**Affine wreath product algebras
with trace maps of generic parity**

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Abstract

MENDONÇA, E. M. **Affine wreath product algebras with trace maps of generic parity**. 2020. 114 f. Thesis (Masters) - Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2020.

The goal of this project is to study the structure and representation theory of affine wreath product algebras $\mathcal{A}_n(F)$. These algebras appear naturally in Heisenberg categorification and generalize many important algebras (degenerate affine Hecke algebras, affine Sergeev algebras and wreath Hecke algebras).

The whole class was introduced by D. Rosso and A. Savage in [RS17]. In [Sav20] the second author studied both structure and representations under the condition that the trace map of F is even. In this project we extend the definition for the case where the trace is odd, obtaining “odd-analogous” results. Since our approach is analogous to Savage’s, we consider the trace map being of arbitrary parity and unify statements and proofs.

Studying the structure theory, we present a basis for $\mathcal{A}_n(F)$ and compute its center. We also introduce Jucys-Murphy and intertwining elements. Considering an equivalence of categories, we describe the simple $\mathcal{A}_n(F)$ -modules in terms of simple representations of degenerate affine Hecke algebras, affine Sergeev algebras and wreath product algebras. We define the cyclotomic quotients $\mathcal{A}_n^{\mathbf{C}}(F)$ for $\mathcal{A}_n(F)$ and we show that these are Frobenius algebras with an appropriately chosen trace map. We also state a cyclotomic Mackey Theorem and show that $\mathcal{A}_n^{\mathbf{C}}(F)$ is a Frobenius extension of $\mathcal{A}_{n-1}^{\mathbf{C}}(F)$.

Keywords: wreath product algebra, Frobenius algebra, Hecke algebra, Sergeev algebra, Mackey Theorem, cyclotomic quotient.

Resumo

MENDONÇA, E. M. **Álgebras afins de produto entrelaçado com traços de paridade genérica**. 2010. 114 f. Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2010.

O objetivo desse projeto é estudar teoria estrutural e representações de álgebras afins de produto entrelaçado $\mathcal{A}_n(F)$. Tais álgebras aparecem naturalmente em categorificações Heisenberg e generaliza outras importantes álgebras (álgebras de Hecke afim degenerada, álgebras de Sergeev afim e álgebras entrelaçadas de Hecke).

A classe de álgebras foi introduzida por D. Rosso e A. Savage em [RS17]. Em [Sav20], o segundo autor estudou teoria estrutural e de representações sobre a condição de que o traço de F fosse par. Nesse projeto estendemos a definição para o caso de traço ímpar, obtendo resultados análogos “ímpares”. Como nossa abordagem análoga a de Savage, nós consideramos os traços com paridade arbitrária e unificamos enunciados e demonstrações.

Estudando a teoria estrutural, nós apresentamos uma base para $\mathcal{A}_n(F)$ e calculamos seu centro. Também introduzimos elementos de Jucys-Murphy e elementos entrelaçados. Considerando uma equivalência de categorias, descrevemos os $\mathcal{A}_n(F)$ -módulos simples em função de representações simples das álgebras de Hecke afim degenerada, álgebras de Sergeev afim e álgebras de produto entrelaçantes. Definimos os quocientes ciclotômicos $\mathcal{A}_n^C(F)$ de $\mathcal{A}_n(F)$ e mostramos que essas álgebras são álgebras de Frobenius com uma apropriada escolha de traço. Enunciamos um ciclotômico Mackey teorema e mostramos que $\mathcal{A}_n^C(F)$ é uma extensão de Frobenius de $\mathcal{A}_{n-1}^C(F)$.

Palavras-chave: álgebras de produto entrelaçado, álgebras de Frobenius, álgebras de Hecke, álgebras de Sergeev, Mackey teorema, quociente ciclotômico.

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List of Symbols

Notation	Meaning	Definition
$\bar{a}$	parity of a	p. 5
$ a $	$\mathbb{Z}$ -degree of a	p. 5
$Z(A)$	center of A	(2.1.1)
$\text{HOM}_A(M, N)$	graded hom space	(2.1.5)
$\text{Hom}_A(M, N)$	homogeneous hom space	(2.1.5)
$\mathcal{Cl}$	2-dimensional Clifford algebra	Ex. 2.1.4
$\cong$	isomorphism (not nec. preserving $\mathbb{Z} \times \mathbb{Z}_2$ -grading)	p. 7
$\simeq$	even isomorphism	p. 7
$\sim$	odd isomorphism	p. 7
Π	parity shift functor	(2.1.6)
$\{n, \epsilon\}$	degree and parity shift functor combined	(2.1.8)
$\mathcal{S}(A)$	set of even iso. classes of A -mod up to degree shift	p. 8
${}^\alpha V$	twisted module	(2.1.9)
$\boxtimes$	outer tensor product of modules	p. 9
$\otimes$	simple tensor product	p. 11
F	graded Frobenius superalgebra	p. 27
tr	trace map of F	p. 27
ψ	Nakayama automorphism of F	p. 20
$\mathcal{Cl}_\tau$	Clifford algebra $\mathcal{Cl}$ equipped with trace map of parity τ	Ex. 2.2.17
D	algebra of Dual numbers ($\mathbb{k}[u]/(u^2)$)	Ex. 2.2.19
$\tau, -\delta$	parity and degree of tr , respectively	p. 27
θ	order of ψ	p. 27
$B, B^\vee$	basis of F and its left dual basis with respect to tr	p. 27
$\rtimes, \ltimes$	smash product, wreath product algebra	§2.3
πa	action of $\pi \in S_n$ on a by superpermutation	p. 26
$f_i, f \in F$	$1^{\otimes(i-1)} \otimes f \otimes 1^{n-i}$	p. 27
ψ_i	$\text{Id}^{\otimes(i-1)} \otimes \psi \otimes \text{Id}^{n-i}$	p. 27

Notation	Meaning	Definition
$\mathcal{A}_n(F)$	affine wreath product algebra	Def. 3.1.1
$t_{i,j}$	$\sum_{b \in B} (-1)^{\tau \bar{b}} b_i b_j^\vee$	(3.1.8)
$t_{i,j}^{(k)}$	$\sum_{b \in B} (-1)^{\tau k \bar{b}} b_i \left(\sum_{l=0}^{k-1} (-1)^{\tau l} x_j^{k-1-l} x_i^l \right) b_j^\vee$	(3.3.1)
P_n	supercommutative polynomial algebra in the variables $x_1, \dots, x_n$ of parity τ	(4.1.1)
$P_n(F)$	$(\mathbb{k}[x] \ltimes F)^{\otimes n}$, with x of parity τ	(4.1.3)
Δ_i	deformed divided difference operator	Def. 4.1
∂_i	divided difference operator (of parity τ)	Def. 4.1.14
x^α	$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$	(4.2.1)
${}^k L$	twisted module ${}^{\psi^k}(\Pi^{ k } L)$	p.56
s_L	smallest positive integer such that ${}^{s_L} L \simeq L$	p.56
σ_L	even F -module isomorphism $L \xrightarrow{\simeq} {}^{s_L} L$	(5.1.1)
flip	$\text{flip}(v_1 \otimes v_2) = (-1)^{\bar{v}_1 \bar{v}_2} v_2 \otimes v_1$	(5.2.1)
$L(a)$	see definition	(5.1.3)
$\rho_{k,\ell}$	superpermutation of k -th and ℓ -th factors	(5.2.3)
N	number of ψ -orbits in $\mathcal{S}(F)$	p. 63
$\mathbf{L}_1, \dots, \mathbf{L}_N$	representatives of ψ -orbits in $\mathcal{S}(F)$	(5.3.1)
s_k, σ_k	$r_{\mathbf{L}_k}, \sigma_{\mathbf{L}_k} : L_k \xrightarrow{\simeq} {}^{s_L} L_k$	p. 63, p. 64
$\mathcal{A}_n(F)\text{-mod}^s$	category of f.d. $\mathcal{A}_n(F)$ -modules semisimple as $F^{\otimes n}$ -modules	p. 64
$\mathcal{C}_n$	set of compositions of n of length at most N	(5.3.2)
S_μ	Young subgroup of S_n	(5.3.3)
$\mathcal{A}_\mu(F)$	parabolic subalgebra of $\mathcal{A}_n(F)$	p. 63
Ind_μ^n	induction functor for parabolic subalgebras	(5.3.4)
$\mathbf{L}(\mu)$	$\mathbf{L}_1(\mathbf{a}_1)^{\boxtimes \mu_1} \boxtimes \cdots \boxtimes \mathbf{L}_N(\mathbf{a}_N)^{\boxtimes \mu_N}$	(5.3.6)
$\mathbf{E}(\mu)$	$\text{END}_{F^{\otimes n}}^{\text{op}} \mathbf{L}(\mu)$	(5.3.7)
$\mathcal{H}_n^\ell$	see definition	(5.3.10)
$\mathcal{R}_\mu, \mathcal{R}_n$	$\mathcal{R}_\mu = \mathcal{H}_{\mu_1}^1 \otimes \cdots \otimes \mathcal{H}_{\mu_N}^N, \mathcal{R}_n = \bigoplus_{\mu \in \mathcal{C}_n} \mathcal{R}_\mu$	(5.3.11)
$y_1, \dots, y_n$	even polynomial generators of $\mathcal{R}_n$	p. 65
$J_{\mathbf{C}}, \chi_{\mathbf{C}}$	ideal $J_{\mathbf{C}}$ of $\mathcal{A}_n(F)$ generated by the cyclotomic polynomial $\chi_{\mathbf{C}}$	(6.2.1)
$\mathcal{A}_n^{\mathbf{C}}(F)$	cyclotomic wreath product algebra	(6.2.2)
$d = d_{\mathbf{C}}$	level of $\mathbf{C}$	(6.2.3)
$\text{tr}_{\mathbf{C}}$	trace map for $\mathcal{A}_n^{\mathbf{C}}(F)$	(6.4.2)
${}^{\mathbf{C}}\text{Ind}_n^{n+1}, {}^{\mathbf{C}}\text{Res}_n^{n+1}$	induction and restriction functors for cyclotomic quotients	p. 93

Chapter 1

Introduction

1.1 Motivation

1.1.1 From symmetric group to categorification

The symmetric group and its representation theory have a rich history with many implications in combinatorics, representation theory, and mathematical physics. One important result in this theory was due to Geissinger, in [Gei77], who gave a realization of the bialgebra of symmetric functions Sym by considering representations of symmetric group S_n , for all $n \in \mathbb{N}$.

In particular, he constructed an isomorphism of bialgebras

$$\text{Sym} \simeq \bigoplus_{n \in \mathbb{N}} \mathcal{K}_0(\mathbb{k}S_n\text{-mod}),$$

where $\mathbb{k}S_n$ is the group algebra of S_n over a field $\mathbb{k}$ of characteristic 0 and $\mathcal{K}_0(\mathcal{C})$ denotes the Grothendieck group of an abelian category $\mathcal{C}$ (i.e, the free abelian group generated by isomorphism classes $[S]$ of all simple objects in $\mathcal{C}$, modulo the subgroup generated by $[S \oplus T] - [S] - [T]$, for all objects S, T in $\mathcal{C}$). The multiplication and comultiplication in $\bigoplus_{n \in \mathbb{N}} \mathcal{K}_0(\mathbb{k}S_n\text{-mod})$ are given by considering, respectively, the induction and restriction functors

$$\begin{aligned} \text{Ind}_{n,m}^{n+m} &:= \mathbb{k}S_{n+m} \otimes_{\mathbb{k}S_n \otimes \mathbb{k}S_m} - : \mathbb{k}S_n\text{-mod} \times \mathbb{k}S_m\text{-mod} \rightarrow \mathbb{k}S_{n+m}\text{-mod}, \\ \text{Res}_{n,m}^{n+m} &:= {}_{\mathbb{k}S_n \otimes \mathbb{k}S_m} \mathbb{k}S_{n+m} \otimes_{\mathbb{k}S_{n+m}} - : \mathbb{k}S_{n+m}\text{-mod} \rightarrow \mathbb{k}S_n\text{-mod} \times \mathbb{k}S_m\text{-mod}, \end{aligned}$$

where $\mathbb{k}S_n \otimes \mathbb{k}S_m$ is identified as a subalgebra of $\mathbb{k}S_{n+m}$ in the usual way (S_n acting on the first n elements and S_m acting on the last m elements). Through the isomorphism, the elementary, the complete symmetric and the Schur functions correspond to the sign, trivial and Specht modules of S_n .

Geissinger's construction allowed to bring combinatoric results of symmetric functions, such as Young's rules and Littlewood–Richardson rules, into representation theory of symmetric groups. More precisely, it allowed to describe how a representation of symmetric group is decomposed into combination of irreducible representations of symmetric groups of small ranks, giving new proofs of branching rules and Frobenius character formula (see [Ful97] and [Mac15]). Furthermore, the approach also applies the other way around. In fact, the correspondence between Schur functions and Specht modules, which form a canonical basis for

$\mathcal{K}_0(\mathbb{k}S_n\text{-mod})$, shows easily that Schur functions also form a basis for the ring Sym of symmetric functions.

The approach taken by Geissinger is a good illustration for what is now called *categorification*. It is the process of replacing set-theoretic concepts by category-theoretic analogues, with the goal to reveal hidden properties in the starting object. Categorification replaces sets by categories, functions with functors, and equations by natural isomorphisms of functors satisfying additional properties.

1.1.2 Heisenberg categorification and affine wreath product algebras

Heisenberg algebras play a fundamental role in mathematics and mathematical physics. The Heisenberg algebra $\text{Heis}_{\mathbb{k}}$ is the unital associative $\mathbb{k}$ -algebra with generators $p_n, p_n^*, n \in I$, and relations

$$p_n p_m^* = p_m^* p_n + \delta_{n,m} 1, \quad p_n p_m = p_m p_n, \quad p_n^* p_m^* = p_m^* p_n^*, \quad m, n \in I.$$

The size of the index set I is called the rank of the Heisenberg algebra. The first relation is often called the canonical commutation relation in the physics literature, where the generators p_n^* and p_n correspond to position and momentum operators in a single particle system with $|I|$ degrees of freedom. Consider here $I = \mathbb{N}$.

It turns out that Heisenberg algebras are related with the two objects appearing in Geissinger's construction. It is not difficult to show that there is a natural isomorphism of functors

$$\text{Ind}_{n-1}^n \circ \text{Res}_{n-1}^n \simeq \text{Res}_n^{n+1} \circ \text{Ind}_n^{n+1} \oplus \text{Id},$$

where $\text{Ind}_m^{m+1}: S_m\text{-mod} \rightarrow S_{m+1}\text{-mod}$ and $\text{Res}_m^{m+1}: S_{m+1}\text{-mod} \rightarrow S_m\text{-mod}$ are the induction and restriction functors obtained from the inclusion $S_m \subseteq S_{m+1}$. This isomorphism is a categorical analogue of the canonical commutation relation in the Heisenberg algebra's relations. Furthermore, the Heisenberg algebras are connected with the ring of symmetric functions by the Fock space representations and the Heisenberg double of Sym (See [SY15] for more details).

Therefore, in an attempt to expand Geissinger's construction, such relations turn the presence of Heisenberg algebras more natural. In that sense, a substantial contribution in strengthening Geissinger's work was due to Khovanov. In a landmark paper [Kho14], Khovanov defined a monoidal category $\mathcal{H}$ that acts naturally on the category of representations of symmetric groups. The tensor product in $\mathcal{H}$ equips its Grothendieck group $\mathcal{K}_0(\mathcal{H})$ with a multiplication, making it an algebra that contains the Heisenberg algebra Heis . Khovanov conjectured that in fact $\mathcal{K}_0(\mathcal{H})$ and Heis are isomorphic. Brundan, Savage and Webster proved this conjecture in [BSW18].

The category $\mathcal{H}$ was constructed in terms of planar braid-like diagrams allowing graphical calculations. Furthermore, the degenerate affine Hecke algebra appeared in the endomorphism algebras of certain objects of $\mathcal{H}$. That way, many important concepts in the representation theory of symmetric groups, such as Jucys-Murphy elements and the center of the group algebra $\mathbb{k}S_n$, have natural interpretations in the graphical calculus of $\mathcal{H}$.

Khovanov's construction has led to various generalizations, including q -deformations, categorification of vertex operators and categorified braid group actions. Recently, Rosso and Savage in [RS17] generalized the existing graphical Heisenberg categorifications by associating a monoidal category $\mathcal{H}_F$ defined in terms of planar diagram to any $\mathbb{N}$ -graded Frobenius super-

algebra F . In their approach, the wreath product $F^{\otimes n} \rtimes \mathbb{k}S_n$ plays a role in a equivalent way as the symmetric group plays in Kovanov's construction. As the degenerate affine Hecke algebra appeared in Kovanov's Heisenberg categorification, an affinization of the wreath product algebras $F^{\otimes n} \rtimes \mathbb{k}S_n$ appeared naturally in the endomorphism space of some objects of $\mathcal{H}_F$. This algebra, denoted by $\mathcal{A}_n(F)$, is defined by Savage in [Sav20] and is called *affine wreath algebra*.

More recently, based on the cyclotomic quotients $\mathcal{A}_n^C(F)$ for $\mathcal{A}_n(F)$, Savage improved his construction and defined in [Sav19] a new graphical category. This category specialize to his first construction when the associated cyclotomic quotient is level one. It also specialize to the higher level Heisenberg categories of [MS18], which include Khovanov's original category.

1.1.3 Generalization of degenerate affine Hecke algebra analogues

Affine Hecke algebras and their degenerate analogs play a vital role in the representation theory of Lie algebras. Over the past decades, several modified versions of these algebras have appeared, in different context of study. Remarkably, the affine wreath product algebras $\mathcal{A}_n(F)$ turn out to provide a good unification of various of those generalizations. Precisely:

1. When $F = \mathbb{k}$, $\mathcal{A}_n(F)$ is the degenerate affine Hecke algebra. (See 3.2.1).
2. When F is the group algebra of a group G , $\mathcal{A}_n(F)$ is the wreath Hecke algebra defined by Wan and Wang [WW08]. (See 3.2.1)
3. When F is the two-dimensional Clifford superalgebra $\mathcal{Cl}$, $\mathcal{A}_n(F)$ is the affine Sergeev algebra (also called the degenerate affine Hecke–Clifford algebra) of Nazarov [Naz97], introduced in the study of projective representation theory of symmetric group. (See 3.2.2)

In [Sav20], Savage showed that, despite their great level of generality, the setting of wreath product algebras turned out to be very tractable. It is possible to deduce a great deal of the structure and representation theory and, by specializing the Frobenius algebra F , to recover known results, but with unified and simplified proofs.

1.2 Goals and structure of the thesis

In his work [Sav20], Savage considered Frobenius superalgebras with parity-preserving trace map only. The goal of this work is to extend his study for parity-reversing trace maps obtaining “odd” analogs of his results. Since the approaches are analogous to that used by [Sav20], with similar proofs, we decided in this work to consider a trace map with an arbitrary parity τ , unifying the proofs.

We start by discussing, in Section 2, some necessary definitions and results for the work. We give a background in superalgebras and supermodules, and we define bimodules. We then define Frobenius extensions, which extend the notion of Frobenius algebras, and we give some examples. Finally we introduce smash product and wreath product of algebras.

Then, in Section 3, we define affine wreath product algebras for a Frobenius algebra F with trace map of parity $\tau \in \mathbb{Z}_2$, and give some relations and examples. In this section, we prove (see Lemma 3.1.3) that the affine wreath product algebra is independent (up to isomorphism) of the trace map. Despite this result, the generalization made in this file still proves important,

since there are Frobenius algebras with only odd traces, as in Example 3.2.3, or because it allows different presentations of already known algebras, as in Example 3.2.2.

In Section 4, we study the structure theory of the affine wreath algebras $\mathcal{A}_n(F)$, describing an explicit basis of $\mathcal{A}_n(F)$ and its center. We also introduce deformed divided difference operators in $(\mathbb{k}[x] \rtimes F)^{\otimes n}$ (where x has odd parity) and prove that they satisfy similar relations as the usual divided differential operators. We then introduce analogs of parity τ of Jucys–Murphy elements. Such elements are the generalization of the usual Jucys–Murphy elements for degenerated affine Hecke algebras. We use the Jucys–Murphy elements to construct a surjective homomorphism from $\mathcal{A}_n(F)$ to the wreath product algebra $F^{\otimes n} \rtimes \mathbb{k}S_n$. We finish by defining intertwining elements for $\mathcal{A}_n(F)$.

In Section 5, we study the representation theory for $\mathcal{A}_n(F)$. By giving an explicit equivalence of categories, we classify the simple $\mathcal{A}_n(F)$ -modules when $\mathbb{k}$ is an algebraically closed field (Theorem 5.5.2), reducing the classification to the known cases of simple representations of degenerate affine Hecke algebras, affine Sergeev algebras, and wreath product algebras.

Finally, in Section 6, we define cyclotomic quotients $\mathcal{A}_n^{\mathbf{C}}(F)$ of $\mathcal{A}_n(F)$ and find explicit basis of these quotients. We prove that the cyclotomic quotients are themselves $\mathbb{N}$ -graded Frobenius superalgebras, and give an explicit description of the associated Nakayama automorphism. We then state a cyclotomic Mackey Theorem. To finish, by restricting the cyclotomic quotient parameter $\mathbf{C}$ to be in $F \times \mathbb{k}^{\otimes(n-1)} \subseteq F^{\otimes n}$ we view $\mathcal{A}_n^{\mathbf{C}}(F)$ as a subalgebra of $\mathcal{A}_{n+1}^{\mathbf{C}}(F)$. Then we prove that $\mathcal{A}_n^{\mathbf{C}}(F)$ is a Frobenius extension of $\mathcal{A}_{n-1}^{\mathbf{C}}(F)$, implying that the induction functor is both left and right adjoint to the restriction up to degree shift. This property plays a vital role in categorification.

Chapter 2

Preliminaries

2.1 Graded superalgebras and their modules

We recall some basic facts about graded associative superalgebras and supermodules. We use as the main reference [Kle05, § 12] and all the results in this section can be found there. Another good reference on the subject is [J88].

Let Γ be an additive monoid. A Γ -graded space is a $\mathbb{k}$ -vector space V with a decomposition $V = \bigoplus_{\gamma \in \Gamma} V_\gamma$. For $\gamma \in \Gamma$, we call the elements of V_γ to be Γ -homogeneous with Γ -degree γ . A subspace W of V is a Γ -graded subspace if it is homogeneous, that is, if W has a Γ -graded space structure such that $W_\gamma = V_\gamma \cap W$, for all $\gamma \in \Gamma$. Given two Γ -graded spaces V and W , the set $\text{HOM}_{\mathbb{k}}(V, W)$ of linear map from V to W , has a natural structure of Γ -graded space by defining:

$$\text{HOM}_{\mathbb{k}}(V, W)_\gamma = \{f \in \text{HOM}_{\mathbb{k}}(V, W) | f(V_\xi) \subseteq W_{\xi+\gamma}, \forall \xi \in \Gamma\}.$$

We write $\text{Hom}_{\mathbb{k}}(V, W) = \text{HOM}_{\mathbb{k}}(V, W)_\epsilon$, where $\epsilon \in \Gamma$ is the identity element. Finally, the direct sum and the tensor product $V \otimes W$ inherit Γ -grading by defining:

$$(V \oplus W)_\gamma = V_\gamma \oplus W_\gamma, \quad (V \otimes W)_\gamma = \bigoplus_{\xi+\zeta=\gamma} V_\xi \otimes W_\zeta.$$

A Γ -graded associative algebra is an associative $\mathbb{k}$ -algebra A with a structure Γ -graded space $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, satisfying $A_\gamma A_\xi \subseteq A_{\gamma+\xi}$, for all $\gamma, \xi \in \Gamma$. The notion of Γ -homogeneous elements, Γ -degree, Γ -graded subalgebra, Γ -graded ideal, Γ -graded algebras homomorphism are defined analogously as before. By considering $\Gamma = \mathbb{Z} \times \mathbb{Z}_2$, we define:

Definition 2.1.1 (Graded superalgebra and superspace). A *graded superalgebra* (respectively, a *graded superspace*) is an associative algebra A (resp., $\mathbb{k}$ -vector space) equipped with a structure of $\mathbb{Z} \times \mathbb{Z}_2$ -graded algebra (resp., space).

We use the terms “degree” and “graded” to refer to the $\mathbb{Z}$ -grading and “parity” and “super” to refer to the $\mathbb{Z}_2$ -grading. For a homogeneous element a of a graded superalgebra, we denote its degree by $|a|$ and its parity by $\bar{a}$, which we call *even* if it is $0 \in \mathbb{Z}_2$ and *odd* otherwise. In the text, when considering the degree and parity of an element a in an equation, we will implicitly assume the element to be homogeneous. Also, throughout this text, we equip $\mathbb{k}$ with the trivial grading, i.e., it is concentrated in degree and parity zero.

The *center* of a graded superalgebra A is the subalgebra

$$Z(A) = \text{span}\{a \in A \mid ab = (-1)^{\bar{a}\bar{b}}ba, \forall b \in A\}, \quad (2.1.1)$$

consisting of all elements that *supercommute* with all other elements.

The *opposite* graded superalgebra A^{op} is the graded superalgebra with same underlying superspace as A , but with multiplication given by

$$a \cdot b = (-1)^{\bar{a}\bar{b}}ba, \quad \forall a, b \in A, \quad (2.1.2)$$

where juxtaposition denotes the original multiplication in A .

Given two graded superalgebras A_1 and A_2 , we view the tensor product $A_1 \otimes A_2$ of graded superspaces as a graded superalgebra with the multiplication given by

$$(a_1 \otimes a_2)(a'_1 \otimes a'_2) = (-1)^{\bar{a}'_1 \bar{a}_2} (a_1 a'_1 \otimes a_2 a'_2),$$

for all homogeneous $a_1, a'_1 \in A_1$ and $a_2, a'_2 \in A_2$. We define the n^{th} tensor power of A to be

$$A^{\otimes n} := A \otimes A \otimes \cdots \otimes A \quad (n \text{ factors}).$$

Three important examples of superalgebras are the following:

Example 2.1.2 (Superalgebra $\mathcal{M}(m|n)$). Let V be a superspace with decomposition $V = V_0 \oplus V_1$ such that $\dim V_0 = m$ and $\dim V_1 = n$. Define $\mathcal{M}(V)$ to be $\text{HOM}_{\mathbb{K}}(V, V) = \text{END}_{\mathbb{K}}(V)$.

Since, up to isomorphism, the algebra $\mathcal{M}(V)$ does not depend on the superspace V , but only on its superdimension $(m, n) \in \mathbb{N}^2$, we can identify $\mathcal{M}(V)$ with the superalgebra $\mathcal{M}(m|n)$, i.e., the superalgebra of all block matrices of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C and D are matrices with entry in $\mathbb{K}$ and size $m \times m$, $m \times n$, $n \times m$ and $n \times n$, respectively.

Example 2.1.3 (Superalgebra $\mathcal{Q}(n)$). Let $V = V_0 \oplus V_1$ be a vector superspace with $\dim V_0 = \dim V_1 = n$ and let J be an odd involution in $\text{HOM}_{\mathbb{K}}(V, V)$. We define the superalgebra $\mathcal{Q}(V, J)$ to be:

$$\mathcal{Q}(V, J) = \{f \in \text{Hom}_{\mathbb{K}}(V, V) \mid fJ = (-1)^{\bar{f}}Jf\}.$$

Some calculations easily show that, up to isomorphism of superalgebras, $\mathcal{Q}(V, J)$ depends only of the superdimension of V . In particular we can identify the superalgebra $\mathcal{Q}(V, J)$ with the superalgebra $\mathcal{Q}(n)$ of all block matrices of the form

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix},$$

with A, B matrices with entry in $\mathbb{K}$ and size $n \times n$. Also, if $\mathbb{K}$ is algebraically closed, it can be proved that the following map from $\mathcal{Q}(1) \otimes \mathcal{Q}(1)$ to $\mathcal{M}(1|1)$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \otimes \begin{bmatrix} a' & b' \\ -b' & a' \end{bmatrix} \mapsto \begin{bmatrix} aa' - \sqrt{-1}bb' & a'b - \sqrt{-1}ab' \\ -a'b - \sqrt{-1}ab' & aa' + \sqrt{-1}bb' \end{bmatrix}, \quad (2.1.3)$$

defines an isomorphism.

Example 2.1.4 (Clifford superalgebra Cl_n). Define the *Clifford superalgebra* Cl_n to be the superalgebra given by odd generators $c_1, \dots, c_n$, subject to the relations

$$c_i^2 = 1 \quad (1 \leq i \leq n), \quad c_i c_j = -c_j c_i \quad (1 \leq i \neq j \leq n).$$

Denote Cl_1 only by Cl . One can prove that $Cl_n \cong Cl^{\otimes n}$, by identifying $c_i = 1 \otimes \dots \otimes c_1 \otimes \dots \otimes 1 \in Cl^{\otimes n}$, where c_1 is placed in the i^{th} -coordinate. Furthermore, a straightforward calculation shows that $Cl \cong \mathcal{Q}(1)$ through the map

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad c \mapsto \sqrt{-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (2.1.4)$$

It follows, from (2.1.3), that $Cl \otimes Cl \cong \mathcal{M}(1|1)$.

Definition 2.1.5. (Graded supermodules) Let A be a graded superalgebra. A left (respectively, right) graded supermodule over A is a left (resp., right) A -module M with $\mathbb{Z} \times \mathbb{Z}_2$ -grading satisfying $A_{i,\tau} M_{j,\sigma} \subseteq M_{i+j,\tau+\sigma}$ (resp., $M_{j,\sigma} A_{i,\tau} \subseteq M_{i+j,\tau+\sigma}$), for all $(i, \tau), (j, \sigma) \in \mathbb{Z} \times \mathbb{Z}_2$.

A *graded subsupermodule* is a submodule that is also a $\mathbb{Z} \times \mathbb{Z}_2$ -graded subspace. A graded supermodule is called *simple* (or *irreducible*) if it contains no non-trivial graded subsupermodule. An graded A -supermodule M is called *completely reducible* if any graded subsupermodule of M admits a complementary subsupermodule.

Let M and N be left graded supermodules over a graded superalgebra A . For $i \in \mathbb{Z}$ and $\varepsilon \in \mathbb{Z}_2$, we consider

$$\text{HOM}_A(M, N)_{i,\varepsilon} = \{\alpha \in \text{HOM}_{\mathbb{k}}(M, N)_{i,\varepsilon} \mid \alpha(am) = (-1)^{\varepsilon \bar{a}} a\alpha(m) \forall m \in M, a \in A\}.$$

We then define

$$\text{HOM}_A(M, N) = \bigoplus_{i \in \mathbb{Z}, \varepsilon \in \mathbb{Z}_2} \text{HOM}_A(M, N)_{i,\varepsilon}, \quad \text{Hom}_A(M, N) = \text{HOM}_A(M, N)_{0,0}. \quad (2.1.5)$$

We will use the term *a homomorphism* from M to N to mean elements of $\text{HOM}_A(M, N)$, while the term *an even homomorphism* refer to elements of $\text{Hom}_A(M, N)$. The notations END and End of *endomorphism* and *even endomorphism* are defined similarly. We use the symbol $\cong$ to denote an isomorphism, $\simeq$ to denote an *even* isomorphism and $\approx$ to denote an *odd* isomorphism. We let $A\text{-mod}$ denote the category of finitely-generated left graded supermodules over A with even homomorphisms.

We have a parity shift functor

$$\Pi: A\text{-mod} \rightarrow A\text{-mod}, \quad M \mapsto \Pi M \quad (2.1.6)$$

that switches the $\mathbb{Z}_2$ -grading, that is, $(\Pi M)_{i,\varepsilon} = M_{i,\varepsilon+1}$. The action of A on ΠM is given by $a \cdot m = (-1)^{\bar{a}} am$, where am is the action on M . The parity shift functor is defined in the morphism by fixing the linear map between the underlying modules, i.e.,

$$(f: M \rightarrow N) \mapsto (\Pi f: \Pi M \rightarrow \Pi N, \quad m \mapsto f(m)).$$

A homogeneous homomorphism $f \in \text{HOM}_A(M, N)$ is odd if, and only if, $f: M \rightarrow \Pi N$ is an

even homomorphism of graded A -supermodules. For all $r \in \mathbb{Z}$ we also have the degree shift functor

$$\{r\}: A\text{-mod} \rightarrow A\text{-mod}, \quad M \mapsto \{r\}M, \quad (2.1.7)$$

such that $\{r\}M_{i,\varepsilon} = M_{i-r,\varepsilon}$ with same action of A . Finally, given $(r, \pi) \in \mathbb{Z} \times \mathbb{Z}_2$, we define the functor

$$\{r, \pi\} = \{r\} \circ \Pi^\pi. \quad (2.1.8)$$

By abuse of notation we will denote also by $\{r, \pi\}$ the morphism $M \rightarrow \{r, \pi\}M$ that is the identity map on the sets.

We let $\mathcal{S}(A)$ denote the set of isomorphism classes of simple objects in $A\text{-mod}$ up to degree shift. By a slight abuse of notation, we will also use the notation $\mathcal{S}(A)$ to denote a set of representatives of these isomorphism classes, where we shift the $\mathbb{Z}$ -grading so that each representative has nonzero degree zero piece and is concentrated in nonnegative degree.

Given a graded A -supermodule V and an automorphism α of A , we define the twisted graded supermodule ${}^\alpha V$ to be the graded A -supermodule with underlying superspace V and with action

$$a \cdot v = \alpha^{-1}(a)v, \quad a \in A, v \in V. \quad (2.1.9)$$

We take the inverse automorphism to have ${}^{\alpha_1}({}^{\alpha_2}V) \simeq {}^{\alpha_1\alpha_2}V$ for $\alpha_1, \alpha_2 \in \text{Aut } A$. Thus we have an action of $\text{Aut } A$ on the class of A -modules. Here and in what follows we use “.” to denote the twisted action and juxtaposition to denote the original (untwisted) action. Since ${}^\alpha V$ is simple if, and only if, V is, the twisting induces a permutation of $\mathcal{S}(A)$.

Let A be a graded superalgebra. By forgetting the $\mathbb{Z} \times \mathbb{Z}_2$ -grading we may consider A as usual algebra. This algebra will be denoted $|A|$. Analogously, by forgetting the $\mathbb{Z} \times \mathbb{Z}_2$ -grading of a graded A -supermodule M we may consider M as an usual $|A|$ -module, which will be denoted by $|M|$.

A simple graded A -supermodule M is said to be of type $\mathbf{M}$ if $|M|$ is irreducible as $|A|$ -module. Otherwise it is said to be of type $\mathbf{Q}$. In fact it is possible to show that:

Lemma 2.1.6. *Let M be a finite dimensional simple graded A -supermodule of type $\mathbf{Q}$. Then there exists a decomposition of $|A|$ -modules $|M| = X_1 \oplus X_2$, where X_1 and X_2 are non-isomorphic simple $|A|$ -modules with same dimension. Furthermore, there exists an odd endomorphism $\widetilde{J}_M \in \text{END}_A(M)$ such that $\widetilde{J}_M^2 = -\text{Id}$.*

Therefore, from the previous lemma we have that a simple A -supermodule is of type $\mathbf{Q}$ if it is evenly isomorphic to its own parity shift. Now we can state the following:

Theorem 2.1.7 (Schur’s Lemma for superalgebras). *Let M and N be finite dimensional simple graded A -supermodules with $M \not\cong N$. Then, $\text{HOM}_A(M, N) = 0$. On the other hand, if $\mathbb{k}$ is an algebraically closed field, then*

$$\text{END}_A(M) \simeq \begin{cases} \mathbb{k} & \text{if } M \text{ is of type } \mathbf{M}, \\ \mathcal{Cl} & \text{if } M \text{ is of type } \mathbf{Q}. \end{cases} \quad (2.1.10)$$

The isomorphism, in the type $\mathbf{Q}$ case, is given by mapping $c \in \mathcal{Cl}$ into $J_M = \sqrt{-1}\widetilde{J}_M$, where $\widetilde{J}_M$ is the odd involution given by Lemma 2.1.6.

When M is of type $\mathbf{Q}$, the isomorphism (2.1.10) is unique up to composition with the endomorphism of $\mathcal{Cl}$ determined by $c \mapsto -c$. In this case, we fix an isomorphism (2.1.10) and use it to define an action of $\mathcal{Cl}$ on M .

A graded superalgebra A is called *semisimple* if its regular left graded A -supermodule is completely reducible. We have the following lemmas:

Lemma 2.1.8. *Let M be a graded A -supermodule. Then M is completely reducible if, and only if, $|M|$ is completely reducible as $|A|$ -module. In particular, a graded superalgebra A is semisimple if, and only if, $|A|$ is a semisimple algebra.*

Lemma 2.1.9. *Let A be a finite dimension graded superalgebra. Then the Jacobson radical $J(|A|)$ can be characterized as the unique smallest superideal $J(A)$ of A such that $A/J(A)$ is a semisimple superalgebra.*

From the previous lemmas, it follows:

Theorem 2.1.10 (Wedderburn's Theorem). *Any finite dimensional simple superalgebra is isomorphic to some $\mathcal{M}(m|n)$ or $\mathcal{Q}(n)$. Moreover, the following conditions on a finite dimensional superalgebra are equivalent:*

1. A is semisimple;
2. every graded A -supermodule is completely reducible;
3. $J(A) = 0$.

Unadorned tensor products will be understood to be over $\mathbb{k}$. The outer tensor product of an A_1 -module M and an A_2 -module N is denoted $M \boxtimes N$. As $\mathbb{k}$ -modules, we have $M \boxtimes N = M \otimes N$. The action is given by

$$(a_1 \otimes a_2)(m \otimes n) = (-1)^{\bar{a}_2 \bar{m}} a_1 m \otimes a_2 n, \quad a_1 \in A_1, \ a_2 \in A_2, \ m \in M, \ n \in N.$$

On morphisms, the outer tensor product of a homomorphisms of A_1 -modules $f: M \rightarrow M'$ and homomorphism of A_2 -modules $g: N \rightarrow N'$ is:

$$f \otimes g: M \boxtimes N \rightarrow M' \boxtimes N', \quad (f \otimes g)(v \otimes w) = (-1)^{\bar{g} \bar{v}} f(v) \otimes g(w),$$

for all $v \in M$ and $w \in N$. In particular, we have:

Lemma 2.1.11. *Let M and N be two finite dimensional graded supermodules over A_1 and A_2 , respectively. Then we have that*

$$- \otimes -: \text{END}_{A_1}(M) \otimes \text{END}_{A_2}(N) \rightarrow \text{END}_{A_1 \otimes A_2}(M \boxtimes N). \quad (2.1.11)$$

is a isomorphism of $\mathbb{k}$ -superalgebras.

Proof. By dimension comparison, it is easy to see that

$$- \otimes -: \text{END}_{\mathbb{k}}(M) \otimes \text{END}_{\mathbb{k}}(N) \rightarrow \text{END}_{\mathbb{k}}(M \otimes N)$$

is an isomorphism of graded superalgebras, with inverse φ . It is sufficient to prove that

$$\varphi(\text{END}_{A_1 \otimes A_2}(M \otimes N)) \subseteq \text{END}_{A_1}(M) \otimes \text{END}_{A_2}(N).$$

Let $f \otimes g \in \text{END}_{\mathbb{k}}(M) \otimes \text{END}_{\mathbb{k}}(N)$ be a non-zero homogeneous element such that $f \otimes g$ is an endomorphism of graded $A_1 \otimes A_2$ -supermodules. Then, a straightforward calculation shows that

$$\begin{aligned} (-1)^{\bar{g}\bar{m}+\bar{b}(\bar{f}+\bar{m})}af(m) \otimes bg(n) &= (a \otimes b)(f \otimes g)(m \otimes n) \\ &= (-1)^{(\bar{a}+\bar{b})(\bar{f}+\bar{g})+\bar{b}\bar{m}+\bar{g}(\bar{a}+\bar{m})}f(am) + g(bn), \end{aligned}$$

for all homogeneous $a \in A_1$, $b \in A_2$, $m \in M$, $n \in N$. Therefore

$$af(m) \otimes bg(n) = ((-1)^{\bar{a}\bar{f}}f(am)) \otimes ((-1)^{\bar{b}\bar{g}}g(bn)).$$

By setting $b = 1$ and taking n such that $g(n) \neq 0$, we have that $(af(m) - (-1)^{\bar{a}\bar{f}}f(am)) \otimes g(n) = 0$, which follows that $f \in \text{END}_{A_1}(M)$. Analogously, by setting $a = 1$ and taking m such that $f(m) \neq 0$, we conclude that $g \in \text{END}_{A_2}(N)$. $\square$

For the remainder of this subsection, we assume $\mathbb{k}$ is an algebraically closed field.

Theorem 2.1.12. *Let M and N be finite dimensional simple graded supermodules over finite dimensional graded superalgebras A_1 and A_2 , respectively.*

1. *If M or N is of type $\mathbb{M}$, then $M \boxtimes N$ is a simple graded $A_1 \otimes A_2$ -supermodule.*
2. *If M and N is of type $\mathbb{Q}$, then there exists a simple graded $A_1 \otimes A_2$ -supermodule X , such that $M \boxtimes N \simeq X \oplus \Pi X$.*

Furthermore, all simple graded $A_1 \otimes A_2$ -supermodule arise as constituents of $M \boxtimes N$ for some choice of irreducibles M and N .

Proof. By 2.1.10 and Nakayama's lemma for modules, we can reduce the theorem to the cases where A_1 and A_2 are semisimple.

Now let $M \boxtimes N = X \oplus Y$ be a decomposition and let $\phi \in \text{END}_{A_1 \otimes A_2}(M \boxtimes N)$ be the homogeneous projection in X . Then ϕ is an idempotent endomorphism, i.e., $\phi^2 = \phi$.

By 2.1.7 and 2.1.11,

$$\begin{aligned} \text{END}_{A_1 \otimes A_2}(M \boxtimes N) &\simeq \text{END}_{A_1}(M) \otimes \text{END}_{A_2}(N) \\ &\simeq \begin{cases} \mathbb{k}, & M, N \text{ of type } \mathbb{M}, \\ \mathcal{C}l, & \text{either } M \text{ or } N \text{ is of type } \mathbb{Q}, \\ \mathcal{C}l \otimes \mathcal{C}l, & M, N \text{ of type } \mathbb{Q}. \end{cases} \end{aligned}$$

In the first two cases, it follows that ϕ is either zero or a bijection, which implies that either X or Y is zero. Hence $M \boxtimes N$ is simple. In the last case, the only non-zero options of homogeneous idempotent endomorphism for ϕ are the ones corresponding to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

through the isomorphism $\mathcal{C}l \otimes \mathcal{C}l \simeq \mathcal{M}(1|1)$, combining the isomorphisms from (2.1.3), (2.1.4) and 2.1.7. Hence ϕ is one of the following corresponding options:

$$\text{Id}, \quad \frac{1}{2}(\text{Id} - \sqrt{-1}(J_M \otimes \text{Id})(\text{Id} \otimes J_N)) \quad \text{and} \quad \frac{1}{2}(\text{Id} + \sqrt{-1}(J_M \otimes \text{Id})(\text{Id} \otimes J_N)),$$

where J_M and J_N are the odd involution as in 2.1.7. Furthermore, by assuming $\phi \neq \text{Id}$, it follows that $J_M \phi J_M = \text{Id} - \phi$. Hence we have that J_M induces an odd isomorphism $Y = \text{Im}(\text{Id} - \phi) \simeq \text{Im} \phi = X$. Finally, the simplicity of X follows from a similar argument using the endomorphism algebra. $\square$

Given M and N satisfying the Theorem above, and assuming them to be of type $\mathbb{Q}$, we will fix a choice for X to be

$$X = \phi(M \boxtimes N), \quad \phi = \frac{1}{2}(\text{Id} + \sqrt{-1}c_1c_2),$$

where $c_1 = c \otimes 1$, $c_2 = 1 \otimes c$ and $c \in \mathcal{Cl}$ acts in M and N as J_M and J_N , respectively. We will call such $A_1 \otimes A_2$ -subsupermodule by *irreducible tensor* of M and N , and denote it by $M \otimes N$. Hence:

$$M \boxtimes N \simeq (M \otimes N) \oplus \Pi(M \otimes N). \quad (2.1.12)$$

We will also use the notation $M \otimes N$ to denote $M \boxtimes N$ in the case when M or N is of type $\mathbb{M}$.

Now let M_1, M_2 be simple graded A_1 -supermodules, and N_1, N_2 be simple graded A_2 -supermodules. Then for any even linear map $f: M_1 \boxtimes N_1 \rightarrow M_2 \boxtimes N_2$, we have an induced even linear map $\tilde{f}: M_1 \otimes N_1 \rightarrow M_2 \otimes N_2$, by restriction of domain and projection on the simple summand. Clearly, if f is a homomorphism of $A_1 \otimes A_2$ -modules so is $\tilde{f}$. Suppose $M_1 = M_2 = M$ and $N_1 = N_2 = N$ are both of type $\mathbb{Q}$. Then, for any linear map f as above, it follows that

$$f = f \circ \phi + f \circ (\text{Id} - \phi) = f \circ c_1 + f \circ c_1 \circ \phi \circ c_1.$$

In particular, if f is given by an action of a homogeneous element of $A_1 \otimes A_2$, we must have that $f\phi(x) = \tilde{f}\phi(x)$ for all $x \in M \boxtimes N$ and that $f \circ c_1 = (-1)^{\tilde{f}} c_1 \circ f$, since $c_1 \in \text{END}_{A_1 \otimes A_2}(M \boxtimes N)$. Therefore it follows that

$$f = \tilde{f} \circ \phi + (-1)^{\tilde{f}} c_1 \circ \tilde{f} \circ \phi \circ c_1, \quad (2.1.13)$$

and the relation (2.1.13) defines a bijection between the action of a homogeneous element of $A_1 \otimes A_2$ on $M \boxtimes N$ and action on $M \otimes N$.

Now we introduce the concept of graded superbimodule.

Let M and N be two graded *right* supermodule over a graded superalgebra A . For $i \in \mathbb{Z}$ and $\varepsilon \in \mathbb{Z}_2$, we consider

$$\text{HOM}_A^r(M, N)_{i,\varepsilon} = \{\alpha \in \text{HOM}_{\mathbb{k}}(M, N)_{i,\varepsilon} \mid \alpha(ma) = \alpha(m)a, \forall m \in M, a \in A\},$$

and then define the set of homomorphism of graded *right* A -supermodules to be:

$$\text{HOM}_A^r(M, N) = \bigoplus_{i \in \mathbb{Z}, \varepsilon \in \mathbb{Z}_2} \text{HOM}_A^r(M, N)_{i,\varepsilon}, \quad \text{Hom}_A^r(M, N) = \text{HOM}_A^r(M, N)_{0,0}.$$

Definition 2.1.13. Let A_1 and A_2 be two graded superalgebras, then a graded (A_1, A_2) -superbimodule is a graded superspace M that is simultaneously left graded A_1 -supermodule and right graded A_2 -supermodule such that

$$a_1(ma_2) = (a_1m)a_2, \text{ for all } a_1 \in A_1, a_2 \in A_2.$$

A homogeneous homomorphism $f: M_1 \rightarrow M_2$ between two graded (A_1, A_2) -superbimodules is an linear function that is simultaneously homomorphism of left grade A_1 -supermodules and right graded A_2 -supermodules, i.e., that satisfies

$$f(a_1 m) = (-1)^{\bar{f}\bar{a}_1} a_1 f(m) \quad \text{and} \quad f(m a_2) = f(m) a_2,$$

for all $a_1 \in A_1$, $a_2 \in A_2$ and $m \in M_1$.

Lemma 2.1.14. *Let M be a graded (A_1, A_2) -superbimodule and N be a graded (A_1, A_1) -superbimodule, then $\text{HOM}_{A_1}(M, N)$ has a structure of graded (A_2, A_1) -superbimodule given by*

$$(a_2 \cdot f)(m) = (-1)^{\bar{a}_2(\bar{f} + \bar{m})} f(m a_2) \quad \text{and} \quad (f \cdot a_1)(m) = (-1)^{\bar{a}_1 \bar{m}} f(m) a_1,$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $m \in M$ and $f \in \text{HOM}_{A_1}(M, N)$.

Proof. First notice that, for any homogeneous element $x \in A_1$ it follows that

$$\begin{aligned} x(a_2 \cdot f)(m) &= (-1)^{\bar{a}_2(\bar{f} + \bar{m})} x f(m a_2) = (-1)^{\bar{a}_2(\bar{f} + \bar{m}) + \bar{x}\bar{f}} f(x(m a_2)) \\ &= (-1)^{\bar{a}_2(\bar{f} + \bar{m} + \bar{x}) + \bar{x}(\bar{f} + \bar{a}_2)} f((x m) a_2) = (-1)^{\bar{x}(\bar{f} + \bar{a}_2)} (a_2 \cdot f)(x m), \end{aligned}$$

for all $a_2 \in A_2$, $m \in M$ and $f \in \text{HOM}_{A_1}(M, N)$. Analogously, we have that

$$\begin{aligned} x(f \cdot a_1)(m) &= (-1)^{\bar{a}_1 \bar{m}} x(f(m) a_1) = (-1)^{\bar{a}_1 \bar{m} + \bar{x}\bar{f}} f(x m) a_1 \\ &= (-1)^{\bar{a}_1(\bar{m} + \bar{x}) + \bar{x}(\bar{f} + \bar{a}_1)} f(x m) a_1 = (-1)^{\bar{x}(\bar{f} + \bar{a}_1)} (f \cdot a_1)(x m), \end{aligned}$$

for all $x, a_1 \in A_1$, $m \in M$ and $f \in \text{HOM}_{A_1}(M, N)$. Thus the actions are well defined. Now, the following proves that it defines a graded (A_2, A_1) -superbimodule structure on $\text{HOM}_{A_1}(M, N)$:

$$\begin{aligned} (a_2 \cdot (f \cdot a_1))(m) &= (-1)^{(\bar{f} + \bar{a}_1 + \bar{m})\bar{a}_2} (f \cdot a_1)(m a_2) \\ &= (-1)^{(\bar{f} + \bar{a}_1 + \bar{m})\bar{a}_2 + \bar{a}_1(\bar{m} + \bar{a}_2)} f(m a_2) a_1 \\ &= (-1)^{\bar{a}_1 \bar{a}_2 + \bar{a}_1(\bar{m} + \bar{a}_2)} (a_2 \cdot f)(m) a_1 \\ &= (-1)^{\bar{a}_1 \bar{m}} (a_2 \cdot f)(m) a_1 = ((a_2 \cdot f) \cdot a_1)(m), \end{aligned}$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $m \in M$ and $f \in \text{HOM}_{A_1}(M, N)$. □

Note: For the remainder of this text, we will use the terms *algebra* and *module* to mean $\mathbb{Z} \times \mathbb{Z}_2$ -graded associative $\mathbb{k}$ -algebra and $\mathbb{Z} \times \mathbb{Z}_2$ -graded module, respectively. Furthermore, all modules are left modules unless otherwise specified.

