An algebraic framework to a theory of sets based on the surreal numbers

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Resumo

RANGEL D.R. **Um contexto algébrico para uma teoria de conjuntos baseada nos números surreais**. 2018. 116+vi f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2018.

A noção de número surreal foi introduzida por J.H. Conway em meados da década de 1970: os números surreais constituem uma classe (própria) linearmente ordenada No contendo a classe de todos os números ordinais (On) e que, trabalhando dentro da base conjuntista NBG, pode ser definida por uma recursão na classe On. Desde então, apareceram muitas construções desta classe e foi isolada uma axiomatização completa desta noção que tem sido objeto de estudo devido ao grande número de propriedades interessantes, incluindo entre elas resultados modelos-teóricos. Tais construções sugerem fortes conexões entre a classe No de números surreais e as classes de todos os conjuntos e todos os números ordinais.

Na tentativa de codificar o universo dos conjuntos diretamente na classe de números surreais, encontramos algumas pistas que sugerem que esta classe não é adequada para esse fim. O presente trabalho é uma tentativa de se obter uma "teoria algébrica (de conjuntos) para números surreais" na linha da Teoria dos Algébrica dos Conjuntos - uma teoria categorial de conjuntos introduzida nos anos 1990: estabelecer links abstratos e gerais entre a classe de todos números surreais e um universo de "conjuntos surreais" semelhantes às relações entre a classe de todos os ordinais (On) e a classe de todos os conjuntos (V), que também respeite e expanda os links entre as classes linearmente ordenadas de todos ordinais e de todos os números surreais.

Introduzimos a noção de álgebra surreal (parcial) (SUR-álgebra) e exploramos algumas das suas propriedades categoriais, incluindo SUR-álgebras (relativamente) livres (SA, ST). Nós estabelecemos links, em ambos os sentidos, entre SUR-álgebras e álgebras ZF (a pedra angular da Teoria Algébrica dos Conjuntos). Desenvolvemos os primeiros passos de um determinado tipo de teoria de conjuntos baseada (ou ranqueada) em números surreais, que expande a relação entre V e On.

Palavras-chave: Números surreais; Teoria Algébrica dos Conjuntos; SUR-álgebras.

Abstract

RANGEL D.R. **An algebraic framework to a theory of sets based on the surreal numbers**. 2018. 116+vi pp. PhD thesis- Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2018.

The notion of surreal number was introduced by J.H. Conway in the mid 1970's: the surreal numbers constitute a linearly ordered (proper) class No containing the class of all ordinal numbers (On) that, working within the background set theory NBG, can be defined by a recursion on the class On. Since then, have appeared many constructions of this class and was isolated a full axiomatization of this notion that been subject of interest due to large number of interesting properties they have, including model-theoretic ones. Such constructions suggests strong connections between the class No of surreal numbers and the classes of all sets and all ordinal numbers.

In an attempt to codify the universe of sets directly within the surreal number class, we have founded some clues that suggest that this class is not suitable for this purpose. The present work is an attempt to obtain an "algebraic (set) theory for surreal numbers" along the lines of the Algebraic Set Theory - a categorial set theory introduced in the 1990's: to establish abstract and general links between the class of all surreal numbers and a universe of "surreal sets" similar to the relations between the class of all ordinals (On) and the class of all sets (V), that also respects and expands the links between the linearly ordered class of all ordinals and of all surreal numbers.

We have introduced the notion of (partial) surreal algebra (SUR-algebra) and we explore some of its category theoretic properties, including (relatively) free SUR-algebras (SA, ST). We have established links, in both directions, between SUR-algebras and ZF-algebras (the keystone of Algebraic Set Theory). We develop the first steps of a certain kind of set theory based (or ranked) on surreal numbers, that expands the relation between V and On.

Keywords: Surreal numbers; Algebraic Set Theory; SUR-algebras.

Contents

Introduction

The notion of surreal number was introduced by J.H. Conway in the mid 1970's: the surreal numbers constitute a linearly ordered (proper) class No containing the class of all ordinal numbers (On) that, working within the background set theory NBG, can be defined by a recursion on the class On. Since then, have appeared many constructions of this class and was isolated a full axiomatization of this notion.

Surreal numbers have been subject of interest in many areas of Mathematics due to large number of interesting properties that they have:
- In Algebra, through the concept of field of Hahn series and variants (see for instance [Mac39], [All62], [Sco69], [vdDE01], [KM15]);
- In Analysis (see the book [All87]);
- In Foundations of Mathematics, particularly in Model Theory, since the surreal number line is for proper class linear orders what the rational number line $\mathbb{Q}$ is for the countable: surreal numbers are the (unique up to isomorphism) proper class Fraïssé limit of the finite linearly ordered sets, they are set-homogeneous and universal for all proper class linear orders.

The plethora of aspects and applications of the surreals maintain the subject as an active research field. To make a point, the 2016 edition of the "Joint Mathematics Meetings AMS" –the largest Math. meeting in the world– have counted 14 talks in its "AMS-ASL Special Session on Surreal Numbers".
http://jointmathematicsmeetings.org/meetings/national/jmm2016/2181_program_ss16.html

In our thesis we try to develop, from scratch, a new (we hope!) and complementary foundational aspect of the Surreal Number Theory: to establish, in some sense, a set theory based on the class of surreal numbers.

Set/class theories are one of the few fundamental mathematical theories that holds the power to base other notions of mathematics (such as points, lines, and real numbers). This is basically due to two aspects of these theories: the first is that the basic entities and relations are very simple in nature, relying only on the primitive notions of set/class and a (binary) membership relation ("$X \in Y$"), the second aspect is the possibility that this theory can perform constructions of sets by several methods. This combination of factors allows to achieve a high degree of flexibility, in such a way that virtually all mathematical objects can be realized as being of some kind of set/class, and it has the potential to define arrows (category) as entities of the theory. In particular, the set/class theories traditionally puts as a principle (the Axiom of Infinity) the existence of an "infinite set" - the smallest of these would be the set of all natural numbers - thus, such numbers are a derived (or "a posteriori) notion, which encodes the essence of the notion of "to be finite", that is apparently more intuitive.

The usual set/class theories (as ZFC or NBG) have the power of "encode" (syntactically) its Model Theory: constructions of models of set theory by the Cohen forcing method or through boolean valued models method are done by a convenient encoding of the fundamental binary relations $\in$ and $=$.

Let us list below some other fundamental theories:

• Set theories with additional predicates for non-Standard Analysis, as the E. Nelson's set theory named IST.

• P. Aczel's "Non-well-founded sets" ([Acz88]), where sets and proper classes are replaced by directed graphs (i.e., a class of vertices endowed with a binary relation)

• K. Lopez-Escobar "Second Order Propositional Calculus" ([LE09]), a theory with three primitive terms, that encodes the full Second Order Intuitionistic Propositional Calculus also includes Impredicative Set Theory.

• Toposes, a notion isolated in the 1970's by W. Lawvere and M. Tierney, provide generalized set theories, from the category-theoretic point of view.

• Algebraic Set Theory (AST), another categorial approach to set/class theory, introduced in the 1990's by A. Joyal and I. Moerdijk ([JM95]).
Algebraic Set Theory replaces the traditional use of axioms on pertinence by categorial relations, proposing the general study of "abstract class categories" endowed with a notion of "small fibers maps". In the same way that the notion of " being finite " is given a posteriori in ZFC, after guaranteeing an achievement of the Peano axioms - which axiomatizes the algebraic notion of free monoid in 1 generator - the notions of "to be a set" and "be an ordinal" are given a posteriori in AST. The class of all sets is determined by a universal property, that of free ZF-algebra, whereas the class of all ordinals is characterized globally by the property of constituting free ZF-algebra with inflationary/monotonous successor function. In the same direction, the (small fibers) rank map, $\rho : V \to On$, is determined by the universal property of V, and the inclusion map, $i : On \to V$, is given by an adjunction condition.

