**A generic multiplicity result
for the Yamabe equation
in a Riemannian product
via Morse theory**

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Resumo

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Sejam M^n uma variedade fechada com $n \geq 3$ e (N, h) uma variedade Riemanniana com curvatura escalar constante. Provamos que a equação de Yamabe no produto Riemanniano $(M \times N, g + \epsilon^2 h)$ dependendo apenas de fatores conformes $u: M \rightarrow \mathbb{R}$ tem pelo menos $P_1(M)$ soluções positivas de pequena energia para (ϵ, g) genérico, onde $\epsilon > 0$ é pequeno o suficiente e g é uma métrica Riemanniana em M de classe C^k , $k < \infty$. Esse resultado é obtido através da adaptação de técnicas empregadas por Micheletti e Pistoia (2009), Ghimenti e Micheletti (2011) e Jimmy Petean (2016).

Palavras-chave: problema de Yamabe, curvatura escalar, pontos críticos não-degenerados, teoria de Morse, teoria dos pontos críticos.

Abstract

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Let M^n be a closed manifold with $n \geq 3$ and (N, h) be a Riemannian manifold with constant scalar curvature. We prove that the Yamabe equation on the Riemannian product $(M \times N, g + \epsilon^2 h)$ depending only on conformal factors $u: M \rightarrow \mathbb{R}$ has at least $P_1(M)$ positive solutions with small energy for generic (ϵ, g) , where $\epsilon > 0$ is sufficiently small and g is a Riemannian metric on M of class C^k , $k < \infty$. This result is obtained by adaptating techniques employed by Micheletti & Pistoia (2009), Ghimenti & Micheletti (2011) and Jimmy Petean (2016).

Keywords: Yamabe problem, scalar curvature, nondegenerate critical points, Morse theory, critical point theory.

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Chapter 1

Introduction

Let (M^n, g) be closed Riemannian manifold with $n \geq 3$. The original Yamabe problem refers to the question: is there a scalar factor $u \in C^\infty(M,]0, \infty[)$ for which $(M, u^{4/(n-2)}g)$ admits constant scalar curvature?

One can prove that $\text{scal}(u^{4/(n-2)}g) \equiv \lambda \in \mathbb{R}$ if, and only if, u is a solution for the Yamabe equation

$$-a_n \Delta_g u + \text{scal}(g)u = \lambda u^{p_n-1},$$

where $a_n = 4(n-1)/(n-2)$ and $p_n = 2n/(n-2)$. Therefore, when we talk about multiplicity of solutions for the Yamabe equation, we refer to the number of solutions for such PDE.

In this context, it is known that the Yamabe problem admits (at least) one solution. This result was obtained with the combined effort of Yamabe [Yam60], Trudinger [Tru68], Schoen [Sch84] and Aubin [Aub86].

Recently, the Yamabe problem in product-like structures became a matter of research. For instance, K. Akutagawa, L. Florit and J. Petean showed that the Yamabe constants of the Riemannian products $(M^m \times N^n, g + rh)$ as r goes to infinity equals the Yamabe constant of $(M^m \times \mathbb{R}^n, [g + g_E])$. ([AFP06])

As for multiplicity of solutions, R. Bettiol and P. Piccione proved that, under certain conditions, there are at least 3 unit volume constant scalar curvature metrics in the conformal class $[g_t]$ for infinitely many t s accumulating at 0, where g_t is a family of constant scalar curvature metrics on the total space of a Riemannian submersion obtained by shrinking the fibers of an original metric g . ([BP13])

In this dissertation, we are interested in the Yamabe problem in the Riemannian product $(M^n \times N^m, g + \epsilon^2 h)$, where $n \geq 3$, (N, h) has constant positive scalar curvature and we limit our search of solutions to conformal factors $u: M \rightarrow]0, \infty[$. Considering such setting and a few normalization hypotheses, the Yamabe equation becomes

$$-\epsilon^2 \Delta_g u + \left[\epsilon^2 \frac{\text{scal}^M(g)}{a_{m+n}} + 1 \right] u = u^{p_{m+n}-1}.$$

The growth of the term on the right-hand side of the PDE is sub-critical, while it is critical in the first context presented. This is due to the increase in dimension from n to $n + m$.

Furthermore, this fact permits us to use standard variational theory to treat the problem. (most importantly, we can use the Sobolev embedding theorem)

This exact same setting was studied by J. Petean in [Pet19] and by C. Rey and J. Ruiz in [RR18]. In [Pet19], J. Petean proves that the Yamabe problem in our context admits at least $\text{Cat}(M) + 1$ solutions for small enough $\epsilon > 0$, where $\text{Cat}(M)$ is the Lusternik-Schnirelmann category of M . More recently, J. N. A. Neto generalized this result to the context of Riemannian submersions with minimal fibers in [dAN20]. In [RR18], C. Rey and J. Ruiz build k -peak solutions for the Yamabe problem concentrating around minima of $\text{scal}(g)$.

The objective of this dissertation is to prove a result of generic multiplicity of positive solutions with small energy for the Yamabe equation. More specifically, denote

- $p \in [2, p_n[$;
- $\mathcal{S}^k$ the Banach space of symmetric $(0, 2)$ -tensors over M of class C^k ;
- $\mathcal{M}^k \subset \mathcal{S}^k$ the set of Riemannian metrics over M of class C^k ;
- $\mathcal{B}_\rho$ an open ball in $\mathcal{S}^k$ centered at 0 with radius ρ ;
- $m_\infty = J_\infty(U) > 0$. (see section 5.6.3)

Our main result reads as

Theorem. Given $g_0 \in \mathcal{M}^k$, there are $\hat{\epsilon} > 0$ and $\hat{\rho} > 0$ for which

$$\{(\epsilon, h) \in]0, \hat{\epsilon}[\times \mathcal{B}_{\hat{\rho}} : -\epsilon^2 \Delta_{g_0+h} u + \left[\epsilon^2 \text{scal}(g_0 + h)/a + 1 \right] u = (u^+)^{p-1} \text{ has at least } P_1(M) \text{ nonconstant solutions } u \text{ with } J_{\epsilon,h}(u) < 2m_\infty\}$$

is an open dense subset of $]0, \hat{\epsilon}[\times \mathcal{B}_{\hat{\rho}}$, where

$$J_{\epsilon,h}(u) = \epsilon^{-n} \int_M \frac{\epsilon^2}{2} |\nabla_{g_0+h} u|_{g_0+h}^2 + \frac{1}{2} \left[\epsilon^2 \frac{\text{scal}(g_0 + h)}{a} + 1 \right] u^2 - \frac{1}{p} (u^+)^p \, d\mu_{g_0+h}$$

and $P_1(M) = \sum_k \dim H_k(M)$. (see definition 2.21)

To prove such result, we follow the general argument of M. Ghimenti and A. M. Micheletti in [GM11], we adjust the techniques of A. M. Micheletti and A. Pistoia in [MP09] to prove the genericity of nondegeneracy of solutions u to the PDE in question and we use the extension of the concept of Riemannian center of mass due to J. Petean in [Pet19] to construct a certain continuous transformation intrinsically on M .

As for the structure of the text: chapters 2 to 4 are concerned with the prerequisites to understand our main result, which is presented in chapter 5. More specifically, chapter 2 aims to fix the notation and remind the reader about basic constructions and important results. Chapter 3 develops the operations on tensors in a Riemannian manifold at greater detail. Finally, chapter 4 presents the functional framework under which we work during chapter 5. In chapter 5, we begin by presenting the Yamabe problem and results related to it. This is followed by a contextualization of our main result, then a sketch of its proof, and finally its details.

Along the text we try to fix the notation in the beginning of each section, before stating results. This choice was made to avoid unnecessary repetition. We also avoid changing

notation between chapters. (unless when it was really necessary) We felt that this choice enables the reader to quickly understand the ideas behind proofs once he gets used to the general notation employed in the text.

Throughout our entire text,

- we adopt the Einstein notation: given $\{a_i, b^i : i \in I\}$, the expression $a_i b^i$ means $\sum_{i \in I} a_i b^i$;
- we only consider vector spaces over $\mathbb{R}$.

Throughout chapters 2 to 4,

- $p \in [1, \infty[$;
- k, l are non-negative integers.

Chapter 2

Notation, conventions and reminders

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2.1 Linear algebra

When dealing with linear algebra,

- V is a finite-dimensional vector space;
- $h: V \times V \rightarrow \mathbb{R}$ is an inner product;
- We denote

$$\bigotimes^k V = \underbrace{V \otimes \dots \otimes V}_{k \text{ times}},$$

where “0 times” is interpreted as meaning the vector space $\mathbb{R}$, as usual;

- Similarly, for $(v_1, \dots, v_k) \in V^k$:

$$\bigotimes_{i=1}^k v_i = v_1 \otimes \dots \otimes v_k;$$

- We denote

$$V^{k,l} = \bigotimes^k V \otimes \bigotimes^l V^*.$$

Consider m an arbitrary positive integer. Recall the universal property of tensor products:

Lemma 2.1 (Universal property of tensor products). Given a vector space U and an m -multilinear map φ , there exists a unique linear $\bar{\varphi}$ such that the diagram below commutes

$$\begin{array}{ccc} & \bigotimes^m V & \\ \uparrow \otimes^m & \searrow \exists! \bar{\varphi} & \\ V^m & \xrightarrow{\varphi} & U \end{array}$$

where

$$\begin{aligned} \otimes^m : V^m &\rightarrow \bigotimes^m V \\ (v_1, \dots, v_m) &\mapsto v_1 \otimes \dots \otimes v_m. \end{aligned}$$

As a simple application of this universal property, one may induce an inner product on the tensor product of inner product spaces: (cor. D.35 in [Lee09])

Corollary 2.2. Let $\{(V_i, h_i) : i = 1, \dots, m\}$ be a set of inner product spaces. Then $h_1, \dots, h_m$ induce a unique inner product

$$\bigotimes_{i=1}^m V_i \times \bigotimes_{i=1}^m V_i \rightarrow \mathbb{R}$$

such that given $\{(v_i, u_i) \in V_i \times V_i : 1 \leq i \leq m\}$;

$$\left(\bigotimes_{i=1}^m v_i, \bigotimes_{i=1}^m u_i \right) \mapsto h_1(v_1, u_1) \dots h_m(v_m, u_m).$$

2.2 Completion of normed/inner product spaces

For details on the results of this section, we refer the reader to sections 2.3 and 3.2 in [Kre89].

Normed spaces

Let $(V, \|\cdot\|_V)$ be a normed space. (possibly with $\dim V = \infty$)

The **completion** of V is a pair $((W, \|\cdot\|_W), \varphi)$ where $(W, \|\cdot\|_W)$ is a Banach space, $\varphi : V \rightarrow W$ is an isometric embedding and $\varphi(V)$ is dense in W . Recall the following classical result:

Theorem 2.3. $(V, \|\cdot\|_V)$ admits a completion $((W, \|\cdot\|_W), \varphi)$. If $((\tilde{W}, \|\cdot\|_{\tilde{W}}), \tilde{\varphi})$ is a completion of $(V, \|\cdot\|_V)$, then $((W, \|\cdot\|_W), \varphi)$ is linearly isometric to $((\tilde{W}, \|\cdot\|_{\tilde{W}}), \tilde{\varphi})$.

We shall commit the following abuses of language: V is isometric to $\varphi(V) \subset W$, so we identify V with $\varphi(V)$. We then consider that $V \subset W$. Given $v \in V$, $\|v\|_W = \|v\|_V$. Therefore, we do not distinguish between $\|\cdot\|_V$ and $\|\cdot\|_W$, so we simply denote both these norms by $\|\cdot\|$. It is an easy exercise to show the following:

Lemma 2.4 (Universal property of completions). Let $\hat{W}$ be a Banach space and $\varphi: V \rightarrow \hat{W}$ be an isometric embedding.

$(\hat{W}, \varphi)$ is a completion of V if, and only if, given a Banach space U and a linear continuous $f: V \rightarrow U$, there exists a unique linear continuous $\hat{f}: \hat{W} \rightarrow U$ such that the diagram below commutes

$$\begin{array}{ccc} & \hat{W} & \\ \varphi \uparrow & \searrow \exists! \hat{f} & \\ V & \xrightarrow{f} & U \end{array} \quad (2.1)$$

From now on, we denote the unique linear continuous extension of $f: V \rightarrow U$ as in the lemma simply by $f: \hat{W} \rightarrow U$.

The following result will be extensively used:

Proposition 2.5. Let $\|\cdot\|, \|\cdot\|'$ be equivalent norms on V . Let $(W, \|\cdot\|), (W', \|\cdot\|')$ be respective completions of $(V, \|\cdot\|), (V, \|\cdot\|')$. Then $W = W'$ as sets, and the inclusions $W \hookrightarrow W', W' \hookrightarrow W$ are linearly continuous.

Its proof is lengthy but straightforward: it relies on the construction of the completion as a set of equivalence classes of Cauchy sequences. Equivalent norms imply that Cauchy sequences with respect to $\|\cdot\|$ are exactly the Cauchy sequences with respect to $\|\cdot\|'$. The fact that inclusions are continuous is a corollary of the fact that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

To finish the discussion on normed spaces, we would like to obtain a result similar to lemma 2.4 in the context of multilinear maps. First, let us introduce the context: consider a finite number of normed spaces $V_1, \dots, V_n$ and its respective completions $W_1, \dots, W_n$. Denote by $\tilde{W}$ the normed vector space $(W_1 \times \dots \times W_n, \|\cdot\|_p)$, where $\|\cdot\|_p$ is a p -norm which considers each component W_i . Denote by V the normed vector space $(V_1 \times \dots \times V_n, \|\cdot\|_p)$, and by W a completion of V . We observe the following:

Lemma 2.6. W is isometric to $\tilde{W}$.

Arguing similarly as when proving lemma 2.4, we obtain:

Corollary 2.7. Given a Banach space U and a continuous n -multilinear $f: V \rightarrow U$, there exists a unique continuous n -multilinear $\bar{f}: \hat{W} \rightarrow U$ such that the diagram (2.1) commutes.

Inner product spaces

Let $(V, \langle \cdot, \cdot \rangle_V)$ be an inner product space. (possibly with $\dim V = \infty$)

The following analogue of theorem 2.3 is another classical result:

Theorem 2.8. If $(V, \langle \cdot, \cdot \rangle_V)$ is an inner product space, it admits a **completion** $((W, \langle \cdot, \cdot \rangle_W), \varphi)$, where $(W, \langle \cdot, \cdot \rangle_W)$ is a Hilbert space, $\varphi: V \rightarrow W$ is a linearly isometric embedding which is compatible with inner products and $\varphi(V)$ is dense in W . If $((\tilde{W}, \langle \cdot, \cdot \rangle_{\tilde{W}}), \tilde{\varphi})$ is a completion of $(V, \langle \cdot, \cdot \rangle_V)$, then $(W, \langle \cdot, \cdot \rangle_W)$ is isomorphic to $(\tilde{W}, \langle \cdot, \cdot \rangle_{\tilde{W}})$ as inner product spaces.

When considering this context, we shall make abuses of language analogous to the ones in the case of normed spaces.

Consider the norms induced by inner products. The Hilbert space $(W, \langle \cdot, \cdot \rangle_W)$ admits the Banach space structure $(W, \|\cdot\|_W)$. This Banach space is isometric to the completion of $(V, \|\cdot\|_V)$. This shows that results concerning the Banach space structure of completions of normed spaces are still valid for the Hilbert spaces obtained as completions.

2.3 Manifolds

For details on this section, we refer the reader to [Lan88].

Definition

Given a Banach space X , a **topological manifold** M modeled on X is a Hausdorff topological space for which any $x \in M$ admits an open neighborhood which is homeomorphic to an open subset of X . From now on, let X be a Banach space and let M be a topological manifold modeled on X .

If $U \subset M$, $V \subset X$ are open subsets and $\varphi: U \rightarrow V$ is a homeomorphism, we call either φ or the pair (U, φ) a **local chart** on M . Two local charts (U_1, φ_1) , (U_2, φ_2) are **smoothly compatible** when $U_1 \cap U_2 = \emptyset$ or, if $U_1 \cap U_2 \neq \emptyset$,

$$\begin{cases} \varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2) \\ \varphi_1 \circ \varphi_2^{-1}: \varphi_2(U_1 \cap U_2) \rightarrow \varphi_1(U_1 \cap U_2) \end{cases} \text{ are smooth.}$$

(our notion of differentiability in a Banach space is given by the Fréchet derivative)

A collection $\mathcal{U}$ of smoothly compatible local charts (U, φ) such that $M = \bigcup_{(U, \varphi) \in \mathcal{U}} U$ is called a **smooth atlas**. One may define an equivalence relation $\sim$ between smooth atlases by $\mathcal{U}_1 \sim \mathcal{U}_2$ if, and only if, given $(U_1, \varphi_1) \in \mathcal{U}_1$ and $(U_2, \varphi_2) \in \mathcal{U}_2$, (U_1, φ_1) and (U_2, φ_2) are smoothly compatible. An equivalence class of smooth atlases is called a **smooth structure**. In such context, we can finally define

Definition 2.9. A **(smooth) manifold** M modeled on X is a topological manifold modeled on X endowed with a smooth structure.

Remark. X has a canonical manifold structure in which it is modeled on itself.

Most importantly, the smooth structure allows us to develop a notion of C^k maps between manifolds for any integer $0 \leq k < \infty$, or $k = \infty$: such map is of class C^k when its local expression is of class C^k . When we do not specify the regularity of a map between manifolds, it is supposed to be smooth.

The **dimension** of M is the dimension of X . In the next section, we detail hypotheses and the general notation that we adopt when we deal with finite-dimensional manifolds.

Analogously to the finite-dimensional setting, manifolds of arbitrary dimension admit vector bundles and vector fields. We shall briefly mention such objects along this text.

Submanifolds

Let M be a manifold modeled on X .

A subset $N \subset M$ is **locally closed** when any $x \in N$ admits a neighborhood $U \subset M$ such that $U \cap N$ is a closed subset of M .

Definition 2.10. A locally closed $N \subset M$ is **submanifold** of M when given $x \in N$,

1. there is a local chart (U, φ) on M for which $x \in U$, $\varphi: U \rightarrow U_1 \times U_2$, where $U_1 \subset X_1$ and $U_2 \subset X_2$ are open subsets of Banach spaces X_1, X_2 ;
2. for the local chart above, $\varphi(U \cap N) = X_1 \times \{x_2\}$ for a certain $x_2 \in U_2$.

In fact, the only kind of submanifold we mention along the text are Nehari manifolds. These are submanifolds of a Hilbert space which are obtained by the (infinite-dimensional) regular value theorem.

General notation

Unless mentioning otherwise, we denote

- M a manifold;
- d the derivative.
- $x \in M$ a generic point;
- X a vector field on M ;
- $C^k(M; N)$ the set of maps from M to N of class C^k ; ($N = \mathbb{R}$ when N is absent)
- $\mathcal{D}(M)$ the set of compactly-supported smooth functions from M to $\mathbb{R}$.

Specially when $\dim M = \infty$, we adopt the following conventions for any $J \in C^1(M)$:

- given $c \in \mathbb{R}$, $J^c = J^{-1}([-\infty, c]) = \{x \in M : J(x) \leq c\}$;
- given $x \in M$, $J(x)$ is called the **energy** of x ;
- a **ground-state** for J is an $x \in M$ for which $J(x) = \inf_M J$;
- given $x \in M$, the derivative of J at x is denoted $dJ_x: y \mapsto dJ_x[y]$;
- a **critical point** for J is an $x \in M$ for which $dJ_x = 0$;
- if $M = X$, we also denote the Fréchet derivative as d . In particular: if J is k times differentiable, the k^{th} derivative of J is denoted $d^k J$;
- for a positive integer k , manifolds $M_1, \dots, M_k$ and a differentiable

$$J: M_1 \times \dots \times M_k \rightarrow \mathbb{R},$$

we denote the partial derivative with respect to the j^{th} coordinate at

$$x = (x_1, \dots, x_k) \in M_1 \times \dots \times M_k$$

by

$$d_j J_x: T(M_j)_{x_j} \rightarrow \mathbb{R}.$$

2.4 Finite-dimensional manifolds

For a detailed account on finite-dimensional manifolds, we refer the reader to [Lee13].

We suppose that any finite-dimensional manifold is **second-countable**, i.e., its topology admits a countable basis. By **closed manifold**, we mean compact finite-dimensional connected manifold without boundary.

In the context of this section, we denote:

- M^n a manifold of positive integer dimension n ;
- d the exterior derivative;
- $(U, (x_1, \dots, x_n))$ a local chart on M centered at x ;
- $(E_1, \dots, E_n)$ a smooth frame on U ;
- $(\sigma^1, \dots, \sigma^n)$ its dual co-frame.

Partitions of unity

A set $\mathcal{A} \subset C^\infty(M)$ is said to be **locally finite** when any $x \in M$ admits a neighborhood $U \subset M$ such that

$$\left| \left\{ f \in \mathcal{A} : f|_U \neq 0 \right\} \right| < \infty.$$

A **partition of unity** for M is a locally finite collection $\mathcal{R} = \{\rho_r\} \subset C^\infty(M, [0, 1])$ such that $\sum_r \rho_r \equiv 1$.

Given $\mathcal{U} = \{(U_r, \varphi_r) : r \in L\}$ a smooth atlas for M , a partition of unity $\mathcal{R}$ is said to be **subordinate to $\mathcal{U}$** when

1. $\mathcal{R} = \{\rho_r : r \in L\}$;
2. $\forall r \in L$ $[\text{supp } \rho_r \subset U_r]$.

We denote:

- $\mathcal{U} = \{(U_r, \varphi_r)\}$ is a smooth atlas for M ;
- $\mathcal{R} = \{\rho_r\}$ is a partition of unity subordinate to $\mathcal{U}$;
- $V_r = \varphi_r(U_r) \subset \mathbb{R}^n$.

Definition 2.11. $\mathcal{U}$ is a **nice smooth atlas** for M when $\mathcal{U}$ is a finite smooth atlas and given $(U_r, \varphi_r) \in \mathcal{U}$, there is a local chart (W_r, ψ_r) such that $\overline{U_r} \subset W_r$ and $\varphi_r = \psi_r|_{U_r}$.

Vector bundles

When considering vector bundles,

- (E^k, M, π) denotes a smooth vector bundle, or simply **VB**, with a total space E , base space M and projection map $\pi : E \rightarrow M$ with fiber dimension $\dim E_x = k$.

When clear from the context, we shall refer to such VB as E or E^k ;

- The **space of sections** of E is denoted:

$$\Gamma(E) = \{s \in C^\infty(M; E) : \pi \circ s = id_M\};$$

- The **support** of $s \in \Gamma(E)$ is

$$\text{supp } s = \overline{\{p \in M : s(p) \neq 0 \in E_p\}};$$

- The **space of sections with compact support** is:

$$\Gamma_c(E) = \{s \in \Gamma(E) : \text{supp } s \subset M \text{ is compact}\}.$$

Tensor bundles

As for tensor bundles,

- $T_l^k M$ is the VB over M of (k, l) -**tensors**, or **tensors of type (k, l)** . Specifically, its fibers are $(T_l^k M)_x = (T M_x)^{k, l}$;
- As usual, we shall often refer to an $S \in \Gamma(T_l^k M)$ as a (k, l) -tensor.

We make the identifications

$$\begin{cases} C^\infty(M) = \Gamma(T_0^0 M) \\ \mathfrak{X}(M) = \Gamma(T_0^1 M) \\ \Omega^1(M) = \Gamma(T_1^0 M) \end{cases}.$$

Last of all, the next result will be extensively used : (2.18 in [War11])

Proposition 2.12. The $C^\infty(M)$ -module $\Gamma(T_l^k M)$ is isomorphic to the $C^\infty(M)$ -module of $C^\infty(M)$ -multilinear maps

$$\underbrace{\Omega^1(M) \times \dots \times \Omega^1(M)}_{k \text{ times}} \times \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{l \text{ times}} \rightarrow C^\infty(M).$$

Due to this result, we can (and will) look at a (k, l) -tensor as a $C^\infty(M)$ -multilinear map or as an element of $\Gamma(T_l^k M)$ whenever one or the other characterization is more convenient for a certain occasion.

2.5 Riemannian manifolds

Whenever we mention a Riemannian manifold or a Riemannian metric, we refer to a finite-dimensional manifold. For details on Riemannian manifolds, we recommend [dC15] and [Pet18].

Consider

- M^n a finite-dimensional manifold;
- g a Riemannian metric on M .