2.2 Frobenius extensions and algebras

In this section we recall Frobenius extensions and the adjunction of induction and restriction functors. Then we define Frobenius algebra and study the case where two Frobenius algebras are also Frobenius extension of each other. The approach is based on [PS16], but here we do not consider twisted actions in the modules involved.

Throughout this section, let A be an algebra and $B \subseteq A$ be a fixed subalgebra. Fix $r \in \mathbb{Z}$ and $\pi \in \mathbb{Z}_2$.

2.2.1 Frobenius extensions

Denote by ${}_A A_B$ and ${}_B A_A$, the regular (A, B) -bimodule and regular (B, A) -bimodule, respectively.

Definition 2.2.1. We call A a *Frobenius extension* of B of degree $-r$ and parity π , if it satisfies

E1 A is finitely generated and projective left B -module;

E2 ${}_A A_B \simeq \text{HOM}_B({}_B A, \{r, \pi\}B)$ as (A, B) -bimodule.

Proposition 2.2.2. Suppose that condition **E1** is satisfied. Then A is a Frobenius extension of B of parity π and degree $-r$ if, and only if, there exists a homomorphism of (B, B) -bimodules $\text{tr}: A \rightarrow B$, of parity π and degree $-r$, satisfying

T1 if $\text{tr}(Aa) = 0$ for some $a \in A$, then $a = 0$.

T2 for all homogeneous $\phi \in \text{HOM}_B({}_B A, B)$ there exists a homogeneous element $a \in A$ such that

$$\phi(x) = (-1)^{\bar{x}\bar{a}} \text{tr}(xa),$$

for all $x \in A$.

Proof. Suppose **E2** is satisfied and let Φ be an isomorphism ${}_A A_B \simeq \text{HOM}_B({}_B A, \{r, \pi\}B)$. Define $\text{tr} = \Phi(1)$. Then tr is a homomorphism of left B -module of parity π and degree $-r$. For all $x \in A$ and $b \in B$ we have:

$$\begin{aligned} \text{tr}(xb) &= \Phi(1)(xb) = (-1)^{\bar{b}\bar{x}}(b \cdot \Phi(1))(x) \\ &= (-1)^{\bar{b}\bar{x}}(\Phi(b))(x) = (-1)^{\bar{b}\bar{x}}(\Phi(1) \cdot b)(x) \\ &= \Phi(1)(x)b = \text{tr}(x)b, \end{aligned}$$

where $\cdot$ denotes the (right and left) actions of B on $\text{HOM}_B({}_B A, \{r, \pi\}B)$. Then tr is a homomorphism of (B, B) -bimodule.

Now, suppose that $\text{tr}(Aa) = 0$ for some $a \in A$. Then, it follows

$$0 = \text{tr}(xa) = \Phi(1)(xa) = (-1)^{\bar{a}\bar{x}}\Phi(a)(x),$$

for all homogeneous $x \in A$. Hence $\Phi(a)$ is the null map. Then, since Φ is an isomorphism, it follows that $a = 0$, and tr satisfies **T1**.

Finally, given a homogeneous $\phi \in \text{HOM}_B({}_B A, B)$, let $a \in A$ such that $\Phi(a) = \phi$. Then, for all $x \in A$ it follows:

$$\phi(x) = \Phi(a)(x) = (a \cdot \Phi(1))(x) = (-1)^{\bar{x}\bar{a}}\Phi(1)(xa) = (-1)^{\bar{x}\bar{a}} \text{tr}(xa).$$

Hence tr satisfies **T2**.

Conversely, if we have tr that satisfies the conditions from the statement, we define the linear map $\Phi: A \rightarrow \text{HOM}_B({}_B A, \{-r, \pi\}B)$ by

$$\Phi(a)(x) = (-1)^{\bar{x}\bar{a}} \text{tr}(xa),$$

for all $a, x \in A$. Then Φ is an isomorphism of (A, B) -bimodule. In fact, from the fact that tr is homomorphism of (B, B) -bimodules, it follows that Φ is a homomorphism of (A, B) -bimodules. The injectivity and surjectivity of Φ follow from the property **T1** and **T2**, respectively. $\square$

Definition 2.2.3. Then, if $\text{tr}: A \rightarrow B$ is a homomorphism of (B, B) -bimodules satisfying conditions **T1** and **T2** from Proposition 2.2.2, we call tr a *trace map* for the Frobenius extension A of B .

Remark 2.2.4. Suppose tr and tr' are two trace map for A . Then there exists an invertible homogeneous element $u \in A$ such that $\text{tr}'(x) = \text{tr}(xu)$, $\forall x \in A$.

Indeed, by property **T2**, there exists $u, u' \in A$ such that

$$\text{tr}'(x) = \text{tr}(xu) \quad \text{and} \quad \text{tr}(x) = \text{tr}'(xu'), \quad \forall x \in A.$$

It follows that $\text{tr}(x) = \text{tr}'(xu') = \text{tr}(xu'u)$, for all $x \in A$. Then, $\text{tr}(A(1 - u'u)) = 0$. Hence, by property **T1**, it follows that $1 = u'u$. Analogously, $\text{tr}'(A(1 - uu')) = 0$, which implies $1 = uu'$. Hence u is an invertible element.

Next we prove a necessary and sufficient condition for a trace map:

Lemma 2.2.5. Fix a (B, B) -bimodule homomorphism tr of parity π and degree $-\lambda$. Then, A is a Frobenius extension of B with trace map tr if, and only if, there exists a pair of finite sets $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ of homogeneous elements in A such that $(|y_i|, \bar{y}_i) = (\lambda - |x_i|, \pi + \bar{x}_i)$, for all $i \in I$, satisfying the following:

$$a = (-1)^{\pi \bar{a}} \sum_{i \in I} (-1)^{\pi \bar{x}_i} \text{tr}(ay_i)x_i = \sum_{i \in I} y_i \text{tr}(x_i a), \quad \forall a \in A. \quad (2.2.1)$$

Proof. Suppose that A is a Frobenius extension of B with trace map tr . Let $\{x_i\}_{i \in I}$ be a finite set of homogeneous elements that generates A as B -module and consider the surjective homomorphism of B -modules

$$g: \bigoplus_{i \in I} \{|x_i|, \bar{x}_i\}B \rightarrow A, \quad (\{|x_i|, \bar{x}_i\}\beta_i)_{i \in I} \mapsto \sum_{i \in I} (-1)^{\bar{\beta}_i \bar{x}_i} \beta_i x_i.$$

The fact that g is a homomorphism follows from:

$$\begin{aligned} g(\gamma(\{|x_i|, \bar{x}_i\}\beta_i)_{i \in I}) &= g(((-1)^{\bar{x}_i \bar{\gamma}} \{|x_i|, \bar{x}_i\}\gamma\beta_i)_{i \in I}) \\ &= \sum_{i \in I} (-1)^{\bar{x}_i \bar{\gamma} + \bar{\gamma} \bar{\beta}_i \bar{x}_i} \gamma \beta_i x_i \\ &= \sum_{i \in I} (-1)^{\bar{x}_i \bar{\gamma} + (\bar{\gamma} + \bar{\beta}_i) \bar{x}_i} \gamma \beta_i x_i \\ &= \gamma \sum_{i \in I} (-1)^{\bar{\beta}_i \bar{x}_i} \beta_i x_i = \gamma g((\{|x_i|, \bar{x}_i\}\beta_i)_{i \in I}), \end{aligned}$$

for all homogeneous $\gamma \in B$, $\beta_i \in B$, $i \in I$.

Then, since A is projective as B -module, there exists an even homomorphism of B -module $\tilde{\phi}: A \rightarrow \bigoplus_{i \in I} \{|x_i|, \bar{x}_i\}B$, such that $g \circ \tilde{\phi} = \text{Id}_A$. For all $j \in I$, consider the projection

$$\pi_j: \bigoplus_{i \in I} \{|x_i|, \bar{x}_i\}B \rightarrow B, \quad (\{|x_i|, \bar{x}_i\}\beta_i)_{i \in I} \mapsto \beta_j.$$

Notice that $\{\pi_i\}_{i \in I}$ are homogeneous homomorphisms of B -modules of parity $\bar{x}_i$ and degree $-|x_i|$ such that $\tilde{\phi} = \bigoplus_{i \in I} (\pi_i \circ \tilde{\phi})$. Then, defining $\phi_i = \pi_i \circ \tilde{\phi} \in \text{HOM}_B(A, B)$, for all $i \in I$, we have

$$a = \sum_{i \in I} (-1)^{\bar{a}\bar{x}_i} \phi_i(a) x_i, \quad (2.2.2)$$

for all $a \in A$. Furthermore $(|\phi_i|, \bar{\phi}_i) = (-|x_i|, \bar{x}_i)$, for all $i \in I$.

Now, by property **T2**, there exists a set $\{y_i\}_{i \in I}$ of homogeneous elements of A satisfying

$$(-1)^{\pi\bar{x}_i} \phi_i(a) = (-1)^{\bar{a}\bar{y}_i} \text{tr}(ay_i), \quad (2.2.3)$$

for all $a \in A$ and $i \in I$. Furthermore, since tr has parity π and degree $-\lambda$, it follows that $(|y_i|, \bar{y}_i) = (\lambda - |x_i|, \pi + \bar{x}_i)$. Hence, by (2.2.2) and (2.2.3) we obtain that

$$a = \sum_{i \in I} (-1)^{\bar{a}(\bar{x}_i + \bar{y}_i) + \pi\bar{x}_i} \text{tr}(ay_i) x_i = (-1)^{\pi\bar{a}} \sum_{i \in I} (-1)^{\pi\bar{x}_i} \text{tr}(ay_i) x_i,$$

for all $a \in A$, satisfying the first equation equality of (2.2.1).

Finally, for all homogeneous $a \in A$, it follows that

$$\begin{aligned} \text{tr}(za) &= \text{tr} \left((-1)^{\pi\bar{z}} \sum_{i \in I} (-1)^{\pi\bar{x}_i} \text{tr}(zy_i) x_i a \right) = \sum_{i \in I} (-1)^{\pi\bar{z} + \pi\bar{x}_i} \text{tr}(\text{tr}(zy_i) x_i a) \\ &= \sum_{i \in I} (-1)^{\pi\bar{z} + \pi\bar{x}_i + \pi(\bar{z} + \bar{y}_i + \pi)} \text{tr}(zy_i) \text{tr}(x_i a) = \sum_{i \in I} (-1)^{\pi(\bar{x}_i + \bar{y}_i + \pi)} \text{tr}(zy_i \text{tr}(x_i a)) \\ &= \sum_{i \in I} \text{tr}(zy_i \text{tr}(x_i a)), \end{aligned}$$

for all $z \in A$. Then,

$$\text{tr} \left(z \left(a - \sum_{i \in I} y_i \text{tr}(x_i a) \right) \right) = 0, \quad \forall z \in A,$$

and by property **T1** of tr , we obtain the second equality of (2.2.1).

Reciprocally, suppose $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ be a pair of sets satisfying conditions of the statement. Defining

$$\tilde{\phi}: A \rightarrow \bigoplus_{i \in I} \{ |x_i|, \bar{x}_i \} B, \quad a \mapsto \left(\{ |x_i|, \bar{x}_i \} (-1)^{\pi\bar{x}_i + \bar{a}\bar{y}_i} \text{tr}(ay_i) \right),$$

we have an injection that makes A a direct summand of a free left B -module. Hence, A is a projective left B -module.

Suppose $a \in A$ such that $\text{tr}(za) = 0$, for all $z \in A$. Then, by the second equality of (2.2.1), it follows that $a = 0$.

Finally, given a homogeneous (B, B) -module homomorphism $f: A \rightarrow B$, define

$$a = \sum_{i \in I} (-1)^{(\bar{f} + \pi)\bar{x}_i} y_i f(x_i).$$

Then, for all $z \in A$, it follows:

$$\begin{aligned}
(-1)^{\bar{z}\bar{a}} \operatorname{tr}(za) &= (-1)^{\bar{z}(\pi+\bar{f})} \operatorname{tr} \left(z \sum_{i \in I} (-1)^{(\bar{f}+\pi)\bar{x}_i} y_i f(x_i) \right) \\
&= \sum_{i \in I} (-1)^{\bar{z}(\pi+\bar{f})+(\bar{f}+\pi)\bar{x}_i} \operatorname{tr}(zy_i) f(x_i) \\
&= \sum_{i \in I} (-1)^{\bar{z}(\pi+\bar{f})+\bar{f}(\pi+\bar{z}+\bar{y}_i)+(\bar{f}+\pi)\bar{x}_i} f(\operatorname{tr}(zy_i)x_i) \\
&= f \left((-1)^{\pi\bar{z}} \sum_{i \in I} (-1)^{\pi\bar{x}_i} \operatorname{tr}(zy_i)x_i \right) = f(z).
\end{aligned}$$

Therefore, tr is a trace map for A . $\square$

For the remainder of this section, assume A is a Frobenius extension of B with fixed trace map tr .

Let the subalgebra

$$C_A(B) = \{a \in A \mid ab = (-1)^{\bar{a}\bar{b}} ba, \forall b \in B\}$$

be the centralizer of B in A . For all homogeneous $c \in C_A(B)$, define the linear map

$$\phi_c: A \rightarrow B, \quad x \mapsto \operatorname{tr}(cx).$$

Notice that ϕ_c is a homomorphism of (B, B) -bimodules of parity $\pi + \bar{c}$ and degree $-r + |c|$. Indeed, ϕ_c is clearly a homomorphism of right B -modules. Also, for all $x \in A$, $b \in B$, it follows that

$$\phi_c(bx) = \operatorname{tr}(cbx) = (-1)^{\bar{b}\bar{c}} \operatorname{tr}(bcx) = (-1)^{\bar{b}(\bar{c}+\pi)} b \operatorname{tr}(cx) = (-1)^{\bar{b}(\bar{c}+\pi)} b \phi_c(x).$$

Thus, ϕ_c is also a homomorphism of left B -modules.

Then, by properties **T1** and **T2**, for each $c \in C_A(B)$, there exists unique $\psi(c) \in A$ such that $\phi_c(x) = (-1)^{\bar{x}\bar{\psi}(c)} \operatorname{tr}(x\psi(c))$, for all homogeneous $x \in A$, which defines an even linear map $\psi: C_A(B) \rightarrow A$. Now to show that $\psi(C_A(B)) = C_A(B)$ and that $\psi: C_A(B) \rightarrow C_A(B)$ is an isomorphism of algebras:

Let $c \in C_A(B)$ and $b \in B$. Then, it follows that

$$\begin{aligned}
\operatorname{tr}(xb\psi(c)) &= (-1)^{(\bar{x}+\bar{b})\bar{c}} \operatorname{tr}(cxb) = (-1)^{(\bar{x}+\bar{b})\bar{c}} \operatorname{tr}(cx)b \\
&= (-1)^{\bar{b}\bar{c}} \operatorname{tr}(x\psi(c))b = \operatorname{tr}(x(-1)^{\bar{b}\bar{c}}\psi(c)b),
\end{aligned}$$

for all $x \in A$. Then, by property **T1**, it follows that $b\psi(c) = (-1)^{\bar{b}\bar{c}}\psi(c)b$, and hence $\operatorname{Im} \psi \subseteq C_A(B)$. To prove the other inclusion, we need to state an equivalent definition of Frobenius extension (where A is a project *right* module over B), which escape the main topic of this work. Therefore we refer to [PS16, 5.1] for the prove that $C_A(B) \subseteq \psi(C_A(B))$. Finally, for all $c_1, c_2 \in C_A(B)$, it follows that

$$\operatorname{tr}(x\psi(c_1c_2)) = (-1)^{\bar{x}(\bar{c}_1+\bar{c}_2)} \operatorname{tr}(c_1c_2x) = (-1)^{(\bar{c}_1+\bar{x})\bar{c}_2} \operatorname{tr}(c_2x\psi(c_1)) = \operatorname{tr}(x\psi(c_1)\psi(c_2)),$$

for all $x \in A$. Thus $\psi(c_1c_2) = \psi(c_1)\psi(c_2)$, for all $c_1, c_2 \in C_A(B)$, which implies that ψ a

homomorphism of algebras.

Definition 2.2.6. We call the map $\psi: C_A(B) \rightarrow C_A(B)$ characterized by

$$\mathrm{tr}(ca) = (-1)^{\bar{c}\bar{a}} \mathrm{tr}(a\psi(c)), \quad \forall a \in A, c \in C_A(B).$$

the *Nakayama automorphism* associated to tr .

2.2.2 Adjointness properties

We now study the relation between Frobenius extensions and induction/restriction functors. First we recall the definition of adjoint functors:

Definition 2.2.7 (Adjoint functors). Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We say that a functor $F: \mathcal{D} \rightarrow \mathcal{C}$ is *left adjoint* to a functor $G: \mathcal{C} \rightarrow \mathcal{D}$ if, for all objects X in $\mathcal{C}$ and Y in $\mathcal{D}$, there exists a bijection

$$\Phi_{X,Y}: \mathrm{HOM}_{\mathcal{C}}(FY, X) \rightarrow \mathrm{HOM}_{\mathcal{D}}(Y, GX),$$

such that this family of bijections is natural in X and Y . In that case, we also call G *right adjoint* to F and we write $F \dashv G$.

There exist several equivalent definitions of adjoint functors, which can be found in any book of Category theory, and we recall the one we use.

Proposition 2.2.8 (Adjunction by unit and counit, see [ML98, Theorems 1.1 and 1.2, Chapter IV]). *Let $F: \mathcal{D} \rightarrow \mathcal{C}$ and $G: \mathcal{C} \rightarrow \mathcal{D}$ be two functors. Then F is left adjoint to G if, and only if, there exist two natural transformations*

$$\varepsilon: FG \Rightarrow 1_{\mathcal{C}} \quad \text{and} \quad \eta: 1_{\mathcal{D}} \Rightarrow GF,$$

such that, for all objects X in $\mathcal{C}$ and Y in $\mathcal{D}$, the following compositions

$$GX \xrightarrow{\eta_{GX}} GFGX \xrightarrow{G(\varepsilon_X)} GX \quad \text{and} \quad FY \xrightarrow{F(\eta_Y)} FGFY \xrightarrow{\varepsilon_{FY}} FY, \quad (2.2.4)$$

are the identity morphisms Id_{FY} and Id_{GX} , respectively. We call the natural transformations ε and η , respectively, *unit* and *counit* of the adjunction.

Fix an algebra A with subalgebra B and consider the following functors:

$$\mathrm{Ind}_B^A = {}_A A_B \otimes_B -: B\text{-mod} \rightarrow A\text{-mod} \quad \text{and} \quad \mathrm{Res}_B^A = {}_B A_A \otimes_A -: A\text{-mod} \rightarrow B\text{-mod}.$$

It is a fact that Ind_B^A is always left adjoint to Res_B^A . Let M be a B -mod and N be an A -mod. The adjunction is given by the function

$$\mathrm{HOM}_A(\mathrm{Ind}_B^A M, N) \rightarrow \mathrm{HOM}_B(M, \mathrm{Res}_B^A N), \quad f \mapsto \left(\tilde{f}: m \mapsto f(1 \otimes m) \right),$$

with inverse given by

$$\mathrm{HOM}_B(M, \mathrm{Res}_B^A N) \rightarrow \mathrm{HOM}_A(\mathrm{Ind}_B^A M, N), \quad f \mapsto \left(\hat{f}: a \otimes m \mapsto f(am) \right).$$

A natural question to be answered is when there exists the opposite adjunction. In what follows we will see that it exists, up to parity shifting, exactly when A is a Frobenius extension of B .

Theorem 2.2.9. *For $\lambda \in \mathbb{Z}$ and $\pi \in \mathbb{Z}_2$, consider the functor*

$$\mathrm{Res}_B^{\{-\lambda, \pi\}A} = \{-\lambda, \pi\}_B A_A \otimes_A - : A\text{-mod} \rightarrow B\text{-mod}.$$

Then Ind_B^A is right adjoint to $\mathrm{Res}_B^{\{-\lambda, \pi\}A}$ if, and only if, A is an Frobenius extension of B of parity π and degree $-\lambda$.

Proof. Suppose that A is an Frobenius extension of B with trace map tr , of parity π and degree $-\lambda$, and fix a pair of finite sets $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ as in Lemma 2.2.5.

Define the even linear maps

$$\begin{aligned} \eta : A &\rightarrow A \otimes_B \{-\lambda, \pi\}A, \quad a \mapsto \sum_{i \in I} a y_i \otimes \{-\lambda, \pi\}x_i, \\ \varepsilon : A \otimes_A \{-\lambda, \pi\}A &\rightarrow B, \quad \{-\lambda, \pi\}a_1 \otimes a_2 \mapsto \mathrm{tr}(a_1 a_2). \end{aligned}$$

We claim that η and ε are homomorphisms of (A, A) -bimodules and (B, B) -bimodules, respectively. Indeed, by the fact that tr is a (B, B) -bimodule homomorphism of parity π , a simple computation shows that ε is homomorphism of (B, B) -bimodule. Now, for all $a \in A$, it is easy to see that $\eta(a) = a\eta(1)$. Also, it follows that

$$\begin{aligned} \eta(a) &= \sum_{i \in I} a y_i \otimes \{-\lambda, \pi\}x_i \stackrel{(2.2.1)}{=} \sum_{i, j \in I} y_j \mathrm{tr}(x_j a y_i) \otimes \{-\lambda, \pi\}x_i \\ &= \sum_{i, j \in I} y_j \otimes (-1)^{\pi(\bar{a} + \bar{y}_i + \bar{x}_j + \pi)} \{-\lambda, \pi\} \mathrm{tr}(x_j a y_i) x_i \\ &= \sum_{j \in I} y_j \otimes \{-\lambda, \pi\} \left((-1)^{\pi(\bar{a} + \bar{x}_j)} \sum_{i \in I} (-1)^{\pi \bar{x}_i} \mathrm{tr}(x_j a y_i) x_i \right) \\ &\stackrel{(2.2.1)}{=} \sum_{j \in I} y_j \otimes \{-\lambda, \pi\} x_j a = \eta(1)a. \end{aligned}$$

Hence, η is an even (A, A) -bimodule homomorphism.

Now, for all left B -module M and left A -module N , define the following maps

$$\begin{aligned} \eta_N : N &\rightarrow A \otimes_B \{-\lambda, \pi\}A \otimes_A N, \quad n \mapsto \eta(1) \otimes n = \sum_{i \in I} y_i \otimes \{-\lambda, \pi\}x_i \otimes n, \\ \varepsilon_M : \{-\lambda, \pi\}A \otimes_A A \otimes_B M &\rightarrow M, \quad \{-\lambda, \pi\}a_1 \otimes a_2 \otimes m \mapsto \varepsilon(\{-\lambda, \pi\}a_1 \otimes a_2)m = \mathrm{tr}(a_1 a_2)m. \end{aligned}$$

Clearly η_N and ε_M are homomorphisms of (A, A) -bimodules and (B, B) -bimodules, respectively, since η and ε are, which defines two natural transformations

$$\tilde{\varepsilon} : \mathrm{Res}_B^{\{-\lambda, \pi\}A} \mathrm{Ind}_B^A \Rightarrow 1_{B\text{-mod}} \quad \text{and} \quad \tilde{\eta} : 1_{A\text{-mod}} \Rightarrow \mathrm{Ind}_B^A \mathrm{Res}_B^{\{-\lambda, \pi\}A}.$$

We claim that $\tilde{\varepsilon}$ and $\tilde{\eta}$ are the unit and counit for the adjunction $\mathrm{Res}_B^{\{-\lambda, \pi\}A} \dashv \mathrm{Ind}_B^A$. In fact,

fixing M and N we have

$$\begin{aligned}
& \varepsilon_{\text{Res}_B^{\{-\lambda, \pi\}A} N} \circ \text{Res}_B^{\{-\lambda, \pi\}A} (\eta_N)(\{-\lambda, \pi\}a \otimes n) \\
&= \varepsilon_{\text{Res}_B^{\{-\lambda, \pi\}A} N} \left(\sum_{i \in I} \{-\lambda, \pi\}a \otimes y_i \otimes \{-\lambda, \pi\}x_i \otimes n \right) \\
&= \sum_{i \in I} \text{tr}(ay_i) \{-\lambda, \pi\}x_i \otimes n \\
&= \{-\lambda, \pi\} \left((-1)^{\pi \bar{a}} \sum_{i \in I} (-1)^{\pi \bar{x}_i} \text{tr}(ay_i) x_i \right) \otimes n \\
&\stackrel{(2.2.1)}{=} \{-\lambda, \pi\}a \otimes n,
\end{aligned}$$

for all $a \in A$ and $n \in N$. And, for all $a \in A$ and $m \in M$ it follows

$$\begin{aligned}
& \text{Ind}_B^A(\varepsilon_M) \circ \eta_{\text{Ind}_B^A M}(a \otimes m) = \text{Id}_A \otimes \varepsilon_M(\eta(1) \otimes a \otimes m) \\
&= \text{Id}_A \otimes \varepsilon_M \left(\sum_{i \in I} y_i \otimes \{-\lambda, \pi\}x_i \otimes a \otimes m \right) \\
&= \sum_{i \in I} y_i \otimes \text{tr}(x_i a) m \\
&= \left(\sum_{i \in I} y_i \text{tr}(x_i a) \right) \otimes m \stackrel{(2.2.1)}{=} a \otimes m.
\end{aligned}$$

Then $\tilde{\varepsilon}$ and $\tilde{\eta}$ satisfy the unit and counit relations (2.2.4). Therefore $\text{Res}_B^{\{-\lambda, \pi\}A} \dashv \text{Ind}_B^A$.

Conversely, let $\tilde{\varepsilon}$ and $\tilde{\eta}$ be, respectively, the unit and counit of the adjunction $\text{Res}_B^{\{-\lambda, \pi\}A} \dashv \text{Ind}_B^A$ and define

$$\varepsilon = \tilde{\varepsilon}_B: \{-\lambda, \pi\}A \otimes_A A \otimes_B B \rightarrow B \quad \text{and} \quad \eta = \tilde{\eta}_A: A \rightarrow A \otimes_B \{-\lambda, \pi\}A \otimes_A A.$$

Now, write $\eta(1) = \sum_{i \in I} y_i \otimes \{-\lambda, \pi\}x_i \otimes 1$ for a finite set I and define the map

$$\text{tr}: A \rightarrow B, \quad a \mapsto \varepsilon(\{-\lambda, \pi\}a \otimes 1 \otimes 1).$$

Notice that $(|y_i| + |x_i|, \bar{y}_i + \bar{x}_i) = (\lambda, \pi)$, for all $i \in I$, and that tr has parity π and degree $-\lambda$. By unit and counit properties, we have that

$$\tilde{\varepsilon}_{\{-\lambda, \pi\}A \otimes_A A} \circ (\text{Id} \otimes \eta)(\{-\lambda, \pi\}a \otimes 1) = \{-\lambda, \pi\}a \otimes 1 \quad (\text{Id} \otimes \varepsilon) \circ \tilde{\eta}_{A \otimes_B B}(a \otimes 1) = a \otimes 1,$$

for all $a \in A$. Then, by reversing the previous computations, we conclude that the pair of finite sets $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$, with the map tr , satisfy the conditions from Lemma 2.2.5. Therefore A is a Frobenius extension of B of parity π and degree $-\lambda$. $\square$

2.2.3 Frobenius algebras

Definition 2.2.10 (Frobenius algebra). A *Frobenius algebra* F of parity τ and degree $-\delta$ is a Frobenius extension of $\mathbb{k}$ of the same parity and degree. Equivalently, an algebra F is a

Frobenius algebra if there exist a homogeneous linear map $\text{tr}: F \rightarrow \mathbb{k}$ of parity τ and degree $-\delta$ such that the map

$$F \rightarrow \text{Hom}_{\mathbb{k}}(F, \mathbb{k}), \quad f \mapsto \left(g \mapsto (-1)^{\bar{f}\bar{g}} \text{tr}(gf) \right), \quad (2.2.5)$$

is a homogeneous isomorphism as left F module. We call tr a *trace map* of F .

Denote by $\psi: F \rightarrow F$ the *Nakayama automorphism* of tr , i.e., the algebra automorphism such that

$$\text{tr}(fg) = (-1)^{\bar{f}\bar{g}} \text{tr}(g\psi(f)) \quad \text{for all } f, g \in F.$$

(See 2.2.1 for the construction of such map. Notice that $C_A(\mathbb{k}) = A$, since A is a trivial $\mathbb{k}$ -module). A Frobenius algebra with Nakayama automorphism equal to the identity map, is called a *Symmetric algebra*.

It follows from Remark 2.2.4 the following lemma:

Lemma 2.2.11. *Let $\text{tr}, \text{tr}': F \rightarrow \mathbb{k}$ be two homogeneous trace maps. There is an invertible homogeneous element $u \in F$ such that $\text{tr}'(f) = \text{tr}(fu)$, for all $f \in F$.*

Remark 2.2.12 (Nakayama automorphism of different trace). Notice that, denoting the Nakayama automorphism of F related to tr and tr' respectively by ψ and ψ' , we have

$$\text{tr}'(ab) = \text{tr}(abu) = (-1)^{(\bar{b}+\bar{u})\bar{a}} \text{tr}(bu\psi(a)) = (-1)^{\bar{b}\bar{a}} \text{tr}'(b(-1)^{\bar{u}\bar{a}} u\psi(a)u^{-1}),$$

for all $a, b \in F$. Therefore ψ' is given by

$$\psi': a \mapsto (-1)^{\bar{u}\bar{a}} u\psi(a)u^{-1}, \quad \text{for all } a \in F. \quad (2.2.6)$$

Now, let $\mathcal{B}$ be a basis of F . For all $b \in \mathcal{B}$, consider the linear map

$$\phi_b: F \rightarrow \mathbb{k}, \quad c \mapsto \delta_{b,c}, \quad \forall c \in \mathcal{B}.$$

Then, **T2** implies that there exists $\tilde{b} \in F$, for all $b \in \mathcal{B}$, such that

$$\phi_b(x) = (-1)^{\bar{b}\bar{x}} \text{tr}(x\tilde{b}), \quad \forall x \in F.$$

Property **T1** of the trace map implies that $\tilde{b}$ is unique. Define $b^\vee = \psi^{-1}(\tilde{b})$, for all $b \in \mathcal{B}$. Since $\{\phi_b\}_{b \in \mathcal{B}}$ defines a basis for $\text{Hom}_{\mathbb{k}}(F, \mathbb{k})$, clearly $\{\tilde{b}\}_{b \in \mathcal{B}}$ is a basis for F , and hence so is $\{b^\vee\}_{b \in \mathcal{B}}$.

Definition 2.2.13 (Left dual basis). Let F be a Frobenius algebra and let $\mathcal{B}$ be a basis of F consisting of homogeneous elements. Its *left dual basis*, $\mathcal{B}^\vee = \{b^\vee \mid b \in \mathcal{B}\}$, is basis of F such that:

$$\text{tr}(b^\vee c) = \delta_{b,c} \quad b, c \in \mathcal{B}.$$

Given a basis $\mathcal{B}$ of a Frobenius algebra F , and its left dual basis $\mathcal{B}^\vee$, it is easy to see that:

$$\sum_{b \in \mathcal{B}} b \text{tr}(b^\vee f) = f, \quad (2.2.7) \quad \sum_{b \in \mathcal{B}} \text{tr}(fb)b^\vee = f. \quad (2.2.8)$$

$$(b^\vee)^\vee = (-1)^{\bar{b}+\tau\bar{b}} \psi^{-1}(b). \quad (2.2.9)$$

Now, let B be a subalgebra of a finite dimensional algebra A and suppose they are Frobenius algebras with trace maps $\text{tr}_B: B \rightarrow \mathbb{k}$ and $\text{tr}_A: A \rightarrow \mathbb{k}$, respectively. Let π_A and π_B be the

parities of tr_A and tr_B , respectively, and $-\lambda_A$ and $-\lambda_B$ their respective degrees. Let ψ_A and ψ_B be the Nakayama automorphisms of tr_A and tr_B , respectively. Fix a basis $\mathcal{B}$ for B and let $\mathcal{B}^\vee$ be its left dual basis.

Proposition 2.2.14. *Suppose ψ_B is the restriction of ψ_A to B and assume that ${}_B A$ is projective left B -module. Then A is a Frobenius extension of B of parity $\pi_A + \pi_B$ and degree $\lambda_B - \lambda_A$, with trace map given by*

$$\text{tr}: A \rightarrow B, \quad a \mapsto \sum_{b \in \mathcal{B}} (-1)^{(\pi_A + \pi_B)\bar{a}} \text{tr}_A(b^\vee a)b.$$

Proof. First we prove that tr is a homomorphism of (B, B) -bimodules of parity $\pi = \pi_A + \pi_B$. For all $\beta \in B$ and $z \in A$ we have:

$$\begin{aligned} \text{tr}(\beta z) &= \sum_{b \in \mathcal{B}} (-1)^{\pi(\bar{\beta} + \bar{z})} \text{tr}_A(b^\vee \beta z)b \stackrel{(2.2.8)}{=} (-1)^{\pi\bar{\beta}} \sum_{b \in \mathcal{B}} (-1)^{\pi\bar{z}} \text{tr}_A \left(\sum_{c \in \mathcal{B}} \text{tr}_B(b^\vee \beta c)c^\vee z \right) b \\ &= (-1)^{\pi\bar{\beta}} \sum_{c \in \mathcal{B}} (-1)^{\pi\bar{z}} \text{tr}_A(c^\vee z) \left(\sum_{b \in \mathcal{B}} \text{tr}_B(b^\vee \beta c)b \right) \stackrel{(2.2.8)}{=} (-1)^{\pi\bar{\beta}} \sum_{c \in \mathcal{B}} (-1)^{\pi\bar{z}} \text{tr}_A(c^\vee z)\beta c \\ &= (-1)^{(\pi_A + \pi_B)\bar{\beta}} \beta \text{tr}(z). \end{aligned}$$

Hence tr is a homomorphism of left B -modules of parity $\pi_A + \pi_B$. For all $\beta \in B$ and $z \in A$, we have

$$\begin{aligned} \text{tr}(z\beta) &= \sum_{b \in \mathcal{B}} (-1)^{\pi(\bar{\beta} + \bar{z})} \text{tr}_A(b^\vee z\beta)b = \sum_{b \in \mathcal{B}} (-1)^{\pi(\bar{\beta} + \bar{z}) + \bar{\beta}(\bar{b}^\vee + \bar{z})} \text{tr}_A(\psi_A^{-1}(\beta)b^\vee z)b \\ &\stackrel{(2.2.8)}{=} \sum_{b, c \in \mathcal{B}} (-1)^{\pi(\bar{\beta} + \bar{z}) + \bar{\beta}(\bar{b}^\vee + \bar{z})} \text{tr}_A(\text{tr}_B(\psi_B^{-1}(\beta)b^\vee c)c^\vee z)b \\ &= \sum_{b, c \in \mathcal{B}} (-1)^{\pi(\bar{\beta} + \bar{z}) + \bar{\beta}(\bar{b}^\vee + \bar{z}) + \bar{\beta}(\bar{b}^\vee + \bar{c})} \text{tr}_A(c^\vee z) \text{tr}_B(b^\vee c\beta)b \\ &\stackrel{(2.2.8)}{=} \sum_{c \in \mathcal{B}} (-1)^{\pi(\bar{\beta} + \bar{z}) + \bar{\beta}(\bar{b}^\vee + \bar{c})} \text{tr}_A(c^\vee z)c\beta. \end{aligned}$$

Notice that, if $\text{tr}_A(c^\vee z) \neq 0$, then $\pi_A = \overline{c^\vee z} = \bar{c} + \pi_B + \bar{z}$, since tr_A has parity π_A . Hence,

$$(-1)^{\bar{\beta}(\bar{c} + \bar{z})} \text{tr}_A(c^\vee z) = (-1)^{\bar{\beta}\pi} \text{tr}_A(c^\vee z)$$

for all $c \in \mathcal{B}$, which implies that

$$\text{tr}(z\beta) = \sum_{c \in \mathcal{B}} (-1)^{\pi(\bar{\beta} + \bar{z}) + \bar{\beta}\pi} \text{tr}_A(c^\vee z)c\beta = \text{tr}(z)\beta.$$

Therefore, tr is a homomorphism of (B, B) -bimodules.

Now, Let $f \in \text{HOM}_B(A, B)$ be homogeneous. Then, there exists $a \in A$ homogeneous such that

$$f(x) = (-1)^{\bar{x}\bar{a}} \text{tr}(xa),$$

for all $x \in A$. Indeed, consider $\tilde{f}: A \rightarrow \mathbb{k}$ defined by $\tilde{f}(x) = (-1)^{\pi(\pi + \bar{f} + \bar{x})} \text{tr}_B(f(x))$ for all

$x \in A$. Then, from **T1**, there exists $a \in A$ homogeneous of parity $\pi + \bar{f}$ such that $\tilde{f}(x) = (-1)^{\bar{x}\bar{a}} \text{tr}_A(xa)$, for all $x \in A$, i.e.,

$$(-1)^{\pi(\bar{x}+\bar{a})} \text{tr}_B(f(x)) = (-1)^{\bar{x}\bar{a}} \text{tr}_A(xa), \quad \forall x \in A. \quad (2.2.10)$$

Hence

$$\begin{aligned} f(x) &\stackrel{(2.2.7)}{=} \sum_{b \in \mathcal{B}} \text{tr}_B(b^\vee f(x))b = \sum_{b \in \mathcal{B}} (-1)^{\bar{f}\bar{b}^\vee} \text{tr}_B(f(b^\vee x))b \\ &\stackrel{(2.2.10)}{=} \sum_{b \in \mathcal{B}} (-1)^{(\bar{a}+\pi)\bar{b}^\vee + \pi(\bar{b}^\vee + \bar{x} + \bar{a}) + \bar{a}(\bar{b}^\vee + \bar{x})} \text{tr}_A(b^\vee xa)b \\ &= (-1)^{\bar{a}\bar{x}} \sum_{b \in \mathcal{B}} (-1)^{\pi(\bar{x}+\bar{a})} \text{tr}_A(b^\vee(xa))b = (-1)^{\bar{a}\bar{x}} \text{tr}(xa), \end{aligned}$$

for all homogeneous $x \in A$. Therefore tr satisfies **T2**. Finally, we verify **T1**. Suppose that $\text{tr}(Az) = 0$ for some $z \in A$. Then, for all $x \in A$ we have that

$$0 = \text{tr}(xz) = \sum_{b \in \mathcal{B}} (-1)^{\pi(\bar{x}+\bar{z})} \text{tr}_A(b^\vee xz)b.$$

Since $\mathcal{B}$ is a basis for B , it follows that

$$0 = \text{tr}_A(b^\vee xz) = (-1)^{\bar{b}^\vee(\bar{x}+\bar{z})} \text{tr}(xz\psi_A(b^\vee)),$$

for all $x \in A$ and $b \in \mathcal{B}$. Hence $z\psi(b^\vee) = 0$, for all $b \in \mathcal{B}$, which implies that $z = 0$, since $\{\psi(b^\vee)\}_{b \in \mathcal{B}}$ is a basis for B . Therefore tr satisfies **T1**, and the proof is finished. $\square$

2.2.4 Examples

In this subsection we provide several examples of Frobenius algebras which we will use to illustrate the results in the following sections. We also show that these algebras are indeed Frobenius. Since the degree of homogeneous elements does not make difference in our argumentation, in the following we will “forget” the $\mathbb{Z}$ -grading.

Let Γ be a signed group, i.e., a group equipped with a homomorphism of group $\text{sgn}: \Gamma \rightarrow \mathbb{Z}_2$. A Γ -graded algebra is an algebra A with decomposition $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ such that $A_\gamma A_\sigma \subseteq A_{\gamma\sigma}$, for all $\gamma, \sigma \in \Gamma$. An element of A is called Γ -homogeneous if it belongs to A_γ for some $\gamma \in \Gamma$. A Γ -graded algebra A naturally inherits a $\mathbb{Z}_2$ -grading by defining the elements a_γ of A of parity $\bar{a}_\gamma = \text{sgn}(\gamma)$, for all $\gamma \in \Gamma$.

We have the following:

Lemma 2.2.15. *Let A be a Γ -graded algebra. Assume that A is finitely generated and that there exist $\gamma_0 \in \Gamma$ and non-zero $a_0 \in A_{\gamma_0}$ satisfying the following properties:*

- a) $A_{\gamma_0} = \mathbb{k}a_0$;
- b) For any Γ -homogeneous element $a \in A$, there exist unique solutions for the equations $xa = a_0$ and $ay = a_0$, which we denote by ℓ_a, r_a , respectively.

Then A is a Frobenius algebra with trace map $\text{tr}: A \rightarrow \mathbb{k}$, of parity $\text{sgn}(\gamma_0)$, defined by

$$\text{tr}(a_0) = 1, \quad \text{and} \quad \text{tr}(A_\sigma) = 0, \quad \forall \sigma \in \Gamma \setminus \{\gamma_0\},$$

and Nakayama autormorphism $\psi: A \rightarrow A$, such that

$$\psi(a) = (-1)^{\bar{a}\bar{r}_a} r_{r_a} = (-1)^{\bar{a}(\text{sgn}(\gamma_0)+1)} r_{r_a},$$

for all nonzero homogeneous $a \in A$.

Proof. Since A is finitely generated, it is enough to show that tr satisfies **T1** and **T2** (by proposition 2.2.2).

By property **b)**, it is easy to see that $A_{\gamma_0} \subseteq Aa$, for any non-zero homogeneous $a \in A$, hence tr satisfies **T1**.

Now, let $\phi \in \text{HOM}_{\mathbb{k}}(A, \mathbb{k})$ be a nonzero Γ -homogeneous element. Since $\ker(\phi)$ has co-dimension one, there exist $\sigma \in \Gamma$ and $a_\sigma \in A_\sigma$, such that $\phi(a_\sigma) = 1$ and $\phi(A_\gamma) = 0$, for all $\gamma \in \Gamma \setminus \{\sigma\}$. Then, let $r_{a_\sigma} \in A_{\sigma^{-1}\gamma_0}$ be the unique element such that $a_\sigma r_{a_\sigma} = a_0$, and define $b = (-1)^{\text{sgn}(\sigma)(\text{sgn}(\gamma_0)+1)} r_{a_\sigma}$. We claim that

$$\phi(x) = (-1)^{\bar{x}\bar{b}} \text{tr}(xb),$$

for all homogeneous $x \in A$. It is clearly true for $x \in A_\gamma$, where $\gamma \neq \sigma$, since $xb \in A_{\gamma\sigma^{-1}\gamma_0} \neq A_{\gamma_0}$. Now, let $x \in A_\sigma$. It follows that $x = \alpha a_\sigma + y$, where

$$y = 0 \quad \text{or} \quad \{y, a_\sigma\} \text{ are linear independent}, \quad (2.2.11)$$

which implies that $\phi(y) = 0$. Also, by property **a)**, we must have $yb = \lambda a_0$, for some scalar $\lambda \in \mathbb{k}$. Then, by the uniqueness in the property **b)**, it follows that either y is a nonzero multiple of a_σ or $\lambda = 0$. But the first case is a contradiction to (2.2.11), so we have $\lambda = 0$. Hence $\text{tr}(yb) = 0$. From the calculation

$$\begin{aligned} \phi(x) &= \alpha\phi(a_\sigma) + \phi(y) = \alpha + (-1)^{\bar{y}\bar{b}} \text{tr}(yb) = \alpha \text{tr}(a_0) + (-1)^{\bar{y}\bar{b}} \text{tr}(yb) = \\ &= (-1)^{\bar{a}_\sigma \bar{r}_{a_\sigma}} \text{tr}(\alpha a_\sigma (-1)^{\text{sgn}(\sigma)(\text{sgn}(\gamma_0)+1)} r_{a_\sigma}) + (-1)^{\bar{y}\bar{b}} \text{tr}(yb) = (-1)^{\bar{x}\bar{b}} \text{tr}(xb), \end{aligned}$$

it follows that

$$\phi(\alpha a_\sigma + y) = (-1)^{\overline{\alpha a_\sigma + y}\bar{b}} \text{tr}((\alpha a_\sigma + y)b),$$

and hence tr satisfies **T2**.

Finally, let $a \in A_\gamma$, $b \in A_\sigma$ be homogeneous non zero elements. If $\sigma \neq \gamma^{-1}\gamma_0$, we have that $\text{tr}(ab) = 0 = \text{tr}(br_{r_a}) = (-1)^{\bar{a}\bar{b}} \text{tr}(b\psi(a))$, where the second equality follows from definition of the trace map and from the fact that $r_{r_a} \in A_{\gamma_0^{-1}\gamma\gamma_0}$. If $\sigma = \gamma^{-1}\gamma_0$, property **a)** implies that $ab = \lambda a_0$, for $\lambda \in \mathbb{k}$. If $\lambda \neq 0$, then $b = \lambda r_a$, by property **b)**. It follows that

$$\text{tr}(ab) = \lambda = \text{tr}(\lambda r_a r_{r_a}) = (-1)^{\bar{a}\bar{b}} \text{tr}(b\psi(a)).$$

(Notice that $\bar{b} = \bar{r}_a = \text{sgn}(\gamma_0) + \bar{a}$). For the last case $\lambda = 0$, we have that $br_{r_a} = 0$, otherwise, since $br_{r_a} \in A_{\gamma_0}$, by properties **a)** and **b)**, we would have b as a nonzero multiple of r_a , contradicting the hypothesis $ab = 0$. Hence $\text{tr}(ab) = 0 = (-1)^{\bar{a}\bar{b}} \text{tr}(b\psi(a))$. And the poof is done. $\square$

Furthermore, notice that, a basis $\mathcal{B}$ for A , of Γ -homogeneous elements, has left dual basis

$$\mathcal{B}^\vee = \{b^\vee = \ell_b | b \in \mathcal{B}\}.$$

Now, we can give more specific examples of Frobenius algebras.

Example 2.2.16 (Group (super)algebra $\mathbb{k}G$). Let G be a finitely generated group equipped with a homomorphism of groups $\text{sgn}: G \rightarrow \mathbb{Z}_2$ and consider the G -grading in $A = \mathbb{k}G$ given by $A_g = \mathbb{k}g$, for all $g \in G$. For any $\tau \in \mathbb{Z}_2$ and fixed $g_\tau \in \text{sgn}^{-1}(\tau)$ we have that A satisfies the hypothesis of Lemma 2.2.15 (with $\ell_h = g_\tau h^{-1}$ and $r_h = h^{-1}g_\tau$, for all $h \in G$). Then $\mathbb{k}G$ is a Frobenius algebra with trace map of parity τ given by

$$\text{tr}(h) = \delta_{hg_\tau}, \quad \forall h \in G,$$

and Nakayama automorphism given by

$$\psi(h) = (-1)^{\bar{h}(1+\tau)} g_\tau^{-1} h g_\tau, \quad \forall h \in G.$$

Example 2.2.17 (Clifford algebra $\mathcal{C}l$). Let $\mathcal{C}l$ be the *Clifford algebra*, i.e, the algebra generated by the odd element c , under the relation $c^2 = 1$. It is easy to see that $\mathcal{C}l$ is isomorphic to $\mathbb{k}\mathbb{Z}_2$, by identifying the class of $1 \in \mathbb{Z}_2$ with c . Then $\mathcal{C}l$ can be viewed as a particular case of example 2.2.16 with two possible trace maps, respectively, even and odd:

$$\text{tr}_0: 1 \mapsto 1, c \mapsto 0 \quad \text{tr}_1: 1 \mapsto 0, c \mapsto 1.$$

Their respective Nakayama automorphisms are:

$$\psi_0(c) = -c, \quad \psi_1 = \text{Id}_{\mathcal{C}l}.$$

We will write $\mathcal{C}l_\tau$ for the Clifford algebra equipped with the trace map tr_τ , for $\tau \in \{0, 1\}$.

Example 2.2.18 (Truncated skew polynomial algebra $T(A, \sigma, n)$). Let A be a Γ -graded algebra such as in Lemma 2.2.15 (with $\gamma_0 \in \Gamma$ and $a_0 \in A_{\gamma_0}$ such as in the statement) and let $\sigma \in \text{End}(A)$ be a Γ -homogeneous endomorphism. Consider the skew polynomial ring $A[y, \sigma]$, i.e, the free $\mathbb{k}$ -algebra with basis $\{ay^n | a \in A, n \in \mathbb{N}\}$ under the relation $ya = \sigma(a)y$, for all homogeneous $a \in A$. For any $n \in \mathbb{N}, n \geq 1$, we define the *truncated skew polynomial algebra* to be $T(A, \sigma, n) = \frac{A[y, \sigma]}{(y^n)}$. In the following, by abuse of notation, we will write y to refer to its class $y + (y^n) \in T(A, \sigma, n)$.

Notice that we have a natural $\Gamma \times \mathbb{N}$ -grading

$$T(A, \sigma, n) = \bigoplus_{(\gamma, i) \in \Gamma \times \mathbb{N}} A_\gamma y^i.$$

Such grading satisfies $A_{\gamma_0} y^{n-1} = \mathbb{k}a_0 y^{n-1}$, hence property **a)** of the Lemma 2.2.15. Furthermore, given $\gamma \in \Gamma$, $a \in A_\gamma$ and $0 \leq i \leq n-1$,

$$\tilde{\ell}_{a,i} = \ell_{\sigma^{n-1-i}(a)} y^{n-1-i} \quad \text{and} \quad \tilde{r}_{a,i} = \sigma^{-i}(r_a) y^{n-1-i}$$

are the unique solutions for $x(ay^i) = a_0 y^{n-1}$ and $(ay^i)x = a_0 y^{n-1}$, respectively. Indeed, for all $\gamma \in \Gamma$, $a \in A_\gamma$ and $0 \leq i \leq n-1$, it follows

$$\begin{aligned} \ell_{\sigma^{n-1-i}(a)} y^{n-1-i} ay^i &= \ell_{\sigma^{n-1-i}(a)} \sigma^{n-1-i}(a) y^{n-1-i} y^i = a_0 y^{n-1}, \\ ay^i \sigma^{-i}(r_a) y^{n-1-i} &= ar_a y^i y^{n-1-i} = a_0 y^{n-1}. \end{aligned}$$

Their uniqueness follows from the uniqueness in the property **b)** of A .

Hence $T(A, n, \sigma)$ satisfies property **b)** of the Lemma 2.2.15. Therefore, $T(A, \sigma, n)$ is a Frobenius algebra with trace map

$$\text{tr}: T(A, \sigma, n) \rightarrow \mathbb{k}, \quad ay^i \mapsto \delta_{a, a_0} \delta_{i, n-1},$$

for $0 \leq i \leq n-1$ and Γ -homogeneous $a \in A$. By declaring the parity of y to be τ , we have that tr has parity $\text{sgn}(\gamma_0) + \tau(n-1)$.

Example 2.2.19 (Algebra of Dual numbers D). A particular case of example 2.2.18 is $T(\mathbb{k}, \text{Id}, 2)$ (here Γ is the trivial group) with the element y being of odd parity. Such algebra is isomorphic to the algebra of dual numbers D , i.e, the algebra freely generated by the odd element u under the relation $u^2 = 0$.

