**Genericity of Bumpy Metrics, Bifurcation and
Stability in Free Boundary CMC Hypersurfaces**

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Genericidade das Métricas Bumpy, Bifurcação e Estabilidade em Hipersuperfícies de CMC e Fronteira Livre

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Abstract

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In this thesis we prove the genericity of the set of metrics on a manifold with boundary M^{n+1} , such that all free boundary constant mean curvature (CMC) embeddings $\varphi : \Sigma^n \rightarrow M^{n+1}$, being Σ a manifold with boundary, are non-degenerate (Bumpy Metrics), (Theorem 2.4.1). We also give sufficient conditions to obtain a free boundary CMC deformation of a CMC immersion (Theorems 3.2.1 and 3.2.2), and a stability criterion for this type of immersions (Theorem 3.3.3 and Corollary 3.3.5). In addition, given a one-parametric family, $\{\varphi_t : \Sigma \rightarrow M\}_{t \in I}$, of free boundary CMC immersions, we give criteria for the existence of smooth bifurcated branches of free boundary CMC immersions for the family $\{\varphi_t\}$, via the implicit function theorem when the kernel of the Jacobi operator J is non-trivial, (Theorems 4.2.3 and 4.3.2), and we study stability and instability problems for hypersurfaces in this bifurcated branches (Theorems 5.3.1 and 5.3.3).

Keywords: Constant Mean Curvature, Free Boundary, Bumpy Metrics, Jacobi Operator, Stability, Bifurcation.

Resumo

RODRÍGUEZ, C. **Genericidade das Métricas Bumpy, Bifurcação e Estabilidade em Hipersuperfícies de CMC e Fronteira Livre**. 2018. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2018.

Nesta tese, provamos a genericidade do conjunto de métricas em uma variedade com fronteira M^{n+1} , de modo que todos os mergulhos de curvatura média constante (CMC) e fronteira livre $\varphi : \Sigma^n \rightarrow M^{n+1}$, sendo Σ uma variedade com fronteira, sejam não-degenerados (Métricas Bumpy), (Teorema 2.4.1). Nós também fornecemos condições suficientes para obter uma deformação CMC e fronteira livre de uma imersão CMC (Teoremas 3.2.1 and 3.2.2), e um critério de estabilidade para este tipo de imersões (Teorema 3.3.3 and Corolário 3.3.5). Além disso, dada uma família 1-paramétrica, $\{\varphi_t : \Sigma \rightarrow M\}_{t \in I}$, de imersões de CMC e fronteira livre, damos os critérios para a existência de ramos de bifurcação suaves de imersões CMC e fronteira livre para a família $\{\varphi_t\}$, por meio de o teorema da função implícita quando o kernel do operador Jacobi J é não-trivial, (Teoremas 4.2.3 and 4.3.2), e estudamos o problema da estabilidade e instabilidade para hipersuperfícies em naqueles ramos de bifurcação (Teoremas 5.3.1 and 5.3.3).

Palavras-chave: Curvatura Média Constante, Fronteira Livre, Métricas Bumpy, Operador de Jacobi, Estabilidade, Bifurcação.

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Introduction

In variational calculus there exists a class of problems called *isoperimetric*; one of them is to find the minimum area among all hypersurfaces of a Riemannian manifold enclosing the same volume. We know that solutions to this problem are hypersurfaces with constant mean curvature (in short CMC). More precisely, if $\varphi : \Sigma \rightarrow M$ is an immersion of an orientable n -dimensional compact manifold Σ into the $(n+1)$ -dimensional Riemannian manifold M , the condition that φ has nonzero constant mean curvature H_0 is equivalent to the fact that φ is a critical point of such isoperimetric problem (see [6]).

If φ_t is a smooth variation of φ , $t \in (-\epsilon, \epsilon)$, $\varphi_0 = \varphi$, such that $V_t = V_0$, for all $t \in (-\epsilon, \epsilon)$, where V_t is the volume of φ_t , a standard approach to find the solution of such a isoperimetric problem is by looking at the critical points of functional $f(t) = A_t + \lambda V_t$, where A_t is the area of φ_t and λ is constant, which is analogous to Lagrange multipliers method. When $\lambda = nH_0$, we have the equivalence aforementioned.

In the case where M is a manifold with boundary, the isoperimetric problem can be described as follows. One wants to minimize the area between all compact hypersurfaces in M with boundary contained in the boundary of M and whose interior lies in the interior of M ($\varphi(\Sigma) \cap \partial M = \varphi(\partial \Sigma)$), which divides M in two regions such that the closure of one of them is compact and with prescribed volume. Solutions of this problem are the orthogonal constant mean curvature hypersurfaces, which will be called *free boundary CMC hypersurfaces*. Let H_0 denote the value of the (constant) mean curvature of one such hypersurface. If $H_0 = 0$ then we say that $\varphi(\Sigma)$ is a *orthogonal free boundary minimal hypersurface*. A. Ros and E. Vergasta obtain results on the stability of solutions of this isoperimetric problem in the case where M is compact and convex, see [24]. There are well-known results obtained by Abraham [1] and Anosov [5] which are related to properties of geodesic flows for generic Riemannian metrics on a closed smooth manifold.

In this dissertation we study four problems related to free boundary CMC hypersurfaces: i) The genericity of the metrics that makes these hypersurfaces are non-degenerate critical points of the functional area (Chapter 2); ii) The existence and stability of variations of this type of hypersurfaces (Chapter 3); iii) The existence of branches of bifurcation in families of immersions with these characteristics (Chapters 4); and iv) The stability in those branches of bifurcation (Chapters 5). We now turn to explain the content of each chapter and the most important results obtained.

In Chapter 1 we define the objects and notation that we will use. We are going to give some basic definitions; some results about the function spaces, operators and applications that are necessary for the work. In particular, given a orthogonal CMC immersion $\varphi_0 : \Sigma^n \rightarrow M^{n+1}$, we define

the Banach space of the Hölder functions in Σ , with regularity $C^{j,\alpha}$, and satisfying $g(\nabla f, \vec{\eta}_{\partial M}) + \mathbb{I}^{\partial M}(\vec{\eta}_{\Sigma_0}, \vec{\eta}_{\Sigma_0})f = 0$, where g is a background riemannian metric in M , ∇ is the gradient in Σ defined by the pullback metric of g generated by φ_0 , $\vec{\eta}_{\partial M}$ and $\vec{\eta}_{\Sigma_0}$ are the orthonormal vectors of ∂M and Σ_0 respectively, and $\mathbb{I}^{\partial M}$ the second fundamental form of the boundary of M , we denote as $C_{\partial}^{j,\alpha}(\Sigma)$. There is an identification of $C_{\partial}^{j,\alpha}(\Sigma)$ with the set of orthogonal immersions in M , $f \mapsto \varphi_f$, defined by $\varphi_f(p) := \exp_{\varphi_0(p)}(f(p)\vec{\eta}_{\Sigma_0}(p))$, where $\exp$ is the exponential map in M defined by the metric g (see Proposition 2.3.1). We will define the Jacobi Operator of φ_0 as $J_{\varphi_0}(f) = \Delta_{\Sigma_0}f - (||\mathbb{I}^{\Sigma_0}||_{HS}^2 + Ric_g(\eta_{\Sigma_0}, \eta_{\Sigma_0}))f$, where $\varphi_0(\Sigma) = \Sigma_0$, which, restricted to space $C_{\partial}^{j,\alpha}(\Sigma)$, is a Fredholm operator of zero index, extremely important condition in the development of all this work. Also, we give a characterization of free boundary CMC hypersurfaces non-degenerate in terms of the Jacobi Operator kernel.

In Chapter 2, we prove the genericity of Riemannian metrics of M for which every free boundary orthogonal minimal immersion $\varphi : \Sigma^n \rightarrow M^{n+1}$ is non-degenerate, which we will call (M, Σ) -bumpy metrics. The group of diffeomorphisms of Σ acts freely on the set of embeddings of Σ into M . We denote by $[\varphi]$ its equivalence class with respect to this action and as $\mathcal{E}_{\partial, \gamma}^{\perp}(\Sigma, M)$ the set of $[\varphi]$ such that $\varphi(\Sigma) \cap \partial M = \varphi(\partial \Sigma)$ and φ is an γ -orthogonal embedding with CMC. So, our main theorem is as follows:

Theorem 2.4.1 *Let M^{n+1} be a differential manifold with smooth boundary $\partial M \neq \emptyset$, and Σ^n a compact differential manifold with smooth boundary $\partial \Sigma \neq \emptyset$. $Met^k(M)$ the set of all C^k metric tensors in M , $k \geq 2$, $\Gamma \subset Met^k(M)$ be an open subset with a structure of separable Banach space. We define the following set*

$$\mathcal{M} = \{(\gamma, [\varphi]) \in \Gamma \times \mathcal{E}_{\partial}(\Sigma, M) : [\varphi] \in \mathcal{E}_{\partial, \gamma}^{\perp}(\Sigma, M), \varphi \text{ is } \gamma\text{-minimal}\}.$$

Then,

- (1) $\mathcal{M}$ is a separable Banach manifold modelled on Γ .
- (2) $\Pi : \mathcal{M} \rightarrow \Gamma$, defined by $\Pi(\gamma, [\varphi]) = \gamma$, is a Fredholm map with index 0.
- (3) γ_0 is critical value of Π if and only if there is a γ_0 -minimal embedding $\varphi_0 : \Sigma \rightarrow M$ which is degenerate.

In (2) we see that Π meets the conditions of the Sard-Smale theorem (see [26]), which we use in (3) to test the genericity of regular points of Π . We note that, when M is noncompact, $Met^k(M)$ has no Banach space structure; so, we have to choose a suitable subset $\Gamma \subset Met^k(M)$ where a Banach structure can be found. Typically, Γ consist of metrics satisfying some growth control at infinity (see definition 2.2.1). This theorem is the version for minimal hypersurfaces with free boundary analogous to results of Brian White [28], and Biliotti-Javaloyes-Piccione [9]. This result is valid also for the case where the mean curvature is a non-zero constant (see section 2.4.3).

In Chapter 3 we find results concerning to existence and uniqueness of CMC deformation of free boundary CMC immersions, $\{\varphi_t\}_{t \in I}$, in addition to some criteria of stability for this type of deformations. Within the Theory of Optimization and Stability in different areas of science and technology is important criteria that serves as tools for study of phenomena such as energy conservation, minimization of materials, optimization of resources, etc. The term stability refers

to the fact that surfaces with constant mean curvature must minimize the area of enclosed volume among all nearby surfaces that enclose the same volume. A hypersurface with constant mean curvature is stable if the second variation of the functional area is nonnegative. This is a version for free boundary CMC hypersurfaces in n -dimension obtained by Miyuki Koiso (see [16]), who study the case for surfaces in $\mathbb{R}^3$ with fixed boundary.

To obtain these results we have based on the properties of the eigenvalues and eigenfunctions of the Jacobi Operator J_{φ_0} , which is interpreted as the linearization of the mean curvature H_{φ_t} at φ_0 (see Section 1.3). So, the first perturbation existence theorem, when the eigenvalues of J_{φ_0} are non-zero, is the following:

Theorem 3.2.1. *Let $\varphi_0 \in C^{j+1,\alpha}(\Sigma, M)$ a free boundary CMC immersion, with mean curvature H_0 . Suppose that $\dim(\ker(J_{\varphi_0})) = 0$, this is, the eigenvalues of problem $J_{\varphi_0}(f) = \lambda f$, $f \in C_{\partial}^{j,\alpha}(\Sigma_0)$, are nonzero. Then, there is a neighborhood $\hat{I}$ of $H_0 \in \mathbb{R}$ and a unique injective C^1 mapping, $\zeta : \hat{I} \rightarrow C_{\partial}^{j,\alpha}(\Sigma_0)$, such that $\zeta(H_0) = 0$ and $\varphi_{\zeta(H)}$ is a free boundary CMC immersion with mean curvature H .*

Moreover, if $\psi : \Sigma \rightarrow M$ is an free boundary CMC immersion sufficiently close to φ_0 , in the topology of $C^{j,\alpha}$, then ψ must be equal (up to diffeomorphisms) to some $\varphi_{\zeta(H)}$.

Now, if J_{φ_0} has some eigenvalue equal to zero, then we have the following theorem:

Theorem 3.2.2 *Let $\varphi_0 \in C^{j+1,\alpha}(\Sigma, M)$ a free boundary CMC immersion, with mean curvature H_0 . Suppose that:*

- $\dim(\ker(J_{\varphi_0})) = 1$. This is, $\lambda = 0$ is an eigenvalue of multiplicity 1 for J_{φ_0} , and
- $\int_{\Sigma_0} f_0 \operatorname{vol}_{\varphi_0^*(g)} \neq 0$ for some $f_0 \in \ker(J_{\varphi_0}) - \{0\}$.

Then there exist a neighborhood $W \subset \ker(J_{\varphi_0})$ of 0 and a unique injective map C^1

$$(\xi, \eta) : W \longrightarrow (C_{\partial}^{j,\alpha}(\Sigma_0) \cap \ker(J_{\varphi_0})^{\perp}) \times \mathbb{R},$$

such that $(\xi, \eta)(0) = (0, H_0)$ and such that $\varphi_{f+\xi(f)} : \Sigma \rightarrow M$, with $f \in W$, is an free boundary CMC immersion, with mean curvature $\eta(f)$.

Moreover, if $\psi : \Sigma \rightarrow M$ is an free boundary CMC immersion sufficiently close to φ_0 , in the topology of $C^{j,\alpha}$, then Y must be equal (up to diffeomorphisms) to some $\varphi_{f+\xi(f)}$.

In both cases, we also obtain uniqueness in this perturbation, up to parameterizations.

Now, regarding the stability of the free boundary CMC hypersurfaces, we obtained the following result, where $\lambda_1 < \lambda_2 \leq \dots$ are the eigenvalues of J_{φ_0} (see remark 3.3.1):

Theorem 3.3.3. *Let $\varphi_0 : \Sigma \rightarrow M$ a free boundary CMC immersion. Let λ_i , $i \geq 1$ be the eigenvalues J_{φ_0} .*

- (1) *If $\lambda_1 \geq 0$, then φ_0 is stable.*
- (2) *If $\lambda_1 < 0 < \lambda_2$, then there is a unique function $\kappa \in C^{j,\alpha}(\Sigma)$ such that $J(\kappa) = 1$ and we have that:*
 - (2-a) *If $\int_{\Sigma} \kappa \operatorname{vol}_{\varphi_0^*(g)} \leq 0$, then φ_0 is stable.*
 - (2-b) *If $\int_{\Sigma} \kappa \operatorname{vol}_{\varphi_0^*(g)} > 0$, then φ_0 is unstable.*

(3) If $\lambda_1 < 0 = \lambda_2$, then we have:

(3-a) If there exist a λ_2 -eigenfunction f_2 such that $\int_{\Sigma} f_2 \operatorname{vol}_{\varphi_0^*(g)} \neq 0$, then φ_0 is unstable.

(3-b) If $\int_{\Sigma} h_2 \operatorname{vol}_{\varphi_0^*(g)} = 0$ for all λ_2 -eigenfunction h_2 , then there exist a unique function $\bar{h}_2 \in (\ker(J_{\varphi_0}))^{\perp}$ such that $J(\bar{h}_2) = 1$ and

(3-b-i) If $\int_{\Sigma} \bar{h}_2 \operatorname{vol}_{\varphi_0^*(g)} \leq 0$, then φ_0 is stable.

(3-b-ii) If $\int_{\Sigma} \bar{h}_2 \operatorname{vol}_{\varphi_0^*(g)} > 0$, then φ_0 is unstable

(4) If $\lambda_2 < 0$, then φ_0 is unstable.

For cases (2) and (3) the calculation of eigenvalues and integrals of the eigenfunctions can be difficult. In this sense, the following criterion gives us a geometric interpretation of stability based on the existence of a deformation family.

Corollary 3.3.5. *Let $\varphi : \Sigma \rightarrow M$ a free boundary CMC immersion of class $C^{j+1,\alpha}$. We assume that $\lambda_1 < 0 \leq \lambda_2$. If there exist a deformation φ_t of φ , $-\epsilon < t < \epsilon$, with $\varphi_0 = \varphi$, such that φ_t is a free boundary CMC immersion of class $C^{j,\alpha}$ for all $t \in (-\epsilon, \epsilon)$, and such that $\frac{dH_t}{dt}|_{t=0} = H'_0 = \text{constant} \neq 0$, where H_t is the constant mean curvature of φ_t and V_t is the volume of φ_t , we have that:*

(1) If $H'_0 V'_0 \leq 0$, then φ is stable,

(2) if $H'_0 V'_0 > 0$, then φ is unstable.

If there is no such deformation, then φ is unstable

In Chapter 4, we study the existence of bifurcation points in a 1-parametric family $\{\varphi_t\}_{-\epsilon < t < \epsilon}$ of free boundary CMC hypersurfaces. A bifurcation occurs when a small smooth change made to the parameter values of a system causes a sudden qualitative or topological change in its behaviour. Natural phenomena such as the buckling of the Euler rod, the appearance of Taylor vortex and the start of oscillations in an electric circuit, have a common cause: a specific physical parameter that exceeds a threshold, which forces the system to organize a new state that is considerably different from the one initially observed (see [15]). The Theory of Bifurcation explains there different phenomena. Examples of bifurcation appear experimentally long ago; Plateau's research on Delaunay surfaces shows how surfaces belonging to a certain family were spontaneously transformed into surfaces of an adjacent family when a certain stability limit was reached (see [23]). We prove the existence of smooth bifurcating families of free boundary CMC hypersurfaces from the family $\{\varphi_t\}_{-\epsilon < t < \epsilon}$. For the proof of our results we follow the the approach of Koiso-Palmer-Piccione [18], who made the version for surfaces in $\mathbb{R}^3$ with fixed boundary. We use the theory of stability for smooth variations of free boundary CMC hypersurfaces developed in Chapter 3. These results are based on the Abstract Bifurcation Theory of Crandall and Rabinowitz ([10] and [11]), which can be applied, under appropriate conditions, to produce a smooth family of free boundary CMC hypersurfaces via an Implicit Function Theorem when $\operatorname{Ker}(J_{\varphi}) \neq \{0\}$. Under the conditions of the main problem of Bifurcation Theory: Let F be a mapping of a subset of a Banach space W in a Banach space X and let $\alpha(t)$ be a curve in W such that $F(\alpha(t)) = 0$. Crandall and Rabinowitz imposed simple conditions in order to find a neighborhood U_p of p such that $F^{-1}(0) \cap U_p$, is topologically (or diffeomorphaly) equivalent to $(-1, 1) \times \{0\} \cup \{0\} \times (-1, 1)$. Also they study the behavior of

F in $F^{-1}(0) \cap U_p$ and the topological structure of the same set. This theory is developed in the framework of the so-called “Theory of Bifurcation from a simple eigenvalue” and offer a wide spectrum of examples, including some generalizations of known results related to problems of non-linear eigenvalues for partial and ordinary differential equations.

We present two results on the existence of bifurcations. The first result provides a smooth bifurcating family of free boundary CMC hypersurfaces, whose mean curvatures coincide with the mean curvatures of the original family, that is the bifurcation parameter is the mean curvature,

Theorem 4.2.3. *Let $\{\varphi_t\}_{-\epsilon < t < \epsilon}$ a family of free boundary CMC hypersurfaces. H_t the constant mean curvature of φ_t . $H'_0 \neq 0$. $\dim(\ker(J_{\varphi_0}|_{C^{j,\alpha}_{\partial}(\Sigma_0)})) = 1$. Then,*

- (1) $\int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} = 0$, $e \neq 0$, $e \in \ker(J_{\varphi_0}|_{C^{j,\alpha}_{\partial}(\Sigma_0)})$, and there exists a differentiable map $\lambda : (-\epsilon_0, \epsilon_0) \rightarrow \mathbb{R}$, $0 < \epsilon_0 \leq \epsilon$, such that $\lambda(0) = \lambda_0 = 0$, $\lambda(t) = \lambda_t$ is a simple eigenvalue of J_t , and there is no other eigenvalue of J_t near 0.
- (2) Assume further that $\lambda'(0) \neq 0$ holds. Then, φ_0 is a bifurcation point with respect to $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, where the bifurcation branch is an analytic family of free boundary CMC immersions. More precisely, there exist an open interval $\hat{I} \subset \mathbb{R}$, $0 \in \hat{I}$, and C^1 functions $\zeta : \hat{I} \rightarrow E^{\perp}$ and $t : \hat{I} \rightarrow \mathbb{R}$, such that $t(0) = 0$, $\zeta(0) = 0$, and

$$\psi_s = \exp_{\varphi_{t(s)}}([\phi_{t(s)} + se + s\zeta(s)]\vec{n}_{\varphi_{t(s)}(\Sigma)}),$$

is a free boundary CMC immersion with mean curvature $\hat{H}_s = H_{t(s)}$.

- (3) Every free boundary CMC immersion sufficiently close, in the topology of $C^{j,\alpha}$, to φ_0 , is equal, up to parameterization, to some element of families $\{\varphi_t\}_{t \in I}$, $0 \in I \subset (-\epsilon_0, \epsilon_0)$, or $\{\psi_s\}_{s \in \hat{I}}$. Furthermore, the surfaces $\{\varphi_t\}_{t \in I}$ and $\{\psi_s\}_{s \in \hat{I}}$ are pairwise distinct except for $\varphi_0 = \psi_0$.

The second result provides a smooth bifurcating branch of free boundary CMC hypersurfaces, whose volumes coincide with the volumes of surfaces in the original family, that is the bifurcation parameter is the volume.

Theorem 4.3.2. *Let $\{\varphi_t\}_{-\epsilon < t < \epsilon}$ a family of free boundary CMC hypersurfaces. H_t and $V(t)$ the constant mean curvature and the volume of φ_t respectively. $H'_0 \neq 0$ and $dV(0) \neq 0$. $\dim(\ker(J_{\varphi_0}|_{C^{j,\alpha}_{\partial}(\Sigma_0)})) = 1$. Then,*

- (1) $\int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} = 0$, $e \neq 0$, $e \in \ker(J_{\varphi_0}|_{C^{j,\alpha}_{\partial}(\Sigma_0)})$, and there exists a differentiable map $\tilde{\lambda} : (-\epsilon_0, \epsilon_0) \rightarrow \mathbb{R}$, $0 < \epsilon_0 \leq \epsilon$, such that $\tilde{\lambda}(0) = \tilde{\lambda}_0 = 0$, $\tilde{\lambda}(t) = \tilde{\lambda}_t$ is a simple eigenvalue of $\tilde{J}_t = \frac{\partial}{\partial h} \bar{F}(t, 0)$, and there is no other eigenvalue of $\tilde{J}_t$ near 0.
- (2) Assume further that $\frac{d\tilde{\lambda}_t}{dt}|_{t=0} = \tilde{\lambda}'_0 \neq 0$ holds. Then, φ_0 is a bifurcation point with respect to $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, where the bifurcation branch is a smooth family of free boundary CMC immersions. More precisely, there exist an open interval $\hat{I} \subset \mathbb{R}$, $0 \in \hat{I}$, and C^1 functions $\eta : \hat{I} \rightarrow B_0$ and $\tau : \hat{I} \rightarrow \mathbb{R}$, such that $\tau(0) = 0$, $\eta(0) = 0$, and

$$\psi_s = \text{Exp}_{\varphi_0}(f_{(\tau(s), se + s\eta(s))}\vec{n}_{\varphi_0}),$$

is a free boundary CMC immersion with volume $\tilde{V}(s) = V(\tau(s))$.

- (3) Every free boundary CMC immersion sufficiently close, in the topology of $C^{j,\alpha}$, to φ_0 , is equal, up to parameterization, to some element of families $\{\varphi_t\}_{t \in I}$, $0 \in I \subset (-\epsilon_0, \epsilon_0)$, or $\{\psi_s\}_{s \in \tilde{I}}$. Furthermore, the surfaces $\{\varphi_t\}_{t \in I}$ and $\{\psi_s\}_{s \in \tilde{I}}$ are pairwise distinct except for $\varphi_0 = \psi_0$.

In Chapter 5, we study stability of free boundary CMC hypersurfaces in the bifurcation branches that we presented in the previous chapter. Bifurcation for a family $\{\varphi_t\}$ only occurs in the case where $\lambda_k = 0$, for some $k \geq 2$, where $\{\lambda_k\}$ is the countable set of eigenvalues of J (see remark 3.3.1). Therefore, in view of results of Chapter 3, we only need to study the case where $\lambda_2 = 0$. If φ_0 is a stable bifurcation point in the family $\{\varphi_t\}$, then only three types of bifurcation occurs: Supercritical pitchfork bifurcation (see Figure 5.1); Subcritical pitchfork bifurcation (see Figure 5.2) and Transcritical bifurcation (see Figure 5.3). In the proof of these stability criteria we use again the Theory of Crandall and Rabinowitz (see [11]). Crandall-Rabinowitz estimated the eigenvalue of the minimum module in the spectrum of the Frechet derivative of the nonlinear operator F along of a bifurcated curve $\alpha(t)$, where $F(\alpha(t)) = 0$. Being X and Y Banach spaces and $F : \mathbb{R} \times X \rightarrow Y$ twice continuously differentiable, they show that the zero eigenvalue of $F_x(t_0, 0)$ corresponds to small real values $\lambda(t)$ of $F_x(t, 0)$ and $\mu(s)$ of $F_x(t(s), x(s))$, where $\lambda(t_0) = 0 = \mu(0)$. The main interest is studying the relationship between $t(s)$, $\lambda(t)$ and $\mu(s)$. They show that if $F_x(t_0, 0)$ satisfies the condition of simplicity, then $\mu(s)$ and $-st'(s)\lambda'(t_0)$ have the same zeros and, in the case where $\mu(s) \neq 0$, the same sign. We make a generalization of these Crandall-Rabinowitz results for the case when F is in terms of the mean curvatures of $\{\varphi_t\}$, $F_x(t_0, 0)$ is the Jacobi Operator, $X = C_{\partial}^{j,\alpha}(\Sigma)$ and $Y = C^{j-2,\alpha}(\Sigma)$ (see Lemmas 5.2.4 and 5.2.3).

When the bifurcation parameter is the volume, we provide two stability criteria:

The first is given in terms of the sign of the derivative of the volume function of hypersurfaces in the bifurcation branch, in addition to the sign of the derivative of the function of the first eigenvalue of $\tilde{J}_t$, where $\tilde{J}_t$ is the restriction of Jacobi operator J_t to scalar fields with integral zero,

Theorem 5.3.1 *Let $\{\psi_s\}_{s \in \tilde{I}}$ be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.3.2. $H'_0 \neq 0$. $\ker(J_{f_0}|_{C_{\partial}^{j,\alpha}(\Sigma)}) = \text{Span}\{e \neq 0\}$. $\lambda_2 = 0$ (Equivalently $\tilde{\lambda}_1 = 0$). $\tilde{\lambda}'_1(0) \neq 0$ ($\tilde{\lambda}_1(t)$ the first eigenvalue of $\tilde{J}_t = J_t - \frac{1}{|\Sigma|} \int_{\Sigma} J_t \text{ vol}_{\varphi_0^*(g)}$). If moreover $V'(0) > 0$ (if necessary, change the parameter t to $-t$). Then,*

- (1) In the case $H'_0 < 0$ (in this case φ_0 is stable by proposition 5.1.2 and $\tilde{\lambda}_1 = 0$)

(1-a) If $\hat{V}'(s) = 0$ for s near 0 (that is, if $\hat{V}$ is locally constant), then, ψ_s is stable for s near 0.

(1-b) If $\hat{V}'(s) \neq 0$, $s \neq 0$, s near 0, then, for a sufficiently small $s_0 > 0$, in each interval $[-s_0, 0)$ and $(0, s_0]$ we have:

(1-b-i) If $\tilde{\lambda}'_1(0)s\hat{V}'(s) < 0$, then, ψ_s is stable.

(1-b-ii) If $\tilde{\lambda}'_1(0)s\hat{V}'(s) > 0$, then, ψ_s is unstable.

In particular, supercritical and subcritical pitchfork bifurcations are presented where $s\hat{V}'(s)$ does not change sign at $s = 0$, and transcritical bifurcation occurs when $s\hat{V}'(s)$ changes sign at $s = 0$.

- (2) In the case $H'_0 > 0$ (in this case φ_0 is unstable by proposition 5.1.2 and $\tilde{\lambda}_2 = 0$), we have that ψ_s is unstable for small $|s|$

The Second is given in terms of the sign of the first and second derivative of the volume function of the hypersurfaces in the bifurcation branch, by fixing the sign positively for the derivative of the first eigenvalue function of $\tilde{J}_t$.