Considering (M, g) ,

- ∇ denotes its **Levi-Civita connection**;
- The **divergence** of X is defined

$$\operatorname{div} X = \operatorname{tr} \nabla X;$$

(this definition is generalized for symmetric 2-tensors in chapter 3)

- The **Laplacian**, or **Laplace-Beltrami operator**, is defined for $f \in C^\infty(M)$:

$$\Delta f = \operatorname{div} \operatorname{grad} f;$$

- We adopt the following as the **Riemannian curvature tensor**:

$$R(X, Y)Z = \nabla_{[X, Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z;$$

- We adopt the following as the **Ricci tensor**:

$$\operatorname{Ric}(X, Z) = \operatorname{tr} g(R(X, \cdot)Z, \cdot);$$

- The **scalar curvature** is:

$$\operatorname{scal} = \operatorname{tr} \operatorname{Ric}.$$

Recall that given $u, v \in TM_x$ which are linearly independent, the **sectional curvature** of

$$\operatorname{span} \{u, v\} \subset TM_x$$

is

$$K(u, v) = \frac{g(R(u, v)u, v)}{g(u, u)g(v, v) - g(u, v)^2}.$$

In chapter 5, it will be important to specify the metric to which we refer when considering the operations above. In these occasions, we shall add a subscript with the metric in question next to each operator.

Metric space structure

Denote $I = [0, 1]$ and suppose M is connected.

Given $\gamma \in C^\infty(I; M)$, denote

$$L(\gamma) = \int_I \sqrt{g(\gamma'(t), \gamma'(t))} dt.$$

Given $x, y \in M$, consider the **set of curves which connect** x to y ,

$$\Omega_{x, y} = \{\gamma \in C^\infty(I; M) : \gamma(0) = x, \gamma(1) = y\}.$$

One may prove that $d(x, y) = \inf_{\Omega_{x, y}} L$ is a distance function on M . Furthermore, the topology induced on M by d is the same topology which M had as a manifold.

A **geodesic ball** is an open ball $B_g(x, r)$ on M obtained when we consider such metric space structure.

Geodesics, the exponential map and normal neighborhoods

Let $I \subset \mathbb{R}$ be an open interval. A **geodesic** is a curve $\gamma \in C^\infty(I; M)$ for which $\nabla_{\gamma'} \gamma' \equiv 0$. Given an initial condition, geodesics are uniquely defined:

Proposition 2.13. Given $x \in M$, there are an open set $U \subset M$, $\delta > 0$, $\epsilon > 0$ and a smooth map $\gamma: (-\delta, \delta) \times TU \rightarrow M$ with the property: given $v \in TU$ with $|v|_g < \epsilon$, $\gamma(\cdot, v)$ is the only geodesic on M which satisfies the initial condition $\gamma'(0) = v \in TM_{\pi(v)}$.

Such result permits us to define the **exponential map**:

Definition 2.14. In the context of the last proposition, we define $\exp_x: v \mapsto \gamma(1, v)$. Note that $\exp_x$ is defined on the open ball $B(0, \delta\epsilon) \subset TM_x$.

In fact, the exponential map is a local diffeomorphism:

Proposition 2.15. There is $\epsilon > 0$ such that $\exp_x: B(0, \epsilon) \rightarrow M$ is a diffeomorphism.

If $U \subset M$ is a neighborhood of x for which the map $(\exp_x)^{-1}(U) \rightarrow U$ given by $v \mapsto \exp_x(v)$ is a diffeomorphism, U is called a **normal neighborhood** of x . A geodesic ball $B_g(x, \epsilon)$ which is a normal neighborhood of x is called a **normal ball**.

Given $x, y \in M$, a curve γ which connects x to y , is said to be **minimizing** when $L(\gamma) = d(x, y)$. In fact, any minimizing curve is a geodesic. A corollary of that is the equality $B_g(x, \epsilon) = \exp_x(B(0, \epsilon))$ for sufficiently small $\epsilon > 0$.

Segments of geodesics which begin at the center of a normal ball and whose image resides in such ball are (the unique) minimizing curves:

Proposition 2.16. Let $B_g(x, \epsilon)$ be a normal ball, $y \in B_g(x, \epsilon)$ and $\gamma: [0, 1] \rightarrow B_g(x, \epsilon)$ be a geodesic which connects x to y . Then given a curve $c: [0, 1] \rightarrow M$ which connects x to y , we have $L(\gamma) \leq L(c)$. Furthermore, if $L(c) = L(\gamma)$ then $\gamma([0, 1]) = c([0, 1])$.

In a complete Riemannian manifold, the **injectivity radius** i_M is a nonnegative number for which $\epsilon \in]0, i_M[$ implies that given $x \in M$, $\exp_x: B(0, \epsilon) \rightarrow M$ is a diffeomorphism with its image. In particular, any compact Riemannian manifold has positive injectivity radius.

Volume comparison

The following volume comparison result is important for section 5.6.5: (sec. 1.4, thm. 1.1 in [Heb00])

Theorem 2.17 (Bishop-Gromov volume comparison). Let (M, g) be a smooth, complete Riemannian n -manifold whose Ricci curvature satisfies $\text{Ric} \geq (n-1)kg$ as bilinear forms, for some k real. Then, for any $0 < r < R$ and any $x \in M$,

$$\frac{\mu_g(B_x(R))}{\mu_g(B_x(r))} \leq \frac{V_k(R)}{V_k(r)}$$

where $\mu_g(B_x(t))$ denotes the volume of the geodesic ball of center x and radius t , and where $V_k(t)$ denotes the volume of a ball of radius t in the complete simply connected Riemannian n -manifold of constant sectional curvature k . In particular, for any $r > 0$ and any $x \in M$, $\mu_g(B_x(r)) \leq V_k(r)$.

2.6 Variational methods

Ekeland's variational principle

(thm. 5.1 in [Str10])

Theorem 2.18. Let X be a complete metric space with metric d , and let $J : X \rightarrow \mathbb{R} \cup \{\infty\}$ be lower semi-continuous, bounded from below and $\neq \infty$. Then given $\alpha, \beta > 0$ and $x \in X$ with $J(x) \leq \inf_X J + \alpha$, there is an element $y \in X$ strictly minimizing the functional $J_y(z) = J(z) + \alpha d(z, y)/\beta$. Moreover, we have $J(y) \leq J(x)$, $d(x, y) \leq \beta$.

Natural constraints and the Nehari manifold

Consider

- X a Hilbert space;
- M a submanifold of X ;
- $J \in C^1(X; \mathbb{R})$.

Denote $\mathcal{B}_\rho$ an open ball on X centered at 0 with radius $\rho \geq 0$.

Definition 2.19 (Natural constraint). Let $\tilde{J} = J|_M$. M is a **natural constraint** for J when

$$\forall x \in M \quad \left(dJ_x = 0 \iff d\tilde{J}_x = 0 \right).$$

If J is of class C^2 and satisfies certain conditions, the **Nehari manifold**

$$N = \{u \in X \setminus \{0\} : dJ_u[u] = 0\}$$

is an example of natural constraint. More precisely: (prop. 6.7 in [AM07])

Proposition 2.20. Suppose J is of class C^2 and $N \neq \emptyset$. Assume that $\mathcal{B}_\rho \cap N = \emptyset$ for a certain $\rho > 0$ and that given $u \in N$, $d^2 J_u[u, u] \neq 0$. Then N is a natural constraint for J .

2.7 Algebraic topology

Consider the topological spaces X, Y .

We denote $H_k(X, Y)$ the k^{th} homology group of the couple (X, Y) with coefficients in some field.

Definition 2.21. The **Poincaré series** $P_t(X, Y)$ is defined as

$$P_t(X, Y) = \sum_k \dim H_k(X, Y) t^k.$$

If there is $k_0 \in \mathbb{N}$ for which $k > k_0$ implies $\dim H_k(X, Y) = 0$, $P_t(X, Y)$ is said to be a **Poincaré polynomial**. (as is the case when X is a closed manifold)

Remark. A basic result in Homology is that $H_k(X, \emptyset) \simeq H_k(X)$, so we set

$$P_t(X) = P_t(X, \emptyset).$$

Definition 2.22. Two continuous maps $f: X \rightarrow Y$ and $g: X \rightarrow Y$ are **homotopic** when there exists a continuous $H: [0, 1] \times X \rightarrow Y$ such that $H(0, \cdot) = f$ and $H(1, \cdot) = g$. In such case, we write $f \simeq g$.

Remark. If $f: X \rightarrow Y$ and $g: Y \rightarrow X$ are continuous maps and $g \circ f$ is homotopic to id_X , then $P_t(Y) = P_t(X) + Z(t)$, where $Z(t)$ is a polynomial with non-negative coefficients.

2.8 Morse theory

For an account on Morse theory, we refer the reader to part II in [PT88].

Consider

- M a manifold modeled on a Hilbert space X ;
- $J \in C^1(M; \mathbb{R})$.

Definition 2.23. J satisfies the **Palais-Smale condition** if any sequence (x_n) in M for which $\{J(x_n)\} \subset \mathbb{R}$ is bounded and $dJ_{x_n} \rightarrow 0$ admits a convergent subsequence.

From now on, suppose that J is of class C^2 . If $x \in M$ is a critical point of J , we can define a symmetric bilinear form

$$\text{Hess}(J)_x: TM_x \times TM_x \rightarrow \mathbb{R}$$

such that given $u, v \in TM_x$, $\text{Hess}(J)_x(u, v) = X_x(Y(J))$, where X, Y are any vector fields in M for which $X_x = u, Y_x = v$.

Definition 2.24. A critical point $x \in M$ of J is said to be **nondegenerate** when $\text{Hess}(J)_x$ is a nondegenerate bilinear form. J is said to be a **Morse function** when every critical point of J is nondegenerate.

Definition 2.25. If $x \in X$ is an isolated critical point of J with $J(x) = c$, the **polynomial Morse index** $i_t(x)$ is the series

$$i_t(x) = \sum_k \dim H_k(J^c, J^c \setminus \{x\}) t^k.$$

Definition 2.26. If $x \in X$ is an isolated critical point of J , the **numerical Morse index** $\mu(x)$ is the dimension of the maximal subspace on which $\text{Hess}(J)_x$ is negative definite.

Remark. If x is a nondegenerate critical point of J , then $i_t(x) = t^{\mu(x)}$.

The following is a corollary of Morse lemma: (9.4.8 in [PT88])

Lemma 2.27. If J is a Morse function which satisfies the Palais-Smale condition, then given a finite interval $[a, b] \subset \mathbb{R}$, the set of critical points x of J with $J(x) \in [a, b]$ is finite.

Chapter 3

Operations on tensors

Contents

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It is essential to get acquainted with tensor concepts such as the application of the metric g to arbitrary tensor fields to deal with Sobolev spaces in Riemannian manifolds. As such, this chapter aims to we recall some tensor concepts of Riemannian geometry.

In each section (with the sole exception of the covariant differentiation), we first show that these concepts stem from considerations on (multi)linear algebra done on a fixed vector field V and on the tensor products of V and V^* . Following these considerations, we use the Riemannian metric g to extend these concepts to the tensor bundle $T_l^k M$. This approach allows us to always obtain global objects. Therefore, so we only use local coordinates when necessary to prove properties of these objects or to provide examples.

This exposition is loosely based on the one found in [Lee09]. One can find further information about the results of this section on [Pet18].

3.1 Norm and inner product of tensor fields

Linear algebraic considerations

Let us observe that $h: V \times V \rightarrow \mathbb{R}$ induces a linear isomorphism $h^\flat: V \rightarrow V^*$ given by $h(v) = h(v, \cdot)$. Denote $h^\sharp = (h^\flat)^{-1}$. h then induces an inner product $h^*: V^* \times V^* \rightarrow \mathbb{R}$ given by

$$h^*(\alpha, \beta) = h\left(h^\sharp(\alpha), h^\sharp(\beta)\right).$$

$h^\flat$ is a linear isomorphism, so the definition above makes (V, h) and (V^*, h^*) isomorphic as inner product spaces. Considering corollary 2.2, we have:

1. if $k > 0$, h induces an inner product h^k on $\bigotimes^k V$;

2. if $l > 0$, h^* induces an inner product $(h^*)^l = h_l$ on $\bigotimes^l V^*$.

Whenever $k + l > 0$, we can use the method above to put an inner product on $V^{k,l}$, which we shall denote by h_l^k .

Application to tensors

Suppose $k + l > 0$. Taking into account the last observations on linear algebra, we can endow each fiber $(T_l^k M)_x$ with an inner product $(g_x)_l^k$. In fact, g induces a bundle metric g_l^k on $T_l^k M$. The context will tell the type of tensor we are considering. Therefore, we shall often denote g_l^k simply by g .

3.2 Musical isomorphisms and type change

Linear algebraic considerations

By lemma 2.1, $h^\flat$ induces a unique linear map $V^{k+1,l} \rightarrow V^{k,l+1}$ for which given $v \in \bigotimes^k V$, $u \in V$, $\alpha \in \bigotimes^l V^*$;

$$v \otimes u \otimes \alpha \mapsto v \otimes h^\flat(u) \otimes \alpha.$$

We can clearly apply an analogous procedure with $h^\sharp$ to obtain a unique linear map $V^{k,l+1} \rightarrow V^{k+1,l}$. One may check these maps are actually linear isomorphisms. If we apply such isomorphisms repeatedly, we see that

$$\bigotimes^{k+l} V \simeq V^{k,l} \simeq V^{r,s} \simeq \bigotimes^{k+l} V^*$$

for any nonnegative integers r, s such that $r + s = k + l$.

Application to tensors

The inner product space (TM_x, g_x) satisfies the conditions considered in the last subsection, so we have the analogous isomorphisms

$$(g_x)^\flat: TM_x \rightarrow (TM_x)^* \qquad (g_x)^\sharp: (TM_x)^* \rightarrow TM_x$$

We are considering a generic point in M . Therefore, we obtain VB isomorphisms

$$\begin{aligned} \flat: TM &\rightarrow T^*M & \sharp: T^*M &\rightarrow TM \\ v &\mapsto (g_{\pi(v)})^\flat(v) & \alpha &\mapsto (g_{\pi^*(\alpha)})^\sharp(\alpha) \end{aligned}$$

The map $\flat$ is called **flat**, and $\sharp$ is called **sharp**. It might not seem so obvious why these maps are smooth. Their local expressions shall promptly convince us: if $v = v^i (E_i)_{\pi(v)} \in TM$, then

$$\flat(v) = v^i g_{ij} (\pi(v)) (\sigma^j)_{\pi(v)}.$$

Analogously, if $\alpha = \alpha_i (\sigma^i)_{\pi^*(\alpha)} \in T^*M$, then

$$\sharp(\alpha) = \alpha_i g^{ij} (\pi^*(\alpha)) (E_j)_{\pi^*(\alpha)}.$$

It is easy to see that these maps are VB isomorphisms, the so-called **musical isomorphisms**. We can easily generalize this idea to obtain a 1-form from any $X \in \mathfrak{X}(M)$: we consider the **flat of** X ,

$$\begin{aligned} X^\flat &= \flat \circ X: TM \rightarrow \mathbb{R} \\ u &\mapsto g(X_{\pi(u)}, u). \end{aligned}$$

Analogously, we can obtain a vector field from any $\alpha \in \Omega^1(M)$: we consider the **sharp of** α ,

$$\begin{aligned} \alpha^\sharp &= \sharp \circ \alpha: M \rightarrow TM \\ x &\mapsto (g_x)^\sharp(\alpha_x). \end{aligned}$$

In fact, the maps below are isomorphisms of $C^\infty(M)$ -modules

$$\begin{aligned} \flat^*: \mathfrak{X}(M) &\rightarrow \Omega^1(M) & \sharp^*: \Omega^1(M) &\rightarrow \mathfrak{X}(M) \\ X &\mapsto X^\flat & \alpha &\mapsto \alpha^\sharp \end{aligned}$$

Up to now, we have generalized the idea from linear algebra of using an inner product to obtain a linear isomorphism. Its generalization is using the metric tensor g to obtain a $C^\infty(M)$ -module isomorphisms $\mathfrak{X}(M) \rightarrow \Omega^1(M)$. Due to lemma 2.1, this idea may be further generalized to change the type of tensors.

Let r, s be nonnegative integers such that $r + s = k + l$. The conditions of the last section are verified, so we have

$$(TM_x)^{k+l} \simeq (T_l^k M)_x \simeq (T_s^r M)_x \simeq (T^* M_x)^{k+l}.$$

Generalizing in an analogous manner as before, we even have the VB isomorphisms

$$T_0^{k+l} M \simeq T_l^k M \simeq T_s^r M \simeq T_{k+l}^0 M.$$

This consideration permits us to obtain a C^∞ -module isomorphism

$$\Gamma(T_l^{k+1} M) \rightarrow \Gamma(T_{l+1}^k M).$$

Explicitly, if $R \in \Gamma(T_l^{k+1} M)$ is locally written as

$$R \Big|_U = R_{j_1 \dots j_l}^{i_1 \dots i_k i_{k+1}} \bigotimes_{q=1}^{k+1} E_{i_q} \otimes \bigotimes_{r=1}^l \sigma^{j_r},$$

then its image by the isomorphism has local expression

$$R_{j_1 \dots j_l}^{i_1 \dots i_k i_{k+1}} g_{i_{k+1} j_{l+1}} \bigotimes_{q=1}^k E_{i_q} \otimes \bigotimes_{r=1}^{l+1} \sigma^{j_r}.$$

Note that this map is the one induced by the flat of the component E_{k+1} . Furthermore, we could have placed σ^{l+1} anywhere after the contravariant components, i.e., the isomorphism shown above is not the only one possible.

Such operation of transforming a $(k+1, l)$ -tensor into a $(k, l+1)$ -tensor is called **lowering an index**. A map $\Gamma(T_{l+1}^k M) \rightarrow \Gamma(T_l^{k+1} M)$ may be obtained in an analogous manner, and such operation is called **raising an index**. Any combination of those operations is called a **type change**.

Due to type changes, we can mistake (k, l) -tensors for (r, s) -tensors when $r + s = k + l$. More importantly, our construction is such that we have the following lemma:

Lemma 3.1. Let S_1, S_2 be (k, l) -tensors. Let $\tilde{S}_1, \tilde{S}_2$ be (r, s) -tensors obtained respectively from S_1, S_2 by applying the same type change. Then $g_l^k(S_1, S_2) = g_s^r(\tilde{S}_1, \tilde{S}_2)$.

3.3 Contractions

Linear algebraic considerations

Consider the bilinear form $\varphi: V \times V^* \rightarrow \mathbb{R}$ given by $\varphi(v, \alpha) = \alpha(v)$. Due to lemma 2.1, there exists a unique linear form $\bar{\varphi}: V \otimes V^* \rightarrow \mathbb{R}$ such that given $v \in V$ and $\alpha \in V^*$, $\bar{\varphi}(v \otimes \alpha) = \alpha(v)$. Suppose that k, l are positive integers and $1 \leq r \leq k, 1 \leq s \leq l$. We can consider a similar bilinear function

$$\begin{aligned} \varphi_s^r: V^k \times (V^*)^l &\rightarrow V^{k-1, l-1} \\ (v, \alpha) &\mapsto \alpha^s(v_r) \bigotimes_{\substack{1 \leq i \leq k \\ i \neq r}} v_i \otimes \bigotimes_{\substack{1 \leq j \leq l \\ j \neq s}} \alpha^j, \end{aligned}$$

where $v = (v_1, \dots, v_k) \in V^k$ and $\alpha = (\alpha^1, \dots, \alpha^l) \in (V^*)^l$. Once again, there exists a unique linear map $\bar{\varphi}_s^r: V^{k, l} \rightarrow V^{k-1, l-1}$ such that given $v = (v_1, \dots, v_k) \in V^k$ and $\alpha = (\alpha^1, \dots, \alpha^l) \in (V^*)^l$,

$$\bar{\varphi}_s^r \left(\bigotimes_{i=1}^k v_i \otimes \bigotimes_{j=1}^l \alpha^j \right) = \alpha^s(v_r) \bigotimes_{\substack{1 \leq i \leq k \\ i \neq r}} v_i \otimes \bigotimes_{\substack{1 \leq j \leq l \\ j \neq s}} \alpha^j.$$

Application to tensor fields

$(T_1^1 M)_x = TM_x \otimes (TM_x)^*$, so we have a linear form $\bar{\varphi}_x: (T_1^1 M)_x \rightarrow \mathbb{R}$ as described in the last subsection in each fiber of $T_1^1 M$. This allows us to define the **trace**

$$\text{tr}: \Gamma(T_1^1 M) \rightarrow C^\infty(M),$$

where given $S \in \Gamma(T_1^1 M)$,

$$\begin{aligned} \text{tr } S: M &\rightarrow \mathbb{R} \\ x &\mapsto \overline{\varphi_x}(S_x). \end{aligned}$$

Unfortunately, this abstract definition does not make it clear that the function $\text{tr } S$ we obtain is smooth. However, this is easy to verify when we consider its local expression:

$$S \Big|_U = S_j^i E_i \otimes \sigma^j \implies \text{tr } S \Big|_U = S_i^i.$$

tr is easily extended to symmetric $(0, 2)$ or $(2, 0)$ -tensors. Let S be one such tensor. S is symmetric so any type change of S to a $(1, 1)$ -tensor results in the same tensor. $\text{tr } S$ is then defined as the trace of S as a $(1, 1)$ -tensor.

Example 3.2 (Scalar curvature). If we write the Ricci tensor locally as

$$\text{Ric} \Big|_U = \text{Ric}_{ab} \sigma^a \otimes \sigma^b,$$

a type change of it to a $(1, 1)$ -tensor is rewritten

$$\begin{aligned} \text{Ric} \Big|_U &= \text{Ric}_{ab} g^{ac} E_c \otimes \sigma^b \\ &= \text{Ric}_b^c E_c \otimes \sigma^b. \end{aligned}$$

That's what you get:

$$\text{scal} \Big|_U = \text{Ric}_i^i = \text{Ric}_{ac} g^{ac}.$$

Let k, l be positive integers and $1 \leq r \leq k, 1 \leq s \leq l$. The **contraction** C_s^r is a map

$$C_s^r: \Gamma(T_l^k M) \rightarrow \Gamma(T_{l-1}^{k-1} M),$$

where given $S \in \Gamma(T_l^k M)$,

$$\begin{aligned} C_s^r S: M &\rightarrow T_{l-1}^{k-1} M \\ x &\mapsto \overline{(\varphi_s^r)_x}(S_x). \end{aligned}$$

3.4 Covariant differentiation

In this section, we momentarily ignore our convention of considering the fiber $(T_l^k M)_x$ with the contravariant part written first. We may see an affine connection ∇ on M as an $\mathbb{R}$ -linear map

$$\nabla: \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) \otimes \Omega^1(M) = \Gamma(T_1^1 M)$$

which takes $X \in \mathfrak{X}(M)$ to $\nabla X: \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ which satisfies Leibniz' rule: given $f \in C^\infty(M)$, $\nabla(fX) = f\nabla X + \nabla X \otimes df$. We denote $\nabla X(Y) = \nabla_Y X$.

In other words, ∇ is $\mathbb{R}$ -linear map which takes the vector field, or $(1, 0)$ -tensor, X to the $(1, 1)$ -tensor ∇X . Inspired by this, we wish to extend ∇ to take a (k, l) -tensor to a $(k, l+1)$ -tensor. Furthermore, we do not want to lose the tensorial generalization of

Leibniz' rule: $\nabla(R \otimes S) = R \otimes (\nabla S) + (\nabla R) \otimes S$. Such extension of ∇ will be called the **covariant derivative**.

We can easily define $\nabla = d$ for functions $f \in C^\infty(M)$. Next, suppose $k+l > 0$. We extend ∇ to differentiate $S \in \Gamma(T_l^k M)$ as follows: let ∇S be such that given $\alpha^1, \dots, \alpha^k \in \Omega^1(M)$ and $X_1, \dots, X_{l+1} \in \mathfrak{X}(M)$,

$$\begin{aligned} \nabla_{X_1} S(\alpha^1, \dots, \alpha^k, X_2, \dots, X_{l+1}) &= \nabla_{X_1} \left(S(\alpha^1, \dots, \alpha^k, X_2, \dots, X_{l+1}) \right) + \\ &\quad - \sum_{i=1}^k S(\alpha^1, \dots, \alpha^{i-1}, \nabla_{X_1} \alpha^i, \alpha^{i+1}, \dots, \alpha^k, X_2, \dots, X_{l+1}) + \\ &\quad - \sum_{j=2}^{l+1} S(\alpha^1, \dots, \alpha^k, X_2, \dots, X_{j-1}, \nabla_{X_1} X_j, X_{j+1}, \dots, X_{l+1}) \end{aligned}$$

Note that this definition was chosen precisely so that the Leibniz' rule is respected.

Chapter 4

Functional framework

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In this chapter, we develop the functional framework under which we shall work on the next chapter. Overall, most of the constructions that we adopt are heavily influenced by section 2.1 of [Nic13].

