**Anosov Families: Structural Stability,
Invariant Manifolds and Entropy for
Non-Stationary Dynamical Systems**

Jeovanny de Jesus Muentes Acevedo

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Resumo

ACEVEDO, J. J. M. **Famílias Anosov: Estabilidade Estrutural, Variedades Invariantes e Entropia de Sistemas Dinâmicos Não-Estacionários**. 2017. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2017.

As famílias Anosov foram introduzidas por P. Arnoux e A. Fisher, motivados por generalizar a noção de difeomorfismo de Anosov. A grosso modo, as famílias Anosov são sequências de difeomorfismos $(f_i)_{i \in \mathbb{Z}}$ definidos em uma sequência de variedades Riemannianas compactas $(M_i)_{i \in \mathbb{Z}}$, em que $f_i : M_i \rightarrow M_{i+1}$ para todo $i \in \mathbb{Z}$, tal que a composição $f_{i+n} \circ \dots \circ f_i$, para $n \geq 1$, tem comportamento assintoticamente hiperbólico. Esta noção é conhecida como um sistema dinâmico não-estacionário ou um sistema dinâmico não-autônomo. Sejam $\mathbf{M}$ a união disjunta de cada M_i , para $i \in \mathbb{Z}$, e $\mathcal{F}^m(\mathbf{M})$ o conjunto consistente das famílias de difeomorfismos $(f_i)_{i \in \mathbb{Z}}$ de classe C^m definidos na sequência $(M_i)_{i \in \mathbb{Z}}$. O propósito principal deste trabalho é mostrar algumas propriedades das famílias Anosov. Em particular, mostraremos que o conjunto destas famílias é aberto em $\mathcal{F}^m(\mathbf{M})$, em que $\mathcal{F}^m(\mathbf{M})$ é munido da topologia forte (ou topologia Whitney); a estabilidade estrutural de certa classe de famílias Anosov, considerando conjugações topológicas uniformes; e várias versões para os Teoremas de variedades estáveis e instáveis. Os resultados que serão apresentados aqui generalizam alguns outros resultados obtidos em Sistemas Dinâmicos Aleatórios, os quais serão mencionados ao longo do trabalho. Além do anterior, será introduzida a entropia topológica para elementos em $\mathcal{F}^m(\mathbf{M})$ e mostraremos algumas das suas propriedades. Provaremos que esta entropia é contínua em $\mathcal{F}^m(\mathbf{M})$ munido da topologia forte. Porém, ela é descontínua em cada elemento de $\mathcal{F}^m(\mathbf{M})$ munido da topologia produto. Também apresentaremos um resultado que pode ser uma ferramenta de muita utilidade no estudo da continuidade da entropia topológica de difeomorfismos definidos em variedades compactas. Finalizaremos o trabalho dando uma lista de problemas que surgiram ao longo desta pesquisa e que serão analisados em um trabalho futuro.

Palavras-chave: Família Anosov, difeomorfismo de Anosov, sistemas dinâmicos não-estacionários, sistemas dinâmicos não-autônomos, sistemas dinâmicos aleatórios, entropia topológica, topologia forte.

Abstract

ACEVEDO, J. J. M. **Anosov Families: Structural Stability, Invariant Manifolds and Entropy for Non-Stationary Dynamical Systems**. 2017. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2017.

Anosov families were introduced by P. Arnoux and A. Fisher, motivated by generalizing the notion of Anosov diffeomorphisms. Roughly, Anosov families are sequences of diffeomorphisms $(f_i)_{i \in \mathbb{Z}}$ defined on a sequence of compact Riemannian manifolds $(M_i)_{i \in \mathbb{Z}}$, where $f_i : M_i \rightarrow M_{i+1}$ for all $i \in \mathbb{Z}$, such that the composition $f_{i+n} \circ \cdots \circ f_i$, for $n \geq 1$, has asymptotically hyperbolic behavior. This notion is known as a non-stationary dynamical system or a non-autonomous dynamical system. Let $\mathbf{M}$ be the disjoint union of each M_i , for each $i \in \mathbb{Z}$, and $\mathcal{F}^m(\mathbf{M})$ the set consisting of families of C^m -diffeomorphisms $(f_i)_{i \in \mathbb{Z}}$ defined on the sequence $(M_i)_{i \in \mathbb{Z}}$. The main goal of this work is to explore some properties of Anosov families. In particular, we will show that the set consisting of these families is open in $\mathcal{F}^m(\mathbf{M})$, where $\mathcal{F}^m(\mathbf{M})$ is endowed with the strong topology (or Whitney topology); the structural stability of a certain class of Anosov families, considering uniform topological conjugacies; and some versions of stable and unstable manifold theorems. The results that will be presented here generalize some results obtained in Random Dynamical Systems, which will be mentioned throughout the work. In addition to the above mentioned theorems, the topological entropy for elements in $\mathcal{F}^m(\mathbf{M})$ will be introduced, and we will show some of its properties. We will prove that this entropy is continuous on $\mathcal{F}^m(\mathbf{M})$ endowed with strong topology. However, it is discontinuous at each element of $\mathcal{F}^m(\mathbf{M})$ endowed with the product topology. We will also present a result that can be a very useful tool in the study of the continuity of the topological entropy of diffeomorphisms defined on compact manifolds. We will finish the work by giving a list of problems that have arisen throughout this research and that will be analyzed in a future work.

Keywords: Anosov family, Anosov diffeomorphism, non-stationary dynamical systems, non-autonomous dynamical systems, random dynamical systems, topological entropy, strong topology.

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List of Abbreviations

s.p.a (*Satisfies the property of angles*)
n.s.d.s. (*Non-stationary dynamical system*)

List of Symbols

$\mathbf{M}$	Disjoint union of the manifolds M_i
$\mathbf{d}$	The metric on $\mathbf{M}$
$\mathcal{F}^m(\mathbf{M})$	Set consisting of sequences of m -diffeomorphisms on $\mathbf{M}$
$D(f)_p$	Derivative of f at p
$\mathbb{Z}$	Set consisting of integer numbers
$\mathbb{R}$	Set consisting of real numbers
$\mathbb{N}$	Set consisting of natural numbers
$\mathbb{S}^1$	The circle
$\mathbb{T}^m$	The m -torus
$\mathbb{C}$	Set consisting of complex numbers
$\mathbb{Q}$	Set consisting of rational numbers
E_p^s	Stable subspace at p
E_p^u	Unstable subspace at p
TM	Tangent bundle of M
T_pM	Tangent space of M at p
$\mathcal{A}^m(\mathbf{M})$	The set consisting of C^m -Anosov families on $\mathbf{M}$
$\mathcal{A}_b^m(\mathbf{M})$	Set of Anosov families s.p.a. with bounded second derivative
$\mathcal{CF}^m(\mathbf{M})$	Set consisting of constant families
$\text{Diff}^m(M)$	Set consisting of C^m - diffeomorphisms on M
$d^m(\cdot, \cdot)$	C^m -metric on $C(X_1, X_2)$ induced by the Riemannian metric on X_2
$\mathbf{D}_i^m$	Set consisting of C^m -diffeomorphisms on M_i to M_{i+1}
$B^m(\phi, \tau)$	Ball in $\mathbf{D}_i^m$ with center ϕ and radius τ
$B(x, \varepsilon)$	Ball with center at the point x and radius ε
$B^m(\phi, (\epsilon_i)_{i \in \mathbb{Z}})$	strong basic neighborhood of $\mathbf{f}$
τ_{prod}	Product topology on $\mathcal{F}^m(\mathbf{M})$
τ_{str}	Strong topology on $\mathcal{F}^m(\mathbf{M})$
τ_{unif}	Uniform topology on $\mathcal{F}^m(\mathbf{M})$

$\mathcal{V}^s(x, \phi)$	Stable set for ϕ at x
$\mathcal{V}_\varepsilon^s(x, \phi)$	Local stable set for ϕ at x
$\mathcal{V}^u(x, \phi)$	Unstable set for ϕ at x
$\mathcal{V}_\varepsilon^u(x, \phi)$	Local stable set for ϕ at x
$\mathcal{N}^s(x, (\varepsilon_i)_{i \in \mathbb{Z}})$	Local stable set for families
$\mathcal{N}^u(x, (\varepsilon_i)_{i \in \mathbb{Z}})$	Local unstable set for families
$\mathbf{f}_\phi$	Constant family associated to ϕ
$N(\mathcal{A})$	Number of sets in a finite subcover of $\mathcal{A}$ with smallest cardinality
$H(\mathcal{A})$	$\log N(\mathcal{A})$
$\mathcal{A} \vee \mathcal{B}$	$\{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$
$\bigvee_{m=1}^k \mathcal{A}^m$	$\{A_1 \cap \dots \cap A_k : A_i \in \mathcal{A}_i\}$
$H_i(\mathbf{f}, \mathcal{A})$	$\lim_{n \rightarrow +\infty} \frac{1}{n} H\left(\bigvee_{k=0}^{n-1} (\mathbf{f}_i^k)^{-1}(\mathcal{A})\right)$
$H(\mathbf{f}, \mathcal{A})$	$(H_i(\mathbf{f}, \mathcal{A}))_{i \in \mathbb{Z}}$
$H(\mathbf{f})$	$(H_i(\mathbf{f}))_{i \in \mathbb{Z}}$
$H_i(\mathbf{f})$	$\sup\{H_i(\mathbf{f}, \mathcal{A}) : \mathcal{A} \text{ is an open cover of } M\}$
$r[n, i](\varepsilon, \mathbf{f})$	The smallest cardinality of any (n, ε) -span of M_i with respect to $\mathbf{f}$
$r[i](\varepsilon, \mathbf{f})$	$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log r[n, i](\varepsilon, \mathbf{f})$
$\mathbf{H}(\mathbf{f})$	$(\mathbf{H}_i(\mathbf{f}))_{i \in \mathbb{Z}}$
$\mathbf{H}_i(\mathbf{f})$	$\lim_{\varepsilon \rightarrow 0} r[i](\varepsilon, \mathbf{f})$ for each $i \in \mathbb{Z}$
$s[n, i](\varepsilon, \mathbf{f})$	The largest cardinality of any (n, ε) -separated subset of M_i with respect $\mathbf{f}$
$s[i](\varepsilon, \mathbf{f})$	$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log s[n, i](\varepsilon, \mathbf{f})$
$\mathcal{H}(\phi)$	Topological entropy of a single map ϕ
$\exp_p$	Exponential application at p
$D^r(\phi, \delta)$	Set consisting of diffeomorphisms ψ such that $d^r(\phi, \psi) \leq \delta$
$\mathcal{G}(\phi)$	Graph of an application ϕ
$SL(\mathbb{Z}, m)$	Special linear group of mxm matrices with integer entries
$\text{Lip}(\phi)$	Lipchitz constant of ϕ
$K_{\alpha, \mathbf{f}, p}^s$	Stable α -cone of $\mathbf{f}$ at p
$K_{\alpha, \mathbf{f}, p}^u$	Unstable α -cone of $\mathbf{f}$ at p
ϱ_p	Injectivity radius of $\exp_p$ at p
$\mathcal{U}(\mathbf{M}, \langle \cdot, \cdot \rangle)$	set consisting of Riemannian metric uniformly equivalent to $\langle \cdot, \cdot \rangle$ on $\mathbf{M}$

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Introduction

Anosov families, which will be defined in Chapter 4, were introduced by P. Arnoux and A. Fisher in [AF05], motivated by generalizing the notion of Anosov diffeomorphisms. Roughly, an Anosov family is a two-sided sequence of diffeomorphisms $f_i : M_i \rightarrow M_{i+1}$ defined on a two sided sequence of compact Riemannian manifolds M_i , for $i \in \mathbb{Z}$, having similar behavior to an Anosov diffeomorphism: there is a splitting of the tangent bundle $TM_i = E_i^s \oplus E_i^u$, invariant by the derivative $D(f_i)$ (that is, $D(f_i)(E_i^s) = E_{i+1}^s$ and $D(f_i)(E_i^u) = E_{i+1}^u$ for any $i \in \mathbb{Z}$), and there exist constants $\lambda \in (0, 1)$ and $c > 0$ such that for $n \geq 1$, $p \in M_i$, we have:

$$\|D(f_{i+n-1} \circ \dots \circ f_i)_p(v)\| \leq c\lambda^n \|v\| \quad \text{for each vector } v \in E_p^s$$

and

$$\|D(f_{i-n}^{-1} \circ \dots \circ f_{i-1}^{-1})_p(v)\| \leq c\lambda^n \|v\| \quad \text{for each vector } v \in E_p^u.$$

The subspaces E_p^s and E_p^u are called the stable and unstable subspaces at p , respectively.

The main goal of this work is to verify some properties of Anosov families which are satisfied by Anosov diffeomorphisms (openness, structural stability and the existence of stable and unstable manifolds). On the other hand, a notion of topological entropy can be defined for sequences of diffeomorphisms. We will examine the continuity of this entropy at each sequence defined on a compact Riemannian manifold (see Chapter 2).

Time-dependent dynamical systems are known as *non-stationary dynamical systems*, *non-autonomous dynamical systems*, *sequence of maps*, among others names (see [KL16], [KS96], [KMS99], [SSZ16], [ZC09], [ZZH06]). We will use these names throughout this thesis. Some results in the case in which these kinds of systems have hyperbolic behavior can be found in [Ste11], [Bak95a] and [Bak95b]. Another important approach (and to which the results obtained in this thesis can be applied) is when the maps f_i are small random perturbations of a fixed map. This represents a specific type of random dynamical systems (see [Arn13], [Bog92], [Liu98], [LQ06], [You86]).

One difference between the notion to be considered in this thesis and the considered in the above mentioned works is that the Anosov families are not necessarily sequences of Anosov diffeomorphisms (see [AF05], Example 3). Furthermore, each M_i , although they are diffeomorphic, could have different Riemannian structures and therefore the hyperbolicity of the sequence $(f_i)_{i \in \mathbb{Z}}$ could be influenced by the Riemannian metric (see Example 3.2.1).

Other interesting class of examples to consider are the flow families given by *non-autonomous differential equations*, where the orbits are integral curves of time-varying vector fields (see [KR11]) as well as many examples of *random dynamical systems* (see [Liu98], [Arn13], among other works).

In this thesis we will also give examples obtained from *skew product transformations* or *linear cocycles* (see Definition 3.2.4 and Example 3.2.5), which are a type of random dynamical systems.

Let M be a compact Riemannian manifold and $\text{Diff}^1(M)$ be the set consisting of diffeomorphisms defined on M , endowed with the C^1 topology (see [Hir12]). The set $\mathcal{A}(M)$ consisting of Anosov diffeomorphisms on M is open in $\text{Diff}^1(M)$, that is, for any Anosov diffeomorphism $\phi : M \rightarrow M$, there exists an open set of diffeomorphisms $\mathcal{O} \subseteq \text{Diff}^1(M)$ such that $\phi \in \mathcal{O} \subseteq \mathcal{A}(M)$ (see [Shu13]). Furthermore, it is possible to take the set $\mathcal{O}$ such that for any $\psi \in \mathcal{O}$ there exists a homeomorphism $h : M \rightarrow M$ (depending on ψ), such that $\phi \circ h = h \circ \psi$, that is, ϕ and ψ are *conjugate* (this fact was proved by D. Anosov in [Ano67]). We will obtain analogous versions of these facts for Anosov families. Let us to talk a few about these results. Consider

$$\mathbf{M} = \coprod_{i \in \mathbb{Z}} M_i = \bigcup_{i \in \mathbb{Z}} M_i \times i.$$

The M_i 's will be called the *components* of $\mathbf{M}$, the *total space*. In (1.1.1) we will give a metric for $\mathbf{M}$. For $m \geq 1$, set

$$\mathcal{F}^m(\mathbf{M}) = \{(f_i)_{i \in \mathbb{Z}} : f_i : M_i \rightarrow M_{i+1} \text{ is a } C^m\text{-diffeomorphism for each } i\}.$$

We consider three different topologies on $\mathcal{F}^m(\mathbf{M})$: the *compact topology*, *uniform topology* and the *strong topology* (or *Whitney topology*) (see Definitions 1.3.2, 1.3.3 and 1.3.4).

In Theorem 4.3.5 we prove that the collection of C^1 -Anosov families, denoted by $\mathcal{A}^1(\mathbf{M})$, is open in $\mathcal{F}^1(\mathbf{M})$ endowed with the strong topology. As we said above, the set consisting of Anosov diffeomorphisms on a compact Riemannian manifold is open. Theorem 4.3.5 is an analogue of this fact for Anosov families. The most important implication of this result is the great variety of non-trivial examples that it provides (we will show many non-trivial examples of Anosov families in Section 3.2, thus Theorem 4.3.5 proves that, in a certain way, these examples are not isolated), since we only ask that the family be Anosov and we do not ask for any additional condition. This author has submitted a paper titled “Openness of Anosov families”, which contains the mentioned above result, to the *Journal of the Korean Mathematical Society* (see [Ace17d]). This work has been accepted for publication.

Young in [You86] proved that families consisting of C^{1+1} random small perturbations of an Anosov diffeomorphism of class C^2 are *uniformly hyperbolic sequences*, that is, are Anosov families (see Remark 3.2.7). Let $\mathcal{A}_b^2(\mathbf{M})$ be the set consisting of C^2 Anosov families whose second derivative is bounded and such that the angles between the unstable and stable subspaces are bounded away from 0 (see (6.0.1) and (3.1.4)). In Section 6.1 we will show that for any $\mathbf{f} = (f_i)_{i \in \mathbb{Z}} \in \mathcal{A}_b^2(\mathbf{M})$ there exists $\delta > 0$ such that if $\mathbf{g} = (g_i)_{i \in \mathbb{Z}}$ with $d^1(f_i, g_i) < \delta$ for all $i \in \mathbb{Z}$, then $\mathbf{g}$ is an Anosov family. That is, $\mathcal{A}_b^2(\mathbf{M})$ is open in $\mathcal{F}^2(\mathbf{M})$ endowed with the uniform topology. This is a generalization of the Young's result, since Anosov families are not necessarily sequences of (small perturbations of) Anosov diffeomorphisms.

Non-stationary dynamical systems are classified by uniform conjugacy, which is defined in Definition 1.2.4. Structural stability of non-stationary dynamical systems will be stated in Definition 1.3.5. In Theorem 6.3.9 we prove that all elements of $\mathcal{A}_b^2(\mathbf{M})$ are structurally stable in $\mathcal{F}^2(\mathbf{M})$

endowed with the uniform topology. This result is a generalization of Theorem 1.1 in [Liu98], which proves the structural stability of random small perturbations of hyperbolic diffeomorphisms. This author wrote a paper, which contains these results, titled “Structural stability of Anosov families” and has been submitted to a journal (see [Ace17e]).

Another approach on the stability of non-stationary hyperbolic dynamical systems can be found in [CRV17] and [Fra74].

Let $\phi : M \rightarrow M$ be an Anosov diffeomorphism. Hirsch and Pugh in [HP70] proved that there exists $\varepsilon > 0$ such that, for each $x \in M$, the stable and unstable sets at x (see Definition 1.4.2), which will be denoted by $\mathcal{V}_\varepsilon^s(x, \phi)$ and $\mathcal{V}_\varepsilon^u(x, \phi)$ respectively, are differentiable submanifolds of M . Furthermore we have that $T_x \mathcal{V}_\varepsilon^s(x, \phi) = E_x^s$ and $T_x \mathcal{V}_\varepsilon^u(x, \phi) = E_x^u$, that is, $\mathcal{V}_\varepsilon^s(x, \phi)$ and $\mathcal{V}_\varepsilon^u(x, \phi)$ are tangent to the stable and unstable subspaces at x , respectively. ϕ is a contraction on $\mathcal{V}_\varepsilon^s(x, \phi)$ and ϕ^{-1} is a contraction on $\mathcal{V}_\varepsilon^u(x, \phi)$. For Anosov families, Example 3.2.2 proves that the above properties are not always valid. In Definition 5.1.4 we will give a notion of stable and unstable sets which works better for non-stationary dynamical systems than the sets given in Definition 1.4.2. We will prove that, with some conditions on the family (see (5.2.2)), these subsets are differentiable.

In [Pes76], Pesin proved the existence of invariant manifolds for diffeomorphisms of a compact smooth manifold onto a set where at least one Lyapunov characteristic exponent is nonzero (see [BP07], [KH97], [Via14]). That theory, which is well-known as “Pesin’s Theory”, has been used to show the existence of families of invariant manifolds for sequences of random diffeomorphisms defined on a compact manifold (see [Arn13], [LQ06], [QQX03]). These results are *probabilistic*: the invariant manifolds exist at almost every point in the support of a chosen invariant measure. Results to be obtained in this thesis are *deterministic*: we have an Anosov family $(f_i)_{i \in \mathbb{Z}}$, where each f_i is fixed, and we give conditions (which depend on the derivative of each f_i) to obtain invariant manifolds along the orbit of a given point $p \in M_0$. Pesin’s Theory has been used to build invariant manifolds for a *two-sided sequence of non-uniformly hyperbolic sequences of diffeomorphisms* (see Theorem 7.3.9 in [BP07] or Theorem 6.2.8 in [KH97]). This is also known as *The Hadamard-Perron Theorem*. In Proposition 5.2.5 we will show a generalization of the Hadamard-Perron Theorem. The essence of the proof of our result is the same as that given in [BP07] and [KH97], except that we have weakened the hypotheses (see Remark 5.2.6). We have written an article titled “Local stable and unstable invariant manifolds for Anosov families”, containing the above mentioned facts, submitted for publication (see [Ace17b]).

Topological entropy is a non-negative real number (possibly $+\infty$) associated to a dynamical system. It was introduced by R. L. Adler, A.G. Konheim and M. H. McAndrew in [AKM65]. It is a good tool to classify dynamical systems, since it is invariant with respect to topological conjugacy. Let us recall now some known results on the continuity of topological entropy (see [AKM65], [Wal00]). In [New89], Newhouse proved that the topological entropy of C^∞ -diffeomorphisms on a compact Riemannian manifold is an upper semicontinuous map. Furthermore, if M is a surface, this map is continuous. The entropy for any homeomorphism of the circle $\mathbb{S}^1$ is zero. Therefore, it depends continuously on homeomorphisms of $\mathbb{S}^1$. In contrast, if we consider all the continuous maps defined on $\mathbb{S}^1$, this entropy is not a continuous map (see [Yan80]). It is clear that the entropy is continuous at each structurally stable diffeomorphism (a diffeomorphism $\phi : M \rightarrow M$ is *structurally stable* if there exist an open neighborhood $\mathcal{O}$ of ϕ such that all the elements in $\mathcal{O}$ are topologically

conjugate to ϕ). In Remark 2.3.5 we will demonstrate an interesting observation which could be an useful tool to prove the continuity of the topological entropy of a single diffeomorphism.

Kolyada and Snoha in [KS96] introduced a notion of topological entropy for non-stationary dynamical systems (see Section 3.1). In [KL16], [KMS99], [Kus67], [SSZ16], [ZC09], [ZZH06], one can find some properties, estimations, formulas and bounds on the topological entropy for non-stationary dynamical systems. We will prove that this entropy depends continuously on each element of $\mathcal{F}^1(\mathbf{M})$ endowed with the strong topology (see Theorem 2.4.5). In contrast, if we consider the product topology on $\mathcal{F}^1(\mathbf{M})$, then the entropy is discontinuous at each sequence (see Proposition 2.3.1). Other results on the continuity of the entropy on $\mathcal{F}^1(\mathbf{M})$ with respect to the uniform topology will be given in Proposition 2.3.4. In summary, we will give properties of the continuity of the entropy considering three different topologies on $\mathcal{F}^1(\mathbf{M})$ (see Remark 2.4.6). These results were published by the author in the *Bulletin of the Brazilian Mathematical Society, New Series*, in an article titled “On the continuity of the topological entropy of non-autonomous dynamical systems” (see [Ace17c]).

Some results about the metric entropy for random dynamical systems can be found in [Bog92], [LY88], [QQX03] and [Rue97a]).

Next, we will describe the structure of this work.

In Chapter 1 we will introduce the class of objects to be studied in this thesis. We define the law of composition for a non-stationary dynamical system, the strong, uniform and product topologies for the set consisting of families of diffeomorphisms and uniform conjugacy, which works properly for classify the non-stationary dynamical systems. We will finish this chapter by giving some properties preserved by uniform conjugacy.

Chapter 2 will be devoted to examining the topological entropy for non-stationary dynamical systems. In this case, we will take $M_i = M \times \{i\}$ for each $i \in \mathbb{Z}$, where M is a compact Riemannian manifold. This entropy will be built via open partitions of M (see Definition 2.1.3) and also via separated and spanning sets (see Definitions 2.1.5 and 2.1.6). These definitions coincide, as in the case of single maps (see Proposition 2.2.1). This fact can be proved similarly to the case of a single map. Some properties of this entropy will be given in Section 3.2. These properties generalize those for the entropy of a single map. One of the most important properties to be shown is that this entropy is an invariant by uniform conjugacy (see Theorem 2.2.5). In Section 3.3 we will see that this entropy is discontinuous at any sequence if we consider the product topology on $\mathcal{F}^m(\mathbf{M})$. In contrast, it is continuous on $\mathcal{F}^m(\mathbf{M})$ endowed with the strong topology if $m \geq 1$.

In Chapter 3 we will introduce the notion of Anosov family and we show some examples and properties of such families. It is important to keep fixed the Riemannian metric on each component, since the notion of Anosov family depends on the metric defined on each M_i (see Example 3.2.1). However, Proposition 3.3.1 proves that this notion does not depend on uniformly equivalent metrics defined on the total space (see Definition 1.1.2). Proposition 3.3.4 shows an analogous version of the Lemma of Mather adapted to Anosov families (see [Shu13]). The Lemma of Mather for Anosov diffeomorphisms on a compact manifold consists of constructing a Riemannian metric on the manifold such that, with this metric, the expansion (contraction) of the unstable (stable) subspaces by the derivative of the diffeomorphism is seen after only one iteration. By compactness of the manifold, this metric is uniformly equivalent to the Riemannian metric that was considered a priori.

This metric, obtained in Proposition 3.3.4, is not necessarily uniformly equivalent to the original metric on $\mathbf{M}$; the total space $\mathbf{M}$ is never compact. The uniform equivalence depends on the angles between the stable and unstable subspaces of the splitting of the tangent bundle on each component (see Corollary 3.3.5). In the case of Anosov diffeomorphisms defined on compact manifolds those angles are uniformly bounded away from zero. In the case of families, those angles may decrease arbitrarily.

Openness for the set consisting of Anosov families in $\mathcal{F}^1(\mathbf{M})$ with the strong topology will be proved in Chapter 4 (see Theorem 4.3.5). We will show that if $(f_i)_{i \in \mathbb{Z}}$ is an Anosov family, then there exists a two-sided sequence of positive numbers $(\delta_i)_{i \in \mathbb{Z}}$ such that if $(g_i)_{i \in \mathbb{Z}} \in \mathcal{F}^1(\mathbf{M})$ and $d^1(f_i, g_i) < \delta_i$, then $(g_i)_{i \in \mathbb{Z}}$ is an Anosov family. In that case, it is not always possible to take $(\delta_i)_{i \in \mathbb{Z}}$ bounded away from zero. In Sections 4.4 and 6.1 we will see that, with some conditions on the norm of the second derivative of the sequence and the angles between the stable and unstable subspaces, then the sequence $(\delta_i)_{i \in \mathbb{Z}}$ of the neighborhood can be taken bounded away from zero, that is, a neighborhood in the uniform topology. In order to prove this openness, we will use the method of *invariant cones* (see [BP07], [KH97]). First we show the particular case in which the angles between the stable and unstable subspaces are uniformly bounded away from zero and then, in the final of Section 4.3, we consider the general case.

The existence of stable and unstable manifolds for Anosov families will be examined in Chapter 5. We show in Theorems 5.2.10 and 5.2.11 a generalized version of the Hadamard-Perron Theorem. In our case, stable and unstable subspaces of an Anosov family are not necessarily orthogonal. We prove that, with some conditions, there exists a family of submanifolds invariant by the derivative of the family and show that we can control the expansion or contraction of the submanifolds by the family. The expansion or contraction of these submanifolds depends also on the angle between the stable and unstable subspaces (see (5.2.12)). In Section 5.4 we will obtain the unstable and stable manifold theorems for Anosov family (the Theorems 5.3.5 and 5.3.6). In the Lemmas 5.3.2 and 5.3.3 we give conditions with which the submanifolds obtained in the Theorems 5.2.10 and 5.2.11 coincide with the stable and unstable subsets for an Anosov family.

In Chapter 6 we will prove that $\mathcal{A}_b^2(\mathbf{M})$ is *uniformly structurally stable* in $\mathcal{F}^2(\mathbf{M})$, that is, for any $\mathbf{f} = (f_i)_{i \in \mathbb{Z}} \in \mathcal{A}_b^2(\mathbf{M})$ there exists $\delta > 0$ such that, if $g_i : M_i \rightarrow M_{i+1}$ is a C^2 -diffeomorphism and $d^2(f_i, g_i) < \delta$ for each $i \in \mathbb{Z}$, then $(g_i)_{i \in \mathbb{Z}}$ is an Anosov family and is conjugate to $\mathbf{f}$ (see Theorem 6.3.9). In Section 6.1 we will show that $\mathcal{A}_b^2(\mathbf{M})$ is open in $\mathcal{F}^2(\mathbf{M})$ endowed with the uniform topology: we have that for any $(f_i)_{i \in \mathbb{Z}} \in \mathcal{A}_b^2(\mathbf{M})$ there exists $\delta > 0$ such that, if $g_i : M_i \rightarrow M_{i+1}$ is a C^2 -diffeomorphism and $d^2(f_i, g_i) < \delta$ for each $i \in \mathbb{Z}$, then $(g_i)_{i \in \mathbb{Z}}$ is an Anosov family satisfying the property of angles, that is, the basic neighborhood can be taken uniform (see Definition 4.4.1). In Section 6.2 we will prove that each element in $\mathcal{A}_b^2(\mathbf{M})$ admits stable and unstable manifolds.

We will finish this thesis in Chapter 7, where we will leave a list of problems that arose throughout this study and that will be analyzed in a future work.

Chapter 1

Non-Stationary Dynamical Systems, Uniform Conjugacy and Strong Topology

In this chapter we will introduce the objects to be studied in this work. We review some well-known notions from Dynamical Systems, General Topology, Riemannian Geometry and Differential Topology. For readers who wish to know more about these topics, the author recommends, for instances, the texts [dC92], [Eng89], [Hir12], [KH97] and [Shu13].

1.1 Non-Stationary Dynamical Systems

Given a sequence of compact metric spaces M_i , we will consider the *disjoint union*

$$\mathbf{M} = \coprod_{i \in \mathbb{Z}} M_i = \bigcup_{i \in \mathbb{Z}} M_i \times \{i\}.$$

The set $\mathbf{M}$ will be called *total space* and the M_i will be called *components of $\mathbf{M}$* . If $d_i(\cdot, \cdot)$ is the metric on M_i , the total space is endowed with the metric

$$\mathbf{d}(x, y) = \begin{cases} \min\{1, d_i(x, y)\} & \text{if } x, y \in M_i \\ 1 & \text{if } x \in M_i, y \in M_j \text{ and } i \neq j. \end{cases} \quad (1.1.1)$$

We sometimes use the notation $(\mathbf{M}, \mathbf{d})$ to indicate we are considering the metric $\mathbf{d}$ given in (1.1.1).

Definition 1.1.1. Two metrics $\hat{d}$ and $\tilde{d}$ on a topological space M are *uniformly equivalent* if there exist positive numbers k and K such that

$$k\tilde{d}(x, y) \leq \hat{d}(x, y) \leq K\tilde{d}(x, y) \quad \text{for all } x, y \in M.$$

It is clear that if $\hat{d}$ and $\tilde{d}$ are uniformly equivalent metrics on M , then, for $\hat{\mathbf{d}}$ and $\tilde{\mathbf{d}}$ defined on $\mathbf{M}$, the disjoint union of each $M_i = M \times \{i\}$ for $i \in \mathbb{Z}$, obtained as in (1.1.1), generate the same topology on $\mathbf{M}$ and, furthermore, they are uniformly equivalent on $\mathbf{M}$. On the other hand, if $\hat{d}_i$ and $\tilde{d}_i$ are uniformly equivalent metrics on M_i for each $i \in \mathbb{Z}$, then the metrics $\hat{\mathbf{d}}$ and $\tilde{\mathbf{d}}$, defined as in (1.1.1), generate the same topology on the total space, but they are not necessarily uniformly equivalent on $\mathbf{M}$ (since $\mathbf{M}$ is not compact).

If M_i is a compact Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle_i$ for $i \in \mathbb{Z}$, we endow the total space $\mathbf{M}$ with the Riemannian metric $\langle \cdot, \cdot \rangle$ induced by $\langle \cdot, \cdot \rangle_i$ setting

$$\langle \cdot, \cdot \rangle|_{M_i} = \langle \cdot, \cdot \rangle_i \quad \text{for } i \in \mathbb{Z}, \quad (1.1.2)$$

and we will use the notation $(\mathbf{M}, \langle \cdot, \cdot \rangle)$ to indicate that we are considering the Riemannian metric given in (1.1.2). In that case, on each M_i , we will consider the metric d_i induced by $\langle \cdot, \cdot \rangle_i$.

Definition 1.1.2. Let $\langle \cdot, \cdot \rangle_i$ and $\langle \cdot, \cdot \rangle_i^*$ be Riemannian metrics on M_i and let $\| \cdot \|_i$ and $\| \cdot \|_i^*$ be the Riemannian norms induced by $\langle \cdot, \cdot \rangle_i$ and $\langle \cdot, \cdot \rangle_i^*$, respectively. We say that $\langle \cdot, \cdot \rangle_i$ and $\langle \cdot, \cdot \rangle_i^*$ (or that $\| \cdot \|_i$ and $\| \cdot \|_i^*$) are *uniformly equivalent on M_i* if there exist positive numbers k_i and K_i such that

$$k_i \|v\|_i^* \leq \|v\|_i \leq K_i \|v\|_i^* \quad \text{for all } v \in TM_i,$$

where TM_i is the tangent bundle of M_i . If there exist k and K such that

$$k \|v\|_i^* \leq \|v\|_i \leq K \|v\|_i^* \quad \text{for all } v \in TM_i, i \in \mathbb{Z},$$

that is, k and K does not depend on i , then we say that $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^*$ are *uniformly equivalent on $\mathbf{M}$* , where $\langle \cdot, \cdot \rangle^*$ is obtained similarly as in (1.1.2) with the Riemannian metrics $\langle \cdot, \cdot \rangle_i^*$.

Let $\hat{d}_i$ and $\tilde{d}_i$ be the metrics induced by $\langle \cdot, \cdot \rangle_i$ and $\langle \cdot, \cdot \rangle_i^*$, respectively, and $\hat{\mathbf{d}}$ and $\tilde{\mathbf{d}}$ defined on $\mathbf{M}$ from $\hat{d}_i$ and $\tilde{d}_i$ as in (1.1.1). These metrics are not necessarily uniformly equivalent on $\mathbf{M}$, since, for instance, k_i could converge to zero or K_i could converge to $+\infty$ when $i \rightarrow \pm\infty$.

Definition 1.1.3. A *non-stationary dynamical system* (or *n.s.d.s.*) $\mathbf{f}$ on $\mathbf{M}$ is a map $\mathbf{f} : \mathbf{M} \rightarrow \mathbf{M}$, such that, for each $i \in \mathbb{Z}$, $\mathbf{f}|_{M_i} = f_i : M_i \rightarrow M_{i+1}$ is a homeomorphism. Sometimes we use the notation $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$. The n -th composition is defined, for $i \in \mathbb{Z}$, to be

$$\mathbf{f}_i^n := \begin{cases} f_{i+n-1} \circ \cdots \circ f_i : M_i \rightarrow M_{i+n} & \text{if } n > 0 \\ f_{i-n}^{-1} \circ \cdots \circ f_{i-1}^{-1} : M_i \rightarrow M_{i-n} & \text{if } n < 0 \\ I_i : M_i \rightarrow M_i & \text{if } n = 0, \end{cases}$$

where I_i is the identity on M_i .

We will use the notation $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ for a n.s.d.s. $\mathbf{f}$ defined on $\mathbf{M}$ endowed with the Riemannian metric $\langle \cdot, \cdot \rangle$.

Remark 1.1.4. The notion above can be found in the literature under several different names: *non-autonomous dynamical systems*, *non-autonomous discrete systems*, *sequence of maps*, *time dependent diffeomorphisms* and *mapping families* (see [AF05], [ZC09], [KR11], [Fra74], [KS96], [SSZ16], [Ste11], among others).

Since f_i is a homeomorphism, the components M_i are homeomorphic metric spaces, however, they are not the “same object” (see Figure 1.1.1). For instance, M_i can be the same Riemannian manifold, and the metrics $\langle \cdot, \cdot \rangle_i$ can change with i , or the M_i can be the same surfaces with different *fractal structures*, *Thurston corrugations*, etc. (see [BJLT12]).

Figure 1.1.1: A n.s.d.s. on a sequence of 2-torus endowed with different Riemannian metrics.

A simple example of a non-stationary dynamical system is the *constant family* associated to a homeomorphism:

Example 1.1.5. Let $\phi : M \rightarrow M$ be a homeomorphism defined on a compact Riemannian manifold M with metric $\langle \cdot, \cdot \rangle$. Take $\mathbf{M} = \coprod M_i$, where $M_i = M \times \{i\}$ for each $i \in \mathbb{Z}$ with the same metric $\langle \cdot, \cdot \rangle$. We define the *constant family* $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ associated to ϕ as the n.s.d.s. $(f_i)_{i \in \mathbb{Z}}$, where $f_i : M_i \rightarrow M_{i+1}$ is defined by $f_i(x, i) = (\phi(x), i + 1)$ for each $x \in M, i \in \mathbb{Z}$.

Another non-stationary dynamical system we consider in this thesis is obtained from a given family in the following way.

Definition 1.1.6. Let $\mathbf{f}$ and $\tilde{\mathbf{f}}$ be non-stationary dynamical systems on $\mathbf{M}$ and $\tilde{\mathbf{M}}$, respectively. We say that $\tilde{\mathbf{f}}$ is a *gathering* of $\mathbf{f}$ if there exists a strictly increasing sequence of integers $(n_i)_{i \in \mathbb{Z}}$ such that $\tilde{M}_i = M_{n_i}$ and $\tilde{f}_i = f_{n_{i+1}-1} \circ \cdots \circ f_{n_i+1} \circ f_{n_i}$:

$$\cdots M_{n_{i-1}} \xrightarrow{\tilde{f}_{i-1} = f_{n_i-1} \circ \cdots \circ f_{n_{i-1}}} M_{n_i} \xrightarrow{\tilde{f}_i = f_{n_{i+1}-1} \circ \cdots \circ f_{n_i}} M_{n_{i+1}} \cdots$$

If $\tilde{\mathbf{f}}$ is a gathering of $\mathbf{f}$, we say that $\mathbf{f}$ is a *dispersal* of $\tilde{\mathbf{f}}$.

In [AF05], Proposition 2.5, it is proved that any non-stationary dynamical system has a dispersal, which has a gathering, which is equal to the constant family associated to the identity on M_0 .

1.2 Uniform Conjugacy Between Non-Stationary Dynamical Systems

In this section we will talk about the morphisms between non-stationary dynamical systems.

Definition 1.2.1. Two continuous maps $\phi_1 : X_1 \rightarrow X_1$ and $\phi_2 : X_2 \rightarrow X_2$, where X_1 and X_2 are topological spaces, are *topologically conjugate* if there exists a homeomorphism $h : X_1 \rightarrow X_2$ such that $h \circ \phi_1 = \phi_2 \circ h$. In that case, h is called a *topological conjugacy between ϕ_1 and ϕ_2* .

Take

$$\mathbf{N} = \coprod_{i \in \mathbb{Z}} N_i,$$

where N_i is a metric space with metric fixed $\hat{d}_i$. Let $\hat{\mathbf{d}}$ be the metric on $\mathbf{N}$ defined as in (1.1.1) with the $\hat{d}_i$'s. Throughout this chapter, $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ and $\mathbf{g} = (g_i)_{i \in \mathbb{Z}}$ will denote a non-stationary dynamical system defined on $(\mathbf{M}, \mathbf{d})$ and $(\mathbf{N}, \hat{\mathbf{d}})$, respectively.

Definition 1.2.2. A *topological conjugacy* between $\mathbf{f}$ and $\mathbf{g}$ is a map $\mathbf{h} : \mathbf{M} \rightarrow \mathbf{N}$, such that, for each $i \in \mathbb{Z}$, $\mathbf{h}|_{M_i} = h_i : M_i \rightarrow N_i$ is a homeomorphism and

$$h_{i+1} \circ f_i = g_i \circ h_i,$$

that is, the following diagram commutes:

$$\begin{array}{ccccccc} M_{-1} & \xrightarrow{f_{-1}} & M_0 & \xrightarrow{f_0} & M_1 & \xrightarrow{f_1} & M_2 \\ \cdots \downarrow h_{-1} & & \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 \cdots \\ N_{-1} & \xrightarrow{g_{-1}} & N_0 & \xrightarrow{g_0} & N_1 & \xrightarrow{g_1} & N_2 \end{array}$$

It is clear that the topological conjugacies define an equivalence relation on the set consisting of the non-stationary dynamical systems on $\mathbf{M}$. However:

Lemma 1.2.3. *If M_0 and N_0 are homeomorphic then there exists a topological conjugacy between $\mathbf{f}$ and $\mathbf{g}$.*

Proof. Let h_0 be a homeomorphism between M_0 and N_0 . It is clear that the map $\mathbf{h} : \mathbf{M} \rightarrow \mathbf{N}$ defined by

$$h_i = \begin{cases} h_0 & \text{if } i = 0 \\ g_{i-1} \circ \cdots \circ g_0 \circ h_0 \circ f_0^{-1} \circ \cdots \circ f_{i-1}^{-1} & \text{if } i > 0 \\ g_i^{-1} \circ \cdots \circ g_{-1}^{-1} \circ h_0 \circ f_{-1} \circ \cdots \circ f_i & \text{if } i < 0, \end{cases}$$

is a conjugacy between $\mathbf{f}$ and $\mathbf{g}$. □

One type of conjugacy that works for the class of non-stationary dynamical systems is *uniform topological conjugacy*:

Definition 1.2.4. We say that a topological conjugacy $\mathbf{h} : \mathbf{M} \rightarrow \mathbf{N}$ between $\mathbf{f}$ and $\mathbf{g}$ is *uniform* if $(h_i : M_i \rightarrow N_i)_{i \in \mathbb{Z}}$ and $(h_i^{-1} : N_i \rightarrow M_i)_{i \in \mathbb{Z}}$ are equicontinuous families (that is, $\mathbf{h}$ and $\mathbf{h}^{-1}$ are uniformly continuous maps). In that case we will say that the families are *uniformly conjugate*.

Since the composition of uniformly continuous functions is uniformly continuous, the class consisting of non-stationary dynamical systems becomes a category, where the objects are the non-stationary dynamical systems and the morphisms are uniform conjugacies.

Another possible definition of conjugacy for non-stationary dynamical systems is the following:

Definition 1.2.5. A *positive (negative) uniform conjugacy* between $\mathbf{f}$ and $\mathbf{g}$ is a sequence of homeomorphisms $h_i : M_i \rightarrow N_i$ for $i \geq 0$ (for $i \leq 0$) such that $(h_i)_{i \geq 0}$ and $(h_i^{-1})_{i \geq 0}$ ($(h_i)_{i \leq 0}$ and $(h_i^{-1})_{i \leq 0}$) are equicontinuous and

$$h_{i+1} \circ f_i = g_i \circ h_i : M_i \rightarrow N_{i+1}, \quad \text{for every } i \geq 0 \text{ (for every } i \leq -1).$$

That is, $(f_i)_{i \geq 0}$ and $(g_i)_{i \geq 0}$ ($(f_i)_{i \leq 0}$ and $(g_i)_{i \leq 0}$) are uniformly conjugate.

It is clear that the conjugacy given in Definition 1.2.5 is weaker than the conjugacy given in Definition 1.2.4.

