Schrödinger-Bopp-Podolsky system in $\mathbb{R}^3$

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Resumo

MASCARO, B. **Sistema do tipo Schrödinger-Bopp-Podolsky em** $\mathbb{R}^3$. 2022. 82 f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

Neste trabalho estudamos o seguinte sistema perturbado do tipo Schrödinger-Bopp-Podolsy em $\mathbb{R}^3$

$$\begin{cases} -\varepsilon^2 \Delta w + V(x)w + \psi w = f(w) \\ -\varepsilon^2 \Delta \psi + \varepsilon^4 \Delta^2 \psi = 4\pi\varepsilon w^2 \end{cases} \tag{P_ε}$$

e usando métodos variacionais e a teoria de Ljusternik-Schnirelmann, nós mostramos uma cota inferior para o número de soluções para tal sistema.

Ao longo do trabalho, são apresentadas algumas noções preliminares e o desenvolvimento do sistema, juntamente com algumas notas históricas sobre o arcabouço físico do problema.

Palavras-chave: Schrödinger-Bopp-Podolsky, Sistema de equações diferenciais parciais, Ljusternik-Schnirelmann.

Abstract

MASCARO, B. **Schrödinger-Bopp-Podolsky system in** $\mathbb{R}^3$. 2022. 82 f. Thesys (Doctorate) - Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2022.
In this work we study the following perturbed Schrödinger-Bopp-Podolsky system in $\mathbb{R}^3$

$$\begin{cases} -\varepsilon^2 \Delta w + V(x)w + \psi w = f(w) \\ -\varepsilon^2 \Delta \psi + \varepsilon^4 \Delta^2 \psi = 4\pi\varepsilon w^2 \end{cases} \qquad (P_\varepsilon)$$

and using variational methods and the Ljusternik-Schnirelmann theory, we show a lower bound for the number of solutions of such system.

Along the work, some preliminaries notions are presented and the development of the system, together with brief historical notes about the physical framework of the problem.

Keywords: Schrödinger-Bopp-Podolsky, System of partial differential equations, Ljusternik-Schnirelmann.

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Function spaces

$L(X,Y)$	$= \{A : X \to Y : A \text{ linear and continuous map}\}$
$Inv(X,Y)$	$= \{A \in L(X,Y) : A \text{ is invertible}\}$
$C^k(X,Y)$	$= \{A : X \to Y : A \text{ is k times differentiable}\}$
$\mathcal{L}_2(X,Y)$	$= \{A : X \times X \to Y : A \text{ is bilinear and continuous map}\}$
$L^p(\Omega)$	$= \{f : \Omega \to \mathbb{R} : f \text{ is measurable and } \|f\|_p < \infty\}$
$L^p_{loc}(\Omega)$	$= \{f : \Omega \to \mathbb{C} : f_{\|K} \in L^p(K), \forall K \subset \Omega, K\text{compact}\}$
$W^k(\Omega)$	$= \{u \in L^1_{loc}(\Omega) : \partial^\alpha u \in L^1_{loc}(\Omega) \text{ for all } \|\alpha\| \leq k\}$
$W^{k,p}(\Omega)$	$= \{u \in W^k(\Omega) : \partial^\alpha u \in L^p(\Omega) \text{ for all } 0 \leq \alpha \leq k\}$
$\mathcal{D}^{m,p}(\Omega)$	is the completion of $C^\infty_c(\Omega)$ relative to the norm

$$\|u\|_{\mathcal{D}^{m,p}} = \left(\sum_{|\alpha|=m} \|\partial^\alpha u\|^p_{L^p}\right)^{\frac{1}{p}}$$

Chapter 1

Introduction

Problems concerning the bilaplacian operator have been studied by many authors in the past years, by various methods of resolution, like variational, topological and the sub-super solutions. In this work we use a mix of methods, like variational methods and a topological constant named *Ljusternik-Schnirelmann* category of a set. Starting from the problem presented by [12], the called *Schrödinger-Bopp-Podolsky* system

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 \end{cases}$$

was studied in the whole $\mathbb{R}^3$, and $a, \omega > 0$. The authors have proved the existence and the nonexistence depending on the parameters q, p.

In the second chapter we present some historical notes about the characters that give our system it's name, and a bit of physical discussion is done without claiming to be complete. Along with that, there is the development of the (SBP) system.

In this work we study a variation of the *Schrödinger-Bopp-Podolsky* above. We replace the parameter ω by a potential $V : \mathbb{R}^3 \to \mathbb{R}$, we consider $q = 1$, we introduce a perturbation ε on the both equations, precisely in the laplacian and the bilaplacian operators, and finally we threat with a more general nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfying some assumptions. Precisely, our problem is written

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = f(u) \\ -\varepsilon^2\Delta\phi + \varepsilon^4\Delta^2\phi = 4\pi\varepsilon u^2 \end{cases} \tag{P_ε}$$

and we prove that the number of solutions is estimated below by the *Ljusternik-Schnirelmann* category of M, the set of minima of the potential V. The function f and the potential V satisfy the following conditions

(V1) $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that

$$0 < \min_{\mathbb{R}^3} V := V_0 < V_\infty := \liminf_{|x|\to+\infty} V \in (V_0, +\infty],$$

with $M = \{x \in \mathbb{R}^3 : V(x) = V_0\}$ smooth and bounded,

(f1) $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 and $f(t) = 0$ for $t \leq 0$,

(f2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$,

(f3) there exists $q_0 \in (3, 2^* - 1)$ such that $\lim_{t\to+\infty} \frac{f(t)}{t^{q_0}} = 0$, where $2^* = 6$,

(f4) there exists $K > 4$ such that $0 < KF(t) := K \int_0^t f(\tau)d\tau \leq tf(t)$ for all $t > 0$,

(f5) the function $t \mapsto \frac{f(t)}{t^3}$ is strictly increasing in $(0, +\infty)$.

In fact, our main result is

Theorem 1.0.1. *Under the above assumptions (V1), (f1)-(f5), there exists an $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$, problem (P_ε) possesses at least catM positive solutions. Moreover, if catM > 1, then (for a suitably small ε) there exist at least catM $+ 1$ positive solutions.*

To demonstrate this, we define a functional associated to (P_ε), which give us the notion of weak solutions for the problem, namely

$$\mathcal{I}_\varepsilon(u, \phi) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int V(\varepsilon x)u^2 + \frac{1}{2}\int \varepsilon\phi u^2 - \frac{1}{16\pi}\|\nabla\phi\|_2^2 - \frac{1}{16\pi}\|\Delta\phi\|_2^2 - \int F(u).$$

All the development is made in the third chapter, where we define the Nehari manifold associated, we set the variational setting for our problem, we prove some compactness properties for the functionals involved, and via the definition of the barycenter map we can find a sublevel of Nehari which give us the existence of another different solution for (P_ε).

Chapter 2

Preliminaries

In this chapter we recall some results needed to do this work. Here we present some well-known results on the theory of Sobolev Spaces, more generally about Functional Analysis and some theory of solutions of Partial Differential Equations.

2.1 Calculus on Banach Spaces

2.1.1 The notion of differentiability in Banach spaces

In this section it will be showed some preliminaries tools of the calculus in infinite dimension. A wide literature exists about the theme of this section, so just some theorems and definitions will be listed below. If the reader wants to know more about, he can see the references [5], [10] and the references therein.

From now on, X, Y, denotes real Banach spaces, and U an open subset of X.

Definition 2.1.1 (Fréchet-differentiability). *Let $u \in U$ and consider a map $F : U \to Y$. We say that F is (Fréchet-) differentiable at u if there exists $A \in L(X,Y)$ such that, if we set*

$$R(h) = F(u+h) - F(u) - A(h),$$

then it results

$$R(h) = o(\|h\|),$$

that is

$$\frac{\|R(h)\|}{\|h\|} \to 0 \ \ as \ \|h\| \to 0.$$

Such A is uniquely determined and will be called the (Fréchet) differential of F at u, and will be denoted by

$$A = dF(u).$$

If F is differentiable at all $u \in U$, we say that F is differentiable in U.

The Fréchet differential is a natural extension of the usual concept of differential in Euclidean spaces, to Banach spaces. In the Euclidean spaces, both definitions coincide.

Definition 2.1.2 (Fréchet-derivative)**.** *Let $F : U \to Y$ be a differentiable function in U. The map*

$$F' : U \to L(X, Y), \quad F' : u \mapsto dF(u),$$

is called the (Fréchet) derivative of F.

Definition 2.1.3 (Gâteux-differential)**.** *Let $F : U \to Y$ be given and let $x \in U$. We say that F is Gâteux-differentiable (or just G-differentiable) at u if there exists $A \in L(X, Y)$ such that for all $h \in X$ we have*

$$\frac{F(u + \varepsilon h) - F(u)}{\varepsilon} \to Ah \ \ as \ \varepsilon \to 0.$$

Those are the notions of differentiability in Banach spaces adopted here, and when there is no possible misunderstanding, it will be referred just as differentiability.

High order derivatives

Like in the regular calculus over Euclidean spaces, we can define high order derivatives in infinite dimension spaces, like Banach spaces, in a such straightforward way.

Definition 2.1.4 (Second Fréchet differential)**.** *Let $u^* \in U$. F is twice (Fréchet-) differentiable at u^* if F' is differentiable at u^*. The second (Fréchet) differential of F at u^* is defined as*

$$d^2F(u^*) = dF'(u^*).$$

If F is twice differentiable in U, the second (Fréchet) derivative of F is the map $F'' : U \to \mathcal{L}_2(X,Y)$,

$$F'' : u \mapsto d^2F(u),$$

and if F'' is continuous from U to $\mathcal{L}_2(X,Y)$ we say that $F \in \mathcal{C}^2(U,Y)$.

The first proposition of our chapter is a very useful way to evaluate $d^2F(u)$.

Proposition 2.1.5. *Let $F : U \to Y$ be twice differentiable at $u^* \in U$. Then for all fixed $h \in X$ the map $F_h : X \to Y$ defined by setting*

$$F_h(u) = dF(u)h$$

is differentiable at u^* and $dF_h(u^*)k = F''(u^*)[h,k]$.

As the last part of our tools of calculus in Banach spaces, we can define partial derivatives similarly as for Euclidean spaces.

Let us consider two Banach spaces X, Y and let $(u^*, v^*) \in X \times Y$. Define mappings $\sigma_{v^*} : X \to X \times Y$ and $\tau_{u^*} : Y \to Y \times X$ as follow

$$\sigma_{v^*}(u) = (u, v^*);$$

$$\tau_{u^*}(v) = (u^*, v).$$

Notice that the derivatives of σ_{v^*} and τ_{u^*} are respectively, the linear maps

$$\sigma := d\sigma_{v^*} : h \to (h, 0);$$

$$\tau := d\tau_{u^*} : k \to (0, k).$$

Definition 2.1.6. *If the map $F \circ \sigma_{v^*}$ is differentiable at u^* we say that F is differentiable with respect to u^* at (u^*, v^*). The linear map $d[F \circ \sigma_{v^*}](u^*)$ is called the partial derivative of F at (u^*, v^*) with respect to u and denoted by $d_uF(u^*, v^*)$.*

Similarly we can define $d_vF(u^*, v^*)$.

As in the calculus in the Euclidean space, we have

Proposition 2.1.7. *If F is differentiable at (u^*, v^*) then F has partial derivatives with respect to u and v at (u^*, v^*) and we have*

$$d_u F(u^*, v^*)(h) = dF(u^*, v^*)\sigma(h) = dF(u^*, v^*)(h, 0),$$

$$d_v F(u^*, v^*)(k) = dF(u^*, v^*)\tau(k) = dF(u^*, v^*)(0, k).$$

Local inversions results are valid as in the regular calculus in the Euclidean space, given the straight forward changes due the dimension of the spaces here.

We say that maps $F \in C(X, Y)$, where X, Y are Banach spaces, whose defined on an open subset of X could be treated with minor changes only.

Definition 2.1.8. *Let $A \in L(X, Y)$ where L is the spaces of linear continuous maps. We say that A is invertible if there exists $B \in L(Y, X)$ such that*

$$B \circ A = Id_X,$$

$$A \circ B = Id_Y.$$

Moreover, the map B is unique and will be denoted by A^{-1}.

Jsut for simplicity, we define two sets

$$Inv(X, Y) = \{A \in L(X, Y) : A \text{ is invertible}\}.$$

Let $U \subset X$ be an open and $V \subset Y$ be open sets. We say that $F \in Hom(U, V)$ if there exists a map $G : V \to U$ such that

$$G(F(u)) = u \ \ \forall u \in U,$$

$$F(G(v)) = v \ \ \forall v \in V.$$

Then we say that $F \in C(X, Y)$ is locally invertible at $u^* \in X$ if there exists open subsets $u^* \in U \subset X$ and $v^* \in V \subset Y$ and $F(u^*) = v^* \in V$ such that $F \in Hom(X, Y)$.

Theorem 2.1.9 (Local Inversion Theorem)**.** *Suppose $F \in C^1(X,Y)$ and $F'(u^*) \in Inv(X,Y)$. Then F is locally invertible at u^* with C^1 inverse. More precisely, there exist opens U of u^* and V of $v^* = F(u^*)$ such that*

(i) $F \in Hom(U,V)$

(ii) $F^{-1} \in C^1(V,X)$ *and for all $v \in V$ there results*

$$dF^{-1}(v) = (F'(u))^{-1}, u = F^{-1}(v)$$

(iii) if $F \in C^k(X,Y), k > 1$, then $F^{-1} \in C^k(V,X)$.

As important as the Theorem above, is the Implicit Function Theorem and all the consequences and wide applicability to the studies of solutions of elliptic equations and systems, so we give its statement here, splitted in two important parts. Here we consider maps $F : \Lambda \times U \to Y$, where Λ and U are open subsets of Banach spaces T and X, respectively, and Y is a Banach space too.

Lemma 2.1.10. *Let $(\lambda^*, u^*) \in \Lambda \times U$. Suppose that*

(i) F is continuous and F has the u-partial derivative in $\Lambda \times U$ and $F_u : \Lambda \times U \to L(X,Y)$ is continuous.

(ii) $F_u(\lambda^*, u^*) \in L(X,Y)$ is invertible.

Then the map $\Psi : \Lambda \times U \to T \times Y$ given by

$$\Psi(\lambda, u) = (\lambda, F(\lambda, u)),$$

is locally invertible at (λ^*, u^*) with continuous inverse Φ.

In addition, if $F \in C^1(\Lambda \times U, Y)$ then $\Phi \in C^1$.

Theorem 2.1.11 (Implicit Function Theorem)**.** *Let $F \in C^k(\Lambda \times U, Y)$, $k \geq 1$, where Y is a Banach space and Λ and U are open subsets of the Banach spaces T and X, respectively. Suppose that $F(\lambda^*, u^*) = 0$ and that $F_u(\lambda^*, u^*) \in Inv(X,Y)$.*

Then there exist open subsets Θ *of* λ^* *in* T *and* U^* *of* u^* *in* X *and a map* $g \in C^k(\Theta, X)$ *such that*

(i) $F(\lambda, g(\lambda)) = 0$ *for all* $\lambda \in \Theta$,

(ii) $F(\lambda, u) = 0,\ (\lambda, u) \in \Theta \times U^*$, *implies* $u = g(\lambda)$,

(iii) $g'(\lambda) = -[F_u(p)]^{-1} \circ F_\lambda(p)$, *where* $p = (\lambda, g(\lambda))$ *and* $\lambda \in \Theta$.

2.2 Basic results about L^p spaces and integration

The L^p spaces are a crucial ingredient for the theory of Sobolev spaces, defined forward, which is used often in this work.

Definition 2.2.1 (L^p spaces)**.** *Let* $p \in \mathbb{R}$ *with* $1 \le p < \infty$. *We set*

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R}; f \text{ is measurable and } \|f\|_p < \infty\}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p d\mu\right]^{\frac{1}{p}}.$$

It is a well known fact that $\|.\|_p$ *is a norm.*

Theorem 2.2.2 (Hölder's inequality)**.** *Assume that* $f \in L^p$ *and* $g \in L^{p'}$ *with* $1 \le p \le \infty$. *Then* $fg \in L^1$ *and*

$$\int |fg| \le \|f\|_p \|g\|_{p'},$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Remark: In particular, if $f \in L^p \cap L^q$ with $1 \le p \le q \le \infty$, then $f \in L^r$ for all $r, p \le r \le q$, and the following interpolation inequality holds:

$$\|f\|_r \le \|f\|_p^\alpha \|f\|_q^{1-\alpha},$$

where

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \quad 0 \leq \alpha \leq 1.$$

Theorem 2.2.3 (Lebesgue or dominated convergence)**.** *Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) $f_n(x) \to f(x)$ *a.e. on* Ω*,*

(b) there is a function $g \in L^1$ *such that for all* $n, |f_n(x)| \leq g(x)$ *a.e. on* Ω*.*

Then $f \in L^1$ *and* $\|f_n - f\|_1 \to 0$*.*

Theorem 2.2.4 (Fatou's lemma)**.** *Let (f_n) be a sequence of function in L^1 that satisfy*

(a) for all $n, f_n \geq 0$ *a.e.*

(b) $\sup_n \int f_n < \infty$*.*

For almost all $x \in \Omega$ we set $f(x) = \liminf_{n\to\infty} f_n(x) \leq \infty$. Then $f \in L^1$ and

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

2.3 Functional Analysis

In this section we present some basic results on functional analysis, that will be useful in the development of this work. For the proofs, we indicate the reading of [10]

The concept of convergence plays a central role on our main results.

Definition 2.3.1 (Strong convergence)**.** *Let X be a normed space and $(u_k) \subset X$ be a sequence. The sequence (u_k) is said strongly convergent if there exists an $u \in X$ such that $\|u_k - u\|_X \to 0$, when $k \to \infty$. We write $u_k \to u$ and u is named the strong limit of u_k.*

Definition 2.3.2 (Weak convergence)**.** *Let X a normed space and X' it's dual. A sequence (u_k) in X is said weakly convergent for a limit u in X if for every $f \in X'$, we have $f(u_k) \to f(u)$, when $k \to \infty$. We write $u_k \rightharpoonup u$ and we say that (u_k) weakly converges to u.*

An important tool which plays a central role in the study of existence of solutions (weak solution - defined forward) is the Lax-Milgram theorem and the Riesz Representation, stated below

Theorem 2.3.3 (Lax-Milgram)**.** *Let $\alpha : H \times H \to \mathbb{R}$ be a continuous and coercive bilinear form in H, where H is a Hilbert space. Then for every $\phi \in H'$, where H' is the topological dual of H, exists a unique element $u \in H$ such that*

$$\alpha(u, v) = \phi[v] \;\; \forall v \in H.$$

Moreover, if α is symmetric, then u is characterized by

$$\frac{1}{2}\alpha(u, u) - \phi[u] = \min_{v \in H} \left\{ \frac{1}{2}\alpha(v, v) - \phi[v] \right\}.$$

Theorem 2.3.4 (Riesz representation)**.** *Let H be a Hilbert space and H' be the topological dual of H. Then for every $f \in H'$ there exists a unique $u_f \in H$ such that*

$$f[v] = (u_f, v)_H, \;\; \forall v \in H.$$

Moreover, $\|u_f\|_H = \|f\|_{H'}$ and the linear application

$$R : H' \to H$$

$$R : f \mapsto u_f$$

is called Riesz isomorphism.