The first section aims to give an overview of the main results and constructions which permit us to integrate real-valued functions on (M, g) . We adopt the measure-theoretic point of view: to integrate functions defined on M , we consider the measurable space $(M, \mathcal{B}(M))$, where $\mathcal{B}(M)$ is the sigma-algebra of the Borel sets in M . In its turn, the Riemannian metric g induces a measure μ_g , so we obtain a measure space $(M, \mathcal{B}(M), \mu_g)$ in which we can use standard definitions and results from measure and integration theory. We note that this approach is equivalent to the one with volume densities. A reference we recommend for such section is chp. 10 in [LS14].

In the second section, we define the L^p -spaces of tensors. The third section aims to introduce the concepts of differential operators and formal adjoints, loosely following section 4.C.2 in [GHL08].

We introduce the Sobolev spaces $W^{k,p}(M, g)$ in the fourth section, presenting its most common definition, as found in [Heb00] or [Aub11]. Our next step is a comment on the concept of weak covariant derivatives – an analogue to the weak derivatives of the Sobolev spaces $W^{k,p}(\mathbb{R}^n)$. We end this section by explaining the concept of weak solutions of PDEs.

In the fifth section, we collect important results on Sobolev spaces which hold on closed (Riemannian) manifolds. The first result is the Sobolev embedding theorem. Later, we show that the L^p -spaces of tensors do not depend on the metric. This is followed by the introduction of the uniform Sobolev spaces $\tilde{W}^{k,p}(M)$ – an alternative definition which does not depend on a metric. We end this section by showing that the Sobolev spaces $W^{k,p}(M, g)$ do not depend on the metric g .

4.1 Integration and L^p -spaces

4.1.1 The measure space $(M, \mathcal{B}(M), \mu_g)$

Recall the following result: (thm. 7.2 in [Fol11])

Theorem 4.1 (Riesz representation theorem). Let X be a locally compact Hausdorff space and I be a positive linear functional on $C_c(X)$, positive meaning that if $f \geq 0$ then $I(f) \geq 0$. Then there exists a unique Radon measure μ on X such that

$$\forall f \in C_c(X) \quad \left[I(f) = \int f \, d\mu \right].$$

Fix $\mathcal{U} = \{(U_r, \varphi_r)\}$ a finite smooth atlas for M and fix $\mathcal{R} = \{\rho_r\}$ a partition of unity subordinate to $\mathcal{U}$. Denote $V_r = \varphi_r(U_r) \subset \mathbb{R}^n$. g induces the positive linear functional

$$f \mapsto \sum_r \int_{V_r} \left[\rho_r f \sqrt{|\det(g_{ij})|} \right] \circ \varphi_r^{-1} \, dx_1 \dots dx_n$$

on $C_c(M)$. Using the theorem above, this positive linear functional induces a measure μ_g in the Borel sets of M . It is worth pointing out that the divergence theorem holds in this context:

Theorem 4.2 (Divergence theorem). If $X \in \mathfrak{X}_c(M)$, then $\int_M \operatorname{div} X \, d\mu_g = 0$.

4.1.2 The spaces $L^p(M, g)$

The basic theory of L^p -spaces can be used in the measure space $(M, \mathcal{B}(M), \mu_g)$. To highlight the fact that we consider a metric μ_g induced by g , we shall denote such normed spaces as $L^p(M, g)$. Recall the following abstract result: (prop. 3.41 in [Aub11])

Theorem 4.3. Let X be a locally compact separable metric space and μ a Radon measure on X . Then $C_c(X)$ is dense in $L^p(X)$.

Due to theorem 4.3, we see that $C_c(M)$ is dense in $L^p(M, g)$. This allows us to describe the Banach space $L^p(M, g)$ in an equivalent way:

Corollary 4.4. $L^p(M, g)$ is the closure of $C_c(M)$ with respect to the norm $\|\cdot\|_{L^p(M, g)}$.

4.2 L^p -spaces of tensors

4.2.1 The spaces $L^p(T_l^k M, g)$

In section 3.1, we have explained how to use the Riemannian metric g to induce a bundle metric g_l^k on $T_l^k M$. The restriction of the map below to each fiber $(T_l^k M)_x$ yields a norm:

$$\begin{aligned} |\cdot|_g : T_l^k M &\rightarrow [0, \infty[\\ v &\mapsto \sqrt{g_l^k(v, v)}. \end{aligned}$$

Consider the vector space of the **p -integrable tensors** of $T_l^k M$:

$$\Gamma^p(T_l^k M, g) = \left\{ S \in \Gamma(T_l^k M) : \int_M |S|_g^p d\mu_g < \infty \right\}.$$

We may consider the norm $\|S\|_{g,p} = (\int_M |S|_g^p d\mu_g)^{1/p}$ on $\Gamma^p(T_l^k M, g)$. In this context, we define

Definition 4.5 (L^p -spaces of tensors). $L^p(T_l^k M, g)$ is the closure of $\Gamma^p(T_l^k M, g)$ with respect to $\|\cdot\|_{g,p}$.

4.2.2 The Hilbert space $L^2(T_l^k M, g)$

The case $p = 2$ deserves special attention. The function below is clearly an inner product on $\Gamma^2(T_l^k M, g)$:

$$(S_1, S_2) \mapsto \langle S_1, S_2 \rangle_g = \int_M g_l^k(S_1, S_2) d\mu_g.$$

As remarked in section 2.2, $(\Gamma^2(T_l^k M, g), \langle \cdot, \cdot \rangle_g)$ admits a Hilbert space completion

$$(L^2(T_l^k M, g), \langle \cdot, \cdot \rangle_g).$$

Consider the bilinear functional below

$$\begin{aligned} \Gamma^2(T_l^k M, g) \times \Gamma^2(T_l^k M, g) &\rightarrow L^1(M, g) \\ (S_1, S_2) &\mapsto g_l^k(S_1, S_2). \end{aligned}$$

Due to Cauchy-Schwarz, this functional is continuous. Considering corollary 2.7, we extend it to a continuous bilinear functional

$$L^2(T_l^k M, g) \times L^2(T_l^k M, g) \rightarrow L^1(M, g).$$

We can then see that there is no abuse of language in the expression below

$$\begin{aligned} L^2(T_l^k M, g) \times L^2(T_l^k M, g) &\rightarrow \mathbb{R} \\ (S_1, S_2) &\mapsto \langle S_1, S_2 \rangle_g = \int_M g_l^k(S_1, S_2) d\mu_g. \end{aligned}$$

4.3 Differential operators and formal adjoints

Consider

- a, b non-negative integers;
- $T_l^k M, T_b^a M$ tensor bundles over M .

Definition 4.6. A **differential operator** from $T_l^k M$ to $T_b^a M$ is a linear map

$$P: \Gamma(T_l^k M) \rightarrow \Gamma(T_b^a M)$$

such that given $s \in \Gamma(T_l^k M)$, $\text{supp}(Ps) \subset \text{supp } s$.

Consider two differential operators $P: \Gamma(T_l^k M) \rightarrow \Gamma(T_b^a M)$ and $Q: \Gamma(T_b^a M) \rightarrow \Gamma(T_l^k M)$. P and Q are said to be **formally adjoint** when given $s_1 \in \Gamma_c(T_l^k M)$, $s_2 \in \Gamma_c(T_b^a M)$ we have $\langle Ps_1, s_2 \rangle_g = \langle s_1, Qs_2 \rangle_g$. It is easy to prove that when P admits a formal adjoint, it is unique. We shall then denote this formal adjoint by P^* .

Example 4.7. The **codifferential** $\delta: \Omega_c^1(M) \rightarrow \mathcal{D}(M)$ is the formal adjoint of d restricted to $\mathcal{D}(M)$. Let us compute how δ operates. Let ω be a compactly supported one-form and $f \in \mathcal{D}(M)$. By theorem 4.2:

$$\begin{aligned} \langle \omega, df \rangle_g &= \int_M g(\omega, df) d\mu_g \\ &= - \int_M f \operatorname{div} \omega d\mu_g. \end{aligned}$$

Therefore,

$$\delta\omega = -\operatorname{div} \omega.$$

Example 4.8. If S is a symmetric $(0, 2)$ -tensor, it is easy to give a meaningful definition of δS . Denote $\tilde{S}(X, Y, Z) = (\nabla_X S)(Y, Z)$.

Perform the type change which takes the first covariant coordinate of $\tilde{S}$ and transforms it into a contravariant coordinate so we obtain a $(1, 2)$ -tensor $\hat{S}$. S is symmetric, so $C_1^1 \hat{S} = C_2^1 \hat{S}$. We then define δS as the 1-form $C_1^1 \hat{S} = C_2^1 \hat{S}$.

4.4 Sobolev spaces

4.4.1 The spaces $W^{k,p}(M, g)$

Define

$$C^{k,p}(M, g) = \left\{ u \in C^\infty(M) : \forall l = 0, \dots, k \int_M |\nabla^l u|_g^p d\mu_g < \infty \right\}.$$

We denote a generic element by $u \in C^{k,p}(M, g)$. We endow this vector space with the **Sobolev norm**:

$$\|u\|_{k,p,g} = \left(\sum_{l=0}^k \int_M |\nabla^l u|_g^p d\mu_g \right)^{1/p}.$$

It is now possible to define:

Definition 4.9 (Sobolev spaces). $W^{k,p}(M, g)$ is the completion of $C^{k,p}(M, g)$ with respect to the norm $\|\cdot\|_{k,p,g}$.

We denote the Hilbert space $W^{k,2}(M, g)$ by $H^k(M, g)$.

4.4.2 Weak covariant derivatives

Suppose that

- $k \geq 1$;
- $1 \leq l \leq k$.

The Sobolev space $W^{k,p}(\mathbb{R}^n)$ is a subspace of $L^p(\mathbb{R}^n)$ whose elements admit weak derivatives. Let us briefly recall the definition of weak derivatives in $\mathbb{R}^n$:

Definition 4.10. Let $u \in W^{k,p}(\mathbb{R}^n)$. The functions $v_1, \dots, v_n \in L^p(\mathbb{R}^n)$ are **weak derivatives** of u when:

$$\forall i = 1, \dots, n \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n) \quad \int_{\mathbb{R}^n} u \frac{\partial \varphi}{\partial x_i} dx = - \int_{\mathbb{R}^n} v_i \varphi dx. \quad (4.1)$$

The Sobolev spaces $W^{k,p}(M, g)$ provide the appropriate setting in which one can consider **weak covariant derivatives**, which are an intrinsic equivalent of the weak derivatives on $W^{k,p}(\mathbb{R}^n)$. Let us explain how this construction is done.

$\|\cdot\|_{k,p,g}$ ensures that the linear operator $\nabla_g^l: C^{k,p}(M, g) \rightarrow L^p(T_l^0 M, g)$ is continuous. By lemma 2.4, it admits a unique linear continuous extension $\nabla_g^l: W^{k,p}(M, g) \rightarrow L^p(T_l^0 M, g)$. In particular, we have the following equality analogous to eq. (4.1):

Corollary 4.11. Let $u \in W^{k,p}(M)$. Then

$$\forall S \in \Gamma_c(T_l^0 M) \quad \left[\int_M g(\nabla^l u, S) d\mu_g = (-1)^l \int_M g(u, (\nabla^*)^l S) d\mu_g \right],$$

where ∇^* is the formal adjoint of ∇ .

Proof. Fix $S \in \Gamma_c(T_l^0 M)$. Let $1/p + 1/p' = 1$. Consider the linear functional

$$\begin{aligned} & C^{k,p}(M, g) \rightarrow \mathbb{R} \\ & u \mapsto \int_M g(\nabla^l u, S) d\mu_g. \end{aligned}$$

We have:

$$\begin{aligned}
 \left| \int_M g(\nabla^l u, S) \, d\mu_g \right| &\leq \int_M |g(\nabla^l u, S)| \, d\mu_g \\
 &\leq \int_M |\nabla^l u|_g |S|_g \, d\mu_g \\
 &\leq \|\nabla^l u\|_{L^p(T_l^0 M, g)} \|S\|_{L^{p'}(T_l^0 M, g)}. \quad (\text{by Hölder})
 \end{aligned}$$

Therefore, this linear functional is continuous. Now, consider the linear functional

$$\begin{aligned}
 C^{k,p}(M, g) &\rightarrow \mathbb{R} \\
 u &\mapsto (-1)^l \int_M g(u, (\nabla^*)^l S) \, d\mu_g.
 \end{aligned}$$

An analogous argument shows that this functional is also continuous.

By lemma 2.4, they admit a unique extension to $W^{k,p}(M, g)$. Also, theorem 4.2 guarantees that they coincide in the dense subset $C^{k,p}(M, g)$, so they also coincide in $W^{k,p}(M, g)$ due to the uniqueness of extension. $\square$

4.4.3 Weak solutions of PDEs

Consider a differential operator $A: C^\infty(M) \rightarrow C^\infty(M)$ which can be written in **divergence form**: locally,

$$A(u)|_U = \sum_{0 \leq l, m \leq k} \nabla_{i_1 \dots i_m} \left(a_{m,l}^{i_1 \dots i_m j_1 \dots j_l} \nabla_{j_1 \dots j_l} u \right) + \sum_{l=0}^k b_l \nabla^l u,$$

where $a_{k,l}$ are $(k+l, 0)$ -tensors and b_l are $(l, 0)$ -tensors.

Consider the PDE $A(u) = f$ where $f \in L^1(M, g)$. $u \in W^{k,p}(M, g)$ is a **weak solution** for this equation when given $\varphi \in \mathcal{D}(M)$:

$$\sum_{0 \leq m, l \leq k} (-1)^m \int_M a_{m,l} \nabla^l u \nabla^m \varphi \, d\mu_g + \sum_{l=0}^k \varphi b_l \nabla^l u \, d\mu_g = \int_M f \varphi \, d\mu_g,$$

where the $a_{k,l}, b_l$ are measurable and locally bounded.

4.5 Closed Riemannian manifolds

In this section, suppose that:

- (M^n, g) is a closed Riemannian manifold;
- $\mathcal{U} = \{(U_r, \varphi_r)\}$ is a finite smooth atlas for M ;
- $\mathcal{R} = \{\rho_r\}$ is a partition of unity subordinate to $\mathcal{U}$;
- $V_r = \varphi_r(U_r) \subset \mathbb{R}^n$.

4.5.1 The spaces $L^p(M)$

Remarkably, the setting of M as a closed manifold implies that $L^p(M, g)$ is independent of the Riemannian metric g . Due to this result, we shall often drop g when referring to $L^p(M, g)$. More specifically,

Lemma 4.12. Given $\tilde{g}$ a Riemannian metric on M , the norms $\|\cdot\|_{p,g}$ and $\|\cdot\|_{p,\tilde{g}}$ are equivalent. In particular, $L^p(M, g) = L^p(M, \tilde{g})$.

Proof. Due to theorem 4.3, it suffices to show that $\|\cdot\|_{p,g}$ and $\|\cdot\|_{p,\tilde{g}}$ are equivalent in $C_c(M)$.

$\text{supp } \rho_r$ is a compact subset of U_r , so there is $0 < d_r \leq D_r < \infty$ for which

$$d_r \leq \sqrt{|\det(g_{ij})|} \Big|_{\text{supp } \rho_r} \leq D_r.$$

Let $d = \min\{d_r\}$ and $D = \max\{D_r\}$. Given $u \in C_c(M)$,

$$d \sum_r \int_{V_r} |u \circ \varphi_r^{-1}|^p \, dx_1 \dots dx_n \leq \|u\|_{p,g}^p \leq D \sum_r \int_{V_r} |u \circ \varphi_r^{-1}|^p \, dx_1 \dots dx_n.$$

□

4.5.2 Uniform Sobolev spaces $\tilde{W}^{k,p}(M)$

Denote $\mathcal{V} = (\mathcal{U}, \mathcal{R})$. Given $u \in C^\infty(M)$, we define

$$\nu_{k,p,\mathcal{V}}(u) = \left[\sum_r \|(\rho_r u) \circ \varphi_r^{-1}\|_{W^{k,p}(V_r)}^p \right]^{1/p},$$

where

$$\|(\rho_r u) \circ \varphi_r^{-1}\|_{W^{k,p}(V_r)}^p = \int_{V_r} |(\rho_r u) \circ \varphi_r^{-1}|^p \, dx_1 \dots dx_n.$$

This map defines a norm on $C^\infty(M)$ which depends on the Sobolev norms $\|\cdot\|_{W^{k,p}(V_r)}$ of the local form of the terms $(\rho_r u)$. In this context, we define:

Definition 4.13 (Uniform Sobolev space). The **uniform Sobolev space** $\tilde{W}^{k,p}(M, \mathcal{V})$ is the completion of $C^\infty(M)$ with respect to the norm $\nu_{k,p,\mathcal{V}}$.

To define the uniform Sobolev spaces, we have made an arbitrary choice of $\mathcal{V}$. In fact, the definition does not depend on this choice. Due to this result, we shall drop $\mathcal{V}$ from the notation. More specifically,

Theorem 4.14. If $\tilde{\mathcal{V}} = (\tilde{\mathcal{U}}, \tilde{\mathcal{R}})$, where $\tilde{\mathcal{U}}$ is a finite smooth atlas for M and $\tilde{\mathcal{R}}$ is a partition of unity subordinate to $\tilde{\mathcal{U}}$, then $\tilde{W}^{k,p}(M, \mathcal{V}) = \tilde{W}^{k,p}(M, \tilde{\mathcal{V}})$.

Proof. 1. We vary the partition of unity.

Let $\tilde{\mathcal{R}} = \{\tau_r\}$ be a partition of unity subordinate to $\mathcal{U}$. Clearly, $\tau_s = \sum_r \rho_r \tau_s$. We have:

$$\begin{aligned} \|(\tau_s u) \circ \varphi_s^{-1}\|_{W^{k,p}(V_s)}^p &= \left\| \left(\sum_r \rho_r \tau_s u \right) \circ \varphi_s^{-1} \right\|_{W^{k,p}(V_s)}^p \\ &\leq \sum_r \|(\rho_r \tau_s u) \circ \varphi_s^{-1}\|_{W^{k,p}(V_s)}^p. \end{aligned} \quad (4.2)$$

Due to the supports of ρ_r and τ_s ,

$$\|(\rho_r \tau_s u) \circ \varphi_s^{-1}\|_{W^{k,p}(V_s)}^p = \|(\rho_r \tau_s u) \circ \varphi_s^{-1}\|_{W^{k,p}(\varphi_s(U_s \cap U_r))}^p.$$

Due to variable change,

$$\|(\rho_r \tau_s u) \circ \varphi_s^{-1}\|_{W^{k,p}(\varphi_s(U_s \cap U_r))}^p = \| |\det d(\varphi_s \circ \varphi_r^{-1})| (\rho_r \tau_s u) \circ \varphi_r^{-1} \|_{W^{k,p}(\varphi_r(U_s \cap U_r))}^p. \quad (4.3)$$

Due to the supports of ρ_r and τ_s ,

$$\begin{aligned} \| |\det d(\varphi_s \circ \varphi_r^{-1})| (\rho_r \tau_s u) \circ \varphi_r^{-1} \|_{W^{k,p}(\varphi_r(U_s \cap U_r))}^p &= \\ &= \| |\det d(\varphi_s \circ \varphi_r^{-1})| (\rho_r \tau_s u) \circ \varphi_r^{-1} \|_{W^{k,p}(V_r)}^p. \end{aligned}$$

Let $f_{r,s} = (\tau_s \circ \varphi_r^{-1}) |\det d(\varphi_s \circ \varphi_r^{-1})|$.

$f_{r,s} \Big|_{\text{supp } (\rho_r \tau_s) \circ \varphi_r^{-1}}$ is bounded above by a certain $C_{s,r} > 0$. Then

$$\|(\rho_r \tau_s u) \circ \varphi_s^{-1}\|_{W^{k,p}(V_s)}^p \leq C_{s,r} \|(\rho_r u) \circ \varphi_r^{-1}\|_{W^{k,p}(V_r)}^p.$$

Let $C_s = \max_r C_{s,r}$. Considering eq. (4.2) and the last inequality,

$$\begin{aligned} \|(\tau_s u) \circ \varphi_s^{-1}\|_{W^{k,p}(V_s)}^p &\leq C_s \sum_r \|(\rho_r u) \circ \varphi_r^{-1}\|_{W^{k,p}(V_r)}^p \\ &\leq C_s \nu_{k,p,\mathcal{V}}(u). \end{aligned}$$

Defining $C = \max_s C_s$, we have

$$\sum_s \|(\tau_s u) \circ \varphi_s^{-1}\|_{W^{k,p}(V_s)}^p \leq |\mathcal{U}| C \nu_{k,p,\mathcal{V}}(u).$$

We may apply the same argument to obtain the reciprocal inequality which guarantees the equivalence of the norms which $\mathcal{R}$ and $\tilde{\mathcal{R}}$ induce.

2. We vary the finite smooth atlas.

Consider $\tilde{\mathcal{U}} = \{(\tilde{U}_s, \chi_s)\}$ finite smooth atlas, $\tilde{\mathcal{R}} = \{\tau_s\}$ partition of unity subordinate to $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}} = (\tilde{\mathcal{U}}, \tilde{\mathcal{R}})$.

Consider also $\mathcal{A} = \{(U_r \cap \tilde{U}_s, \varphi_r|_{U_r \cap \tilde{U}_s})\}$, $\mathcal{B} = \{\rho_r \tau_s\}$ partition of unity subordinate to $\mathcal{A}$ and $\mathcal{W} = (\mathcal{A}, \mathcal{B})$.

$\nu_{k,p,\mathcal{W}}$ is equivalent to $\nu_{k,p,\mathcal{V}}$. One of the inequalities is evident and the other follows from eq. (4.2).

Consider now $\tilde{\mathcal{A}} = \{(U_r \cap \tilde{U}_s, \chi_s|_{U_r \cap \tilde{U}_s})\}$ and $\tilde{\mathcal{W}} = (\tilde{\mathcal{A}}, \mathcal{B})$.

By the same argument as before, $\nu_{k,p,\tilde{\mathcal{W}}}$ is equivalent to $\nu_{k,p,\tilde{\mathcal{V}}}$.

To conclude, it suffices to show that $\nu_{k,p,\mathcal{W}}$ and $\nu_{k,p,\tilde{\mathcal{W}}}$ are equivalent. In order to do so, we must perform a variable change as in eq. (4.3) and argue analogously as in the first part of the proof.

□

4.5.3 $W^{k,p}(M, g) = \tilde{W}^{k,p}(M)$

So far, we have defined two norms $\|\cdot\|_{k,p,g}$ and $\nu_{k,p}$ for the space $C^\infty(M) = C^{k,p}(M, g)$ and considered the completions of its corresponding normed spaces, obtaining respectively the Sobolev space $W^{k,p}(M, g)$ and the uniform Sobolev space $\tilde{W}^{k,p}(M)$.

We might then ask what is the relation between the spaces $W^{k,p}(M, g)$ and $\tilde{W}^{k,p}(M)$.

Theorem 4.15 (Equivalence of definitions).

$$W^{k,p}(M, g) = \tilde{W}^{k,p}(M).$$

In general, $W^{k,p}(M, g)$ depends on the Riemannian metric g considered. On the other hand, we suppose that M is closed and the definition of the uniform Sobolev space $\tilde{W}^{k,p}(M)$ does not make any mention to a Riemannian metric on M . In particular,

Corollary 4.16 (Independence of metric). If $\tilde{g}$ is a Riemannian metric on M , then

$$W^{k,p}(M, g) = W^{k,p}(M, \tilde{g}).$$

This corollary is mentioned without a proof in section 2.2 of chapter 2 in [Heb00]. It is also part of thm. 2.20 in [Aub11]. (this last reference presupposes the characterization of Sobolev spaces as uniform Sobolev spaces for closed Riemannian manifolds)

Unfortunately, the proof of the last theorem is rarely found in its entirety in the literature. Our proof presented next is based on the outline of section 3 in [AIN06].

Proof. We shall prove that $\|\cdot\|_{k,p,g}$ and $\nu_{k,p}$ are equivalent, hence the completion of $C^\infty(M)$ endowed with any of them yields the same space.

For that, we proceed in two steps:

1. we define an intermediary norm λ and we prove that $\|\cdot\|_{k,p,g}$ is equivalent to λ ;
2. we show that λ is equivalent to $\nu_{k,p}$.

Both steps of the proof are completely analogous yet a direct proof is difficult due to the complexity of the terms involved.