Corollary 5.3.2 *Let $\{\psi_s\}_{s \in \hat{I}}$ be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.3.2. $H'_0 < 0$, $V'_0 > 0$, $\tilde{\lambda}'_1(0) > 0$. Then, there exist positives constants $t_0 \in (0, \epsilon)$ and $s_0 \in \hat{I}$ such that:*

- (1) φ_t is stable for all $t \in [0, t_0]$ and unstable for all $t \in [-t_0, 0)$.
- (2) If $\hat{V}'(0) \neq 0$, then we have transcritical bifurcation for $\{\psi_s\}_{s \in [-s_0, s_0]}$. This is,
 - (2-a) If $\hat{V}'(0) > 0$, then ψ_s is stable for $s \in [-s_0, 0]$ and unstable for $s \in (0, s_0]$.
 - (2-b) If $\hat{V}'(0) < 0$, then ψ_s is stable for $s \in [0, s_0]$ and unstable for $s \in [-s_0, 0)$.
- (3) If $\hat{V}'(0) = 0$ and there is the second derivative of $\hat{V}$ in $s = 0$, we have
 - (3-a) If $\hat{V}''(0) < 0$, then ψ_s is stable for all $s \in [-s_0, s_0]$. Here is a supercritical pitchfork bifurcation.
 - (3-b) If $\hat{V}''(0) > 0$, then ψ_s is unstable for all $s \in [-s_0, 0) \cup (0, s_0]$. Here is a subcritical pitchfork bifurcation.

If $H'_0 > 0$ and $V'_0 < 0$, inverting the parameterization of φ_t , $t \mapsto -t$, similar conclusions are obtained.

Now, when the bifurcation parameter is mean curvature, we also provide two stability criteria similar to the previous case and whose proofs are similar (see Theorema 5.3.3 and Corollary 5.3.4). The first is given in terms of the sign of derivative of mean curvature function of the hypersurfaces in the bifurcation branch, in addition to the sign of derivative of the function of second eigenvalue of J_t . The second is given in terms of the sign of the first and second derivative of the mean curvature function of the hypersurfaces in the bifurcation branch, by fixing the sign positively for the derivative of the second eigenvalue function of J_t .

In Chapter 6 we write the conclusions, obtained results and some statements of possible examples of the impact that our results could have on the study and research that occur in other areas of knowledge, such as Physics, Engineering, Chemistry, Economics, Biology, Medicine, etc. We also raised some problems to be solved in this theory. Among other things, the implementation of an example of a family of free boundary CMC hypersurfaces that has a bifurcation point and the description of the type of hypersurfaces in the bifurcation branch.

Although in the case of fixed boundary CMC surfaces there are examples of families with and without bifurcation points (see [17] and [18]), for our case (free boundary) to present an example it is more complex. The difficulty is the following: the bifurcation point must be a degenerated hypersurface, that is, the class $C^{j,\alpha}$ function associated this hypersurface must be a solution of a differential equation that may contain variable coefficients, such as the norm of the Second Fundamental Form and the Ricci curvature. In addition, this same function must comply with the linearized free boundary condition. Additionally, we also intend studying, in the near future, the same results but in the case of semi-Riemannian metrics.

Now we present a simple example of a family of compact surfaces with constant mean curvature and fixed boundary without bifurcation.

Fixed a circle in a plane of $\mathbb{R}^3$, we consider the set of the truncated spheres contained in one of the hemispheres and whose edge is the fixed circle (see Figure 1)

Figure 1: *Family of truncated spheres with fixed border.*

Clearly, each surface has constant mean curvature equals $-\frac{1}{r}$, where r is the radius of the sphere. Since the mean curvature function defined in this family has a minimum in the Semi-Sphere with center in the center of the fixed circle (see Figure 2), then, clearly, this surface is stable.

Figure 2: *Semi-sphere, point of minimum of the mean curvature function.*

The geometrical reason why there is no bifurcation points in this family of truncated spheres is the following:

In 1841 Delaunay proved that the only surfaces of revolution with constant mean curvature are the surfaces obtained by rotating the roulettes of the conics (see [12]). These are the plane, cylinder, sphere, the catenoid, the unduloid and nodoid (see Figure 3).

By definition, surfaces in the bifurcation branch are not congruent with any of the original branch, except at the bifurcation point. That is, if $\{\varphi_t : \Sigma \rightarrow \mathbb{R}^3\}_{-\epsilon_0 < t < \epsilon_0}$ is the original family and

Figure 3: *Delaunay surfaces: cylinder, unduloid, nodoid, catenoid.*

$\{\psi_s : \Sigma \rightarrow \mathbb{R}^3\}_{-\epsilon_1 < s < \epsilon_1}$ is the bifurcation branch, we have $\varphi_0 = \psi_0$, but there is no diffeomorphism $\phi : \Sigma \rightarrow \Sigma$ such that $\psi_s = \varphi_t \circ \phi$ for $t, s \neq 0$. However, all the elements of the family of truncated spheres considered are homotopically equivalent to a sphere without a point, while compact portions of cylinders, catenoids, unduloids and nodoids are homotopically equivalent to a sphere without two points. Therefore, none of the truncated spheres can be congruent to any of these other Delaunay surfaces, that is, there is no bifurcation point.

Chapter 1

Preliminaries

Throughout this work we will consider M as a $(n + 1)$ -dimensional differential manifold with smooth boundary $\partial M \neq \emptyset$ and Σ as n -dimensional differential manifold with smooth boundary $\partial \Sigma \neq \emptyset$. In this chapter we introduce the concepts of admissibility and orthogonality of hypersurfaces with boundary in a manifold with smooth boundary, we will give the definition of mean curvature and free boundary CMC (Constant Mean Curvature) hypersurface. Also, we give the meaning of nondegeneracy of hypersurfaces with CMC, under the Hölder condition of regularity $C^{j,\alpha}$, through Jacobi operator restricted to the Banach space of the Hölder functions defined on a manifold with boundary that fulfill the linearized free boundary condition.

1.1 Orthogonal sub-manifolds and mean curvature

Definition 1.1.1. Let $\varphi : \Sigma \rightarrow M$ be an embedding. We identify φ with its image $\varphi(\Sigma) \subset M$. $\vec{\eta}_{\partial M}$ is the outer unit normal field along the boundary of M . We call φ *admissible* if it satisfies (a) and (b), and *normal (orthogonal)* if it also satisfies (c):

- (a) $\varphi(\Sigma) \cap \partial M = \varphi(\partial \Sigma)$,
- (b) the normal bundle $T(\varphi(\Sigma))^\perp$ is orientable,
- (c) and for each point $p \in \varphi(\partial \Sigma)$, $\vec{\eta}_{\partial M}(p) \in T_p \varphi(\Sigma)$,

The admissible hypersurface $\varphi(\Sigma)$ is said to *bound a finite volume* if

- (d) $M \setminus \varphi(\Sigma) = \Omega_1 \cup \Omega_2$, with $\bar{\Omega}_1$ compact and $\Omega_1 \cap \Omega_2 = \emptyset$.

If $\varphi : \Sigma \rightarrow M$ is an orthogonal admissible embedding, then $\varphi(\Sigma)$ is compact and $\varphi(\Sigma)$ and ∂M are transverse sub-manifolds (see Definition A.0.1). We say that $\varphi(\Sigma)$ is a *orthogonal sub-manifold* of M (see Figure 1.1).

Let g be a Riemannian metric on M and $\varphi_0 : \Sigma \rightarrow M$ an orthogonal immersion. And write $\Sigma_0 := \varphi_0(\Sigma)$. We define the *second fundamental form* on Σ_0 as

$$\mathbb{I}^{\Sigma_0}(X, Y) := g(\nabla_X Y, \vec{\eta}_{\Sigma_0}), \quad (1.1.1)$$

where $\vec{\eta}_{\Sigma_0}$ is the unit normal vector field to Σ_0 in the orientable normal bundle, ∇ is the Levi-Civita connection in M , and X, Y are vector fields in $T\Sigma_0$.

Figure 1.1: *Orthogonal admissible embedding.*

The *mean curvature function* $H_{\Sigma_0} : \Sigma_0 \rightarrow \mathbb{R}$ is defined as trace of the second fundamental form $\mathbb{I}^{\Sigma_0}$. The *mean curvature vector* of Σ_0 is defined as $\vec{H}_{\Sigma_0} = H_{\Sigma_0} \vec{\eta}_{\Sigma_0}$. If H_{Σ_0} is constant, Σ_0 is called a *constant mean curvature hypersurface* (or CMC hypersurface), and if $H_{\Sigma_0} = 0$, Σ_0 is a *minimal hypersurface*.

1.2 Variational problem

In the theory of variational problems is known that the hypersurfaces with CMC of M minimize the area among all hypersurfaces enclosing a fixed volume. In the case where $\partial\Sigma$ is allowed to move freely along ∂M the variational problem is called *free boundary CMC problem*. The solutions of this problem are orthogonal hypersurfaces with CMC which are called *free boundary CMC hypersurfaces*. We introduce the following notation:

- $\text{Emb}_{\partial}(\Sigma, M)$ be the space of admissible embeddings of Σ in M ,
- $\text{Emb}_{\partial\perp}(\Sigma, M) \subset \text{Emb}_{\partial}(\Sigma, M)$ the sub-space of normal admissible embeddings and bounding a finite volume.

We have (see for example the article by Barbosa-do Carmo [6]) that $\varphi_0 \in \text{Emb}_{\partial\perp}(\Sigma, M)$ have CMC H if only if is a critical point of funtional $\mathfrak{f}_H : \text{Emb}_{\partial\perp}(\Sigma, M) \rightarrow \mathbb{R}$, defined by

$$\mathfrak{f}_H(\varphi) = \int_{\Sigma} \text{vol}_{\varphi^*(g)} - H \int_{\Omega_1} \text{vol}_g, \quad (1.2.1)$$

Note that if $H = 0$ then $\varphi_0(\Sigma)$ has the minimal volume over all hypersurfaces $\varphi(\Sigma)$, $\varphi \in \text{Emb}_{\partial}(\Sigma, M)$. In this case, φ_0 is said to be a *free boundary minimal hypersurface*.

We say that $\varphi_0 : \Sigma \rightarrow M$ is *non-degenerate* if φ_0 is a non-degenerate critical point of $\mathfrak{f}_H$.

Definition 1.2.1. A metric g on M is called “ (Σ, M) -Bumpy”, if all $\varphi \in \text{Emb}_{\partial\perp}(\Sigma, M)$, with CMC in the metric g , is non-degenerate.

We discuss below some other characterizations of non-degenerate hypersurfaces.

1.3 Jacobi operator

One of the most important elements for the study of our theory is the Jacobi Operator. The properties of this operator will allow us to prove different results concerning the genericity, bifurcation and stability of the free boundary CMC hypersurfaces. Let $\Sigma_0 = \varphi_0(\Sigma)$ be a orthogonal CMC hypersurface, $\varphi_0 \in \text{Emb}_{\partial\perp}(\Sigma, M)$. $C^j(\Sigma_0)$ is the set of functions $f : \Sigma_0 \rightarrow \mathbb{R}$ with continuous derivatives to j order, j could be infinite. The second-order linear differential operator $J_{\varphi_0} : C^j(\Sigma_0) \rightarrow C^{j-2}(\Sigma_0)$, $j \geq 2$, defined by

$$J_{\varphi_0}(f) := \Delta_{\Sigma_0} f - (\|\mathbb{I}^{\Sigma_0}\|_{HS}^2 + \text{Ric}_g(\eta_{\Sigma_0}, \eta_{\Sigma_0}))f, \quad (1.3.1)$$

is called *Jacobi operator*, where Δ_{Σ_0} is the (nonnegative) laplacian of (Σ_0, γ) and $\|\mathbb{I}^{\Sigma_0}\|_{HS}^2$ is the square of Hilbert-Schmidt norm (see Definition A.0.3) of the second fundamental form of φ_0 . A *Jacobi scalar field* along of φ_0 is a smooth function $f \in C^j(\Sigma_0)$ such that $J_{\varphi_0}(f) = 0$.

We consider a smooth variation of φ_0 as follows:

$$\Phi : \Sigma \times (-\epsilon, \epsilon) \rightarrow M, \quad \epsilon > 0, \quad (1.3.2)$$

such that $\Phi(\Sigma, s) = \varphi_s(\Sigma) = \Sigma_s \subset M$, $\varphi_s \in \text{Emb}_{\partial}(\Sigma, M)$ with CMC H_s . Let $V = \frac{\partial}{\partial s}|_{s=0} \Phi$ be the corresponding variational vector field. Then $\xi_0 = g(V, \vec{\eta}_{\Sigma_0})$ satisfies

$$\frac{d}{ds}\Big|_{s=0} H_s = \Delta_{\Sigma_0} \xi_0 - (\|\mathbb{I}^{\Sigma_0}\|_{HS}^2 + \text{Ric}_g(\eta_{\Sigma_0}, \eta_{\Sigma_0}))\xi_0 = J_{\varphi_0}(\xi_0), \quad (1.3.3)$$

(see [29], A.2). Then J_{φ_0} represents the second variation $d^2\mathfrak{f}_H(\varphi_0)$ of $\mathfrak{f}_H$ at the critical point φ_0 , with respect to L^2 inner product.

Remark 1.3.1. Note that ξ_0 is a Jacobi field exactly when $H_s \equiv H_0$.

Lemma 1.3.2. *If each φ_s is normal, that is $\varphi_s \in \text{Emb}_{\partial\perp}(\Sigma, M)$, with CMC, then ξ_0 satisfies the so-called linearized free boundary condition*

$$g(\nabla \xi_0, \vec{\eta}_{\partial M}) + \mathbb{I}^{\partial M}(\vec{\eta}_{\Sigma_0}, \vec{\eta}_{\Sigma_0})\xi_0 = 0, \quad (1.3.4)$$

where $\nabla \xi_0$ is the g -gradient of ξ_0 in Σ_0 .

Proof. We can decompose V in its tangent and normal components

$$V = V^T + \xi_0 \vec{\eta}_{\Sigma_0},$$

and

$$\nabla_V \vec{\eta}_{\Sigma_0} = \nabla_{V^T} \vec{\eta}_{\Sigma_0} - \nabla \xi_0,$$

(see Proposition 15 of Ambrozio, [3]). So, if $V^T = 0$ then

$$\begin{aligned}
Vg(\vec{\eta}_{\Sigma_0}, \vec{\eta}_{\partial M}) &= g(\nabla_V \vec{\eta}_{\Sigma_0}, \vec{\eta}_{\partial M}) + g(\vec{\eta}_{\Sigma_0}, \nabla_V \vec{\eta}_{\partial M}) \\
&= g(\nabla_{V^T} \vec{\eta}_{\Sigma_0} - \nabla \xi_0, \vec{\eta}_{\partial M}) + g(\vec{\eta}_{\Sigma_0}, \nabla_V \vec{\eta}_{\partial M}) \\
&= -g(\nabla \xi_0, \vec{\eta}_{\partial M}) + g(\vec{\eta}_{\Sigma_0}, \nabla_{\xi_0} \vec{\eta}_{\partial M}) \\
&= -\frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} + g(\vec{\eta}_{\Sigma_0}, \nabla_{\vec{\eta}_{\Sigma_0}} \vec{\eta}_{\partial M}) \xi_0.
\end{aligned}$$

Thus

$$\frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} = g(\vec{\eta}_{\Sigma_0}, \nabla_{\vec{\eta}_{\Sigma_0}} \vec{\eta}_{\partial M}) \xi_0 - Vg(\vec{\eta}_{\Sigma_0}, \vec{\eta}_{\partial M}).$$

Therefore, if each φ_s is a free boundary CMC hypersurface

$$\frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} = g(\vec{\eta}_{\Sigma_0}, \nabla_{\vec{\eta}_{\Sigma_0}} \vec{\eta}_{\partial M}) \xi_0.$$

Then,

$$\begin{aligned}
g(\nabla \xi_0, \vec{\eta}_{\partial M}) + \mathbb{I}^{\partial M}(\vec{\eta}_{\Sigma_0}, \vec{\eta}_{\Sigma_0}) \xi_0 &= \frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} + g(\nabla_{\vec{\eta}_{\Sigma_0}} \vec{\eta}_{\Sigma_0}, \vec{\eta}_{\partial M}) \xi_0 \\
&= \frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} - g(\vec{\eta}_{\Sigma_0}, \nabla_{\vec{\eta}_{\Sigma_0}} \vec{\eta}_{\partial M}) \xi_0 \\
&= \frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} - \frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} \\
&= 0.
\end{aligned}$$

□

1.4 Regularity

Sard's theorem or Sard-Smale theorem in the case of infinite dimensions (see [26]) is the main tool when it comes to solving problems of genericity of regular points for a certain map between Banach manifolds. Said map needs the condition of being Fredholm with a certain index (For the Fredholm operator definition see A.0.5). To obtain this condition it is necessary to establish a regularity condition type Hölder, $C^{j,\alpha}$ (see definition A.0.4), for our embeddings. We can endow space of functions defined from Σ to $\mathbb{R}$, $C^{j,\alpha}(\Sigma)$, with regularity $C^{j,\alpha}$, with the following norm:

$$\|f\|_{C^{j,\alpha}} = \|f\|_{C^j} + \max_{|\beta|=j} |D^\beta f|_{C^{0,\alpha}}, \quad (1.4.1)$$

where β ranges over multi-indices and

$$\|f\|_{C^j} = \max_{|\beta| \leq j} \sup_{x \in \Sigma} |D^\beta f(x)|, \quad |Df|_{C^{0,\alpha}} = \sup_{x \neq y \in T\Sigma} \frac{|Df(x) - Df(y)|}{\|x - y\|^\alpha}.$$

It is well-known that $C^{j,\alpha}(\Sigma)$ endowed with this norm is a (non separable) Banach space.

Remark 1.4.1. When the operator $\mathfrak{f}_H$ defined in (1.2.1) is considered on the space of $C^{j,\alpha}$ -

embeddings, the Jacobi operator acts on the corresponding tangent space at φ_0 , which can be identified with $C^{j,\alpha}(\Sigma_0)$ (see Proposition 2.3.1).

We define the following space,

$$C_{\partial}^{j,\alpha}(\Sigma_0) := \{f \in C^{j,\alpha}(\Sigma_0) : g(\nabla f, \eta_{\partial M}) + \mathbb{I}^{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0})f = 0\}. \quad (1.4.2)$$

The restriction of $J_{\varphi_0} : C_{\partial}^{j,\alpha}(\Sigma) \rightarrow C^{j-2}(\Sigma)$ is a Fredholm operator of index zero (see [21, section 2]).

Since J_{φ_0} is the representation of the second variation of the area functional, the following proposition is immediately.

Proposition 1.4.2. The embedding $\varphi_0 \in \text{Emb}_{\partial\perp}(\Sigma, M)$ with g -CMC is called *non-degenerate* if $J_{\varphi_0}|_{C_{\partial}^{j,\alpha}(\Sigma)}$ is an isomorphism of Banach spaces, i.e. $\ker J_{\varphi_0} \cap C_{\partial}^{j,\alpha}(\Sigma) = \emptyset$.

The degenerate embeddings will be very important objects in the following chapters: for example, they are a fundamental part of the demonstration of the main theorem of Chapter 2 on the genericity of the bumpy metrics. We will also see that from degenerate free boundary CMC embeddings we can decide about the existence of perturbations and also about the generation of bifurcations branch and changes in the stability of families of free boundary CMC hypersurfaces, as will be seen in Chapters 3, 4 and 5.

Chapter 2

Genericity Of Nondegenerate Free Boundary CMC Embeddings

In this chapter we prove two important theorems aimed at demonstrating the genericity of the Bumpy metrics (defined in 1.2.1). The first, Theorem 2.1.1, is a result given in general terms over Banach spaces Γ , X and Y , where it is proved that the kernel of a certain application $H : \Gamma \times X \rightarrow Y$, defined from a functional $A : \Gamma \times X \rightarrow \mathbb{R}$, is a Banach manifold and the first projection Π , defined on that manifold is a Fredholm operator of zero index, whose critical points (γ_0, u_0) are elements such that u_0 is a degenerate critical point of $A(\gamma_0, \cdot)$. The second, Theorem 2.4.1, where we prove that for the spaces $\Gamma = \mathcal{W} \cap \text{Met}^k(M)$, ($\mathcal{W}$ is a Banach subspace of type C^k -Whitney of the symmetric tensor fields on M , defined in the section 2.2), $X = C^{j,\alpha}(\Sigma)$ and $Y = C^{j-2,\alpha}(\Sigma)$, the functional area $A : \Gamma \times X \rightarrow \mathbb{R}$ and the mean curvature of the operator $H : \Gamma \times X \rightarrow Y$, the conditions of the Theorem 2.1.1 are fulfilled. Thus, as an immediate consequence of the Theorem 2.4.1 and the Sard-Smale Theorem, we obtained the genericity of non-degenerate free boundary CMC embeddings, Corollary 2.4.6.

Definition 2.0.1. A subset of metrical space is said to be *generic* if it is the countable intersection of dense open subsets. By Baires's theorem, a generic set is dense.

2.1 Important Theorem

The Theorem 2.1.1 is of great importance for us since it shows the genericness of the regular points of a defined mapping between Banach spaces in general terms and will be used in the proof of the main theorem of this Chapter. It will be supposed that $A : \Gamma \times X \rightarrow \mathbb{R}$ is a map of class C^j , with $j \geq 2$, where Γ and X are Banach spaces; we also assume that X has an inner product. We require a condition of transversality (see Definition A.0.2) between the function $\frac{\partial A}{\partial x} : \Gamma \times X \rightarrow TX^*$ and the zero section of TX^* , this is equivalent to saying that $\forall(\gamma_0, x_0)$, $\frac{\partial A}{\partial x}(\gamma_0, x_0)$ and $w \neq 0$, $w \in \ker(\frac{\partial^2 A}{\partial x^2})$, $\exists v \in T_{\gamma_0}\Gamma$ such that $\frac{\partial^2 A}{\partial \gamma \partial x}(\gamma_0, x_0)(w, v) \neq 0$ (see [9, Proposition 3.1]).

It is necessary to give the following definition of locally fibered sub-manifold that is imposed as a condition in the Theorem 2.1.1

Definition 2.1.1. Let Γ and X be Banach spaces and $\Pi : \Gamma \times X \rightarrow \Gamma$ the first projection. Let $\mathfrak{X} \subset \Gamma \times X$ be a sub-manifold. We say that $\mathfrak{X}$ is a locally fibered if it meets the following condition: For each $(u_0, v_0) \in \mathfrak{X}$, there is an open $U \subset \Gamma$ and a closed subspace $W \subset X$, with $u_0 \in U$

and $v_0 \in W$, and a diffeomorphism $\varphi : U \times W \rightarrow \Pi^{-1}(U)$, such that the following diagram is commutative

$$\begin{array}{ccc} \Pi^{-1}(U) & \xleftarrow{\varphi} & U \times W \\ & \searrow \Pi & \downarrow \text{proj}_1 \\ & & U \end{array}$$

e.i., for all $(u, w) \in U \times W$, $\Pi \circ \varphi(u, w) = u$.

Theorem 2.1.1. *Let Γ , X and Y be Banach spaces, $\mathcal{H}$ a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $X \subset Y \subset \mathcal{H}$. Let*

$$A : \Gamma \times X \rightarrow \mathbb{R} \quad (2.1.1)$$

a C^j function, $j \geq 2$. Suppose that there exists a map $H : \Gamma \times X \rightarrow Y$ such that

$$\left. \frac{d}{dt} \right|_{t=0} A(\gamma, u + tv) = \langle H(\gamma, u), v \rangle, \quad (2.1.2)$$

for all $\gamma \in \Gamma$ and $u, v \in X$.

Let $\mathfrak{X} \subset \Gamma \times X$ be a locally fibrated sub-manifold such that for all $(\gamma_0, u_0) \in \mathfrak{X}$ the operator

$$\left. \frac{\partial H}{\partial u}(\gamma_0, u_0) \right|_{T_{u_0} X} : T_{u_0} X \rightarrow Y \quad (2.1.3)$$

is a Fredholm with index zero ($T_{u_0} X \cong X$).

Furthermore, suppose that for all $k \in \text{Ker} \left. \frac{\partial H}{\partial u}(\gamma_0, u_0) \right|_{T_{u_0} X}$, $k \neq 0$, there exists a family $(\gamma(s), u(t)) \in \mathfrak{X}$, such that $\gamma(0) = \gamma_0$, $u(0) = u_0$, $u'(0) = k$ and

$$\left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} A(\gamma(s), u(t)) \neq 0. \quad (2.1.4)$$

Then

- (1) $H|_{\mathfrak{X}} : \mathfrak{X} \rightarrow Y$ is a submersion near (γ_0, u_0) , so there exists a neighborhood $W \subset \mathfrak{X}$, $(\gamma_0, u_0) \in W$, such that

$$\mathcal{M} = \{(\gamma, u) \in W : H(\gamma, u) = 0\}$$

is a sub-manifold of $\mathfrak{X}$, and

$$T_{(\gamma, u)} \mathcal{M} = \text{Ker} \left(dH(\gamma, u) \Big|_{T_{(\gamma, u)} \mathfrak{X}} \right).$$

- (2) *The projection*

$$\Pi : \mathcal{M} \rightarrow \Gamma, \quad \Pi(\gamma, u) = \gamma$$

is an Fredholm operator with index zero.

- (3) *The critical points of $\Pi|_{\mathcal{M}}$ are elements $(\gamma_0, u_0) \in \mathcal{M}$ such that u_0 is a degenerate critical point of the functional $A(\gamma_0, \cdot)$.*

Proof. (1) In order to simplify the notation we write $J = \left. \frac{\partial H}{\partial u}(\gamma_0, u_0) \right|_{T_{u_0} \mathfrak{X}_{\gamma_0}}$. We show that J is symmetric with respect to the product in $\mathcal{H}$. Indeed,

$$\begin{aligned}
 \frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} A(\gamma_0, u_0 + sv + tw) &= \frac{d}{ds} \Big|_{s=0} \left(\frac{d}{dt} \Big|_{t=0} A(\gamma_0, u_0 + sv + tw) \right) \\
 &= \frac{d}{ds} \Big|_{s=0} \langle H(\gamma_0, u_0 + sv), w \rangle \\
 &= \langle Jv, w \rangle.
 \end{aligned}$$

On the other hand

$$\frac{\partial^2}{\partial t \partial s} \Big|_{s=t=0} A(\gamma_0, u_0 + sv + tw) = \langle Jw, v \rangle.$$

Now, for all $u \in \ker J$ and $v \in \mathfrak{X}_{\gamma_0}$, we have $\langle Ju, v \rangle = \langle u, Jv \rangle = 0$. Thus, $\text{im}(J) \subset (\ker J)^\perp$. Since J is Fredholm with index 0, we obtain

$$\text{im}(J) = (\ker J)^\perp. \quad (2.1.5)$$

In order to prove that $H : \mathfrak{X} \rightarrow Y$ is a submersion we have to prove that $dH(\gamma_0, u_0) \Big|_{T_{(\gamma_0, u_0)} \mathfrak{X}}$ is surjective and its kernel is complemented.

First we prove surjectivity. Let $(\gamma(s), u(t))$ be a family compatible with $\mathfrak{X}$, such that $\gamma(0) = \gamma_0$, $u(0) = u_0$, $u'(0) = k \neq 0$, $k \in \text{Ker } J$ (for example, take $u(t) = u_0 + tk$). Note that $\gamma'_0 = \frac{d}{ds} \Big|_{s=0} \gamma(s)$, $\mathbb{R} \cdot \gamma'_0 \subset T_{\gamma_0} \Gamma$ and $(\mathbb{R} \cdot \gamma'_0) \times \{0\} \subset T_{(\gamma_0, u_0)} \mathfrak{X}$. So,

$$\begin{aligned}
 0 &\neq \frac{\partial^2}{\partial s \partial t} A(\gamma(s), u(t)) \\
 &= \frac{d}{ds} \Big|_{s=0} \frac{d}{dt} \Big|_{t=0} A(\gamma(s), u(t)) \\
 &= \frac{d}{ds} \Big|_{s=0} \langle H(\gamma(s), u_0), k \rangle \\
 &= \left\langle \frac{\partial H}{\partial \gamma}(\gamma_0, u_0) \gamma'_0, k \right\rangle.
 \end{aligned}$$

Since $\frac{\partial H}{\partial \gamma}(\gamma_0, u_0) \gamma'_0 \in \text{Im}(dH(\gamma_0, u_0) \Big|_{T_{(\gamma_0, u_0)} \mathfrak{X}})$, we use Lemma B.0.5, with $V = Y$, $W = \text{Ker } J$ and $Z = \text{Im}(dH(\gamma_0, u_0) \Big|_{T_{(\gamma_0, u_0)} \mathfrak{X}})$, which shows that $dH(\gamma_0, u_0)$ restricted to $T_{(\gamma_0, u_0)} \mathfrak{X}$ is surjective.

Now, we have

$$\ker(dH(\gamma_0, u_0) \Big|_{T_{(\gamma_0, u_0)} \mathfrak{X}}) \subset \ker J,$$

and from the fact that J is Fredholm we infer that

$$\text{Dim}(\ker(dH(\gamma_0, u_0) \Big|_{T_{(\gamma_0, u_0)} \mathfrak{X}})) \leq \text{Dim}(\ker J) < \infty.$$

Therefore $\ker(dH(\gamma_0, u_0) \Big|_{T_{(\gamma_0, u_0)} \mathfrak{X}})$ is complemented in $T_{(\gamma_0, u_0)} \mathfrak{X}$.