Dynamical systems are classified by topological conjugacy. Uniform topological conjugacies are very suitable for classifying non-stationary dynamical systems, random dynamical systems, discrete time process generated by non-autonomous differential equation, among others systems (see [AF05], [KR11], [Liu98], and [Arn13] for more details).

1.3 Compact and Strong Topologies

Let X_1 and X_2 be compact Riemannian manifolds. For $r = 1, 2$, let $\|\cdot\|_r$ be the Riemannian norm on X_r and $\text{dist}_r(\cdot, \cdot)$ the metric induced by $\|\cdot\|_r$ on X_r . Consider two homeomorphisms $\phi : X_1 \rightarrow X_2$ and $\psi : X_1 \rightarrow X_2$. The d^0 metric induced by $\text{dist}(\cdot, \cdot)$ on

$$\text{Hom}(X_1, X_2) = \{h : X_1 \rightarrow X_2 : h \text{ is a homeomorphism}\}$$

is given by

$$d^0(\phi, \psi) = \max_{x \in X_1} \text{dist}_2(\phi(x), \psi(x)) + \max_{y \in X_2} \text{dist}_1(\phi^{-1}(y), \psi^{-1}(y)).$$

If ϕ and ψ are diffeomorphisms of class C^1 , the d^1 metric on

$$\text{Diff}^1(X_1, X_2) = \{\phi : X_1 \rightarrow X_2 : \phi \text{ is a } C^1\text{-diffeomorphism}\}$$

is given by

$$d^1(\phi, \psi) = d^0(\phi, \psi) + \max_{x \in X_1} \|D\phi_x - D\psi_x\|_2 + \max_{y \in X_2} \|D(\phi)_y^{-1} - D(\psi)_y^{-1}\|_1,$$

where $D\phi_x$ and $D\psi_x$ are the derivatives of ϕ and ψ at $x \in X_1$, respectively. If ϕ and ψ are diffeomorphisms of class C^2 , the d^2 metric on

$$\text{Diff}^2(X_1, X_2) = \{\phi : X_1 \rightarrow X_2 : \phi \text{ is a } C^2\text{-diffeomorphism}\}$$

is given by

$$d^2(\phi, \psi) = d^1(\phi, \psi) + \max_{x \in X_1} \|D^2\phi_x - D^2\psi_x\|_2 + \max_{y \in X_2} \|D^2(\phi)_y^{-1} - D^2(\psi)_y^{-1}\|_1,$$

where $D^2\phi_x$ and $D^2\psi_x$ are the second derivatives of ϕ and ψ at $x \in X_1$, respectively.

Definition 1.3.1. Suppose that M_i is a Riemannian manifold with Riemannian norm $\|\cdot\|_i$ and d_i is the metric induced by $\|\cdot\|_i$. For $m \geq 0$ and $\tau > 0$, set:

- i. $\mathbf{D}_i^m = \{\phi : M_i \rightarrow M_{i+1} : \phi \text{ is a } C^m\text{-diffeomorphism}\};$
- ii. $B^m(\phi, \tau) = \{\psi \in \mathbf{D}_i^m : d^m(\phi, \psi) < \tau\}, \text{ for } \phi \in \mathbf{D}_i^m;$
- iii. $\mathcal{F}^m(\mathbf{M}) = \{\mathbf{f} = (f_i)_{i \in \mathbb{Z}} : f_i \in \mathbf{D}_i^m \text{ for each } i \in \mathbb{Z}\}.$

Note that

$$\mathcal{F}^m(\mathbf{M}) = \prod_{i=-\infty}^{+\infty} \mathbf{D}_i^m.$$

Definition 1.3.2 (Product Topology). The *product topology* on $\mathcal{F}^m(\mathbf{M})$ is generated by the subsets

$$\mathcal{U} = \prod_{i < -j} \mathbf{D}_i^m \times \prod_{i = -j}^j [U_i] \times \prod_{i > j} \mathbf{D}_i^m,$$

where U_i is an open subset of $\mathbf{D}_i^m$, for $-j \leq i \leq j$, for some $j \in \mathbb{N}$. The space $\mathcal{F}^m(\mathbf{M})$ endowed with the product topology will be denoted by $(\mathcal{F}^m(\mathbf{M}), \tau_{\text{prod}})$.

Definition 1.3.3 (Uniform topology). Given $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ and $\mathbf{g} = (g_i)_{i \in \mathbb{Z}}$ in $\mathcal{F}^m(\mathbf{M})$, take

$$d_{\text{norm}}^m(\mathbf{f}, \mathbf{g}) = \sup_{i \in \mathbb{Z}} (\min\{d^m(f_i, g_i), 1\}).$$

The *uniform topology* on $\mathcal{F}^m(\mathbf{M})$ is spanned by $d_{\text{norm}}^m(\cdot, \cdot)$. Let τ_{unif} be the uniform topology on $\mathcal{F}^m(\mathbf{M})$.

We can also endow $\mathcal{F}^m(\mathbf{M})$ with the *C^m -strong topology* (or *Whitney topology*): for each $\mathbf{f} \in \mathcal{F}^m(\mathbf{M})$ and a sequence of positive numbers $(\epsilon_i)_{i \in \mathbb{Z}}$, a *strong basic neighborhood* of $\mathbf{f}$ is the set

$$B^m(\mathbf{f}, (\epsilon_i)_{i \in \mathbb{Z}}) = \{\mathbf{g} \in \mathcal{F}^m(\mathbf{M}) : g_i \in B^m(f_i, \epsilon_i) \text{ for all } i\}.$$

Definition 1.3.4 (Strong Topology). The *C^m -strong topology* on $\mathcal{F}^m(\mathbf{M})$ is generated by the strong basic neighborhood of each $\mathbf{f} \in \mathcal{F}^m(\mathbf{M})$. Thus, a subset $\mathcal{A}$ of $\mathcal{F}^m(\mathbf{M})$ is open if for all $\mathbf{f} \in \mathcal{A}$, there exists a strong basic neighborhood $B^m(\mathbf{f}, (\epsilon_i)_{i \in \mathbb{Z}})$ of $\mathbf{f}$, such that $B^m(\mathbf{f}, (\epsilon_i)_{i \in \mathbb{Z}}) \subseteq \mathcal{A}$. The space $\mathcal{F}^m(\mathbf{M})$ endowed with the strong topology will be denoted by $(\mathcal{F}^m(\mathbf{M}), \tau_{\text{str}})$.

Unless stated otherwise, we are considering the strong topology on $\mathcal{F}^m(\mathbf{M})$. For simplicity, the τ_{str} will be omitted.

Definition 1.3.5. We say that $\mathbf{f} \in \mathcal{F}^m(\mathbf{M})$ is *structurally stable* in $\mathcal{F}^m(\mathbf{M})$ if there exists a strong basic neighborhood $B^m(\mathbf{f}, (\epsilon_i)_{i \in \mathbb{Z}})$ of $\mathbf{f}$, such that each $\mathbf{g} \in B^m(\mathbf{f}, (\epsilon_i)_{i \in \mathbb{Z}})$ is uniformly conjugate to $\mathbf{f}$. A subset $\mathcal{A}$ of $\mathcal{F}^m(\mathbf{M})$ is *structurally stable* if every element in $\mathcal{A}$ is structurally stable.

Note that

$$\tau_{\text{prod}} \subset \tau_{\text{unif}} \subset \tau_{\text{str}}.$$

1.4 Some Properties of the Uniform Conjugacy

From now on, if we do not say otherwise, $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ and $\mathbf{g} = (g_i)_{i \in \mathbb{Z}}$ will represent two non-stationary dynamical systems defined on $\mathbf{M}$ and $\mathbf{N}$, respectively. By simplicity, we will denote by $\mathbf{d}$ the metric on both $\mathbf{M}$ and $\mathbf{N}$.

The following lemma it is clear and therefore we will omit the proof.

Lemma 1.4.1. *$\mathbf{f}$ and $\mathbf{g}$ are positive (negative) uniformly conjugate if, and only if, for any $i_0 \in \mathbb{Z}$ there exists a family of homeomorphisms $(h_i)_{i \geq i_0}$ ($(h_i)_{i \leq i_0}$) such that $(h_i)_{i \geq i_0}$ and $(h_i^{-1})_{i \geq i_0}$ ($(h_i)_{i \leq i_0}$ and $(h_i^{-1})_{i \leq i_0}$) are equicontinuous and $h_{i+1} \circ f_i = g_i \circ h_i : M_i \rightarrow N_{i+1}$, for every $i \geq i_0$ (for every $i \leq i_0$).*

Definition 1.4.2. Let $\phi : X \rightarrow X$ be a homeomorphism on a metric space X with metric ρ . For $x \in X$ and $\varepsilon > 0$, set:

1. $\mathcal{V}^s(x, \phi) = \{y \in X : \rho(\phi^n(x), \phi^n(y)) \rightarrow 0 \text{ when } n \rightarrow +\infty\} :=$ the *stable set* for ϕ at x ;
2. $\mathcal{V}_\varepsilon^s(x, \phi) = \{y \in X : \rho(\phi^n(x), \phi^n(y)) < \varepsilon, \text{ for all } n \geq 0\} :=$ the *local stable set* for ϕ at x ;
3. $\mathcal{V}^u(x, \phi) = \{y \in X : \rho(\phi^n(x), \phi^n(y)) \rightarrow 0 \text{ when } n \rightarrow -\infty\} :=$ the *unstable set* for ϕ at x ;
4. $\mathcal{V}_\varepsilon^u(x, \phi) = \{y \in X : \rho(\phi^n(x), \phi^n(y)) < \varepsilon, \text{ for all } n \leq 0\} :=$ the *local stable set* for ϕ at x .

Next we prove that stable and unstable sets for non-stationary dynamical systems are preserved by uniform conjugacy:

Proposition 1.4.3. Suppose $h = (h_i)_{i \in \mathbb{Z}}$ is a uniform conjugacy between f and g . For each $x \in M_i$, we have

$$h_i(\mathcal{V}^s(x, f)) = \mathcal{V}^s(h_i(x), g) \quad \text{and} \quad h_i(\mathcal{V}^u(x, f)) = \mathcal{V}^u(h_i(x), g).$$

Proof. Let $y \in \mathcal{V}^s(x, f)$ and $\varepsilon > 0$. There exists $\delta > 0$ such that, for all $j \in \mathbb{Z}$, if $z_1, z_2 \in M_j$ with $\mathbf{d}(z_1, z_2) < \delta$, then $\mathbf{d}(h_j(z_1), h_j(z_2)) < \varepsilon$. Since $\mathbf{d}(f_i^n(x), f_i^n(y)) \rightarrow 0$ when $n \rightarrow +\infty$, there exists $n \in \mathbb{N}$ such that, for all $n \geq N$, $\mathbf{d}(f_i^n(x), f_i^n(y)) < \delta$. Consequently, for all $n \geq N$,

$$\varepsilon > \mathbf{d}(h_{n+i}(f_i^n(x)), h_{n+i}(f_i^n(y))) = \mathbf{d}(g_i^n h_i(x), g_i^n h_i(y)).$$

Thus $h_i(y) \in \mathcal{V}^s(h_i(x), g)$. Now, using the equicontinuity of $(h_i^{-1})_{i \in \mathbb{Z}}$, we can prove that

$$h_i^{-1}(\mathcal{V}^s(h_i(x), g)) \subseteq \mathcal{V}^s(x, f).$$

The proof of the unstable case is similar and therefore we omit it. □

For the local stable and unstable sets we have:

Proposition 1.4.4. Suppose $h = (h_i)_{i \in \mathbb{Z}}$ is a uniform conjugacy between f and g . Fix $x \in M_i$. For $r = s, u$, there exist positive numbers ε_r , δ_r and ϵ_r such that

$$h_i(\mathcal{V}_{\epsilon_r}^r(x, f)) \subseteq \mathcal{V}_{\delta_r}^r(h_i(x), g) \subseteq h_i(\mathcal{V}_{\varepsilon_r}^r(x, f)).$$

Proof. We will prove the stable case. Since $(h_i^{-1})_{i \in \mathbb{Z}}$ is an equicontinuous family, given $\varepsilon_s > 0$, there exists $\delta_s > 0$ such that, for any $i \in \mathbb{Z}$, if $v, w \in N_i$ and $\mathbf{d}(v, w) < \delta_s$, then $\mathbf{d}(h_i^{-1}(v), h_i^{-1}(w)) < \varepsilon_s$. Fix $x \in M_i$. We will show that if $y \in \mathcal{V}_{\delta_s}^s(h_i(x), g)$ then $h_i^{-1}(y) \in \mathcal{V}_{\varepsilon_s}^s(x, f)$. Indeed, since $y \in \mathcal{V}_{\delta_s}^s(h_i(x), g)$, $\mathbf{d}(g_i^n h_i(x), g_i^n(y)) < \delta_s$ for all $n \geq 0$. Thus, for all $n \geq 0$ we have

$$\varepsilon_s > \mathbf{d}(h_{n+i}^{-1} g_i^n h_i(x), h_{n+i}^{-1} g_i^n(y)) = \mathbf{d}(f_i^n h_i^{-1} h_i(x), f_i^n h_i^{-1}(y)) = \mathbf{d}(f_i^n(x), f_i^n h_i^{-1}(y)),$$

and hence, $h_i^{-1}(y) \in \mathcal{V}_{\varepsilon_s}^s(x, f)$. Therefore,

$$h_i^{-1}(\mathcal{V}_{\delta_s}^s(h_i(x), g)) \subseteq \mathcal{V}_{\varepsilon_s}^s(x, f).$$

Since $(h_i)_{i \in \mathbb{Z}}$ is an equicontinuous family, analogously we can prove that there exists $\epsilon_s > 0$ such that

$$h_i(\mathcal{V}_{\epsilon_s}^s(x, f)) \subseteq \mathcal{V}_{\delta_s}^s(h_i(x), g),$$

which proves the proposition. $\square$

Take two homeomorphisms $\phi : X_1 \rightarrow X_1$ and $\psi : X_2 \rightarrow X_2$ defined on two compact metric spaces X_1 and X_2 , respectively. We will denote by $\mathbf{f}_\phi$ and $\mathbf{f}_\psi$ the constant families associated, respectively, to ϕ and to ψ . It is clear that if ϕ and ψ are topologically conjugate then $\mathbf{f}_\phi$ and $\mathbf{f}_\psi$ are uniformly conjugate. In the next proposition we prove the converse is not necessarily true.

Take $\mathbb{S}^1 = \{z \in \mathbb{C} : \|z\| = 1\}$, with $R_\alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ the circle rotation by a number $\alpha \in [0, 1]$, that is, $R_\alpha(z) = e^{2\pi\alpha i}z$ for $z \in \mathbb{S}^1$. It is well-known that if $\alpha_1 \in \mathbb{Q} \cap (0, 1)$ and $\alpha_2 \in [\mathbb{R} \setminus \mathbb{Q}] \cap (0, 1)$ then R_{α_1} and R_{α_2} are not topologically conjugate. However:

Proposition 1.4.5. *Given $\alpha_1, \alpha_2 \in [0, 1]$, the constant families associated to R_{α_1} and R_{α_2} , respectively, are uniformly conjugate.*

Proof. It is sufficient to prove that, for any $\alpha \in [0, 1]$, the constant families associated to R_α and to the identity on $\mathbb{S}^1$, respectively, are uniformly conjugate. Let I be the identity on $\mathbb{S}^1$. Consider the family of homeomorphisms $(h_k : \mathbb{S}^1 \rightarrow \mathbb{S}^1)_{k \in \mathbb{Z}}$, where $h_k(z) = e^{2k\pi(1-\alpha)i}z$ for all $k \in \mathbb{Z}$. Thus, for $k \in \mathbb{Z}$ and $z \in \mathbb{S}^1$,

$$R_\alpha(h_{k+1}(z)) = e^{2\pi\alpha i}e^{2(k+1)\pi(1-\alpha)i}z = e^{2k\pi(1-\alpha)i}z = h_k(I(z)),$$

i. e., $(h_k)_{k \in \mathbb{Z}}$ is a conjugacy between $\mathbf{f}_{R_\alpha}$ and $\mathbf{f}_I$. If $z_1, z_2 \in \mathbb{S}^1$ and $\mathbf{d}(z_1, z_2) \leq \frac{1}{2} \min\{\alpha, 1-\alpha\}$, then $\mathbf{d}(R_{(1-\alpha)}(z_1), R_{(1-\alpha)}(z_2)) = \mathbf{d}(z_1, z_2)$. Consequently, if $\mathbf{d}(z_1, z_2) \leq \frac{1}{2} \min\{\alpha, 1-\alpha\}$, for all $k \in \mathbb{Z}$, we have

$$\mathbf{d}(h_k(z_1), h_k(z_2)) = \mathbf{d}(R_{(1-\alpha)}^k(z_1), R_{(1-\alpha)}^k(z_2)) = \mathbf{d}(z_1, z_2).$$

This fact proves that $(h_k)_{k \in \mathbb{Z}}$ is equicontinuous. Analogously we can prove that $(h_k^{-1})_{k \in \mathbb{Z}}$ is equicontinuous. Hence, $\mathbf{f}_{R_\alpha}$ and $\mathbf{f}_I$ are uniformly conjugate. $\square$

From Proposition 1.4.5 we have also that the uniform conjugacy does not preserve the rotation number of a homeomorphism, i. e. there exist homeomorphisms on the circle with different rotation numbers whose associated constant families are uniformly conjugate.

Definition 1.4.6. A map $\phi : X \rightarrow X$, on a metric space X with metric ρ , is called an *isometry* if $\rho(\phi(x), \phi(y)) = \rho(x, y)$ for all $x, y \in M$.

Any circle rotation is an isometry. In the following proposition we will see a more general result than that obtained in Proposition 1.4.5.

Proposition 1.4.7. *If $\phi : X \rightarrow X$ is an isometry, then $\mathbf{f}_\phi$ is uniformly conjugate to the constant family associated to the identity on X .*

Proof. Consider the family $(h_k)_{k \in \mathbb{Z}}$, where $h_k(x) = \phi^{-k}(x)$ for every $x \in M_k = X \times \{k\}$. It is clear that $(h_k)_{k \in \mathbb{Z}}$ is a topological conjugacy between $\mathbf{f}_\phi$ and $\mathbf{f}_I$, where I is the identity on X . Since ϕ is an isometry, the family $(h_k)_{k \in \mathbb{Z}}$ is equicontinuous. $\square$

It follows from Proposition 1.4.7 that all the constant families associated to any isometry on M are uniformly conjugate.

We will finish this chapter with the following proposition.

Proposition 1.4.8. *If f and g are uniformly conjugate, then the gatherings $\tilde{f}$ and $\tilde{g}$ obtained, respectively, from f and g by a sequence of integers $(n_i)_{i \in \mathbb{Z}}$ (see Definition 1.1.6), are uniformly conjugate.*

Proof. If f and g are uniformly conjugate by $h = (h_i)_{i \in \mathbb{Z}}$, then $\tilde{f}$ and $\tilde{g}$ are uniformly conjugate by the family $\tilde{h} = (\tilde{h}_{n_i})_{i \in \mathbb{Z}}$:

$$\begin{array}{ccccccc}
 M_{n_{i-1}} & \xrightarrow{f_{n_{i-1}}} & \dots & \xrightarrow{f_{n_i-1}} & M_{n_i} & \xrightarrow{f_{n_i}} & \dots & \xrightarrow{f_{n_{i+1}-1}} & M_{n_{i+1}} \\
 \dots \downarrow h_{n_{i-1}} & & & & \downarrow h_{n_i} & & & & \downarrow h_{n_{i+1}} \dots \\
 M_{n_{i-1}} & \xrightarrow{g_{n_{i-1}}} & \dots & \xrightarrow{g_{n_i-1}} & M_{n_i} & \xrightarrow{g_{n_i}} & \dots & \xrightarrow{g_{n_{i+1}-1}} & M_{n_{i+1}}
 \end{array}$$

It is clear that $\tilde{h} = (\tilde{h}_{n_i})_{i \in \mathbb{Z}}$ is equicontinuous. □

Chapter 2

Entropy for Non-Stationary Dynamical Systems

In [AKM65], R. L. Adler, A. G. Konheim and M. H. McAndrew introduced the topological entropy of a continuous map $\phi : X \rightarrow X$ on a compact topological space X via open covers of X . Roughly, the topological entropy is the exponential growth rate of the number of essentially different orbit segments of length n . In 1971, R. Bowen defined the topological entropy of a uniformly continuous map on an arbitrary metric space via spanning and separated sets, which, when the space is compact, coincides with the topological entropy as defined by Adler, Konheim and McAndrew. Both definitions can be found in [Wal00].

S. Kolyada and L. Snoha, in [KS96], introduced a notion of topological entropy for non-autonomous dynamical systems, which generalizes the notion of entropy for single dynamical systems. They considered only sequences of type $(f_i)_{i \geq 0}$ and the entropy for this sequence was a single number (possibly $+\infty$). In this chapter we will extend this idea to sequences of type $(f_i)_{i \in \mathbb{Z}}$. We can define a different entropy for the same sequence by considering the composition of the inverse of each f_i for $i \rightarrow -\infty$ (see Remark 2.2.11).

First, the entropy of a non-stationary dynamical system $(f_i)_{i \in \mathbb{Z}}$ will be defined as a sequence of non-negative numbers $(a_i)_{i \in \mathbb{Z}}$, where each a_i depends only on f_j for $j \geq i$. Then we will see that $(a_i)_{i \in \mathbb{Z}}$ is a constant sequence (see Corollary 2.2.7). Consequently, this common number will be considered to be the entropy of $(f_i)_{i \in \mathbb{Z}}$. As a consequence, we will also see the entropy of a non-stationary dynamical system can be considered as the topological entropy of a single homeomorphism defined on the total space $\mathbf{M}$ (see Remark 2.2.9).

The main goal of this chapter is to show that, if $m \geq 1$ and we consider the strong topology on $\mathcal{F}^m(\mathbf{M})$, the entropy depends continuously on each non-stationary dynamical system in $\mathcal{F}^m(\mathbf{M})$. In contrast, with the product topology on $\mathcal{F}^m(\mathbf{M})$, the entropy is discontinuous for any non-stationary dynamical system. To prove that the entropy is continuous on $\mathcal{F}^m(\mathbf{M})$ with the uniform topology is equivalent to prove the continuity of the entropy for single maps (see Proposition 2.3.4). The present chapter is a work published by the author in the Bulletin of the Brazilian Mathematical Society, New Series (see [Ace17c]).

Throughout this chapter, we will take

$$M_i = M \times \{i\} \quad \text{and} \quad N_i = N \times \{i\}, \quad \text{for each } i \in \mathbb{Z},$$

where M and N are compact Riemannian manifolds.

2.1 Definition of Entropy for Non-Stationary Dynamical Systems

In this section we will introduce the notion of topological entropy for a non-stationary dynamical system $(f_i)_{i \in \mathbb{Z}}$, generalizing the topological entropy for a single map (see [Wal00]). Firstly, this entropy will not be a single positive number but a sequence of non-negative numbers $(a_i)_{i \in \mathbb{Z}}$, where each a_i depends only on f_j for each $j \geq i$. In Corollary 2.2.7 we will see that this sequence is constant.

We consider the following definitions: an *open cover* of M is a collection of open subsets of M , $\mathcal{A} = \{A_\lambda\}_{\lambda \in \Lambda}$, such that $M = \bigcup_\lambda A_\lambda$. In this section, $\mathcal{A}$ and $\mathcal{B}$ will denote open covers of M . Since $M_i = M \times \{i\}$, if $\mathcal{A}$ is an open cover of M , then $\mathcal{A}_i = \mathcal{A} \times \{i\}$ is an open partition of M_i . By abuse of notation, we will omit the sub index i of $\mathcal{A}_i$ for covers of M_i .

Definition 2.1.1. Let $N(\mathcal{A})$ be the number of sets in a finite subcover of $\mathcal{A}$ with smallest cardinality. The *entropy* of $\mathcal{A}$ is the number

$$H(\mathcal{A}) := \log N(\mathcal{A}).$$

The proof of the following statements can be found in [Wal00] for the case of a single map. Such proofs can be adapted for non-stationary dynamical systems and therefore we omit them. We will consider the following notations: For each $i \in \mathbb{Z}$ and $n \geq 0$, set

$$(f_i^n)^{-1}(\mathcal{A}) = \{(f_{i+n-1} \circ \dots \circ f_i)^{-1}(A) : A \in \mathcal{A}\}.$$

Set

$$\mathcal{A} \vee \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Inductively we define $\bigvee_{m=1}^k \mathcal{A}^m$ for a collection of open covers $\mathcal{A}^1, \dots, \mathcal{A}^k$ of M .

We say $\mathcal{B}$ is a *refinement* of $\mathcal{A}$ if each element of $\mathcal{B}$ is contained in some element of $\mathcal{A}$.

Proposition 2.1.2. *The entropy satisfies the following properties:*

- i. $H(\mathcal{A} \vee \mathcal{B}) \leq H(\mathcal{A}) + H(\mathcal{B})$.
- ii. If $\mathcal{B}$ is a refinement of $\mathcal{A}$ then $H(\mathcal{A}) \leq H(\mathcal{B})$.
- iii. $H(\mathcal{A}) = H((f_i^k)^{-1}(\mathcal{A}))$ for each $i \in \mathbb{Z}$ and $k \geq 0$.
- iv. $H(\bigvee_{k=0}^{n-1} (f_i^k)^{-1}(\mathcal{A})) \leq nH(\mathcal{A})$, for each $i \in \mathbb{Z}$ and $n \geq 1$.
- v. The limit

$$H_i(\mathbf{f}, \mathcal{A}) = \lim_{n \rightarrow +\infty} \frac{1}{n} H \left(\bigvee_{k=0}^{n-1} (f_i^k)^{-1}(\mathcal{A}) \right) \quad (2.1.1)$$

exists and is finite, for each $i \in \mathbb{Z}$.

Definition 2.1.3 (Topological entropy). We define the *entropy of $\mathbf{f}$ relative to $\mathcal{A}$* as the sequence

$$H(\mathbf{f}, \mathcal{A}) = (H_i(\mathbf{f}, \mathcal{A}))_{i \in \mathbb{Z}}.$$

The *topological entropy of $\mathbf{f}$* is the sequence

$$H(\mathbf{f}) = (H_i(\mathbf{f}))_{i \in \mathbb{Z}},$$

where

$$H_i(\mathbf{f}) = \sup\{H_i(\mathbf{f}, \mathcal{A}) : \mathcal{A} \text{ is an open cover of } M\}.$$

From now on, X will represent a compact metric space. We recall that the *topological entropy* of a homeomorphism $\phi : X \rightarrow X$, which we denote by $\mathcal{H}(\phi)$, is defined considering open covers of X . The above definition only makes sense when $\mathcal{A}$ is an open cover of M instead of a general open cover of $\mathbf{M}$. If we consider arbitrary collections of open covers of each M_i , the limit (2.1.1) could be infinite (we can take open covers $\mathcal{A}_i$ of each M_i with $N(\mathcal{A}_i)$ arbitrarily large, for each i).

Now we introduce the definition of topological entropy using span and separated subsets. That entropy will be called *$\star$ -topological entropy* for differentiate it from the topological entropy. As in the case of a single homeomorphism, we will see in Theorem 2.2.1 that the topological entropy coincides with $\star$ -topological entropy for non-stationary dynamical systems.

Definition 2.1.4. Let $n \in \mathbb{N}$, $\varepsilon > 0$ and $i \in \mathbb{Z}$ be given. We say that a compact subset $K \subseteq M_i$ is a (n, ε) -*spanning* of M_i with respect $\mathbf{f}$ if for each $x \in M_i$ there exists $y \in K$ such that $\max_{0 \leq j < n} \mathbf{d}(\mathbf{f}_i^j(x), \mathbf{f}_i^j(y)) < \varepsilon$, i. e.,

$$M_i \subseteq \bigcup_{y \in K} \bigcap_{k=0}^{n-1} (\mathbf{f}_i^k)^{-1}(\overline{B(\mathbf{f}_i^k(y), \varepsilon)}),$$

where $\overline{B(\mathbf{f}_i^k(y), \varepsilon)}$ is the closed ball with center $\mathbf{f}_i^k(y) \in M_{i+k}$ and radius ε .

Denote by $r[n, i](\varepsilon, \mathbf{f})$ the smallest cardinality of any (n, ε) -span of M_i with respect $\mathbf{f}$. Since M_i is compact, we have $r[n, i](\varepsilon, \mathbf{f}) < \infty$ for each $i \in \mathbb{Z}$ and $n \geq 1$. Set

$$r[i](\varepsilon, \mathbf{f}) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log r[n, i](\varepsilon, \mathbf{f}).$$

Definition 2.1.5. The *$\star$ -topological entropy of $\mathbf{f}$* is the sequence

$$\mathbf{H}(\mathbf{f}) = (\mathbf{H}_i(\mathbf{f}))_{i \in \mathbb{Z}}, \quad \text{where } \mathbf{H}_i(\mathbf{f}) = \lim_{\varepsilon \rightarrow 0} r[i](\varepsilon, \mathbf{f}) \text{ for each } i \in \mathbb{Z}.$$

Now we define the entropy using separated subsets and we will prove that the entropy considering span subsets coincide with the entropy considering separated subsets.

Definition 2.1.6. Let $n \in \mathbb{N}$, $\varepsilon > 0$ and $i \in \mathbb{Z}$ be fixed. A subset $E \subseteq M_i$ is called (n, ε) -*separated* with respect to $\mathbf{f}$ if given $x, y \in E$, with $x \neq y$, we have $\max_{0 \leq j < n} \mathbf{d}(\mathbf{f}_i^j(x), \mathbf{f}_i^j(y)) > \varepsilon$, i. e., if for all $x \in E$, the set

$$\bigcap_{k=0}^{n-1} (\mathbf{f}_i^k)^{-1}(\overline{B(\mathbf{f}_i^k(x), \varepsilon)})$$

contains no other point of E .

Denote by $s[n, i](\varepsilon, \mathbf{f})$ the largest cardinality of any (n, ε) -separated subset of M_i with respect to $\mathbf{f}$. Set

$$s[i](\varepsilon, \mathbf{f}) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log s[n, i](\varepsilon, \mathbf{f}).$$

Proposition 2.1.7. *Given $\varepsilon > 0$ and $i \in \mathbb{Z}$ we have:*

- i. $r[n, i](\varepsilon, \mathbf{f}) \leq s[n, i](\varepsilon, \mathbf{f}) \leq r[n, i](\varepsilon/2, \mathbf{f})$, for all $n > 0$.
- ii. $r[i](\varepsilon, \mathbf{f}) \leq s[i](\varepsilon, \mathbf{f}) \leq r[i](\varepsilon/2, \mathbf{f})$, for all $n > 0$.

Proof. The proposition is proved in [Wal00], Chapter 7, for a single map. That proof works for non-stationary dynamical systems and, therefore, we omit the proof. $\square$

From Proposition 2.1.7 we have

$$\mathbf{H}_i(\mathbf{f}) = \lim_{\varepsilon \rightarrow 0} s[i](\varepsilon, \mathbf{f}) \quad \text{for all } i \in \mathbb{Z}.$$

Consequently, $\mathbf{H}_i(\mathbf{f})$ can be defined using either span or separated subsets.

Notice that if $\mathbf{f}$ is a constant family associated to a homeomorphism $\phi : X \rightarrow X$, then it is clear that

$$\mathbf{H}_i(\mathbf{f}) = \mathcal{H}(\phi), \quad \text{for all } i \in \mathbb{Z}. \quad (2.1.2)$$

Therefore, $\mathbf{H}$ extends the notion of topological entropy for single homeomorphisms.

Some estimations and properties of the topological entropy for non-stationary dynamical systems can be found in [ZC09], [Kaw17], [KMS99], [KS96], [SSZ16] and [ZZH06]. In [KL16], C. Kawan and Y. Latushkin give a formula for the topological entropy of a *non-stationary subshift of finite type*, which were introduced by Fisher and Arnoux in [AF05]. Regarding the metric entropy for non-autonomous dynamical systems the author recommends C. Kawan's works (see [Kaw14] and references there).

In [Bog92], [Kus67], [LY88], [Liu98], [Rue97a], is defined a *measure-theoretic entropy for random dynamical systems* (see [Arn13]). In these papers we can also found analogous versions for n.s.d.s. of the thermodynamic formalism of dynamical systems (see [Rue97b]). In [QQX03] some relations between the entropy for random dynamical systems and the *Lyapunov exponents* and the *Pesin's formula* are given.

2.2 Some Properties of Entropy for Non-Stationary Dynamical Systems

In this section we will see some properties of this topological entropy for non-stationary dynamical systems. Some are analogous to the well-known properties of entropy for a single map (see [Wal00]). For single maps, the topological entropy is invariant for topological conjugacy. The main result of this section is to prove the analogous version for non-stationary dynamical systems, that is, this entropy is invariant for uniform conjugacy (see Theorem 2.2.5). This result will be fundamental to show the continuity of the entropy in the following section (see Theorem 2.4.5).

As we mentioned, the notions of entropy for non-stationary dynamical systems, considering either open covers or separated subsets, coincide. This fact can be proved analogously to the case of single homeomorphisms (see [Wal00], Chapter 7, Section 2):

Proposition 2.2.1. *For each $i \in \mathbb{Z}$ we have $H_i(\mathbf{f}) = \mathbf{H}_i(\mathbf{f})$.*

From now on, we will use the notation $\mathbf{H}_i(\mathbf{f})$.

The topological entropy $\mathcal{H}(\phi)$ of a single homeomorphism $\phi : X \rightarrow X$ satisfies $\mathcal{H}(\phi^n) = |n|\mathcal{H}(\phi)$, for $n \in \mathbb{Z}$. For non-stationary dynamical systems we have:

Proposition 2.2.2. *Suppose $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ is an equicontinuous sequence. Fix $n \geq 1$. Let $\tilde{\mathbf{f}} \in \mathcal{F}^1(\tilde{M})$ be the gathering obtained of $\mathbf{f}$ by the sequence $(ni)_{i \in \mathbb{Z}}$, that is, $\tilde{M}_i = M_{ni}$ and $\tilde{f}_i = f_{n(i+1)-1} \circ \dots \circ f_{ni}$;*

$$\dots M_{n(i-1)} \xrightarrow{\tilde{f}_{i-1} = f_{ni-1} \circ \dots \circ f_{n(i-1)}} M_{ni} \xrightarrow{\tilde{f}_i = f_{n(i+1)-1} \circ \dots \circ f_{ni}} M_{n(i+1)} \dots$$

Thus, for each $i \in \mathbb{Z}$ we have

$$\mathbf{H}_i(\tilde{\mathbf{f}}) = n\mathbf{H}_{in}(\mathbf{f}).$$

Proof. For $i \in \mathbb{Z}$, $x, y \in M_{ni}$ and $m > 0$, we have

$$\max_{0 \leq k < m} \mathbf{d}(\tilde{f}_i^k(x), \tilde{f}_i^k(y)) = \max_{0 \leq k < m} \mathbf{d}(f_{ni}^{nk}(x), f_{ni}^{nk}(y)) \leq \max_{0 \leq j < nm} \mathbf{d}(f_{ni}^j(x), f_{ni}^j(y)).$$

This fact proves that, for all $\varepsilon > 0$, each (nm, ε) -span subset K of M_{ni} with respect to $\mathbf{f}$ is a (m, ε) -span subset of M_{ni} with respect to $\tilde{\mathbf{f}}$. Consequently, we obtain $r[m, ni](\varepsilon, \tilde{\mathbf{f}}) \leq r[nm, ni](\varepsilon, \mathbf{f})$. Hence,

$$\mathbf{H}_i(\tilde{\mathbf{f}}) \leq n\mathbf{H}_{in}(\mathbf{f}).$$

On the other hand, since $\mathbf{f}$ is equicontinuous, we can prove that $(f_{ni})_{i \in \mathbb{Z}}$, $(f_{ni}^2)_{i \in \mathbb{Z}}$, $\dots$, $(f_{ni}^{n-1})_{i \in \mathbb{Z}}$ is an equicontinuous collection of families. Consequently, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\max_{\substack{1 \leq k < n \\ j \in \mathbb{Z}}} \{\mathbf{d}(f_{nj}^{k+1}(x), f_{nj}^{k+1}(y)) : x, y \in M_{nj}, \mathbf{d}(x, y) < \delta\} < \varepsilon.$$

Now, if K is a (m, δ) -span of M_{ni} with respect to $\tilde{\mathbf{f}}$, then, for all $x \in M_{ni}$, there exists $y \in K$ such that

$$\max\{\mathbf{d}(x, y), \mathbf{d}(f_{ni}^n(x), f_{ni}^n(y)), \dots, \mathbf{d}(f_{ni}^{(m-1)n}(x), f_{ni}^{(m-1)n}(y))\} < \delta.$$

Thus,

$$\begin{aligned} \max_{0 \leq k < n} \{\mathbf{d}(f_{ni}^k(x), f_{ni}^k(y))\} &< \varepsilon, \\ \max_{0 \leq k < n} \{\mathbf{d}(f_{n(i+1)}^k \circ f_{ni}^n(x), f_{n(i+1)}^k \circ f_{ni}^n(y))\} &< \varepsilon, \\ &\vdots \\ \max_{0 \leq k < n} \{\mathbf{d}(f_{n(i+m-1)}^k \circ f_{ni}^{(m-1)n}(x), f_{n(i+m-1)}^k \circ f_{ni}^{(m-1)n}(y))\} &< \varepsilon. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \max_{0 \leq k < n} \{d(f_{ni}^k(x), f_{ni}^k(y))\} &< \varepsilon, \\ \max_{0 \leq k < n} \{d(f_{ni}^{n+k}(x), f_{ni}^{n+k}(y))\} &< \varepsilon, \\ &\vdots \\ \max_{0 \leq k < n} \{d(f_{ni}^{(m-1)n+k}(x), f_{ni}^{(m-1)n+k}(y))\} &< \varepsilon. \end{aligned}$$

Therefore,

$$\max\{d(f_{ni}^k(x), f_{ni}^k(y)) : k = 0, \dots, mn - 1\} < \varepsilon,$$

that is, K is a (mn, ε) -span of M_{ni} with respect to f . Consequently, we have $r[m, ni](\varepsilon, \tilde{f}) \geq r[nm, ni](\varepsilon, f)$ and, therefore,

$$H_i(\tilde{f}) \geq nH_{in}(f),$$

which proves the proposition. $\square$

From the proof of Proposition 2.2.2, we have always the inequality

$$H_i(\tilde{f}) \leq nH_{in}(f).$$

In [KS96] can be found an example of a general n.s.d.s. where the above inequality is strict.

Proposition 2.2.3. *Suppose $f = (f_i)_{i \in \mathbb{Z}}$ is a sequence consisting of isometries, that is, $f_i : M_i \rightarrow M_{i+1}$ is an isometry for all i . Thus $H_i(f) = 0$, for all $i \in \mathbb{Z}$.*

Proof. This follows directly from Definition 2.1.5. $\square$

Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the circle endowed with the Riemannian metric inherited from $\mathbb{R}^2$. For homeomorphisms ϕ on $\mathbb{S}^1$ we have that $\mathcal{H}(\phi) = 0$ (see [Wal00]). This property is also valid for n.s.d.s:

Proposition 2.2.4. *Suppose that $M_i = \mathbb{S}^1 \times \{i\}$ for each $i \in \mathbb{Z}$ endowed with the Riemannian metric inherited from $\mathbb{R}^2$. If f is a non-stationary dynamical system on M , then $H_i(f) = 0$, for all $i \in \mathbb{Z}$.*

Proof. See [KS96], Theorem D. $\square$

In the following theorem we will see that this entropy for non-stationary dynamical systems is an invariant for uniform conjugacy. This result generalizes the fact that the topological entropy of homeomorphisms defined on compact metric spaces is an invariant for topological conjugacy.

Theorem 2.2.5. *If f and g are positively uniformly conjugate, then*

$$H_i(f) = H_i(g) \quad \text{for all } i \in \mathbb{Z}.$$

Proof. Fix $i \in \mathbb{Z}$. It follows from Lemma 1.4.1 that there exists a uniform conjugacy $(h_j)_{j \geq i}$ between $(f_j)_{j \geq i}$ and $(g_j)_{j \geq i}$. Since $(h_j)_{j \geq i}$ is equicontinuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $j \geq i$, if $x, y \in M_j$ and $d(x, y) < \delta$, then $d(h_j(x), h_j(y)) < \varepsilon$. Let K be a (m, δ) -span of M_i

with respect to $\mathbf{f}$. Thus, for all $x \in M_i$ there exists $y \in K$ such that $\max_{0 \leq j < m} \mathbf{d}(\mathbf{f}_i^j(x), \mathbf{f}_i^j(y)) < \delta$. Consequently, if $0 \leq j < m$,

$$\varepsilon > \max_{0 \leq j < m} \mathbf{d}(h_{i+j} \circ \mathbf{f}_i^j(x), h_{i+j} \circ \mathbf{f}_i^j(y)) = \max_{0 \leq j < m} \mathbf{d}(\mathbf{g}_i^j \circ h_i(x), \mathbf{g}_i^j \circ h_i(y)).$$

This fact proves that $r[m, i](\varepsilon, \mathbf{f}) \geq r[m, i](\delta, \mathbf{g})$. Hence,

$$\mathbf{H}_i(\mathbf{f}) \geq \mathbf{H}_i(\mathbf{g}).$$

Since $(h_j^{-1})_{j \geq i}$ is equicontinuous, analogously we prove

$$\mathbf{H}_i(\mathbf{f}) \leq \mathbf{H}_i(\mathbf{g}),$$

which proves the theorem. $\square$

It follows from the proof of the above theorem that if $(f_i)_{i \geq i_0}$ and $(g_i)_{i \geq i_0}$ are uniformly conjugate then $\mathbf{H}_{i_0}(\mathbf{f}) = \mathbf{H}_{i_0}(\mathbf{g})$. Furthermore, entropy depends only on the future:

Corollary 2.2.6. *Suppose that there exists $i_0 \in \mathbb{Z}$ such that $f_j = g_j$ for all $j \geq i_0$. Then we have*

$$\mathbf{H}_i(\mathbf{f}) = \mathbf{H}_i(\mathbf{g}) \quad \text{for all } i \in \mathbb{Z}.$$

Proof. It is clear that $(f_j)_{j \geq i_0}$ and $(g_j)_{j \geq i_0}$ are uniformly conjugate (take $h_j = Id$ for each $j \geq i_0$). It follows from Lemma 1.4.1 that $(f_j)_{j \geq 0}$ and $(g_j)_{j \geq 0}$ are uniformly conjugate. By Theorem 2.2.5 we have $\mathbf{H}_i(\mathbf{f}) = \mathbf{H}_i(\mathbf{g})$ for all $i \in \mathbb{Z}$. $\square$

Corollary 2.2.7. *For all $i, j \in \mathbb{Z}$ we have $\mathbf{H}_i(\mathbf{f}) = \mathbf{H}_j(\mathbf{f})$.*

Proof. It is sufficient to prove that $\mathbf{H}_i(\mathbf{f}) = \mathbf{H}_{i+1}(\mathbf{f})$ for all $i \in \mathbb{Z}$. Fix $i \in \mathbb{Z}$. Take the family $\mathbf{g} = (g_j)_{j \in \mathbb{Z}}$, where $g_j = I_j : M_j \rightarrow M_{j+1}$ for each $j \leq i$ is the identity, modulo the identification $M_i = M$ (remember that $M_i = M \times \{i\}$), and $g_j = f_j$ for $j > i$. Thus $\mathbf{H}_i(\mathbf{f}) = \mathbf{H}_i(\mathbf{g})$. For each $x, y \in M_i$ and $n \geq 2$ we have

$$\max_{0 \leq j < n} \mathbf{d}(\mathbf{g}_i^j(x), \mathbf{g}_i^j(y)) = \max_{0 \leq j < n-1} \mathbf{d}(\mathbf{g}_{i+1}^j(x), \mathbf{g}_{i+1}^j(y)).$$

Using this fact we can prove that $\mathbf{H}_i(\mathbf{g}) = \mathbf{H}_{i+1}(\mathbf{g})$. Consequently, we have that

$$\mathbf{H}_i(\mathbf{f}) = \mathbf{H}_{i+1}(\mathbf{f}),$$

for any $i \in \mathbb{Z}$. $\square$

Remark 2.2.8. From now on we will omit the index i of $\mathbf{H}_i$ and we will consider the entropy of a non-stationary dynamical system as a single number, as a consequence of Corollary 2.2.7.

