $A \subseteq H \setminus H^+$ because all the points in A are internal points. Notice that, since $Q \subseteq \tilde{S}$ and $0, 1 \notin \text{ran}(\theta)$, it follows that $\lim_{s \in S} F_s(x_\beta) \neq 0$ for all $x_\beta \in Q$. Thus

$$(H \setminus H^+) \cap Q = \left\{ x \in Q \cap (H \setminus H^+) : \lim_{s \in S} F_s(x) \neq 0 \right\}$$

is countable by Proposition 4.2.12 and Corollary 4.2.10. Thus π_1 has the c_0 -EP and it follows from Theorem 3.3.7 that $[0, 1]^\omega$ is not separably determined. $\square$

We want to finish this section with a small discussion about what we have done here. It was well known that c_0 -EP and $c_0(I)$ -EP are not equivalent for arbitrary compact Hausdorff spaces. For instance, it was proved in [ACG⁺02, Theorem 1.1.b] that there exists a non-complemented copy of $c_0(I)$ in a WCG Banach space $C(K)$, with $|I| = \beth_\omega$. In particular, there exists a $c_0(I)$ -valued operator that cannot be extended. On the other hand, it is discussed that when $C(K)$ is WCG, then it has the c_0 -EP.

We wanted to know if there was an example of this kind in the class of linear compact spaces, and at the same time we provide a construction, we were able to lower the condition on the cardinality of I rather drastically: our construction is done with $|I| = \omega_1$.

Chapter 5

Weak normality in Ψ -spaces

In general topology, Ψ -spaces are topological spaces associated to almost disjoint families in ω . This class of spaces is due to Mrówka [Mr655] and Isbell, credited in [GJ60]. It is sometimes called Mrówka-Isbell space in the literature.

Over the years Ψ -spaces have proved to be rather useful tools and a versatile source of examples in many areas of topology, ranging from Fréchet and sequential spaces, compactifications, hyper-spaces and functional analysis. For a reference in this field of study and related applications, we have the surveys [Hru14] and [HHH18].

Regarding notation, we define some of the set theoretical topological and cardinal characteristics concepts as we need them, for undefined concepts we refer (resp.) to [Kun11], [E⁺77] and [Bla10].

5.1 Almost disjoint families and Ψ -spaces

In this section we provide basic results and definitions regarding almost disjoint families and the Ψ -spaces associated to them. The concepts range from combinatorics properties involving almost disjoint families, topological properties of the Ψ -spaces and the cardinal characteristics associated with the previous topics.

Our terminology is mostly standard: $A \subseteq^* B$ means that A is almost contained in B , that is, $A \setminus B$ is finite, $A =^* B$ means $A \subseteq^* B$ and $B \subseteq^* A$.

Definition 5.1.1. Let N a countable infinite set such that $N \cap [N]^\omega = \emptyset$. An almost disjoint family (over N) is an infinite collection $\mathcal{A}$ of infinite subsets of N such that for all distinct $a, b \in \mathcal{A}$, $a \cap b$ is finite. It is maximal almost disjoint (MAD) if it is not properly included in any larger almost disjoint family.

Remark. Although we consider almost disjoint families over arbitrary sets N , in the vast majority of cases we are restricted to almost disjoint families over ω and we omit the set N when it is clear from context.

We present the following folklore results regarding almost disjoint families:

Proposition 5.1.2. (a) *There exists an almost disjoint family of size $\mathfrak{c}$;*

(b) *Every MAD family is uncountable;*

(c) *Every almost disjoint family can be extended to a MAD family.*

Proof. (a): For each $x \in \mathbb{R}$, fix an infinite sequence $S_x \subseteq \mathbb{Q}$ convergent to x . It follows that $\{S_x : x \in \mathbb{R}\}$ is an almost disjoint and identifying $\mathbb{Q}$ with ω we have an almost disjoint family of size $\mathfrak{c}$.

(b): Let $\{a_n : n \in \omega\}$ an almost disjoint family. Fix $k_0 \in a_0$ and, for each $n \geq 1$, let $k_n \in a_n \setminus \bigcup_{j < n} a_j$. It follows that $\{k_n : n \in \omega\}$ is almost disjoint from all a_n .

(c): Follows directly from Zorn's Lemma. □

Definition 5.1.3. The almost disjointness number $\mathfrak{a}$ is the smallest cardinality of any MAD family over ω , more formally:

$$\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ is MAD}\}.$$

It follows from Proposition 5.1.2(c) and the definition of the disjointness number that $\omega_1 \leq \mathfrak{a} \leq \mathfrak{c}$ and it is well known that there are models of set theory in which all possibilities of the inequalities hold.

Definition 5.1.4. Given an almost disjoint family $\mathcal{A}$, the Ψ -space associated to $\mathcal{A}$, denoted by $\Psi(\mathcal{A})$, is defined as follows: the underlying set is $\omega \cup \mathcal{A}$, all elements of ω are isolated and the basic neighborhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$, where $F \subseteq \omega$ is finite.

It follows immediately from the definition that $\Psi(\mathcal{A})$ is Hausdorff, separable, scattered, zero-dimensional, first countable, locally compact and regular. Since $\mathcal{A}$ is infinite, closed and discrete, it follows that $\Psi(\mathcal{A})$ is not countably compact. In addition, $\Psi(\mathcal{A})$ is metrizable if, and only if, $\mathcal{A}$ is countable.