Thus $H : \mathfrak{X} \rightarrow Y$ is a submersion. Whence there is a neighborhood U of (γ_0, u_0) such that

$$\mathcal{M} = H^{-1}(0) \cap U$$

is a sub-manifold and

$$T_{(\gamma_0, u_0)}\mathcal{M} = \ker(dH(\gamma_0, u_0)|_{T_{(\gamma_0, u_0)}\mathfrak{X}})$$

(2) Let

$$\begin{aligned} \Pi : \Gamma \times X &\longrightarrow \Gamma \\ (\gamma, u) &\longmapsto \gamma \end{aligned}$$

be the projection on the first factor. Take the restriction of Π to $\mathcal{M}$, $\Pi|_{\mathcal{M}}$. We have to prove that $\dim(\ker(d\Pi|_{T_{(\gamma_0, u_0)}\mathcal{M}})) < \infty$ and $\text{im}(d\Pi|_{T_{(\gamma_0, u_0)}\mathcal{M}})$ is closed and has finite co-dimension.

We have that

$$\begin{aligned} \ker(d\Pi|_{T_{(\gamma_0, u_0)}\mathcal{M}}) &= \ker \Pi \cap T_{(\gamma_0, u_0)}\mathcal{M} \\ &= (\{0\} \times X) \cap \ker(dH(\gamma_0, u_0)|_{T_{(\gamma_0, u_0)}\mathfrak{X}}) \\ &= \{0\} \times \ker J, \end{aligned}$$

which is of finite dimension.

Now, by hypothesis about $\mathfrak{X}$, there is a neighborhood $U \subset \Gamma$, $\gamma \in U$, and a diffeomorphism $\varphi : U \times W \rightarrow \Pi^{-1}(U)$, $W \subset X$ a closed sub-space, such that the follow diagram is commutative

$$\begin{array}{ccc} \Pi^{-1}(U) & \xleftarrow{\varphi} & U \times W \\ & \searrow \Pi & \downarrow \text{proy}_1 \\ & & U \end{array}$$

So, locally Π is as a projection from a product space.

In particular for γ_0 fixed, we can take φ such that $\varphi(\gamma_0, u) = (\gamma_0, u)$, for all $u \in W$. Set

$$\overline{H} := H \circ \varphi : U \times W \longrightarrow Y.$$

Then

$$\begin{aligned} \frac{\partial \overline{H}}{\partial u}(\gamma_0, u_0) &= \frac{d}{dt} \Big|_{t=0} \overline{H}(\gamma_0, u_0 + tv) \\ &= \frac{d}{dt} \Big|_{t=0} H(\varphi(\gamma_0, u_0 + tv)) \\ &= \frac{d}{dt} \Big|_{t=0} H(\gamma_0, u_0 + tv) \\ &= \frac{\partial H}{\partial u}(\gamma_0, u_0). \end{aligned}$$

And we have

$$\ker(d\overline{H}(\gamma_0, u_0)) = \{(\xi, \omega) : \frac{\partial \overline{H}}{\partial \gamma}(\gamma_0, u_0)\xi + \frac{\partial \overline{H}}{\partial u}(\gamma_0, u_0)\omega = 0\}$$

and $\xi \in [\frac{\partial \overline{H}}{\partial \gamma}(\gamma_0, u_0)]^{-1}(\text{im} J)$.

We prove that $\text{im}(d\Pi|_{T_{(\gamma_0, u_0)}\mathcal{M}})$ has finite co-dimension.

$$\begin{aligned}
 \text{im}(d\Pi|_{T_{(\gamma_0, u_0)}\mathcal{M}}) &= \Pi(T_{(\gamma_0, u_0)}\mathcal{M}) \\
 &= \Pi(\ker(d\bar{H}(\gamma_0, u_0))) \\
 &= \left[\frac{\partial \bar{H}}{\partial \gamma}(\gamma_0, u_0) \right]^{-1} (\text{im}(\frac{\partial \bar{H}}{\partial u}(\gamma_0, u_0))) \\
 &= \left[\frac{\partial \bar{H}}{\partial \gamma}(\gamma_0, u_0) \right]^{-1} (\text{im}J).
 \end{aligned} \tag{2.1.6}$$

The Fredholmness of J implies that its image is closed and finite co-dimensional. Thus, we use Lemma B.0.6 with $U = T_{\gamma_0}\Gamma$, $V = Y$, $S = \text{im}\frac{\partial \bar{H}}{\partial u}(\gamma_0, u_0)$ and $L = \frac{\partial \bar{H}}{\partial \gamma}(\gamma_0, u_0)$. Now, we have

$$\begin{aligned}
 \text{im}\frac{\partial \bar{H}}{\partial \gamma}(\gamma_0, u_0) + \text{im}\frac{\partial \bar{H}}{\partial u}(\gamma_0, u_0) &= \text{im}(d\bar{H}(\gamma_0, u_0)) \\
 &= \text{im}(dH(\gamma_0, u_0))\Big|_{T_{(\gamma_0, u_0)}\mathfrak{X}},
 \end{aligned}$$

but $dH(\gamma_0, u_0)\Big|_{T_{(\gamma_0, u_0)}\mathfrak{X}}$ is surjective, so

$$\text{Codim}_Y \left(\text{im}\frac{\partial \bar{H}}{\partial \gamma}(\gamma_0, u_0) + \text{im}\frac{\partial \bar{H}}{\partial u}(\gamma_0, u_0) \right) = 0$$

and

$$\text{Codim}_Y \text{im}\left(\frac{\partial \bar{H}}{\partial u}(\gamma_0, u_0)\right) = \text{Codim}_{T_{\gamma_0}\Gamma} \left(\left[\frac{\partial \bar{H}}{\partial \gamma}(\gamma_0, u_0) \right]^{-1} (\text{im}J) \right).$$

On the other hand,

$$\text{Codim}_Y \text{im}\left(\frac{\partial \bar{H}}{\partial u}(\gamma_0, u_0)\right) = \text{Dim}(\text{im}J)^\perp = \ker(J)$$

Thus, Π is Fredholm with index 0.

(3) Recall that (γ_0, u_0) is a regular point of $\Pi|_{\mathcal{M}}$ if

$$d\Pi(\gamma_0, u_0)\Big|_{T_{(\gamma_0, u_0)}\mathcal{M}}$$

is surjective, and also remember that

$$\text{im}(d\Pi|_{T_{(\gamma_0, u_0)}\mathcal{M}}) = \left[\frac{\partial \bar{H}}{\partial \gamma}(\gamma_0, u_0) \right]^{-1} \left(\text{Im}\left(\frac{\partial \bar{H}}{\partial u}(\gamma_0, u_0)\right) \right),$$

as we see in 2.1.6. Then, (γ_0, u_0) is a regular point of $\Pi|_{\mathcal{M}}$ if and only if

$$\text{im}\left(\frac{\partial \bar{H}}{\partial \gamma}(\gamma_0, u_0)\right) \subset \text{im}\left(\frac{\partial \bar{H}}{\partial u}(\gamma_0, u_0)\right),$$

but

$$\operatorname{im}(d\bar{H}(\gamma_0, u_0)) = \operatorname{im}\left(\frac{\partial \bar{H}}{\partial \gamma}(\gamma_0, u_0)\right) + \operatorname{im}\left(\frac{\partial \bar{H}}{\partial u}(\gamma_0, u_0)\right) = \operatorname{im}\left(\frac{\partial \bar{H}}{\partial u}(\gamma_0, u_0)\right) = \operatorname{im}J.$$

Now,

$$\operatorname{im}(d\bar{H}(\gamma_0, u_0)) = \operatorname{im}(dH(\gamma_0, u_0))\Big|_{T(\gamma_0, u_0)\mathfrak{X}}$$

and $dH(\gamma_0, u_0)\Big|_{T(\gamma_0, u_0)\mathfrak{X}}$ is surjective. Hence $\ker J = \{0\}$, since J is Fredholm of index 0.

Therefore, $(\gamma_0, u_0) \in \mathcal{M}$ is a regular point to $\Pi|_{\mathcal{M}}$ if and only if $\ker J = \{0\}$. Whence γ_0 is a critical valor of $\Pi|_{\mathcal{M}}$ if and only if there exists u_0 which is a degenerate critical point of $A(\gamma_0, \cdot)$. □

2.2 C^k -Whitney type Banach space of tensor fields

The definitions of this section are taken from [7] and [9]. We denote by $\mathcal{G}_{sym}^k(TM^* \otimes TM^*)$ the vector space of all sections σ of class C^k , $k \geq 2$, of the vector bundle $TM^* \otimes TM^*$ such that $\sigma_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is symmetric for all p . Let $Met^k(M) \subset \mathcal{G}_{sym}^k(TM^* \otimes TM^*)$ be the set of all metric tensors g on M of class C^k . The set $\mathcal{G}_{sym}^k(TM^* \otimes TM^*)$ does not have necessarily a canonical Banach space structure, for example if M is noncompact. In order to give a structure of Banach space to this space of tensors, we introduce the following definition (see [9, section 4.1]).

Definition 2.2.1. A vector subspace $\mathcal{W} \subset \mathcal{G}_{sym}^k(TM^* \otimes TM^*)$ will be called C^k -Whitney type Banach space of tensor fields over M if complies the following conditions:

1. $\mathcal{W}$ contains all tensor fields in $\mathcal{G}_{sym}^k(TM^* \otimes TM^*)$ having compact support;
2. $\mathcal{W}$ has a Banach space norm $\|\cdot\|_{\mathcal{E}}$ with the property that $\|\cdot\|_{\mathcal{E}}$ -convergence of a sequence implies convergence in the weak Whitney C^k -topology¹.

The second condition means that given any sequence $(b_n)_{n \in \mathbb{N}}$ and $b_{\infty} \in \mathcal{W}$ such that $\lim_{n \rightarrow \infty} \|b_n - b_{\infty}\|_{\mathcal{W}} = 0$, then for each compact set $K \subset M$, the restriction $b_n|_K$ converges to $b_{\infty}|_K$ in the C^k -topology as $n \rightarrow \infty$.

We can construct a C^k -Whitney type Banach space of tensors on M by using an auxiliary Riemannian metric g_R on M as follows (see [9, Example 1]). The Levi-Civita connection ∇^R of g_R induces a connection on all vector bundles over M obtained with functorial constructions from the tangent bundle TM . Also for each $r, s \in \mathbb{N}$, g_R induces canonical Hilbert space norms on each tensor bundle $TM^{*(r)} \otimes TM^{(s)}$, which will be denoted $\|\cdot\|_R$. Now, we define $\mathcal{G}_{sym}^k(TM^* \otimes TM^*; g_R)$ as the subset of $\mathcal{G}_{sym}^k(TM^* \otimes TM^*)$ consisting of all sections σ such that

$$\|\sigma\|_k = \max_{i=0, \dots, k} \left[\sup_{x \in M} \|(\nabla^R)^i \sigma(x)\|_R \right] < +\infty. \quad (2.2.1)$$

The norm $\|\cdot\|_k$ in (2.2.1) turns $\mathcal{G}_{sym}^k(TM^* \otimes TM^*; g_R)$ into a separable normed space (see [25]), which is complete if the Riemannian metric g_R is complete. Thus, we have that $\mathcal{G}_{sym}^k(TM^* \otimes TM^*; g_R)$ is a C^k -Whitney type Banach space of tensors.

¹For definition and properties of Whitney C^k -topology see [25]

When M is compact, $\mathcal{G}_{sym}^k(TM^* \otimes TM^*; g_R) = \mathcal{G}_{sym}^k(TM^* \otimes TM^*)$, and $Met^k(M)$ is an open subset.

2.3 Smooth structure of non-degenerate hypersurface

Since the group of diffeomorphisms of Σ acts freely on the set of embeddings of Σ into M , the appropriate setup for studying the set of submanifolds of a given diffeomorphism type is obtained by considering the notion of *unparameterized embeddings*.

Definition 2.3.1. Two embeddings φ_1 and φ_2 from Σ in M will be *equivalent* if there exists a diffeomorphism $\phi : \Sigma \rightarrow \Sigma$ such that $\varphi_2 = \varphi_1 \circ \phi$, i.e., if they are different parametrizations of the same submanifold of M diffeomorphic to Σ . For $\varphi \in \text{Emb}(\Sigma, M)$, we denote by $[\varphi]$ the class of all embedding that are equivalent to φ . We say that $[\varphi]$ is a *unparametrized embedding* of Σ in M .

Definition 2.3.2. We define the following sets:

- $\mathcal{E}(\Sigma, M) := \{[\varphi] : \varphi \text{ is a embedding of order } C^{j,\alpha}\},$
- $\mathcal{E}_\partial(\Sigma, M) := \{[\varphi] \in \mathcal{E}(\Sigma, M) : \varphi(\Sigma) \cap \partial M = \varphi(\partial\Sigma)\},$
- Let $\gamma \in Met(M)$,

$$\mathcal{E}_{\partial,\gamma}^\perp(\Sigma, M) := \{[\varphi] \in \mathcal{E}_\partial(\Sigma, M) : \varphi \text{ is } \gamma - \text{orthogonal}\}.$$

There is a smooth Banach manifold structure, of infinite dimension, for a sufficiently small neighborhood of $[\varphi_0] \in \mathcal{E}_{\partial,\gamma}^\perp(\Sigma, M)$ in some suitable topology.

Proposition 2.3.1. [8, Proposition 4.1] Let Σ be a compact manifold with boundary and $\varphi_0 \in \text{Emb}_\partial(\Sigma, M)$. Let $\mathcal{U} \subset \mathcal{E}_{\partial,\gamma}^\perp(\Sigma, M)$ be a sufficiently small neighborhood of $[\varphi_0]$, then $\mathcal{U}$ can be identified with an infinite-dimensional smooth submanifold $\mathcal{N}$ of Banach space $C^{j,\alpha}(\Sigma)$, with $0 \in \mathcal{N}$ corresponding to $[\varphi_0]$, such that $T_0\mathcal{N} = C_{\partial}^{j,\alpha}(\Sigma)$ (see 1.4.2)

2.4 Genericity of Bumpy metrics

In this section we prove that the set of all Riemannian metrics γ in M for which every minimal free boundary embedding is non-degenerate (Bumpy metrics) is generic in the space of Riemannian metrics of M .

Theorem 2.4.1. Let M be a $(n+1)$ -dimensional differential manifold with smooth boundary $\partial M \neq \emptyset$, and Σ a n -dimensional compact differential manifold with smooth boundary $\partial\Sigma \neq \emptyset$. Let $\mathcal{W} \subset \mathcal{G}_{sym}^k(TM^* \otimes TM^*)$ be a C^k -Whitney type Banach sub-space of the symmetrical tensor fields over M , with $k > j \geq 2$, let $\Gamma \subset \mathcal{W} \cap Met^k(M)$ be an open subset of $\mathcal{W}$. Let $\mathcal{M}$ be the set defined as

$$\mathcal{M} = \{(\gamma, [\varphi]) \in \Gamma \times \mathcal{E}_\partial(\Sigma, M) : [\varphi] \in \mathcal{E}_{\partial,\gamma}^\perp(\Sigma, M), \varphi \text{ is } \gamma\text{-minimal}\}.$$

Then,

- (1) $\mathcal{M}$ is a separable Banach manifold modelled on Γ .

- (2) $\Pi : \mathcal{M} \rightarrow \Gamma$, defined by $\Pi(\gamma, [\varphi]) = \gamma$, is a Fredholm map with index 0.
- (3) γ_0 is critical value of Π if and only if there is a γ_0 -minimal embedding $\varphi_0 : \Sigma \rightarrow M$ which is degenerate.

2.4.1 Analytic preliminaries

We begin by proving some lemmas to clarify the ideas in the proof of Theorem 2.4.1.

Lemma 2.4.2. *Let $\Sigma_0 = \varphi_0(\Sigma)$ be a free boundary minimal surface and $\bar{f} : \Sigma_0 \rightarrow \mathbb{R}$ satisfying the linearized free boundary condition*

$$\gamma_0(\nabla \bar{f}, \vec{\eta}_{\partial M}) + \mathbb{I}^{\partial M}(\vec{\eta}_{\Sigma_0}, \vec{\eta}_{\Sigma_0})\bar{f} = 0.$$

Then, there is a map

$$o : (-\epsilon, \epsilon) \rightarrow C^{j,\alpha}(\Sigma_0),$$

with $\frac{o(t)}{t} \rightarrow 0$ if $t \rightarrow 0$, such that $\varphi_t : \Sigma_0 \rightarrow M$ defined by

$$\varphi_t(p) := \exp_{\varphi_0(p)}([t\bar{f}(p) + o(t)(p)]\vec{n}_{\Sigma_0}(p))$$

is orthogonal.

Proof. By Proposition 2.3.1 there is a bijective correspondence between a neighborhood $\mathcal{U} \subset \mathcal{E}_{\partial}^{\perp}(\Sigma, M)$ of $[\varphi_0]$ and a infinite-dimensional smooth sub-manifold $\mathcal{N}$ of the Banach space $C^{j,\alpha}(\Sigma)$, with $0 \in \mathcal{N}$ corresponding to $[\varphi_0]$ and $T_0\mathcal{N} = C_{\partial}^{j,\alpha}(\Sigma)$. So, there is a diffeomorphism, given by the Inverse Mapping Theorem (see A.0.1), between $\mathcal{U}$ and a neighborhood $V \subset T_0\mathcal{N}$ of 0, such that $\varphi_t \mapsto t\bar{f}$. On the other hand, Exp_{φ_0} also generates a diffeomorphism between $\mathcal{U}$ and some neighborhood $V' \subset T_0\mathcal{N}$, such that $\varphi_t \mapsto t\bar{g}_t$, with $g_0 = 0$ and $g'_0 = \bar{f}$. Since V and V' are diffeomorphic we have $g_t = t\bar{f} + o(t)$, where $o(t)$ is differentiable and $\frac{o(t)}{t} \rightarrow 0$ if $t \rightarrow 0$. $\square$

Lemma 2.4.3. *Let Σ be a n -dimensional compact sub-manifold embedding in M , $f : \Sigma \rightarrow \mathbb{R}$ function, $f \in C^j$, $f \neq 0$ and n_{Σ} the unit normal vector field to Σ . Then, there is a map $\psi : M \rightarrow \mathbb{R}$ such that*

$$i) \ \psi(p) = 0, \text{ for all } p \in \Sigma,$$

$$ii) \ \int_{\Sigma} d\psi(n_{\Sigma}) \cdot f(p) d\Sigma \neq 0$$

Proof. Let $p_0 \in \Sigma$ be such that $f(p_0) > 0$. There is a local coordinate chart around of p_0 , $(U, x = (x_1, \dots, x_{n+1}))$, such that

$$(1) \ x(U) = B_1(0) \subset \mathbb{R}^{n+1},$$

$$(2) \ x|_{U \cap \Sigma} = (x_1, \dots, x_n, 0) \text{ and}$$

$$(3) \ f(p) > 0 \text{ for all } p \in U \cap \Sigma.$$

We have

$$dx_p(\vec{n}_p) = \vec{v}_{x(p)},$$

where $v_{x(p)} = (v_1, \dots, v_{n+1})$ with $v_{n+1} \neq 0$. We may assume that $v_{n+1} > 0$.

Set

$$\begin{aligned} h: B_1(0) &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_{n+1}) &\longmapsto l(x_{n+1}), \end{aligned}$$

where $l: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that

$$(i) \text{ } \text{Supp}(l) \subset [-1, 1],$$

$$(ii) \text{ } l(0) = 0 \text{ and}$$

$$(iii) \text{ } l'(0) > 0.$$

For example l can be taken as follows (see Figure 2.1):

$$l(t) = \begin{cases} -[2(2(-t-1))^3 + 3(2(-t-1))^2], & \text{if } -1 < t \leq -1/2 \\ \sin(\pi t), & \text{if } -1/2 < t < 1/2 \\ 2(2(t-1))^3 + 3(2(t-1))^2, & \text{if } 1/2 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Figure 2.1: Graph of $l(t)$.

Under these conditions we have

$$\begin{aligned} h(B_1(0) \cap \mathbb{R}^n) &= 0 \\ \nabla h_p &= (0, \dots, 0, l'(0)) \text{ for all } p \in B_1(0) \cap \mathbb{R}^n \end{aligned}$$

So, we define $\psi: M \rightarrow \mathbb{R}$ as

$$\psi(p) = \begin{cases} h \circ x(p) & \text{if } p \in U \\ 0 & \text{if } p \in M \setminus U. \end{cases}$$

Therefore, ψ fulfills the affirmation i).

Now, for all $p \in U \cap \Sigma$ we have

$$\begin{aligned} d\psi_p(\vec{n}_p) &= \nabla h_{x(p)} \cdot dx_p(\vec{n}_p) \\ &= \nabla h_{x(p)} \cdot \vec{v}_{x(p)} \\ &= l'(0) \cdot v_{n+1} > 0 \end{aligned}$$

and

$$d\psi_p(\vec{n}_p) = 0 \quad \text{for } p \in \Sigma \setminus U.$$

Then

$$\int_{\Sigma} d\psi_p(\vec{n}_p) \cdot f(p) d\Sigma = \int_{\Sigma \cap U} d\psi_p(\vec{n}_p) \cdot f(p) d\Sigma > 0$$

□

Lemma 2.4.4. *Let (X, d) be a non-separable metric space and let (Y, τ_s) be a separable topological space, with $\emptyset \neq Y \subset X$. Assume that $\tau_d|_Y \subset \tau_s$, where τ_d is the topology generated by d . Then, the closure of Y in τ_d is separable.*

Proof. Let $\bar{Y}$ be the closure of Y in τ_d , and $y \in \bar{Y}$. Let $B_r(y)$ be a d -ball with center y and radio r , thus $B_r(y) \cap Y \in \tau_s$. If D is a dense subset of Y then there is $x \in D \cap [B_r(y) \cap Y]$, so $x \in B_r(y)$. Therefore, $\bar{Y}$ is separable in τ_d . □

2.4.2 Proof of the Genericity Theorem of Metrics Bumpy

In this section will be dedicated to the proof of Theorem 2.4.1, the main result of this chapter. Under suitable constraints on the set of metrics in M and the space of embeddings of Σ in M , the its proof follows from theorem 2.1.1.

Parts (2) and (3) are immediate consequence of Theorem 2.1.1.

To prove (1), let g be a metric in M , g of class C^∞ , such that ∂M is g -totally geodesic. Take $(\gamma_0, [\varphi_0]) \in \mathcal{M}$. Let $\vec{n}_0$ be the unit normal vector field along of $\varphi_0(\Sigma)$. For each $f: \Sigma \rightarrow \mathbb{R}$ of class $C^{k,\alpha}$ sufficiently small, we associate the embedding $\varphi_f: \Sigma \rightarrow M$ defined by

$$\varphi_f(p) := \exp_{\varphi_0(p)}(f(p)\vec{n}_0(p)),$$

where $\exp$ is the exponential map in M defined by g . We set $\vec{n}_0(p) := \vec{n}_{0_{\varphi_0(p)}}$. Since φ_0 is orthogonal then $\vec{n}_0(p) \in T_{\varphi_0(p)}(\partial M)$, for all $p \in \varphi_0^{-1}(\varphi_0(\Sigma) \cap \partial M)$. Hence $\varphi_f(\partial \Sigma) \subset \partial M$ since ∂M is totally geodesic. Note that if $f \equiv 0$ then $\varphi_f = \varphi_0$.

Now, let $U \subset C^{j,\alpha}(\Sigma)$ be a sufficiently small neighborhood of 0 such that $\exp$ is a diffeomorphism in a neighborhood V generated by U ,

$$\begin{aligned} \exp_{\varphi_0(p)}: C^{j,\alpha}(\Sigma) &\longrightarrow M \\ f &\longmapsto \exp(f.\vec{n}_0) =: \varphi_f(p). \end{aligned}$$

Note that the map $f \mapsto [\varphi_f]$ is a diffeomorphism between U and a neighborhood $\tilde{U}$ of $[\varphi_0] \in \mathcal{E}_\partial(\Sigma, M)$.

Defined the following spaces:

$$\begin{aligned} X &= C^{j,\alpha}(\Sigma) \\ Y &= C^{j-2,\alpha}(\Sigma) \\ \mathcal{H} &= L^2(\Sigma), \end{aligned}$$

and the set

$$\mathfrak{X} = \{(\gamma, f) : f \in U, \varphi_f \text{ is } \gamma\text{-orthogonal to } \partial M\}.$$

Let's see that $\mathfrak{X} \subset \Gamma \times X$ is a locally fiber sub-bundle.

Proposition 2.4.5. *$\mathfrak{X}$ is a locally fibered sub-manifold over Γ*

Proof. Let's denote $X = C^{j,\alpha}(\Sigma)$ to simplify writing. We define the following map, clearly differentiable,

$$\begin{aligned} \phi : \Gamma \times X &\longrightarrow C^{j-1,\alpha}(\partial\Sigma) \\ (\gamma, f) &\longmapsto \gamma(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}), \end{aligned}$$

where $\vec{n}_{\varphi_f}$ is a unitary normal field to $\varphi_f(\Sigma)$ and $\vec{n}_{\partial M}$ is a unitary normal field to ∂M . So $\mathfrak{X} = \phi^{-1}(0)$. Therefore, if ϕ is a submersion for all $(\gamma, f) \in \mathfrak{X}$, then $\mathfrak{X}$ is a sub-manifold.

Let $(\gamma_0, f_0) \in \mathfrak{X}$, we have to $T_{\gamma_0}\Gamma \times \{0\} \subset T_{(\gamma_0, f_0)}\mathfrak{X} \subset T_{\gamma_0}\Gamma \times T_{f_0}X$.

Now,

$$d\phi(\gamma, f) : T_\gamma\Gamma \times T_fX \longrightarrow T_{\phi(\gamma, f)}C^{j-1,\alpha}(\partial\Sigma),$$

then, we must prove that $d\phi(\gamma, f)$ is surjective and $\text{Ker}(d\phi(\gamma, f))$ is complemented.

We have that

$$\begin{aligned} d\phi(\gamma, f)(\tilde{\gamma}, \tilde{f}) &= \frac{\partial\phi}{\partial\gamma}\tilde{\gamma} + \frac{\partial\phi}{\partial f}\tilde{f} \\ &= \tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) + \frac{\partial\tilde{f}}{\partial\vec{n}_{\partial M}} + \gamma(\nabla_{\vec{n}_{\varphi_f}}\vec{n}_{\varphi_f}, \vec{n}_{\partial M})\tilde{f} \\ &= \tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) + \frac{\partial\tilde{f}}{\partial\vec{n}_{\partial M}} + \mathbb{I}^{\partial M}(\vec{n}_{\varphi_f}, \vec{n}_{\varphi_f})\tilde{f}. \end{aligned}$$

To prove that $d\phi(\gamma, f)$ is surjective we observe that $\tilde{\gamma} \in T_\gamma\Gamma$ is a symmetric bilinear form, then for any $h \in T_{\phi(\gamma, f)}C^{j-1,\alpha}(\partial\Sigma)$, we choose $\tilde{\gamma}$ such that

$$\tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) = h - \left(\frac{\partial\tilde{f}}{\partial\vec{n}_{\partial M}} + \mathbb{I}^{\partial M}(\vec{n}_{\varphi_f}, \vec{n}_{\varphi_f})\tilde{f} \right).$$

Now, to see that $d\phi(\gamma, f)$ is complemented, notice that the projection in the second factor,

$$P_2 : \text{Ker}(d\phi(\gamma, f)) \longrightarrow T_fX,$$

is surjective. Indeed, if $\tilde{f} \in T_fX$, we can take $\tilde{\gamma}$ such that $\tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) = -\left(\frac{\partial\tilde{f}}{\partial\vec{n}_{\partial M}} + \mathbb{I}^{\partial M}(\vec{n}_{\varphi_f}, \vec{n}_{\varphi_f})\tilde{f} \right)$.

For fixed $\tilde{f} \in T_fX$, we define the following space.

$$\{\tilde{\gamma} : (\tilde{\gamma}, \tilde{f}) \in \text{Ker}(d\phi(\gamma, f))\} = \tilde{\gamma}_{\tilde{f}} + \{\tilde{\gamma} : \tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) = 0\}.$$

where $\tilde{\gamma}_{\tilde{f}} = -\left(\frac{\partial \tilde{f}}{\partial \vec{n}_{\partial M}} + \mathbb{I}^{\partial M}(\vec{n}_{\varphi_f}, \vec{n}_{\varphi_f})\tilde{f}\right)$.

Let's see that

$$\Gamma_0 = \{\tilde{\gamma} : \tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) = 0\}$$

is complemented. In general, if N_1 and N_2 are differentiable fields in M , such that $N_1|_{\partial M} = \vec{n}_{\partial M}$, $N_2|_{\varphi_f} = \vec{n}_{\varphi_f}$ and let $\eta_0 \in \Gamma_{Sym}^{0,2}(M)$ be such that, in a neighborhood of $\partial M \cup \Sigma_f$,

$$\eta_0(N_1, N_2) = 1.$$

Then, for all $\eta \in \Gamma_{Sym}^{0,2}(M)$

$$\eta = \eta_1 + \eta_2,$$

where $\eta_1 = -\eta(N_1, N_2) \cdot \eta_0$ and $\eta_2 = \eta - \eta(N_1, N_2) \cdot \eta_0$. So, η_1 is in the generated by η_0 , $\eta_2(N_1, N_2) = 0$, then $\eta_2 \in \Gamma_0$. Therefore, Γ_0 is complemented.