2.4 Sobolev Spaces

The concept of Sobolev spaces is connected directly with the notion of a weak derivative of a distribution, and this concept plays a central role for our work, since almost all the calculations and conclusions are made working over Sobolev spaces. Here we define the most popular Sobolev Spaces and show some basic and important results about them.

For the completeness of the subject, we give the notion of weak derivative

Definition 2.4.1. *Let Ω be an open set of $\mathbb{R}^n$, a function $g \in L^1_{loc}(\Omega)$ and a multi-index $\alpha \in \mathbb{N}^n$. Then a function $h \in L^1_{loc}(\Omega)$ is a α-th weak derivative of g if*

$$\int_\Omega h\varphi dx = (-1)^{|\alpha|} \int_\Omega g\partial^\alpha \varphi dx, \quad \textit{for every } \varphi \in C_c^{|\alpha|}(\Omega).$$

In this case we write

$$h = \partial^\alpha g.$$

Moreover, as a basic result, the weak derivative is linear and unique.

Here we introduce the space W^k, or just the space of the functions which are k-times weak differentiable.

$$W^k(\Omega) = \{u \in L^1_{loc}(\Omega) : \partial^\alpha u \in L^1_{loc}(\Omega) \text{ for all } |\alpha| \leq k\}$$

Note that $W^1(\Omega)$ is the set of the weak differentiable functions. It is important to see that $C^k(\Omega) \subset W^k(\Omega)$, and so the concept of weak differentiation extends the notion of the classic derivative.

2.4.1 $W^{k,p}$ spaces and the Sobolev embeddings

Here we give the definitions of Sobolev spaces and their embeddings on other important spaces, which is ostensibly used in this work. The proofs of the theorems enunciated here are technical and we refer the reader to [4] and [1].

Definition 2.4.2 (Sobolev Space)**.** *Given $p \in [1, \infty]$ and $k \in \mathbb{N}$, we indicate the space of weak derivatives*

$$W^{k,p}(\Omega) = \{u \in W^k(\Omega) : \partial^\alpha u \in L^p(\Omega) \textit{ for all } 0 \leq \alpha \leq k\},$$

as the Sobolev space $W^{k,p}(\Omega)$.

These spaces are linear spaces and admit the norm

$$\|u\|_{k,p} = \|u\|_{k,p,\Omega} = \|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k} \|\partial^\alpha u\|_p^p\right)^{\frac{1}{p}} = \left(\int_\Omega (\sum |\partial^\alpha u|^p) dx\right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty,$$

$$\|u\|_{k,\infty} = \max_{|\alpha|\leq k} \|\partial^\alpha u\|_\infty.$$

We can see that $W^{0,p}(\Omega) = L^p(\Omega)$, and $W^{k,p}(\Omega) \subset W^k(\Omega)$ for all k, p.

A well know fact is that the linear space $W^{k,p}(\Omega)$ is a Banach space.

Definition 2.4.3 (The Hilbert space H^k)**.** *Let fix $p = 2$. Then we write*

$$W^{k,2}(\Omega) = H^k(\Omega).$$

The space $H^k(\Omega)$ is a Hilbert space with the respective inner product

$$\langle u, v\rangle_k = \sum_{|\alpha|\leq k} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

The spaces defined above are

- separable, if $p \in [1, \infty)$;
- reflexible, if $p \in (1, \infty)$.

Theorem 2.4.4 (Meyers-Serrin)**.** *It is valid that $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

It is natural that given $u \in W^{k,p}(\Omega)$ we would like an extension in $W^{k,p}(\mathbb{R}^n)$ for the results of embedding theorems, but this is not valid in general. Some properties of smoothness are needed in $\partial\Omega$. Since our problem concerns $\mathbb{R}^3$, we enunciate just the theorems where the domain is $\mathbb{R}^n$.

From now on, the exponent

$$p^* = \frac{Np}{N - p}$$

is called the *Sobolev critical exponent*. For the proofs of this part of this work we refer the reader see [18]

2.4.2 The case $kp < N$

Definition 2.4.5. *Let $1 \leq p < \infty$. The Sobolev space $W^{1,p}(\mathbb{R}^N)$ is the space of all functions $u \in L^1_{loc}(\mathbb{R}^N)$ whose distributional gradient ∇u belongs to $L^p(\mathbb{R}^N)$, i.e.*

$$W^{1,p}(\mathbb{R}^n) = \{u \in L^1_{loc}(\mathbb{R}^n) : \nabla u \in L^p(\mathbb{R}^N)\}$$

Theorem 2.4.6 (Sobolev-Gagliardo-Nirenberg's embedding theorem)**.** *Let $1 \leq p < N$. Then there exists a constant $C = C(N,p) > 0$ such that for every function $u \in W^{1,p}(\mathbb{R}^N)$ vanishing at infinity,*

$$\left(\int_{\mathbb{R}^N} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \leq \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

In particular, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $p \leq q \leq p^*$.

Corollary 2.4.7. *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $k \geq 2$ and $kp < N$. We have*

(i) $W^{k+j,p}(\mathbb{R}^N)$ is continuously embedded in $W^{j,q}(\mathbb{R}^n)$ for all $j \in \mathbb{N}$ and for all $p \leq q \leq \frac{Np}{N-kp}$.

(ii) $W^{k,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $p \leq q \leq \frac{Np}{N-kp}$.

2.4.3 The case $p = N$

Theorem 2.4.8. *The space $W^{1,N}(\mathbb{R}^N)$is continuously embedded in the space $L^q(\mathbb{R}^N)$ for all $N \leq q < \infty$.*

Corollary 2.4.9. *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $k \geq 2$ and $kp = N$. We have*

(i) $W^{k+j,p}(\mathbb{R}^N)$ is continuously embedded in $W^{j,q}(\mathbb{R}^n)$ for all $j \in \mathbb{N}$ and for all $p \leq q < \infty$.

(ii) $W^{k,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $p \leq q < \infty$.

2.4.4 The case $p > N$

Theorem 2.4.10 (Morrey)**.** *Let $N < p < \infty$. Then the space $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $C^{0,1-\frac{N}{p}}(\mathbb{R}^N)$. Moreover, if $u \in W^{1,p}(\mathbb{R}^N)$ and $\overline{u}$ is its representative in $C^{0,1-\frac{N}{p}}(\mathbb{R}^N)$,* *then*

$$\lim_{|x|\to\infty} \overline{u}(x) = 0.$$

2.5 Critical Point Theory

The critical point theory is an important technique to solve a lot of problems, and in this work it's not different, we use a tool of this theory, called *Palais-Smale* sequences, a compactness property for sequences which is needed, for example, to use the Mountain Pass Theorem in the infinite dimensional case. We give here the definition and some consequences of it.

2.5.1 The Palais-Smale condition

Definition 2.5.1 (Palais-Smale sequence)**.** *Let E be a Banach space and $J : E \to \mathbb{R}$ be a C^1-functional. We call a sequence $u_n \in E$ a Palais-Smale sequence, or just (PS)-sequence, if $J(u_n)$ is bounded and $J'(u_n) \to 0$ (in the dual E').*

If $J(u_n) \to c$ and $J'(u_n) \to 0$, for some $c \in \mathbb{R}$, we say that u_n is a $(PS)_c$ - sequence.

Definition 2.5.2 (Palais-Smale condition)**.** *Let J and E be as in Definition 2.5.1. We say that J satisfies the Palais-Smale condition on E if every PS-sequence has a converging subsequence.*

Definition 2.5.3 ($(PS)_c$ - condition)**.** *For E, J and $c \in \mathbb{R}$ as in the Definition 2.5.1. We say that J satisfies the local Palais-Smale condition at the level c, if every $(PS)_c$ - sequence has a converging subsequence.*

The Palais-Smale condition is named a *compactness* condition in the following sense, let $\mathbb{K}_c$ be the set of critical points of a functional J at the level c, namely

$$\mathbb{K}_c = \{u \in E : J(u) = c \text{ and } J'(u) = 0\}$$

where E is a Banach space.

The following proposition justifies the idea that the Palais-Smale condition is a condition of compactness.

Proposition 2.5.4. *Suppose $J : E \to \mathbb{R}$ satisfies (PS). Then $\mathbb{K}_c$ is compact for any $c \in \mathbb{R}$.*

2.5.2 Deformations and the Mountain Pass Theorem

In this section we present two essential theorems of the critical point theory, the Deformation Theorem due to [13] and the classical Mountain Pass Theorem, by [6].

Definition 2.5.5. *Let E a Banach space and $B \subset E$ be a subset. A deformation of B is a continuous function $\eta : [0,1] \times B \to B$ such that $\eta(0,u) = u$ for all $u \in B$.*

Definition 2.5.6. *Let E be a Banach space and let $A \subset B \subset E$. We say that B is deformable in A if exists a deformation η of B such that*

$$\eta(t,u) \in A, \forall u \in A, \forall t \in [0,1];$$

$$\eta(1,u) \in A, \forall u \in B.$$

Definition 2.5.7. *Let E be a Banach space and let $J : E \to \mathbb{R}$ be a functional. Given $c \in \mathbb{R}$ we define*

$$J^c = \{x \in E : J(x) \leq c\}$$

Theorem 2.5.8 (Deformation Theorem)**.** *Let E be a Banach space and let $J : E \to \mathbb{R}$ be a C^1-functional satisfying (PS). Given $c \in \mathbb{R}$ and an open neighborhood U of $\mathbb{K}_c$, then there exist $\varepsilon > 0$ and $\eta \in C([0,1] \times E, E)$ such that*

(Df1) $\eta(0,u) = u$ *for all* $u \in E$ *and all* $t \in [0,1]$,

(Df2) $\eta(t,u) = u$ *for all* $u \notin J^{-1}[c-2\varepsilon, c+2\varepsilon]$ *and all* $t \in [0,1]$,

(Df3) $\eta(t,\cdot) : E \to E$ *is a homeomorphism for all* $t \in [0,1]$,

(Df4) $J(\eta(t,u)) \leq J(u)$ *for all* $u \in E$ *and all* $t \in [0,1]$,

(Df5) $\eta(1, J^{c+\varepsilon} \setminus U) \subset J^{c-\varepsilon}$,

(Df6) *if* $\mathbb{K}_c = \emptyset$, *then* $\eta(1, J^{c+\varepsilon} \setminus U) \subset J^{c-\varepsilon}$,

(Df7) *if J is even, then $\eta(t,\cdot)$ is odd in u.*

The Deformation Theorem above is slightly more general for our purpose, but is sufficient for guarantee the Mountain Pass Theorem.

Theorem 2.5.9 (Mountain Pass Theorem)**.** *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfying (PS). Suppose $J(0) = 0$ and*

(MPT1) There are constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$, and

(MPT2) there is an $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.

Then J possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, g(1) = e\}.$$

2.6 The Ljusternik - Schnirelmann category

The main tool that give us the multiplicity result of critical points to our studied problem is the Ljusternik - Schnirelmann category theorem, and this can be viewed by analyzing the "size" of a set. For a subset A of a topological space X, the Ljusternik - Schnirelmann is defined as the least integer k such that A can be covered by k closed sets that are contractible in X. The following properties can be verified when X is a Finsler manifold

(C1) $Cat_X(A) = 0$ if and only if $A = \emptyset$.

(C2) If A_1 and A_2 are closed in X and $\eta : [0,1] \times A_1 \to A_1$ is a continuous deformation of A_1 with $\eta(1, A_1) \subset A_2$ then $Cat_X(A_1) \leq Cat_X(A_2)$.

(C3) For any closed set K, there exists a closed neighborhood $K^\delta = \{x; dist(x, K) \leq \delta\}$ of K so that $Cat_X(K^\delta) = Cat_X(K)$.

(C4) $Cat_X(A_1 \cup A_2) \leq Cat_X(A_1) + Cat_X(A_2)$ for all closed subsets A_1, A_2.

The proof of these properties can be found in several books about the subject, we refer to the reader [17]

Here we give two formulations of Ljusternik - Schnirelmann theorems, which will be useful to our purpose, founded in [4] and [17].

Theorem 2.6.1. *Let $M = G^{-1}(0)$, where $G \in C^{1,1}(E, \mathbb{R})$, E-Banach space, and $G'(u) \neq 0$ for all $u \in M$. Let $J \in C^{1,1}(E, \mathbb{R})$ be bounded from below on M and let J satisfy the Palais-Smale condition.*
Then J has at least $Cat_k(M)$ critical points on M, where

$$Cat_k(M) = \sup\{Cat_M(A) : A \subset M \;\; A \text{ compact}\}.$$

Another version of a Ljusternik - Schnirelmann theorem is

Theorem 2.6.2. *Let ϕ be a bounded from below C^1-functional satisfying the Palais-Smale condition on a C^1-Finsler manifold X. Assume that $Cat_X(X) = N$ and define for $1 \leq n \leq N$, the families*

$$\mathcal{F}_n = \{A : \; A \text{ compact in } X \text{ and } Cat_X(A) \geq n\}.$$

Let

$$c_n = c(\phi, \mathcal{F}_n) = \inf_{A \in \mathcal{F}_n} \max_{u \in A} \phi(u)$$

and assume $c_j = c_{j+p}$ for $1 \leq j \leq j+p \leq N$, then for every min-maxing sequence $(A_n)_n$ in $\mathcal{F}_{j+p}$ we have

$$Cat_X(K_{c_j} \cup A_\infty) \geq p + 1.$$

In particular, ϕ has at least N distinct critical points.

Chapter 3

Physical framework

In this chapter we discuss the motivations and the importance to study problems of type Schrödinger-Bopp-Podolsky, and where they come from, some historical facts and the development of that type of system.

The physics uses few models to illustrate some more general principles, and a standard way to develop a model is via the definition of a action as the integral of a determinated Lagrangian. As an example of this approach, suppose $\Omega \subset \mathbb{R}$ is a interval and f, which we will refer to our action, and a functional of $\varphi \in C^k(\Omega), k \geq 1$, is of the form

$$f(\varphi) = \int_{\Omega} F\left(x, \varphi(x), \frac{d\varphi}{dx}\right) dt$$

and that F is called *Lagrangian* in the classic mechanics and *density* in field theory, associated with f, and we will replace this F by our *Lagrangian*. Then after some calculation, we search for the extremuns of f. Then φ will be a extremun of f if and only if

$$\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\frac{\partial F}{\partial(d\varphi/\partial x)} = 0.$$

A straightforward generalization of the above equation holds if $\Omega \subset \mathbb{R}^n$ and then f has an extremun at φ only if

$$\frac{\partial F}{\partial \varphi} - \sum_k \frac{d}{dx^k}\frac{\partial F}{\partial(\partial\varphi/\partial x^k)} = 0.$$

This is called the ***Euler-Lagrange*** equation in the calculus of variations (in our case, it just make sense for fields), and some generalizations are drawn from it.

And this is not different in this theory. The Bopp-Podolsky theory was developed independently by Bopp and Podolsky. It is a second order gauge theory for the electromagnetic field, and the theory refines the Maxwell theory. The coupling with the Schrödinger equation (as well as in Maxwell-Schrödinger theory), it is used to describe the evolution of a charged nonrelativistic quantum mechanical particle, interacting with its own eletromagnetic field. As the Mie in [21] theory and its generalizations given by Born and Infield in [9] and [20], it was introduced to solve the so called *infinity problem* that appears in the classical Maxwell theory. Above we give some historical notes about the development and their creators. At the end, we develop the Schrödinger-Bopp-Podolsky system using Lagrangians and a functional named total action to give rise to our system.

3.0.1 Erwin Schrödinger

"The task is ... not so much to see what no one has yet seen; but to think what nobody has yet thought, about that which everybody sees."
Erwin Schrödinger
1887 - 1961

Erwin Rudolf Josef Alexander Schrödinger, born in Erdberg, on 12 August 1887, at Vienna, Austria-Hungary.

Just in 1920 he obtained a position equivalent to an a associate professor in Stuttgart. After a long career he retired in 1955, in Dublin.

One of the most famous contributions to physics was the formulation of the Schrödinger equation, in 1926, which describes how the quantum state of a physical system changes with the time.

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t)$$

One of the formulations of the Schrödinger's equation.

Another great contribution was the so-called Cat's Schrödinger experiment, where the objective is to illustrate the necessity of the interaction of an observer with the respective measure that he wants.

In 1933 he shares with Paul Dirac the Nobel Prize in Physics by their solutions and contributions to the atom physics.

Erwin died in 1961 by tuberculosis.

3.0.2 Boris Yakovlevich Podolsky

"I am happy to be able to tell you that I estimate Podolsky's abilities very highly. He is an independent investigator of unquestionable talent." - Einstein about Boris.
Boris Yakovlevich Podolsky
1896-1966

Boris Yakovlevich Podolsky was a American-Russian physicist, born in Tangarog, Russia, in 29 June of 1896. Even in your early years, he moved to the United States, in 1913, where he would receive his Phd in Theoretical Physics from Caltech.

Along his your carrer, Podolsky works with several famous names of the sciences, Albert Einstein, Paul Dirac, Nathan Rosen and Lev Landau are some of them. Here we gave attention to his (supposedly) most famous result, the EPR paradox (Einstein-Podolsky-

Rosen paradox), a thought experimental where the description of physical reality provided by quantum mechanics was incomplete.

Just as a curiosity, some histories about a possible case of espionage by Boris during the second war exists, but we will not enter in this matter.

Boris died in 1966. He worked in the Xavier University, Cincinnati until his death.

3.0.3 Friedrich Arnold "Fritz" Bopp

Friedrich Arnold "Fritz" Bopp
1909 - 1987

Friedrich Arnold "Fritz" Bopp was born in Frankfurt, in the German Empire in 27 December, 1909. He studied physics at the Goethe University Frankfurt and the University of Göttingen. His Diplom thesis, a similar title as the bachelor's degree was obtained, under supervision of the famous mathematician Hermann Weyl. He completed his doctoral in 1937 at Breslau University.

Bopp has worked in many places and projects during his career. For example, he was a staff scientist at the Kaiser-Wilhelm Instituts für Physik (actual Max Planck Institute for Physics), German nuclear energy project, University of Tübingen, was president of the Deutsche Physikalische Gesellschaft, the oldest organization of physicists, and others. He wrote several books, edited and supplemented many others.

Bopp died aged 77, in 1987, at Munich.

3.0.4 Deduction of Schrödinger equation, an example

The strategy is standard in physics. Through a Lagrangian density, we apply the Euler-Lagrange equations and we obtain our system.