Remark 2.2.9. We can consider the system $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ as a homeomorphism $\mathbf{f} : \mathbf{M} \rightarrow \mathbf{M}$. If $\mathbf{f}$ is uniformly continuous, then we can calculate the topological entropy of the single map $\mathbf{f}$, $\mathcal{H}(\mathbf{f})$, via open covers or spanning or separated sets of $\mathbf{M}$. It can be proved that $\mathcal{H}(\mathbf{f}) = \mathbf{H}(\mathbf{f})$.

If we consider another metric $\tilde{d}$ uniformly equivalent to d on M , then the identity

$$I : (\mathbf{M}, \mathbf{d}) \rightarrow (\mathbf{M}, \tilde{\mathbf{d}})$$

$$p \mapsto p$$

is a uniformly continuous map. It follows from Theorem 7.4 in [Wal00] that the topological entropy of $\mathbf{f}$ considering the metric $\tilde{d}$ on M coincides with the topological entropy of $\mathbf{f}$ considering d on M . Consequently, the entropy for a non-stationary dynamical system on $\mathbf{M}$ is the same for equivalent metrics on M .

We can define the inverse of $\mathbf{f}$ as $\mathbf{f}^{-1} = (g_i)_{i \in \mathbb{Z}}$, where $g_i := f_i^{-1} : M_{i+1} \rightarrow M_i$ for each i . In this case, for $n > 0$ we have

$$(\mathbf{f}^{-1})_i^0 := I_{i+1} : M_{i+1} \rightarrow M_{i+1}$$

and

$$(\mathbf{f}^{-1})_i^n := g_{i-n+1} \circ \cdots \circ g_i : M_{i+1} \rightarrow M_{i-n+1}$$

In the case of a single homeomorphism $\phi : M \rightarrow M$, we have $\mathcal{H}(\phi) = \mathcal{H}(\phi^{-1})$ (see [Wal00], Theorem 7.3). The following example proves that, in general, we could have $\mathbf{H}(\mathbf{f}) \neq \mathbf{H}(\mathbf{f}^{-1})$.

Example 2.2.10. Let $I : M \rightarrow M$ be the identity on M and $\phi : M \rightarrow M$ be a homeomorphism on M with non-zero topological entropy. Let $f_i : M_i \rightarrow M_{i+1}$ be the diffeomorphisms defined as $f_i = I$ for $i \geq 0$ and $f_i = \phi$ for $i < 0$ and take $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$. From Corollary 2.2.6 we have $\mathbf{H}(\mathbf{f}) = \mathcal{H}(I) = 0$ and $\mathbf{H}(\mathbf{f}^{-1}) = \mathcal{H}(\phi) \neq 0$, for each $i \in \mathbb{Z}$.

The essence of the above example is that entropy of $\mathbf{f}$ depends only on the future, while the entropy of $\mathbf{f}^{-1}$ depends only on the past.

Remark 2.2.11. As a consequence of Example 2.2.10, we can also consider the entropy $\mathbf{H}(\mathbf{f}^{-1})$, which we denote by $\mathbf{H}^{(-1)}(\mathbf{f})$. All the above results for $\mathbf{H}$ have analog versions for $\mathbf{H}^{(-1)}$.

There are dynamical systems defined on a compact metric space that are not topologically conjugate but have the same topological entropy. Now, from Theorem 2.2.5 we have that two constant families associated to homeomorphisms with different topological entropies cannot be uniformly conjugate. On the other hand, Propositions 1.4.5 and 2.2.4 prove that there are constant families, associated to homeomorphisms with the same topological entropy, that can be uniformly topologically conjugate. One natural question that arises from this notion of entropy is as follows: Suppose that $\mathbf{f}$ and $\mathbf{g}$ are constant families. If $\mathbf{H}(\mathbf{f}) = \mathbf{H}(\mathbf{g})$ then are $\mathbf{f}$ and $\mathbf{g}$ always uniformly conjugate? The answer is negative, as the following example shows:

Example 2.2.12. In this example we consider the stable and unstable sets given in Definition 1.4.2. Proposition 1.4.3 proves that, if $\mathbf{h} = (h_i)_{i \in \mathbb{Z}}$ is a uniform conjugacy between $\mathbf{f}$ and $\mathbf{g}$, then, for each $x \in M_i$, we have

$$h_i(\mathcal{V}^s(x, \mathbf{f})) = \mathcal{V}^s(h_i(x), \mathbf{g}) \quad \text{and} \quad h_i(\mathcal{V}^u(x, \mathbf{f})) = \mathcal{V}^u(h_i(x), \mathbf{g}).$$

Let $M = \mathbb{S}^1$, p_N be the north pole and p_S be the south pole of $\mathbb{S}^1$. Suppose that $\phi : M \rightarrow M$ is a homeomorphism with stable set $\mathcal{V}^s(p_N, \phi) = M \setminus \{p_S\}$ (see Figure 2.2.1). Let $\mathbf{f}$ and $\mathbf{g}$ be the

constant families associated to ϕ and to the identity on M , respectively. Then $\mathbf{H}(\mathbf{f}) = \mathbf{H}(\mathbf{g}) = 0$ for all $i \in \mathbb{Z}$, because all the homeomorphisms on the circle has zero entropy (see (2.1.2)). Note that ϕ and the identity are not conjugate on $\mathbb{S}^1$. On the other hand, we have

$$\mathcal{V}^s((p_N, 0), \mathbf{f}) = [M \setminus \{p_S\}] \times \{0\} \quad \text{and} \quad \mathcal{V}^s((p_N, 0), \mathbf{g}) = \{(p_N, 0)\}.$$

Since uniform conjugacy preserves the stable sets, we have that $\mathbf{f}$ and $\mathbf{g}$ can not be uniformly conjugate.

Figure 2.2.1: Graph of ϕ .

In the above example we have that both ϕ and I have zero entropy. The next example proves that we can have two homeomorphisms ϕ and ψ with $\mathcal{H}(\phi) = \mathcal{H}(\psi) > 0$ and such that their associated constant families are not uniformly conjugate.

Example 2.2.13. Let $A : M \rightarrow M$ be a homeomorphism with a fixed point $z_0 \in M$ and $\mathcal{H}(A) > 0$ (A could be, for instance, an Anosov diffeomorphism induced by a 2x2 hyperbolic matrix defined on the 2-torus) and take ϕ as defined in the above example. Take $\psi = A \times \phi : M \times \mathbb{S}^1 \rightarrow M \times \mathbb{S}^1$, $(p, z) \mapsto (A(p), \phi(z))$ and $\zeta = A \times I : M \times \mathbb{S}^1 \rightarrow M \times \mathbb{S}^1$, $(p, z) \mapsto (A(p), z)$. Thus $\mathcal{H}(\psi) = \mathcal{H}(\zeta) = \mathcal{H}(A) > 0$. We can prove that the constant families associated to ψ and ζ are not uniformly conjugate.

2.3 On the continuity of Entropy for the Product Topology

As we said in the introduction, the main goal of this chapter is prove that topological entropy for non-stationary dynamical systems is a continuous map on $\mathcal{F}^m(\mathbf{M})$ endowed with the strong topology. This will be shown in the next section. In contrast, if we consider the product topology on $\mathcal{F}^m(\mathbf{M})$, we have that:

Proposition 2.3.1. *Suppose that $\mathbf{H}(\mathcal{F}^m(\mathbf{M}))$ has two or more elements. Then*

$$\mathbf{H} : (\mathcal{F}^m(\mathbf{M}), \tau_{prod}) \rightarrow \mathbb{R} \cup \{+\infty\}$$

is discontinuous at any $\mathbf{f} \in \mathcal{F}^m(\mathbf{M})$.

Proof. Let $\mathbf{f} = (f_i)_{i \in \mathbb{Z}} \in \mathcal{F}^m(\mathbf{M})$. Since $\mathbf{H}(\mathcal{F}^m(\mathbf{M}))$ has two or more elements, there exists $\mathbf{g} = (g_i)_{i \in \mathbb{Z}} \in \mathcal{F}^m(\mathbf{M})$ such that $\mathbf{H}(\mathbf{g}) \neq \mathbf{H}(\mathbf{f})$. Let $\mathcal{V} \in \tau_{prod}$ be an open neighborhood of $\mathbf{f}$. For some

$k \in \mathbb{N}$, the family $\mathbf{h} = (h_i)_{i \in \mathbb{Z}}$, defined by

$$h_i = \begin{cases} f_i & \text{if } -k \leq i \leq k \\ g_i & \text{if } i > k \text{ or } i < -k, \end{cases}$$

belongs to $\mathcal{V}$, by definition of τ_{prod} . It follows from Corollary 2.2.6 that

$$\mathbf{H}(\mathbf{h}) = \mathbf{H}(\mathbf{g}),$$

which proves the proposition, since $(\mathcal{F}^m(\mathbf{M}), \tau_{prod})$ a metric space. $\square$

From Proposition 2.2.4 we have that, if $M = \mathbb{S}^1$, then $\mathbf{H}(\mathcal{F}^m(\mathbf{M})) = \{0\}$ for every $m \geq 0$. In this case, $\mathbf{H} : (\mathcal{F}^m(\mathbf{M}), \tau_{prod}) \rightarrow \mathbb{R} \cup \{+\infty\}$ is continuous.

Below we will see some interesting results that are obtained when we consider only the constant families while maintaining the product topology on $\mathcal{F}^m(\mathbf{M})$.

Set

$$\mathcal{CF}^m(\mathbf{M}) = \{\mathbf{f} \in \mathcal{F}^m(\mathbf{M}) : \mathbf{f} \text{ is a constant family}\},$$

$$\tilde{\tau}_{str} = \tau_{str}|_{\mathcal{CF}^m(\mathbf{M})} \text{ and } \tilde{\tau}_{prod} = \tau_{prod}|_{\mathcal{CF}^m(\mathbf{M})}.$$

Proposition 2.3.2. $\tilde{\tau}_{str} = \mathcal{P}(\mathcal{CF}^m(\mathbf{M})) = \{A : A \subseteq \mathcal{CF}^m(\mathbf{M})\}$.

Proof. It is sufficient to prove that each $\{(f_i)_{i \in \mathbb{Z}}\}$, with $(f_i)_{i \in \mathbb{Z}} \in \mathcal{CF}^m(\mathbf{M})$, is open in $\mathcal{CF}^m(\mathbf{M})$. Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be a sequence of positive numbers with $\varepsilon_i \rightarrow 0$ when $|i| \rightarrow \pm\infty$. Then, consider the strong basic neighborhood $B^m((f_i)_{i \in \mathbb{Z}}, (\varepsilon_i)_{i \in \mathbb{Z}}) \subseteq \mathcal{F}^m(\mathbf{M})$ of $(f_i)_{i \in \mathbb{Z}}$. Notice that

$$\{(f_i)_{i \in \mathbb{Z}}\} = B^m((f_i)_{i \in \mathbb{Z}}, (\varepsilon_i)_{i \in \mathbb{Z}}) \cap \mathcal{CF}^m(\mathbf{M}).$$

Consequently, $\{(f_i)_{i \in \mathbb{Z}}\}$ is open in $(\mathcal{CF}^m(\mathbf{M}), \tilde{\tau}_{str})$. $\square$

The map

$$\begin{aligned} \pi_0 : (\mathcal{F}^m(\mathbf{M}), \tau) &\rightarrow (\mathbf{D}_0^m, d^m) \\ (f_i)_{i \in \mathbb{Z}} &\mapsto f_0, \end{aligned}$$

where $\mathbf{D}_i^m = \text{Diff}^m(M_i, M_{i+1})$ for $i \in \mathbb{Z}$, is continuous for $\tau \in \{\tau_{str}, \tau_{prod}\}$, because

$$\pi_0^{-1}(U) = \prod_{i < 0} \mathbf{D}_i^m \times [U \times \{0\}] \times \prod_{i > 0} \mathbf{D}_i^m,$$

for $U \subseteq \mathbf{D}_0^m$, thus, if U is an open subset of $\mathbf{D}_0^m$, then $\pi_0^{-1}(U)$ is open in $(\mathcal{F}^m(\mathbf{M}), \tau_{prod})$. Hence, the restriction

$$\tilde{\pi}_0 = \pi_0|_{\mathcal{CF}^m(\mathbf{M})} : (\mathcal{CF}^m(\mathbf{M}), \tilde{\tau}) \rightarrow (\mathbf{D}_0^m, d^m)$$

is continuous for $\tilde{\tau} \in \{\tilde{\tau}_{str}, \tilde{\tau}_{prod}\}$.

We can identify $(\mathbf{D}_0^m, d^m)$ with $(\text{Diff}^m(M), d^m)$, the space consisting of diffeomorphisms on M endowed with the C^m -metric obtained from the metric d on M (remember that $M_i = M \times \{i\}$).

From now on we will make use of this identification. For a C^m -diffeomorphism $\phi : M \rightarrow M$, we denote the constant family associated to ϕ by $\mathbf{f}_\phi$. Notice that $\tilde{\pi}_0$ is invertible, in fact,

$$\begin{aligned} \tilde{\pi}_0^{-1} : (\text{Diff}^m(M), d^m) &\rightarrow (\mathcal{CF}^m(\mathbf{M}), \tilde{\tau}) \\ \phi &\mapsto \mathbf{f}_\phi. \end{aligned}$$

Clearly, if $\tilde{\tau} = \tilde{\tau}_{str}$, then $\tilde{\pi}_0^{-1}$ is not continuous (see Proposition 2.3.2). On the other hand, we have:

Proposition 2.3.3. *If $\tilde{\tau} = \tilde{\tau}_{prod}$, then $\tilde{\pi}_0^{-1}$ is continuous.*

Proof. All the open subsets of $(\mathcal{CF}^m(\mathbf{M}), \tilde{\tau}_{prod})$ are unions of sets of the form

$$\mathcal{U} = \left(\prod_{i < -j} \mathbf{D}_i^m \times \prod_{i=-j}^j [U_i] \times \prod_{i > j} \mathbf{D}_i^m \right) \cap \mathcal{CF}^m(\mathbf{M}),$$

where U_i is an open subset of $\mathbf{D}_i^m$, for $-j \leq i \leq j$. Notice that

$$(\tilde{\pi}_0^{-1})^{-1}(\mathcal{U}) = \tilde{\pi}_0(\mathcal{U}) = \bigcap_{i=-j}^j U_i,$$

which is an open subset of $\text{Diff}^m(M)$. Thus, $\tilde{\pi}_0^{-1}$ is continuous. $\square$

Consequently, we have:

Proposition 2.3.4. *$\mathbf{H} : (\mathcal{CF}^m(\mathbf{M}), \tilde{\tau}_{prod}) \rightarrow \mathbb{R} \cup \{+\infty\}$, is continuous if, and only if, $\mathcal{H} : (\text{Diff}^m(M), d^m) \rightarrow \mathbb{R} \cup \{+\infty\}$, is continuous.*

Proof. This is clear, because $\mathbf{H} = \mathcal{H} \circ \tilde{\pi}$ and $\tilde{\pi}$ is a homeomorphism. $\square$

Remark 2.3.5. Proposition 2.3.4 could be a useful tool to show the continuity of the topological entropy at some C^m -diffeomorphisms: to show that $\mathcal{H}$ is continuous at $\phi \in \text{Diff}^m(M)$, we could try to prove that $\mathbf{H}|_{\mathcal{CF}^m(\mathbf{M})}$ is continuous at $\mathbf{f}_\phi$. In order to prove this fact, we have to find an open neighborhood $\mathcal{U} \subseteq \text{Diff}^m(M)$ of ϕ , such that each constant family associated to any diffeomorphism in $\mathcal{U}$ is uniformly conjugate to $\mathbf{f}_\phi$. Thus, by Theorem 2.2.5 and (2.1.2), we had that

$$\mathcal{H}(\psi) = \mathbf{H}(\mathbf{f}_\psi) = \mathbf{H}(\mathbf{f}_\phi) = \mathcal{H}(\phi) \quad \text{for any } \psi \in \mathcal{U}.$$

Remember that Proposition 1.4.5 proves that there exist diffeomorphisms ϕ and ψ which are not topologically conjugate, however $\mathbf{f}_\phi$ and $\mathbf{f}_\psi$ could be uniformly conjugate.

2.4 Continuity of Entropy with respect to the Strong Topology

Finally, we will prove the continuity of $\mathbf{H} : (\mathcal{F}^m(\mathbf{M}), \tau_{str}) \rightarrow \mathbb{R} \cup \{+\infty\}$ for the strong topology on $\mathcal{F}^m(\mathbf{M})$, for $m \geq 1$. More specifically, entropy is locally constant, that is, each $(f_i)_{i \in \mathbb{Z}} \in \mathcal{F}^m(\mathbf{M})$ has a strong basic neighborhood in which the entropy is constant. It is sufficient to prove the case when $m = 1$. Remember that we are considering a compact Riemannian manifold M with metric d and we will consider the metric $\mathbf{d}$ on $\mathbf{M}$ given in (1.1.1).

Definition 2.4.1. Let $\varrho > 0$ be such that, for each $p \in M$, the exponential application

$$\exp_p : B(0_p, \varrho) \rightarrow B(p, \varrho)$$

is a diffeomorphism and

$$\|v\| = d(\exp_p(v), p), \text{ for all } v \in B(0_p, \varrho),$$

where 0_p is the zero vector in $T_p M$, the tangent space of M at p . ϱ is called an *injectivity radius*

Figure 2.4.1: Exponential application.

of M at p (see Figure 2.4.1). See [dC92] for more detail.

We will denote by ϱ the injectivity radius of each $p \in M$ and we will suppose that $\varrho < 1/2$. We will fix $\mathbf{f} = (f_i)_{i \in \mathbb{Z}} \in \mathcal{F}^1(\mathbf{M})$. For $\delta > 0$ and $r = 0, 1$, set

$$D^r(I_i, \delta) = \{h \in \text{Hom}(M_i) : h \text{ is a } C^r\text{-diffeomorphism and } d^r(h, I_i) \leq \delta\}$$

$$\text{and } D^1(f_i, \delta) = \{g \in \text{Diff}^1(M_i, M_{i+1}) : d^1(g, f_i) \leq \delta\}.$$

The closure of $D^1(I_i, \delta)$ on $D^0(I_i, \delta)$ will be denoted by $\overline{D^1(I_i, \delta)}$.

Lemma 2.4.2. *There exist two sequences $(r_i)_{i \geq 0}$ and $(\delta_i)_{i \geq 0}$, with $r_i \rightarrow 0$ when $i \rightarrow +\infty$, such that, for each $g \in D^1(f_i, \delta_i)$, the map*

$$\begin{aligned} \tilde{G}_{i+1} : D^r(I_{i+1}, r_{i+1}) &\rightarrow D^r(I_i, r_i) \\ h &\mapsto g^{-1} h f_i \end{aligned}$$

is well-defined for each $i \geq 1$ and $r = 0, 1$. (see Figure 2.4.2).

Proof. Note that if $g \in \text{Diff}^1(M_i, M_{i+1})$ and $h \in \text{Diff}^r(M_{i+1}, M_{i+1})$, we have

$$d^r(g^{-1} h f_i, I_i) \leq d^r(g^{-1} h f_i, g^{-1} f_i) + d^r(g^{-1} f_i, I_i) \quad \text{for } i \geq 0.$$

If h is C^r -close to I_{i+1} , then $g^{-1} h f_i$ is C^r -close to $g^{-1} f_i$ and if g is C^1 -close to f_i , then $g^{-1} f_i$ is C^r -close to I_i . Fix $r_0 \in (0, \varrho/4)$. There exist $r_1 \in (0, r_0/2)$ and $\delta_0 > 0$ such that, if $h_1 \in D^r(I_1, r_1)$ and $g_0 \in D^1(f_0, \delta_0)$, then $g_0^{-1} h_1 f_0 \in D^r(I_0, r_0)$. Take $r_2 \in (0, r_1/2)$ and $\delta_1 > 0$ such that, if $h_2 \in D^r(I_2, r_2)$ and $g_1 \in D^1(f_1, \delta_1)$, then $g_1^{-1} h_2 f_1 \in D^r(I_1, r_1)$. Inductively, we can build two

Figure 2.4.2: Shaded regions represent the discs $D^r(I_i, r_i)$. $\mathcal{G}(\phi)$ is the graph of the map ϕ

sequences $(r_i)_{i \geq 0}$ and $(\delta_i)_{i \geq 0}$, with $r_i \in (0, r_{i-1}/2)$ for each $i \geq 1$, such that if $h_i \in D^r(I_i, r_i)$ and $g_{i-1} \in D^1(f_{i-1}, \delta_{i-1})$, then $g_{i-1}^{-1} h_i f_{i-1} \in D^r(I_{i-1}, r_{i-1})$, which proves the lemma. $\square$

Analogously, we can find a sequence of positive numbers $(\delta_i)_{i \leq 0}$ and $(r_i)_{i \leq 0}$, with $r_i \rightarrow 0$ when $i \rightarrow -\infty$, such that for each $g \in D^1(f_{i-1}, \delta_{i-1})$, the map

$$\begin{aligned} \hat{G}_{i-1} : D^r(I_{i-1}, r_{i-1}) &\rightarrow D^r(I_i, r_i) \\ h &\mapsto f_{i-1} h g^{-1} \end{aligned}$$

is well-defined for each $i \leq 0$ and $r = 0, 1$.

Lemma 2.4.3. *There exist two sequences $\tilde{h} = (\tilde{h}_i)_{i \geq 0} \in \prod_{i \geq 0} D^0(I_i, r_i)$ and $\hat{h} = (\hat{h}_i)_{i \leq 0} \in \prod_{i \leq 0} D^0(I_i, r_i)$ such that*

$$\tilde{G}_{i+1} \tilde{h}_{i+1} = \tilde{h}_i \text{ for all } i \geq 0 \quad \text{and} \quad \hat{G}_{i-1} \hat{h}_{i-1} = \hat{h}_i \text{ for all } i \leq 0.$$

Proof. For each $i > 0$, let $h^i = \tilde{G}_1 \circ \dots \circ \tilde{G}_i(I_i)$. It follows from Lemma 2.4.2 that h^i belongs to $D^1(I_0, r_0)$. Consequently, the sequence $(h^i)_{i \geq 0}$ is equicontinuous, because each h^i is C^1 and the sequence has uniformly bounded derivative. Hence, there exist a subsequence $i_m \rightarrow \infty$ and $\tilde{h}_0 \in D^0(I_0, r_0)$ such that $h^{i_m} \rightarrow \tilde{h}_0$ as $m \rightarrow \infty$. Note that

$$\tilde{G}_1 : \overline{D^1(I_1, r_1)} \rightarrow \tilde{G}_1(\overline{D^1(I_1, r_1)}) \subseteq D^0(I_0, r_0)$$

is invertible, where $\overline{D^1(I_1, r_1)}$ is the closure in $D^0(I_1, r_1)$, and both $\tilde{G}_1$ and $\tilde{G}_1^{-1}$ are continuous. Therefore,

$$G_1(\overline{D^1(I_1, r_1)}) = \overline{G_1(D^1(I_1, r_1))}.$$

Since $\tilde{h}_0 \in \overline{G_1(D^1(I_1, r_1))}$, we have

$$\tilde{h}_1 = \tilde{G}_1^{-1}(\tilde{h}_0) \in \overline{D^1(I_1, r_1)} \subseteq D^0(I_1, r_1).$$

Inductively, we can prove

$$\tilde{h}_i = \tilde{G}_i^{-1} \circ \dots \circ \tilde{G}_1^{-1}(\tilde{h}_0) \in D^0(I_i, r_i) \quad \text{for each } i \geq 1.$$

Take $\tilde{\mathbf{h}} = (\tilde{h}_i)_{i \geq 0}$. It is clear that $\tilde{G}_{i+1}\tilde{h}_{i+1} = \tilde{h}_i$ for all $i \geq 0$.

The proof of the existence of $\tilde{\mathbf{h}}$ is analogous and therefore we omit it. $\square$

Note that $\tilde{h}_0$ is a limit of C^1 -diffeomorphisms, which are $\varrho/4$ -close to I_0 in the C^1 -topology. Consequently, for each $x \in M_0$,

$$[\exp_x^{-1} \circ \tilde{h}_0 \circ \exp_x - \exp_x^{-1} \circ I_0 \circ \exp_x]|_{B(0_x, \varrho)}$$

is $\varrho/4$ -Lipschitz. Since $\varrho < 1$, we can prove that $\tilde{h}_0$ is injective. Furthermore, for each $i \geq 0$ and $x \in M_i$, we have

$$\mathbf{d}(\tilde{h}_i^{-1}(x), x) = \mathbf{d}(\tilde{h}_i^{-1}(x), \tilde{h}_i \tilde{h}_i^{-1}(x)) = \mathbf{d}(y, \tilde{h}_i(y)),$$

where $y = \tilde{h}_i^{-1}(x)$. Hence $d^0(\tilde{h}_i, I_i) = d^0(\tilde{h}_i^{-1}, I_i)$ for each $i \geq 0$.

Analogously, we can prove that $\hat{h}_i$ is invertible and $d^0(\hat{h}_i, I_i) = d^0(\hat{h}_i^{-1}, I_i)$ for each $i \leq 0$.

Lemma 2.4.4. *The families $(\tilde{h}_i)_{i \geq 0}$, $(\tilde{h}_i^{-1})_{i \geq 0}$, $(\hat{h}_i)_{i \leq 0}$ and $(\hat{h}_i^{-1})_{i \leq 0}$ are equicontinuous.*

Proof. Let $\varepsilon > 0$. Since $\tilde{h}_i, \tilde{h}_i^{-1} \in D^0(I_i, r_i)$ and $r_i \rightarrow 0$ when $i \rightarrow +\infty$, there exists $k > 0$ such that, for each $i > k$,

$$\max\{d^0(\tilde{h}_i, I_i), d^0(\tilde{h}_i^{-1}, I_i)\} < \varepsilon/3.$$

Hence, if $i < k$ and $x, y \in M_i$ with $\mathbf{d}(x, y) < \varepsilon/3$, then

$$\mathbf{d}(\tilde{h}_i(x), \tilde{h}_i(y)) \leq \mathbf{d}(\tilde{h}_i(x), I_i(x)) + \mathbf{d}(I_i(x), I_i(y)) + \mathbf{d}(I_i(y), \tilde{h}_i(y)) < \varepsilon$$

and

$$\mathbf{d}(\tilde{h}_i^{-1}(x), \tilde{h}_i^{-1}(y)) \leq \mathbf{d}(\tilde{h}_i^{-1}(x), I_i(x)) + \mathbf{d}(I_i(x), I_i(y)) + \mathbf{d}(I_i(y), \tilde{h}_i^{-1}(y)) < \varepsilon.$$

On the other hand, it is clear that there exists $\delta \in (0, \varepsilon/3)$ such that, if $0 \leq i \leq k$, and $x, y \in M_i$ with $\mathbf{d}(x, y) < \delta$, then

$$\max\{\mathbf{d}(\tilde{h}_i(x), \tilde{h}_i(y)), \mathbf{d}(\tilde{h}_i(x)^{-1}, \tilde{h}_i^{-1}(y))\} < \varepsilon.$$

The facts above prove that for each $i \geq 0$, if $x, y \in M_i$ and $\mathbf{d}(x, y) < \delta$, then

$$\max\{\mathbf{d}(\tilde{h}_i(x), \tilde{h}_i(y)), \mathbf{d}(\tilde{h}_i^{-1}(x), \tilde{h}_i^{-1}(y))\} < \varepsilon.$$

Consequently, $(\tilde{h}_i)_{i \geq 0}$ and $(\tilde{h}_i^{-1})_{i \geq 0}$ are equicontinuous. Analogously we can prove that $(\hat{h}_i)_{i \leq 0}$ and $(\hat{h}_i^{-1})_{i \leq 0}$ are equicontinuous. $\square$

Finally, we have:

Theorem 2.4.5. *For all $m \geq 1$,*

$$H : (\mathcal{F}^m(\mathbf{M}), \tau_{str}) \rightarrow \mathbb{R} \cup \{+\infty\} \quad \text{and} \quad H^{(-1)} : (\mathcal{F}^m(\mathbf{M}), \tau_{str}) \rightarrow \mathbb{R} \cup \{+\infty\}$$

are locally constants.

Proof. Let $\mathbf{f} \in \mathcal{F}^m(\mathbf{M})$. It follows from Lemmas 2.4.3 and 2.4.4 there exists a sequence of positive numbers $(r_i)_{i \in \mathbb{Z}}$ such that every $\mathbf{g} \in B^1(\mathbf{f}, (r_i)_{i \in \mathbb{Z}})$ is positively and negatively uniformly conjugate

to $\mathbf{f}$. Thus, from Theorem 2.2.5 we have

$$\mathbf{H}(\mathbf{g}) = \mathbf{H}(\mathbf{f}) \quad \text{and} \quad \mathbf{H}^{(-1)}(\mathbf{g}) = \mathbf{H}^{(-1)}(\mathbf{f}), \quad \text{for every } \mathbf{g} \in B^1(\mathbf{f}, (r_i)_{i \in \mathbb{Z}}),$$

which proves the theorem. $\square$

Theorem 2.4.5 means that if $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ and $\mathbf{g} = (g_i)_{i \in \mathbb{Z}}$ are two non-autonomous dynamical systems such that $d^1(f_i, g_i) \rightarrow 0$ very quickly as $i \rightarrow +\infty$, then $\mathbf{H}(\mathbf{g}) = \mathbf{H}(\mathbf{f})$. In particular, if $\phi : M \rightarrow M$ is a fixed diffeomorphism and $d^1(f_i, \phi) \rightarrow 0$ very quickly as $i \rightarrow +\infty$, then $\mathbf{H}(\mathbf{f}) = \mathcal{H}(\phi)$. On the other hand, Kolyada and Snoha in [KS96], Theorem E, proved that for any non-autonomous dynamical system $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ such that $d^0(f_i, \phi) \rightarrow 0$ as $i \rightarrow +\infty$, we always have $\mathbf{H}(\mathbf{f}) \leq \mathcal{H}(\phi)$. They gave an example where the inequality is strict.

Remark 2.4.6. Summarizing the results on the entropy for non-stationary dynamical systems shown in this Chapter, we have:

1. If $\mathbf{H}(\mathcal{F}^m(\mathbf{M}))$ has two or more elements, $\mathbf{H} : (\mathcal{F}^m(\mathbf{M}), \tau_{prod}) \rightarrow \mathbb{R} \cup \{+\infty\}$ is discontinuous at any $\mathbf{f} \in \mathcal{F}^m(\mathbf{M})$ (Proposition 2.3.1);
2. $\mathbf{H} : (\mathcal{CF}^m(\mathbf{M}), \tau_{unif}) \rightarrow \mathbb{R} \cup \{+\infty\}$ is continuous if, and only if, $\mathcal{H} : (\text{Diff}^m(M), d^m) \rightarrow \mathbb{R} \cup \{+\infty\}$, is continuous (see Proposition 2.3.4).
3. $\mathbf{H} : (\mathcal{F}^m(\mathbf{M}), \tau_{str}) \rightarrow \mathbb{R} \cup \{+\infty\}$ is a continuous map (Theorem 2.4.5).

Chapter 3

Anosov Families

Anosov families, which will be presented in Definition 3.1.2, were introduced by P. Arnoux and A. Fisher in [AF05], motivated by generalizing the notion of Anosov diffeomorphism (see [KH97], [Shu13], [Via14]). These families are non-stationary dynamical systems with a similar behavior to Anosov diffeomorphisms: the tangent space at each point in the total space has a splitting in two subspaces, one stable and the other unstable. Readers may find, for example, in [Bak95a], [Bak95b], and more recently, in [Ste11], several approaches and results in non-stationary dynamical systems in which each diffeomorphism in the sequence has a hyperbolic behavior. Example 3.2.9 proves that the Anosov families do not necessarily consist of Anosov diffeomorphisms. Random dynamical systems with hyperbolic behavior can be found in [Liu98].

In this chapter we will give the definition of Anosov families and we will also see some interesting examples and some properties that satisfy such families.

3.1 Anosov Families: Definition

In this section we will introduce the notion of Anosov family. Before that we remember the notion of hyperbolic sets and Anosov diffeomorphism:

Definition 3.1.1. Let M be a Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle$ and let $\| \cdot \|$ be the norm induced by $\langle \cdot, \cdot \rangle$ on M . Let $\phi : M \rightarrow M$ be a C^1 -diffeomorphism. A compact subset Λ of M is *hyperbolic* for ϕ if:

- i. The tangent bundle $T\Lambda$ has a continuous splitting $E^s \oplus E^u$ which is $D\phi$ -invariant, that is, for each $p \in \Lambda$, $T_p\Lambda = E_p^s \oplus E_p^u$ with $D_p\phi(E_p^s) = E_{\phi(p)}^s$ and $D_p\phi(E_p^u) = E_{\phi(p)}^u$;
- ii. there exist constants $\lambda \in (0, 1)$ and $c > 0$ such that for each $n \geq 1$, $p \in \Lambda$, we have:

$$\|D_p\phi^{-n}(v)\| \leq c\lambda^n\|v\| \quad \text{for every vector } v \in E_p^u,$$

and

$$\|D_p\phi^n(v)\| \leq c\lambda^n\|v\| \quad \text{for every vector } v \in E_p^s.$$

See Figure 3.1.1.

In the above definition, if $\Lambda = M$, then ϕ is called an *Anosov diffeomorphism*.

Figure 3.1.1: $q = \phi^{-1}(p)$ and $z = \phi(p)$. $D(\phi)_q(A) = B$ and $D(\phi)_p(B) = C$

Definition 3.1.2. From now on, if we do not say otherwise, M_i will be a Riemannian manifold with fixed Riemannian metric $\langle \cdot, \cdot \rangle_i$, for each $i \in \mathbb{Z}$. We denote by $\| \cdot \|_i$ the induced norm by $\langle \cdot, \cdot \rangle_i$ on TM_i and we will take $\| \cdot \|$ defined on $\mathbf{M}$ as $\| \cdot \|_{M_i} = \| \cdot \|_i$ for $i \in \mathbb{Z}$. An *Anosov family* is a non-stationary dynamical system $\mathbf{f} = (f_i)_{i \in \mathbb{Z}} \in \mathcal{F}^1(\mathbf{M})$ such that:

- i. the tangent bundle $T\mathbf{M}$ has a continuous splitting $E^s \oplus E^u$ which is $D\mathbf{f}$ -invariant, i. e., for each $p \in \mathbf{M}$, $T_p \mathbf{M} = E_p^s \oplus E_p^u$ with

$$D\mathbf{f}_p(E_p^s) = E_{\mathbf{f}(p)}^s \quad \text{and} \quad D\mathbf{f}_p(E_p^u) = E_{\mathbf{f}(p)}^u,$$

where $T_p \mathbf{M}$ is the tangent space at p ;

- ii. there exist constants $\lambda \in (0, 1)$ and $c > 0$ such that for each $i \in \mathbb{Z}$, $n \geq 1$, and $p \in M_i$, we have:

$$\|D(\mathbf{f}_i^{-n})_p(v)\| \leq c\lambda^n \|v\| \quad \text{for each vector } v \in E_p^u,$$

and

$$\|D(\mathbf{f}_i^n)_p(v)\| \leq c\lambda^n \|v\| \quad \text{for each vector } v \in E_p^s.$$

Without loss of generality, we can consider $c \geq 1$, since otherwise it can be replaced by 1; if we can take $c = 1$ we say the family is *strictly Anosov*.

Remark 3.1.3. The notion of Anosov family depends on the Riemannian metric considered on each M_i (see Example 3.2.1). We will use the notation $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ for point out we are fixing the Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathbf{M}$.

In Proposition 3.3.4 we will show that there exists a Riemannian metric $\langle \cdot, \cdot \rangle^*$, defined on the total space, equivalent to $\langle \cdot, \cdot \rangle$ on each M_i , such that $(\mathbf{M}, \langle \cdot, \cdot \rangle^*, \mathbf{f})$ is an strictly Anosov family.¹ In the case of a diffeomorphism on a compact Riemannian manifold this fact is known as Lemma of Mather (see [Shu13]).

Notice that for an Anosov family, there are many invariant splittings of the tangent bundle TM_i , which are $D\mathbf{f}$ -invariant. Simply we can choose a splitting of TM_0 and transport it forward and backward to the other components by $D\mathbf{f}$: for instance, if $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ is an Anosov family,

¹The Riemannian metric $\langle \cdot, \cdot \rangle^*$ to be built in the Proposition 3.3.4 is not necessarily uniformly equivalent to $\langle \cdot, \cdot \rangle$ on $\mathbf{M}$.

then for $p \in \mathbf{M}$ and $v \in E_p^s$, the family $F^u = \langle \{Df_p^n(v)\}_{n \in \mathbb{Z}} \rangle \cup E^u$ is Df -invariant and its vectors are expanded by Df^n when $n \rightarrow +\infty$. Analogously, if $w \in E_p^u$, $F^s = \langle \{Df_p^n(w)\}_{n \in \mathbb{Z}} \rangle \cup E^s$ is Df -invariant and its vectors are expanded by Df^{-n} when $n \rightarrow +\infty$. However, these expansions do not satisfy the condition (ii) in Definition 3.1.2, that is, these splittings do not satisfy the condition of *hyperbolicity*. Actually, in [AF05], Proposition 2.12, it is shown for an Anosov family that the splitting $T_p\mathbf{M} = E_p^s \oplus E_p^u$ satisfying the condition of hyperbolicity is unique.

Lemma 3.1.4. *Fix $p \in M_i$.*

i. *Let $v \in E_p^u$. The condition*

$$\|D(f_i^{-n})_p(v)\| \leq c\lambda^n \|v\| \quad \text{for each } i \in \mathbb{Z}, n \geq 1,$$

is equivalent to the condition

$$\|D(f_i^n)_p(v)\| \geq c^{-1}\lambda^{-n} \|v\| \quad \text{for each } i \in \mathbb{Z}, n \geq 1.$$

ii. *Let $v \in E_p^s$. The condition*

$$\|D(f_i^n)_p(v)\| \leq c\lambda^n \|v\| \quad \text{for each } i \in \mathbb{Z}, n \geq 1,$$

is equivalent to

$$\|D(f_i^{-n})_p(v)\| \geq c^{-1}\lambda^{-n} \|v\| \quad \text{for each } i \in \mathbb{Z}, n \geq 1.$$

Proof. See [AF05], Lemma 2.7. □

Lemma 3.1.5. *For each $p \in M_i$ we have*

i. $E_p^s = \{v \in T_p M_i : \|D(f_i^n)_p(v)\| \text{ is bounded, for } n \geq 1\}$.

ii. $E_p^u = \{v \in T_p M_i : \|D(f_i^{-n})_p(v)\| \text{ is bounded, for } n \geq 1\}$.

Proof. Set

$$B_p^s = \{v \in T_p M_i : \|D(f_i^n)_p(v)\| \text{ is bounded, for } n \geq 1\}.$$

It is clear that $E_p^s \subseteq B_p^s$. Suppose that there exists a vector $\tilde{v} \in T_p M_i$ such that $\tilde{v} \notin E_p^s$. Thus $\tilde{v} = av_s + bv_u$, with $bv_u \neq 0$. Therefore

$$\|D(f_i^n)_p(\tilde{v})\| \geq |b| \|D(f_i^n)_p(v_u)\| - |a| \|D(f_i^n)_p(v_s)\| \geq |b| c^{-1} \lambda^{-n} \|v_u\| - |a| c \lambda^n \|v_s\|,$$

where $\|D(f_i^n)_p(\tilde{v})\| \rightarrow +\infty$, that is, $\tilde{v} \notin B_p^s$. Thus $B_p^s \subseteq E_p^s$.

Analogously we can prove ii. □

Remark 3.1.6. One of the conditions in Definition 3.1.2 is the continuity of the splitting $E^s \oplus E^u$ of the tangent bundle $T\mathbf{M}$. This means that the subspaces E_p^s and E_p^u depend continuously on p : a family of subspaces $F_p \subseteq T_p\mathbf{M}$ depends continuously on $p \in \mathbf{M}$ if there are continuous maps

$$F_k : \mathbf{M} \rightarrow T\mathbf{M}$$

$$p \mapsto v_k(p), \quad \text{for } k = 1, \dots, \dim F_p,$$

such that $\{v_1(p), \dots, v_{\dim F_p}(p)\}$ is a basis of F_p for each $p \in \mathbf{M}$.

We will see that the continuity of the splitting $E^s \oplus E^u$ can be obtained from the condition (ii) in the Definition 3.1.2 and using the Df -invariance of the splitting. First we will prove the following lemma:

Lemma 3.1.7. *The dimensions of the subspaces E_p^u and E_p^s are locally constant for $p \in \mathbf{M}$.*

Proof. Let $p \in \mathbf{M}$ and $k = \dim E_p^s$. Suppose by contradiction that there exists a sequence $(p_m)_{m \in \mathbb{N}} \in \mathbf{M}$ converging to p such that $\dim E_{p_m}^s \geq k + 1$. Take a sequence of orthonormal vectors

$$v_1(p_m), \dots, v_k(p_m), v_{k+1}(p_m) \quad \text{in } E_{p_m}^s, \text{ for each } m.$$

Choosing a suitable subsequence, we can suppose that

$$v_1(p_m) \rightarrow v_1 \in T_p \mathbf{M}, \dots, v_{k+1}(p_m) \rightarrow v_{k+1} \in T_p \mathbf{M} \quad \text{as } m \rightarrow \infty.$$

It follows from condition (ii) in Definition 3.1.2 that, for all $n \geq 1$,

$$\|D_p(f^n)(v_s)\| \leq c\lambda^n \|v_s\| \quad \text{for each } s = 1, \dots, k+1. \quad (3.1.1)$$

By Lemma 3.1.5 we obtain $v_1, \dots, v_{k+1} \in E_p^s$. Since $v_1(p_m), \dots, v_k(p_m)$, and $v_{k+1}(p_m)$ are orthonormal for all $n \geq 1$, by the continuity of the Riemannian metric we have that $v_1, \dots, v_{k+1}$ are orthonormal, which contradicts that $\dim E_p^s = k$.

Similarly we can prove that there is not any sequence p_m converging to p with $\dim E_{p_m}^s < k$. Therefore, the dimension of E_p^s is locally constant.

Analogously we obtain that the dimension of E_p^u is locally constant. $\square$

The following proposition is a version for non-stationary dynamical systems of Proposition 5.2.1 in [BS02].

Proposition 3.1.8. *Let $(\mathbf{M}, \langle \cdot, \cdot \rangle, f)$ be a non-stationary dynamical system of class C^1 . Suppose that $T\mathbf{M}$ has a splitting $E^s \oplus E^u$ which is Df -invariant and satisfies the property (ii) from the definition of Anosov family. Thus, the subspaces E_p^s and E_p^u depend continuously on p .*

Proof. Let $(p_m)_{m \in \mathbb{N}}$ be a sequence in $\mathbf{M}$ such that $p_m \rightarrow p \in \mathbf{M}$ as $m \rightarrow \infty$. Without loss of generality, we can suppose that $(p_m)_{m \in \mathbb{N}} \subseteq M_i$ and $p \in M_i$ for some $i \in \mathbb{Z}$ (see (1.1.1)) and furthermore $\dim E_{p_m}^s = \dim E_p^s = k$ for every $m \geq 1$ (see Lemma 3.1.7). Let $\{v_1(p_m), \dots, v_k(p_m)\}$ be an orthonormal basis of $E_{p_m}^s$, for each $m \geq 1$, such that $v_1(p_m) \rightarrow v_1 \in T_p M_i, \dots, v_k(p_m) \rightarrow v_k \in T_p M_i$ as $m \rightarrow \infty$. Thus $v_1, \dots, v_k$ are orthonormal and belong to E_p^s (see (3.1.1)), which proves that E_p^s depends continuously on p . Analogously we can prove that E_p^u depends continuously on p . $\square$

For each $i \in \mathbb{Z}$, set

$$\theta_i = \min_{p \in M_i} \{\theta_p : \theta_p \text{ is the angle between } E_p^s \text{ and } E_p^u\}. \quad (3.1.2)$$

Since for each $p \in M_i$, the subspaces E_p^s and E_p^u are transversal, that is, $E_p^s \oplus E_p^u = T_p M_i$, then, by the compactness of M_i and the continuity of the Riemannian metric and the subspaces E_p^s and

E_p^u , we obtain that $\theta_i > 0$ for each $i \in \mathbb{Z}$. That is, the angles between the subspaces are uniformly bounded away from zero on each component. Hence, there exists $0 < \mu_i < 1$ such that

$$\cos(\theta_i) \in [\mu_i - 1, 1 - \mu_i] \quad \text{for each } i \in \mathbb{Z}. \quad (3.1.3)$$

Notice that the sequence $(\mu_i)_{i \in \mathbb{Z}}$ could have a subsequence converging to 1, that is, there may be a point $p \in M_0$ and a subsequence $(n_k)_{k \in \mathbb{Z}}$, such that the angles between the subspaces $E_{f^{n_k}(p)}^s$ and $E_{f^{n_k}(p)}^u$ converge to 0, as $k \rightarrow \infty$ or $k \rightarrow -\infty$.