Thus, the complement of $\text{Ker}(d\phi(\gamma, f))$ is the set

$$\{(\tilde{f}, \Gamma_0 - \tilde{\gamma}_{\tilde{f}}) : \tilde{f} \in T_f X\}.$$

Now, by the Local Form of the Submersions (see statement in [A.0.3](#)), we get that X is locally fibred. $\square$

Now, we define the function

$$\begin{aligned} A &= \Gamma \times X \longrightarrow \mathbb{R} \\ (\gamma, f) &\longmapsto A(\gamma, f) := \gamma\text{-area of } \varphi_f(\Sigma), \end{aligned}$$

and the operator

$$\begin{aligned} H &= \mathfrak{X} \longrightarrow Y \\ (\gamma, f) &\longmapsto H(\gamma, f) := \gamma\text{-mean curvature of } \varphi_f(\Sigma). \end{aligned}$$

We have therefore that

$$\frac{\partial H}{\partial f}(\gamma_0, 0) = J_{(\gamma_0, \varphi_0)},$$

where $J_{(\gamma_0, \varphi_0)}$ is the Jacobi operator J_{φ_0} defined in the metric γ_0 (see section [1.3](#)), which restricted to $C_{\partial}^{j,\alpha}(\Sigma)$ is Fredholm with index 0.

Also, we have that

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} A(\gamma, f + tl) &= \frac{d}{dt}\Big|_{t=0} \int_{\Sigma} \text{Vol}_{\varphi_{(f+tl)}^*}(\gamma) \\ &= \int_{\Sigma} H(\gamma, f) \cdot l \\ &= \langle H(\gamma, f), l \rangle_{\mathcal{H}} \end{aligned}$$

It remains to show that the condition of transversality defined by equation [2.1.4](#) is fulfilled, in order to verify the hypotheses of Theorem [2.1.1](#).

Let $\bar{f} \in \ker(\frac{\partial H}{\partial f}(\gamma_0, 0)) = \ker(J_{\varphi_0})$ be non-zero. As J_{φ_0} is restricted to $C_{\partial}^{j,\alpha}(\Sigma)$, then $\bar{f}$ satisfied

the linearized free boundary condition. Take a smooth variation of φ_0 , $\varphi_t := \varphi_{f_t}$, $-\epsilon < t < \epsilon$, such that

$$\varphi_t(p) = \text{Exp}_{\varphi_0(p)}([t\bar{f}(p) + o(t)]\vec{n}_0(p)), \quad (2.4.1)$$

where $o(t) : (-\epsilon, \epsilon) \rightarrow C^{k,\alpha}(\Sigma_0)$ is a differential application with $\frac{o(t)}{t} \rightarrow 0$ if $t \rightarrow 0$. By Lemma 2.4.2 φ_t is orthogonal to ∂M .

Now, take a variation γ_s , $-\delta < s < \delta$, of γ_0 by conformal metrics as follows:

$$\gamma_s(q) = (1 + s\psi(q))\gamma_0(q), \quad (2.4.2)$$

where $\psi : M \rightarrow \mathbb{R}$ is a smooth function such that $\psi(q) = 0$ for all $q \in \varphi_0(\Sigma)$, this is, if $q = \varphi_0(p)$, $\psi(\text{Exp}_q(0)) = \psi(q) = 0$.

Let $\{\Omega, (x_1, \dots, x_n)\}$ be a local coordinate chart of Σ , $p \in \Omega$. The volume form associated to the metric γ_s is given by

$$\begin{aligned} \text{vol}_{\varphi_f^*(\gamma_s)}|_p &= \sqrt{\text{Det}[\varphi_f^*(\gamma_s)]} dx^1 \wedge \dots \wedge dx^n|_p \\ &= \sqrt{\text{Det}\left[\gamma_s\left(d\varphi_f\left(\frac{\partial}{\partial x_i}\right), d\varphi_f\left(\frac{\partial}{\partial x_j}\right)\right)\right]} dx^1 \wedge \dots \wedge dx^n|_p \\ &= \sqrt{\text{Det}\left[\left(1 + s\psi(\text{Exp}_{\varphi_0(p)}(f(p)\vec{n}_0(p)))\right)\gamma_0\left(d\varphi_f\left(\frac{\partial}{\partial x_i}\right), d\varphi_f\left(\frac{\partial}{\partial x_j}\right)\right)\right]} \\ &\quad dx^1 \wedge \dots \wedge dx^n \\ &= \left(1 + s\psi(\text{Exp}_{\varphi_0(p)}(f(p)\vec{n}_0(p)))\right)^{n/2} \sqrt{\text{Det}\left[\gamma_0\left(d\varphi_f\left(\frac{\partial}{\partial x_i}\right), d\varphi_f\left(\frac{\partial}{\partial x_j}\right)\right)\right]} \\ &\quad dx^1 \wedge \dots \wedge dx^n \\ &= \left(1 + s\psi(\text{Exp}_{\varphi_0(p)}(f(p)\vec{n}_0(p)))\right)^{n/2} \text{vol}_{\varphi_f^*(\gamma_0)} \end{aligned}$$

To simplify the notation we write

$$f_t = t\bar{f} + o(t)$$

and

$$v_t = f_t\vec{n}_0.$$

Note that $f_0 \equiv 0$, so $v_0 = \vec{0}$. Also take $q = \varphi_0(p)$ and $\varphi_t = \varphi_{f_t}$. Hence, the area function over the variations of φ_0 and γ_0 given by equations 2.4.1 and 2.4.2 have the following form:

$$\begin{aligned} A(\gamma_s, f_t) &= \int_{\Sigma} \text{vol}_{\varphi_t^*(\gamma_s)} \\ &= \int_{\Sigma} \left(1 + s\psi(\text{Exp}_q(v_t(p)))\right)^{n/2} \text{vol}_{\varphi_t^*(\gamma_0)}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
& \frac{\partial^2}{\partial t \partial s} A(\gamma_s, f_t) \Big|_{s=t=0} \\
&= \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \int_{\Sigma} \left(1 + s\psi(\text{Exp}_q(v_t))\right)^{n/2} \text{vol}_{\varphi_t^*(\gamma_0)} \\
&= \frac{d}{dt} \Big|_{t=0} \int_{\Sigma} \frac{n}{2} \left(1 + s\psi(\text{Exp}_q(v_t))\right)^{n/2-1} \Big|_{s=0} \left(\psi(\text{Exp}_q(v_t))\right) \text{vol}_{\varphi_t^*(\gamma_0)} \\
&= \frac{d}{dt} \Big|_{t=0} \int_{\Sigma} \frac{n}{2} \psi(\text{Exp}_q(v_t)) \text{vol}_{\varphi_t^*(\gamma_0)} \\
&= \int_{\Sigma} \frac{n}{2} \left[\frac{d}{dt} \left(\psi(\text{Exp}_q(v_t))\right) \text{vol}_{\varphi_t^*(\gamma_0)} + \psi(\text{Exp}_q(v_t)) \frac{d}{dt} \text{vol}_{\varphi_t^*(\gamma_0)} \right]_{t=0} \\
&= \int_{\Sigma} \frac{n}{2} \left[d\psi(\text{Exp}_q(v_t)) \cdot d\text{Exp}_q(v_t) \left(\frac{d}{dt} v_t\right) \text{vol}_{\varphi_t^*(\gamma_0)} \right]_{t=0} + 0,
\end{aligned}$$

but $\frac{d}{dt}(v_t) = \frac{d}{dt}((t\bar{f} + O(t))\bar{n}_0) = (\bar{f} + O'(t))\bar{n}_0$, then

$$\begin{aligned}
&= \int_{\Sigma} \frac{n}{2} \left[d\psi(\text{Exp}_q(0)) \cdot d\text{Exp}_q(0)(\bar{f}(p)\bar{n}_0(p)) \right] \text{vol}_{\varphi_0^*(\gamma_0)} \\
&= \int_{\Sigma} \frac{n}{2} \left[d\psi(\varphi_0(p)) \cdot \bar{f}(p)\bar{n}_0(p) \right] \text{vol}_{\varphi_0^*(\gamma_0)} \\
&= \int_{\Sigma} \frac{n}{2} \bar{f}(p) \left[d\psi(\varphi_0(p))(\bar{n}_0(p)) \right] \text{vol}_{\varphi_0^*(\gamma_0)}
\end{aligned}$$

so, by lemma 2.4.3 we can to choose ψ such that the last integral be non zero.

Therefore, by Theorem 2.1.1, there is a neighborhood W of $(\gamma_0, 0)$ such that

$$\widetilde{\mathcal{M}} = \{(\gamma, f) \in W : H(\gamma, f) = 0\}$$

is a Banach sub-manifold of $\Gamma \times X$. Now, since $h(\gamma, f) = (\gamma, [\varphi_f])$ is a diffeomorphism between $\widetilde{\mathcal{M}}$ and an open subset of $\mathcal{M}$, we conclude that $\mathcal{M}$ is a Banach sub-manifold of $\Gamma \times \mathcal{E}_{\partial}(\Sigma, M)$.

It remains to show the separability of $\mathcal{M}$. For a coordinate system $(x_1, \dots, x_n)$ in $\varphi_f(\Sigma) = \Sigma_f$, the γ -mean curvature of Σ_f is

$$H(\gamma, f) = \gamma^{ij} \gamma \left(\nabla_{\frac{\partial \varphi_f}{\partial x_i}} \frac{\partial \varphi_f}{\partial x_j}, \vec{\eta}_{\Sigma_f} \right),$$

where $\gamma_{ij} = \gamma(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and $\gamma^{il} \gamma_{lj} = \delta_{ij}$. If $H(\gamma_0, 0) = 0$, then φ_0 is solution of a linear elliptic partial differential equation. So by the Schauder theory for elliptic equations we have that φ_0 is C^k (see [2, Sec. 10]). Let S be the set of γ -orthogonal minimal embeddings of Σ in M . $S \subset C^k(\Sigma, M)$. Since $C^k(\Sigma, M)$ is a separable space and $C^k(\Sigma, M) \subset C^{j,\alpha}(\Sigma, M)$ the Lemma 2.4.4 implies that the closure of S in the topology $C^{j,\alpha}$ is also separable.

Now we state Smale's theorem which is a generalization of Sard's theorem for infinite-dimensional spaces. For a proof, see [26]. We will say *almost all* instead of "except for a set of first category".

Sard-Smale Theorem. *Let V and W be two differentiable Banach manifolds, connected and second-countable. Let $\phi : V \rightarrow W$ be a C^k Fredholm map with $k > \max\{\text{index } \phi, 0\}$. Then, the set of regular values of ϕ is generic in W .*

Whence, as a consequence of Theorem 2.4.1 and the Sard-Smale theorem, we infer the following

result.

Corollary 2.4.6. *Under hypotheses of Theorem 2.4.1, the set of (M, Σ) -Bumpy metrics is generic in Γ .*

□

2.4.3 Nonzero constant mean curvature

Let's see now that the main result is valid also when the mean curvature is a constant different from zero.

Let $\gamma \in \text{Met}(M)$, recall the following notation (see 2.3.2):

$$\text{Emb}_{\partial^\perp}^\gamma(\Sigma, M) := \{\varphi : \Sigma \rightarrow M \text{ is } \gamma\text{-orthogonal admissible} \\ \text{embedding and bounding a finite volume}\}$$

$\varphi \in \text{Emb}_{\partial^\perp}^\gamma(\Sigma, M)$ has CMC $\mathfrak{h} \neq 0$ if and only if it is a critical point for the functional

$$\mathfrak{f}_\mathfrak{h}(\varphi) = \int_\Sigma \text{vol}_{\varphi^*(\gamma)} - \mathfrak{h} \int_{\Omega_\varphi} \text{vol}_\gamma,$$

where Ω_φ is the finite volume that bounded φ .

If in the statement of the main theorem we modify the condition of zero mean curvature by the condition of constant mean curvature $\mathfrak{h} \neq 0$ in the definition of $\mathcal{M}$, that is,

$$\mathcal{M} = \{(\gamma, [\varphi]) \in \Gamma \times \mathcal{E}_\partial(\Sigma, M) : [\varphi] \in \mathcal{E}_{\partial, \gamma}^\perp(\Sigma, M), \varphi \text{ with } \gamma\text{-CMC } \mathfrak{h} \neq 0\},$$

the proof of the theorem is not modified, except for transversality. More explicitly, we must verify that, given $(\gamma_0, [\varphi_0]) \in M$, for all $\bar{f} \in \text{Ker}(\frac{\partial H}{\partial f}(\gamma_0, 0))$, $\bar{f} \neq 0$, there exists a family $(\gamma_s, f_t) \in \mathfrak{X}$, $s, t \in (-\epsilon, \epsilon)$,

$$\mathfrak{X} = \{(\gamma, f) \in \Gamma \times C^{j, \alpha} : f \in U, \varphi_f \in \text{Emb}_{\partial^\perp}^\gamma(\Sigma, M)\},$$

with $f'(0) = \bar{f}$, such that

$$\frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} \bar{A}(\gamma_s, f_t) \neq 0$$

where

$$\bar{A}(\gamma_s, f_t) = \int_\Sigma \text{vol}_{\varphi_t^*(\gamma_s)} - \mathfrak{h} \int_{\Omega_{\varphi_t}} \text{vol}_{\gamma_s}.$$

As before, we take a family of conformal metrics of the form $\gamma_s = (1 + s\psi)\gamma_0$, with $\psi : M \rightarrow \mathbb{R}$ a smooth function such that $\phi(p) = 0$ for all $p \in \Sigma_0 = \varphi_0(\Sigma)$, where

$$\frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} \int_\Sigma \text{vol}_{\varphi_t^*(\gamma_s)} \neq 0,$$

it is sufficient to prove that

$$\frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} \int_{\Omega_{\varphi_t}} \text{vol}_{\gamma_s} = 0.$$

Now,

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} \int_{\Omega_{\varphi_t}} \text{vol}_{\gamma_s} &= \int_{\Sigma_0} \gamma_s \left(\frac{\partial \varphi_t}{\partial t} \Big|_{t=0}, \vec{n}_{\Sigma_0} \right) \text{vol}_{\varphi_0^*(\gamma_s)} \\ &= \int_{\Sigma_0} \gamma_0 \left(\frac{\partial \varphi_t}{\partial t} \Big|_{t=0}, \vec{n}_{\Sigma_0} \right) \text{vol}_{\varphi_0^*(\gamma_0)}. \end{aligned}$$

The last integral does not depend on s , so we get the conclusion.

Corollary 2.4.7. *If in Theorem 2.4.1 we replaces the hypothesis that φ is γ -minimal by φ has nonzero CMC, the set of (M, Σ) -Bumpy metrics is generic in Γ .*

Chapter 3

Deformation of Free Boundary CMC Immersions and Stability Criteria

In this Chapter, given a free boundary CMC immersion $\varphi \in C^{j+1,\alpha}(\Sigma, M)$, we prove, using properties of eigenvalues and eigenfunctions of the Jacobi operator J_φ , that if $\dim(\text{Ker}(J_\varphi)) = 0$, or if $\dim(\text{Ker}(J_\varphi)) = 1$ and, for $f \in \text{Ker}(J_\varphi)$, $f \neq 0$, $\int_\Sigma f \text{vol}_{\varphi^*(g)} \neq 0$, φ has a free boundary CMC deformation and this is unique up to diffeomorphism (see Theorems 3.2.1 and 3.2.2). In addition, we give stability and unstability criteria for free boundary CMC immersions (see Theorem 3.3.3 and Corollary 3.3.5). We have based on the article by Miyuki Koiso (see [16]), who studies the case for CMC surfaces in $\mathbb{R}^3$ with fixed boundary. Using the properties of the eigenvalues and the eigenfunctions of a problem of eigenvalues associated with the second variation of the functional area, Koiso gives sufficient conditions under which it has a CMC deformation that fixes the boundary and also a criterion of stability for the CMC immersions. For a special case, combine these results and obtain a geometric way to study stability.

3.1 Analytical preliminaries

Following with the same notation, Σ^n and M^{n+1} be smooth manifolds, with smooth boundaries $\partial\Sigma$ and ∂M respectively. Let g be a C^∞ riemannian metric in M , and $\varphi_0 : \Sigma \rightarrow M$ an free boundary CMC immersion, H_0 the mean curvature of φ_0 . Let $\vec{n}_{\Sigma_0}$ be the unit normal vector field to $\Sigma_0 = \varphi_0(\Sigma)$ in the orientable normal bundle and $\vec{n}_{\partial M}$ the unit normal vector field to ∂M . $C_\partial^{j,\alpha}(\Sigma_0)$ the space of functions $C^{j,\alpha}$ that complies with the linearized free boundary condition $g(\nabla f, \vec{n}_{\partial M}) + \mathbb{I}^{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0})f = 0$, defined in (1.4.2). We have that $C_\partial^{j,\alpha}(\Sigma_0) \subset L^2(\Sigma_0)$.

Remember that by the Proposition 2.3.1 we have the bijection $f \mapsto \varphi_f$ from a suitable neighborhood U of $0 \in C_\partial^{j,\alpha}(\Sigma_0)$ to a neighborhood V of $\varphi_0 \in C^{j,\alpha}(\Sigma, M)$ defined by

$$\varphi_f(p) := \exp_{\varphi_0(p)}([f(p) + o(f)(p)]\vec{n}_{\Sigma_0}(p)),$$

where $\exp$ is the exponential function of g , $o : U \rightarrow C_\partial^{j,\alpha}(\Sigma_0)$ is such that $\frac{o(f)}{f} \rightarrow 0$ if $f \rightarrow 0$ and such that φ_f is orthogonal to M .

Remember too that restriction of Jacobi operator, define as $J_{\varphi_0}(f) := \Delta_{\Sigma_0}f - (\|\mathbb{I}^{\Sigma_0}\|_{HS}^2 + \text{Ric}_g(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0}))f$, to the closed subspace $C_\partial^{j,\alpha}(\Sigma_0)$ is a Fredholm operator of index zero that takes values in $C_\partial^{j-2,\alpha}(\Sigma_0)$.

To simplify the notation we set

$$J = J_{\varphi_0} : C_{\partial}^{j,\alpha}(\Sigma_0) \rightarrow C^{j-2,\alpha}(\Sigma_0).$$

Let $E = \text{Ker}(J)$ and $E^\perp$ the orthogonal of E in $L^2(\Sigma)$.

By definition $\Delta_{\Sigma_0} f = \text{Div}(-\nabla f)$ and if V is a vector field, then $\text{Div}(V) = \nabla \cdot V$. Now, if h is a scalar field we have to $\text{Div}(hV) = \nabla h \cdot V + h \text{Div}(V)$. On the other hand by the Gauss theorem (divergence theorem)

$$\int_{\Sigma} \text{Div}(hV) d_{\Sigma} = \int_{\partial\Sigma} hV \cdot \vec{n}_{\Sigma} d_{\partial\Sigma}$$

Proposition 3.1.1. *J is symmetric with the $L^2(\Sigma_0)$ product inner.*

Proof. From the definition of J_{φ_0} we see that it is sufficient to prove that Δ_{Σ_0} is symmetric. Let $f, h \in C_{\partial}^{j,\alpha}(\Sigma_0)$. Then, by the divergence theorem

$$\begin{aligned} \int_{\Sigma_0} \Delta_{\Sigma_0}(f)h d_{\Sigma_0} &= \int_{\Sigma_0} \text{Div}(-\nabla f)h d_{\Sigma_0} \\ &= - \int_{\Sigma_0} \nabla f \cdot \nabla h d_{\Sigma_0} - \int_{\partial\Sigma_0} h \left(\frac{\partial f}{\partial \vec{n}_{\partial M}} \right) d_{\partial\Sigma_0} \\ &= - \int_{\Sigma_0} \nabla f \cdot \nabla h d_{\Sigma_0} - \int_{\partial\Sigma_0} h \left(g(\nabla f, \vec{n}_{\partial M}) \right) d_{\partial\Sigma_0} \\ &= - \int_{\Sigma_0} \nabla f \cdot \nabla h d_{\Sigma_0} + \int_{\partial\Sigma_0} h \left(f \mathbb{I}^{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0}) \right) d_{\partial\Sigma_0} \\ &= - \int_{\Sigma_0} \nabla h \cdot \nabla f d_{\Sigma_0} + \int_{\partial\Sigma_0} f \left(h \mathbb{I}^{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0}) \right) d_{\partial\Sigma_0} \\ &= \int_{\Sigma_0} \Delta_{\Sigma_0}(h)f d_{\Sigma_0} \end{aligned}$$

□

Lemma 3.1.2. *Let λ be a real number. Let*

$$J(f) - \lambda f = h \tag{3.1.1}$$

a equation, where $h \in C^{j-2,\alpha}(\Sigma_0)$. (3.1.1) has a solution in the following cases:

- (a) *If λ is not an eigenvalue of J , then (3.1.1) has a unique solution in $C_{\partial}^{j,\alpha}(\Sigma_0)$ for all $h \in C^{j-2,\alpha}(\Sigma_0)$*
- (b) *If λ is an eigenvalue of J , $E_{\lambda} \subset C_{\partial}^{j,\alpha}(\Sigma_0)$ be the eigenspace associated to λ and $E_{\lambda}^{\perp} \subset L^2(\Sigma_0) \cap C_{\partial}^{j,\alpha}(\Sigma_0)$ the orthogonal to E_{λ} , then (3.1.1) have solution in $C_{\partial}^{j,\alpha}(\Sigma_0)$ if and only if $h \in E_{\lambda}^{\perp}$, this is*

$$\int_{\Sigma_0} h \sigma \text{vol}_{\varphi_0^*(g)} = 0,$$

for all $\sigma \in E_{\lambda}$.

Proof. For (a), we have that J is a Fredholm operator of index zero. Then so is the operator $(J - \lambda) : C_{\partial}^{j,\alpha}(\Sigma_0) \rightarrow C^{j-2,\alpha}(\Sigma_0)$ (as the sum of Fredholm and compact operators). If λ is not an eigenvalue then $(J - \lambda)$ is injective and hence also surjective ($\text{Codim}(J - \lambda) = 0$).

For (b), let $h \in C^{j-2,\alpha}(\Sigma_0)$ be such that $h = J(f) - \lambda(f)$ for some $f \in C^{j,\alpha}_\partial(\Sigma_0)$ and let $\sigma \in E_\lambda$. Then,

$$\begin{aligned} \int_{\Sigma_0} h \sigma \, \text{vol}_{\varphi_0^*(g)} &= \int_{\Sigma_0} (J(f) - \lambda f) \sigma \, \text{vol}_{\varphi_0^*(g)} \\ &= \int_{\Sigma_0} J(f) \sigma - \lambda \sigma f \, \text{vol}_{\varphi_0^*(g)} \\ &= \int_{\Sigma_0} J(f) \sigma - J(\sigma) f \, \text{vol}_{\varphi_0^*(g)} \\ &= 0. \end{aligned}$$

The equality holds because J is symmetric (by proposition 3.1.1). Therefore $\text{Im}(J - \lambda) \subset E_\lambda^\perp$.

Again, as $(J - \lambda)$ is Fredholm of index zero, $\text{Im}(J - \lambda)$ is closed and

$$\text{Codim}(\text{Im}(J - \lambda)) = \text{Dim}(\text{Ker}(J - \lambda)) = \text{Dim}(E_\lambda),$$

and since $\text{Im}(J - \lambda) \subset E_\lambda^\perp$, then $\text{Im}(J - \lambda) = E_\lambda^\perp$. $\square$

3.2 Existence and uniqueness of CMC deformation

Next, we enunciate our first perturbation theorem. Here we assume that the kernel of the Jacobi operator is trivial. The perturbation obtained is unique, up to parameterizations.

Theorem 3.2.1. *Let $\varphi_0 \in C^{j+1,\alpha}(\Sigma, M)$ a free boundary CMC immersion, with mean curvature H_0 . Suppose that $\text{Dim}(E) = 0$, this is, the eigenvalues of problem*

$$\begin{cases} J(f) = \lambda f \\ f \in C^{j,\alpha}_\partial(\Sigma_0) \end{cases} \quad (3.2.1)$$

are nonzero. Then, there is a neighborhood $\hat{I}$ of $H_0 \in \mathbb{R}$ and a unique injective C^1 mapping, $\zeta : \hat{I} \rightarrow C^{j,\alpha}_\partial(\Sigma_0)$, such that $\zeta(H_0) = 0$ and $\varphi_{\zeta(H)}$ is a free boundary CMC immersion with mean curvature H .

Moreover, if $\psi : \Sigma \rightarrow M$ is an free boundary CMC immersion sufficiently close to φ_0 , in the topology of $C^{j,\alpha}$, then ψ must be equal (up to diffeomorphisms) to some $\varphi_{\zeta(H)}$.

Proof. Let U a suitably small neighborhood of $0 \in C^{j,\alpha}_\partial(\Sigma_0)$ such that φ_f is orthogonal to M for each $f \in U$.

We define the following map

$$\begin{aligned} F : U \times \mathbb{R} &\longrightarrow C^{j-2,\alpha}(\Sigma_0) \\ (f, H) &\mapsto F(f, H) = H - H_f, \end{aligned}$$

where H_f is the mean curvature of φ_f . Then we have that

i) $F(0, H_0) = 0$ and

ii) φ_f has CMC if and only if $F(f, H) = 0$ for some $H \in \mathbb{R}$.

Now,

$$\frac{\partial F}{\partial f}(0, H_0) : C_{\partial}^{j,\alpha}(\Sigma_0) \rightarrow C^{j-2,\alpha}(\Sigma_0)$$

is given by

$$\frac{\partial F}{\partial f}(0, H_0)(h) = J(h).$$

Let us see that if 0 is not an eigenvalue of $J(f) = \lambda f$ then $\frac{\partial F}{\partial f}(0, H_0)$ is *bijective*. The injectivity is immediate because $\dim(\text{Ker } J) = \dim(E) = 0$. Now, applying part (a) of Lemma 3.1.2 with $\lambda = 0$ the surjectivity is obtained.

Thus, the conditions for applying the Implicit Mapping Theorem (see the statement in A.0.2) are satisfied for F . Then there is a neighborhood $\hat{I}$ of H_0 and a map $\zeta : \hat{I} \rightarrow U$ such that $\zeta(H_0) = 0$ and $F(\zeta(H), H) = 0$. So $\varphi_{\zeta(H)}$ is a free boundary CMC immersion with mean curvature H . $\square$

A second result on existence of perturbation is given below. Here we assume that the Jacobi operator kernel is generated by a single non-zero function in $C_{\partial}^{j+1,\alpha}(\Sigma)$. Again, we get uniqueness up to parameterizations.

Theorem 3.2.2. *Let $\varphi_0 \in C^{j+1,\alpha}(\Sigma, M)$ a free boundary CMC immersion, with mean curvature H_0 . Suppose that:*

1. *$\dim(E) = 1$. This is, $\lambda = 0$ is an eigenvalue of multiplicity 1 for the problem (3.2.1), and*
2. *$\int_{\Sigma_0} f_0 \text{ vol}_{\varphi_0^*(g)} \neq 0$ for some $f_0 \in E - \{0\}$.*

Then there exist a neighborhood $W \subset E$ of 0 and a unique injective map C^1

$$(\xi, \eta) : W \longrightarrow (C_{\partial}^{j,\alpha}(\Sigma_0) \cap E^{\perp}) \times \mathbb{R},$$

such that $(\xi, \eta)(0) = (0, H_0)$ and such that $\varphi_{f+\xi(f)} : \Sigma \rightarrow M$, with $f \in W$, is a free boundary CMC immersion, with mean curvature $\eta(f)$.

Moreover, if $\psi : \Sigma \rightarrow M$ is a free boundary CMC immersion sufficiently close to φ_0 , in the topology of $C^{j,\alpha}$, then Y must be equal (up to diffeomorphisms) to some $\varphi_{f+\xi(f)}$.

Proof. We take $F : U \times \mathbb{R} \longrightarrow C^{j-2,\alpha}(\Sigma_0)$, defined as $F(f, H) = H - H_f$, as in the proof of Theorem 3.2.1. We have to

- i) $F(0, H_0) = 0$ and
- ii) φ_f have CMC if and only if $F(f, H) = 0$ for some $H \in \mathbb{R}$.

Now we defined the following map:

$$\begin{aligned} \bar{F} : (U \cap E) \times (U \cap E^{\perp}) \times \mathbb{R} &\longrightarrow C^{j-2,\alpha}(\Sigma_0) \\ (f_1, f_2, H) &\longmapsto \bar{F}(f_1, f_2, H) = F(f_1 + f_2, H). \end{aligned}$$

Thus,

$$\bar{F}(0, 0, H_0) = 0.$$

Let's see that

$$\frac{\partial \bar{F}}{\partial(f_2, H)}(0, 0, H) : (C_{\partial}^{j, \alpha}(\Sigma_0) \cap E^{\perp}) \times \mathbb{R} \longrightarrow C^{j-2, \alpha}(\Sigma_0)$$

is *bijective*.