The next result is an useful tool to work with subsets of Mrówka-Isbell spaces:

Proposition 5.1.5. *Let $\mathcal{A}$ an almost disjoint family, $U \subseteq \Psi(\mathcal{A})$ and $W \subseteq \omega$. Then:*

- (a) *U is open if, and only if, for all $a \in \mathcal{A}$, $a \subseteq^* U$;*
- (b) *Given $a \in \mathcal{A}$, then $a \in \text{cl}(W)$ if, and only if, $|a \cap W| = \omega$.*

Proof. (a): If U is an open set and $a \in \mathcal{A}$, there exists $n \in \omega$ such that $\{a\} \cup (a \setminus n) \subseteq U$, hence $a \subseteq^* U$. On the other hand, suppose that for all $a \in \mathcal{A}$, $a \subseteq^* U$ and let $x \in U$. If $x \in \omega$, then $\{x\} \subseteq U$. If $x \in \mathcal{A}$, then $x \subseteq^* U$ and there exists $n \in \omega$ such that $x \setminus n \subseteq U$, thus $\{x\} \cup (x \setminus n) \subseteq U$.

(b): If $a \in \text{cl}(W)$, for each $n \in \omega$, since $W \cap (\{a\} \cup (a \setminus n)) \neq \emptyset$, there exists $m \in \omega$, $m \geq n$ such that $m \in a \cap W$. Hence $|a \cap W| = \omega$. On the other hand, if $|a \cap W| = \omega$ and $F \subseteq \omega$ is finite, then there exists $n \in a \cap \omega$, such that $n > \max F$, then $W \cap (\{a\} \cup (a \setminus F)) \neq \emptyset$. Since F is arbitrary, $a \in \text{cl}(W)$. $\square$

Proposition 5.1.6. *For an almost disjoint family $\mathcal{A}$, $\Psi(\mathcal{A})$ is pseudo-compact if, and only if, $\mathcal{A}$ is a MAD family.*

Proof. Suppose that $\mathcal{A}$ is not maximal. Then there exists an infinite set $J = \{x_n : n \in \omega\} \subseteq \omega$ almost disjoint from each element of $\mathcal{A}$. Since J is closed and discrete, the function $g : \Psi(\mathcal{A}) \rightarrow \mathbb{R}$ defined by $g(x_n) = n$, for $n \in \omega$, and $g|_{\Psi(\mathcal{A}) \setminus J} \equiv 0$ is continuous and unbounded, hence $\Psi(\mathcal{A})$ is not pseudo-compact.

On the other hand, suppose by contradiction that $\mathcal{A}$ is maximal and $\Psi(\mathcal{A})$ is not pseudo-compact. There exists $f : \Psi(\mathcal{A}) \rightarrow \mathbb{R}^+$ unbounded, continuous and non-negative. Since ω is dense in $\Psi(\mathcal{A})$, we can construct a sequence $C = \{x_n : n \in \omega\} \subseteq \omega$ such that $x_{n+1} > \max\{n, f(x_n)\}$. Since $\mathcal{A}$ is maximal, there exists $a \in \mathcal{A}$ such that $|a \cap C| = \omega$. Notice that $a \cap C$ is a subsequence of a , converging to a and because f is continuous we have that $f(a) = \lim_{x \in a \cap C} f(x) = +\infty$, a contradiction. $\square$

Corollary 5.1.7. *If $\Psi(\mathcal{A})$ is normal, then $\mathcal{A}$ cannot be a MAD family*

Proof. Suppose towards a contradiction that $\mathcal{A}$ is a MAD family and $\Psi(\mathcal{A})$ is normal. It follows that $|\mathcal{A}| \geq \omega_1$ by Proposition 5.1.2. Hence, $\Psi(\mathcal{A})$ is a normal, pseudocompact space that is not countably compact. $\square$

Although the Mrówka-Isbell spaces satisfies several nice topological properties, it is not an easy task to determine whether the space $\Psi(\mathcal{A})$ is normal or not. In the remaining of this section we devote our attention to analyze under what assumptions the Mrówka-Isbell spaces are normal. The following lemma, due to F.B. Jones is an useful tool to construct examples of non-normal topological spaces:

Lemma 5.1.8 (Jones' Lemma). *If X a normal space, then for every closed discrete subset $S \subseteq X$, $|S| < 2^{|S|} \leq 2^{d(X)}$.*

Proof. Let $D \subseteq X$ a dense subset of size $d(X)$ and $S \subseteq X$ closed discrete. Notice that for each $A \subseteq S$, the sets A and $S \setminus A$ are closed and disjoint. Since X is normal, there exists disjoint open subsets U, V separating A and $S \setminus A$, respectively.

For each $A \subseteq S$, fix U_A, V_A disjoint open subsets of X such that $U_A \supseteq A$ and $V_A \supseteq S \setminus A$. We claim that $f : \mathcal{P}(S) \rightarrow \mathcal{P}(D)$, defined by $f(A) = U_A \cap D$ is one-to-one:

If $A_1 \neq A_2$, **WLOG**, $A_1 \setminus A_2 \neq \emptyset$. Then $U_{A_1} \cap V_{A_2}$ is a nonempty open subset of X and since D is dense, there exists $z \in D \cap U_{A_1} \cap V_{A_2}$ and $z \notin U_{A_2}$ because $U_{A_1} \cap U_{A_2} = \emptyset$. Thus $U_{A_1} \cap D \neq U_{A_2} \cap D$. $\square$

Since the Mrówka-Isbell $\Psi(\mathcal{A})$ is separable and $\mathcal{A}$ is a closed and discrete, we have the following result as an immediate consequence of the Jones' Lemma 5.1.8:

Corollary 5.1.9. *If $\Psi(\mathcal{A})$ is normal, then $|\mathcal{A}| < \mathfrak{c}$.*

The previous result gives a necessary condition to achieve normality in Mrówka-Isbell spaces. Although, the restriction of the size of $|\mathcal{A}|$ alone is not equivalent to normality. To state a characterization for normality of the spaces $\Psi(\mathcal{A})$ we need the following terminology:

Definition 5.1.10. Let $\mathcal{A}$ be an almost disjoint family. We say that $X \subseteq \omega$ is a partitioner for $\mathcal{A}$ if for each $a \in \mathcal{A}$, $a \subseteq^* X$ or $a \cap X =^* \emptyset$.