Then D is a Frobenius algebra with odd trace map

$$\text{tr}: D \rightarrow \mathbb{k}, \quad \text{tr}(u) = 1, \quad \text{tr}(1) = 0.$$

and trivial Nakayama automorphism.

Since, by lemma 2.2.11, all trace maps differ by an invertible element, and D has no odd invertible element, notice that D does not admit even trace map.

Example 2.2.20 (Taft-Hopf algebra $T_{q,\tau}$). Let $q \in \mathbb{N}$, $\tau \in \{0, 1\}$ and let $\omega \in \mathbb{k}$ be a primitive $(1+\tau)q$ -th root of $1 \in \mathbb{k}$ (assuming that $\mathbb{k}$ is algebraically closed). Let $G = C_{q(1+\tau)}$ be the cyclic group of order $(1+\tau)q$ and let g be its generator. Define a sign in G by declaring g to be of parity τ , i.e, define $\text{sgn}: G \rightarrow \mathbb{Z}_2$ such that $\text{sgn}(g) = \tau$. Consider a Frobenius structure on $\mathbb{k}G$, as in example 2.2.16, where $\text{tr}_{\mathbb{k}G}(1) = 1$.

We define the Taft-Hopf algebra $T_{q,\tau}$ by the truncated skew polynomial algebra $T(\mathbb{k}G, \sigma, q(\tau+1))$, where y has parity τ and $\sigma: \mathbb{k}G \rightarrow \mathbb{k}G$ is the automorphism such that $\sigma(g^k) = \omega^k g$, for all $k \in \mathbb{N}$. Then, $T_{q,\tau}$ is an algebra with generators g, y under the relations

$$g^{(1+\tau)q} = 1, \quad yg = \omega gy, \quad y^{(1+\tau)q} = 1,$$

and with trace map of parity τ given by

$$\text{tr}(g^l y^k) = \delta_{l,0} \delta_{k,(1+\tau)q-1},$$

for all $k, l \in \mathbb{N}$. Furthermore, its Nakayama automorphism is given by $\psi(y) = y$ and $\psi(g) = \omega g$.

Remark 2.2.21 (Hopf Algebras). Notice that all specific examples ($\mathcal{C}l_\tau$, D and $T_{q,\tau}$) above are finite dimensional Hopf algebras and have Nakayama automorphism of finite order (although an algebra in Lemma 2.2.15 does not necessarily satisfies such properties). In fact it is a well known result that any finite dimensional Hopf algebra is Frobenius algebra with Nakayama automorphism of finite order. We refer [SY11, Chapter VI, Theorems 3.6, 5.1 and 5.2] for more details.

2.3 Smash products and wreath product algebras

Suppose we have a left action of a group G on a $\mathbb{k}$ -algebra A via algebra automorphism. Then we can form the *smash product algebra* $A \rtimes G$ defined by $A \otimes_{\mathbb{k}} \mathbb{k}G$, as $\mathbb{k}$ -module, with

multiplication given by the fact that A and $\mathbb{k}G$ are subalgebras, and

$$wa = (w \cdot a)w, \quad w \in \mathbb{k}G, \quad a \in A. \quad (2.3.1)$$

Analogously, given a group H acting on A on the right via algebra automorphism, we define the *smash product algebra* $H \ltimes A$ to be $\mathbb{k}H \otimes_{\mathbb{k}} A$, as $\mathbb{k}$ -module, with multiplication such that

$$ah = h(a \cdot h), \quad h \in \mathbb{k}G, \quad a \in A. \quad (2.3.2)$$

As a special case of the above constructions, we have the natural action of S_n on $A^{\otimes n}$ by superpermuting the factors:

$$s_i \cdot (a_1 \otimes \cdots \otimes a_n) = (-1)^{\bar{a}_i \bar{a}_{i+1}} a_1 \otimes \cdots \otimes a_{i-1} \otimes a_{i+1} \otimes a_i \otimes a_{i+2} \otimes \cdots \otimes a_n,$$

where s_i , $1 \leq i \leq n-1$, denotes the simple transposition of i and $i+1$.

Definition 2.3.1 (Wreath product algebra). We call the smash product algebra $F^{\otimes n} \ltimes S_n$, with the action given by superpermutation, the *wreath product algebra* associated to the $\mathbb{N}$ -graded Frobenius superalgebra F .

For all $\pi \in S_n$ and $\mathbf{f} \in F^{\otimes n}$, we denote the action of π on $\mathbf{f}$ by ${}^\pi \mathbf{f}$.

Chapter 3

Affine wreath product algebras

In this section we define affine wreath product algebras and give some examples and relations. Throughout this file, we will call an $\mathbb{N}$ -graded superalgebra algebra, unless otherwise stated.

3.1 Definition

Throughout this text we fix a Frobenius algebra F (see definition 2.2.10) with trace map tr of parity τ and degree $-\delta$. We also make the following assumptions:

- The Nakayama automorphism ψ has finite order θ ,
- The characteristic of $\mathbb{k}$ does not divide θ ,
- The action of the Nakayama automorphism is diagonalizable.

We also fix a basis $\mathcal{B}$ of F consisting of homogeneous elements and its left dual basis, $\mathcal{B}^\vee = \{b^\vee \mid b \in \mathcal{B}\}$.

Fixing $n \in \mathbb{N}_+$, for $f \in F$ and $1 \leq i \leq n$, we define

$$f_i = 1^{\otimes(i-1)} \otimes f \otimes 1^{\otimes(n-i)} \in F^{\otimes n}, \quad (3.1.1)$$

$$\psi_i = \text{Id}^{\otimes(i-1)} \otimes \psi \otimes \text{Id}^{\otimes(n-i)} : F^{\otimes n} \rightarrow F^{\otimes n}. \quad (3.1.2)$$

Let $\mathbb{k}\langle x_1, \dots, x_n \rangle$ be the free associative algebra generated by $x_1, \dots, x_n$, and let $s_i \in S_n$ be the transposition $(i, i+1)$, for $1 \leq i \leq n-1$.

Definition 3.1.1 (Affine wreath product algebra $\mathcal{A}_n(F)$). We define the *affine wreath product algebra* $\mathcal{A}_n(F)$ to be the $\mathbb{N}$ -graded superalgebra that is the free product of $\mathbb{k}$ -algebras

$$\mathbb{k}\langle x_1, \dots, x_n \rangle \star F^{\otimes n} \star \mathbb{k}S_n,$$

modulo the relations

$$x_i x_j = (-1)^\tau x_j x_i, \quad 1 \leq i, j \leq n, \quad i \neq j \quad (3.1.3)$$

$$\mathbf{f} x_i = (-1)^{\tau \bar{\mathbf{f}}} x_i \psi_i(\mathbf{f}), \quad 1 \leq i \leq n, \quad \mathbf{f} \in F^{\otimes n}, \quad (3.1.4)$$

$$s_i x_j = x_j s_i, \quad 1 \leq i \leq n-1, \quad 1 \leq j \leq n, \quad j \neq i, i+1, \quad (3.1.5)$$

$$s_i x_i = x_{i+1} s_i - t_{i,i+1}, \quad 1 \leq i \leq n-1, \quad (3.1.6)$$

$$\pi \mathbf{f} = {}^\pi \mathbf{f} \pi, \quad \pi \in S_n, \quad \mathbf{f} \in F^{\otimes n}, \quad (3.1.7)$$

where

$$t_{i,j} := \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b}} b_i b_j^\vee \quad \text{for } 1 \leq i, j \leq n, \quad i \neq j, \quad (3.1.8)$$

and where we always interpret $b_j^\vee$ as $(b^\vee)_j$. The degree and parity on $\mathcal{A}_n(F)$ are determined by

$$|x_i| = \delta, \quad \bar{x}_i = \tau, \quad 1 \leq i \leq n, \quad (3.1.9)$$

$$|f_i| = |f|, \quad \bar{f}_i = \bar{f}, \quad 1 \leq i \leq n, \quad f \in F, \quad (3.1.10)$$

$$|\pi| = 0, \quad \bar{\pi} = 0, \quad \pi \in S_n. \quad (3.1.11)$$

By convention, we set $\mathcal{A}_0(F) = \mathbb{k}$.

Remark 3.1.2 (Independence of the chosen basis). Notice that the definition of $t_{i,j}$ is independent of the chosen homogeneous basis $\mathcal{B}$. Indeed, let $\mathcal{B} = \{b^{(1)}, \dots, b^{(a)}\}$ and $\mathcal{B}' = \{c^{(1)}, \dots, c^{(a)}\}$ be two homogeneous bases for F . Then there exist $M = (m_{kl})$, $M' = (m'_{kl}) \in M_a(\mathbb{k})$ such that,

$$b^{(k)} = \sum_{l=1}^a m_{kl} c^{(l)} \quad \text{and} \quad b^{(k)\vee} = \sum_{l=1}^a m'_{kl} c^{(l)\vee},$$

for all $k \in \{1, \dots, a\}$.

Notice that, since $\mathcal{B}$ and $\mathcal{B}'$ are homogeneous, for all $k, l \in \{1, \dots, a\}$ we have:

$$\begin{aligned} (-1)^{\tau \bar{b}^{(k)}} \delta_{kl} &= \text{tr}(b^{(l)\vee} (-1)^{\tau \bar{b}^{(k)}} b^{(k)}) = \sum_{r,s=1}^a (-1)^{\tau \bar{c}^{(r)}} m_{kr} m'_{ls} \text{tr}(c^{(s)\vee} c^{(r)}) \\ &= \sum_{r,s=1}^a (-1)^{\tau \bar{c}^{(r)}} m_{kr} m'_{ls} \delta_{sr} \\ &= \sum_{r=1}^a (-1)^{\tau \bar{c}^{(r)}} m_{kr} m'_{lr}. \end{aligned}$$

Thus

$$\sum_{k=1}^a (-1)^{\tau \bar{b}^{(k)}} b_i^{(k)} b_j^{(k)\vee} = \sum_{k,l,r=1}^a (-1)^{\tau \bar{b}^{(k)}} m_{kl} c_i^{(l)} m'_{kr} c_j^{(r)\vee} = \sum_{l=1}^a (-1)^{\tau \bar{c}^{(l)}} c_i^{(l)} c_j^{(l)\vee}.$$

Note that $|t_{i,j}| = \delta$ and $\bar{t}_{i,j} = \tau$ for all i, j . Conjugating by s_i , we see that (3.1.6) is equivalent to

$$s_i x_{i+1} = x_i s_i + t_{i+1,i}. \quad (3.1.12)$$

It follows from Definition 2.3.1 that we have an algebra homomorphism

$$F^{\otimes n} \rtimes S_n \rightarrow \mathcal{A}_n(F), \quad \mathbf{f} \mapsto \mathbf{f}, \quad \pi \mapsto \pi, \quad \mathbf{f} \in F^{\otimes n}, \quad \pi \in S_n. \quad (3.1.13)$$

Lemma 3.1.3. *Up to isomorphism of algebras, $\mathcal{A}_n(F)$ depends only on the underlying algebra F .*

Proof. The proof is analogous to [Sav20, Lemma 3.2]. Let F and F' be two Frobenius algebras with same underlying algebra and let tr , tr' be their respective traces. Let τ and σ be their parities. By Lemma 2.2.11, there exists an invertible homogeneous element $u \in F^\times$ such that $\text{tr}'(f) = \text{tr}(fu)$ for all $f \in F$. It is clear that the parity of u must be $\tau + \sigma$.

Define the homomorphism $\Phi: \mathbb{k}\langle x_1, \dots, x_n \rangle \star F^{\otimes n} \star \mathbb{k}S_n \rightarrow \mathcal{A}_n(F')$ by

$$x_i \mapsto (-1)^{\tau+\sigma} x_i u_i, \quad \mathbf{f} \mapsto \mathbf{f}, \quad \pi \mapsto \pi, \quad 1 \leq i \leq n, \quad \mathbf{f} \in F^{\otimes n}, \quad \pi \in S_n, \quad (3.1.14)$$

and extending it by universal property. We claim that Φ satisfies the defining relations of $\mathcal{A}_n(F)$.

It is easy to check that Φ satisfies the relations (3.1.3), (3.1.5) and (3.1.7). The proof that Φ satisfies (3.1.4) is just a straightforward calculation using Remark 2.2.12.

Now notice that, under tr' , the left dual of the basis $\mathcal{B}$ is $\{(bu)^\vee \mid b \in \mathcal{B}\}$. That is, denoting the dual of b by $b^{\vee'}$ (in the basis $\mathcal{B}$) with respect to tr' , we have that $b^{\vee'} = (bu)^\vee$. Thus, using primes to denote elements in $\mathcal{A}_n(F')$, by Remark 3.1.2, we have:

$$\begin{aligned} t'_{i,j} u_i &= \sum_{b \in \mathcal{B}} (-1)^{\sigma \bar{b}} b_i b_j^{\vee'} u_i \\ &= \sum_{b \in \mathcal{B}} (-1)^{\sigma \bar{b} + \bar{u}(\bar{b} + \sigma)} b_i u_i b_j^{\vee'} \\ &= \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b} + \sigma + \sigma \tau} (bu)_i (bu)_j^\vee \\ &= (-1)^{\tau + \sigma} \sum_{b \in \mathcal{B}} (-1)^{\tau \overline{(bu)}} (bu)_i (bu)_j^\vee = (-1)^{\tau + \sigma} t_{i,j}. \end{aligned}$$

A straightforward calculation shows that Φ satisfies (3.1.6).

Therefore Φ defines an algebra homomorphism from $\mathcal{A}_n(F)$ to $\mathcal{A}_n(F')$ and it is easy to check that it preserves the $\mathbb{Z} \times \mathbb{Z}_2$ -grading. Since u is invertible, Φ is clearly an isomorphism. $\square$

3.2 Examples

We now give examples that have already appeared in the literature (with $F = \mathbb{k}$, $\mathbb{k}G$, or $\mathcal{C}l$), however we admit different $\mathbb{Z}_2$ -gradings and obtain either different presentation or a more general construction.

We also give an example (where $F = D$ the dual algebra) that shows to be interesting by its similarities with the *affine Sergeev algebra*.

Example 3.2.1 (Affine wreath Hecke algebra). Let G be a group with a sign $\text{sgn}: G \rightarrow \mathbb{Z}_2$. Fixing $\tau \in \mathbb{Z}_2$ and $\tilde{g} = g_\tau \in \text{sgn}^{-1}(\tau)$, let $F = \mathbb{k}G$ be the Frobenius algebra as in example 2.2.16. Consider then $\mathcal{A}_n(F)$. Notice that, for a trivial sign and $\tilde{g} = 1$, we have that

$\mathcal{A}_n(\mathbb{k}G)$ is the *wreath Hecke algebra* introduced in [WW08]. Using non-trivial sign and other parity of $\tilde{g}$, we obtain a generalization of such algebra.

More specifically, $\mathcal{A}_n(F)$ is the unital associative $\mathbb{k}$ -algebra generated by $x_1, \dots, x_n$ (of parity τ), $g_i \in (\mathbb{k}G)^{\otimes n}$ (of parity $\bar{g}_i = \bar{g} = \text{sgn}(g)$), for $1 \leq i \leq n$, and even elements in S_n , under the relation:

$$\begin{aligned} x_i x_j &= (-1)^\tau x_j x_i & 1 \leq i, j \leq n, i \neq j, \\ g_j x_i &= (-1)^{\tau \bar{g}} x_i g_j, \quad g_i x_i = (-1)^{\bar{g}} x_i (\tilde{g}^{-1} g \tilde{g})_i & 1 \leq i, j \leq n, i \neq j, g \in G \\ s_i x_j &= x_j s_i & 1 \leq i \leq n-1, 1 \leq j \leq n, j \neq i, i+1, \\ s_i x_i &= x_{i+1} s_i - \tilde{g}_{i+1} \sum_{g \in G} g_i g_{i+1}^{-1}, & 1 \leq i \leq n-1, \\ g_i h_j &= (-1)^{\bar{h} \bar{g}} h_j g_i & 1 \leq i, j \leq n, i \neq j, g, h \in G, \\ \pi g_i &= g_{\pi(i)} \pi & 1 \leq i \leq n, g \in G, \pi \in S_n. \end{aligned}$$

Furthermore, by taking $G = \{1\}$, we have that $\mathcal{A}_n(\mathbb{k}G) = \mathcal{A}_n(\mathbb{k})$ is the *degenerated affine Hecke algebra* (see [Kle05, § 3.1]), i.e, the unital associative $\mathbb{k}$ -algebra, concentrated in even parity, generated by $\mathbb{k}[x_1, \dots, x_n]$ and S_n such that

$$s_i x_j = x_j s_i \quad \text{and} \quad s_i x_{i+1} - x_i s_i = 1 = x_{i+1} s_i - s_i x_i,$$

for all $1 \leq i \leq n-1, 1 \leq j \leq n$ such that $|i-j| > 2$.

Example 3.2.2 (Affine Sergeev algebra). Consider the Clifford superalgebra $\mathcal{Cl}_\tau$ generated by a single odd generator c satisfying $c^2 = 1$ and with trace map tr of parity τ as in example 2.2.17.

For $\tau = 0$, we have that $\mathcal{A}_n(\mathcal{Cl}_0)$ is isomorphic to the degenerate affine Sergeev algebra introduced in [Naz97, §3]. Now consider $\mathcal{Cl}_1$. Then its Nakayama automorphism is the identity map and, if we choose $B = \{1, c\}$, we have $1^\vee = c$ and $c^\vee = 1$. Thus,

$$t_{i,j} = 1_i 1_j^\vee - c_i c_j^\vee = c_j - c_i.$$

By Lemma 3.1.3, we have that $\mathcal{A}_n(\mathcal{Cl}_1)$ is also isomorphic to the degenerate affine Sergeev algebra, but we came up with a different presentation: as the free associative unital algebra generated by $x_1, \dots, x_n, c_1, \dots, c_n$ and $\pi \in S_n$, modulo the ideal generated by the following relations:

$$\begin{aligned} c_i^2 &= 1, & 1 \leq i \leq n \\ x_i x_j &= -x_j x_i & 1 \leq i, j \leq n, i \neq j, \\ c_j x_i &= -x_i c_j & 1 \leq i, j \leq n \\ s_i x_j &= x_j s_i & 1 \leq i \leq n-1, 1 \leq j \leq n, j \neq i, i+1, \\ s_i x_i &= x_{i+1} s_i + c_i - c_{i+1}, & 1 \leq i \leq n-1, \\ c_i c_j &= -c_j c_i & 1 \leq i, j \leq n, i \neq j, \\ \pi c_i &= c_{\pi(i)} \pi & 1 \leq i \leq n, \pi \in S_n. \end{aligned}$$

Example 3.2.3 (Affine Hecke-Grassmann algebra). Let $F = D$ be the Dual algebra as in example 2.2.19, i.e., the superalgebra generated by 1 and the odd element u , such that $u^2 = 0$.

Its Nakayama automorphism is the identity map and, choosing $B = \{1, u\}$ as its basis, we have $1^\vee = u$ and $u^\vee = 1$. We define the *affine Hecke-Grassmann algebra* to be $\mathcal{A}_n(D)$. It is generated by odd elements $x_1, \dots, x_n, u_1, \dots, u_n$ and even element $\pi \in S_n$, satisfying the following relations:

$$\begin{aligned} u_i^2 &= 0, & 1 \leq i \leq n \\ x_i x_j &= -x_j x_i & 1 \leq i, j \leq n, i \neq j, \\ u_j x_i &= -x_i u_j & 1 \leq i, j \leq n \\ s_i x_j &= x_j s_i & 1 \leq i \leq n-1, 1 \leq j \leq n, j \neq i, i+1, \\ s_i x_i &= x_{i+1} s_i + u_i - u_{i+1}, & 1 \leq i \leq n-1, \\ u_i u_j &= -u_j u_i & 1 \leq i, j \leq n, i \neq j, \\ \pi u_i &= u_{\pi(i)} \pi & 1 \leq i \leq n, \pi \in S_n. \end{aligned}$$

Now we point out some properties that might be interesting for further investigations.

First, notice that $D^{\otimes n} \subseteq \mathcal{A}_n(D)$ is isomorphic to $\bigwedge(\mathbb{k}^n)$, the *exterior algebra* of an n -dimensional $\mathbb{k}$ -vector space.

Furthermore, the subalgebra of $\mathcal{A}_n(D)$ generated by

$$\tilde{\partial}_i = (-1)^i (u_i s_i + s_i u_i), \quad 1 \leq i \leq n-1,$$

is a homomorphic image of oN_n , the *odd nilCoxeter algebra* (of type A and degree n), introduced in [EKL14, §2.2]. The algebra oN_n is the associative unital algebra generated by odd elements $\tilde{\partial}_1, \dots, \tilde{\partial}_n$ under the relations:

$$\tilde{\partial}_i^2 = 0, \quad \tilde{\partial}_i \tilde{\partial}_j = -\tilde{\partial}_j \tilde{\partial}_i, \quad \tilde{\partial}_k \tilde{\partial}_{k+1} \tilde{\partial}_k = \tilde{\partial}_{k+1} \tilde{\partial}_k \tilde{\partial}_{k+1},$$

for all $1 \leq i, j, k \leq n$ with $|i - j| \geq 2$ and $k \neq n$. Indeed:

For all $1 \leq i \leq n$, we have that

$$\begin{aligned} ((-1)^i (u_i s_i + s_i u_i))^2 &= u_i s_i u_i s_i + u_i s_i s_i u_i + s_i u_i u_i s_i + s_i u_i s_i u_i \\ &= u_i u_{i+1} + u_{i+1} u_i = 0. \end{aligned}$$

It follows from the commutativity of elements in S_n and anti-commutativity of u_i 's that $\tilde{\partial}_i \tilde{\partial}_j = -\tilde{\partial}_j \tilde{\partial}_i$, for all $1 \leq i, j \leq n$, such that $|i - j| \geq 2$. Finally, for $1 \leq i \leq n-1$, we have:

$$\begin{aligned} \tilde{\partial}_i \tilde{\partial}_{i+1} \tilde{\partial}_i &= (-1)^{i+1} (u_i s_i + s_i u_i) (u_{i+1} s_{i+1} + s_{i+1} u_{i+1}) (u_i s_i + s_i u_i) \\ &= (-1)^{i+1} (u_i s_i u_{i+1} s_{i+1} + u_i s_i s_{i+1} u_{i+1} + s_i u_i u_{i+1} s_{i+1} + s_i u_i s_{i+1} u_{i+1}) (u_i s_i + s_i u_i) \\ &= (-1)^{i+1} (u_i u_{i+2} s_i s_{i+1} + s_i u_i u_{i+1} s_{i+1} + s_i u_i s_{i+1} u_{i+1}) (u_i s_i + s_i u_i) \\ &= (-1)^{i+1} (u_i u_{i+2} u_{i+1} s_i s_{i+1} s_i + u_{i+1} u_i u_{i+2} s_i s_{i+1} s_i) \\ &= (-1)^i (u_i u_{i+1} u_{i+2} s_i s_{i+1} s_i + u_i u_{i+1} u_{i+2} s_i s_{i+1} s_i) \end{aligned}$$

An analogous calculation gives us that

$$\tilde{\partial}_{i+1} \tilde{\partial}_i \tilde{\partial}_{i+1} = (-1)^i (u_i u_{i+1} u_{i+2} s_{i+1} s_i s_{i+1} + u_i u_{i+1} u_{i+2} s_{i+1} s_i s_{i+1}).$$

But, since $(s_i s_{i+1})^3 = 1$, we conclude that $\tilde{\partial}_i \tilde{\partial}_{i+1} \tilde{\partial}_i = \tilde{\partial}_{i+1} \tilde{\partial}_i \tilde{\partial}_{i+1}$.

And finally, $\mathcal{A}_n(D)$ has almost the same relations as $\mathcal{A}_n(\mathcal{Cl}_1) \simeq \mathcal{A}_n(\mathcal{Cl}_0)$, but with nilpotents elements. Such similarity suggests that one can extend the approaches of affine Sergeev algebras to $\mathcal{A}_n(D)$ and find analogous results.

As an example of such approach, we point to [BK01]. In this paper, Brundan and Kleshchev showed that the integrable highest weight $\mathfrak{g}$ -module of highest weight λ , where $\mathfrak{g}$ is the twisted affine Kac-Moody algebra of type $A_{2\ell}^{(2)}$, is isomorphic to the Grothendieck group of modules over the cyclotomic quotients of $\mathcal{A}_n(\mathcal{Cl}_0)$ (given by λ), for all $n \in \mathbb{N}$. The notion of cyclotomic quotient $\mathcal{A}_n^{\mathbf{C}}(D)$ for $\mathcal{A}_n(D)$ is defined in this work (see Section 6) and can be easily restricted for the case where $\mathbf{C}$ is tuples of integers. Therefore, the approach of Brundan and Kleshchev could be applied in the investigation of $\mathcal{A}_n^{\mathbf{C}}(D)$ -mod in order to find similar result.

3.3 Relations

In this subsection we deduce some relations that will be useful.

For $k \in \mathbb{N}^+$, $i, j \in \{1, \dots, n\}$, $i \neq j$, define

$$t_{i,j}^{(k)} \doteq \sum_{b \in \mathcal{B}} (-1)^{\tau k \bar{b}} b_i p_k(x_i, x_j) b_j^\vee, \quad (3.3.1)$$

where p_k is the polynomial defined by

$$p_k(x, y) = \sum_{l=0}^{k-1} (-1)^{\tau l} y^{k-1-l} x^l.$$

Hence we have $|t_{i,j}^{(k)}| = k\delta$, $\overline{t_{i,j}^{(k)}} = k\tau$ and $t_{i,j}^{(1)} = t_{i,j}$.

Lemma 3.3.1. *For all $i \in \{1, \dots, n-1\}$ and $k \in \mathbb{N}_+$, we have:*

$$s_i x_i^k = x_{i+1}^k s_i - t_{i,i+1}^{(k)}, \quad (3.3.2)$$

$$s_i x_{i+1}^k = x_i^k s_i + t_{i+1,i}^{(k)}. \quad (3.3.3)$$

Proof. Notice that, by (3.1.4), for all $i, j \in \{1, \dots, n\}$ with $i \neq j$, we have

$$x_j b_i = (-1)^{\tau \bar{b}} b_i x_j \quad \text{and} \quad b_j^\vee x_i = (-1)^{\tau \bar{b} + \tau} x_i b_j^\vee.$$

It is clear that, for $k = 1$, the relations (3.3.2) and (3.3.3) are exactly the relations (3.1.6) and (3.1.12). By induction, suppose it holds for some k . Then we have

$$\begin{aligned} s_i x_i^{k+1} &= (s_i x_i^k) x_i \\ &= (x_{i+1}^k s_i - t_{i,i+1}^{(k)}) x_i = x_{i+1}^{k+1} s_i - (x_{i+1}^k t_{i,i+1} + t_{i,i+1}^{(k)} x_i) \\ &= x_{i+1}^{k+1} s_i - \sum_{b \in \mathcal{B}} (-1)^{\tau(k+1)\bar{b}} b_i \left(x_{i+1}^k + (-1)^\tau p_k(x_i, x_{i+1}) x_i \right) b_{i+1}^\vee \\ &= x_{i+1}^{k+1} s_i - \sum_{b \in \mathcal{B}} (-1)^{\tau(k+1)\bar{b}} b_i \left(x_{i+1}^k + \sum_{l=0}^{k-1} (-1)^{\tau(l+1)} x_{i+1}^{k-1-l} x_i^{l+1} \right) b_{i+1}^\vee \\ &= x_{i+1}^{k+1} s_i - t_{i,i+1}^{(k+1)}, \end{aligned}$$

and thus (3.3.2) is satisfied.

The proof is analogous for (3.3.3). $\square$

Notice that, by replacing $i + 1$ by any $j \neq i$ in the proof of Lemma 3.3.1, it follows by induction on k that:

$$t_{i,j}^{(k+1)} = x_j^k t_{i,j} + t_{i,j}^{(k)} x_i, \quad 1 \leq i, j \leq n, i \neq j, k \in \mathbb{N}^+. \quad (3.3.4)$$

In particular, we obtain that

$$t_{i,j}^{(k)} = \sum_{l=0}^{k-1} x_j^{k-1-l} t_{i,j} x_i^l, \quad 1 \leq i, j \leq n, i \neq j, k \in \mathbb{N}^+. \quad (3.3.5)$$

The following lemma shows how $t_{i,j}^{(k)}$ commutes with elements of $F^{\otimes n}$:

Lemma 3.3.2. *Let $f \in F$, $i, j, l \in \{1, \dots, n\}$ with $i \neq j$ and $l \neq i, j$, and let $k \in \mathbb{N}_+$. Then the following relations holds:*

$$f t_{i,j}^{(k)} = (-1)^{\tau k \bar{f}} t_{i,j}^{(k)} f_l, \quad (3.3.6)$$

$$f_i t_{i,j}^{(k)} = (-1)^{\tau k \bar{f}} t_{i,j}^{(k)} f_j, \quad (3.3.7)$$

$$f_j t_{i,j}^{(k)} = (-1)^{\tau k \bar{f}} t_{i,j}^{(k)} \psi^k(f)_i, \quad (3.3.8)$$

Furthermore, summing up those relations, we have, for all $\mathbf{f} \in F^{\otimes n}$, $i, j \in \{1, \dots, n\}$ with $i \neq j$,

$$\mathbf{f} t_{i,j}^{(k)} = (-1)^{\tau k \bar{\mathbf{f}}} t_{i,j}^{(k)} \psi_i^k(s_{i,j} \mathbf{f}) = (-1)^{\tau k \bar{\mathbf{f}}} t_{i,j}^{(k)} s_{i,j} \left(\psi_j^k(\mathbf{f}) \right), \quad (3.3.9)$$

where $s_{i,j} \in S_n$ is the transposition of i and j .

Proof. First notice that, by equation (3.3.5), it is enough to show the relations for $k = 1$. In fact, suppose equation (3.3.9) is satisfied for $k = 1$, then it follows

$$\begin{aligned} \mathbf{f} t_{i,j}^{(k)} &\stackrel{(3.3.5)}{=} \sum_{\ell=0}^{k-1} \mathbf{f} x_j^{k-1-\ell} t_{i,j} x_i^\ell \stackrel{(3.1.4)}{=} \sum_{\ell=0}^{k-1} (-1)^{\tau(k-1-\ell)\bar{\mathbf{f}}} x_j^{k-1-\ell} \psi_j^{k-1-\ell}(\mathbf{f}) t_{i,j} x_i^\ell \\ &\stackrel{(3.3.9)}{=} \sum_{\ell=0}^{k-1} (-1)^{\tau(k-\ell)\bar{\mathbf{f}}} x_j^{k-1-\ell} t_{i,j} s_{i,j} \left(\psi_j^{k-\ell}(\mathbf{f}) \right) x_i^\ell = \sum_{\ell=0}^{k-1} (-1)^{\tau(k-\ell)\bar{\mathbf{f}}} x_j^{k-1-\ell} t_{i,j} \psi_i^{k-\ell}(s_{i,j} \mathbf{f}) x_i^\ell \\ &\stackrel{(3.1.4)}{=} \sum_{\ell=0}^{k-1} (-1)^{\tau k \bar{\mathbf{f}}} x_j^{k-1-\ell} t_{i,j} x_i^\ell \psi_i^k(s_{i,j} \mathbf{f}) \stackrel{(3.3.5)}{=} (-1)^{\tau k \bar{\mathbf{f}}} t_{i,j} s_{i,j} \left(\psi_j^k(\mathbf{f}) \right), \end{aligned}$$

for all $\mathbf{f} \in F^{\otimes n}$. We will prove the relations assuming that f is homogeneous; then the general result will follow by linearity.

We have:

$$\begin{aligned} f t_{i,j}^{(1)} &= f t_{i,j} = \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b}} f_l b_i b_j^\vee = \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b} + \tau \bar{f} \bar{b}} b_i f_l b_j^\vee \\ &= \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b} + \tau \bar{f} \bar{b} + \tau \bar{f}(\bar{b} + \tau)} b_i b_j^\vee f_l = (-1)^{\tau \bar{f}} t_{i,j} f_l. \end{aligned}$$

Thus (3.3.6) is satisfied.

Notice that, given homogeneous $a \in F$ and $b \in \mathcal{B}$, we have that:

$$\bar{b} \neq \bar{a} + \tau \Rightarrow \text{tr}(ab) = 0 \quad \text{and} \quad \bar{b} \neq \bar{a} \Rightarrow \text{tr}(b^\vee a) = 0.$$

Combining these equalities with equations (2.2.7) and (2.2.8), respectively, it follows that

$$\sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b}} \text{tr}(b^\vee a) b = (-1)^{\tau \bar{a}} a = \sum_{b \in \mathcal{B}} (-1)^{\tau(\bar{b}+1)} \text{tr}(ab) b^\vee, \quad (3.3.10)$$

for all homogeneous $a \in F$. We obtain:

$$\begin{aligned} f_i t_{i,j} &= \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b}} f_i b_i b_j^\vee \stackrel{(2.2.7)}{=} \sum_{b, c \in \mathcal{B}} (-1)^{\tau \bar{b}} \text{tr}(c^\vee f b) c_i b_j^\vee \\ &= \sum_{c \in \mathcal{B}} c_i \left(\sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b}} \text{tr}(c^\vee f b) b_j^\vee \right) \stackrel{(3.3.10)}{=} \sum_{c \in \mathcal{B}} (-1)^{\tau(\bar{c}^\vee f + 1)} c_i c_j^\vee f_j \\ &= \left(\sum_{c \in \mathcal{B}} (-1)^{\tau \bar{c}} c_i c_j^\vee \right) (-1)^{\tau \bar{f}} f_j = (-1)^{\tau \bar{f}} t_{i,j} f_j. \end{aligned}$$

Thus (3.3.7) is satisfied for $k = 1$.

Finally, for the last equation, it follows:

$$\begin{aligned} f_j t_{i,j} &= \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b} + \bar{f} \bar{b}} b_i f_j b_j^\vee \stackrel{(2.2.7)}{=} \sum_{b, c \in \mathcal{B}} (-1)^{\tau \bar{b} + \bar{f} \bar{b}} b_i \text{tr}(f b^\vee c) c_j^\vee \\ &= \sum_{b, c \in \mathcal{B}} (-1)^{\tau \bar{b} + \bar{f} \bar{b} + \bar{f} \bar{b}^\vee c} \text{tr}(b^\vee c \psi(f)) b_i c_j^\vee = \sum_{b, c \in \mathcal{B}} (-1)^{\tau \bar{b} + \tau \bar{f} + \bar{f} \bar{c}} \text{tr}(b^\vee c \psi(f)) b_i c_j^\vee \\ &\stackrel{(3.3.10)}{=} \sum_{c \in \mathcal{B}} (-1)^{\tau(\bar{f} + \bar{c}) + \tau \bar{f} + \bar{f} \bar{c}} c_i \psi(f)_i c_j^\vee = \sum_{c \in \mathcal{B}} (-1)^{\tau(\bar{c} + \bar{f})} c_i c_j^\vee \psi(f)_i = (-1)^{\tau \bar{f}} t_{i,j} \psi(f)_i. \end{aligned}$$

Thus, equation (3.3.8) is satisfied for $k = 1$.

Let $\mathbf{f} = f_1 \otimes \cdots \otimes f_n \in F^{\otimes n}$. Abusing the notation we write $f_i \in F^{\otimes n}$ and $\mathbf{f} = f_1 \cdots f_n$, then it is easy to see that relations (3.3.6)–(3.3.8) show that (3.3.9) is satisfied. $\square$

Lemma 3.3.3. *For $1 \leq i, j \leq n$, $i \neq j$, we have*

$$\psi_j(t_{i,j}) = (-1)^\tau t_{j,i}. \quad (3.3.11)$$

Proof. We have

$$\begin{aligned} \psi_j^{-1}(t_{j,i}) &= \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b}} \psi^{-1}(b)_j b_i^\vee \\ &= \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b}} b_i^\vee (-1)^{\bar{b} + \bar{b} \tau} \psi^{-1}(b)_j \\ &= (-1)^\tau \sum_{b \in \mathcal{B}} (-1)^{\tau(\bar{b} + \tau)} b_i^\vee (b^\vee)_j^\vee \\ &= (-1)^\tau t_{i,j}. \end{aligned}$$

We have used the fact that the parity of $b^\vee$ is equal $\bar{b} + \tau$ in the second equality, the fact that $(-1)^{\bar{b}+\tau\bar{b}}\psi^{-1}(b) = (b^\vee)^\vee$ in the third equality, and the fact that the definition of $t_{i,j}$ is independent of the basis in the final equality (where we are summing over the basis $\{b^\vee \mid b \in \mathcal{B}\}$). $\square$

Lemma 3.3.4. *For $1 \leq i, j \leq n$, $i \neq j$, $k \in \mathbb{N}_+$, $\pi \in S_n$, and p polynomial in variables $x_1^\theta, \dots, x_n^\theta$, we have*

$$x_l t_{i,j}^{(k)} = (-1)^{\tau k} t_{i,j}^{(k)} x_l, \quad l \neq i, j, \quad (3.3.12)$$

$$x_i t_{i,j} = t_{j,i} x_i, \quad (3.3.13)$$

$$p t_{i,j} = (-1)^{\tau \theta} t_{i,j} p, \quad (3.3.14)$$

$$\pi t_{i,j} = t_{\pi(i), \pi(j)} \pi, \quad (3.3.15)$$

$$s_i t_{i,i+1}^{(k)} = t_{i+1,i}^{(k)} s_i. \quad (3.3.16)$$

Proof. It is easy to see that, for all $l \neq i, j$,

$$x_l p_k(x_i, x_j) = (-1)^{\tau(k-1)} p_k(x_i, x_j) x_l.$$

Then

$$\begin{aligned} x_l t_{i,j}^{(k)} &= \sum_{b \in \mathcal{B}} (-1)^{\tau k \bar{b}} x_l b_i p_k(x_i, x_j) b_j^\vee \\ &= \sum_{b \in \mathcal{B}} (-1)^{\tau k \bar{b} + \tau \bar{b}} b_i x_l p_k(x_i, x_j) b_j^\vee \\ &= \sum_{b \in \mathcal{B}} (-1)^{\tau k \bar{b} + \tau(\bar{b} + k - 1)} b_i p_k(x_i, x_j) x_l b_j^\vee \\ &= \sum_{b \in \mathcal{B}} (-1)^{\tau k \bar{b} + \tau(\bar{b} + k - 1 + \bar{b}^\vee)} b_i p_k(x_i, x_j) b_j^\vee x_l \\ &= (-1)^{\tau k} t_{i,j}^{(k)}, \end{aligned}$$

and relation (3.3.12) follows. To prove (3.3.13), we have

$$\begin{aligned} x_i t_{i,j} &= \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b}} x_i b_i b_j^\vee \stackrel{(3.1.4)}{=} \sum_{b \in \mathcal{B}} \psi_i^{-1}(b_i) x_i b_j^\vee \\ &= \sum_{b \in \mathcal{B}} (-1)^{\tau(\bar{b} + \tau)} \psi_i^{-1}(b_i) b_j^\vee x_i \\ &= (-1)^\tau \psi_i^{-1}(t_{i,j}) x_i \\ &\stackrel{(3.3.11)}{=} t_{j,i} x_i. \end{aligned}$$

Relation (3.3.14) follows immediately from (3.1.4), $\bar{b} + \bar{b}^\vee = \tau$ and the fact that θ is the order of the Nakayama automorphism ψ . Relation (3.1.7) implies (3.3.15). Finally, (3.3.16)

follows from

$$\begin{aligned} s_i x_i^k &= x_{i+1}^k s_i - t_{i,i+1}^{(k)} \\ s_i x_{i+1}^k &= x_i^k s_i + t_{i+1,i}^{(k)} \end{aligned}$$

multiplying the first on the left by s_i , the second on the right by s_i , and adding the resulting two equations. $\square$

Chapter 4

Structure theory

In this section we examine the structure theory of the general affine wreath product algebras. We define the deformed divided difference operators which we use to find a PBW-basis for $\mathcal{A}_n(F)$ and calculate its center. We also define Jucys-Murphy's elements (of parity τ) and the intertwining elements for $\mathcal{A}_n(F)$.

We assume $\mathbb{k}$ to be an arbitrary commutative ring of characteristic not equal to 2.

4.1 Deformed divided difference operators

Let $\mathbb{k}\langle x_1, \dots, x_n \rangle$ be the free unital associative algebra generated by $x_1, \dots, x_n$, and define

$$P_n = \frac{\mathbb{k}\langle x_1, \dots, x_n \rangle}{I}, \quad (4.1.1)$$

where I is the two sided ideal generated by $x_j x_k - (-1)^\tau x_k x_j$ for all $1 \leq j, k \leq n$, with $j \neq k$, i.e., P_n is the super-commutative polynomial algebra on the variables $x_1, \dots, x_n$, of parity $\tau \in \mathbb{Z}_2$.

Let $P_n(F)$ be the free product of $\mathbb{k}$ -algebras $P_n \star F^{\otimes n}$ modulo the relation (3.1.4).

We have a natural action of S_n on $F^{\otimes n}$ by super-commuting the coordinates, which can be extended to $P_n(F)$, by commuting the variables, i.e.,

$${}^\pi x_j = x_{\pi(j)}, \quad {}^\pi(ab) = {}^\pi a {}^\pi b, \quad \text{and} \quad {}^\pi(a+b) = {}^\pi a + {}^\pi b, \quad (4.1.2)$$

for all $1 \leq j \leq n$, $\pi \in S_n$, and $a, b \in P_n(F)$.

Consider $\mathbb{k}[x]$ acting on F from the right by:

$$f \cdot x = (-1)^{\tau \bar{f}} \psi(f), \quad \forall f \in F,$$

and let $\mathbb{k}[x] \ltimes F$ be the smash product of $\mathbb{k}[x]$ and F with respect to such action (see 2.3), i.e, the algebra with $\mathbb{k}[x] \otimes F$ as underlying $\mathbb{k}$ -module and multiplication given by

$$fx = (-1)^{\tau \bar{f}} x \psi(f),$$

for all homogeneous $f \in F$.

Consider the $\mathbb{Z} \times \mathbb{Z}_2$ -grading in $\mathbb{k}[x] \ltimes F$ where $\bar{x} = \tau$ and $|x| = \delta$. We have a natural

isomorphism of algebras

$$P_n(F) \simeq (\mathbb{k}[x] \ltimes F)^{\otimes n}, \quad (4.1.3)$$

and we will often identify them. The parity and degrees of the x_i are τ and δ , respectively. We define the *polynomial degree* of an element of $P_n(F)$ to be its total degree as a polynomial in the x_i .

For $1 \leq i \leq n-1$, we define a skew derivation $\Delta_i: P_n(F) \rightarrow P_n(F)$ as in [Sav20, Section 4.1], the linear map that satisfy

$$\Delta_i(F^{\otimes n}) = 0,$$

$$\Delta_i(x_i) = t_{i,i+1}, \quad \Delta_i(x_{i+1}) = -t_{i+1,i}, \quad \Delta_i(x_j) = 0, \quad j \neq i, i+1, \quad (4.1.4)$$

and, for all $a_1, a_2 \in P_n(F)$,

$$\Delta_i(a_1 a_2) = \Delta_i(a_1) a_2 + {}^{s_i} a_1 \Delta_i(a_2).$$

In particular, it follows

$$\Delta_i(\mathbf{f}a) = {}^{s_i} \mathbf{f} \Delta_i(a), \quad \mathbf{f} \in F^{\otimes n}, \quad a \in P_n(F). \quad (4.1.5)$$

We claim that these operators are well defined in $P_n(F)$. In fact it follows from relations (3.3.12), (3.3.13) and Lemma 3.3.2, as we now check.

Let $1 \leq j, l \leq n$ with $j \neq l$ and $j, l \notin \{i, i+1\}$, then it follows directly by definition that $\Delta_i(x_j x_l) = 0$. We also have that

$$\begin{aligned} \Delta_i(x_i x_j - (-1)^\tau x_j x_i) &= \Delta_i(x_i) x_j - (-1)^\tau x_j \Delta_i(x_i) \\ &= t_{i,i+1} x_j - (-1)^\tau x_j t_{i,i+1} \stackrel{(3.3.12)}{=} t_{i,i+1} x_j - t_{i,i+1} x_j = 0. \end{aligned}$$

Analogously, $\Delta_i(x_{i+1} x_j - (-1)^\tau x_j x_{i+1}) = 0$. Finally, we have that

$$\begin{aligned} \Delta_i(x_i x_{i+1}) &= \Delta_i(x_i) x_{i+1} + {}^{s_i} x_i \Delta_i(x_{i+1}) \\ &= t_{i,i+1} x_{i+1} - x_{i+1} t_{i+1,i} \stackrel{(3.3.13)}{=} t_{i,i+1} x_{i+1} - t_{i,i+1} x_{i+1} = 0, \end{aligned}$$

and analogously, we have that $\Delta_i(x_{i+1} x_i) = 0$. Thus $\Delta_i(x_i x_{i+1} - (-1)^\tau x_{i+1} x_i) = 0$. And Δ_i satisfies relation (3.1.3). Now, for all $\mathbf{f} \in F^{\otimes n}$ and $1 \leq j \leq n$ we have

$$\begin{aligned} \Delta_i(\mathbf{f} x_j) &= {}^{s_i} \mathbf{f} \Delta_i(x_j) = \begin{cases} {}^{s_i} \mathbf{f} t_{i,i+1}, & \text{if } j = i \\ -{}^{s_i} \mathbf{f} t_{i+1,i}, & \text{if } j = i+1 \\ 0, & \text{if } j \notin \{i, i+1\} \end{cases} \\ &\stackrel{3.3.2}{=} \begin{cases} (-1)^{\tau \bar{\mathbf{f}}} t_{i,i+1} \psi_i(\mathbf{f}), & \text{if } j = i \\ -(-1)^{\tau \bar{\mathbf{f}}} t_{i+1,i} \psi_{i+1}(\mathbf{f}), & \text{if } j = i+1 \\ 0, & \text{if } j \notin \{i, i+1\} \end{cases} \\ &= (-1)^{\tau \bar{\mathbf{f}}} \Delta_i(x_j) \psi_j(\mathbf{f}) = \Delta_i((-1)^{\tau \bar{\mathbf{f}}} x_j \psi_j(\mathbf{f})). \end{aligned}$$

Thus Δ_i satisfies (3.1.4). And in conclusion, Δ_i is well defined.

We call such operators *deformed divided difference operators*.

Lemma 4.1.1. *For all $a \in P_n(F)$ and $1 \leq i \leq n-1$, in $\mathcal{A}_n(F)$ we have*

$$s_i a = {}^{s_i} a s_i - \Delta_i(a).$$

Proof. The result for $f \in F$ and $x_1, \dots, x_n$ follows immediately from (3.1.5)–(3.1.7) and (3.1.12). And can be extended to the general case by induction. $\square$

It is easy to see that, by Lemmas 4.1.1 and 3.3.1, for $1 \leq i \leq n-1$ and $k \in \mathbb{N}_+$, we have

$$\Delta_i(x_i^k) = t_{i,i+1}^{(k)} \quad (4.1.6)$$

$$\Delta_i(x_{i+1}^k) = -t_{i+1,i}^{(k)} \quad (4.1.7)$$

In addition, we have the following result.

Proposition 4.1.2. *We have*

$$\Delta_i({}^{s_j} a) = {}^{s_j} \Delta_i(a), \quad 1 \leq i, j \leq n-1, |i-j| > 1, a \in P_n(F), \quad (4.1.8)$$

$$\Delta_i({}^{s_i} a) = -{}^{s_i} \Delta_i(a), \quad 1 \leq i \leq n-1, a \in P_n(F), \quad (4.1.9)$$

$$\Delta_i({}^{s_{i+1}s_i} a) = {}^{s_{i+1}s_i} \Delta_{i+1}(a), \quad 1 \leq i \leq n-2, a \in P_n(F), \quad (4.1.10)$$

$$\Delta_i^2 = 0, \quad 1 \leq i \leq n-1. \quad (4.1.11)$$

$$\Delta_i \Delta_j = \Delta_j \Delta_i, \quad 1 \leq i, j \leq n-1, |i-j| > 1, \quad (4.1.12)$$

$$\Delta_i \Delta_{i+1} \Delta_i = \Delta_{i+1} \Delta_i \Delta_{i+1}, \quad 1 \leq i \leq n-1. \quad (4.1.13)$$

Proof. We prove them by induction on the degrees of $a \in P_n(F)$.

It is easy to see that all relations are satisfied for elements in $F^{\otimes n}$. So we just need to prove the base of induction for x_l , with $1 \leq l \leq n$.

For $1 \leq i, j \leq n-1$, with $|j-i| > 1$, we have:

$$\Delta_i({}^{s_j} x_l) = \begin{cases} 0, & l \notin \{i, i+1\} \\ t_{i,i+1}, & l = i \\ -t_{i+1,i}, & l = i+1 \end{cases} = {}^{s_j} \Delta_i(x_l),$$

for all $1 \leq j \leq n$. Suppose that (4.1.8) is satisfied for $a, b \in P_n(F)$, then ab satisfies (4.1.8) as well:

$$\begin{aligned} \Delta_i({}^{s_j} ab) &= \Delta_i({}^{s_j} a) {}^{s_j} b + {}^{s_i s_j} a \Delta_i({}^{s_j} b) \\ &= {}^{s_j} \Delta_i(a) {}^{s_j} b + {}^{s_j s_i} a {}^{s_j} \Delta_i(b) = {}^{s_j} \Delta_i(ab). \end{aligned}$$

An analogous calculation, together with the braid relation on S_n , proves (4.1.9) and (4.1.10).

Now we prove the remaining relations. Since the deformed divided difference operators decrease the polynomial degree by one, any composition of two operators vanishes when applied on a polynomial of degree at most one. Thus (4.1.11)–(4.1.13) are satisfied for all $x_l \in P_n(F)$.

Now suppose that the relations (4.1.11)-(4.1.13) are true for $a \in P_n(F)$, then it follows:

$$\begin{aligned}\Delta_i \Delta_i(x_l a) &= \Delta_i(\Delta_i(x_l) a + {}^{s_i}x_l \Delta_i(a)) \\ &= \Delta_i \Delta_i(x_l) a + {}^{s_i} \Delta_i(x_l) \Delta_i(a) + \Delta_i({}^{s_i}x_l) \Delta_i(a) + {}^{s_i s_i}x_l \Delta_i \Delta_i(a) \\ &= ({}^{s_i} \Delta_i(x_l) + \Delta_i({}^{s_i}x_l)) \Delta_i(a) = 0,\end{aligned}$$

where we used the induction hypothesis, in the third equality, and, in the last one, the relation (4.1.9). Thus (4.1.11) is proved. The inductive passages for relations (4.1.12) and (4.1.13) are analogous, using instead relations (4.1.8) and (4.1.10) (together with the braid relation on S_n), respectively. $\square$

We define the τ -divided difference operator ∂_i , for $1 \leq i \leq n-1$, to be the linear operators in $\mathbb{k}\langle x_1, \dots, x_n \rangle$ satisfying

$$\partial_i(1) = 0, \quad \partial_i(x_j) = \begin{cases} 1, & \text{if } j = i; \\ (-1)^{\tau+1}, & \text{if } j = i+1; \\ 0, & \text{remaining cases,} \end{cases} \quad (4.1.14)$$

and, for all homogeneous $a, b \in \mathbb{k}\langle x_1, \dots, x_n \rangle$,

$$\partial_i(ab) = \partial_i(a)b + (-1)^{\tau \bar{a}} {}^{s_i}a \partial_i(b). \quad (4.1.15)$$

Some straightforward calculations show us that, for all $1 \leq i \leq n-1$,

$$\partial_i(x_j x_k - (-1)^\tau x_k x_j) = 0,$$

for all $1 \leq j, k \leq n$, with $j \neq k$. Therefore the τ -divided difference operators ∂_i 's are well defined as operators in P_n .