Definition 3.1.9. We say that $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ satisfies the property of angles (s.p.a) if there exists $\mu \in (0, 1)$ such that

$$\cos(\theta_i) \in [\mu - 1, 1 - \mu] \quad \text{for each } i \in \mathbb{Z}. \quad (3.1.4)$$

In Example 3.2.3 we will show that there exist Anosov families that do not satisfy the property of angles.

3.2 Some Examples of Anosov Families

An easy example of an Anosov family is the constant family associated to an Anosov diffeomorphism (see Example 1.1.5). In this section we will see some other examples of Anosov families. The next example, which is due to Arnoux and Fisher [AF05], shows that suitably changing the metric $\langle \cdot, \cdot \rangle_i$ on each M_i , the constant family associated to the identity could become an Anosov family, hence it is important to keep fixed the metrics on each M_i .

Example 3.2.1. For each $i \in \mathbb{Z}$, take $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\mathbf{M}$ be the disjoint union of $M_i = M \times \{i\}$ and $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ be the constant family on $\mathbf{M}$ associated to the identity on $\mathbb{T}^2$. If we endow each M_i with the Riemannian metric $\langle \cdot, \cdot \rangle_i$ induced by the plane, then it is clear that $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ is not an Anosov family, where $\langle \cdot, \cdot \rangle_i = \langle \cdot, \cdot \rangle$ for all i . On the other hand, on each M_i , take the Riemannian metric obtained by the inner product

$$\langle (x_1, y_1), (x_2, y_2) \rangle_i^* = 2^{-2i}x_1x_2 + 2^{2i}y_1y_2 \quad \text{where } (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 \equiv T_pM_i.$$

Hence, the norm induced on M_i by $\langle \cdot, \cdot \rangle_i$ is given by

$$\|(x, y)\|_i^* = \sqrt{(2^{-i}x)^2 + (2^iy)^2} \quad \text{for all } (x, y) \in \mathbb{R}^2 \equiv T_pM_i.$$

Thus, for $p \in M_i$, if $(x, y) \in T_pM_i$, we have $D_p(f_i)(x, y) = (x, y) \in T_pM_{i+1}$. Take E_p^s as being the x -axis and E_p^u as being the y -axis for each $p \in M_i$, $i \in \mathbb{Z}$. Note that, for all $p \in M_i$, $(x, 0) \in E_p^s$ and $(0, y) \in E_{f_i(p)}^u$, we have

$$\|D_p(f_i)(x, 0)\|_{i+1}^* = \|(x, 0)\|_{i+1}^* = \sqrt{(2^{-i-1}x)^2} = \frac{1}{2}\sqrt{(2^{-i}x)^2} = \frac{1}{2}\|(x, 0)\|_i^*$$

and

$$\|D_{f_i(p)}(f_i^{-1})(0, y)\|_i^* = \|(0, y)\|_i^* = \sqrt{(2^iy)^2} = \frac{1}{2}\sqrt{(2^{i+1}y)^2} = \frac{1}{2}\|(0, y)\|_{i+1}^*.$$

Consequently, $(\mathbf{M}, \langle \cdot, \cdot \rangle^*, \mathbf{f})$ is an Anosov family, where $\langle \cdot, \cdot \rangle^*$ is obtained by $\langle \cdot, \cdot \rangle_i^*$ as in (1.1.2).

Notice that $\langle \cdot, \cdot \rangle_i$ and $\langle \cdot, \cdot \rangle_i^*$ in the above example are uniformly equivalent on each M_i , however, $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^*$ are not uniformly equivalent on $\mathbf{M}$.

Another interesting example is the following one:

Example 3.2.2. Let M be a Riemannian manifold with metric $\|\cdot\|$ and $\phi : M \rightarrow M$ is an Anosov diffeomorphism with constant $c \geq 1$ and $\lambda \in (0, 1)$. Take $M_i = M$ for all i with Riemannian norm defined as

$$\|(v_s, v_u)\|_i = \begin{cases} \sqrt{a^{2i}\|v_s\|^2 + b^{2i}\|v_u\|^2} & \text{if } i \geq 0 \\ \|(v_s, v_u)\| & \text{if } i < 0, \end{cases} \quad (3.2.1)$$

where $a, b \in (\lambda, 1/\lambda)$. Consider $\mathbf{M}$ as the disjoint union of each M_i endowed with the metric obtained above. Let $\mathbf{f}$ be the constant family associated to ϕ . We show that $\mathbf{f}$ is an Anosov family with constants $\tilde{c} = c$ and $\tilde{\lambda} = \max\{\lambda, a\lambda, \lambda/b\} < 1$. Indeed, let $TM = E_\phi^s \oplus E_\phi^u$ be the splitting of the tangent bundle TM corresponding to ϕ . Let $v \in E_\phi^s$ and $n \geq 1$. If $i \leq 0$ and $i + n \leq 0$, then

$$\|D\mathbf{f}^n(v)\|_{i+n} = \|D\mathbf{f}^n(v)\|_i \leq c\lambda^n\|v\|_i \leq c\tilde{\lambda}^n\|v\|_i.$$

If $i \leq 0$ and $i + n > 0$, then

$$\|D\mathbf{f}^n(v)\|_{i+n} = a^{n-i}\|D\mathbf{f}^n(v)\| \leq a^{n-i}c\lambda^n\|v\| = c(a\lambda)^{n-i}\lambda^i\|v\|_i \leq c\tilde{\lambda}^n\|v\|_i.$$

If $i > 0$, then for all $n \geq 1$ we have

$$\|D\mathbf{f}^n(v)\|_{i+n} = a^{n+i}\|D\mathbf{f}^n(v)\| \leq a^{n+i}c\lambda^n\|v\| = c(a\lambda)^na^i\|v\| \leq c\tilde{\lambda}^n\|v\|_i.$$

Now, suppose that $v \in E_\phi^u$ and $n \geq 1$. If $i > 0$ and $i - n \geq 0$, then

$$\|D\mathbf{f}^{-n}(v)\|_{i-n} = b^{i-n}\|D\mathbf{f}^{-n}(v)\| \leq b^i c(\lambda/b)^n\|v\| \leq c\tilde{\lambda}^n\|v\|_i.$$

If $i > 0$ and $i - n < 0$, then

$$\|D\mathbf{f}^n(v)\|_{i-n} = \|D\mathbf{f}^n(v)\| \leq c(\lambda/b)^i\lambda^{n-i}b^i\|v\| \leq c\tilde{\lambda}^n\|v\|_i.$$

If $i \leq 0$, then

$$\|D\mathbf{f}^{-n}(v)\|_{i-n} = \|D\mathbf{f}^{-n}(v)\|_i \leq c\lambda^n\|v\|_i \leq c\tilde{\lambda}^n\|v\|_i.$$

Consequently, the splitting $E_\phi^s \oplus E_\phi^u$ induces a splitting of TM with which $\mathbf{f}$ satisfies Definition 3.1.2.

The following example, where we only suitably changed the Riemannian metric on the 2-torus, shows that there exist Anosov families that do not satisfy the property of angles.

Example 3.2.3. Take $M = \mathbb{T}^2$ and let $\phi : M \rightarrow M$ be the Anosov diffeomorphism induced by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues of A are $\lambda = (3 + \sqrt{5})/2 > 1$ and $1/\lambda$. Consider the eigenvectors $v_s = ((1 + \sqrt{5})/2, 1)$ and $v_u = ((1 - \sqrt{5})/2, 1)$ of A associated to λ and $1/\lambda$, respectively (see Figure 3.2.1). Let $(\zeta_i)_{i \in \mathbb{Z}}$

Figure 3.2.1: The square $[0, 1] \times [0, 1]$ is mapped by A to the parallelogram with vertices $(0, 0)$, $(2, 1)$, $(3, 2)$, $(1, 1)$

be a sequence in $[0, 1)$. In the basis $\{v_s, v_u\}$ of $\mathbb{R}^2$, set

$$B_i = \begin{pmatrix} 1 & \zeta_i \\ \zeta_i & 1 \end{pmatrix} \quad i \in \mathbb{Z}.$$

The eigenvalues of B_i are $\alpha_i = 1 + \zeta_i$ and $\beta_i = 1 - \zeta_i$. Since $\zeta_i \in [0, 1)$, the matrix B_i is positive definite. Thus, it induces an inner product $\langle \cdot, \cdot \rangle_i$ on $\mathbb{R}^2$: if $v_1 = av_s + bv_u$, $v_2 = cv_s + dv_u \in \mathbb{R}^2$,

$$\langle v_1, v_2 \rangle_i = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & \zeta_i \\ \zeta_i & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \quad i \in \mathbb{Z}.$$

Notice that the angle between v_s and v_u with the inner product $\langle \cdot, \cdot \rangle_i$ is:

$$\theta_i = \arccos \left(\frac{\langle v_1, v_2 \rangle_i}{\sqrt{\langle v_1, v_1 \rangle_i \cdot \langle v_2, v_2 \rangle_i}} \right) = \zeta_i.$$

Furthermore, if $\|\cdot\|_i$ is the norm induced by $\langle \cdot, \cdot \rangle_i$ and $\|\cdot\|$ is the canonical norm of $\mathbb{R}^2$, we have $\|v_s\|_i = \|v_s\|$ and $\|v_u\|_i = \|v_u\|$ for all $i \in \mathbb{Z}$ (the inner product $\langle \cdot, \cdot \rangle_i$ change only the angles between v_s and v_u). Consequently, $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ is an Anosov family, where $\mathbf{M}$ is the disjoint union of the M_i and $\langle \cdot, \cdot \rangle$ is obtained by $\langle \cdot, \cdot \rangle_i$ as in (1.1.2) and $f_i(x, i) = (\phi(x), i + 1)$ for $x \in M$, $i \in \mathbb{Z}$. If $\zeta_i \rightarrow 0$ as $i \rightarrow \infty$, then $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ is an Anosov family that does not satisfy the property of angles.

Before presenting the next example, we consider the following definition:

Definition 3.2.4 (Linear cocycles). Let X be a compact metric space, $\phi : X \rightarrow X$ a homeomorphism and $A : X \rightarrow SL(\mathbb{Z}, m)$ a continuous application, where $SL(\mathbb{Z}, m)$ is the special linear group of $m \times m$ matrices with integer entries and with determinant 1. The *linear cocycle* defined by A

over ϕ is the transformation

$$F : X \times \mathbb{R}^m \rightarrow X \times \mathbb{R}^m, \quad (x, v) \rightarrow (f(x), A(x)v).$$

The cocycle F is *hyperbolic* if there exist $\lambda \in (0, 1)$ and $c > 0$ such that, for all $x \in M$, there exist subspaces E_x^s and E_x^u of $\mathbb{R}^m$ such that $\mathbb{R}^m = E_x^s \oplus E_x^u$, and furthermore,

- i. $A(x)E_x^s = E_{\phi(x)}^s$ and $A(x)E_x^u = E_{\phi(x)}^u$,
- ii. $\|A^n(x)v_s\| \leq c\lambda^n\|v_s\|$ for $v_s \in E_x^s$ and $\|A^{-n}(x)v_u\| \leq c\lambda^n\|v_u\|$ for $v_u \in E_x^u$,

for all $n \geq 1$, where

$$A^n(x) = A(f^{n-1}(x)) \cdots A(x) \quad \text{and} \quad A^{-n}(x) = A(f^{-n}(x))^{-1} \cdots A(f^{-1}(x))^{-1}.$$

Example 3.2.5. Let F be a hyperbolic linear cocycle defined by $A : X \rightarrow SL(\mathbb{Z}, m)$ over $\phi : X \rightarrow X$, then for each $x \in X$, the family $(A(f^n(x)))_{n \in \mathbb{Z}}$ defined on $M_i = \mathbb{R}^m / \mathbb{Z}^m$, the m -dimensional torus endowed with the Riemannian metric inherited from $\mathbb{R}^m$, induces an Anosov family.

When $m = 2$ we have that the linear cocycle F defined by A over ϕ is hyperbolic if, and only if, there exist constants $\sigma > 1$ and $c > 0$ such that $\|A^n(x)\| \geq c\sigma^n$ for all $x \in X$ and $n \geq 1$ (see [Via14], Proposition 2.1). Let $SL(\mathbb{N}, 2)$ be the set of 2×2 matrices with entries in $\mathbb{N} = \{1, 2, \dots\}$ and with determinant 1. It follows from Proposition 2.7 in [Via14] that if the image of A is in $SL(\mathbb{N}, 2)$, then F is hyperbolic. Notice that, in this case, the image of A is finite, because M is compact. Consequently:

Proposition 3.2.6. *Let $Y = \{F_1, F_2, \dots, F_k\}$ be a subset of $SL(\mathbb{N}, 2)$. Any non-stationary dynamical system $(A_i)_{i \in \mathbb{Z}}$ with values in Y is Anosov.*

Remark 3.2.7. Let $\phi : M \rightarrow M$ be an Anosov diffeomorphism of class C^2 on a compact Riemannian manifold M and $\beta > 0$ such that $\text{Lip}(D\phi) < \beta$, where $\text{Lip}(D\phi)$ is a Lipchitz constant of the derivative application $x \mapsto D\phi_x$. For $\alpha > 0$, take

$$\Omega_{\alpha, \beta}(\phi) = \{\psi \in C^1(M) : d(\phi, \psi) \leq \alpha \text{ and } \text{Lip}(D\psi) \leq \beta\},$$

where $d(\cdot, \cdot)$ is the C^1 -metric on $\text{Diff}^1(M)$. If α is small enough, any sequence $(\psi_i)_{i \in \mathbb{Z}}$ in $\Omega_{\alpha, \beta}(\phi)$ defines an Anosov family in $\mathbf{M} = \coprod_{i \in \mathbb{Z}} M$ (see [You86], Proposition 2.2). Consequently, the set consisting of the constant families associated to Anosov diffeomorphisms of class C^2 is open in $\mathcal{F}^2(\mathbf{M})$.

Using the above fact we have:

Example 3.2.8. Given $\alpha \in \mathbb{R}$, consider $\phi_\alpha : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ defined by

$$\phi_\alpha(x, y) = (2x + y - (1 + \alpha) \sin x \bmod 2\pi, x + y - (1 + \alpha) \sin x \bmod 2\pi).$$

Thus, for all $\alpha \in [-1, 0)$, ϕ_α is an Anosov diffeomorphism (see [BP07]). Notice that ϕ_{-1} is the well-known linear toral automorphism induced by the matrix

$$A_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus we have that given $\alpha^* \in [-1, 0)$ there exists $\varepsilon > 0$ such that, if $(\alpha_i)_{i \in \mathbb{Z}}$ is a sequence in $[-1, 0)$ with $|\alpha_i - \alpha^*| < \varepsilon$, then $(f_i)_{i \in \mathbb{Z}}$ is an Anosov family, where $f_i = \phi_{\alpha_i}$ for each $i \in \mathbb{Z}$.

The next results provide many examples of Anosov families which do not necessarily consist of (perturbations of) Anosov diffeomorphisms. The following example, which was taken of [AF05], proves that the Anosov families do not necessarily consist of Anosov diffeomorphisms.

Example 3.2.9. For any sequence of positive integers $(n_i)_{i \in \mathbb{Z}}$ set

$$A_i = \begin{pmatrix} 1 & 0 \\ n_i & 1 \end{pmatrix} \text{ for } i \text{ even} \quad \text{and} \quad A_i = \begin{pmatrix} 1 & n_i \\ 0 & 1 \end{pmatrix} \text{ for } i \text{ odd}.$$

The family $(A_i)_{i \in \mathbb{Z}}$ is called the *multiplicative family* determined by the sequence $(n_i)_{i \in \mathbb{Z}}$. Since the entries of the matrices A_i are integers and $\det A_i = 1$ for all i , they induce diffeomorphisms f_i on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, that is, each f_i is defined as $\pi A_i = f_i \pi$, where $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ is the canonical projection. Take M_i as $\mathbb{T}^2$ and let $\|\cdot\|$ be the Riemannian metric on $\mathbb{T}^2$ inherited from $\mathbb{R}^2$. For each $i \in \mathbb{Z}$, let $s_i = (a_i, b_i)$, $u_i = (c_i, d_i)$ and $\lambda_i \in (0, 1)$ be such that $a_i d_i + c_i b_i = 1$,

$$\text{for } i \text{ even, } a_i = [n_i n_{i+1} \dots], \quad b_i = 1, \quad \frac{d_i}{c_i} = [n_{i-1} n_{i-2} \dots], \quad \text{and } \lambda_i = a_i,$$

and

$$\text{for } i \text{ odd, } b_i = [n_i n_{i+1} \dots], \quad a_i = 1, \quad \frac{c_i}{d_i} = [n_{i-1} n_{i-2} \dots] \quad \text{and } \lambda_i = b_i.$$

Here, $[n_i n_{i+1} \dots] = \frac{1}{n_i + \frac{1}{n_{i+1} + \dots}}$. Thus, for all $i \in \mathbb{Z}$, $A_i s_i = \lambda_i s_{i+1}$ and $A_i u_i = \lambda_i^{-1} u_{i+1}$ (see [AF05]). Therefore,

$$\|A_i^n s_i\| = \lambda_{i+n-1} \dots \lambda_i \|s_{i+n}\| \leq c \lambda_{i+n-1} \dots \lambda_i \|s_i\|$$

and

$$\|A_i^n u_i\| = \lambda_{i+n-1}^{-1} \dots \lambda_i^{-1} \|u_{i+n}\| \geq c^{-1} \lambda_{i+n-1}^{-1} \dots \lambda_i^{-1} \|u_i\|,$$

where

$$c = \max \left\{ \sup_{i,j} \left\{ \frac{\|s_i\|}{\|s_j\|} \right\}, \sup_{i,j} \left\{ \frac{\|u_i\|}{\|u_j\|} \right\} \right\}$$

($c < \infty$ because $\|v\| \in (1/2, \sqrt{2})$ for all $v \in \{s_i : i \in \mathbb{Z}\} \cup \{u_i : i \in \mathbb{Z}\}$).

Note that, if there exists $\lambda \in (0, 1)$ such that $\lambda_i < \lambda$ for all i , we have

$$\|A_i^n s_i\| \leq c \lambda^n \|s_i\| \quad \text{and} \quad \|A_i^n u_i\| \geq c^{-1} \lambda^{-n} \|u_i\| \quad \text{for all } n \geq 1.$$

This shows that, if there is a $\lambda \in (0, 1)$ such that $\lambda_i \leq \lambda$ for all i , then $f_i : M_i \rightarrow M_{i+1}$ for $i \in \mathbb{Z}$, define an Anosov family, with constants λ and c as defined above, the stable subspaces are spanned by s_i and the unstable subspaces are spanned by u_i . However, we will prove that any multiplicative family is Anosov.

For the rest of this section, we will fix a multiplicative family $(A_i)_{i \in \mathbb{Z}}$ determined by a sequence $(n_i)_{i \in \mathbb{Z}}$. Furthermore, we consider the values c and λ_i defined in Example 3.2.9.

Proposition 3.2.10. $(A_i)_{i \in \mathbb{Z}}$ is an Anosov family with constant $\lambda = \sqrt{2/3}$ and $2c$.

Proof. Notice that, if $\lambda_j \in (2/3, 1)$ for some $j \in \mathbb{Z}$, then $\lambda_{j-1} \in (0, 2/3)$ and $\lambda_{j+1} \in (0, 1/2)$. Indeed, if $\lambda_j = \frac{1}{n_j + \frac{1}{n_{j+1} + \dots}} \in (2/3, 1)$ we must have $n_j = 1$ and $n_{j+1} \geq 2$. Hence,

$$\lambda_{j-1} = \frac{1}{n_{j-1} + \frac{1}{1+\dots}} < \frac{1}{1 + (1/2)} \quad \text{and} \quad \lambda_{j+1} = \frac{1}{n_{j+1} + \frac{1}{n_{j+2} + \dots}} < 1/2.$$

Next, by induction on n , we prove that $c\lambda_{i+n-1} \dots \lambda_i < 2c\lambda^n$, for each $i \in \mathbb{Z}$ and $n \geq 1$. Fix $i \in \mathbb{Z}$. It is clear that if $n = 1, 2$, then $c\lambda_{i+n-1} \dots \lambda_i < 2c\lambda^n$. Let $n \geq 2$ and assume that $c\lambda_{i+m-1} \dots \lambda_i < 2c\lambda^m$ for each $m \in \{1, \dots, n\}$. Clearly, if $\lambda_{i+n+1} \leq 2/3$, then $c\lambda_{i+n+1}\lambda_{i+n} \dots \lambda_i < 2c\lambda^{n+1}$. On the other hand, if $\lambda_{i+n+1} > 2/3$, then $\lambda_{i+n-1} < 1/2$ and by induction assumption we have

$$c\lambda_{i+n}\lambda_{i+n-1} \dots \lambda_i < 2c\lambda^{n-2}\lambda_{i+n}\lambda_{i+n-1} < 2c\lambda^{n-2}\frac{1}{2} < 2c\lambda^{n-2}\lambda^2.$$

It follows from the above facts that

$$\|A_i^n s_i\| \leq 2c\lambda^n \|s_i\| \quad \text{and} \quad \|A_i^n u_i\| \geq (2c)^{-1}\lambda^{-n}\|u_i\|,$$

for each $i \in \mathbb{Z}$ and $n \geq 1$, which proves the proposition. $\square$

It is clear that:

Proposition 3.2.11. *Any gathering of an Anosov family is also an Anosov family (see Definition 1.1.6).*

Example 3.2.12. It follows from Proposition 3.2.11 that if $\phi : M \rightarrow M$ is an Anosov diffeomorphism, then for each sequence of positive integers $(n_i)_{i \in \mathbb{Z}}$, if $f_i = \phi^{n_i}$, then $(f_i)_{i \in \mathbb{Z}}$ is an Anosov family. Moreover, any gathering of a multiplicative family is an Anosov family

If $F_i \in SL(\mathbb{N}, 2)$, then

$$F_i = \begin{pmatrix} 1 & 0 \\ n_{i,k_i} & 1 \end{pmatrix} \begin{pmatrix} 1 & n_{i,k_i-1} \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & 0 \\ n_{i,2} & 1 \end{pmatrix} \begin{pmatrix} 1 & n_{i,1} \\ 0 & 1 \end{pmatrix}, \quad (3.2.2)$$

for some non-negative integers $n_{i,1}, \dots, n_{i,k_i}$, that is, $SL(\mathbb{N}, 2)$ is a semigroup generated by

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(see [AF05], Lemma 3.11).

Corollary 3.2.13. *Consider a sequence $(F_i)_{i \in \mathbb{Z}}$ in $SL(\mathbb{N}, 2)$ and the factorization of each F_i as in (3.2.2). If n_{i,k_i} and $n_{i,1}$ are non-zero for each $i \in \mathbb{Z}$, then $(F_i)_{i \in \mathbb{Z}}$ is an Anosov family.*

Proof. Note that $(F_i)_{i \in \mathbb{Z}}$ is a gathering of an multiplicative family. It follows from Proposition 3.2.10 that $(F_i)_{i \in \mathbb{Z}}$ is an Anosov family. $\square$

Suppose that $M_i = M \times \{i\}$ for each i , where M is a compact Riemannian manifold. Since we are considering the total space as the disjoint union of the M_i 's, the splitting of the tangent spaces at the points (p, i) and (p, j) can be different for $i \neq j$, as we will see in the next remark.

Remark 3.2.14. Set $n_i = 1$ for each integer $i \neq 0$ and $n_0 = 2$. Consider

$$A_i = \begin{pmatrix} 1 & 0 \\ n_i & 1 \end{pmatrix} \text{ for } i \text{ even} \quad \text{and} \quad A_i = \begin{pmatrix} 1 & n_i \\ 0 & 1 \end{pmatrix} \text{ for } i \text{ odd}.$$

From Example 3.2.9 we have

$$\begin{aligned} a_0 &= \frac{1}{2+\phi}, & b_0 &= 1, & c_0 &= -\frac{2+\phi}{2}, & d_0 &= -\phi \frac{2+\phi}{2}, \\ a_1 &= 1, & b_1 &= \frac{1}{\phi}, & c_1 &= \frac{\phi}{1+\phi+\phi^2}, & d_1 &= \frac{\phi(1+\phi)}{1+\phi+\phi^2}, \end{aligned}$$

where $\phi = [111 \dots]$. Thus,

$$s_0 = \left(\frac{1}{2+\phi}, -1\right), \quad u_0 = \left(-\frac{2+\phi}{2}, -\phi \frac{2+\phi}{2}\right), \quad s_1 = \left(1, -\frac{1}{\phi}\right), \quad u_1 = \left(\frac{\phi}{1+\phi+\phi^2}, \frac{\phi(1+\phi)}{1+\phi+\phi^2}\right).$$

This fact shows that we can have two different splitting of the tangent bundle in each component, one is obtained by s_0 and u_0 and other one is obtained by s_1 and u_1 .

Note that in the examples obtained from the results shown above, the families consist of factors of hyperbolic matrices. We have

$$\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m \\ n & nm+1 \end{pmatrix},$$

which is a hyperbolic matrix for $n, m \in \mathbb{N}$. The results to be obtained in Chapter 4 will provide more general examples of Anosov families, which, I doubt it, it be possible that none of the maps f_i is a factor of a hyperbolic matrix (see Problem 7.3.3): in Theorem 4.4.2 we will prove that if an Anosov family $(\mathbf{F}_i)_{i \in \mathbb{Z}}$ consists of matrices, then there exists a $\varepsilon > 0$ such that if a family $(g_i)_{i \in \mathbb{Z}}$ is ε -close to $(\mathbf{F}_i)_{i \in \mathbb{Z}}$, then the family is Anosov. Consequently, we can choose diffeomorphisms in those neighborhood so that none of them are factors of a hyperbolic matrix.

3.3 Lemma of Mather for Anosov Families

A Riemannian metric is *adapted to an hyperbolic set* of a diffeomorphism if, in this metric, the expansion (contraction) of the unstable (stable) subspaces is seen after only one iteration. By Lemma of Mather (see [Shu13], Proposition 4.2 or [BS02], Proposition 5.2.2) each Anosov diffeomorphism on a compact manifold M admits a Riemannian metric adapted to M . By compactness of M , this metric is uniformly equivalent to the Riemannian metric firstly considered. In Proposition 3.3.4, whose proof is based on the proof of Proposition 5.2.2 in [BS02], we will obtain an analogous version of the Lemma of Mather for Anosov families. The Riemannian metric to be obtained in Proposition 3.3.4 is not necessarily uniformly equivalent to the first metric on $\mathbf{M}$.

In this section, $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ will represent an Anosov family with constants $\lambda \in (0, 1)$ and $c \geq 1$. Sometimes we will omit the index i of f_i if it is clear that we are considering the i -th diffeomorphism of $\mathbf{f}$.

The notion of Anosov diffeomorphism on a compact Riemannian manifold does not depend

on the Riemannian metric (see [KH97]). In contrast, the notion of Anosov family depends on the Riemannian metric taken on each M_i (see Example 3.2.1). However, the next proposition proves that the notion of Anosov family does not depend on the Riemannian metrics uniformly equivalent on the total space.

Proposition 3.3.1. *Suppose that $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_\star$ are Riemannian metrics uniformly equivalent on $\mathbf{M}$. Thus, $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ is an Anosov family if, and only if, $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{f})$ is an Anosov family.*

Proof. Let $\|\cdot\|$ and $\|\cdot\|_\star$ be the norms induced by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_\star$, respectively. Let k and K be such that

$$k\|v\|_\star \leq \|v\| \leq K\|v\|_\star \quad \text{for all } v \in TM_i, i \in \mathbb{Z}.$$

Suppose that $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ is an Anosov family with constant $\lambda \in (0, 1)$ and $c \geq 1$. Thus, for $v \in T_p \mathbf{M}, n \geq 1$,

$$\|D_p(\mathbf{f}_i^n)(v)\|_\star \leq (1/k)\|D_p(\mathbf{f}_i^n)(v)\| \leq (c/k)\lambda^n\|v\| \leq (Kc/k)\lambda^n\|v\|_\star.$$

On the other hand,

$$\|D_p(\mathbf{f}_i^{-n})(v)\|_\star \leq (Kc/k)\lambda^{-n}\|v\|_\star, \text{ for } v \in T_p \mathbf{M}, n \geq 1.$$

Therefore, $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{f})$ is an Anosov family with constant λ e $\tilde{c} = Kc/k$.

Similarly we can prove that if $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{f})$ is an Anosov family then $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ is an Anosov family. $\square$

Definition 3.3.2. Set

$$\mathcal{U}(\mathbf{M}, \langle \cdot, \cdot \rangle) = \{\langle \cdot, \cdot \rangle_\star : \langle \cdot, \cdot \rangle_\star \text{ is a Riemannian metric uniformly equivalent to } \langle \cdot, \cdot \rangle \text{ on } \mathbf{M}\}.$$

Proposition 3.3.1 means that if $\langle \cdot, \cdot \rangle_\star \in \mathcal{U}(\mathbf{M}, \langle \cdot, \cdot \rangle)$, then $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ is an Anosov family if, and only if, $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{f})$ is an Anosov family. Therefore, we can redefine the notion of Anosov family on $\mathbf{M}$ as follows:

Definition 3.3.3. $\mathbf{f}$ is an Anosov family on $(\mathbf{M}, \langle \cdot, \cdot \rangle)$ if there exist $\langle \cdot, \cdot \rangle_\star \in \mathcal{U}(\mathbf{M}, \langle \cdot, \cdot \rangle)$ such that $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{f})$ satisfies the conditions in Definition 3.1.2.

Next, we show the Lemma of Mather for Anosov families.

Proposition 3.3.4. *Given $\zeta > 0$, there exists a C^∞ Riemannian metric $\langle \cdot, \cdot \rangle_\star$ and uniformly equivalent to $\langle \cdot, \cdot \rangle$ on each M_i , such that $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{f})$ is a strictly Anosov family with $\lambda' = \lambda + \zeta$. Furthermore, for each $p \in \mathbf{M}$, we have $\langle v_s, v_u \rangle_\star < \varepsilon$ for every unit vectors $v_s \in E_p^s$ and $v_u \in E_p^u$. Consequently, $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{f})$ satisfies the property of angles.*

Proof. For each $p \in \mathbf{M}$, if $(v_s, v_u) \in E_p^s \oplus E_p^u$, take

$$\|(v_s, v_u)\|_\star = \sqrt{\|v_s\|_\star^2 + \|v_u\|_\star^2}, \quad (3.3.1)$$

where

$$\|v_s\|_\star = \sum_{n=0}^{\infty} (\lambda + \zeta)^{-n} \|D(\mathbf{f}^n)_p v_s\| \quad \text{and} \quad \|v_u\|_\star = \sum_{n=0}^{\infty} (\lambda + \zeta)^{-n} \|D(\mathbf{f}^{-n})_p v_u\|.$$

Notice that if $v_s \in E_p^s$ we have

$$\|v_s\|_* = \sum_{n=0}^{\infty} (\lambda + \zeta)^{-n} \|D(f^n)_p v_s\| \leq \sum_{n=0}^{\infty} (\lambda + \zeta)^{-n} c \lambda^n \|v_s\| = \frac{\lambda + \zeta}{\zeta} c \|v_s\|. \quad (3.3.2)$$

Analogously, $\|v_u\|_* \leq \frac{\lambda + \zeta}{\zeta} c \|v_u\|$ for $v_u \in E_p^u$. Consequently the series $\|v_s\|_*$ and $\|v_u\|_*$ converge uniformly.

Let us see that the norm $\|\cdot\|_*$ is uniformly equivalent to the norm $\|\cdot\|$ on each M_i . It is clear that $\|v_s\| \leq \|v_s\|_*$ and $\|v_u\| \leq \|v_u\|_*$. Thus,

$$\|(v_s, v_u)\| \leq \|v_s\| + \|v_u\| \leq 2(\|v_s\|^2 + \|v_u\|^2)^{1/2} \leq 2(\|v_s\|_*^2 + \|v_u\|_*^2)^{1/2} = 2\|(v_s, v_u)\|_*.$$

This fact implies

$$\|v\| \leq 2\|v\|_* \quad \text{for all } v \in T\mathbf{M}. \quad (3.3.3)$$

Let θ_p be the angle between two vectors $v_s \in E_p^s$ and $v_u \in E_p^u$, for $p \in M_i$. Take μ_i as in (3.1.3). Since $(1 - \mu_i)(\|v_s\|^2 + \|v_u\|^2) \geq 2(1 - \mu_i)\|v_s\|\|v_u\|$, we have

$$\|v_s\|^2 + \|v_u\|^2 + 2(\mu_i - 1)\|v_s\|\|v_u\| \geq \mu_i(\|v_s\|^2 + \|v_u\|^2).$$

Therefore

$$\begin{aligned} \|(v_s, v_u)\|^2 &= \|v_s\|^2 + \|v_u\|^2 - 2 \cos \theta_p \|v_s\|\|v_u\| \geq \|v_s\|^2 + \|v_u\|^2 + 2(\mu_i - 1)\|v_s\|\|v_u\| \\ &\geq \mu_i(\|v_s\|^2 + \|v_u\|^2). \end{aligned}$$

Consequently,

$$\|(v_s, v_u)\|_*^2 = \|v_s\|_*^2 + \|v_u\|_*^2 \leq \left(\frac{\lambda + \zeta}{\zeta} c\right)^2 (\|v_s\|^2 + \|v_u\|^2) \leq \frac{1}{\mu_i} \left(\frac{\lambda + \zeta}{\zeta} c\right)^2 \|(v_s, v_u)\|^2.$$

Thus,

$$\|v\|_* \leq \frac{1}{\mu_i} \left(\frac{\lambda + \zeta}{\zeta} c\right)^2 \|v\| \quad \text{for all } v \in TM_i. \quad (3.3.4)$$

From (3.3.3) and (3.3.4) we have that the metrics $\|\cdot\|$ and $\|\cdot\|_*$ are uniformly equivalent on each M_i .

Notice that, if $v_s \in E_p^s$,

$$\begin{aligned} \|Df_p v_s\|_* &= \sum_{n=0}^{\infty} (\lambda + \zeta)^{-n} \|Df_p^{n+1} v_s\| = (\lambda + \zeta) \sum_{n=1}^{\infty} (\lambda + \zeta)^{-n} \|Df_p^n v_s\| \\ &= (\lambda + \zeta) \sum_{n=1}^{\infty} (\lambda + \zeta)^{-n} \|Df_p^n v_s\| + (\lambda + \zeta) \|v_s\| - (\lambda + \zeta) \|v_s\| \\ &= (\lambda + \zeta) \left(\sum_{n=0}^{\infty} (\lambda + \zeta)^{-n} \|Df_p^n v_s\| - \|v_s\| \right) \\ &= (\lambda + \zeta) (\|v_s\|_* - \|v_s\|) \leq (\lambda + \zeta) \|v_s\|_*. \end{aligned}$$

Similarly, we can prove that, if $v_u \in E_p^u$, then

$$\|D(f^{-1})_p v_u\|_* \leq (\lambda + \zeta) \|v_u\|_*.$$

Notice that the norm $\|\cdot\|_*$ comes from an inner product $\langle \cdot, \cdot \rangle_*$, which defines a continuous Riemannian metric on $\mathbf{M}$. Consequently, for each i , we can choose a C^∞ -Riemannian metric $\langle \cdot, \cdot \rangle_{*,i}$ such that $|\langle v, v \rangle_{*,i} - \langle v, v \rangle_*| < \varepsilon$ for each $v \in TM_i$. We take $\langle \cdot, \cdot \rangle_*$ on $\mathbf{M}$, defined on each M_i as $\langle \cdot, \cdot \rangle_{*,i}|_{M_i} = \langle \cdot, \cdot \rangle_{*,i}$. It is clear that $(\mathbf{M}, \langle \cdot, \cdot \rangle_*, f)$ satisfies the property of angles. $\square$

By (3.3.3) and (3.3.4) we have that, for each $i \in \mathbb{Z}$,

$$\left(\frac{1}{\mu_i} \left(\frac{\lambda + \varepsilon}{\varepsilon} c \right)^2 \right)^{-1} \|v\|_* \leq \|v\| \leq 2\|v\|_* \quad \text{for all } v \in TM_i, \quad (3.3.5)$$

where μ_i depends on the angles between the stable and unstable subspaces of the splitting of TM_i . Hence, we have:

Corollary 3.3.5. *Suppose that $(\mathbf{M}, \langle \cdot, \cdot \rangle, f)$ satisfies the property of angles. Then, there exists a C^∞ -Riemannian metric $\langle \cdot, \cdot \rangle_* \in \mathcal{U}(\mathbf{M}, \langle \cdot, \cdot \rangle)$, such that $(\mathbf{M}, \langle \cdot, \cdot \rangle_*, f)$ is a strictly Anosov family which satisfies the property of angles.*

Proof. Let μ be as in (3.1.4). From (3.3.3) and (3.3.4) we have

$$\left(\frac{1}{\mu} \left(\frac{\lambda + \varepsilon}{\varepsilon} c \right)^2 \right)^{-1} \|v\|_* \leq \|v\| \leq 2\|v\|_* \quad \text{for all } v \in TM,$$

where $\|\cdot\|_*$ is a norm defined in (3.3.1). Thus, $\|\cdot\|$ and $\|\cdot\|_*$ are uniformly equivalent. The corollary follows from the proof of the Proposition 3.3.4. $\square$

The metric obtained in Proposition 3.3.4 is not necessarily uniformly equivalent to the original metric on $\mathbf{M}$. The uniform equivalence depends on the angles between the stable and unstable subspaces of the splitting of the tangent bundle on each component. In the case of Anosov diffeomorphisms defined on compact manifolds those angles are uniformly bounded away from 0. In the case of Anosov families, those angles may decrease arbitrarily, however, they can never be zero, by compactness of each component.

3.4 Invariant Cones for Anosov Families

In this section we will prove that f satisfies the property of invariant cones (see [BP07], [KH97]). This fact will be useful to prove the openness of the set consisting of Anosov families. From now on, we will fix $\zeta \in (0, 1 - \lambda)$ and consider the Riemannian metric $\langle \cdot, \cdot \rangle_*$ on $\mathbf{M}$ obtained in Proposition 3.3.4. Hence $(\mathbf{M}, \langle \cdot, \cdot \rangle_*, f)$ is a strictly Anosov family with constant $\tilde{\lambda} = \lambda + \zeta$ and we can suppose the stable and unstable subspaces are orthogonal (see (3.3.1)). Furthermore, $\|\cdot\|_*$ will represent the norm induced by $\langle \cdot, \cdot \rangle_*$.

Definition 3.4.1. For each $p \in \mathbf{M}$ and $\alpha > 0$, set

$$\begin{aligned} K_{\alpha, f, p}^s &= \{(v, w) \in E_p^s \oplus E_p^u : \|w\|_* < \alpha \|v\|_*\} \cup \{(0, 0)\}, \\ K_{\alpha, f, p}^u &= \{(v, w) \in E_p^s \oplus E_p^u : \|v\|_* < \alpha \|w\|_*\} \cup \{(0, 0)\}. \end{aligned}$$

$K_{\alpha, \mathbf{f}, p}^s$ is called the *stable α -cone of $\mathbf{f}$ at p* and $K_{\alpha, \mathbf{f}, p}^u$ the *unstable α -cone of $\mathbf{f}$ at p* (see Figure 3.4.1).

Figure 3.4.1: Stable and unstable α -cones at p .

Notice that $(v, w) \in \overline{K_{\alpha, \mathbf{f}, p}^u}$ if and only if $\|v\|_* \leq \alpha\|w\|_*$ and $(v, w) \in \overline{K_{\alpha, \mathbf{f}, p}^s}$ if and only if $\|w\|_* \leq \alpha\|v\|_*$.

Lemma 3.4.2. Fix $p \in M$ and $\alpha \in (0, 1)$. We have:

- i. Let $v \in T_p M$. Then, $D(\mathbf{f}^n)_p(v) \in K_{\alpha, \mathbf{f}, \mathbf{f}^n(p)}^u$ for all $n \leq 0$ if and only if $v \in E_p^u$.
- ii. Let $v \in T_p M$. Then, $D(\mathbf{f}^n)_p(v) \in K_{\alpha, \mathbf{f}, \mathbf{f}^n(p)}^s$ for all $n \geq 0$ if and only if $v \in E_p^s$.

Proof. We prove i. It is clear that if $v \in E_p^u$ then $D(\mathbf{f}^n)_p(v) \in K_{\alpha, \mathbf{f}, \mathbf{f}^n(p)}^u$ for all $n \leq 0$. Suppose that $v \notin E_p^u$. Thus $v = v_s + v_u$ where $v_s \in E_p^s \setminus \{0\}$ and $v_u \in E_p^u$. By Lemma 3.1.4 we have

$$\|D(\mathbf{f}^n)_p(v_s)\| \geq c^{-1}\lambda^n\|v_s\| \rightarrow +\infty \quad \text{and} \quad \|D(\mathbf{f}^n)_p(v_u)\| \leq c\lambda^n\|v_u\| \rightarrow 0 \quad \text{as } n \rightarrow -\infty.$$

Consequently, we can not have that $D(\mathbf{f}^n)_p(v) \in K_{\alpha, \mathbf{f}, \mathbf{f}^n(p)}^u$ for all $n \leq 0$. □

Lemma 3.4.3. Let $\alpha \in (0, \frac{1-\lambda}{1+\lambda})$ and take $\lambda' = \tilde{\lambda} \frac{1+\alpha}{1-\alpha} < 1$. Thus:

- i. $D\mathbf{f}_p(K_{\alpha, \mathbf{f}, p}^u) \subseteq K_{\alpha, \mathbf{f}, \mathbf{f}(p)}^u$. Furthermore, if $(v, w) \in K_{\alpha, \mathbf{f}, p}^u$, then

$$\|D\mathbf{f}_p(v, w)\|_* \geq (\lambda')^{-1}\|(v, w)\|_*.$$

- ii. $D\mathbf{f}_{\mathbf{f}(p)}^{-1}(K_{\alpha, \mathbf{f}, \mathbf{f}(p)}^s) \subseteq K_{\alpha, \mathbf{f}, p}^s$. Furthermore, if $(v, w) \in K_{\alpha, \mathbf{f}, \mathbf{f}(p)}^s$, then

$$\|D\mathbf{f}_{\mathbf{f}(p)}^{-1}(v, w)\|_* \geq (\lambda')^{-1}\|(v, w)\|_*.$$

See Figure 3.4.2.

Proof. Fix $(v, w) \in K_{\alpha, \mathbf{f}, p}^u$. Thus

$$\|D\mathbf{f}_p(v)\|_* \leq \tilde{\lambda}\|v\|_* \leq \tilde{\lambda}\alpha\|w\|_* \leq \tilde{\lambda}^2\alpha\|D\mathbf{f}_p(w)\|_* \leq \alpha\|D\mathbf{f}_p(w)\|_*.$$

Therefore $D\mathbf{f}_p(K_{\alpha, \mathbf{f}, p}^u) \subseteq K_{\alpha, \mathbf{f}, \mathbf{f}(p)}^u$.