We say that a partitioner X for $\mathcal{A}$ is a partitioner for $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ if $b \subseteq^* X$ and $c \cap X =^* \emptyset$ for each $b \in \mathcal{B}$ and $c \in \mathcal{C}$. When $X \subseteq \omega$ is not necessarily a partitioner, we say that $\mathcal{B}, \mathcal{C}$ are separated by X .

Notice that there exists an one to one correspondence between the clopen subsets of $\Psi(\mathcal{A})$ and the partitioners of $\mathcal{A}$, which is defined as follows: for each $X \subseteq \omega$ consider $\mathcal{B}_X = \{a \in \mathcal{A} : a \subseteq^* X\}$ and $\mathcal{C}_X = \{a \in \mathcal{A} : a \cap X =^* \emptyset\}$. Hence:

- $\mathcal{B}_X \cup X$ and $\mathcal{C}_X \cup (\omega \setminus X)$ are disjoint open subsets of $\Psi(\mathcal{A})$;
- If X is a partitioner for $\Psi(\mathcal{A})$, then $\mathcal{A} = \mathcal{B}_X \cup \mathcal{C}_X$ and $\Psi(\mathcal{A}) = (\mathcal{B}_X \cup X) \cup (\mathcal{C}_X \cup (\omega \setminus X))$ is union of clopen subsets.

Then, denoting by $\text{Part}(\mathcal{A}) = \{X \subseteq \omega : X \text{ is a partitioner for } \mathcal{A}\}$, it easily follows that:

Lemma 5.1.11. *If $\mathcal{A}$ is an almost disjoint family, then $F : \text{Clop}(\Psi(\mathcal{A})) \rightarrow \text{Part}(\mathcal{A})$, defined by $F(W) = W \cap \omega$, is a bijective function, with inverse given by $F^{-1}(X) = \mathcal{B}_X \cup X$.*

In addition, we have the following characterization for normality in Mrówka-Isbell spaces:

Proposition 5.1.12. *Let $\mathcal{A}$ be an almost disjoint family. Then $\mathcal{A}$ is normal if, and only if, for all $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be separated.*

Proof. ($\Rightarrow$): Let $\mathcal{B} \subseteq \mathcal{A}$, since $\Psi(\mathcal{A})$ is normal, there exists disjoint open sets U, V such that $\mathcal{B} \subseteq U$ and $\mathcal{A} \setminus \mathcal{B} \subseteq V$. Consider $X = U \cap \omega$, by Proposition 5.1.5 we have that for all $b \in \mathcal{B}$, $b = b \cap \omega \subseteq^* U \cap \omega = X$. On the other hand, if $a \in \mathcal{A} \setminus \mathcal{B}$ then $a \subseteq^* V$ and it follows that $a \cap X =^* \emptyset$.

($\Leftarrow$): Let F, G disjoint closed sets in $\Psi(\mathcal{A})$ and consider $\mathcal{B} = F \cap \mathcal{A}$, $\mathcal{C} = G \cap \mathcal{A}$. By assumption, there exists $X \subseteq \omega$ such that for all $b \in \mathcal{B}$, $a \in \mathcal{A} \setminus \mathcal{B}$, $b \subseteq^* X$ and $a \cap X =^* \emptyset$. By Proposition 5.1.5, $U = \mathcal{B} \cup X$ and $V = \mathcal{C} \cup (\omega \setminus X)$ are disjoint open sets, thus $\Psi(\mathcal{A})$ is normal. $\square$

One of the first results of structural properties in the class of Mrówka-Isbell spaces was the construction of Luzin:

Theorem 5.1.13 ([Luz47]). *There exists an uncountable almost disjoint family such that no two uncountable subfamilies can be separated.*

Proof. We will recursively define A_α , $\alpha < \omega_1$, as follows:

Let $\{a_n : n \in \omega\}$ be a partition of ω into infinite sets. For $\alpha \geq \omega$, if a_β have been chosen for $\beta < \alpha$, enumerate $\mathcal{A}_\alpha = \{a_\beta : \beta < \alpha\}$ as $\{b_n : n \in \omega\}$ and choose $x_n \subseteq b_n \setminus \left(\bigcup_{k < n} b_k\right)$, with $|x_n| = n$ and let $a_\alpha = \bigcup_{n \in \omega} x_n$. The family $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ has the following property:

$$\forall \alpha < \omega_1, \forall n \in \omega : \quad \{\beta < \alpha : a_\beta \cap a_\alpha \subseteq n\} \text{ is finite.}$$

If $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ are uncountable disjoint subfamilies of $\mathcal{A}$ that can be separated, then there exists $X \subseteq \omega$ such that for all $b \in \mathcal{B}$, $c \in \mathcal{C}$, $b \subseteq^* X$ and $c \cap X =^* \emptyset$.

There exists n_0 and $\mathcal{B}' \subseteq \mathcal{B}$ uncountable, such that $b \setminus X \subseteq n_0$ for all $b \in \mathcal{B}'$. Similarly, there exists n_1 and $\mathcal{C}' \subseteq \mathcal{C}$ uncountable, such that $c \cap X \subseteq n_1$ for all $c \in \mathcal{C}'$. For $m = \max\{n_0, n_1\}$, it follows that $(\bigcup \mathcal{B}') \cap (\bigcup \mathcal{C}') \subseteq m$.

We claim that, since both $\mathcal{B}'$ and $\mathcal{C}'$ are uncountable, there exists an $a_\alpha \in \mathcal{B}'$ such that the set $\{\beta < \alpha : a_\beta \in \mathcal{C}'\}$ is infinite:

Consider an increasing sequence $(\beta_n)_{n \in \omega}$ in ω_1 such that $a_{\beta_n} \in \mathcal{C}'$. Since ω_1 is regular and $\mathcal{B}'$ is uncountable, there exists an $\alpha \in \omega_1$ such that $\alpha > \sup\{\beta_n : n \in \omega\}$ and $a_\alpha \in \mathcal{B}'$.