We have that

$$\frac{\partial \bar{F}}{\partial(f_2, H)}(0, 0, H)(f, H) = H - J(f)$$

and

$$\text{Ker}\left(\frac{\partial \bar{F}}{\partial(f_2, H)}(0, 0, H)\right) = \{(f, H) : H - J(f) = 0\}$$

i.e. if $(f_2, H) \in \text{Ker}\left(\frac{\partial \bar{F}}{\partial(f_2, H)}(0, 0, H)\right)$, then $J(f_2) = H$. Let $f_0 \in E$. Then, by (b) in Lemma 3.1.2 we have

$$H \int_{\Sigma_0} f_0 \text{vol}_{\varphi_0^*(g)} = 0,$$

which implies that $H = 0$, i.e., $f_2 \in E$. But $f_2 \in E^{\perp}$, so $f_2 = 0$. Therefore, $\text{Ker}\left(\frac{\partial \bar{F}}{\partial(f_2, H)}(0, 0, H)\right) = \{(0, 0)\}$ and we have the injectivity.

To prove the surjectivity, we take $h \in C^{j-2, \alpha}(\Sigma_0)$ and set

$$H = \frac{\int_{\Sigma_0} h f_0 \text{vol}_{\varphi_0^*(g)}}{\int_{\Sigma_0} f_0 \text{vol}_{\varphi_0^*(g)}}.$$

Again by (b) in Lemma 3.1.2 there exist $f \in C_{\partial}^{j, \alpha}(\Sigma_0)$ such that

$$J(f) = h - H.$$

Now, we decompose $f = f_1 + f_2$ such that $f_1 \in E$, $f_2 \in E^{\perp}$. So,

$$J(f_2) = h - H.$$

Then,

$$h = H - J(f_2) = \frac{\partial \bar{F}}{\partial(f_2, H)}(0, 0, H)(f_2, H).$$

Now we can use The Implicit Mapping Theorem in $\bar{F} : (U \cap E) \times (U \cap E^{\perp}) \times \mathbb{R} \longrightarrow C^{j-2, \alpha}(\Sigma_0)$. Then, there exist a neighborhood $W \subset E \cap U$ of 0 and a map

$$(\xi, \eta) : W \longrightarrow (U \cap E^{\perp}) \times \mathbb{R},$$

such that

$$(\xi, \eta)(0) = (0, H_0)$$

and such that, for all $f \in W$,

$$0 = \psi(f, (\xi, \eta)(f)) = \psi(f, \xi(f), \eta(f)) = \phi(f + \xi(f), \eta(f)).$$

Thus, $\varphi_{f+\xi(f)}$ have CMC $\eta(f)$. If W is suitable small then (ξ, η) is unique. $\square$

3.3 Stability Criteria

Let $\varphi_t : \Sigma \rightarrow M$ be a variation of φ_0 , $-\epsilon < t < \epsilon$, such that φ_t is a free boundary CMC immersion. Let $A(t)$ and $V(t)$ be the area and volume of φ_t respectively. We will say that φ_t is volume-preserving if $V(t) = V(0)$ for all $t \in (-\epsilon, \epsilon)$. Remember that φ_t have CMC H_t if only if is a critical point of funtional

$$\mathfrak{f}(t) = A(t) - H_t V(t) = \int_{\Sigma} \text{vol}_{\varphi_t^*(g)} - H_t \int_{\Omega_t} \text{vol}_g,$$

where Ω_t is the volume enclosed by φ_t . Let $\xi_t = g(\frac{\partial \varphi_t}{\partial t}, \vec{n}_{\Sigma_t})$, then we have (see [3, Appendix])

$$A'(t) = \int_{\Sigma} H_t \xi_t \text{vol}_{\varphi_t^*(g)}.$$

Then,

$$A''(0) = \int_{\Sigma} J(\xi_0) \xi_0 \text{vol}_{\varphi_0^*(g)}$$

Also,

$$V'(t) = \int_{\Sigma} \xi_t \text{vol}_{\varphi_t^*(g)}$$

Definition 3.3.1. A free boundary CMC immersion $\varphi : \Sigma \rightarrow M$ is said to be *stable* if only if $A''(0)(f) \geq 0$, for all $f \in C_{\partial}^{j,\alpha}(\Sigma)$ such that $\int_{\Sigma} f \text{vol}_{\varphi_0^*(g)} = 0$. When φ is not stable, it is said to be unstable.

To simplify the notation we write

$$I(f) = \int_{\Sigma} J(f) f \text{vol}_{\varphi_0^*(g)}.$$

Since the stability criteria, presented below, depend of eigenvalues of Jacobi operator, it is important to highlight the following in relation to them:

Remark 3.3.1. In the problem (3.2.1), the eigenvalues of $J(f) = \lambda f$ are a countable set (Smale, [27, lemma 1]), such that $\lambda_1 < \lambda_2 \leq \dots$, with $\lambda_i \rightarrow \infty$ if $i \rightarrow \infty$. Each eigenfunction is in $C_{\partial}^{j+1,\alpha}(\Sigma)$ (Gilbarg-Trudinger, [13, Theorem 8.13], and Ladyzhenskaya-Ural'tseva [19, Chap. 3, Theorem 12.1]). And we can choose an orthonormal basis $B = \{f_i\}$ for $L^2(\Sigma)$ where each f_i is associated to λ_i . The first eigenvalue λ_1 , which has a special role in the spectral theory of J , is always simple, i.e., of multiplicity 1, and the λ_1 -eigenfunction are positives. Let f_k be a (non zero) eigenfunction corresponding to the eigenvalue λ_k , for some $k \geq 1$. The connected components of the set $\Sigma \setminus f_k^{-1}(0)$ are called the **nodal domains** of f_k . Then, the number of nodal domains of f_k is less than or equal to k ; this is known as **Courant's nodal domain theorem** (see [4]).

For the demonstration of our stability criteria we need the following Smale's Lemma.

Lemma 3.3.2. [27, lemma 4][Smale Lemma]

- a) Let $\mathcal{H}$ be the prehilbert space $\mathcal{H}_0^k(E)$ with the inner product $\langle \cdot, \cdot \rangle_L$ and B_L quadratic form on $\mathcal{H}$, $B_L(h) = B_L(h, h)$. Then B_L has a minimum λ_1 at f_1 on the unit sphere S of $\mathcal{H}$ and on $\{f_1, \dots, f_{q-1}\}^\perp \cap S$ has its minimum value λ_q and f_q

b) Let $d(v)$ be the minimum of B_L on $V^\perp \cap S$, where V is a finite dimensional subspace of $\mathcal{H}$.
Then

$$\lambda_n = \max_{\dim V < n} d(V)$$

Thus, we can write our first stability criterion in the following way:

Theorem 3.3.3. *Let $\varphi_0 : \Sigma \rightarrow M$ a free boundary CMC immersion. Let $\lambda_i, i \geq 1$ be the eigenvalues J_{φ_0} .*

(1) *If $\lambda_1 \geq 0$, then φ_0 is stable.*

(2) *If $\lambda_1 < 0 < \lambda_2$, then there is a unique function $\kappa \in C^{j,\alpha}(\Sigma)$ such that $J(\kappa) = 1$ and we have that:*

(2-a) *If $\int_\Sigma \kappa \, \text{vol}_{\varphi_0^*(g)} \leq 0$, then φ_0 is stable.*

(2-b) *If $\int_\Sigma \kappa \, \text{vol}_{\varphi_0^*(g)} > 0$, then φ_0 is unstable.*

(3) *If $\lambda_1 < 0 = \lambda_2$, then we have:*

(3-a) *If there exist a λ_2 -eigenfunction f_2 such that $\int_\Sigma f_2 \, \text{vol}_{\varphi_0^*(g)} \neq 0$, then φ_0 is unstable.*

(3-b) *If $\int_\Sigma h_2 \, \text{vol}_{\varphi_0^*(g)} = 0$ for all λ_2 -eigenfunction h_2 , then there exist a unique function $\bar{h}_2 \in (\ker(J_{\varphi_0}))^\perp$ such that $J(\bar{h}_2) = 1$ and*

(3-b-i) *If $\int_\Sigma \bar{h}_2 \, \text{vol}_{\varphi_0^*(g)} \leq 0$, then φ_0 is stable.*

(3-b-ii) *If $\int_\Sigma \bar{h}_2 \, \text{vol}_{\varphi_0^*(g)} > 0$, then φ_0 is unstable*

(4) *If $\lambda_2 < 0$, then φ_0 is unstable.*

Proof. By Lemma 3.3.2 we have that:

$$\lambda_1 = I(f_1) = \int_\Sigma J(f_1) f_1 \, \text{vol}_{\varphi_0^*(g)} = \min \left\{ I(f) : f \in C_{\partial}^{j,\alpha}(\Sigma) \text{ and } \int_\Sigma f^2 \, \text{vol}_{\varphi_0^*(g)} = 1 \right\} \quad (3.3.1)$$

$$\begin{aligned} \lambda_i = I(f_i) &= \int_\Sigma J(f_i) f_i \, \text{vol}_{\varphi_0^*(g)} = \min \left\{ I(f) : f \in C_{\partial}^{j,\alpha}(\Sigma), \right. \\ &\quad \left. \int_\Sigma f^2 \, \text{vol}_{\varphi_0^*(g)} = 1, \text{ and } \int_\Sigma f f_k \, \text{vol}_{\varphi_0^*(g)} = 0, \text{ for } k \in \{1, \dots, i-1\} \right\}, \\ &\quad i = 2, 3, \dots \end{aligned} \quad (3.3.2)$$

So, if $\lambda_1 \geq 0$ we have (I).

Now we assume that $\lambda_1 < 0$. We know that f_1 does not change sign (see Remark 3.3.1), then

$$\int_\Sigma f_1 \, \text{vol}_{\varphi_0^*(g)} \neq 0.$$

For $\kappa \in C_{\partial}^{j+1,\alpha}(\Sigma)$, let

$$a = - \frac{\int_\Sigma \kappa \, \text{vol}_{\varphi_0^*(g)}}{\int_\Sigma f_1 \, \text{vol}_{\varphi_0^*(g)}},$$

and

$$\xi = af_1 + \kappa.$$

Then,

$$\begin{aligned} \int_{\Sigma} \xi \operatorname{vol}_{\varphi_0^*(g)} &= \int_{\Sigma} af_1 + \kappa \operatorname{vol}_{\varphi_0^*(g)} \\ &= \int_{\Sigma} -\frac{\int_{\Sigma} \kappa \operatorname{vol}_{\varphi_0^*(g)} f_1 + \kappa \int_{\Sigma} f_1 \operatorname{vol}_{\varphi_0^*(g)}}{\int_{\Sigma} f_1 \operatorname{vol}_{\varphi_0^*(g)}} \\ &= \frac{1}{\int_{\Sigma} f_1 \operatorname{vol}_{\varphi_0^*(g)}} \left[-\int_{\Sigma} \kappa \operatorname{vol}_{\varphi_0^*(g)} \int_{\Sigma} f_1 \operatorname{vol}_{\varphi_0^*(g)} \right. \\ &\quad \left. + \int_{\Sigma} \kappa \operatorname{vol}_{\varphi_0^*(g)} \int_{\Sigma} f_1 \operatorname{vol}_{\varphi_0^*(g)} \right] \\ &= 0 \end{aligned}$$

By Lemma 3.1.2-(a) (with $\lambda = 0$ and $h = 1$) there exists $\kappa \in C_{\partial}^{j+1,\alpha}(\Sigma)$ such that $J(\kappa) = 1$. Using the symmetry of J we have:

$$\begin{aligned} I(\xi) &= \int_{\Sigma} \xi J(\xi) \operatorname{vol}_{\varphi_0^*(g)} = \int_{\Sigma} (af_1 + \kappa) J(af_1 + \kappa) \operatorname{vol}_{\varphi_0^*(g)} \\ &= \int_{\Sigma} [a^2 f_1 J(f_1) + af_1 J(\kappa) + a\kappa J(f_1) + \kappa J(\kappa)] \operatorname{vol}_{\varphi_0^*(g)} \\ &= a^2 \lambda_1 \int_{\Sigma} f_1^2 \operatorname{vol}_{\varphi_0^*(g)} + 2a \int_{\Sigma} f_1 J(\kappa) \operatorname{vol}_{\varphi_0^*(g)} \\ &\quad + a \int_{\Sigma} [\kappa J(f_1) - f_1 J(\kappa)] \operatorname{vol}_{\varphi_0^*(g)} + \int_{\Sigma} \kappa \operatorname{vol}_{\varphi_0^*(g)} \\ &= a^2 \lambda_1 + 2a \int_{\Sigma} f_1 J(\kappa) \operatorname{vol}_{\varphi_0^*(g)} + \int_{\Sigma} \kappa \operatorname{vol}_{\varphi_0^*(g)} \\ &= a^2 \lambda_1 - \int_{\Sigma} \kappa \operatorname{vol}_{\varphi_0^*(g)}. \end{aligned}$$

Since $\lambda_1 < 0$, then $I(\xi) < 0$ if $\int_{\Sigma} \kappa \operatorname{vol}_{\varphi_0^*(g)} > 0$. Thus, (2-b) is satisfied.

Now, if $\kappa = f_2$, $f_2 \in B$, then

$$\begin{aligned} I(\xi) &= a^2 \lambda_1 \int_{\Sigma} f_1^2 \operatorname{vol}_{\varphi_0^*(g)} + 2a \int_{\Sigma} f_1 J(f_2) \operatorname{vol}_{\varphi_0^*(g)} \\ &\quad + a \int_{\Sigma} [f_2 J(f_1) - f_1 J(f_2)] \operatorname{vol}_{\varphi_0^*(g)} + \int_{\Sigma} f_2 J(f_2) \operatorname{vol}_{\varphi_0^*(g)} \\ &= a^2 \lambda_1 \int_{\Sigma} f_1^2 \operatorname{vol}_{\varphi_0^*(g)} + 2a \lambda_2 \int_{\Sigma} f_1 f_2 \operatorname{vol}_{\varphi_0^*(g)} + \lambda_2 \int_{\Sigma} f_2^2 \operatorname{vol}_{\varphi_0^*(g)} \\ &= \lambda_1 \frac{(\int_{\Sigma} f_2 \operatorname{vol}_{\varphi_0^*(g)})^2}{(\int_{\Sigma} f_1 \operatorname{vol}_{\varphi_0^*(g)})^2} + \lambda_2, \end{aligned}$$

therefore if $\int_{\Sigma} f_2 \operatorname{vol}_{\varphi_0^*(g)} \neq 0$ and $\lambda_2 = 0$, then $I(\xi) < 0$, so we have (3-a), e.i., φ_0 is unstable. If $\lambda_2 < 0$, (4) is fulfilled.

To proof (2-a), we define:

$$E_1 = \{af_1 : a \in \mathbb{R}\}, \quad E_1^{\perp} = \left\{ f \in C_{\partial}^{j+1,\alpha}(\Sigma) : \int_{\Sigma} f_1 f \operatorname{vol}_{\varphi_0^*(g)} = 0 \right\}.$$

Again by Lemma 3.1.2 -(a), there is a function $\kappa \in C_{\partial}^{j+1,\alpha}(\Sigma)$ such that $J(\kappa) = 1$. If $\int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)} \leq 0$, then

$$I(\kappa) = \int_{\Sigma} \kappa J(\kappa) \text{vol}_{\varphi_0^*(g)} = \int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)} \leq 0$$

As

$$\lambda_2 = I(f_2) = \min \left\{ I(f) : f \in C_{\partial}^{j,\alpha}(\Sigma) \cap E_1^{\perp}, \int_{\Sigma} f^2 \text{vol}_{\varphi_0^*(g)} = 1 \right\}$$

and $\lambda_2 > 0$, we have that $\kappa \notin E_1^{\perp}$. Therefore, for any $\xi \in C_{\partial}^{j,\alpha}(\Sigma)$, where $\int_{\Sigma} \xi \text{vol}_{\varphi_0^*(g)} = 0$, we can write $\xi = b\kappa + w$, $b \in \mathbb{R}$, $w \in E_1^{\perp}$. So,

$$\begin{aligned} I(\xi) &= \int_{\Sigma} \xi J(\xi) \text{vol}_{\varphi_0^*(g)} = \int_{\Sigma} (b\kappa + w)[bJ(\kappa) + J(w)] \text{vol}_{\varphi_0^*(g)} \\ &= \int_{\Sigma} [b^2\kappa J(\kappa) + bwJ(\kappa) + b\kappa J(w) + wJ(w)] \text{vol}_{\varphi_0^*(g)} \\ &= b^2 \int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)} + 2b \int_{\Sigma} wJ(\kappa) \text{vol}_{\varphi_0^*(g)} \\ &\quad + b \int_{\Sigma} \kappa J(w) - wJ(\kappa) \text{vol}_{\varphi_0^*(g)} + I(w) \\ &= b^2 \int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)} + 2b \int_{\Sigma} w \text{vol}_{\varphi_0^*(g)} + I(w) \\ &= -b^2 \int_{\Sigma} \kappa J(\kappa) \text{vol}_{\varphi_0^*(g)} + 2b \int_{\Sigma} (b\kappa + w) \text{vol}_{\varphi_0^*(g)} + I(w) \\ &= -b^2 I(\kappa) + I(w) \\ &\geq 0. \end{aligned}$$

Thus, φ_0 is stable.

Now, under the hypotheses of (3-b), $\lambda_2 = 0$ and $\int_{\Sigma} h_2 \text{vol}_{\varphi_0^*(g)} = 0$ for all λ_2 -eigenfunctions h_2 . Let E_0 be the eigenspace associated with $\lambda_2 = 0$, then $E = \text{Ker}(J) = E_0$. By Lemma 3.1.2-(b) (with $h = 1$ and $\lambda = 0$) there exist a unique function $\bar{h}_2 \in E^{\perp} \cap C_{\partial}^{j+1,\alpha}(\Sigma)$ such that $J(\bar{h}_2) = 1$. So,

(i) If $\int_{\Sigma} \bar{h}_2 \text{vol}_{\varphi_0^*(g)} = 0$,

$$I(\bar{h}_2) = \int_{\Sigma} \bar{h}_2 J(\bar{h}_2) \text{vol}_{\varphi_0^*(g)} = \int_{\Sigma} \bar{h}_2 \text{vol}_{\varphi_0^*(g)} = 0 = \lambda_2.$$

assuming that $\bar{h}_2 \in E_1^{\perp}$ and since

$$\lambda_2 = I(f_2) = \min \left\{ I(f) : f \in C_{\partial}^{j,\alpha}(\Sigma) \cap E_1^{\perp}, \int_{\Sigma} f^2 \text{vol}_{\varphi_0^*(g)} = 1 \right\},$$

then $\bar{h}_2$ is a λ_2 -function, e.i., $J(\bar{h}_2) = 0$. This is a contradiction. Therefore, $\bar{h}_2 \notin E_1^{\perp}$. So, stability is proved as in (2-a). Similary, if $\int_{\Sigma} \bar{h}_2 \text{vol}_{\varphi_0^*(g)} < 0$, stability is also proved as in (2-a).

(ii) If $\int_{\Sigma} \bar{h}_2 \text{vol}_{\varphi_0^*(g)} > 0$, then the proof is the same as for (2-b). Thus, φ_0 is unstable.

□

The following proposition shows that when the operator J_{φ_0} has an eigenvalue equal to zero, this is equivalent to the fact that the function of mean curvatures H_t , for a free boundary CMC immersions variation, has a critical point at $t = 0$.

Proposition 3.3.4. *If $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, is a perturbation of φ_0 , such that $\varphi_t : M \rightarrow \Sigma$ is a $C^{j,\alpha}$ free boundary CMC immersion, and $\frac{dH_t}{dt}|_{t=0} = H'_0 = 0$, where H_t is the constant mean curvature of φ_t , then some eigenvalue λ_i of J_{φ_0} must vanish.*

Conversely, if some eigenvalue λ_i of J_{φ_0} vanishes, and some λ_j -eigenfunction f_i satisfies $\int_{\Sigma} f_i \text{vol}_{\varphi_0^(g)} \neq 0$, then for every free boundary CMC perturbation φ_t of φ_0 we have $H'_0 = 0$.*

Proof. Let $\xi_0 = g(\frac{\partial \varphi_t}{\partial t}|_{t=0}, \vec{n}_{\Sigma_0})$. We have that $J(\xi_0) := J_{\varphi_0}(\xi_0) = H'_0$. If $H'_0 = 0$, then $\xi_0 \in \text{Ker}(J)$, $\xi_0 \neq 0$. Thus, there is some eigenvalue $\lambda_i = 0$.

Conversely, if $\lambda_i = 0$ and f_i is a λ_i -eigenfunction such that $\int_{\Sigma} f_i \text{vol}_{\varphi_0^*(g)} = \langle 1, f_i \rangle_{L^2(\Sigma)} \neq 0$, then $1 \notin (\text{Ker}(J))^{\perp}$. Therefore, the only constant function in the image of J must be 0. Given any free boundary CMC perturbation φ_t of φ_0 and $\xi_0 = g(\frac{\partial \varphi_t}{\partial t}|_{t=0}, \vec{n}_{\Sigma_0})$, as $J(\xi_0) = H'_0$ is a constant function in $\text{Im}(J)$ so $H'_0 = 0$. $\square$

The second stability criteria is given in the following corollary. It's possible check the positivity of the first eigenvalue or the negativity of the second eigenvalue in Theorem 3.3.3. However, for cases II and III the calculation of the eigenvalues can be quite difficult. In this sense, the following criterion facilitates the understanding of the geometric meaning for these cases and is based on the existence of families of 1-parameter deformations.

Corollary 3.3.5. *Let $\varphi : \Sigma \rightarrow M$ a free boundary CMC immersion of class $C^{j+1,\alpha}$. We assume that $\lambda_1 < 0 \leq \lambda_2$. If there exists a deformation φ_t of φ , $-\epsilon < t < \epsilon$, with $\varphi_0 = \varphi$, such that φ_t is a free boundary CMC immersion of class $C^{j,\alpha}$ for all $t \in (-\epsilon, \epsilon)$, and such that $\frac{dH_t}{dt}|_{t=0} = H'_0 = \text{constant} \neq 0$, where H_t is the constant mean curvature of φ_t and V_t is the volume of φ_t , we have that:*

(1) *If $H'_0 V'_0 \leq 0$, then φ is stable;*

(2) *if $H'_0 V'_0 > 0$, then φ is unstable.*

If there is no such deformation, then φ is unstable.

Proof. In the case of $\lambda_1 < 0 < \lambda_2$, by Theorem 3.2.1 there exists a strictly monotonous deformation $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, with $H_t = t$. So, $H'_t = 1$, in particular $H'_0 = 1$. We have that $H'_0 = J(\xi_0)$, where $\xi_0 = g(\frac{d\varphi_t}{dt}|_{t=0}, \vec{n}_{\Sigma})$ and $V'_0 = \int_{\Sigma} \xi_0 \text{vol}_{\varphi_0^*(g)}$. That way, the conditions of the Theorem 3.3.3-(2) fulfilled and then we have (1) and (2).

Now, if $\lambda_1 < 0 = \lambda_2$, one of the implications of proposition 3.3.4 says that: If $\lambda_i = 0$, for some $i = 1, 2, \dots$, and, for some $f_i \in E_{\lambda_i}$, $\int_{\Sigma} f_i \text{vol}_{\varphi_0^*(g)} \neq 0$, then, for every perturbation $\{\varphi_t\}_{t \in I}$ of φ_0 we have that $H'_0 = 0$. This is equivalent to: If $\lambda_i = 0$, for some $i = 1, 2, \dots$, and there is a perturbation $\{\varphi_t\}$ of φ_0 such that $H'_0 \neq 0$, then, for all $f_i \in E_{\lambda_i}$, $\int_{\Sigma} f_i \text{vol}_{\varphi_0^*(g)} = 0$. Therefore, since $H'_0 = c \neq 0$, c constant, then the conditions for Theorem 3.3.3-(3-b) are met. So, let's take $h = \frac{1}{c}\xi_0$, we have

$$J(h) = \frac{1}{c}J(\xi_0) = \frac{1}{c}H'_0 = 1.$$

Now, $J(h) = 1$ if only if $h = u + h_0$, where $u \in E^\perp \cap C_{\partial}^{j+1,\alpha}(\Sigma)$ is the only one such that $J(u) = 1$, (by Lemma 3.1.2-(b)), and $h_0 \in E = E_{\lambda_2}$. So, if we take $\bar{h}_2 = h - h_0$, and as $\int_{\Sigma} h_0 \operatorname{vol}_{\varphi_0^*(g)} = 0$, we have

$$\begin{aligned} H'_0 V'_0 &= H'_0 \int_{\Sigma} \xi_0 \operatorname{vol}_{\varphi_0^*(g)} = c \int_{\Sigma} \xi_0 \operatorname{vol}_{\varphi_0^*(g)} \\ &= c^2 \int_{\Sigma} \frac{1}{c} \xi_0 \operatorname{vol}_{\varphi_0^*(g)} - c^2 \int_{\Sigma} h_0 \operatorname{vol}_{\varphi_0^*(g)} \\ &= c^2 \int_{\Sigma} \left(\frac{1}{c} \xi_0 - h_0 \right) \operatorname{vol}_{\varphi_0^*(g)} \\ &= c^2 \int_{\Sigma} \bar{h}_2 \operatorname{vol}_{\varphi_0^*(g)} \end{aligned}$$

Thus, (1) and (2) are equivalent to (i) and (ii) in the part III.b) of Theorem 3.3.3. □

We will use these criteria in Chapter 5 to study the stability in bifurcation branches that will be presented in Chapter 4.

Chapter 4

Bifurcation of Free Boundary CMC Immersions

In this Chapter we present two results on the existence of bifurcation points for a 1-parametric family $\{\varphi_t\}_{t \in I}$ of free boundary CMC hypersurfaces. The first result, Theorem 4.2.3, furnishes a smooth bifurcating family where the parameter is the mean curvature, i.e., a family where mean curvatures coincide with the mean curvatures of the original family. The second result, Theorem 4.3.2, provides a smooth bifurcating branch of free boundary CMC hypersurfaces, whose volumes coincide with the volumes of surfaces in the original family, that is the bifurcation parameter is the volume. In both cases, the conditions are imposed that the kernel of the Jacobi operator is one-dimensional and that the mean curvature function defined on the family $\{\varphi_t\}_{t \in I}$ is strictly monotonous. In the second case, additionally, it is imposed that the function volume defined on $\{\varphi_t\}_{t \in I}$ is also strictly monotonous. For prove of our criteria, we will base ourselves on the Koiso-Palmer-Piccione article [18] and use the Crandall and Rabinowitz bifurcation theory presented in [10] and [11].

4.1 Definition of bifurcation

In this Chapter we will continue using the notation established in the previous Chapters, to remember: Σ^n and M^{n+1} smooth manifolds, with smooth boundaries $\partial\Sigma$ and ∂M respectively; g a C^∞ riemannian metric in M ; $\varphi : \Sigma \rightarrow M$ an free boundary CMC immersion; $\vec{n}_\Sigma$ the unit normal vector field to $\varphi(\Sigma)$ in the orientable normal bundle;

$$C_\partial^{j,\alpha}(\Sigma) := \{f \in C^{j,\alpha}(\Sigma) : g(\nabla f, \vec{n}_{\partial M}) + \mathbb{I}^{\partial M}(\vec{n}_\Sigma, \vec{n}_\Sigma)f = 0\};$$

$$J = J_\varphi|_{C_\partial^{j,\alpha}(\Sigma)} : C_\partial^{j,\alpha}(\Sigma) \rightarrow C^{j-2,\alpha}(\Sigma),$$

the Jacobi operator of φ defined as

$$J_\varphi(f) := \Delta_\Sigma f - (\|\mathbb{I}^\Sigma\|_{HS}^2 + Ric_g(\vec{n}_\Sigma, \vec{n}_\Sigma))f;$$

$E = \text{Ker}(J)$ and $E^\perp = (\text{Ker}(J))^\perp \cap C_\partial^{j,\alpha}(\Sigma)$, this orthogonal with respect to the internal product of $L^2(\Sigma)$; $\{\varphi_t : \Sigma \rightarrow M\}_{-\epsilon < t < \epsilon}$ be a variation of φ , such that $\varphi_0 = \varphi$ and φ_t is a free boundary

CMC immersion, with mean curvature H_t , and defined as

$$\varphi_t(p) = \text{Exp}_{\varphi_0(p)}(\phi_t(p)\vec{n}_{\Sigma_0}(p)),$$

where $\phi_t = t\xi_0 + o(t)$, $\xi_0 = g(\frac{\partial\varphi_t}{\partial t}|_{t=0}, \vec{n}_{\Sigma_0}) \in C^{j,\alpha}_{\partial}(\Sigma)$, $o : (-\epsilon, \epsilon) \rightarrow C^{j,\alpha}(\Sigma)$, $\frac{o(t)}{t} \rightarrow 0$ if $t \rightarrow 0$; J_t the Jacobi operator of φ_t

Next we give the general definition of bifurcation point for an application F defined between general Banach spaces.

Definition 4.1.1. Let W and Y be Banach spaces, $\Omega \subset W$ open and $F : \Omega \rightarrow Y$ a continuous map. Suppose there is a simple arc in Ω parameterized by $\omega : I \rightarrow \Omega$, where I is an interval in $\mathbb{R}$, such that $F(\omega(t)) = 0$ for all $t \in I$. If there exists a number $t_0 \in I$ such that every neighborhood of $w(t_0)$ contains zeros of F that are not in $\text{Im}(\omega)$, then $w(t_0)$ is called a *bifurcation point* for the equation $F = 0$ with respect to curve ω .