To explain this reasoning in a simpler way, first we deduce the Schrödinger equation, cited in 3.0.1.

Example 3.0.1 (Schrödinger's equation). *Let the field $\varphi = \varphi(x,t)$ where x is the position of some particle and t represents the time. Here we will use some abuse of notation, where we will consider $x = (x_1(t), x_2(t), x_3(t))$ and in the Lagrangian density, we will treat the partial derivatives as "variables".*

(1) First, let the Lagrangian density

$$\mathcal{L}\left(x, \varphi, \nabla\varphi\right) = i\hbar(\dot{\varphi}\varphi^* - \varphi\dot{\varphi}^*) - \frac{\hbar^2}{2m}\nabla\varphi^*\nabla\varphi - V\varphi^*\varphi, \tag{3.1}$$

here φ^* denotes the complex conjugate of φ and for the purpose of Lagrange's equations, φ and φ^* will be treated as independents fields.

(2) The Euler-Lagrange equation in the fields theory is given by

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \sum_k \frac{d}{dx_k}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial x_k}\right)}\right). \tag{3.2}$$

This equation holds for φ and φ^*.

(3) Now, our work is to calculate the both sides of (3.2) using (3.1). We will show just in the case of φ^* and we obtain the Schrödinger's equation for φ.

(i)

$$\frac{\partial\mathcal{L}}{\partial\varphi^*} = i\hbar\dot{\varphi} - V\varphi^*$$

(ii)

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial x_k}\right)} = -\frac{\hbar^2}{2m}\frac{\partial\varphi}{\partial x_k} \Rightarrow \frac{d}{dx_k}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial x_k}\right)}\right) = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x_k^2}$$

(4) Now comparing the two expressions above we have

$$i\hbar\dot{\varphi} - V\varphi = \sum_k -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x_k^2},$$

or simply

$$i\hbar\frac{\partial}{\partial t}\varphi(x,t) = -\frac{\hbar^2}{2m}\sum_k \frac{\partial^2\varphi(x,t)}{\partial x_k^2} + V(x,t)\varphi(x,t),$$

i.e. the Schrödinger's equation for the field $\varphi(x,t)$.

3.0.5 The Schrödinger-Bopp-Podolsky system

Here we finally show the deduction of the Schrödinger-Bopp-Posolky system, as showed in [12].

First consider the nonlinear Schrödinger Lagrangian density

$$\mathcal{L}_{Sc} = i\hbar\psi^*\frac{\partial}{\partial t}\psi - \frac{\hbar^2}{2m}|\nabla\psi|^2 + \frac{2}{p}|\psi|^p,$$

where $\psi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}, \hbar, m, p > 0$ and let $(\phi, \mathbf{A})$ be the gauge potential of the electromagnetic field $(\mathbf{E}, \mathbf{H})$ namely $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}, \quad \mathbf{H} = \nabla\times\mathbf{A}.$$

Making use of the minimal coupling rule, we couple the field ψ with the electromagnetic field $(\mathbf{E}, \mathbf{H})$,i.e, the study of the interaction between of ψ with the electromagnetic field generated by itself, just replacing in $\mathcal{L}_{Sc}$ the derivatives $\frac{\partial}{\partial t}$ and ∇ respectively with the covariant ones

$$D_t = \frac{\partial}{\partial t} + \frac{iq}{\hbar}\phi, \quad \mathbf{D} = \nabla - \frac{iq}{\hbar c}\mathbf{A},$$

where q is a *coupling* constant. Then we have the new coupled Lagrangian Schrödinger

$$
\begin{aligned}
\mathcal{L}_{CSc} &= i\hbar\psi^* D_t\psi - \frac{\hbar^2}{2m}|\mathbf{D}\psi|^2 + \frac{2}{p}|\psi|^p \\
&= i\hbar\psi^*\left(\frac{\partial}{\partial t} + \frac{iq}{\hbar}\phi\right)\left|\left(\nabla - \frac{iq}{\hbar c}\mathbf{A}\right)\right|^2 + \frac{2}{p}|\psi|^p.
\end{aligned}
$$

Now, to get the total Lagrangian density, we have to add to $\mathcal{L}_{CSc}$ the Lagrangian density of the electromagnetic field. For this, we make use of the Bopp-Podolsky Lagrangian density, by the Formula (3.9) in [24] we have

$$
\begin{aligned}
\mathcal{L}_{BP} &= \frac{1}{8\pi}\left\{|\mathbf{E}|^2 - |\mathbf{H}|^2 + a^2\left[(\mathrm{div}\mathbf{E})^2 - \left|\nabla\times\mathbf{H} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}\right|^2\right]\right\} \\
&= \frac{1}{8\pi}\left\{|\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}|^2 - |\nabla\times\mathbf{A}|^2\right. \\
&\quad \left. + a^2\left[\left(\Delta\phi + \frac{1}{c}\mathrm{div}\frac{\partial}{\partial t}\mathbf{A}\right)^2 - \left|\nabla\times\nabla\times\mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A})\right|^2\right]\right\}.
\end{aligned}
$$

Now, the total action

$$
\mathcal{S}(\psi,\phi,\mathbf{A}) = \int \mathcal{L}dxdt
$$

will be defined using $\mathcal{L} := \mathcal{L}_{CSc} + \mathcal{L}_{BP}$. Now, the Euler-Lagrange equations of the action $\mathcal{S}$ are given by

$$
\begin{cases}
i\hbar\left(\frac{\partial}{\partial t} + \frac{iq}{\hbar}\phi\right)\psi + \frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar c}\mathbf{A}\right)^2\psi + |\psi|^{p-2}\psi = 0 \\
-\mathrm{div}\left(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}\right) + a^2\left[\Delta\left(\Delta\phi + \frac{1}{c}\mathrm{div}\frac{\partial}{\partial t}\mathbf{A}\right)\right. \\
\qquad \left. -\frac{1}{c}\frac{\partial}{\partial t}\mathrm{div}\left(\nabla\times\nabla\times\mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A})\right)\right] = 4\pi q|\psi|^2 \\
-\frac{\hbar q}{mc}\Im\left[\left(\nabla\psi^* + \frac{iq}{\hbar c}\mathbf{A}\psi^*\right)\psi\right] - \frac{1}{4\pi}\left\{\frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}) + \nabla\times\nabla\times\mathbf{A}\right\} \\
\qquad + \frac{a^2}{4\pi}\left[\frac{1}{c}\nabla\frac{\partial}{\partial t}\left(\Delta\phi + \frac{1}{c}\mathrm{div}\frac{\partial}{\partial t}\mathbf{A}\right) - \nabla\times\nabla\times\nabla\times\nabla\times\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\nabla\times\nabla\times\mathbf{A}\right. \\
\qquad \left. -\frac{1}{c}\nabla\times\nabla\times\frac{\partial}{\partial t}(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}) - \frac{1}{c^3}\frac{\partial^3}{\partial t^3}(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A})\right] = 0
\end{cases}
$$

Widely used in physics, considering ψ as standing waves in the form $\psi(t,x) = e^{iS(t,x)}u(t,x)$, with $S, u : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$, the Euler Lagrange equations would look like

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \left[\frac{\hbar^2}{2m}\left|\nabla S - \frac{q}{\hbar c}\mathbf{A}\right|^2 + \hbar\frac{\partial}{\partial t}S + q\phi\right] u = |u|^{p-2}u \\ \frac{\partial}{\partial t}u^2 + \frac{\hbar}{m}\mathrm{div}\left[\left(\nabla S - \frac{q}{\hbar c}\mathbf{A}\right)u^2\right] = 0 \\ -\mathrm{div}\left(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}\right) + a^2\left[\Delta\left(\Delta\phi + \frac{1}{c}\mathrm{div}\frac{\partial}{\partial t}\mathbf{A}\right)\right. \\ \qquad \left. -\frac{1}{c}\frac{\partial}{\partial t}\mathrm{div}\left(\nabla\times\nabla\times\mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}\right)\right] = 4\pi q|u|^2 \\ \frac{\hbar q}{mc}\left(\nabla S - \frac{q}{\hbar c}\mathbf{A}\right)u^2 - \frac{1}{4\pi}\left\{\frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}) + \nabla\times\nabla\times\mathbf{A}\right\} \\ \qquad + \frac{a^2}{4\pi}\left[\frac{1}{c}\nabla\frac{\partial}{\partial t}\left(\Delta\phi + \frac{1}{c}\mathrm{div}\frac{\partial}{\partial t}\mathbf{A}\right) - \nabla\times\nabla\times\nabla\times\nabla\times\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\nabla\times\nabla\times\mathbf{A}\right. \\ \qquad \left. -\frac{1}{c}\nabla\times\nabla\times\frac{\partial}{\partial t}(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}) - \frac{1}{c^3}\frac{\partial^3}{\partial t^3}(\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A})\right] = 0. \end{cases}$$

Now, instead of using $\psi(x,t) = e^{iS(t,x)}u(t,x)$, we use standing waves of type $\psi(t,x) = e^{\frac{i\omega t}{\hbar}}u(x)$, in the practice, is just change $S(t,x)$ by $\frac{\omega t}{\hbar}$ and $u(t,x)$ by $u(x)$. Then, analyzing the system of equations above, we can notice the following:

(i) In the first equation

$$-\frac{\hbar^2}{2m}\Delta u + \left[\frac{\hbar^2}{2m}\left|\nabla(\frac{\omega t}{\hbar})\underbrace{-\frac{q}{\hbar c}\mathbf{A}}_{=0}\right|^2 + \hbar\frac{\partial}{\partial t}\frac{\omega t}{\hbar} + q\phi\right]u = |u|^{p-2}u$$

$$-\frac{\hbar^2}{2m}\Delta u + \left[\frac{\hbar^2}{2m}\left|\underbrace{\nabla(\frac{\omega t}{\hbar})}_{=0}\right|^2 + \hbar\frac{\partial}{\partial t}\frac{\omega t}{\hbar} + q\phi\right]u = |u|^{p-2}u$$

$$-\frac{\hbar^2}{2m}\Delta u + \omega u + q\phi u = |u|^{p-2}u \tag{3.3}$$

(ii) In the second equation

$$\underbrace{\frac{\partial}{\partial t}u^2}_{0} + \frac{\hbar}{m}\mathrm{div}\left[\left(\underbrace{\nabla\frac{\omega t}{\hbar}}_{=0} - \underbrace{\frac{q}{\hbar c}\mathbf{A}}_{=0}\right)u^2\right] = 0$$

because $u(x)$ does not depends on t. Then the second equation is satisfied.

(iii) In the third equation

$$-\mathrm{div}\left(\nabla\phi+\underbrace{\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}}_{=0}\right)+a^2\left[\Delta\left(\Delta\phi+\underbrace{\frac{1}{c}\mathrm{div}\frac{\partial}{\partial t}\mathbf{A}}_{=0}\right)\right.$$

$$\left.-\frac{1}{c}\frac{\partial}{\partial t}\mathrm{div}\left(\underbrace{\nabla\times\nabla\times\mathbf{A}}_{=0}+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi+\underbrace{\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}}_{=0}\right)\right]=4\pi q|u|^2$$

$$-\mathrm{div}\,(\nabla\phi)+a^2\left[\Delta\,(\Delta\phi)-\frac{1}{c}\frac{\partial}{\partial t}\mathrm{div}\left(\frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi\right)\right]=4\pi q|u|^2$$

$$\underbrace{-\mathrm{div}\,(\nabla\phi)}_{=-\Delta\phi}+a^2\left[\underbrace{\Delta\,(\Delta\phi)}_{=\Delta^2\phi}-\underbrace{\frac{1}{c}\frac{\partial}{\partial t}\mathrm{div}\left(\frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi\right)}_{=0}\right]=4\pi q|u|^2,$$

and then

$$-\Delta\phi+a^2\Delta^2\phi=4\pi qu^2 \tag{3.4}$$

(iv) In the fourth equation we have

$$\frac{\hbar q}{mc}\left(\nabla\frac{\omega t}{\hbar}-\underbrace{\frac{q}{\hbar c}\mathbf{A}}_{=0}\right)u^2-\frac{1}{4\pi}\left\{\frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi+\underbrace{\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}}_{=0})+\underbrace{\nabla\times\nabla\times\mathbf{A}}_{=0}\right\}$$

$$+\frac{a^2}{4\pi}\left[\frac{1}{c}\nabla\frac{\partial}{\partial t}\left(\Delta\phi+\underbrace{\frac{1}{c}\mathrm{div}\frac{\partial}{\partial t}\mathbf{A}}_{=0}\right)-\underbrace{\nabla\times\nabla\times\nabla\times\nabla\times\mathbf{A}}_{=0}-\underbrace{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\nabla\times\nabla\times\mathbf{A}}_{=0}\right.$$

$$\left.-\frac{1}{c}\nabla\times\nabla\times\frac{\partial}{\partial t}(\nabla\phi+\underbrace{\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}}_{=0})-\frac{1}{c^3}\frac{\partial^3}{\partial t^3}(\nabla\phi+\underbrace{\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}}_{=0})\right]=0$$

$$\frac{\hbar q}{mc}\left(\underbrace{\nabla\frac{\omega t}{\hbar}}_{=0}\right)u^2-\frac{1}{4\pi}\left\{\frac{1}{c}\underbrace{\frac{\partial}{\partial t}(\nabla\phi)}_{=0}\right\}+\frac{a^2}{4\pi}\left[\frac{1}{c}\underbrace{\nabla\frac{\partial}{\partial t}\,(\Delta\phi)}_{=0}\right.$$

$$\left.-\frac{1}{c}\underbrace{\nabla\times\nabla\times\frac{\partial}{\partial t}(\nabla\phi)}_{=0}-\frac{1}{c^3}\underbrace{\frac{\partial^3}{\partial t^3}(\nabla\phi)}_{=0}\right]=0.$$

And then the fourth equation is satisfied.

Hence, the remaining equations (3.3) and (3.4) form the system

$$\begin{cases} -\dfrac{\hbar^2}{2m}\Delta u + \omega u + q\phi u = |u|^{p-2}u \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi q u^2 \end{cases} \tag{SBP}$$

which is called the *Schrödinger-Bopp-Podolsky* system.

In our work, we replace the constants $\dfrac{-\hbar^2}{2m}$ and a^2 by perturbations ε and their respective powers. We will also change the number ω by a potential $V : \mathbb{R}^3 \to \mathbb{R}$. In addition to this, we study the case when instead of $|u|^{p-2}u$, our nonlinearity is a given function $f : \mathbb{R} \to \mathbb{R}$ satisfying some assumptions.

Chapter 4

Shrödinger-Bopp-Podolsky system in $\mathbb{R}^3$

In this chapter, by using the ideas developed originally in [7, 8, 28] we show existence and multiplicity of positive solutions for the following problem in $\mathbb{R}^3$

$$\begin{cases} -\varepsilon^2 \Delta w + V(x) w + \psi w = f(w) \\ -\varepsilon^2 \Delta \psi + \varepsilon^4 \Delta^2 \psi = 4\pi\varepsilon w^2 \end{cases} \qquad (P_\varepsilon)$$

whenever $\varepsilon > 0$ is a small parameter and f and V satisfy the assumptions given below.

In the mathematical literature, such a problem was introduced recently in [12] and describes the stationary states of the Schrödinger equation in the generalized electrodynamics developed by Bopp and Podolsky. Roughly speaking, the system appears when one searches for stationary solutions, namely solutions of type $\psi(x,t) = u(x)e^{it}$, of the Schrödinger equation of a moving charged particle which interacts with its own purely electrostatic field, in the case in which the generalized electromagnetic theory of Bopp-Podolsky is considered. The reason to prefer the Bopp-Podolsky theory to the classical and more studied Maxwell theory, is that in the first case the energy associated to a charged particle is finite. In fact in the Bopp-Podolsky generalized electrodynamic the equation of the electrostatic field generated by a charge particle (let us say at rest in the origin) is

$$-\Delta\psi + \Delta^2\psi = \delta$$

and the fundamental solution $\mathcal{K}(x) = \frac{1-e^{-1/|x|}}{|x|}$ has finite energy, being $\int |\nabla \mathcal{K}|^2 + \int |\Delta \mathcal{K}|^2 <$

$+\infty$. On the other hand in the Maxwell theory the equation of the electrostatic field is

$$-\Delta\psi = \delta$$

and the fundamental solution $\mathcal{A}(x) = \frac{1}{|x|}$ satisfies $\int |\nabla \mathcal{A}|^2 = +\infty$, giving rise to the so-called *infinity problem*. Then the equations in system (P_ε) have to be interpreted as

- a Schrödinger type equation (the first one) in presence of a fixed external potential V and an "internal" potential ψ, and
- an equation (the second one) which says that the potential ψ has as source the same wave function, being $|\psi| = u^2$, justifying the term "internal".

For the mathematical derivation of such a system and some related results concerning different conditions, we refer the reader to the recent papers [2, 12, 16, 27] where the problem is studied under various conditions or even in a bounded domain.

In this work we assume that V and the nonlinearity f satisfy

(V1) $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that

$$0 < \min_{\mathbb{R}^3} V := V_0 < V_\infty := \liminf_{|x|\to+\infty} V \in (V_0, +\infty],$$

with $M = \{x \in \mathbb{R}^3 : V(x) = V_0\}$ smooth and bounded,

(f1) $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 and $f(t) = 0$ for $t \leq 0$,

(f2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$,

(f3) there exists $q_0 \in (3, 2^* - 1)$ such that $\lim_{t\to+\infty} \frac{f(t)}{t^{q_0}} = 0$, where $2^* = 6$,

(f4) there exists $K > 4$ such that $0 < KF(t) := K\int_0^t f(\tau)d\tau \leq tf(t)$ for all $t > 0$,

(f5) the function $t \mapsto \frac{f(t)}{t^3}$ is strictly increasing in $(0, +\infty)$.

The assumptions on the nonlinearity f are standards in order to work with variational methods, use the Nehari manifold and deal with the Palais-Smale condition.

The assumption (V1) will be fundamental in order to estimate the number of positive solutions and also to recover some compactness.

The main result of this work is:

Theorem 4.0.1. *Under the above assumptions* (V1), (f1)-(f5), *there exists an* $\varepsilon^* > 0$ *such that for every* $\varepsilon \in (0, \varepsilon^*]$, *problem* (P_ε) *possesses at least* $catM$ *positive solutions. Moreover, if* $catM > 1$, *then (for a suitably small* ε*) there exist at least* $catM + 1$ *positive solutions.*

In particular among these solutions there is the ground state, namely the solution with minimal energy; this will be evident by the proof. Here $\mathrm{cat} M := \mathrm{cat}_M M$ is the Ljusternik-Schnirelmann category and by positive solutions we mean a pair (u, ψ) with u positive, since ψ will be automatically positive.