Remark 4.1.3 (Equivalent definitions of divided difference operators). Notice that the definition of ∂_i 's above unifies the definition for Demazure operators, given by

$$D_i: p \mapsto \frac{p - {}^{s_i}p}{x_i - x_{i+1}}, \quad \forall p \in \mathbb{k}[x_1, \dots, x_n], \quad 1 \leq i \leq n-1,$$

and its *odd* version, given in [EKL14, 2.2.1].

When $\tau = 0$, the equivalence between both definitions easily follows from the facts:

$$D_i(x_i) = 1 = -D_i(x_{i+1}), \quad D_i(x_j) = 0, \quad \text{and}$$

$$D_i(pq) = \frac{pq - {}^{s_i}pq}{x_i - x_{i+1}} = \frac{p - {}^{s_i}p}{x_i - x_{i+1}} q + {}^{s_i}p \frac{q - {}^{s_i}q}{x_i - x_{i+1}} = D_i(p)q + {}^{s_i}p D_i(q),$$

for all $p, q \in \mathbb{k}[x_1, \dots, x_n]$, $1 \leq i, j \leq n$, $i < n-1$, $i \neq j$.

In the case $\tau = 1$, we have that P_n is the odd polynomial ring on the skew-commutative variables $x_1, \dots, x_n$ as defined in [EKL14, 2.1.1]. In the paper, the authors defined the divided difference operator by considering a signed action of S_n on P_n (given by $\pi \cdot x_i = (-1)^{\text{sgn}(\pi)} x_{\pi(i)}$, for all $\pi \in S_n$ and $1 \leq i \leq n$), and declaring

$$\partial_i(ab) = \partial_i(a)b - s_i \cdot a \partial_i(b), \quad \forall a, b \in P_n, \quad (4.1.16)$$

in the place of (4.1.15). But an easy argument can show that $s_i \cdot a = (-1)^{\tau \bar{a}} s_i a$, for all $a \in P_n$. Therefore (4.1.16) is in fact (4.1.15), and hence the equivalence between definitions.

Now we can state the following:

Lemma 4.1.4. *Suppose $\psi = \text{Id}_F$, i.e., the Frobenius algebra F is symmetric. Then, for all $1 \leq i \leq n-1$,*

$$\Delta_i(\mathbf{f}p) = {}^{s_i}\mathbf{f}\partial_i(p), \quad \forall \mathbf{f} \in F^{\otimes n}, p \in P_n.$$

Proof. First, notice that, since $\psi = \text{Id}_F$, it follows from (3.3.11), (3.3.12) and (3.3.13), that $x_l t_{i,j} = (-1)^{\tau} t_{i,j} x_l$, for all $1 \leq l \leq n$. Thus, together with Lemma 3.3.2, it follows that

$$a t_{i,j} = (-1)^{\tau a} t_{i,j} a, \quad (4.1.17)$$

for all homogeneous $a \in P_n(F)$ with polynomial degree at most one.

Now we prove the lemma by induction on the polynomial degree. Clearly we have

$$\Delta_i(\mathbf{f}) = 0 = {}^{s_i}\mathbf{f}\partial_i(0),$$

$$\Delta_i(x_l) = \begin{cases} 0, & l \notin \{i, i+1\} \\ t_{i,i+1}, & l = i \\ -t_{i+1,i} \stackrel{(3.3.11)}{=} (-1)^{\tau+1} t_{i,i+1}, & l = i+1 \end{cases} = t_{i,i+1} \partial_i(x_l),$$

$$\text{and } \Delta_i(\mathbf{f}x_l) \stackrel{(4.1.5)}{=} {}^{s_i}\mathbf{f}\Delta_i(x_l) = {}^{s_i}\mathbf{f}t_{i,i+1} \partial_i(x_l),$$

for all $\mathbf{f} \in F^{\otimes n}$ and $1 \leq l \leq n$. Finally, supposing the lemma is true for $\mathbf{f}p, q$ with $\mathbf{f} \in F^{\otimes n}$ and homogeneous $p, q \in P_n$, it follows:

$$\begin{aligned} \Delta_i(\mathbf{f}pq) &= \Delta_i(\mathbf{f}p)q + {}^{s_i}\mathbf{f}p\Delta_i(q) \\ &= {}^{s_i}\mathbf{f}t_{i,i+1} \partial_i(p)q + {}^{s_i}\mathbf{f}p {}^{s_i}\partial_i(q) \\ &\stackrel{(4.1.17)}{=} {}^{s_i}\mathbf{f}t_{i,i+1} \partial_i(p)q + (-1)^{\tau \bar{p}} {}^{s_i}\mathbf{f}t_{i,i+1} {}^{s_i}p \partial_i(q) \\ &= {}^{s_i}\mathbf{f}t_{i,i+1} (\partial_i(p)q + (-1)^{\tau \bar{p}} {}^{s_i}p \partial_i(q)) = {}^{s_i}\mathbf{f}t_{i,i+1} \partial_i(pq). \end{aligned}$$

This complete the proof. $\square$

Remark 4.1.5 (NilWreath algebra). An easy calculation shows that, as operator in $P_n(F)$, we have the following relations:

$$\Delta_i x_i = x_{i+1} \Delta_i + t_{i,i+1} \text{Id}, \quad \Delta_i x_{i+1} = x_i \Delta_i - t_{i+1,i} \text{Id},$$

for all $1 \leq i \leq n-1$.

Such relations, together with (4.1.5) and Proposition 4.1.2, suggest a more general definition for *nilHecke algebras* related to a Frobenius algebra F . Then one could define the *nilWreath algebra* $\mathcal{NW}_n(F)$ to be the associative unital algebra generated by $P_n(F)$ and even elements $\omega_1, \dots, \omega_{n-1}$ under the relations

$$\omega_i^2 = 0, \quad \omega_i \omega_j = \omega_j \omega_i, \quad \omega_k \omega_{k+1} \omega_k = \omega_{k+1} \omega_k \omega_{k+1},$$

$$\omega_i \mathbf{f} = {}^{s_i}\mathbf{f} \omega_i, \quad \omega_i x_j = x_j \omega_i, \quad \omega_i x_i = x_{i+1} \omega_i + t_{i,i+1} \text{Id}, \quad \omega_i x_{i+1} = x_i \omega_i - t_{i+1,i} \text{Id},$$

for all $1 \leq i, j, k \leq n$, $k < n-1$, $|i-j| > 1$, and $\mathbf{f} \in F^{\otimes n}$. Then, $P_n(F)$ has a natural structure of $\mathcal{NW}_n(F)$ -module, by letting ω_i acting as Δ_i .

Notice that $\mathcal{NW}_n(\mathbb{k})$ is isomorphic to the standard *nilHecke algebra* (see [Lau10]) and that $x_1, \dots, x_n, \partial_i := t_{i,i+1}^{-1} \Delta_i$, for $1 \leq i \leq n-1$, generate a homomorphic image of the *odd nilHecke algebra* (see [EKL14]) in $\mathcal{NW}_n(\mathcal{Cl}_1)$. Then, it would be interesting to investigate *nilWreath algebras*, together with analogous concepts of symmetric and Schubert polynomials for $P_n(F)$, with the aim of replicating roles played by the traditional nilHecke algebra in categorifications.

4.2 Basis theorem

In this section we determine basis for $\mathcal{A}_n(F)$. Our approach is analogous to [Sav20, § 4.2], which is pretty similar to the approach done in [KM19, § 3].

Let $\leq$ denote the strong Bruhat order on S_n . Then, for $\pi, \sigma \in S_n$, we have that

$$\sigma \leq \pi \Leftrightarrow \sigma = s_{j_1} \cdots s_{j_l},$$

where $\pi = s_{i_1} \cdots s_{i_k}$ is a reduced expression and $(j_1, \dots, j_l)$ is a substring of $(i_1, \dots, i_k)$. If $\sigma \leq \pi$ and $\sigma \neq \pi$, we simply write $\sigma < \pi$. For a more detailed explanation of Bruhat order we refer to [BB05, § 2].

Lemma 4.2.1. *For $\pi \in S_n$ and $a \in P_n(F)$, we have*

$$\pi a = {}^\pi a \pi + \sum_{\sigma < \pi} a_\sigma \sigma \quad \text{and} \quad a \pi = \pi \left({}^{\pi^{-1}} a \right) + \sum_{\sigma < \pi} \sigma a'_\sigma,$$

for some $a_\sigma, a'_\sigma \in P_n(F)$ of polynomial degree less than the polynomial degree of a .

Proof. Let $\pi \in S_n$ with length $\ell(\pi) = 1$. Then $\pi = s_i$, for some $1 \leq i \leq n-1$. Thus the first and second equation follows from Lemma 4.1.1 by applying to a and ${}^{s_i}a$, respectively.

Suppose the lemma is true for any element of S_n with length $\leq k$ and let $\pi = s_{i_1} \cdots s_{i_{k+1}}$ be a reduced expression. Call $\pi' = s_{i_1} \pi = s_{i_2} \cdots s_{i_{k+1}}$, then it follows that, by induction hypothesis:

$$\begin{aligned} \pi a &= s_{i_1} \pi' a = s_{i_1} \left({}^{\pi'} a \pi' + \sum_{\sigma' < \pi'} a_{\sigma'} \sigma' \right) \\ &= \left((s_{i_1} {}^{\pi'} a s_{i_1} + \Delta_{i_1}({}^{s_{i_1}} a)) \pi' + \sum_{\sigma' < \pi'} (s_{i_1} a_{\sigma'} s_{i_1} + \Delta_{i_1}(a_{\sigma'})) \sigma' \right) \\ &= \left({}^\pi a \pi + \Delta_{i_1}({}^{s_{i_1}} a) \pi' + \sum_{\sigma < \pi} (s_{i_1} a_{s_{i_1} \sigma} s_{i_1} + \Delta_{i_1}(a_{s_{i_1} \sigma})) s_{i_1} \sigma \right) \\ &= \left({}^\pi a \pi + \sum_{\sigma < \pi} a'_\sigma \sigma \right), \end{aligned}$$

where the polynomial degree of each a'_σ is less or equals to the polynomial degree of $a_{s_{i_1} \sigma}$, which is smaller than the polynomial degree of a . Then the first equation is true for elements of S_n with any length.

The proof of the second equation is analogous.

□

Proposition 4.2.2. *Let V denote $P_n(F) \otimes \mathbb{k}S_n$, considered as a graded $\mathbb{k}$ -supermodule. Then V is an $\mathcal{A}_n(F)$ -module under the action*

$$z \cdot (a \otimes w) = za \otimes w, \quad s_i \cdot (a \otimes w) = {}^{s_i}a \otimes s_i w - \Delta_i(a) \otimes w,$$

for all $z, a \in P_n(F)$, $w \in \mathbb{k}S_n$, and $1 \leq i \leq n-1$.

Proof. It follows by a straightforward calculation. It is clear that the action satisfies the relations (3.1.3) and (3.1.4). The proof that the other relations (relations (3.1.5)-(3.1.7) and relations in S_n) are satisfied is exactly as in [Sav20, Theorem 4.4].

□

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we let

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}. \quad (4.2.1)$$

We define the *graded dimension* $\text{grdim } V$ of a $(\mathbb{Z} \times \mathbb{Z}_2)$ -graded $\mathbb{k}$ -module, with finite-dimensional graded components, to be

$$\text{grdim } V = \sum_{i \in \mathbb{Z}, \varepsilon \in \mathbb{Z}_2} (\dim V_{i,\varepsilon}) q^i \varpi^\varepsilon \in \mathbb{N}[q^{\pm 1}, \varpi] / (\varpi^2 - 1).$$

Theorem 4.2.3. *The map*

$$\begin{aligned} P_n(F) \otimes \mathbb{k}S_n &\longrightarrow \mathcal{A}_n(F) \\ a \otimes w &\longmapsto aw \end{aligned}$$

is an isomorphism of graded $\mathcal{A}_n(F)$ -supermodules.

Furthermore, the algebra $\mathcal{A}_n(F)$ has graded dimension equal to

$$\text{grdim } \mathcal{A}_n(F) = n! \left(\frac{\text{grdim } F}{1 - q^\delta \varpi^\tau} \right)^n.$$

Proof. Let

$$\begin{aligned} \widetilde{\mathcal{B}}_1 &= \{x^\alpha \mathbf{b} \otimes \pi \mid \alpha \in \mathbb{N}^n, \mathbf{b} \in \mathcal{B}^{\otimes n}, \pi \in S_n\} \subseteq V, \\ \widetilde{\mathcal{B}}_2 &= \{x^\alpha \mathbf{b} \pi \mid \alpha \in \mathbb{N}^n, \mathbf{b} \in \mathcal{B}^{\otimes n}, \pi \in S_n\} \subseteq \mathcal{A}_n(F), \end{aligned}$$

where $\mathcal{B}$ is a basis for F of homogeneous elements.

First, notice that $\widetilde{\mathcal{B}}_1$ is a $\mathbb{k}$ -basis for $P_n(F) \otimes \mathbb{k}S_n$, since $\{x^\alpha \mathbf{b}\}_{\alpha \in \mathbb{N}^n, \mathbf{b} \in \mathcal{B}^{\otimes n}}$ and S_n are $\mathbb{k}$ -bases for $P_n(F)$ and $\mathbb{k}S_n$, respectively. We claim that $\widetilde{\mathcal{B}}_2$ is a basis for $\mathcal{A}_n(F)$. Indeed, it follows from Lemma 4.2.1 that $\widetilde{\mathcal{B}}_2$ spans $\mathcal{A}_n(F)$ and, the facts that $x^\alpha \mathbf{b} \pi (1 \otimes 1) = x^\alpha \mathbf{b} \otimes \pi$ and $\widetilde{\mathcal{B}}_1$ is $\mathbb{k}$ -linear independent, imply that $\widetilde{\mathcal{B}}_2$ is also linear independent. Then V is a cyclic $\mathcal{A}_n(F)$ -module generated by 1×1 , hence the homomorphism from the statement is well defined and, since it maps $\widetilde{\mathcal{B}}_2$ in $\widetilde{\mathcal{B}}_1$, it is an isomorphism.

Then, it follows that $\text{grdim } \mathcal{A}_n(F) = \text{grdim}(P_n(F) \otimes \mathbb{k}S_n) = n! \text{grdim } P_n(F)$. But

$$\begin{aligned} \text{grdim } P_n(F) &= \text{grdim}(\mathbb{k}[x] \ltimes F)^{\otimes n} \\ &= (\text{grdim}(\mathbb{k}[x] \ltimes F))^n \\ &= (\text{grdim } \mathbb{k}[x] \text{grdim } F)^n \\ &= \left(\frac{\text{grdim } F}{1 - q^\delta \varpi^\tau} \right)^n. \end{aligned}$$

And we finish the proof. $\square$

Corollary 4.2.4 (Basis theorem for $\mathcal{A}_n(F)$). *The sets*

$$\{x^\alpha \mathbf{b}\pi \mid \alpha \in \mathbb{N}^n, \mathbf{b} \in \mathcal{B}^{\otimes n}, \pi \in S_n\} \quad \text{and} \quad \{\pi x^\alpha \mathbf{b} \mid \alpha \in \mathbb{N}^n, \mathbf{b} \in \mathcal{B}^{\otimes n}, \pi \in S_n\},$$

are $\mathbb{k}$ -bases for $\mathcal{A}_n(F)$.

Proof. It follows from the proof of Theorem 4.2.3 and from Lemma 4.2.1, using induction on the length of $\pi \in S_n$. $\square$

Corollary 4.2.5. *As $(\mathbb{Z} \times \mathbb{Z}_2)$ -graded $\mathbb{k}$ -modules, we have*

$$\mathcal{A}_n(F) \simeq P_n \otimes (F^{\otimes n} \rtimes_\rho S_n).$$

The multiplication in $\mathcal{A}_n(F)$ is determined by the fact that the factors P_n and $F^{\otimes n} \rtimes_\rho S_n$ are subalgebras and by imposing relations (3.1.4)–(3.1.6).

It follows from Theorem 4.2.3 and Corollary 4.2.5 that P_n , $F^{\otimes n}$, $\mathbb{k}S_n$, $F^{\otimes n} \rtimes_\rho S_n$ and $P_n(F)$ can be identifying as subalgebras of $\mathcal{A}_n(F)$, which we will do in the remainder of the text.

Example 4.2.6. By specialising F we obtain:

- (*Classical results*) Corollary 4.2.4 recovers [Kle05, Th. 14.2.2] and [WW08, Th. 2.8], respectively for $F = \mathcal{C}l_0$ (see example 3.2.2) and $F = \mathbb{k}G$, where G is a finite group G concentrated in even parity (see 3.2.1).

The case where F is a purely even symmetric algebra, Theorem 4.2.3 recover [KM19, Th. 3.8];

- (*Affine Hecke-Grassmann algebra*) For $F = D$ (see example 3.2.3), from Corollary 4.2.4 we have that the following are bases for $\mathcal{A}_n(F)$

$$\{x^\alpha \mathbf{u}^\beta \pi \mid \alpha \in \mathbb{N}^n, \beta \in \{0, 1\}^n, \pi \in S_n\} \quad \text{and} \quad \{\pi x^\alpha \mathbf{u}^\beta \mid \alpha \in \mathbb{N}^n, \beta \in \{0, 1\}^n, \pi \in S_n\},$$

where $\mathbf{u}^\beta = u^{\beta_1} \otimes \cdots \otimes u^{\beta_n}$, for $\beta = (\beta_1, \dots, \beta_n) \in \{0, 1\}^n$. And, from Theorem 4.2.3, we have that

$$\text{grdim } \mathcal{A}_n(D) = n! \left(\frac{1 + q^\delta \varpi}{1 - q^\delta \varpi} \right)^n = n! \left(1 + 2 \sum_{\ell > 1} (q^\delta \varpi)^\ell \right)^n,$$

by assuming $|u| = \delta \neq 0$.

4.3 Description of the center

For all $k \in \mathbb{Z}$, let

$$F_{\psi,k} := \{f \in F \mid \psi(f) = (-1)^{k\tau} f\}, \quad (4.3.1)$$

$$F^{(k)} := \{f \in F \mid gf = (-1)^{\bar{f}\bar{g}} f \psi^k(g) \text{ for all } g \in F\} \quad \text{and} \quad (4.3.2)$$

$$F_{\psi}^{(k)} := F^{(k)} \cap F_{\psi,k}. \quad (4.3.3)$$

Note that, for all $k \in \mathbb{Z}$, $F^{(k\theta)} = Z(F)$ is the center of F .

It is clear that

$$F^{(k)} F^{(\ell)} \subseteq F^{(k+\ell)} \quad \text{and} \quad F_{\psi}^{(k)} F_{\psi}^{(\ell)} \subseteq F_{\psi}^{(k+\ell)}, \quad k, \ell \in \mathbb{Z}. \quad (4.3.4)$$

Lemma 4.3.1. *The center of $\mathbb{k}[x] \ltimes F$ is*

$$Z(\mathbb{k}[x] \ltimes F) = \bigoplus_{k=0}^{\infty} x^k F_{\psi}^{(-k)}.$$

Proof. For all $x^k f \in x^k F_{\psi}^{(-k)}$, with $f \in F_{\psi}^{(-k)}$ homogeneous, we have:

$$\begin{aligned} g(x^k f) &= (-1)^{k(\tau\bar{g})} x^k \psi^k(g) f \\ &\stackrel{(4.3.3)}{=} (-1)^{k(\tau\bar{g})} x^k (-1)^{\bar{f}\bar{g}} f g \\ &= (-1)^{(k\tau+\bar{f})\bar{g}} x^k f g \\ &= (-1)^{\overline{x^k f \bar{g}}} (x^k f) g, \end{aligned}$$

for all $g \in F$ homogeneous. Also, since $\bar{x} = \tau$, we have,

$$x(x^k f) = x^k (-1)^{\tau\bar{f}} \psi^{-1}(f) x \stackrel{(4.3.1)}{=} (-1)^{\tau(\bar{f}+k\tau)} (x^k f) x = (-1)^{\overline{x^k f \bar{x}}} (x^k f) x.$$

Therefore $\bigoplus_{k=0}^{\infty} x^k F_{\psi}^{(-k)} \subseteq Z(\mathbb{k}[x] \ltimes F)$.

The other inclusion follows at the same way as in [Sav20, Lemma 4.10], but with $\bar{x} = \tau$ and $F_{\psi,k}$ instead F_{ψ} :

Let $z = \sum_{k \in \mathbb{N}, b \in \mathcal{B}} a_{k,b} x^k b$ (with $a_{k,b} \in \mathbb{k}$ and $\mathcal{B}$ a basis for F) be a homogeneous element in $Z(\mathbb{k}[x] \ltimes F)$. Notice that, if $a_{k,b} \neq 0$, then $\bar{z} = k\tau + \bar{b}$. Then it follows:

$$\begin{aligned} 0 &= zx - (-1)^{\tau\bar{z}} xz \\ &= \sum_{k,b} \left(a_{k,b} (-1)^{\tau\bar{b}} x^{k+1} \psi(b) \right) - (-1)^{\tau\bar{z}} \sum_{k,b} \left(a_{k,b} x^{k+1} b \right) \\ &= \sum_{k,b} (-1)^{\tau\bar{b}} \left(a_{k,b} x^{k+1} (\psi - (-1)^{k\tau} \text{Id})(b) \right). \end{aligned}$$

Assuming, without loss of generality, that $\mathcal{B}$ is a basis of F consisting of eigenvectors for ψ (see 3.1), it follows from the equation above that $a_{k,b} = 0$, for all $b \notin \mathcal{B} \cap F_{\psi,k}$. Therefore we may write $z = \sum_{k \in \mathbb{N}} x^k f_k$, with $f_k \in F_{\psi,k}$, for all k . Notice that, if $f_k \neq 0$, we have $\bar{z} = k\tau + \bar{f}_k$.

Then it follows, for all homogeneous $g \in F$,

$$\begin{aligned} 0 &= gz - (-1)^{\bar{g}\bar{z}}zg \\ &= \sum_k (-1)^{k\tau\bar{g}} x^k \psi^k(g) f_k - (-1)^{\bar{g}\bar{z}} \sum_k x^k f_k g \\ &= \sum_k (-1)^{k\tau\bar{g}} x^k (\psi^k(g) f_k - (-1)^{\bar{g}\bar{f}_k} f_k g). \end{aligned}$$

Hence, from Theorem 4.2.3, it follows that $f_k \in F^{(-k)}$. □

Example 4.3.2. (a) For $F = Cl_\tau$ (see Example 3.2.2) we have

$$(Cl_\tau)_{\psi,k} = \begin{cases} \mathbb{k}, & \tau = 0 \\ \delta_{\bar{k},0} Cl_1, & \tau = 1 \end{cases} \quad \text{and} \quad (Cl_\tau)^{(-k)} = \begin{cases} \delta_{\bar{k},0} \mathbb{k}, & \tau = 0 \\ \mathbb{k}, & \tau = 1. \end{cases}$$

Therefore $(Cl_\tau)_{\psi}^{(-k)} = \delta_{\bar{k},0} \mathbb{k}$, for all $k \in \mathbb{Z}$.

(b) When F is a symmetric algebra we obtain, for all $k \in \mathbb{N}$:

$$F_{\psi,k} = \delta_{\tau\bar{k},0} F \quad \text{and} \quad F^{(-k)} = Z(F) \quad \Rightarrow \quad F_{\psi}^{(-k)} = \delta_{\tau\bar{k},0} Z(F)$$

where $Z(F)$ is the (super)center of the superalgebra F .

(c) In particular, for $F = D$ (see Example 3.2.1), it follows from (b) that

$$D_{\psi}^{(-k)} = \delta_{\bar{k},0} Z(D) = \delta_{\bar{k},0} D,$$

for all $k \in \mathbb{N}$.

Lemma 4.3.3. *The centralizer of P_n in $\mathcal{A}_n(F)$ is contained in the subalgebra $P_n(F)$.*

Proof. The proof is equal to [Sav20, Lem. 4.12], but using $x_i^{2\theta}$ instead x_i^θ :

Let $z = \sum_{\pi \in S_n} z_\pi \pi$ be in the centralizer of P_n in $\mathcal{A}_n(F)$, where $z_\pi \in P_n(F)$ for all $\pi \in S_n$.

Let $\pi \in S_n$ be maximal with respect to the strong Bruhat order such that $z_\pi \neq 0$. Assume $\pi \neq 1$ and choose $i \in \{1, 2, \dots, n\}$ such that $\pi(i) \neq i$. Then, by Lemma 4.2.1, and the fact that $x_i^{2\theta}$ is even, we obtain

$$x_i^{2\theta} z - z x_i^{2\theta} = \left(x_i^{2\theta} - x_{\pi(i)}^{2\theta} \right) z_\pi \pi + \sum_{\sigma: \sigma \not\geq \pi} z'_\sigma \sigma,$$

for some $z'_\sigma \in P_n(F)$. Thus, by Theorem 4.2.3, $x_i^{2\theta} - x_{\pi(i)}^{2\theta} \neq 0$ that implies that z is not central, which is a contradiction. Therefore $\pi = 1$. □

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, let

$$\mathbf{F}^{(\alpha)} := F^{(\alpha_1)} \otimes \dots \otimes F^{(\alpha_n)} \quad \text{and} \quad \mathbf{F}_{\psi}^{(\alpha)} := F_{\psi}^{(\alpha_1)} \otimes \dots \otimes F_{\psi}^{(\alpha_n)}. \quad (4.3.5)$$

It follows from (4.3.4) that

$$\bigoplus_{\alpha \in \mathbb{N}^n} x^\alpha \mathbf{F}^{(-\alpha)} \quad \text{and} \quad \bigoplus_{\alpha \in \mathbb{N}^n} x^\alpha \mathbf{F}_\psi^{(-\alpha)}$$

are subalgebras of $\mathcal{A}_n(F)$. Note that $x^\alpha \mathbf{F}^{(-\alpha)} = (x^{\alpha_1} F^{(-\alpha_1)}) \otimes \cdots \otimes (x^{\alpha_n} F^{(-\alpha_n)})$.

Theorem 4.3.4. *The center of $\mathcal{A}_n(F)$ is $\left(\bigoplus_{\alpha \in \mathbb{N}^n} x^\alpha \mathbf{F}_\psi^{(-\alpha)}\right)^{S_n}$, the subalgebra of $\bigoplus_{\alpha \in \mathbb{N}^n} x^\alpha \mathbf{F}_\psi^{(-\alpha)}$ invariant under the action of S_n by superpermutation. In other words, the center of $\mathcal{A}_n(F)$ consists of finite sums of the form*

$$\sum_{\alpha \in \mathbb{N}^n} x^\alpha f_\alpha, \quad f_\alpha \in \mathbf{F}_\psi^{(-\alpha)}, \quad (4.3.6)$$

such that $f_{\pi \cdot \alpha} = \pi f_\alpha$ for all $\alpha \in \mathbb{N}^n$ and $\pi \in S_n$.

Proof. It follows immediately from Lemmas 4.3.1 and 4.3.3, and the fact that $Z(A_1 \otimes A_2) = Z(A_1) \otimes Z(A_2)$ for $\mathbb{k}$ -algebras A_1 and A_2 , that

$$Z(\mathcal{A}_n(F)) \subseteq \bigoplus_{\alpha \in \mathbb{N}^n} x^\alpha \mathbf{F}_\psi^{(-\alpha)},$$

where $Z(\mathcal{A}_n(F))$ denote the center of $\mathcal{A}_n(F)$.

Now, let $z \in Z(\mathcal{A}_n(F))$ and write $z = \sum_{\alpha \in \mathbb{N}^n} x^\alpha \mathbf{f}_\alpha$ for some $\mathbf{f}_\alpha \in \mathbf{F}_\psi^{(-\alpha)}$. For $i \in \{1, \dots, n-1\}$, it follows from Lemma 4.2.1 that

$$0 = s_i z - z s_i = (s_i z - z) s_i + a,$$

for some $a \in P_n(F)$. It implies from Theorem 4.2.3 that $a = 0$ and z is invariant under the superpermutation action of S_n .

It remains to prove that elements of the form (4.3.6) commute with elements of S_n . Let $1 \leq i \leq n-1$. Since s_i commutes with x_j and f_j for all $j \neq i, i+1$ and $f \in F$, it suffices to check that s_i commutes with elements of the form

$$x_i^q x_{i+1}^r \mathbf{f} + (-1)^{\tau r q} x_i^r x_{i+1}^q s_i \mathbf{f}, \quad q, r \in \mathbb{N}, \mathbf{f} \in \mathbf{F}_\psi^{(-\alpha)},$$

where $\alpha = (0, \dots, 0, q, r, 0, \dots, 0)$, with q appearing in the i -th component.

By (3.3.2) and (3.3.3), we just need to show that

$$0 = x_{i+1}^q t_{i+1,i}^{(r)} \mathbf{f} s_i - t_{i,i+1}^{(q)} x_{i+1}^r \mathbf{f} s_i + (-1)^{\tau q r} \left(x_{i+1}^r t_{i+1,i}^{(q)} s_i \mathbf{f} s_i - t_{i,i+1}^{(r)} x_{i+1}^q s_i \mathbf{f} s_i \right). \quad (4.3.7)$$

Now

$$x_{i+1}^q t_{i+1,i}^{(r)} \mathbf{f} = \sum_{b \in \mathcal{B}} \sum_{\ell=0}^{r-1} (-1)^{\tau(r\bar{b}+\ell)} x_{i+1}^q b_{i+1} x_i^{r-\ell-1} x_{i+1}^\ell b_i^\vee \mathbf{f}$$

$$\begin{aligned}
&= \sum_{b \in \mathcal{B}} \sum_{\ell=0}^{r-1} (-1)^{\tau(r\bar{b}+\ell+r\ell)} x_{i+1}^q b_{i+1} x_{i+1}^\ell x_i^{r-\ell-1} b_i^\vee \mathbf{f} \\
&\stackrel{(3.1.4)}{=} \sum_{b \in \mathcal{B}} \sum_{\ell=0}^{r-1} (-1)^{\tau(r\ell-\bar{b}+r-1)} x_{i+1}^{q+\ell} \psi^\ell(b)_{i+1} \psi^{\ell-r+1}(b^\vee)_i x_i^{r-\ell-1} \mathbf{f} \\
&\stackrel{(3.1.4)}{=} \sum_{b \in \mathcal{B}} \sum_{\ell=0}^{r-1} (-1)^{\tau(r\ell-\bar{b}+r-1+q)} x_{i+1}^\ell \psi^{\ell-q}(b)_{i+1} \psi^{\ell-r+1}(b^\vee)_i x_{i+1}^q x_i^{r-\ell-1} \mathbf{f} \\
&\stackrel{\mathbf{f} \in \mathbf{F}_\psi^{(-\alpha)}}{=} \sum_{b \in \mathcal{B}} \sum_{\ell=0}^{r-1} (-1)^{\tau((r-q)\ell-\bar{b}+r-1+(r+q-\ell)\bar{\mathbf{f}})} x_{i+1}^\ell \mathbf{f} \psi^{\ell-q-r}(b)_{i+1} \psi^{\ell-q-r+1}(b^\vee)_i x_{i+1}^q x_i^{r-\ell-1} \\
&= \sum_{\ell=0}^{r-1} (-1)^{\tau((r-q)\ell+r-1+(r+q-\ell)\bar{\mathbf{f}})} x_{i+1}^\ell \mathbf{f} \psi_{i+1}^{-1}(t_{i+1,i}) x_{i+1}^q x_i^{r-\ell-1} \quad (\text{see explanation below}) \\
&\stackrel{(3.3.11)}{=} \sum_{\ell=0}^{r-1} (-1)^{\tau((r-q)\ell+r+(r+q-\ell)\bar{\mathbf{f}})} x_{i+1}^\ell \mathbf{f} t_{i,i+1} x_{i+1}^q x_i^{r-\ell-1} \\
&\stackrel{(3.3.9)}{=} \sum_{\ell=0}^{r-1} (-1)^{\tau((r-q)\ell+(r+q-\ell+1)\bar{\mathbf{f}})} x_{i+1}^\ell t_{i,i+1}^{s_i} \mathbf{f} x_{i+1}^q x_i^{r-\ell-1} \quad \left(\text{since } \mathbf{f} \in \mathbf{F}^{(-\alpha)} \right) \\
&= \sum_{\ell=0}^{r-1} (-1)^{\tau(-q\ell+q)} x_{i+1}^\ell t_{i,i+1} x_{i+1}^q x_i^{r-\ell-1} s_i \mathbf{f} \quad \left(\text{since } s_i \mathbf{f} \in \mathbf{F}^{(-s_i \cdot \alpha)} \right) \\
&= \sum_{\ell=0}^{r-1} (-1)^{\tau q r} x_{i+1}^\ell t_{i,i+1} x_i^{r-\ell-1} x_{i+1}^q s_i \mathbf{f} \\
&= (-1)^{\tau q r} \sum_{b \in \mathcal{B}} \sum_{\ell=0}^{r-1} (-1)^{\tau(r\bar{b}+r-\ell-1)} b_i x_{i+1}^\ell x_i^{r-\ell-1} b_{i+1}^\vee x_{i+1}^q s_i \mathbf{f} \\
&= (-1)^{\tau q r} t_{i,i+1}^{(r)} x_{i+1}^q s_i \mathbf{f},
\end{aligned}$$

where, in the sixth equality, we used the fact that the definition of $t_{i,j}$ is independent of the choice of basis and thus we summed over the basis $\{\psi^{\ell-q-r+1}(b) \mid b \in \mathcal{B}\}$ (and abused notation by calling this new basis $\mathcal{B}$ again.)

Thus, the first and last terms in (4.3.7) cancel. The proof that the second and third terms cancel is similar. $\square$

Example 4.3.5. For the following specializations of F , Theorem 4.3.4 implies:

- (a) For $F = \mathcal{Cl}_\tau$ (see Example 3.2.2 and 4.3.2(a)) it follows that

$$Z(\mathcal{A}_n(\mathcal{Cl}_\tau)) = (\mathbb{k}[x_1^2, \dots, x_n^2])^{S_n},$$

i.e., the center of $\mathcal{A}_n(F)$ is the algebra of symmetric polynomial in the variables $x_1^2, \dots, x_n^2$.

- (b) When F is a symmetric algebra (see Example 4.3.2(b)) the center

$$Z(\mathcal{A}_n(F)) = (Z(F)[x_1^{\tau+1}, \dots, x_n^{\tau+1}])^{S_n},$$

i.e., algebra of symmetric polynomial in the variables $x_1^{\tau+1}, \dots, x_n^{\tau+1}$ and coefficients in $Z(F)$, which recovers [EKL14, Th. 3.12], when F is purely even.

- (c) In particular, for $F = D$ (see Example 3.2.1 and 4.3.2(c)), it follows from (b) and the fact that D is supercommutative, that $Z(\mathcal{A}_n(D))$ is the algebra of symmetric polynomial in the variables $x_1^2, \dots, x_n^2$ and coefficients in D .

Corollary 4.3.6. *The center of $\mathcal{A}_n(F)$ contains the ring of symmetric polynomials in $x_1^{2\theta}, \dots, x_n^{2\theta}$. In particular, $\mathcal{A}_n(F)$ is finitely generated as a module over its center.*

Proof. That the center contains the ring of symmetric polynomials in $x_1^{2\theta}, \dots, x_n^{2\theta}$ follows from Theorem 4.3.4 and the fact that $\mathbb{k} \subseteq F_\psi^{(-2k\theta)}$ for all $k \in \mathbb{N}$. Thus $\mathcal{A}_n(F)$ is finitely generated as a module over its center by Theorem 4.2.3. Precisely, the set

$$\{x^\alpha \mathbf{b} \pi \mid \alpha \in \{0, \dots, 2\theta - 1\}^n, \mathbf{b} \in \mathcal{B}^{\otimes n}, \pi \in S_n\}$$

is a finite generator of $\mathcal{A}_n(F)$ as $Z(\mathcal{A}_n(F))$ -module. $\square$

4.4 Jucys–Murphy’s elements

Consider the Jucys–Murphy’s elements for $\mathbb{k}S_n$:

$$X_1 = 0, \quad X_k = (1 \ k - 1) + \dots + (k - 1 \ k), \quad 2 \leq k \leq n,$$

where $(i \ j)$ are the simple transposition which exchanges i and j . Such elements generates the Gelfand–Tsetlin subalgebra of the family of group algebras $\mathbb{k}S_1 \subseteq \mathbb{k}S_2 \subseteq \dots \subseteq \mathbb{k}S_n$. Roughly speaking, the Jucys–Murphy’s elements generate a subalgebra of $\mathbb{k}S_n$ that plays a role of a Cartan subgroup, allowing a “Cartan’s highest weight theory” for $\mathbb{k}S_n$ -modules. See [Vok04] for more details.

Therefore, Jucys–Murphy’s elements play a fundamental role in the the representation theory of the symmetric group. Hence, once defined a “deformed version” of $\mathbb{k}S_n$ (such as wreath product algebras), it is important to define analogous elements for the new algebra.

In [Kle05, §13.2], Kleshchev define the analogous elements for the Sergeev algebra. We now extend these definitions for the wreath product algebra $F^{\otimes n} \rtimes_\rho S_n$.

Definition 4.4.1. We define the *Jucys–Murphy elements* of $F^{\otimes n} \rtimes_\rho S_n$ to be

$$J_1 = 0, \quad J_k = \sum_{i=1}^{k-1} t_{i,k} s_{i,k}, \quad 2 \leq k \leq n, \quad (4.4.1)$$

where $s_{i,k} \in S_n$ is the transposition of i and k .

The definition is the same from [Sav20, §4.5], but here they have parity τ (the same parity of elements $t_{i,j}$).

Proposition 4.4.2. *There exists a surjective homomorphism of algebra $\mathcal{A}_n(F) \twoheadrightarrow F^{\otimes n} \rtimes_\rho S_n$ such that*

$$\mathbf{f} \mapsto \mathbf{f}, \quad \pi \mapsto \pi, \quad x_i \mapsto J_i, \quad \text{for all } \mathbf{f} \in F^{\otimes n}, \pi \in S_n \text{ and } 1 \leq i \leq n.$$

Proof. The given map is clearly linear and surjective, thus, to show that it is an algebra homomorphism, it suffices to prove that it respects the relations of $\mathcal{A}_n(F)$.

Notice that

$$s_1 J_1 = 0 = J_2 s_1 - t_{1,2}.$$

Thus relation (3.1.6) is satisfied, by the fact that, for $1 < k \leq n$

$$\begin{aligned} s_k J_k s_k + t_{k,k+1} s_k &= \sum_{i=1}^{k-1} s_k t_{i,k} s_{i,k} s_k + t_{k,k+1} s_k \stackrel{(3.3.15)}{=} \sum_{i=1}^{k-1} t_{i,k+1} s_k s_{i,k} s_k + t_{k,k+1} s_k \quad (4.4.2) \\ &= \sum_{i=1}^{k-1} t_{i,k+1} s_{i,k+1} + t_{k,k+1} s_k = J_{k+1}. \end{aligned}$$

To prove that s_l and J_k commute, for all $|k - l| \leq 2$, we use induction on k . For $k = 1$ it is trivial. Now, let $l \in \{1, \dots, n\}$ such that $|k - l| \leq 2$. If $l \leq k + 3$, then s_l commutes with $s_{i,k}$ and $t_{i,k}$ for all $1 \leq i \leq k$, and the relation follows. For $l \leq k - 1$, we have:

$$\begin{aligned} s_l J_{k+1} &= s_l (s_k J_k s_k + t_{k,k+1} s_k) \stackrel{(3.3.15)}{=} s_k s_l J_k s_k + t_{k,k+1} s_l s_k \\ &= s_k J_k s_l s_k + t_{k,k+1} s_k s_l = J_{k+1} s_l, \end{aligned}$$

where the third equality follows from induction hypothesis. Finally, for $l = k - 1$, we have:

$$\begin{aligned} s_{k-1} J_{k+1} &= s_{k-1} \left(\sum_{i=1}^{k-2} t_{i,k+1} s_{i,k+1} + t_{k-1,k+1} s_{k-1} s_k s_{k-1} + t_{k,k+1} s_k \right) \\ &\stackrel{(3.3.15)}{=} \sum_{i=1}^{k-2} t_{i,k+1} s_{k-1} s_{i,k+1} + t_{k,k+1} s_{k-1} s_{k-1} s_k s_{k-1} + t_{k-1,k+1} s_{k-1} s_k \\ &= \left(\sum_{i=1}^{k-2} t_{i,k+1} s_{i,k+2} + t_{k,k+1} s_k + t_{k-1,k+1} s_{k-1} s_k s_{k-1} \right) s_{k-1} \\ &= J_{k+1} s_{k-1}. \end{aligned}$$

Thus the map satisfies (3.1.5).

It remains to prove that Jucys–Murphy elements are supercommutative. We prove it by induction on $1 \leq k \leq n$, that

$$J_k J_l = (-1)^\tau J_l J_k, \quad 1 \leq l \leq k - 1.$$

Case $k = 1$ is trivial. Now, let $l \leq k - 1$, then it follows that

$$\begin{aligned}
 J_l J_{k+1} &= J_l(s_k J_k s_k + t_{k,k+1} s_k) = (-1)^\tau s_k J_k s_k J_l + \sum_{i=1}^{l-1} t_{i,l} s_{i,l} t_{k,k+1} s_k \\
 &= (-1)^\tau s_k J_k s_k J_l + \sum_{i=1}^{l-1} t_{i,l} t_{k,k+1} s_k s_{i,l} = (-1)^\tau s_k J_k s_k J_l + (-1)^\tau \sum_{i=1}^{l-1} t_{k,k+1} t_{i,l} s_k s_{i,l} \\
 &= (-1)^\tau (s_k J_k s_k J_l + t_{k,k+1} s_k) \sum_{i=1}^{l-1} t_{i,l} s_{i,l} = (-1)^\tau J_{k+1} J_l,
 \end{aligned}$$

where, in the second equality, we use the fact that $s_k J_l = J_l s_k$, for $|k - l| \geq 2$, together with induction hypothesis and, in the forth equality, the fact that $t_{i,l} t_{k,k+1} = (-1)^\tau t_{k,k+1} t_{i,l}$ for $i, l \notin \{k, k+1\}$. When $l = k$, we use the fact that J_{k+1} and s_{k-1} commute and induction hypothesis to conclude:

$$J_k J_{k+1} = (s_{k-1} J_{k-1} s_{k-1} + t_{k-1,k} s_{k-1}) J_{k+1} = (-1)^\tau J_{k+1} s_{k-1} J_{k-1} s_{k-1} + t_{k-1,k} J_{k+1} s_{k-1}.$$

Now by Lemma 3.3.2 and (3.3.15) it follows:

$$\begin{aligned}
 t_{k-1,k} J_{k+1} &= \sum_{i=1}^k t_{k-1,k} t_{i,k+1} s_{i,k+1} \\
 &= (-1)^\tau \sum_{i=1}^{k-2} (t_{i,k+1} s_{i,k+1} t_{k-1,k}) + t_{k-1,k} t_{k-1,k+1} s_{k-1,k+1} + t_{k-1,k} t_{k,k+1} s_k \\
 &= (-1)^\tau \sum_{i=1}^{k-2} (t_{i,k+1} s_{i,k+1} t_{k-1,k}) + (-1)^\tau t_{k-1,k+1} \psi_{k-1}^{(s_{k-1,k+1} t_{k-1,k})} s_{k-1,k+1} \\
 &\quad + (-1)^\tau t_{k,k+1} \psi_k^{(s_{k,k+1} t_{k-1,k})} s_k \\
 &= (-1)^\tau \left(\sum_{i=1}^{k-2} (t_{i,k+1} s_{i,k+1} t_{k-1,k}) + t_{k-1,k+1} t_{k+1,k} s_{k-1,k+1} + t_{k,k+1} t_{k-1,k+1} s_k \right) \\
 &\stackrel{(3.3.15)}{=} (-1)^\tau \left(\sum_{i=1}^{k-2} (t_{i,k+1} s_{i,k+1} t_{k-1,k}) + t_{k-1,k+1} s_{k-1,k+1} t_{k-1,k} + t_{k,k+1} s_k t_{k-1,k} \right) \\
 &= (-1)^\tau J_{k+1} t_{k-1,k}.
 \end{aligned}$$

Then we have:

$$\begin{aligned}
 J_k J_{k+1} &= (-1)^\tau J_{k+1} s_{k-1} J_{k-1} s_{k-1} + t_{k-1,k} J_{k+1} s_{k-1} \\
 &= (-1)^\tau J_{k+1} (s_{k-1} J_{k-1} s_{k-1} + t_{k-1,k} s_{k-1}) = (-1)^\tau J_{k+1} J_k.
 \end{aligned}$$

Therefore the map satisfies relation (3.1.3).

Finally, Lemma 3.3.2 and (3.1.7) implies

$$\begin{aligned} \mathbf{f}J_k &= \sum_{i=1}^{k-1} \mathbf{f}t_{i,k}s_{i,k} = (-1)^{\tau\bar{\mathbf{f}}} \sum_{i=1}^{k-1} t_{i,k}^{s_{i,k}}(\psi_k(\mathbf{f}))s_{i,k} \\ &\stackrel{(3.1.7)}{=} (-1)^{\tau\bar{\mathbf{f}}} \sum_{i=1}^{k-1} t_{i,k}s_{i,k}\psi_k(\mathbf{f}) = (-1)^{\tau\bar{\mathbf{f}}} J_k\psi_k(\mathbf{f}). \end{aligned}$$

Thus the map satisfies the relation (3.1.4), and the proof is finished. $\square$

Example 4.4.3. 1. Suppose $F = \mathbb{k}G$ is the Frobenius algebra such that G is equipped with a pair $(\text{sgn}: G \rightarrow \mathbb{Z}_2, \tilde{g} = g_\tau \in \text{sgn}^{-1}(\tau))$ and the trace map tr is defined by $\text{tr}(g) = \delta_{\tilde{g},g}$, for all $g \in G$ (see Example 2.2.16 for more details). In that case we have that

$$J_1 = 0 \quad \text{and} \quad J_k = \sum_{i=1}^{k-1} \tilde{g}_k \sum_{h \in G} h_1 h_k^{-1} s_{i,k} = \sum_{i=1}^{k-1} \tilde{g}_k \sum_{h \in G} h_1 s_{i,k} h_1^{-1}, \quad 1 < k \leq n.$$

In particular, when $\text{sgn} = 0$, i.e. $\mathbb{k}G$ is concentrated in even parity, and $\tilde{g} = 1$, we recovered the definition of (generalized) Jucys–Murphy elements of $(\mathbb{k}G)^{\otimes n} \rtimes \mathbb{k}S_n$ (see [WW08, § 2.2]). Moreover, in the case where G is trivial, we recovered the traditional Jucys–Murphy elements.

2. When $F = \mathcal{C}l_\tau$, i.e. the Clifford algebra with trace map $\text{tr}(c^n) = \delta_{0, n+\tau}$, for all $n \in \mathbb{N}$, (see Examples 2.2.17 and 3.2.2). For $\tau = 0$ we have that

$$J_1 = 0 \quad \text{and} \quad J_k = \sum_{i=1}^{k-1} (1 - c_i c_k) s_{i,k}, \quad 1 < k \leq n,$$

which recovers the definition of Jucys–Murphy elements of the Sergeev algebra (see [Kle05, § 13.2]). For $\tau = 1$, we have:

$$J_1 = 0 \quad \text{and} \quad J_k = \sum_{i=1}^{k-1} (c_k - c_i) s_{i,k}, \quad 1 < k \leq n.$$

3. When $F = D$ the Dual algebra (see Example 2.2.19) we have $J_k = \sum_{i=1}^{k-1} (u_k - u_i) s_{i,k}$, $1 < k \leq n$.

4.5 Intertwining elements

Intertwining elements play a fundamental role in the treatment of integral modules for degenerate affine Hecke algebras (see [Kle05, §3.8]), affine Sergeev algebras (see [Kle05, §14.8]), and wreath Hecke algebras (see [WW08, §5.2]). A more general definition of intertwining elements for affine wreath product algebra $\mathcal{A}_n(F)$ was defined by Savage in [Sav20]. We now recall the definition and extend it for more general cases where the trace map tr of F has arbitrary parity τ .

Definition 4.5.1 (Intertwining elements). Let $\vartheta = (1 + \tau)\theta$, where we view $\tau \in \{0, 1\}$ and θ is the order of the Nakayama automorphism ψ . Then, for all $i \in \{1, \dots, n-1\}$ we define

$$\Omega_i = x_{i+1}^\vartheta s_i - s_i x_{i+1}^\vartheta.$$

Note that, from equations (3.3.2) and (3.3.3), it follows that

$$\Omega_i = s_i(x_i^\vartheta - x_{i+1}^\vartheta) + t_{i,i+1}^{(\vartheta)} \quad \text{and} \quad \Omega_i = (x_{i+1}^\vartheta - x_i^\vartheta)s_i - t_{i+1,i}^{(\vartheta)} \quad (4.5.1)$$

Example 4.5.2 (Classical definitions). For the case where F is $\mathbb{k}$, $\mathcal{Cl}_0$ (with trace map $\text{tr}(1) = 1$ and $\text{tr}(c) = 0$), or a group algebra $\mathbb{k}G$ (concentrated in even parity) (see Examples 3.2.2 and 3.2.1), our definition recovers the definitions of [Kle05, §3.8], [Kle05, §14.8] and [WW08, §5.2]). Precisely, the intertwining elements are, respectively:

$$\begin{aligned} \Omega_i &= s_i(x_i - x_{i+1}) + 1, & (F = \mathbb{k}) \\ \Omega_i &= s_i(x_i^2 - x_{i+1}^2) + (x_{i+1} + x_i) + c_i(x_{i+1} + x_i)c_{i+1} \\ &= s_i(x_i^2 - x_{i+1}^2) + (x_i + x_{i+1}) + c_i c_{i+1}(x_i - x_{i+1}), & (F = \mathcal{Cl}) \\ \Omega_i &= s_i(x_i - x_{i+1}) + t_{i,i+1} \\ &= s_i(x_i - x_{i+1}) + \sum_{g \in G} g_i g_{i+1}^{-1}, & (F = \mathbb{k}G), \end{aligned}$$

for all $i \in \{1, \dots, n-1\}$.