1. First step:

Given $u \in C^\infty(M)$, define

$$\lambda(u)^p = \sum_r \|\rho_r u\|_{k,p,g}^p.$$

$\{|\nabla^l \rho_r| : \forall r, 0 \leq l \leq k\}$ is a finite set of continuous functions in the compact space M . Therefore, there is $C > 0$ such that

$$\forall r \forall l = 0, \dots, k \|\nabla^l \rho_r\|_{\infty,g} \leq C. \quad (4.4)$$

In similar fashion, there is $L > 0$ such that

$$\forall l = 1, \dots, k \|\cdot\|_{1,\mathbb{R}^l} \leq L \|\cdot\|_{p,\mathbb{R}^l}.$$

Indeed: if l is a positive integer, then $\|\cdot\|_{1,\mathbb{R}^l} \leq l^{1-1/p} \|\cdot\|_{p,\mathbb{R}^l}$.

Last of all, let

$$K = \binom{k}{\lfloor k/2 \rfloor}.$$

Note that

$$\forall l = 0, \dots, k \forall m = 0, \dots, l \binom{l}{m} \leq K.$$

Therefore,

$$\begin{aligned} |\nabla^l(\rho_r u)|_g &= \left| \sum_{m=0}^l \binom{l}{m} \nabla^m \rho_r \otimes \nabla^{l-m} u \right|_g \\ &\leq \sum_{m=0}^l \binom{l}{m} |\nabla^m \rho_r|_g |\nabla^{l-m} u|_g \\ &\leq B \left(\sum_{m=0}^l |\nabla^m u|_g^p \right)^{1/p}, \end{aligned}$$

where $B = CLK$.

Hence

$$\begin{aligned} \|\rho_r u\|_{k,p,g}^p &\leq B^p \sum_{l=0}^k \int_M \sum_{m=0}^l |\nabla^m u|_g^p d\mu_g \\ &\leq B^p \sum_{l=0}^k \|u\|_{l,p,g}^p \\ &\leq (k+1) B^p \|u\|_{k,p,g}^p. \end{aligned}$$

Also $\lambda(u)^p \leq A \|u\|_{k,p,g}^p$, where $A = (k+1)NB^p$. We just obtained our first inequality.

Now, let $T > 0$ be such that $\|\cdot\|_{p,\mathbb{R}^N} \geq T\|\cdot\|_{1,\mathbb{R}^N}$. We obtain

$$\begin{aligned}\lambda(u) &= \left(\sum_r \|\rho_r u\|_{k,p,g}^p \right)^{1/p} \\ &\geq T \sum_r \|\rho_r u\|_{k,p,g} \\ &\geq T \left\| \sum_r \rho_r u \right\|_{k,p,g} \\ &\geq T \|u\|_{k,p,g}.\end{aligned}$$

Woah, we're halfway there. In fact, the second step is way shorter.

2. Second step:

To repeat all that we have done in the first step, we need to analyse the local form of the covariant derivatives of functions $u \in C^\infty(M)$.

Lemma 4.17. Let k be a positive integer, $l \in \{1, \dots, k\}$ and $(\tilde{U}, \tilde{\varphi})$ be a local chart of M . Then there exists a set indexed by multi-indices α

$$\{P_\alpha : 1 \leq |\alpha| \leq k\} \subset C^\infty(\tilde{U})$$

such that the k^{th} covariant derivative of $u \in C^\infty(M)$ can be written at $\tilde{U}$ as

$$\nabla^k u = \sum_{1 \leq |\alpha| \leq k} (D^\alpha u) P_\alpha \, dx_{i_1} \otimes \dots \otimes dx_{i_l},$$

where $\alpha = (i_1, \dots, i_l)$ is a multi-index. More specifically,

$$P_\alpha = S_\alpha \left((g_{ij}), (g^{ij}) \right),$$

where $\{S_\alpha : 1 \leq |\alpha| \leq k\}$ is a set of linear partial differential operators and S_α has order $|S_\alpha| \leq k - |\alpha|$.

This lemma can be easily proved with an induction.

Let $\mathcal{V} = (\mathcal{U}, \mathcal{R})$. Due to theorem 4.14, we can suppose without loss of generality that $\mathcal{U}$ is a nice smooth atlas. (see definition 2.11)

Denote the functions P_α of the remark on the local chart (V_r, ψ_r) by $P_{\alpha,r}$. The functions $P_{\alpha,r}$ and their derivatives of order up to k are bounded in U_r , because $\overline{U_r} \subset V_r$. The set $\{P_{\alpha,r} : 1 \leq |\alpha| \leq k, \forall r\}$ is finite, so it admits an upper bound analogous to the one in eq. (4.4). We can repeat the arguments in the first step to prove that

$$\exists C > 0 \, \lambda \leq C \nu_{k,p,\mathcal{V}}.$$

The other inequality follows trivially as in the first step.

□

From this last proof, we derive the following corollary

Corollary 4.18 (Equivalence of Sobolev norms). If $\tilde{g}$ is a Riemannian metric on M , then

$$\|\cdot\|_{k,p,g} \text{ is equivalent to } \|\cdot\|_{k,p,\tilde{g}}.$$

4.5.4 Sobolev embeddings

A basal result in the theory of Sobolev spaces is: (thm. 2.9 in [Heb00])

Theorem 4.19 (Sobolev embedding theorem). Let (M, g) be a smooth, compact Riemannian n -manifold. For any integers $j \geq 0$ and $m \geq 1$, any real number $q \geq 1$, and any real number p such that $1 \leq p < nq/(n - mq)$, the embedding of $W^{j+m,q}(M)$ in $W^{j,p}(M)$ is compact. In particular, for any $q \in [1, n)$ real and any $p \geq 1$ such that $1/p > 1/q - 1/n$, the embedding of $W^{1,q}(M)$ in $L^p(M)$ is compact.

Chapter 5

The Yamabe problem and main result

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5.1 The Yamabe problem

Consider the following context:

- (M^n, g) is a closed Riemannian manifold with $n \geq 3$;
- $[g] = \{(ug) : u \in C^\infty(M,]0, \infty[)\}$ is the **conformal class** of g ;
- u is a function on $C^\infty(M,]0, \infty[)$.

In this context, we can state the original problem posed by Yamabe:

Problem 5.1. Find a Riemannian metric $\tilde{g}$ conformal to g with constant scalar curvature.

Let us analyse the general formulation of the Yamabe problem. If we consider $\tilde{g} = u^{4/(n-2)}g$, we have

$$-a_n \Delta_g u + \text{scal}(g)u = \text{scal}(\tilde{g})u^{p_n-1}, \quad (5.1)$$

where

- $a_n = 4(n-1)/(n-2)$;
- $p_n = 2n/(n-2)$.

For details, see section 8E in chp. 8 of [WH15]. In particular, $\text{scal}(\tilde{g}) \equiv \lambda \in \mathbb{R}$ if, and only if,

$$-a_n \Delta_g u + \text{scal}(g)u = \lambda u^{p_n-1}. \quad (5.2)$$

Eq. 5.2 is called the **Yamabe equation**. In fact, this PDE admits a variational characterization. Specifically, u is a solution of eq. (5.2) if, and only if, u is a critical point of the **Yamabe functional**

$$J_g(u) = \left(\int_M u^{p_n} d\mu_g \right)^{-2/p_n} \int_M a_n |\nabla_g u|_g^2 + \text{scal}(g)u^2 d\mu_g.$$

Define the **Yamabe constant** $Y(M, g) = \inf_{C^\infty(M,]0, \infty[)} J_g$. In fact, the Yamabe constant only depends on $[g]$, the conformal class of g . Indeed, suppose that $\tilde{g} = u^{4/(n-2)}g$. Clearly, $d\mu_{\tilde{g}} = u^{p_n} d\mu_g$. Therefore,

$$\begin{aligned} \int_M a_n |\nabla_{\tilde{g}} v|_{\tilde{g}}^2 + \text{scal}(\tilde{g})v^2 d\mu_{\tilde{g}} &= \int_M u^{p_n} \left[a_n |\nabla_{\tilde{g}} v|_{\tilde{g}}^2 + \text{scal}(\tilde{g})v^2 \right] d\mu_g \\ &= \int_M a_n u^2 |\nabla_g v|_g^2 + u^{p_n} \text{scal}(\tilde{g})v^2 d\mu_g \end{aligned}$$

Taking into account eq. (5.1),

$$\int_M a_n |\nabla_{\tilde{g}} v|_{\tilde{g}}^2 + \text{scal}(\tilde{g})v^2 d\mu_{\tilde{g}} = \int_M a_n u^2 |\nabla_g v|_g^2 + uv^2 [-a_n \Delta_g u + \text{scal}(g)u] d\mu_g$$

Considering the chain rule and the divergence theorem, we conclude that $J_g(uv) = J_{\tilde{g}}(v)$. This clearly implies $J_g(C^\infty(M,]0, \infty[)) = J_{\tilde{g}}(C^\infty(M,]0, \infty[))$.

For a Riemannian metric h , define the **normalized Hilbert-Einstein functional**

$$E(h) = \left(\int_M d\mu_h \right)^{-2/p_n} \int_M \text{scal}(h) d\mu_h.$$

Due to the discussion above, $E(\tilde{g}) = J_{\tilde{g}}(1) = J_g(u)$. If we restrict our study to metrics of class C^k where $1 \leq k < \infty$, we have that E is a map of class C^1 . Therefore, we may recover solutions for eq. (5.2) by looking at metrics h which are critical points of $E|_{[g]}$ and determining their conformal factors u such that $h = ug$.

The combined effort of Yamabe [Yam60], Trudinger [Tru68], Schoen [Sch84] and Aubin [Aub86] resulted in the theorem that problem 5.1 admits a solution which minimizes J_g .

A key step to that result is the following:

Theorem 5.2. If (M, g) is not conformally equivalent to the round sphere $\mathbb{S}^n$, then $Y(M, g) < Y(\mathbb{S}^n)$.

Also, the Yamabe problem admits a maximum principle. Specifically: if $Y(M, g) \leq 0$, the Yamabe problem has a unique solution. If $Y(M, g) > 0$, uniqueness fails.

In this context, there appears a first example of multiplicity related to the Yamabe problem: considering the constant curvature metric on the sphere $\mathbb{S}^n$, eq. (5.2) is invariant under conformal transformations. ([Oba71])

5.2 Setting, motivation and result

Consider the following context:

- (M^n, g) is a closed Riemannian manifold with scalar curvature $\text{scal}^M(g)$, where $n \geq 3$;
- (N^m, h) is a Riemannian manifold with positive constant scalar curvature $\text{scal}^N(h)$.

We are interested in the Yamabe equation in the Riemannian product $(M \times N, g + \epsilon^2 h)$, where $\epsilon > 0$ and we only consider scalar factors $u: M \rightarrow]0, \infty[$. In such setting, u is a scalar factor such that

$$\text{scal}^{M \times N} \left(u^{p_{m+n}-2} (g + \epsilon^2 h) \right) \equiv 1$$

if, and only if,

$$-a_{m+n} \Delta_g u + \left[\text{scal}^M(g) + \frac{\text{scal}^N(h)}{\epsilon^2} \right] u = u^{p_{m+n}-1}. \quad (5.3)$$

During the last years, a number of articles was centered on the Yamabe problem in product-like structures. For instance, multiplicity of solutions for the Yamabe problem in such setting was the object of study of R. Bettiol and P. Piccione in [BP13]. Closer to our context, the exact same setting which we consider was studied by J. Petean in [Pet19] and by C. Rey and J. Ruiz in [RR18].

In [Pet19], J. Petean explicits a connection between the number of solutions of eq. (5.3) and the topology of M by showing that it admits at least $\text{Cat}(M) + 1$ solutions for small enough $\epsilon > 0$, where $\text{Cat}(M)$ is the Lusternik-Schnirelmann category of M , i.e., the least amount of contractible closed sets needed to cover M . In particular, this article establishes a connection between the topology of M and the number of solutions for eq. (5.3). More recently, J. N. A. Neto generalized this result to the context of Riemannian submersions with minimal fibers in [dAN20].

In [RR18], C. Rey and J. Ruiz build k -peak solutions for eq. (5.3) concentrating around $x_0 \in M$ an isolated local minimum of $\text{scal}(g)$ for small $\epsilon > 0$ and any positive integer k . In particular, such result allows us to relate the number of solutions for eq. (5.3) and the number of isolated minima of $\text{scal}(g)$. In particular, we deduce a connection between the number of solutions for eq. (5.3) and the geometry of (M, g) .

Specifically, we proved a generic result of multiplicity of solutions with small energy for

eq. (5.3), where genericity concerns the parameters $(\epsilon, g) \in]0, \infty[\times \mathcal{M}^k$. ($\mathcal{M}^k$ is the set of Riemannian metrics of class C^k , with k a positive integer) The bound we obtain is in terms of $P_1(M)$, strengthening the link between the topology of M and the multiplicity of solutions. For that purpose, we follow the general argument of M. Ghimenti and A. M. Micheletti in [GM11], we adjust the techniques of A. M. Micheletti and A. Pistoia in [MP09] to prove the genericity of nondegeneracy of solutions u to eq. (5.3) and we use the adaptation of the concept of center of mass due to J. Petean in [Pet19] to construct certain continuous transformations intrinsically on M .

Normalize h such that $\text{scal}^N(h) = a_{m+n}$. Then finding solutions of eq. (5.3) is equivalent to finding solutions of

$$-\epsilon^2 \Delta_g u + \left[\epsilon^2 \frac{\text{scal}^M(g)}{a_{m+n}} + 1 \right] u = u^{p_{m+n}-1}. \quad (5.4)$$

It will be clear from the proof that the only property of p_{m+n} which matters is the fact that p_{m+n} is subcritical. In that context, we can substitute p_{m+n} by any sub-critical p , i.e., $p \in]2, p_n[$. We would also like to introduce the following notation:

- p' is the real number such that $1/p + 1/p' = 1$;
- $a = a_{m+n}$;
- $\text{scal} = \text{scal}^M$.

With this notation in mind, we reach the PDE that we wish to study:

$$\begin{cases} -\epsilon^2 \Delta_g u + [1 + \epsilon^2 \text{scal}(g)/a] u = (u^+)^{p-1} \\ u \in H^1(M) \end{cases}. \quad (5.5)$$

As we supposed beforehand that u was a positive function, there was a term u^{p-1} at the right-hand side of the PDE. We substitute the term u^{p-1} by $(u^+)^{p-1}$ because

$$\begin{aligned} -\epsilon^2 \Delta_g u + \left[1 + \epsilon^2 \frac{\text{scal}(g)}{a} \right] u &= (u^+)^{p-1} \implies \\ \implies -\epsilon^2 (u^-) \Delta_g u + \left[1 + \epsilon^2 \frac{\text{scal}(g)}{a} \right] (u^-) u &= (u^-) (u^+)^{p-1} \implies \\ \implies \epsilon^2 \left| \nabla_g (u^-) \right|_g^2 + \left[1 + \epsilon^2 \frac{\text{scal}(g)}{a} \right] (u^-)^2 &= 0, \end{aligned}$$

where we use the fact that $(\nabla_g u)^- = \nabla_g (u^-)$. Therefore, if $(\epsilon, g) \in]0, \infty[\times \mathcal{M}^k$ is such that $[1 + \epsilon^2 \text{scal}(g)/a]$ is positive in M , then any solution u of eq. (5.5) is automatically non-negative.

Denote also:

- $\mathcal{S}^k$ the Banach space of symmetric $(0, 2)$ -tensors of class C^k ;
- $\mathcal{M}^k \subset \mathcal{S}^k$ the set of Riemannian metrics of class C^k ;
- $\mathcal{B}_\rho$ an open ball in $\mathcal{S}^k$ centered at 0 with radius ρ ;
- $m_\infty = J_\infty(U) > 0$. (see section 5.6.3)

In this context, our main result reads as

Theorem 5.3. Given $g_0 \in \mathcal{M}^k$, there are $\hat{\epsilon} > 0$ and $\hat{\rho} > 0$ for which

$$\{(\epsilon, h) \in]0, \hat{\epsilon}[\times \mathcal{B}_{\hat{\rho}} : -\epsilon^2 \Delta_{g_0+h} u + \left[\epsilon^2 \text{scal}(g_0 + h)/a + 1 \right] u = (u^+)^{p-1} \text{ has at least } P_1(M) \text{ nonconstant solutions } u \text{ with } J_{\epsilon, h}(u) < 2m_\infty\}$$

is an open dense subset of $]0, \hat{\epsilon}[\times \mathcal{B}_{\hat{\rho}}$, where

$$J_{\epsilon, h}(u) = \epsilon^{-n} \int_M \frac{\epsilon^2}{2} |\nabla_{g_0+h} u|_{g_0+h}^2 + \frac{1}{2} \left[\epsilon^2 \frac{\text{scal}(g_0 + h)}{a} + 1 \right] u^2 - \frac{1}{p} (u^+)^p d\mu_{g_0+h}$$

and $P_1(M) = \sum_k \dim H_k(M)$. (see definition 2.21)

5.3 Preliminary definitions and notation

Consider a closed manifold M^n , where $n \geq 3$.

Let us fix

- $\mathcal{U} = \{(U_r, \varphi_r)\}$ a nice smooth atlas for M ; (see definition 2.11)
- $k \geq 2$ a positive integer.

Definition 5.4. $\mathcal{C}^k$ is the Banach space of real-valued functions on M of class C^k endowed with the norm

$$\|u\|_{\mathcal{C}^k} = \sum_r \sum_{|\alpha| \leq k} \sup_{\varphi_r(U_r)} \left| \frac{\partial^\alpha u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right|.$$

Definition 5.5. $\mathcal{S}^k$ is the Banach space of symmetric covariant 2-tensors on M of class C^k endowed with the norm

$$\|h\|_{\mathcal{S}^k} = \sum_r \sum_{|\alpha| \leq k} \sum_{i,j} \sup_{\varphi_r(U_r)} \left| \frac{\partial^\alpha h_{ij}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right|.$$

Definition 5.6. $\mathcal{M}^k \subset \mathcal{S}^k$ is the (open) cone of Riemannian metrics of class C^k .

Fix $g_0 \in \mathcal{M}^k$.

We adopt

- $\|u\|_{L^p} = \left(\int_M |u|^p d\mu_{g_0} \right)^{1/p}$ as the norm on $L^p(M)$;
- $\langle u, v \rangle_{H^1} = \int_M g_0(\nabla_{g_0} u, \nabla_{g_0} v) + uv d\mu_{g_0}$ as the inner product on $H^1(M)$.

We shall employ the following notation:

- $B(0, R)$ is the ball in $\mathbb{R}^n$ of center 0 and radius R ;
- $B_g(x, R)$ is the geodesic ball in the Riemannian manifold (M, g) , where $g \in \mathcal{M}^k$, of center x and radius R ;
- $\mathcal{B}_\rho$ is the ball in $\mathcal{S}^k$ of center 0 and radius ρ ;

- $I(u, \rho)$ is the ball in $H^1(M)$ of center u and radius ρ .

5.4 Sketch of the proof

In order to obtain our main result, we need to consider the following theorem of genericity of nondegeneracy:

Theorem 5.7. Let $\mathcal{A} \subset H^1(M)$ be a bounded open set. Suppose that

$$\lambda = \left\{ 2a / [2a + 2(2-p)(n-1)] \right\}^{1/(p-2)}$$

does not belong to $\mathcal{A}$. Then there exists $\tilde{\epsilon} > 0$ and $\tilde{\rho} > 0$ for which the set

$$\mathcal{D} = \{(\epsilon, h) \in]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}} : \text{any } u \in \mathcal{A} \text{ solution of} \\ -\epsilon^2 \Delta_{g_0+h} u + \left[\epsilon^2 \text{scal}(g_0+h)/a + 1 \right] u = (u^+)^{p-1} \text{ is nondegenerate} \}$$

is an open dense subset of $]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}}$.

This theorem is an adaptation of the main result in [MP09] to the context we presently consider. More specifically, the cited article concerns a result of generic nondegeneracy of solutions $u \in \mathcal{A}$ for the equation

$$-\epsilon^2 \Delta_g u + u = (u^+)^{p-1} \quad (5.6)$$

in a closed Riemannian manifold (M^n, g) , where $\mathcal{A} \subset H^1(M)$ is a bounded open subset for which $1 \notin \mathcal{A}$ and $g \in \mathcal{M}^k$.

In the following, we shall outline the proof of theorem 5.7. Its fine details are explained in section 5.5.

Genericity of nondegeneracy

The arguments employed in [MP09] reside on the fact that eq. (5.6) is a variational PDE. More specifically: given $g \in \mathcal{M}^k$ and $\epsilon > 0$, its solutions are critical points of the functional

$$H^1(M) \rightarrow \mathbb{R} \\ u \mapsto \epsilon^{-n} \int_M \frac{\epsilon^2}{2} |\nabla_g u|_g^2 + \frac{1}{2} u^2 - \frac{1}{p} (u^+)^p \, d\mu_g$$

In the context we presently consider, the functional which will play such role is

$$J_{\epsilon, h} : H^1(M) \rightarrow \mathbb{R} \\ u \mapsto \frac{1}{2} E_{\epsilon, h}(u, u) - \frac{1}{p} \|u^+\|_{L_{\epsilon, h}^p}^p \quad (5.7)$$

where $E_{\epsilon, h}$ is a specific inner product in $H^1(M)$ (definition 5.14), $\|\cdot\|_{L_{\epsilon, h}^p}$ is a specific norm in $L^p(M)$ (definition 5.13) and $h \in \mathcal{B}_{\tilde{\rho}} \subset \mathcal{S}^k$ is a perturbation of $g_0 \in \mathcal{M}^k$.

In lemma 5.20, we construct a function $F:]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}} \times H^1(M) \rightarrow H^1(M)$ for which $F(\epsilon_0, h_0, u_0) = 0$ implies $u_0 \geq 0$ and

$$-\epsilon_0^2 \Delta_{g_0+h_0} u_0 + \left[\epsilon_0^2 \text{scal}(g_0 + h_0) / a + 1 \right] u_0 = u_0^{p-1}.$$

Recall that: given Banach spaces X, Y ; a **Fredholm operator** is a bounded linear operator $T: X \rightarrow Y$ with finite-dimensional kernel $\ker T$ and finite-dimensional cokernel $\text{coker } T = Y/(\text{im } T)$. Given a Fredholm operator $T: X \rightarrow Y$, its **index** is defined $\text{ind } T = \dim \ker T - \dim \text{coker } T$.

Theorem 5.7 is then a corollary of the following transversality theorem applied to F :

Theorem 5.8 (Abstract transversality theorem). Let X, Y, Z be real Banach spaces and $U \subset X, V \subset Y$ be open subsets. Let $F: V \times U \rightarrow Z$ be a C^m -map, where $m \geq 1$, such that

1. for any $y \in V$, $F(y, \cdot): x \mapsto F(x, y)$ is a Fredholm map of index l where $l \leq m$;
2. z_0 is a regular value of F , that is, the operator $dF_{(y_0, x_0)}: Y \times X \rightarrow Z$ is onto at any point (y_0, x_0) such that $F(y_0, x_0) = z_0$;
3. the set of $x \in U$ such that $F(y, x) = z_0$ with y in a compact set of X is relatively compact in U . (its closure is a compact subset of U)

Then the set $\{y \in V : z_0 \text{ is a regular value of } F(y, \cdot)\}$ is a dense open subset of V .

We left many details behind: we did not explain why we could apply such theorem to F , nor did we explain why F satisfies such conditions. Last of all, we did not even explain whether there is a map F with the properties we listed before the transversality theorem.

For a while, suppose that there exists a map F with the properties listed before the theorem. The fact that F satisfies conditions 1 and 3 are corollaries of the following lemmas: (for details, we refer the reader to lemmas 4.1 and 4.2 in [MP09])

Lemma 5.9. Given $(\epsilon, h) \in]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}}$, $u \mapsto F(\epsilon, h, u)$ with $u \in \mathcal{A}$ is a Fredholm map of index 0.

Lemma 5.9 is a consequence of the fact that $dF_{(\epsilon_0, h_0, u_0)}$ is of the form $(I - K)$, where $I: H^1(M) \rightarrow H^1(M)$ is the identity, and $K: H^1(M) \rightarrow H^1(M)$ is a compact operator.

Lemma 5.10. The set

$$\{u \in \mathcal{A} : F(\epsilon, h, u) = 0, (\epsilon, h) \text{ belongs to a compact subset of }]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}}\}$$

is relatively compact in $\mathcal{A}$.

Lemma 5.10 is easily proved with the basic tools of calculus in Banach spaces.

Finally, the missing ingredient is proving that F satisfies condition 2, which is the objective of section 5.5.3.