We point out that the assumption on the potential V is not too restrictive since it is satisfied by an interesting class which appear in physical models, such as the confining potentials. There is then an interesting relation between the topology of the set of minima of V and the number of solutions.
For example, for a potential of type

$$V(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ |x|^2 & \text{otherwise} \end{cases}$$

the theorem states the existence of (at least) one solution for small ε, being M the unit ball $\{x \in \mathbb{R}^3 : |x| \leq 1\}$ and $catM = 1$. On the other hand with the following double-well potential

$$V(x) = \begin{cases} 1 & \text{if } \ \pi/2 \leq |x| \leq 3\pi/2 \quad \text{and} \quad 5\pi/2 \leq |x| \leq 7\pi/2 \\ 2 + \cos|x| & \text{if } \ 0 \leq |x| \leq \pi/2 \quad \text{and} \quad 3\pi/2 \leq |x| \leq 5\pi/2 \\ |x|^2 + 1 - 49\pi^2/4 & \text{if } \ |x| \geq 7\pi/2 \end{cases}$$

the theorem states that there are at least three solutions for small ε, since M is the union of the annuli $\{x \in \mathbb{R}^3 : \pi/2 \leq |x| \leq 3\pi/2\}$ and $\{x \in \mathbb{R}^3 : 5\pi/2 \leq |x| \leq 7\pi/2\}$ and $catM = 2$.

As a matter of notations, all the integrals, unless otherwise specified, are understood on $\mathbb{R}^3$ with the Lebesgue measure. We denote with $\|\cdot\|_p$ the usual L^p norm. Finally $o_n(1)$ denotes a vanishing sequence and we use the letter C to denote a positive constant whose value does not matter and can vary from line to line.

4.1 Preliminaries

Let us start by recalling some results that will be useful for our work. For more details see [12].

Let $\mathcal{D}$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by

$$< \phi, \psi >_{\mathcal{D}} = \int \nabla\phi\nabla\psi + \int \Delta\phi\Delta\psi.$$

The space $\mathcal{D}$ is an Hilbert space continuously embedded into $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and consequently in $L^6(\mathbb{R}^3)$. Moreover this space is embedded also into $L^\infty(\mathbb{R}^3)$.

The following lemmas are used to justify a "reduction method" in order to deal with just one equation.

Lemma 4.1.1. *The space $C_0^\infty(\mathbb{R}^3)$ is dense in*

$$A = \{\phi \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3)\}$$

normed by $\sqrt{< \phi, \phi >_{\mathcal{D}}}$ and, therefore, $\mathcal{D} = A$.

Proof. The proof here follows the same steps of [12].

Let $\phi \in A$ and define $\rho \in C_0^\infty(\mathbb{R}^3, \mathbb{R}_+)$, such that $\|\rho\|_1 = 1$, and $\{\rho_n\} \subset C_0^\infty$ the sequence of mollifiers given by $\rho_n(x) = n^3\rho(nx)$. Define $\phi_n := \rho_n * \phi \in C^\infty(\mathbb{R}^3)$. Now using the properties of mollifiers we have

$$\partial_i\phi_n = \rho_n * \partial_i\phi \in L^2(\mathbb{R}^3), \ \ 1 \le i \le 3, \ \ \Delta\phi_n = \rho_n * \Delta\phi \in L^2(\mathbb{R}^3)$$

and

$$\|\nabla\phi_n - \nabla\phi\|_2 \to 0, \ \ \|\Delta\phi_n - \Delta\phi\|_2 \to 0$$

we have

$$\phi_n \in C^\infty(\mathbb{R}^3) \cap A \ \text{ and} \|\phi_n - \phi\|_{\mathcal{D}} \to 0. \tag{4.1}$$

Let now $\xi \in C^\infty \cap A, \zeta \in C_0^\infty(\mathbb{R}^3; [0,1])$ with $\zeta(x) = 1$ in $B(0,1), supp(\zeta) \subset B(0,2)$ and define

$$\xi_n := \zeta\left(\frac{\cdot}{n}\right)\xi \in C_0^\infty(\mathbb{R}^3).$$

Differentiating we have

$$\nabla\xi_n = \zeta\left(\frac{\cdot}{n}\right)\nabla\xi + \frac{1}{n}\xi\nabla\zeta\left(\frac{\cdot}{n}\right),$$

$$\Delta\xi_n = \zeta\left(\frac{\cdot}{n}\right)\Delta\xi + \frac{2}{n}\nabla\xi\nabla\zeta\left(\frac{\cdot}{n}\right) + \frac{1}{n^2}\xi\Delta\zeta\left(\frac{\cdot}{n}\right).$$

Evaluating the integral

$$\frac{1}{n^2}\int \xi^2(x)\left|\nabla\zeta\left(\frac{x}{n}\right)\right|^2 \le \frac{1}{n^2}\left(\int_{|x|\ge n}\xi^6\right)^{\frac{1}{3}}\left(\int\left|\nabla\zeta\left(\frac{x}{n}\right)\right|^3\right)^{\frac{2}{3}} = C\left(\int_{|x|\ge n}\xi^6\right)^{\frac{1}{3}} \to 0$$

we can do the same calculation to conclude

$$\frac{2}{n}\nabla\xi\nabla\zeta\left(\frac{\cdot}{n}\right), \frac{1}{n^2}\xi\Delta\zeta\left(\frac{\cdot}{n}\right) \to 0 \ \text{ in } L^2(\mathbb{R}^3),$$

as $n \to +\infty$, then

$$\|\nabla\xi - \nabla\xi_n\|_2^2 \le 2\|(1-\zeta\left(\frac{\cdot}{n}\right))\partial_i\xi\|_2^2 + o_n(1)$$

$$\|\Delta\xi - \Delta\xi_n\|_2^2 \le 2\|(1-\zeta\left(\frac{\cdot}{n}\right))\Delta\xi\|_2^2 + o_n(1)$$

which shows that $\|\xi_n - \xi\|_{\mathcal{D}} \to 0$. This convergence joint with (4.1.3) concludes the proof. ■

For every fixed $u \in H^1(\mathbb{R}^3)$, the Riesz theorem implies that there exists a unique solution $\phi_{\varepsilon,u} \in \mathcal{D}$, for the second equation in (P_ε). Such a solution is given by $\phi_{\varepsilon,u} = K * u^2$, where

$$K(x) = \varepsilon\frac{1-e^{-|x|}}{|x|}.$$

To simplify the notation, now we refer $\phi_{\varepsilon,u} := \phi_u$. The next two lemmas plays an important role about properties of our solution ϕ_u.

Lemma 4.1.2. *For every $u \in H^1(\mathbb{R}^3)$ we have:*

i) $\forall y \in \mathbb{R}^3, \phi_{u(.+y)} = \phi_u(.+y)$;

ii) $\phi_u \geq 0$;

iii) $\forall s \in (3, +\infty], \phi_u \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;

iv) $\forall s \in (3/2, +\infty], \nabla\phi_u = \nabla K * u^2 \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;

v) $\|\phi_u\|_6 \leq C\|u\|^2$;

vi) ϕ_u *is the unique minimizer of the functional*

$$E(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{1}{2}\|\Delta\phi\|_2^2 - \int \phi u^2, \qquad \phi \in \mathcal{D}.$$

Moreover,

vii) if $v_n \rightharpoonup v$ *in* $H^1(\mathbb{R}^3)$, *then* $\phi_{v_n} \rightharpoonup \phi_v$ *in* $\mathcal{D}$.

Proof. See [12]

■

Arguing like in [22], we can define the map

$$T : u \in H^1(\mathbb{R}^3) \mapsto \int \phi_u u^2 \in \mathbb{R},$$

where $\phi_u = K * u^2$.

Then

$$|T(u)| \leq \varepsilon C\|u\|^4. \tag{4.2}$$

The next lemma is enunciated originally in [22], and an analogous result can be proved for our problem. The proof follow the same steps.

Lemma 4.1.3. *The following proposition hold.*

i) T is of class C^2 and for every $u, v, w \in H^1(\mathbb{R}^3)$

$$T'(u)[v] = 4\int_{\mathbb{R}^3} \phi_u uv, \quad T''(u)[v,w] = 4\int_{\mathbb{R}^3} \phi_u vw + 8\int_{\mathbb{R}^3} \phi_{u,w} uv,$$

ii) if $u_n \to u$ in $L^r(\mathbb{R}^3)$, with $2 \le r < 2^*$, then $T(u_n) \to T(u)$;

iii) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ then $T(u_n - u) = T(u_n) - T(u) + o_n(1)$.

Proof. See [22]

■

4.2 The variational setting

After the change of variables $u(x) := w(\varepsilon x)$, $\phi(x) := \psi(\varepsilon x)$ our problem can be written as

$$\begin{cases} -\Delta u + V(\varepsilon x)u + \phi u = f(u), \\ -\Delta\phi + \Delta^2\phi = 4\pi\varepsilon u^2. \end{cases} \tag{P_ε^*}$$

Hence the critical points of the functional

$$\mathcal{I}_\varepsilon(u,\phi) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int V(\varepsilon x)u^2 + \frac{1}{2}\int \varepsilon\phi u^2 - \frac{1}{16\pi}\|\nabla\phi\|_2^2 - \frac{1}{16\pi}\|\Delta\phi\|_2^2 - \int F(u)$$

in $H^1(\mathbb{R}^3) \times \mathcal{D}$ are easily seen to be weak solutions of (P_ε^*); indeed such a critical point $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}$ satisfies

$$0 = \partial_u \mathcal{I}_\varepsilon(u,\phi)[v] = \int \nabla u \nabla v + \int V(\varepsilon x)uv + \int \varepsilon\phi uv - \int f(u)u, \qquad v \in H^1(\mathbb{R}^3),$$

$$0 = \partial_\phi \mathcal{I}_\varepsilon(u,\phi)[\xi] = \frac{1}{2}\int \varepsilon u^2\xi - \frac{1}{8\pi}\int \nabla\phi\nabla\xi - \frac{1}{8\pi}\int \Delta\phi\Delta\xi, \qquad \xi \in \mathcal{D}.$$

The next step is the usual reduction argument in order to deal with a one variable functional. Noting that $\partial_\phi \mathcal{I}_\varepsilon$ is a C^1 function and defining G_Φ as the graph of the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto$

$\phi_u \in \mathcal{D}$, an application of the Implicit Function Theorem gives

$$G_\Phi = \{(u,\phi) \in H^1(\mathbb{R}^3) \times D : \partial_\phi \mathcal{I}_\varepsilon(u,\phi) = 0\}, \qquad \Phi \in C^1(H^1(\mathbb{R}^3), \mathcal{D}).$$

Then

$$0 = \partial_\phi \mathcal{I}_\varepsilon(u,\Phi(u)) = \frac{1}{2}\int \phi_u u^2 - \frac{1}{8\pi}\|\nabla\phi_u\|_2^2 - \frac{1}{8\pi}\|\Delta\phi_u\|_2^2$$

and substituting

$$-\frac{1}{4}\int \phi_u u^2 = -\frac{1}{16\pi}\|\nabla\phi_u\|_2^2 - \frac{1}{16\pi}\|\Delta\phi_u\|_2^2$$

in the expression of $\mathcal{I}_\varepsilon$ we obtain the functional

$$I_\varepsilon(u) := \mathcal{I}_\varepsilon(u,\Phi(u)) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int V(\varepsilon x)u^2 + \frac{1}{4}\int \phi_u u^2 - \int F(u).$$

This functional is of class C^1 in $H^1(\mathbb{R}^3)$ and, for all $u, v \in H^1(\mathbb{R}^3)$:

$$\begin{aligned} I'_\varepsilon(u)[v] &= \partial_u \mathcal{I}_\varepsilon(u,\Phi(u))[v] + \partial_\phi \mathcal{I}_\varepsilon(u,\Phi(u)) \circ \Phi'(u)[v] \\ &= \partial_u \mathcal{I}_\varepsilon(u,\Phi(u))[v] \\ &= \int \nabla u \nabla v + \int V(\varepsilon x)uv + \int \phi_u uv - \int f(u)u. \end{aligned}$$

Then it is easy to see that the following statements are equivalents:

i) the pair $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}$ is a critical point of $\mathcal{I}_\varepsilon$, i.e. (u,ϕ) is a solution of (P_ε^*);

ii) u is a critical point of I_ε and $\phi = \phi_u$.

Then, solving (P_ε^*) is equivalent to find critical points of I_ε, i.e., to solve

$$-\Delta u + V(\varepsilon x)u + \phi_u u = f(u) \qquad \text{in} \quad \mathbb{R}^3.$$

Let us define the Hilbert space

$$W_\varepsilon = \left\{ u \in H^1(\mathbb{R}^3) : \int V(\varepsilon x)u^2 < +\infty \right\},$$

endowed with the scalar product and (squared) norm given by

$$(u,v)_\varepsilon = \int \nabla u \nabla v + \int V(\varepsilon x) uv,$$

and

$$\|u\|_\varepsilon^2 = \int |\nabla u|^2 + \int V(\varepsilon x) u^2.$$

We will find the critical points of I_ε in W_ε.

Defining the Nehari manifold associated to I_ε,

$$\mathcal{N}_\varepsilon = \{u \in W_\varepsilon \setminus \{0\} : J_\varepsilon(u) = 0\},$$

where

$$J_\varepsilon(u) = I'_\varepsilon(u)[u] = \|u\|_\varepsilon^2 + \int \phi_u u^2 - \int f(u)u,$$

we have the following lemma.

Lemma 4.2.1. *For every $u \in \mathcal{N}_\varepsilon$, $J'_\varepsilon(u)[u] < 0$ and there are positive constants $h_\varepsilon, k_\varepsilon$, such that $\|u\|_\varepsilon \geq h_\varepsilon, I_\varepsilon(u) \geq k_\varepsilon$. Moreover, $\mathcal{N}_\varepsilon$ is diffeomorphic to the set*

$$\mathcal{S}_\varepsilon = \{u \in W_\varepsilon : \|u\|_\varepsilon = 1, u > 0 \quad a.e.\}.$$

Proof. Let‘s split the proof in two parts:

Part 1: $J'_\varepsilon(u)[u] < 0$.

We notice first that

$$meas\{x \in \mathbb{R}^3 : u(x) > 0\} > 0,$$

in fact, for every $u \leq 0$, by $(f1)$ and $J_\varepsilon(u) = 0$, we have

$$\begin{aligned} J_\varepsilon(u) = 0 \quad &\iff \quad \int |\nabla u|^2 + \int V(\varepsilon x)u^2 + \int \phi_u u^2 - \int \underbrace{f(u)u}_{=0} = 0 \\ &\iff \quad \int |\nabla u|^2 + \int V(\varepsilon x)u^2 + \int \phi_u u^2 = 0 \end{aligned}$$

and hence $u = 0$ almost everywhere.

Now,

$$u \in \mathcal{N}_\varepsilon \iff J_\varepsilon(u) = 0 \iff 4\int \phi_u u^2 = 4\int f(u)u - 4\|u\|_\varepsilon^2 \tag{4.3}$$

Then,

$$\begin{aligned}
J_\varepsilon'(u)[u] &= 2\|u\|_\varepsilon^2 + \underbrace{4\int \phi_u u^2}_{(4.3)} - \int f'(u)u^2 - \int f(u)u \\
&= 2\|u\|_\varepsilon^2 + 4\int f(u)u - 4\|u\|_\varepsilon^2 - \int f'(u)u^2 - \int f(u)u \\
&= -2\|u\|_\varepsilon^2 + 3\int f(u)u - \int f'(u)u^2 \\
&\leq 3\int f(u)u - \int f'(u)u^2 \\
&= \int \left[3f(u)u - f'(u)u^2\right] \qquad (4.4)
\end{aligned}$$

For the completeness of the argument, the hypothesis $(f5)$ can be interpreted as

$$\frac{d}{dt}\left(\frac{f(t)}{t^3}\right) > 0 \text{ for } t > 0,$$

and developing this inequality, we have

$$3f(t)t - f'(t)t^2 < 0 \text{ for } t > 0.$$

So we conclude (4.4) < 0, or just

$$J_\varepsilon'(u)[u] < 0.$$

Part 2: The majorants and the diffeomorphism.

For all $\bar{u} \in \mathcal{S}_\varepsilon$ let $\alpha_\varepsilon(\bar{u})$ the positive number that reaches the maximum of the function $\lambda \mapsto I_\varepsilon(\lambda\bar{u})$ defined in $\mathbb{R}^+$. Now we show that $\alpha_\varepsilon(\bar{u})$ is well defined. In fact , $\max_{\lambda\in\mathbb{R}^+} I_\varepsilon(\lambda\bar{u})$ is achieved, by $(f2)$ and $(f3)$, $I_\varepsilon(u)$ has a local minimum in 0 and by $(f4)$, we have $F(t) \geq Ct^K, C > 0, K > 4$, and then

$$\lim_{\lambda\to\infty} I_\varepsilon(\lambda\bar{u}) = -\infty.$$

Then the maximum is achieved.

Now, the uniqueness of $\alpha_\varepsilon(\bar{u})$ follows by

$$\begin{aligned} 0 &= \frac{\partial}{\partial\lambda} I_\varepsilon(\lambda\bar{u}) = \lambda\|\bar{u}\|_\varepsilon^2 + \lambda^3 \int \phi_{\bar{u}}\bar{u}^2 - \int f(\lambda\bar{u})\bar{u} \\ &\iff \frac{1}{\lambda^2}\|\bar{u}\|_\varepsilon^2 + \int \phi_{\bar{u}}\bar{u}^2 = \frac{1}{\lambda^3}\int f(\lambda\bar{u})\bar{u}. \end{aligned}$$

By $(f5)$, $\frac{1}{\lambda^3} f(\lambda\bar{u})\bar{u}$ is strictly increasing and the uniqueness is guaranteed.

Clearly $\alpha_\varepsilon(\bar{u})\bar{u} \in \mathcal{N}_\varepsilon$. Consequently, $\mathcal{N}_\varepsilon$ is the image of the function $\psi_\varepsilon : \mathcal{S}_\varepsilon \to \mathcal{N}_\varepsilon$ defined by

$$\psi_\varepsilon(u) = \alpha_\varepsilon(\bar{u})\bar{u}.$$

By the implicit function theorem and the fact of $J'_\varepsilon(u)[u] < 0$, ψ_ε and α_ε are functions of class C^1. The fact of $\|u\|_\varepsilon^2 \geq h_\varepsilon$ is a consequence of the definition of α_ε and the fact that 0 is a local minimum for I_ε.