Example 4.5.3 (Symmetric algebra). Let F be a symmetric Frobenius algebra, with $\psi = \text{Id}$. We have $\theta = 1$ and $\vartheta = (\tau + 1)$. Then, the intertwining elements of $\mathcal{A}_n(F)$ are:

$$\begin{aligned} \Omega_i &= s_i(x_i^{\tau+1} - x_{i+1}^{\tau+1}) + \sum_{b \in \mathcal{B}} (-1)^{\tau(\tau+1)\bar{b}} b_i(x_{i+1}^\tau - x_i^\tau) b_{i+1}^\vee \\ &\stackrel{(3.1.4)}{=} s_i(x_i^{\tau+1} - x_{i+1}^{\tau+1}) + (x_{i+1}^\tau - x_i^\tau) \sum_{b \in \mathcal{B}} (-1)^{\tau\bar{b}} b_i b_{i+1}^\vee \\ &= s_i(x_i^{\tau+1} - x_{i+1}^{\tau+1}) + (x_{i+1}^\tau - x_i^\tau) t_{i,i+1}, \end{aligned}$$

for all $i \in \{1, \dots, n-1\}$.

Example 4.5.4 (Affine Sergeev algebra with odd presentation). Let $F = \mathcal{Cl}_1$ with trace map $\text{tr}(1) = 0$ and $\text{tr}(c) = 1$, as in example 3.2.2. Since $\mathcal{Cl}_1$ is a symmetric algebra, it follows from the previous example that

$$\Omega_i = s_i(x_i^2 - x_{i+1}^2) + (x_{i+1} - x_i)(c_{i+1} - c_i), \quad 1 \leq i \leq n-1.$$

Example 4.5.5. Let $F = D$ be the Dual algebra, as in example 2.2.19 (see also 3.2.3. Since D is a symmetric algebra, it follows that

$$\Omega_i = s_i(x_i^2 - x_{i+1}^2) + (x_{i+1} - x_i)(u_{i+1} - u_i), \quad 1 \leq i \leq n-1.$$

Now we show that intertwining elements satisfy analogous relations as the classical ones.

Lemma 4.5.6. *For all $1 \leq i, j \leq n-1$ we have*

$$\Omega_i^2 = \left(t_{i,i+1}^{(\vartheta)}\right)^2 - \left(x_i^\vartheta - x_{i+1}^\vartheta\right)^2, \quad (4.5.2)$$

$$\Omega_i f_j = f_{s_i(j)} \Omega_i, \quad \forall f \in F, \quad (4.5.3)$$

$$\Omega_i x_j = x_{s_i(j)} \Omega_i, \quad (4.5.4)$$

$$\Omega_i \Omega_j = \Omega_j \Omega_i, \quad \text{if } |i-j| \geq 2. \quad (4.5.5)$$

Proof. We have

$$\Omega_i^2 = s_i(x_i^\vartheta - x_{i+1}^\vartheta) s_i(x_i^\vartheta - x_{i+1}^\vartheta) + s_i(x_i^\vartheta - x_{i+1}^\vartheta) t_{i,i+1}^{(\vartheta)} + t_{i,i+1}^{(\vartheta)} s_i(x_i^\vartheta - x_{i+1}^\vartheta) + \left(t_{i,i+1}^{(\vartheta)}\right)^2.$$

Now, using (4.5.1), we have

$$(x_i^\vartheta - x_{i+1}^\vartheta) s_i = s_i(x_{i+1}^\vartheta - x_i^\vartheta) - t_{i,i+1}^{(\vartheta)} - t_{i+1,i}^{(\vartheta)}.$$

Using this and the fact that

$$s_i t_{i+1,i}^{(\vartheta)} \stackrel{(3.3.3)}{=} s_i(s_i x_{i+1}^\vartheta - x_i^\vartheta s_i) = (x_{i+1}^\vartheta s_i s_i - s_i x_i^\vartheta) s_i \stackrel{(3.3.2)}{=} t_{i,i+1}^{(\vartheta)} s_i,$$

we obtain:

$$\begin{aligned} \Omega_i^2 &= -(x_{i+1}^\vartheta - x_i^\vartheta)^2 - s_i t_{i,i+1}^{(\vartheta)} (x_i^\vartheta - x_{i+1}^\vartheta) + s_i (x_i^\vartheta - x_{i+1}^\vartheta) t_{i,i+1}^{(\vartheta)} + \left(t_{i,i+1}^{(\vartheta)}\right)^2 \\ &\stackrel{(3.3.14)}{=} \left(t_{i,i+1}^{(\vartheta)}\right)^2 - (x_{i+1}^\vartheta - x_i^\vartheta)^2. \end{aligned}$$

Thus (4.5.2) is true. Relation (3.1.4) and the fact that ϑ is a multiple of the order of ψ easily imply relation (4.5.3).

To prove (4.5.4), we compute:

$$\begin{aligned} \Omega_i x_i &= (x_{i+1}^\vartheta - s_i x_{i+1}^\vartheta) x_i \stackrel{(3.1.6)}{=} x_{i+1}^{\vartheta+1} s_i - x_{i+1}^\vartheta t_{i,i+1} - s_i x_i x_{i+1}^\vartheta \\ &\stackrel{(3.3.13)}{=} x_{i+1}^{\vartheta+1} s_i - (t_{i,i+1} + s_i x_i) x_{i+1}^\vartheta \\ &\stackrel{(3.1.6)}{=} x_{i+1}^{\vartheta+1} s_i - x_{i+1} s_i x_{i+1}^\vartheta \\ &= x_{i+1} (x_{i+1}^\vartheta s_i - s_i x_{i+1}^\vartheta) = x_{i+1} \Omega_i, \end{aligned}$$

and

$$\begin{aligned} \Omega_i x_{i+1} &= (x_{i+1}^\vartheta - s_i x_{i+1}^\vartheta) x_{i+1} \stackrel{(3.1.12)}{=} x_{i+1}^\vartheta x_i s_i + x_{i+1}^\vartheta t_{i+1,i} - s_i x_{i+1}^{\vartheta+1} \\ &\stackrel{(3.3.13)}{=} x_i x_{i+1}^\vartheta s_i + (t_{i,i+1} - s_i x_{i+1}) x_{i+1}^\vartheta \\ &\stackrel{(3.1.12)}{=} x_i x_{i+1}^\vartheta s_i - x_i s_i x_{i+1}^\vartheta \\ &= x_i (x_{i+1}^\vartheta s_i - s_i x_{i+1}^\vartheta) = x_i \Omega_i. \end{aligned}$$

Furthermore it is easy to see that (4.5.4) is satisfied for $|i-j| > 1$. This completes the proof of (4.5.4). Relation (4.5.5) is straightforward, by the fact that x_k^ϑ is even, for all $1 \leq k \leq n$. $\square$

Chapter 5

Representation theory

In this section we classify the simple modules over $\mathcal{A}_n(F)$. In our approach, we study the simple modules step by step over each “pre-stage” of the construction of $\mathcal{A}_n(F)$. Precisely, we:

1. Classify the $P_1(F)$ -modules, obtaining (by outer-tensoring the $P_1(F)$ -modules) the classification of simple $P_n(F)$ -modules;
2. Study the action of $t_{k,l}$ in simple $P_n(F)$ -modules;
3. Define auxiliary algebras $\mathcal{R}_n$ and $\mathcal{H}_n$ in terms of endomorphism algebras of simple $P_n(F)$ -modules and the action of $t_{k,l}$;
4. Construct an equivalence of categories between $\mathcal{R}_n\text{-mod}$ and $\mathcal{A}_n(F)\text{-mod}^s$, the full subcategory of $\mathcal{A}_n(F)\text{-mod}$ where the objects are semisimple as $F^{\otimes n}$ -modules (see Theorem 5.4.7);
5. Classify the simple $\mathcal{A}_n(F)$ -modules in terms of simple $\mathcal{H}_n$ -modules (see Theorem 5.5.2).

Before start with the classification, let prove the important statement:

Lemma 5.0.1. *All simple $\mathcal{A}_n(F)$ -modules are finite-dimensional.*

Proof. First we recall the following general result for algebras (considering the classical definition, i.e., without considering $\mathbb{Z}_2$ -grading):

If A is an algebra, over an algebraically closed field, finitely generated as a module over its center, then all simple A -modules are finite dimensional.

Indeed, let V be a simple A -module. Without loss of generality we may assume that the annihilator of V in A is zero. Let Z denote the center of A . Since V is a simple A -module, it is generated (as an A -module) by any nonzero element. Since A is finitely generated as a Z -module, it follows that V is finitely generated as a Z -module. Since multiplication by elements of Z commutes with the A -action, it follows from Schur’s Lemma (and the fact that the annihilator of V is zero) that every nonzero element of Z acts on A as a nonzero scalar. Now, let $z \in Z$, $z \neq 0$. Then, $zV = V$ and so, by Nakayama’s Lemma, there exists $a \in 1 + zZ$ such that $aV = 0$. This implies that $a = 0$, and therefore z is invertible. Hence Z is a field. Since $\mathbb{k}$ is algebraically closed, we have $Z = \mathbb{k}$. Thus V is finitely generated over $\mathbb{k}$.

Let $|\mathcal{A}_n(F)|$ be the algebra $\mathcal{A}_n(F)$ forgetting the $\mathbb{Z}_2$ -grading and let $\mathcal{A}_n(F)_0$ be the even part of $\mathcal{A}_n(F)$. Notice that the proof of Corollary 4.3.6 shows that there exists a finite subset $X \subseteq \mathcal{A}_n(F)$ and subalgebra A of $\mathcal{A}_n(F)_0 \cap Z(\mathcal{A}_n(F))$, such that $AX = \mathcal{A}_n(F)$. Then, since

$\mathcal{A}_n(F)_0 \cap Z(\mathcal{A}_n(F)) \subseteq Z(|\mathcal{A}_n(F)|)$ we have that $Z(|\mathcal{A}_n(F)|)X = AX = |\mathcal{A}_n(F)|$. Therefore, from the fact above, all simple $|\mathcal{A}_n(F)|$ -modules are finite dimensional. Now, since any simple $\mathcal{A}_n(F)$ -module is isomorphic to either V or $V \oplus \Pi V$, for some simple $|\mathcal{A}_n(F)|$ -modules V , is proved the lemma. $\square$

In this section we assume that $\mathbb{k}$ is an algebraically closed field of characteristic not equal to two. We also continue to assume that its characteristic does not divide the order θ of the Nakayama automorphism ψ . Throughout this section we look the parity τ of tr as a number in $\{0, 1\}$.

In this section we classify the simple $\mathcal{A}_n(F)$ -modules.

5.1 Simple $P_1(F)$ -modules

Giving an automorphism of any algebra A , $\varphi: A \rightarrow A$, and fixing $\varepsilon \in \mathbb{Z}_2$, we define ε -twist by φ to be the endofunctor in $F\text{-mod}$

$$\varphi(\Pi^\varepsilon(-)): M \mapsto \varphi(\Pi^\varepsilon(M)) = \begin{cases} \varphi M & \text{if } \varepsilon = 0, \\ \varphi(\Pi M) & \text{if } \varepsilon = 1, \end{cases}$$

that simultaneously twist the module by ϕ and shift its parity by ε (see Section 2.1).

Now, fix a simple F -module L . For $k \in \mathbb{Z}$, write ${}^k L$ for ${}^{\psi^k}(\Pi^{\tau|k|}(L))$, i.e., the module obtained by τ -shifting $|k|$ times by ψ (notice that the twisting commutes with parity shifting). Define $s_L \in \mathbb{N}$ to be the number of distinct isomorphism classes of modules obtained from L by τ -twisting by ψ . It follows that ${}^{s_L} L \simeq L$. Let m_L be the smallest *positive* integer such that $\psi^{m_L}(f)(v) = f(v)$ for all $f \in F$ and $v \in L$. Then m_L divides θ , and since ${}^{(1+\tau)m_L} L = \psi^{(1+\tau)m_L} L \simeq L$, we have that s_L divides $(1+\tau)m_L$.

Example 5.1.1. (a) Let F be a symmetric Frobenius algebra (i.e, a Frobenius algebra with Nakayama automorphism equals to identity map) and let L be a simple F -module. Then $1 = m_L = \theta$. For $\tau = 0$ we have that $s_L = 1$. If $\tau = 1$, then s_L is 1, if L is of type $\mathbb{Q}$, and 2 otherwise.

(b) If $F = Cl_\tau$ (see 3.2.2), and L is its unique simple module (that is the regular module $Cl Cl$), then $s_L = 1$ (since there is only one simple module and it is of type $\mathbb{Q}$). Since $\psi(c) = (-1)^{(1+\tau)}c$, we obtain

$$m_L = \theta = 2, \text{ if } \tau = 0, \quad \text{and} \quad m_L = \theta = 1, \text{ if } \tau = 1.$$

(c) Let $F = D$ is the dual algebra with basis $\{1, u\}$, as defined in Example 2.2.19. Then, since u is nilpotent, F has only two simple modules L (equal to $\mathbb{k}$ with x acting by zero) and ΠL , both of type $\mathbb{M}$. Since F is symmetric, we have $s_L = 2$ and $m_L = 1 = \theta$.

(d) Let $F = T_{q,\tau}$ be the (q, τ) -Taft Hopf algebra as defined in Example 2.2.20. Let $\omega \in \mathbb{k}$ be the $(1+\tau)q$ -root of $1 \in \mathbb{k}$ in the definition of $T_{q,\tau}$. Then, since y is nilpotent, simple F -modules are equivalent to simple $F/\langle y \rangle$ -modules. Further $F/\langle y \rangle$ is isomorphic to the group superalgebra of the cyclic group of order $(1+\tau)q$, with g of parity τ .

For $\tau = 0$, any F -module is of type $\mathbf{M}$ and isomorphic to L_k , for some $k \in \mathbb{Z}_{q-1}$, where L_k is the one dimension superspace spanned by the even vector v_0 . The action of g in L_k is given by multiplication by ω^k . We also have that g acts in ${}^1L_k = {}^\psi L_k$ by

$$g \cdot v_0 = \psi^{-1}(g)v_0 = \omega^{-1}gv_0 = \omega^{k-1}v_0.$$

Hence ${}^1L_k \simeq L_{k-1}$.

For $\tau = 1$, we have that any simple F -module is of type $\mathbf{Q}$ and isomorphic to L_k , for some $k \in \mathbb{Z}_{2q-1}$, where L_k is a two-dimension super space spanned by $\{v_0, v_1\}$ (where $\bar{v}_i = i \in \mathbb{Z}_2$) with the action of g given by

$$g \cdot v_0 = \omega^k v_1 \quad \text{and} \quad g \cdot v_1 = \omega^k v_0.$$

We have that 1L_k is the super space spanned by $\{\Pi(v_0), \Pi(v_1)\}$ (where $\Pi(v_i)$ is the image of the parity-shift in v_i) with the action of g given by

$$\begin{aligned} g \cdot \Pi(v_0) &= (-1)^{\bar{g}} \Pi(\psi^{-1}(g) \cdot v_0) = -\omega^{-1} \Pi(g \cdot v_0) \\ &= -\omega^{k-1} \Pi(v_1) = \omega^{q+k-1} \Pi(v_1), \end{aligned} \quad (-1 = \omega^q).$$

Analogously, $g \cdot \Pi(v_1) = \omega^{q+k-1} \Pi(v_0)$. It is easy to see that ${}^1L_k \simeq L_{k+q-1}$.

For both case, τ equals to 0 and 1, we have that s_{L_k} satisfies condition $(1+\tau)q | s_{L_k}(q-1)$. Since $(1+\tau)q$ and $q-1$ are co-prime, it implies that $s_{L_k} = (1+\tau)q$. Therefore $\theta = m_{L_k} = (1+\tau)q = s_{L_k}$.

By Schur's Lemma, there is an even F -module isomorphism

$$\sigma_L: L \rightarrow {}^{s_L}L, \quad (5.1.1)$$

which is unique up to a nonzero scalar.

Then, denoting the action on ${}^{s_L}L$ by “ $\cdot$ ”, and the action on L by juxtaposition, we have, for all $f \in F$ and $v \in L$:

$$\sigma_L(fv) = f \cdot \sigma_L(v) = (-1)^{\tau s_L \bar{f}} \psi^{-s_L}(f) \sigma_L(v).$$

By induction, it is easy to see that, for all $k \in \mathbb{N}$, we have $\sigma_L^k(fv) = (-1)^{\tau k s_L \bar{f}} \psi^{-k s_L}(f) \sigma_L^k(v)$, for all $f \in F$ and $v \in L$. In particular, for $k = (1+\tau)m_L/s_L$, we have:

$$\sigma_L^{(1+\tau)m_L/s_L}(fv) = f \sigma_L^{(1+\tau)m_L/s_L}(v),$$

for all $f \in F$ and $v \in V$. Thus $\sigma_L^{(1+\tau)m_L/s_L}$ is an even isomorphism $L \simeq L$. Then, by Schur's Lemma $\sigma_L^{(1+\tau)m_L/s_L}$ is a multiple of the identity map by a nonzero scalar.

Rescaling σ_L if necessary, we may assume that

$$\sigma_L^{(1+\tau)m_L/s_L} = \text{Id}_L \quad (5.1.2)$$

Now, for $a \in \mathbb{k}$ we define an $P_1(F)$ -module $L(a)$ as follows. We let

$$L(a) := \bigoplus_{\ell=0}^{s_L-1} {}^\ell L, \quad (5.1.3)$$

as F -modules. The action of x is given by

$$x(v_0, \dots, v_{s_L-1}) = (v_1, v_2, \dots, v_{s_L-1}, a\sigma_L(v_0)), \quad v_i \in {}^i L, \quad 1 \leq i \leq s_L - 1.$$

Lemma 5.1.2. *Let $a \in \mathbb{k}^\times$, then $L(a)$ is a simple $P_1(F)$ -module.*

Proof. Let $s = s_L$. First of all, we claim that the action of x is of parity τ . Let $i \in \{0, \dots, s-1\}$ and let $\mu_i: {}^i L \rightarrow L(a)$ be the canonical inclusion. Then for all homogeneous $v_i \in {}^i L$ (with $i \geq 1$), we have

$$\overline{x\mu_i(v_i)} + \tau = \overline{\mu_{i-1}(v_i)} + \tau = \overline{\mu_i(v_i)} = \bar{v}_i.$$

For a homogeneous $v_0 \in L$, notice that the parity of $\mu_s(v_0)$ is $\bar{v}_0 + \tau(s-1)$. Also, considering σ_L as a linear map $L \rightarrow L$ of parity τs , we have that $\overline{\sigma_L(v_0)} = \bar{v}_0 + \tau s$. Then

$$\overline{x\mu_0(v_0)} = \overline{\mu_s(a\sigma_L(v_0))} = \bar{v}_0 + \tau(s-1) + \tau s = \bar{v}_0 - \tau.$$

Therefore the action of x is of parity τ .

For $f \in F$, we have

$$\begin{aligned} (-1)^{\tau\bar{f}} x\psi(f)(v_0, \dots, v_{s-1}) &= (-1)^{\tau\bar{f}} x \left(\psi(f)v_0, (-1)^{\tau\bar{f}} f v_1, \dots, (-1)^{\tau(s-1)\bar{f}} \psi^{2-s}(f)v_{s-1} \right) \\ &= \left(f v_1, (-1)^{\tau\bar{f}} \psi^{-1}(f)v_2, \dots, (-1)^{\tau(s-2)\bar{f}} \psi^{2-s}(f)v_{s-1}, (-1)^{\tau\bar{f}} a\sigma_L(\psi(f)v_0) \right) \end{aligned}$$

and

$$\begin{aligned} f x(v_0, \dots, v_{s-1}) &= f(v_1, v_2, \dots, v_{s-1}, a\sigma_L(v_0)) \\ &= \left(f v_1, (-1)^{\tau\bar{f}} \psi^{-1}(f)v_2, \dots, (-1)^{\tau(s-2)\bar{f}} \psi^{2-s}(f)v_{s-1}, (-1)^{(s-1)\bar{f}} \psi^{1-s}(f)a\sigma_L(v_0) \right) \\ &= \left(f v_1, (-1)^{\tau\bar{f}} \psi^{-1}(f)v_2, \dots, (-1)^{\tau(s-2)\bar{f}} \psi^{2-s}(f)v_{s-1}, a(-1)^{\tau\bar{f}} \sigma_L(\psi(f)v_0) \right). \end{aligned}$$

And hence the action of x satisfies the relation (3.1.4), for all $f \in F$. Therefore $L(a)$ is a $P_1(F)$ -module.

To see that $L(a)$ is simple when $a \in \mathbb{k}^\times$, suppose V is a nonzero $P_1(F)$ -submodule of $L(a)$. Since V is an F -submodule, it is a sum of some of the summands ${}^\ell L$. But x acts on ${}^\ell L$ by, simultaneously, parity shift and twist by ψ . Hence, the invariance under the action of x implies that V must contain *all* the summands ${}^\ell L$ (whose quantity is exactly s_L , by definition). $\square$

Remark 5.1.3. Note that the indexing $L(a)$ depends on the choice of σ_L . In light of the condition (5.1.2), the choice of σ_L is unique up to multiplication by an $((1+\tau)m_L/s_L)$ -th root of unity. Different choices of σ_L simply shift the parameter a multiplying by this root of unity.

Example 5.1.4. (a) Let F be a symmetric Frobenius algebra, L be a simple F -module and $a \in \mathbb{k}^\times$. We have $m_L = \theta = 1$.

If $\tau = 0$, then $s_L = 1$ and we have that $L(a)$ is equal L as F -module, with x acting as a .

If $\tau = 1$, we have two cases. If L is of type $\mathbf{M}$, then $s_L = 2$ and $L(a)$ is equal to $L \oplus \Pi L$ as F -module and action of x is given by $x(v_0, v_1) = (v_1, av_0)$, for all $v_0 \in L$ and $v_1 \in \Pi L$. If L is of type $\mathbf{Q}$, then $s_L = 1$ and $L(a)$ is equal to L as F -module, while x acting as $a\phi_L$, where $\phi_L: L \rightarrow L$ is an odd involution.

- (b) Let $F = D$ is the dual algebra with basis $\{1, u\}$, as defined in Example 2.2.19, and let L be a simple F -module and $a \in \mathbb{k}^\times$. Since u is nilpotent, we have that L is of type $\mathbf{M}$ and is isomorphic to either $\mathbb{k}$ or $\Pi\mathbb{k}$, with u acting by zero. Since D is a symmetric algebra we have that $L(a)$ is equal to $L \oplus \Pi L$ as F -module with x acting by $x(v_0, v_1) = (v_1, av_0)$, for all $v_0 \in L$ and $v_1 \in \Pi L$. Clearly $L(a) \simeq \Pi L(a)$.
- (c) Let $F = Cl_\tau$ as in Example 3.2.2. Its unique simple F -module L is isomorphic to the regular F -module ${}_C Cl$. Then $\sigma_L: L \rightarrow {}^\psi L$ is an isomorphism of parity τ such that $\sigma^2 = \sigma_L^{(1+\tau)m_L/s_L} = \text{Id}$ and we can choose $\sigma_L(1) = (\sqrt{-1}c)^\tau$ and $\sigma_L(c) = -(\sqrt{-1}c)^\tau c$. Then, for all $a \in \mathbb{k}^\times$, $L(a)$ is equal to L as F -module, while x acting as

$$x \cdot v = (-1)^{\bar{v}} a (\sqrt{-1}c)^\tau v,$$

for all homogeneous $v \in L$.

- (d) Let $F = T_{q,\tau}$ be the (q, τ) -Taft Hopf algebra and L_k be a simple F -module as defined in Example 5.1.1(d). Then $\sigma_{L_k}: L_k \rightarrow L_k$ is an even isomorphism such that $\text{Id}_{L_k} = \sigma_{L_k}^{(1+\tau)m_L/s_L} = \sigma_{L_k}^{(1+\tau)}$. Thus $\sigma_{L_k} = \text{Id}_{L_k}$. Therefore, for all $a \in \mathbb{k}$, the $P_1(F)$ -module $L_k(a)$ decompose

$$L_k(a) = \bigoplus_{\ell=0}^{(1+\tau)q-1} {}^\ell L_k = \bigoplus_{\ell=0}^{(1+\tau)q-1} L_{k+\ell(q-1)},$$

as F -module, with action of x given by cyclic permutation to the left of the coordinates, after multiply the first coordinates by a .

Lemma 5.1.5. *Suppose L and L' are simple F -modules and $a, b \in \mathbb{k}^\times$. The $P_1(F)$ modules $L(a)$ and $L'(b)$ are evenly isomorphic if and only if $a = b$ and $L' \simeq {}^\ell L$ as F -modules for some $\ell \in \{0, \dots, s_L - 1\}$.*

Proof. Suppose $L(a) \simeq L'(b)$ are evenly isomorphic as $P_1(F)$ -modules. Considering the decomposition as F -modules, we see that $s_L = s_{L'}$ and $L' \simeq {}^\ell L$ for some $\ell \in \{0, \dots, s_L - 1\}$. Relabelling if necessary, we may assume $L = L'$. Then, since x^{s_L} acts on $L(a)$ as $a\sigma_L$ and on $L(b)$ as $b\sigma_L$, it follows that $a = b$. The converse statement is clear. $\square$

For an F -module V , let $V(0)$ denote the $P_1(F)$ -module equals to V and with x acting by zero. Clearly $V(0)$ is simple if V is, and $V(0) \simeq V'(0)$ as $P_1(F)$ -modules if and only if $V \simeq V'$ as F -modules.

Proposition 5.1.6. *The modules*

$$L(a), \quad L \in \mathcal{S}(F), \quad a \in \mathbb{k},$$

exhaust, up to even isomorphism and degree shift, all simple $P_1(F)$ -modules. Furthermore,

- if $a \neq b$, then $L(a) \not\simeq L(b)$,

- $L(0) \simeq L'(0)$ if and only if $L \simeq L'$, and
- for $a \neq 0$, $L(a) \simeq L'(a)$ if and only if $L' \simeq {}^\ell L$ for some $\ell \in \mathbb{Z}$.

Proof. We have to prove that every simple $P_1(F)$ -module is evenly isomorphic to $L(a)$ for some simple F -module L and $a \in \mathbb{k}$.

Let V be a simple $P_1(F)$ -module. Shifting the degree if necessary, we assume that V is concentrated in nonnegative degree with nonzero degree zero piece. Since the center of $P_1(F)$ contains $\mathbb{k}[x^{2\theta}]$, and $P_1(F)$ is of finite rank over $\mathbb{k}[x^{2\theta}]$, it follows that V is finite dimensional. Now let L be a simple F -submodule of V . Then it follows by

$$x^k(fv) = (x^k f)v = ((-1)^{\tau k \bar{f}} \psi^{-k}(f)x)v = (-1)^{\tau k \bar{f}} \psi^{-k}(f)(xv) = f \cdot (xv),$$

that $x^k L$ is either zero or is simple F -submodule of V evenly isomorphic to ${}^\ell L$.

Since $\sum_{k=0}^{\infty} x^k L$ is a $P_1(F)$ -submodule of V and V is simple as a $P_1(F)$ -module, we have $V = \sum_{k=0}^{\infty} x^k L$ (with the sum actually being finite by the finite-dimensionality of V), and so V is semisimple as an F -module.

It follows that

$$V = \bigoplus_{\ell=0}^{s_L-1} V_\ell, \quad (5.1.4)$$

where V_ℓ is the ${}^\ell L$ -isotypic component of V , i.e., the sum of all submodules of V isomorphic to ${}^\ell L$.

Now, if x acts by zero on any F -submodule of a summand V_ℓ in (5.1.4), that submodule would be a $P_1(F)$ -submodule, and hence equals to all of V . Since V is simple it implies that $V_\ell \simeq {}^\ell L$. Thus $V \simeq {}^\ell L(0)$.

Now assume that x does not act by zero on any F -submodule of a summand in (5.1.4). It follows from Schur's Lemma that the action of x induces linear isomorphisms $V_\ell \rightarrow V_{\ell+1}$ for all $0 \leq \ell \leq s_L - 1$. In addition, x^{s_L} induces a linear automorphism of V_0 . Let $v \in V_0$ be an eigenvector of x^{s_L} . Relation (3.1.4) implies that x^{s_L} leaves Fv invariant. Then $\bigoplus_{\ell=0}^{s_L-1} x^\ell Fv$ is a $P_1(F)$ -submodule of V , with $x^\ell Fv \subseteq V_\ell$. Since V is simple, it follows that $Fv = V_0$. So V_0 is simple and hence isomorphic to L .

It follows from the definition of s_L that the action of x^{s_L} induces an even isomorphism $L \simeq {}^{s_L} L$. Thus there exists some nonzero scalar $a \in \mathbb{k}^\times$ such that x^{s_L} acts by $a s_L$. Then clearly $V \simeq L(a)$. $\square$

5.2 Action of $t_{k,\ell}$

For $\mathbb{Z}$ -graded superspaces V_1 and V_2 define the following map:

$$\text{flip}: V_1 \otimes V_2 \rightarrow V_2 \otimes V_1, \quad \text{flip}(v_1 \otimes v_2) = (-1)^{\bar{v}_1 \bar{v}_2} v_2 \otimes v_1. \quad (5.2.1)$$

In particular, $\text{flip}: F^{\otimes 2} \rightarrow F^{\otimes 2}$ is an algebra automorphism, and for F -modules V_1 and V_2 , we can consider ${}^{\text{flip}}(V_1 \boxtimes V_2)$. It follows that

$$\text{flip}: {}^{\text{flip}}(V_1 \boxtimes V_2) \rightarrow V_2 \boxtimes V_1 \quad (5.2.2)$$

is an even isomorphism of $F^{\otimes 2}$ -modules.

Lemma 5.2.1. *Suppose L and L' are simple F -modules.*

- (a) *The element $t_{1,2} \in F^{\otimes 2}$ acts as zero on the $F^{\otimes 2}$ -module $L \otimes L'$ unless ${}^1L \simeq L \cong L'$ and $\delta = 0$ (i.e. F is concentrated in degree zero).*
- (b) *If ${}^1L \simeq L$ as F -modules and $\delta = 0$, then $t_{1,2}$ acts on $L \otimes L$ either as zero or as $\text{flip} \circ (\sigma \otimes \text{Id})$, where $\sigma: L \rightarrow {}^\psi L$ is an isomorphism of F -modules of parity τ .*

Proof. By Lemma 3.3.2, we have

$$\begin{aligned} t_{1,2}(f \otimes g) &= (-1)^{\bar{f}\bar{g} + \tau(\bar{f} + \bar{g})} (g \otimes \psi^{-1}(f)) t_{1,2} \\ &= (-1)^{\tau(\bar{f} + \bar{g})} (\text{Id} \otimes \psi^{-1}) \circ \text{flip}(f \otimes g) t_{1,2}, \quad f, g \in F. \end{aligned}$$

Thus, $t_{1,2}$ induces an homomorphism of $F^{\otimes 2}$ -modules

$$L \boxtimes L' \rightarrow {}^{\text{flip}}(L \boxtimes {}^\psi L'),$$

which is of parity τ (the parity of $t_{1,2}$). By (5.2.2), we then have a homomorphism of $F^{\otimes 2}$ -modules of parity τ

$$\text{flip} \circ t_{1,2}: L \boxtimes L' \rightarrow {}^\psi L' \boxtimes L.$$

But, since $\Pi(V \boxtimes W) \simeq (\Pi V) \boxtimes W$, for any supermodules V and W , we have that

$$\text{flip} \circ t_{1,2}: L \boxtimes L' \rightarrow ({}^1 L') \boxtimes L.$$

is an *even* homomorphism of $F^{\otimes 2}$ -modules. Then since L and L' are simple F -modules, $\text{flip} \circ t_{1,2}$ induces an even homomorphism of $F^{\otimes 2}$ -modules (see Section 2.1):

$$\text{flip} \circ t_{1,2}: L \otimes L' \rightarrow ({}^1 L') \otimes L.$$

Thus, by Schur's Lemma, $t_{1,2}$ acts as zero unless $L \otimes L' \simeq ({}^1 L') \otimes L$ as $F^{\otimes 2}$ -modules.

But $L \otimes L' \simeq ({}^1 L') \otimes L$ holds if and only if $L \cong {}^1 L' = {}^\psi(\Pi^\tau(L'))$ and $L' \cong L$, and these isomorphism have both the same parity, and that condition is equivalent to ${}^1 L \simeq L \cong L'$

Furthermore, since $|t_{1,2}| = \delta$, it follows that $t_{1,2}$ must act as zero if $\delta > 0$ (since isomorphisms live in degree zero). This proves part (a).

Now suppose $\sigma: L \rightarrow {}^\psi L$ is an isomorphism of F -modules of parity τ and $\delta = 0$. Then

$$(\sigma^{-1} \otimes \text{Id}) \circ \text{flip} \circ t_{1,2}: L \otimes L \rightarrow L \otimes L$$

is an even homomorphism of $F^{\otimes 2}$ -modules and by Schur's Lemma must be multiplication by a scalar. If this scalar is nonzero, we may rescale σ so that this scalar equals to one. And this completes the prove. $\square$

For $k, \ell \in \{1, \dots, n\}$, $k < \ell$ and $\mathbb{Z}$ -graded vector spaces $V_1, \dots, V_n$, define the linear map

$$\rho_{k,\ell}: V_1 \otimes \dots \otimes V_n \rightarrow V_1 \otimes \dots \otimes V_\ell \otimes \dots \otimes V_k \otimes \dots \otimes V_n, \quad (5.2.3)$$

that supercommutes the k -th and ℓ -th factors. In particular $\rho_{k,\ell}: F^{\otimes n} \rightarrow F^{\otimes n}$ is algebra automorphism and, for F -modules $V_1, \dots, V_n$ and $k, \ell \in \{1, \dots, n\}$, $k < \ell$, we have that

$$\rho_{k,\ell}: V_1 \boxtimes \dots \boxtimes V_n \rightarrow \rho_{k,\ell}(V_1 \boxtimes \dots \boxtimes V_\ell \boxtimes \dots \boxtimes V_k \boxtimes \dots \boxtimes V_n) \quad (5.2.4)$$

is an even isomorphism for $F^{\otimes n}$ -modules.

Corollary 5.2.2. *Let $L_1, \dots, L_n$ be simple F -modules. Suppose $k, \ell \in \{1, \dots, n\}$ and $k < \ell$. If $L_k = L_\ell$ and $\delta = 0$, then $t_{k,\ell}$ acts on $L_1 \otimes \dots \otimes L_n$ either as zero or as*

$$\rho_{k,\ell} \circ \left(\text{Id}^{\otimes(k-1)} \otimes \sigma \otimes \text{Id}^{\otimes(n-k)} \right),$$

where $\sigma: L_k \rightarrow {}^\psi L_k$ is an isomorphism of F -modules of parity τ . Otherwise, if $L_k \not\cong L_\ell$ or $\delta \neq 0$, then $t_{k,\ell}$ acts as zero.

Proof. By Lemma 3.3.2, we have that $t_{k,\ell}$ induces an homomorphism of $F^{\otimes n}$ -modules of parity τ :

$$\rho_{k,\ell}(L_1 \boxtimes \dots \boxtimes L_n) \rightarrow L_1 \boxtimes \dots \boxtimes {}^\psi L_\ell \boxtimes \dots \boxtimes L_k \boxtimes \dots \boxtimes L_n. \quad (5.2.5)$$

Now, notice that $\rho_{j,j+1} = \text{Id}^{\otimes(j-1)} \otimes \text{flip} \otimes \text{Id}^{\otimes(n-j-1)}$ for all $j \in \{1, \dots, n\}$. Since we have that

$$\text{flip}: {}^{\text{flip}}(V_1 \boxtimes V_2) \rightarrow \Pi^\tau(V_2) \boxtimes V_1$$

is a isomorphism of parity τ , for all F -modules V_1, V_2 , and since

$$\rho_{k,\ell} = \rho_{\ell-1,\ell} \circ \rho_{\ell-2,\ell-1} \circ \dots \circ \rho_{k,k+1} \circ \rho_{k+1,k+2} \circ \dots \circ \rho_{\ell-1,\ell},$$

it follows that

$$\rho_{k,\ell}: L_1 \boxtimes \dots \boxtimes L_n \rightarrow \rho_{k,\ell}(L_1 \boxtimes \dots \boxtimes \Pi^\tau(L_\ell) \boxtimes \dots \boxtimes L_k \boxtimes \dots \boxtimes L_n)$$

is an isomorphism of $F^{\otimes n}$ -modules of parity τ .

Then, by (5.2.5) and the arguments in 2.1,

$$\rho_{k,\ell} \circ t_{k,\ell}: L_1 \otimes \dots \otimes L_n \rightarrow L_1 \otimes \dots \otimes {}^1 L_\ell \otimes \dots \otimes L_k \otimes \dots \otimes L_n$$

is an *even* isomorphism of $F^{\otimes n}$ -modules. To finish the proof we use argument from Lemma 5.2.1. $\square$

Example 5.2.3. (a) Suppose F is a symmetric Frobenius algebra of parity τ and let $L_1, \dots, L_n \in \mathcal{S}(F)$. Then $t_{k,\ell}$ acts on $L_1 \otimes \dots \otimes L_n$ as zero, unless $L_k \cong L_\ell$ and $\delta = 0$, which, in this case, $t_{k,\ell}$ acts as zero or as

$$\rho_{k,\ell} \circ \left(\text{Id}^{\otimes(k-1)} \otimes \sigma \otimes \text{Id}^{\otimes(n-k)} \right),$$

where $\sigma: L_k \rightarrow L_k$ is an isomorphism of F -modules of parity τ .

(b) If $F = D$ the dual algebra (see 2.2.19), we have that $t_{k,\ell} = u_\ell - u_k$. Since u acts as zero for any simple D -module, it follows that $t_{k,\ell}$ acts as zero in $L_1 \otimes \dots \otimes L_n$, for all $L_i \in \mathcal{S}(D)$.

- (c) Let $F = \mathcal{Cl}_\tau$ and $\delta = 0$. Consider $L = {}_{\mathcal{Cl}}\mathcal{Cl}$ the unique simple $\mathcal{Cl}$ -module. We have that $\sigma(1) = (\sqrt{-1}c)^\tau$, $\sigma(c) = -(\sqrt{-1}c)^\tau c$ define an isomorphism $L \cong L$ of parity τ such that $\sigma^2 = \text{Id}$. Then $t_{1,2}$ acts on $L \otimes L$ as a nonzero scalar multiple of $\text{flip} \circ (\sigma \otimes \text{Id})$.

Actually, choosing $L \otimes L$ to be isomorphic to the submodule of $L \boxtimes L$ generated by

$$v_0 = \frac{1}{2}(1 \otimes 1 + \sqrt{-1}(c \otimes c)), \quad \text{and} \quad v_1 = \frac{1}{2}(c \otimes 1 - \sqrt{-1}(1 \otimes c)),$$

(see Section 2.1), we have that $t_{1,2}$ acts on $L \otimes L$ as $(-1)^\tau(1 + \sqrt{-1})\text{flip} \circ (\sigma \otimes \text{Id})$. Indeed, since $t_{1,2} = 1 \otimes c^\tau - c \otimes c^{\tau+1}$, we have that:

$$\begin{aligned} 2t_{1,2}v_0 &= (1 \otimes c^\tau - c \otimes c^{\tau+1})(1 \otimes 1 + \sqrt{-1}(c \otimes c)) \\ &= 1 \otimes c^\tau + (-1)^\tau \sqrt{-1}(c \otimes c^{\tau+1}) - c \otimes c^{\tau+1} - \sqrt{-1}(-1)^{\tau+1}(1 \otimes c^\tau) \\ &= (1 + (-1)^\tau \sqrt{-1})(1 \otimes c^\tau + (-1)^\tau \sqrt{-1}(c \otimes c^{\tau+1})) \\ &= (1 + (-1)^\tau \sqrt{-1})\text{flip}(c^\tau \otimes 1 - \sqrt{-1}(c^{\tau+1} \otimes c)) \\ &= \frac{(1 + (-1)^\tau \sqrt{-1})}{\sqrt{-1}^\tau} \text{flip}(\sigma(1) \otimes 1 + \sqrt{-1}(\sigma(c) \otimes c)) \\ &= 2(-1)^\tau(1 + \sqrt{-1})\text{flip} \circ (\sigma \otimes \text{Id})(v_0). \end{aligned}$$

A calculation shows $(-1)^\tau(1 + \sqrt{-1})\text{flip} \circ (\sigma \otimes \text{Id})(v_1) = t_{1,2}v_1$.

5.3 Associated algebras

Recall that ψ acts on the set $\mathcal{S}(F)$ by τ -twisting (see page 56). Let N be the number of ψ -orbits in $\mathcal{S}(F)$, and let

$$\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_N \tag{5.3.1}$$

be a set of representatives of these orbits. For $k \in \{1, 2, \dots, N\}$, let $s_k := s_{\mathbf{L}_k}$. Then

$$\mathbf{L}_1, {}^1\mathbf{L}_1, \dots, {}^{s_1-1}\mathbf{L}_1, \dots, \mathbf{L}_N, {}^1\mathbf{L}_N, \dots, {}^{s_N-1}\mathbf{L}_N,$$

is an enumeration of the elements of $\mathcal{S}(F)$.

Let

$$\mathcal{C}_n := \{\mu = (\mu_1, \dots, \mu_N) \mid \mu_1, \dots, \mu_N \in \mathbb{N}, \mu_1 + \dots + \mu_N = n\} \tag{5.3.2}$$

denote the set of compositions of n of length at most N and, for $\mu \in \mathcal{C}_n$, let

$$S_\mu = S_{\mu_1} \times \dots \times S_{\mu_N} \tag{5.3.3}$$

denote the corresponding Young subgroup of S_n .

Then, for $\mu \in \mathcal{C}_n$, let $\mathcal{A}_\mu(F)$ denote the *parabolic subalgebra* of $\mathcal{A}_n(F)$ generated by $F^{\otimes n}$, $x_1, \dots, x_n$, and S_μ . So we have an isomorphism of graded superalgebras

$$\mathcal{A}_\mu(F) \cong \mathcal{A}_{\mu_1}(F) \otimes \dots \otimes \mathcal{A}_{\mu_N}(F),$$

and an even isomorphism of $(\mathbb{Z} \times \mathbb{Z}_2)$ -graded $\mathbb{k}$ -modules

$$\mathcal{A}_\mu(F) \simeq P_n(F) \otimes \mathbb{k}S_\mu.$$

We denote by Ind_μ^n the induction functor

$$\text{Ind}_\mu^n := \mathcal{A}_n(F) \otimes_{\mathcal{A}_\mu(F)} - : \mathcal{A}_\mu(F)\text{-mod} \rightarrow \mathcal{A}_n(F)\text{-mod}. \quad (5.3.4)$$

Let $\mathcal{A}_n(F)\text{-mod}^s$ (respectively, $\mathcal{A}_\mu(F)\text{-mod}^s$, for $\mu \in \mathcal{C}_n$) denote the full subcategory of $\mathcal{A}_n(F)\text{-mod}$ (resp., $\mathcal{A}_\mu(F)\text{-mod}$) consisting of finite-dimensional $\mathcal{A}_n(F)$ -modules that are semisimple as $F^{\otimes n}$ -modules.

For $\mu \in \mathcal{C}_n$ and $1 \leq k \leq n$, let ℓ_k denote the unique integer such that

$$\mu_1 + \cdots + \mu_{\ell_k-1} < k \leq \mu_1 + \cdots + \mu_{\ell_k}. \quad (5.3.5)$$

For $1 \leq k \leq N$, let $\sigma_k = \sigma_{\mathbf{L}_k}$ (see (5.1.1)). If $t_{1,2}$ acts as zero on $\mathbf{L}_k \otimes \mathbf{L}_k$, let $\mathbf{a}_k = 1$. (This choice is not crucial; we choose $\mathbf{a}_k = 1$ for simplicity). Otherwise, let $b_k \in \mathbb{K}^\times$ such that $t_{1,2}$ acts on $\mathbf{L}_k \otimes \mathbf{L}_k$ as $\text{flip} \circ (b_k \sigma_k \otimes \text{Id})$ (see Lemma 5.2.1), and define:

$$\mathbf{a}_k = \begin{cases} b_k & \text{if } \mathbf{L}_k \text{ is of type } \mathbf{M}, \\ \frac{(1-\sqrt{-1})}{2} b_k & \text{if } \mathbf{L}_k \text{ is of type } \mathbf{Q}. \end{cases}$$

Let

$$\mathbf{L}(\mu) := \mathbf{L}_1(\mathbf{a}_1)^{\boxtimes \mu_1} \boxtimes \cdots \boxtimes \mathbf{L}_N(\mathbf{a}_N)^{\boxtimes \mu_N}. \quad (5.3.6)$$

Thus $\mathbf{L}(\mu)$ is a $P_n(F)$ -module on which the action of x_k is invertible for all $1 \leq k \leq n$. By a slight abuse of notation, we will write x_k^{-1} for the endomorphism of $\mathbf{L}(\mu)$ that is the inverse of the action of x_k .

Consider

$$\mathbf{E}(\mu) = \text{END}_{F^{\otimes n}}^{\text{op}} \mathbf{L}(\mu) \stackrel{2.1.11}{\simeq} (\text{END}_F^{\text{op}} \mathbf{L}_1(\mathbf{a}_1))^{\otimes \mu_1} \otimes \cdots \otimes (\text{END}_F^{\text{op}} \mathbf{L}_N(\mathbf{a}_N))^{\otimes \mu_N}. \quad (5.3.7)$$

Then $\mathbf{L}(\mu)$ is naturally a right $\mathbf{E}(\mu)$ -module via the action

$$vz = (-1)^{\bar{v}\bar{z}} zv, \quad z \in \mathbf{E}(\mu), \quad v \in \mathbf{L}(\mu).$$

Let $1 \leq k \leq N$. Since

$$\text{END}_F \mathbf{L}_k(\mathbf{a}_k) \simeq (\text{END}_F \mathbf{L}_k)^{\oplus r_k},$$

we have a natural right action of $\mathbb{K}[y_k]$ on $\text{END}_F^{\text{op}} \mathbf{L}_k(\mathbf{a}_k)$ given by

$$z \cdot y = (-1)^{\tau \bar{z}} x^{-1} z x, \quad z \in \text{END}_F^{\text{op}} \mathbf{L}_k(\mathbf{a}_k), \quad (5.3.8)$$

where x denotes the endomorphism of $\mathbf{L}_k(\mathbf{a}_k)$ defined by the action of x , and x^{-1} denotes its inverse. Thus we can form the smash product $\mathbb{K}[y] \ltimes \text{END}_F^{\text{op}} \mathbf{L}_k(\mathbf{a}_k)$ (see Section 2.3). In particular, it is satisfied the following relation:

$$zy = y(z \cdot y), \quad \forall z \in \text{END}_F^{\text{op}} \mathbf{L}_k(\mathbf{a}_k). \quad (5.3.9)$$

Now, notice that, when $t_{1,2}$ does not act as zero on $\mathbf{L}_\ell \otimes \mathbf{L}_\ell$, it follows that $\mathbf{L}_\ell \simeq {}^1\mathbf{L}_\ell$, and thus $s_\ell = 1$. In particular

$$\text{END}_F^{\text{op}} \mathbf{L}_\ell = \text{END}_F^{\text{op}} \mathbf{L}_\ell(\mathbf{a}_\ell) \simeq \begin{cases} \mathbb{K} & \text{if } \mathbf{L}_\ell \text{ is of type } \mathbf{M}, \\ \mathcal{C}\ell & \text{if } \mathbf{L}_\ell \text{ is of type } \mathbf{Q}, \end{cases}$$

which, in both case, have a Frobenius algebra structure (with trace of even parity).

Then, for $1 \leq \ell \leq N$ and $m \in \mathbb{N}$, we define

$$\mathcal{H}_m^\ell = \begin{cases} (\mathbb{k}[y] \rtimes \text{END}_F^{\text{op}} \mathbb{L}(\mathfrak{a}_\ell))^{\otimes m} \rtimes_\rho S_m & \text{if } t_{1,2} \text{ acts as zero on } L_\ell \otimes L_\ell, \\ \mathcal{A}_m(\text{END}_F^{\text{op}} \mathbb{L}_\ell) & \text{if } t_{1,2} \text{ does not act as zero on } L_\ell \otimes L_\ell, \end{cases} \quad (5.3.10)$$

and

$$\mathcal{R}_n = \bigoplus_{\mu \in \mathcal{C}_n} \mathcal{R}_\mu, \quad \mathcal{R}_\mu = \mathcal{H}_{\mu_1}^1 \otimes \cdots \otimes \mathcal{H}_{\mu_N}^N. \quad (5.3.11)$$

We will denote the polynomial generators in $\mathcal{R}_n$ by $y_1, \dots, y_n$ to avoid confusion with the generators $x_1, \dots, x_n \in \mathcal{A}_n(F)$. Furthermore, when it makes sense, we will denote by $\tilde{t}_{k,\ell} \in \mathcal{R}_\mu$ the element that comes from the definition of affine wreath product algebra contained in $\mathcal{R}_\mu$. Note that $\mathbb{E}(\mu)$ can naturally be viewed as a subalgebra of $\mathcal{R}_\mu$ and that $\mathcal{R}_\mu$ is generated as an algebra by $\mathbb{E}(\mu)$, $y_1, \dots, y_n$, and S_μ .

Example 5.3.1. Specialising F , we have the following associated algebras:

- (a) For $F = Cl_\tau$ (see 2.2.17), we have the regular Cl -module $L = {}_{Cl}Cl$ as the unique simple Cl -module and it is of type $\mathbb{Q}$. Then it follows:

$$N = 1, \quad \mathcal{C}_n = \{(n)\}, \quad \ell_k = 1, 1 \leq k \leq n.$$

We have also that $t_{1,2}$ acts on $L \otimes L$ by $\text{flip} \circ ((-1)^\tau (1 + \sqrt{-1})\sigma_L \otimes \text{Id})$, where σ_L is the odd automorphism of L , satisfying σ_L^2 (see Example 5.2.3(c) for more details). Thus by setting

$$a_1 = (-1)^\tau \frac{1 - \sqrt{-1}}{2} (1 + \sqrt{-1}) = (-1)^\tau,$$

we have that $\mathbb{L}(n) = L(a_1)^{\boxtimes n} = L((-1)^\tau)^{\boxtimes n}$, which is equal to $Cl^{\otimes n}$ as $Cl^{\otimes n}$ -module, with x_i acting by

$$x_i \cdot 1 = (-\sqrt{-1}c_i)^\tau, \quad x_i \cdot c_i = -(-\sqrt{-1}c_i)^\tau c_i,$$

(see 5.1.4(c) for more details).