Defining F is the objective of section 5.5.2. In fact, such construction is only possible due to the existence of an auxiliary self-adjoint map $A_{\epsilon, h}: L_{\epsilon, h}^{p'}(M) \rightarrow H_{\epsilon, h}^1(M)$ of class C^1 . (definition 5.17)

This map $A_{\epsilon, h}$ satisfies $\langle A_{\epsilon, h} u, v \rangle_{H_{\epsilon, h}^1} = \langle u, v \rangle_{L_{\epsilon, h}^2}$. This permits us to exploit the fact that

$(L^p(M))^* \simeq L^{p'}(M) \hookrightarrow (L^2(M))^*$ by $u \mapsto \langle u, \cdot \rangle_{L^2}$. In its turn, we can only define such map A due to the Sobolev embedding theorem – the details of such construction being found in section 5.5.1.

Multiplicity of solutions

Our proof is based on the one provided in [GM11]. In fact, the cited article provides a multiplicity result analogous to theorem 5.3, but for eq. (5.6). The notable difference in our approach is the intrinsic definition of the map $c_{\epsilon,h}$, adapting the technique adopted by [Pet19].

We denote

- $N_{\epsilon,h}$ the Nehari manifold of $J_{\epsilon,h}$; (see section 2.6)
- given $d \in \mathbb{R}$, $\Sigma_{\epsilon,h}^d = J_{\epsilon,h}^d \cap N_{\epsilon,h}$;
- $m_{\epsilon,h} = \inf \{ J_{\epsilon,h}(u) : u \in N_{\epsilon,h} \}$.

The map $\mathcal{M}^k \rightarrow \mathbb{R}$ given by $g \mapsto \mu_g(M)$ is continuous, so there is $\hat{\rho} > 0$ for which

$$\forall h \in \mathcal{B}_{\hat{\rho}} \quad \left[J_{\epsilon,h}(\lambda) > \frac{p-2}{2p} \frac{\lambda}{\epsilon^n} \frac{\mu_{g_0}(M)}{2} \right],$$

where $\lambda \in \mathbb{R}$ is the constant introduced in theorem 5.7. A direct consequence of this is that there exists $\hat{\epsilon} > 0$ for which $\epsilon \in]0, \hat{\epsilon}[$ implies $J_{\epsilon,h}(\lambda) > 2m_\infty$.

Lemma 5.11. There exist $\hat{\epsilon} \in]0, 1[$, $\hat{\rho} > 0$ and $\alpha > 0$ such that given $\epsilon \in]0, \hat{\epsilon}[$ and $h \in \mathcal{B}_{\hat{\rho}}$,

$$A(h, \epsilon) = \{ u \in N_{\epsilon,h} : J_{\epsilon,h}(u) \leq 2m_\infty \} \subset I(0, \alpha) \setminus \{ \lambda \}.$$

The main facts used to prove this last lemma are the remark that precedes it, the fact that the norms $\|\cdot\|_{H_{\epsilon,h}^1}$ and $\|\cdot\|_{H^1}$ are equivalent (see lemma 5.16) and the fact that there is $\hat{\rho} > 0$ for which $\lim_{\epsilon \rightarrow 0^+} m_{\epsilon,h} = m_\infty$ uniformly with respect to $h \in \mathcal{B}_{\hat{\rho}}$. (see section 5.6.5) Per se, its full proof is not hard or interesting, so we refer the intrigued reader to lemma 2.7 in [GM11] for details.

Let $\hat{\epsilon}, \hat{\rho} > 0$ be such that the last lemma holds and for which $\mathcal{D} \subset]0, \hat{\epsilon}[\times \mathcal{B}_{\hat{\rho}}$ is an open dense subset, where $\mathcal{D}$ was defined in theorem 5.7, where we take $\mathcal{A} = I(0, \alpha) \setminus \{ \lambda \}$. Clearly, the following set is an open dense subset of $]0, 1[\times \mathcal{B}_{\hat{\rho}}$:

$$\begin{aligned} D(\hat{\epsilon}, \hat{\rho}) = \{ (\epsilon, h) \in]0, \hat{\epsilon}[\times \mathcal{B}_{\hat{\rho}} : \text{any non-constant solution } u \text{ of} \\ -\epsilon^2 \Delta_{g_0+h} u + \left[\epsilon^2 \text{scal}(g_0 + h)/a + 1 \right] u = (u^+)^{p-1} \\ \text{such that } J_{\epsilon,h}(u) < 2m_\infty \text{ is nondegenerate} \} \end{aligned}$$

Suppose that $\hat{\epsilon} > 0$ is small enough so that given $h \in \mathcal{B}_{\hat{\rho}}$, $0 < 3m_\infty/4 < m_{\epsilon,h} < 5m_\infty/4$. (see section 5.6.5)

From now on, we suppose that $(\epsilon, h) \in]0, \hat{\epsilon}[\times \mathcal{B}_{\hat{\rho}}$.

M is compact, so $J_{\epsilon,h}$ satisfies the Palais-Smale condition. $J_{\epsilon,h}$ admits a finite number of critical points u with $J_{\epsilon,h}(u) < 2m_\infty$. (see lemma 2.27) Therefore: given $\delta \in]0, m_\infty/4[$, we

can assume that $m_\infty + \delta$ is not a critical value of $J_{\epsilon,h}$. The following relation holds:

$$P_t(J_{\epsilon,h}^{m_\infty+\delta}, J_{\epsilon,h}^{m_\infty-\delta}) = tP_t(\Sigma_{\epsilon,h}^{m_\infty+\delta}) \quad (5.8)$$

This relation is a consequence of the Thom isomorphism theorem and of usual arguments of Morse theory involving descending gradient flow. For details, we refer the reader to lemma 5.2 in [BC94].

We shall show that there are continuous maps $i_{\epsilon,h}: M \rightarrow \Sigma_{\epsilon,h}^{m_\infty+\delta}$ and $c_{\epsilon,h}: \Sigma_{\epsilon,h}^{m_\infty+\delta} \rightarrow M$ such that $c_{\epsilon,h} \circ i_{\epsilon,h} \simeq \text{id}_M$. In such context, we can use the remark in section 2.7 to conclude that

$$P_t(\Sigma_{\epsilon,h}^{m_\infty+\delta}) = P_t(M) + Z(t),$$

where $Z(t)$ is a polynomial with nonnegative integer coefficients.

$J_{\epsilon,h}$ satisfies the Palais-Smale condition and the critical points $u \in I(0, \alpha) \setminus \{\lambda\}$ of $J_{\epsilon,h}$ for which $J_{\epsilon,h}(u) < m_\infty + \delta$ are nondegenerate. By a result in section 2.8:

$$\sum_{u \in C} i_t(u) = \sum_{u \in C} t^{\mu(u)} = P_t(J_{\epsilon,h}^{m_\infty+\delta}, J_{\epsilon,h}^{m_\infty-\delta}),$$

where $C = \{u \in H^1(M) : d(J_{\epsilon,h})_u = 0; J_{\epsilon,h}(u) \in]m_\infty - \delta, m_\infty + \delta[\}$.

This permits us to prove theorem 5.3.

Remark. $J_{\epsilon,h}$ has at least $P_1(M)$ nonconstant critical points in $N_{\epsilon,h}$. Indeed, the only case in which eq. (5.5) admits a constant solution is when $\text{scal}(g_0)$ is constant. In such context, the discussion in section 5.6.2 implies that eq. (5.5) admits a unique non-zero constant solution.

The construction of the maps $i_{\epsilon,h}$ and $c_{\epsilon,h}$ are the main concern of the following sections, in which we explicit the details of the proof. To comment on that, the construction of the map $i_{\epsilon,h}$ is done as follows:

1. we consider the non-zero ground-state solution U of the limiting PDE $-\Delta U + U = U|U|^{p-2}$ in $\mathbb{R}^n$. It is known that U is radially symmetric and radially decreasing. Without loss of generality, we can suppose that U is radially symmetric around $0 \in \mathbb{R}^n$. U_ϵ given by $U_\epsilon(x) = U(x/\epsilon)$ is then a solution of $-\epsilon^2 \Delta U_\epsilon + U_\epsilon = U_\epsilon |U_\epsilon|^{p-2}$.

The details about this step are found in section 5.6.3;

2. recall that we consider small $\epsilon > 0$, so we can take geodesic open balls $B_g(x, \epsilon) \subset M$ to open balls $B(0, \epsilon) \subset \mathbb{R}^n$ through the exponential map $\exp_x$. Using this idea, $i_{\epsilon,h}$ is obtained by associating $x \in M$ to $(\varphi_R \circ |\exp_x^{-1}|)(U_\epsilon \circ \exp_x^{-1})$ multiplied by a specific constant, where φ_R is a continuous cut-off function.

The details of such construction are done in section 5.6.4.

In other words, $i_{\epsilon,h}$ associates $x \in M$ to a continuous function $i_{\epsilon,h}(x) \in \Sigma_{\epsilon,h}^{m_\infty+\delta}$ with support $\text{supp } i_{\epsilon,h}(x) \subset B_g(x, \epsilon)$. In particular, $i_{\epsilon,h}(x)$ is “radially symmetric” around x , and it decreases as the distance to x increases, so $[i_{\epsilon,h}(x)](x) = \max_M i_{\epsilon,h}(x)$.

What about $c_{\epsilon,h}$? If we wanted to define a $c_{\epsilon,h}$ such that $c_{\epsilon,h} \circ i_{\epsilon,h} = \text{id}_M$ we might want to retrieve an a point $x_u \in M$ for which $u(x_u) = \max_M u$ for any $u \in \Sigma_{\epsilon,h}^{m_\infty+\delta}$. This idea would work for functions which are defined pointwise, yet recall that $\Sigma_{\epsilon,h}^{m_\infty+\delta} \subset H^1(M)$.

In particular, this means that a function $u \in \Sigma_{\epsilon,h}^{m_\infty+\delta}$ is only defined almost everywhere, so we cannot speculate about the value of u at specific points on M . In fact, the property which interests us is the concentration of the integral of $[i_{\epsilon,h}(x)]^+$ in a sense which is precised along the proof. In section 5.6.5, we show that any $u \in \Sigma_{\epsilon,h}^{m_\infty+\delta}$ admits (at least) a point $x_u \in M$ which similarly concentrates the integral of $(u^+)^p$.

In this context: given $u \in \Sigma_{\epsilon,h}^{m_\infty+\delta}$, the map $c_{\epsilon,h}$ retrieves one such point $x_u \in M$. If $u = i_{\epsilon,h}(x)$, then x_u is “close enough” to x for them to be linked by a unique minimizing geodesic. This is the fundamental idea behind the definition of $c_{\epsilon,h}$. Technically, the construction of such map depends on an extension of the concept of Riemannian center of mass, and is done in section 5.6.6.

5.5 Proof of genericity of nondegeneracy

5.5.1 Context and preliminaries

The following lemma is a consequence of the local form of scal and the fact that M is compact, so we can cover it with a finite number of local charts.

Lemma 5.12. $\text{scal}: \mathcal{M}^k \rightarrow \mathcal{C}^{k-2}$ is a function of class C^1 .

Let $\tilde{\rho} > 0$ be such that

$$\begin{cases} (g_0 + \mathcal{B}_{\tilde{\rho}}) \subset \mathcal{M}^k; \\ \forall h \in \mathcal{B}_{\tilde{\rho}} \quad \|\text{scal}(g_0 + h)\|_{\mathcal{C}^0} \leq 2\|\text{scal}(g_0)\|_{\mathcal{C}^0}. \end{cases}$$

(there is such $\tilde{\rho} > 0$ because $\mathcal{M}^k$ is an open subset of $\mathcal{S}^k$ and scal is continuous)

Let $\tilde{\epsilon} > 0$ be such that $\tilde{\epsilon}^2 < a\|\text{scal}(g_0)\|_{\mathcal{C}^0}/2$. From now on,

- ϵ is any element in $]0, \tilde{\epsilon}[$;
- h is any element in $\mathcal{B}_{\tilde{\rho}}$;
- for an $h \in \mathcal{B}_{\tilde{\rho}}$, $g = (g_0 + h) \in (g_0 + \mathcal{B}_{\tilde{\rho}}) \subset \mathcal{M}^k$.

Due to the hypothesis $(\epsilon, h) \in]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}}$, any $f(\epsilon, h)$ is a positive function on M , where

$$f(\epsilon, h) = \epsilon^2 \frac{\text{scal}(g)}{a} + 1.$$

Definition 5.13. $L_{\epsilon,h}^p(M)$ is the Banach space given by the completion of $C_c(M)$ with respect to the norm below:

$$\|u\|_{L_{\epsilon,h}^p} = \left(\epsilon^{-n} \int_M |u|^p d\mu_g \right)^{1/p}$$

$\|\cdot\|_{L_{\epsilon,h}^p}$ is equivalent to $\|\cdot\|_{L^p}$, so $L_{\epsilon,h}^p(M) = L^p(M)$ as sets. As for their Banach space structures, the inclusions $L_{\epsilon,h}^p(M) \hookrightarrow L^p(M)$ and $L^p(M) \hookrightarrow L_{\epsilon,h}^p(M)$ are bounded linear injections. (see section 2.2)

Definition 5.14. Define the following bilinear forms:

1. On $C^\infty(M)$,

$$E(\epsilon, h)(u, v) = E_{\epsilon, h}(u, v) = \epsilon^{-n} \int_M \epsilon^2 g(\nabla_g u, \nabla_g v) + f(\epsilon, h) uv \, d\mu_g$$

2. On $C_c(M)$,

$$G(\epsilon, h)(u, v) = G_{\epsilon, h}(u, v) = \epsilon^{-n} \int_M uv \, d\mu_g$$

It is easy to check that those bilinear forms are inner products.

Definition 5.15. Define the Hilbert space $H_{\epsilon, h}^1(M)$ given by the completion of $C^\infty(M)$ with respect to the norm below:

$$\|u\|_{H_{\epsilon, h}^1}^2 = E_{\epsilon, h}(u, u) = \epsilon^{-n} \int_M \epsilon^2 g(\nabla_g u, \nabla_g v) + f(\epsilon, h) uv \, d\mu_g$$

In particular, we shall denote equivalently $E_{\epsilon, h} = \langle \cdot, \cdot \rangle_{H_{\epsilon, h}^1}$.

Lemma 5.16. $\|\cdot\|_{H^1}$ is equivalent to $\|\cdot\|_{H_{\epsilon, h}^1}$ in $C^\infty(M)$.

Proof. Let $u = f(\epsilon, h)$. u is a continuous positive function on the compact space M , so

$$0 < \min_M u \leq \max_M u < \infty.$$

Let $c = \min_M u$ and $C = \max_M u$. Clearly,

$$\min\{1, c/\epsilon^2\} \|\cdot\|_{H^1}^2 \leq \epsilon^{n-2} \|\cdot\|_{H_{\epsilon, h}^1}^2 \leq \max\{1, C/\epsilon^2\} \|\cdot\|_{H^1}^2$$

□

Due to the lemma above, $H_{\epsilon, h}^1(M) = H^1(M)$ as sets. Once again, the inclusions

$$H_{\epsilon, h}^1(M) \hookrightarrow H^1(M) \text{ and } H^1(M) \hookrightarrow H_{\epsilon, h}^1(M)$$

are bounded linear injections.

Due to the Sobolev embedding theorem (theorem 4.19), there are compact embeddings $H^1(M) \hookrightarrow L^p(M)$. The composition of a compact linear operator with a bounded linear operator is a compact linear operator, so we have a compact embedding

$$H_{\epsilon, h}^1(M) \hookrightarrow L_{\epsilon, h}^p(M).$$

Definition 5.17 (The map A). $A_{\epsilon, h}: L_{\epsilon, h}^{p'}(M) \rightarrow H_{\epsilon, h}^1(M)$ is the adjoint of the compact embedding above.

It is important to note that:

$$E_{\epsilon, h}(A_{\epsilon, h} u, v) = G_{\epsilon, h}(u, v). \quad (5.9)$$

The three following lemmas are consequences of the characterization of Fréchet differentiability as Gâteaux differentiability whose differential is continuous. For detailed proofs

of analogous results, we refer the reader to lemmas 2.1, 2.2 and 2.3 in [MP09].

Lemma 5.18. $E:]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}} \rightarrow \text{Bil}(H^1(M) \times H^1(M), \mathbb{R})$ is of class C^1 , where $E(\epsilon, h) = E_{\epsilon, h}$.

In particular:

$$\begin{aligned} dE_{(\epsilon_0, h_0)}[\epsilon, h](u, v) &= \frac{(-n)}{\epsilon_0^{n+1}} \epsilon \int_M \epsilon_0^2 g(\nabla_g u, \nabla_g v) d\mu_g + \frac{2}{\epsilon_0^n} \int_M \epsilon_0 \epsilon g(\nabla_g u, \nabla_g v) d\mu_g + \\ &+ \frac{(-n)}{\epsilon_0^{n+1}} \epsilon \int_M f(\epsilon_0, h_0) uv d\mu_g + \frac{1}{\epsilon_0^n} \int_M 2\epsilon_0 \epsilon \frac{\text{scal}(g)}{a} uv d\mu_g + \frac{\epsilon_0^2}{\epsilon_0^n} \int_M b(h)(\nabla_g u, \nabla_g v) d\mu_g + \\ &+ \frac{\epsilon_0^2}{\epsilon_0^N} \int_M \frac{d(\text{scal})_g[h]}{a} uv d\mu_g + \frac{1}{2\epsilon_0^n} \int_M f(\epsilon_0, h_0) uv \text{tr}_g h d\mu_g, \end{aligned}$$

where $b(h)$ is the 2-covector given locally by

$$b(h)_{ij} = \frac{1}{2} g^{ij} \text{tr}_g h - g^{ia} h_{ab} g^{bj}.$$

Lemma 5.19. $G:]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}} \rightarrow \text{Bil}(H^1(M) \times H^1(M), \mathbb{R})$ is of class C^1 , where $G(\epsilon, h) = G_{\epsilon, h}$.

In particular:

$$dG_{(\epsilon_0, h_0)}[\epsilon, h](u, v) = \frac{(-n)}{\epsilon_0^{n+1}} \epsilon \int_M uv d\mu_g + \frac{1}{2\epsilon_0^n} \int_M uv \text{tr}_g h d\mu_g.$$

Lemma 5.20. $A:]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}} \rightarrow B(L^{p'}(M), H^1(M))$ is of class C^1 , where $A(\epsilon, h) = A_{\epsilon, h}$.

In particular:

$$dE_{(\epsilon_0, h_0)}[\epsilon, h](A_{\epsilon_0, h_0} u, v) + E_{\epsilon_0, h_0} \left(dA_{(\epsilon_0, h_0)}[\epsilon, h] u, v \right) = dG_{(\epsilon_0, h_0)}[\epsilon, h](u, v).$$

Note that the equality in the last lemma is simply the differentiation of eq. (5.9).

5.5.2 The map F

Let $\mathcal{A}$ be an open bounded subset of $H^1(M)$. Suppose that $\lambda \notin \mathcal{A}$, where

$$\lambda = \left[\frac{2a}{2a + 2(2-p)(n-1)} \right]^{1/(p-2)}.$$

Definition 5.21.

$$\begin{aligned} F:]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}} \times H^1(M) &\rightarrow H^1(M) \\ (\epsilon, h, u) &\mapsto u - A_{\epsilon, h}(u^+)^{p-1}. \end{aligned}$$

A is a map of class C^1 , so F is obviously of class C^1 .

Remark. If $F(\epsilon_0, h_0, u_0) = 0$, then $-\epsilon_0^2 \Delta_{g_0+h_0} u_0 + f(\epsilon_0, h_0) u_0 = u_0^{p-1}$.

Proof. Suppose $F(\epsilon_0, h_0, u_0) = 0$.

Recall that E_{ϵ_0, h_0} is an inner product in $H^1(M)$. Let $v \in H^1(M)$. On one hand,

$$E_{\epsilon_0, h_0} \left(v, u_0 - A_{\epsilon_0, h_0}(u_0^+)^{p-1} \right) = 0.$$

On the other hand,

$$\begin{aligned} E_{\epsilon_0, h_0} \left(v, u_0 - A_{\epsilon_0, h_0}(u_0^+)^{p-1} \right) &= E_{\epsilon_0, h_0}(v, u_0) - E_{\epsilon_0, h_0} \left(v, A_{\epsilon_0, h_0}(u_0^+)^{p-1} \right) \\ &= E_{\epsilon_0, h_0}(v, u_0) - G_{\epsilon_0, h_0} \left(v, (u_0^+)^{p-1} \right) \end{aligned}$$

because of eq. (5.9). Then:

$$\begin{aligned} E_{\epsilon_0, h_0} \left(v, u_0 - A_{\epsilon_0, h_0}(u_0^+)^{p-1} \right) &= \\ &= \epsilon_0^{-n} \int_M \epsilon_0^2 (g_0 + h_0) (\nabla_{g_0+h_0} u_0, \nabla_{g_0+h_0} v) + f(\epsilon_0, h_0) u_0 v - (u_0^+)^{p-1} v \, d\mu_{g_0+h_0}. \end{aligned}$$

Comparing both expressions and taking into account that v is arbitrary, we conclude that:

$$-\epsilon_0^2 \Delta_{g_0+h_0} u_0 + f(\epsilon_0, h_0) u_0 = u_0^{p-1}.$$

□

As explained in section 5.2, a solution for eq. (5.5) is automatically non-negative, so $F(\epsilon_0, h_0, u_0) = 0$ also implies $u_0 \geq 0$. To conclude, we point out that due to regularity theory, we also know that $u_0 \in C^\infty(M)$.

5.5.3 Missing ingredient

Lemma 5.22. Let $(\epsilon_0, h_0, u_0) \in]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{p}} \times \mathcal{A}$ be such that $F(\epsilon_0, h_0, u_0) = 0$.

Suppose that

$$\left\{ \begin{array}{l} w \in \ker d_3 F_{(\epsilon_0, h_0, u_0)} \\ \forall \epsilon \in \mathbb{R} \, \forall h \in \mathcal{S}^k \left[\langle (d_{12} F)_{(\epsilon_0, h_0, u_0)}[\epsilon, h], w \rangle_{H_{\epsilon_0, h_0}^1} = 0 \right] \end{array} \right. .$$

Then $w = 0$.

Proof. During the proof,

- (ϵ, h) is any element in $\mathbb{R} \times \mathcal{S}^k$;
- $g = g_0 + h_0$.