Now considering $u \in \mathcal{N}_\varepsilon$, and by $(f4), K > 4$, we have

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\|u\|_\varepsilon^2 + \frac{1}{4}\int \phi_u u^2 - \int F(u) \\ &\geq \frac{1}{2}\|u\|_\varepsilon^2 + \frac{1}{4}\int \phi_u u^2 - \frac{1}{K}\int f(u)u \\ &= \frac{1}{2}\|u\|_\varepsilon^2 + \frac{1}{4}\int \phi_u u^2 - \frac{1}{K}\|u\|_\varepsilon^2 - \frac{1}{K}\int \phi_u u^2 \\ &= \left(\frac{1}{2} - \frac{1}{K}\right)\|u\|_\varepsilon^2 + \left(\frac{1}{4} - \frac{1}{K}\right)\int \phi_u u^2 \\ &\geq \left(\frac{1}{2} - \frac{1}{K}\right)\|u\|_\varepsilon^2 \geq k_\varepsilon. \end{aligned}$$

And the proof is finished. ■

By the assumptions on f, the functional I_ε has the Mountain Pass geometry shown below:

(MP1) $I_\varepsilon(0) = 0$;

(MP2) due to (f2) and (f3), for all $\xi > 0$ there exists $M_\xi > 0$ such that

$$F(u) \leq \xi u^2 + M_\xi |u|^{q_0+1}.$$

Knowing that $\phi_u > 0$ (for $u \neq 0$)

$$\begin{aligned} I_\varepsilon(u) &\geq \frac{1}{2}\|u\|_\varepsilon^2 - \int F(u) \geq \frac{1}{2}\|u\|_\varepsilon^2 - \xi\|u\|_2^2 - M_\xi\|u\|_{q_0+1}^{q_0+1} \\ &\geq \frac{1}{2}\|u\|_\varepsilon^2 - \xi C_1\|u\|_\varepsilon^2 - M_\xi C_2\|u\|_\varepsilon^{q_0+1}, \end{aligned}$$

and then, for $\|u\|_\varepsilon^2 = \rho$ small enough, we conclude that I_ε has a strict local minimum at $u = 0$.

(MP3) By (f4) we have $F(t) \geq Ct^K$ where $C > 0$ and $K > 4$. Fixed $v \in C_0^\infty(\mathbb{R}^3), v > 0$, we have $\phi_{tv} = t^2\phi_v$ and then

$$I_\varepsilon(tv) = \frac{t^2}{2}\|v\|_\varepsilon^2 + \frac{t^4}{4}\int \phi_v v^2 - \int F(tv) \leq \frac{t^2}{2}\|v\|_\varepsilon^2 + \frac{t^4}{4}\int \phi_v v^2 - Ct^K \int v^K.$$

So, with t big enough, we get $I_\varepsilon(tv) < 0$.

Denoting by

$$c_\varepsilon = \inf_{\gamma\in\mathcal{H}_\varepsilon} \sup_{t\in[0,1]} I_\varepsilon(\gamma(t)), \quad \mathcal{H}_\varepsilon = \{\gamma \in C([0,1], W_\varepsilon) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\},$$

the Mountain Pass level, and with

$$m_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u)$$

the ground state level, we know by [29] that

$$c_\varepsilon = m_\varepsilon = \inf_{u\in W_\varepsilon\setminus\{0\}} \sup_{t\geq 0} I_\varepsilon(tu). \tag{4.5}$$

4.2.1 The problem at infinity

Let us consider the "limit" problem (the autonomous problem) associated to (P_ε^*), that is,

$$-\Delta u + \mu u = f(u) \quad \text{in } \mathbb{R}^3 \tag{A_μ}$$

where $\mu > 0$ is a constant. The solutions are critical points of the functional

$$E_\mu(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{\mu}{2}\int u^2 - \int F(u),$$

in $H^1(\mathbb{R}^3)$. We will denote with $H^1_\mu(\mathbb{R}^3)$ simply the space $H^1(\mathbb{R}^3)$ endowed with the (equivalent squared) norm

$$\|u\|^2_{H^1_\mu(\mathbb{R}^3)} := \|\nabla u\|_2^2 + \mu\|u\|_2^2.$$

By the assumptions of the nonlinearity f, it is easy to see that the functional E_μ has the Mountain Pass geometry (similarly to I_ε), with Mountain Pass level

$$c_\mu^\infty := \inf_{\gamma\in\mathcal{H}_\mu}\sup_{t\in[0,1]} E_\mu(\gamma(t)),$$

$$\mathcal{H}_\mu := \left\{\gamma \in C([0,1], H^1_\mu(\mathbb{R}^3)) : \gamma(0) = 0, E_\mu(\gamma(1)) < 0\right\}.$$

Introducing the set

$$\mathcal{M}_\mu := \left\{u \in H^1(\mathbb{R}^3)\setminus\{0\} : \|u\|^2_{H^1_\mu} = \int f(u)u\right\},$$

it is standard to see that (like in Lemma 4.2.1):

- $\mathcal{M}_\mu$ has the structure of a differentiable manifold (said the Nehari manifold associated to E_μ);
- $\mathcal{M}_\mu$ is bounded away from zero and radially homeomorphic to the subset of positive functions on the unit sphere (a kind of $\mathcal{S}_\varepsilon$, see Lemma 4.2.1);
- the Mountain Pass level c_μ^∞ coincides with the ground state level

$$m_\mu^\infty := \inf_{u\in\mathcal{M}_\mu} E_\mu(u) > 0.$$

In the next sections, we will mainly deal with $\mu = V_0$ and $\mu = V_\infty$, when finite. It is easy to see that $m_\varepsilon \geq m_{V_0}^\infty$.

4.3 Compactness properties for I_ε, E_μ and the existence of a ground state solution

Let us start by showing the boundedness of the Palais-Smale sequences for E_μ in $H^1_\mu(\mathbb{R}^3)$ and I_ε in W_ε. Let $\{u_n\} \subset H^1_\mu(\mathbb{R}^3)$ be a Palais-Smale sequence for E_μ, that is, $|E_\mu(u_n)| \leq C$ and $E'_\mu(u_n) \to 0$. Then, for large n,

$$\begin{aligned} E_\mu(u_n) - \frac{1}{K}E'_\mu(u_n)[u_n] &= \frac{1}{2}\|u_n\|^2_{H^1_\mu} - \int F(u_n) - \frac{1}{K}\|u_n\|^2_{H^1_\mu} + \frac{1}{K}\int f(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{K}\right)\|u_n\|^2_{H^1_\mu} + \frac{1}{K}\int [f(u_n)u_n - KF(u_n)] \\ &\geq \left(\frac{1}{2} - \frac{1}{K}\right)\|u_n\|^2_{H^1_\mu}. \end{aligned}$$

Since, on the other hand

$$\begin{aligned} \left|E_\mu(u_n) - \frac{1}{K}E'_\mu(u_n)[u_n]\right| &\leq |E_\mu(u_n)| + \frac{1}{K}|E'_\mu(u_n)|\|u_n\|_{H^1_\mu} \\ &< C + \frac{1}{K}|E'_\mu(u_n)|\|u_n\|_{H^1_\mu}, \end{aligned}$$

we conclude that $\{u_n\}$ is bounded.

Arguing similarly we conclude that any Palais-Smale sequence $\{u_n\}$ for I_ε is bounded in W_ε.

In order to prove compactness, we need some preliminaries lemmas. The next lemma is a technical lemma, mostly known by Lions Lemma, for completenes of this work, we give the proof of this famous result.

Lemma 4.3.1. *If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and for some $R > 0$ and $2 \leq q < 2^* = 6$, if we have*

$$\sup_{x\in\mathbb{R}^3} \int_{B_R(x)} |u_n|^q \to 0 \quad as \quad n \to \infty,$$

then $u_n \to 0$ in $L^p(\mathbb{R}^3)$ for $2 < p < 2^*$.

Proof. Let $q < s < 2^*$ and $u \in H^1(\mathbb{R}^3)$. By the interpolation inequality (Hölder) and the

Sobolev embeddings

$$\begin{aligned}\|u\|_{L^s(B_R(x))} &\leq \|u\|^{1-\lambda}_{L^q(B_R(x))}\|u\|^{\lambda}_{L^{2^*}(B_R(x))}\\ &\leq c\|u\|^{1-\lambda}_{L^q(B_R(x))}\left[\int_{B_R(x)}(|u|^2+|\nabla u|^2)\right]^{\frac{\lambda}{2}}\\ &\leq c\|u\|^{1-\lambda}_{L^q(B_R(x))}\|u\|^{\frac{\lambda}{2}}_{H^1(B_R(x))}\end{aligned}$$

where

$$\frac{1}{s}=\frac{\lambda}{2^*}+\frac{1-\lambda}{q}\Longrightarrow \lambda=\frac{2^*}{s}\left(\frac{s-q}{2^*-q}\right).$$

Choosing $\lambda=\frac{2}{s}$ and taking the s- power in both sides we obtain

$$\int_{B_R(x)}|u|^s\leq c^s\|u\|^{(1-\lambda)s}_{L^q(B_R(x))}\|u\|^2_{H^1(B_R(x))}.$$

Now covering $\mathbb{R}^3$ by balls of radius R, in such a way that each point of $\mathbb{R}^3$ is contained in at most 4 balls, we find

$$\int_{\mathbb{R}^3}|u|^s\leq 4c^s\sup_{x\in\mathbb{R}^3}\left[\int_{B_R(x)}|u|^q\right]^{(1-\lambda)\frac{s}{q}}\int_{\mathbb{R}^3}(|u|^2+|\nabla u|^2).$$

Under the assumption of the lemma, we have

$$\int_{\mathbb{R}^3}|u|^s\leq 0\Rightarrow u_n\to 0 \text{ in } L^s(\mathbb{R}^3).$$

Since $2<s<2^*$, then $u_n\to 0$ in $L^p(\mathbb{R}^3)$ for $2<p<2^*$. ■

The next results are proved as in [22].

Lemma 4.3.2. *Let $\{u_n\}\subset W_\varepsilon$ be bounded and such that $I'_\varepsilon(u_n)\to 0$. Then, we have either*

a) $u_n\to 0$ in W_ε, or

b) there exist a sequence $\{y_n\}\subset\mathbb{R}^3$ and constants $R,c>0$ such that

$$\liminf_{n\to+\infty}\int_{B_R(y_n)}u_n^2\geq c>0.$$

Proof. Let‘s suppose that $b)$ does not occur. Then we have the boundedness of $\{u_n\}$ in $L^2(\mathbb{R}^3)$.

We know that $I'_\varepsilon(u_n) \to 0$, then we are in the assumptions of Lemma 4.3.1, hence $u_n \to 0$ in $L^p(\mathbb{R}^3)$ for $p \in (2, 2^*)$.

Now using

$$\forall \xi > 0, \exists M_\xi > 0 : \int f(u)u \leq \xi \int u^2 + M_\xi \int |u|^{q_0+1}, \forall u \in H^1(\mathbb{R}^3) \tag{4.6}$$

and the fact of $u_n \to 0$ in L^{q_0+1}, does not exist a sequence such that $\liminf \int u_n^2 > 0$, then

$$0 \leq \int f(u_n)u_n \leq o_n(1) \longrightarrow f(u_n)u_n \to 0, \quad n \to +\infty$$

consequently

$$\|u_n\|_\varepsilon^2 - \int f(u_n)u_n \leq \|u_n\|_\varepsilon^2 + \int \phi_u u^2 - \int f(u_n)u_n = I'_\varepsilon(u_n)[u_n] = o_n(1). \tag{4.7}$$

Then $u_n \to 0$ in W_ε.

■

In the rest of this work, we assume, without loss of generality, that $0 \in M$, that is, $V(0) = V_0$.

Lemma 4.3.3. *Assume that $V_\infty < \infty$ and let $\{v_n\} \subset W_\varepsilon$ be a $(PS)_d$ sequence for I_ε such that $v_n \rightharpoonup 0$ in W_ε. Then $v_n \nrightarrow 0$ in W_ε implies $d \geq m_{V_\infty}^\infty$.*

Proof. Observe that by the condition $(V1)$:

$$\forall \xi > 0, \exists \tilde{R} = R_\xi > 0 : V(\varepsilon x) > V_\infty - \xi, \quad \forall x \notin B_{\tilde{R}}.$$

Let $\{t_n\} \subset (0, +\infty)$ be such that $\{t_n v_n\} \subset \mathcal{M}_{V_\infty}$. We begin showing the following

Claim: The sequence $\{t_n\}$ satisfies $\limsup_{n\to\infty} t_n \leq 1$.

Let’s suppose by contradiction that this claim does not hold, then there exists $\delta > 0$ and a subsequence $\{t_{n_j}\}$ such that

$$t_{n_j} \geq 1 + \delta, \quad \forall n \in \mathbb{N}. \tag{4.8}$$

Since we know that $\{v_n\}$ is bounded, a $(PS)_d$ sequence for I_ε, and $I'_\varepsilon(v_n)[v_n] = o_n(1)$, that is

$$\|v_n\|_\varepsilon^2 + \int \phi_{v_n} v_n^2 = \int f(v_n)v_n + o_n(1).$$

Furthermore, $\{t_n v_n\} \subset \mathcal{M}_{V_\infty}$, then

$$\|t_{n_j} v_n\|^2_{H^1_{V_\infty}} = \int f(t_{n_j} v_n) t_{n_j} v_n \iff \int |\nabla v_n|^2 + \int V_\infty v_n^2 = \int \frac{f(t_{n_j} v_n) v_n}{t_{n_j}}$$

Subtracting the equations above we have

$$\begin{aligned}
\int [f(t_{n_j} v_n) t_{n_j} v_n - f(v_n) v_n] &= \|t_{n_j} v_n\|^2_{H^1_{V_\infty}} - \|v_n\|_\varepsilon^2 - \int \phi_{v_n} v_n^2 + o_n(1) \\
\int [f(t_{n_j} v_n) t_{n_j} - f(v_n)] v_n &= t_{n_j}^2 \left[\int |\nabla v_n|^2 + V_\infty \int v_n^2 \right] - \int |\nabla v_n|^2 - \int V(\varepsilon x) u^2 - \int \phi_{v_n} + o_n(1) \\
\int \left[\frac{f(t_{n_j} v_n)}{t_{n_j}} - f(v_n) \right] v_n &= \int (V_\infty - V(\varepsilon x)) v_n^2 - \int \phi_{v_n} v_n^2 + o_n(1) \qquad (4.9)
\end{aligned}$$

Since $\int \phi_{v_n} v_n^2 \geq 0$, then

$$\int \left[\frac{f(t_{n_j} v_n)}{t_{n_j}} - f(v_n) \right] v_n \leq \int (V_\infty - V(\varepsilon x)) v_n^2 + o_n(1)$$

Using (4.8), the fact of $v_n \to 0$ in $L^2(B_{\tilde{R}})$ and $\{v_n\}$ be bounded in W_ε, by (4.9) we conclude

$$\forall \xi > 0 : \int \left[\frac{f(t_{n_j} v_n)}{t_{n_j}} - f(v_n) \right] v_n \leq \xi C + o_n(1). \qquad (4.10)$$

How $v_n \nrightarrow 0$ in W_ε, by the Lemma 4.3.2 we can obtain $\{y_n\} \subset \mathbb{R}^3, R, c > 0$, such that

$$\int_{B_R(y_n)} v_n^2 \geq c > 0 \qquad (4.11)$$

Defining $\tilde{v_{n_j}} := (. + y_n)$, we can suppose, up to a subsequence, $\tilde{v_{n_j}} \rightharpoonup \tilde{v_n}$ in $H^1(\mathbb{R}^3)$ and by (4.11) that there exists $\Omega \subset \mathbb{R}^3$ with positive measure such that $\tilde{v} > 0$ in Ω. By $(f5)$, (4.11), (4.10) can be rewriten as

$$0 < \int \left[\frac{f((1+\delta)\tilde{v_n})}{(1+\delta)\tilde{v_n}} - \frac{f(\tilde{v_n})}{\tilde{v_n}}\right] \tilde{v_n}^2 \leq \xi C + o_n(1).$$

Taking the limit in n and applying the Fatou's lemma, we have with $\forall \xi > 0$

$$0 < \int \left[\frac{f((1+\delta)\tilde{v})}{(1+\delta)\tilde{v}} - \frac{f(\tilde{v})}{\tilde{v}}\right] \tilde{v}^2 \leq \xi C + o_n(1),$$

an absurd, which proves the claim.

Case 1: $\limsup_{n\to\infty} t_n = 1$

We can suppose, up to a subsequence, that $t_n \to 1$. Then

$$d + o_n(1) = I_\varepsilon(v_n).$$

Using that $m^{\infty}_{V_\infty} - E_{V_\infty}(t_n v_n) < 0$:

$$d + o_n(1) = I_\varepsilon(v_n) \geq m^{\infty}_{V_\infty} + I_\varepsilon(v_n) - E_{V_\infty}(t_n v_n) \tag{4.12}$$

Furthermore

$$\begin{aligned} I_\varepsilon(v_n) - E_{V_\infty}(t_n v_n) &= \frac{1}{2}\int |\nabla v_n|^2 + \frac{1}{2}\int V(\varepsilon x) v_n^2 + \frac{1}{4}\int \phi_{v_n} v_n^2 - \int F(v_n) \\ &\quad - \frac{1}{2}\int |\nabla t_n v_n|^2 - \frac{V_\infty}{2}\int t_n^2 v_n^2 + \int F(t_n v_n) \\ &= \underbrace{\frac{(1-t_n^2)}{2}\int |\nabla v_n|^2}_{\to 0} + \frac{1}{2}\int (V(\varepsilon x) - t_n^2 V_\infty) v_n^2 + \underbrace{\frac{1}{4}\int \phi_{v_n} v_n^2}_{\geq 0} \\ &\quad + \int [F(t_n v_n) - F(v_n)] \end{aligned}$$

We will majorate $\frac{1}{2}\int(V(\varepsilon x)-t_n^2V_\infty)v_n^2$ using $V(\varepsilon x)>V_\infty-\xi, \forall x\notin B_{\tilde{R}}$:

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^3}(V(\varepsilon x)-t_n^2V_\infty)v_n^2 &= \int_{\mathbb{R}^3}V(\varepsilon x)v_n^2 - t_n^2\int_{\mathbb{R}^3}V_\infty v_n^2\\
&\geq \int_{B_{\tilde{R}}^c}(V_\infty-\xi)v_n^2+\underbrace{\int_{B_{\tilde{R}}}V(\varepsilon x)v_n^2}_{o_n(1)}-t_n^2\int_{B_{\tilde{R}}}V_\infty v_n^2-t_n^2\int_{B_{\tilde{R}}^c}V_\infty v_n^2\\
&= \underbrace{\int_{B_{\tilde{R}}^c}V_\infty(1-t_n^2)v_n^2}_{\to 0}+o_n(1)-\xi\int_{B_{\tilde{R}}^c}v_n^2-\underbrace{t_n^2\int_{B_{\tilde{R}}}V_\infty v_n^2}_{>-c}\\
&\geq o_n(1)-\xi\int_{B_{\tilde{R}}^c}v_n^2\\
&\geq -\xi C.
\end{aligned}$$

Then holds

$$I_\varepsilon(v_n)-E_{V_\infty}(t_nv_n)\geq o_n(1)-C\xi+\int\big(F(t_nv_n)-F(v_n)\big),$$

by the Mean Value Theorem for integrals

$$\int\big(F(t_nv_n)-F(v_n)\big)=o_n(1),$$

then (4.12) have the form

$$d+o_n(1)\geq m_{V_\infty}^\infty-C\xi+o_n(1)$$

and how ξ is arbitrary, taking the limit in n we have

$$d\geq m_{V_\infty}^\infty.$$

Case 2: $\limsup_{n\to\infty}t_n=t_0<1$

We can assume that $t_n\to t_0$ and $t_n<1$.