What is sought when studying Bifurcation Theory is to find the bifurcation points for the equation $F = 0$ with respect to curve ω and to understand the structure of $F^{-1}\{0\}$ around such points. In our case W is going to be of the form $I \times X$, where X is a real Banach space and the curve will be of the form $\text{Im}(\omega) = \{(t, 0) : t \in I, 0 \in X\}$.

In our context, φ_0 is a bifurcation point with respect to $\{\varphi_t\}_{t \in (-\epsilon, \epsilon)}$ if and only if, for all $s > 0$, given a s -neighborhood of φ_0 , $V_s \subset C^{j,\alpha}(\Sigma, M)$, there exists $\psi_s \in V_s$, such that, ψ_s is a free boundary CMC immersion and $\psi_s(\Sigma)$ is not congruent with any $\varphi_t(\Sigma)$.

4.2 Existence of Bifurcation with parameter the mean curvature

Next, we will give the definition of K -simple own value of T , where K and T are bounded linear operators. Also, a lemma and a theorem that are part of the Crandall-Rabinowitz Theory that are important in the development of the demonstration of our criteria.

Definition 4.2.1. Let X and Y be Banach Spaces. Let $B(X, Y)$ denote the set of bounded linear maps of X into Y . Let $T, K \in B(X, Y)$. Then $\mu \in \mathbb{R}$ is a K -simple eigenvalue of T if

- (i) $\dim(\text{Ker}(T - \mu K)) = \text{Codim}(\text{Im}(T - \mu K)) = 1$,
and, if $\text{Ker}(T - \mu K) = \text{Span}\{x_0\}$,
- (ii) $K(x_0) \notin \text{Im}(T - \mu K)$.

The next lemma is about of existence of K -simple eigenvalues for operators T nearby to the fixed operator T_0 .

Lemma 4.2.1. (*[11, Lemma 1.3] Crandall-Rabinowitz*) Let $T_0, K \in B(X, Y)$ and assume that r_0 is a K -simple eigenvalue of T_0 . Then there exists a value $\delta > 0$ such that whenever $T \in B(X, Y)$ and $\|T - T_0\| < \delta$, there is a unique $r(T) \in \mathbb{R}$ satisfying $|r(T) - r_0| < \delta$ for which $T - r(T)K$ is singular. The map $T \mapsto r(T)$ is analytic and $r(T)$ is a K -simple eigenvalue of T . Finally, if $\text{Ker}(T_0 - r_0 K) = \text{Span}\{x_0\}$ and Z is a complement of $\text{Span}\{x_0\}$ in X , there is a unique null vector $x(T)$ of $T - r(T)K$ satisfying $x(T) - x_0 \in Z$. The map $T \mapsto x(T)$ is also analytic.

The following theorem guarantees the existence of bifurcation branches, under some conditions, for a function $F : I \times V \rightarrow Y$, where $I \subset \mathbb{R}$, $V \subset X$, X, Y Banach spaces in general.

Theorem 4.2.2. ([10, Teorema 1.7] Crandall-Rabinowitz) *Let X, Y be Banach spaces, V a neighborhood of 0 in X , $I = (-\epsilon, \epsilon)$ be a non-empty open interval, and let*

$$F : I \times V \longrightarrow Y$$

with the following properties

- (1) $F(t, 0) = 0$ for all $t \in I$,
- (2) The partial derivatives F_t, F_x and F_{tx} exist and are continuous,
- (3) $\dim(\text{Ker}(F_x(0, 0))) = \text{Codim}(\text{Im}(F_x(0, 0))) = 1$,
- (4) If $\text{Ker}(F_x(0, 0)) = \text{Span}\{x_0\}$,

$$F_{tx}(0, 0)(x_0) \notin \text{Im}(F_x(0, 0))$$

If Z is any complement $\text{Ker}(F_x(0, 0))$ in X , then there is a neighborhood U of $(0, 0)$ in $\mathbb{R} \times X$, an interval $(-a, a)$, and continuous functions $\zeta : (-a, a) \rightarrow Z$ and $\tau : (-a, a) \rightarrow \mathbb{R}$, such that $\tau(0) = 0$, $\zeta(0) = 0$, and

$$F^{-1}(0) \cap U = \{(\tau(s), sx_0 + s\zeta(s)) : -a < s < a\} \cup \{(t, 0) : (t, 0) \in U\}.$$

If F_{xx} is also continuous, the functions τ and ζ are once continuously differentiable.

Now, we give our first criterion of existence of bifurcation for a family of free boundary CMC hypersurfaces, so that the mean curvatures of the hypersurfaces in the bifurcation branch are equal to those of the original family.

Theorem 4.2.3. *Let $\{\varphi_t : \Sigma \rightarrow M\}_{-\epsilon < t < \epsilon}$ be a smooth variation of φ , such that $\varphi_0 = \varphi$ and φ_t is a free boundary CMC immersion, with mean curvature H_t and:*

- (i) $\frac{dH_t}{dt}|_{t=0} = H'_0 \neq 0$
- (ii) $\dim(E) = 1$. That is, $E = \{be : b \in \mathbb{R}\}$, for some $e \in C_0^{j,\alpha}(\Sigma_0)$, $e \neq 0$.

Then,

- (1) $\int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} = 0$, and there exists a differentiable map $\lambda : (-\epsilon_0, \epsilon_0) \rightarrow \mathbb{R}$, $0 < \epsilon_0 \leq \epsilon$, such that $\lambda(0) = \lambda_0 = 0$, $\lambda(t) = \lambda_t$ is a simple eigenvalue of J_t , and there is no other eigenvalue of J_t near 0.
- (2) Assume further that $\lambda'(0) \neq 0$ holds. Then, φ_0 is a bifurcation point with respect to $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, where the bifurcation branch is an analytic family of free boundary CMC immersions. More precisely, there exist an open interval $\hat{I} \subset \mathbb{R}$, $0 \in \hat{I}$, and C^1 functions $\zeta : \hat{I} \rightarrow E^\perp$ and $t : \hat{I} \rightarrow \mathbb{R}$, such that $t(0) = 0$, $\zeta(0) = 0$, and

$$\psi_s = \text{Exp}_{\varphi_{t(s)}}([\phi_{t(s)} + se + s\zeta(s)]\vec{n}_{\varphi_{t(s)}(\Sigma)}),$$

is a free boundary CMC immersion with mean curvature $\hat{H}_s = H_{t(s)}$.

- (3) Every free boundary CMC immersion sufficiently close, in the topology of $C^{j,\alpha}$, to φ_0 , is equal, up to parameterization, to some element of families $\{\varphi_t\}_{t \in I}$, $0 \in I \subset (-\epsilon_0, \epsilon_0)$, or $\{\psi_s\}_{s \in \hat{I}}$. Furthermore, the surfaces $\{\varphi_t\}_{t \in I}$ and $\{\psi_s\}_{s \in \hat{I}}$ are pairwise distinct except for $\varphi_0 = \psi_0$.

Proof. Let $v_0 = \frac{\partial \varphi_t}{\partial t} \Big|_{t=0}$ be the variational field of φ_t in φ_0 , and $\xi_0 = g(v_0, \vec{n}_{\Sigma_0})$. We have that

$$J(\xi_0) = \frac{d}{dt} \Big|_{t=0} H_t = H'_0 \neq 0.$$

So, as $e \in E = \text{Ker}(J)$

$$H'(0) \int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} = \int_{\Sigma} e J(\xi_0) \, \text{vol}_{\varphi_0^*(g)} = \int_{\Sigma} \xi_0 J(e) \, \text{vol}_{\varphi_0^*(g)} = 0,$$

then

$$\int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} = 0$$

Now, we must verify that the conditions to apply the Lemma 4.2.1 are met with $X = C_{\partial}^{j,\alpha}(\Sigma)$, $Y = C^{j,\alpha}(\Sigma)$, $Z = E^{\perp}$ and let $\iota : X \hookrightarrow Y$ the inclusion map, and thus complete the proof of (1).

Now, by the Fredholm Alternative Theorem

$$y \in \text{Im}(J) \Leftrightarrow y = J(x) \text{ for some } x \in X \Leftrightarrow \int_{\Sigma} y e \, \text{vol}_{\varphi_0^*(g)} = 0 \Leftrightarrow y \in E^{\perp}.$$

So,

$$\text{Im}(J) = E^{\perp}.$$

Therefore $\text{Codim}((\text{Im})(J)) = 1$ in $C_{\partial}^{j,\alpha}(\Sigma)$ and $\iota(e) = e \notin \text{Im}(J)$. Thus, 0 is a ι -simple eigenvalue of J . Then, by Lemma 4.2.1, there exist a differentiable map $\lambda : (-\epsilon_0, \epsilon_0) \rightarrow \mathbb{R}$, such that $\lambda_0 = 0$, λ_t is a simple eigenvalue of J_t and there is no other eigenvalue of J_t near to 0.

For the proof of (2), recall from Proposition 2.3.1 that we have the bijection $f \mapsto \varphi_f$ from a suitable neighborhood U of $0 \in C^{j,\alpha}(\Sigma_0)$ to a neighborhood W of $\varphi_0 \in C^{j,\alpha}(\Sigma, M)$ defined by

$$\varphi_f(p) := \text{Exp}_{\varphi_0(p)}([f(p) + o(f)(p)]\vec{n}_{\Sigma_0}(p)),$$

where Exp is the exponential function generated by g , $o : U \rightarrow C^{j,\alpha}(\Sigma_0)$ is such that $\frac{o(f)}{f} \rightarrow 0$ if $f \rightarrow 0$ and such that φ_f is orthogonal to M . Let H_f be the mean curvature of φ_f . Now, we define

$$\begin{aligned} F : I \times U &\longrightarrow C^{j-2,\alpha}(\Sigma_0) \\ (t, f) &\longmapsto F(t, f) = H_{\phi_t+f} - H_{\phi_t} \end{aligned}$$

where $I = (-\epsilon, \epsilon)$, $\phi_t = t\xi_0 + o(t)$, $\xi_0 \in C_{\partial}^{j,\alpha}(\Sigma_0)$ fixed, $o : I \rightarrow C^{j,\alpha}(\Sigma_0)$, $\frac{o(t)}{t} \rightarrow 0$ if $t \rightarrow 0$. We want to prove that F fulfills the conditions of Theorem 4.2.2.

It is clear that F is a twice continuously Fréchet differentiable mapping by the smoothness of the variation.

It is also evident that $F(t, 0) = 0$ for all $t \in I$.

Now, if $F(t, f) = 0$ then $H_{\phi_t+f} = H_{\phi_t} = \text{const.}$

Conversely, if for some $h \in C_{\partial}^{j,\alpha}(\Sigma)$, $H_h = c$, c a constant, and c is near enough to H_0 , then there is some $t_c \in I$ such that $H_{\phi_{t_c}} = c$. The latter because $H'_0 \neq 0$. If we do $f = h - \phi_{t_c}$ we have to $H_{\phi_{t_c}+f} = c$, thus $F(t_c, f) = 0$. Therefore, in a neighborhood of φ_0 any free boundary CMC immersion is obtained as a solution of $F(t, f) = 0$.

Let $T_h U$ be the tangent space to U in $h \in U$. We have that $T_h U \cong C_{\partial}^{j,\alpha}(\Sigma)$. So,

$$\frac{\partial}{\partial f} F(t, 0) : C_{\partial}^{j,\alpha}(\Sigma) \rightarrow C^{j-2,\alpha}(\Sigma),$$

and

$$\frac{\partial}{\partial f} F(t, 0)(u) = J_t(u),$$

where J_t is the Jacobi operator of φ_t and $J_0 = J$.

Thus, $\text{Ker}(\frac{\partial}{\partial f} F(0, 0)) = \text{Ker}(J) = E$ and therefore one-dimensional. Also, since J is Fredholm with zero index, then $\text{Codim}(\text{Im}(J)) = \text{Codim}(\text{Im}(\frac{\partial}{\partial f} F(0, 0))) = 1$.

It remains to prove that

$$\frac{\partial^2}{\partial t \partial f} F(0, 0)(e) \notin \text{Im}\left(\frac{\partial}{\partial f} F(0, 0)\right).$$

From part (1) we have

$$\frac{\partial}{\partial f} F(t, 0)(e_t) = J_t(e_t) = \lambda_t e_t.$$

e_t the unit eigenvector associated with λ_t . If we differentiate the second equation with respect to t and evaluate at $t = 0$ then we have

$$J'_0(e_0) + J_0(e'_0) = \lambda'_0 e_0,$$

Remember that $\lambda_0 = 0$, $e_0 = e$ and $J_0 = J$. Further, $\lambda'_0 \neq 0$. So $\lambda'_0 e \in E - \{0\}$, that is $\lambda'_0 e \notin E^\perp = \text{Im}(J) = \text{Im}\left(\frac{\partial}{\partial f} F(0, 0)\right)$. Thus,

$$\frac{\partial^2}{\partial t \partial f} F(0, 0)(e) = J'_0(e) = \lambda'_0 e - J(e'_0) \notin \text{Im}\left(\frac{\partial}{\partial f} F(0, 0)\right).$$

Then, by Theorem 4.2.2, there exist a neighborhood $\tilde{U}$ of $(0, 0)$ in $\mathbb{R} \times C_{\partial}^{j,\alpha}(\Sigma)$, an interval $\hat{I} = (-a, a) \subset \mathbb{R}$ and continuous functions $\zeta : \hat{I} \rightarrow E^\perp$ $t : \hat{I} \rightarrow \mathbb{R}$, such that $\zeta(0) = 0$, $t(0) = 0$ and

$$F^{-1}\{0\} \cap \tilde{U} = \{(t(s), se + s\zeta(s)) : s \in \hat{I}\} \cup \{(t, 0) : t \in I, 0 \in C_{\partial}^{j,\alpha}(\Sigma)\}.$$

Therefore, we have the existence of unique free boundary CMC immersions, with mean curvature $H_{t(s)}$, of the form

$$\psi_s = \text{Exp}_{\varphi_{t(s)}}([\phi_{t(s)} + se + s\zeta(s)]\vec{n}_{\varphi_{t(s)}(\Sigma)}).$$

Now, for (3) the variational vector field of φ_t at φ_0 is $v_0 \vec{n}_{\Sigma_0}$, while the variational vector field of ψ_s at φ_0 is $(t'(0)v_0 + e) \vec{n}_{\Sigma_0}$. Since $J(v_0) = \text{const} \neq 0$ and $J(e) = 0$, then v_0 and e are linearly independent. So, the families $\{\varphi_t : t \in I\}$ and $\{\psi_s : s \in \hat{I}\}$ have non equivalent elements, except for $\varphi_0 = \psi_0$. $\square$

4.3 Existence of Bifurcation with parameter the volume

In this section we will prove a second smooth bifurcation result of the hypersurfaces with CMC and free boundary, whose volumes coincide with the volumes of the surfaces in the original family. For this we need to define some objects and develop some preliminaries.

For simplicity we denote:

$$B = C_{\partial}^{j,\alpha}(\Sigma),$$

$$B_0 = \left\{ h \in B : \int_{\Sigma} h \, \text{vol}_{\varphi_0^*(g)} = 0 \right\}$$

and

$$V : B \longrightarrow \mathbb{R}$$

$$f \longmapsto V(f),$$

where $V(f)$ is the volume of $\varphi_f : \Sigma \rightarrow M$. Also, let

$$l := dV(0) : B \longrightarrow \mathbb{R} \tag{4.3.1}$$

$$w \longmapsto l(w) = dV(0)(w) = \int_{\Sigma} w \, \text{vol}_{\varphi_0^*(g)}.$$

Let $w_0 \in B$ be such that $l(w_0) = 1$ ($w_0 = \frac{1}{dV(0)(\xi_0)} \xi_0$).

Now, we define

$$P : B \longrightarrow \mathbb{R} \times B_0 \tag{4.3.2}$$

$$f \longmapsto (V(f), f - dV(0)(f)w_0).$$

We have that $P(0) = (V(0), 0)$. Let us see that $dP(0)$ is an isomorphism:

$$dP(0) : B \rightarrow \mathbb{R} \times B_0,$$

where

$$dP(0)(w) = (dV(0)(w), w - dV(0)(w)w_0)$$

$$= (l(w), w - l(w)w_0)$$

Therefore, $\text{Ker}(dP(0)) = \{0\}$. So, $dP(0)$ is injective.

Now, to prove that is surjective, let $a \in \mathbb{R}$ and $h \in B_0$. There exist $f \in B$ such that $l(f) = a$ and $h = f - aw_0$. In effect, if we take $f = h + aw_0$, then

$$\int_{\Sigma} h \, \text{vol}_{\varphi_0^*(g)} = \int_{\Sigma} f \, \text{vol}_{\varphi_0^*(g)} - a \int_{\Sigma} w_0 \, \text{vol}_{\varphi_0^*(g)} = a - a = 0.$$

So,

$$dP(0)f = (l(f), f - l(f)w_0) = (a, h + aw_0 - aw_0) = (a, h).$$

Then, by the Inverse Mapping Theorem (see Theorem A.0.1), there exists a neighborhood of

0 where P is invertible. Thus, for a neighborhood U_0 of 0 in B_0 and an interval $\hat{I} \subset \mathbb{R}$ such that $V(0) \in \hat{I}$, we can define

$$Q = P^{-1} : \hat{I} \times U_0 \longrightarrow B. \quad (4.3.3)$$

For $(t, h) \in I \times U$, with suitable neighborhoods I of $0 \in \mathbb{R}$ and U of $0 \in B_0$, we write

$$f_{(t,h)} = Q(V(\phi_t), \gamma(t) + h), \quad (4.3.4)$$

where

$$\gamma(t) = \phi_t - l(\phi_t)w_0, \quad (4.3.5)$$

$\phi_t = t\xi_0 + o(t)$ as defined before. As $\phi'_0 = \xi_0$ and $\gamma'(t) = \phi'_t - \frac{\xi_0}{\int_{\Sigma} \xi_0 \text{vol}_{\varphi_0^*(g)}} l(\phi'_t)$, then

$$\gamma(0) = 0 \quad \text{and} \quad \gamma'(0) = 0 \quad (4.3.6)$$

We can observe that, fixing t and varying h , $\varphi_{f_{(t,h)}}$ are immersions that have the same volume as $\varphi_t = \varphi_{\phi_t}$. In particular $f_{(t,0)} = \phi_t$.

Let

$$C_*^{j-2,\alpha}(\Sigma) := \left\{ f \in C^{j-2,\alpha}(\Sigma) : \int_{\Sigma} f \text{vol}_{\varphi_0^*(g)} = 0 \right\},$$

and define

$$\begin{aligned} F : I \times U &\rightarrow C_*^{j-2,\alpha}(\Sigma) \\ (t, h) &\mapsto F(t, h) = H_{f_{(t,h)}} - H_{\phi_t}, \end{aligned} \quad (4.3.7)$$

where $I \times U$ a suitable neighborhood of $(0, 0)$ in $\mathbb{R} \times B_0$,

and

$$\begin{aligned} \kappa_0 : C_*^{j-2,\alpha}(\Sigma) &\rightarrow C_*^{j-2,\alpha}(\Sigma) \\ f &\mapsto f - \frac{1}{|\Sigma|} \int_{\Sigma} f \text{vol}_{\varphi_0^*(g)}, \end{aligned} \quad (4.3.8)$$

where $|\Sigma| := \int_{\Sigma} \text{vol}_{\varphi_0^*(g)}$ the area of Σ . We can see that

$$\text{Ker}(\kappa_0) = \{\text{The constant functions in } C_*^{j-2,\alpha}(\Sigma)\}.$$

Also, clearly κ_0 is an surjective linear operator.

Then, since $F(t, 0) = 0$ and $\kappa_0(0) = 0$, the composition

$$\bar{F} := \kappa_0 \circ F : I \times U \rightarrow C_*^{j-2,\alpha}(\Sigma) \quad (4.3.9)$$

satisfies

$$\bar{F}(t, 0) = 0. \quad (4.3.10)$$

The following lemma characterizes the free boundary CMC immersions that have fixed volume ν , as solutions of the equation $\bar{F}(t, h) = 0$ and that are represented in terms of the functions $f_{(t,h)}$ defined in (4.3.4).

Lemma 4.3.1. *Let Ω be a neighborhood of 0 in B and $f \in \Omega$. Suppose that $dV(0) \neq 0$. Then:*

- 1) *If there exist $(t, h) \in I \times U$ such that $f = f_{(t,h)}$, $V(\phi_t) = \nu$ and $\bar{F}(t, h) = 0$, then $H_{f_{(t,h)}}$ is constant and $V(f_{(t,h)}) = \nu$. Here $H_{f_{(t,h)}}$ is the mean curvature of $\varphi_{f_{(t,h)}} : \Sigma \rightarrow M$.*
- 2) *Conversely, if H_f is constant and $V(f) = \nu$, then there exist $(t, h) \in I \times U$ such that $f = f_{(t,h)}$, $V(\phi_t) = \nu$ and $\bar{F}(t, h) = 0$.*

Proof. 1) If

$$0 = \bar{F}(t, h) = \kappa_0(F) = \kappa_0(H_{f_{(t,h)}} - H_{\phi_t}),$$

then $H_{f_{(t,h)}} - H_{\phi_t}$ is constant, because $\text{Ker}(\kappa_0)$ consists of constant functions. As H_{ϕ_t} is constant we have that $H_{f_{(t,h)}}$ is constant. Now, if t is fixed, the functions $f_{(t,h)} = Q(V(\phi_t), \gamma(t) + h)$ have the same volume $V(\phi_t)$. Therefore, $V(f_{(t,h)}) = V(\phi_t) = \nu$.

- 2) Being Ω a neighborhood of 0, small enough, and $f \in \Omega$, we have that $P(f) = (\nu, \eta)$ is near $(V(0), 0)$ (P was defined in 4.3.2). As $dV(0) \neq 0$ there is a unique t close to 0 such that $V(\phi_t) = \nu$. Now, $f = P^{-1}(V(\phi_t), \eta)$, $\eta \in B_0$. It is enough to see that η is of the form $\eta = \gamma(t) + h$ ($h \in B_0$, $\gamma(t)$ was defined in 4.3.5). Indeed, it is enough to take $h = \eta - \gamma(t)$. Therefore, $f = P^{-1}(V(\phi_t), \gamma(t) + h) = Q(V(\phi_t), \gamma(t) + h) = f_{(t,h)}$. Also, by hypothesis, $H_{f_{(t,h)}} = H_f$ is constant. So, $H_{f_{(t,h)}} - H_{\phi_t}$ is constant. Then

$$\bar{F}(t, h) = \kappa_0(H_{f_{(t,h)}} - H_{\phi_t}) = 0$$

□

Now, we present our second criterion of existence of bifurcation, the case in which the parameter is the volume. Again, the idea of the proof is to see that the conditions of Crandall-Rabinowitz Theorema are met (see Theorema 4.2.2).

Theorem 4.3.2. *Let $\{\varphi_t : \Sigma \rightarrow M\}_{-\epsilon < t < \epsilon}$ be a smooth variation of φ , such that $\varphi_0 = \varphi$ and φ_t is a free boundary CMC immersion, with mean curvature H_t and:*

- (i) $\frac{dH_t}{dt}|_{t=0} = H'_0 \neq 0$
- (ii) $\frac{dV(\phi_t)}{dt}|_{t=0} = dV(0) \neq 0$
- (iii) $\text{Dim}(E) = 1$. That is, $E = \{be : b \in \mathbb{R}\}$, for some $e \in C^{j,\alpha}_0(\Sigma_0)$, $e \neq 0$.

Then,

- (1) $\int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} = 0$, there exists a differentiable map $\tilde{\lambda} : (-\epsilon_0, \epsilon_0) \rightarrow \mathbb{R}$, $0 < \epsilon_0 \leq \epsilon$, such that $\tilde{\lambda}(0) = \tilde{\lambda}_0 = 0$, $\tilde{\lambda}(t) = \tilde{\lambda}_t$ is a simple eigenvalue of $\tilde{J}_t = \frac{\partial}{\partial h} \bar{F}(t, 0)$, and there is no other eigenvalue of $\tilde{J}_t$ near 0.
- (2) Assume further that $\frac{d\tilde{\lambda}_t}{dt}|_{t=0} = \tilde{\lambda}'_0 \neq 0$ holds. Then, φ_0 is a bifurcation point with respect to $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, where the bifurcation branch is a smooth family of free boundary CMC immersions. More precisely, there exist an open interval $\hat{I} \subset \mathbb{R}$, $0 \in \hat{I}$, and C^1 functions $\eta : \hat{I} \rightarrow B_0$ and $\tau : \hat{I} \rightarrow \mathbb{R}$, such that $\tau(0) = 0$, $\eta(0) = 0$, and

$$\psi_s = \exp_{\varphi_0}(f_{(\tau(s), s e + s \eta(s))} \vec{n}_{\varphi_0}),$$

is a free boundary CMC immersion with volume $\tilde{V}(s) = V(\tau(s))$.

- (3) Every free boundary CMC immersion sufficiently $C^{j,\alpha}$ -close, to φ_0 , is equal, up to reparameterization, to some element of families $\{\varphi_t\}_{t \in I}$, $I \subset (-\epsilon_0, \epsilon_0)$, or $\{\psi_s\}_{s \in \hat{I}}$. Furthermore, the hypersurfaces in the families $\{\varphi_t\}_{t \in I}$ and $\{\psi_s\}_{s \in \hat{I}}$ are different, except for $\varphi_0 = \psi_0$.

Proof. (1) The proof of $\int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} = 0$ is equal to the proof that was made in the first part of Theorem 4.2.3. Now,

$$\begin{aligned} \tilde{J}_t &= \frac{\partial}{\partial h} \bar{F}(t, 0) : B_0 \longrightarrow C_*^{j-2, \alpha}(\Sigma) \\ \tilde{J}_t &= \frac{\partial}{\partial h} \bar{F}(t, 0) = \frac{\partial}{\partial h} \kappa_0(F(t, 0)) = \kappa_0\left(\frac{\partial}{\partial h} F(t, 0)\right) \\ &= \kappa_0\left(\frac{\partial}{\partial h} (H_{f(t, h)} - H_{\phi_t})|_{h=0}\right) \end{aligned}$$

So,

$$\tilde{J}_t(u) = \kappa_0\left(J_t\left(\frac{\partial f(t, 0)}{\partial h} u\right)\right).$$

Since PQ is the identity in $\hat{I} \times U_0$, we have

$$P(f_{(t, h)}) = PQ(V(\phi(t)), \gamma(t) + h) = (V(\phi(t)), \gamma(t) + h). \quad (4.3.11)$$

Differentiating (4.3.11) with respect to h and evaluating at $u \in B_0$, we have:

$$dP(f_{(t, h)})\left(\frac{\partial f(t, h)}{\partial h}\right)(u) = (0, u).$$

On the other hand, from the definition of P (see 4.3.2) we have

$$\begin{aligned} dP(f_{(t, h)})\left(\frac{\partial f(t, h)}{\partial h}\right)(u) &= \left(\int_{\Sigma} \frac{\partial f(t, h)}{\partial h}(u) \xi_t \, \text{vol}_{\varphi_{f(t, h)}^*(g)} \right), \\ &\quad \left(\frac{\partial f(t, h)}{\partial h}(u) - \left(\int_{\Sigma} \frac{\partial f(t, h)}{\partial h}(u) \, \text{vol}_{\varphi_0^*(g)} \right) w_0 \right), \end{aligned}$$

where $\xi_t = g\left(\frac{\partial \varphi_{f(t, h)}}{\partial t}, \vec{n}_{f(t, h)}\right)$, $\vec{n}_{f(t, h)}$ the orthonormal field to $\varphi_{f(t, h)}$. So, for any $u \in B_0$

$$\int_{\Sigma} \frac{\partial f(t, h)}{\partial h}(u) \xi_t \, \text{vol}_{\varphi_{f(t, h)}^*(g)} = 0,$$

and

$$\left(\frac{\partial f(t, h)}{\partial h}\right)(u) - \left(\int_{\Sigma} \frac{\partial f(t, h)}{\partial h}(u) \, \text{vol}_{\varphi_0^*(g)}\right) w_0 = u.$$

Therefore, in the particular case $(t, h) = (0, 0)$, we have, for any $u \in B_0$,

$$\int_{\Sigma} \frac{\partial f(0, 0)}{\partial h}(u) \, \text{vol}_{\varphi_0^*(g)} = 0 \quad \text{and} \quad \frac{\partial f(0, 0)}{\partial h}(u) = u.$$

Thus,

$$\tilde{J}_0(u) = \frac{\partial \bar{F}}{\partial h}(0, 0)(u) = \kappa_0(J_0(u)) = J_0(u) - \frac{1}{|\Sigma|} \int_{\Sigma} J_0(u) \, \text{vol}_{\varphi_0^*(g)},$$

here $J_0 = J$. Then, $\tilde{J}_0(u) = 0$ if and only if $J(u)$ is a constant.