By the definition of F ,

$$(d_{12} F)_{(\epsilon_0, h_0, u_0)}[\epsilon, h] = -dA_{(\epsilon_0, h_0)}[\epsilon, h](u_0^{p-1}).$$

Considering lemma 5.20,

$$\begin{aligned} \langle (d_{12}F)_{(\epsilon_0, h_0, u_0)}[\epsilon, h], w \rangle_{H_{\epsilon_0, h_0}^1} &= -\langle dA_{(\epsilon_0, h_0)}[\epsilon, h](u_0^{p-1}), w \rangle_{H_{\epsilon_0, h_0}^1} \\ &= -dG_{(\epsilon_0, h_0)}[\epsilon, h](u_0^{p-1}, w) + dE_{(\epsilon_0, h_0)}[\epsilon, h](A_{\epsilon_0, h_0}u_0^{p-1}, w). \end{aligned}$$

Take $\epsilon = 0$. We have $F(\epsilon_0, h_0, u_0) = 0$, so $A_{\epsilon_0, h_0}u_0^{p-1} = u_0$. Considering lemmas 5.18 and 5.19:

$$\begin{aligned} \langle (d_{12}F)_{(\epsilon_0, h_0, u_0)}[0, h], w \rangle_{H_{\epsilon_0, h_0}^1} &= -\frac{1}{2} \frac{1}{\epsilon_0^n} \int_M u_0^{p-1} w \operatorname{tr}_g h \, d\mu_g \\ &\quad + \frac{\epsilon_0^2}{\epsilon_0^n} \int_M b(h)(\nabla_g u_0, \nabla_g w) \, d\mu_g + \frac{\epsilon_0^2}{\epsilon_0^n} \int_M \frac{d(\operatorname{scal})_g[h]}{a} u_0 w \, d\mu_g + \\ &\quad + \frac{1}{2} \frac{1}{\epsilon_0^n} \int_M f(\epsilon_0, h_0) u_0 w \operatorname{tr}_g h \, d\mu_g. \end{aligned}$$

The variation of the scalar curvature is: (lemma 3.7 in [CK04])

$$d(\operatorname{scal})_g[h] = -\Delta_g(\operatorname{tr}_g h) + \delta_g \delta_g h - g(h, \operatorname{Ric}(g)).$$

(see example 4.8)

Let $h = \alpha g$ where $\alpha \in C^\infty(M)$. Then

$$d(\operatorname{scal})_g[\alpha g] = (1 - n)\Delta_g \alpha - \alpha \operatorname{scal}(g).$$

We have:

$$\begin{aligned} \langle (d_{12}F)_{(\epsilon_0, h_0, u_0)}[0, \alpha g], w \rangle_{H_{\epsilon_0, h_0}^1} &= -\frac{n}{2} \frac{1}{\epsilon_0^n} \int_M \alpha u_0^{p-1} w \, d\mu_g + \\ &\quad + \frac{n-2}{2} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M \alpha g(\nabla_g u_0, \nabla_g w) \, d\mu_g + \frac{1}{a} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M [(1-n)\Delta_g \alpha - \alpha \operatorname{scal}(g)] u_0 w \, d\mu_g + \\ &\quad + \frac{n}{2} \frac{1}{\epsilon_0^n} \int_M f(\epsilon_0, h_0) \alpha u_0 w \, d\mu_g. \end{aligned}$$

$-\epsilon_0^2 \Delta_g u_0 + f(\epsilon_0, h_0) u_0 - u_0^{p-1} = 0$. As $(\alpha w) \in H^1(M)$, we have:

$$\int_M \epsilon_0^2 g(\nabla_g u_0, \nabla_g(\alpha w)) + f(\epsilon_0, h_0) u_0 \alpha w \, d\mu_g = \int_M u_0^{p-1} \alpha w \, d\mu_g. \quad (5.10)$$

Back to our computation,

$$\begin{aligned} \langle (d_{12}F)_{(\epsilon_0, h_0, u_0)}[0, h], w \rangle_{H_{\epsilon_0, h_0}^1} &= -\frac{n}{2} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M g(\nabla_g u_0, \nabla_g(\alpha w)) \, d\mu_g + \\ &\quad + \frac{n-2}{2} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M \alpha g(\nabla_g u_0, \nabla_g w) \, d\mu_g + \frac{1}{a} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M [(1-n)\Delta_g \alpha - \alpha \operatorname{scal}(g)] u_0 w \, d\mu_g. \end{aligned}$$

By lemma 5.24:

$$\begin{aligned} \langle (d_{12}F)_{(\epsilon_0, h_0, u_0)}[0, h], w \rangle_{H_{\epsilon_0, h_0}^1} &= -\frac{n}{2} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M \alpha g(\nabla_g u_0, \nabla_g w) \, d\mu_g + \\ &\quad - \frac{n}{2} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M w g(\nabla_g u_0, \nabla_g \alpha) \, d\mu_g + \frac{n-2}{2} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M \alpha g(\nabla_g u_0, \nabla_g w) \, d\mu_g + \\ &\quad + \frac{1}{a} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M [(1-n)\Delta_g \alpha - \alpha \operatorname{scal}(g)] u_0 w \, d\mu_g. \end{aligned}$$

Obviously,

$$\begin{aligned} \langle (d_{12}F)_{(\epsilon_0, h_0, u_0)}[0, h], w \rangle_{H_{\epsilon_0, h_0}^1} &= -\frac{n}{2} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M w g(\nabla_g u_0, \nabla_g \alpha) \, d\mu_g + \\ &\quad - \frac{\epsilon_0^2}{\epsilon_0^n} \int_M \alpha g(\nabla_g u_0, \nabla_g w) \, d\mu_g + \frac{1}{a} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M [(1-n)\Delta_g \alpha - \alpha \operatorname{scal}(g)] u_0 w \, d\mu_g. \end{aligned}$$

By corollary 4.11,

$$\begin{aligned} \langle (d_{12}F)_{(\epsilon_0, h_0, u_0)}[0, h], w \rangle_{H_{\epsilon_0, h_0}^1} &= -\frac{n}{2} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M w g(\nabla_g u_0, \nabla_g \alpha) \, d\mu_g + \\ &\quad - \frac{\epsilon_0^2}{\epsilon_0^n} \int_M \alpha g(\nabla_g u_0, \nabla_g w) \, d\mu_g + \frac{n-1}{a} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M g(\nabla_g(u_0 w), \nabla_g \alpha) \, d\mu_g + \\ &\quad - \frac{1}{a} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M \alpha \operatorname{scal}(g) u_0 w \, d\mu_g. \end{aligned}$$

By lemma 5.24:

$$\begin{aligned} \langle (d_{12}F)_{(\epsilon_0, h_0, u_0)}[0, h], w \rangle_{H_{\epsilon_0, h_0}^1} &= -\frac{n}{2} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M w g(\nabla_g u_0, \nabla_g \alpha) \, d\mu_g + \\ &\quad - \frac{\epsilon_0^2}{\epsilon_0^n} \int_M \alpha g(\nabla_g u_0, \nabla_g w) \, d\mu_g + \frac{n-1}{a} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M u_0 g(\nabla_g w, \nabla_g \alpha) \, d\mu_g + \\ &\quad + \frac{n-1}{a} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M w g(\nabla_g u_0, \nabla_g \alpha) \, d\mu_g - \frac{1}{a} \frac{\epsilon_0^2}{\epsilon_0^n} \int_M \alpha \operatorname{scal}(g) u_0 w \, d\mu_g. \end{aligned}$$

$$\langle (d_{12}F)_{(\epsilon_0, h_0, u_0)}[0, h], w \rangle_{H_{\epsilon_0, h_0}^1} = 0, \text{ so}$$

$$\begin{aligned} 0 &= [2(n-1) - an] \epsilon_0^2 \int_M w g(\nabla_g u_0, \nabla_g \alpha) \, d\mu_g - 2a \epsilon_0^2 \int_M \alpha g(\nabla_g u_0, \nabla_g w) \, d\mu_g + \\ &\quad + 2(n-1) \epsilon_0^2 \int_M u_0 g(\nabla_g w, \nabla_g \alpha) \, d\mu_g - 2 \epsilon_0^2 \int_M \alpha \operatorname{scal}(g) u_0 w \, d\mu_g. \quad (5.11) \end{aligned}$$

By lemma 5.25, $-\epsilon_0^2 \Delta_g w + f(\epsilon_0, h_0)w = (p-1)wu_0^{p-2}$. So

$$\int_M \epsilon_0^2 g(\nabla_g w, \nabla_g(\alpha u_0)) + f(\epsilon_0, h_0) \alpha u_0 w - (p-1) \alpha u_0^{p-1} w \, d\mu_g = 0.$$

By the product rule:

$$\int_M \epsilon_0^2 u_0 g(\nabla_g w, \nabla_g \alpha) d\mu_g = - \int_M \epsilon_0^2 \alpha g(\nabla_g w, \nabla_g u_0) + f(\epsilon_0, h_0) \alpha u_0 w - (p-1) \alpha u_0^{p-1} w d\mu_g. \quad (5.12)$$

Substituting eq. (5.10) and eq. (5.12) on eq. (5.11), we see that

$$\begin{aligned} \int_M \alpha \left\{ [an - 4(n-1) - 2a] \epsilon_0^2 g(\nabla_g u_0, \nabla_g w) + \right. \\ \left. + [an - 4(n-1)] f(\epsilon_0, h_0) - 2\epsilon_0^2 \text{scal}(g) \right\} u_0 w + [2p(n-1) - an] u_0^{p-1} w \Big\} d\mu_g = 0. \end{aligned}$$

α is arbitrary, so we have the following equality μ_g -a.e.:

$$\begin{aligned} [2a + 4(n-1) - an] \epsilon_0^2 w \Delta_g u_0 + \left\{ [an - 4(n-1)] f(\epsilon_0, h_0) - 2\epsilon_0^2 \text{scal}(g) \right\} u_0 w + \\ + [2p(n-1) - an] u_0^{p-1} w = 0. \end{aligned} \quad (5.13)$$

$$-\epsilon_0^2 \Delta_g u_0 + f(\epsilon_0, h_0) u_0 - u_0^{p-1} = 0, \text{ so}$$

$$-[2a + 4(n-1) - an] \epsilon_0^2 w \Delta_g u_0 + [2a + 4(n-1) - an] f(\epsilon_0, h_0) u_0 w - [2a + 4(n-1) - an] u_0^{p-1} w = 0. \quad (5.14)$$

Summing eq. (5.13) with eq. (5.14), we obtain that μ_g -a.e.:

$$\left\{ 2a u_0 - [2a + 2(2-p)(n-1)] u_0^{p-1} \right\} w = 0.$$

Let $\lambda = \{(2a)/[2a + 2(2-p)(n-1)]\}^{1/(p-2)}$. $u_0 > 0$ is continuous and not identically λ , so

$$\left\{ 2a - [2a + 2(2-p)(n-1)] u_0^{p-2} \right\} \neq 0.$$

In particular, $w = 0$ in a non-empty open $I \subset M$ so w has a zero of infinite order. Consider the following unique continuation theorem ([AKS62]):

Theorem 5.23. Let M^n be a Riemannian manifold with a Lipschitz metric. If an exterior differential form u of rank p has a zero of infinite order at a point x of the manifold, and in each compact part of any coordinate patch the components of the differential du and of the codifferential δu are majorated by a constant times the sum of absolute values of components of u , then $u = 0$ on the whole manifold.

Considering such theorem, we conclude that $w = 0$. □

Taking into account lemma 5.22, we can provide a direct proof that F satisfies condition 2 in theorem 5.8. We reproduce the argument here for completeness, as it corresponds to the same employed in [MP09].

Let $(\epsilon_0, h_0, u_0) \in]0, \tilde{\epsilon}[\times \mathcal{B}_{\tilde{\rho}} \times \mathcal{A}$ be such that $F(\epsilon_0, h_0, u_0) = 0$ and $b \in H^1(M)$. $d_3 F_{(\epsilon_0, h_0, u_0)}$ is self-adjoint and Fredholm of index 0, so

$$\text{im } d_3 F_{(\epsilon_0, h_0, u_0)} \oplus \ker d_3 F_{(\epsilon_0, h_0, u_0)} = H^1(M).$$

Take a basis $\{w_1, \dots, w_m\}$ for $\ker d_3 F_{(\epsilon_0, h_0, u_0)}$. For $i = 1, \dots, m$; consider the linear functionals

$$f_i(\epsilon, h) = E_{\epsilon, h} \left(d_3 F_{(\epsilon_0, h_0, u_0)}[\epsilon, h], w_i \right).$$

By lemma 5.22, $\{f_1, \dots, f_m\}$ is linearly independent. Then there are linearly independent $\{(\epsilon_1, h_1), \dots, (\epsilon_m, h_m)\} \subset \mathbb{R} \times \mathcal{S}^k$ for which given $i = 1, \dots, m$; $f_i(\epsilon_i, h_i) = 1$.

5.5.4 Technical details

Lemma 5.24. If $u \in C^2(M)$ and $v \in H^1(M)$, then $uv \in H^1(M)$. In such case,

$$\nabla_{g_0}(uv) = u\nabla_{g_0}v + v\nabla_{g_0}u.$$

Proof. $H^1(M)$ is the completion of $C^\infty(M)$ with respect to $\|\cdot\|_{H^1}$.

Let (v_n) be a Cauchy sequence in $C^\infty(M)$ such that $v_n \xrightarrow{H^1} v$.

$$\begin{aligned} \|u(v_n - v_m)\|_{H^1}^2 &= \langle u - \Delta_{g_0}u, v_n - v_m \rangle_{L^2} \\ &\leq \|u - \Delta_{g_0}u\|_{L^2} \|v_n - v_m\|_{L^2} \end{aligned}$$

So (uv_n) is a Cauchy sequence in $H^1(M)$ and $uv \in H^1(M)$.

$\nabla_{g_0}(uv) = u\nabla_{g_0}v + v\nabla_{g_0}u$: this equality clearly holds for v_n .

Note that

$$v_n \xrightarrow{H^1} v \implies \nabla_{g_0}v_n \xrightarrow{L^2(T^*M)} \nabla_{g_0}v.$$

The other convergence comes from the DCT and the fact that $\mu_{g_0}(M) < \infty$ so $L^2(M) \subset L^1(M)$. $\square$

Lemma 5.25. Suppose $F(\epsilon_0, h_0, u_0) = 0$ and $w \in \ker d_3 F_{(\epsilon_0, h_0, u_0)}$. Then

$$-\epsilon_0^2 \Delta_g w + f(\epsilon_0, h_0)w = (p-1)wu_0^{p-2}.$$

Proof. $d_3 F_{(\epsilon_0, h_0, u_0)}[u] = u - A_{\epsilon_0, h_0} \left((p-1)uu_0^{p-2} \right)$, so $w = (p-1)A^{\epsilon_0, h_0}(wu_0^{p-2})$.

Definition 5.17 implies that given $\varphi \in C^\infty(M)$,

$$\begin{aligned} \langle w, \varphi \rangle_{H_{\epsilon_0, h_0}^1} &= \langle (p-1)A_{\epsilon_0, h_0}(wu_0^{p-2}), \varphi \rangle_{H_{\epsilon_0, h_0}^1} \\ &= G_{\epsilon_0, h_0} \left((p-1)wu_0^{p-2}, \varphi \right). \end{aligned}$$

That is, $-\epsilon_0^2 \Delta_g w + f(\epsilon_0, h_0)w = (p-1)wu_0^{p-2}$. $\square$

5.6 Proof of multiplicity of solutions

5.6.1 Context and preliminaries

Let $\hat{\rho} > 0$ and $\hat{\epsilon} > 0$ satisfy the conditions at the beginning of section 5.5.1. We still denote:

- $\epsilon \in]0, \hat{\epsilon}[$;
- $h \in \mathcal{B}_{\hat{\rho}}$;
- $g = g_0 + h \in (g_0 + \mathcal{B}_{\hat{\rho}})$.

The general form of the results which we shall consider is “there are $\hat{\epsilon}, \hat{\rho} > 0$ for which a certain property holds when $(\epsilon, h) \in]0, \hat{\epsilon}[\times \mathcal{B}_{\hat{\rho}}$ ”. The general idea as we go from one result to another is to always consider the minimum between the last $\hat{\epsilon}, \hat{\rho} > 0$ obtained and the new ones so that all results considered beforehand hold.

The remark below is a corollary of the fact that $\{U_r\}$ is finite and that given r ,

$$v \mapsto g_0(v, v) / |\mathrm{d}(\varphi_r)(v)|^2$$

is bounded at U_r :

Remark. There are $0 < c \leq C < \infty$ such that

$$\forall r \forall \xi \in T(U_r) \quad c |\mathrm{d}(\varphi_r)(\xi)|^2 \leq g_0(\xi, \xi) \leq C |\mathrm{d}(\varphi_r)(\xi)|^2. \quad (5.15)$$

Let us recall the following classical result in [Ehr74]:

Theorem 5.26. Let M be compact. Then the map $g \mapsto i_g(M)$, which takes a smooth Riemannian metric g to the injectivity radius of (M, g) , is continuous with respect to the C^2 topology on the set of smooth Riemannian metrics.

Due to this theorem, there is $\hat{\rho} > 0$ for which $R_0 = \inf_{h \in \mathcal{B}_{\hat{\rho}}} i_g(M) > 0$. Fix $R \in]0, R_0/2[$.

5.6.2 The functional $J_{\epsilon, h}$ and the Nehari manifold $N_{\epsilon, h}$

As noted in section 5.4, we shall use the fact that solutions of

$$-\epsilon^2 \Delta_g u + \left[\epsilon^2 \frac{\mathrm{scal}(g)}{a} + 1 \right] u = (u^+)^{p-1}$$

are precisely the critical points of

$$\begin{aligned} J_{\epsilon, h}(u) &= \epsilon^{-n} \int_M \frac{\epsilon^2}{2} |\Delta_g u|_g^2 + \frac{1}{2} f(\epsilon, h) u - \frac{1}{p} (u^+)^p \, \mathrm{d}\mu_g \\ &= \frac{1}{2} \|u\|_{H_{\epsilon, h}^1}^2 - \frac{1}{p} \|u^+\|_{L_{\epsilon, h}^p}^p. \end{aligned}$$

If we prove that the Nehari manifold

$$N_{\epsilon, h} = \left\{ u \in H^1(M) \setminus \{0\} : \mathrm{d}(J_{\epsilon, h})_u[u] = 0 \right\}$$

is a natural constraint for $J_{\epsilon,h}$, we conclude that a critical point of $J_{\epsilon,h}|_{N_{\epsilon,h}}$ is a critical point of $J_{\epsilon,h}$. Let us verify that $N_{\epsilon,h}$ verifies the conditions of proposition 2.20.

First of all, given $u \in H^1(M)$ for which $u^+ \neq 0$, there is a unique $\lambda_{\epsilon,h}(u) \in]0, \infty[$ such that $\lambda_{\epsilon,h}(u) \in N_{\epsilon,h}$. Indeed, consider the polynomial in t :

$$d(J_{\epsilon,h})_{tu}[tu] = t^2 E_{\epsilon,h}(u, u) - t^p \|u^+\|_{L_{\epsilon,h}^p}^p.$$

This polynomial has a unique nonzero root $\lambda_{\epsilon,h}(u)$. More specifically,

Definition 5.27.

$$\lambda_{\epsilon,h}(u)^{p-2} = \left[\int_M (u^+)^p d\mu_g \right]^{-1} \int_M \epsilon^2 |\nabla_g u|_g^2 + f(\epsilon, h) u^2 d\mu_g.$$

A corollary of this discussion is that $N_{\epsilon,h} \neq \emptyset$.

On one hand,

$$\begin{aligned} J_{\epsilon,h}(\lambda_{\epsilon,h}(u)u) &= \frac{1}{2} \lambda_{\epsilon,h}(u)^2 \|u\|_{H_{\epsilon,h}^1}^2 - \frac{1}{p} \lambda_{\epsilon,h}(u)^p \|u^+\|_{L_{\epsilon,h}^p}^p \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) \lambda_{\epsilon,h}(u)^2 \|u\|_{H_{\epsilon,h}^1}^2 && \text{(because } u \in N_{\epsilon,h} \text{)} \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) \left(\frac{\|u\|_{H_{\epsilon,h}^1}}{\|u^+\|_{L_{\epsilon,h}^p}} \right)^{p/(p-2)}. \end{aligned}$$

On the other hand, we proved in section 5.5 that the inclusion $H_{\epsilon,h}^1 \hookrightarrow L_{\epsilon,h}^p$ is bounded. Therefore, for some $K > 0$:

$$\|u\|_{L_{\epsilon,h}^p} \leq K \|u\|_{H_{\epsilon,h}^1}.$$

$u \in N_{\epsilon,h}$ implies $u = \lambda_{\epsilon,h}(u)u$. Therefore, for $u \in N_{\epsilon,h}$:

$$\begin{aligned} J_{\epsilon,h}(u) &= J_{\epsilon,h}(\lambda_{\epsilon,h}(u)u) \\ &\geq \left(\frac{1}{2} - \frac{1}{p} \right) K^{-p/(p-2)}. \end{aligned}$$

I.e., $N_{\epsilon,h}$ is bounded away from 0. To conclude, $J_{\epsilon,h}$ is of class C^2 because $p > 2$ and a simple calculation shows that given $u \in N_{\epsilon,h}$,

$$d^2(J_{\epsilon,h})_u[u, u] = (p-2) \|u\|_{H_{\epsilon,h}^1}^2 > 0.$$

In this context, we define

$$m_{\epsilon,h} = \inf_{N_{\epsilon,h}} J_{\epsilon,h} > 0.$$

5.6.3 Limiting equation and solution on $\mathbb{R}^n$

It is known that there exists a unique (up to translation) non-zero ground-state solution U for $-\Delta U + U = U^{p-1}$ on $\mathbb{R}^n$. Furthermore, such solution is positive, spherically symmetric and decreases exponentially at infinity. Note that such PDE is variational with respect to the functional

$$J_\infty: H^1(\mathbb{R}^n) \rightarrow \mathbb{R}$$

$$v \mapsto \int_{\mathbb{R}^n} \frac{1}{2} |\nabla v|^2 + \frac{1}{2} v^2 - \frac{1}{p} (v^+)^p \, dx.$$

In such context, define

$$m_\infty = \min \left\{ v \in H^1(\mathbb{R}^n) \setminus \{0\} : d(J_\infty)_v[v] = 0 \right\}.$$

We clearly have $J_\infty(U) = m_\infty$. Given $\epsilon > 0$, we define $U_\epsilon(x) = U(x/\epsilon)$. Note that U_ϵ is a solution of $-\epsilon^2 \Delta U_\epsilon + U_\epsilon = U_\epsilon^{p-1}$.

5.6.4 The map $i_{\epsilon,h}$

$(\mathbb{R}^n, \langle \cdot, \cdot \rangle) \simeq (TM_x, g_x)$ as normed vector spaces, so we identify these in the following.

Let $\psi_R: [0, \infty[\rightarrow [0, 1]$ be such that

$$\begin{cases} \psi_R(t) = 1 & (t \in [0, R]) \\ \psi'_R(t) = -1/R & (t \in]R, 2R[) \\ \psi_R(t) = 0 & (t \in [2R, \infty[) \end{cases}.$$

For $x \in M$, $\epsilon > 0$ and $h \in \mathcal{B}_{\hat{\rho}}$, define

$$w_{\epsilon,h}^x(y) = \begin{cases} \psi_R(|\exp_x^{-1}(y)|) U_\epsilon \circ \exp_x^{-1}(y) & (y \in B_g(x, 2R)) \\ 0 & (\text{otherwise}) \end{cases}.$$

Note that

$$\begin{cases} w_{\epsilon,h}^x \circ \exp_x^{-1} = U_\epsilon & \text{in } B_g(x, R) \\ 0 < w_{\epsilon,h}^x \circ \exp_x^{-1} < U_\epsilon & \text{in } B_g(x, 2R) \setminus B_g(x, R) \\ w_{\epsilon,h}^x \circ \exp_x^{-1} = 0 & \text{in } y \in \mathbb{R}^n \setminus B_g(x, 2R) \end{cases}.$$

Definition 5.28 (The map i).

$$i_{\epsilon,h}: M \rightarrow N_{\epsilon,h}$$

$$x \mapsto \lambda_{\epsilon,h}(w_{\epsilon,h}^x) w_{\epsilon,h}^x.$$

Proposition 5.29. The following hold:

1. $i_{\epsilon,h}$ is continuous;
2. Given $\delta > 0$, there is $\hat{\epsilon} > 0$ for which $\forall \epsilon \in]0, \hat{\epsilon}[\, \forall x \in M \, \forall h \in \mathcal{B}_{\hat{\rho}} \, [i_{\epsilon,h}(x) \in \Sigma_{\epsilon,h}^{m_\infty + \delta}]$.

Let us comment on the proof of such result. For more details, we refer the reader to proposition 4.1 in [GM11].

We are interested in integrals whose domains are geodesic balls of small radius $\epsilon > 0$, so we can use geodesic coordinates to compute these integrals as integrals in $\mathbb{R}^n$. In this context, the idea of the proof is that uniformly on $h \in \mathcal{B}_{\hat{\rho}}$ and on $x \in M$,

$$\begin{cases} \epsilon^{-n} \|w_{\epsilon,h}^x\|_{L^2(M,g)}^2 \xrightarrow{\epsilon \rightarrow 0^+} \|U\|_{L^2(\mathbb{R}^n)}^2 \\ \epsilon^{-n} \|w_{\epsilon,h}^x\|_{L^p(M,g)}^p \xrightarrow{\epsilon \rightarrow 0^+} \|U\|_{L^p(\mathbb{R}^n)}^p \\ \epsilon^{2-n} \|\nabla_g(w_{\epsilon,h}^x)\|_{L^2(M,g)}^2 \xrightarrow{\epsilon \rightarrow 0^+} \|\nabla U\|_{L^2(\mathbb{R}^n)}^2 \end{cases}.$$

These imply $\lambda_{\epsilon,h}(w_{\epsilon,h}^x) \xrightarrow{\epsilon \rightarrow 0^+} 1$. We conclude that uniformly on $h \in \mathcal{B}_{\hat{\rho}}$ and on $x \in M$,

$$J_{\epsilon,h} \circ i_{\epsilon,h}(x) \xrightarrow{\epsilon \rightarrow 0^+} m_{\infty}.$$

Remark. uniformly with respect to $h \in \mathcal{B}_{\hat{\rho}}$,

$$\limsup_{\epsilon \rightarrow 0^+} m_{\epsilon,h} \leq m_{\infty}$$

5.6.5 Concentration

As explained in the sketch of the proof (section 5.4), we can only construct the application $c_{\epsilon,h}: N_{\epsilon,h} \rightarrow M$ because every $u \in \Sigma_{\epsilon,h}^{m_{\infty}+\delta}$ admits (at least) a point $x_u \in M$ which concentrates the integral of $(u^+)^p$. The following result is the precise statement which corresponds to such idea:

Theorem 5.30. Given $\eta \in]0, 1[$, there are $\hat{\rho}, \hat{\epsilon}, \hat{\delta} > 0$ for which given $h \in \mathcal{B}_{\hat{\rho}}$, $\epsilon \in]0, \hat{\epsilon}[$, $\delta \in]0, \hat{\delta}[$ and $u \in \Sigma_{\epsilon,h}^{m_{\epsilon,h}+\delta}$, there is $x \in M$ such that:

$$\int_{B_g(x,R)} (u^+)^p d\mu_g \geq \eta \int_M (u^+)^p d\mu_g.$$

The main results in this section are this theorem and the fact that there is $\hat{\rho} > 0$ for which $\rho \in]0, \hat{\rho}[$ implies $m_{\epsilon,h} \xrightarrow{\epsilon \rightarrow 0^+} m_{\infty}$ uniformly with respect to $h \in \mathcal{B}_{\hat{\rho}}$.