Notice that, since $\{\beta < \alpha : a_\beta \in \mathcal{C}'\}$ is infinite, for one of these β , $a_\beta \cap a_\alpha \not\subseteq m$, which a contradiction. $\square$

The previous theorem yields the following definition:

Definition 5.1.14. An almost disjoint family $\mathcal{A}$ is a Luzin family (or a Luzin gap) if it can be enumerated as $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ such that:

$$\forall n \in \omega, \forall \alpha < \omega_1 : \quad \{\beta < \alpha : a_\alpha \cap a_\beta \subseteq n\} \text{ is finite.}$$

It follows from the Proposition 5.1.12 that Mrówka-Isbell spaces associated with Luzin families are non-normal, showing that the converse in Corollary 5.1.9 is not true.

To state a new characterization of normality in Ψ -spaces, we need the following terminology, from [HHH05]:

Definition 5.1.15. The set 2^ω is endowed with the product topology, i.e., the topology with basic open sets $[s] = \{x \in 2^\omega : s \subseteq x\}$, where $s \in 2^{<\omega}$.

Given $X \subseteq 2^\omega$, the almost disjoint family over $2^{<\omega}$ induced by X is the family:

$$\mathcal{A}_X = \{A_x : x \in X\},$$

where $A_x = \{x|_n : n \in \omega\}$ for each $x \in X$.

Given $F \subseteq 2^\omega$, $\hat{F} = \{x|_n : n \in \omega, x \in F\}$ and we say that an uncountable set $Y \subseteq 2^\omega$ is a Q -set iff every subset of Y is F_σ in Y .

The next proposition is a folklore result that can be found in [HHH05] and we give the full proof of this fact for the sake of completeness:

Proposition 5.1.16. Given an uncountable $X \subseteq 2^\omega$, $\Psi(\mathcal{A}_X)$ is normal iff X is a Q -set.

Proof. ($\implies$) Let $F \subseteq X$ and consider $\mathcal{B} = \{A_x \in \mathcal{A}_X : x \in F\}$. By Proposition 5.1.12, there exists $J \subseteq 2^{<\omega}$ such that for each $A_x \in \mathcal{B}$, $A_x \subseteq^* J$ and for each $A_y \in \mathcal{A}_X \setminus \mathcal{B}$, $A_y \cap J =^* \emptyset$. It follows that:

$$\mathcal{B} = \{A_x \in \mathcal{A}_X : A_x \subseteq^* J\} = \bigcup_{n \in \omega} \bigcap_{m \geq n} \{A_x \in \mathcal{A}_X : x|_m \in J\}.$$

Thus, it follows that F is F_σ in X because:

$$F = \{x \in X : A_x \in \mathcal{B}\} = \bigcup_{n \in \omega} \bigcap_{m \geq n} \underbrace{\{x \in X : x|_m \in J\}}_{\text{closed in } X}.$$

($\Leftarrow$) Let $\mathcal{B} \subseteq \mathcal{A}_X$ and consider $\tilde{B} = \{x \in X : A_x \in \mathcal{B}\}$. Since X is a Q -set, there exist closed subsets F_n and G_n of X such that $\tilde{B} = \bigcup_{n \in \omega} F_n$ and $X \setminus \tilde{B} = \bigcup_{n \in \omega} G_n$. We proceed with a standard shoelace argument, defining $J_0 = \hat{F}_0$, $K_0 = \hat{G}_0 \setminus \hat{F}_0$, $J_n = \hat{F}_n \setminus (\bigcup_{i < n} \hat{G}_i)$, $K_n = \hat{G}_n \setminus (\bigcup_{i \leq n} \hat{F}_i)$. Let $J = \bigcup_{n \in \omega} J_n$. It follows that $J \cap K_m = \emptyset$ for all $m \in \omega$ and we claim that J is a partitioner.

If $A_x \in \mathcal{B}$ there exists $n \in \omega$ such that $x \in F_n$. Since $\bigcup_{i < n} G_i$ is closed in X and $\bigcup_{i < n} G_i \cap \tilde{B} = \emptyset$, there exists $k \in \omega$ such that $[x]_k \cap \bigcup_{i < n} G_i = \emptyset$. Thus $A_x \subseteq^* J_n \subseteq J$. Similarly, if $A_x \in \mathcal{A}_X \setminus \mathcal{B}$, $A_x \cap J =^* \emptyset$.

By Proposition 5.1.12, it suffices to show that $\Psi(\mathcal{A}_X)$ is normal. $\square$

The existence of Q -sets is independent of and consistent with the Axioms of ZFC and is equivalent to the existence of an uncountable normal Ψ -space. We cite [HHH05] and [FM80] as references for this field of study.

5.2 Weaker forms of normality in Ψ -spaces

In the previous section were given basic results about almost disjoint families and the class of Mrówka-Isbell topological spaces associated to them and we have explored classical results connecting combinatorial properties of $\mathcal{A}$ and the topological properties of $\Psi(\mathcal{A})$. In this section we provide an overview on different forms of normality that have risen in the study of Mrówka-Isbell spaces.

Weak forms of normality have been arisen in the literature in late 60's and recently much work has been done in this field of study. For instance, L. Kalantan builds a Ψ -space which is not mildly-normal in [Kal02] and, in [Kal08] under CH constructs a Mrówka-Isbell space that is quasi-normal. In [GBS20] S.Garcia and P. Szeptycki give a consistent example of a Luzin MAD family whose associated Ψ -space is quasi-normal.