Figure 3.4.2: Stable and unstable invariant α -cones. $q = f(p)$

On the other hand,

$$\|Df_p(v, w)\|_* \geq \|Df_p(w)\|_* - \|Df_p(v)\|_* \geq (1 - \alpha)\|Df_p(w)\|_* \geq \frac{1 - \alpha}{\tilde{\lambda}(1 + \alpha)}\|(v, w)\|_*,$$

and this fact proves i.

The statement (ii) can be proved analogously. $\square$

An equivalent way to prove the above lemma is by using coordinate charts for open sets of M_i , as we will see below. In this case we will use the exponential charts (see Definition 2.4.1). For each $p \in \mathbf{M}$, let $\varrho_p > 0$ be the injectivity radius of $\exp_p$ at p . Take $\varepsilon_p = \varrho_p/2$. Let $\delta_p > 0$ be small enough such that

$$\tilde{f}_p = \exp_{f(p)}^{-1} \circ f \circ \exp_p : B(0_p, \delta_p) \rightarrow B(0_{f(p)}, \varepsilon_{f(p)})$$

is well defined. It is clear that δ_p depends on both ϱ_p and f . Notice that $D(f)_p = D(\tilde{f}_p)_{0_p}$.

Remark 3.4.4. For each $n \in \mathbb{Z}$, consider $S_n = \max_{p \in M_n} \|D(f_n)_p\|$. Notice that if, for each p ,

$$\delta_p \leq \min\{\varepsilon_p, \varepsilon_{f(p)} / \max\{S_n, 1\}\},$$

then, for all $x \in B(0_p, \delta_p)$, we have $\tilde{f}_p(x) \in B(0_{f(p)}, \varepsilon_{f(p)})$. Consequently, if $M_n = M \times \{n\}$, where M is a compact Riemannian manifold, $\langle \cdot, \cdot \rangle_n = \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on M , and $(S_n)_{n \in \mathbb{Z}}$ is bounded, then we can find a uniform δ with which $\tilde{f}_p$ is well-defined for each $p \in \mathbf{M}$, that is, there exists $\delta > 0$ such that, considering $\delta_p = \delta$ for each $p \in \mathbf{M}$, $\tilde{f}_p$ is well-defined.

Definition 3.4.5. For $z \in T_p \mathbf{M}$, we denote by z_s and z_u the orthogonal projections of z on E_p^s and E_p^u , respectively, and hence $z = (z_s, z_u)$. If $(z_s, z_u) \in B^s(0_p, \delta_p) \times B^u(0_p, \delta_p)$, where $B^s(0_p, \delta_p)$ is the ball in E_p^s and $B^u(0_p, \delta_p)$ is the ball in E_p^u , then

$$\tilde{f}_p(z_s, z_u) = (a_p(z_s, z_u) + A_p(z_s), b_p(z_s, z_u) + B_p(z_u)), \quad (3.4.1)$$

where

$$\begin{aligned} a_p &: B^s(0_p, \delta_p) \times B^u(0_p, \delta_p) \rightarrow E^s, \quad a_p(z_s, z_u) = (\tilde{f}_p(z_s, z_u))_s - D(\tilde{f}_p)_{0_p}(z_s), \\ b_p &: B^s(0_p, \delta_p) \times B^u(0_p, \delta_p) \rightarrow E^u, \quad b_p(z_s, z_u) = (\tilde{f}_p(z_s, z_u))_u - D(\tilde{f}_p)_{0_p}(z_u), \\ A_p &: B^s(0_p, \delta_p) \rightarrow E^s, \quad A_p(z_s) = D(\tilde{f}_p)_{0_p}(z_s), \text{ and} \\ B_p &: B^u(0_p, \delta_p) \rightarrow E^u, \quad B_p(z_u) = D(\tilde{f}_p)_{0_p}(z_u). \end{aligned}$$

Notice that $a_p(0_p) = b_p(0_p) = D(a_p)_{0_p} = D(b_p)_{0_p} = 0$. Set

$$\mu_p = \sup_{v \in E_p^s} \frac{\|A_p v\|_\star}{\|v\|_\star} \quad \text{and} \quad \kappa_p = \sup_{v \in E_{\mathbf{f}(p)}^u} \frac{\|B_p^{-1} v\|_\star}{\|v\|_\star}. \quad (3.4.2)$$

It is clear that $\max\{\mu_p, \kappa_p\} \leq \tilde{\lambda}$.

Definition 3.4.6. For each p consider $\epsilon_p > 0$ small enough such that

$$(\tilde{f}_p)^{-1} = \exp_p^{-1} \circ \mathbf{f}^{-1} \circ \exp_{\mathbf{f}(p)} : B^s(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)}) \times B^u(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)}) \rightarrow B^s(0_p, \epsilon_p)$$

is well-defined. For $(z_s, z_u) \in B^s(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)}) \times B^u(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)})$, set

$$(\tilde{f}_p)^{-1}(z_s, z_u) = (c_p(z_s, z_u) + C_p(z_s), d_p(z_s, z_u) + D_p(z_u)),$$

where

$$\begin{aligned} c_p &: B(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)}) \rightarrow E^s, \quad c_p(z_s, z_u) = (\tilde{f}_p)_s^{-1}(z_s, z_u) - C_p(z_s); \\ d_p &: B(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)}) \rightarrow E^u, \quad d_p(z_s, z_u) = (\tilde{f}_p)_u^{-1}(z_s, z_u) - D_p(z_u); \\ C_p &: B^s(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)}) \rightarrow E^s, \quad C_p(z_s) = D(\tilde{f}_p)_0^{-1}(z_s); \\ D_p &: B^u(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)}) \rightarrow E^u, \quad D_p(z_u) = D(\tilde{f}_p)_0^{-1}(z_u). \end{aligned}$$

Notice that

$$\sup_{v \in E_p^s} \frac{\|C_p^{-1} v\|_\star}{\|v\|_\star} = \sup_{v \in E_p^s} \frac{\|A_p v\|_\star}{\|v\|_\star} = \mu_p \quad \text{and} \quad \sup_{v \in E_{\mathbf{f}(p)}^u} \frac{\|D_p v\|_\star}{\|v\|_\star} = \sup_{v \in E_{\mathbf{f}(p)}^u} \frac{\|B_p^{-1} v\|_\star}{\|v\|_\star} = \kappa_p.$$

Definition 3.4.7. Set

$$\sigma_p(\delta_p) = \sup_{z \in B^s(0_p, \delta_p) \times B^u(0_p, \delta_p)} \{\|D(a_p, b_p)_z\|_\star\}$$

and

$$\rho_p(\epsilon_{\mathbf{f}(p)}) = \sup_{z \in B^s(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)}) \times B^u(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)})} \{\|D(c_p, d_p)_z\|_\star\}.$$

Remark 3.4.8. Notice that

$$(a_p(z), b_p(z)) = \tilde{f}_p(z) - D(\tilde{f}_p)_0(z) \quad \text{for } z \in B^s(0_p, \delta_p) \times B^u(0_p, \delta_p).$$

Hence, for each $z \in B^s(0_p, \delta_p) \times B^u(0_p, \delta_p)$ we have

$$\|D(\tilde{f}_p)_z - D(\tilde{f}_p)_0\|_* = \|D[\tilde{f}_p - D(\tilde{f}_p)_0]_z\|_* = \|D(a_p, b_p)_z\|_* \leq \sigma_p(\delta_p).$$

For $q \in B(p, \delta_p)$, let $z = \exp_p^{-1}(q)$, $\tilde{E}_z^s = D(\exp_p^{-1})_q(E_q^s)$, $\tilde{E}_z^u = D(\exp_p^{-1})_q(E_q^u)$,

$$\begin{aligned} K_{\alpha, \mathbf{f}, z}^s &= \{(v_s, v_u) \in \tilde{E}_z^s \oplus \tilde{E}_z^u : \|v_u\|_* < \alpha\|v_s\|_*\} \cup \{(0, 0)\}, \\ \text{and } K_{\alpha, \mathbf{f}, z}^u &= \{(v_s, v_u) \in \tilde{E}_z^s \oplus \tilde{E}_z^u : \|v_s\|_* < \alpha\|v_u\|_*\} \cup \{(0, 0)\}. \end{aligned}$$

We have

$$\begin{aligned} D(\tilde{f}_n)_z(\tilde{E}_z^s) &= D(\exp_{\mathbf{f}^{n+1}(p)}^{-1} f_n \exp_{\mathbf{f}^n(p)})_z D(\exp_{\mathbf{f}^n(p)}^{-1})_q(E_q^s) \\ &= D(\exp_{\mathbf{f}^{n+1}(p)}^{-1} f_n \exp_{\mathbf{f}^n(p)} \circ \exp_{\mathbf{f}^n(p)}^{-1})_q(E_q^s) \\ &= D(\exp_{\mathbf{f}^{n+1}(p)}^{-1} f_n)_q(E_q^s) = D(\exp_{\mathbf{f}^{n+1}(p)}^{-1})_{f_n(q)} \circ D(f_n)_q(E_q^s) \\ &= D(\exp_{\mathbf{f}^{n+1}(p)}^{-1})_{f_n(q)}(E_{f_n(q)}^s) = \tilde{E}_{f_n(z)}^s. \end{aligned}$$

Analogously, we can prove that

$$D(\tilde{f}_n)_z(\tilde{E}_z^u) = \tilde{E}_{f_n(z)}^u.$$

For the next two lemmas we will suppose that $\delta_p > 0$ and $\epsilon_p > 0$ are small enough such that

$$\sigma_p(\delta_p) < \frac{(\kappa_p^{-1} - \mu_p)\alpha}{(1 + \alpha)^2} \quad \text{and} \quad \rho_p(\epsilon_{\mathbf{f}(p)}) < \frac{(\mu_p^{-1} - \kappa_p)\alpha}{(1 + \alpha)^2}.$$

Lemma 3.4.9. *We have*

- i. $D(\tilde{f}_p)_z(\overline{K_{\alpha, \mathbf{f}, z}^u}) \subseteq K_{\alpha, \mathbf{f}, \tilde{f}_p(z)}^u$, if $z \in B^s(0_p, \delta_p) \times B^u(0_p, \delta_p)$;
- ii. $D(\tilde{f}_p)_{\tilde{f}_p(z)}^{-1}(\overline{K_{\alpha, \mathbf{f}, \tilde{f}_p(z)}^s}) \subseteq K_{\alpha, \mathbf{f}, z}^s$, if $\tilde{f}_p(z) \in B^s(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)}) \times B^u(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)})$.

Proof. If $(v_s, v_u) \in \overline{K_{\alpha, \mathbf{f}, z}^u}$, then $\|v_s\|_* \leq \alpha\|v_u\|_*$. We have

$$D(\tilde{f}_p)_z(v_s, v_u) = (D(a_p)_z(v_s, v_u) + A_p(v_s), D(b_p)_z(v_s, v_u) + B_p(v_u)).$$

Now,

$$\|D(b_p)_z(v_s, v_u) + B_p(v_u)\|_* > -\sigma_p\|(v_s, v_u)\|_* + \kappa_p^{-1}\|v_u\|_* \geq (\kappa_p^{-1} - \sigma_p(1 + \alpha))\|v_u\|_*$$

and therefore

$$\begin{aligned} \|D(a_p)_z(v_s, v_u) + A_p(v_s)\|_* &\leq \sigma_p\|(v_s, v_u)\|_* + \mu_p\|v_s\|_* \leq ((1 + \alpha)\sigma_p + \alpha\mu_p)\|w\|_* \\ &\leq \frac{(1 + \alpha)\sigma_p + \alpha\mu_p}{\kappa_p^{-1} - \sigma_p(1 + \alpha)}\|D(b_p)_z(v_s, v_u) + B_p(v_u)\|_*. \end{aligned}$$

Since $\sigma_p < \frac{(\kappa_p^{-1} - \mu_p)\alpha}{(1 + \alpha)^2}$, we have $\frac{\alpha\mu_p + \sigma_p(1 + \alpha)}{\kappa_p^{-1} - \sigma_p(1 + \alpha)} \leq \alpha$. This fact proves $D(\tilde{f}_p)_z(v_s, v_u) \in K_{\alpha, \mathbf{f}, \tilde{f}_p(z)}^u$.

Analogously we can prove that $D(\tilde{f}_p)_{\tilde{f}_p(z)}^{-1}(\overline{K_{\alpha, \mathbf{f}, \tilde{f}_p(z)}^s}) \subseteq K_{\alpha, \mathbf{f}, z}^s$. □

Lemma 3.4.10. *We have*

i. *if $z \in B^s(0_p, \delta_p) \times B^u(0_p, \delta_p)$, then*

$$\|D(\tilde{f}_p)_z(v)\|_\star \geq \frac{\kappa_p^{-1} - \sigma_p(1 + \alpha)}{1 + \alpha} \|v\|_\star \text{ for each } v \in \overline{K_{\alpha, \mathbf{f}, \tilde{f}_p(z)}^u};$$

ii. *if $\tilde{f}_p(z) \in B^s(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)}) \times B^u(0_{\mathbf{f}(p)}, \epsilon_{\mathbf{f}(p)})$, then*

$$\|D(\tilde{f}_p)_z^{-1}(v)\|_\star \geq \frac{\mu_p^{-1} - \rho_p(1 + \alpha)}{1 + \alpha} \|v\|_\star \text{ for each } v \in \overline{K_{\alpha, \mathbf{f}, \tilde{f}_p(z)}^s}.$$

Proof. If $(v_s, v_u) \in \overline{K_{\alpha, \mathbf{f}, z}^u}$, then

$$\|D(\tilde{f}_p)_z(v_s, v_u)\|_\star \geq (\kappa_p^{-1} - \sigma_p(1 + \alpha)) \|v_u\|_\star \geq \frac{\kappa_p^{-1} - \sigma_p(1 + \alpha)}{1 + \alpha} \|(v_s, v_u)\|_\star.$$

Analogously we can prove (ii). □

Chapter 4

Openness for Anosov Families

A well-known fact is that the set consisting of diffeomorphisms Anosov on a compact Riemannian manifold is open (see, for example, [Shu13]). Set

$$\mathcal{A}^1(\mathbf{M}) = \{g \in \mathcal{F}^1(\mathbf{M}) : g \text{ is an Anosov family}\}.$$

The goal of this chapter is to show the analogous result for Anosov families, that is, we will prove that $\mathcal{A}^1(\mathbf{M})$ is an open subset of $(\mathcal{F}^1(\mathbf{M}), \tau_{str})$ (remember that $(\mathcal{F}^1(\mathbf{M}), \tau_{str})$ is the set consisting of non-stationary dynamical systems on $(\mathbf{M}, \langle \cdot, \cdot \rangle)$ endowed with the C^1 -strong topology). It is important to point out that the openness of $\mathcal{A}^1(\mathbf{M})$ provides a great variety of non-trivial examples of Anosov families, because the examples given in Section 3.2, besides are not trivial, they are not isolated in a certain way. We do not ask for additional conditions on each element $f \in \mathcal{A}^1(\mathbf{M})$ to prove that there is a strong basic neighborhood of f contained in $\mathcal{A}^1(\mathbf{M})$. For this reason, that basic neighborhood is not necessarily uniform (see 4.4.1). In Theorems 6.1.7 and 4.4.2 we will see that, with some conditions on the family, that neighborhood can be taken uniform, that is, a neighborhood in the uniform topology.

Young in [You86] proves that families consisting of random small perturbations of an Anosov diffeomorphism of class C^2 are Anosov families (see Remark 3.2.7). Theorem 6.1.7 is a generalization of this result. To prove Theorem 6.1.7 we use the same method to be used in this chapter (the method of invariant cones).

On the other hand, let X be a compact metric space, $\phi : X \rightarrow X$ a homeomorphism and $A : X \rightarrow SL(\mathbb{Z}, m)$ a continuous application such that the linear cocycle F defined by A over ϕ is hyperbolic. Thus, there exists an $\varepsilon > 0$ such that, if $B : X \rightarrow SL(\mathbb{Z}, m)$ is continuous and $\|A(x) - B(x)\| < \varepsilon$ for all $x \in X$, then the linear cocycle G defined by B over ϕ is hyperbolic (see [Via14]). This fact shows the openness of Anosov families that are obtained by hyperbolic cocycles. This is another particular case of Theorems 4.3.5 and 6.1.7.

As a particular case, in Section 5.4 we will prove that if the family consists of $m \times m$ -matrices acting on the m -torus, then the stability is uniform (see Definition 4.4.1).

4.1 Method of Invariant Cones

In order to prove the openness of $\mathcal{A}^1(\mathbf{M})$, we use the *method of invariant cones* (see [BP07] and [KH97]). The results to be given in this section and Section 6.2 are versions for Anosov families of some results given in [BP07], Chapter 7, where is considered *nonuniformly hyperbolic sequences of diffeomorphisms* on open subsets of $\mathbb{R}^m$ (see Definition 7.7.7 in [BP07]). We have adapted those ideas to Anosov families. In the case of Anosov families, the injectivity radius of the exponential application at each point of M_i could decrease when $|i|$ increases, since the M_i 's are not necessarily the same Riemannian manifold. We need a radius small enough such that the inequality in (4.1.3) be valid, and it depends also on the behavior of each f_i in the family.

In this section we will prove that there exists a sequence of positive numbers $(\xi_i)_{i \in \mathbb{Z}}$ and $\eta \in (0, 1)$ such that: given $\mathbf{g} \in B^1(\mathbf{f}, (\xi_i)_{i \in \mathbb{Z}})$, then $\mathbf{g}$ satisfies the property of invariant cones, that is, if $p \in \mathbf{M}$, then

$$\begin{aligned} D\mathbf{g}_p(\overline{K_{\alpha, \mathbf{f}, p}^u}) &\subseteq K_{\alpha, \mathbf{f}, \mathbf{g}(p)}^u \text{ and } \|D\mathbf{g}_p v\| \geq \frac{1}{\eta} \|v\| \text{ if } v \in \overline{K_{\alpha, \mathbf{f}, p}^u} \text{ and} \\ D\mathbf{g}_p^{-1}(\overline{K_{\alpha, \mathbf{f}, \mathbf{g}(p)}^s}) &\subseteq K_{\alpha, \mathbf{f}, p}^s \text{ and } \|D\mathbf{g}_p^{-1} v\| \geq \frac{1}{\eta} \|v\| \text{ if } v \in \overline{K_{\alpha, \mathbf{f}, \mathbf{g}(p)}^s}, \end{aligned}$$

where $K_{\alpha, \mathbf{f}, p}^u$ and $K_{\alpha, \mathbf{f}, p}^s$ are the α -cones of $\mathbf{f}$ at p (see Definition 3.4.1). These properties will be proved in the Lemmas 4.1.3-4.1.5.

In the rest of this section, $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ will represent an Anosov family on $(\mathbf{M}, \langle \cdot, \cdot \rangle)$ which satisfies the property of angles. We will use the notations given in Section 4.4. We can choose $\beta_i > 0$, with $\beta_i < \min\{\varrho_{i-1}, \varrho_i, \varrho_{i+1}\}/2$, such that, if $p \in M_i$,

$$f_i(B(p, 2\beta_i)) \subseteq B(f_i(p), \varrho_{i+1}/2) \quad \text{and} \quad f_i^{-1}(B(f_i(p), 2\beta_{i+1})) \subseteq B(p, \varrho_i/2).$$

Thus, if $\mathbf{g} = (g_i)_{i \in \mathbb{Z}} \in B^1(\mathbf{f}, (\beta_i)_{i \in \mathbb{Z}})$, we have

$$g_i(B(p, \beta_i)) \subseteq B(f_i(p), \varrho_{i+1}) \text{ and } g_i^{-1}(B(f_i(p), \beta_{i+1})) \subseteq B(p, \varrho_i). \quad (4.1.1)$$

Consider a linear isomorphism $\tau_p : T_p \mathbf{M} \rightarrow \mathbb{R}^m$ which maps an orthonormal basis of E_p^s to an orthonormal basis of $\mathbb{R}^k$ and maps an orthonormal basis of E_p^u to an orthonormal basis of $\mathbb{R}^{m-k}$, for each $p \in \mathbf{M}$, where k is the dimension of E_p^s and m the dimension of each M_i .

Remark 4.1.1. Since $\mathbf{f}$ satisfies the property of angles, the norm

$$\|(v_s, v_u)\|_* = \sqrt{\|v_s\|_*^2 + \|v_u\|_*^2}, \quad \text{for } (v_s, v_u) \in E^s \oplus E^u,$$

defined in (3.3.1), is uniformly equivalent to the norm $\|\cdot\|$ on $\mathbf{M}$ (Corollary 3.3.5). Furthermore, we have $\|\tau_q(v)\|_* = \|v\|_*$ for all $v \in T_q \mathbf{M}$, $q \in \mathbf{M}$.

Without losing generality, in this chapter, if we do not say otherwise, we will suppose that $\mathbf{f}$ is strictly Anosov and satisfies the property of angles with the norm $\|\cdot\|$ on $\mathbf{M}$.

Definition 4.1.2. For $\mathbf{g} = (g_i)_{i \in \mathbb{Z}} \in B^1(\mathbf{f}, (\beta_i)_{i \in \mathbb{Z}})$ and $p \in M_i$ we set

$$\hat{g}_p = \tau_{f_i(p)} \circ \exp_{f_i(p)}^{-1} \circ g_i \circ \exp_p \circ \tau_p^{-1} : B(0, \beta_i) \rightarrow B(0, \varrho_{i+1})$$

and

$$\hat{g}_p^{-1} = \tau_p \circ \exp_p^{-1} \circ g_i^{-1} \circ \exp_{f_i(p)} \circ \tau_{f_i(p)}^{-1} : B(0, \beta_{i+1}) \rightarrow B(0, \varrho_i),$$

which are well-defined as a consequence of (4.1.1).

For $x \in \mathbb{R}^m$, we denote by x_1 and x_2 the orthogonal projections of x on $\mathbb{R}^k$ and $\mathbb{R}^{m-k}$, respectively. If $(v, w) \in B^k(0, \beta_i) \times B^{m-k}(0, \beta_i)$, then

$$\hat{f}_p(v, w) = (\hat{a}_p(v, w) + \hat{A}_p(v), \hat{b}_p(v, w) + \hat{B}_p(w)),$$

where

$$\hat{a}_p = \tau_{f_i(p)} \circ a_p \circ \tau_p^{-1}, \quad \hat{b}_p = \tau_{f_i(p)} \circ b_p \circ \tau_p^{-1}, \quad \hat{A}_p = \tau_{f_i(p)} \circ A_p \circ \tau_p^{-1} \quad \text{and} \quad \hat{B}_p = \tau_{f_i(p)} \circ B_p \circ \tau_p^{-1}$$

(see (3.4.1)).

Analogously, if $(v, w) \in B^k(0, \beta_{i+1}) \times B^{m-k}(0, \beta_{i+1})$, we have

$$\hat{f}_p^{-1}(v, w) = (\hat{c}_p(v, w) + \hat{C}_p(v), \hat{d}_p(v, w) + \hat{D}_p(w)),$$

where

$$\hat{c}_p = \tau_p \circ c_p \circ \tau_{f_i(p)}^{-1}, \quad \hat{d}_p = \tau_p \circ d_p \circ \tau_{f_i(p)}^{-1}, \quad \hat{C}_p(v) = \tau_p \circ C_p \circ \tau_{f_i(p)}^{-1} \quad \text{and} \quad \hat{D}_p(w) = \tau_p \circ D_p \circ \tau_{f_i(p)}^{-1}$$

Take

$$\vartheta_p(\beta_i) = \sup\{\sigma_p(\beta_i), \rho_p(\beta_i)\}, \tag{4.1.2}$$

(see Definition 3.4.7).

Lemma 4.1.3. Fix $\alpha \in (0, \frac{1-\lambda}{1+\lambda})$ and $\xi > 0$. For each $i \in \mathbb{Z}$, there exist $X_i = \{p_{1,i}, \dots, p_{m_i,i}\} \subseteq M_i$ and $\beta_i > 0$ such that $M_i = \cup_{j=1}^{m_i} B(p_{j,i}, \beta_i)$ and

$$\vartheta_i := \max_{q \in X_i} \vartheta_q(\beta_i) \leq \xi.$$

Proof. Since $D(\hat{a}_p)_0 = 0$, $D(\hat{b}_p)_0 = 0$, $D(\hat{c}_p)_0 = 0$, $D(\hat{d}_p)_0 = 0$, each f_i is of class C^1 and M_i is compact, it follows that for each i we can choose $X_i = \{p_{1,i}, \dots, p_{m_i,i}\} \subseteq M_i$ and $\beta_i > 0$ small enough such that $M_i = \cup_{j=1}^{m_i} B(p_{j,i}, \beta_i)$ and $\max_{q \in X_i} \vartheta_q(\beta_i) \leq \xi$. $\square$

We will consider $X_i \subseteq M_i$ and β_i obtained from Lemma 4.1.3, with $\beta_i > 0$ small enough such that

$$\vartheta_i \leq \min \left\{ \frac{(\lambda^{-1} - \lambda)\alpha}{2(1+\alpha)^2}, \frac{\lambda^{-1}(1-\alpha) - (1+\alpha)\alpha}{2(1+\alpha)} \right\} \quad \text{for each } i \in \mathbb{Z}. \tag{4.1.3}$$

Since $\alpha \in (0, \frac{1-\lambda}{1+\lambda})$, the minimum in (4.1.3) is positive.

Set

$$\begin{aligned} K_\alpha^s &= \{(v, w) \in \mathbb{R}^k \oplus \mathbb{R}^{m-k} : \|w\| < \alpha\|v\|\} \quad \text{and} \\ K_\alpha^u &= \{(v, w) \in \mathbb{R}^k \oplus \mathbb{R}^{m-k} : \|v\| < \alpha\|w\|\}. \end{aligned}$$

Lemma 4.1.4. *Let $\alpha \in (0, \frac{1-\lambda}{1+\lambda})$ and β_i be such that (4.1.3) be valid. Thus, there exists $\xi_i > 0$ for each $i \in \mathbb{Z}$ such that, if $g \in B^1(f, (\xi_i)_{i \in \mathbb{Z}})$, for all $p \in X_i$ we have:*

- i. $D(\hat{g}_p)_z(\overline{K_\alpha^u}) \subseteq K_\alpha^u$ for all $z \in B^k(0, \beta_i) \times B^{m-k}(0, \beta_i)$, and
- ii. $D(\hat{g}_p^{-1})_z(\overline{K_\alpha^s}) \subseteq K_\alpha^s$ for all $z \in B^k(0, \beta_{i+1}) \times B^{m-k}(0, \beta_{i+1})$.

Proof. Let us take $\xi_i < \min\{\beta_i, \beta_{i+1}, \vartheta_i\}$. Fix $z \in B^k(0, \beta_i) \times B^{m-k}(0, \beta_i)$. If $(x, y) \in \overline{K_\alpha^u} \setminus \{(0, 0)\}$, then

$$\begin{aligned} \|(D(\hat{g}_p)_z(x, y))_1\| &\leq \|(D(\hat{g}_p)_z(x, y))_1 - (D(\hat{f}_p)_z(x, y))_1\| + \|(D(\hat{f}_p)_z(x, y))_1\| \\ &\leq \xi_i \|(x, y)\| + \|D(\hat{a}_p)_z(x, y)\| + \|\hat{A}_p(x)\| \\ &\leq \vartheta_i(\alpha \|y\| + \|y\|) + \vartheta_i \|(x, y)\| + \lambda \|x\| \\ &\leq \vartheta_i(\alpha + 1) \|y\| + \vartheta_i(\alpha + 1) \|y\| + \lambda \alpha \|y\| \\ &= ((\alpha + 1)2\vartheta_i + \lambda \alpha) \|y\|. \end{aligned}$$

Therefore

$$\|(D(\hat{g}_p)_z(x, y))_1\| \leq ((\alpha + 1)2\vartheta_i + \lambda \alpha) \|y\|. \quad (4.1.4)$$

On the other hand, we have

$$\begin{aligned} \|(D(\hat{g}_p)_z(x, y))_2\| &\geq \|(D(\hat{f}_p)_z(x, y))_2\| - \|(D(\hat{g}_p)_z(x, y))_2 - (D(\hat{f}_p)_z(x, y))_2\| \\ &\geq \|\hat{B}_p y\| - \|D(\hat{b}_p)_z(x, y)\| - \xi_i \|(x, y)\| \\ &\geq \lambda^{-1} \|y\| - \vartheta_i(\alpha + 1) \|y\| - \vartheta_i(\alpha + 1) \|y\| \\ &\geq (\lambda^{-1} - 2\vartheta_i(\alpha + 1)) \|y\|, \end{aligned}$$

that is,

$$\|(D(\hat{g}_p)_z(x, y))_2\| \geq (\lambda^{-1} - 2\vartheta_i(\alpha + 1)) \|y\|. \quad (4.1.5)$$

Since $\vartheta_i < \frac{\alpha(\lambda^{-1}-\lambda)}{2(1+\alpha)^2}$, we can prove that $\frac{(\alpha+1)2\vartheta_i+\lambda\alpha}{\lambda^{-1}-2\vartheta_i(\alpha+1)} < \alpha$. Consequently, from (4.1.4) and (4.1.5) we have

$$\|(D(\hat{g}_p)_z(x, y))_1\| < \alpha \|(D(\hat{g}_p)_z(x, y))_2\|,$$

that is, $D(\hat{g}_p)_z(x, y) \in K_\alpha^u$. Thus, $D(\hat{g}_p)_z(\overline{K_\alpha^u}) \subseteq K_\alpha^u$ for all $z \in B^k(0, \beta_i) \times B^{m-k}(0, \beta_i)$.

Analogously we can prove ii. □

Lemma 4.1.5. *If $\xi_i < \min\{\beta_i, \beta_{i+1}, \vartheta_i\}$ for each $i \in \mathbb{Z}$, there exists $\eta < 1$ such that, if $g \in B^1(f, (\xi_i)_{i \in \mathbb{Z}})$, then, for $p \in X_i$, $z \in B^k(0, \beta_i) \times B^{m-k}(0, \beta_i)$, we have*

- i. $\|D(\hat{g}_p)_z(x, y)\| \geq \eta^{-1} \|(x, y)\|$ if $(x, y) \in \overline{K_\alpha^u}$;
- ii. $\|D(\hat{g}_p^{-1})_z(x, y)\| \geq \eta^{-1} \|(x, y)\|$ if $(x, y) \in \overline{K_\alpha^s}$.

Proof. We will prove i. since the proof of ii. is analogous. Fix $p \in X_i$ and take $(x, y) \in \overline{K_\alpha^u}$. By Lemma 4.1.4 we have

$$\|(D(\hat{f}_p)_z(x, y))_1\| \leq \alpha \|(D(\hat{f}_p)_z(x, y))_2\| \quad \text{for } z \in B^k(0, \beta_i) \times B^{m-k}(0, \beta_i).$$

Thus,

$$\begin{aligned}
\|D(\hat{g}_p)_z(x, y)\| &\geq \|D(\hat{f}_p)_z(x, y)\| - \|D(\hat{f}_p)_z(x, y) - D(\hat{g}_p)_z(x, y)\| \\
&\geq \|(D(\hat{f}_p)_z(x, y))_2\| - \|(D(\hat{f}_p)_z(x, y))_1\| - \xi_i\|(x, y)\| \\
&\geq (1 - \alpha)\|(D(\hat{f}_p)_z(x, y))_2\| - \vartheta_i\|(x, y)\| \\
&\geq (1 - \alpha)(\|\hat{B}_p(y)\| - \|D(\hat{b}_p)_z(x, y)\|) - \vartheta_i\|(x, y)\| \\
&\geq (1 - \alpha)(\lambda^{-1}\|y\| - \vartheta_i\|(x, y)\|) - \vartheta_i\|(x, y)\| \\
&\geq (1 - \alpha)\left(\frac{\lambda^{-1}}{1 + \alpha}\|(x, y)\| - \vartheta_i\|(x, y)\|\right) - \vartheta_i\|(x, y)\| \\
&= ((1 - \alpha)\left(\frac{\lambda^{-1}}{1 + \alpha} - \vartheta_i\right) - \vartheta_i)\|(x, y)\| = \frac{1}{\eta}\|(x, y)\|,
\end{aligned}$$

where $\frac{1}{\eta} := (1 - \alpha)\left(\frac{\lambda^{-1}}{1 + \alpha} - \vartheta_i\right) - \vartheta_i > 1$, because $\vartheta_i < \frac{(1 - \alpha)\lambda^{-1} - (1 + \alpha)}{2(1 + \alpha)}$. $\square$

For each $i \in \mathbb{Z}$, let us take the set of charts

$$\phi_{j,i} : B^k(0, \beta_i) \times B^{m-k}(0, \beta_i) \rightarrow B(p_{j,i}, \beta_i) \quad \text{where } \phi_{j,i} = \exp_{p_{j,i}} \circ \tau_{p_{j,i}}^{-1},$$

where $p_{j,i} \in X_i$, for $j = 1, \dots, m_i$. It follows from Lemmas 4.1.4 and 4.1.5 that there exists a strong basic neighborhood $B^1(\mathbf{f}, (\xi_i)_{i \in \mathbb{Z}})$ of $\mathbf{f}$ such that, if $\mathbf{g} \in B^1(\mathbf{f}, (\xi_i)_{i \in \mathbb{Z}})$, then:

Lemma 4.1.6. *$\{(B(p_{j,i}, \beta_i), \phi_{j,i}) : j = 1, \dots, m_i, i \in \mathbb{Z}\}$ is an Euclidean atlas for $\mathbf{M}$ such that, for all $i \in \mathbb{Z}$ and $j = 1, \dots, m_i$:*

- i. $\phi_{j,i+1}^{-1} \mathbf{g} \phi_{j,i}(B^k(0, \beta_i) \times B^{m-k}(0, \beta_i)) \subseteq B^k(0, \delta_{i+1}) \times B^{m-k}(0, \delta_{i+1})$.
- ii. $\phi_{j,i}^{-1} \mathbf{g}^{-1} \phi_{j,i+1}(B^k(0, \beta_{i+1}) \times B^{m-k}(0, \beta_{i+1})) \subseteq B^k(0, \delta_i) \times B^{m-k}(0, \delta_i)$.
- iii. For all $z \in B^k(0, \beta_i) \times B^{m-k}(0, \beta_i)$, if $x \in \overline{K_\alpha^u}$, we have

$$D(\phi_{j,i+1}^{-1} \mathbf{g} \phi_{j,i})_z(\overline{K_\alpha^u}) \subseteq K_\alpha^u \quad \text{and} \quad \|D(\phi_{j,i+1}^{-1} \mathbf{g} \phi_{j,i})_z(x)\| \geq \eta^{-1}\|x\|.$$

- iv. For all $z \in B^k(0, \beta_{i+1}) \times B^{m-k}(0, \beta_{i+1})$, if $x \in \overline{K_\alpha^s}$, we have

$$D(\phi_{j,i}^{-1} \mathbf{g}^{-1} \phi_{j,i+1})_z(\overline{K_\alpha^s}) \subseteq K_\alpha^s \quad \text{and} \quad \|D(\phi_{j,i}^{-1} \mathbf{g}^{-1} \phi_{j,i+1})_z(x)\| \geq \eta^{-1}\|x\|.$$

Since $D_0 \exp_p = Id_{T_p M}$, $\hat{g}_p = \tau_{f(p)} \circ \exp_{f(p)}^{-1} \circ g_i \circ \exp_p \circ \tau_p^{-1}$ and τ_p is an isometry, by choosing β_i sufficiently small, the Lemmas 4.1.4 and 4.1.5 are valid for $\mathbf{g}$ and some $\eta' \in (0, 1)$ (which we will continue calling by η). That is, since $M_i = \bigcup_{j=1}^{m_i} \phi_{j,i}(B^k(0, \beta_i) \times B^{m-k}(0, \beta_i))$, if $p \in M_i$, then

$$p \in \phi_{j,i}(B^k(0, \beta_i) \times B^{m-k}(0, \beta_i)) \quad \text{for some } j \in \{1, \dots, m_i\},$$

and therefore:

Lemma 4.1.7. *There exists $\eta \in (0, 1)$ such that, if $\mathbf{g} \in B^1(\mathbf{f}, (\xi_i)_{i \in \mathbb{Z}})$, for each $p \in \mathbf{M}$ we have:*

- i. $D\mathbf{g}_p(K_{\alpha, \mathbf{f}, p}^u) \subseteq K_{\alpha, \mathbf{f}, \mathbf{g}(p)}^u$. Furthermore,

$$\|D\mathbf{g}_p(v)\| \geq \eta^{-1}\|v\| \quad \text{for any } v \in K_{\alpha, \mathbf{f}, p}^u.$$

ii. $D(\mathbf{g}^{-1})_{\mathbf{g}(p)}(K_{\alpha,\mathbf{f},\mathbf{g}(p)}^s) \subseteq K_{\alpha,\mathbf{f},p}^s$. Furthermore,

$$\|D(\mathbf{g}^{-1})_{\mathbf{g}(p)}(v)\| \geq \eta^{-1}\|v\| \quad \text{for any } v \in K_{\alpha,\mathbf{f},\mathbf{g}(p)}^s.$$

4.2 Openness for Anosov Families with the Property of Angles

First we prove that the set consisting of Anosov families satisfying the property of the angles is open. In the next section we will show the general case. We will consider $(\xi_i)_{i \in \mathbb{Z}}$ as in Lemma 4.1.7 and fix $\mathbf{g} \in B^1(\mathbf{f}, (\xi_i)_{i \in \mathbb{Z}})$. Using the results obtained in the previous section, we will build families of subspaces F_p^s and F_p^u of $T_p\mathbf{M}$, for each $p \in \mathbf{M}$, with which $\mathbf{g}$ satisfies the conditions from Definition 3.1.2.

Lemma 4.2.1. *For each $p \in M$, take*

$$F_p^s = \bigcap_{n=0}^{\infty} D\mathbf{g}_{\mathbf{g}^n(p)}^{-n}(\overline{K_{\alpha,\mathbf{f},\mathbf{g}^n(p)}^s}) \quad \text{and} \quad F_p^u = \bigcap_{n=0}^{\infty} D\mathbf{g}_{\mathbf{g}^{-n}(p)}^n(\overline{K_{\alpha,\mathbf{f},\mathbf{g}^{-n}(p)}^u}). \quad (4.2.1)$$

Thus, the families F_p^s and F_p^u are $D\mathbf{g}$ -invariant (see Figure 4.2.1).

Figure 4.2.1: $F_{p,n}^r = \bigcap_{k=1}^n D\mathbf{g}_{\mathbf{g}^{\pm k}(p)}^{\pm k}(\overline{K_{\alpha,\mathbf{f},\mathbf{g}^{\pm k}(p)}^r})$, for $r = s, u$ and $n = 1, 2, 3$.

Proof. By Lemma 4.1.4 we have for all $p \in \mathbf{M}$

$$D\mathbf{g}_{\mathbf{g}(p)}^{-1}(\overline{K_{\alpha,\mathbf{f},\mathbf{g}(p)}^s}) \subseteq K_{\alpha,\mathbf{f},p}^s \quad \text{and} \quad D\mathbf{g}_p(\overline{K_{\alpha,\mathbf{f},p}^u}) \subseteq K_{\alpha,\mathbf{f},\mathbf{g}(p)}^u.$$

Therefore

$$\begin{aligned} D\mathbf{g}_{\mathbf{g}(p)}^{-1}(F_{\mathbf{g}(p)}^s) &= \bigcap_{n=0}^{\infty} D\mathbf{g}_{\mathbf{g}(p)}^{-1}(D\mathbf{g}_{\mathbf{g}^{n+1}(p)}^{-n}(\overline{K_{\alpha,\mathbf{f},\mathbf{g}^{n+1}(p)}^s})) = \bigcap_{n=0}^{\infty} D\mathbf{g}_{\mathbf{g}^{n+1}(p)}^{-n-1}(\overline{K_{\alpha,\mathbf{f},\mathbf{g}^{n+1}(p)}^s}) \\ &\subseteq \bigcap_{n=0}^{\infty} D\mathbf{g}_{\mathbf{g}^n(p)}^{-n}(\overline{K_{\alpha,\mathbf{f},\mathbf{g}^n(p)}^s}) = F_p^s. \end{aligned}$$

Now,

$$\begin{aligned} Dg_p(F_p^s) &= Dg_p(\overline{K_{\alpha, \mathbf{f}, p}^s}) \cap \bigcap_{n=1}^{\infty} Dg_p(Dg_{g^n(p)}^{-n}(\overline{K_{\alpha, \mathbf{f}, g^n(p)}^s})) \subseteq \bigcap_{n=1}^{\infty} Dg_{g^n(p)}^{-n+1}(\overline{K_{\alpha, \mathbf{f}, g^n(p)}^s}) \\ &= \bigcap_{n=0}^{\infty} Dg_{g^{n+1}(p)}^{-n}(\overline{K_{\alpha, \mathbf{f}, g^{n+1}(p)}^s}) = F_{g(p)}^s. \end{aligned}$$

The above facts show that $Dg_p(F_p^s) = F_{g(p)}^s$. Analogously we can show that

$$Dg_p(F_p^u) = F_{g(p)}^u,$$

which proves the lemma. $\square$

Inductively we have that for all $n \geq 1$

$$Dg_p^n(F_p^s) = F_{g^n(p)}^s \quad \text{and} \quad Dg_p^n(F_p^u) = F_{g^n(p)}^u. \quad (4.2.2)$$

Since $F_p^r \subseteq K_{\alpha, \mathbf{f}, p}^r$ for $r = s, u$, it follows from Lemma 4.1.7 that for all $n \geq 1$

$$\|Dg_p^{-n}v\| \geq \frac{1}{\eta^n}\|v\| \text{ if } v \in F_p^s \quad \text{and} \quad \|Dg_p^n v\| \geq \frac{1}{\eta^n}\|v\| \text{ if } v \in F_p^u. \quad (4.2.3)$$

Lemma 4.2.2. F_p^s and F_p^u given in (4.2.1) are vectorial subspaces and furthermore $T_p \mathbf{M} = F_p^s \oplus F_p^u$, for each $p \in \mathbf{M}$.

Proof. Fix $p \in \mathbf{M}$. Since $E_q^s \subseteq K_{\alpha, \mathbf{f}, q}^s$ for each $q \in \mathbf{M}$, we can choose an orthonormal basis $\{v_{1,p}^n, \dots, v_{k,p}^n\}$ of $Dg_{g^n(p)}^{-n}(E_{\mathbf{f}^n(p)}^s) \subseteq F_p^s$ for each $n \geq 0$. We can find a subsequence n_i , with $n_i \rightarrow \infty$ when $i \rightarrow \infty$, such that $v_{i,p}^{n_i} \rightarrow v_{i,p}$, where $v_{i,p} \in F_p^s$, for each $i = 1, \dots, k$. This fact proves that F_p^s contains a k -dimensional vectorial subspace J_p^s , the subspace spanned by $\{v_{1,p}, \dots, v_{k,p}\}$. Analogously, we can prove that F_p^u contains a $(m-k)$ -dimensional vectorial subspace J_p^u . Remember that $\alpha < \frac{1-\lambda}{1+\lambda} < 1$. Therefore,

$$\overline{K_{\alpha, \mathbf{f}, p}^s} \cap \overline{K_{\alpha, \mathbf{f}, p}^u} = \{0_p\}.$$

Since $J_p^s \subseteq \overline{K_{\alpha, \mathbf{f}, p}^s}$ and $J_p^u \subseteq \overline{K_{\alpha, \mathbf{f}, p}^u}$, we have $J_p^s \cap J_p^u = \{0_p\}$. Consequently,

$$J_p^s \oplus J_p^u = T_p \mathbf{M}.$$

Next, we prove that $J_p^s = F_p^s$ and $J_p^u = F_p^u$. Let $v \in F_p^s$. Since $T_p \mathbf{M} = J_p^s \oplus J_p^u$, $v = v_s + v_u$, where $v_s \in J_p^s$ and $v_u \in J_p^u$. Notice that $v - v_s \in K_{\alpha, \mathbf{f}, p}^s$ because $K_{\alpha, \mathbf{f}, p}^s$ is a cone. It follows from (4.2.2) and (4.2.3) that, for each $n \geq 1$,

$$\|v_u\| \leq \eta^n \|Dg_p^n(v - v_s)\| \leq \eta^{2n} \|v - v_s\|.$$

Since $\eta < 1$, we have that $v_u = 0$. Hence $v \in J_p^s$ and therefore $J_p^s = F_p^s$. Analogously we can prove that $J_p^u = F_p^u$. $\square$

Consequently,

Proposition 4.2.3. *If $g \in B^1(\mathbf{f}, (\xi_i)_{i \in \mathbb{Z}})$, then $g \in \mathcal{A}^1(\mathbf{M})$ and satisfies the property of angles.*

Proof. From Lemmas 4.1.7, 4.2.1 and 4.2.2 we have that, considering the splitting $T_p\mathbf{M} = F_p^s \oplus F_p^u$, for each $p \in \mathbf{M}$, $\mathbf{g}$ has hyperbolic behaviour. We can prove that this splitting is unique (see Lemma 3.1.5) and depends continuously on p (see Proposition 3.1.8). Consequently, $\mathbf{g}$ is an Anosov family. Finally, since $F_p^s \subseteq K_{\alpha, \mathbf{f}, p}^s$ and $F_p^u \subseteq K_{\alpha, \mathbf{f}, p}^u$ for all p and $\alpha < \frac{1-\lambda}{1+\lambda} < 1$, we have that $\mathbf{g}$ s. p. a. $\square$

From Proposition 4.2.3 we obtain the set consisting of the Anosov families that satisfy the property of angles is open in $\mathcal{F}^1(\mathbf{M})$.