Since the application $t\mapsto\frac{1}{4}f(t)t-F(t)>0$ by $(f4)$ in $(0,\infty)$ (remembering that $\{t_nv_n\}\subset\mathcal{M}_{V_\infty}$):

$$\begin{aligned}
m_{V_\infty}^{\infty} &\leq E_{V_\infty}(t_n v_n) = \underbrace{\frac{1}{2}\int |\nabla t_n v_n|^2 + \frac{V_\infty}{2}\int t_n^2 v_n^2}_{=\frac{1}{2}\int f(t_n v_n) t_n v_n} - \int F(t_n v_n) \\
&= \int \left[\frac{1}{2} f(t_n v_n) t_n v_n - F(t_n v_n)\right] \\
&= \underbrace{\int \frac{1}{4} f(t_n v_n) t_n v_n}_{\{t_n v_n\}\subset \mathcal{M}_{V_\infty}\downarrow} + \int \left[\frac{1}{4} f(t_n v_n) t_n v_n - F(t_n v_n)\right] \\
&= \frac{1}{4}\|t_n v_n\|^2_{H^1_{V_\infty}} + \int \left[\frac{1}{4} f(t_n v_n) t_n v_n - F(t_n v_n)\right] \\
&\leq \frac{1}{4}\|t_n v_n\|^2_{H^1_{V_\infty}} + \int \left[\frac{1}{4} f(v_n) v_n - F(v_n)\right] \text{ since } t_n v_n \leq v_n \qquad (4.13)
\end{aligned}$$

But

$$\|t_n v_n\|^2_{H^1_{V_\infty}} = \int |\nabla t_n v_n|^2 + V_\infty \int t_n^2 v_n^2 \overset{(t_n v_n < v_n)}{\leq} \int |\nabla v_n|^2 + \int t_n^2 V_\infty v_n^2 \qquad (4.14)$$

Using (4.8)

$$t_n^2 V_\infty - \xi < V_\infty - \xi < V(\varepsilon x) \quad \text{for } x \notin B_{\tilde{R}},$$

then

$$\begin{aligned}
\int_{\mathbb{R}^3} t_n^2 V_\infty v_n^2 &= \int_{B_{\tilde{R}}} t_n^2 V_\infty v_n^2 + \int_{B^c_{\tilde{R}}} t_n^2 V_\infty v_n^2 \\
&\leq \int_{B^c_{\tilde{R}}} V_\infty v_n^2 + \int_{B^c_{\tilde{R}}} V(\varepsilon x) v_n^2 + \xi \int_{B^c_{\tilde{R}}} v_n^2 \quad \text{, using the boundedness of } v_n \\
&\leq o_n(1) + \int_{\mathbb{R}^3} V(\varepsilon x) v_n^2 + C\xi.
\end{aligned}$$

Putting the inequality above with (4.14) we have

$$\|t_n v_n\|^2_{H^1_{V_\infty}} \leq \|v_n\|^2_\varepsilon + C\xi + o_n(1).$$

And finally using (4.13)

$$\begin{aligned} m_{V_\infty}^\infty &\leq \frac{1}{4}\|v_n\|_\varepsilon^2 + \int \left[\frac{1}{4} f(v_n)v_n - F(v_n)\right] + C\xi + o_n(1) \\ &= \frac{1}{4}\int |\nabla v_n|^2 + \frac{1}{4}\int V(\varepsilon x) v_n^2 + \frac{1}{4}\int f(v_n)v_n - \int F(v_n) + C\xi + o_n(1) \\ &= \underbrace{I(v_n)}_{\to d} - \frac{1}{4}\underbrace{I_\varepsilon'(v_n)[v_n]}_{\to 0} + C\xi + o_n(1) \\ &= d + C\xi + o_n(1) \end{aligned}$$

And we conclude

$$m_{V_\infty}^\infty \leq d.$$

■

Then the Palais-Smale condition holds:

Proposition 4.3.4. *The functional I_ε in W_ε satisfies the $(PS)_c$ condition*

1. *at any level $c < m_{V_\infty}^\infty$, if $V_\infty < \infty$,*

2. *at any level $c \in \mathbb{R}$, if $V_\infty = \infty$.*

Proof. Let $\{u_n\} \subset W_\varepsilon$ such that $I_\varepsilon(u_n) \to c$ and $I_\varepsilon'(u_n) \to 0$. We already saw that $\{u_n\}$ is bounded in W_ε. Then exists $u \in W_\varepsilon$ such that, up to a subsequence, $u_n \rightharpoonup u$ in W_ε. Note that $I_\varepsilon'(u) = 0$. By the lemma 4.1.3, item (i) we have for all $w \in W_\varepsilon$

$$(u_n, w)_\varepsilon \to (u, w)_\varepsilon, \quad A'(u_n)[w] \to A'(u)[w] \quad \text{and} \quad \int f(u_n)w \to \int f(u)w.$$

Defining $v_n := u_n - u$

$$\int F(v_n) = \int F(u) - \int F(u_n) + o_n(1)$$

Arguing as in [11], Lemma 3.3 and [3] Proposition 2.1, we have

$$I_\varepsilon'(v_n) = 0$$

Furthermore

$$I_\varepsilon(v_n) = I_\varepsilon(u_n) - I_\varepsilon(u) + o_n(1) = c - I_\varepsilon(u) + o_n(1) := d + o_n(1).$$

Then, $I_\varepsilon(v_n) \to d, I_\varepsilon(v_n) \to 0$, lead us to conclude that $\{v_n\}$ is a $(PS)_d$ sequence for I_ε. Now

$$\begin{aligned} I_\varepsilon(u) &= \underbrace{I_\varepsilon(u) - \frac{1}{4} I'_\varepsilon(u)[u]}_{\text{it is valid since } I'_\varepsilon(u)=0} \\ &= \underbrace{\frac{1}{4}\|u\|_\varepsilon^2}_{\geq 0} + \int \left(\frac{1}{4} f(u)u - F(u)\right) \\ &\geq \frac{1}{4}\int (f(u)u - 4F(u)) \\ &\geq 0. \end{aligned}$$

Then $I_\varepsilon(u) \geq 0$ implies $d \leq c$.

1. If $V_\infty < \infty$ and $c < m^\infty_{V_\infty}$ we have

$$d \leq c < m^\infty_{V_\infty}.$$

   In this case Lemma 4.3.3 guarantees that $v_n \to 0$, or just

$$u_n \to u \quad \text{in } W_\varepsilon,$$

   i.e., $u_n \to u$, $I_\varepsilon(u_n) \to c$ and $I'_\varepsilon(u_n) \to 0$.

   Then u_n possesses a convergent subsequence, so it is a $(PS)_c$ sequence for I_ε.

2. If $V_\infty = \infty$, from the compact immersion $W_\varepsilon \hookrightarrow\hookrightarrow L^r(\mathbb{R}^3), 2 \leq r < 2^*$ up to a subsequence we have $v_n \to 0$ in $L^r(\mathbb{R}^3)$, then $I'_\varepsilon(v_n) \to 0$ leads us to

$$I'_\varepsilon(v_n)[v_n] = \underbrace{\|v_n\|_\varepsilon^2}_{\to 0} + \underbrace{\int \phi_{v_n} v_n^2}_{\to 0} - \underbrace{\int f(v_n)v_n}_{\to 0} = o_n(1). \tag{4.15}$$

By the Lemma 4.1.3, (ii), $A(v_n) \to A(v) = A(0)$.

Now $A(v_n) = \int \phi_{\varepsilon,v_n} v_n^2 = o_n(1)$, and

$$\int f(v_n)v_n \leq \xi \int v_n^2 + M_\xi \int |v_n|^{q_0+1}, \tag{4.16}$$

putting (4.15) and (4.16), we conclude $\|v_n\|_\varepsilon^2 = o_n(1)$, i.e. $u_n \to u$ in W_ε.

Consequently, $\{u_n\}$ is a $(PS)_c$ sequence, for all $c \in \mathbb{R}$.

■

Then we have

Proposition 4.3.5. *The functional I_ε restricted to $\mathcal{N}_\varepsilon$ satisfies the $(PS)_c$ condition:*

1. *at any level $c < m_{V_\infty}^\infty$, if $V_\infty < \infty$,*
2. *at any level $c \in \mathbb{R}$, if $V_\infty = \infty$.*

Moreover the constrained critical points of the functional I_ε on $\mathcal{N}_\varepsilon$ are critical points of I_ε in W_ε, hence solution of (P_ε^*).

In order to prove our main result, we recall the lemma contained in [15] about the problem (A_μ):

Lemma 4.3.6 (Ground state for the autonomous problem)**.** *Let $\{u_n\} \subset \mathcal{M}_\mu$ be a sequence satisfying $E_\mu(u_n) \to m_\mu^\infty$. Then, up to subsequences the following alternative holds:*

a) $\{u_n\}$ strongly converges in $H^1(\mathbb{R}^3)$;

b) there exists a sequence $\{\widetilde{y}_n\} \subset \mathbb{R}^3$ such that $u_n(.+\widetilde{y}_n)$ strongly converges in $H^1(\mathbb{R}^3)$.

In particular, there exists a minimizer $\mathfrak{m}_\mu \geq 0$ for m_μ^∞.

Now we can prove the existence of a ground state solution for our problem. This is a result like [15, Theorem 1].

Theorem 4.3.7. *Suppose that V and f verify (V1) and (f1)-(f5). Then there exists a ground state solution $\mathfrak{u}_\varepsilon \in W_\varepsilon$ of (P_ε^*):*

1. *for every* $\varepsilon \in (0, \bar{\varepsilon}]$*, for some* $\bar{\varepsilon} > 0$*, if* $V_\infty < \infty$*;*

2. *for every* $\varepsilon > 0$*, if* $V_\infty = \infty$*.*

Proof. As we see I_ε has the Mountain Pass geometry in W_ε, then exists $\{u_n\} \subset W_\varepsilon$ satisfying $I_\varepsilon(u_n) \to c_\varepsilon, \;\; I'_\varepsilon(u_n) \to 0.$

1. If $V_\infty < \infty$, by Proposition 4.3.4 we just have to show that $c_\varepsilon < m^\infty_{V_\infty}$, for all $\varepsilon > 0$ less than $\bar{\varepsilon}$.

   Let $\mu \in (V_0, V_\infty)$, then

$$m^\infty_{V_0} < m^\infty_\mu < m^\infty_{V_\infty} \tag{4.17}$$

   For $r > 0$, let η_r be a smooth cut-off function in $\mathbb{R}^3$ witch is equal 1 in B_r and with support in B_{2r}.

   Let $\omega_r := \eta_r \mathfrak{m}_\mu$ and $s_r > 0$ be such that $s_r\omega_r \in \mathcal{M}_\mu$.

   So, for all $r > 0$

$$\begin{cases} E_\mu(s_r\omega_r) \geq m^\infty_{V_\infty}; \\ \omega_r \to \mathfrak{m}_\mu \quad \text{in} \quad H^1(\mathbb{R}^3), \; r \to \infty \quad \Rightarrow m^\infty_{V_\infty} \leq \liminf_{r\to\infty} E_\mu(s_r\omega_r) = E_\mu(\mathfrak{m}_\mu) = m^\infty_\mu; \\ s_r \to 1 \end{cases}$$

   which contradicts the Inequality (4.17).

   This means that there exists $\bar{r} > 0$ such that $\omega := s_{\bar{r}}\omega_{\bar{r}} = s_{\bar{r}}\eta_{\bar{r}}\mathfrak{m}_{\bar{r}} \in \mathcal{M}_\mu$ satisfying

$$E_\mu(\omega) < m^\infty_{V_\infty}. \tag{4.18}$$

   Let $\varepsilon > 0$ and $t_\varepsilon > 0$ be the numbers such that $t_\varepsilon\omega \in \mathcal{N}_\varepsilon$, and consequently

$$t^2_\varepsilon\|\omega\|^2_\varepsilon + t^4_\varepsilon \int \phi_\omega\omega^2 = t_\varepsilon \int f(t_\varepsilon\omega)\omega,$$

   what implies

$$\begin{aligned} \frac{\|\omega\|_\varepsilon^2}{t_\varepsilon^2} + \int \phi_\omega \omega^2 &= \int \frac{f(t_\varepsilon \omega)}{(t_\varepsilon \omega)^3}\omega^4 \\ &\geq \int_{B_{\bar r}} \frac{f(t_\varepsilon \omega)}{(t_\varepsilon \omega)^3}\omega^4. \end{aligned} \tag{4.19}$$

Then we claim the existence of a $T > 0$ such that $\limsup_{\varepsilon\to 0^+} t_\varepsilon \leq T$. If by contradiction there exists an $\varepsilon_n \to 0^+$ with $t_{\varepsilon_n} \to \infty$, so by (4.19) and $(f5)$

$$\frac{\|\omega\|_{\varepsilon_n}^2}{t_{\varepsilon_n}^2} + \int \phi_\omega \omega^2 \geq \frac{f(t_{\varepsilon_n}\omega(\bar x))}{(t_{\varepsilon_n}\omega(\bar x))^3} \int_{B_{\bar r}} \omega^4, \tag{4.20}$$

where $\omega(\bar x) := \min_{B_{\bar r}} \omega(x) > 0$.

Passing to the limit in n, by $(f4)$ the right hand side of (4.20) tends to ∞, while the left side is limited, witch is an absurd.

As a consequence, there exist $\varepsilon_1 > 0$ such that

$$\forall \varepsilon \in (0, \varepsilon_1] : \ t_\varepsilon \in (0, T]. \tag{4.21}$$

The condition $(V1)$ implies the existence of a $\varepsilon_2 > 0$ such that

$$\forall \varepsilon \in (0, \varepsilon_2] : \ V(\varepsilon x) \leq \frac{V_0 + \mu}{2}, \ \forall x \in \operatorname{supp} \omega \tag{4.22}$$

Now, define

$$\varepsilon_3 := \frac{(\mu - V_0)|\omega|_2^2}{CT^2\|\omega\|^4}$$

where the constant C is the same of (4.2). Hence in particular

$$\forall \varepsilon \in (0, \varepsilon_3] : \int \phi_\omega \omega^2 \leq C\|\omega\|^4 \tag{4.23}$$

and

$$T^2 C\|\omega\|^4 \leq (\mu - V_0) \int \omega^2 \tag{4.24}$$

Defining $\bar{\varepsilon} := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and make use of (4.21) - (4.24) we have for every $\varepsilon \in (0, \bar{\varepsilon}]$:

$$\begin{aligned}\int V(\varepsilon x)\omega^2 + \frac{t_\varepsilon^2}{2}\int \phi_\omega \omega^2 &\leq \frac{V_0+\mu}{2}|\omega|_2^2 + \frac{1}{2}T^2 C\|\omega\|^4 \\ &\leq \frac{V_0+\mu}{2}|\omega|_2^2 + \frac{\mu - V_0}{2}|\omega|_2^2 = \mu\int \omega^2\end{aligned}$$

from what we have $I_\varepsilon(t_\varepsilon \omega) \leq E_\mu(t_\varepsilon \omega)$. Consequently, using (4.5) and (4.18),

$$c_\varepsilon \leq I_\varepsilon(t_\varepsilon\omega) \leq E_\mu(t_\varepsilon\omega) \leq E_\mu(\omega) < m_{V_\infty}^\infty$$

and we conclude $c_\varepsilon < m_{V_\infty}^\infty$, what concludes the proof of the case 1.

2. If $V_\infty = \infty$, by proposition 4.3.4 the sequence $\{u_n\}$ strongly converges for some $\mathfrak{u}_\varepsilon$ in $H^1(\mathbb{R}^3)$, satisfying

$$I_\varepsilon(\mathfrak{u}_\varepsilon) = c_\varepsilon$$

and

$$I'_\varepsilon(\mathfrak{u}_\varepsilon) = 0$$

i.e. $\mathfrak{u}_\varepsilon$ is the ground state solution we are searching for.

Thereby the proof is complete for both cases.

■

4.4 Proof of Theorem 4.0.1

We follow the steps as in [15], to which we refer for the proofs. Let us start with a fundamental result.

Lemma 4.4.1. *Let $\varepsilon_n \to 0^+$ and $u_n \in \mathcal{N}_{\varepsilon_n}$ be such that $I_{\varepsilon_n}(u_n) \to m_{V_\infty}^\infty$. Then there exists a sequence $\{\widetilde{y}_n\} \subset \mathbb{R}$ such that $u_n(. + \widetilde{y}_n)$ has a convergent subsequence in $H^1(\mathbb{R}^3)$. Moreover, up to a subsequence, $y_n := \varepsilon_n \widetilde{y}_n \to y \in M$.*

Proof. We begin showing that $\{u_n\}$ is bounded in $H^1_{V_0}(\mathbb{R}^3)$. Recall that $\|u\|^2_{H^1_{V_0}} = |\nabla u|_2^2 + V_0|u|_2^2$.

By assumptions we have $I'_{\varepsilon_n}(u_n)[u_n] = 0$ and $I_{\varepsilon_n}(u_n) \to m^{\infty}_{V_\infty}$. Then

$$I'_{\varepsilon_n}(u_n)[u_n]0 \iff \|u_n\|^2_{\varepsilon_n} + \int \phi_{u_n} u_n^2 = \int f(u_n)u_n, \tag{4.25}$$

and

$$I_{\varepsilon_n}(u_n) \to m^{\infty}_{V_0} \iff \frac{1}{2}\|u_n\|^2_{\varepsilon_n} + \frac{1}{4}\int \phi_{u_n} u_n^2 - \int F(u_n) = m^{\infty}_{V_0} + o_n(1)$$

which combined

$$\frac{1}{4}\int f(u_n)u_n = \frac{1}{4}\|u_n\|^2_{\varepsilon_n} + \frac{1}{4}\int \phi_{u_n} u_n^2$$

$$\frac{1}{4}\int f(u_n)u_n - \int F(u_n) = \frac{1}{4}\left(\|u_n\|^2_{\varepsilon_n} + \int \phi_{u_n} u_n^2\right) - \int F(u_n).$$

Since

$$\frac{1}{4}\left(\|u_n\|^2_{\varepsilon_n} + \int \phi_{u_n} u_n^2\right) - \int F(u_n) \le \frac{1}{2}\|u_n\|^2_{\varepsilon_n} + \frac{1}{4}\int \phi_{u_n} u_n^2 - \int F(u_n)$$

we have

$$\frac{1}{4}\int f(u_n)u_n - \int F(u_n) \le m^{\infty}_{V_0} + o_n(1)$$

Now using $(f4)$ we get

$$0 \le \left(\frac{1}{4} - \frac{1}{K}\right)\int f(u_n)u_n \le m^{\infty}_{V_0} + o_n(1)$$

and coming back to (4.25), we have for some positive constant C, independent on n

$$\|u_n\|_{H^1_{V_0}} \le \|u_n\|_{\varepsilon_n} \le C. \tag{4.26}$$

We prove the following

Claim 1: There exists $\{\tilde{y}_n\} \subset \mathbb{R}^3$, and $R, c > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} u_n^2 \ge c > 0.$$

If it was not the case then

$$lim_{n\to\infty} sup_{y\in\mathbb{R}^3} \int_{B_R(y)} u_n^2 = 0, \quad \text{for every} \quad R > 0.$$

By Lemma 4.3.2, we have $u_n \to 0$ in $L^p(\mathbb{R}^3)$ for $2 < p < 2^*$, then

$$\int_{\mathbb{R}^3} f(u_n)u_n \to 0.$$

Therefore, $\|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = o_n(1)$, and we have too

$$0 \leq \int F(u_n) \leq \frac{1}{K} \int f(u_n)u_n.$$

then we conclude $\int F(u_n) = o_n(1)$. Consequently by our first assumption

$$\lim_{n\to\infty} I_{\varepsilon_n}(u_n) = m_{V_0}^\infty = 0,$$

which is a contradiction, and the claim is proved.