The main aim of this work is to obtain an "algebraic theory for surreals" along the lines of the Algebraic Set Theory: to establish abstract and general links between the class of all surreal numbers and a universe of "surreal sets" similar to (but expanding it) the (ZF-algebra) relations between the classes On and V, giving the first steps towards a certain kind of (alternative) "relative set theory" (see [Fre16] for another presentation of this general notion).

In more details:

We want to perform a construction (within the background class theory NBG) of a "class of all surreal sets", V^*, that satisfies, as far as possible, the following requirements:

• V^* is an expansion of the class of all sets V, via a map $j^* : V \to V^*$.

• V^* is ranked in the class of surreal numbers No, in an analogous fashion that V is ranked in the class of ordinal numbers On. I.e., expand, through the injective map $j : On \to No$, the traditional set theoretic relationship $V \underset{i}{\overset{\rho}{\rightleftarrows}} On$ to a new setting $V^* \underset{i^*}{\overset{\rho^*}{\rightleftarrows}} No$.

Noting that:
(i) the (injective) map $j : On \to No$, is a kind of "homomorphism", partially encoding the ordinal arithmetic;
(ii) the traditional set-theoretic constructions (in V) keep some relation with its (ordinal) complexity (e.g., $x \in y \;\; \to \;\; \rho(x) < \rho(y)$, $\rho(\{x\}) = \rho(P(x)) = \rho(x) + 1$, $\rho(\bigcup_{i \in I} x_i) = \bigcup_{i \in I} \rho(x_i)$);
then we wonder about the possibility of this new structured domain V^* determines a category, by the encoding of arrows (and composition) as objects of V^*, in an analogous fashion to the way that the class V of all sets determines a category, i.e. by the encoding of some notion of "function" as certain surreal set (i.e. an objects of V^*); testing, in particular, the degree of compatibility of such constructions with the map $j^* : V \to V^*$ and examining if this new expanded domain could encode homomorphically the cardinal arithmetic.

We list below 3 instances of communications that we have founded in our bibliographic research on possible themes relating surreal numbers and set theory: we believe that they indicate that we are pursuing a right track.

(I) The Hypnagogic digraph and applications
J. Hamkins have defined in [Ham13] the notion of "hypnagogic digraph", $(Hg, \rightharpoonup)$, an acyclic digraph graded on $(No, <)$, i.e., it is given a "rank" function $v : Hg \to No$ such that for each $x, y \in Hg$, if $x \rightharpoonup y$, then $v(x) < v(y)$. The hypnagogic digraph is a proper-class analogue the countable random $\mathbb{Q}$-graded digraph: it is the Fraïssé limit of the class of all finite No-graded digraphs. It is simply the On-saturated No-graded class digraph, making it set-homogeneous and universal for all class acyclic digraphs. Hamkins have applied this structure, and some relativized versions, to prove interesting results concerning models of ZF set theory.

(II) Surreal Numbers and Set Theory
https://mathoverflow.net/questions/70934/surreal-numbers-and-set-theory
Asked July 21, 2011, by Alex Lupsasca:
I looked through MathOverflow's existing entries but couldn't find a satisfactory answer to the following question:
What is the relationship between No, Conway's class of surreal numbers, and V, the Von Neumann set-theoretical universe?
In particular, does V contain all the surreal numbers? If so, then is there a characterization of the surreal numbers as sets in V? And does No contain large cardinals?
I came across surreal numbers recently, but was surprised by the seeming lack of discussion of their relationship to traditional set theory. If they are a subclass of V, then I suppose that could explain why so few people are studying them.

(III) Surreal Numbers as Inductive Type?
https://mathoverflow.net/questions/63375/surreal-numbers-as-inductive-type?rq=1
Asked in April 29, 2011, by Todd Trimble:
Prompted by James Propp's recent question about surreal numbers, I was wondering whether anyone had investigated the idea of describing surreal numbers (as ordered class) in terms of a universal property, roughly along the following lines.
In categorical interpretations of type theories, it is common to describe inductive or recursive types as so-called initial algebras of endofunctors. The most famous example is the type of natural numbers, which is universal or initial among all sets X which come equipped with an element x and an function $f : X \to X$. In other words, initial among sets X which come equipped with functions $1 + X \to X$ (the plus is coproduct); we say such sets are algebras of the endofunctor F defined by $F(X) = 1 + X$. Another example is the type of binary trees, which can be described as initial with respect to sets that come equipped with an element and a binary operation, or in other words the initial algebra for the endofunctor $F(X) = 1 + X^2$.
In their book Algebraic Set Theory, Joyal and Moerdijk gave a kind of algebraic description of the cumulative hierarchy V. Under some reasonable assumptions on a background category (whose objects may be informally regarded as classes, and equipped with a structure which allows a notion of "smallness"), they define a ZF-structure as an ordered object which has small sups (in particular, an empty sup with which to get started) and with a "successor" function. Then, against such a background, they define the cumulative hierarchy V as the initial ZF-structure, and show that it satisfies the axioms of ZF set theory (the possible backgrounds allow intuitionistic logic). By tweaking the assumptions on the successor function, they are able to describe other set-theoretic structures; for example, the initial ZF-structure with a monotone successor gives On, the class of ordinals, relative to the background.
Now it is well-known that surreal numbers generalize ordinals, or rather that ordinals are special numbers where player R has no options, or in different terms, where there are no numbers past the "Dedekind cut" which divides L options from R options. In any case, on account of the highly recursive nature of surreal numbers, it is extremely tempting to believe that they too can be described as a recursive type, or as an initial algebraic structure of some sort (in a background category along the lines given by Joyal-Moerdijk, presumably). But what would it

be exactly?
I suppose that if I knocked my head against a wall for a while, I might be able to figure it out or at least make a strong guess, but maybe someone else has already worked through the details?

Chronology of the work:

This works begins at the second semester of 2014, dedicated to the study of the basic results on surreal numbers exposed on books and trying to prospect a new/complementary approach to this vast theory with connections in many branches of Mathematics.

In the first half of 2015 we wonder if No is such a fundamental object, it should be some way a basis for a Set Theory. To aid us to materialize such idea we have dedicated almost all the semester to the study of the principles of Algebraic Set Theory [JM95].

At the 2^{nd} semester 2015 we begin the preparation for qualification exam, that has occurred in November 2015, thus we begin the search of articles and the partially study them. In this meantime we came across Hamkin's paper [Ham13] and two Mathoverflow links mentioned above that have indicated that we are pursuing a right track.

In first semester 2016 we focus on the main theme of the thesis according its title "An algebraic framework to a theory of sets based on the surreal numbers", and effectively begins the novelties of work, as the definition of the concept of SUR-algebras and its morphisms and the SA example, that we continue until the first months of 2018, simultaneously with the begin of the typing process.

Overview of this thesis:

Chapter 1:
This initial chapter establishes the notations and contains the preliminary results needed for the sequel of this thesis. It begins establishing our set theoretic backgrounds – that we will use freely in the text without further reference – which is founded in NBG class theory, and contains mainly the definitions and basic results on some kinds of binary relations, in particular on well-founded relations, and "cuts" as certain pairs of subsets of a class endowed with a binary irreflexive relation. After, in the second section, we introduce briefly a (categorically naive) version of ZF-algebras, a notion introduced in the 1990's in the setting of Algebraic Set Theory, in particular we introduce the concept of *standard ZF-algebra*, very useful to developments occurred in the Chapter 3. The last section is dedicated to introduce the linearly ordered class of surreal numbers under many equivalent constructions and to present a characterization and some of its main structure, including its algebraic structure and its relations with the class (or ZF-algebra) of all ordinal numbers.