To prove our main theorem, we follow three subsections:

1. in the first subsection, we prove a result (proposition 5.31) of concentration of a fixed part of the total integral of $(u^+)^p$ in a certain $x \in M$;
2. in the second subsection, we prove the uniform limit above;
3. in the third subsection, we develop preliminary lemmas.

Concentration of a fixed part of the total integral of $(u^+)^p$

Proposition 5.31. There are $\hat{\rho} > 0$, $\gamma > 0$ and $\hat{\epsilon} > 0$ for which given $\epsilon \in]0, \hat{\epsilon}[$, $\delta > 0$, $h \in \mathcal{B}_{\hat{\rho}}$ and $u \in \Sigma_{\epsilon, h}^{m_\infty + \delta}$, there is $x \in M$ for which

$$\epsilon^{-n} \int_{B_g(x, 2\zeta(\epsilon))} (u^+)^p d\mu_g \geq \gamma,$$

where $\zeta(\epsilon) = \sqrt{\rho/c + 1\epsilon}$.

Suppose that there are $K \in \mathbb{N}$, $\hat{\rho} > 0$ and $\hat{\epsilon} > 0$ for which given $\epsilon \in]0, \hat{\epsilon}[$, there is a finite collection $\{P_i\}$ of closed subsets of M with properties:

1. $M = \bigcup_i P_i$;
2. $\forall i \neq j \quad [P_i \cap P_j \subset \partial(P_i) \cap \partial(P_j)]$;
3. $\exists 0 < \xi(\epsilon) < \zeta(\epsilon) \quad \forall i \quad \exists x_i \in P_i \quad \forall h \in \mathcal{B}_{\hat{\rho}} \quad [B_g(x_i, \xi(\epsilon)) \subset P_i \subset B_g(x_i, 2\zeta(\epsilon))]$;
4. $\forall x \in M \quad \forall h \in \mathcal{B}_{\hat{\rho}} \quad \left| \left\{ i : x \in B_g(x_i, 3\zeta(\epsilon)) \right\} \right| \leq K$.

The details on this hypothesis are found at section 5.6.7. Such collection $\{P_i\}$ puts us in a context similar to the one considered in prop. 3.6 in [Pet19], so we can prove proposition 5.31 using the same techniques.

Proof. Given a borel set $U \subset M$, we denote its characteristic function by χ_U .

Fix $\epsilon \in]0, \hat{\epsilon}[$, $\delta > 0$, $h \in \mathcal{B}_{\hat{\rho}}$ and $u \in \Sigma_{\epsilon, h}^{m_\infty + \delta}$. Let us show that there is $\gamma > 0$ as in the statement of the proposition. Let $u_i = u^+ \chi_{P_i}$. Then

$$\begin{aligned} \epsilon^{-n} \int_M \epsilon^2 |\nabla_g u|_g^2 + f(\epsilon, h) u^2 d\mu_g &= \epsilon^{-n} \int_M (u^+)^p d\mu_g \\ &= \sum_i \left(\epsilon^{-n(p-2)/p} \|u_i\|_{L^p(M, g)}^{p-2} \right) \left(\epsilon^{-2n/p} \|u_i\|_{L^p(M, g)}^2 \right) \\ &\leq \max_i \left\{ \epsilon^{-n(p-2)/p} \|u_i\|_{L^p(M, g)}^{p-2} \right\} \sum_i \epsilon^{-2n/p} \|u_i\|_{L^p(M, g)}^2. \end{aligned} \quad (5.16)$$

Let ϕ_ϵ be the continuous cut-off function in $\mathbb{R}^n$ which is 1 in $B(0, 2\zeta(\epsilon))$, vanishes away from $B(0, 3\zeta(\epsilon))$ and such that $|\nabla \phi_\epsilon| = 1/\zeta(\epsilon)$ in the intermediate annulus. Define

$$u_{i, \epsilon} = \phi_\epsilon \circ d_g(\cdot, x_i) u^+.$$

Considering that $u_i \leq u_{i, \epsilon}$ and the Sobolev embedding theorem (theorem 4.19), we obtain that for some $L > 0$:

$$\epsilon^{-2n/p} \|u_i\|_{L^p(M, g)}^2 \leq \epsilon^{-2n/p} \|u_{i, \epsilon}\|_{L^p(M, g)}^2 \leq L \epsilon^{-n} \int_M \epsilon^2 |\nabla_g(u_{i, \epsilon})|_g^2 + f(h, \epsilon) u_{i, \epsilon}^2 d\mu_g. \quad (5.17)$$

As $u_{i,\epsilon} \leq u^+$, we have the following inequality on $B_g(x_i, 3\zeta(\epsilon)) \setminus P_i$:

$$\epsilon^2 |\nabla_g(u_{i,\epsilon})|_g^2 \leq 2(u^+)^2 + 2\epsilon^2 |\nabla_g(u^+)|_g^2. \quad (5.18)$$

Indeed,

$$\begin{aligned} \nabla_g(u_{i,\epsilon}) &= \nabla_g(\phi_\epsilon \circ d(\cdot, x_i)(u^+)) \\ &= u^+ \nabla_g(\phi_\epsilon \circ d(\cdot, x_i)) + \phi_\epsilon \circ d(\cdot, x_i) \nabla_g(u^+). \end{aligned}$$

Then

$$\begin{aligned} |\nabla_g(u_{i,\epsilon})|_g^2 &= (u^+)^2 \left| \nabla_g(\phi_\epsilon \circ d(\cdot, x_i)) \right|_g^2 + \\ &\quad + 2\phi_\epsilon \circ d(\cdot, x_i) u^+ g\left(\nabla_g(\phi_\epsilon \circ d(\cdot, x_i)), \nabla_g(u^+)\right) + [\phi_\epsilon \circ d(\cdot, x_i)]^2 |\nabla_g(u^+)|_g^2. \end{aligned}$$

Inequality (5.18) is then a consequence of the construction of ϕ_ϵ and the obvious inequality:

$$\left| u^+ \nabla_g(\phi_\epsilon \circ d(\cdot, x_i)) - \phi_\epsilon \circ d(\cdot, x_i) \nabla_g(u^+) \right|_g^2 \geq 0.$$

Together, inequalities (5.16), (5.17) and (5.18) imply:

$$\begin{aligned} \epsilon^{-n} \int_M \epsilon^2 |\nabla_g u|_g^2 + f(\epsilon, h) u^2 \, d\mu_g &\leq \Lambda L \epsilon^{-n} \sum_i \left\{ \int_{P_i} \epsilon^2 |\nabla_g(u_i)|_g^2 + f(\epsilon, h) u_i^2 \, d\mu_g + \right. \\ &\quad \left. + \int_{C_i} 2 \left[\epsilon^2 |\nabla_g(u^+)|_g^2 + (u^+)^2 \right] + f(\epsilon, h) (u^+)^2 \, d\mu_g \right\}, \end{aligned}$$

where $\Lambda = \max_i \left\{ \epsilon^{-n(p-2)/p} \|u_i\|_{L^p(M,g)}^{p-2} \right\}$ and $C_i = B_g(x_i, 3\zeta(\epsilon)) \setminus P_i$.

Each $x \in M$ can be in at most K of the C_i s, so

$$\begin{aligned} \epsilon^{-n} \int_M \epsilon^2 |\nabla_g u|_g^2 + f(\epsilon, h) u^2 \, d\mu_g &\leq \Lambda L \epsilon^{-n} \int_M \epsilon^2 |\nabla_g(u^+)|_g^2 + f(\epsilon, h) (u^+)^2 \, d\mu_g + \\ &\quad + K \int_M 2 \left[\epsilon^2 |\nabla_g(u^+)|_g^2 + (u^+)^2 \right] + f(\epsilon, h) (u^+)^2 \, d\mu_g, \end{aligned}$$

which implies $\Lambda L(1 + 4K) \geq 1$. The proposition follows with

$$\gamma = \left[\frac{1}{L(1 + 4K)} \right]^{p/(p-2)}.$$

Remark. γ depends on K . Due to corollary 5.48, we see that γ does not depend on $\epsilon \in]0, \hat{\epsilon}[$.

□

$\lim_{\epsilon \rightarrow 0^+} m_{\epsilon, h} = m_\infty$ **uniformly with respect to** $h \in \mathcal{B}_\rho$

Proposition 5.32. Given $\eta \in]0, 1[$, there are $\hat{\delta}, \hat{\rho}, \hat{\epsilon} > 0$ for which given $\delta \in]0, \hat{\delta}[$, $\epsilon \in]0, \hat{\epsilon}[$, $h \in \mathcal{B}_{\hat{\rho}}$ and $u \in \Sigma_{\epsilon, h}^{m_\infty + \delta}$, there is $x \in M$ for which

$$\frac{p-2}{2p} \epsilon^{-n} \int_{B_g(x, R)} (u^+)^p d\mu_g \geq (1-\eta)m_\infty$$

Proof. It suffices to suppose that $u \in \Sigma_{\epsilon, h}^{m_{\epsilon, h} + 2\delta}$. Indeed, $\lim_{(\epsilon, h) \rightarrow (0, 0)} m_{\epsilon, h} = m_\infty$, so it suffices to take sufficiently small $\hat{\rho}, \hat{\epsilon} > 0$.

Suppose the proposition does not hold. In particular, there are sequences (δ_l) and (ϵ_l) in $]0, 1[$, (h_l) in $\mathcal{S}^k$, $(u_l) \in H^1(M)$ such that $\delta_l, \epsilon_l, h_l \rightarrow 0$, $u_l \in \Sigma_{\epsilon_l, h_l}^{m_{\epsilon_l, h_l} + 2\delta_l}$ and for all $x \in M$,

$$K_l(x) = \frac{p-2}{2p} \epsilon_l^{-n} \int_{B_{g_l}(x, R)} (u_l^+)^p d\mu_{g_l} < (1-\eta)m_\infty.$$

To simplify the notation, write:

- J_l for J_{ϵ_l, h_l} ;
- N_l for N_{ϵ_l, h_l} ;
- H_l^1 for H_{ϵ_l, h_l}^1 .

Let $\kappa_l = \min\{\lambda_l^2, \delta_l\}$, where $\lambda_l > 0$ is such that

$$\|u_l - v\|_{H^1} \leq \lambda_l \implies \|(u_l^+)^p - (v^+)^p\|_{L^1} < \left[(1-\eta)m_\infty - \max_M K_l \right].$$

There is such λ_l because convergence in H^1 implies convergence in L^p by the Sobolev embedding theorem. Moreover, $\max_M K_l < (1-\eta)m_\infty$.

By Ekeland's variational principle (theorem 2.18) with $\alpha = \kappa_l$, $\beta = \sqrt{\kappa_l}$, we obtain $v_l \in N_l$ such that

$$\begin{cases} \|u_l - v_l\|_{H^1} \leq \sqrt{\kappa_l} \leq \sqrt{\delta_l} \\ J_l(v_l) \leq J_l(u_l) \\ \|d(J_l)_{v_l}\| \leq \sqrt{\kappa_l} \leq \sqrt{\delta_l} \end{cases}.$$

Note that $\forall \xi \in H^1(M)$ $\left[\|d(J_l)_{v_l}[\xi]\| \leq \sqrt{\delta_l} \|\xi\|_{H_l^1} \right]$. By proposition 5.31, there is a sequence (x_l) for which

$$\epsilon_l^{-n} \int_{B_{g_l}(x_l, 2\zeta(\epsilon_l))} (v_l^+)^p d\mu_{g_l} \geq \gamma.$$

Define

$$\begin{aligned} w_l &: \mathbb{R}^n \rightarrow \mathbb{R} \\ z &\mapsto v_l \circ \exp_{x_l}(2\zeta(\epsilon_l)z) \psi_R(2\zeta(\epsilon_l)|z|). \end{aligned}$$

$\text{supp } w_l \subset B(0, r_l)$, where $r_l = R/\zeta(\epsilon_l)$, so $w_l \in H^1(\mathbb{R}^n)$.

Lemmas 5.33, 5.34 are technical and are proved using the same techniques as lemmas 5.4 and 5.5 in [GM11]. In such context, we only comment on the ideas used in their proofs.

Lemma 5.33. There exists $w \in H^1(\mathbb{R}^n)$ such that $w_l \xrightarrow{H^1} w$ and $w_l \xrightarrow{L^p_{loc}} w$. (up to subsequence)

The ideas behind the proof of this lemma are to use the remark after proposition 5.29 and to consider the construction of the w_l s in order to prove that (w_l) is bounded in $H^1(\mathbb{R}^n)$. A bounded sequence in a Banach space admits a weak limit and the strong convergence in L^p_{loc} is a consequence of the Sobolev embedding theorem in $\mathbb{R}^n$.

Lemma 5.34. $w \in H^1(\mathbb{R}^n)$ of last lemma is a positive weak solution of $-\Delta w + w = |w|^{p-2}w$.

To prove this lemma: first, we want to show that given $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $d(J_\infty)_w[\varphi] = 0$. Due to the application of Ekeland's variational principle, $d(J_l)_{v_l} \rightarrow 0$. When we analyse the integral which is the definition of the J_l s in open balls of small radius in local coordinates, we see that $d(J_l)_{v_l}[\hat{\varphi}_l] \rightarrow d(J_\infty)_w[\varphi]$ where

$$\hat{\varphi}_l: x \mapsto \varphi \left(1/[2\zeta(\epsilon_l)] \exp_{x_l}^{-1}(x) \right).$$

To finish, $w \neq 0$ due to proposition 5.31.

Let us continue with the proof of the main result. Given $\sigma \in]0, \eta[$ and $T > 0$, we have for large enough l :

$$\frac{p-2}{2p} \epsilon_l^{-n} \int_{B(0,T)} |w_l^+(x)|^p dx < \frac{1-\eta}{1-\sigma} m_\infty.$$

By lemma 5.34:

$$\frac{p-2}{2p} \|w\|_{H^1}^2 = \frac{p-2}{2p} \|w\|_{L^p}^p \geq m_\infty.$$

By lemma 5.33, for large enough $T > 0$ and l :

$$\frac{p-2}{2p} \int_{B(0,T)} |w_l^+(x)|^p dx > \frac{1-\eta}{1-\sigma} m_\infty.$$

Contradiction. □

Corollary 5.35. There is $\hat{\rho} > 0$ for which $\lim_{\epsilon \rightarrow 0^+} m_{\epsilon,h} = m_\infty$ uniformly with respect to $h \in \mathcal{B}_{\hat{\rho}}$.

Preliminary lemmas

This subsection aims to adapt the results in section 3 of [Pet19] to our current setting. In particular, the first three results hold without any adaptation.

The following proposition is a direct consequence of the definitions of $N_{\epsilon,h}$ and of $m_{\epsilon,h}$:

Proposition 5.36. Let $u \in N_{\epsilon,h}$ be of the form $u = u_1 + u_2$, where u_1, u_2 have disjoint

supports. For $i = 1, 2$; let

$$\begin{cases} \|(u_i)^+\|_{L^p(M,g)}^p = a_i \epsilon^n \\ \int_M \epsilon^2 |\nabla_g(u_i)|_g^2 + f(\epsilon, h) u_i^2 d\mu_g = b_i \epsilon^n \end{cases}.$$

Then

$$J_{\epsilon,h}(u) \geq m_{\epsilon,h} \left[\left(\frac{a_1}{b_1} \right)^{2/(p-2)} + \left(\frac{a_2}{b_2} \right)^{2/(p-2)} \right].$$

Lemma 5.37. Let $r > 0$ and x_1, x_2, y_1, y_2 be positive numbers st $x_1 + x_2 = y_1 + y_2 \in [r, 2r]$. Then

$$\varphi(x_1, x_2, y_1, y_2) = \left(\frac{a_1}{b_1} \right)^{2/(p-2)} + \left(\frac{a_2}{b_2} \right)^{2/(p-2)} > 1.$$

For any $\delta \in]0, 1[$, consider

$$A_\delta = \left\{ (x_1, x_2, y_1, y_2) \in]0, \infty[^4 : x_1 + x_2 = y_1 + y_2 \in [r, 2r]; x_1, x_2 \geq \delta r \right\}.$$

Let $\Psi(\delta) = \inf_{A_\delta} \varphi$. Then $\Psi:]0, 1[\rightarrow]1, \infty[$ is a continuous function which does not depend on r .

Corollary 5.38. Let $u \in N_{\epsilon,h}$ be of the form $u = u_1 + u_2$, where u_1, u_2 have disjoint supports. Suppose that $J_{\epsilon,h}(u) \in [m_{\epsilon,h}, 2m_{\epsilon,h}]$ and that there is $\delta > 0$ for which given $i = 1, 2$;

$$\|(u_i)^+\|_{L^p(M,g)}^p \geq \delta \epsilon^n.$$

Then $J_{\epsilon,h}(u) \geq \Psi(\delta) m_{\epsilon,h}$.

The last three results concern functions $u \in N_{\epsilon,h}$ of the form $u = u_1 + u_2$, where u_1, u_2 have disjoint supports. Fix $\delta > 0$. Now, we shall develop a method of associating a given $u \in \Sigma_{\epsilon,h}^{m_{\epsilon,h}+\delta}$ to a $\bar{u} \in \Sigma_{\epsilon,h}^{m_{\epsilon,h}+2\delta}$ of the form $\bar{u} = u_1 + u_2$, where u_1, u_2 have disjoint supports.

Fix $l > 1$ and $i < l$ positive integers.

Let $\varphi_{i,l}: [0, \infty[\rightarrow [0, 1]$ be smooth such that

$$\begin{cases} \varphi_{i,l}(t) = 1 & (t \in [0, R(i-1)/l]) \\ \varphi_{i,l}(t) = 0 & (t \in [Ri/l, \infty[) \end{cases}.$$

Let $\sigma_{i,l}: [0, \infty[\rightarrow [0, 1]$ be smooth such that

$$\begin{cases} \sigma_{i,l}(t) = 0 & (t \in [0, Ri/l]) \\ \sigma_{i,l}(t) = 1 & (t \in [R(i+1)/l, \infty[) \end{cases}.$$

Define

$$\begin{cases} \varphi_{i,l}^{x,h} = \varphi_{i,l} \circ d_g(x, \cdot) \\ \sigma_{i,l}^{x,h} = \sigma_{i,l} \circ d_g(x, \cdot) \end{cases}.$$

Given $u \in N_{\epsilon,h}$, define

$$\begin{cases} (u_1)_{i,l}^{x,h} = u \varphi_{i,l}^{x,h} \\ (u_2)_{i,l}^{x,h} = u \sigma_{i,l}^{x,h} \end{cases}.$$

Note that with such definition,

$$\begin{cases} u = (u_1)_{i,l}^{x,h} & \text{in } B_g(x, R(i-1)/l) \\ u = (u_2)_{i,l}^{x,h} & \text{in } M \setminus B_g(x, R(i+1)/l) \end{cases}.$$

Let

$$\bar{u}_{i,l}^{x,h} = \lambda_{\epsilon,h} \left((u_1)_{i,l}^{x,h} + (u_2)_{i,l}^{x,h} \right) \left[(u_1)_{i,l}^{x,h} + (u_2)_{i,l}^{x,h} \right] \in N_{\epsilon,h}.$$

We just explained how to associate a given $u \in \Sigma_{\epsilon,h}^{m_{\epsilon,h}+\delta}$ to a $\bar{u} \in N_{\epsilon,h}$. But how to ensure that $J_{\epsilon,h}(\bar{u}) \leq m_{\epsilon,h} + 2\delta$?

Lemma 5.39. Given $\delta > 0$, there is $\hat{\epsilon} > 0$ for which given $\epsilon \in]0, \hat{\epsilon}[$ and $h \in \mathcal{B}_{\hat{\rho}}$, there is $l \in \mathbb{N}$ such that given $u \in \Sigma_{\epsilon,h}^{m_{\epsilon,h}+\delta}$ and $x \in M$, there is an integer $0 < i < l$ for which $\bar{u}_{i,l}^x \in \Sigma_{\epsilon,h}^{m_{\epsilon,h}+2\delta}$.

Remark. The crucial point of this adaptation of lemma 3.4 in [Pet19] is the fact that $\hat{\epsilon} > 0$ does not depend on $h \in \mathcal{B}_{\hat{\rho}}$.

Let us comment on the proof of such lemma.

$\lim_{\epsilon \rightarrow 0+} m_{\epsilon,h} = m_\infty$ uniformly with respect to $h \in \mathcal{B}_{\hat{\rho}}$, so there is $\hat{\epsilon} > 0$ for which $\epsilon \in]0, \hat{\epsilon}[$ implies $\sup_{h \in \mathcal{B}_{\hat{\rho}}} m_{\epsilon,h} < \infty$. It is easy to see that there is $\eta \in]0, 1[$ such that

$$\forall h \in \mathcal{B}_{\hat{\rho}} \left[(m_{\epsilon,h} + \delta_0) \left(\frac{4-3\eta}{\eta} \right)^{2/(p-2)} (4-3\eta) < (m_{\epsilon,h} + 2\delta_0) \right].$$

Fix $h \in \mathcal{B}_{\hat{\rho}}$. First, take a positive even integer l st $6/l < 1 - \eta$. To find appropriate $i \in]0, l[\cap \mathbb{N}$, we consider a partition of $B_g(x, R)$ by annular regions A_j of radius $(2/l)R$, with $j = 1, \dots, l/2$. Take $i \in]0, l[\cap \mathbb{N}$ for which

$$\begin{cases} \|u_{1,i,l}^x + u_{2,i,l}^x\|_{L^p(M,g)}^p \geq \eta \|u\|_{L^p(M,g)}^p \\ \int_{A_i} |\nabla_g u|_g^2 d\mu_g \leq (1-\eta) \int_M |\nabla_g u|_g^2 d\mu_g \end{cases}.$$

Consider the following facts:

1. $|\varphi'_{i,l}|, |\sigma'_{i,l}| \leq l/R$;
2. $\exists \hat{\epsilon} > 0$ for which $\epsilon \in]0, \hat{\epsilon}[\implies \forall h \in \mathcal{B}_{\hat{\rho}} \left[4\epsilon^2 l^2 / R^2 \leq 3(1-\eta) \left(\frac{\text{scal}(g)}{a} \epsilon^2 + 1 \right) \right]$.
(indeed, $g \mapsto \text{scal}(g)$ is continuous as a map to $L^\infty(M)$)

We can use these inequalities to deduce the upper bound

$$\lambda_{\epsilon,h} \left((u_1)_{i,l}^{x,h} + (u_2)_{i,l}^{x,h} \right) \leq \left(\frac{4-3\eta}{\eta} \right)^{1/(p-2)}.$$

This gives us the desired upper bound for $J_{\epsilon,h}(\bar{u}_{i,l}^x)$.

Proof of theorem 5.30

Suppose otherwise. In particular, there are sequences (δ_l) and (ϵ_l) in $]0, 1[$, (h_l) in $\mathcal{S}^k$, $(u_l) \in H^1(M)$ such that

$$\begin{cases} \delta_l, \epsilon_l, h_l \rightarrow 0 \\ u_l \in \Sigma_{\epsilon_l, h_l}^{m_{\epsilon_l, h_l} + \delta_l} \end{cases}.$$

and for all $x \in M$,

$$\int_{B_{g_l}(x, R)} (u_l^+)^p d\mu_{g_l} < \eta \int_M (u_l^+)^p d\mu_{g_l}.$$

By proposition 5.31, there is a sequence (x_l) in M for which

$$\epsilon_l^{-n} \int_{B_{g_l}(x_l, \zeta(\epsilon_l))} (u_l^+)^p d\mu_{g_l} \geq \gamma. \quad (5.19)$$

By lemma 5.39, we obtain functions $\Sigma_{\epsilon_l, h_l}^{m_{\epsilon_l, h_l} + 2\delta_l} \ni \bar{u}_l = u_{l,1} + u_{l,2}$ where $u_{l,1}$ is supported in a ball centered at x_l ; $u_{l,1}$, $u_{l,2}$ have disjoint support and $u_l = \bar{u}_l$ in $B_{g_l}(x_l, \zeta(\epsilon_l))$ and in $M \setminus B_{g_l}(x_l, R)$.