By (4.26) the sequence $v_n := u_n(. + \tilde{y}_n)$ is also bounded in $H^1(\mathbb{R}^3)$ and

$$v_n \rightharpoonup v \neq 0 \quad \text{in} \quad H^1(\mathbb{R}^3) \tag{4.27}$$

because the claim 1 give us

$$\int_{B_R} v^2 = \liminf_{n\to\infty} \int_{B_R} v_n^2 = \liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} u_n^2 \geq c > 0.$$

Define now $t_n > 0$ be such that $\tilde{v}_n := t_n v_n \in \mathcal{M}_{V_0}$. The next step is to prove

$$E_{V_0}(\tilde{v}_n) \to m_{V_0}^\infty. \tag{4.28}$$

First note that

$$\begin{aligned} m_{V_0}^{\infty} \le E_{V_0}(\tilde{v}_n) &= \frac{1}{2}\|\tilde{v}_n\|_{H_{V_0}^1}^2 - \int F(\tilde{v}_n) \\ &= \frac{t_n^2}{2}\int [|\nabla u_n(x+\tilde{y}_n)|^2 + V_0 u_n^2(x+\tilde{y}_n)]dx - \int F(t_n u_n(x+\tilde{y}_n))dx \\ &= \frac{t_n^2}{2}\int |\nabla u_n(z)|^2 dz + \frac{t_n^2}{2}\int V_0 u_n^2(z)dz - \int F(t_n u_n(z))dz \\ &\le \frac{t_n^2}{2}\int |\nabla u_n|^2 + \frac{t_n^2}{2}\int V(\varepsilon_n z)u_n^2 + \frac{t_n^4}{4}\int \phi_{u_n} u_n^2 - \int F(t_n u_n) \\ &= I_{\varepsilon_n}(t_n u_n), \end{aligned}$$

and then

$$m_{V_0}^{\infty} \le E_{V_0}(\tilde{v}_n) \le I_{\varepsilon_n}(t_n u_n) \le I_{\varepsilon_n}(u_n) = m_{V_0}^{\infty} + o_n(1),$$

thereby (4.28) is proved.

Now we have to prove

$$v_n \to v \ \text{ in } \ H^1(\mathbb{R}^3.$$

As in the first part of this demonstration, by (4.26) we proved the boundedness of $\{u_n\}$ in $H_{V_0}^1(\mathbb{R}^3)$, arguing similarly we have

$$\{\tilde{v}_n\} \subset \mathcal{M}_{V_0} \ \text{ and } \ E_{V_0}(\tilde{v}_n) \to m_{V_0}^{\infty} \Rightarrow \|\tilde{v}_n\|_{H_{V_0}^1} \le C,$$

and just like the claim 1, it holds for the sequence $\{\tilde{v}_n\}$. Then $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H_{V_0}^1(\mathbb{R}^3)$ and exists $\delta > 0$ such that

$$0 < \delta \le \|v_n\|_{H_{V_0}^1}. \tag{4.29}$$

This implies for $t_n > 0$

$$0 < t_n\delta \le \|t_n v_n\|_{H_{V_0}^1} = \|\tilde{v}_n\|_{H_{V_0}^1} \le C,$$

showing that, up to subsequence, $t_n \to t_0 \ge 0$.

If $t_0 = 0$ using (4.26), we have

$$0 \leq \|\tilde{v}_n\|_{H^1_{V_0}} = t_n \|v_n\|_{H^1_{V_0}} \leq t_n C \to 0,$$

and then $\tilde{v}_n \to 0$ in $H^1_{V_0}(\mathbb{R}^3)$.

This fact, together with (4.28) leads us to $m^\infty_{V_0} = 0$, which is an absurd. So t_0 must be greater than zero.

Then

$$t_n v_n \rightharpoonup t_0 \tilde{v} =: \tilde{v} \ \text{ in } \ H^1(\mathbb{R}^3).$$

By (4.29) $\tilde{v} \neq 0$. Applying lemma 4.12 to $\{\tilde{v}_n\}$, we get $\tilde{v}_n \to \tilde{v}$ in $H^1(\mathbb{R}^3)$ and then

$$v_n \to \tilde{v}.$$

Therefore by (4.27) we deduce

$$v_n \to v$$

and the existence of convergent subsequence of the first part of the proposition is proved.

Let us proceed with the second part.

Claim 2: $\{y_n\}$ is bounded in $\mathbb{R}^3$.

First, fix $y_n = \varepsilon_n \tilde{y}_n$ with $\tilde{y}_n$ given in the previous claim. Assume the contrary, then we split in two cases

1. if $V_\infty < \infty$, since $\tilde{v}_n \to \tilde{v}$ in $H^1(\mathbb{R}^3)$ and $V_0 < V_\infty$ we have

$$\begin{aligned}
m^\infty_{V_0} &= \frac{1}{2}\|\tilde{v}\|^2_{H^1_{V_0}} - \int F(\tilde{v}) \\
&< \frac{1}{2}\|\tilde{v}\|_{H^1_{V_0}} - \int F(\tilde{v}) \\
&\leq \liminf_{n\to\infty} \frac{1}{2}\int |\nabla \tilde{v}_n|^2 + \lim_{n\to\infty}\left(\frac{1}{2}\int V(\varepsilon_n x + y_n)\tilde{v}_n^2(x)dx - \int F(\tilde{v}_n)\right) \\
&= \liminf_{n\to\infty}\left(\frac{t_n^2}{2}\int |\nabla u_n|^2 + \frac{t_n^2}{2}\int V(\varepsilon_n z)u_n^2 - \int F(t_n u_n)\right) \\
&\leq \liminf_{n\to\infty}\left(\frac{1}{2}\|t_n u_n\|^2_{\varepsilon_n} - \int F(t_n u_n) + \frac{t_n^4}{4}\int \phi_{u_n} u_n^2\right)
\end{aligned}$$

and then

$$m_{V_0}^{\infty} < \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = m_{V_0}^{\infty}$$

which is an absurd. Then if $V_\infty < \infty$, $\{y_n\}$ is bounded in $\mathbb{R}^3$.

2. If $V_\infty = \infty$ we have since $u_n \in \mathcal{N}_{\varepsilon_n}$

$$\begin{aligned} \int V(\varepsilon_n x + y_n) v_n^2(x) dx &\leq \int |\nabla v_n(x)|^2 + \int V(\varepsilon_n x + y_n) v_n^2(x) dx + \int \phi_{v_n} v_n^2(x) dx \\ &= \int f(v_n(x)) v_n(x) dx, \end{aligned}$$

and using the Fatou's lemma we obtain

$$\int \underbrace{\liminf_{n\to\infty} V(\varepsilon_n x + y_n)}_{\to\infty} v_n^2 \leq \liminf_{n\to\infty} \int f(v_n) v_n$$

and then

$$\infty = \liminf_{n\to\infty} \int f(v_n) v_n = \int f(v) v,$$

which is an absurd.

Then in both cases $\{y_n\}$ is bounded and the claim 2 is proved.

Consequently we can assume $y_n \to y \in \mathbb{R}^3$.

If $y \notin M$, then $V_0 < V(y)$, we just replace V_∞ by $V(y)$ in the computation of case 1 of the previous claim and we will have a contradiction.

Hence $y \in M$ and the proof of the second part of the proposition is complete.

■

4.4.1 The barycenter map

To define the barycenter map, we first define for $\delta > 0$ (later on it will be fixed conveniently), a smooth nonincreasing cut-off function η in $C_0^\infty(\mathbb{R}^3, [0,1])$ such that

$$\eta(s) = \begin{cases} 1, & \text{if} \quad 0 \leq s \leq \delta/2, \\ 0, & \text{if} \quad s \geq \delta. \end{cases}$$

Let $\mathfrak{w}_{V_0}$ be a ground state solution of problem (A_μ) with $\mu = V_0$. For any $y \in M$, let us define

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\mathfrak{w}_{V_0}\left(\frac{\varepsilon x - y}{\varepsilon}\right).$$

Now, let $t_\varepsilon > 0$ verifying $\max_{t\geq 0} I_\varepsilon(t\Psi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y})$, by the properties of the Nehari manifold, $t_\varepsilon\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$, and define the map $\Phi_\varepsilon : y \in M \mapsto t_\varepsilon\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$.

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$ and $\Phi_\varepsilon(y)$ is continuous. The next result will help us to define a map from M to a suitable sublevel in the Nehari manifold.

Lemma 4.4.2. *The function Φ_ε satisfies*

$$\lim_{\varepsilon \to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0}^\infty,$$

uniformly in $y \in M$.

Proof. Suppose by contradiction that the lemma is false. Then there exist $\delta > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0^+$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - m_{V_0}^\infty| \geq \delta_0 \tag{4.30}$$

Using the Dominated Convergence Theorem of Lebesgue, we have the following

$$\begin{aligned} \lim_{n\to\infty} \|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n} &= \|\mathfrak{m}_{V_0}\|^2_{H^1_{V_0}}, \\ \lim_{n\to\infty} \int F(\Psi_{\varepsilon_n,y_n}) &= \int F(\mathfrak{m}_{V_0}) \\ \lim_{n\to\infty} \|\Psi_{\varepsilon_n,y_n}\|^2_{H^1_{V_0}} &= \|\mathfrak{m}_{V_0}\|^2_{H^1_{V_0}} \end{aligned} \tag{4.31}$$

By the last convergence, we conclude that the sequence $\{\|\Psi_{\varepsilon_n,y_n}\|\}$ is bounded. From (4.2) we have

$$\int \phi_{\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n} \leq \varepsilon_n C \|\Psi_{\varepsilon_n,y_n}\|^4$$

and then

$$\lim_{n\to\infty} \int \phi_{\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n} = 0. \tag{4.32}$$

Using the fact that $t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n} \in \mathcal{N}_{\varepsilon_n}$, i. e. $I_{\varepsilon_n}(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})[t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}] = 0$, it means that

$$\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} + t^2_{\varepsilon_n} \int \phi_{\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n} = \int \frac{f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})}{t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n}, \tag{4.33}$$

simply dividing both sides of $I'_{\varepsilon_n}(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})[t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}]$ by $t^2_{\varepsilon_n}$ and multiplying the right hand side of the interior term of the integral by $\frac{\Psi_{\varepsilon_n,y_n}}{\Psi_{\varepsilon_n,y_n}}$.

We now prove the following:

Claim: $\lim_{n\to\infty} t_{\varepsilon_n} = 1$.

First we will show that the sequence $\{t_{\varepsilon_n}\}$ is bounded.

Since $\varepsilon_n \to 0^+$, we can assume $\frac{\delta}{2} < \frac{\delta}{2(\varepsilon_n)}$ and from (4.33), using the property $(f5)$, and making the change of variable $z := \frac{(\varepsilon_n x - y_n)}{\varepsilon_n}$, we have

$$\begin{aligned} \frac{\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n}}{t_{\varepsilon_n}} + \int \phi_{\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n} &= \int \frac{f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})}{(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})^3} \Psi^4_{\varepsilon_n,y_n} \\ &\geq \frac{f(t_{\varepsilon_n}\mathfrak{m}_{V_0}(\bar{z}))}{(t_{\varepsilon_n}\mathfrak{m}_{V_0}(\bar{z}))^3} \int_{B_{\frac{\delta}{2}}} \mathfrak{m}^4_{V_0}(z), \end{aligned} \tag{4.34}$$

where $\mathfrak{m}_{V_0}(\bar{z}) := \min_{B_{\frac{\delta}{2}}} \mathfrak{m}_{V_0}(z) > 0$.

If $\{t_{\varepsilon_n}\}$ were unbounded, passing to the limit in n, by (4.31) and (4.32) we have

$$\frac{\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n}}{t^2_{\varepsilon_n}} \to 0 \quad \text{and} \quad \int \phi_{\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n} \to 0,$$

then the left hand side of (4.34) tends to 0, while the right hand side tends to ∞, which is an absurd. So we can assume $t_{\varepsilon_n} \to t_0 \geq 0$.

Let $\xi > 0$, by (4.6), there exists $M_\xi > 0$ such that

$$\int \frac{f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})}{t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n} \leq \xi \int \Psi^2_{\varepsilon_n,y_n} + M_\xi t^{q-1}_{\varepsilon_n} \int \Psi^{q+1}_{\varepsilon_n,y_n} \tag{4.35}$$

Since $\{\Psi_{\varepsilon_n,y_n}\}$ is bounded in $H^1(\mathbb{R}^3)$, if $t_0 = 0$, from (4.35) we can deduce

$$\lim_{n\to\infty} \int \frac{f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})}{t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n} = 0$$

which joint (4.32) and (4.33), we have

$$\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} + t^2_{\varepsilon_n} \int \underbrace{\phi_{\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n}}_{\to 0} = \int \underbrace{\frac{f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})}{t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}}}_{\to 0} \Psi^2_{\varepsilon_n,y_n}$$

then

$$\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} \to 0,$$

contradicting (4.31). Then $t_{\varepsilon_n} \to t_0$.

Now, passing the limit in n in (4.33), we have

$$\|\mathfrak{m}^\infty_{H^1_{V_0}}\| = \int \frac{f(t_0\mathfrak{m}_{V_0})}{t_0} \mathfrak{m}_{V_0},$$

and since $\mathfrak{m}_{V_0} \in \mathcal{M}_{V_0}$, it has to be $t_0 = 1$, and the claim is proved.

Finally

$$\begin{aligned} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t^2_{\varepsilon_n}}{2} \int |\nabla \Psi_{\varepsilon_n,y_n}|^2 + \frac{t^2_{\varepsilon_n}}{2} \int V(\varepsilon_n x)\Psi^2_{\varepsilon_n,y_n} \\ &\quad + \frac{t^4_{\varepsilon_n}}{4} \int \phi_{\Psi_{\varepsilon_n,y_n}} \Psi^2_{\varepsilon_n,y_n} - \int F(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}), \end{aligned}$$

and using the claim we have

$$\begin{aligned} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{1}{2}\int |\nabla \mathfrak{w}_{V_0}|^2 + \frac{V_0}{2}\int \mathfrak{w}_{V_0}^2 - \int F(\mathfrak{w}_{V_0}) \\ &= E_{V_0}(\mathfrak{w}_{V_0}) = m_{V_0}^\infty. \end{aligned}$$

which contradicts (4.30).

Then

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0}^\infty.$$

■

By the previous lemma, $h(\varepsilon) := |I_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}^\infty| = o(1)$ for $\varepsilon \to 0^+$ uniformly in y, and then $I_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}^\infty \le h(\varepsilon)$. In particular, the sublevel set in the Nehari

$$\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)} := \left\{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \le m_{V_0}^\infty + h(\varepsilon)\right\}$$

is not empty, since for sufficiently small ε,

$$\forall y \in M : \Phi_\varepsilon(y) \in \mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}. \tag{4.36}$$

Now, we fix the $\delta > 0$ mentioned before such that M and

$$M_{2\delta} := \{x \in \mathbb{R}^3 : d(x, M) \le 2\delta\}$$

are homotopically equivalent.

Take $\rho = \rho(\delta) > 0$ such that $M_{2\delta} \subset B_\rho$ and define $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ as follows

$$\chi(x) = \begin{cases} x, & \text{if} \quad |x| \le \rho, \\ \rho\frac{x}{|x|}, & \text{if} \quad |x| \ge \rho. \end{cases}$$

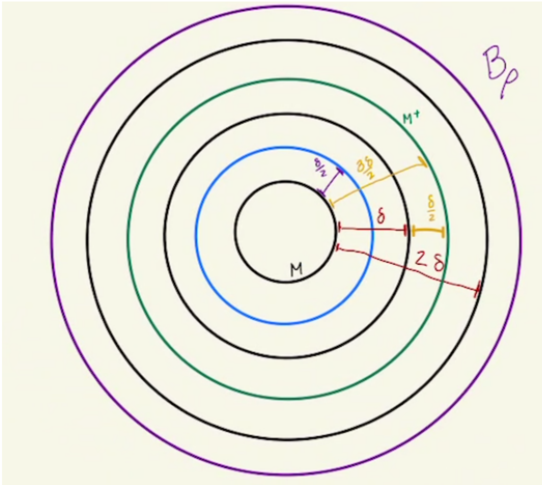

Figure 4.1: *Representation of B_ρ and the set M.*

The barycenter map β_ε is defined as

$$\beta_\varepsilon(u) := \frac{\displaystyle\int \chi(\varepsilon x) u^2(x)}{\displaystyle\int u^2} \in \mathbb{R}^3,$$

for all $u \in W_\varepsilon$ with compact support. Some technical lemmas are stated now. For the proofs see e.g. [15].

Lemma 4.4.3. *The function β_ε satisfies*

$$\lim_{\varepsilon \to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y$$

uniformly in $y \in M$.