Chapter 2:
Motivated by properties of the linearly ordered class $(No, <)$, we introduce the notion

of *Surreal Algebra (SUR-algebra)*: an structure $\mathcal{S} = (S, <, *, -, t)$, where $<$ is an acyclic relation on S, $*$ is a distinguished element of S, $-$ is an involution of S and t is a function that chooses an intermediary member between each small (Conway) cut in $(S, <)$, satisfying some additional compatibility properties between them. Every SUR-algebra turns out to be a proper class. We verify that No provides naturally a SUR-algebra and present new relevant examples: the free surreal algebra (SA) and the free transitive SUR-algebra (ST). In the sequel, a section is dedicated to a generalization of this new concept: we introduce the notion of partial SUR-algebra and consider two kinds on morphisms between them. This relaxation is needed to perform constructions as products, sub partial-SUR-algebra and certain kinds of directed colimits. As an application of the latter construction, we are able to prove some universal properties satisfied by SA and ST (and generalizations), that justifies its names of (relatively) free SUR-algebras.

Chapter 3:
This chapter is dedicated to established links, in both directions, between certain classes of (equipped) SUR-algebras and certain classes of ZF-algebras (the keystone of Algebraic Set Theory), that "explains" and "expands" the relationship $On \overset{j}{\underset{b}{\rightleftarrows}} No$. Even if some general definitions and results are given in this chapter, its main goal is not develop an extensive and systematic study of the introduced concepts (we intend dedicate to this in the future) but, instead, to provide means to understand and appreciate the content of the main diagram presented in the Section 2, that summarizes the relationship between the structured classes On, V, No, SA.
Inspired by the fundamental properties of the "birthday" function on surreal number, $b : No \twoheadrightarrow On$, we introduce, in the first section, the notion of *anchored SUR-algebra*, $(\mathcal{S}, \mathbf{b})$, where $\mathbf{b} : S \to C$ is a certain function from the SUR algebra $\mathcal{S}$ onto a (rooted) well-founded class $\mathcal{C} = (C, \prec)$, satisfying some convenient properties. This induces an useful well-founded relation $\prec_{\mathbf{b}}$ in S and a recursively-defined subclass of $HP_{\mathbf{b}}(\mathcal{S}) \subseteq S$ of *hereditary positive members* that, under additional conditions, provides an induced ZF-algebra structure (e.g., $HP_{b^*}(SA) \cong V, HP_b(No) \cong On$) and also axiomatizations results for SA and ST, analogous to the axiomatization of No, describing them up to isomorphisms.
In the Section 2 we present the "main diagram" of this thesis, that summarizes the relationship between the structured classes On, V, No, SA and will be the basis for the development of Chapter 4: (i) the maps $\rho : V \to On$ and $\rho^* : SA \to No$ connect objects of a same category (of ZF-algebras in the former case and of SUR-algebras in the latter); (ii) the pairs of arrows $On \overset{j}{\underset{b}{\rightleftarrows}} No$ and $V \overset{j^*}{\underset{b^*}{\rightleftarrows}} SA$ are "Chimera morphisms", i.e. each pair establishes connections between ZF-algebras and SUR-algebras.
In the two last sections, we introduce the concept of hereditary positive members of a SUR-algebra endowed with certain well-founded relation and associate to some (well-founded) ZF-algebras Z corresponding SUR-algebra **space of signs** $Sig(Z)$ (e.g. $Sig(V) \cong$

$SA, Sig(On) \cong No$). Every SUR-algebra "space of signs" is anchored on its underlying well-founded ZF-algebra, that is its class of hereditary positive members.

Chapter 4:
We develop here the first steps of a certain kind of set theory based (or ranked) on surreal numbers, that expands the relation between V and On.
For us, there are three main requirements for a theory deserves be named a "Set Theory": (i) have the potential to define arrows (category) as entities of the theory, through a fundamental binary relation; (ii) be the "derived set theory" of a free object in a category (like in ZF-algebra setting); (iii) its "internal" category is a topos-like category.
In fact, we work out a "positive set theory" on SA ranked on No, that expands the ZF-algebra relationship $V \to On$ through the "positive" map $j^+ : V \to SA$, given by $j^+(X) = j^*(X) = \langle j^+[X], \emptyset \rangle$, $X \in V$. Thus the free/initial SUR-algebra SA supports, in many senses, an expansion of the free/initial ZF-algebra V and its underlying set theory. To accomplish this, we use the functions j^* and b^* that establishes connections between V and SA. Under logical and set-theoretical perspective, the map $j^+ : V \to SA$ preserves and reflects many constructions. On the category-theoretic perspective, the map j^+ defines a full, faithful and logical functor $j^+ : Set \to Cat^+(SA)$, from the topos Set associated to V into the topos $Cat^+(SA)$ associated to SA.

Chapter 5:
In the this last chapter, we present a (non-exhaustive) list of questions that have occurred to us during the elaboration of this thesis, that we can not be able to deal in the present work by lack of time and/or of skills, but that we intend to address in the future.

Chapter 1

Preliminaries

This chapter establishes the notations and contains the preliminary results needed for the sequel of this thesis. It begins establishing our set theoretic backgrounds – that we will use freely in the text without further reference – which is founded in NBG class theory, and contains mainly the definitions and basic results on some kinds of binary relations. After, we introduce briefly a (categorically naive) version of ZF-algebras, a notion introduced in the 1990's in the setting of Algebraic Set Theory. Finally, we present the class of surreal numbers, and some of its main structure, under many equivalent constructions.

1.1 Set theoretic backgrounds

This preliminary section is devoted to establishing our set theoretic backgrounds which is founded in NBG class theory[1], and contains mainly the definitions and basic results on the binary relations that will appear in the sequel of this work as: (strict) partial order relations, acyclic relations, extensional relations, well founded relations, and "cuts" as certain pairs of subsets of a class endowed with a binary irreflexive relation.

1.1.1 NBG

In this work, we will adopt the (first-order, with equality) theory NBG as our background set theory, where the unique symbol in the language is the binary relation $\in$. We will use freely the results of NBG, in the sequel, we just recall below some notions and notations. We recall also the basic notions and results on some kinds of binary relations needed for

[1]In some parts of the thesis, we will need some category-theoretic tools and reasoning, thus we will use an expansion of NBG by axioms asserting the existence of Grothendieck universes.

the development of this thesis. Standard references of set/class theory are [Jec03] and [Kun13].

1. On NBG:

Recall that the primitive concept of NBG is the notion of *class*. A class is *improper* when it is a member of some class, otherwise the class is *proper*. The notion of *set* in NBG is defined: a set is a improper class.

We will use V to denote the universal class – whose members are all sets – On will stand for the class of all ordinal numbers and Tr denote the class of all transitive sets. $On \subseteq Tr \subseteq V$ and all the three are proper classes.

Given classes $\mathcal{C}$ and $\mathcal{D}$, then $\mathcal{C}$ is a subclass of $\mathcal{D}$ (notation: $\mathcal{C} \subseteq \mathcal{D}$), when all members of $\mathcal{C}$ are also members of $\mathcal{D}$. Classes that have the same members are equal. Every subclass of a set is a set.

$\emptyset$ stands for the unique class without members. $\emptyset$ is a set.

Given a class $\mathcal{C}$, denote $P_s(\mathcal{C})$ the class whose members are all the *subsets* of $\mathcal{C}$. If $\mathcal{C}$ is a set, then $P_s(\mathcal{C})$ is a set too. There is no class that has as members all the *subclasses* of a proper class[2].

Given classes $\mathcal{C}$ and $\mathcal{D}$, and a function $f : \mathcal{C} \to \mathcal{D}$, then the (direct) image $f[\mathcal{C}] = \{d \in \mathcal{D} : \exists c \in \mathcal{C}, d = f(c)\}$ is a subset of D, whenever $\mathcal{C}$ is a set.

Since NBG satisfies the axiom of global choice (i.e., there is a choice function on $V \setminus \{\emptyset\}$) and then every class (proper or improper) can be well-ordered, which implies nice cardinality results: as in ZFC, any set X is equipotent to a unique cardinal number (= initial ordinal), called the its cardinality of X (notation: $card(X)$); moreover, all the proper classes are equipotent – we will denote $card(C) = \infty$ the cardinality of the proper class C – ∞ can be seen as a representation of the well-ordered the proper class On.

□

2. Binary relations:

A relation R is a class whose members are ordered pairs. The domain (respect., range) of R is the class of all first (respect., second) components of the ordered pair in the relation. The support of the relation R (notation: $supp(R)$) is the class obtained by the reunion of its domain and range. We will say that a binary relation is defined *on/over* its support class.