We say that a subset A of a topological space X is regularly closed if $A = \text{cl}(\text{int}(A))$ and a set will be called π -closed if it is a finite intersection of regularly closed sets. Two subsets $A, B \subseteq X$ are said to be separated if there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Definition 5.2.1. A topological space X is said to be:

- π -normal if any two disjoint sets $A, B \subseteq X$, where A is a closed set and B is π -closed set, are separated;
- almost-normal if any two disjoint sets $A, B \subseteq X$, where A is a closed set and B is a regularly closed set, are separated;
- quasi-normal if any two disjoint π -closed sets $A, B \subseteq X$ are separated;
- partly-normal if any two disjoint sets $A, B \subseteq X$, where A is a regularly closed set and B is a π -closed set, are separated;
- mildly-normal if any two disjoint regularly closed sets $A, B \subseteq X$ are separated;
- semi-normal if for every closed set F and every open set U containing F there exists a regularly open set V such that $F \subseteq V \subseteq U$.

In [GBS20] the authors prove that π -normality and almost-normality are equivalent conditions. Furthermore, we have the following chain of implications:

$$\text{normal} \implies \text{almost-normal} \implies \text{quasi-normal} \implies \text{partly-normal} \implies \text{mildly-normal}.$$

We also state the following result, that relates some of the normality concepts previously defined:

Proposition 5.2.2 ([SA70]). *A topological space is normal iff it is almost-normal and semi-normal.*

In the view of the previous proposition, the study of semi-normality and almost-normality in Mrówka-Isbell spaces ought to be useful to better comprehend normality in such spaces. From now on, we will focus our attention to semi-normality property in Ψ -spaces, presenting some of the results of [RR20]. We also recommend this article, and the others previously cited, as a source of results regarding almost normality in Mrówka-Isbell spaces.

5.3 Semi-normality in Isbell-Mrówka spaces

Semi-normality can be translated in combinatorial terms for Mrówka-Isbell spaces. In the end, it follows that semi-normality is equivalent to a weaker form of separation, which was considered by Dow in [Dow97] and Brendle in [Bre99] and we state next:

Definition 5.3.1. Let $\mathcal{A}$ an almost disjoint family (over N) and two subfamilies $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, we say that a set $X \subseteq N$ weakly separates $\mathcal{B}$ and $\mathcal{C}$ if for all $b \in \mathcal{B}$ and $c \in \mathcal{C}$, $|X \cap b| < \omega$ and $|X \cap c| = \omega$.

We say that $\mathcal{A}$ is weakly separated if for every $\mathcal{B} \subseteq \mathcal{A}$, the pair $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be weakly separated.

Now we are ready to present an original result regarding the combinatorial characterization of semi-normality in Isbell-Mrówka spaces. (2) is a combinatorial property that looks like semi-normality. This is one of the main contributions of the paper [RR20]:

Proposition 5.3.2. *Let $\mathcal{A}$ be an almost disjoint family. The following are equivalent:*

- (1) $\Psi(\mathcal{A})$ is semi-normal;
- (2) For each $\mathcal{B} \subseteq \mathcal{A}$ and each $W \subseteq N$ such that $b \subseteq^* W$ for all $b \in \mathcal{B}$, there exists $W_0 \subseteq W$ satisfying the following:

$$\text{for all } a \in \mathcal{A} : \quad a \in \mathcal{B} \iff a \subseteq^* W_0. \quad (5.3.1)$$

- (3) $\mathcal{A}$ is weakly separated.

Proof. (1) $\implies$ (2): Fix $\mathcal{B} \subseteq \mathcal{A}$ and $W \subseteq N$ such that each $b \in \mathcal{B}$, $b \subseteq^* W$. Since $\mathcal{B}$ is closed and $\mathcal{B} \cup W$ is open, there exists a regular open set such that $\mathcal{B} \subseteq V \subseteq \mathcal{B} \cup W$. We claim that (5.3.1) holds for $W_0 = V \cap N$:

If $b \in \mathcal{B}$, then $b \in V$, so $b \subseteq^* W_0$. On the other hand, if $b \in \mathcal{A}$ is such that $b \subseteq^* W_0$, then $b \in \text{cl}(W_0) = \text{cl}(V)$. Since $b \subseteq^* V$, it follows that $b \in \text{int}(\text{cl}(V)) = V \subseteq \mathcal{B} \cup W$, hence $b \in \mathcal{B}$.

(2) $\implies$ (3): Fix $\mathcal{B} \subseteq \mathcal{A}$. By hypothesis, there exists $W_0 \subseteq N$ satisfying (5.3.1). We claim that $X = N \setminus W_0$ weakly separates $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$. Indeed, if $b \in \mathcal{B}$ then $b \subseteq^* W_0$, so $b \cap (N \setminus W_0) = b \cap X$ is finite. On the other hand, if $a \in \mathcal{A} \setminus \mathcal{B}$, then $a \setminus W_0 = a \cap X$ is infinite since $a \not\subseteq^* W_0$.

(3) $\implies$ (1): Let F closed and U open such that $F \subseteq U$, then $F = \mathcal{B} \cup K$ where $\mathcal{B} = F \cap \mathcal{A}$ and $K = F \cap N$. By hypothesis, there exists $X \subseteq N$ such that for each $b \in \mathcal{B}$, $|b \cap X| < \omega$ and for each $a \in \mathcal{A} \setminus \mathcal{B}$, $|a \cap X| = \omega$. Consider $W = U \cap N$ and let $V = \mathcal{B} \cup K \cup (W \setminus X)$, then $F \subseteq V \subseteq U$. We claim that V is a regular open set.