4.3 Openness of Anosov Families: General Case

Finally will show that the set consisting of all the Anosov families is open in $\mathcal{F}^1(\mathbf{M})$. Indeed, we will suppose that $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f}) \in \mathcal{A}^1(\mathbf{M})$ does not satisfy the property of angles with the Riemannian metric $\langle \cdot, \cdot \rangle$. Let $\langle \cdot, \cdot \rangle_\star$ be the Riemannian metric obtained in Proposition 3.3.4. Thus $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{f})$ is a strictly Anosov family that satisfy the property of angles.

Definition 4.3.1. Let μ_i be as in (3.1.3). Fix $\zeta > 0$. We will consider $\alpha > 0$ such that $1 - \alpha c \frac{\lambda + \zeta}{\zeta} > 0$. For each $i \in \mathbb{Z}$, set

$$\Delta_i = \frac{1}{\mu_i} \left(\frac{\lambda + \zeta}{\zeta} c \right)^2.$$

By (3.3.3) and (3.3.4) we have

$$\Delta_i^{-1} \|v\|_\star \leq \|v\| \leq 2\|v\|_\star \quad \text{for all } v \in TM_i, i \in \mathbb{Z}.$$

Let $d^\star$ be the metric on $\mathbf{D}_i^1$ obtained considering the Riemannian metric $\langle \cdot, \cdot \rangle_\star$ on each M_i (see Definition 1.3.4). For a sequence of positive numbers $(\epsilon_i)_{i \in \mathbb{Z}}$, let $B^\star(\mathbf{f}, (\epsilon_i)_{i \in \mathbb{Z}})$ be the strong basic neighborhood of $\mathbf{f}$ in $\mathcal{F}^1(\mathbf{M})$ considering the metric $d^\star$ on each $\mathbf{D}_i^1$.

From Proposition 4.2.3 it follows that:

Lemma 4.3.2. *There exists a sequence $(\xi_i)_{i \in \mathbb{Z}}$ such that, if $\mathbf{g} = (g_i)_{i \in \mathbb{Z}} \in B^\star(\mathbf{f}, (\xi_i)_{i \in \mathbb{Z}})$, then $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{g})$ is an Anosov family.*

In that case, we have that $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{g})$ is an Anosov family with constants $\tilde{c} = 1$ and $\tilde{\lambda} = \eta \in (0, 1)$ (see (4.2.3)).

Let $(\tilde{\xi}_i)_{i \in \mathbb{Z}}$, where $\tilde{\xi}_i = \xi_i / \Delta_i$ for each $i \in \mathbb{Z}$. Notice that if $\mathbf{g} \in B^1(\mathbf{f}, (\tilde{\xi}_i)_{i \in \mathbb{Z}})$, then $\mathbf{g} \in B^\star(\mathbf{f}, (\xi_i)_{i \in \mathbb{Z}})$. Consequently, if $\mathbf{g} \in B^1(\mathbf{f}, (\tilde{\xi}_i)_{i \in \mathbb{Z}})$, then $(\mathbf{M}, \langle \cdot, \cdot \rangle_\star, \mathbf{g})$ is an Anosov family.

Remark 4.3.3. In the following lemma we will show that each non-stationary dynamical system in $B^1(\mathbf{f}, (\tilde{\xi}_i)_{i \in \mathbb{Z}})$ is an Anosov family with the metric $\langle \cdot, \cdot \rangle$. This fact is not immediate, since $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_\star$ are not necessarily uniformly equivalent on $\mathbf{M}$ (Example 3.2.1 proves that the notion of Anosov family depends on the metric on the total space).

Lemma 4.3.4. *If $\mathbf{g} \in B^1(\mathbf{f}, (\tilde{\xi}_i)_{i \in \mathbb{Z}})$, then $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{g})$ is an Anosov family.*

Proof. Consider the stable subspace $E_{\mathbf{g}, p}^s$ of $\mathbf{g}$ at p with respect to the metric $\langle \cdot, \cdot \rangle_\star$. We have from Lemma 4.1.5 that $E_{\mathbf{g}, p}^s$ is contained in the stable α -cone of $\mathbf{f}$ at p . If $v \in E_{\mathbf{g}, p}^s$, then $v = v_s + v_u$, where $v_s \in E_{\mathbf{f}, p}^s$ and $v_u \in E_{\mathbf{f}, p}^u$. It follows from (3.3.2) that

$$\|v_s\| \leq \|v_s + v_u\| + \alpha \|v_s\|_\star \leq \|v\| + \alpha \frac{\lambda + \zeta}{\zeta} c \|v_s\|.$$

Thus, by (3.3.3),

$$\begin{aligned} \|Dg_p^n(v)\| &\leq 2\|Dg_p^n(v)\|_* \leq 2\eta^n\|v\|_* \leq 2\eta^n(\|v_s\|_* + \|v_u\|_*) \leq 2\eta^n(\|v_s\|_* + \alpha\|v_s\|_*) \\ &= 2\eta^n(1+\alpha)\|v_s\|_* \leq 2\eta^n(1+\alpha)\frac{\lambda+\zeta}{\zeta}c\|v_s\| \leq 2\eta^n(1+\alpha)\frac{\lambda+\zeta}{\zeta}c(1-\alpha\frac{\lambda+\zeta}{\zeta}c)^{-1}\|v\| \\ &= c'\eta^n\|v\|, \end{aligned}$$

where $c' = 2(1+\alpha)\frac{\lambda+\zeta}{\zeta}c(1-\alpha\frac{\lambda+\zeta}{\zeta}c)^{-1}$.

Analogously we obtain that if $v \in E_{g,p}^u$ then $\|Dg_p^{-n}(v)\| \leq c'\eta^n\|v\|$.

Consequently, $(\mathbf{M}, \langle \cdot, \cdot \rangle, g)$ is an Anosov family with constants $\eta \in (0, 1)$ and c' . $\square$

From Proposition 4.2.3 and Lemma 4.3.4 we have:

Theorem 4.3.5. *Let $f \in \mathcal{A}^1(\mathbf{M})$. There exists a sequence of positive numbers $(\xi_i)_{i \in \mathbb{Z}}$ such that, if $g \in B^1(f, (\xi_i)_{i \in \mathbb{Z}})$, then $(\mathbf{M}, \langle \cdot, \cdot \rangle, g)$ is an Anosov family. Consequently, $\mathcal{A}^1(\mathbf{M})$ is open in $\mathcal{F}^1(\mathbf{M})$.*

Proof. The theorem follows from Proposition 4.2.3 and from Lemmas 4.3.2 and 4.3.4. $\square$

Notice that for the basic neighborhood $B^1(f, (\xi_i)_{i \in \mathbb{Z}})$ of f the ξ_i could be arbitrarily small for $|i|$ large. In that case, if $(g_i)_{i \in \mathbb{Z}} \in B^1(f, (\xi_i)_{i \in \mathbb{Z}})$, then f_i and g_i are C^1 -closer for $|i|$ large.

4.4 Openness for Anosov Families consisting of Matrices

In this section, we will suppose that $F = (F_i)_{i \in \mathbb{Z}}$ is an Anosov family satisfying the property of angles, where $F_i : \mathbb{T}^m \rightarrow \mathbb{T}^m$ is a diffeomorphism on $\mathbb{T}^m$ induced by a matrix $A_i : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$ is the m -torus endowed with the Riemannian metric inherited from $\mathbb{R}^m$. Hence, the diagram

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{A_i} & \mathbb{R}^m \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T}^m & \xrightarrow{F_i} & \mathbb{T}^m \end{array}$$

commutes for each $i \in \mathbb{Z}$, where π is the projection map.

Definition 4.4.1. For $\varepsilon > 0$ and $r \geq 0$, a *uniform basic neighborhood* of $f = (f_i)_{i \in \mathbb{Z}}$ is the set

$$B^r(f, \varepsilon) = \{(g_i)_{i \in \mathbb{Z}} \in \mathcal{F}^r(\mathbf{M}) : d^r(f_i, g_i) < \varepsilon \text{ for all } i \in \mathbb{Z}\}.$$

The goal of this section is to prove the following theorem:

Theorem 4.4.2. *There exists $\xi > 0$ such that, if $g \in B^1(F, \xi)$, then $g \in \mathcal{A}^1(\mathbf{M})$.*

Since f satisfies the property of angles, the Riemannian metric $\|\cdot\|_*$ obtained in Proposition 3.3.4 is uniformly equivalent to $\|\cdot\|$. It is clear that all the uniform basic neighborhood considering the metric $\|\cdot\|_*$ contains a uniform basic neighborhood considering $\|\cdot\|$ and viceversa. Consequently, without loss of generality, we can suppose that f is strictly Anosov with the metric $\|\cdot\|$ (f is strictly Anosov with $\|\cdot\|_*$).

Remark 4.4.3. Notice that in this case, $a_p(z) = b_p(z) = c_p(z) = d_p(z) = 0$ for each $p \in \mathbb{T}^m$ and $z \in \mathbb{R}^m$ (see (3.4.1)). Therefore, $\sigma_p(\delta) = \rho_p(\delta) = 0$ for each $p \in \mathbb{T}^m$ and $\delta > 0$ (see Definition 3.4.7).

For $\alpha \in (0, \frac{1-\lambda}{1+\lambda})$, set

$$\vartheta = \min \left\{ \frac{(\lambda^{-1} - \lambda)\alpha}{2(1+\alpha)^2}, \frac{\lambda^{-1}(1-\alpha) - (1+\alpha)\alpha}{2(1+\alpha)} \right\} \quad \text{for each } i \in \mathbb{Z}.$$

Let $X = \{p_1, \dots, p_k\} \subseteq \mathbb{T}^m$ be such that $\mathbb{T}^m = \cup_{i=1}^k B(p_i, \vartheta)$.

Lemma 4.4.4. *Fix $\xi \in (0, \vartheta)$. If $g \in B^1(\mathbf{F}, \xi)$, then for all $p \in X$ we have:*

- i. $D(\hat{g}_p)_z(\overline{K_\alpha^u}) \subseteq K_\alpha^u$ for all $z \in B(p, \vartheta)$, and
- ii. $D(\hat{g}_p^{-1})_z(\overline{K_\alpha^s}) \subseteq K_\alpha^s$ for all $z \in B(p, \vartheta)$.

Proof. See Lemma 4.1.4. □

Lemma 4.4.5. *If $\xi \in (0, \vartheta)$, there exists $\eta < 1$ such that, if $g \in B^1(\mathbf{F}, \xi)$, then, for $p \in X$, $z \in B(p, \vartheta)$, we have*

- i. $\|D(\hat{g}_p)_z(w)\| \geq \eta^{-1}\|w\|$ if $w \in \overline{K_\alpha^u}$;
- ii. $\|D(\hat{g}_p^{-1})_z(w)\| \geq \eta^{-1}\|w\|$ if $w \in \overline{K_\alpha^s}$.

Proof. See Lemma 4.1.5. □

Proof of Theorem 4.4.2. Fix $g \in B^1(\mathbf{F}, \xi)$. Following the Lemmas 4.1.7 and 4.2.1 we have that for each $p \in \mathbf{M}$, the subspaces F_p^s and F_p^u , given in Lemma 4.2.1, are Dg -invariants. By Lemma 4.2.2 we have that F_p^s and F_p^u are vectorial subspaces and furthermore $T_p\mathbf{M} = F_p^s \oplus F_p^u$, for each $p \in \mathbf{M}$. This facts prove Theorem 4.4.2. □

There are many examples of Anosov families consisting of sequences of matrices, as we saw in Section 3. Any multiplicative family satisfies the property of angles, consequently they are uniformly stable. On the other hand, since any gathering of an Anosov family is Anosov, it can be proved that if a family satisfies the property of angles, any gathering satisfies the property of angles. Thus, any family obtained from a gathering of a multiplicative family satisfies the property of angles.

Chapter 5

Stable and Unstable Manifolds

Let $\phi : M \rightarrow M$ be a diffeomorphism on a compact Riemannian manifold M with metric ρ . Let $x \in M$ and $\varepsilon > 0$. Let $\mathcal{V}_\varepsilon^s(x, \phi)$ and $\mathcal{V}_\varepsilon^u(x, \phi)$ be the local stable and unstable sets of ϕ at x , respectively (see Definition 1.4.2). If ϕ is an Anosov diffeomorphism, there exists $\varepsilon > 0$ such that, for every $x \in M$, $\mathcal{V}_\varepsilon^s(x, \phi)$ and $\mathcal{V}_\varepsilon^u(x, \phi)$ are differentiable submanifolds of M , tangent to the stable and unstable subspaces at x , respectively (see [Shu13]). The goal of this chapter is to show a similar version of this property for Anosov families.

5.1 Stable and Unstable Sets

As we mentioned above, if $\phi : M \rightarrow M$ is an Anosov diffeomorphism, then there is $\varepsilon > 0$ such that, for each $x \in M$, the stable and unstable sets at x are differentiable submanifolds of M tangent to the stable and unstable subspaces at x , respectively. In that case, ϕ is a contraction on $\mathcal{V}_\varepsilon^s(x, \phi)$ (that is, there exists a $\nu \in (0, 1)$ such that $d(\phi(z), \phi(y)) \leq \nu d(z, y)$ for all $z, y \in \mathcal{V}_\varepsilon^s(x, \phi)$) and ϕ^{-1} is a contraction on $\mathcal{V}_\varepsilon^u(x, \phi)$. Furthermore, $\phi(\mathcal{V}_\varepsilon^s(x, \phi)) \subseteq \mathcal{V}_\varepsilon^s(\phi(x), \phi)$ and $\phi^{-1}(\mathcal{V}_\varepsilon^u(\phi(x), \phi)) \subseteq \mathcal{V}_\varepsilon^u(x, \phi)$ for each $x \in M$. If we consider the stable and unstable subsets for non-stationary dynamical systems as in Definition 1.4.2, the facts above are not always valid for Anosov families (see Example 3.2.2). In Definition 5.1.4 we will give a notion of stable and unstable sets which works better for non-stationary dynamical systems than the sets given in Definition 1.4.2.

Definition 5.1.1. A homeomorphism $\phi : X \rightarrow X$ on the metric space (X, ρ) is *expansive* on a subset Y of X if there is $\varepsilon > 0$ such that

$$\sup_{n \in \mathbb{Z}} \{\rho(\phi^n(x), \phi^n(y)) : x \in X, y \in Y, x \neq y\} > \varepsilon.$$

It is well known that if $\Lambda \subseteq X$ is a compact hyperbolic subset for a C^1 -diffeomorphism $\phi : X \rightarrow X$, then ϕ is expansive on Λ . In the following example we will see that there are Anosov families that are not expansive.

Example 5.1.2. Take $a, b \in (\lambda, 1)$ in the Example 3.2.2. For $x, y \in M_0$ we obtain $\mathbf{d}(f^n(x), f^n(y)) \rightarrow 0$ as $n \rightarrow +\infty$ ¹. Therefore, the stable set of $\mathbf{f}$ at any point x in M_0 is the whole M_0 . Thus, for all

¹Notice that the volume of each M_i with the Riemannian metric $\|\cdot\|_i$ defined in (3.2.1) is decreasing, for $i \geq 1$, if $a, b \in (\lambda, 1)$ (see Figure 5.1.1).

$x \in M_0$ and $\varepsilon > 0$, $\mathcal{V}_\varepsilon^s(x, \phi) \cap \mathcal{V}_\varepsilon^u(x, \phi) = \mathcal{V}_\varepsilon^u(x, \phi)$. On the other hand, if $y \in \mathcal{V}^u(x, \phi)$, we obtain

$$\mathbf{d}(\mathbf{f}^{-n}(x), \mathbf{f}^{-n}(y)) = \mathbf{d}(\phi^{-n}(x), \phi^{-n}(y)) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Consequently, $\mathbf{f}$ is not expansive, because $\mathbf{d}(\mathbf{f}^n(x), \mathbf{f}^n(y)) \rightarrow 0$ as $n \rightarrow \pm\infty$.

Figure 5.1.1: $M_1, M_2, M_3, \dots$, endowed with the metric given in (3.2.1), for $a, b \in (\lambda, 1)$.

Next we define a notion about stables and unstable sets which work better for non-stationary dynamical systems than the sets given in Definition 1.4.2. These sets consist of the points whose orbits approach exponentially to the orbits of a given point.

Definition 5.1.3. Given two points $p, q \in \mathbf{M}$, set

$$\Theta_{p,q} = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{d}(\mathbf{f}_i^n(q), \mathbf{f}_i^n(p)) \quad \text{and} \quad \Delta_{p,q} = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{d}(\mathbf{f}_i^{-n}(q), \mathbf{f}_i^{-n}(p)).$$

Definition 5.1.4. Let $\varepsilon = (\varepsilon_i)_{i \in \mathbb{Z}}$ be a sequence of positive numbers. Fix $p \in M_i$. Set

$$\mathcal{N}^s(p, \varepsilon) = \{q \in B(p, \varepsilon_i) : \mathbf{f}_i^n(q) \in B(\mathbf{f}_i^n(p), \varepsilon_{i+n}) \text{ for } n \geq 1 \text{ and } \Theta_{p,q} < 0\}.$$

$\mathcal{N}^s(p, \varepsilon)$ will be called the *local stable set* for $\mathbf{f}$ at p ;

$$\mathcal{N}^u(p, \varepsilon) = \{q \in B(p, \varepsilon_i) : \mathbf{f}_i^{-n}(q) \in B(\mathbf{f}_i^{-n}(p), \varepsilon_{i-n}) \text{ for } n \geq 1 \text{ and } \Delta_{p,q} < 0\}.$$

$\mathcal{N}^u(p, \varepsilon)$ will be called the *local unstable set* for $\mathbf{f}$ at p .

5.2 Hadamard-Perron Theorem for Anosov Families

In this section we will give conditions for obtain invariant manifolds at each point of the total space, whose expansion or contraction by each $\mathbf{f}_i$ can be controlled (see Theorems 5.2.10 and 5.2.11). This result is a generalized version of the Hadamard-Perron Theorem for obtain local stable and unstable manifold for Anosov families (see [BP07], [KH97]). In our case, stable and unstable subspaces are not necessarily orthogonal. Therefore, the size of the manifolds to be obtained could decay along the orbits.

We will fix an Anosov family $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ with constant $\lambda \in (0, 1)$ and $c \geq 1$.

Remark 5.2.1. If $c > 1$, let n be the minimum positive integer such that $c\lambda^n \leq \lambda$. Hence the gathering $\tilde{\mathbf{f}}$ obtained of $\mathbf{f}$ by the sequence $(ni)_{i \in \mathbb{Z}}$ is a strictly Anosov family with constant λ . Thus, considering a gathering of $\mathbf{f}$ if necessary, we can assume that the family is strictly Anosov.

Let us fix $p \in \mathbf{M}$. Without loss of generality, we can assume that $p \in M_0$ (if $p \notin M_0$, $q = f^n(p) \in M_0$ for some $n \in \mathbb{Z}$, then consider q instead of p). By simplicity, throughout this chapter we will consider the following notations:

Definition 5.2.2. For $\varepsilon > 0$ and $n \in \mathbb{Z}$, set

- i. $B_n(\varepsilon)$ the ball in $T_{f_0^n(p)}^s$ with radius ε and center $0_{f_0^n(p)} \in T_{f_0^n(p)}^s \mathbf{M}$,
- ii. $B_n^s(\varepsilon)$ the ball in $E_{f_0^n(p)}^s$ with radius ε and center $0_{f_0^n(p)} \in E_{f_0^n(p)}^s$,
- iii. $B_n^u(\varepsilon)$ the ball in $E_{f_0^n(p)}^u$ with radius ε and center $0 \in E_{f_0^n(p)}^u$,
- iv. $a_n = a_{f^n(p)}$, $b_n = b_{f^n(p)}$, $A_n = A_{f^n(p)}$, $B_n = B_{f^n(p)}$, $C_n = C_{f^n(p)}$, $D_n = D_{f^n(p)}$, $c_n = c_{f^n(p)}$, $d_n = d_{f^n(p)}$, $\kappa_n = \kappa_{f^n(p)}$, $\mu_n = \mu_{f^n(p)}$, $\sigma_n = \sigma_{f^n(p)}$ and $\rho_n = \rho_{f^n(p)}$ (see Section 4.4).

For each $n \in \mathbb{Z}$, let $\varrho_n > 0$ be the injectivity radius of $\exp_{f_0^n(p)}$ at $f_0^n(p)$ (see Definition 2.4.1). Take $\varepsilon_n = \varrho_n/2$ and let $\delta_n > 0$ be small enough such that

$$\tilde{f}_n = \exp_{f_{n+1}(p)}^{-1} \circ f_n \circ \exp_{f^n(p)} : B_n^s(\delta_n) \times B_n^u(\delta_n) \rightarrow B_{n+1}(\varepsilon_{n+1})$$

is well defined, for each $n \in \mathbb{Z}$.

Definition 5.2.3. Let $\alpha \in (0, 1)$ and $(\gamma_n)_{n \in \mathbb{Z}}$ be a sequence of positive numbers. Define:

- i. $\Gamma_n^u(\alpha, \gamma_n) = \{\phi : B_n^u(\gamma_n) \rightarrow B_n^s(\gamma_n) : \phi \text{ is } \alpha\text{-Lipschitz and } \phi(0) = 0\}$.
- ii. $\Gamma^u(\alpha, (\gamma_n)_n) = \{\phi = (\phi_n)_{n \in \mathbb{Z}} : \phi_n \in \Gamma_n^u(\alpha, \delta_n)\}$.

If $\phi = (\phi_n)_{n \in \mathbb{Z}}$, $\psi = (\psi_n)_{n \in \mathbb{Z}} \in \Gamma^u(\alpha, (\gamma_n)_n)$, define the metric

$$d_{\Gamma^u}(\phi, \psi) = \sup_{n \in \mathbb{Z}} \left\{ \sup_{x \in B_n^u(\gamma_n) \setminus \{0\}} \frac{\|\phi_n(x) - \psi_n(x)\|}{\|x\|} \right\}.$$

It is not difficult to prove the following proposition:

Proposition 5.2.4. $(\Gamma^u(\alpha, (\gamma_n)_n), d_{\Gamma^u})$ is a complete metric space.

For an application $F : X \rightarrow Y$, we will denote by $\mathcal{G}(F)$ the set $\{(F(x), x) : x \in X\}$. Throughout this section, we will fix $\alpha \in (0, 1)$, $\gamma \in (\lambda^2, 1)$ and

$$\tilde{\sigma}_n = \min \left\{ \frac{(\kappa_n^{-1} - \mu_n)\alpha}{(1 + \alpha)^2}, \frac{(\gamma\kappa_n^{-1} - \mu_n)}{(1 + \alpha)(1 + \gamma)} \right\}. \quad (5.2.1)$$

Proposition 5.2.5. Suppose that for each $n \leq -1$ we can choose the δ_n 's such that

$$\frac{\kappa_n^{-1} + \alpha\mu_n}{1 + \alpha}\delta_n \geq \delta_{n+1} \quad \text{for } n \leq -1 \quad \text{and} \quad \sigma_n < \tilde{\sigma}_n. \quad (5.2.2)$$

Then, there exists a sequence of positive numbers $(\delta_n)_{n \geq 0}$ such that, for each $n \in \mathbb{Z}$, if $\phi_n \in \Gamma_n^u(\alpha, \delta_n)$, we have that

$$\{\tilde{f}_n(\phi_n(w), w) : w \in B_n^u(\delta_n)\} \cap B_{n+1}^s(\delta_{n+1}) \times B_{n+1}^u(\delta_{n+1})$$

is the $\mathcal{G}$ of an application ψ_{n+1} in $\Gamma_{n+1}^u(\alpha, \delta_{n+1})$ (see Figure 5.2.1).

Figure 5.2.1: $\mathcal{G}(\psi_{n+1}) = \tilde{f}_n \mathcal{G}(\phi_n)$. Shaded regions represent the unstable α -cones.

Proof. Inductively, for each $n \geq 0$ we can choose $\delta_n > 0$ such that $\sigma_n < \omega_n$ and if $\phi_{n-1} \in \Gamma_{n-1}^u(\alpha, \delta_{n-1})$, then $\{\tilde{f}_{n-1}(\phi_{n-1}(w), w) : w \in B_{n-1}^u(\delta_{n-1})\} \cap B_n^s(\delta_n) \times B_n^u(\delta_n)$ is the $\mathcal{G}$ of an application ψ_n in $\Gamma_n^u(\alpha, \delta_n)$.

Now, fix $\phi_n \in \Gamma_n^u(\alpha, \delta_n)$. For $w \in B_n^u(\delta_n)$, let

$$r_n(w) = B_n w + b_n(\phi_n(w), w). \quad (5.2.3)$$

If $w, z \in B_n^u(\delta_n)$ we have

$$\begin{aligned} \|r_n(w) - r_n(z)\| &\geq \|B_n w - B_n z\| - \|b_n(\phi_n(w), w) - b_n(\phi_n(z), z)\| \\ &\geq \kappa_n^{-1} \|w - z\| - \sigma_n(1 + \alpha) \|w - z\| \end{aligned}$$

(from the second inequality in (5.2.2) we have $\kappa_n^{-1} - \sigma_n(1 + \alpha) > 0$). Thus,

$$\|r_n(w) - r_n(z)\| \geq (\kappa_n^{-1} - \sigma_n(1 + \alpha)) \|w - z\| \quad (5.2.4)$$

and therefore r_n is injective. From (5.2.2) and (5.2.4) we obtain

$$B_n^u(\delta_{n+1}) \subseteq r_n(B_n^u(\delta_n)). \quad (5.2.5)$$

Consequently, we can define the map $\psi_{n+1} : B_{n+1}^u(\delta_{n+1}) \rightarrow E_{n+1}^s$, as

$$\psi_{n+1}(w) = A_n \phi_n(r_n^{-1}(w)) + a_n(\phi_n(r_n^{-1}(w)), r_n^{-1}(w)) \quad (5.2.6)$$

for $w \in B_{n+1}^u(\delta_{n+1})$. Now, if $x = r_n(w), y = r_n(z) \in B_{n+1}^u(\delta_{n+1})$, it follows from (5.2.4) that

$$\begin{aligned} \|\psi_{n+1}(x) - \psi_{n+1}(y)\| &\leq \|A_n(\phi_n(w) - \phi_n(z))\| + \|a_n(\phi_n(w), w) - a_n(\phi_n(z), z)\| \\ &\leq \alpha \mu_n \|w - z\| + \sigma_n(1 + \alpha) \|w - z\| \\ &\leq \frac{\alpha \mu_n + \sigma_n(1 + \alpha)}{\kappa_n^{-1} - \sigma_n(1 + \alpha)} \|r_n(w) - r_n(z)\| \\ &\leq \alpha \|r_n(w) - r_n(z)\| = \alpha \|x - y\|. \end{aligned}$$

Thus,

$$\|\psi_{n+1}(x) - \psi_{n+1}(y)\| \leq \alpha \|x - y\|, \quad (5.2.7)$$

that is, ψ_{n+1} is α -Lipschitz. It is clear that $\psi_{n+1}(0) = 0$ and, since $\alpha < 1$, it follows from (5.2.7) that $\psi_{n+1}(B_{n+1}^u(\delta_{n+1})) \subseteq B_{n+1}^s(\delta_{n+1})$. Consequently, $\psi_{n+1} \in \Gamma_{n+1}^u(\alpha, \delta_{n+1})$. On the other hand, if $x = r_n(w) \in B_{n+1}^u(\delta_{n+1})$ we have

$$(\psi_{n+1}(x), x) = (A_n \phi_n(w) + a_n(\phi_n w, w), B_n(w) + b_n(\phi_n w, w)) = \tilde{f}_n(\phi_n w, w). \quad (5.2.8)$$

Therefore, $\{\tilde{f}_n(\phi_n(w), w) : w \in B_n^u(\delta_n)\} \cap B_{n+1}^s(\delta_{n+1}) \times B_{n+1}^u(\delta_{n+1})$ is the $\mathcal{G}$ of ψ_{n+1} . This fact proves the proposition. $\square$

Remark 5.2.6. Proposition 5.2.5 is shown in [BP07], Proposition 7.3.5, when there exists $\delta > 0$ such that, considering $\delta_n = \delta$ for all $n \in \mathbb{Z}$, $\sigma_n < \tilde{\sigma}$ for a small enough $\tilde{\sigma} > 0$. We have adapted that proof to obtain a more general result, in which δ_n may vary with n but satisfying the first condition in (5.2.2) (this condition means that δ_n must not decay very quickly when $n \rightarrow -\infty$) and σ_n could increase but not more than $\tilde{\sigma}_n$, which could be very large. Notice that

$$\frac{\kappa_n^{-1} + \alpha \mu_n}{1 + \alpha} > 1 \quad \text{for each } n \in \mathbb{Z}.$$

On the other hand, from Remark 3.4.8 we have that

$$\|D(\tilde{f}_n)_z - D(\tilde{f}_n)_0\| = \|D(a_n, b_n)_z\| \quad \text{for each } z \in B_n(\delta_n) \text{ and } n \in \mathbb{Z}.$$

Therefore, the assumption of Proposition 5.2.5 means that the sequence $(\delta_n)_{n \in \mathbb{Z}}$ must not decay quickly to zero (the decay of δ_n is controlled by $\frac{\kappa_n^{-1} + \alpha \mu_n}{1 + \alpha}$) and

$$\|D(\tilde{f}_n)_z - D(\tilde{f}_n)_0\| < \tilde{\sigma}_n \quad \text{for each } z \text{ such that } \|z\| < \delta_n.$$

Remark 5.2.7. Since $a_n(0) = b_n(0) = D(a_n)_0 = D(b_n)_0 = 0$, we always can chose a sequence δ_n satisfying the second condition in (5.2.2). If each f_i is C^2 and the second derivative $p \rightarrow D^2 f_p$, for $p \in \mathbf{M}$, is bounded, then we can find an uniform δ satisfying (5.2.2). We will explain these facts with more detail in Chapter 7.

From Proposition 5.2.5 we have the application

$$\begin{aligned} \mathbf{G} : \Gamma^u(\alpha, (\delta_n)_n) &\rightarrow \Gamma^u(\alpha, (\delta_n)_n) \\ (\phi_n)_{n \in \mathbb{Z}} &\mapsto (\psi_{n-1})_{n \in \mathbb{Z}}, \end{aligned}$$

where ψ_n is given in (5.2.6), is well defined. The following proposition, whose proof is based on the proof of Proposition 7.3.6 in [BP07], shows that $\mathbf{G} : \Gamma^u(\alpha, (\delta_n)_n) \rightarrow \Gamma^u(\alpha, (\delta_n)_n)$ is a contraction.

Proposition 5.2.8. $\mathbf{G} : \Gamma^u(\alpha, (\delta_n)_n) \rightarrow \Gamma^u(\alpha, (\delta_n)_n)$ is a contraction.

Proof. Fix $\phi = (\phi_n)_{n \in \mathbb{Z}}, \varphi = (\varphi_n)_{n \in \mathbb{Z}} \in \Gamma^u(\alpha, (\delta_n)_n)$. Let

$$\psi = (\psi_n)_{n \in \mathbb{Z}} = \mathbf{G}(\phi) \quad \text{and} \quad \chi = (\chi_n)_{n \in \mathbb{Z}} = \mathbf{G}(\varphi).$$

For $x \in B_{n+1}^u(\delta_n)$,

$$\psi_{n+1}(x) = A_n \phi_n((r_\phi)_n^{-1}(x)) + a_n(\phi_n((r_\phi)_n^{-1}(x)), (r_\phi)_n^{-1}(x))$$

and

$$\chi_{n+1}(x) = A_n \varphi_n((r_\varphi)_n^{-1}(x)) + a_n(\varphi_n((r_\varphi)_n^{-1}(x)), (r_\varphi)_n^{-1}(x))$$

where

$$(r_\phi)_n(w) = B_n w + b_n(\phi_n(w), w) \quad \text{and} \quad (r_\varphi)_n(w) = B_n w + b_n(\varphi_n(w), w).$$

for $w \in B_n^u(\delta_n)$ (see (5.2.3)). If $x \in B_{n+1}^u(\delta_{n+1})$, there exists $w \in B_n^u(\delta_n)$ such that $x = (r_\phi)_n w$. Thus

$$\begin{aligned} \|\psi_{n+1}(x) - \chi_{n+1}(x)\| &= \|\psi_{n+1}(r_\phi)_n w - \chi_{n+1}(r_\phi)_n w\| \\ &\leq \|\psi_{n+1}(r_\phi)_n w - \chi_{n+1}(r_\varphi)_n w\| + \|\chi_{n+1}(r_\varphi)_n w - \chi_{n+1}(r_\phi)_n w\| \\ &\leq \|A_n \phi_n(w) + a_n(\phi_n(w), w) - A_n \varphi_n(w) - a_n(\varphi_n(w), w)\| + \alpha \|(r_\varphi)_n w - (r_\phi)_n w\| \\ &\leq \|A_n \phi_n(w) - A_n \varphi_n(w)\| + \|a_n(\phi_n(w), w) - a_n(\varphi_n(w), w)\| \\ &\quad + \alpha \|B_n w + b_n(\varphi_n(w), w) - B_n w - b_n(\phi_n(w), w)\| \\ &\leq \mu_n \|\phi_n(w) - \varphi_n(w)\| + \sigma_n \|(\phi_n(w), w) - (\varphi_n(w), w)\| \\ &\quad + \alpha \sigma_n \|(\varphi_n(w), w) - (\phi_n(w), w)\| \\ &= (\mu_n + \sigma_n(1 + \alpha)) \|\phi_n(w) - \varphi_n(w)\|. \end{aligned}$$

From (5.2.4) we have

$$\|x\| = \|(r_\phi)_n w\| \geq (\kappa_n^{-1} - \sigma_n(1 + \alpha)) \|w\|.$$

Therefore

$$\frac{\|\psi_{n+1}(x) - \chi_{n+1}(x)\|}{\|x\|} \leq \frac{\mu_n + \sigma_n(1 + \alpha)}{\kappa_n^{-1} - \sigma_n(1 + \alpha)} \frac{\|\phi_n(w) - \varphi_n(w)\|}{\|w\|}.$$

Since $\sigma_n < \frac{\gamma \kappa_n^{-1} - \mu_n}{(1 + \alpha)(1 + \gamma)}$ (see (5.2.2)), then

$$\frac{\mu_n + \sigma_n(1 + \alpha)}{\kappa_n^{-1} - \sigma_n(1 + \alpha)} < \gamma < 1.$$

Consequently,

$$\begin{aligned} d_{\Gamma^u}(\mathbf{G}(\phi), \mathbf{G}(\varphi)) &= \sup_{n \in \mathbb{Z}} \left\{ \sup_{x \in B_n^u(\delta_{n+1}) \setminus \{0\}} \frac{\|\psi_{n+1}(x) - \chi_{n+1}(x)\|}{\|x\|} \right\} \\ &\leq \sup_{n \in \mathbb{Z}} \left\{ \sup_{w \in B_n^u(\delta_n) \setminus \{0\}} \gamma \frac{\|\phi_n(w) - \varphi_n(w)\|}{\|w\|} \right\} \\ &= \gamma \sup_{n \in \mathbb{Z}} \left\{ \sup_{w \in B_n^u(\delta_n) \setminus \{0\}} \frac{\|\phi_n(w) - \varphi_n(w)\|}{\|w\|} \right\} \\ &= \gamma d_{\Gamma^u}(\phi, \varphi). \end{aligned}$$

This fact proves that $\mathbf{G}$ is a contraction. □

Since $\Gamma^u(\alpha, (\delta_n)_n)$ is a complete metric space, by the Banach fixed-point Theorem we have there exists a unique $\phi^* \in \Gamma^u(\alpha, (\delta_n)_n)$ such that

$$\mathbf{G}(\phi^*) = \phi^*.$$

In other words, for each $n \in \mathbb{Z}$, there exists a unique $\phi_n^* \in \Gamma_n^u(\alpha, \delta_n)$ such that

$$\tilde{f}_n(\phi_n^* w, w) = (\phi_{n+1}^* r_n(w), r_n(w)), \quad \text{for all } w \in B_n^u(\delta_n)$$

(see (5.2.8)). Consequently, if

$$W_n(\delta_n) = \{(\phi_n^* w, w) : w \in B_n^u(\delta_n)\}, \quad (5.2.9)$$

we have $W_{n+1}(\delta_{n+1}) \subseteq \tilde{f}_n(W_n(\delta_n))$, because $B_{n+1}^u(\delta_{n+1}) \subseteq r_n(B_n^u(\delta_n))$.

Notice that the size of α only changes the diameter of each $W_n(\delta_n)$. This is because the uniqueness of the fixed point of a contraction and, furthermore, if $\tilde{\alpha} \leq \alpha$, then

$$\Gamma^u(\tilde{\alpha}, (\tilde{\delta}_n)_n) \subseteq \Gamma^u(\alpha, (\delta_n)_n) \quad \text{and} \quad \Gamma^s(\tilde{\alpha}, (\tilde{\epsilon}_n)_n) \subseteq \Gamma^s(\alpha, (\epsilon_n)_n)$$

for appropriate $\tilde{\delta}_n$ and $\tilde{\epsilon}_n$. We will assume that $\alpha = (\lambda^{-1} - 1)/2$. Fix $\tau \in (\frac{1+\lambda}{2}, 1)$. From now on, we will suppose that, for each $n \in \mathbb{Z}$,

$$\sigma_n < \min \left\{ \tilde{\sigma}_n, \frac{2\lambda\tau\kappa_n^{-1} - 1 - \lambda}{1 + \lambda} \right\}.$$

Hence,

$$\tau_n := \frac{1 + \alpha}{\kappa_n^{-1} - \sigma_n(1 + \alpha)} = \frac{1 + \lambda}{2\lambda\kappa_n^{-1} - \sigma_n(1 + \lambda)} < \tau \quad \text{for each } n \in \mathbb{Z}. \quad (5.2.10)$$

The sets $W_n(\delta_n)$ are topological submanifolds of M_n , because they are graphs of α -Lipschitz maps. Next, we prove that $W_n(\delta_n)$ is differentiable (that proof is taken from [BP07], p. 201).

Lemma 5.2.9. *$W_n(\delta_n)$ is differentiable and $T_0 W_n(\delta_n) = E_{\mathbf{f}^n(p)}^u$.*

Proof. For $x, y \in B_n^u(\delta_n)$, with $x \neq y$, and $\phi \in \Gamma_n^u(\alpha, \delta_n)$, set

$$\Psi_{x,y}(\phi) = \frac{(\phi(x), x) - (\phi(y), y)}{\|(\phi(x), x) - (\phi(y), y)\|},$$

$$\omega_x(\phi) = \{w \in T_{(\phi(x), x)} B_n(\delta_n) : \Psi_{x, x_i}(\phi) \rightarrow w \text{ for some sequence } x_i \rightarrow x\}$$

$$\text{and} \quad \varpi_x(\phi) = \{tw : t \in \mathbb{R} \text{ and } w \in \omega_x(\phi)\}.$$

We have that ϕ is differentiable at x if, and only if, $\varpi_x(\phi)$ is a $\dim(E_{(\phi(x), x)}^u)$ -dimensional subspace. Hence, we will prove that $\varpi_x(\phi)$ is a $\dim(E_{(\phi(x), x)}^u)$ -dimensional subspace.

For $z = \exp_{\mathbf{f}^n(p)}^{-1}(q)$, set

$$\begin{aligned} \tilde{E}_z^s &= D(\exp_{\mathbf{f}^n(p)}^{-1})_q(E_q^s), & \tilde{E}_z^u &= D(\exp_{\mathbf{f}^n(p)}^{-1})_q(E_q^u), \\ \tilde{K}_{\alpha, \mathbf{f}, z}^s &= D(\exp_{\mathbf{f}^n(p)}^{-1})_q(K_{\alpha, \mathbf{f}, q}^s) & \text{and} & \quad \tilde{K}_{\alpha, \mathbf{f}, z}^u = D(\exp_{\mathbf{f}^n(p)}^{-1})_q(K_{\alpha, \mathbf{f}, q}^u). \end{aligned}$$

Since ϕ_n^* is α -Lipschitz, for each $x \in B_n^u(\delta_n)$ we have

$$\varpi_x(\phi_n^*) \subseteq \overline{K_{\alpha, \mathbf{f}, (\phi_n^*(x), x)}^u}.$$

Suppose that a sequence $(x_i)_{i \in \mathbb{N}} \subseteq B_n^u(\delta_n)$ converges to $x \in B_n^u(\delta_n)$ as $i \rightarrow \infty$. Let $w_i, w \in B_{n+1}^u(\delta_{n+1})$ be such that $(\phi_{n+1}^*(w_i), w_i) = f_n(\phi_n^*(x_i), x_i)$ and $(\phi_{n+1}^*(w), w) = f_n(\phi_n^*(x), x)$. Thus, $\Psi_{x, x_i}(\phi_n) \rightarrow z$ as $i \rightarrow \infty$, if, and only if

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{(\phi_{n+1}^*(w_i), w_i) - (\phi_{n+1}^*(w), w)}{\|(\phi_{n+1}^*(w_i), w_i) - (\phi_{n+1}^*(w), w)\|} &= \lim_{i \rightarrow \infty} \frac{f_n(\phi_n^*(x_i), x_i) - f_n(\phi_n^*(x), x)}{\|f_n(\phi_n^*(x_i), x_i) - f_n(\phi_n^*(x), x)\|} \\ &= \lim_{i \rightarrow \infty} \frac{\frac{f_n(\phi_n^*(x_i), x_i) - f_n(\phi_n^*(x), x)}{\|(\phi_n^*(x_i), x_i) - (\phi_n^*(x), x)\|}}{\frac{\|f_n(\phi_n^*(x_i), x_i) - f_n(\phi_n^*(x), x)\|}{\|(\phi_n^*(x_i), x_i) - (\phi_n^*(x), x)\|}} \\ &= \frac{D(f_n)_{(\phi_n^*(x), x)}(z)}{\|D(f_n)_{(\phi_n^*(x), x)}(z)\|}. \end{aligned}$$

This fact implies that, for all $n \in \mathbb{Z}, x \in B_n^u(\delta_n)$ and $w \in B_{n+1}^u(\delta_{n+1})$ with $(\phi_{n+1}^*(w), w) = f_n(\phi_n^*(x), x)$, we have

$$D(f_n)_{(\phi_n^*(x), x)} \varpi_x(\phi_n^*) = \varpi_w(\phi_{n+1}^*).$$

Since $\varpi_x(\phi_n^*) \subseteq \overline{K_{\alpha, \mathbf{f}, (\phi_n^*(x), x)}^u}$, by Lemma 3.4.2 we have $\varpi_x(\phi_n^*) \subseteq E_{(\phi_n^*(x), x)}^u$.