A relation R is reflexive when $(x, x) \in R$ for each x in the support of R; on the other hand, R is irreflexive, when $(x, x) \notin R$ for each x in the support of R. We will use $<, \prec, \lhd$ to denote general irreflexive relations; $\leq, \preceq, \sqsubseteq$ will stand for reflexive relations.

[2] This is a "metaclass" in NBG, i.e., an equivalence class of formulae in one free variable, modulo the NBG-theory: any such formula is not collectivizing.

A pre-order is a reflexive and transitive relation. A partial order is a antisymmetric pre-order. A *strict* partial order is a irreflexive and transitive relation. There are well known processes of: obtain a strict partial order from a partial order and conversely.

Let R be a binary relation and let $s, s' \in supp(R)$. Then s and s' are R-comparable when: $s = s'$ or $(s, s') \in R$ or $(s', s) \in R$. A relation R is total or linear when every pair of members of its support are comparable.

Every pre-order relation $\preceq$ on a class $\mathcal{C}$ gives rise to an equivalence relation $\sim$ on the same support: for each $c, c' \in \mathcal{C}$, $c \sim c'$ iff $c \preceq c'$ and $c' \preceq c$.

Let $n \in \mathbb{N}$, a n-cycle of the relation R is a $n+1$-tuple $(x_0, x_1, \cdots, x_n)$ such that $x_n = x_0$ and, for each $i < n$, $(x_i, x_{i+i}) \in R$. A relation is *acyclic* when it does not have cycles. Every acyclic relation is irreflexive. A binary relation is a strict partial order iff it is a transitive and acyclic relation. Note that a binary relation is acyclic and total iff it is a strict linear order.

□

3. Induced binary relations:

Given a binary relation R on a class $\mathcal{C}$. For each $c \in \mathcal{C}$, denote $c^R := \{d \in C : (d, c) \in R\}$.

Define a new binary relation on $\mathcal{C}$: for each $c, c' \in \mathcal{C}$, $c \sqsubseteq_R c'$ iff holds $\forall x((x, c) \in R \to (x, c') \in R)$ or, equivalently, $c^R \subseteq c'^R$. Clearly, $\sqsubseteq_R$ is pre-order relation on $\mathcal{C}$.

Denote $\equiv_R$, the equivalence relation associated to the pre-order $\sqsubseteq_R$. We will say that the binary R is extensional when $\equiv_R$ is the identity relation on $\mathcal{C}$ or, equivalently, when $\sqsubseteq_R$ is a partial order. The axiom of extensionality in NBG ensures that $(V, \in_{\restriction V \times V})$ is a class endowed with an extensional relation and, since members of ordinal numbers are ordinal numbers[3], then $(On, \in_{\restriction On \times On})$ is class endowed with an extensional relation.

□

1.1.2 Well founded relations

In this subsection we recall basic properties and constructions concerning general well-founded relations. Also, we introduce a special kind of well-founded relation suitable for our purposes in Chapter 3.

4. On well-founded relations:

Recall that a binary relation $\prec$ on a class $\mathcal{C}$ is well-founded when:
(i) The subclass $x^{\prec} = \{y \in \mathcal{C} : y \prec x\}$ is a set.

[3] If $\alpha \in On$, then $\alpha^{\in} = \{\beta \in On : \beta \in \alpha\} = \{x \in V : x \in \alpha\} = \alpha$.)

(ii) For each $X \subseteq \mathcal{C}$ that is a non-empty subset, there is $u \in X$ that is a $\prec$-minimal member of X (i.e., $\forall v \in \mathcal{C}$, $v \prec u \Rightarrow v \notin X$).[4]

Let $\prec$ be an well-founded relation on a class $\mathcal{C}$. Since for each $n \in \mathbb{N}$, the (non-empty) subset $\{x_0, \cdots, x_n\} \subseteq \mathcal{C}$ has a $\prec$-minimal member, then $\prec$ is an acyclic relation and, in particular, $\prec$ is irreflexive.

If $\mathcal{D} \subseteq \mathcal{C}$, then $(\mathcal{D}, \prec_{\restriction \mathcal{D} \times \mathcal{D}})$ is an well-founded class.

An well-founded relation that is a strict linear/total order is a well-order relation.

The axiom of regularity in NBG, guarantees that the binary relation $\in$ over the universal class V is an well-founded relation. $(On, \in)$ is an well-ordered proper class.

Let $\prec$ be an well-founded relation on a class $\mathcal{C}$. Then it holds:
The induction principle: Let $X \subseteq \mathcal{C}$ be a subclass. If, for each $c \in \mathcal{C}$, the inclusion $c^{\prec} \subseteq X$ entails $c \in \mathcal{C}$, then $X = \mathcal{C}$.
The recursion theorem: Let H be a (class) function such that $H(c, g)$ is defined for each $c \in \mathcal{C}$ and g a (set) function with domain $c^{\prec}$. Then there is a unique (class) function F with domain $\mathcal{C}$ such that $F(c) = H(c, F_{\restriction c^{\prec}})$, for each $c \in \mathcal{C}$.

$\square$

5. Rooted well-founded relations:

Remark: Let $(\mathcal{C}, \prec)$ be a well-founded class; the subclass *root*$(\mathcal{C})$ of its *roots* has as members its $\prec$-minimal members. Note that:
• If $\mathcal{C}$ is a non-empty class, then *root*$(\mathcal{C})$ is a non-empty class.
• If $\sqsubseteq$ denotes the pre-order on $\mathcal{C}$ associated to $\prec$ (i.e., $c \sqsubseteq d$ iff $\forall x \in \mathcal{C}(x \prec c \Rightarrow x \prec d)$, then: $root(\mathcal{C}) = \{a \in C : a \sqsubseteq c, \text{ for all } c \in \mathcal{C}\}$.

Definition: A well-founded class $(\mathcal{C}, \prec)$ will be called *rooted*, when it has a unique root Φ. If it is the case, then the structure $(\mathcal{C}, \prec, \Phi)$ will be called a rooted well-founded class.

If $\prec$ is an extensional well-founded relation on a non-empty class $\mathcal{C}$, then $(\mathcal{C}, \prec)$ is rooted: indeed, if $r, r' \in root(\mathcal{C})$, then $r \sqsubseteq r'$ and $r' \sqsubseteq r$, thus $r = r'$. However, to emphasize the distinguished element in a structure of rooted well-founded class, we will employ the redundant expression "rooted extensional well-founded class".

Examples and counter-examples:

$(V, \in, \emptyset)$ is a rooted extensional well-founded class

$(On, \in, \emptyset)$ is a rooted extensional well-ordered class.

Every well-ordered set $(X, \leq)$ gives rise to a rooted extensional well-ordered set $(X, <, \Phi)$, where $\Phi = \perp$ is the least element of X and the strict relation, $<$, associated to $\leq$,

[4]By the global axiom of choice (for classes), this condition is equivalent of a apparently stronger one: (ii)' For each $X \subseteq \mathcal{C}$ that is a non-empty *subclass*, there is $u \in X$ that is a $\prec$-minimal member of X.

is an well-founded relation, since for each $x, y \in X$, $x^< \subseteq y^<$ iff $x \leq y$.

$(\mathbb{N}, \leq, 0)$ is a well-ordered set, thus it gives rise to a rooted extensional well-ordered set. $(\mathbb{N} \setminus \{0\}, |, 1)$ is determines a rooted well-founded set that is not extensional. Note that $(\mathbb{N} \setminus \{0, 1\}, |)$ is an well-founded set that is not rooted since its subset of minimal elements is the infinite set of all prime numbers.

□

1.1.3 Cuts and densities

Many of useful variants of the concept of Dedekind cut were already been defined on the setting strict linear order on a given set (see for instance [All87]). In this preliminary subsection we present expansions of these notions in two different directions: we consider binary relations that are only irreflexive (instead of being a strict linear order) and defined on general classes instead of improper classes (= sets). We also generalize the notions of density à la Hausdorff to this new setting.