Clearly, V is open. For the regularity, let $x \in \text{int}(\text{cl}(V))$. If $x \in N$, then $x \in K \cup (W \setminus X) \subseteq V$. If $x \in \mathcal{A}$, then $x \subseteq^* \text{cl}(V)$ and it follows that $x \subseteq^* K \cup (W \setminus X)$. In the case of $x \cap K$ is infinite, $x \in F \subseteq V$ since F is closed, otherwise $x \subseteq^* (W \setminus X)$ and it follows that $x \cap X$ is finite, thus $x \in \mathcal{B} \subseteq V$. $\square$

Definition 5.3.3. A subset $\mathcal{A}$ of $[\omega]^\omega$ is said to be centered (or to have the strongly finite intersection property, SFIP) if every finite subset of $\mathcal{A}$ has infinite intersection. An infinite set X is called a pseudointersection of $\mathcal{A}$ if $X \subseteq^* A$ whenever $A \in \mathcal{A}$.

The pseudointersection number $\mathfrak{p}$ is defined as the least size of a centered subset of $[\omega]^\omega$ which does not admit a pseudointersection, more precisely:

$$\mathfrak{p} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ has the SFIP but no pseudo-intersection}\}.$$

It is well known that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{a}$ [Bla10]. Furthermore, in [Bre99], Brendle defines $\mathfrak{ap}$ as the smallest cardinal κ for which there exists an almost disjoint family $\mathcal{A}$ of size κ that is not weakly separated and observes that $\mathfrak{p} \leq \mathfrak{ap}$. The last inequality can be proved by applying the following result, that is stated as a conjunction of classic Solovay's Lemma ([MS70, Lemma 3]), that relies on $MA(\kappa)$, and Bell's theorem ([Bel81, Theorem 1.2]) that proves that $\mathfrak{p} = \mathfrak{m}_\sigma$ so it can be written in ZFC:

Proposition 5.3.4. *Given $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(N)$ such that $\max\{|\mathcal{C}|, |\mathcal{D}|\} < \mathfrak{p}$ and for all $x \in \mathcal{D}$ and $F \in [\mathcal{C}]^{<\omega}$, $|x \setminus \bigcup F| = \omega$. Then, there exists $d \subseteq N$, such that for each $x \in \mathcal{C}$, $|d \cap x| < \omega$ and for each $y \in \mathcal{D}$, $|d \cap y| = \omega$.*

Proof. To ease the burden on the reader, the proof can be found in [Kun80, Theorem 2.15], where it is defined a σ -centered order, so the hypothesis about $MA(\kappa)$ can be replaced by limiting the size of $\mathcal{C}$ and $\mathcal{D}$ by $\mathfrak{p}$, using [Kun11, Theorem III.3.61]. $\square$

Combining Proposition 5.3.2 and the inequality $\mathfrak{p} \leq \mathfrak{ap}$ gives us a family of uncountable almost disjoint families for which normality and almost-normality are the same:

Corollary 5.3.5. *If $\mathcal{A}$ is an almost disjoint family with $|\mathcal{A}| < \mathfrak{p}$, then $\mathcal{A}$ is semi-normal.*

Corollary 5.3.6. *If $\mathcal{A}$ is an almost disjoint family with $|\mathcal{A}| < \mathfrak{p}$, then $\mathcal{A}$ is normal iff $\mathcal{A}$ is almost-normal.*

In particular, assuming $\mathfrak{p} > \omega_1$, then no Luzin family is almost-normal.

Proposition 5.3.7. *If an almost disjoint family $\mathcal{A}$ is semi-normal, then $2^{|\mathcal{A}|} = \mathfrak{c}$. In particular, almost disjoint families of cardinality $\mathfrak{c}$ are not semi-normal.*

Proof. If $\mathcal{A}$ is semi-normal, then $\mathcal{A}$ is weakly separated by Proposition 5.3.2. But then we may inject $\mathcal{P}(\mathcal{A})$ into $\mathcal{P}(N)$ by letting, for each $\mathcal{B} \subseteq \mathcal{A}$, $X_{\mathcal{B}}$ be a subset of $\mathbb{N}$ such that, for all $a \in \mathcal{A}$, $|a \cap X_{\mathcal{B}}| = \omega$ iff $a \in \mathcal{B}$. $\square$

By Proposition 5.3.2, one can define (equivalently) $\mathfrak{ap}$ as the least cardinality of a non semi-normal almost disjoint family. In [Bre99], Brendle proved that $\mathfrak{ap} \leq \min\{\text{add}(\mathcal{M}), \mathfrak{q}\}$, where $\mathfrak{q}$ is the least cardinality of a subset of 2^ω which is not a Q -set and $\text{add}(\mathcal{M})$ is the least cardinality of a collection of meager subsets of $\mathbb{R}$ whose union is not meager. This inequality yields the following result:

Corollary 5.3.8. *Assuming $\text{add}(\mathcal{M}) = \omega_1$, there exists a non semi-normal almost disjoint family of size ω_1 .*

5.4 Strongly $(\aleph_0, < \mathfrak{c})$ -separated almost disjoint families

In this section of the work we will define a new topological separation property for Ψ -spaces that generalizes the concept of strongly $\aleph_0$ -separated almost disjoint family, defined in [GBS20], that is related to almost normality.

Definition 5.4.1. [GBS20] An almost disjoint family $\mathcal{A}$ is said to be strongly $\aleph_0$ -separated if for each pair of disjoint countable subfamilies $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, there exists a partitioner X for $\mathcal{B}$ and $\mathcal{C}$.

In [GBS20], they show that every almost-normal almost disjoint family is strongly $\aleph_0$ -separated and that strongly $\aleph_0$ -separated MAD families exist under CH. Although their paper does not investigate the converse, we are going to argue to be consistently false. For this purpose, first we need a stronger separation concept:

Definition 5.4.2. We say that an almost disjoint family $\mathcal{A}$ is strongly $(\aleph_0, < \mathfrak{c})$ -separated iff for every two disjoint subfamilies $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, with $\mathcal{B}$ countable and $|\mathcal{C}| < \mathfrak{c}$, there exists a partitioner $X \subseteq \omega$ for $\mathcal{A}$ and $\mathcal{B}$.