On the other hand, for any $v \in B_n^u(\delta_n)$, we can choose a sequence $(t_m)_{m \in \mathbb{Z}}$ converging to 0, such that $\Phi_{(\phi_n^*(x), x)}(\phi^*)$ converges as $m \rightarrow \infty$. This fact implies that $\varpi_x(\phi_n^*)$ projects onto $\mathbb{R}^k$. The above facts imply that $\varpi_x(\phi_n^*) = E_{(\phi_n^*(x), x)}^u$.

Therefore ϕ_n^* is differentiable at x . Taking $x = 0$ we have

$$T_0 W_n(\delta_n) = E_{(\phi_n^*(0), 0)}^u = E_{\mathbf{f}^n(p)}^u.$$

which proves the lemma. $\square$

The angles between the stable and unstable subspaces will be important to control the contraction or expansion of the submanifolds by $\mathbf{f}$. Notice that, if the angles θ_n (see (3.1.2)) decay when $n \rightarrow \pm\infty$, the vectors in $B_n^s(\delta_n)$ and in $B_n^u(\delta_n)$ are ever closer. From (3.3.5), we have

$$\Delta_n \|v\|_* \leq \|v\| \leq 2\|v\|_*, \quad \text{for } v \in TM_n, \quad (5.2.11)$$

where $\Delta_n = \left(\frac{1}{1 - \cos(\theta_n)} \left(\frac{\lambda + \zeta}{\zeta} c \right)^2 \right)^{-1}$. Hence, if $w \in B_n^u(\delta_n)$ for $n \in \mathbb{Z}$, by (5.2.4), (5.2.8) and (5.2.11) we have

$$\begin{aligned} \|\tilde{f}_n(\phi_n^* w, w)\| &\geq \Delta_{n+1} \|\tilde{f}_n(\phi_n^* w, w)\|_* \geq \Delta_{n+1} \|r_n(w)\|_* \geq \Delta_{n+1} \frac{\kappa_n^{-1} - \sigma_n(1 + \alpha)}{1 + \alpha} \|(\phi_n^* w, w)\|_* \\ &\geq \frac{\Delta_{n+1}}{2} \frac{\kappa_n^{-1} - \sigma_n(1 + \alpha)}{1 + \alpha} \|(\phi_n^* w, w)\|. \end{aligned}$$

Consequently, since $(\tilde{f}_n)^{-1}(W_{n+1}(\delta_{n+1})) \subseteq W_n(\delta_n)$, for any $z \in B_{n+1}^u(\delta_{n+1})$, there exists $w \in B_n^u(\delta_n)$ such that $\tilde{f}_n(\phi_n^* w, w) = (\phi_{n+1}^* z, z)$. Thus we have that

$$\|(\tilde{f}_n)^{-1}(\phi_{n+1}^* z, z)\| \leq \frac{2}{\Delta_{n+1}} \tau \|(\phi_{n+1}^* z, z)\|, \quad \text{for } z \in B_{n+1}^u(\delta_{n+1})$$

(see (5.2.10)). Analogously, there exists $y \in B_{n-1}^u(\delta_{n-1})$ such that $\tilde{f}_{n-1}(\phi_{n-1}^* y, y) = (\phi_n^* w, w)$. Therefore,

$$\begin{aligned} \|(\phi_{n+1}^* z, z)\| &\geq \Delta_{n+1} \tau^{-1} \|(\phi_n^* w, w)\|_* \geq \Delta_{n+1} \tau^{-1} \tau^{-1} \|(\phi_{n-1}^* y, y)\|_* \geq \frac{\Delta_{n+1}}{2} \tau^{-2} \|(\phi_{n-1}^* y, y)\| \\ &= \frac{\Delta_{n+1}}{2} \tau^{-2} \|(\tilde{f}_{n-1})^{-1}(\phi_n^* w, w)\| = \frac{\Delta_{n+1}}{2} \tau^{-2} \|(\tilde{f}_{n-1})^{-1}(\tilde{f}_n)^{-1}(\phi_{n+1}^* z, z)\|. \end{aligned}$$

Inductively we can prove for $k \geq 0$ that

$$\|(\tilde{f}_{n-k})^{-1} \circ \dots \circ (\tilde{f}_n)^{-1}(\phi_{n+1}^* z, z)\| \leq \frac{2}{\Delta_{n+1}} \tau^{k+1} \|(\phi_{n+1}^* z, z)\| \text{ for } z \in B_{n+1}^u(\delta_{n+1}). \quad (5.2.12)$$

Theorem 5.2.10. Fix $p \in M_0$. Suppose that the Anosov family $(\mathbf{M}, \langle \cdot, \cdot \rangle, \mathbf{f})$ admits a sequence of positive numbers $\delta = (\delta_n)_{n \in \mathbb{Z}}$ as in Proposition 5.2.5. Thus, there exists a two-sided sequence

$$\{\mathcal{W}^u(\mathbf{f}_0^n(p), \delta) : n \in \mathbb{Z}\},$$

where $\mathcal{W}^u(\mathbf{f}_0^n(p), \delta)$ is a differentiable submanifold of M_n with size δ_n , such that for $n \in \mathbb{Z}$:

- i. $\mathbf{f}_0^n(p) \in \mathcal{W}^u(\mathbf{f}_0^n(p), \delta)$ and $T_{\mathbf{f}_0^n(p)} \mathcal{W}^u(\mathbf{f}_0^n(p), \delta) = E_{\mathbf{f}_0^n(p)}^u$,
- ii. $\mathbf{f}_{n-1}^{-1}(\mathcal{W}^u(\mathbf{f}_0^n(p), \delta)) \subseteq \mathcal{W}^u(\mathbf{f}_0^{n-1}(p), \delta)$, and furthermore
- iii. if $q \in \mathcal{W}^u(p_{n+1}, \delta)$, where $p_n = \mathbf{f}_0^n(p)$, and $k \geq 0$ we have

$$\mathbf{d}(\mathbf{f}_{n+1}^{-(k+1)}(q), \mathbf{f}_{n+1}^{-(k+1)}(p_{n+1})) \leq \frac{2}{\Delta_{n+1}} \tau^{k+1} \mathbf{d}(q, p_{n+1}). \quad (5.2.13)$$

Proof. Let $W_n(\delta_n)$ be as in (5.2.9) and take

$$\mathcal{W}^u(\mathbf{f}_0^n(p), \delta) = \exp_{\mathbf{f}_0^n(p)}(W_n(\delta_n)) \quad \text{for each } n \in \mathbb{Z}.$$

The statements (i) and (ii) of the theorem are clear.

For (iii); if $q \in \mathcal{W}^u(p_n, \delta)$, for each $k \geq 1$ there exists a unique $v_{n-k+1} \in T_{\mathbf{f}_{n+1}^{-k}(p)} \mathbf{M}$ such that $\exp_{\mathbf{f}_{n+1}^{-k}(p)}(v_{n-k+1}) = \mathbf{f}_{n+1}^{-k}(q)$ and $\|v_{n-k+1}\| = \mathbf{d}(\mathbf{f}_{n+1}^{-k}(p), \mathbf{f}_{n+1}^{-k}(q))$. By (5.2.12) and the invariance of $\mathcal{W}^u(\mathbf{f}_0^n(p), \delta)$ by $\mathbf{f}$ we have (5.2.13). $\square$

Theorem 5.2.10 is a more generalized version of the Hadamard-Perron Theorem adapted to Anosov families for the unstable case, since the angles between the stable and unstable subspace could be arbitrarily small and, furthermore, the sequence (δ_n) satisfying the condition (5.2.2) is not necessarily bounded away from zero.

Analogously we can obtain a more generalized version of the Hadamard-Perron Theorem adapted to Anosov families for the stable case. Indeed, set

$$\tilde{\rho}_n = \min \left\{ \frac{(\mu_n^{-1} - \kappa_n)\alpha}{(1 + \alpha)^2}, \frac{(\gamma\mu_n^{-1} - \kappa_n)}{(1 + \alpha)(1 + \gamma)}, \frac{2\lambda\tau\mu_n^{-1} - 1 - \lambda}{1 + \lambda} \right\} \quad \text{for } n \in \mathbb{Z}. \quad (5.2.14)$$

Suppose that for each $n \geq 0$ there exists $\epsilon_n > 0$ satisfying

$$\epsilon_{n-1} \leq \frac{\mu_n^{-1} + \alpha\kappa_n}{1 + \alpha} \epsilon_n \quad \text{and} \quad \rho_n(\epsilon_{n+1}) < \tilde{\rho}_n \quad (5.2.15)$$

(see Definition 3.4.7). Thus, there exists a sequence of positive numbers $(\epsilon_n)_{n < 0}$ such that, considering $\epsilon = (\epsilon_n)_{n \in \mathbb{Z}}$, we have:

Theorem 5.2.11. *There exists a two-sided sequence of differentiable submanifold*

$$\{\mathcal{W}^s(\mathbf{f}_0^n(p), \epsilon) : n \in \mathbb{Z}\},$$

such that

- i. $\mathbf{f}_0^n(p) \in \mathcal{W}^s(\mathbf{f}_0^n(p), \epsilon)$ and $T_{\mathbf{f}_0^n(p)}\mathcal{W}^s(\mathbf{f}_0^n(p), \epsilon) = E_{\mathbf{f}_0^n(p)}^s$,
- ii. $f_n(\mathcal{W}^s(\mathbf{f}_0^n(p), \epsilon)) \subseteq \mathcal{W}^s(\mathbf{f}_0^{n+1}(p), \epsilon)$, and furthermore
- iii. if $q \in \mathcal{W}^s(p_n, \epsilon)$ and $k \geq 1$ we have

$$d(\mathbf{f}_n^k(q), \mathbf{f}_n^k(p_n)) \leq \frac{2}{\Delta_n} \tau^k d(q, p_n). \quad (5.2.16)$$

Definition 5.2.12. We will call $\{\mathcal{W}^s(\mathbf{f}_0^n(p), \epsilon) : n \in \mathbb{Z}\}$ as a *family of admissible (s, α, ϵ) -manifold at p* and $\{\mathcal{W}^u(\mathbf{f}_0^n(p), \delta) : n \in \mathbb{Z}\}$ as a *family of admissible (u, α, δ) -manifold at p* (see [BP07]).

In the next section we will see that with some conditions, the family of admissible manifolds coincide with the sets given in Definition 5.1.4.

The first inequality (5.2.2) means that the radius δ_n of the balls $B_n^u(\delta_n)$ must not decrease very fast when $n \rightarrow -\infty$. This condition is sufficient to have the invariance of the admissible manifolds by $\mathbf{f}$ obtained in Theorem 5.2.10n (see (5.2.4)). Remember we have considered exponential charts to work on the “ambient Euclidian” and each δ_n depends on both $\mathbf{f}$ and ϱ_n , the injectivity radius of the exponential map at $\mathbf{f}_0^n(p)$. This fact is of great importance to the construction of unstable (stable) manifolds, because the expansions (contractions) of each manifold could not be caused by the family but by the geometry of each component (see Example 3.2.2).

5.3 Local Stable and Unstable Manifolds for Anosov Families

In the previous section we obtained admissible manifolds for Anosov families whose expansion or contraction are controlled by the Δ_k ’s, τ_k ’s and ς_k ’s. In this section we will give certain conditions with which the stable and unstable sets (see Definition 5.1.4) coincide with the admissible manifolds (see Lemmas 5.3.2 and 5.3.3).

We had talked about the importance of maintaining the metrics established, because the notion of Anosov family depends on the Riemannian metrics on each M_n . Changing the metrics on each component we could get very different stable (unstable) sets (see Example 3.2.2). However, these sets don’t depend on the metrics if they are uniformly equivalent on $\mathbf{M}$, only that they can change the diameter:

Proposition 5.3.1. *Let $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ be uniformly equivalent Riemannian metrics on $\mathbf{M}$. Fix $p \in M_0$. Given a sequence of positive numbers small enough $\varepsilon = (\varepsilon_i)_{i \in \mathbb{Z}}$, there exist sequences of positive numbers $\varepsilon' = (\varepsilon'_i)_{i \in \mathbb{Z}}$ and $\tilde{\varepsilon} = (\tilde{\varepsilon}_i)_{i \in \mathbb{Z}}$ such that, for $r = u, s$,*

$$\mathcal{N}^r(p, \varepsilon, \langle \cdot, \cdot \rangle) \subseteq \mathcal{N}^r(p, \varepsilon', \langle \cdot, \cdot \rangle') \subseteq \mathcal{N}^r(p, \tilde{\varepsilon}, \langle \cdot, \cdot \rangle).$$

Proof. We will show only the stable case, since the unstable case is analogous. Consider $\mathbf{N} = \mathbf{M}$ with the Riemannian metric $\langle \cdot, \cdot \rangle'$, that is, $N_i = M_i$ with the Riemannian metric $\langle \cdot, \cdot \rangle'_i := \langle \cdot, \cdot \rangle|_{M_i}$ for each $i \in \mathbb{Z}$. Let $I_i : (M_i, \langle \cdot, \cdot \rangle) \rightarrow (M_i, \langle \cdot, \cdot \rangle')$ be the identity. For each $n \geq 0$ we can find a $\varepsilon_n > 0$ small enough such that

$$\text{diam}[B(\mathbf{f}_0^n(p), \varepsilon_n, \langle \cdot, \cdot \rangle)] < \varrho_n \quad \text{and} \quad \text{diam}[I_n(B(\mathbf{f}_0^n(p), \varepsilon_n, \langle \cdot, \cdot \rangle))] < \varrho'_n$$

(we use the notations $B(\mathbf{f}_i^n(p), \varepsilon_n, \langle \cdot, \cdot \rangle)$ for the ball in $(M_n, \langle \cdot, \cdot \rangle)$ and ϱ'_n for the injectivity radius of the exponential map at $\mathbf{f}_0^n(p)$ considering the metric $\langle \cdot, \cdot \rangle'$ on $\mathbf{M}$). For each $n \geq 0$, take

$$\varepsilon'_n = \frac{1}{2} \text{diam}[I_n(B(\mathbf{f}_0^n(p), \varepsilon_n, \langle \cdot, \cdot \rangle))].$$

Fix $q \in \mathcal{N}^s(p, \varepsilon, \langle \cdot, \cdot \rangle)$. Thus $I_n(\mathbf{f}_0^n(q)) \in B(\mathbf{f}_0^n(p), \varepsilon'_n, \langle \cdot, \cdot \rangle')$ for all $n \geq 0$. Let $v \in T_p M_0$ be such that $\exp_p(v) = q$. Since $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ are uniformly equivalent, there exist positive numbers k, K such that $k\|v\|' \leq \|v\| \leq K\|v\|'$, for all $v \in T_{\mathbf{f}^n(p)} M_n$, $n \geq 0$, where $\|\cdot\|$ and $\|\cdot\|'$ are the norms induced by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$, respectively. Thus,

$$\frac{1}{n} \log \mathbf{d}(\mathbf{f}^n(p), \mathbf{f}^n(q)) = \frac{1}{n} \log \|\tilde{f}_{i+n-1} \circ \cdots \circ \tilde{f}_i(v)\| \geq \frac{1}{n} \log k \|\tilde{f}_{i+n-1} \circ \cdots \circ \tilde{f}_i(v)\|'.$$

Therefore, $q \in \mathcal{N}^s(p, \varepsilon', \langle \cdot, \cdot \rangle')$. Thus, $\mathcal{N}^s(p, \varepsilon, \langle \cdot, \cdot \rangle) \subseteq \mathcal{N}^s(p, \varepsilon', \langle \cdot, \cdot \rangle')$. Analogously we can prove the existence of the sequence $\tilde{\varepsilon} = (\tilde{\varepsilon}_i)_{i \in \mathbb{Z}}$ such that

$$\mathcal{N}^s(p, \varepsilon', \langle \cdot, \cdot \rangle') \subseteq \mathcal{N}^s(p, \tilde{\varepsilon}, \langle \cdot, \cdot \rangle),$$

which proves the proposition. $\square$

From now on we assume that $\mathbf{f}$ admits sequences of positive numbers $\delta = (\delta_n)_{n \in \mathbb{Z}}$ and $\epsilon = (\epsilon_n)_{n \in \mathbb{Z}}$ which satisfy the conditions of Theorems 5.2.10 and 5.2.11. In the following lemma we show the inclusions $\mathcal{W}^u(p, \delta) \subseteq \mathcal{N}^u(p, \delta)$ and $\mathcal{W}^s(p, \epsilon) \subseteq \mathcal{N}^s(p, \epsilon)$. In Lemma 5.3.3 we will see a condition to obtain the reverse inclusion.

Lemma 5.3.2. *For each $p \in M_0$ we have*

$$\mathcal{W}^u(p, \delta) \subseteq \mathcal{N}^u(p, \delta) \quad \text{and} \quad \mathcal{W}^s(p, \epsilon) \subseteq \mathcal{N}^s(p, \epsilon).$$

Proof. We will prove $\mathcal{W}^u(p, \delta) \subseteq \mathcal{N}^u(p, \delta)$. Take $q \in \mathcal{W}^u(p, \delta)$. By Theorem 5.2.10, we have $\mathbf{f}_0^{-n}(q) \in B(\mathbf{f}_0^{-n}(p), \delta_{-n})$ and

$$\mathbf{d}(\mathbf{f}_0^{-n}(q), \mathbf{f}_0^{-n}(p)) \leq \frac{2}{\Delta_0} \tau^n \mathbf{d}(p, q) \quad \text{for each } n \geq 1.$$

Therefore,

$$\frac{1}{n} \log \mathbf{d}(\mathbf{f}_0^{-n}(q), \mathbf{f}_0^{-n}(p)) \leq \frac{1}{n} \log \frac{2}{\Delta_0} + \frac{\log \mathbf{d}(p, q)}{n} + \frac{1}{n} \log \tau^n.$$

Since $\tau < 1$, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{d}(\mathbf{f}_0^{-n}(q), \mathbf{f}_0^{-n}(p)) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \tau^n = \log \tau < 0.$$

Hence, $q \in \mathcal{N}^s(p, \delta)$. Consequently, $\mathcal{W}^s(p, \delta) \subseteq \mathcal{N}^s(p, \delta)$. $\square$

Lemma 5.3.3. *Set*

$$\begin{aligned} \Omega &= \limsup_{n \rightarrow -\infty} \theta_n & \tilde{\Omega} &= \liminf_{n \rightarrow \infty} \left(\frac{1}{n} \log(1 - \cos \theta_{-n}) \right) \\ \Theta &= \limsup_{n \rightarrow \infty} \theta_n & \tilde{\Theta} &= \liminf_{n \rightarrow \infty} \left(\frac{1}{n} \log(1 - \cos \theta_n) \right). \end{aligned}$$

Thus

- i. Assume that we can choose the δ_n 's such that $\delta_n \leq \epsilon_n$ for each $n \leq 0$. If $\Omega > 0$ and $\tilde{\Omega} \geq \log \tau$, then there exists a sequence of positive numbers $\delta' = (\delta'_n)_{n \in \mathbb{Z}}$ such that $\mathcal{N}^u(p, \delta') \subseteq \mathcal{W}^u(p, \delta')$.
- ii. Assume that we can choose the ϵ_n 's such that $\epsilon_n \leq \delta_n$ for each $n \geq 0$. If $\Theta > 0$ and $\tilde{\Theta} \geq \log \tau$, then there exists a sequence of positive numbers $\epsilon' = (\epsilon'_n)_{n \in \mathbb{Z}}$ such that $\mathcal{N}^s(p, \epsilon') \subseteq \mathcal{W}^s(p, \epsilon')$.

Proof. We will prove (i). Fix $\nu \in (0, \Omega)$. Let $(n_i)_{i \in \mathbb{N}}$ be a sequence of natural numbers, with $0 = n_0 < n_1 < \dots < n_m < \dots$, and $\theta_{n_i} \geq \nu$ for each $i \geq 0$. Since $\delta_n \leq \epsilon_n$ and $\Omega > 0$, we can choose $\delta'_n \leq \delta_n/3$ small enough such that $\delta' = (\delta'_n)_{n \in \mathbb{Z}}$ satisfies (5.2.2) and

$$B(\mathbf{f}_0^{-n_i}(p), \delta'_{-n_i}) \subseteq \mathcal{W}^s(\mathbf{f}_0^{-n_i}(p), \epsilon) \times \mathcal{W}^u(\mathbf{f}_0^{-n_i}(p), \delta), \quad \text{for each } i \geq 0.$$

By Theorem 5.2.10, we have a family

$$\{\mathcal{W}^u(\mathbf{f}_0^n(p), \delta') : n \in \mathbb{Z}\}$$

of admissible (u, α, δ') -manifold. Next, we prove that $\mathcal{N}^u(p, \delta') \subseteq \mathcal{W}^u(p, \delta')$. Indeed, suppose there exists $q \in \mathcal{N}^u(p, \delta') \setminus \mathcal{W}^u(p, \delta')$. Since $\mathbf{f}_0^{-n_i}(q) \in B(\mathbf{f}_0^{-n_i}(p), \delta'_{-n_i})$, we have

$$(\tilde{f}_{-n_i})^{-1} \circ \dots \circ (\tilde{f}_{-1})^{-1}(\exp_p^{-1}(q)) = (x_{-n_i}, y_{-n_i}) \in \mathcal{W}^s(\mathbf{f}_0^{-n_i}(p), \epsilon) \times \mathcal{W}^u(\mathbf{f}_0^{-n_i}(p), \delta),$$

for all $i \geq 0$, where $x_{-n_i} \in \mathcal{W}^s(\mathbf{f}_0^{-n_i}(p), \epsilon) \setminus \{0\}$ and $y_{-n_i} \in \mathcal{W}^u(\mathbf{f}_0^{-n_i}(p), \delta)$. We can obtain from (5.2.13) and (5.2.16) that

$$\|(x_{-n_i}, y_{-n_i})\| \geq \|x_{-n_i}\| - \|y_{-n_i}\| \geq \frac{\Delta_{-n_i}}{2} \tau^{-n_i} \|x_0\| - \frac{2}{\Delta_0} \tau^{n_i} \|y_0\|.$$

We have $\frac{2}{\Delta_0} \tau^{n_i} \|y_0\| \rightarrow 0$ as $i \rightarrow +\infty$. Consequently,

$$\begin{aligned} \limsup_{i \rightarrow \infty} \frac{1}{n_i} \log \|(x_{-n_i}, y_{-n_i})\| &\geq \limsup_{i \rightarrow \infty} \frac{1}{n_i} \log \frac{\Delta_{-n_i}}{2} \tau^{-n_i} \|x_0\| = \limsup_{i \rightarrow \infty} \left(\frac{1}{n_i} \log \Delta_{-n_i} - \log \tau \right) \\ &= \limsup_{i \rightarrow \infty} \left(\frac{1}{n_i} \log \Delta_{-n_i} \right) - \log \tau = \limsup_{i \rightarrow \infty} \left(\frac{1}{n_i} \log(1 - \cos \theta_{-n_i}) \right) - \log \tau \\ &\geq \liminf_{n \rightarrow \infty} \left(\frac{1}{n} \log(1 - \cos \theta_{-n}) \right) - \log \tau \geq 0. \end{aligned}$$

This fact contradicts that $q \in \mathcal{N}^s(p, \delta')$. $\square$

From now on we will assume that for each $p \in M_0$ we can choose the sequences $(\delta_n)_{n \in \mathbb{Z}}$ and $(\epsilon_n)_{n \in \mathbb{Z}}$ as in Theorems 5.2.10 and 5.2.11, such that

$$\delta_n = \epsilon_n, \text{ for each } n, \quad \log \tilde{\lambda} \leq \min\{\tilde{\Omega}, \tilde{\Theta}\} \quad \text{and} \quad 0 < \min\{\Omega, \Theta\}. \quad (5.3.1)$$

Therefore we have by Lemmas 5.3.2 and 5.3.3 that there exist two sequences of positive numbers $\delta' = (\delta'_n)_{n \in \mathbb{Z}}$ and $\epsilon' = (\epsilon'_n)_{n \in \mathbb{Z}}$ such that

$$\{\mathcal{W}^u(f_0^n(p), \delta') : n \in \mathbb{Z}\} \quad \text{and} \quad \{\mathcal{W}^s(f_0^n(p), \epsilon') : n \in \mathbb{Z}\}$$

are two-sided sequences of admissible manifolds at p , and furthermore

$$\mathcal{N}^u(p, \delta') = \mathcal{W}^u(p, \delta') \quad \text{and} \quad \mathcal{N}^s(p, \epsilon') = \mathcal{W}^s(p, \epsilon').$$

Next we will prove that $\mathcal{N}^u(p, \eta)$ and $\mathcal{N}^s(p, \eta)$ depend continuously on p .

Lemma 5.3.4. *Let $(p_m)_{m \in \mathbb{N}}$ be a sequence in M_0 converging to $p \in M_0$ as $m \rightarrow \infty$. If $q_m \in \mathcal{N}^r(p_m, \eta)$ converges to $q \in B(p, \eta_0)$ as $m \rightarrow \infty$, then $q \in \mathcal{N}^r(p, \eta)$, for $r = s, u$.*

Proof. We will prove only the stable case. Set

$$\omega = \sup_{p_m, m \geq 0} \left\{ \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{d}(f_0^n(q'), f_0^n(p_m)) : q' \in \mathcal{N}^s(p_m, \eta) \right\}.$$

By compactness of M_0 , we have $\omega \leq 0$. Let $\beta \in (0, \exp(\omega))$. For each $n \in \mathbb{N}$, take $m_n \in \mathbb{N}$, with $m_1 < \dots < m_n < \dots$, such that

$$\mathbf{d}(f_0^n(p_{m_n}), f_0^n(p)) < \beta^n \quad \text{and} \quad \mathbf{d}(f_0^n(q_{m_n}), f_0^n(q)) < \beta^n.$$

For every n we have

$$\frac{1}{n} \log(\mathbf{d}(f_0^n(q), f_0^n(p))) \leq \frac{1}{n} \log[\mathbf{d}(f_0^n(q), f_0^n(q_{m_n})) + \mathbf{d}(f_0^n(q_{m_n}), f_0^n(p_{m_n})) + \mathbf{d}(f_0^n(p_{m_n}), f_0^n(p))].$$

Since

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log(a_n + b_n) = \max\left\{ \limsup_{n \rightarrow \infty} \frac{1}{n} \log(a_n), \limsup_{n \rightarrow \infty} \frac{1}{n} \log(b_n) \right\}$$

for any sequence of positive numbers a_n and b_n , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log(\mathbf{d}(f_0^n(q), f_0^n(p))) \leq \max\{\log(\beta), \omega\} = \omega.$$

Consequently, $q \in \mathcal{N}^s(p, \eta)$. $\square$

Finally, by Theorems 5.2.10 and 5.2.11 and Lemmas 5.3.2-5.3.4, we obtain the following local unstable and stable manifold theorems for Anosov families:

Theorem 5.3.5 (Local unstable manifold for Anosov families). *If $(M, \langle \cdot, \cdot \rangle, f)$ admits a sequence of positive numbers $(\delta_n)_{n \in \mathbb{Z}}$ satisfying (5.3.1) for each $p \in M_0$, then there exists a sequence of positive numbers $\eta = (\eta_n)_{n \in \mathbb{Z}}$, such that $\mathcal{N}^u(f_0^n(p), \eta)$ is a differentiable submanifold of M_n with:*

- i. $T_{f^n(p)}\mathcal{N}^u(f_0^n(p), \eta) = E_{f^n(p)}^u$;
- ii. $f_{n-1}^{-1}(\mathcal{N}^u(f_0^n(p), \eta)) \subseteq \mathcal{N}^u(f_0^{n-1}(p), \eta)$;
- iii. if $q \in \mathcal{N}^u(p_{n+1}, \delta)$, where $p_n = f_0^n(p)$, and $k \geq 0$ we have

$$d(f_{n+1}^{-(k+1)}(q), f_{n+1}^{-(k+1)}(p_{n+1})) \leq \frac{2}{\Delta_{n+1}} \tau^{k+1} d(q, p_{n+1}).$$

- iv. $\mathcal{N}^u(p, \eta)$ depends continuously on p .

Theorem 5.3.6 (Local stable manifold for Anosov families). *If $(M, \langle \cdot, \cdot \rangle, f)$ admits a sequence of positive numbers $(\epsilon_n)_{n \in \mathbb{Z}}$ satisfying (5.3.1) for each $p \in M_0$, then there exists a sequence of positive numbers $\eta = (\eta_n)_{n \in \mathbb{Z}}$, such that $\mathcal{N}^s(f_0^n(p), \eta)$ is a differentiable submanifold of M_n with:*

- i. $T_{f^n(p)}\mathcal{N}^s(f_0^n(p), \eta) = E_{f^n(p)}^s$;
- ii. $f_n(\mathcal{N}^s(f_0^n(p), \eta)) \subseteq \mathcal{N}^s(f_0^{n+1}(p), \eta)$;
- iii. if $q \in \mathcal{N}^s(p_n, \epsilon)$, where $p_n = f_0^n(p)$, and $k \geq 1$ we have

$$d(f_n^k(q), f_n^k(p_n)) \leq \frac{2}{\Delta_n} \varsigma^k d(q, p_n),$$

- iv. $\mathcal{N}^s(p, \eta)$ depends continuously on p .

5.4 Stable and unstable manifolds for Anosov Families consisting of matrices

In this section, we will see that any Anosov family $F = (F_n)_{n \in \mathbb{Z}}$, where $F_n : \mathbb{T}^m \rightarrow \mathbb{T}^m$ is a diffeomorphism induced by a matrix $A_n : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $\mathbb{T}^m$ is the m -torus endowed with the Riemannian metric inherited from $\mathbb{R}^m$, admits stable and unstable admissible manifolds. This is a particular case of the theorems obtained in the previous sections. As we saw in Remark 4.4.3, $a_p(z) = b_p(z) = c_p(z) = d_p(z) = 0$ for each $p \in \mathbb{T}^m$ and $z \in \mathbb{R}^m$ (see (3.4.1)). Therefore, $\sigma_p(\delta) = \rho_p(\delta) = 0$ for each $p \in \mathbb{T}^m$ and $\delta > 0$ (see Definition 3.4.7).

Inductively we can choose a two-sided sequence $\delta = (\delta_n)_{n \in \mathbb{Z}}$ of positive numbers such that $F_n(B^s(\delta_n)) \subseteq B_n^s(\delta_{n+1})$ for each $n \in \mathbb{Z}$. Take

$$\mathcal{W}^s(F_0^n(p), \delta) = \{F_0^n(p) + tv_s : v_s \in E_{F_0^n(p)}^s \text{ and } t \in (-\delta_n, \delta_n)\} \text{ for } p \in M_0. \quad (5.4.1)$$

The two-sided sequence $\{\mathcal{W}^s(F_0^n(p), \delta) : n \in \mathbb{Z}\}$ satisfies the properties i., ii. and iii. from Theorem 5.2.10 (in this case, we can take $\tau = \lambda$ and take $2c$ instead of $\frac{2}{\Delta_n}$). Consequently:

Theorem 5.4.1. *If $F = (F_n)_{n \in \mathbb{Z}}$ is an Anosov family where $F_n : \mathbb{T}^m \rightarrow \mathbb{T}^m$ is a diffeomorphism induced by a matrix $A_n : \mathbb{R}^m \rightarrow \mathbb{R}^m$, then the local stable manifolds for F consist of vectors obtained as in (5.4.1).*

Analogously we can obtain:

Theorem 5.4.2. *If $\mathbf{F} = (F_n)_{n \in \mathbb{Z}}$ is an Anosov family where $F_n : \mathbb{T}^m \rightarrow \mathbb{T}^m$ is a diffeomorphism induced by a matrix $A_n : \mathbb{R}^m \rightarrow \mathbb{R}^m$, then for any $p \in M_0$ we have*

$$\mathcal{W}^u(\mathbf{F}_0^n(p), \delta) = \{\mathbf{F}_0^n(p) + tv_s : v_s \in E_{\mathbf{F}_i^n(p)}^u \text{ and } t \in (-\delta_n, \delta_n)\},$$

are the unstable manifolds of $\mathbf{F}$ at p , for some two-sided sequence of positive numbers $(\delta_n)_{n \in \mathbb{Z}}$.

Chapter 6

Structural Stability for Anosov Families

As noted above, when $\mathbf{f}$ is the constant family associated to an Anosov diffeomorphism, Theorem 4.3.5 is valid for a uniform basic strong neighborhood of $\mathbf{f}$ (see Definition 4.4.1). It is also possible to find a uniform neighborhood of $\mathbf{f}$ if each f_i is a small perturbations of an Anosov diffeomorphism. In general it is not possible to find a uniform neighborhood of an Anosov family such that each family in that neighborhood is Anosov. For example, if the injectivity radius of the exponential application at each point in M_i is arbitrarily small for $|i|$ large, or if the angle between the stable and unstable subspace of the splitting of the tangent bundle decays, or if we can not get the inequality (4.1.3) with a uniform β_i , etc., it is necessary to take each ξ_i ever smaller.

Set

$$\mathcal{A}_b^2(\mathbf{M}) = \{\mathbf{f} \in \mathcal{F}^2(\mathbf{M}) : \mathbf{f} \text{ is Anosov, s.p.a. and } S_{\mathbf{f}} < \infty\}, \quad (6.0.1)$$

where

$$S_{\mathbf{f}} := \sup_{i \in \mathbb{Z}} \|f_i\|_{C^2} = \sup_{i \in \mathbb{Z}} \{\max \{\|Df_i\|, \|Df_i^{-1}\|, \|D^2f_i\|, \|D^2f_i^{-1}\|\}\}.$$

The main goal of this chapter is to prove that $\mathcal{A}_b^2(\mathbf{M})$ is *uniformly structurally stable* in $\mathcal{F}^2(\mathbf{M})$, that is, for each $\mathbf{f} \in \mathcal{A}_b^2(\mathbf{M})$, there exists $\varepsilon > 0$ such that any $\mathbf{g} \in B^2(\mathbf{f}, \varepsilon)$ is Anosov and uniformly conjugate to $\mathbf{f}$ (see Theorems 6.1.7 and 6.3.9). This fact generalizes Theorem 1.1 in [Liu98], which proves the structural stability of random small perturbations of hyperbolic diffeomorphisms. Furthermore, in Section 6.2 we will prove another version of Theorems 5.3.5 and 5.3.6 for elements in $\mathcal{A}_b^2(\mathbf{M})$. We will prove that every element in $\mathcal{A}_b^2(\mathbf{M})$ admits stable and unstable manifolds, and, furthermore, the size of these stable and unstable manifolds is the same along the orbits (see Theorems 6.2.2 and 6.2.3).

For each $n \in \mathbb{Z}$, let $\varrho_n > 0$ be an injectivity radius of each $p \in M_n$. Throughout this chapter, we will suppose that

$$\varrho := \inf_{n \in \mathbb{Z}} \varrho_n$$

is positive.

6.1 Openness of $\mathcal{A}_b^2(\mathbf{M})$

In Chapter 4 we showed that $\mathcal{A}^1(\mathbf{M})$ is open in $\mathcal{F}^1(\mathbf{M})$ endowed with the strong topology, that is, for each $\mathbf{g} \in \mathcal{A}^1(\mathbf{M})$ there exists a strong basic neighborhood $B^1(\mathbf{g}, (\xi_i)_{i \in \mathbb{Z}})$ of $\mathbf{g}$ such

that $B^1(\mathbf{g}, (\xi_i)_{i \in \mathbb{Z}}) \subseteq \mathcal{A}^1(\mathbf{M})$. In that case, it is not always possible to take the sequence $(\xi_i)_{i \in \mathbb{Z}}$ bounded away from zero, that is, ξ_i could decay as $i \rightarrow \pm\infty$. The goal of this section is to show that for the case of Anosov families in $\mathcal{A}_b^2(\mathbf{M})$, this basic neighborhood can be taken uniform (see Definition 4.4.1), as in the case where the family consists of matrices acting on the torus, as we saw in Section 4.4. That is, $\mathcal{A}_b^2(\mathbf{M})$ is open in $\mathcal{F}^2(\mathbf{M})$ endowed with the uniform topology. This fact is a generalization of the Young's result in [You86] (see Remark 3.2.7).

From now on, $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ will be a fixed element in $\mathcal{A}_b^2(\mathbf{M})$.

Remark 6.1.1. The Riemannian norm $\|\cdot\|_*$, given in (3.3.1), is uniformly equivalent to $\|\cdot\|$ on $\mathbf{M}$, since $\mathbf{f}$ s.p.a. By Proposition 3.3.4, $\mathbf{f}$ is strictly Anosov with the norm $\|\cdot\|_*$. Clearly, this norm is uniformly equivalent to the norm on $T\mathbf{M}$ given by

$$\|(v_s, v_u)\|_* = \max\{\|v_s\|_*, \|v_u\|_*\}, \text{ for } (v_s, v_u) \in E_p^s \oplus E_p^u, p \in \mathbf{M}. \quad (6.1.1)$$

Consequently, there exists $C \geq 1$ such that

$$(1/C)\|v\|_* = \|v\| \leq C\|v\|_*, \text{ for every } v \in E_p^s \oplus E_p^u, p \in \mathbf{M}. \quad (6.1.2)$$

From now on we will consider the metric given in (6.1.1). Take $r \in (0, \varrho/20S_{\mathbf{f}})$. Fix $\mathbf{g} = (g_i)_{i \in \mathbb{Z}} \in B^2(\mathbf{g}, r)$. If $p, q \in M_i$ and $\mathbf{d}(p, q) \leq r$, then $\mathbf{d}(f_i(p), g_i(q)) < \varrho/2$ and $\mathbf{d}(f_i^{-1}(p), g_i^{-1}(q)) < \varrho/2$. Therefore, $g_i(\overline{B(p, r)}) \subseteq B(f_i(p), \varrho/2)$ and $g_i^{-1}(\overline{B(f_i(p), r)}) \subseteq B(p, \varrho/2)$. Consequently,

$$\begin{aligned} \tilde{g}_p &= \exp_{\mathbf{f}(p)}^{-1} \circ \mathbf{g} \circ \exp_p : B(0_p, r) \rightarrow B(0_{\mathbf{f}(p)}, \varrho/2) \\ \text{and } \tilde{g}_p^{-1} &= \exp_p^{-1} \circ \mathbf{g}^{-1} \circ \exp_{\mathbf{f}(p)} : B(0_{\mathbf{f}(p)}, r) \rightarrow B(0_p, \varrho/2), \end{aligned}$$

are well-defined for each $p \in \mathbf{M}$.

Definition 6.1.2. For $\tilde{r} \leq r$, consider

$$\sigma_p(\tilde{r}, \mathbf{g}) = \sup_{z \in B(0_p, \tilde{r})} \|D(D(\tilde{f}_p)_0 - \tilde{g}_p)_z\|_* \quad \text{and} \quad \rho_p(\tilde{r}, \mathbf{g}) = \sup_{z \in B(0_{\mathbf{f}(p)}, \tilde{r})} \|D(D(\tilde{f}_p^{-1})_0 - \tilde{g}_p^{-1})_z\|_*.$$

Proposition 6.1.3. Fix $\tau > 0$. There exist $\tilde{r} > 0$, $\tilde{\delta} > 0$ and $X_i = \{p_{1,i}, \dots, p_{m_i,i}\} \subseteq M_i$ for each $i \in \mathbb{Z}$, such that $M_i = \cup_{j=1}^{m_i} B(p_{j,i}, \tilde{r})$ for each $i \in \mathbb{Z}$ and, furthermore, for every $\mathbf{g} \in B^2(\mathbf{f}, \tilde{\delta})$, we have that

$$\vartheta(\tilde{r}, \mathbf{g}) = \sup_{p \in X_i, i \in \mathbb{Z}} \{\sigma_p(\tilde{r}, \mathbf{g}), \rho_p(\tilde{r}, \mathbf{g})\} < \tau.$$

Proof. Fix $p \in M_i$. Let $z \in B(0_p, r)$ and $(v, w) \in E^s \oplus E^u$. There exists $K > 0$ (which does not depend on p), such that

$$\|D(\tilde{f}_p)_0(v, w) - D(\tilde{f}_p)_z(v, w)\|_* \leq K[1 + \|Df_i\|_*]\|D^2f_i\|_*\|z\|_*(v, w)\|_*$$

(see [LQ06], p. 50). Thus,

$$\begin{aligned} \|D(D(\tilde{f}_p)_0 - \tilde{g}_p)_z(v, w)\|_* &= \|D(\tilde{f}_p)_0(v, w) - D(\tilde{g}_p)_z(v, w)\|_* \\ &\leq \|D(\tilde{f}_p)_0(v, w) - D(\tilde{f}_p)_z(v, w)\|_* + \|D(\tilde{f}_p - \tilde{g}_p)_z(v, w)\|_* \\ &\leq K[1 + S_{\mathbf{f}}]S_{\mathbf{f}}\|z\|_*(v, w)\|_* + d^2(f_i, g)\|(v, w)\|_*. \end{aligned}$$

Therefore, if $\tilde{\delta}_1 < \tau/2$ and $\tilde{r}_1 < \tau/2(K(1 + S_{\mathbf{f}})S_{\mathbf{f}})$, then for each $\mathbf{g} \in B^1(\mathbf{f}, \tilde{\delta}_1)$ and $z \in B(0_p, \tilde{r}_1)$, we have $\|D(D(\tilde{f}_p)_0 - \tilde{g}_p)_z\|_* < \tau$.

Analogously we can prove there exist $\tilde{\delta}_2 > 0$ and $\tilde{r}_2 > 0$ such that, if $\mathbf{g} \in B^2(\mathbf{f}, \tilde{\delta}_2)$ and $z \in B(0_{f_i(p)}, \tilde{r}_2)$, then $\|D(D(\tilde{f}_p^{-1})_0 - \tilde{g}_p^{-1})_z\|_* \leq \tau$.

Take $\tilde{r} = \min\{\tilde{r}_1, \tilde{r}_2\}$ and $\tilde{\delta} = \min\{\tilde{\delta}_1, \tilde{\delta}_2\}$. Notice that neither $\tilde{r}$ nor $\tilde{\delta}$ depend on p . Since M_i is compact, we can choose a finite subset $X_i = \{p_{1,i}, \dots, p_{m_i,i}\} \subseteq M_i$ such that $M_i = \cup_{j=1}^{m_i} B(p_{j,i}, \tilde{r})$ for each $i \in \mathbb{Z}$, which proves the proposition. $\square$

We will fix $\alpha \in (0, \frac{1-\tilde{\lambda}}{1+\tilde{\lambda}})$ and furthermore, we will suppose that $\tilde{\delta} \in (0, \varrho/20S_{\mathbf{f}})$ and $\tilde{r}$ are small enough such that

$$\vartheta(\tilde{r}, \mathbf{g}) < \sigma_A := \min \left\{ \frac{(\tilde{\lambda}^{-1} - \tilde{\lambda})\alpha}{2(1+\alpha)^2}, \frac{\tilde{\lambda}^{-1}(1-\alpha) - (1+\alpha)\alpha}{2(1+\alpha)} \right\}. \quad (6.1.3)$$

For each $i \in \mathbb{Z}$, let $X_i = \{p_{1,i}, \dots, p_{m_i,i}\} \subseteq M_i$ be such that $M_i = \cup_{j=1}^{m_i} B(p_{j,i}, \tilde{r})$. Note that, for each $i \in \mathbb{Z}$,

$$\vartheta := \max_{q \in X_i} \vartheta_q(\tilde{r}) \leq \vartheta(\tilde{r}, \mathbf{f}) < \sigma_A \quad (\text{see Lemma 4.1.3}).$$

The next two lemmas can be showed analogously to the Lemmas 4.1.4 and 4.1.5, respectively.

Lemma 6.1.4. *If $\delta \leq \frac{1}{2} \min\{\tilde{r}, \tilde{\delta}, \vartheta\}$, then, for $\mathbf{g} \in B^2(\mathbf{f}, \delta)$ and $p \in \mathbf{M}$ we have:*

- i. $D(\hat{g}_p)_z(\overline{K_\alpha^u}) \subseteq K_\alpha^u$ for all $z \in B^k(0, \tilde{r}) \times B^{m-k}(0, \tilde{r})$, and
- ii. $D(\hat{g}_p)_z^{-1}(\overline{K_\alpha^s}) \subseteq K_\alpha^s$ for all $z \in B^k(0, \tilde{r}) \times B^{m-k}(0, \tilde{r})$.