Through this subsection, $\mathcal{C}$ denote a class and $<$ stands for a irreflexive binary relation whose support is $\mathcal{C}$.

6. Cuts

A **Conway cut** in $(\mathcal{C}, <)$ is a pair (A, B) of *arbitrary subclasses*[5] of $\mathcal{C}$ such that $\forall a \in A, \forall b \in B, a < b$ (notation: $A < B$). Since $<$ is a irreflexive relation on $\mathcal{C}$, then $A \cap B = \emptyset$. A Conway cut (A, B) will be called *small*, when A and B are *subsets* of $\mathcal{C}$. We can define in theory NBG the class $C_s(\mathcal{C}, <) := \{(A, B) \in P_s(\mathcal{C}) \times P_s(\mathcal{C}) : A < B\}$, formed by all the small Conway cuts in $(\mathcal{C}, <)$.

A **Cuesta-Dutari cut** in $(\mathcal{C}, <)$ is a Conway cut (A, B) such that $A \cup B = \mathcal{C}$. Note that $(\emptyset, \mathcal{C})$ and $(\mathcal{C}, \emptyset)$ are always Cuesta-Dutari cuts in $(\mathcal{C}, <)$. On the other hand, if $\mathcal{C}$ is a proper class, then the class $CD_s(\mathcal{C}, <)$ of all small Cuesta-Dutari cuts in $(\mathcal{C}, <)$ is the empty class.

A **Dedekind cut** in $(\mathcal{C}, <)$ is a Cuesta-Dutari cut (A, B) such that A and B are non-empty subclasses. If $\mathcal{C}$ is a set, then (A, B) is a Dedekind cut in $(\mathcal{C}, <)$ iff (A, B) is a Conway Cut such that the set $\{A, B\}$ is a partition of $\mathcal{C}$.

□

7. Densities

Let α be an ordinal number. Then $(\mathcal{C}, <)$ will be called an η_α-class, when for each *small Conway cut* (A, B) in $(\mathcal{C}, <)$, such that $card(A), card(B) < \aleph_\alpha$, there is some $t \in \mathcal{C}$ such that $\forall a \in A, \forall b \in B$, $(a < t,\ t < b)$ (notation: $A < t < B$).

[5] A and/or B could be the empty set.

Let $(\mathcal{C}, <)$ be an η_α-class. Taking cuts $(\emptyset, \{c\})$ (respectively $(\{c\}, \emptyset)$), for all $c \in \mathcal{C}$, we can conclude that an η_α-class $(\mathcal{C}, <)$ does not have $<$-minimal (respec. $<$-maximal) elements. Taking cuts $(\emptyset, X)$ (or $(X, \emptyset)$), for all $X \subseteq \mathcal{C}$ such that $card(X) < \aleph_\alpha$, we see that an η_α-class $(\mathcal{C}, <)$ has $card(\mathcal{C}) \geq \aleph_\alpha$.

An η_0-class $(\mathcal{C}, <)$ is just a "dense and without extremes" class.

If $(\mathcal{C}, <)$ is an η_α-class and $\beta \in On$ is such that $\beta \leq \alpha$, then clearly $(\mathcal{C}, <)$ is an η_β-class.

$(\mathcal{C}, <)$ will be called an η_∞-class, when it is an η_α-class for all ordinal number α: this means that for each small Conway cut (A, B) in $(\mathcal{C}, <)$ there is some $t \in \mathcal{C}$ such that $A < t < B$. Every η_∞-class is a proper class. We will see in the Section 3 in this Chapter that the strictly linearly ordered proper class of all surreal numbers $(No, <)$ is η_∞. We will introduce in Chapter 2 the notion of SUR-algebra: every such structure is a η_∞-class.

□

From now on, we will use the notion of Conway cut only in the *small* sense.

1.2 A glance on Algebraic Set Theory

In this section, we introduce briefly a (categorially naive) version of ZF-algebras, a notion introduced in the 1990's in the setting of the so called "Algebraic Set Theory" (AST), see [JM95]. Our main goal here is to provide an intermediary step between the usual axiomatic set/class theories and our algebraic approach to obtain something like "a set theory based on surreal numbers", materialized by the concept of "surreal-algebra" (SUR-algebra), introduced in Chapter 2 and further developed in Chapter 3 (in connection with ZF-algebras).

An analogy can be useful to understand the point of the AST approach: as in traditional set/class theories, the notion "be a infinite set" is defined by a convenient universal property (ω is the $\subseteq$-least inductive set) and, after that, the notion of "a set x is finite" can be formalized (x is equipotent to a member of the set ω), in algebraic set theory, the notion of "be a set" or "be an ordinal" is given a priori, through convenient universal properties among the ZF-algebras, and the notion of "membership" is defined a posteriori.

1.2.1 Set theory and ZF-algebras

The NBG class theory provides, naturally, a grading by size of the objects of the theory: the proper classes are the "large" ones (all of them have the same size) and the sets or improper classes are the "small" objects[6].

A similar size distinction can be adapted to the arrows in the theory:

8. Small functions in NBG:

Definition: Let $f : C \to D$ be a class function.
(i) f is called *locally small* when, for each *subset* $D' \subseteq D$, $f^{-1}[D'] \subseteq C$ is also a subset.
(ii) f has *small fibers* when, for each $d \in D$, $f^{-1}[\{d\}]$ is a subset of C.

Remark: Since $f^{-1}[D'] = \bigcup_{d \in D'} f^{-1}[\{d\}]$, a function is locally small iff it has small fibers.

Example: The rank function $\rho : V \to On$ has small fibers.
Recall the recursively defined Von Neumann cumulative hierarchy of *sets*: for each ordinal α, $V_\alpha = \bigcup_{\beta \in \alpha} P_s(V_\beta)$. By the regularity axiom in NBG, $V = \bigcup_{\alpha \in On} V_\alpha$. For each set x, $\rho(x) := min\{\alpha \in On : x \subseteq V_\alpha\} = min\{\alpha \in On : x \in V_{\alpha+1}\}$, thus $\rho^{-1}[\{\alpha\}] = V_{\alpha+1} \setminus V_\alpha$ is a set, thus $\rho : V \to On$ has small fibers. Recall that ρ is a retraction of the (small fibers) inclusion $i : On \hookrightarrow V$.

□

Instead of the general (and original) higher technical categorial setting of AST – a Heyting pretopos with natural numbers object (to represent a category whose objects are "big sets") endowed with a class of arrows satisfying the axioms for small maps –, we adopt the below described "naive" approach to algebraic set theory: our base category is the category whose objects are classes[7] and the small maps are just the ones that we have considered above.

9. ZF-algebras and its morphisms:

V **as a ZF-algebra:** We can arrive to the concept of ZF-algebra regarding some distinctive properties of the universal class V:
- $P_s(V)$ is a "large" (= proper class) small-complete sup-lattice by the $\subseteq$-relation, where the small suprema are given by reunions;
- $V = P_s(V)$;
- $u : V \to P_s(V)$ $x \mapsto \{x\}$ is an *endofunction.*

Definition: A ZF-algebra is a structure $\mathcal{L} = (L, \bigvee, s)$, where L is a class, $(L, \bigvee)$ is a small-complete sup-lattice, and $s : L \to L$ is an endofunction, called *successor function* of the ZF-algebra $\mathcal{L}$.

[6] We will use the index "s", as in $P_s(X)$, to indicate the small/set character of an object of the theory.
[7] To be more precise, a set theoretical setting to formalize such "very-large" categories (the category of classes is a very-large category, since it has a metaclass of objects and a metaclass of arrows) include, for instance, NBG (or ZFC) plus axioms asserting the existence of at least two distinct Grothendieck universes U_0 and U_1, lets say $U_0 \in U_1$, then: the members of U_0 are (represent the) "sets", the subsets of U_0 are classes (they belong to U_1) and the "meta-classes" are the subsets of U_1.