Finally, we have that

$$\mathbb{E}(n) = \text{END}_{Cl}^{\text{op}}(L((-1)^\tau))^{\otimes n} \simeq Cl^{\otimes n} \quad \text{and} \quad \mathcal{R}_n = \mathcal{H}_n^1 = \mathcal{A}_n(Cl_0).$$

- (b) For $F = D$ the Dual algebra (see 2.2.19), the only two simple D -modules are $L = \mathbb{k}$ and ΠL , where u acts by zero. Then L represents the unique orbit in $\mathcal{S}(D)$ by odd-twisting. Then we obtain:

$$N = 1, \quad \mathcal{C}_n = \{(n)\}, \quad \ell_k = 1, 1 \leq k \leq n.$$

We also have that L is of type $\mathbb{M}$ and that $t_{1,2}$ acts by zero in $L \otimes L = L \boxtimes L$, thus $a_1 = 1$. Then $\mathbb{L}(n) = L(1)^{\boxtimes n}$, which is equal to $(\mathbb{k} \oplus \Pi \mathbb{k})^{\boxtimes n}$ as $D^{\otimes n}$ -module. Write

$$e_1 = (1, 0) \in \mathbb{k} \oplus \Pi(\mathbb{k}), \quad e_{-1} = (0, \Pi(1)) \in \mathbb{k} \oplus \Pi(\mathbb{k}),$$

$$\text{and } \mathbf{e}_{(\alpha_1, \dots, \alpha_n)} = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n} \in (\mathbb{k} \oplus \Pi \mathbb{k})^{\boxtimes n},$$

for all $(\alpha_1, \dots, \alpha_n) \in \{1, -1\}^n$. Then the action of x_i in $\mathbf{L}(n)$ is

$$x_i \cdot \mathbf{e}_{(\alpha_1, \dots, \alpha_n)} = \alpha_1 \cdots \alpha_{i-1} \mathbf{e}_{(\alpha_1, \dots, -\alpha_i, \dots, \alpha_n)},$$

(see 5.1.4(b)).

Finally, we have that

$$\mathbf{E}(n) = (\text{END}_D^{\text{op}}(L(-1)))^{\otimes n} = (\text{END}_{\mathbb{K}}^{\text{op}}(\mathbb{K} \oplus \Pi \mathbb{K}))^{\otimes n} \stackrel{\phi^{\otimes n}}{\simeq} \mathcal{M}(1|1)^{\otimes n},$$

and $\mathcal{R}_n = \mathcal{H}_n^1 = (\mathbb{K}[y] \ltimes \mathcal{M}(1|1))^{\otimes n} \rtimes_{\rho} S_n$, with the action of y in $\mathcal{M}(1|1)$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Here we used the fact that ϕ maps the action of x in $L(1)$ to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

In the next lemma we describe how $t_{k,k+1}$ acts on $\mathbf{L}(\mu)$:

Lemma 5.3.2. *If $\ell_k = \ell_{k+1}$ (i.e. $s_k \in S_{\mu}$), as an operator on $\mathbf{L}(\mu)$, we have*

$$t_{k,k+1} = \begin{cases} 0 & \text{if } t_{1,2} \text{ acts as zero on } \mathbf{L}_{\ell_k} \otimes \mathbf{L}_{\ell_k}, \\ \tilde{t}_{k,k+1} \rho_{k,k+1} x_k & \text{if } t_{1,2} \text{ does not act as zero on } \mathbf{L}_{\ell_k} \otimes \mathbf{L}_{\ell_k}. \end{cases} \quad (5.3.12)$$

Proof. At first, notice that $t_{k,k+1}$ acts on $\mathbf{L}(\mu)$ by

$$\text{Id}^{\otimes(k-1)} \otimes t_{1,2} \otimes \text{Id}^{\otimes(n-k-1)},$$

where we view $t_{1,2}$ as an endomorphism in $\text{END}_{F^{\otimes 2}}(\mathbf{L}_{\ell_k}(\mathbf{a}_k) \boxtimes \mathbf{L}_{\ell_{k+1}}(\mathbf{a}_{k+1}))$. Thus we just need to understand how $t_{1,2}$ acts on $\mathbf{L}_{\ell_k}(\mathbf{a}_k) \boxtimes \mathbf{L}_{\ell_{k+1}}(\mathbf{a}_{k+1})$.

If $t_{1,2}$ acts as zero on $\mathbf{L}_{\ell_k} \otimes \mathbf{L}_{\ell_k}$, then it acts as zero on $\mathbf{L}_{\ell_k}(\mathbf{a}_k) \boxtimes \mathbf{L}_{\ell_{k+1}}(\mathbf{a}_{k+1})$, and therefore $t_{k,k+1}$ acts as zero on $\mathbf{L}(\mu)$.

Suppose that $t_{1,2}$ does not act as zero on $\mathbf{L}_{\ell_k} \otimes \mathbf{L}_{\ell_k}$. Then $s_k = 1$ by Lemma 5.2.1, and thus $\mathbf{L}_{\ell_k}(\mathbf{a}_k) = \mathbf{L}_{\ell_k}$ as F -modules, with x acting on the $\mathbb{K}[x] \ltimes F$ -module $\mathbf{L}_{\ell_k}(\mathbf{a}_k)$ as $\mathbf{a}_k \sigma_k$. By Corollary 5.2.2 and definition of $\mathbf{L}(\mu)$, we have that $t_{1,2}$ acts as $\text{flip} \circ b_k \sigma_k$ on $\mathbf{L}_{\ell_k} \otimes \mathbf{L}_{\ell_k}$.

If $\mathbf{L}_{\ell_k}$ is of type **M**, then $\mathbf{L}_{\ell_k} \otimes \mathbf{L}_{\ell_k} = \mathbf{L}_{\ell_k} \boxtimes \mathbf{L}_{\ell_k}$, $\mathbf{a}_k = b_k$ and clearly $t_{1,2}$ acts on $\mathbf{L}_{\ell_k} \boxtimes \mathbf{L}_{\ell_k}$ as $(\text{flip } x_1)$, which implies that $t_{k,k+1}$ acts on $\mathbf{L}(\mu)$ as $\rho_{k,k+1} x_k$.

Now suppose that $\mathbf{L}_{\ell_k}$ is of type **Q**. Fix an isomorphism $\text{END}_F(\mathbf{L}_{\ell_k}) \simeq \mathcal{Cl}$, which we will view as equality. Then, the isomorphism

$$\text{END}_F(\mathbf{L}_{\ell_k}) \simeq \text{END}_F({}^{\psi}\mathbf{L}_{\ell_k}), \quad z \mapsto (-1)^{\tau \bar{z}} \sigma_k z \sigma_k^{-1},$$

gives us an isomorphism $\text{END}_F({}^{\psi}\mathbf{L}_{\ell_k}) \simeq \mathcal{Cl}$, which we will also view as equality. Thus we have actions of $\mathcal{Cl}$ on $\mathbf{L}_{\ell_k}$ and ${}^{\psi}\mathbf{L}_{\ell_k}$, that super commutes with σ_k . It follows that $\text{Id}_{\mathbf{L}_{\ell_k}}$ is an even isomorphism of $\mathcal{Cl}$ -modules $\mathbf{L}_{\ell_k} \simeq {}^1\mathbf{L}_{\ell_k}$, which we will view as equality. Let $\phi = \frac{1}{2}(\text{Id} + \sqrt{-1}c_1c_2) \in \text{END}_{F^{\otimes 2}}(\mathbf{L}_{\ell_k} \boxtimes \mathbf{L}_{\ell_k})$, where we denote by c the odd generator of $\mathcal{Cl}$ such

that $c^2 = 1$, $c_1 = c \otimes \text{Id}$ and $c_2 = \text{Id} \otimes c$. Then we have

$$\begin{aligned} \mathbb{L}_{\ell_k} \otimes \mathbb{L}_{\ell_k} &= \text{Im}(\phi), \quad \mathbb{L}_{\ell_k} \boxtimes \mathbb{L}_{\ell_k} = \text{Im}(\phi) \oplus \text{Im}(\text{Id} - \phi), \quad c_1 \phi c_1 = \text{Id} - \phi \\ &\text{and } c_1 : \text{Im}(\text{Id} - \phi) \xrightarrow{\sim} \text{Im}(\phi), \end{aligned}$$

(see Section 2.1). Furthermore, it follows that, as an operator in $L_{\ell_k} \boxtimes L_{\ell_k}$,

$$t_{1,2} \stackrel{(2.1.13)}{=} \text{flip} \circ b_k \sigma_k \phi + (-1)^\tau c_1 \text{flip} \circ b_k \sigma_k \phi c_1,$$

since $t_{1,2}$ is an element of $F \otimes F$ of parity τ and acts as $\text{flip} \circ b_k \sigma_k$ on $\text{Im}(\phi)$. Here σ_k acts on the first factor. Finally, we have:

$$\begin{aligned} t_{1,2} &= \text{flip} b_k \sigma_k \phi + (-1)^\tau c_1 \text{flip} b_k \sigma_k \phi c_1 \\ &= \frac{b_k}{2} (\text{flip} \sigma_k + \sqrt{-1} \text{flip} \sigma_k c_1 c_2 + (-1)^\tau (c_1 \text{flip} \sigma_k c_1 + \sqrt{-1} c_1 \text{flip} \sigma_k c_1 c_2 c_1)) \\ &= \frac{b_k}{2} (\text{flip} \sigma_k + \sqrt{-1} \text{flip} c_1 c_2 \sigma_k + c_1 \text{flip} c_1 \sigma_k - \sqrt{-1} c_1 \text{flip} c_2 \sigma_k) \\ &= \frac{b_k}{2} (\text{flip} \sigma_k + \sqrt{-1} c_2 c_1 \text{flip} \sigma_k + c_1 c_2 \text{flip} \sigma_k - \sqrt{-1} \text{flip} \sigma_k) \\ &= \frac{(1 - \sqrt{-1})b_k}{2} (1 + c_1 c_2) \text{flip} \sigma_k = (1 + c_1 c_2) \text{flip} a_k \sigma_k. \end{aligned}$$

Therefore, as an operator on $\mathbb{L}(\mu)$, we have $t_{k,k+1} = (\text{Id} + c_k c_{k+1}) \rho_{k,k+1} x_k$. $\square$

5.4 An equivalence of categories

Fix $\mu \in \mathcal{C}_n$. In this subsection we prove an equivalence of category between $\mathcal{A}_n(F)\text{-mod}^s$ and $\mathcal{R}_\mu\text{-mod}$.

Note that S_μ acts on $\mathbb{L}(\mu)$ by superpermuting the tensor factors. For $\pi \in S_\mu$, we will denote this action on $\mathbb{L}(\mu)$ by ρ_π , to avoid confusion with the action of π on $\mathcal{A}_n(F)$ -modules.

Proposition 5.4.1. *Let $V \in \mathcal{A}_n(F)\text{-mod}^s$. Then $\text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), V)$ is an $\mathcal{R}_\mu$ -module under the action*

$$\begin{aligned} \pi \diamond \phi &= \pi \phi \rho_\pi^{-1}, \quad \pi \in S_\mu, \\ y_k \diamond \phi &= (-1)^{\tau \bar{\phi}} x_k \phi x_k^{-1}, \quad 1 \leq k \leq n, \\ z \diamond \phi &= (-1)^{\bar{\phi} \bar{z}} \phi z, \quad z \in \mathbb{E}(\mu), \end{aligned}$$

for $\phi \in \text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), V)$. Thus, we have a functor

$$\text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), -) : \mathcal{A}_n(F)\text{-mod}^s \rightarrow \mathcal{R}_\mu\text{-mod}.$$

Proof. First we show the actions are well defined. Suppose $v \in \mathbb{L}(\mu)$ and $\mathbf{f} \in F^{\otimes n}$. For $1 \leq$

$k \leq n$, we also have

$$\begin{aligned} (y_k \diamond \phi)(\mathbf{f}v) &= (-1)^{\tau\bar{\phi}} x_k \phi x_k^{-1}(\mathbf{f}v) = (-1)^{\tau(\bar{\phi}+\bar{\mathbf{f}})} x_k \phi \psi_k(\mathbf{f}) x_k^{-1}(v) \\ &= (-1)^{\tau(\bar{\phi}+\bar{\mathbf{f}})+\bar{\phi}\bar{\mathbf{f}}} x_k \psi_k(\mathbf{f}) \phi x_k^{-1}(v) = (-1)^{\tau\bar{\phi}+\bar{\phi}\bar{\mathbf{f}}} \mathbf{f} x_k \phi x_k^{-1}(v) = (-1)^{\bar{\mathbf{f}}(\overline{y_k \diamond \phi})} \mathbf{f}(y_k \diamond \phi)(v). \end{aligned}$$

Thus $y_k \diamond \phi$ is an $F^{\otimes n}$ -module homomorphism. For $\pi \in S_\mu$ and $z \in \mathbf{E}(\mu)$, it is straightforward to verify the $\pi \diamond \phi$ and $z \diamond \phi$ are $F^{\otimes n}$ -module homomorphisms.

Now suppose $1 \leq k \leq n-1$ such that $\ell_k = \ell_{k+1}$ (i.e. such that $s_k \in S_\mu$). Then we have

$$\begin{aligned} (s_k y_k) \diamond \phi &= (-1)^{\tau\bar{\phi}} s_k x_k \phi x_k^{-1} \rho_{k,k+1} \quad \text{and} \\ (y_{k+1} s_k) \diamond \phi &= (-1)^{\tau\bar{\phi}} x_{k+1} s_k \phi \rho_{k,k+1} x_{k+1}^{-1} = (-1)^{\tau\bar{\phi}} x_{k+1} s_k \phi x_k^{-1} \rho_{k,k+1}. \end{aligned}$$

Thus

$$\begin{aligned} (s_k y_k - y_{k+1} s_k) \diamond \phi &= (-1)^{\tau\bar{\phi}} (s_k x_k - x_{k+1} s_k) \phi x_k^{-1} \rho_{k,k+1} \stackrel{(3.1.6)}{=} -(-1)^{\tau\bar{\phi}} t_{k,k+1} \phi x_k^{-1} \rho_{k,k+1} \\ &= -\phi t_{k,k+1} x_k^{-1} \rho_{k,k+1} \stackrel{(5.3.12)}{=} -\phi \tilde{t}_{k,k+1} \rho_{k,k+1} x_k x_k^{-1} \rho_{k,k+1} = -\phi \tilde{t}_{k,k+1} = -\tilde{t}_{k,k+1} \diamond \phi, \end{aligned}$$

where in the third equality we use the fact that $t_{k,k+1}$ is an element of parity τ .

For $z \in \mathbf{E}(\mu)$, we also have

$$\begin{aligned} (z y_k) \diamond \phi &= (-1)^{\bar{\phi}\bar{z}+\tau\bar{\phi}} x_k \phi x_k^{-1} z \\ &\stackrel{(5.3.8)}{=} (-1)^{\bar{\phi}\bar{z}+\tau(\bar{\phi}+\bar{z})} x_k \phi (z \cdot y_k) x_k^{-1} = (-1)^{\tau(\bar{\phi}+\bar{z})} x_k ((z \cdot y_k) \diamond \phi) x_k^{-1} = (y_k(z \cdot y_k)) \diamond \phi. \end{aligned}$$

Remaining relations, involving only elements of S_μ , only elements of $\mathbf{E}(\mu)$, or only the y_k , are straightforward to verify. $\square$

Proposition 5.4.2. *Suppose M is an $\mathcal{R}_\mu$ -module. Then $\mathbf{L}(\mu) \otimes_{\mathbf{E}(\mu)} M$ is an $\mathcal{A}_\mu(F)$ -module under the action*

$$\begin{aligned} \mathbf{f} * (w \otimes v) &= \mathbf{f} w \otimes v, \quad \mathbf{f} \in F^{\otimes n}, \\ \pi * (w \otimes v) &= \rho_\pi w \otimes \pi v, \quad \pi \in S_\mu, \\ x_k * (w \otimes v) &= x_k w \otimes y_k v, \end{aligned}$$

for $w \in \mathbf{L}(\mu)$, $v \in M$. Thus, we have a functor

$$\mathbf{L}(\mu) \otimes_{\mathbf{E}(\mu)} -: \mathcal{R}_\mu\text{-mod} \rightarrow \mathcal{A}_\mu(F)\text{-mod}.$$

Proof. First we show that the given actions are well-defined on the tensor product. Let $\mathbf{f} \in F^{\otimes n}$, $w \in \mathbf{L}(\mu)$, $v \in M$, $\pi \in S_\mu$, and $z \in \mathbf{E}(\mu)$. Then

$$\mathbf{f} * (wz \otimes v) = \mathbf{f} w z \otimes v = \mathbf{f} w \otimes z v = \mathbf{f} * (w \otimes z v),$$

and

$$\begin{aligned}\pi * (wz \otimes v) &= \rho_\pi(wz) \otimes \pi v = (-1)^{\bar{w}\bar{z}} \rho_\pi(zw) \otimes \pi v = (-1)^{\bar{w}\bar{z}} \rho_\pi(z) \rho_\pi(w) \otimes \pi v \\ &= \rho_\pi(w) \rho_\pi(z) \otimes \pi v = \rho_\pi(w) \otimes \rho_\pi(z) \pi v = \rho_\pi(w) \otimes \pi zv = \pi * (w \otimes zv).\end{aligned}$$

And finally,

$$\begin{aligned}x_k * (wz \otimes v) &= (-1)^{\bar{w}\bar{z}} x_k * (zw \otimes v) = (-1)^{\bar{w}\bar{z}} x_k zw \otimes y_k v \stackrel{(5.3.8)}{=} (-1)^{\bar{w}\bar{z} + \tau\bar{z}} (z \cdot y_k^{-1})(x_k w) \otimes y_k v \\ &= x_k w (z \cdot y_k^{-1}) \otimes y_k v = (x_k w) \otimes (z \cdot y_k^{-1}) y_k v \stackrel{(5.3.9)}{=} (x_k w) \otimes y_k zv = x_k * (w \otimes zv),\end{aligned}$$

where in the fourth equality we use the fact that $\overline{(x_k w)} = \bar{w} + \tau$, and y_k^{-1} denotes the inverse of the action by y_k in $E(\mu)$. Therefore the actions of $\mathbf{f}$, π and x_k are balanced with respect to the tensor product.

The relations involving only elements of $F^{\otimes n}$, only elements of S_μ , or only x_k are clear. If $1 \leq k \leq n$, $\mathbf{f} \in F$, $w \in L(\mu)$, and $v \in M$, then

$$(\mathbf{f}x_k) * (w \otimes v) = \mathbf{f}x_k w \otimes y_k v = (-1)^{\tau\bar{\mathbf{f}}} x_k \psi_k(\mathbf{f}) w \otimes y_k v = (-1)^{\tau\bar{\mathbf{f}}} (x_k \psi_k(\mathbf{f})) * (w \otimes v).$$

If $1 \leq k \leq n-1$ and $\ell_k = \ell_{k+1}$ (i.e. $s_k \in S_\mu$), we also have

$$\begin{aligned}(s_k x_k) * (w \otimes v) &= \rho_{k,k+1} x_k w \otimes s_k y_k v, \\ (x_{k+1} s_k) * (w \otimes v) &= x_{k+1} \rho_{k,k+1} w \otimes y_{k+1} s_k v = \rho_{k,k+1} x_k w \otimes y_{k+1} s_k v.\end{aligned}$$

Thus we have

$$\begin{aligned}(s_k x_k - x_{k+1} s_k) * (w \otimes v) &= \rho_{k,k+1} x_k w \otimes (s_k y_k - y_{k+1} s_k) v = -\rho_{k,k+1} x_k w \otimes \tilde{t}_{k,k+1} v \\ &= -\rho_{k,k+1} x_k w \tilde{t}_{k,k+1} \otimes v = -\tilde{t}_{k,k+1} (\rho_{k,k+1} x_k w) \otimes v \stackrel{(5.3.12)}{=} -t_{k,k+1} * (w \otimes v). \quad \square\end{aligned}$$

The following lemma will be our maining tool to prove the equivalence of categories $\mathcal{R}_n\text{-mod} \cong \mathcal{A}_n(F)\text{-mod}^s$.

Lemma 5.4.3. *Suppose A is a graded superalgebra and V is a finite-dimensional semisimple A -module. Let $\mathcal{C}_V$ denote the full subcategory of $A\text{-mod}$ whose objects are evenly isomorphic to finite direct sums of degree shifts of simple submodules of V .*

Regard V as an $(A, \text{END}_A^{\text{op}} V)$ -bimodule, with right action given by

$$vz = (-1)^{\bar{v}\bar{z}} z(v), \quad \forall z \in \text{END}_A^{\text{op}} V.$$

For all A -module U , regard $\text{HOM}_A(V, U)$ as left $\text{END}_A^{\text{op}} V$ -module with action given in Lemma 2.1.14.

Then the functors

$$V \otimes_{\text{END}_A^{\text{op}} V} -: (\text{END}_A^{\text{op}} V)\text{-mod} \rightarrow \mathcal{C}_V, \quad \text{HOM}_A(V, -): \mathcal{C}_V \rightarrow (\text{END}_A^{\text{op}} V)\text{-mod},$$

yield an equivalence of categories between $\mathcal{C}_V$ and $(\text{END}_A^{\text{op}} V)\text{-mod}$.

Proof. Suppose $U \in (\text{END}_A^{\text{op}} V)\text{-mod}$ and define

$$\Phi_U: U \rightarrow \text{HOM}_A(V, V \otimes_{\text{END}_A^{\text{op}} V} U), \quad \Phi_U(u): v \mapsto (-1)^{\bar{u}\bar{v}} v \otimes u, \quad \forall u \in U, v \in V.$$

We claim that Φ_U is an isomorphism of $(\text{END}_A^{\text{op}} V)\text{-modules}$. Indeed, for all homogeneous $z \in \text{END}_A^{\text{op}} V$, $u \in U$ and $v \in V$, we have that

$$\begin{aligned} \Phi_U(zu)(v) &= (-1)^{\bar{v}(\bar{u}+\bar{z})} v \otimes zu = (-1)^{(\bar{v}+\bar{u})\bar{z}+(\bar{v}+\bar{z})\bar{u}} v z \otimes u \\ &= (-1)^{(\bar{v}+\bar{u})\bar{z}} \Phi_U(u)(vz) = (z\Phi_U(u))(v), \end{aligned} \quad (5.4.1)$$

thus Φ_U is a homomorphism of $(\text{END}_A^{\text{op}} V)\text{-modules}$. It is a straightforward calculation to show that the following defines the inverse of Φ_U :

$$\text{HOM}_A(V, V \otimes_{\text{END}_A^{\text{op}} V} U) \ni f \mapsto \sum_{j=1}^n \pi_j \left(\sum_{i=0}^n u_{j,i} \right) \in U,$$

where $\{v_i, \dots, v_n\}$ is a basis of V , $u_{j,i} \in U$ are such that $f(v_i) = \sum_{j=1}^n v_j \otimes u_{j,i}$ and $\pi_i: V \rightarrow V$ is the projection on $\mathbb{k}v_i$, for all $1 \leq i, j \leq n$. It is also clear that Φ_U is natural on U . Therefore $\{\Phi_U\}_{U \in (\text{END}_A^{\text{op}} V)\text{-mod}}$ defines a natural isomorphism

$$\Phi: \text{Id} \Longrightarrow \text{HOM}_A(V, V \otimes_{\text{END}_A^{\text{op}} V} -).$$

On the other hand, for $W \in \mathcal{C}_V$, we have an even homomorphism of $A\text{-modules}$

$$\Upsilon_W: V \otimes_{\text{END}_A^{\text{op}} V} \text{HOM}_A(V, W) \rightarrow W, \quad \Upsilon_W(v \otimes \phi) \mapsto (-1)^{\bar{\phi}\bar{v}} \phi(v).$$

Write

$$V = \bigoplus_{i=1}^n V_i \quad \text{and} \quad W = \bigoplus_{i=1}^n V_i^{\oplus m_i},$$

where $m_i \in \mathbb{N}$ and V_i is simple for all $1 \leq i \leq n$. Let $\{v_1^i, \dots, v_{n_i}^i\}$ be a basis for V_i and let the morphisms

$$\mu_i^m: V_i \rightarrow W \quad (1 \leq m \leq m_i), \quad \text{and} \quad \pi_i: V \rightarrow V_i,$$

be respectively the inclusion on the m^{th} -copy of V_i (and the zero map, if $m_i = 0$) on W and the projection of V onto V_i . Notice that $\{\mu_i^m(v_j^i) \mid 1 \leq i \leq n, 1 \leq j \leq n_i, 1 \leq m \leq m_i\}$ forms a basis for W . Then a straightforward calculation shows that

$$\mu_i^m(v_j^i) \mapsto v_j^i \otimes (\mu_i^m \circ \pi_i), \quad 1 \leq i \leq n, 1 \leq j \leq n_i, 1 \leq m \leq m_i,$$

defines the inverse map for Υ_W . Hence Υ_W is an isomorphism. Furthermore, it is clearly natural in W . Therefore $\{\Upsilon_W\}_{W \in \mathcal{C}_V}$ defines a natural isomorphism

$$\Upsilon: V \otimes_{\text{END}_A^{\text{op}} V} \text{HOM}_A(V, -) \Longrightarrow \text{Id}.$$

□

Then we have the following:

Proposition 5.4.4. *Let M be an $\mathcal{R}_\mu$ -module. Then*

$$\Phi: M \rightarrow \text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), \mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M), \quad \Phi(v)(w) = (-1)^{\bar{v}\bar{w}} w \otimes v, \quad v \in M, \quad w \in \mathbb{L}(\mu),$$

is an even isomorphism of $\mathcal{R}_\mu$ -modules and it define a natural transformation

$$\Phi: \text{Id} \Rightarrow \text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), \mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} -).$$

Furthermore, $\mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M$ is a simple $\mathcal{A}_\mu(F)$ -module if and only if M is a simple $\mathcal{R}_\mu$ -module.

Proof. First notice that $\Phi(v): \mathbb{L}(\mu) \rightarrow \mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M$ is a $F^{\otimes n}$ -homomorphism for all $v \in M$. Indeed, for all $\mathbf{f} \in F^{\otimes n}$ and $w \in \mathbb{L}(\mu)$ we have

$$\Phi(v)(\mathbf{f}w) = (-1)^{\bar{v}(\bar{\mathbf{f}}+\bar{w})} \mathbf{f}w \otimes v = (-1)^{\bar{v}\bar{\mathbf{f}}} \mathbf{f} * ((-1)^{\bar{v}\bar{w}} w \otimes v) = (-1)^{\bar{\Phi(v)}\bar{\mathbf{f}}} \mathbf{f} * \Phi(v)(w).$$

It is also straightforward to verify that Φ is a homomorphism of $\mathcal{R}_\mu$ -modules: Let $\pi \in S_\mu$, $1 \leq k \leq n$, and $z \in \mathbb{E}(\mu)$, we have

$$\begin{aligned} \Phi(\pi v)(w) &= (-1)^{\bar{v}\bar{w}} w \otimes \pi v = (-1)^{\bar{v}\bar{w}} \pi * (\rho_\pi^{-1} w \otimes v) = (\pi \diamond \Phi(v))(w), \\ \Phi(y_k v)(w) &= (-1)^{\bar{v}\bar{w}} w \otimes y_k v = (-1)^{\bar{v}\bar{w}} x_k * (x_k^{-1} w \otimes v) = (y_k \diamond \Phi(v))(w), \\ \Phi(zv)(w) &= (-1)^{(\bar{z}+\bar{v})\bar{w}} w \otimes zv = (-1)^{\bar{v}\bar{w}} zw \otimes v = (z \diamond \Phi(v))(w), \end{aligned}$$

for all $w \in \mathbb{L}(\mu)$.

By Lemma 5.4.3, setting $A = F^{\otimes n}$ and $V = \mathbb{L}(\mu)$, we obtain that Φ is an isomorphism of $\mathbb{E}(\mu)$ -modules. Therefore, it is bijective and hence an isomorphism of $\mathcal{R}_\mu$ -modules.

Suppose that $\mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M$ is a simple $\mathcal{A}_\mu(F)$ -module and W is a nonzero $\mathcal{R}_\mu$ -submodule of M . Then $\mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} W$ is an $\mathcal{A}_\mu(F)$ -submodule of $\mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M$, which is simple. Thus $\mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M = \mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} W$. Then

$$M \simeq \text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), \mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M) = \text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), \mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} W) \simeq W.$$

Hence $W = M$. So M is simple.

Conversely, suppose that M is a simple $\mathcal{R}_\mu$ -module and that W is a nonzero $\mathcal{A}_\mu(F)$ -submodule of $\mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M$. Then, by Proposition 5.4.1, $\text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), W)$ is a nonzero $\mathcal{R}_\mu$ -submodule of $\text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), \mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M) \simeq M$, which is simple. Hence $\text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), W) \simeq M$. Then, by Lemma 5.4.3, it follows that as $F^{\otimes n}$ -module:

$$W \simeq \mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} \text{HOM}_{F^{\otimes n}}(\mathbb{L}(\mu), W) \simeq \mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M.$$

So $\mathbb{L}(\mu) \otimes_{\mathbb{E}(\mu)} M$ is simple. □

For $V \in \mathcal{A}_n(F)\text{-mod}^s$, let $I_\mu V$ be the sum of all $F^{\otimes n}$ -submodules of V evenly isomorphic to degree shifts of submodules of $\mathbb{L}(\mu)$ and define

$$V_\mu := \sum_{\pi \in S_n} \pi(I_\mu V). \tag{5.4.2}$$

As discussed in Section 5.1, it follows from (3.1.4) that x_i maps a simple $F^{\otimes n}$ -submodule W of V either to zero or to a submodule evenly isomorphic to 1W , the module obtained from

W by τ -twisting by ψ . Hence, by the definition of $L(\mu)$, it follows that $I_\mu V$ is invariant under the action of $P_n(F)$. It also follows from (3.1.7) that $\pi \in S_\mu$ preserves simple $F^{\otimes n}$ -submodule isomorphic to simple summand of $L(\mu)$, up to isomorphism. Thus $I_\mu V$ is an $\mathcal{A}_\mu(F)$ -submodule of V and I_μ defines a functor

$$I_\mu: \mathcal{A}_n(F)\text{-mod}^s \rightarrow \mathcal{A}_\mu(F)\text{-mod}.$$

Immediately from the definition, we obtain that V_μ is an $\mathcal{A}_n(F)$ -submodule of V .

Lemma 5.4.5. *Let $V \in \mathcal{A}_n(F)\text{-mod}^s$. Then $V_\mu \simeq \text{Ind}_\mu^n(I_\mu V)$ and we have a decomposition $V = \bigoplus_{\mu \in \mathcal{C}_n} V_\mu$ in $\mathcal{A}_n(F)\text{-mod}^s$.*

Proof. If $I_\mu V = 0$, clearly $V_\mu \simeq 0 \simeq \text{Ind}_\mu^n(I_\mu V)$.

So assume that $I_\mu V \neq 0$. Then the inclusion $I_\mu V \hookrightarrow \text{Res}_\mu^n V_\mu$ is a nonzero even homomorphism of $\mathcal{A}_\mu(F)$ -modules. Since induction is left adjoint to restriction, we have a nonzero even homomorphism $\phi: \text{Ind}_\mu^n(I_\mu V) \rightarrow V_\mu$ of $\mathcal{A}_n(F)$ -modules. It follows from the definition of V_μ that ϕ is surjective. Furthermore, if X is a complete set of representatives of left cosets of S_μ in S_n , we have

$$\text{Ind}_\mu^n I_\mu V = \mathcal{A}_n(F) \otimes_{\mathcal{A}_\mu(F)} I_\mu V = \bigoplus_{\pi \in X} \pi \otimes I_\mu V \quad \text{and} \quad V_\mu = \bigoplus_{\pi \in X} \pi(I_\mu V).$$

Therefore, $\text{Ind}_\mu^n(I_\mu V)$ and V_μ have the same dimension, and so ϕ is injective. Hence the isomorphism.

To prove the decomposition of V , recall that it is semisimple as an $F^{\otimes n}$ -module. For $\mu \in \mathcal{C}_n$, V_μ is the direct sum of the isotypic components corresponding to simple $F^{\otimes n}$ -modules containing

$$\left(\ell_{\pi(1)} L_{\pi(1)} \right)^{\boxtimes \mu_{\pi(1)}} \boxtimes \cdots \boxtimes \left(\ell_{\pi(N)} L_{\pi(N)} \right)^{\boxtimes \mu_{\pi(N)}},$$

with $\pi \in S_N$ and $1 \leq \ell_i \leq s_i$ for all $1 \leq i \leq N$. Then, by varying $\mu \in \mathcal{C}_n$, we exhausted all possible isotypic components of V . Therefore

$$V = \bigoplus_{\mu \in \mathcal{C}_n} V_\mu.$$

□

Proposition 5.4.6. *Suppose $V \in \mathcal{A}_n(F)\text{-mod}^s$. Then*

$$\Upsilon: L(\mu) \otimes_{E(\mu)} \text{HOM}_{F^{\otimes n}}(L(\mu), I_\mu V) \rightarrow I_\mu V, \quad v \otimes \varphi \mapsto (-1)^{\bar{v}\bar{\varphi}} \varphi(v),$$

is an even isomorphism of $\mathcal{A}_\mu(F)$ -modules, which defines a natural transformation

$$\Upsilon: L(\mu) \otimes_{E(\mu)} \text{HOM}_{F^{\otimes n}}(L(\mu), I_\mu -) \Rightarrow I_\mu.$$

Proof. By Propositions 5.4.1 and 5.4.2, $L(\mu) \otimes_{E(\mu)} \text{HOM}_{F^{\otimes n}}(L(\mu), I_\mu V)$ is an $\mathcal{A}_n(F)$ -module.

We verify that Υ is $E(\mu)$ -balanced. For $v \in L(\mu)$, $\varphi \in \text{HOM}_{F^{\otimes n}}(L(\mu), I_\mu V)$, and $z \in E(\mu)$,

we have

$$\begin{aligned}\Upsilon(vz \otimes \varphi) &= (-1)^{\bar{v}\bar{z}}\Upsilon(zv \otimes \varphi) = (-1)^{\bar{v}\bar{z}+(\bar{v}+\bar{z})\bar{\varphi}}\varphi(zv) \\ &= (-1)^{\bar{v}(\bar{z}+\bar{\varphi})}(z \diamond \varphi)(v) = \Upsilon(v \otimes (z \diamond \varphi)).\end{aligned}$$

We now shows that Υ is a even homomorphism of $\mathcal{A}_\mu(F)$ -modules. Let $v \in \mathbf{L}(\mu)$ and $\varphi \in \text{HOM}_{F^{\otimes n}}(\mathbf{L}(\mu), I_\mu V)$. For $\mathbf{f} \in F^{\otimes n}$, $\pi \in S_\mu$, and $1 \leq k \leq n$, we have

$$\begin{aligned}\Upsilon(\mathbf{f} * (v \otimes \varphi)) &= \Upsilon(\mathbf{f}v \otimes \varphi) = (-1)^{(\bar{\mathbf{f}}+\bar{v})\bar{\varphi}}\varphi(\mathbf{f}v) = (-1)^{\bar{v}\bar{\varphi}}\mathbf{f}\varphi(v) = \mathbf{f}\Upsilon(v \otimes \varphi), \\ \Upsilon(\pi * (v \otimes \varphi)) &= \Upsilon(\rho_\pi v \otimes \pi \diamond \varphi) = \Upsilon(\rho_\pi v \otimes \pi \varphi \rho_\pi^{-1}) = (-1)^{\bar{v}\bar{\varphi}}\pi \varphi(v) = \pi \Upsilon(v),\end{aligned}$$

$$\begin{aligned}\Upsilon(x_k * (v \otimes \varphi)) &= \Upsilon(x_k v \otimes y_k \diamond \varphi) \\ &= (-1)^{\tau \bar{\varphi}}\Upsilon(x_k v \otimes x_k \varphi x_k^{-1}) \stackrel{(\bar{x}_k=\tau)}{=} (-1)^{\tau \bar{\varphi}+(\bar{v}+\tau)\bar{\varphi}}x_k \varphi(v) = x_k \Upsilon(v \otimes \varphi).\end{aligned}$$

Now, as an $F^{\otimes n}$ -module, $I_\mu V$ is isomorphic to a finite direct sum of modules evenly isomorphic to degree shifts of simple summands of $\mathbf{L}(\mu)$. It follows from Lemma 5.4.3 (by setting $A = F^{\otimes n}$ and $V = \mathbf{L}(\mu)$) that Υ is bijective. $\square$

Finally we prove the main theorem of this section.

Theorem 5.4.7. *The functor $\mathbf{F}: \mathcal{A}_n(F)\text{-mod}^s \rightarrow \mathcal{R}_n\text{-mod}$ defined by*

$$\mathbf{F}(V) = \bigoplus_{\mu \in \mathcal{C}_n} \text{HOM}_{F^{\otimes n}}(\mathbf{L}(\mu), I_\mu V)$$

is an equivalence of categories with inverse $\mathbf{G}: \mathcal{R}_n\text{-mod} \rightarrow \mathcal{A}_n(F)\text{-mod}^s$ given by

$$\mathbf{G}\left(\bigoplus_{\mu \in \mathcal{C}_n} M_\mu\right) = \bigoplus_{\mu \in \mathcal{C}_n} \text{Ind}_\mu^n(\mathbf{L}(\mu) \otimes_{\mathbf{E}(\mu)} M_\mu).$$

Proof. The map Φ of Proposition 5.4.4 is natural in M , and the map Υ of Proposition 5.4.6 is natural in V .

We have

$$\begin{aligned}\mathbf{F}\mathbf{G}\left(\bigoplus_{\mu \in \mathcal{C}_n} M_\mu\right) &= \bigoplus_{\mu \in \mathcal{C}_n} \text{HOM}_{F^{\otimes n}}\left(\mathbf{L}(\mu), I_\mu\left(\bigoplus_{\mu \in \mathcal{C}_n} \text{Ind}_\mu^n(\mathbf{L}(\mu) \otimes_{\mathbf{E}(\mu)} M_\mu)\right)\right) \\ &\simeq \bigoplus_{\mu \in \mathcal{C}_n} \text{HOM}_{F^{\otimes n}}(\mathbf{L}(\mu), \mathbf{L}(\mu) \otimes_{\mathbf{E}(\mu)} M_\mu) \\ &\stackrel{\Phi}{\simeq} \bigoplus_{\mu \in \mathcal{C}_n} M_\mu,\end{aligned}$$

where the first even isomorphism follows from Lemma 5.4.5, and the last even isomorphism follows from Proposition 5.4.4.

On the other hand, we have

$$\begin{aligned} \mathbf{GF}(V) &= \bigoplus_{\mu \in \mathcal{C}_n} \text{Ind}_{\mu}^n (\mathbf{L}(\mu) \otimes_{\mathbf{E}(\mu)} \text{HOM}_{F^{\otimes n}} (\mathbf{L}(\mu), I_{\mu} V)) \\ &\stackrel{\mathfrak{r}}{\simeq} \bigoplus_{\mu \in \mathcal{C}_n} \text{Ind}_{\mu}^n (I_{\mu} V) \\ &\simeq V, \end{aligned}$$

where the first even isomorphism follows from Proposition 5.4.6, and the second even isomorphism follows from Lemma 5.4.5. $\square$

5.5 Simple $\mathcal{A}_n(F)$ -modules

We are now ready to classify $\mathcal{A}_n(F)$ -modules.

Proposition 5.5.1. *Every simple $\mathcal{A}_n(F)$ -module is semisimple as an $F^{\otimes n}$ -module. In particular, the categories $\mathcal{A}_n(F)\text{-mod}^s$ and $\mathcal{A}_n(F)\text{-mod}$ have the same class of simple modules.*

Proof. Suppose V is a simple $\mathcal{A}_n(F)$ -module. Let W be a simple $P_n(F)$ -submodule of V . Then $\sum_{\pi \in S_n} \pi W$ is an $\mathcal{A}_n(F)$ -submodule of V and hence $V = \sum_{\pi \in S_n} \pi W$ since V is simple. By Proposition 5.1.6, $W \simeq L_1(a_1) \otimes \cdots \otimes L_n(a_n)$, where $L_1, \dots, L_n$ are simple F -modules and $a_1, \dots, a_n \in \mathbb{k}$. Thus each πW is semisimple as an $F^{\otimes n}$ -module, and so V is semisimple as an $F^{\otimes n}$ -module. $\square$

Theorem 5.5.2. *Every simple $\mathcal{A}_n(F)$ -module is evenly isomorphic to a module of the form*

$$\text{Ind}_{\mu}^n (\mathbf{L}(\mu) \otimes_{\mathbf{E}(\mu)} (V_1 \otimes \cdots \otimes V_N)), \quad (5.5.1)$$

where $\mu = (\mu_1, \dots, \mu_N) \in \mathcal{C}_n$, and V_k is a simple $\mathcal{H}_{\mu_k}^{\ell}$ -module for $1 \leq \ell \leq N$. Furthermore, the above modules (over all $\mu \in \mathcal{C}_n$ and V_{ℓ} , $1 \leq \ell \leq N$, running over a set of representatives of even isomorphism classes, up to degree shift) form a complete set of pairwise not evenly-isomorphic simple $\mathcal{A}_n(F)$ -modules, up to degree shift.

Proof. This follows immediately from Proposition 5.5.1 and Theorem 5.4.7. $\square$

Theorem 5.5.2 reduces the study of simple $\mathcal{A}_n(F)$ -modules to the study of simple modules for $\mathcal{H}_m^{\ell}$.

Notice that, if $t_{1,2}$ does not act as zero on $\mathbf{L}_{\ell} \otimes \mathbf{L}_{\ell}$, then $\mathcal{H}_m^{\ell}$ is either a degenerate affine Hecke algebra $\mathcal{A}_m(\mathbb{k})$ (when $\mathbf{L}_{\ell}$ is of type $\mathbf{M}$) or an affine Sergeev algebra $\mathcal{A}_m(\mathcal{Cl}_0)$ (when $\mathbf{L}_{\ell}$ is of type $\mathbf{Q}$), as explained in Examples 3.2.1 and 3.2.2. The simple modules for these algebras have been classified (see, for example, [Kle05]).

On the other hand, if $t_{1,2}$ acts as zero on $\mathbf{L}_{\ell} \otimes \mathbf{L}_{\ell}$, then $\mathcal{H}_m^{\ell}$ is a wreath product algebra of the form $(\mathbb{k}[y] \rtimes A)^{\otimes m} \rtimes_{\rho} S_m$, for a finite-dimensional algebra A . Modules for the class of wreath product algebras can be classified using the results of Section 5.1 and Clifford theory (e.g., see [RS17, §4] for the characteristic zero case).

In this way, Theorem 5.5.2 provides a complete classification of the simple $\mathcal{A}_n(F)$ -modules.

Example 5.5.3. Recall Examples 5.3.1 and use Theorem 5.5.2 to describe $\mathcal{A}_n(F)\text{-mod}$:

1. Specialising $F = \mathcal{C}l_1$, we obtain that every simple $\mathcal{A}_n(\mathcal{C}l_1)$ -module is of the form

$$L(-1)^{\boxtimes n} \otimes_{\mathcal{C}l^{\otimes n}} V,$$

where the $\mathcal{A}_n(\mathcal{C}l_1)$ acts as defined by 5.4.2 and V is a simple module of the *affine Sergeev algebra* $\mathcal{A}_n(\mathcal{C}l_0)$.

Let now $\Phi: \mathcal{A}_n(\mathcal{C}l_0) \rightarrow \mathcal{A}_n(\mathcal{C}l_1)$ be the algebra isomorphism of Lemma 3.1.3, satisfying

$$\Phi(c_i) = c_i, \quad \Phi(\pi) = \pi, \quad \Phi(y_i) = -x_i c_i,$$

for all $1 \leq i \leq n$ and $\pi \in S_n$. Fix $V \in \mathcal{S}(\mathcal{A}_n(\mathcal{C}l_0))$ and consider the $\mathcal{A}_n(\mathcal{C}l_0)$ -module

$$\tilde{V} = {}^{\Phi^{-1}} \left(L(-1)^{\boxtimes n} \otimes_{\mathcal{C}l^{\otimes n}} V \right).$$

We have that, for all $1 \otimes v \in \tilde{V}$:

$$y_i \cdot (1 \otimes v) = \Phi(y_i) \star (1 \otimes v) = -x_i c_i \star (1 \otimes v) = -(x_i c_i \otimes y_i v) = -\sqrt{-1}(1 \otimes y_i v),$$

for all $1 \leq i \leq n$. Hence, $\tilde{V} \simeq V$. Therefore, as expected, any simple $\mathcal{A}_n(\mathcal{C}l_1)$ -module is isomorphic to a simple module V of the affine Sergeev algebra, but with action of $\mathcal{A}_n(\mathcal{C}l_1)$ twisted by Φ .

2. Specialising $F = D$ the Dual algebra, we have that every simple $\mathcal{A}_n(D)$ -module is of the form

$$L(1)^{\boxtimes n} \otimes_{\mathcal{M}(1|1)^{\otimes n}} V,$$

where the $\mathcal{A}_n(D)$ acts as defined by Proposition 5.4.2 and V is a simple module over the wreath product algebra $(\mathbb{K}[y] \ltimes \mathcal{M}(1|1))^{\otimes n} \rtimes_{\rho} S_n$.

Chapter 6

Cyclotomic quotients

When studying the representation theory of degenerate affine Hecke algebra $\mathcal{A}_n(\mathbb{C})$ (see 3.2.1), it is easy to show that the action of x_1 in any finite dimensional representation V has a minimal polynomial $p \in \mathbb{k}[x]$. Then V inherits a structure of module over the quotient $\mathcal{A}_n(\mathbb{C})/p(x_1)$. In fact, any finite dimensional representation of $\mathcal{A}_n(\mathbb{C})$ can be obtained from an inflation of an $\mathcal{A}_n(\mathbb{C})/q(x_1)$ -module, for some $q \in \mathbb{k}[x]$. This class of quotients is called cyclotomic quotients and has interesting application. For instance, its representation theory allows a connection of $\mathcal{A}_n(\mathbb{C})$ -modules and representation theory of affine Lie algebras. See [Kle05, Part I] for more details.

In this section we define and study cyclotomic quotients of affine wreath product algebras $\mathcal{A}_n(F)$. In its full generality, we will allow in the definition a quotient by a polynomial “with coefficients in $F^{\otimes n}$ ”. We generalize the definition given in [Sav20, Section 6] for any Frobenius algebra F with trace map of arbitrary parity τ .

We study the Frobenius algebra structure that appears in the cyclotomic quotients, describing explicitly its trace map and its Nakayama automorphism. We also study the induction and restriction functors on the level of cyclotomic quotients, proving a Cyclotomic Mackey Theorem and showing its adjunction (up to degree shift).

6.1 Shifting homomorphisms

The idea behind the generalization of cyclotomic quotients for $\mathcal{A}_n(F)$ is allowing quotients of polynomial with “roots in $F^{\otimes n}$ ”. We call such “roots” the cyclotomic elements for the cyclotomic quotients. But, to the definition of cyclotomic quotients works well, we need to impose some properties on the cyclotomic elements. In this section we investigate this properties.

Recall the definitions (4.3.1), (4.3.3) and (4.3.5). Then, for $1 \leq i \leq n$ and $k \in \mathbb{Z}$, define

$$\mathbf{F}_i^{(k)} := \left(\mathbf{F}_\psi^{(0^{i-1}, k, 0^{n-i})} \right)^{S_n^i} \subseteq F^{\otimes n} \subseteq \mathcal{A}_n(F), \quad \text{where } S_n^i = \{\pi \in S_n \mid \pi i = i\}. \quad (6.1.1)$$

Here we denoted $(0^{i-1}, k, 0^{n-i}) = (0, \dots, 0, k, 0, \dots, 0)$, where the k appears in the i -th place. Thus, for all $\mathbf{f} \in \mathbf{F}_i^{(k)}$, $\mathbf{g} \in F^{\otimes n}$, $\pi \in S_n^i$ and $j \in \{1, \dots, n\}$, we have

$$\mathbf{g}\mathbf{f} = (-1)^{\bar{\mathbf{f}}\mathbf{g}} \mathbf{f} \psi_i^k(\mathbf{g}), \quad \psi_j(\mathbf{f}) = (-1)^{\delta_{ij}k\tau} \mathbf{f}, \quad \pi \mathbf{f} = \mathbf{f}, \quad (6.1.2)$$

where δ_{ij} is the Kronecker delta.

Example 6.1.1. Specializing F , it follows:

- (a) In the case when $F = Cl_\tau$, we have that, for $k \in \mathbb{Z}$, $F_\psi^{(k)} = \delta_{0,\bar{k}} \mathbb{k}$ (see Example 4.3.2(a)) and hence $\mathbf{F}_i^{(k)} = \delta_{0,\bar{k}} \mathbb{k}$, for all $1 \leq i \leq n$.
- (b) In the case when F is a symmetric algebra with trace map of parity τ , we have that, for $k \in \mathbb{Z}$, $F_\psi^{(k)} = \delta_{0,\tau\bar{k}} Z(F)$ (see Example 4.3.2(b)) and hence

$$\mathbf{F}_i^{(k)} = \delta_{0,\tau\bar{k}} (Z(F)^{\otimes n})^{S_n^i}, \quad 1 \leq i \leq n.$$

- (c) When $F = D$ the dual algebra, since D is a symmetric algebra with trace map of odd parity and $Z(D) = D$, we have that $\mathbf{F}_i^{(k)} = \delta_{0,\bar{k}} (D^{\otimes n})^{S_n^i}$, for all $k \in \mathbb{Z}$, $1 \leq i \leq n$.

Lemma 6.1.2. For $1 \leq i \leq n$ and $\pi \in S_n$, we have $\pi \mathbf{F}_i^{(k)} \pi^{-1} = \mathbf{F}_{\pi i}^{(k)}$.

Proof. It suffices to show that $\pi \mathbf{F}_i^{(k)} \pi^{-1} \subseteq \mathbf{F}_{\pi i}^{(k)}$. The reverse inclusion then will follow by considering π^{-1} instead π . First, it is clear that

$$\pi \mathbf{F}_i^{(k)} \pi^{-1} \stackrel{(3.1.7)}{=} \pi \left(\mathbf{F}_i^{(k)} \right) \subseteq \mathbf{F}_\psi^{(0^{\pi i-1}, k, 0^{n-\pi i})},$$

where the last inclusion follows from the fact that ${}^\pi \left(\mathbf{F}_\psi^{(\alpha)} \right) = \mathbf{F}_\psi^{(\pi \cdot \alpha)}$ for $\alpha \in \mathbb{Z}^n$. Now suppose $\mathbf{f} \in \mathbf{F}_i^{(k)}$. It rest to proof that ${}^{\pi_1} \pi \mathbf{f} \pi^{-1} = \pi \mathbf{f} \pi^{-1}$, for all $\pi_1 \in S_n^i$. But we have:

$${}^{\pi_1} (\pi \mathbf{f} \pi^{-1}) \stackrel{(3.1.7)}{=} \pi_1 \pi \mathbf{f} \pi^{-1} \pi_1^{-1} = \pi (\pi^{-1} \pi_1 \pi) \mathbf{f} (\pi^{-1} \pi_1 \pi)^{-1} \pi^{-1} = \pi \left({}^{\pi^{-1} \pi_1 \pi} \mathbf{f} \right) \pi^{-1} = \pi \mathbf{f} \pi^{-1},$$

where the last equality follows from the fact that $\pi^{-1} \pi_1 \pi \in S_n^i$. An the proof is done. $\square$

Proposition 6.1.3. Suppose $\mathbf{c} \in \mathbf{F}_1^{(1)}$ is of parity τ and of degree δ . Then

$$x_i \mapsto x_i + s_{i-1} \cdots s_1 \mathbf{c} s_1 \cdots s_{i-1}, \quad \mathbf{f} \mapsto \mathbf{f}, \quad \pi \mapsto \pi, \quad 1 \leq i \leq n, \quad \mathbf{f} \in F^{\otimes n}, \quad \pi \in S_n,$$

determines an algebra automorphism $\zeta_{\mathbf{c}}: \mathcal{A}_n(F) \rightarrow \mathcal{A}_n(F)$.