Clearly, every strongly $(\aleph_0, < \mathfrak{c})$ -separated almost disjoint family is strongly $\aleph_0$ -separated and, under CH, these concepts are equivalent.

Definition 5.4.3. Given $f, g \in \omega^\omega$, we say that g dominates f , denoted by $f <^* g$, when the set $\{n \in \omega : f(n) \geq g(n)\}$ is finite. A family $\mathcal{B} \subseteq \omega^\omega$ is said to be unbounded if for every $f \in \omega^\omega$ there exists $g \in \mathcal{B}$ such that $g \not<^* f$. The bounding number $\mathfrak{b}$ is defined as follows:

$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \omega^\omega \text{ is unbounded}\}.$$

Definition 5.4.4. We say that a set $X \subseteq \omega$ splits an infinite set $Y \subset \omega$ if both $X \cap Y$ and $Y \setminus X$ are infinite. A family $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is a splitting family if each $X \in [\omega]^\omega$ is split by at least one $A \in \mathcal{S}$. The splitting number $\mathfrak{s}$ is defined as follows:

$$\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq \mathcal{P}(\omega) \text{ is splitting}\}.$$

It is well known that $\mathfrak{p} \leq \mathfrak{s} \leq \mathfrak{c}$ and that $\mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{a}$, and all inequalities are consistent to be strict [Bla10]. Now we are ready to study the relations between the $(\aleph_0, < \mathfrak{c})$ -separation property and the cardinal characteristics defined throughout this chapter.

In [vD84] it is proved that $\mathfrak{b}$ is the least size of a non pseudonormal first countable topological space (Theorem 12.2) and that there exists a Ψ -space of size $\mathfrak{b}$ that is not pseudonormal (Proposition 11.4(c)). We have proved this fact independently in [RR20] and we state the result here as well for the sake of completeness:

Lemma 5.4.5. *Let $\mathcal{A}$ an almost disjoint family with $|\mathcal{A}| < \mathfrak{b}$. If $\mathcal{B} \subset \mathcal{A}$ is a countable set, then there exists a partitioner for $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$. In particular, $\mathcal{A}$ is strongly $(\aleph_0, < \mathfrak{c})$ -separated.*

Proof. Let $\mathcal{B} = \{b_n : n \in \omega\}$ list all elements of $\mathcal{B}$. For each $a \in \mathcal{A} \setminus \mathcal{B}$, consider the function $f_a \in \omega^\omega$ defined by $f_a(n) = \sup(a \cap b_n)$. Since $\mathcal{F} = \{f_a \in \omega^\omega : a \in \mathcal{A} \setminus \mathcal{B}\}$ is family of functions with $|\mathcal{F}| < \mathfrak{b}$, there exists $g \in \omega^\omega$ such that $f_a <^* g$, for all $a \in \mathcal{A} \setminus \mathcal{B}$.

Let $X = \bigcup_{n \in \omega} (b_n \setminus g(n))$. We claim that X is a separator for $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$.

Clearly, we have that $b_n \subseteq^* X$, for all $n \in \omega$. Given $a \in \mathcal{A} \setminus \mathcal{B}$, since $g >^* f_a$, there exists $k \in \omega$ such that $b_n \setminus g(n) = \emptyset$ for all $n \geq k$, thus $X \cap a =^* \emptyset$. $\square$

Corollary 5.4.6. *If $\mathfrak{p} > \omega_1$, there exists a strongly $(\aleph_0, < \mathfrak{c})$ -separated almost disjoint family which is not almost-normal.*

Proof. By Corollary 5.3.6 and Lemma 5.4.5, every non normal almost disjoint family of size ω_1 is strongly $(\aleph_0, < \mathfrak{c})$ -separated and is not almost-normal. So consider any non-normal almost disjoint family of size ω_1 (e.g. any Luzin family). $\square$

Definition 5.4.7 ([GS07]). Given an almost disjoint family $\mathcal{A}$, we define the family of the positive sets in respect to $\mathcal{A}$ as follows:

$$\mathcal{J}^+(\mathcal{A}) = \{X \in \omega : |\{a \in \mathcal{A} : |a \cap X| = \omega\}| \geq \omega\}.$$

An almost disjoint family is said to be completely separable when, for each $X \in \mathcal{J}^+(\mathcal{A})$ there exists $a \in \mathcal{A}$ such that $a \subseteq X$.

The reader may notice that the family of big sets $\mathcal{J}^+(\mathcal{A})$ is the family of stationary sets induced by the ideal $\mathcal{J}(\mathcal{A}) = \{X \in \omega : |\{a \in \mathcal{A} : |a \cap X| = \omega\}| < \omega\}$. Moreover, completely separable almost disjoint families exist in ZFC [GS07], however, although they exist in most models [Hru14], it remains open if completely separable MAD families exist in ZFC.

Definition 5.4.8. [GGHMRG17] Let P be a property of MAD families. We say that MAD families with property P exist generically if all almost disjoint families of size less than $\mathfrak{c}$ can be extended to a MAD family with the property P .

Now we have all the tools needed to modify the technique presented in [GBS20, Proposition 3.3] to be able to weaken the CH hypothesis:

Theorem 5.4.9 ($\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$). *Completely separable MAD families which are strongly $(\aleph_0, < \mathfrak{c})$ -separated exist generically.*

Proof. Let $\mathcal{A}'$ be an infinite almost disjoint family of size $\kappa < \mathfrak{c}$ and write $\mathcal{A}' = \{a_\gamma : \gamma < \kappa\}$ so that $a_\gamma \neq a_\nu$ whenever $\gamma \neq \nu$.

Let $\{B_\beta \in [\mathfrak{c}]^\omega : \kappa \leq \beta < \mathfrak{c}\}$ list all countable subsets of $\mathfrak{c}$ such that for each β , $B_\beta \subseteq \beta$ and, for all $B \in [\mathfrak{c}]^\omega$, $|\{\beta : B_\beta = B\}| = \mathfrak{c}$ and list $[\omega]^\omega = \{Y_\alpha : \kappa \leq \alpha < \mathfrak{c}\}$.