Lemma 6.1.5. *If $\delta \leq \frac{1}{2} \min\{\tilde{r}, \tilde{\delta}, \vartheta\}$, there exists $\eta \in (0, 1)$ such that, if $\mathbf{g} \in B^1(\mathbf{f}, \delta)$ then, for $p \in \mathbf{M}$ and $z \in B^k(0, \tilde{r}) \times B^{m-k}(0, \tilde{r})$, we have*

- i. $\|D(\hat{g}_p)_z(v)\| \geq \eta^{-1}\|v\|$ if $v \in \overline{K_\alpha^u}$;
- ii. $\|D(\hat{g}_p^{-1})_z(v)\| \geq \eta^{-1}\|v\|$ if $v \in \overline{K_\alpha^s}$.

Now, take $\xi = \frac{1}{2C} \min\{\tilde{r}, \tilde{\delta}, \vartheta\}$ (see (6.1.2)). Using the Lemmas 6.1.4 and 6.1.5 we can prove that:

Lemma 6.1.6. *Let $\mathbf{g} = (g_i)_{i \in \mathbb{Z}} \in B^2(\mathbf{f}, \xi)$. For each $p \in \mathbf{M}$, take*

$$F_p^s = \bigcap_{n=0}^{\infty} D(\mathbf{g}^{-n})_{\mathbf{g}^n(p)}(\overline{K_{\alpha, \mathbf{f}, \mathbf{g}^n(p)}^s}) \quad \text{and} \quad F_p^u = \bigcap_{n=0}^{\infty} D(\mathbf{g}^n)_{\mathbf{g}^{-n}(p)}(\overline{K_{\alpha, \mathbf{f}, \mathbf{g}^{-n}(p)}^u}).$$

The families F_p^s and F_p^u are $D\mathbf{g}$ -invariant subspaces with which $\mathbf{g}$ satisfies the conditions in Definition 3.1.2. Furthermore, $\mathbf{g}$ satisfies the property of angles.

Consequently, we have:

Theorem 6.1.7. *For each $\mathbf{f} \in \mathcal{A}_b^2(\mathbf{M})$, there exists $\xi > 0$ such that $B^2(\mathbf{f}, \xi) \subseteq \mathcal{A}_b^2(\mathbf{M})$. That is, $\mathcal{A}_b^2(\mathbf{M})$ is open in $\mathcal{F}^2(\mathbf{M})$ with the uniform topology.*

6.2 Local Stable and Unstable Manifolds for Elements in $\mathcal{A}_b^2(\mathbf{M})$

In Theorems 5.3.5 and 5.3.6 we gave conditions for obtain stable and unstable manifolds at each point in each component M_i . In that case, the diameter of each manifold ($2\epsilon_i$ and $2\delta_i$) could decay when $i \rightarrow \pm\infty$. In Theorems 6.2.2 and 6.2.3 we will see that each $\mathbf{f} \in \mathcal{A}_b^2(\mathbf{M})$ admits stable and unstable manifold with the same size at each point.

Fix $\gamma \in (\tilde{\lambda}^2, 1)$ and consider $\tilde{\sigma}_n$ and $\tilde{\rho}_n$ as in (5.2.1) and (5.2.14), respectively.

Remark 6.2.1. It is clear that

$$\tilde{\sigma} := \min \left\{ \frac{(\tilde{\lambda}^{-1} - \tilde{\lambda})\alpha}{(1 + \alpha)^2}, \frac{\gamma\tilde{\lambda} - \tilde{\lambda}}{(1 + \alpha)(1 + \gamma)}, \frac{1 - \tilde{\lambda}}{1 + \tilde{\lambda}} \right\} \leq \min\{\tilde{\sigma}_n, \tilde{\rho}_n\}.$$

Take $\tilde{\delta}$ small enough such that $\max\{\sigma_p(\tilde{\delta}, \mathbf{f}), \rho_p(\tilde{\delta}, \mathbf{f})\} \leq \tilde{\sigma}$ (see Definition 6.1.2). Thus, $\mathbf{f}$ satisfies the assumption of Proposition 5.2.5, considering $\delta_n = \tilde{\delta}$ for every $n \in \mathbb{Z}$.

Therefore, there exist $\varepsilon > 0$ and $\tau \in (0, 1)$ such that:

Theorem 6.2.2. *For each $p \in \mathbf{M}$, $\mathcal{N}^u(p, (\varepsilon)_{i \in \mathbb{Z}})$ is a differentiable submanifold of $\mathbf{M}$ and there exists $K^u > 0$ such that:*

- (i) $T_p \mathcal{N}^u(p, (\varepsilon)_{i \in \mathbb{Z}}) = E_p^u$,
- (ii) $\mathbf{f}^{-1}(\mathcal{N}^u(p, (\varepsilon)_{i \in \mathbb{Z}})) \subseteq \mathcal{N}^u(\mathbf{f}^{-1}(p), (\varepsilon)_{i \in \mathbb{Z}})$,
- (iii) if $q \in \mathcal{N}^u(p, (\varepsilon)_{i \in \mathbb{Z}})$ and $n \geq 1$ we have

$$d(\mathbf{f}^{-n}(q), \mathbf{f}^{-n}(p)) \leq K^u \tau^n d(q, p).$$

- (iv) Let $(p_m)_{m \in \mathbb{N}}$ be a sequence in M_i converging to $p \in M_i$ when $m \rightarrow \infty$. If $q_m \in \mathcal{N}^u(p_m, (\varepsilon)_{i \in \mathbb{Z}})$ converges to $q \in B(p, \varepsilon)$ as $m \rightarrow \infty$, then $q \in \mathcal{N}^u(p, (\varepsilon)_{i \in \mathbb{Z}})$.

Proof. Theorems 5.2.10 and 5.3.5. □

Analogously, we have:

Theorem 6.2.3. *For each $p \in \mathbf{M}$, $\mathcal{N}^s(p, (\varepsilon)_{i \in \mathbb{Z}})$ is a differentiable submanifold of $\mathbf{M}$ and there exists $K^s > 0$ such that:*

- (i) $T_p \mathcal{N}^s(p, (\varepsilon)_{i \in \mathbb{Z}}) = E_p^s$,
- (ii) $\mathbf{f}(\mathcal{N}^s(p, (\varepsilon)_{i \in \mathbb{Z}})) \subseteq \mathcal{N}^s(\mathbf{f}(p), (\varepsilon)_{i \in \mathbb{Z}})$,
- (iii) if $q \in \mathcal{N}^s(p, (\varepsilon)_{i \in \mathbb{Z}})$ and $n \geq 1$ we have

$$d(\mathbf{f}^n(q), \mathbf{f}^n(p)) \leq K^s \zeta^n d(q, p).$$

- (iv) Let $(p_m)_{m \in \mathbb{N}}$ be a sequence in M_i converging to $p \in M_i$ when $m \rightarrow \infty$. If $q_m \in \mathcal{N}^s(p_m, (\varepsilon)_{i \in \mathbb{Z}})$ converges to $q \in B(p, \varepsilon)$ as $m \rightarrow \infty$, then $q \in \mathcal{N}^s(p, (\varepsilon)_{i \in \mathbb{Z}})$.

Proof. See Theorems 5.2.11 and 5.3.6. □

K^u and K^s from Theorems 6.2.2 and Theorems 6.2.3, respectively, depend on the constant c of $\mathbf{f}$, on the constant C in (6.1.2) and on the minimum angle between the stable and unstable subspaces of the splitting $T\mathbf{M} = E^s \oplus E^u$, which is positive because we are supposing that $\mathbf{f}$ satisfies the property of angles.

6.3 Structural Stability of $\mathcal{A}_b^2(\mathbf{M})$

In this section we will show that $\mathcal{A}_b^2(\mathbf{M})$ is uniformly structurally stable in $\mathcal{F}^2(\mathbf{M})$: for each $\mathbf{f} \in \mathcal{A}_b^2(\mathbf{M})$ there exists a uniform basic neighborhood $B^2(\mathbf{f}, \delta)$ of $\mathbf{f}$ such that, each $\mathbf{g} \in B^2(\mathbf{f}, \delta)$ is uniformly conjugate to $\mathbf{f}$. Since $\|\cdot\|$ and $\|\cdot\|_*$ are uniformly equivalent (see (6.1.2)) and $\mathbf{f}$ is strictly Anosov with $\|\cdot\|_*$, we can suppose, without loss of generality, that $\mathbf{f}$ is strictly Anosov with constant $\lambda \in (0, 1)$ considering the norm $\|\cdot\|$ on $\mathbf{M}$. Furthermore, we can suppose that the stable and unstable subspaces are orthogonal.

To prove the structural stability of Anosov families in $\mathcal{A}_b^2(\mathbf{M})$ we have adapted the Shub's ideas in [Shu13] to prove the structural stability of Anosov diffeomorphisms on compact Riemannian manifolds. We will divide the proof of this fact into a series of lemmas and propositions. Throughout this section, we will consider $\tilde{r} > 0$, $\xi > 0$ and $\eta \in [\lambda, 1)$ as in Section 4.3.

Definition 6.3.1. For $\tau > 0$ and $i \in \mathbb{Z}$, set:

- (i) $D(I_i, \tau) = \{h : M_i \rightarrow M_i : h \text{ is } C^0 \text{ and } d(h(p), I_i(p)) \leq \tau \text{ for any } p \in M_i\}$;
- (ii) $\mathcal{D}(\tau) = \{(h_i)_{i \in \mathbb{Z}} : h_i \in D(I_i, \tau) \text{ for any } i \in \mathbb{Z}\}$;
- (iii) $\Gamma(M_i) = \{\sigma : M_i \rightarrow TM_i : \sigma \text{ is a continuous section}\}$;
- (iv) $\Gamma_\tau(M_i) = \{\sigma \in \Gamma(M_i) : \sup_{p \in M_i} \|\sigma(p)\| \leq \tau\}$;
- (v) $\Gamma(\mathbf{M}) = \{(\sigma_i)_{i \in \mathbb{Z}} : \sigma_i \in \Gamma(M_i) \text{ and } \sup_{i \in \mathbb{Z}} \|\sigma_i\|_{\Gamma_i} < \infty\}$;
- (vi) $\Gamma_\tau(\mathbf{M}) = \{(\sigma_i)_{i \in \mathbb{Z}} \in \Gamma(\mathbf{M}) : \sigma_i \in \Gamma_\tau(M_i) \text{ for each } i \in \mathbb{Z}\}$.

$\Gamma(M_i)$ is a Banach space with the norm $\|\sigma\|_{\Gamma_i} = \sup_{p \in M_i} \|\sigma(p)\|$. Therefore:

Lemma 6.3.2. $\Gamma(\mathbf{M})$ is a Banach space endowed with the norm $\|(\sigma_i)_{i \in \mathbb{Z}}\|_\infty = \sup_{i \in \mathbb{Z}} \|\sigma_i\|_{\Gamma_i}$.

Let $0 < \varepsilon \leq \varrho/2$. We can identify $\mathcal{D}(\varepsilon)$ with $\Gamma_\varepsilon(\mathbf{M})$ by the homeomorphism

$$\begin{aligned} \Phi : \mathcal{D}(\varepsilon) &\rightarrow \Gamma_\varepsilon(\mathbf{M}) \\ (h_i)_{i \in \mathbb{Z}} &\mapsto (\Phi_i(h_i))_{i \in \mathbb{Z}}, \end{aligned}$$

where $\Phi_i(h_i)(p) = \exp_p^{-1}(h_i(p))$, for $p \in M_i$. Note that $\Phi_i(I_i)$ is the zero section in $\Gamma(M_i)$.

Next we will prove the following lemma:

Lemma 6.3.3. Fix $\kappa > 0$. There exist $\xi' \in (0, \xi]$ and $r' \in (0, \tilde{r}/3]$ such that, if $\mathbf{g} = (g_i)_{i \in \mathbb{Z}} \in B^2(\mathbf{f}, \xi')$, then the map

$$\begin{aligned} \mathcal{G} : \mathcal{D}(r') &\rightarrow \mathcal{D}(\kappa) \\ (h_i)_{i \in \mathbb{Z}} &\mapsto (g_{i-1} \circ h_{i-1} \circ f_{i-1}^{-1})_{i \in \mathbb{Z}} \end{aligned}$$

is well-defined.

Proof. It is sufficient to prove that there exist $\xi' > 0$ and $r' > 0$ such that, for every $i \in \mathbb{Z}$, if $g_i : M_i \rightarrow M_{i+1}$ is a diffeomorphism with $d^1(g_i, f_i) < \xi'$ and $h \in D(I_i, r')$, then $g_i h f_i^{-1} \in D(I_{i+1}, \kappa)$. For each continuous map $h : M_i \rightarrow M_i$, we have

$$d(g_i h f_i^{-1}(p), I_{i+1}(p)) \leq d(g_i h f_i^{-1}(p), g_i f_i^{-1}(p)) + d(g_i f_i^{-1}(p), I_{i+1}(p)).$$

Let $\mathcal{S} := \sup_{i \in \mathbb{Z}} \|Df_i\| < \infty$. Consider $r' < \kappa/2(\mathcal{S} + 1)$ and $\xi' < \min\{1, \kappa/2\}$. For $i \in \mathbb{Z}$, if

$$g_i \in B^1(f_i, \xi') = \{g \in \text{Diff}^1(M_i, M_{i+1}) : \max\{d^1(g, f_i), d^1(g^{-1}, f_i^{-1})\} < \xi'\}$$

we have

$$d(g_i f_i^{-1}(p), I_{i+1}(p)) = d(g_i f_i^{-1}(p), f_i f_i^{-1}(p)) \leq d^1(g_i, f_i) < \xi' < \kappa/2.$$

Furthermore, if $h \in D(I_i, r')$ and $g_i \in B^1(f_i, \xi')$, then

$$d(g_i h f_i^{-1}(p), g_i f_i^{-1}(p)) \leq \|Dg_i\| d(h f_i^{-1}(p), f_i^{-1}(p)) \leq (\xi' + \mathcal{S})r' < \kappa/2.$$

Therefore, if $h \in D(I_i, r')$ and $g_i \in B^1(f_i, \xi')$, we have $g_i \circ h \circ f_i^{-1} \in D(I_{i+1}, \kappa)$, which proves the lemma. $\square$

Fix $\zeta \in (0, \min\{1 - \lambda, 1 - \eta, \varrho/2\})$. It follows from Lemma 6.3.3 that there exist

$$r' \in (0, \tilde{r}/3) \quad \text{and} \quad \xi' \in (0, \min\{\xi, r'(1 - \lambda - \zeta)\}), \quad (6.3.1)$$

such that if $\mathbf{g} = (g_i)_{i \in \mathbb{Z}} \in B^2(\mathbf{f}, \xi')$ then

$$\begin{aligned} \mathbf{G} : \Gamma_{r'}(\mathbf{M}) &\rightarrow \Gamma_\zeta(\mathbf{M}), \\ (\sigma_i)_{i \in \mathbb{Z}} &\mapsto (\Phi_i \mathcal{G}_{i-1} \Phi_{i-1}^{-1}(\sigma_{i-1}))_{i \in \mathbb{Z}} \end{aligned}$$

is well-defined, where $\mathcal{G}_{i-1}(h) = g_{i-1} \circ h \circ f_{i-1}^{-1}$, for $h \in D(I_{i-1}, r')$. Consequently,

$$\Phi_i \mathcal{G}_{i-1} \Phi_{i-1}^{-1}(\sigma)(p) = \exp_p^{-1} \circ g_{i-1} \circ \exp_{f_{i-1}^{-1}(p)} \sigma(f_{i-1}^{-1}(p)) \quad \text{for } p \in M_i, \sigma \in \Gamma_{r'}(M_{i-1}).$$

Set

$$\Gamma^s(M_i) = \{\sigma \in \Gamma(M_i) : \text{the image of } \sigma \text{ is contained in } E^s\}$$

and

$$\Gamma^u(M_i) = \{\sigma \in \Gamma(M_i) : \text{the image of } \sigma \text{ is contained in } E^u\}$$

where E^s and E^u are the stable and unstable subbundles induced by $\mathbf{f}$. It is clear that

$$\Gamma(M_i) = \Gamma^s(M_i) \oplus \Gamma^u(M_i).$$

Proposition 6.3.4. *The map*

$$\begin{aligned} \mathbf{F} : \Gamma(\mathbf{M}) &\rightarrow \Gamma(\mathbf{M}) \\ (\sigma_i)_{i \in \mathbb{Z}} &\mapsto (\mathbf{F}_{i-1}(\sigma_{i-1}))_{i \in \mathbb{Z}} \end{aligned}$$

is a hyperbolic bounded linear operator, where $\mathbf{F}_i : \Gamma(M_i) \rightarrow \Gamma(M_{i+1})$ is defined by the formula $\mathbf{F}_i(\sigma)(p) = D(f_i)_{f_i^{-1}(p)}(\sigma(f_i^{-1}(p)))$, for $p \in M_{i+1}, \sigma \in \Gamma(M_i)$.

Proof. It is not difficult to prove that $\mathbf{F}$ is a linear operator and

$$\|\mathbf{F}\| = \sup_{i \in \mathbb{Z}} \{\|\mathbf{F}_i(\sigma_i)\|_{\Gamma_i} : \|(\sigma_i)\|_\infty = 1\} \leq \sup_i \|Df_i\|_* < \infty.$$

Thus, $\mathbf{F}$ is a bounded linear operator. It is clear that $\mathbf{F}_i(\Gamma^s(M_i)) = \Gamma^s(M_{i+1})$ and $\mathbf{F}_i(\Gamma^u(M_i)) = \Gamma^u(M_{i+1})$. Furthermore,

$$\|\mathbf{F}_i(\sigma)\|_{\Gamma_{i+1}} \leq \lambda \|\sigma\|_{\Gamma_i} \text{ for } \sigma \in \Gamma^s(M_i) \quad \text{and} \quad \|\mathbf{F}_i^{-1}(\sigma)\|_{\Gamma_i} \leq \lambda \|\sigma\|_{\Gamma_{i+1}} \text{ for } \sigma \in \Gamma^u(M_{i+1}).$$

For $t = s, u$, set

$$\Gamma^t(\mathbf{M}) = \{(\sigma_i)_{i \in \mathbb{Z}} \in \Gamma(\mathbf{M}) : \sigma_i \in \Gamma^t(M_i) \text{ for each } i \in \mathbb{Z}\}.$$

We can prove that $\Gamma(\mathbf{M}) = \Gamma^s(\mathbf{M}) \oplus \Gamma^u(\mathbf{M})$ and $\mathbf{F}$ is a hyperbolic linear operator with respect to this splitting. $\square$

Lemma 6.3.5. *There exist $\xi' \in (0, \xi]$ and $r' \in (0, \tilde{r}/3]$ such that, if $\mathbf{g} \in B^2(\mathbf{f}, \xi')$, then*

$$\text{Lip}([\mathbf{F} - \mathbf{G}]|_{\Gamma_{r'}(\mathbf{M})}) < \zeta,$$

where Lip denotes a Lipschitz constant.

Proof. For $\sigma \in \Gamma_{r'}(M_{i-1})$ and $p \in M_i$ we have

$$[\mathbf{F}_{i-1} - \Phi_i \mathcal{G}_{i-1} \Phi_{i-1}^{-1}](\sigma)(p) = [D(f_{i-1})_{f_{i-1}^{-1}(p)} - \exp_p^{-1} \circ g_{i-1} \circ \exp_{f_{i-1}^{-1}(p)}](\sigma(f_{i-1}^{-1}(p))).$$

Let $q = f_{i-1}^{-1}(p)$. Note that

$$D(f_{i-1})_q = D(\exp_p^{-1} \circ f_{i-1} \circ \exp_q)_{0_q} \quad \text{where } 0_q \text{ is the zero vector in } T_q M.$$

For any $v \in T_q M$ with $\|v\| < \xi$, we have

$$\begin{aligned} D[D(f_{i-1})_q - \exp_p^{-1} \circ g_{i-1} \circ \exp_q]v &= D[D(\exp_p^{-1} \circ f_{i-1} \circ \exp_q)_{0_q} - \exp_p^{-1} \circ g_{i-1} \circ \exp_q]v \\ &= D(\exp_p^{-1} \circ f_{i-1} \circ \exp_q)_{0_q} - D(\exp_p^{-1} \circ g_{i-1} \circ \exp_q)_v. \end{aligned}$$

As we saw in the proof of Proposition 6.1.3,

$$\|D(\exp_p^{-1} \circ f_{i-1} \circ \exp_q)_{0_q} - D(\exp_p^{-1} \circ g_{i-1} \circ \exp_q)_v\| \leq K[1 + S_{\mathbf{f}}]S_{\mathbf{f}}\|v\| + d^2(f_{i-1}, g_{i-1}).$$

Hence if $\sigma = (\sigma_i)_{i \in \mathbb{Z}}, \tilde{\sigma} = (\tilde{\sigma}_i)_{i \in \mathbb{Z}} \in \Gamma_{r'}(M_{i-1})$ and $p \in M_i$ we have

$$\begin{aligned} \|[\mathbf{F}_{i-1} - \Phi_i \mathcal{G}_{i-1} \Phi_{i-1}^{-1}](\sigma_{i-1})(p) - [\mathbf{F}_{i-1} - \Phi_i \mathcal{G}_{i-1} \Phi_{i-1}^{-1}](\tilde{\sigma}_{i-1})(p)\| &\leq \mathcal{J} \|\sigma_{i-1}(p) - \tilde{\sigma}_{i-1}(p)\| \\ &\leq \mathcal{J} \|\sigma_{i-1} - \tilde{\sigma}_{i-1}\|_{\Gamma_i} \\ &\leq \mathcal{J} \|\sigma - \tilde{\sigma}\|_{\infty}, \end{aligned}$$

where $\mathcal{J} = K[1 + S_f]S_f\|v\| + d^2(f_{i-1}, g_{i-1})$. We can choose r' and ξ' small enough such that $\mathcal{J} < \zeta$. Thus

$$\|[\mathbf{F}_{i-1} - \Phi_i \mathcal{G}_{i-1} \Phi_{i-1}^{-1}](\sigma_{i-1}) - [\mathbf{F}_{i-1} - \Phi_i \mathcal{G}_{i-1} \Phi_{i-1}^{-1}](\tilde{\sigma}_{i-1})\|_{\Gamma_i} \leq \zeta \|\sigma - \tilde{\sigma}\|_{\infty},$$

and therefore

$$\|[\mathbf{F} - \mathbf{G}](\sigma) - [\mathbf{F} - \mathbf{G}](\tilde{\sigma})\|_{\infty} \leq \zeta \|\sigma - \tilde{\sigma}\|_{\infty},$$

which proves the lemma. $\square$

From now on we will suppose that ξ' and r' satisfy (6.3.1) and Lemma 6.3.5. Furthermore, we will fix $\mathbf{g} = (g_i)_{i \in \mathbb{Z}} \in B^2(\mathbf{f}, \xi')$. To prove the following lemma, we have based on the proof of the Proposition 7.7 in [Shu13].

Lemma 6.3.6. $\mathbf{G}|_{\Gamma_{r'}(\mathbf{M})}$ has a fixed point in $\Gamma_{r'}(\mathbf{M})$.

Proof. Since $\Gamma(\mathbf{M}) = \Gamma^s(\mathbf{M}) \oplus \Gamma^u(\mathbf{M})$, each $\sigma = (\sigma_i)_{i \in \mathbb{Z}} \in \Gamma_{r'}(\mathbf{M})$ can be written as $\sigma = \sigma_s + \sigma_u$, where $\sigma_s = (\sigma_{i,s})_{i \in \mathbb{Z}} \in \Gamma^s(\mathbf{M})$ and $\sigma_u = (\sigma_{i,u})_{i \in \mathbb{Z}} \in \Gamma^u(\mathbf{M})$. Let $\tilde{\mathbf{G}}$ be defined on $\Gamma_{r'}(\mathbf{M})$ as

$$\tilde{\mathbf{G}}(\sigma) = (\mathbf{G}(\sigma))_s + (\mathbf{F}^{-1}[\sigma_u + \mathbf{F}(\sigma_u) - (\mathbf{G}(\sigma))_u])_u.$$

If $\sigma \in \Gamma_{r'}(\mathbf{M})$ is a fixed point of $\tilde{\mathbf{G}}$, we have

$$(\mathbf{G}(\sigma))_s = \sigma_s \quad \text{and} \quad (\mathbf{G}(\sigma))_u = \sigma_u,$$

that is, σ is a fixed point of $\mathbf{G}$. Therefore, in order to prove the lemma, it is sufficient to find a fixed point of $\tilde{\mathbf{G}}$. First we prove that $\tilde{\mathbf{G}}$ is a contraction. Take $\sigma = (\sigma_{i,s})_{i \in \mathbb{Z}} + (\sigma_{i,u})_{i \in \mathbb{Z}}$ and $\tilde{\sigma} = (\tilde{\sigma}_{i,s})_{i \in \mathbb{Z}} + (\tilde{\sigma}_{i,u})_{i \in \mathbb{Z}}$ in $\Gamma_{r'}(\mathbf{M})$, where $\sigma_s, \tilde{\sigma}_s \in \Gamma^s(\mathbf{M})$ and $\sigma_u, \tilde{\sigma}_u \in \Gamma^u(\mathbf{M})$. Thus,

$$\begin{aligned} \|\tilde{\mathbf{G}}(\sigma)_{i+1} - \tilde{\mathbf{G}}(\tilde{\sigma})_{i+1}\|_{\Gamma_{i+1}} &= \|\tilde{\mathbf{G}}_i(\sigma_i) - \tilde{\mathbf{G}}_i(\tilde{\sigma}_i)\|_{\Gamma_{i+1}} \\ &\leq \max\{\|(\mathbf{F}^{-1}[(\sigma_{i,u} - \tilde{\sigma}_{i,u}) + ((\mathbf{F} - \mathbf{G})(\sigma))_{i,u} - ((\mathbf{F} - \mathbf{G})(\tilde{\sigma}))_{i,u}])_u]\|, \|(\mathbf{G}(\sigma - \tilde{\sigma}))_{i,s}\|\} \\ &\leq \max\{\lambda(1 + \zeta)\|\sigma - \tilde{\sigma}\|_{\infty}, (\lambda + \zeta)\|\sigma - \tilde{\sigma}\|_{\infty}\} = (\lambda + \zeta)\|\sigma - \tilde{\sigma}\|_{\infty}. \end{aligned}$$

Hence, $\|\tilde{\mathbf{G}}(\sigma) - \tilde{\mathbf{G}}(\tilde{\sigma})\|_{\infty} \leq (\lambda + \zeta)\|\sigma - \tilde{\sigma}\|_{\infty}$. Since $\lambda + \zeta < 1$, $\tilde{\mathbf{G}}$ is a contraction. Now we prove that $\tilde{\mathbf{G}}(\Gamma_{r'}(\mathbf{M})) \subseteq \Gamma_{r'}(\mathbf{M})$. If $0 = (0_i)_{i \in \mathbb{Z}}$ is the sequence of the zero sections, we have that

$$\begin{aligned} \|\tilde{\mathbf{G}}(\sigma)_{i+1}\|_{\Gamma_{i+1}} &\leq (\lambda + \zeta)\|\sigma\|_{\Gamma_i} + \|\tilde{\mathbf{G}}(0)_{i+1}\|_{\Gamma_{i+1}} \\ &\leq (\lambda + \zeta)\|\sigma\|_{\infty} + \max\{\|(\mathbf{F}^{-1}(\mathbf{G}(0))_u)_{i+1}\|_{\Gamma_{i+1}}, \|(\mathbf{G}(0))_s\|_{\Gamma_{i+1}}\} \\ &\leq (\lambda + \zeta)\|\sigma\|_{\infty} + \max\{\lambda\|(\mathbf{G}(0))_u\|_{\Gamma_{i+1}}, \|(\mathbf{G}(0))_s\|_{\Gamma_{i+1}}\} \\ &\leq (\lambda + \zeta)\|\sigma\|_{\infty} + \|\mathbf{G}(0)\|_{\infty}, \end{aligned}$$

thus $\|\tilde{\mathbf{G}}(\sigma)\|_\infty \leq (\lambda + \zeta)\|\sigma\|_\infty + \|\mathbf{G}(0)\|_\infty$. Now, for each $i \in \mathbb{Z}$, $p \in M_i$, we have

$$\begin{aligned} \|\mathbf{G}_i(0_i)(p)\| &= \|\exp_p^{-1} \circ g_i \circ \exp_{f_i^{-1}(p)}(0_{f_i^{-1}(p)})\| = \|\exp_p^{-1}(g_i f_i^{-1}(p))\| \\ &= \mathbf{d}(g_i f_i^{-1}(p), p) = \mathbf{d}(g_i f_i^{-1}(p), f_i f_i^{-1}(p)) < \delta' \end{aligned}$$

Consequently, if $\sigma \in \Gamma_{r'}(\mathbf{M})$, then $\|\tilde{\mathbf{G}}(\sigma)\|_\infty < (\lambda + \zeta)r' + \delta' < r'$, that is, $\tilde{\mathbf{G}}(\sigma) \in \Gamma_{r'}(\mathbf{M})$. Therefore, $\tilde{\mathbf{G}}$ has a fixed point in $\Gamma_{r'}(\mathbf{M})$. $\square$

If $(\sigma_i)_{i \in \mathbb{Z}}$ is the fixed point of $\mathbf{G}$, then, considering $h_i = \Phi_i^{-1}(\sigma_i)$ for each $i \in \mathbb{Z}$, we have that $(h_i)_{i \in \mathbb{Z}}$ is a fixed point of $\mathcal{G}$. Hence, Lemma 6.3.6 implies that $\mathcal{G}|_{\mathcal{D}(r')}$ has a fixed point in $\mathcal{D}(r')$.

Lemma 6.3.7. Fix $\mathbf{g} = (g_i)_{i \in \mathbb{Z}} \in B^2(\mathbf{f}, \delta')$. Let $p \in \mathbf{M}$ and $v = (v_s, v_u), w = (w_s, w_u) \in B^s(0, \tilde{r}) \times B^u(0, \tilde{r})$. If $\|v_s - w_s\| \leq \|v_u - w_u\|$, then

$$\|(\tilde{g}_p(v))_s - (\tilde{g}_p(w))_s\| \leq (\eta^{-1} - \zeta)\|v_u - w_u\| \leq \|(\tilde{g}_p(v))_u - (\tilde{g}_p(w))_u\|.$$

On the other hand, if $\|v_u - w_u\| \leq \|v_s - w_s\|$, then

$$\|(\tilde{g}_p^{-1}(v))_u - (\tilde{g}_p^{-1}(w))_u\| \leq (\eta^{-1} - \zeta)\|v_s - w_s\| \leq \|(\tilde{g}_p^{-1}(v))_s - (\tilde{g}_p^{-1}(w))_s\|.$$

Proof. See [Shu13], Lemma II.1. $\square$

Let $v = (v_s, v_u) \in B^s(0, \tilde{r}) \times B^u(0, \tilde{r})$ and $q = \exp_p(v)$. Suppose that $\mathbf{d}(\mathbf{g}^n(p), \mathbf{g}^n(q)) < \tilde{r}$ for $n = \pm 1$. Thus, if $\|v_s\| \leq \|v_u\|$, by Lemma 6.3.7 we have

$$\begin{aligned} \mathbf{d}(q, p) &= \|v\| \leq C\|v\| = C\|v_u\| \leq C(\eta^{-1} - \zeta)^{-1}\|(\tilde{g}_p(v))_u - (\tilde{g}_p(0_p))_u\| \\ &\leq C^2(\eta^{-1} - \zeta)^{-1}\|\tilde{g}_p(v) - \tilde{g}_p(0_p)\| \leq 2C^2(\eta^{-1} - \zeta)^{-1}\mathbf{d}(\mathbf{g}(p), \mathbf{g}(q)) \\ &\leq 2C^2(\eta^{-1} - \zeta)^{-1}\tilde{r}. \end{aligned}$$

Analogously if $\|v_u\| \leq \|v_s\|$, then

$$\mathbf{d}(q, p) \leq 2C^2(\eta^{-1} - \zeta)^{-1}\tilde{r}.$$

Inductively, we can prove that:

Proposition 6.3.8. For each $i \in \mathbb{Z}$, if $p, q \in M_i$ and $\mathbf{d}(\mathbf{g}_i^n(p), \mathbf{g}_i^n(q)) < \tilde{r}$ for each $n \in [-N, N]$, then

$$\mathbf{d}(q, p) \leq 2C^2(\eta^{-1} - \zeta)^{-N}\tilde{r}.$$

Finally,

Theorem 6.3.9. $\mathcal{A}_b^2(\mathbf{M})$ is uniformly structurally stable.

Proof. Take $(g_i)_{i \in \mathbb{Z}} \in B^2(\mathbf{f}, \delta')$. It follows from Lemma 6.3.6 that there exists $\mathbf{h} = (h_i)_{i \in \mathbb{Z}}$, with $h_i \in D(I_i, r')$ for each i , such that

$$g_i \circ h_i = h_{i+1} \circ f_i \quad \text{for each } i \in \mathbb{Z}.$$

We will prove that $\mathbf{h}$ is equicontinuous and each h_i is injective. Let $\alpha > 0$. Take $N > 0$ such that $2C^2(\eta^{-1} - \zeta)^{-N}\tilde{r} < \alpha$. Since $(f_i)_{i \in \mathbb{Z}}$ is equicontinuous, the family of sequences

$$\{(\mathbf{f}_i^n)_{i \in \mathbb{Z}} : n \in [-N, N]\}$$

is equicontinuous. Consequently, there exists $\beta > 0$ such that, for each $i \in \mathbb{Z}$, if $x, y \in M_i$ and $\mathbf{d}(x, y) < \beta$, then $\mathbf{d}(\mathbf{f}_i^n(x), \mathbf{f}_i^n(y)) < \tilde{r}/3$ for any $n \in [-N, N]$. Hence, for each $i \in \mathbb{Z}$ and $n \in [-N, N]$, if $x, y \in M_i$ and $\mathbf{d}(x, y) < \beta$, then

$$\begin{aligned} \mathbf{d}(\mathbf{g}_i^n h_i(x), \mathbf{g}_i^n h_i(y)) &\leq \mathbf{d}(\mathbf{g}_i^n h_i(x), \mathbf{f}_i^n(x)) + \mathbf{d}(\mathbf{f}_i^n(x), \mathbf{f}_i^n(y)) + \mathbf{d}(\mathbf{f}_i^n(y), \mathbf{g}_i^n h_i(y)) \\ &< r' + \tilde{r}/3 + r' \leq \tilde{r}. \end{aligned}$$

It follows from Proposition 6.3.8 that $\mathbf{d}(h_i(x), h_i(y)) < \alpha$. This fact proves that $(h_i)_{i \in \mathbb{Z}}$ is an equicontinuous family. Notice that if $h_i(x) = h_i(y)$ for some $x, y \in M_i$, then $\mathbf{d}(\mathbf{g}_i^n h_i(x), \mathbf{g}_i^n h_i(y)) < \tilde{r}$ for any $n \in \mathbb{Z}$. Thus $x = y$ and therefore h_i is injective. Analogously we can prove that $(h_i^{-1})_{i \in \mathbb{Z}}$ is equicontinuous. Consequently, $\mathcal{A}_b^2(\mathbf{M})$ is uniformly structurally stable. $\square$

Chapter 7

Some Other Problems That Arose From This Work

We will finish this work by presenting some problems that arose from this thesis which we hope to work in future projects.

7.1 Another Classification of Dynamical Systems on the Circle

In Remark 2.3.5 we gave a motivation to study the classification of constant families by uniform conjugacies. Now, take $\mathbb{S}^1 = \{z \in \mathbb{R}^2 : \|z\| = 1\}$, endowed with the Riemannian metric inherited from $\mathbb{R}^2$. For a homeomorphism $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, we denote by $\mathbf{f}_\phi$ the constant family associated to ϕ . In Proposition 1.4.5 we prove that there exist homeomorphisms ϕ and ψ on $\mathbb{S}^1$, which are not topologically conjugate, with $\mathbf{f}_\phi$ and $\mathbf{f}_\psi$ uniformly conjugate. Uniform conjugacies provide an equivalence relation. It might be interesting to try to classify the homeomorphisms on the circle with respect to this new equivalence relation.

Notice that, by the Poincaré Classification Theorem we have that if ϕ is *topologically transitive*, then it is topologically conjugate to a rotation (see [KH97], Theorem 11.2.7). Therefore, every constant family associated to a topologically transitive homeomorphism on the circle is uniformly conjugate to the constant family associated to the identity on the circle (see Proposition 1.4.5). For more details about this problem, see [Ace17a].

7.2 Entropy for Non-Stationary Dynamical Systems: Further Generalizations

In this work was proved the continuity of the entropy of non-stationary dynamical systems as long as each diffeomorphism f_i is of class C^m with $m \geq 1$. A very interesting project would be to study the continuity of this entropy for sequences of homeomorphisms or Hölder continuous maps. In this case, M could be a general metric space, that is, not necessarily a differentiable manifold. A series of results that could be very useful to work on this problem can be found in [ZC09], [KL16], [KMS99], [KS96], [Yan80], [ZZH06], among others papers.

The entropy built here, was for a fixed metric space M , with each map $f_i : M \rightarrow M$ a homeomorphism. This notion could be extended considering, for each $i \in \mathbb{Z}$, a more general metric

space M_i , with a fixed metric d_i , and each f_i a continuous map on M_i to M_{i+1} , not necessarily a homeomorphism. Another interesting project would be to study the properties of this entropy.

7.3 Existence and classification of Anosov Families

The existence of Anosov diffeomorphisms $\phi : M \rightarrow M$ imposes strong restrictions on the manifold M . All known examples of Anosov diffeomorphisms are defined on *infranilmanifolds* (see [BP07], [Shu13], [Via14]). If M is a parallelizable Riemannian manifold¹, suitably changing the metrics on each component $M_i = M \times \{i\}$ we can build an Anosov family in the disjoint union of M_i , taking f_i as the identity $I_i : M_i \rightarrow M_{i+1}$ (see Example 3.2.1). As we saw in Example 3.2.9, an Anosov family does not necessarily consist of Anosov diffeomorphisms. We say that a Riemannian manifold M with Riemannian metric $\langle \cdot, \cdot \rangle$, admits an Anosov family if there exists an Anosov family on $\mathbf{M} = \bigcup_{i \in \mathbb{Z}} M \times \{i\}$, considering on each $M \times \{i\}$ the Riemannian metric $\langle \cdot, \cdot \rangle_i = \langle \cdot, \cdot \rangle$ for all $i \in \mathbb{Z}$. A natural question that arises from the above facts is: if M is a parallelizable Riemannian manifold, is there an Anosov family on $\mathbf{M}$? Since the constant family associated to an Anosov diffeomorphism is an Anosov family, each manifold admitting an Anosov diffeomorphism admits an Anosov family.

It is well-known that there are not Anosov diffeomorphisms on the circle $\mathbb{S}^1$. Next we prove that the circle does not admit Anosov families.

Proposition 7.3.1. *Take $M_i = \mathbb{S}^1 \times \{i\}$ with Riemannian metric inherited from $\mathbb{R}^2$ and $\mathbf{M}$ the disjoint union of the M_i . Thus, there is not any Anosov family on $\mathbf{M}$. More specifically, it does not exist a contractive or expansive family on $\mathbb{S}^1$.*

Proof. Suppose that $(f_i)_{i \in \mathbb{Z}}$ is an Anosov family on $\mathbf{M}$. Fix $p \in M_0$. Since the circle is one-dimensional, then, either $\|D(f_0^n)_p(v)\| \leq c\lambda^n \|v\|$ for all $n \geq 1$, $v \in T_p \mathbb{S}^1$ or $\|D(f_0^{-n})_p(v)\| \leq c\lambda^n \|v\|$ for all $n \geq 1$, $v \in T_p \mathbb{S}^1$. Without loss of generality we can assume that $\|D(f_0^n)_p(v)\| \leq c\lambda^n \|v\|$ for all $n \geq 1$, $v \in T_p \mathbb{S}^1$. Let n be large enough such that $c\lambda^n < 1/2$. Take $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ as $\phi = f_{n-1} \circ \dots \circ f_0$. Thus, if $p \in \mathbb{S}^1$, then $\|D\phi_p(v)\| \leq (1/2)\|v\|$ for all $v \in T_p \mathbb{S}^1$. Since ϕ is bijective, this is impossible. $\square$

From the previous proposition we get that if M is $\mathbb{S}^1$ then the answer to the above question is “no”. Non-parallelizable manifolds cannot admit Anosov families. $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ are the only spheres which are parallelizable (see [Ker58]). Therefore the n -sphere $\mathbb{S}^n$ does not admit Anosov families for $n \in \{1, 2, 4, 5, 6, 8, 9, \dots\}$. On the other hand, non-orientable manifolds are non-parallelizable. Consequently, non-orientable manifolds do not admit Anosov families.

Another natural question is:

Problem 7.3.2. Does M admit an Anosov family if and only if M admits an Anosov diffeomorphism?

Results obtained in Section 4.4 provide of a great variety of examples of Anosov families. However, in all these examples, the diffeomorphisms f_i live in a small neighborhood of an (factor of an) Anosov diffeomorphism. Hence, another problem is the following one:

¹A manifold M is called *parallelizable* if there exist differentiable vector fields $X_1, \dots, X_m$ on M such that $\{X_1(x), \dots, X_m(x)\}$ is a basis for the tangent space $T_x M$, for all $x \in M$.

Problem 7.3.3. Given an Anosov family $\mathbf{f} = (f_i)_{i \in \mathbb{Z}}$ defined on M , for each (or some) $i \in \mathbb{Z}$ there exists $n > 0$ such that $\mathbf{f}_i^n$ is (homotopic to) an Anosov diffeomorphism?

Next, it follows from J. Franks [Fra69] and A. Manning [Man74] that:

Theorem 7.3.4. *If $\phi : \mathbb{T}^m \rightarrow \mathbb{T}^m$ is an Anosov diffeomorphism defined on the m -torus $\mathbb{T}^m$, then ϕ is topologically conjugate to a hyperbolic toral automorphism $A : \mathbb{T}^m \rightarrow \mathbb{T}^m$, which belongs to the same homotopy class of ϕ .*

Hence all the Anosov diffeomorphisms on $\mathbb{T}^m$ are classified. It is possible to generalize this result for Anosov families:

Problem 7.3.5. Let $(f_i)_{i \in \mathbb{Z}}$ be an Anosov family, where each $f_i : \mathbb{T}^m \rightarrow \mathbb{T}^m$ is an Anosov diffeomorphism in the same homotopy class of a hyperbolic toral automorphism $A : \mathbb{T}^m \rightarrow \mathbb{T}^m$. Are $(f_i)_{i \in \mathbb{Z}}$ and $(A)_{i \in \mathbb{Z}}$ uniformly conjugate?

Theorem 6.3.9 provides a particular case of Problem 7.3.5.

7.4 Hölder Continuity of the Subbundles

Let H be a Hilbert space with norm $\|\cdot\|$. Given two subspaces E and F of H , set

$$\Gamma(E, F) := \sup_{\substack{v \in E \\ \|v\|=1}} \inf_{w \in F} \|v - w\|.$$

$\Gamma(E, F)$ is called the *aperture between E and F* .

Definition 7.4.1. Let X be a metric space with metric d . A family $\{E_x\}_{x \in X}$ of subspaces of H is called *Hölder continuous with exponent $\alpha \in (0, 1]$ and constant $L > 0$* , if for any $x, y \in X$ we have

$$\rho(E_x, E_y) := \max\{\Gamma(E_x, E_y), \Gamma(E_y, E_x)\} < L[d(x, y)]^\alpha.$$

A first problem would be to find conditions to obtain the Hölder continuity of the subbundles E^s and E^u .

On the other hand, the absolute continuity of the stable and unstable manifolds could also be studied. The author recommends viewing [BS02], Chapter 6 and [LQ06], Chapter 3 to address these problems.

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