If $J(u) = c$, c a constant, $\xi_0 = g(\frac{\partial \varphi_0}{\partial t}|_{t=0}, \vec{n}_0)$ and as $H'_0 \neq 0$ we have to

$$J(u - \frac{c}{H'_0} \xi_0) = J(u) - \frac{J(u)}{H'_0} J(\xi_0) = 0,$$

since $J(\xi_0) = H'_0$. Therefore, $u - \frac{c}{H'_0} \xi_0 = ae$, $a \in \mathbb{R}$. So, $u = ae + \frac{c}{H'_0} \xi_0$. Moreover, if $u \in B_0$, that is

$$\begin{aligned} 0 &= \int_{\Sigma} u \, \text{vol}_{\varphi_0^*(g)} = \int_{\Sigma} ae + \frac{c}{H'_0} \xi_0 \, \text{vol}_{\varphi_0^*(g)} \\ &= a \int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} + \frac{c}{H'_0} \int_{\Sigma} \xi_0 \, \text{vol}_{\varphi_0^*(g)} \\ &= \frac{c}{H'_0} dV(0), \end{aligned}$$

since $\int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} = 0$. Now, as $dV(0) \neq 0$ by hypothesis, then $c = J(u) = 0$. Thus,

$$\text{Ker}(\tilde{J}_0) = \text{Ker}\left(\frac{\partial \bar{F}}{\partial h}(0, 0)\right) = \text{Ker}(J) \quad (4.3.12)$$

So,

$$\text{Dim}(\text{Ker}(\tilde{J}_0)) = 1 \quad (4.3.13)$$

We have $J : B \rightarrow C^{j-2, \alpha}(\Sigma)$ is a linear Fredholm operator with index $i_F(J) = 0$. This is $\text{Codim}(\text{Im}(J)) = 1$. Now, as $\int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} = 0$, so,

$$\text{Ker}(J) \subset B_0.$$

Since $\text{Codim}_B(B_0) = 1$, then $\text{Codim}(J(B_0)) = 2$. Let $\iota : B_0 \hookrightarrow B$ be the inclusion map. ι is Fredholm with index $i_F(\iota) = -1$. On the other hand $\text{Ker}(\kappa_0)$ consist of the constant functions. So, $\text{Dim}(\text{Ker}(\kappa_0)) = 1$. As κ_0 is surjective, then $\text{Codim}(\text{Im}(\kappa_0)) = 0$. Thus, the Fredholm index of κ_0 is $i_F(\kappa_0) = 1$. Also, $J \circ \iota : B_0 \rightarrow C^{j-2, \alpha}(\Sigma)$ have $\text{Ker}(J \circ \iota) = \text{Ker}(J)$. So $\text{Dim}(\text{Ker}(J \circ \iota)) = 1$. Therefore, $\text{Codim}(J \circ \iota) = -1$.

Thus

$$\tilde{J}_0 = \frac{\partial \bar{F}}{\partial h}(0, 0) = \kappa_0 \circ J \circ \iota$$

is Fredholm with index $i_f(\tilde{J}_0) = i_F(\kappa_0 \circ J \circ \iota) = i_F(\kappa_0) + i_F(J) + i_f(\iota) = 0$.

So,

$$\text{Codim}(\text{Ker}(\tilde{J}_0)) = 1 \quad (4.3.14)$$

Now, let $\iota_0 : B_0 \hookrightarrow C_*^{j-2, \alpha}(\Sigma)$ be the inclusion map. We must see that $\iota_0(e) \notin \text{Im}(\tilde{J}_0)$. We know that $e \notin J(B) = E^\perp$, then $e \notin J(B_0)$. $\kappa_0(e) = e - \frac{1}{|\Sigma|} \int_{\Sigma} e \, \text{vol}_{\varphi_0^*(g)} = e$. Now, suppose there is $u \in B_0$ such that $\kappa_0(J(u)) = e$. This is, $0 = \kappa_0(J(u)) - e = \kappa_0(J(u)) - \kappa_0(e) =$

$\kappa_0(J(u) - e)$. So, $J(u) - e = c$, where c is a constant. $J(u) = c + e$.

$$\begin{aligned} 0 &< \int_{\Sigma} e^2 \operatorname{vol}_{\varphi_0^*(g)} = c \int_{\Sigma} e \operatorname{vol}_{\varphi_0^*(g)} + \int_{\Sigma} e^2 \operatorname{vol}_{\varphi_0^*(g)} \\ &= \int_{\Sigma} e(c + e) \operatorname{vol}_{\varphi_0^*(g)} \\ &= \int_{\Sigma} eJ(u) \operatorname{vol}_{\varphi_0^*(g)} \\ &= \int_{\Sigma} uJ(e) \operatorname{vol}_{\varphi_0^*(g)} = 0, \end{aligned}$$

which is a contradiction. Therefore, $\iota_0(e) = e \notin \operatorname{Im}(T_0)$. Thus, 0 is a ι_0 -simple eigenvalue of T_0 .

Then, by Lemma 4.2.1 there exists ι_0 -simple eigenvalues $\tilde{\lambda}_t$ of $\tilde{J}_t = \frac{\partial \bar{F}}{\partial h}(t, 0)$, where $\tilde{\lambda}_0 = 0$ and $\tilde{\lambda}_t$ is the only one eigenvalue of $\tilde{J}_t$ near to 0.

- (2) To prove the existence of a bifurcation we are going to apply again the Crandall-Rabinowitz Theorem (see Theorem 4.2.2) in the spaces $X = B_0$, $Y = C_*^{j-2, \alpha}(\Sigma)$, $Z = E^\perp \cap B_0$ and to the operator $\bar{F} : I \times U \rightarrow C_*^{j-2, \alpha}(\Sigma)$. It remains to prove that

$$\frac{\partial^2 \bar{F}}{\partial t \partial h}(0, 0)(e) \notin \operatorname{Im}\left(\frac{\partial \bar{F}}{\partial h}(0, 0)\right).$$

Let $\tilde{e}_t$ be the unitary eigenvector of T_t associated to eigenvalue $\tilde{\lambda}_t$. So,

$$\tilde{J}_t(\tilde{e}_t) = \tilde{\lambda}_t \tilde{e}_t. \quad (4.3.15)$$

Deriving 4.3.15 with respect to t at $t = 0$, we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \tilde{J}_t(\tilde{e}_t) &= \frac{\partial^2 \bar{F}}{\partial t \partial h}(0, 0)(\tilde{e}_0) + \tilde{J}_0(\tilde{e}'_0) \\ &= \tilde{\lambda}'_0 \tilde{e}_0 + \tilde{\lambda}_0 \tilde{e}'_0 \end{aligned}$$

where $\tilde{e}_0 = e$ and $\tilde{\lambda}_0 = 0$. Thus,

$$\frac{\partial^2 \bar{F}}{\partial t \partial h}(0, 0)(e) = \tilde{\lambda}'_0 e - \tilde{J}_0(e'_0).$$

If we assume that $\tilde{\lambda}'_0 \neq 0$, $\tilde{\lambda}'_0 e \neq 0$ and $\tilde{\lambda}'_0 e \notin \operatorname{Im}(\tilde{J}_0)$. So,

$$\frac{\partial^2 \bar{F}}{\partial t \partial h}(0, 0)(e) \notin \operatorname{Im}\left(\frac{\partial \bar{F}}{\partial h}(0, 0)\right).$$

Then, by Theorem 4.2.2 there exist a neighborhood $\tilde{U}$ of $(0, 0)$ in $\mathbb{R} \times B_0$, an interval $\hat{I} = (-a, a) \subset \mathbb{R}$ and continuous functions $\eta : \hat{I} \rightarrow E^\perp \cap B_0$, $\tau : \hat{I} \rightarrow \mathbb{R}$, such that, $\eta(0) = 0$, $\tau(0) = 0$ and

$$\bar{F}^{-1}(0) \cap \tilde{U} = \{(\tau(s), se + s\eta(s)) : s \in \hat{I}\} \cup \{(t, 0) : (t, 0) \in \tilde{U}\}.$$

Therefore, we have the existence of unique free boundary CMC immersions with volume $\tilde{V}(s) = V(\phi_{\tau(s)})$, which are of the form

$$\psi_s = \exp_{\varphi_0}(f_{(\tau(s), se+s\eta(s))}\vec{n}_{\varphi_0}).$$

- (3) As we had seen before, the variational vector field of φ_t in φ_0 is $v_0\vec{n}_{\varphi_0}$, $v_0 = \frac{\partial\varphi_t}{\partial t}|_{t=0}$. Now let us see how is the variational vector field of ψ_s in φ_0 .

Differentiating (4.3.11) with respect to t and calculating in $r \in \mathbb{R}$ we have:

$$dP(f_{(t,h)})\left(\frac{\partial f_{(t,h)}}{\partial t}\right)(r) = \left(\frac{dV(\phi(t))}{dt}(\tau), \frac{d\gamma(t)}{dt}(r)\right).$$

On the other hand

$$\begin{aligned} dP(f_{(t,h)})\left(\frac{\partial f_{(t,h)}}{\partial t}\right)(r) &= \left(\int_{\Sigma} \frac{\partial f_{(t,h)}}{\partial t}(r)\xi_t \operatorname{vol}_{\varphi_{f_{(t,h)}}^*}(g), \right. \\ &\quad \left. \left(\frac{\partial f_{(t,h)}}{\partial t}\right)(r) - \left(\int_{\Sigma} \frac{\partial f_{(t,h)}}{\partial t}(r) \operatorname{vol}_{\varphi_0^*}(g)\right)w_0\right). \end{aligned}$$

That is, for all $r \in \mathbb{R}$,

$$\int_{\Sigma} \frac{\partial f_{(t,h)}}{\partial t}(r)\xi_t \operatorname{vol}_{\varphi_{f_{(t,h)}}^*}(g) = \frac{dV(\phi(t))}{dt}(r),$$

and

$$\left(\frac{\partial f_{(t,h)}}{\partial t}\right)(r) - \left(\int_{\Sigma} \frac{\partial f_{(t,h)}}{\partial t}(r) \operatorname{vol}_{\varphi_0^*}(g)\right)w_0 = \frac{d\gamma(t)}{dt}(r)$$

In the particular case $(t, h) = (0, 0)$, as $\gamma'(0) = 0$ (see 4.3.6), then

$$\int_{\Sigma} \frac{\partial f_{(0,0)}}{\partial t}(r) \operatorname{vol}_{\varphi_0^*}(g) = dV(0)(\xi_0)(r),$$

and

$$\left(\frac{\partial f_{(0,0)}}{\partial t}\right)(r) - dV(0)(\xi_0)(r)w_0 = 0.$$

But, as $w_0 = \frac{1}{dV(0)(\xi_0)}\xi_0$ (see 4.3.1), then

$$\left(\frac{\partial f_{(0,0)}}{\partial t}\right)(r) = r\xi_0$$

So, using ??, we have

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0}(f_{(\tau(s), se+s\eta(s))}) &= \frac{d}{ds}\Big|_{s=0}(Q(\phi_{\tau(s)}, \gamma(\tau(s)) + se + s\eta(s))) \\ &= \frac{\partial Q}{\partial t}(V_0, 0)\tau'(0) + \frac{\partial Q}{\partial h}(V_0, 0)(\gamma'(0)\tau'(0) + e) \\ &= \frac{\partial f_{(0,0)}}{\partial t}(\tau'(0)) + \frac{\partial f_{(0,0)}}{\partial h}(\tau'(0))(e) \\ &= \tau'(0)\xi_0 + e \end{aligned}$$

So, the variational field of ψ_s in φ_0 is $(\tau'(0)\xi_0 + e)\vec{n}_{\varphi_0}$. Since $J(\xi_0) = H'_0 \neq 0$ and $J(e) = 0$,

then ξ_0 and e are linearly independent. Thus, the families $\{\varphi_t : t \in I\}$ and $\{\psi_s : s \in \hat{I}\}$ are different except for $\varphi_0 = \psi_0$.

□

Chapter 5

Study of Stability in the Bifurcation Branches

Given the results on the existence of bifurcation in the Theorems 4.2.3 and 4.3.2 in the previous chapter, we are now going to study the stability of the immersions in these branches of bifurcation. One in terms of the sign of derivative of the mean curvature function (or volume function, according to the case) of the hypersurfaces in the bifurcation branch, in addition to the sign of the derivative of the function first eigenvalue of $\tilde{J}_t$; and other in terms of the sign of first and second derivative of the mean curvature function (or volume function, according to the case) of the hypersurfaces in the bifurcation branch. Again, in this study we based on Koiso-Palmer-Piccione [14] and Crandall-Rabinowitz [9] and [10].

5.1 A first result in stability

Recall from Remark 3.3.1, let $\lambda_1 < \lambda_2 \leq \dots, \lambda_n \rightarrow \infty (n \rightarrow \infty)$ the eigenvalues of J . When Theorem 3.2.2, of existence and uniqueness of deformation, is applied in particular to the case $\lambda_1 = 0$ we have that $E_{\lambda_1} = E = \text{Ker}(J)$. In this case the multiplicity of λ_1 is 1 and of functional analysis we know that the λ_1 -eigenfunctions do not change of sign. In particular, we have that $\int_{\Sigma} e \text{vol}_{\varphi_0^*(g)} \neq 0$, for $e \in E$. Therefore, the bifurcation for a family $\{\varphi_t\}$ only occurs in the case where $\lambda_k = 0$, for some $k \geq 2$.

Now, remember that in Theorem 4.3.2 we had defined the operator $\tilde{J}_0 : B_0 \rightarrow C_*^{j-2,\alpha}(\Sigma)$ ($\tilde{J}_0 = \tilde{J}$ for simplicity) as

$$\tilde{J}(u) = \frac{\partial \bar{F}}{\partial h}(0,0)(u) = J(u) - \frac{1}{|\Sigma|} \int_{\Sigma} J(u) \text{vol}_{\varphi_0^*(g)}.$$

We can see that $\tilde{J}$ is the representation operator of the restriction of $A''(0)$ to B_0 . As $u \mapsto \frac{1}{|\Sigma|} \int_{\Sigma} J(u) \text{vol}_{\varphi_0^*(g)}$ is an operator of rank 1, then $\tilde{J}$ is a symmetric compact finite-rank perturbation of J . Thus, $\tilde{J}$ admits a sequence of eigenvalues, $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \tilde{\lambda}_3, \dots, \tilde{\lambda}_n \rightarrow \infty (n \rightarrow \infty)$; where each eigenspace $E_{\tilde{\lambda}_i}$ is finite-dimensional, and there is a sequence of eigenfunctions, $\{\tilde{e}_i : \tilde{J}(\tilde{e}_i) = \tilde{\lambda}_i \tilde{e}_i, i = 1, 2, \dots\}$, that form an orthonormal basis for B_0 . In that way, a CMC immersion is stable if and only if $A''(0)$ is positive semi-definite on B_0 (see [6]).

Thus, the following Lemma provides us a characterization of the stability of a CMC immersion from the first eigenvalue of $\tilde{J}$.

Lemma 5.1.1. φ_0 is stable if and only if $\tilde{\lambda}_1 \geq 0$

Proof.

$$A''(0)\Big|_{B_0}(\tilde{e}_1) = \int_{\Sigma} \tilde{J}(\tilde{e}_1)\tilde{e}_1 \, vol_{\varphi_0^*(g)} = \int_{\Sigma} \tilde{\lambda}_1 \tilde{e}_1^2 \, vol_{\varphi_0^*(g)}.$$

Therefore, φ_0 is stable if only if $\tilde{\lambda}_1 \geq 0$. □

An immediate consequence of the second criterion of stability, Corollary 3.3.5, is the following proposition, a first result in the stability of φ_0 when $\lambda_2 = 0$ holds (equivalently, $\tilde{\lambda}_1 = 0$ holds).

and under the conditions of Theorem 4.3.2 we obtain in the following proposition

Proposition 5.1.2. *Under the conditions of Theorem 4.3.2 and assuming that:*

- (i) $\frac{dH_t}{dt}\Big|_{t=0} = H'_0 \neq 0$.
- (ii) $E = \{be : b \in \mathbb{R}\}$, for some $e \in C_{\partial}^{j,\alpha}(\Sigma_0)$, $e \neq 0$.
- (iii) $\lambda_2 = 0$.

Then,

- (1) If $H'_0 V'_0 \leq 0$, then φ is stable and $\tilde{\lambda}_1 = 0$.
- (2) If $H'_0 V'_0 > 0$, then φ is unstable and $\tilde{\lambda}_2 \leq 0$

□

Proof. From (1) and (2) we have that there exist $j \geq 2$ and $k \geq 1$ such that $\lambda_j = \tilde{\lambda}_k = 0$. Taking into account the Remark 3.3.1 and the variational characterization of the eigenvalues of Jacobi operator J , presented in (3.3.1) and (3.3.2), we can write

$$\lambda_k = \max_{\substack{\text{Dim}(V)=k \\ V \subset C_{\partial}^{j,\alpha}(\Sigma)}} \left[\min_{u \in V \setminus \{0\}} \frac{\int_{\Sigma} \tilde{J}(u)u \, vol_{\varphi_0^*(g)}}{\int_{\Sigma} |u|^2 \, vol_{\varphi_0^*(g)}} \right], \quad (5.1.1)$$

and

$$\tilde{\lambda}_k = \max_{\substack{\text{Dim}(W)=k \\ W \subset B_0}} \left[\min_{u \in W \setminus \{0\}} \frac{\int_{\Sigma} \tilde{J}(u)u \, vol_{\varphi_0^*(g)}}{\int_{\Sigma} |u|^2 \, vol_{\varphi_0^*(g)}} \right] \quad (5.1.2)$$

Now, as

$$\begin{aligned} \int_{\Sigma} \tilde{J}(u)u \, vol_{\varphi_0^*(g)} &= \int_{\Sigma} \left[J(u) - \frac{1}{|\Sigma|} \int_{\Sigma} J(u) \, vol_{\varphi_0^*(g)} \right] u \, vol_{\varphi_0^*(g)} \\ &= \int_{\Sigma} J(u)u \, vol_{\varphi_0^*(g)} - c \int_{\Sigma} u \, vol_{\varphi_0^*(g)} \\ &= \int_{\Sigma} J(u)u \, vol_{\varphi_0^*(g)} \end{aligned}$$

Therefore, $\tilde{\lambda}_k \leq \lambda_k$ □

5.2 Preliminaries for stability on Bifurcation

For the study of stability at the branches of bifurcation of free boundary CMC hypersurfaces, it is necessary to enunciate some theorems and lemmas that are based on the Crandall-Rabinowitz theory.

Corollary 5.2.1. (*[11, Corollary 1.13] Crandall-Rabinowitz*) *Let X and Y be Banach Spaces. $B(X, Y)$ denote the set of bounded linear maps of X into Y . Let $K \in B(X, Y)$ and 0 a K -simple eigenvalue of $F_x(t_0, 0)$ (F as in the Theorem 4.2.2). $\text{Ker}(F_x(t_0, 0)) = \text{Span}\{x_0\}$. Then, there exist open intervals $\tilde{L}, \tilde{I} \in \mathbb{R}$ and continuously differential functions $\lambda : \tilde{L} \rightarrow \mathbb{R}$, $\mu : \tilde{I} \rightarrow \mathbb{R}$, $u : \tilde{L} \rightarrow X$, $w : \tilde{I} \rightarrow X$, such that*

$$(i) \quad F_x(t, 0)u(t) = \lambda(t)K(u(t)), \text{ for } t \in \tilde{L}$$

$$(ii) \quad F_x(\tau(s), x(s))w(s) = \mu(s)K(w(s)), \text{ for } s \in \tilde{I}. \text{ Where } x(s) = sx_0 + s\zeta(s), \zeta \text{ and } \tau \text{ as in Theorem 4.2.2.}$$

Moreover

$$\lambda(t_0) = \mu(0) = 0, \quad u(t_0) = x_0 = w(0),$$

and

$$u(t) - x_0 \in Z, \quad w(s) - x_0 \in Z.$$

The main result relating τ , μ and λ is

Theorem 5.2.2. (*[11, Theorem 1.16] Crandall-Rabinowitz*) *We assume the same hypotheses and the notation of corollary 5.2.1. Then, $\lambda'(t_0) \neq 0$ and near to $s = 0$ the functions $\mu(s)$ and $-s\tau'(s)\lambda(s)$ have the same zeroes, and, whenever $\mu(s) \neq 0$, the same sign. More precisely,*

$$\lim_{\substack{s \rightarrow 0 \\ \mu(s) \neq 0}} \frac{-s\tau'(s)\lambda'(t_0)}{\mu(s)} = 1.$$

Moreover, there is a constant c such that

$$\|x'(s) - w(s)\| \leq c \text{Min}\{|s\tau'(s)|, |\mu(s)|\}$$

near $s = 0$

The following lemmas provide us with the existence of eigenvalues for the Jacobi operator associated with the hypersurfaces that belong to the bifurcation branches.

Lemma 5.2.3. *assuming the hypotheses of the Theorem 4.2.3. Let $H'_0 \neq 0$, $E = \text{Span}\{e\}$, $e \in C^{j,\alpha}_{\partial}(\Sigma)$, $e \neq 0$. Then, there exist an interval $\tilde{I} \subset \hat{I}$, $0 \in \tilde{I}$ and continuously differentiable functions $\mu : \tilde{I} \rightarrow \mathbb{R}$ and $w : \tilde{I} \rightarrow C^{j,\alpha}_{\partial}(\Sigma)$, whit $\mu_0 = 0$ and $w_s \neq 0$ for all $s \in \tilde{I}$, such that:*

$$(1) \quad J_{\psi_s}(w_s) = \mu_s w_s, \text{ for all } s \in \tilde{I}. \quad (J_{\psi_s} \text{ is the Jacobi operator of } \psi_s, \psi_s \text{ the free boundary CMC immersion provided by Theorem 4.2.3.})$$

$$(2) \quad \lim_{\substack{s \rightarrow 0 \\ \mu(s) \neq 0}} \frac{-st'(s)\lambda'(0)}{\mu_s} = 1. \quad (t(s) \text{ and } \lambda(s) \text{ as in the Theorem 4.2.3.})$$

(3) $\|x'(s) - w_s\| \leq c \min\{|st'(s)|, |\mu_s|\}$, for all $s \in \tilde{I}$. Where $x(s) = se + s\zeta(s)$ and c is a positive constant. ($\zeta(s)$ as in the Theorem 4.2.3). We can observe, from the previous limit, that $\mu(s)$ and

$$-st'(s)\lambda'(0) = \frac{-s\hat{H}'_s}{H'_{t(s)}}\lambda'(0)$$

have the same zeroes and, where $\mu_s \neq 0$, the same sign. ($\hat{H}_s$ is the constant mean curvature of ψ_s .)

Proof. In the proof of Theorem 4.2.3 we saw that for the spaces $X = C^{j,\alpha}_{\partial}(\Sigma)$, $Y = C^{j-2,\alpha}_{\partial}(\Sigma)$ and $Z = E^{\perp}$, together with the inclusion map $\iota : X \hookrightarrow Y$ and the operator $F : I \times U \rightarrow Y$, $0 \in I \subset \mathbb{R}$, $U \subset X$, $F(t, f) = H_{\phi_t+f} - H_{\phi_t}$; the conditions of Theorem 4.2.2 (Crandall-Rabinowitz) are met. Also, $\frac{\partial F}{\partial f}(t, 0)(u) = J_t(u)$ and 0 is a ι -simples eigenvalue of $J = \frac{\partial F}{\partial f}(0, 0)$.

We have that

$$\hat{H}_s = H_{t(s)} = H_{\phi_{t(s)}+se+s\zeta(s)} = H_{\phi_{t(s)}+x(s)}.$$

Now, for $(t_0, f_0) \in I \times U$.

$$\begin{aligned} \frac{\partial F}{\partial f}(t_0, f_0) : T_{f_0} &\rightarrow Y, \\ \frac{\partial F}{\partial f}(t_0, f_0)u &= \lim_{r \rightarrow \infty} \frac{H_{\phi_{t_0}+f_0+ru} + H_{\phi_{t_0}} - H_{\phi_{t_0}+f_0} + H_{\phi_{t_0}}}{r} \\ &= \lim_{r \rightarrow \infty} \frac{H_{\phi_{t_0}+f_0+ru} - H_{\phi_{t_0}+f_0}}{r} \\ &= J_{\phi_{t_0}+f_0}(u) \end{aligned}$$

So,

$$\frac{\partial F}{\partial f}(t(s), x(s)) = J_{\phi_{t(s)}+x(s)} = J_{\psi_s}.$$

Then, by Corollary 5.2.1 (ii), with $K = \iota$, there exist an interval $\tilde{I} \subset \hat{I}$, $0 \in \tilde{I}$ and functions $\mu : \tilde{I} \rightarrow \mathbb{R}$, $w : \tilde{I} \rightarrow X$ such that

$$J_{\psi_s}(w_s) = \mu_s w_s, \quad \text{for all } s \in \tilde{I}.$$

Thus, we have the part (1).

Now, $\hat{H}'_s = H'_{t(s)}t'(s)$. So $t'(s) = \frac{\hat{H}'_s}{H'_{t(s)}}$. Thus, parts (2) and (3) follow from Theorem 5.2.2 (Crandall-Rabinowitz). $\square$

Lemma 5.2.4. *We assume the hypotheses of Theorem 4.3.2. Let $H'_0 \neq 0$, $E = \text{Span}\{e\}$, $e \in C^{j,\alpha}_{\partial}(\Sigma)$, $e \neq 0$, J_{ψ_s} the Jacobi operator of ψ_s the free boundary CMC immersion provided by Theorem 4.3.2. Then, there exist an interval $\tilde{I} \subset \hat{I}$, $0 \in \tilde{I}$ and continuously differentiable functions $\mu : \tilde{I} \rightarrow \mathbb{R}$ and $w : \tilde{I} \rightarrow C^{j,\alpha}_{\partial}(\Sigma)$, with $\mu_0 = 0$ and $w_s \neq 0$ for all $s \in \tilde{I}$, such that:*

(1) $J_{\psi_s}(w_s) = \mu_s w_s + c_s$, for all $s \in \tilde{I}$, c_s a constant that depends on s .

(2) $\lim_{\substack{s \rightarrow 0 \\ \mu(s) \neq 0}} \frac{-s\tau'(s)\tilde{\lambda}'(0)}{\mu_s} = 1$. ($\tau(s)$ and $\tilde{\lambda}(s)$ as in the Theorem 4.3.2.)

(3) $\|x'(s) - w_s\| \leq c \min\{|s\tau'(s)|, |\mu_s|\}$, for all $s \in \tilde{I}$. Where $x(s) = se + s\eta(s)$ and c is a positive constant. ($\eta(s)$ as in the Theorem 4.3.2). We can observe, from the previous limit, that μ_s

and

$$-s\tau'(s)\tilde{\lambda}'(0) = \frac{-s\hat{V}'_s}{V'_{\tau(s)}}\tilde{\lambda}'(0)$$

have the same zeroes and, where $\mu_s \neq 0$, the same sign. ($\hat{V}_s$ is the volume of ψ_s .)

Proof. In the proof of Theorem 4.3.2 we observed that for the spaces $X = B_0$, $Y = C_*^{j-2,\alpha}(\Sigma)$ and $Z = E^\perp \cap B_0$, together with the inclusion map $\iota_0 : B_0 \hookrightarrow Y$ and the operator $\bar{F} : I \times U \rightarrow Y$, $0 \in I \subset \mathbb{R}$, the conditions of Theorem 4.2.2 (Crandall-Rabinowitz) are satisfied, where $U \subset B_0$, $\bar{F}(t, h) = \kappa_0(F(t, h)) = F(t, h) - \frac{1}{|\Sigma|} \int_\Sigma F(t, h) \text{vol}_{\varphi_0^*(g)}$ and $F(t, h) = H_{f(t, h)} - H_{\phi_t}$. We have that

$$\frac{\partial}{\partial h} \bar{F}(t, 0) = \kappa_0 \left(\frac{\partial}{\partial h} F(t, 0) \right).$$

So,

$$\frac{\partial}{\partial h} \bar{F}(0, 0)u = J(u) - \frac{1}{|\Sigma|} \int_\Sigma J(u) \text{vol}_{\varphi_0^*(g)},$$

and 0 is a ι_0 -simple eigenvalue of $\frac{\partial}{\partial h} \bar{F}(0, 0)$. Then, again by corollary 5.2.1 (ii), whit $K = \iota_0$, there exist an interval $\tilde{I} \subset \hat{I}$, $0 \in \tilde{I}$ and functions $\mu : \tilde{I} \rightarrow \mathbb{R}$, $w : \tilde{I} \rightarrow X$ such that

$$\frac{\partial}{\partial h} \bar{F}(\tau(s), x(s))(w_s) = \mu_s w_s, \quad \text{for all } s \in \tilde{I}.$$

Now,

$$\frac{\partial}{\partial h} \bar{F}(\tau(s), x(s))(u) = J_{\psi_s}(u) - \frac{1}{|\Sigma|} \int_\Sigma J_{\psi_s}(u) \text{vol}_{\varphi_0^*(g)}.$$

So,

$$J_{\psi_s}(w_s) = \mu_s w_s + \frac{1}{|\Sigma|} \int_\Sigma J_{\psi_s}(w(s)) \text{vol}_{\varphi_0^*(g)}.$$

Thus, we have the part (1).