We will define recursively almost disjoint families $\mathcal{A}_\alpha$, $a_\alpha \in [\omega]^\omega$ and $X_\alpha \subseteq \omega$, for $\kappa \leq \alpha < \mathfrak{c}$ such that:

- (1) $\mathcal{A}_\beta = \{a_\gamma : \gamma < \beta\}$;
- (2) $\mathcal{A}_\kappa = \mathcal{A}'$.
- (3) $\forall \beta < \mathfrak{c} : \forall \gamma \in B_\beta, a_\gamma \subseteq^* X_\beta$;
- (4) $\forall \beta < \mathfrak{c} : \forall \gamma \in \beta \setminus B_\beta, a_\gamma \cap X_\beta =^* \emptyset$;
- (5) $\forall \gamma < \mathfrak{c} : \forall \beta \leq \gamma, a_\gamma \subseteq^* X_\beta$ or $a_\gamma \cap X_\beta =^* \emptyset$;
- (6) if $Y_\beta \in \mathcal{J}^+(\mathcal{A}_\beta)$, $a_\beta \subseteq Y_\beta$ is an infinite subset.
- (7) $\forall \eta < \gamma < \mathfrak{c} : a_\eta \cap a_\gamma$ is finite.

Fix $\alpha < \mathfrak{c}$ and suppose that X_β and a_β are defined for $\kappa \leq \beta < \alpha$. Since B_α is countable and $|\alpha| < \mathfrak{c} = \mathfrak{b}$, using Lemma 5.4.5, let $X_\alpha \subseteq \omega$ be a partitioner for $\{a_\xi : \xi \in B_\alpha\}$ and $\{a_\xi : \xi \in \alpha \setminus B_\alpha\}$.

To define a_α , notice that since $|\alpha| < \mathfrak{a}$ there exists an infinite $Y \subseteq \omega$ almost disjoint from a_β , for all $\beta < \alpha$. In addition, if $Y_\alpha \in \mathcal{J}^+(\mathcal{A}_\alpha)$, we can take $Y \subseteq Y_\alpha$:

Indeed, if $\{\gamma < \alpha : |a_\gamma \cap Y_\alpha| = \omega\}$ is finite, take $Y \doteq Y_\alpha \setminus \bigcup \{a_\gamma : \gamma < \alpha \wedge |a_\gamma \cap Y_\alpha| = \omega\}$. Otherwise, note that $\mathcal{B} = \{a_\gamma \cap Y_\alpha : \gamma < \alpha \wedge |a_\gamma \cap Y_\alpha| = \omega\}$ is an almost disjoint family in Y_α . Since $|\mathcal{B}| < \mathfrak{a}$, there exists $Y \subseteq Y_\alpha$ almost disjoint from each element of $\mathcal{B}$.

Since $\mathfrak{s} = \mathfrak{c}$, $\{X_\gamma \cap Y : \gamma \leq \alpha\}$ is not a splitting family in Y . Thus, there exists $a_\alpha \subseteq Y$ such that for all $\gamma \leq \alpha$, $a_\alpha \cap X_\gamma =^* \emptyset$ or $a_\alpha \subseteq^* X_\gamma$.

Notice that $\mathcal{A}$ is an almost disjoint family extending $\mathcal{A}'$ by (2) and (7)

We show that for every infinite $Y \subseteq \omega$, either $Y \notin \mathcal{J}^+(\mathcal{A})$ or there exists $\alpha < \mathfrak{c}$ such that $a_\alpha \subseteq Y$, thus proving $\mathcal{A}$ is MAD and completely separable. If $Y \in \mathcal{J}^+(\mathcal{A})$, let α be such that $Y = Y_\alpha$. Then $Y \in \mathcal{J}^+(\mathcal{A}_\alpha)$, thus, by (6), $a_\alpha \subseteq Y_\alpha$.

Finally, we prove that $\mathcal{A}$ is $(\aleph_0, < \mathfrak{c})$ -separated. Given an infinite countable set $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{C} \in [\mathcal{A}]^{< \mathfrak{c}}$ such that $\mathcal{B} \cap \mathcal{C} = \emptyset$, let $B = \{\alpha < \mathfrak{c} : a_\alpha \in \mathcal{B}\}$ and $C = \{\alpha < \mathfrak{c} : a_\alpha \in \mathcal{C}\}$. Notice that B is infinite and countable, $|C| < \mathfrak{c}$, and $B \cap C = \emptyset$. Let $\alpha_0 = \sup C$, which is less than $\mathfrak{c}$ since $\mathfrak{c} = \mathfrak{b}$ is regular, and let $\alpha > \alpha_0$ be such that $B_\alpha = B$. In particular, $B = B_\alpha \subseteq \alpha$ and $C \subseteq \alpha \setminus B_\alpha$. By (3), for all $b \in \mathcal{B}$, $b \subseteq^* X_\alpha$.

By (4), for all $c \in \mathcal{C}$, $c \cap X_\alpha =^* \emptyset$

By (3) and (4) together by using α in the place of β , we see that for every $\gamma < \alpha$, $a_\gamma \subseteq^* X_\alpha$ or $a_\gamma \cap X_\alpha =^* \emptyset$. If $\gamma > \alpha$, we apply (5) for this γ and α in the place of β to conclude that $a_\gamma \subseteq^* X_\alpha$ or $a_\gamma \cap X_\alpha =^* \emptyset$. $\square$

It is well known that the cardinal characteristic $\mathfrak{par}$, as defined in [Bla10], equals the minimum of $\mathfrak{b}, \mathfrak{s}$. Thus, the previous theorem could have its hypothesis replaced by $\mathfrak{par} = \mathfrak{c}$.

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