Proof. Suppose by contradiction, that the lemma is false. Then exists $\delta_0 > 0$, $\{y_n\} \subset M$, and $\varepsilon_n \to 0^+$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0.$$

Then we have

$$
\begin{aligned}
\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \beta_{\varepsilon_n}(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}) \\
&= \frac{\displaystyle\int \chi(\varepsilon_n x) t_{\varepsilon_n}^2 \Psi_{\varepsilon_n,y_n}^2(x)}{\displaystyle\int t_{\varepsilon_n}^2 \Psi_{\varepsilon_n,y_n}^2(x)} \\
&= \frac{\displaystyle\int \chi(\varepsilon_n x) t_{\varepsilon_n}^2 \eta^2(|\varepsilon_n x - y_n|)\mathfrak{m}_{V_0}^2\left(\frac{\varepsilon_n x - y_n}{\varepsilon_n}\right)}{\displaystyle\int t_{\varepsilon_n}^2 \eta^2(|\varepsilon_n x - y_n|)\mathfrak{m}_{V_0}^2\left(\frac{\varepsilon_n x - y_n}{\varepsilon_n}\right)} \\
&= \frac{\displaystyle\int \chi(\varepsilon_n z + y_n) t_{\varepsilon_n}^2 |\eta(|\varepsilon_n z|)\mathfrak{m}_{V_0}(z)|^2}{\displaystyle\int t_{\varepsilon_n}^2 |\eta(|\varepsilon_n z|)\mathfrak{m}_{V_0}(z)|^2} + y_n - y_n \\
&= y_n + \frac{\displaystyle\int \chi(\varepsilon_n z + y_n) t_{\varepsilon_n}^2 |\eta(|\varepsilon_n z|)\mathfrak{m}_{V_0}(z)|^2}{\displaystyle\int t_{\varepsilon_n}^2 |\eta(|\varepsilon_n z|)\mathfrak{m}_{V_0}(z)|^2} - \frac{\displaystyle\int y_n t_{\varepsilon_n}^2 |\eta(|\varepsilon_n z|)\mathfrak{m}_{V_0}(z)|^2}{\displaystyle\int t_{\varepsilon_n}^2 |\eta(|\varepsilon_n z|)\mathfrak{m}_{V_0}(z)|^2},
\end{aligned}
$$

and

$$
\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\displaystyle\int [\chi(\varepsilon_n z + y_n) - y_n]|\eta(|\varepsilon_n z|)\mathfrak{m}_{V_0}(z)|^2}{\displaystyle\int |\eta(|\varepsilon_n z|)\mathfrak{m}_{V_0}(z)|^2}.
$$

Now

$$
|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = \frac{\displaystyle\int [\chi(\varepsilon_n z + y_n) - y_n]|\eta(|\varepsilon_n z|)\mathfrak{m}_{V_0}(z)|^2}{\displaystyle\int |\eta(|\varepsilon_n z|)\mathfrak{m}_{V_0}(z)|^2}.
$$

Using that $\varepsilon_n \to 0^+$, $\chi(\varepsilon_n \ + y_n) \to y_n$, and $\eta(|\varepsilon_n z|) \to 1$, we have

$$
|\beta_{\varepsilon_n}(\Psi_{\varepsilon_n}(y_n)) - y_n| = o_n(1).
$$

which is an absurd.

Then the lemma holds. ∎

Lemma 4.4.4. *We have*

$$
\lim_{\varepsilon\to 0^+} \sup_{u\in\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}} \inf_{y\in M} |\beta_\varepsilon(u) - y| = 0.
$$

Proof. Let $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0^+$. For each $n \in \mathbb{N}$, exist $u_n \in \mathcal{N}_{\varepsilon_n}^{m_{V_0}^{\infty}+h(\varepsilon_n)}$ such that (by (4.36))

$$\inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| = \sup_{u \in \mathcal{N}_\varepsilon^{m_{V_0}^{\infty}+h(\varepsilon)}} \inf_{y \in M_\delta} |\beta_\varepsilon(u) - y| + o_n(1).$$

Then it is sufficient to find a sequence $\{y_n\} \subset M_\delta$ such that

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(u_n) - y| = 0. \tag{4.37}$$

Remember that $u_n \in \mathcal{N}^{m_{V_0}^{\infty}+h(\varepsilon_n)}$, we have

$$m_{V_0}^{\infty} \le c_{\varepsilon_n} \le I_{\varepsilon_n}(u_n) \le m_{V_0}^{\infty} + h(\varepsilon_n).$$

Consequently by the lemma 4.4.2 we conclude

$$I_{\varepsilon_n}(u_n) \to m_{V_0}^{\infty}.$$

By Lemma 4.4.1, we can obtain a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that $v_n := u_n(. + \tilde{y}_n)$ converging in $H^1(\mathbb{R}^3)$ for some v and $\{y_n\} := \{\varepsilon_n \tilde{y}_n\} \subset M_\delta$, for n sufficiently large, then

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\displaystyle\int [\chi(\varepsilon_n z + y_n) - y_n] v_n^2(z)}{\displaystyle\int v_n^2(z)}.$$

And then, by $v_n \to v$ in $H^1(\mathbb{R}^3)$, and $\chi(\varepsilon_n z + y_n) \to y_n$, so we conclude

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

■

Now the proof of our main result can be finished. In virtue of the above lemmas, there exists $\varepsilon^* > 0$ such that

$$\forall \varepsilon \in (0, \varepsilon^*] : \sup_{u \in \mathcal{N}_\varepsilon^{m_{V_0}^{\infty}+h(\varepsilon)}} d(\beta_\varepsilon(u), M_\delta) < \frac{\delta}{2}.$$

Let $M^+ := \{x \in \mathbb{R}^3 : d(x, M) \le 3\delta/2\}$. It is homotopically equivalent to M. Now, reducing

$\varepsilon^* > 0$ if necessary, we can assume that the above lemmas and (4.36) hold. So the composed map

$$M \xrightarrow{\Phi_\varepsilon} \mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)} \xrightarrow{\beta_\varepsilon} M^+$$

is homotopic to the inclusion map. Or just $\beta_\varepsilon \circ \Phi_\varepsilon \simeq j$ where j is the inclusion map.

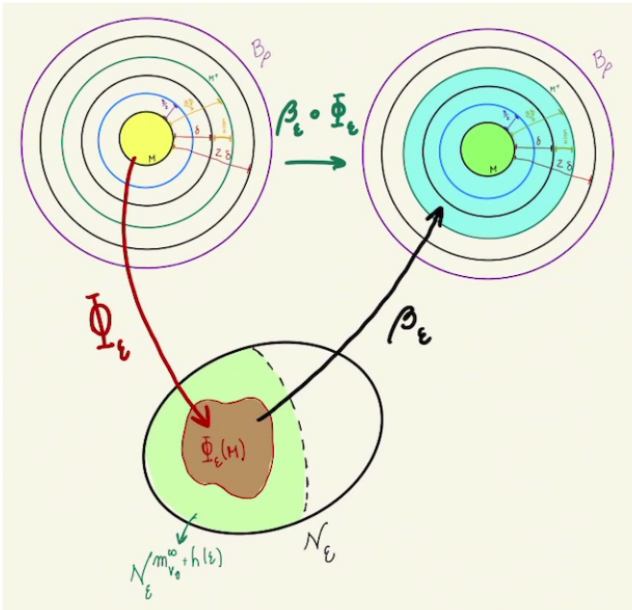

Figure 4.2: *Representation of the composition map*

In the case $V_\infty < \infty$ we eventually reduce ε^* in such a way that also the Palais-Smale condition is satisfied in the interval $(m_{V_0}^\infty, m_{V_0}^\infty + h(\varepsilon))$.

From the properties of the Ljusternick-Schnirelamnn category we have

$$cat(\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}) \geq cat_{M^+}(M), \tag{4.38}$$

For the completennes of the argument, we write here the lines that justify the inequality (4.38).

Suppose $cat_{\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}}(\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}) = n$, then exists

$$\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)} \subset A_1 \cup A_2 \cup \cdots \cup A_n,$$

where A_i, $i = 1, \ldots, n$ are closed and contractible in $\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}$, there exists

$$\mathcal{H}_i \in C([0,1] \times A_i, \mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}), \quad i = 1, \ldots, n$$

and $w_i \in \mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}$, such that

$$\begin{aligned}\mathcal{H}_i(0,u) &= u, \ \forall u \in A_i \\ \mathcal{H}_i(1,u) &= w_i, \ \forall u \in A_i.\end{aligned}$$

Now let $K_i = \Phi_\varepsilon^{-1}(A_i)$, where K_i is closed in M and

$$M \subset K_1 \cup \cdots \cup K_n.$$

Then we can define

$$h_i(t,x) = \beta_\varepsilon \circ \mathcal{H}_i(t, \Phi_\varepsilon(x)).$$

Consequently we have

$$\begin{aligned}&h_i \in C([0,1] \times K_i, M^+) \\ &h_i(0,x) = \beta_\varepsilon \circ \mathcal{H}_i(0, \Phi_\varepsilon(x)) = \beta_\varepsilon \circ \Phi_\varepsilon(x) = x, \ \ \forall x \in K_i \\ &h_i(1,x) = \beta_\varepsilon \circ \mathcal{H}_i(1, \Phi_\varepsilon(x)) = \beta_\varepsilon \circ w_i = \beta_\varepsilon(w_i) = x_i \in M^+ \ \ \forall x \in K_i,\end{aligned}$$

and then $cat_{M^+}(M) = cat_M(M)$ and for a suitable ε we conclude (4.38).

Hence the Ljusternik-Schnirelmann theory ensures the existence of at least $\mathrm{cat}_{M^+}(M) = \mathrm{cat}(M)$ critical points of I_ε constrained in $\mathcal{N}_\varepsilon$, which are then solutions of our problem.

If $\mathrm{cat}M > 1$, the existence of another critical point of I_ε in $\mathcal{N}_\varepsilon$ follows from the ideas used in [28]. The strategy is to exhibit a subset $\mathcal{A} \subset \mathcal{N}_\varepsilon$ such that

1. $\mathcal{A}$ is not contractible in $\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}$,

2. $\mathcal{A}$ is contractible $\mathcal{N}_\varepsilon^{\bar{c}} = \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leq \bar{c}\}$ for some $\bar{c} > m_{V_0}^\infty + h(\varepsilon)$.

This would imply, since the Palais-Smale holds, the existence of a critical level between $m_{V_0}^\infty + h(\varepsilon)$ and $\bar{c}$.

Take $\mathcal{A} := \Phi_\varepsilon(M)$ which is not contractible in $\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}$. Let $t_\varepsilon(u) > 0$ the unique positive number such that $t_\varepsilon(u)u \in \mathcal{N}_\varepsilon$.

Choosing a function $u^* \in W_\varepsilon$ such that $u^* \geq 0, I_\varepsilon(t_\varepsilon(u^*)u^*) > m_{V_0}^\infty + h(\varepsilon)$ and considering the compact and contractible cone

$$\mathfrak{C} := \{tu^* + (1-t)u : t \in [0,1], u \in \mathcal{A}\},$$

we observe that, since the functions in $\mathfrak{C}$ have to be positive on a set of nonzero measure, it has to be $0 \notin \mathfrak{C}$.

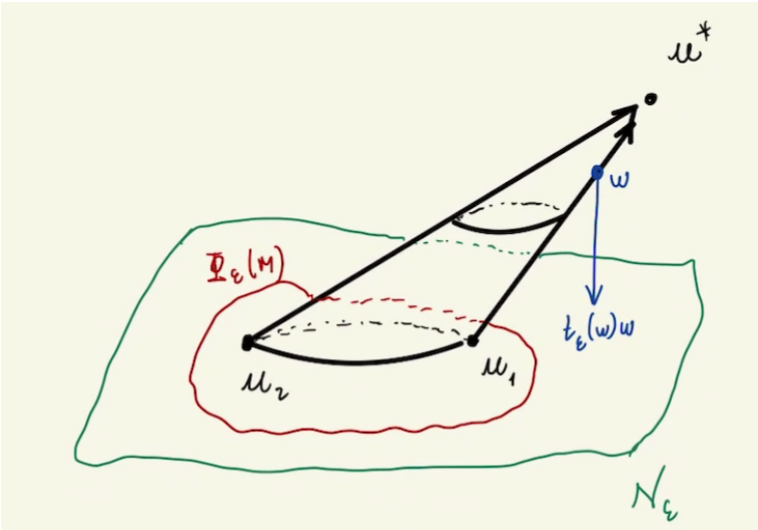

Figure 4.3: *A representation of* $\mathfrak{C}$.

Then let $t_\varepsilon(\mathfrak{C}) = \{t_\varepsilon(w)w : w \in \mathfrak{C}\} \subset \mathcal{N}_\varepsilon$ and

$$\bar{c} := \max_{t_\varepsilon(\mathfrak{C})} I_\varepsilon > m_{V_0}^\infty + h(\varepsilon).$$

It follows that $\mathcal{A} \subset t_\varepsilon(\mathfrak{C}) \subset \mathcal{N}_\varepsilon$ and $t_\varepsilon(\mathfrak{C})$ is contractible in $\mathcal{N}_\varepsilon^{\bar{c}}$.

Then there is a critical level for I_ε greater than $m_{V_0}^\infty + h(\varepsilon)$, hence different from the previous one.

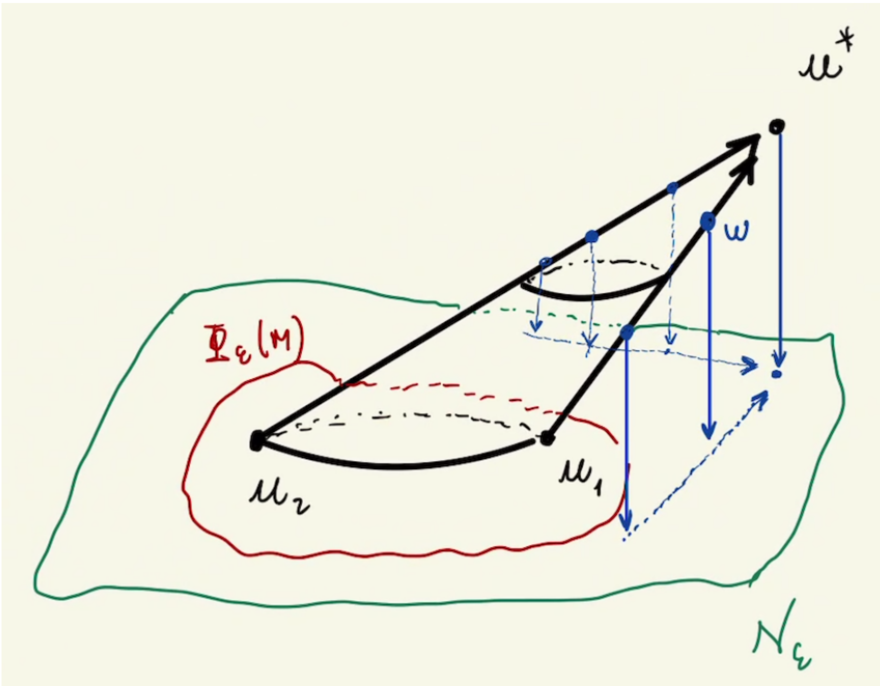

Figure 4.4: *A representaion of* $t_\varepsilon(\mathfrak{C})$.

Bibliography

[1] R. A. Adams and J. J. F. Fournier. *Sobolev Spaces.* Academic Press, 2008. 11

[2] D. G. Afonso and G. Siciliano. Normalized solutions to a Schrödinger-Bopp-Podolsky system under neumann boundary conditions. *to appear on Comm. Contemporary Math.* 30

[3] C. O. Alves and G. M. Figueiredo. Existence and multiplicity of positive solutions to a p-laplacian equation in $\mathbb{R}^n$. *Differential and Integral Equations*, 19:143–162, 2006. 49

[4] A. Ambrosetti, A. Malchiodi, et al. *Nonlinear analysis and semilinear elliptic problems*, volume 104. Cambridge University Press, 2007. 11, 17

[5] A. Ambrosetti and G. Prodi. *A primer of nonlinear analysis*, volume 1. Cambridge University Press, 1993. 3

[6] A. Ambrosetti and P. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Funct. Anal.*, 14:349–381, 1973. 15

[7] V. Benci and G. Cerami. The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems. *Arch. Rat. Mech. Anal.*, pages 79–93, 1991. 29

[8] V. Benci and G. Cerami. Multiple positive solutions of some elliptic problems via the morse theory and the domain topology. *Calculus of Variations and Partial Differential Equations*, pages 29–48, 1994. 29

[9] M. Born. Modified fileld equations with a finite radius of the electron. *Nature*, 132:282, 1933. 20

[10] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Springer, 2011. 3, 9

[11] P. C. C. C. O. Alves and E. S. Medeiros. Multiplicity of solutions for a class od quasilinear problem in exterior domains with neumann conditions. *Abstract and Applied Analisys*, 03:251–268, 2004. 49

[12] P. d'Avenia and G. Siciliano. Nonlinear Schrödinger equation in the Bopp-Podolsky electrodynamics: solutions in the eletrostatic case. *J. Differential Equations*, 267:1025–1065, 2019. 1, 24, 29, 30, 32, 34

[13] D.C.Clarck. A variant of the ljusternick-schnirelman theory and nonlinear eigenvalue problems. *Mathematical Annals*, 199:55–72, 1972. 15

[14] E. V. E. Di Nezza, G. Palatucii. Hitchhiker‘s guide to the fractional sobolev spaces. *Bull. Sci. Math.*, 136:512–573, 2012.

[15] G. Figueiredo and G. Siciliano. A multiplicity result via Ljusternick-Schnirelmann category and Morse theory for a fractional Schrödinger equation in $\mathbb{R}^{\mathbb{N}}$. *Nonlinear Differential Equations and Applications NoDEA*, 2016. 51, 54, 64

[16] G. M. Figueiredo and G. Siciliano. Multiple solutions for a Schrödinger-Bopp-Podolsky system with positive potentials. *Math. Nachrichten, to appear.* 30

[17] N. Ghoussoub. *Duality and pertubation methods in critical point theory.* Cambridge University Press, 1993. 17

[18] G. Leoni. *A first course in Sobolev spaces*, volume 105. Graduate studies in mathematics, 2009. 12

[19] P. L. Lions. The concentration-compactness principle in the Calculus of Variations. the locally compact case, part 2. *Annales de l'I. H. P., section C, tome 1*, 4:223–283, 1948.

[20] L. I. M. Born. Foundations of the new field theory. *Proc. R. Soc. Lond. Ser. A. Math. Phys. Eng. Sci.*, 144:425–451, 1934. 20

[21] G. Mie. Grundlagen einer theorie der materie. *Ann. Phys.*, 345:1–66, 1913. 20

[22] E. G. Murcia and G. Siciliano. Positive semiclassical states for a fractional Schrödinger-Poisson system. *Differential and Integral Equations*, 3-4:231–258, 2017. 34, 35, 43

[23] G. S. P. d'Avenia and M. Squassina. On fractional choquard equations. *Math. Models Appl. Sci.*, 25:521–573, 2012.

[24] B. Podolsky. A generalized electrodynamics. *Phys. Rev.*, pages 68–71, 1942. 25

[25] T. R. R. Abraham, J.E. Marsden. *Manifolds, Tensor Analysis, and Applications*, volume 75. Applied Mathematical Sciences, 1988.

[26] P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65. Reg. Conf. Ser. Math., 1986.

[27] G. Siciliano and K. Silva. The fibering method approach for a non-linear Schrödinger equation coupled with the electromagnetic field. *Publ. Mat.*, 64:373–390, 2020. 30

[28] G. C. V. Benci and D. Passaseo. On the number of the positive solutions of some nonlinear elliptic problems. *Synthesis Lectures on Mathematics and Statistics*, pages 93–107, 1991. 29, 68

[29] M. Willem. *Minimax Theorems.* Birkhäuser, 1996. 40