We have

$$\epsilon_l^{-n} \int_M (u_{l,1}^+)^p d\mu_{g_l} \geq \gamma$$

and

$$\epsilon_l^{-n} \int_M (u_{l,2}^+)^p d\mu_{g_l} > \epsilon_l^{-n} (1 - \eta) \int_M (u_l^+)^p d\mu_{g_l} \geq (1 - \eta) \frac{2p}{p-2} m_{\epsilon_l, h_l}. \quad (5.20)$$

$\lim_{\epsilon \rightarrow 0^+} m_{\epsilon, h} = m_\infty$ uniformly with respect to $h \in \mathcal{B}_{\hat{p}}$. For sufficiently big l , we conclude that:

$$(1 - \eta) \frac{2p}{p-2} m_{\epsilon_l, h_l} \geq \frac{1 - \eta}{2} \frac{2p}{p-2} m_\infty.$$

Considering eq. (5.19) and eq. (5.20), we get $\|u_{l,1}\|_{L^p(M, g_l)}^p, \|u_{l,2}\|_{L^p(M, g_l)}^p \geq \delta \epsilon_l^n$ for sufficiently big l , where

$$\delta = \min \left\{ \gamma, \frac{1 - \eta}{2} \frac{2p}{p-2} m_\infty \right\}.$$

Furthermore, if l is sufficiently big so that $|m_{\epsilon_l, h_l} - m_\infty| < m_\infty/2$ and $\delta_l \in]0, m_\infty/4[$, we guarantee that $J_{\epsilon_l, h_l}(\bar{u}_l) \in [m_{\epsilon_l, h_l}, 2m_{\epsilon_l, h_l}]$.

We are in conditions to use corollary 5.38, so $J_{\epsilon_l, h_l}(\bar{u}_l) \geq \Psi(\delta) m_{\epsilon_l, h_l}$. $\Psi(\delta) > 1$. Therefore, for large enough l , $m_{\epsilon_l, h_l} + 2\delta_l < \Psi(\delta) m_{\epsilon_l, h_l}$.

Contradiction.

5.6.6 Center of mass and the map $c_{\epsilon,h}$

Let $u \in L^1(M)$ be nonnegative. Consider

$$P_u^g(x) = \int_M d_g(x, y) u(y) d\mu_g(y).$$

In [GK73] and [Kar77], H. Karcher and K. Grove developed the concept of **Riemannian center of mass** with respect to g of a function u supported in a geodesic ball $B_g(x, r)$ for sufficiently small $r > 0$ as the unique global minimum of the function P_u^g . In section 5 of [Pet19], J. Petean extends such notion to functions which are concentrated in small geodesic balls, yet are not necessarily supported in such balls.

In this section, we outline the construction done by Petean and we show that it applies to our context. In particular, we do not provide proofs for lemma 5.42 and theorem 5.43, as they follow directly from the arguments by Petean.

Definition 5.40. Given $u \in L^1(M) \setminus \{0\}$ and $r > 0$, the (u, r) -**concentration function** with respect to $(g_0 + h) = g \in \mathcal{M}^k$ is

$$C_{u,r}^g(x) = \|u\|_{L^1(M,g)}^{-1} \int_{B_g(x,r)} |u| d\mu_g.$$

Note that $C_{u,r}^g: M \rightarrow [0, 1]$ is a continuous function.

Definition 5.41. Given $u \in L^1(M) \setminus \{0\}$ and $r > 0$, the r -**concentration coefficient** of u with respect to $(g_0 + h) = g \in \mathcal{M}^k$ is

$$C_r^g(u) = \sup_{x \in M} \|u\|_{L^1(M,g)}^{-1} \int_{B_g(x,r)} |u| d\mu_g.$$

Consider the set of functions u whose r -concentration coefficient with respect to g is greater than $\mu \in (0, 1)$:

$$L_{r,\mu}^1(M, g) = \left\{ u \in L^1(M) : C_r(u) > \mu \right\}.$$

For $\eta \in (1/2, 1)$, define the piecewise affine map $\phi_\eta: \mathbb{R} \rightarrow [0, 1]$ given by:

$$\begin{cases} \phi_\eta(t) = 0 & (t \leq 1 - \eta) \\ \phi'_\eta(t) = (2\eta - 1)^{-1} & (1 - \eta < t < \eta) \\ \phi_\eta(t) = 1 & (t \geq \eta) \end{cases}.$$

Given $u \in L_{r,\eta}^1(M, g)$, let $\Phi_{r,\eta}^g(u)(x) = \phi_\eta \circ C_{u,r}^g(x)u(x)$. Considering the construction of $\Phi_{r,\eta}^g$, we have the following lemma:

Lemma 5.42. Given $u \in L_{r,\eta}^1(M, g)$, the support of $\Phi_{r,\eta}^g(u)$ is contained in a geodesic ball of radius $2r$. (centered at a point of maximal concentration)

The following theorem is the key result to associate a $u = i_{\epsilon,h}(x)$ which is concentrated enough to an $x_u \in M$ which is “close enough” to x for them to be connected by a unique minimizing geodesic whose image is contained in a normal ball:

Theorem 5.43. Let r_0 be the injectivity radius of (M, g) . For any $r < (1/2)r_0$ and

$\eta > 1/2$, there exists a continuous function $\mathbf{Cm}^g(r, \eta): L^1_{r, \eta}(M, g) \rightarrow M$ for which $C^g_{r, u}(x) > \eta$ implies $\mathbf{Cm}^g(r, \eta)(u) \in B_g(x, 2r)$.

In such context, $\mathbf{Cm}^g(r, \eta)(u)$ is called the (r, η) -**Riemannian center of mass** of u with respect to g .

Theorem 5.44.

$$c_{\epsilon, h} \circ i_{\epsilon, h} \simeq \text{id}_M.$$

Proof. Fix $\eta \in]1/2, 1[$. Take $\hat{\delta}, \hat{\epsilon}, \hat{\rho} > 0$ which satisfy the conditions of theorem 5.30.

Fix $\delta \in]0, \hat{\delta}[$. By corollary 5.35, we can suppose that $\hat{\epsilon}, \hat{\rho} > 0$ are small enough to have:

$$\forall h \in \mathcal{B}_{\hat{\rho}} \left[|m_{\epsilon, h} - m_{\infty}| < \delta/2 \right]. \quad (5.21)$$

Likewise, suppose that $\hat{\epsilon} > 0$ is small enough so that proposition 5.29 holds for $\delta/2$. Given $(\epsilon, h) \in]0, \hat{\epsilon}[\times \mathcal{B}_{\hat{\rho}}$, we have $i_{\epsilon, h}: M \rightarrow \Sigma^{m_{\infty} + \delta/2}_{\epsilon, h}$. Due to ineq. 5.21, we have $m_{\infty} + \delta/2 < m_{\epsilon, h} + \delta$ so $\Sigma^{m_{\infty} + \delta/2}_{\epsilon, h} \subset \Sigma^{m_{\epsilon, h} + \delta}_{\epsilon, h}$. Therefore, we have a continuous map $i_{\epsilon, h}: M \rightarrow \Sigma^{m_{\epsilon, h} + \delta}_{\epsilon, h}$.

By theorem 5.30, $u \in \Sigma^{m_{\epsilon, h} + \delta}_{\epsilon, h}$ implies $(u^+)^p \in L^1_{R, \eta}(M, g)$. By theorem 5.43, we have a continuous map

$$\begin{aligned} c_{\epsilon, h}: \Sigma^{m_{\epsilon, h} + \delta}_{\epsilon, h} &\rightarrow M \\ u &\mapsto \mathbf{Cm}^g(R, \eta) \left((u^+)^p \right). \end{aligned}$$

$c_{\epsilon, h} \circ i_{\epsilon, h}(x) \in B_g(x, 2R)$ so we have a homotopy between $c_{\epsilon, h} \circ i_{\epsilon, h}(x)$ to x by following the unique minimizing geodesic from $c_{\epsilon, h} \circ i_{\epsilon, h}(x)$ to x . $\square$

5.6.7 The collection $\{P_i\}$

In what follows, we explain how to obtain the collection $\{P_i\}$ described in section 5.6.5 in a series of steps. This will require us to expand on the concepts reviewed on section 2.5 to the context of a varying Riemannian metric of class C^k .

Denote $d: M \times M \rightarrow [0, \infty[$ the distance function on M induced by g_0 .

Lemma 5.45. The function $\mathcal{M}^k \rightarrow C(M \times M)$ which associates a metric g of class C^k to the distance function d_g it induces in M is continuous.

Proof. Let $x, y \in M$ be distinct points. Let $h \in \mathcal{B}_{\hat{\rho}}$.

There are two possible cases:

1. $d_g(x, y) \geq d(x, y)$: let $\gamma: [0, d(x, y)] \rightarrow M$ be a minimizing geodesic from x to y with respect to g_0 . We have:

$$d_g(x, y) - d(x, y) \leq \int_0^{d(x, y)} \sqrt{g(\gamma'(t), \gamma'(t))} - \sqrt{g_0(\gamma'(t), \gamma'(t))} dt.$$

Indeed, γ is a curve which links x to y but is not necessarily a (minimizing) geodesic with respect to g . Then:

$$\begin{aligned} d_g(x, y) - d(x, y) &\leq \int_0^{d(x, y)} \sqrt{1 + |h(\gamma'(t), \gamma'(t))|} - 1 \, dt \\ &\leq \int_0^{d(x, y)} \sqrt{1 + \|h\|_{S^0}/c} - 1 \, dt, \end{aligned}$$

which holds because ineq. 5.15 implies the following inequality for any given $\xi \in TM$:

$$\begin{aligned} h(\xi, \xi) &= \xi^i \xi^j h_{ij} \\ &\leq \left(\max_l |\xi^l| \right)^2 \sum_{i,j} |h_{ij}| \\ &\leq \|h\|_{S^0} g_0(\xi, \xi)/c. \end{aligned} \tag{5.22}$$

Therefore,

$$d_g(x, y) - d(x, y) \leq \left(\sqrt{\|h\|_{S^0}/c + 1} - 1 \right) d(x, y).$$

2. $d_g(x, y) < d(x, y)$: this time, take $\gamma: [0, d_g(x, y)] \rightarrow M$ a minimizing geodesic from x to y with respect to g . One can argue analogously and conclude that:

$$d(x, y) - d_g(x, y) \leq \left(\sqrt{\|h\|_{S^0}/c + 1} - 1 \right) d_g(x, y) < \left(\sqrt{\|h\|_{S^0}/c + 1} - 1 \right) d(x, y).$$

In any of the cases,

$$|d(x, y) - d_g(x, y)| \leq \left(\sqrt{\|h\|_{S^0}/c + 1} - 1 \right) d(x, y).$$

M is compact so $\text{diam}(M) < \infty$. (Hopf-Rinow) We then conclude that:

$$\|d - d_g\|_{L^\infty} \leq \left(\sqrt{\|h\|_{S^0}/c + 1} - 1 \right) \text{diam}(M).$$

In particular, the proof also shows that:

$$\frac{d(x, y)}{\sqrt{\|h\|_{S^0}/c + 1}} \leq d_g(x, y) \leq \sqrt{\|h\|_{S^0}/c + 1} d(x, y). \tag{5.23}$$

□

Lemma 5.46. Let

$$\mathcal{D} = \left\{ \mathcal{C} \subset M : \forall x, y \in \mathcal{C} \ [x \neq y \rightarrow B_{g_0}(x, \epsilon) \cap B_{g_0}(y, \epsilon) = \emptyset] \right\}.$$

I.e., $\mathcal{D}$ is the collection of subsets $\mathcal{C}$ of M for which geodesic balls in (M, g_0) centered in elements $x \in \mathcal{C}$, all with radius ϵ , are all pairwise disjoint. If $\mathcal{C} \in \mathcal{D}$ is maximal with

respect to $\subset$, then

$$|\mathcal{C}| \leq \left\lfloor \frac{\mu_{g_0}(M)}{\inf_M \mu_{g_0}(B_{g_0}(\cdot, \epsilon))} \right\rfloor < \infty.$$

Remark. The fact that there exists $N \in \mathbb{N}$ for which $\mathcal{C}$ maximal in $\mathcal{D}$ implies $|\mathcal{C}| \leq N$ does not really depend on considering a measure space structure. One may alternatively prove such fact by considering that M is a compact metric space and that the ϵ -packing number of a compact metric space is necessarily finite.

Remark. If $\{x_i\} \in \mathcal{D}$ is maximal, then $\{B_{g_0}(x_i, 2\epsilon)\}$ is an open cover for M .

Proof. Suppose otherwise. Let $x \in M \setminus \bigcup_i B_{g_0}(x_i, 2\epsilon)$. Given i ,

$$d(x, x_i) > 2\epsilon \implies B_{g_0}(x, \epsilon) \cap B_{g_0}(x_i, \epsilon) = \emptyset.$$

Then $\{x_i\} \cup \{x\} \in \mathcal{D}$. Contradiction with the maximality of $\{x_i\}$. $\square$

One can easily apply induction to construct maximal sets $\{x_i\} \in \mathcal{D}$. A consequence of Bishop-Gromov (theorem 2.17) is:

Lemma 5.47. There exist $K \in \mathbb{N}$ and $\hat{\epsilon} > 0$ for which given $\epsilon \in]0, \hat{\epsilon}[$, a maximal $\{x_i\} \in \mathcal{D}$ and $x \in M$, x is contained in at most K geodesic balls $B_{g_0}(x_i, 2\epsilon)$. I.e.,

$$\left| \{x_i : x \in B_{g_0}(x_i, 2\epsilon)\} \right| \leq K.$$

Proof. Suppose $B_{g_0}(x_i, 2\epsilon) \cap B_{g_0}(x_j, 2\epsilon) \neq \emptyset$. We have $B_{g_0}(x_j, \epsilon) \subset B_{g_0}(x_i, 5\epsilon)$, because

$$B_{g_0}(x_i, 2\epsilon) \cap B_{g_0}(x_j, 2\epsilon) \neq \emptyset \implies d(x_i, x_j) < 4\epsilon.$$

We know that

$$i \neq j \implies B_{g_0}(x_i, \epsilon) \cap B_{g_0}(x_j, \epsilon) = \emptyset.$$

We can estimate the number of indices j for which $B_{g_0}(x_i, 2\epsilon) \cap B_{g_0}(x_j, 2\epsilon) \neq \emptyset$ as follows:

$$\begin{aligned} \left| \{x_j : B_{g_0}(x_i, 2\epsilon) \cap B_{g_0}(x_j, 2\epsilon) \neq \emptyset\} \right| &\leq \frac{\mu_{g_0}(B_{g_0}(x_i, 5\epsilon))}{\min_j \mu_{g_0}(B_{g_0}(x_j, \epsilon))} \\ &\leq \max_j \frac{\mu_{g_0}(B_{g_0}(x_j, 9\epsilon))}{\mu_{g_0}(B_{g_0}(x_j, \epsilon))}, \end{aligned}$$

because

$$B_{g_0}(x_i, 2\epsilon) \cap B_{g_0}(x_j, 2\epsilon) \neq \emptyset \implies B_{g_0}(x_i, 5\epsilon) \subset B_{g_0}(x_j, 9\epsilon).$$

By Bishop-Gromov:

$$\left| \{x_j : B_{g_0}(x_i, 2\epsilon) \cap B_{g_0}(x_j, 2\epsilon) \neq \emptyset\} \right| \leq \frac{V_{\tilde{k}}(9\epsilon)}{V_{\tilde{k}}(\epsilon)},$$

where $\tilde{k} \in \mathbb{R}$ is such that $\text{Ric} \geq (n-1)\tilde{k}$ on (M, g_0) . Note the right-hand side term tends to a positive quantity as $\epsilon \rightarrow 0^+$. (it suffices to think about why this happens in $\mathbb{R}^n$) $\square$

Corollary 5.48. There exists $K \in \mathbb{N}$, $\hat{\rho} > 0$ and $\hat{\epsilon} > 0$ for which given $\epsilon \in]0, \hat{\epsilon}[$, a maximal $\{x_i\} \in \mathcal{D}$ and $x \in M$,

$$\forall h \in \mathcal{B}_{\hat{\rho}} \left| \left\{ x_i : x \in B_g(x_i, 3\zeta(\epsilon)) \right\} \right| \leq K ,$$

where $\zeta(\epsilon) = \sqrt{\rho/c + 1}\epsilon$.

Proof. Let $T^1M = \{v \in TM : g_0(v, v) = 1\}$. Given $v \in T^1M$ and $g \in \mathcal{M}^k$,

$$\left| (\text{Ric}(g) - \text{Ric}(g_0))(v, v) \right| \leq \|\text{Ric}(g) - \text{Ric}(g_0)\|_{\mathcal{S}^0} / c. \text{ (see ineq. 5.22)}$$

Given $\tilde{k} \in \mathbb{R}$, the map below is continuous:

$$\begin{aligned} \mathcal{M}^k &\rightarrow \mathcal{S}^{k-2} \\ g &\mapsto \text{Ric}(g) - (n-1)\tilde{k}g. \end{aligned}$$

Indeed, it suffices to look at the local form of Ric in terms of g to conclude that it is even smooth. Therefore, it is easy to see that there are $K \in \mathbb{N}$ and $\hat{\rho} > 0$ for which $h \in \mathcal{B}_{\hat{\rho}}$ implies $\text{Ric}(g) \geq K(n-1)g$. $\square$

Finally, let us construct our collection $\{P_i\}$.

Let $K \in \mathbb{N}$, $\hat{\rho} > 0$ and $\hat{\epsilon} > 0$ be such that the condition of corollary 5.48 is satisfied.

Fix $\epsilon \in]0, \hat{\epsilon}[$, a maximal $\{x_i\} \in \mathcal{D}$ and let $P_i = \{x \in M : \forall j \ d(x, x_i) \leq d(x, x_j)\}$.

Let us verify that this collection satisfies the following properties:

1. $M = \bigcup_i P_i$;
2. $\forall i \neq j \ [P_i \cap P_j \subset \partial(P_i) \cap \partial(P_j)]$;
3. $\exists 0 < \xi(\epsilon) < \zeta(\epsilon) \ \forall i \ \exists x_i \in P_i \ \forall h \in \mathcal{B}_{\hat{\rho}} \ [B_g(x_i, \xi(\epsilon)) \subset P_i \subset B_g(x_i, 2\zeta(\epsilon))]$;
4. $\forall x \in M \ \forall h \in \mathcal{B}_{\hat{\rho}} \ \left| \left\{ i : x \in B_g(x_i, 3\zeta(\epsilon)) \right\} \right| \leq K$.

$\{B_{g_0}(x_i, \epsilon)\}$ is disjoint, so $B_{g_0}(x_i, \epsilon) \subset P_i$. It also holds that $P_i \subset B_{g_0}(x_i, 2\epsilon)$. Indeed, suppose that $x \in P_i \setminus B_{g_0}(x_i, 2\epsilon)$. By the definition of P_i , we get

$$\forall j \ [\epsilon \leq d(x, x_i) \leq d(x, x_j)] .$$

Contradiction with the maximality of $\{x_i\}$ in $\mathcal{D}$.

Let $\xi(\epsilon) = \epsilon / \sqrt{\hat{\rho}/c + 1}$. Considering ineq. 5.23, we see that

$$\forall h \in \mathcal{B}_{\hat{\rho}} \ [B_g(x_i, \xi(\epsilon)) \subset B_{g_0}(x_i, \epsilon)] ,$$

hence $B_g(x_i, \xi(\epsilon)) \subset P_i$. Let $\zeta(\epsilon) = \sqrt{\hat{\rho}/c + 1}\epsilon$. Clearly,

$$\forall h \in \mathcal{B}_{\hat{\rho}} \ [B_{g_0}(x_i, 2\epsilon) \subset B_g(x_i, 2\zeta(\epsilon))] ,$$

so $P_i \subset B_g(x_i, 2\zeta(\epsilon))$.

Bibliography

- [AFP06] Kazuo Akutagawa, Luis A. Florit, and Jimmy Petean. On yamabe constants of riemannian products, 2006.
- [AIN06] Bernd Ammann, Alexandru D. Ionescu, and Victor Nistor. Sobolev spaces on lie manifolds and regularity for polyhedral domains. *Documenta mathematica*, 2006.
- [AKS62] N. Aronszajn, A. Krzywicki, and J. Szarski. A unique continuation theorem for exterior differential forms on riemannian manifolds. *Arkiv för Matematik*, 4(5):417–453, 1962.
- [AM07] Antonio Ambrosetti and Andrea Malchiodi. *Nonlinear analysis and semilinear elliptic problems*. Cambridge Univ. Press, 2007.
- [Aub86] Thierry Aubin. Le problème de yamabe concernant la courbure scalaire. *Lecture Notes in Mathematics Differential Geometry Peñíscola 1985*, page 66–72, 1986.
- [Aub11] Thierry Aubin. *Some nonlinear problems in Riemannian geometry*. Springer, 2011.
- [BC94] Vieri Benci and Giovanna Cerami. Multiple positive solutions of some elliptic problems via the morse theory and the domain topology. *Calculus of Variations and Partial Differential Equations*, 2(1):29–48, 1994.
- [BP13] Renato Bettiol and Paolo Piccione. Multiplicity of solutions to the yamabe problem on collapsing riemannian submersions. *Pacific Journal of Mathematics*, 266(1):1–21, Sep 2013.
- [CK04] Bennett Chow and Dan Knopf. *The Ricci flow: an introduction*. American Mathematical Society, 2004.
- [dAN20] Joao Nunes de Araujo Neto. Numero minimo de solucoes para equacao de yamabe subcritica modelada em um fibrado riemanniano com fibras minimas, 2020.
- [dC15] Manfredo Perdigao do Carmo. *Riemannian geometry*. Birkhaeuser, 2015.
- [Ehr74] Paul E. Ehrlich. Continuity properties of the injectivity radius function. *Compositio Mathematica*, 29(2):151–178, 1974.
- [Fol11] Gerald Budge Folland. *Real analysis: modern techniques and their applications*. John Wiley and Sons, 2011.
- [GHL08] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry*. World Pub. Corp., 2008.

- [GK73] Karsten Grove and Hermann Karcher. How to conjugate c1-close group actions. *Mathematische Zeitschrift*, 132:11–20, 1973.
- [GM11] Marco Ghimenti and Anna Maria Micheletti. Positive solutions for a singularly perturbed nonlinear elliptic problem on manifolds via morse theory. *Complex Variables and Elliptic Equations*, 56(12):1115–1127, 2011.
- [Heb00] Emmanuel Hebey. *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*. American Mathematical Society, 2000.
- [Kar77] H. Karcher. Riemannian center of mass and mollifier smoothing. *Communications on Pure and Applied Mathematics*, 30(5):509–541, 1977.
- [Kre89] Erwin Kreyszig. *Introductory functional analysis with applications*. Wiley, 1989.
- [Lan88] Serge Lang. *Differential manifolds*. Springer, 1988.
- [Lee09] Jeffrey M. Lee. *Manifolds and differential geometry*. American Mathematical Society, 2009.
- [Lee13] John M. Lee. *Introduction to smooth manifolds*. Springer, 2013.
- [LS14] Lynn H. Loomis and Shlomo Sternberg. *Advanced calculus*. Jones and Barlett Publ., 2014.
- [MP09] Anna Maria Micheletti and Angela Pistoia. Generic properties of singularly perturbed nonlinear elliptic problems on riemannian manifold. *Advanced Nonlinear Studies*, 9:803–813, 2009.
- [Nic13] Liviu I. Nicolaescu. Notes on the atiyah-singer index theorem, 2013.
- [Oba71] Morio Obata. The conjectures on conformal transformations of riemannian manifolds. *J. Differential Geom.*, 6(2):247–258, 12 1971.
- [Pet18] Peter Petersen. *Riemannian geometry*. Springer, 2018.
- [Pet19] Jimmy Petean. Multiplicity results for the yamabe equation by luster-
nik–schnirelmann theory. *Journal of Functional Analysis*, 276:1788–1805, 2019.
- [PT88] Richard S. Palais and Chuu-lian Terng. *Critical point theory and submanifold geometry*. Richard S. Palais, 1988.
- [RR18] Carolina A. Rey and Juan Miguel Ruiz. Multipeak solutions for the yamabe equation, 2018.
- [Sch84] Richard Schoen. Conformal deformation of a riemannian metric to constant scalar curvature. *J. Differential Geom.*, 20(2):479–495, 1984.
- [Str10] Michael Struwe. *Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems*. Springer, 2010.
- [Tru68] Neil S. Trudinger. Remarks concerning the conformal deformation of riemannian structures on compact manifolds. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 3, 22(2):265–274, 1968.
- [War11] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*. Springer-Verlag, 2011.

- [WH15] Kuehnel Wolfgang and Bruce Hunt. *Differential geometry curves, surfaces, manifolds*. American Mathematical Society, 2015.
- [Yam60] Hidehiko Yamabe. On a deformation of riemannian structures on compact manifolds. *Osaka Math. J.*, 12(1):21–37, 1960.