Derived binary relations: Let $\mathcal{L} = (L, \bigvee, s)$ be a ZF-algebra, then there are two useful binary relations on L naturally defined:
- partial order relation: $x \leq y$ iff $\bigvee\{x, y\} = y$, $x, y \in L$;
- "membership" relation: $x \varepsilon y$ iff $s(x) \leq y$, $x, y \in L$.

ZF-algebra morphisms: Let $\mathcal{L} = (L, \bigvee, s)$ and $\mathcal{L}' = (L', \bigvee', s')$ be ZF-algebras. A ZF-algebra morphism $h : \mathcal{L} \to \mathcal{L}'$ is a (class) function $h : L \to L'$ such that h preserves small suprema and preserves successor functions (i.e., $h \circ s = s' \circ h$).

The category $ZF - alg$**:** Of course, we can define a "very-large" category $ZF - alg$, whose objects are the ZF-algebras and the arrows are the ZF-algebra morphisms, with obvious composition and identities.

□

10. On special kinds of successor function: Let $\mathcal{L} = (L, \bigvee, s)$ be a ZF-algebra.

Definition: The successor function $s : L \to L$ will be called an:
- irreducible successor: when, for all $x \in L$ and all small family $\{y_i : i \in I\} \subseteq L$, $s(x) \leq \bigvee_{i \in I} y_i$ iff $\exists i \in I, s(x) \leq y_i$;
- inflationary successor: when $\forall x \in L, x \leq s(x)$;
- order-successor: when $\forall x \in L$, $x < y$ iff $s(x) \leq y$.

Remark:
(i) If s is an inflationary successor, then ε is a transitive relation. Indeed: suppose that $y \varepsilon x$ and let $z \in y^\varepsilon$, then $s(z) \leq y \leq s(y) \leq x$, thus $z \in x^\varepsilon$ and $y^\varepsilon \subseteq x^\varepsilon$.
(ii) Suppose that s represents an order-successor. Then:
- s is inflationary and ε is transitive;
- $\forall w, z \in L$ $(w < z \Leftrightarrow w \varepsilon z)$.
(iii) Suppose that s represents an order-successor and let $z \in L$. Then the following conditions on $z \in L$ are equivalent:
(1) z is not a successor (i.e. $z \neq s(y), \forall y \in L$);
(2) z is a limit $\forall y \in L$, $(s(y) \leq z \Rightarrow s(y) < z)$;
(3) $\forall y \in L$, $(y < z \Leftrightarrow s(y) < z)$;
(4) $z^\varepsilon \subseteq \bigcup_{y \varepsilon z} y^\varepsilon$;
(5) $z^\varepsilon = \bigcup_{y \varepsilon z} y^\varepsilon$.
Indeed, for instance (2) $\Rightarrow$ (4):
Let $x \in z^\varepsilon$, then $s(x) \leq z$ and, by (2), $s(x) < z$. Since s is an order-sucessor, $s(s(x)) \leq z$, and applying again the hypothesis $s(s(s(x)))) \leq z$. Set $y := s(s(x))$, then $y \varepsilon z$ (since $s(y) \leq z$) and $x \in y^\varepsilon$ (since $s(x) \leq y$), thus $x \in \bigcup_{y \varepsilon z} y^\varepsilon$.
(iv) Suppose that s represents an order-successor and $\leq$ is a linear order, then:
- $\forall w, z \in L$ $(w \sqsubseteq z \Leftrightarrow w \varepsilon z$ or $w = z)$.
- s satisfies the condition: $\forall x \in L$, $x < s(y)$ iff $x \leq y$.

□

Example 11.

(i) $\mathcal{V} = (V, \bigcup, s)$, $s(x) = \{x\}$ is a ZF-algebra, since:
- by the pair axiom, if x is set, then $\{x\}$ is a set too;
- by the reunion axiom, the reunion of a set of members of V is a set (i.e., is a member of V) and is the $\subseteq$-least set that contains all of its members.

(ii) $\mathcal{O}n = (On, \bigcup, s)$, $s(x) = x^+ = x \cup \{x\}$ is a ZF-algebra, since:
- if x is a ordinal number, then x^+ is a ordinal number too;
- every set of ordinal numbers has a $\subseteq$-supreme in On that coincide with the reunion of that set.

(iii) $\mathcal{T}r = (Tr, \bigcup, s)$, $s(x) = x^+ = x \cup \{x\}$ is a ZF-algebra, since:
- if x is a transitive set, then x^+ is a transitive too;
- every set of transitive sets has a $\subseteq$-supreme in Tr that coincide with the reunion of that set.

Note that, the map $x \mapsto x^+$ is, in general, $\subseteq$-inflationary ($\forall x \in V$, $x \subseteq x^+$) and, restricted to On, it is $\subseteq$-increasing ($\forall x, y \in On$, $x \subseteq y \Rightarrow x^+ \subseteq y^+$, since $x \subsetneq y$ iff $x \in y$).

□

It is clear that, between the inclusion maps $\mathcal{O}n \hookrightarrow \mathcal{T}r, \mathcal{O}n \hookrightarrow \mathcal{V}, \mathcal{T}r \hookrightarrow \mathcal{V}$, only the first one is a ZF-algebra morphism.

It is well-known that the rank function $\rho : V \to On$ can be characterized (or, alternatively, $\in$-recursively defined) by $\rho(x) = \bigcup\{\rho(y)^+ : y \in x\}$, $\forall x \in V$. The (proof of the) general result below shows, in particular, that ρ is a ZF-algebra morphism $\rho : \mathcal{V} \to \mathcal{O}$ (in fact, it is the unique such morphism).

Proposition 12. *The ZF-algebra* $\mathcal{V} = (V, \bigcup, \{\ \})$ *is the initial (or relatively free) ZF-algebra.*

Proof. Let $\mathcal{L} = (L, \bigvee, s)$ be a ZF-algebra. We must show, that there is a unique ZF-algebra morphism $h : \mathcal{V} \to \mathcal{L}$.

Candidate and uniqueness:
Since $x = \bigcup_{y \in x}\{y\}$, $\forall x \in V$, there is at most one ZF-algebra morphism $h : \mathcal{V} \to \mathcal{L}$ such that $h(x) = h(\bigcup_{y \in x}\{y\}) = \bigvee_{y \in x} h(\{y\}) = \bigvee_{y \in x} s(h(y)) = \bigvee Range(s \circ h_{\restriction x^{\in}})$.

Existence:
Define, by recursion on the well-founded relation $\in$, a function $h : V \to L$ by $h(x) := \bigvee Range(s \circ h_{\restriction x^{\in}})$, $x \in V$ (this sup exists, since $Range(s \circ h_{\restriction x^{\in}})$ is a set). Then:
- For each small family $\{x_i : i \in I\}$ in V, $h(\bigcup_{i \in I} x_i) = \bigvee Range(s \circ h_{\restriction (\bigcup_{i \in I} x_i)^{\in}}) = \bigvee\{s(h(y)) : y \in x_i, \text{ for some } i \in I\} = \bigvee_{i \in I}(\bigvee_{y \in x_i} s(h(y))) = \bigvee_{i \in I} h(x_i)$, thus h preserves small suprema.
- For each $y \in V$, $h(\{y\}) = \bigvee Range(s \circ h_{\restriction \{y\}^{\in}}) = \bigvee\{s(h(y))\} = s(h(y))$, thus h preserves successors.

□

Proposition 13. $\mathcal{O}n = (On, \bigcup, (\ )^+)$ *is the initial (or relatively free) ZF-algebra in the category of ZF-algebras with* inflationary *successor map.*

Proof. Let $\mathcal{L} = (L, \bigvee, s)$ be a ZF-algebra such that $\forall l \in L, l \leq s(l)$, for instance, $\mathcal{L} = \mathcal{O}n, \mathcal{T}r$. We must show, that there is a unique ZF-algebra morphism $h : \mathcal{O} \to \mathcal{L}$.