Proof. It suffices to show that $\zeta_{\mathbf{c}}$ is an algebra homomorphism, since it is then clearly invertible, with inverse $\zeta_{-\mathbf{c}}$.

Write $\mathbf{c}^{(i)} = s_{i-1} \cdots s_1 \mathbf{c} s_1 \cdots s_{i-1}$, for all $i \in \{1, \dots, n\}$.

By Lemma 6.1.2, we have $\mathbf{c}^{(i)} \in \mathbf{F}_{s_{i-1} \cdots s_1(1)}^{(1)} = \mathbf{F}_i^{(1)}$. It follows from relations 6.1.2 that, for all $1 \leq i, j \leq n$ and $\mathbf{f} \in F^{\otimes n}$:

$$\mathbf{c}^{(i)} \mathbf{c}^{(j)} = (-1)^{\tau(1+\delta_{i,j})} \mathbf{c}^{(i)} \mathbf{c}^{(j)}, \quad x_i \mathbf{c}^{(j)} = (-1)^{\tau(1+\delta_{i,j})} \mathbf{c}^{(j)} x_i, \quad \mathbf{f} \mathbf{c}^{(i)} = (-1)^{\tau \bar{\mathbf{f}}^{(i)}} \psi_i(\mathbf{f}). \quad (6.1.3)$$

And, for $|i - j| \geq 2$, we have $s_i \in S_n^j$ and thus

$$s_i \mathbf{c}^{(j)} = \mathbf{c}^{(j)} s_i. \quad (6.1.4)$$

Clearly $\zeta_{\mathbf{c}}$ is a homomorphism when restricted to $F^{\otimes n}$ and $\mathbb{k}S_n$ and it satisfies relation (3.1.7). It follows from the above relations that

$$\begin{aligned}\zeta_{\mathbf{c}}(x_i)\zeta_{\mathbf{c}}(x_j) &= (x_i + \mathbf{c}^{(i)})(x_j + \mathbf{c}^{(j)}) \\ &= x_i x_j + \mathbf{c}^{(i)} x_j + x_i \mathbf{c}^{(j)} + \mathbf{c}^{(i)} \mathbf{c}^{(j)} \\ &= (-1)^{\tau(1+\delta_{ij})} (x_j x_i + x_j \mathbf{c}^{(i)} + \mathbf{c}^{(j)} x_i + \mathbf{c}^{(j)} \mathbf{c}^{(i)}) \\ &= (-1)^{\tau(1+\delta_{ij})} \zeta_{\mathbf{c}}(x_j) \zeta_{\mathbf{c}}(x_i),\end{aligned}$$

for all $i, j \in \{1, \dots, n\}$, and hence $\zeta_{\mathbf{c}}$ satisfies relation (3.1.3). Also, from (6.1.3) and (6.1.4) it is easy to verify that $\zeta_{\mathbf{c}}$ satisfies relations (3.1.5) and (3.1.4). It remain to show that $\zeta_{\mathbf{c}}$ preserves (3.1.6). It follows from the computation:

$$\zeta(s_i x_i) = s_i x_i + s_i \mathbf{c}^{(i)} \stackrel{(3.1.6)}{=} x_{i+1} s_i - t_{i,i+1} + s_i \mathbf{c}^{(i)} = x_{i+1} s_i + \mathbf{c}^{(i+1)} s_i - t_{i,i+1} = \zeta(x_{i+1} s_i - t_{i,i+1}),$$

for all $i \in \{1, \dots, n\}$. □

6.2 Cyclotomic wreath product algebras

Fix $\vartheta = (1 + \tau)\theta$, where $\tau \in \{0, 1\}$ is the parity of the trace map of F . For $k \in \{1, \dots, \vartheta\}$, choose $e_k \in \mathbb{N}$ elements

$$\mathbf{c}^{(k,1)}, \dots, \mathbf{c}^{(k,e_k)} \in \mathbf{F}_1^{(k)},$$

of parity $k\tau$ and degree $k\delta$. Define

$$\mathbf{C} = \left(\mathbf{c}^{(1,1)}, \dots, \mathbf{c}^{(1,e_1)}, \dots, \mathbf{c}^{(\vartheta,1)}, \dots, \mathbf{c}^{(\vartheta,e_{\vartheta})} \right).$$

Let $J_{\mathbf{C}}$ be the two-sided ideal in $\mathcal{A}_n(F)$ generated by the homogeneous element

$$\chi_{\mathbf{C}} = \prod_{k=1}^{\vartheta} \prod_{j=1}^{e_k} \left(x_1^k - \mathbf{c}^{(k,j)} \right). \quad (6.2.1)$$

Note that we chose the elements $\mathbf{c}^{(k,j)}$ so that they commute with elements of $\mathcal{A}_n(F)$ in the same way that x_1^k does. Then it follows that $\chi_{\mathbf{C}}$ is independent of the order of the factors in (6.2.1).

We define the *cyclotomic wreath product algebra* to be the quotient

$$\mathcal{A}_n^{\mathbf{C}}(F) := \mathcal{A}_n(F) / J_{\mathbf{C}}. \quad (6.2.2)$$

By convention, we set $\mathcal{A}_0^{\mathbf{C}}(F) = \mathbb{k}$. Since $\chi_{\mathbf{C}}$ is homogeneous, $\mathcal{A}_n^{\mathbf{C}}(F)$ inherits the structure of a graded superalgebra. We define the *level* of $\mathbf{C}$ and the corresponding algebra $\mathcal{A}_n^{\mathbf{C}}(F)$ to be the polynomial degree of $\chi_{\mathbf{C}}$, and we denote it by $d_{\mathbf{C}}$. Thus

$$d = d_{\mathbf{C}} = \sum_{k=1}^{\vartheta} k e_k. \quad (6.2.3)$$

For any element of $\mathcal{A}_n(F)$, we will denote its canonical image in $\mathcal{A}_n^C(F)$ by the same symbol.

Example 6.2.1 (Recovering others definitions). By specializing F , definition (6.2.2) recover definitions that already appeared in the literature. Precisely:

1. In the case when $F = \mathcal{Cl}_\tau$, is the Clifford algebra with trace map of parity τ , for all $m \in \mathbb{N}$ (see 2.2.17, we have that $\mathbf{F}_1^{(k)} = \delta_{0,\bar{k}}\mathbb{k}$, for all $k \in \mathbb{Z}$. Hence $\mathbf{C} \in \mathbb{k}^{d_2}$ and $\chi_{\mathbf{C}}$ are of the form

$$\chi_{\mathbf{C}} = x_1^{d_1} \prod_{i=1}^{d_2} (x_1^2 - \alpha_i), \quad d_1, d_2 \in \mathbb{Z}, \mathbf{C} = (\alpha_1, \dots, \alpha_d).$$

Therefore, assuming $\tau = 0$ and restricting $\mathbf{C}$ to be of the form $(i_1(i_1 + 1), \dots, i_d(i_d + 1))$ where $i_j \in \mathbb{N}$, for all $1 \leq j \leq d$, we have that $\mathcal{A}_n^C(\mathcal{Cl}_0)$ is a *cyclotomic affine Sergeev algebra* by the definition from [Kle05, 15.3].

2. In the case when F is a symmetric algebra with trace map of parity τ , we have that $\mathbf{F}_1^{(k)} = \delta_{0,\tau\bar{k}} \left(Z(F) \otimes (Z(F)^{\otimes n-1})^{S_{n-1}} \right)$, for all $k \in \mathbb{Z}$. Hence $\chi_{\mathbf{C}}$ are of the form, for $d_1, d_2 \in \mathbb{Z}$,

$$\chi_{\mathbf{C}} = x_1^{d_1} \prod_{i=1}^{d_2} (x_1^{\tau+1} - \mathbf{c}^i), \quad \mathbf{C} = (\mathbf{c}^1, \dots, \mathbf{c}^{d_2}) \in \left(Z(F) \otimes (Z(F)^{\otimes n-1})^{S_{n-1}} \right)^{d_2},$$

where each $\mathbf{c}^i$ is of parity $\tau + 1$ and degree δ .

When $\tau = 0$, and restricting $\mathbf{C} \in \left((Z(F)^{\otimes n})^{S_n} \right)^{d_2} \subseteq \left(Z(F) \otimes (Z(F)^{\otimes n-1})^{S_{n-1}} \right)^{d_2}$ we have that $\mathcal{A}_n^C(F)$ is a *level $(d_1 + d_2)$ cyclotomic quotient algebra* by the definition from [EKL14, 3.6].

Example 6.2.2 ($\mathcal{A}_n^C(D)$). In the case when $F = D$ the dual algebra, we have that

$$\mathbf{F}_1^{(k)} = \delta_{0,\bar{k}} (D^{\otimes n})^{S_n^1} = \delta_{0,\bar{k}} \left(D \otimes (D^{\otimes n-1})^{S_{n-1}} \right),$$

for all $\mathbb{k} \in \mathbb{Z}$. Notice that, by considering the isomorphism $D^{\otimes m} \simeq \bigwedge \mathbb{k}^m$ (see 3.2.3), we have that

$$(D^{\otimes m})^{S_m} = \text{span}_{\mathbb{k}} \{1, u_1 + u_2 + \dots + u_m\}.$$

Hence it follows that if $\mathbf{c} \in \mathbf{F}_1^{(2)}$ is a nonzero even element, then it is a multiple of either $u_1 \sum_{i=2}^n u_i$ or 1. Then, for $d_1, d_2 \in \mathbb{N}$ the *level $d = d_1 + 2d_2$ cyclotomic quotient algebra* $\mathcal{A}_n^C(D)$ is $\mathcal{A}_n(D)$ modulo the two sided ideal generated by

$$\begin{aligned} \chi_{\mathbf{C}} &= x_1^{d_1} \prod_{i=1}^{d_2} (x_1^2 - (\alpha_i + \beta_i u_1(u_2 + \dots + u_n))), \\ \mathbf{C} &= ((d_1, d_2), ((\alpha_1, \beta_1), \dots, (\alpha_{d_2}, \beta_{d_2}))) \in \mathbb{N}^2 \times (\mathbb{k}^2)^{d_2}. \end{aligned}$$

Now we prove a result that shows to be a motivation to study cyclotomic wreath product algebras:

Proposition 6.2.3. *If $\delta > 0$ or $\mathbb{k}$ is an algebraically closed field, then every finite-dimensional $\mathcal{A}_n(F)$ -module is the inflation of an $\mathcal{A}_n^C(F)$ -module for some $\mathbf{C}$.*

Proof. Let V be a finite-dimensional $\mathcal{A}_n(F)$ -module.

If $\delta > 0$, then the action of x_1 on V is nilpotent by degree considerations. Then let k be the first positive integer such that x_1^k act as zero. Clearly V is isomorphic to an inflation of an $\mathcal{A}_n^{\mathbf{C}}(F)$ -module where $J_{\mathbf{C}}$ is the ideal generated by x^k .

Now, if $\delta = 0$, we take $p \in \mathbb{k}[t]$ to be the minimal polynomial of $x_1^{1+\tau}$ on V . If $\mathbb{k}$ is algebraically we have that

$$p(t) = \prod_{j=1}^q (t - a_j),$$

for $q \in \mathbb{N}$ and $a_i \in \mathbb{k}$, $i \in \{1, \dots, q\}$. Since $\mathbb{k} \subseteq \mathbf{F}_1^{(1+\tau)}$, we have that p is of the form (6.2.1). Thus we can define $\mathbf{C}$ such that $\chi_{\mathbf{C}} = p(x_1^{1+\tau})$, which follows that V is isomorphic to an inflation of an $\mathcal{A}_n^{\mathbf{C}}(F)$ -module. $\square$

6.3 Basis theorem

We now describe an explicit basis for $\mathcal{A}_n^{\mathbf{C}}(F)$. Our approach is the same from [Sav20], which were inspired by the methods of [Kle05, §7.5, §15.4]. Precisely, we:

1. Compute a basis of $\mathcal{A}_n(F)$ as $(F^{\otimes n} \rtimes_{\rho} S_n)$ -modules, in terms of the “polynomial”

$$\chi_{\mathbf{C}} \quad s_1 \chi_{\mathbf{C}} s_1 \quad \dots \quad s_{n-1} s_{n-2} \dots s_1 \chi_{\mathbf{C}} s_1 \dots s_{n-1}. \quad (6.3.1)$$

See Lemma 6.3.5;

2. Explicitly calculate the ideal $J_{\mathbf{C}}$ in terms of the elements of $\mathcal{A}_n(F)$ listed in (6.3.1);
3. Using the previous results, we show (in 6.3.8) that

$$\{x^{(\alpha_1, \dots, \alpha_n)} \mathbf{b} \pi \mid \mathbf{b} \in \mathcal{B}^{\otimes n}, \pi \in S_n, \alpha_i < d_{\mathbf{C}}\}$$

is a basis for $\mathcal{A}_n^{\mathbf{C}}(F)$.

Finally, with a basis theorem, we obtain the expected (and important) corolary: every level one cyclotomic quotient is isomorphic to the wreath product algebra $F^{\otimes n} \rtimes_{\rho} S_n$.

Fix $\mathbf{C}$ and denote by $d = d_{\mathbf{C}}$ its level. Let $\chi_1 := \chi_{\mathbf{C}}$ and, for $i = 2, \dots, n$, define

$$\chi_i = s_{i-1} \dots s_1 \chi_1 s_1 \dots s_{i-1}.$$

Lemma 6.3.1. *For $f \in F$ and $1 \leq i, j \leq n$, we have*

$$f_i \chi_j = \begin{cases} (-1)^{\tau d \bar{f}} \chi_j f_i & \text{if } i \neq j, \\ (-1)^{\tau d \bar{f}} \chi_j \psi^d(f)_i & \text{if } i = j. \end{cases}$$

In particular,

$$F^{\otimes m} \chi_j = \chi_j F^{\otimes m}, \quad 1 \leq j \leq n \text{ and } 1 \leq m \leq n, \quad (6.3.2)$$

where we identify $F^{\otimes m}$ with $F^{\otimes m} \otimes \mathbb{k}^{\otimes n-m} \subseteq F^{\otimes n}$.

Proof. The case $j = 1$ follows immediately from the computation:

$$f_i(x_1^k - \mathbf{c}^{(k)}) = (-1)^{\tau k \bar{f}} (x_1^k - \mathbf{c}^{(k)}) \psi_1^k(f_i),$$

for all $\mathbf{c}^{(k)} \in \mathbf{F}_1^{(k)}$ of parity $k\tau$. Now let $\pi = s_{j-1} \cdots s_1$. The general case $j \in \{1, \dots, n\}$ follows from the first case and relation (3.1.7):

$$\begin{aligned} f_i \chi_j &= f_i \pi \chi_1 \pi^{-1} = \pi^{\pi^{-1}} f_i \chi_1 \pi^{-1} \\ &= (-1)^{\tau d \bar{f}} \pi \chi_1 \psi_1^d(\pi^{-1} f_i) \pi^{-1} \\ &= (-1)^{\tau d \bar{f}} \chi_1^\pi \left(\psi_1^d(\pi^{-1} f_i) \right) = \begin{cases} (-1)^{\tau d \bar{f}} \chi_j f_i & \text{if } i \neq j, \\ (-1)^{\tau d \bar{f}} \chi_j \psi^d(f)_i & \text{if } i = j. \end{cases} \quad \square \end{aligned}$$

Lemma 6.3.2. *We have $x_1 \chi_1 = \chi_1 x_1$. For $1 \leq i < j \leq n$, we also have $(-1)^{\tau d} x_j \chi_i = \chi_i x_j$.*

Proof. The first statement follows from the fact that, for all $\mathbf{c}^{(k)} \in \mathbf{F}_1^{(k)}$ of parity $k\tau$, we have:

$$x_1(x_1^k - \mathbf{c}^{(k)}) = x_1^{k+1} - x_1 \mathbf{c}^{(k)} \stackrel{(3.1.4)}{=} x_1^k x_1 - (-1)^{\tau \overline{\mathbf{c}^{(k)}}} \psi_1^{-1}(\mathbf{c}^{(k)}) x_1 \stackrel{(6.1.2)}{=} (x_1^k - \mathbf{c}^{(k)}) x_1.$$

Now, for $1 \leq j \leq n$ and $\mathbf{c}^{(k)} \in \mathbf{F}_1^{(k)}$ of parity $k\tau$, we have

$$x_j(x_1^k - \mathbf{c}^{(k)}) = (-1)^{k\tau} x_1^k x_j - (-1)^{\tau \overline{\mathbf{c}^{(k)}}} \psi_j^{-1}(\mathbf{c}^{(k)}) x_j = (-1)^{k\tau} (x_1^k - \mathbf{c}^{(k)}) x_j.$$

And hence, we have that

$$x_j \chi_1 = (-1)^{\tau d} \chi_1 x_j. \quad (6.3.3)$$

Finally, for $1 \leq i < j \leq n$, the second statement follows from:

$$x_j \chi_i \stackrel{(3.1.5)}{=} s_{i-1} \cdots s_1 x_j \chi_1 s_1 \cdots s_{i-1} \stackrel{(6.3.3)}{=} (-1)^{\tau d} s_{i-1} \cdots s_1 \chi_1 x_j s_1 \cdots s_{i-1} \stackrel{(3.1.5)}{=} \chi_i x_j. \quad \square$$

Lemma 6.3.3. *For all $i \in \{1, \dots, n\}$, we have $S_{i-1} \chi_i = \chi_i S_{i-1}$. In particular, we have*

$$F^{\otimes n} S_{n-1} \chi_n F^{\otimes n} S_n = \chi_n F^{\otimes n} S_n. \quad (6.3.4)$$

Proof. First notice that, for all $\mathbf{c}^{(k)} \in \mathbf{F}_1^{(k)}$ and $j > 1$, we have:

$$s_j(x_1^k - \mathbf{c}^{(k)}) = (x_1^k - s_j \mathbf{c}^{(k)}) s_j,$$

since $s_j \in S_n^1$. Thus, for all $j \in \{2, \dots, n-1\}$ we have that s_j commutes with χ_1 .

Now, for $1 \leq j \leq i-2$, we have

$$\begin{aligned} s_j \chi_i &= s_j s_{i-1} \cdots s_1 \chi_1 s_1 \cdots s_{i-1} \\ &= s_{i-1} \cdots s_j s_{j+1} s_j s_{j-1} \cdots s_1 \chi_1 s_1 \cdots s_{n-1} \\ &= s_{i-1} \cdots s_{j+1} s_j s_{j+1} s_{j-1} \cdots s_1 \chi_1 s_1 \cdots s_{n-1} \\ &= s_{i-1} \cdots s_1 s_{j+1} \chi_1 s_1 \cdots s_{n-1} \\ &= s_{i-1} \cdots s_1 \chi_1 s_{j+1} s_1 \cdots s_{n-1} \\ &= s_{i-1} \cdots s_1 \chi_1 s_1 \cdots s_{j+1} s_j s_{j+1} s_{j+2} \cdots s_{n-1} \\ &= s_{i-1} \cdots s_1 \chi_1 s_1 \cdots s_j s_{j+1} s_j s_{j+2} \cdots s_{n-1} \\ &= s_{n-1} \cdots s_1 \chi_1 s_1 \cdots s_{n-1} s_j = \chi_i s_j. \end{aligned}$$

Therefore $S_{i-1} \chi_i = S_{i-1} \chi_i$. Then, equation (6.3.4) follows from (6.3.2) by relations in $\mathcal{A}_n(F)$

□

The following results are the analogous from [Sav20, Section 6.2], but without admit that $\mathbf{F}_1^{(k)} \subseteq F_\psi^{(k)}$. To obtain a more general result, it was necessary to add a more complicated notation in the proof.

Lemma 6.3.4. *For all $i \in \{1, \dots, n\}$, we have*

$$\chi_i - x_i^d \in \sum_{e=0}^{d-1} P_{i-1} x_i^e F^{\otimes n} S_i.$$

Proof. The case $i = 1$ is immediate. Assuming the result for some $1 \leq i < n$, we have

$$\begin{aligned} \chi_{i+1} - x_{i+1}^d &\stackrel{(3.3.2)}{=} s_i \chi_i s_i - s_i x_i^d s_i - t_{i,i+1}^{(d)} s_i = s_i (\chi_i - x_i^d) s_i - t_{i,i+1}^{(d)} s_i \\ &\in s_i \sum_{e=0}^{d-1} P_{i-1} x_i^e F^{\otimes n} S_i s_i - t_{i,i+1}^{(d)} s_i \subseteq \sum_{e=0}^{d-1} P_i x_{i+1}^e F^{\otimes n} S_{i+1}. \quad \square \end{aligned}$$

For $Z = \{z_1 < \dots < z_k\} \subseteq \{1, \dots, n\}$, let $\chi_\emptyset = 1$ and

$$\chi_Z := \chi_{z_1} \chi_{z_2} \cdots \chi_{z_k} \in \mathcal{A}_n(F).$$

We also define, for all $m \in \mathbb{N}$

$$\begin{aligned} \Pi_m &:= \{(\alpha, Z) \mid Z \subseteq \{1, \dots, m\}, \alpha \in \mathbb{N}^m, \alpha_i < d \text{ whenever } i \notin Z\}, \\ \Pi_m^+ &:= \{(\alpha, Z) \in \Pi_m \mid Z \neq \emptyset\}. \end{aligned}$$

Lemma 6.3.5. *$\mathcal{A}_n(F)$ is a free right $F^{\otimes n} \rtimes_\rho S_n$ -module on the basis*

$$\{x^\alpha \chi_Z \mid (\alpha, Z) \in \Pi_n\}.$$

Furthermore, for all $1 \leq m \leq n$ and $\beta \in \mathbb{N}^m$,

$$x^\beta \in \sum_{(\alpha', Z') \in \Pi_m} x^{\alpha'} \chi_{Z'}' F^{\otimes n} S_m. \quad (6.3.5)$$

Proof. Consider the ordering $\prec$ on $\mathbb{N}^n$ given by $\alpha \prec \alpha'$ if and only if

$$\alpha_n = \alpha'_n, \quad \dots, \quad \alpha_{k+1} = \alpha'_{k+1}, \quad \alpha_k < \alpha'_k$$

for some $k \in \{1, \dots, n\}$. Define the following class of functions, for $m \in \{1, \dots, n\}$:

$$\gamma_m: \Pi_m \rightarrow \mathbb{N}^n, \quad \gamma(\alpha, Z) := (\gamma_1, \dots, \gamma_m, 0, \dots, 0), \quad \text{such that } \gamma_i = \begin{cases} \alpha_i & \text{if } i \notin Z, \\ \alpha_i + d & \text{if } i \in Z. \end{cases}$$

Let $\varsigma_1: \Pi_1 \rightarrow \mathbb{N}$ be the map identical to zero and let define, inductively, the class of function, for $m \in \{2, \dots, n\}$:

$$\varsigma_m: \Pi_m \rightarrow \mathbb{N}, \quad \varsigma_m(\alpha, Z) = \varsigma_{m-1}(\alpha', Z') + \alpha_m |Z'| d,$$

where $\alpha' = (\alpha_1, \dots, \alpha_{m-1})$ and $Z' = Z \setminus \{m\}$. We prove, by induction on m , that

$$x^\alpha \chi_Z - (-1)^{\tau_{\varsigma_m}(\alpha, Z)} x^{\gamma_m(\alpha, Z)} \in \sum_{\beta \prec \gamma_m(\alpha, Z)} x^\beta F^{\otimes n} S_m. \quad (6.3.6)$$

Since γ_m is a bijection between Π_m and $\mathbb{N}^m \hookrightarrow \mathbb{N}^n$, equation (6.3.5) follows from (6.3.6).

Consider the base case $m = 1$. If $Z = \emptyset$, then $\gamma_1(\alpha, Z) = (\alpha, 0, \dots, 0) \in \mathbb{N}^n$, and the claim is clearly true. If $Z = \{1\}$, then $\gamma(\alpha, Z) = (\alpha + d, 0, \dots, 0)$ and we have

$$x^\alpha \chi_Z - x^{\gamma(\alpha, Z)} = x_1^\alpha \chi_1 - x_1^{\alpha+d} = x_1^\alpha (\chi_1 - x_1^d) \stackrel{6.3.4}{\in} \sum_{e=0}^{d-1} x_1^{e+\alpha} F^{\otimes n} S_1 \subseteq \sum_{\beta \prec \gamma_1(\alpha, Z)} x^\beta F^{\otimes n} S_1.$$

Thus, the claim is true for $n = 1$.

Now let $m > 1$ and suppose that (6.3.6) holds for $m - 1$. Let $(\alpha, Z) \in \Pi_m$. To simplify the reading, denote

$$\varsigma_m(\alpha, Z), \quad \varsigma_{m-1}(\alpha', Z'), \quad \alpha_1 + \dots + \alpha_{m-1},$$

by

$$\varsigma_m, \quad \varsigma_{m-1}, \quad |\alpha'|,$$

respectively. Then, if $Z = Z'$, we have:

$$\begin{aligned} x^\alpha \chi_Z - (-1)^{\tau_{\varsigma_m}(\alpha, Z)} x^{\gamma_m(\alpha, Z)} &= x^{\alpha'} x_m^{\alpha_m} \chi_Z - (-1)^{\tau(\varsigma_{m-1} + \alpha_m |Z|d)} x^{\gamma_{m-1}} x_m^{\alpha_m} \\ &= (-1)^{\tau_{\alpha_m} |\alpha'|} x_m^{\alpha_m} (x^{\alpha'} \chi_Z - (-1)^{\tau(\varsigma_m + \alpha_m |\alpha'|)} x^{\gamma_{m-1}}) \\ &\in (-1)^{\tau_{\alpha_m} |\alpha'|} \sum_{\beta \prec \gamma_{m-1}(\alpha', Z)} x_m^{\alpha_m} x^\beta F^{\otimes n} S_n \\ &= \sum_{\beta \prec \gamma_m(\alpha', Z)} x^\beta x_m^{\alpha_m} F^{\otimes n} S_n \\ &\subseteq \sum_{\beta \prec \gamma_m(\alpha, Z)} x^\beta F^{\otimes n} S_n. \end{aligned}$$

On the other hand, if $m \in Z$, then, setting $Z' = Z \setminus \{m\}$, we have

$$\begin{aligned} x^\alpha \chi_Z - (-1)^{\tau_{\varsigma_m}(\alpha, Z)} x^{\gamma_m(\alpha, Z)} &= x^{\alpha'} x_m^{\alpha_m} \chi_{Z'} \chi_m - (-1)^{\tau(\varsigma_{m-1} + \alpha_m |Z'|d)} x^{\gamma_{m-1}} x_m^{\alpha_m + d} \\ &= (-1)^{\tau_{\alpha_m} |\alpha'|} x_m^{\alpha_m} \left(x^{\alpha'} \chi_{Z'} \chi_m - (-1)^{\tau_{\varsigma_{m-1}}} x^{\gamma_{m-1}} x_m^d \right) \\ &= (-1)^{\tau_{\alpha_m} |\alpha'|} x_m^{\alpha_m} \left((x^{\alpha'} \chi_{Z'} - (-1)^{\tau_{\varsigma_{m-1}}} x^{\gamma_{m-1}}) \chi_m + (-1)^{\tau_{\varsigma_{m-1}}} x^{\gamma_{m-1}} (\chi_m - x_m^d) \right) \\ &\in x_m^{\alpha_m} \sum_{\beta \prec \gamma_{m-1}(\alpha', Z')} x^\beta F^{\otimes n} S_{m-1} \chi_m + x_m^{\alpha_m} x^{\gamma_{m-1}} \sum_{e=0}^{d-1} P_{m-1} x_m^e F^{\otimes n} S_m \\ &= \sum_{\beta \prec \gamma_{m-1}(\alpha', Z')} x^\beta x_m^{\alpha_m} F^{\otimes n} \chi_m S_{m-1} + \sum_{e=0}^{d-1} P_{m-1} x_m^{\alpha_m + e} F^{\otimes n} S_m \\ &\subseteq \sum_{\beta \prec \gamma_m(\alpha, Z)} x^\beta F^{\otimes n} S_m, \end{aligned}$$

where in the fourth line we used the induction hypothesis and Lemma 6.3.4, in the fifth we used Lemma 6.3.3, and in the last we used equation (6.3.2).

In particular we have, for all $(\alpha, Z) \in \Pi_n$:

$$x^\alpha \chi_Z - \xi x^{\gamma_n(\alpha, Z)} \in \sum_{\beta \prec \gamma_n(\alpha, Z)} x^\beta F^{\otimes n} S_n, \quad \text{for some } \xi \in \{1, -1\}.$$

Now, $\gamma_n: \Pi_n \rightarrow \mathbb{N}^n$ is a bijection and, by Theorem 4.2.3, $\{x^\alpha \mid \alpha \in \mathbb{N}^n\}$ is a basis for $\mathcal{A}_n(F)$ viewed as right $F^{\otimes n} \rtimes_\rho S_n$ -module. Thus, the lemma follows from (6.3.6). $\square$

Lemma 6.3.6. *For all $m \in \{1, \dots, n\}$, define $J_C^{(m)} := \mathcal{A}_m(F) \chi_1 \mathcal{A}_m(F)$, where $\mathcal{A}_m(F) \subseteq \mathcal{A}_n(F)$ is a subalgebra in the canonical way. Then*

$$J_C^{(m)} = \sum_{i=1}^m P_m \chi_i F^{\otimes m} S_m,$$

for all $1 \leq m \leq n$.

Proof. We have

$$J_C^{(m)} = \mathcal{A}_m(F) \chi_1 \mathcal{A}_m(F) = \mathcal{A}_m(F) \chi_1 P_m F^{\otimes m} S_m \stackrel{6.3.2}{=} \mathcal{A}_m(F) \chi_1 F^{\otimes m} S_m.$$

Then, since for all $\pi \in S_m$ there exists a unique pair $u \in S_m^1$ and $i \in \{1, \dots, m\}$ such that $\pi = s_{i-1} \cdots s_1 u$, it follows that

$$\begin{aligned} \mathcal{A}_m(F) \chi_1 F^{\otimes m} S_m &= P_m F^{\otimes m} S_m \chi_1 F^{\otimes m} S_m \\ &= \sum_{i=1}^m \sum_{u \in S_m^1} P_m F^{\otimes m} s_{i-1} \cdots s_1 u \chi_1 F^{\otimes m} S_m \\ &= \sum_{i=1}^m \sum_{u \in S_m^1} P_m F^{\otimes m} s_{i-1} \cdots s_1 \chi_1 u F^{\otimes m} S_m \\ &= \sum_{i=1}^m P_m F^{\otimes m} s_{i-1} \cdots s_1 \chi_1 s_1 \cdots s_{i-1} F^{\otimes m} S_m \\ &= \sum_{i=1}^m P_m F^{\otimes m} \chi_i F^{\otimes m} S_m \\ &\stackrel{(6.3.2)}{=} \sum_{i=1}^m P_m F^{\otimes m} \chi_i F^{\otimes m} S_m, \end{aligned}$$

where in the third equality we use the fact that ${}^u(\mathbf{c}^{(k)}) = \mathbf{c}^{(k)}$, for all $\mathbf{c}^{(k)} \in \mathbf{F}_1^{(k)}$. Therefore the lemma is proved. $\square$

Lemma 6.3.7. *For $d > 0$, we have $J_C = \sum_{(\alpha, Z) \in \Pi_n^+} x^\alpha \chi_Z F^{\otimes n} S_n$.*

Proof. Let $J_{\mathbf{C}}^{(m)}$ defined in Lemma 6.3.6, we will prove, by induction on m , that

$$J_{\mathbf{C}}^{(m)} \subseteq \sum_{(\alpha, Z) \in \Pi_m^+} x^\alpha \chi_Z F^{\otimes n} S_m, \quad (6.3.7)$$

where here we see $\alpha \in \mathbb{N}^n$ in the natural way.

When $n = 1$, it follows from (6.3.2) and Lemma 6.3.2:

$$\sum_{k \in \mathbb{N}} x_1^k \chi_1 F = P_1 \chi_1 F \stackrel{(6.3.2)}{=} A_1(F) \chi_1 F \stackrel{6.3.2}{=} J_{\mathbf{C}}.$$

Suppose $m > 1$ and let $J_m = \sum_{(\alpha, Z) \in \Pi_m^+} x^\alpha \chi_Z F^{\otimes n} S_m$. Then, by Lemma 6.3.6, it suffices to prove that

$$x^\alpha \chi_i F^{\otimes m} S_m \subseteq J_m,$$

for all $\alpha \in \mathbb{N}^m$ and $1 \leq i \leq m$.

Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ and $\beta = (\alpha_1, \dots, \alpha_{m-1}, 0)$, so that $x^\alpha = \xi x_m^{\alpha_m} x^\beta$, where $\xi \in \{-1, 1\}$.

For $i = m$, it follows by expanding x^β from equation (6.3.5) and by applying Lemma 6.3.3:

$$\begin{aligned} x^\alpha \chi_m F^{\otimes m} S_m &\stackrel{(6.3.5)}{\subseteq} \sum_{(\alpha', Z') \in \Pi_{m-1}} x_m^{\alpha_m} x^{\alpha'} \chi_{Z'} F^{\otimes n} S_{m-1} \chi_m F^{\otimes m} S_m \\ &\stackrel{6.3.3}{\subseteq} \sum_{(\alpha', Z') \in \Pi_{m-1}} x_m^{\alpha_m} x^{\alpha'} \chi_{Z'} \chi_m F^{\otimes n} S_m \subseteq J. \end{aligned}$$

Now suppose $1 \leq i < n$. By the induction hypothesis, we have

$$x^\alpha \chi_i F^{\otimes m} S_m = x_m^{\alpha_m} x^\beta \chi_i F^{\otimes m} S_m \subseteq \sum_{(\alpha', Z') \in \Pi_{m-1}^+} x_m^{\alpha_m} x^{\alpha'} \chi_{Z'} F^{\otimes n} S_m.$$

Now we prove by induction on α_m that $x_m^{\alpha_m} x^{\alpha'} \chi_{Z'} F^{\otimes n} S_m \subseteq J_m$ for all $(\alpha', Z') \in \Pi_{m-1}^+$. If $\alpha_m < d$, this follows immediately from the definition of Π_m^+ . The case $\alpha_m \geq d$ follows from Lemmas 6.3.2 and 6.3.4 and induction hypotheses in (6.3.7) and α_m . Indeed:

$$\begin{aligned} x_m^{\alpha_m} x^{\alpha'} \chi_{Z'} F^{\otimes n} S_m &= x_m^{\alpha_m-d} x^{\alpha'} \chi_{Z'} x_m^d F^{\otimes n} S_m \\ &\stackrel{6.3.4}{\subseteq} x_m^{\alpha_m-d} x^{\alpha'} \chi_{Z'} \left(\chi_m + \sum_{e=0}^{d-1} P_{m-1} x_m^e F^{\otimes n} S_m \right) F^{\otimes n} S_m \\ &\stackrel{6.3.2}{\subseteq} x_m^{\alpha_m-d} x^{\alpha'} \chi_{Z'} \chi_m F^{\otimes n} S_m + \sum_{e=0}^{d-1} x_m^{\alpha_m-d+e} x^{\alpha'} \chi_{Z'} F^{\otimes n} S_m, \end{aligned}$$

where, all signs coming from commutations of elements are incorporated in $F^{\otimes n}$. It is easy to see, by the definition of J_m and Π_m , that $x_m^{\alpha_m-d} x^{\alpha'} \chi_{Z'} \chi_m F^{\otimes n} S_m \in J_m$. Now, by (6.3.7), for

$0 \leq e < d$, we have

$$x_m^{\alpha_m - d + e} J_{\mathbf{C}}^{(m)} F^{\otimes n} S_m \subseteq \sum_{(\alpha', Z') \in \Pi_{m-1}^+} x_m^{\alpha_m - d + e} x^{\alpha'} \chi_{Z'} F^{\otimes n} S_m.$$

Since $0 \leq \alpha_m - d + e < \alpha_m$, each term in the above sum is contained in J_m by induction α_m .

In particular $J_{\mathbf{C}} = J_{\mathbf{C}}^{(n)} \subseteq J_n$. Since it is clear that $J_n \subseteq J_{\mathbf{C}}$ the proof is done. $\square$

Theorem 6.3.8 (Basis theorem for cyclotomic quotients). *The canonical images of the elements*

$$\{x^\alpha \mathbf{b}\pi \mid \alpha \in \mathbb{N}^n \text{ with } \alpha_1, \dots, \alpha_n < d_{\mathbf{C}}, \mathbf{b} \in \mathcal{B}^{\otimes n}, \pi \in S_n\}$$

form a basis for $\mathcal{A}_n^{\mathbf{C}}(F)$.

Proof. By Lemmas 6.3.5 and 6.3.7, the elements $\{x^\alpha \chi_Z \mid (a, Z) \in \Pi_n^+\}$ form a basis for $J_{\mathbf{C}}$ viewed as a right $F^{\otimes n} \rtimes_{\rho} S_n$ -module. Thus Lemma 6.3.5 implies that

$$\{x^\alpha \mid \alpha \in \mathbb{N}^n \text{ with } \alpha_1, \dots, \alpha_n < d\}$$

is a basis for a complement to $J_{\mathbf{C}}$ in $\mathcal{A}_n(F)$, viewed as a right $F^{\otimes n} \rtimes_{\rho} S_n$ -module. The theorem follows. $\square$

Corollary 6.3.9. *Every level one cyclotomic wreath product algebra is isomorphic to*

$$F^{\otimes n} \rtimes_{\rho} S_n.$$

Proof. Suppose $\chi_{\mathbf{C}} = x_1$. Then, by Proposition 4.4.2, there is a surjective homomorphism $\mathcal{A}_n(F) \twoheadrightarrow F^{\otimes n} \rtimes_{\rho} S_n$, with kernel containing $J_{\mathbf{C}}$, which induces a surjective homomorphism

$$\phi: \mathcal{A}_n^{\mathbf{C}}(F) \rightarrow F^{\otimes n} \rtimes_{\rho} S_n.$$

Then, by Theorem 6.3.8 and by comparing dimension, it follows that ϕ is an isomorphism.

The general case follows from Proposition 6.1.3. $\square$

6.4 Frobenius algebra structure

Notice that $F^{\otimes n}$ is a Frobenius algebra with trace map given by

$$\mathrm{tr}^{\otimes n}: \mathbf{f} = f_1 \cdots f_n \mapsto \mathrm{tr}(f_1) \cdots \mathrm{tr}(f_n).$$

It is easy to see that such trace map has parity τn and degree $-\delta n$.

Furthermore, since tr has parity τ , we have that

$$\mathrm{tr}^{\otimes n}(\sigma \mathbf{f}) = \mathrm{sgn}(\sigma)^{\tau} \mathrm{tr}^{\otimes n}(\mathbf{f}), \quad \mathbf{f} \in F^{\otimes n}, \sigma \in S_n, \quad (6.4.1)$$

where $\mathrm{sgn}: S_n \rightarrow \{-1, 1\}$ is the sign map for S_n .

Then, by Theorem 6.3.8, we can define an even linear map $\mathrm{tr}_{\mathbf{C}}: \mathcal{A}_n^{\mathbf{C}}(F) \rightarrow \mathbb{k}$ by setting, for all $\mathbf{f} \in F^{\otimes n}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\alpha_1, \dots, \alpha_n < d_{\mathbf{C}}$,

$$\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{f}\pi) = \delta_{\alpha, (d-1, \dots, d-1)} \mathrm{tr}^{\otimes n}(\mathbf{f}) \delta_{\pi, 1}, \quad (6.4.2)$$

and extending by linearity.

Theorem 6.4.1. *The cyclotomic quotient $\mathcal{A}_n^{\mathbf{C}}(F)$ is an $\mathbb{N}$ -graded Frobenius superalgebra with trace map $\mathrm{tr}_{\mathbf{C}}$ (of parity $nd\tau$ and degree $-nd\delta$).*

Proof. Consider the strong Bruhat order on S_n , and the total order on $\mathbb{N}^n$ given by

$$\alpha < \beta \iff \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i \text{ for some } 1 \leq i \leq n. \quad (6.4.3)$$

Consider $\mathcal{B}_n^{\mathbf{C}}$ of $\mathcal{A}_n^{\mathbf{C}}(F)$ given in Theorem 6.3.8.

To prove that

$$a \mapsto \left(b \mapsto (-1)^{\bar{a}\bar{b}} \mathrm{tr}_{\mathbf{C}}(ba) \right)$$

defines an isomorphism from $\mathcal{A}_n^{\mathbf{C}}(F)$ to $\mathcal{A}_n^{\mathbf{C}}(F)^{\vee}$, it suffices to prove that, for each basis element $x^{\alpha}\mathbf{b}\pi \in \mathcal{B}_n^{\mathbf{C}}$, we can find $z \in \mathcal{A}_n^{\mathbf{C}}(F)$ such that $\mathrm{tr}_{\mathbf{C}}(zx^{\alpha}\mathbf{b}\pi) = \pm 1$ and $\mathrm{tr}_{\mathbf{C}}(zx^{\alpha'}\mathbf{b}'\pi') = 0$ for all

$$\pi' \not\leq \pi \text{ or } (\pi' = \pi, \alpha' < \alpha) \text{ or } (\pi = \pi', \alpha = \alpha', \mathbf{b}' \neq \mathbf{b}).$$

Indeed, if this is true, one can find a dual basis by inverting a unitriangular matrix. By ordering the basis $\mathcal{B}$ of F and by ordering incomparable elements of S_n , one can define an ordering $\{a_1, \dots, a_m\} = \mathcal{B}_n^{\mathbf{C}}$ and the elements $z_1, \dots, z_m \in \mathcal{A}_n^{\mathbf{C}}(F)$ satisfying

$$\mathrm{tr}_{\mathbf{C}}(z_i a_j) = \begin{cases} \pm 1, & \text{if } j = i \\ 0, & \text{if } j < i. \end{cases}$$

It implies that $\{z_1, \dots, z_m\}$ is a basis for $\mathcal{A}_n^{\mathbf{C}}(F)$ and that $\Phi: a \mapsto \left(b \mapsto (-1)^{\bar{a}\bar{b}} \mathrm{tr}_{\mathbf{C}}(ba) \right)$ can be identified with the triangular matrix $(\Phi(a_i)(z_j))_{i,j}$, of no zero determinant. Thus Φ is isomorphism.

Fix a basis element $x^{\alpha}\mathbf{b}\pi \in \mathcal{B}_n^{\mathbf{C}}$, and define:

$$\mathbf{g} = (\psi^{-\alpha_1} \otimes \dots \otimes \psi^{-\alpha_n})(\mathbf{b}^{\vee}), \quad \beta = (d-1-\alpha_1, \dots, d-1-\alpha_n), \quad z = \xi \zeta \pi^{-1} x^{\beta} \mathbf{g},$$

where $\xi, \zeta \in \{1, -1\}$ satisfy $\mathbf{g}x^{\alpha} = \zeta x^{\alpha}\mathbf{b}^{\vee}$ and $x^{\beta}x^{\alpha} = \xi x^{\gamma}$ (where $\gamma = (d-1, \dots, d-1)$).

It follows that

$$\begin{aligned} \mathrm{tr}_{\mathbf{C}}(zx^{\alpha}\mathbf{b}\pi) &= \mathrm{tr}_{\mathbf{C}}(\pi^{-1}\xi x^{\beta}x^{\alpha}\mathbf{b}^{\vee}\mathbf{b}\pi) = \mathrm{tr}_{\mathbf{C}}(\pi^{-1}x^{\gamma}\mathbf{b}^{\vee}\mathbf{b}\pi) \\ &\stackrel{4.2.1}{=} \mathrm{tr}_{\mathbf{C}}(x^{\gamma} \pi^{-1}(\mathbf{b}^{\vee}\mathbf{b})) + \sum_{\sigma < \pi^{-1}} \mathrm{tr}_{\mathbf{C}}(a_{\sigma}\sigma\pi), \end{aligned}$$

for elements $a_{\sigma} \in P_n(F)$. Clearly, $\mathrm{tr}_{\mathbf{C}}(x^{\gamma} \pi^{-1}(\mathbf{b}^{\vee}\mathbf{b})) = \mathrm{tr}^{\otimes}(\pi^{-1}(\mathbf{b}^{\vee}\mathbf{b})) = \pm 1$. Also, since $\sigma < \pi^{-1}$ implies $\sigma\pi \neq 1$, we have $\sum_{\sigma < \pi^{-1}} \mathrm{tr}_{\mathbf{C}}(a_{\sigma}\sigma\pi) = 0$. Therefore $\mathrm{tr}_{\mathbf{C}}(zx^{\alpha}\mathbf{b}\pi) = 1$.

Now, let $x^{\alpha'}\mathbf{b}'\pi' \in \mathcal{B}_n^{\mathbf{C}}$. Then by Lemma 4.2.1, there exist $b_{\sigma} \in P_n(F)$ such that

$$\mathrm{tr}_{\mathbf{C}}(zx^{\alpha'}\mathbf{b}'\pi') = \mathrm{tr}_{\mathbf{C}}(\lambda x^{\gamma'} \pi^{-1}(\mathbf{b}^{\vee}\mathbf{b}')\pi^{-1}\pi') + \sum_{\sigma < \pi^{-1}} \mathrm{tr}_{\mathbf{C}}(b_{\sigma}\sigma\pi'),$$

where $\gamma' = \beta + \alpha'$ and $\lambda \in \{-1, 1\}$. The first term clearly vanishes if

$$\pi' \not\leq \pi \text{ or } (\pi' = \pi, \alpha' < \alpha) \text{ or } (\pi = \pi', \alpha = \alpha', \mathbf{b}' \neq \mathbf{b}),$$

since such conditions imply, respectively, $\pi^{-1}\pi' \neq 1$, $\gamma' \neq \gamma$ and $\text{tr}^{\otimes n}(\mathbf{b}^\vee \mathbf{b}') = 0$. For the second term, notice that $\sigma < \pi^{-1}$ implies $\sigma\pi' \neq 1$, for all $\pi' = \pi$ or $\pi' \not\leq \pi$. Therefore $\text{tr}_{\mathbf{C}}(zx^{\alpha'} \mathbf{b}' \pi') = 0$ and the proof is complete. $\square$

Theorem 6.4.2. *The Nakayama automorphism $\psi_{\mathbf{C}}$ for $(\mathcal{A}_n^{\mathbf{C}}(F), \text{tr}_{\mathbf{C}})$ is given by*

$$\mathbf{f} \mapsto \left(\psi^{d_{\mathbf{C}}} \right)^{\otimes n}(\mathbf{f}), \quad \pi \mapsto \text{sgn}(\pi)^{\tau} \pi, \quad x_i \mapsto (-1)^{\tau d} x_i,$$

for all $1 \leq i \leq n$, $\mathbf{f} \in F^{\otimes n}$, $\pi \in S_n$, where sgn is the sign map for S_n .

Proof. Let $\alpha \in \mathbb{N}^n$ with $\alpha_1, \dots, \alpha_n < d$, $\mathbf{b} = b_1 \otimes \dots \otimes b_n \in \mathcal{B}^{\otimes n}$, $\pi \in S_n$.

Fix $f \in F$ and $1 \leq i \leq n$. If $\pi \neq 1$ or $\alpha_k \neq d-1$ for some $1 \leq k \leq n$, we have

$$\text{tr}^{\mathbf{C}}(f_i x^{\alpha} \mathbf{b} \pi) = 0 = \text{tr}^{\mathbf{C}}(x^{\alpha} \mathbf{b} \pi f_i).$$

Now suppose $\alpha = (d-1, \dots, d-1)$. Then

$$\begin{aligned} \text{tr}_{\mathbf{C}}(f_i x^{\alpha} \mathbf{b}) &= \text{tr}_{\mathbf{C}} \left((-1)^{\tau n(d-1)} \bar{f} x^{\alpha} \psi^{d-1}(f)_i \mathbf{b} \right) \\ &= (-1)^{\tau n(d-1)} \bar{f} (-1)^{(\bar{b}_1 + \dots + \bar{b}_{i-1}) \bar{f}} \text{tr}(b_1) \dots \text{tr}(b_{i-1}) \text{tr} \left(\psi^{d-1}(f) b_i \right) \text{tr}(b_{i+1}) \dots \text{tr}(b_n) \\ &= (-1)^{\tau n(d-1)} \bar{f} (-1)^{(\bar{b}_1 + \dots + \bar{b}_i) \bar{f}} \text{tr}(b_1) \dots \text{tr}(b_{i-1}) \text{tr} \left(b_i \psi^d(f) \right) \text{tr}(b_{i+1}) \dots \text{tr}(b_n) \\ &= (-1)^{(\tau n(d-1) + \bar{b}) \bar{f}} \text{tr}_{\mathbf{C}} \left(x^{\alpha} \mathbf{b} \psi^d(f_i) \right). \end{aligned}$$

Since $\tau n(d-1) + \bar{b}$ is the parity of $x^{\alpha} \mathbf{b}$, it proves that the Nakayama automorphism of $\mathcal{A}_n^{\mathbf{C}}(F)$ maps f_i to $\psi^d(f)_i$. It follows that it maps $\mathbf{f} \in F^{\otimes n}$ to $(\psi^d)^{\otimes n}(\mathbf{f})$.

To show that is the given map $\psi_{\mathbf{C}}$ on S_n , it is enough to show that $\psi_{\mathbf{C}}(s_i) = (-1)^{\tau} s_i$, for all $1 \leq i \leq n-1$. Since $\text{tr}_{\mathbf{C}}$ involves projection onto the highest polynomial degree monomial $x_1^{d-1} \dots x_n^{d-1}$ in the x_i with respect to the basis of Theorem 6.3.8, it follows from Lemma 4.2.1 and (6.4.1) that

$$\begin{aligned} \text{tr}_{\mathbf{C}}(s_i x^{\alpha} \mathbf{b} \pi) &= \text{tr}_{\mathbf{C}}((s_i x^{\alpha})(s_i \mathbf{b}) s_i \pi + a \pi) & (a \in P_n(F) \text{ with pol. degree } < \alpha) \\ &= \text{tr}_{\mathbf{C}}((s_i x^{\alpha})(s_i \mathbf{b}) s_i \pi) + \text{tr}_{\mathbf{C}}(a \pi) \\ &= \delta_{s_i \pi, 1} \delta_{\alpha, (d-1, \dots, d-1)} \text{tr}^{\otimes n}(s_i \mathbf{b}) \\ &= (-1)^{\tau} \delta_{\pi s_i, 1} \delta_{\alpha, (d-1, \dots, d-1)} \text{tr}^{\otimes n}(\mathbf{b}) \\ &= (-1)^{\tau} \text{tr}_{\mathbf{C}}(x^{\alpha} \mathbf{b} \pi s_i). \end{aligned}$$

Since it is an automorphism, it therefore maps π to $\text{sgn}(\pi)^{\tau} \pi$ for all $\pi \in S_n$.