We have that $\hat{V}(s) = V(\tau(s))$. Then $\hat{V}'(s) = V'(\tau(s))\tau'(s)$. So, $\tau'(s) = \frac{\hat{V}'(s)}{V'(\tau(s))}$. Thus, the parts (2) and (3) follow from Theorem 5.2.2. $\square$

5.3 Stability in the Bifurcation Branch

Now we are going to present the stability criteria in the bifurcation branches. If a free boundary immersion φ_0 is stable, then only three types of bifurcation can occur, which we will enunciate following. In part (1) of Theorem 5.3.1 we will prove this. Recall that φ_0 is stable if $H'_0 V(0)' < 0$. The same three types of bifurcation occur when $\hat{H}'(s) \neq 0$, as we can see in part 2 of Theorem 5.3.3.

- a) Supercritical pitchfork bifurcation; if $\hat{V}'(0) = 0$ and $\hat{V}''(0) < 0$ then ψ_s is stable for $s \in [-s_0, s_0]$ (Figure 5.1)
- b) Subcritical pitchfork bifurcation; if $\hat{V}'(0) = 0$ and $\hat{V}''(0) > 0$ then ψ_s is unstable for $s \in [-s_0, 0] \cup (0, s_0]$ (Figure 5.2)
- c) Transcritical bifurcation; If $\hat{V}'(0) > 0$ then ψ_s is stable for $s \in [-s_0, 0]$ and unstable for $(0, s_0]$ (Figure 5.3)

Figure 5.1: Supercritical pitchfork bifurcation.

Figure 5.2: Subcritical pitchfork bifurcation.

Similar conclusions hold in the case when $H'_0 < 0$ and $V(0) < 0$, by reversing the parameterization of φ_t , $t \mapsto -t$.

We start with the case when the bifurcation parameter is the volume. That is, the bifurcation branch obtained in the theorem 4.3.2.

Theorem 5.3.1. *Let $\{\psi_s\}_{s \in \hat{I}}$ be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.3.2. $H'_0 \neq 0$. $\ker(J_{f_0}|_{C^{j,\alpha}_g(\Sigma)}) = \text{Span}\{e \neq 0\}$. $\lambda_2 = 0$ (Equivalently $\tilde{\lambda}_1 = 0$). $\tilde{\lambda}'_1(0) \neq 0$ ($\tilde{\lambda}_1(t)$ the first eigenvalue of $\tilde{J}_t = J_t - \frac{1}{|\Sigma|} \int_{\Sigma} J_t \text{vol}_{\varphi_0^*(g)}$). If moreover $V'(0) > 0$ (if necessary, change the parameter t to $-t$). Then,*

- (1) In the case $H'_0 < 0$ (in this case φ_0 is stable by proposition 5.1.2 and $\tilde{\lambda}_1 = 0$)
 - (1-a) If $\hat{V}'(s) = 0$ for s near 0 (that is, if $\hat{V}$ is locally constant), then, ψ_s is stable for s near 0.
 - (1-b) If $\hat{V}'(s) \neq 0$, $s \neq 0$, s near 0, then, for a sufficiently small $s_0 > 0$, in each interval $[-s_0, 0)$ and $(0, s_0]$ we have:
 - (1-b-i) If $\tilde{\lambda}'_1(0)s\hat{V}'(s) < 0$, then, ψ_s is stable.
 - (1-b-ii) If $\tilde{\lambda}'_1(0)s\hat{V}'(s) > 0$, then, ψ_s is unstable.

In particular, supercritical and subcritical pitchfork bifurcations are presented where $s\hat{V}'(s)$ does not change sign at $s = 0$, and transcritical bifurcation occurs when $s\hat{V}'(s)$ changes sign at $s = 0$.

- (2) In the case $H'_0 > 0$ (in this case φ_0 is unstable by proposition 5.1.2 and $\tilde{\lambda}_2 = 0$), we have that ψ_s is unstable for small $|s|$

Proof. (1) If $H'_0 V'(0) < 0$, then $\tilde{\lambda}_1 = 0$ by proposition 5.1.2 ($\tilde{\lambda}_1$ eigenvalue of operator $J - \frac{1}{|\Sigma|} \int_{\Sigma} J \text{vol}_{\varphi_0^*(g)}$). Let μ_s be the eigenvalue of $\tilde{J}_{\psi_s} = J_{\psi_s} - \frac{1}{|\Sigma|} \int_{\Sigma} J_{\psi_s} \text{vol}_{\varphi_0^*(g)}$ obtained in Lemma 5.2.4. So, for s sufficiently small, μ_s is the eigenvalue closest to 0 of $\tilde{J}_{\psi_s}$. This is because the condition $\tilde{\lambda}_2 > 0$ is open in the space of Jacobi operators $(J_{\varphi}(f) = \Delta_{\Sigma} f -$

Figure 5.3: Transcritical bifurcation.

($\|\mathbb{I}^\Sigma\|_{HS} + Ric(\vec{n}_\Sigma, \vec{n}_\Sigma))f$, is the sum of positive operator and compact operator). So, $\mu_s = \tilde{\lambda}_1(s)$. Thus, ψ_s is stable if only if $\mu_s \geq 0$.

How are we assuming that $V'_0 > 0$, also by Lemma 5.2.4 we have that μ_s and $-s\hat{V}'(s)\tilde{\lambda}'_1(0)$ have the same zeros and, where $\mu_s \neq 0$, have the same sign. As we also assume that $\tilde{\lambda}'(0) \neq 0$, so

- (1-a) If $\hat{V}'(s) = 0$, then $\mu_s = 0$ and ψ_s is stable.
- (1-b) If $\hat{V}'(s) \neq 0$, let $s_0 > 0$ sufficiently small, then, in each interval $[-s_0, 0)$ and $(0, s_0]$, we have the following options:
 - 1-b-i If $s\hat{V}'(s)\tilde{\lambda}'_1(0) < 0$, then $\mu_s > 0$. Therefore, ψ_s is stable.
 - 1-b-ii If $s\hat{V}'(s)\tilde{\lambda}'_1(0) > 0$, then $\mu_s < 0$. Therefore, ψ_s is unstable.
- (2) If $H'_0 V'(0) > 0$, then $\tilde{\lambda}_2 = 0$ by proposition 5.1.2. So, $\tilde{\lambda}_1(s) < 0$ for s enough small (the condition $\tilde{\lambda}_1 < 0$ is open in the set of Jacobi operators). Therefore, ψ_s is unstable.

□

As a consequence of the previous theorem we obtain the following stability criterion, where we put stronger conditions for H'_0 , V'_0 and $\tilde{\lambda}'_1(0)$ and the criterion is given in terms of the second derivative of $\hat{V}$. We obtain conditions for pitchfork or transcritical bifurcation.

Corollary 5.3.2. *Let $\{\psi_s\}_{s \in \hat{I}}$ be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.3.2. Under the assumptions of the theorem 5.3.1, we also assume that $H'_0 < 0$, $V'_0 > 0$ and $\tilde{\lambda}'_1(0) > 0$. Then, there exist positive constants $t_0 \in (0, \epsilon)$ and $s_0 \in \hat{I}$ such that:*

- (1) φ_t is stable for all $t \in [0, t_0]$ and unstable for all $t \in [-t_0, 0)$.
- (2) If $\hat{V}'(0) \neq 0$, then we have transcritical bifurcation for $\{\psi_s\}_{s \in [-s_0, s_0]}$ (see Figure 5.3). This is,
 - (2-a) If $\hat{V}'(0) > 0$, then ψ_s is stable for $s \in [-s_0, 0]$ and unstable for $s \in (0, s_0]$.
 - (2-b) If $\hat{V}'(0) < 0$, then ψ_s is stable for $s \in [0, s_0]$ and unstable for $s \in [-s_0, 0)$.
- (3) If $\hat{V}'(0) = 0$ and there is the second derivative of $\hat{V}$ in $s = 0$, we have
 - (3-a) If $\hat{V}''(0) < 0$, then ψ_s is stable for all $s \in [-s_0, s_0]$. Here is a supercritical pitchfork bifurcation (see Figure 5.1).
 - (3-b) If $\hat{V}''(0) > 0$, then ψ_s is unstable for all $s \in [-s_0, 0) \cup (0, s_0]$. Here is a subcritical pitchfork bifurcation (see Figure 5.2).

If $H'_0 > 0$ and $V'_0 < 0$, inverting the parameterization of φ_t , $t \mapsto -t$, similar conclusions are obtained.

Proof. (1) Since φ_0 is stable if and only if $\tilde{\lambda}_1 \geq 0$, this is by lemma 5.1.1, and additionally, by part (1) of proposition 5.1.2, $\tilde{\lambda}_1 = 0$, and by the hypothesis that $\tilde{\lambda}'_1(0) > 0$, we have the first statement.

(2) Part (2) follows immediately from I.2. of theorem 5.3.1.

(3) For part (3) we must see the sign of $s\hat{V}'(s)$ and use again I.2. of Theorem 5.3.1.

(3-a) If $\hat{V}''(0) < 0$, then $\hat{V}'(s)$ is decreasing in $[-s_0, s_0]$. So, for $s \in [-s_0, 0)$, $s\hat{V}'(0) < 0$, thus ψ_s is stable. For $s \in (0, s_0]$, $s\hat{V}'(0) < 0$, thus ψ_s is stable too. Therefore, ψ_s is stable for $s \in [-s_0, s_0]$

(3-b) If $\hat{V}''(0) > 0$, then $\hat{V}'(s)$ is increasing in $[-s_0, s_0]$. So, for $s \in [-s_0, 0)$, $s\hat{V}'(0) > 0$, thus ψ_s is unstable. For $s \in (0, s_0]$, $s\hat{V}'(0) > 0$, thus ψ_s is unstable too.

□

Now, we going to study the case when then bifurcation parameter is the mean curvature. That is, stability in the bifurcation branch obtained in the theorem 4.2.3. Here also we obtain conditions on the eigenvalues of the Jacobi operator for pitchfork or transcritical bifurcation.

Theorem 5.3.3. *Let $\{\psi_s\}_{s \in \hat{I}}$ be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.2.3. Under the same conditions and the same notation:*

- (i) $H'_0 \neq 0$.
- (ii) $E = \text{Span}\{e\}$, for some $e \in C^{j,\alpha}_0(\Sigma_0)$, $e \neq 0$.
- (iii) $\lambda_2 = 0$.
- (iv) $\lambda'_2(0) \neq 0$. ($\lambda_2(t)$ the second eigenvalue of J_t .)

And additionally assuming that $H'_0 < 0$ holds, (if necessary, change the parameter t to $-t$). Let μ_s be the eigenvalue of J_{ψ_s} obtained in Lemma 5.2.3. Then,

- (1) If $\hat{H}'_s = 0$ for s near to 0 (that is, if $\hat{H}$ is locally constant, $\hat{H}_s$ the constant mean curvature of ψ_s), then $\mu_s = 0$ for s near 0.
- (2) If $\hat{H}'_s \neq 0$ for s near to 0, $s \neq 0$, then, for $s_0 > 0$, sufficiently small, in each interval $[-s_0, 0)$ and $(0, s_0]$,
 - (2-a) $\mu_s > 0$ if $s\hat{H}'_s\lambda'_2(0) > 0$
 - (2-b) $\mu_s < 0$ if $s\hat{H}'_s\lambda'_2(0) < 0$

In particular, supercritical and subcritical pitchfork bifurcations correspond to the cases where $s\hat{H}'_s$ does not change sign at $s = 0$, and transcritical bifurcation occurs when $s\hat{H}'_s$ changes sign at $s = 0$.

Proof. By Lemma 5.2.3, for s enough close to 0, μ_s and $\frac{-s\hat{H}'_s\lambda'_2(0)}{H'_{\tau(s)}}$ have the same zeros and, where $\mu_s \neq 0$, have the same sign. Therefor, having to $H'_{\tau(s)} < 0$

- (1) If $\hat{H}'_s = 0$, then $\mu_s = 0$.

(2) If $\hat{H}'_s \neq 0$, let $s_0 > 0$, sufficiently small. So, in each interval $[-s_0, 0)$ and $(0, s_0]$, we have the following options:

(2-a) if $s\hat{H}'_s\lambda'_2(0) > 0$, then $\mu_s > 0$

(2-b) if $s\hat{H}'_s\lambda'_2(0) < 0$, then $\mu_s < 0$

□

Again we have that ψ_s is stable if and only if $\mu_s \geq 0$. Therefore, we have the following corollary with results analogous to Corollary 5.3.2

Corollary 5.3.4. *Let $\{\psi_s\}_{s \in \hat{I}}$ be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.2.3. Under the conditions of the theorem 5.3.3, we assuming that $\lambda'_2(0) > 0$. Then, there exist positives constants $t_0 \in (0, \epsilon)$ and $s_0 \in \hat{I}$ such that:*

(1) φ_t is stable for all $t \in [0, t_0]$ and unstable for all $t \in [-t_0, 0)$.

(2) If $\hat{H}'_s(0) \neq 0$, then we have transcritical bifurcation for $\{\psi_s\}_{s \in [-s_0, s_0]}$ (see Figure 5.3). This is,

(2-a) If $\hat{H}'_s(0) < 0$, then ψ_s is stable for $s \in [-s_0, 0]$ and unstable for $s \in (0, s_0]$.

(2-b) If $\hat{H}'_s(0) > 0$, then ψ_s is stable for $s \in [0, s_0]$ and unstable for $s \in [-s_0, 0)$.

(3) If $\hat{H}'_s(0) = 0$ and there is the second derivative of $\hat{H}$ in $s = 0$, we have

(3-a) If $\hat{H}''_s(0) > 0$, then ψ_s is stable for all $s \in [-s_0, s_0]$. Here is a supercritical pitchfork bifurcation (see Figure 5.1).

(3-b) If $\hat{H}''_s(0) < 0$, then ψ_s is unstable for all $s \in [-s_0, 0) \cup (0, s_0]$. Here is a subcritical pitchfork bifurcation (see Figure 5.2).

If $H'_0 > 0$, inverting the parameterization of φ_t , $t \mapsto -t$, similar conclusions are obtained.

Proof. The proof is completely analogous to of Corollary 5.3.2. □

Chapter 6

Conclusions

6.1 Obtained results and impacts

Variational problems are commonly known in areas such as Physics, Physics-Chemistry, Engineering, Economics, Biology, Medicine, etc. Some very specific examples in these areas range from Determining the advance of invasive biological species; The healing of a tissue of the corneal epithelium after the occurrence of a wound; The front of propagation of natural phenomena such as tsunamis, forest fires and avalanches; Speculation on the rise and fall of prices in financial events; Even theoretical-physical phenomena such as the boundary that separates the plasma and vacuum zone inside certain types of experimental reactors of nuclear fusion by magnetic confinement (see [22]).

In this type of problems, involved solutions must satisfy certain conditions in the limits of a known domain. They are called Boundary Problems. In many cases, the domain boundary in which is defined the solution functions is unknown. Therefore, these solutions must also satisfy additional conditions at the boundary. Two types of solutions can be imposed in this case, solutions with fixed boundary and solutions with free boundary. The latter are precisely those concerning this dissertation.

Genericity, existence of perturbation, stability and bifurcation are important characteristics to study when natural phenomena of this nature occurs. Our study in variations of free boundary CMC hypersurfaces generates tools for the study of this type of variational problems. We mainly use the ideas from White (see [28]), Biliotti-Javaloyes-Piccione (see [9]), Bettiol-Piccione-Santoro (see [8]), Koiso (see [16]), Koiso-Palmer-Piccione (see [18]), Crandall-Rabinowitz (see [10] and [11]), to obtain generalizations and results in the aspects mentioned above in families of free boundary CMC hypersurfaces.

We establish results in genericity of Bumpy metrics (Chapter 2). In fact, we prove that the set of Riemannian metrics, which makes all orthogonal embedding with CMC non-degenerate, is generic; in other words, almost all Riemannian metrics meet this condition.

We establish criteria for the existence of perturbation or deformation with free boundary hypersurfaces with CMC from a given free boundary hypersurface $\varphi_0 : \Sigma \rightarrow M$, by setting conditions at the Jacobi Operator's kernel in φ_0 and at the derivation of the volume function in φ_0 (Chapter 3). Given a deformation of this type, we establish criteria to study the stability of φ_0 , that is, by testing if φ_0 corresponds to a critical point of the area functional where there is a minimum. These criteria are established from the sign of the eigenvalues of the Jacobi Operator in φ_0 and of the

derivatives of the volume and mean curvature functions in φ_0 .

We also establish criteria for the existence of bifurcation points in a free boundary CMC hypersurfaces family, for the following cases: firstly, the elements of the branch of bifurcation have the same values in the constant mean curvatures as the elements of the family original; secondly, the volumes of elements of the branch of bifurcations coincide with those of the original branch (Chapter 4).

Finally, given the existence of bifurcation branches of free boundary CMC hypersurfaces, we study the stability of the elements of these branches (Chapter 5), by using the results of Chapter 3.

6.2 Open problems

- In the theory developed by Koiso-Palmer-Piccione there are specific examples of bifurcation and stability in families of fixed boundary CMC surfaces in $\mathbb{R}^3$ (see [18] and [17]).

For our case (free boundary), examples are difficult to construct. It requires the solution of systems of differential equations that involve variable coefficients, such as the norm of the second fundamental form and the Ricci curvature of the ambient manifold,

$$\Delta_{\Sigma_0} f - (||\mathbb{I}^{\Sigma_0}||_{HS}^2 + Ric_g(\eta_{\Sigma_0}, \eta_{\Sigma_0}))f = 0,$$

and that also meet with the linearized free boundary condition

$$g(\nabla f, \eta_{\partial M}) + \mathbb{I}^{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0})f = 0.$$

Therefore, it remains finding a specific example of a free boundary CMC hypersurfaces family having a degenerate point where there is a bifurcation and we can study the stability of the members of this family.

- Additionally, we still need to study in this theory if the same results are obtained in genericity, bifurcation and stability in the case of Semi-Riemannian metrics.

Appendix A

Appendix A

In this appendix we will give some definitions and important results of Functional Analysis, which were necessary during the development of the whole thesis.

Definition A.0.1. If $f : N \rightarrow M$ is a smooth map and $S \subset M$ is an embedded sub-manifold, we say that f is transverse to S if, for every $p \in f^{-1}(S)$, $T_{f(p)}M = T_{f(p)}S + df_p(T_pN)$.

Definition A.0.2. The definition of transversality between a map $F : \mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{Z} \subset \mathcal{Y}$ a smooth submanifold, where $\mathcal{X}$ and $\mathcal{Y}$ are Banach manifolds, is that presented in the Definition A.0.1 but with the additional assumption that $dF_{\mathbf{r}_0}^{-1}(T_{F(\mathbf{r}_0)}\mathcal{Z})$ is a complemented subspace of $T_{\mathbf{r}_0}\mathcal{X}$, i.e., there is a subspace $\mathcal{V} \subset T_{\mathbf{r}_0}\mathcal{X}$ such that $T_{\mathbf{r}_0}\mathcal{X} = dF_{\mathbf{r}_0}^{-1}(T_{F(\mathbf{r}_0)}\mathcal{Z}) \oplus \mathcal{V}$.

Definition A.0.3. If A is a bounded operator in a Hilbert space H and $\{e_i : i \in I\}$ is a orthonormal bases for H , is defined the Hilbert-Schmidt norm as $\|A\|_{HS}^2 = \text{Tr}(A^*A) = \sum_{i \in I} \|Ae_i\|_H^2$, where $\|\cdot\|_H$ is the norm of H .

Definition A.0.4. The Hölder space $C^{j,\alpha}(\Omega)$, where Ω is an open subset of some Euclidean space and $j \geq 0$ an integer, consists of those functions on Ω having continuous derivatives up to order j and such that the j -th partial derivatives are Hölder continuous with exponent α , where $0 < \alpha \leq 1$. A real valued function f on n -dimensional Euclidean space is Hölder continuous, when there are nonnegative real constants c , such that

$$|f(x) - f(y)| \leq c\|x - y\|^\alpha$$

Definition A.0.5. A linear continuous operator $T : E \rightarrow F$ between normed spaces is Fredholm if $\text{Ker } T$ is finite dimensional and $\text{Im } T$ is close and finite codimensional, the index of T is $\text{ind } T = \dim \text{Ker } T - \dim \text{coker } T$. A Fredholm map is a C^1 map $f : M \rightarrow N$, M and N being differentiable Banach manifolds, such that for each $x \in M$, the derivative $df_x : T_x(M) \rightarrow T_{f(x)}(N)$ is a Fredholm operator. The index of f is defined to be the index of df_x for some x . The definition doesn't depend on x , see [14].

Theorem A.0.1. [20, The Inverse Mapping Theorem, 5.2] Let E, F Banach spaces, U an open sub-set of E , and Let $f : U \rightarrow F$ a C^p -morphism with $p \geq 1$. Assume that for some point $x_0 \in U$ The derivative $f'(x_0) : E \rightarrow F$ is a toplinear isomorphism. Then f is a local C^p -isomorphism at x_0 .

Theorem A.0.2. [20, The Implicit Mapping Theorem, 5.9] Let U, V be open sets in Banach Spaces $\mathbf{E}, \mathbf{F}$ respectively, and set $f : U \times V \rightarrow G$ be a C^p mapping. Let $(a, b) \in U \times V$, and assume that $D_2f(a, b) : \mathbf{F} \rightarrow G$ is an isomorphism. Let $f(a, b) = 0$. Then there exist a continuous map $g : U_0 \rightarrow V$ defined on an open neighborhood U_0 of a such that $g(a) = b$ and such that $f(x, g(x)) = 0$ for all $x \in U_0$. If U_0 is taken to be a sufficiently small ball, then g is uniquely determined, and is also of class C^p .

Theorem A.0.3 (Local Form of the Submersions). Let X and Y be Banach spaces and let $f : X \rightarrow Y$ be a submersion in x_0 , e.i, $df(x_0) : T_{x_0} \rightarrow T_{f(x_0)}Y$ is surjective and $\text{Ker}(df(x_0))$ is complemented. Then, there are open sets $U \subset X$ and $V \subset \text{Ker}(df(x_0))$, with $x_0 \in U$ and $0 \in V$, and a diffeomorphism $\varphi : V \times W \rightarrow U$, $W \subset Y$ closed sub-space, such that $f \circ \varphi(x, w) = f(x_0)$, for all $(x, w) \in V \times W$

Appendix B

Appendix B

In this appendix we prove some lemmas of linear algebra in spaces of infinite dimension that are necessary in the proof of the genericity of the Bumpy Metrics in Chapter 2

Lemma B.0.4. *Let V be a infinite dimensional vector space with inner product $\langle, \rangle$ and $W \subset V$ an finite dimensional sub-space. Let's suppose that $V = W \oplus W^\perp$. Let $Z \subset V$ be a sub-space, $Z \supset W^\perp$. Then*

$$1) (W^\perp)^\perp = W.$$

$$2) V = Z \oplus Z^\perp$$

Proof. 1) Clearly $W \subset (W^\perp)^\perp$. Now, if $w \in (W^\perp)^\perp$, we can write $w = w_1 + w_2$, with $w_1 \in W$ and $w_2 \in W^\perp$. From the fact $W \subset (W^\perp)^\perp$, we have $w_1 \in (W^\perp)^\perp$. Thus $w_2 = w + w_1 \in (W^\perp)^\perp$ (recall that $(W^\perp)^\perp$ is a closed subspace of V). Whence $w_2 \in W^\perp \cap (W^\perp)^\perp = \{0\}$, that is, $w = w_1 \in W$. So, we conclude that $(W^\perp)^\perp \subset W$.

2) Note that the quotient spaces $Z/W^\perp$ and $V/W^\perp$ are both finite dimensional and $Z/W^\perp \subset V/W^\perp$. Given $v \in V$, there are unique $v_1 \in W$ and $v_2 \in W^\perp$ such that $v = v_1 + v_2$. Therefore the map $\pi_1 : V/W^\perp \rightarrow W$, $\pi_1(v + W^\perp) = v_1$, is an isomorphism. Let $i : Z/W^\perp \hookrightarrow V/W^\perp$ be the inclusion map. Consider $L = \pi_1 \circ i : Z/W^\perp \rightarrow W$ and let $\tilde{Z} = \text{Im} L \subset W$. Thus $\tilde{Z} = \{z - z_2 : z \in Z, z = z_1 + z_2, z_1 \in W, z_2 \in W^\perp\} \subset Z$ and $W = \tilde{Z} \oplus \tilde{Z}^\perp$. Let $\{x_1, \dots, x_k\} \subset W$ be a basis for $\tilde{Z}^\perp$. We claim that $x_1, \dots, x_k \in Z^\perp$. Let $z + W^\perp \in Z/W^\perp$, $z = z_1 + z_2 \in Z$, $z_1 \in W$ and $z_2 \in W^\perp$. We have

$$\begin{aligned} \langle z, x_i \rangle &= \langle z, x_i \rangle + \langle W^\perp, x_i \rangle \\ &= \langle z + W^\perp, x_i \rangle \\ &= \langle z_1 + z_2 + W^\perp, x_i \rangle \\ &= \langle z_1 + W^\perp, x_i \rangle \\ &= \langle z_1, x_i \rangle + \langle W^\perp, x_i \rangle = 0, \end{aligned}$$

since $z_1 \in \tilde{Z}$, $x_i \in \tilde{Z}^\perp \subset W$, $i = 1, \dots, k$.

Now we prove that $\text{Span}\{Z, x_1, \dots, x_k\} = V$. Let $v = v_1 + v_2 \in V$ be given, with $v_1 \in W$ and $v_2 \in W^\perp \subset Z$. There holds that $v_1 = \tilde{z}_1 + \sum_{i=1}^k a_i x_i$, $\tilde{z}_1 \in \tilde{Z}$, where a_i , $i = 1, \dots, k$, are scalars. Whence $v = (\tilde{z}_1 + v_2) + \sum_{i=1}^k a_i x_i$.

It remains to show that $\text{Span}\{x_1, \dots, x_k\} = Z^\perp$. In fact, it suffices to show that $\text{Span}\{x_1, \dots, x_k\} \supset Z^\perp$. Let

$$i : Z^\perp \hookrightarrow V = Z \oplus \text{Span}\{x_1, \dots, x_k\}$$

be the inclusion map, and

$$\pi_2 : Z \oplus \text{Span}\{x_1, \dots, x_k\} \rightarrow \text{Span}\{x_1, \dots, x_k\}$$

the natural projection. We have that $\text{Ker}(\pi_2 \circ i) = \{0\}$, so $\pi_2 \circ i$ is injective, thus $\text{Dim}(Z^\perp) \leq k$. $\square$

Lemma B.0.5. *Let V be a infinite-dimensional vector space with inner product $\langle, \rangle$ and $W \subset V$ an finite dimensional subspace. Suppose that $V = W \oplus W^\perp$. Let $Z \subset V$ be a subspace, $Z \supset W^\perp$, such that for all $w \in W \setminus \{0\}$, there exist $z \in Z$ with $\langle z, w \rangle \neq 0$. Then $Z = V$*

Proof. We show that

$$(\forall w \in W \setminus \{0\})(\exists z \in Z)(\langle z, w \rangle \neq 0) \Leftrightarrow W \cap Z^\perp = \{0\}. \quad (\text{B.0.1})$$

First suppose that $(\forall w \in W \setminus \{0\})(\exists z \in Z)(\langle z, w \rangle \neq 0)$. Let $w \in W \cap Z^\perp$ be given, then $\forall z \in Z$, $\langle z, w \rangle = 0$, so $w = 0$, that is, $W \cap Z^\perp = \{0\}$.

Now suppose that $W \cap Z^\perp = \{0\}$ and let $w \in W \setminus \{0\}$, then $w \notin Z^\perp$, so there is $z \in Z$ such that $\langle z, w \rangle \neq 0$.

Since $Z \supset W^\perp$, by 1) of Lemma B.0.4 we have $Z^\perp \subset (W^\perp)^\perp = W$. From the fact $W \cap Z^\perp = \{0\}$, we deduce that $Z^\perp = \{0\}$. Also by 2) of Lemma B.0.4 we have $V = Z \oplus Z^\perp$. So, $V = Z$. $\square$

The next lemma and its proof were taken from [9, Lemma 2.2].

Lemma B.0.6. *Let $L : U \rightarrow V$ be a linear map between vector spaces, and let $S \subset V$ be a subspace of finite codimension. Then, $L^{-1}(S)$ is finite co-dimensional in U , and*

$$\text{Codim}_U(L^{-1}(S)) = \text{Codim}_V(S) - \text{Codim}_V(\text{Im}(L) + S)$$

.

Proof. If $\pi : V \rightarrow V/S$ is the canonical projection, the linear map $\pi \circ L : U \rightarrow V/S$ has kernel $L^{-1}(S)$. Hence, $\pi \circ L$ defines an injective linear map from $U/L^{-1}(S)$ to V/S , and we have

$$\begin{aligned} \text{Codim}_V(S) &= \text{Dim}(V/S) \\ &= \text{Dim}(U/L^{-1}(S)) + \text{Codim}_{V/S}(\text{Im}(\pi \circ L)) \\ &= \text{Codim}_U(L^{-1}(S)) + \text{Codim}_V(\text{Im}(L) + S). \end{aligned} \quad \square$$

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