Candidate and uniqueness:
By an analysis of cases of ordinal numbers –zero, limit or successor– (or recalling that every member of a ordinal number is a ordinal and that $\rho : V \to On$ is a retraction of the inclusion $i : On \hookrightarrow V$), for every $x \in On$, we have $x = \bigcup_{y \in x} y^+$. Then there is at most one ZF-algebra morphism $h : \mathcal{O}n \to \mathcal{L}$ such that $h(x) = h(\bigcup_{y \in x} y^+) = \bigvee_{y \in x} h(y^+) = \bigvee_{y \in x} s(h(y)) = \bigvee Range(s \circ h_{\restriction x^{\in}})$.

Existence:
Define, by recursion on the well-founded relation $\in$, a function $h : On \to L$ by $h(x) := \bigvee Range(s \circ h_{\restriction x^{\in}})$, $x \in On$ (this sup exists, since $Range(s \circ h_{\restriction x^{\in}})$ is a set). Then:
• For each small family $\{x_i : i \in I\}$ in On, $h(\bigcup_{i \in I} x_i) = \bigvee Range(s \circ h_{\restriction (\bigcup_{i \in I} x_i)^{\in}}) = \bigvee \{s(h(y)) : y \in x_i, \text{ for some } i \in I\} = \bigvee_{i \in I}(\bigvee_{y \in x_i} s(h(y))) = \bigvee_{i \in I} h(x_i)$, thus h preserves small suprema.
• For each $y \in On$, $h(y^+) = \bigvee Range(s \circ h_{\restriction (y^+)^{\in}}) = \bigvee_{z \in y \cup \{y\}} s(h(z)) = \bigvee_{z \in y} s(h(z)) \vee \bigvee \{s(h(y))\} = h(y) \vee s(h(y)) = s(h(y))$ (since s is assumed to be inflationary), thus h preserves successors.

□

Remark 14.

(i) In the general categorial setting for algebraic set theory, but adding adequate hypothesis (see for instance [JM95]), it can be proved the existence of a free ZF-algebra $\mathcal{V}$ satisfying a universal property even stronger than in Proposition 12 and many ZF-algebras of "ordinal numbers" (free ZF-algebras with successor map inflationary or increasing). In this upside down (abstract class) theory, there is a "derived" set theory – the theory of the objects in free model (= free ZF-algebra) $\mathcal{V}$, where $x \varepsilon y$ is **defined** as $s(x) \leq y$, for each $x, y \in V$ – it is an intuitionist fragment of ZF set-theory.

(ii) Note that the inclusion map $i : On \hookrightarrow V$ is a section of the (unique) ZF-algebra morphism $\rho : V \to On$ (i.e. $\rho \circ i = Id_{On}$), that preserves arbitrary suprema (= reunions), preserves and reflects the binary relations = and ε (= $\in$) and such that $\forall x \in V, \forall \beta \in On$, $x \in i(\beta)$ iff $x = i(\alpha)$ for some and unique $\alpha \in On$. But $i : On \hookrightarrow V$ does not preserves successors. An analogous relation holds for the inclusion $i : Tr \hookrightarrow V$ (apart from that it is not a section of ρ). In fact, as a consequence of the universal property of V, there is no ZF-algebra morphisms $h : On \to V$, $g : Tr \to V$.

□

1.2.2 Standard ZF-algebras

In this subsection we introduce a special kind of ZF-algebra suitable for our purposes in Chapter 3.

Definition 15. *A ZF-algebra* $(A, \bigvee, s)$ *is a* **standard ZF-algebra** *iff it verifies:*

(s0) A has a ε*-minimal member.*

(s1) ε *is an well founded relation.*

(s2) $\forall z \in A$, $z = \bigvee_{x \varepsilon z} s(x)$ [8]

□

Proposition 16. *Let* $(A, \bigvee, s)$ *be a ZF-algebra that satisfies (s1), then are equivalent:*

(i) $\forall z \in A\ z = \bigvee_{x \varepsilon z} s(x)$

(ii) $\forall w, z \in A\ (w \leq z \Leftrightarrow w \sqsubseteq z)$

Proof.

(i) ⇒ (ii). Suppose $w \leq z$ and take any $x \varepsilon w$, then $s(x) \leq w \leq z$, thus $x \varepsilon z$: this means $w \sqsubseteq z$. Conversely, suppose $w \sqsubseteq z$, thus $w^{\varepsilon} \subseteq z^{\varepsilon}$ and $w = \bigvee_{x \varepsilon w} s(x) \leq \bigvee_{x \varepsilon z} s(x) = z$.

(ii) ⇒ (i). Suppose that ε is an well founded relation. Let $z \in A$ and define $w := \bigvee_{x \varepsilon z} s(x)$. Since $s(x) \leq z$, for all $x \varepsilon z$, then $w = \bigvee_{x \varepsilon z} s(x) \leq z$. Let $y \in A$ such that $y \varepsilon z$, then $s(y) \leq w$, thus $y \varepsilon w$: this means $z \sqsubseteq w$ and, by (ii), we have $z \leq w$. Since since $\leq$ is antisymmetric, we have $z = w = \bigvee_{x \varepsilon z} s(x)$.

□

Items (c), (d), (e) below are obtained by specifying 10 to the standard ZF-algebra setting in the following:

Remark 17. Let $(A, \bigvee, s)$ is a standard ZF-algebra. Then:

(a) (A, ε) is a rooted extensional well-founded class: since $\leq$ is antisymmetric, by (ii) in the Proposition 16 above, ε is extensional. Thus, by (s0) and the remarks on 5, (A, ε) is a rooted (extensional) well-founded class.

(b) Since ε is an acyclic relation, there is no $x \in A$ such that $s(x) \leq x$.

(c) If $s : A \to A$ is inflationary (i.e. $\forall x \in A, x \leq s(x)$), then ε is a transitive relation (i.e. $y \varepsilon x \Rightarrow y \sqsubseteq x$, for all $x, y \in A$).

(d) Suppose that $s : A \to A$ represents an order-successor (i.e. $\forall x \in A$, $x < y$ iff $s(x) \leq y$), in particular $s : A \to A$ is inflationary and ε is transitive. Then:
(d1) $\forall w, z \in A\ (w < z \Leftrightarrow w \varepsilon z)$.

[8] Since ε is well-founded, z^{ε} is a *subset* of A, thus this sup exists.

(d2) Consider the following conditions on $z \in A$:
(i) z is not a successor (i.e. $z \neq s(y), \forall y \in A$);
(ii) $z = \bigvee_{y \varepsilon z} y$ Then $(i) \Rightarrow (ii)$. Indeed: if for all x, $s(x) \leq z \Rightarrow s(x) < z$, then $s(x) < s(s(x)) \leq z$; since $z = \bigvee_{x \varepsilon z} s(x)$ and $s(x) \leq z$, then $y = s(s(x)) \leq z$ and we have $z = \bigvee_{y \varepsilon z} y$.
Moreover, if $s : A \to A$ is an irreducible successor[9], then $(i) \Leftrightarrow (ii)$.

(e) Suppose that s represents an order-successor and $\leq$ is a linear order, then:
- $\forall w, z \in A$ $(w \sqsubseteq z \Leftrightarrow w \varepsilon z$ or $w = z)$.
- $s : A \to A$ satisfies the condition: $\forall x \in A$, $x < s(y)$ iff $x \leq y$.

□

18. Examples and counter-examples:

(i) $\mathcal{V} = (V, \bigcup, s)$, where $s(x) = \{x\}$, for all $x \in V$, is a standard ZF-algebra. Indeed:
- $\forall x, y \in V$, $x \varepsilon y$ iff $s(x) \subseteq y$ iff $\{x\} \subseteq y$ iff $x \in y$;
- $(V, \in)$ is an well-founded (extensional) class;
- $z = \bigcup_{x \in z} \{x\}$, $\forall z \in V$;
- $\emptyset \in V$ is the unique root of $(V, \in)$.