It remain to compute $\psi_{\mathbf{C}}$ on $x_1, \dots, x_n$. Notice that $\text{tr}(\psi(b^{\vee})\psi(c)) = \text{tr}(\psi(b^{\vee}c)) = \text{tr}(b^{\vee}c) = \delta_{b,c}$, for all $b, c \in \mathcal{B}$. Thus $\psi(b)^{\vee} = \psi(b^{\vee})$, for all $b \in \mathcal{B}$, and then

$$\psi_{\mathbf{C}}(t_{i,j}) = \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b}} \psi(b)_i \psi(b^{\vee})_j = \sum_{b \in \mathcal{B}} (-1)^{\tau \bar{b}} \psi(b)_i \psi(b)_j^{\vee} = t_{i,j},$$

where in the last equality follows from independence of $t_{i,j}$ from the chosen basis B . Then, by

(3.1.6) we have that

$$\begin{aligned}\psi_{\mathbf{C}}(x_{i+1}) &= \psi_{\mathbf{C}}(s_i x_i s_i + t_{i,i+1} s_i) = \\ &= s_i \psi_{\mathbf{C}}(x_i) s_i + \psi_{\mathbf{C}}(t_{i,i+1})((-1)^{\tau d}(s_i)) = s_i \psi_{\mathbf{C}}(x_i) s_i + (-1)^{\tau d} t_{i,i+1} s_i, \quad (6.4.4)\end{aligned}$$

for $1 \leq i \leq n-1$. Thus, if $\psi_{\mathbf{C}}(x_i) = (-1)^{\tau d} x_i$, for some $1 \leq i \leq n-1$, by 6.4.4 we have:

$$\psi_{\mathbf{C}}(x_{i+1}) = s_i \psi_{\mathbf{C}}(x_i) s_i + (-1)^{\tau d} t_{i,i+1} s_i = (-1)^{\tau d} (s_i x_i s_i + t_{i,i+1} s_i) = (-1)^{\tau d} x_{i+1}.$$

Therefore we just need to prove that $\psi_{\mathbf{C}}(x_1) = (-1)^{\tau d} x_1$.

First suppose $\pi(1) = 1$ and $\alpha_1 < d-1$ then

$$\begin{aligned}\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} \pi x_1) &= (-1)^{\tau(\alpha_2 + \dots + \alpha_n) + \tau \bar{\mathbf{b}}} \mathrm{tr}_{\mathbf{C}}(x_1 x^\alpha \psi_1(\mathbf{b}) \pi) \\ &= (-1)^{\tau(\alpha_2 + \dots + \alpha_n) + \tau \bar{\mathbf{b}}} \delta_{\alpha, (d-2, d-1, \dots, d-1)} \mathrm{tr}(\psi(b_1)) \cdots \mathrm{tr}(b_n) \delta_{1, \pi} \\ &= (-1)^{\tau(\alpha_2 + \dots + \alpha_n) + \tau \bar{\mathbf{b}}} \delta_{\alpha, (d-2, d-1, \dots, d-1)} \mathrm{tr}^{\otimes n}(\mathbf{b}) \delta_{1, \pi} \\ &= (-1)^{\tau(\overline{x^\alpha \mathbf{b} \pi}) + \tau \alpha_1} \delta_{\alpha, (d-2, d-1, \dots, d-1)} \mathrm{tr}^{\otimes n}(\mathbf{b}) \delta_{1, \pi}.\end{aligned}$$

Thus, if $\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} \pi x_1) \neq 0$ we have $\alpha_1 = d-2$. It follows then

$$\mathrm{tr}_{\mathbf{C}}(x_1 x^\alpha \mathbf{b} \pi) = (-1)^{\tau(\overline{x^\alpha \mathbf{b} \pi})} \mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} \pi ((-1)^{\tau d} x_1)).$$

Now, assume $\alpha_1 = d-1$. From the definition of $\chi_{\mathbf{C}}$ and the fact that $F_1^{(k)} F_1^{(l)} \subseteq F_1^{(k+l)}$, for all $k, l \in \mathbb{N}$, it follows that

$$x_1^d = \sum_{i=0}^{d-1} \mathbf{f}_{(i)} x_1^i, \quad \mathbf{f}_{(i)} \in F_1^{(d-i)}, \quad 0 \leq i \leq d-1. \quad (6.4.5)$$

Then, by writing $\alpha' = (0, \alpha_2, \alpha_3, \dots, \alpha_n)$ and $|\alpha'| = \alpha_2 + \dots + \alpha_n$, we have

$$\begin{aligned}\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} \pi x_1) &= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}})} (\mathrm{tr}_{\mathbf{C}}(x_1^d x^{\alpha'} \psi_1(\mathbf{b}) \pi)) \\ &= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}})} \sum_{i=0}^{d-1} \mathrm{tr}_{\mathbf{C}}(\mathbf{f}_{(i)} x_1^i x^{\alpha'} \psi_1(\mathbf{b}) \pi) \\ &= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}})} \sum_{i=0}^{d-1} \mathrm{tr}_{\mathbf{C}}((-1)^{\tau i \bar{\mathbf{f}}_{(i)}} x_1^i \psi_1^i(\mathbf{f}_{(i)}) x^{\alpha'} \psi_1(\mathbf{b}) \pi) \\ &= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}})} \sum_{i=0}^{d-1} \mathrm{tr}_{\mathbf{C}}((-1)^{\tau(|\alpha'| + i) \bar{\mathbf{f}}_{(i)}} x_1^i x^{\alpha'} \psi_1^i \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n}(\mathbf{f}_{(i)}) \psi_1(\mathbf{b}) \pi).\end{aligned}$$

Since $\mathbf{f}_{(i)} \in F_1^{(d-i)}$, $0 \leq i \leq d-1$, it follows that

$$\psi_1^i \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n}(\mathbf{f}_{(i)}) = \psi_1^i(\mathbf{f}_{(i)}) = (-1)^{\tau i(d-i)} \mathbf{f}_{(i)}.$$

And since $\text{tr}_{\mathbf{C}}(a\pi) = 0$, if $a \in P_n(F)$ has degree of x_1 less than $d - 1$, it follows:

$$\begin{aligned}
\text{tr}_{\mathbf{C}}(x^\alpha \mathbf{b}\pi x_1) &= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}}) + \tau(|\alpha'| + d - 1)\bar{\mathbf{f}}_{(d-1)}} \text{tr}_{\mathbf{C}}\left(x_1^{d-1} x^{\alpha'} (-1)^{\tau(d-1)} \mathbf{f}_{(d-1)} \psi_1(\mathbf{b})\pi\right) \\
&= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}}) + \tau(|\alpha'| + d - 1)\bar{\mathbf{f}}_{(d-1)}} \delta_{\alpha_2, d-1} \cdots \delta_{\alpha_n, d-1} \delta_{\pi, 1} \text{tr}^{\otimes n}\left((-1)^{\tau(d-1)} \mathbf{f}_{(d-1)} \psi_1(\mathbf{b})\right) \\
&= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}}) + \tau(|\alpha'| + d - 1)\bar{\mathbf{f}}_{(d-1)}} \delta_{\alpha_2, d-1} \cdots \delta_{\alpha_n, d-1} \delta_{\pi, 1} \text{tr}^{\otimes n}\left((-1)^{\tau d} \psi_1(\mathbf{f}_{(d-1)} \mathbf{b})\right) \\
&= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}} + d) + \tau(|\alpha'| + d - 1)\bar{\mathbf{f}}_{(d-1)}} \delta_{\alpha_2, d-1} \cdots \delta_{\alpha_n, d-1} \delta_{\pi, 1} \text{tr}^{\otimes n}(\mathbf{f}_{(d-1)} \mathbf{b}) \\
&= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}} + d)} \sum_{i=0}^{d-1} \text{tr}_{\mathbf{C}}\left((-1)^{\tau(|\alpha'| + i)\bar{\mathbf{f}}_{(i)}} x_1^i x^{\alpha'} \mathbf{f}_{(i)} \mathbf{b}\pi\right) \\
&= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}} + d)} \sum_{i=0}^{d-1} \text{tr}_{\mathbf{C}}\left(\psi_1^{-i} \psi_2^{-\alpha_2} \cdots \psi_n^{-\alpha_n}(\mathbf{f}_{(i)}) x_1^i x^{\alpha'} \mathbf{b}\pi\right) \\
&= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}} + d)} \sum_{i=0}^{d-1} \text{tr}_{\mathbf{C}}\left((-1)^{\tau(d-i)i} \mathbf{f}_{(i)} x_1^i x^{\alpha'} \mathbf{b}\pi\right).
\end{aligned}$$

Since $\text{tr}_{\mathbf{C}}\left((-1)^{\tau(d-i)i} \mathbf{f}_{(i)} x_1^i x^{\alpha'} \mathbf{b}\pi\right) = 0 = \text{tr}_{\mathbf{C}}\left((-1)^{\tau(d-1)} \mathbf{f}_{(i)} x_1^i x^{\alpha'} \mathbf{b}\pi\right)$, for $1 \leq i < d - 1$, it follows that

$$\begin{aligned}
\text{tr}_{\mathbf{C}}(x^\alpha \mathbf{b}\pi x_1) &= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}} + d) + \tau(d-1)} \sum_{i=0}^{d-1} \text{tr}_{\mathbf{C}}\left(\mathbf{f}_{(i)} x_1^i x^{\alpha'} \mathbf{b}\pi\right) \\
&= (-1)^{\tau(|\alpha'| + \bar{\mathbf{b}} + d) + \tau(d-1)} \text{tr}_{\mathbf{C}}\left(x_1^d x^{\alpha'} \mathbf{b}\pi\right) \\
&= (-1)^{\tau(\overline{x^\alpha \mathbf{b}\pi}) + \tau d} \text{tr}_{\mathbf{C}}(x_1 x^\alpha \mathbf{b}\pi).
\end{aligned}$$

This proves that $\psi_{\mathbf{C}}(x_1) = (-1)^{\tau d} x_1$ in the case when $\pi(1) = 1$.

Now, consider the case where $\pi = s_1 \pi'$, for some $\pi' \in S_n$ with $\pi'(1) = 1$. Then, by using (6.4.5) and the fact that $s_1 \pi' \neq 1$, it follows that $\text{tr}_{\mathbf{C}}(x_1 x^\alpha \mathbf{b} s_1 \pi') = 0$. Proving that $\text{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} s_1 \pi' x_1) = 0$, the result will follow for that case. First, notice that

$$\text{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} s_1 \pi' x_1) = \text{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} s_1 x_1 \pi') = \text{tr}_{\mathbf{C}}(x^\alpha \mathbf{b}(x_2 s_1 - t_{1,2}) \pi'). \quad (6.4.6)$$

If $\alpha \neq (d-1, \dots, d-1)$, it follows that $\text{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} x_2 s_1 \pi') = 0$, since $s_1 \pi' \neq 1$, and $\text{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} t_{1,2} \pi') = 0$, by the degree of α . Then suppose $\alpha = (d-1, \dots, d-1)$. It follows

$$\begin{aligned}
x_2^d &\stackrel{(3.3.2)}{=} s_1 x_1^d s_1 + t_{1,2}^{(d)} s_1 \stackrel{(6.4.5)}{=} \sum_{i=0}^{d-1} (s_1 \mathbf{f}_{(i)} s_1 x_1^i s_1) + t_{1,2}^{(d)} s_1 \stackrel{(3.3.2)}{=} \sum_{i=0}^{d-1} (s_1 \mathbf{f}_{(i)} (x_2^i - t_{1,2}^{(i)} s_1)) + t_{1,2}^{(d)} s_1. \\
&\hspace{25em} (6.4.7)
\end{aligned}$$

Now, let $\beta = (d-1, 0, d-1, \dots, d-1) \in \mathbb{N}^n$ and let $\zeta \in \{1, -1\}$ such that $x^\alpha = \zeta x^\beta x_2^{d-1}$.

Then, it follows:

$$\begin{aligned} \mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} x_2 s_1 \pi') &= (-1)^{\tau \bar{\mathbf{b}}} \mathrm{tr}_{\mathbf{C}}(x^\alpha x_2 \psi_2(\mathbf{b}) s_1 \pi') = \zeta(-1)^{\tau \bar{\mathbf{b}}} \mathrm{tr}_{\mathbf{C}}(x^\beta x_2^d \psi_2(\mathbf{b}) s_1 \pi') \\ &\stackrel{(6.4.7)}{=} \zeta(-1)^{\tau \bar{\mathbf{b}}} \left(\sum_{i=0}^{d-1} \mathrm{tr}_{\mathbf{C}} \left(x^\beta \left(s_1 \mathbf{f}_{(i)} \right) (x_2^i - t_{1,2}^{(i)} s_1) \psi_2(\mathbf{b}) s_1 \pi' \right) + \mathrm{tr}_{\mathbf{C}}(x^\beta t_{1,2}^{(d)} s_1 \psi_2(\mathbf{b}) s_1 \pi') \right). \end{aligned}$$

By considering the power of x_2 , and using the fact that the biggest power of x_2 in $t_{1,2}^{(k)}$ and $k-1$, we conclude that

$$\begin{aligned} 0 &= \mathrm{tr}_{\mathbf{C}} \left(x^\beta \left(s_1 \mathbf{f}_{(i)} \right) x_2^i \psi_2(\mathbf{b}) s_1 \pi' \right), & 1 \leq i < d-1 \quad \text{and} \\ 0 &= \mathrm{tr}_{\mathbf{C}} \left(x^\beta \left(s_1 \mathbf{f}_{(i)} \right) t_{1,2}^{(i)} s_1 \psi_2(\mathbf{b}) s_1 \pi' \right), & 1 \leq i \leq d-1. \end{aligned}$$

Thus:

$$\begin{aligned} \mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} x_2 s_1 \pi') &= \zeta(-1)^{\tau \bar{\mathbf{b}}} \left(\mathrm{tr}_{\mathbf{C}} \left(x^\beta \left(s_1 \mathbf{f}_{(i)} \right) x_2^{d-1} \psi_2(\mathbf{b}) s_1 \pi' \right) + \mathrm{tr}_{\mathbf{C}}(x^\beta t_{1,2}^{(d)} s_1 \psi_2(\mathbf{b}) s_1 \pi') \right) \\ &= \zeta(-1)^{\tau \bar{\mathbf{b}}} \left(\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{g} s_1 \pi') + \mathrm{tr}_{\mathbf{C}}(x^\beta t_{1,2}^{(d)} s_1 \psi_2(\mathbf{b}) \pi') \right), \end{aligned}$$

for some $\mathbf{g} \in F^{\otimes n}$. But, since $s_1 \pi' \neq 1$, it follows that $\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{g} s_1 \pi') = 0$. Then, we have

$$\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} x_2 s_1 \pi') = \zeta(-1)^{\tau \bar{\mathbf{b}}} \mathrm{tr}_{\mathbf{C}}(x^\beta t_{1,2}^{(d)} s_1 \psi_2(\mathbf{b}) \pi').$$

Now, notice that $t_{1,2}^{(d)} = x_2^{d-1} t_{1,2} + a$, where $a \in P_n(F)$ has polynomial degree less than α . Thus, it follows that

$$\begin{aligned} \mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} x_2 s_1 \pi') &= \zeta(-1)^{\tau \bar{\mathbf{b}}} \mathrm{tr}_{\mathbf{C}}(x^\beta x_2^{d-1} t_{1,2} s_1 \psi_2(\mathbf{b}) \pi') \\ &\stackrel{3.3.2}{=} \zeta \mathrm{tr}_{\mathbf{C}}(x^\beta x_2^{d-1} \mathbf{b} t_{1,2} \pi') = \mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} t_{1,2} \pi'), \end{aligned}$$

which implies that $\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} (x_2 s_1 - t_{1,2}) \pi') = 0$ and, by (6.4.6), it follows

$$\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{f} s_1 \pi' x_1) = 0, \quad \alpha \in \mathbb{N}, \mathbf{f} \in F^{\otimes n}, \pi' \in S_n, \text{ such that } \pi'(1) = 1. \quad (6.4.8)$$

And we conclude that $\psi_{\mathbf{C}}(x_1) = (-1)^{\tau d} x_1$ in this case.

Now, consider the remaining case where $\pi(1) \neq 1$, and write $\pi = \pi_1 s_1 \pi_2$, with $\pi_1(1) = \pi_2(1) = 1$ (take $\pi_1 = s_{\pi(1),2}$ and $\pi_2 = s_1 \pi_1 \pi$). Then, for $\alpha_1 < d-1$, clearly $\mathrm{tr}_{\mathbf{C}}(x_1 x^\alpha \mathbf{b} \pi_1 s_1 \pi_2) = 0$, since $\pi \neq 1$. And, for $\alpha_1 = d-1$, using the same argument as before, and the relation (6.4.5), it follows that $\mathrm{tr}_{\mathbf{C}}(x_1 x^\alpha \mathbf{b} \pi_1 s_1 \pi_2) = 0$. It remained to prove that $\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} \pi_1 s_1 \pi_2 x_1) = 0$. Writing $\mathbf{g} = \pi_1^{-1} \mathbf{b}$, it follows from Lemma 4.2.1 that:

$$\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} \pi_1 s_1 \pi_2 x_1) = \mathrm{tr}_{\mathbf{C}}(x^\alpha \pi_1 \mathbf{g} s_1 \pi_2 x_1) = \mathrm{tr}_{\mathbf{C}}(\pi_1 x^{\pi_1^{-1}(\alpha)} \mathbf{g} s_1 \pi_2 x_1) + \sum_{\sigma < \pi_1} \mathrm{tr}_{\mathbf{C}}(\sigma p_\sigma \mathbf{g} s_1 \pi_2 x_1),$$

where $p_\sigma \in P_n(F)$ with polynomial degree less than $\pi_1^{-1}(\alpha)$. By the computation of $\psi_{\mathbf{C}}$ on

elements of S_n , it follows that there exists $\zeta, \zeta_\sigma \in \{-1, 1\}$, for all $\sigma < \pi_1$, such that

$$\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} \pi_1 s_1 \pi_2 x_1) = \zeta \mathrm{tr}_{\mathbf{C}}(x^{\pi_1^{-1}(\alpha)} \mathbf{g} s_1 \pi_2 x_1 \pi_1) + \sum_{\sigma < \pi_1} \zeta_\sigma \mathrm{tr}_{\mathbf{C}}(p_\sigma \mathbf{g} s_1 \pi_2 x_1 \sigma).$$

Notice that the fact that $\sigma < \pi_1$ and $\pi_1(1) = 1$ implies that $\sigma(1) = 1$. Then, from (6.4.8), by using $\pi' = s_1 \pi_2 \pi_1$ and $\pi' = s_1 \pi_2 \sigma$, it follows that

$$\mathrm{tr}_{\mathbf{C}}(x^\alpha \mathbf{b} \pi_1 s_1 \pi_2 x_1) = \zeta \mathrm{tr}_{\mathbf{C}}(x^{\pi_1^{-1}(\alpha)} \mathbf{g} s_1 \pi_2 \pi_1 x_1) + \sum_{\sigma < \pi_1} \zeta_\sigma \mathrm{tr}_{\mathbf{C}}(p_\sigma \mathbf{g} s_1 \pi_2 \sigma x_1) = 0.$$

Therefore, $\psi_{\mathbf{C}}(x_1) = (-1)^{\tau d} x_1$. And the proof is complete. $\square$

6.5 Cyclotomic Mackey Theorem

For the remainder of this section, we assume

$$\mathbf{c}^{(k,j)} \in F_\psi^{(k)} \quad \text{for all } 1 \leq k \leq \theta, \ 1 \leq j \leq e_k, \quad (6.5.1)$$

where we identify F with the first factor of $F \otimes \mathbb{k}^{\otimes(n-1)} \subseteq F^{\otimes n}$ for each $n \geq 1$. Hence, fixed $\mathbf{C}$, $\mathcal{A}_n^{\mathbf{C}}(F)$ is defined for all $n \in \mathbb{N}$.

Theorem 6.3.8 implies that the subalgebra of $\mathcal{A}_{n+1}^{\mathbf{C}}(F)$ generated by $x_1, \dots, x_n, F^{\otimes n} \otimes 1$, and S_n is isomorphic to $\mathcal{A}_n^{\mathbf{C}}(F)$. Thus we have induction and restriction functors

$${}^{\mathbf{C}}\mathrm{Ind}_n^{n+1}: \mathcal{A}_n^{\mathbf{C}}(F)\text{-mod} \rightarrow \mathcal{A}_{n+1}^{\mathbf{C}}(F)\text{-mod}, \quad (6.5.2)$$

$${}^{\mathbf{C}}\mathrm{Res}_n^{n+1}: \mathcal{A}_{n+1}^{\mathbf{C}}(F)\text{-mod} \rightarrow \mathcal{A}_n^{\mathbf{C}}(F)\text{-mod}, \quad (6.5.3)$$

respectively.

Proposition 6.5.1. *Recall that $d = d_{\mathbf{C}}$ is the level of $\mathbf{C}$ (see (6.2.3)).*

(a) $\mathcal{A}_{n+1}^{\mathbf{C}}(F)$ is a free right $\mathcal{A}_n^{\mathbf{C}}(F)$ -module with basis

$$\{x_j^a b_j s_j \cdots s_n \mid 0 \leq a < d, \ b \in \mathcal{B}, \ 1 \leq j \leq n+1\}.$$

(b) The following is a decomposition of $(\mathcal{A}_n^{\mathbf{C}}(F), \mathcal{A}_n^{\mathbf{C}}(F))$ -bimodules

$$\mathcal{A}_{n+1}^{\mathbf{C}}(F) = \mathcal{A}_n^{\mathbf{C}}(F) s_n \mathcal{A}_n^{\mathbf{C}}(F) \oplus \bigoplus_{0 \leq a < d, \ b \in \mathcal{B}} x_{n+1}^a b_{n+1} \mathcal{A}_n^{\mathbf{C}}(F).$$

(c) If $0 \leq a < d$ and $f \in F$, there exist isomorphisms of $(\mathcal{A}_n^{\mathbf{C}}(F), \mathcal{A}_n^{\mathbf{C}}(F))$ -bimodules

$$\begin{aligned} \mathcal{A}_n^{\mathbf{C}}(F) s_n \mathcal{A}_n^{\mathbf{C}}(F) &\simeq \mathcal{A}_n^{\mathbf{C}}(F) \otimes_{\mathcal{A}_{n-1}^{\mathbf{C}}(F)} \mathcal{A}_n^{\mathbf{C}}(F), \\ x_{n+1}^a f_{n+1} \mathcal{A}_n^{\mathbf{C}}(F) &\simeq \Pi^{\bar{f} + \tau a} \mathcal{A}_n^{\mathbf{C}}(F). \end{aligned}$$

Proof. (a) It follows from Theorem 6.3.8 that the $\mathbb{k}$ -dimension of $\mathcal{A}_n^{\mathbf{C}}(F)$ is $n!(d \dim F)^n$. Define:

$$B_{n+1}^{\mathbf{C}} = \{x_j^a b_j s_j \cdots s_n \mid 0 \leq a < d, \ b \in \mathcal{B}, \ 1 \leq j \leq n+1\}.$$

Then $X = B_{n+1}^{\mathbf{C}} \mathcal{A}_n^{\mathbf{C}}(F)$ is a submodule of $\mathcal{A}_{n+1}^{\mathbf{C}}(F)$ with dimension not bigger than

$$n!(d \dim(F))^n |B_{n+1}^{\mathbf{C}}| = (n+1)!(d \dim(F))^{n+1} = \dim(\mathcal{A}_{n+1}^{\mathbf{C}}(F)),$$

and we have equality of dimension only if $B_{n+1}^{\mathbf{C}}$ is linear independent. So it is enough to prove that $X = \mathcal{A}_{n+1}^{\mathbf{C}}(F)$.

First, notice that $1^{\otimes n} \otimes F \subseteq X$ and $s_n \in X$. Now, fix $1 \leq j \leq n$, and $0 \leq a < d$. We claim that

$$x_j^a b_j s_j \cdots s_n x_k \equiv \pm x_j^a x_{k+1} b_j s_j \cdots s_n \pmod{X}, \quad (6.5.4)$$

for all $b \in \mathcal{B}$ and $j < k \leq n$. Indeed, for $k = n$, it follows:

$$\begin{aligned} x_j^a b_j s_j \cdots s_n x_n &\stackrel{(3.1.6)}{=} \pm x_j^a x_{n+1} b_j s_j \cdots s_n - x_j^a b_j s_j \cdots s_{n-1} t_{n,n+1} \\ &\stackrel{(3.3.15)}{=} \pm x_j^a x_{n+1} b_j s_j \cdots s_n - x_j^a b_j t_{j,n+1} s_j \cdots s_{n-1} \\ &= \pm x_j^a x_{n+1} b_j s_j \cdots s_n - \sum_{c \in \mathcal{B}} \pm x_j^a b_j c_j c_{n+1}^{\vee} s_j \cdots s_{n-1} \\ &= \pm x_j^a x_{n+1} b_j s_j \cdots s_n - \sum_{c \in \mathcal{B}} \pm \underbrace{c_{n+1}^{\vee}}_{\in X} \underbrace{x_j^a b_j c_j s_j \cdots s_{n-1}}_{\in \mathcal{A}_n^{\mathbf{C}}(F)} \\ &\equiv \pm x_j^a x_{n+1} b_j s_j \cdots s_n \pmod{X}. \end{aligned}$$

For $j \leq k \leq n-1$, it follows:

$$\begin{aligned} x_j^a b_j s_j \cdots s_n x_k &= \pm x_j^a x_{k+1} b_j s_j \cdots s_n - x_j^a b_j s_j \cdots s_{k-1} t_{k,k+1} s_{k+1} \cdots s_n \\ &\stackrel{(3.3.15)}{=} \pm x_j^a x_{n+1} b_j s_j \cdots s_n - x_j^a b_j (s_j \cdots s_{k-1}) (s_{k+1} \cdots s_n) t_{k,k+1} \\ &= \pm x_j^a x_{n+1} b_j s_j \cdots s_n - x_j^a b_j (s_{k+1} \cdots s_n) (s_j \cdots s_{k-1}) t_{k,k+1} \\ &= \pm x_j^a x_{n+1} b_j s_j \cdots s_n - (s_{k+1} \cdots s_n) x_j^a b_j t_{j,k+1} (s_j \cdots s_{k-1}) \\ &= \pm x_j^a x_n b_j s_j \cdots s_n - \sum_{c \in \mathcal{B}} \pm (s_{k+1} \cdots s_n) x_j^a b_j c_j c_{k+1}^{\vee} s_j \cdots s_{k-1} \\ &= \pm x_j^a x_n b_j s_j \cdots s_n - \sum_{c \in \mathcal{B}} \pm \underbrace{c_{k+1}^{\vee} s_{k+1} \cdots s_n}_{\in X} \underbrace{x_j^a b_j c_j s_j \cdots s_{k-1}}_{\in \mathcal{A}_n^{\mathbf{C}}(F)} \\ &\equiv \pm x_j^a x_{k+1} b_j s_j \cdots s_n \pmod{X}. \end{aligned}$$

Now, notice that, for any $1 \leq j \leq n$, and $0 \leq a < d$

$$x_j^a b_j s_j \cdots s_n x_k \equiv \pm x_k x_j^a b_j s_j \cdots s_n \pmod{X}, \quad (6.5.5)$$

for all $b \in \mathcal{B}$ and $1 \leq k < j$. Thus, from (6.5.4) and (6.5.5), it follows that

$$x^\alpha b_j s_j \cdots s_n \in X, \quad \text{for all } b \in \mathcal{B}, \alpha \in \mathbb{N}^{n+1}.$$

Now, let $b, c \in \mathcal{B}$, $1 \leq k \leq n$, with $k \neq j$. It is easy to see that

$$x_\alpha b_j s_j \cdots s_n b_k = \begin{cases} \pm x^\alpha b_k b_j s_j \cdots s_n, & \text{if } k < j, \\ \pm x^\alpha b_j b_{k+1} s_j \cdots s_n, & \text{if } k > j. \end{cases}$$

Thus, we conclude that

$$x^\alpha \mathbf{b} s_j \cdots s_n \in X, \quad \text{for all } \mathbf{b} \in \mathcal{B}^{\otimes(n+1)}, \alpha \in \mathbb{N}^{n+1}.$$

Furthermore, since for every $\pi \in S_{n+1} \setminus S_n$ there exist $j \in \{1, \dots, n\}$ and $\pi' \in S_n$ such that $\pi = s_j \cdots s_n \pi'$, it follows that

$$x^\alpha \mathbf{b} \pi \in X, \quad \text{for all } \mathbf{b} \in \mathcal{B}^{\otimes(n+1)}, \alpha \in \mathbb{N}^{n+1}, \pi \in S_{n+1} \setminus S_n.$$

And, for $\pi \in S_n$, we have that

$$x^\alpha \mathbf{b} \pi = \pm \underbrace{x_{n+1}^{\alpha_{n+1}} b_{n+1}}_{\in X} \underbrace{x_1^{\alpha_1} \cdots x_n^{\alpha_n} b_1 \cdots b_n}_{\in \mathcal{A}_n^{\mathbf{C}}(F)} \pi \in X,$$

for all $\mathbf{b} \in \mathcal{B}^{\otimes(n+1)}, \alpha \in \mathbb{N}^{n+1}$. And the proof of the first item is complete.

(b) By part (a), it follows

$$\begin{aligned} \mathcal{A}_{n+1}^{\mathbf{C}}(F) &= \mathcal{A}_n^{\mathbf{C}}(F) B_{n+1}^{\mathbf{C}} \mathcal{A}_n^{\mathbf{C}}(F) \\ &= \mathcal{A}_n^{\mathbf{C}}(F) B_n^{\mathbf{C}} \mathcal{A}_n^{\mathbf{C}}(F) \oplus \mathcal{A}_n^{\mathbf{C}}(F) (B_{n+1}^{\mathbf{C}} \setminus B_n^{\mathbf{C}}) \mathcal{A}_n^{\mathbf{C}}(F) \\ &= \mathcal{A}_n^{\mathbf{C}}(F) s_n \mathcal{A}_n^{\mathbf{C}}(F) \oplus \bigoplus_{1 \leq a < d, b \in \mathcal{B}} x_{n+1}^a b_{n+1} \mathcal{A}_n^{\mathbf{C}}(F). \end{aligned}$$

(c) The second isomorphism follows immediately from part (a), the fact that x has parity τ and choosing the basis B of F to contain the element f . To prove the first isomorphism, note that the map

$$\mathcal{A}_n^{\mathbf{C}}(F) \times \mathcal{A}_n^{\mathbf{C}}(F) \rightarrow \mathcal{A}_n^{\mathbf{C}}(F) s_n \mathcal{A}_n^{\mathbf{C}}(F), \quad (u, v) \mapsto u s_n v,$$

is $\mathcal{A}_{n-1}^{\mathbf{C}}(F)$ -balanced, and hence induces an even homomorphism

$$\Phi: \mathcal{A}_n^{\mathbf{C}}(F) \otimes_{\mathcal{A}_{n-1}^{\mathbf{C}}(F)} \mathcal{A}_n^{\mathbf{C}}(F) \rightarrow \mathcal{A}_n^{\mathbf{C}}(F) s_n \mathcal{A}_n^{\mathbf{C}}(F)$$

of $(\mathcal{A}_n^{\mathbf{C}}(F), \mathcal{A}_n^{\mathbf{C}}(F))$ -bimodules.

Since $\mathcal{A}_n^{\mathbf{C}}(F) \otimes_{\mathcal{A}_{n-1}^{\mathbf{C}}(F)} \mathcal{A}_n^{\mathbf{C}}(F)$ are generated by $\mathcal{A}_n^{\mathbf{C}}(F) \otimes_{\mathcal{A}_{n-1}^{\mathbf{C}}(F)} 1$ as right $\mathcal{A}_n^{\mathbf{C}}(F)$ -module, and, by part (a), $\mathcal{A}_n^{\mathbf{C}}(F)$ has basis $B_n^{\mathbf{C}}$ as right $\mathcal{A}_{n-1}^{\mathbf{C}}(F)$ -module, it follows that $\mathcal{A}_n^{\mathbf{C}}(F) \otimes_{\mathcal{A}_{n-1}^{\mathbf{C}}(F)} \mathcal{A}_n^{\mathbf{C}}(F)$ is a free right $\mathcal{A}_n^{\mathbf{C}}(F)$ -module with basis

$$\{x_j^a b_j s_j \cdots s_{n-1} \otimes 1 \mid 0 \leq a < d, b \in \mathcal{B}, 1 \leq j \leq n\}.$$

Since Φ maps these elements to $B_n^{\mathbf{C}}$, which is a basis for $\mathcal{A}_n^{\mathbf{C}}(F) s_n \mathcal{A}_n^{\mathbf{C}}(F)$ as a right $\mathcal{A}_n^{\mathbf{C}}(F)$ -module, it follows that Φ is an isomorphism. $\square$

For $\sigma \in \{0, 1\}$, define

$$n_\sigma = \left\lfloor \frac{d+1}{2} \right\rfloor \dim F_\sigma + \left\lfloor \frac{d}{2} \right\rfloor \dim F_{\sigma+\tau},$$

where F_σ is the part of F of parity $\sigma \pmod 2$, and $\lfloor m \rfloor = \max\{a \in \mathbb{N} \mid a \leq m\}$, for all $m \in \mathbb{N}$.

Theorem 6.5.2 (Cyclotomic Mackey Theorem). *Recall that $d = d_{\mathbf{C}}$ is the level of $\mathbf{C}$ (see (6.2.3)). Then, for $n \in \mathbb{N}_+$, there exists a natural isomorphism of functors*

$${}^{\mathbf{C}}\text{Res}_n^{n+1} {}^{\mathbf{C}}\text{Ind}_n^{n+1} \cong \text{Id}^{\oplus n_0} \oplus \Pi^{\oplus n_1} \oplus {}^{\mathbf{C}}\text{Ind}_{n-1}^n {}^{\mathbf{C}}\text{Res}_{n-1}^n.$$

Proof. Let M be a $(\mathcal{A}_n^{\mathbf{C}}(F), \mathcal{A}_n^{\mathbf{C}}(F))$ -bimodule. Proposition 6.5.1, it follows that

$$\begin{aligned} {}^{\mathbf{C}}\text{Res}_n^{n+1} {}^{\mathbf{C}}\text{Ind}_n^{n+1} M &= {}^{\mathbf{C}}\text{Res}_n^{n+1} \left(\mathcal{A}_{n+1}^{\mathbf{C}}(F) \otimes_{\mathcal{A}_n^{\mathbf{C}}(F)} M \right) \\ &\cong {}^{\mathbf{C}}\text{Res}_n^{n+1} \left(\left(\mathcal{A}_n^{\mathbf{C}}(F) \otimes_{\mathcal{A}_{n-1}^{\mathbf{C}}(F)} \mathcal{A}_n^{\mathbf{C}}(F) \right) \otimes_{\mathcal{A}_n^{\mathbf{C}}(F)} M \oplus \bigoplus_{\substack{0 \leq a < d, \\ b \in \mathcal{B}}} \left(\Pi^{\bar{b} + \tau a} \mathcal{A}_n^{\mathbf{C}}(F) \right) \otimes_{\mathcal{A}_n^{\mathbf{C}}(F)} M \right) \end{aligned}$$

But, we have that

$$\begin{aligned} \left(\mathcal{A}_n^{\mathbf{C}}(F) \otimes_{\mathcal{A}_{n-1}^{\mathbf{C}}(F)} \mathcal{A}_n^{\mathbf{C}}(F) \right) \otimes_{\mathcal{A}_n^{\mathbf{C}}(F)} M &\cong {}^{\mathbf{C}}\text{Ind}_{n-1}^n {}^{\mathbf{C}}\text{Res}_{n-1}^n \left(\mathcal{A}_n^{\mathbf{C}}(F) \otimes_{\mathcal{A}_n^{\mathbf{C}}(F)} M \right) \\ &\cong {}^{\mathbf{C}}\text{Ind}_{n-1}^n {}^{\mathbf{C}}\text{Res}_{n-1}^n M, \end{aligned}$$

and

$$\begin{aligned} \bigoplus_{\substack{0 \leq a < d, \\ b \in \mathcal{B}}} \left(\Pi^{\bar{b} + \tau a} \mathcal{A}_n^{\mathbf{C}}(F) \right) \otimes_{\mathcal{A}_n^{\mathbf{C}}(F)} M &\cong \bigoplus_{\substack{0 \leq a < d, \\ b \in \mathcal{B}}} \Pi^{\bar{b} + \tau a} \left(\mathcal{A}_n^{\mathbf{C}}(F) \otimes_{\mathcal{A}_n^{\mathbf{C}}(F)} M \right) \\ &\cong \bigoplus_{\substack{0 \leq a < d, \\ b \in \mathcal{B}}} \Pi^{\bar{b} + \tau a} M \cong (\text{Id}^{\oplus n_0} \oplus \Pi^{\oplus n_1}) M. \end{aligned}$$

Which follows the proof. $\square$

When the trace map of F is even, notice that

$$n_0 = d \dim F_{\text{even}}, \quad \text{and} \quad n_1 = d \dim F_{\text{odd}}$$

Then, Theorem 6.5.2 recovers the Cyclotomic Mackey Theorem for (even) affine wreath algebras (see [Sav20, Theo 8.18]).

6.6 Frobenius extension structure

We continue to make the assumption (6.5.1). Let

$$F_{n+1} = 1^{\otimes n} \otimes F = \text{span}_{\mathbb{k}}\{f_{n+1} \mid f \in F\} \subseteq F^{\otimes(n+1)}.$$

By Proposition 6.5.1(b), we have a decomposition of $(\mathcal{A}_n^{\mathbf{C}}(F), \mathcal{A}_n^{\mathbf{C}}(F))$ -bimodules

$$\mathcal{A}_{n+1}^{\mathbf{C}}(F) = x_{n+1}^{d-1} F_{n+1} \mathcal{A}_n^{\mathbf{C}}(F) \oplus \bigoplus_{a=0}^{d-2} x_{n+1}^a F_{n+1} \mathcal{A}_n^{\mathbf{C}}(F) \oplus \mathcal{A}_n^{\mathbf{C}}(F) s_n \mathcal{A}_n^{\mathbf{C}}(F). \quad (6.6.1)$$

Then, the projection onto the first summand of (6.6.1) defines a homomorphism of $(\mathcal{A}_n^{\mathbf{C}}(F), \mathcal{A}_n^{\mathbf{C}}(F))$ -bimodules:

$$p_{n+1}: \mathcal{A}_{n+1}^{\mathbf{C}}(F) \rightarrow x_{n+1}^{d-1} F_{n+1} \mathcal{A}_n^{\mathbf{C}}(F).$$

Now define the linear map

$$T_{n+1}: x_{n+1}^{d-1} F_{n+1} \mathcal{A}_n^{\mathbf{C}}(F) \rightarrow \mathcal{A}_n^{\mathbf{C}}(F), \quad x_{n+1}^{d-1} f_{n+1} y \mapsto \text{tr}(f)y, \quad f \in F, y \in \mathcal{A}_n^{\mathbf{C}}(F). \quad (6.6.2)$$

We claim that T_{n+1} is an homomorphism of $(\mathcal{A}_n^{\mathbf{C}}(F), \mathcal{A}_n^{\mathbf{C}}(F))$ -bimodules of parity τd . Indeed, first notice that for all $f \in F$, such that $f \neq \tau$, we have that $\text{tr}(f) = 0$ and thus $T_{n+1}(x_{n+1}^{d-1} f_{n+1} y) = 0$, for all $y \in \mathcal{A}_n^{\mathbf{C}}(F)$. Then fix $f \in F$ of parity τ . For all $g \in F$, and $1 \leq i \leq n$, it follows:

$$\begin{aligned} T_{n+1}(g_i x_{n+1}^{d-1} f_{n+1} y) &= (-1)^{\tau(d-1)\bar{g} + \bar{f}\bar{g}} T_{n+1}(x_{n+1}^{d-1} f_{n+1} g_i y) \\ &= (-1)^{\tau d \bar{g}} \text{tr}(f) g_i y \\ &= (-1)^{\overline{T_{n+1} \bar{g}}} g_i T_{n+1}(x_{n+1}^{d-1} f_{n+1} y). \end{aligned}$$

Clearly, for all $\pi \in S_n$, we have that

$$T_{n+1}(\pi x_{n+1}^{d-1} f_{n+1} y) = T_{n+1}(x_{n+1}^{d-1} f_{n+1} \pi y) = \pi T_{n+1}(x_{n+1}^{d-1} f_{n+1} y).$$

Finally, for all $1 \leq i \leq n$, it follows:

$$\begin{aligned} T_{n+1}(x_i x_{n+1}^{d-1} f_{n+1} y) &= (-1)^{\tau(d-1) + \bar{f}} T_{n+1}(x_{n+1}^{d-1} f_{n+1} x_i y) \\ &= (-1)^{\tau d} \text{tr}(f) x_i y \\ &= (-1)^{\overline{T_{n+1} \bar{x}}} x_i T_{n+1}(x_{n+1}^{d-1} f_{n+1} y). \end{aligned}$$

Now define the homomorphism of $(\mathcal{A}_n^{\mathbf{C}}(F), \mathcal{A}_n^{\mathbf{C}}(F))$ -bimodules

$$\text{tr}_{n+1}^{\mathbf{C}} = (-1)^{\tau n} T_{n+1} \circ p_{n+1}: \mathcal{A}_{n+1}^{\mathbf{C}}(F) \rightarrow \mathcal{A}_n^{\mathbf{C}}(F). \quad (6.6.3)$$

Note that the degree of $\text{tr}_{n+1}^{\mathbf{C}}$ is $-d\delta$. Since, in the current setup, the notation $\text{tr}_{\mathbf{C}}$ is ambiguous, let $\text{tr}_{\mathbf{C}}^n: \mathcal{A}_n(F) \rightarrow \mathbb{k}$ denote the trace map (6.4.2).

Lemma 6.6.1. *It follows that*

$$\text{tr}_{\mathbf{C}}^{n+1} = \text{tr}_{\mathbf{C}}^n \circ \text{tr}_{n+1}^{\mathbf{C}} = \text{tr}_1^{\mathbf{C}} \circ \text{tr}_2^{\mathbf{C}} \circ \cdots \circ \text{tr}_{n+1}^{\mathbf{C}}.$$

Proof. Notice that $\text{tr}_{\mathbf{C}}^{n+1}$ and $\text{tr}_{n+1}^{\mathbf{C}}$ vanish on $\bigoplus_{a=0}^{d-2} x_{n+1}^a F_{n+1} \mathcal{A}_n^{\mathbf{C}}(F) \oplus \mathcal{A}_n^{\mathbf{C}}(F) s_n \mathcal{A}_n^{\mathbf{C}}(F)$, since the power of x_{n+1} is always different then $d-1$. Thus we just need to check the equality for elements on $x_{n+1}^{d-1} F_{n+1} \mathcal{A}_n^{\mathbf{C}}(F)$.

Let $\mathbf{f} \in F^{\otimes n}$, $g \in F$, $\pi \in S_n$ and $\alpha \in \mathbb{N}^n$. It follows that:

$$\begin{aligned} \mathrm{tr}_{\mathbf{C}}^{n+1}(x_{n+1}^{d-1}g_{n+1}x^\alpha\mathbf{f}\pi) &= (-1)^{(\tau|\alpha|+\bar{\mathbf{f}})\bar{g}+\tau|\alpha|(d-1)} \mathrm{tr}_{\mathbf{C}}^{n+1}(x^\alpha x_{n+1}^{d-1}\mathbf{f}g_{n+1}\pi) \\ &= (-1)^{(\tau|\alpha|+\bar{\mathbf{f}})\bar{g}+\tau|\alpha|(d-1)} \mathrm{tr}_{\mathbf{C}}^n(x^\alpha\mathbf{f}\pi) \mathrm{tr}(g) \\ &= (-1)^{(\tau|\alpha|+\bar{\mathbf{f}})\bar{g}+\tau|\alpha|(d-1)} \mathrm{tr}_{\mathbf{C}}^n(\mathrm{tr}(g)x^\alpha\mathbf{f}\pi) \\ &= (-1)^{(\tau|\alpha|+\bar{\mathbf{f}})\bar{g}+\tau|\alpha|(d-1)+\tau n} \mathrm{tr}_{\mathbf{C}}^n(\mathrm{tr}_{n+1}^{\mathbf{C}}(x_{n+1}^{d-1}g_{n+1}x^\alpha\mathbf{f}\pi)). \end{aligned}$$

But, $\mathrm{tr}_{\mathbf{C}}^{n+1}(x_{n+1}^{d-1}g_{n+1}x^\alpha\mathbf{f}\pi) \neq 0$ only if $\bar{g} = \tau$, $\bar{\mathbf{f}} = n\tau$ and $\alpha = (d-1, \dots, d-1)$. Then, it follows that

$$\mathrm{tr}_{\mathbf{C}}^{n+1} = (-1)^{(\tau n(d-1)+n\tau)\tau+\tau n(d-1)(d-1)+\tau n} \mathrm{tr}_{\mathbf{C}}^n \circ \mathrm{tr}_{n+1}^{\mathbf{C}} = \mathrm{tr}_{\mathbf{C}}^n \circ \mathrm{tr}_{n+1}^{\mathbf{C}}.$$

□

Theorem 6.6.2. *The cyclotomic quotient $\mathcal{A}_{n+1}^{\mathbf{C}}(F)$ is a Frobenius extension of $\mathcal{A}_n^{\mathbf{C}}(F)$ with trace map $\mathrm{tr}_{n+1}^{\mathbf{C}}$.*

Proof. First, notice that $\mathcal{A}_{n+1}^{\mathbf{C}}(F)$ is a free left $\mathcal{A}_n^{\mathbf{C}}(F)$ -module with basis

$$\{x_{n+1}^a b_{n+1} \sigma \mid 0 \leq a \leq d-1, b \in \mathcal{B}, \sigma \in X\},$$

where X is a set of representatives of right cosets of S_n in S_{n+1} . Therefore $\mathcal{A}_{n+1}^{\mathbf{C}}(F)$ is a projective left $\mathcal{A}_n^{\mathbf{C}}(F)$ -module.

Let $\mathcal{B}_n^{\mathbf{C}}$ be a basis of $\mathcal{A}_n(F)$. By Theorem 6.4.1, Proposition 2.2.14, $\mathcal{A}_{n+1}(F)$ is a Frobenius extension of $\mathcal{A}_n(F)$ with trace map

$$z \mapsto \sum_{b \in \mathcal{B}_n^{\mathbf{C}}} (-1)^{\tau d(\bar{z}+(n+1))} \mathrm{tr}_{\mathbf{C}}^{n+1}(b^\vee z) b.$$

(Notice that we multiplied the trace of Proposition 2.2.14 by an constant, which do not change the statement.)

The proof follows from the following:

$$\begin{aligned} \sum_{b \in \mathcal{B}_n^{\mathbf{C}}} (-1)^{\tau d(\bar{z}+(n+1))} \mathrm{tr}_{\mathbf{C}}^{n+1}(b^\vee z) b &= \sum_{b \in \mathcal{B}_n^{\mathbf{C}}} (-1)^{\tau d(\bar{z}+(n+1))} \mathrm{tr}_{\mathbf{C}}^n \left(\mathrm{tr}_{n+1}^{\mathbf{C}}(b^\vee z) \right) b \\ &= \sum_{b \in \mathcal{B}_n^{\mathbf{C}}} (-1)^{\tau d(\bar{z}+(n+1)+\bar{b}^\vee)} \mathrm{tr}_{\mathbf{C}}^n \left(b^\vee \mathrm{tr}_{n+1}^{\mathbf{C}}(z) \right) b = \sum_{b \in \mathcal{B}_n^{\mathbf{C}}} \mathrm{tr}_{\mathbf{C}}^n \left(b^\vee \mathrm{tr}_{n+1}^{\mathbf{C}}(z) \right) b \stackrel{(2.2.7)}{=} \mathrm{tr}_{n+1}^{\mathbf{C}}(z), \end{aligned}$$

for all homogeneous $z \in \mathcal{A}_{n+1}(F)$, where we used, in the forth equality, the fact that

$$\mathrm{tr}_{\mathbf{C}}^n \left(b^\vee \mathrm{tr}_{n+1}^{\mathbf{C}}(z) \right) \neq 0 \quad \Rightarrow \quad \bar{b}^\vee + \bar{z} + \tau d = \overline{b^\vee \mathrm{tr}_{n+1}^{\mathbf{C}}(z)} = \overline{\mathrm{tr}_{\mathbf{C}}^n} = \tau n d,$$

for all $z \in \mathcal{A}_{n+1}(F)$ and $b \in \mathcal{B}_n^{\mathbf{C}}$. Therefore $\mathrm{tr}_{n+1}^{\mathbf{C}}$ is a trace map. □

Corollary 6.6.3. *The exact functor ${}^{\mathbf{C}}\mathrm{Ind}_n^{n+1}$ is left adjoint to ${}^{\mathbf{C}}\mathrm{Res}_n^{n+1}$ and right adjoint to ${}^{\mathbf{C}}\mathrm{Res}_n^{n+1}$ up to parity and degree shift by $d\tau$ and $-d\delta$, respectively.*

Proof. It follows from section 2.2.2, Theorem 6.6.2 and Theorem 2.2.9. $\square$

As mentioned in the introduction, the constructions of a generalization of Heisenberg categorification was one of the motivation to study affine wreath product algebras. Furthermore, Savage create in [Sav19] a graphical supercategory $\mathcal{H}eis_{F,k}$, depending of a frobenius algebra F with *even* trace map, that specialize (for $F = \mathbb{k}$) to the higher level Heisenberg categories of [MS18], which include Khovanov's original category. Such category has deep connections with the cyclotimic affine wreath algebras. Precisely, one can define a strict $\mathbb{k}$ -linear supermonoidal functor

$$\psi_{\mathbf{C}}: \mathcal{H}eis_{F^{\text{op}},k} \rightarrow \text{END}_{\mathbb{k}} \left(\bigoplus_{n \geq 0} \mathcal{A}_n^{\mathbf{C}}(F)\text{-mod} \right) \quad (6.6.4)$$

to the graded monoidal supercategory of endofunctors of the sum of the categories of left $\mathcal{A}_n^{\mathbf{C}}(F)$ -modules. In such functor, the cups and caps that appears in the definition of $\mathcal{H}eis_{F,k}$ (see [Sav19, 1.1]) are mapped to the unity and counit, respectively, of the adjunction $({}^{\mathbf{C}}\text{Ind}_n^{n+1}, {}^{\mathbf{C}}\text{Res}_n^{n+1})$. Therefore, Corolary 6.6.3 shows to be essential for an improvement of the functor (6.6.4) to a generic setting where F has generic trace maps.

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