(ii) $\mathcal{O}n = (On, \bigcup, s)$, where $s(x) = x^+ = x \cup \{x\}$, for all $x \in On$, is a standard ZF-algebra. Indeed:
- $\forall x, y \in On$, $x \varepsilon y$ iff $s(x) \subseteq y$ iff $x^+ \subseteq y$ iff $x \in y$;
- $(On, \in_\restriction)$ is an well-founded (extensional) class;
- $z = \bigcup_{x \in_\restriction z} x^+$, $\forall z \in On$;
- $\emptyset \in On$ is the unique root of $(On, \in_\restriction)$.

(iii) $\mathcal{T}r = (Tr, \bigcup, s)$, where $s(x) = x^+ = x \cup \{x\}$, for all $x \in On$, is a ZF-algebra that satisfies (s0) and (s1) but does not satisfies (s2). Indeed:
- $\forall x, y \in Tr$, $x \varepsilon y$ iff $s(x) \subseteq y$ iff $x^+ \subseteq y$ iff $x \in y$;
- $(Tr, \in_\restriction)$ is an well-founded (extensional) class;
- $\emptyset \in Tr$ is the unique root of $(Tr, \in_\restriction)$;
- $(Tr, \in_\restriction)$ is not extensional. In fact:
$3 = \{0, \{0\}, \{0, \{0\}\}\}$ and
$V_3 = \{0, \{0\}, \{0, \{0\}\}, \{\{0\}\}\}$,
are distinct members of Tr with the same transitive members.
- Since $(Tr, \in_\restriction)$ is not extensional thus, by the items above and Remark 17.(a), it does not satisfies (s2).

□

[9] I.e. for all $x \in L$ and all small family $\{y_i : i \in I\} \subseteq L$, $s(x) \leq \bigvee_{i \in I} y_i$ iff $\exists i \in I, s(x) \leq y_i$.

1.3 On Surreal Numbers

This section is dedicated to present the class of surreal numbers – a concept introduced by J.H. Conway in the mid 1970's – under many (equivalent) constructions within the background set class theory NBG, its order and algebraic structure and its relations with the class (or the ZF-algebra) of all ordinal numbers.

1.3.1 Constructions

1.3.1.1 The Conway's construction formalized in NBG

We begin with the Conway's construction following the appendix of his book [Con01], in which he gave a more formal construction.

We start defining, recursively, the sets G_α in order to define class of games.

(i) $G_0 = \{\langle \emptyset, \emptyset \rangle\}$

(ii) $G_\alpha = \{\langle A, B \rangle : A, B \subseteq \bigcup_{\beta<\alpha} G_\beta\}$

The class G of Conway games is given by $G = \bigcup_{\alpha<On} G_\alpha$.

We can define a preorder $\leqslant$ in G. Let $x = \langle L_x, R_x \rangle$ and $y = \langle L_y, R_y \rangle$.

$$x \leqslant y \text{ iff no } x^L \text{ satisfies } x^L \geqslant y \text{ and no } y^R \text{ satisfies } x \geqslant y^R,$$

were $x^L \in L_x$ and $y^R \in R_y$.

The second step of the construction is the definition of the class of pre-numbers. We will again define the ordinal steps P_α recursively:

- $P_0 = \{\langle \emptyset, \emptyset \rangle\}$
- $P_\alpha = \{\langle A, B \rangle : A, B \in \bigcup_{\beta<\alpha} P_\beta \text{ and } B \not\geqslant A\}$

The class P of the pre-numbers is given by $P = \bigcup_{\alpha<On} P_\alpha$.

Finally, the class No is defined as the quotient of the class of prenumbers by the equivalence relation induced by $\leqslant$. To avoid problems with equivalence classes that are proper classes, we can make a Scott's Trick.

Following Conway's notation, we will denote a class $(X,Y)/\sim$ by $\{X|Y\}$ and given a surreal number x, we will denote $x = \{L_x|R_x\}$, where $L_x|R_x$ is a prenumber that represents x. We will also use the notation x^L for an element of L_x and x^R for an element of R_x.

The *birth* function b is defined as $b(x) = min\{\alpha : \exists(L,R) \in P_\alpha \; x = \{L|R\}\}$

We can also define, for any ordinal α, the sets O_α, N_α and M_α ("old", "made" and "new"):

- $O_\alpha = \{x \in No : b(x) < \alpha\}$
- $N_\alpha = \{x \in No : b(x) = \alpha\}$
- $M_\alpha = \{x \in No : b(x) \leqslant \alpha\}$

To end this subsection we will now define, recursively, the operations $+, -, \cdot$ in P:

- $x + y = \{x^L + y, x + y^L | x^R + y, x + y^R\}$;
- $xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R | x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}$;
- $-x = \{-x^R | -x^L\}$;
- $0 = \{\emptyset|\emptyset\}$;
- $1 = \{0|\emptyset\}$.

Proposition 19. *With this operations, No is a real-closed Field. In addition, every (set) ordered field has an isomorphic copy inside No. If Global Choice is assumed, this is valid also for class ordered fields.*

1.3.1.2 The Cuesta-Dutari cuts construction

Given an strict linear order $(T,<)$, we can make a Cuesta "completion" of T, denoted by $\chi(T)$, wich is defined by

$$\chi(T) = (T \cup CD(T), <'),$$

with $<'$ defined as follows:

(i) If $x, y \in T$, then the order is the same as in T;

(ii) If $x = (L,R), y = (L',R') \in CD(T)$, then $x <' y$ if $L \subsetneq L'$;

(iii) If $x \in T$ and $y = (L,R) \in CD(T)$, then $x <' y$ if $x \in L$ or $y <' x$ if $x \in R$.

The idea of that construction is basically the iteration of the Cuesta-Dutari completion starting from the empty set until the we obtain a η_∞ class.

By recursion we define the sets T_α:

- $T_0 = \emptyset$;
- $T_{\alpha+1} = \chi(T_\alpha)$;
- $T_\gamma = \bigcup_{\beta<\gamma} T_\beta$.

And finally we have

$$No := \bigcup_{\alpha \in On} T_\alpha$$

In that construction, the *birth* function b is given by the map that assigns to each surreal number x, the ordinal $b(x)$ which corresponds to the least set $T_b(x)$ that x belongs.

Note that in this construction the sets "old", "made" and "new" can be presented in a simpler way:
- $O_\alpha = \bigcup_{\beta<\alpha} T_\beta$
- $M_\alpha = T_\alpha$
- $N_\alpha = T_\alpha \setminus O_\alpha$

1.3.1.3 The binary tree construction or the space of signs construction

Consider the class $\Sigma = \{f : \alpha \to \{-, +\} : \alpha \in On\}$. We can define in this class an relation $<$ as follows:

- $f < g \iff f(\alpha) < g(\alpha)$, where α is the least ordinal such that f and g differs, with the convention $- < 0 < +$ ($f(\alpha) = 0$ iff f is not defined in α).

With this relation, Σ is a strict linearly ordered class isomorphic to $(No, <)$.

In this construction, the birth function is given by the map $b : \Sigma \to On$, $f \mapsto \operatorname{dom} f$.

1.3.2 The axiomatic approach

It is an well-known fact that the notion of real numbers ordered field can be completely described (or axiomatized) as a certain structure –of complete ordered field – and every pair of such kind of structure are isomorphic under a unique ordered field isomorphism (in fact, there is a unique ordered field morphism between each pair of complete ordered fields and it is, automatically, an isomorphism). In this subsection, strongly based on section 3 of the chapter 4 in [All87], we present a completely analogous description for the ordered class (or ordered field) of surreal numbers.

Definition 20. *A full class of surreal numbers is a structure* $\mathcal{S} = (S, <, b)$ *such that:*
(i) $(S, <)$ *is a strictly linearly ordered class;*

(ii) $b : S \to On$ *is a surjective function;*
(iii) For each (small) Conway cut (L, R) *in* $(S, <)$*, the class* $I_S(L, R) = \{x \in S : L < \{x\} < R\}$ *is non-empty and its subclass* $m_S(L, R) = \{x \in I_S(L, R) : \forall y \in S, b(y) < b(x) \to y \notin I_S(L, R)\}$ *is a singleton;*
(iv) For each Conway cut (L, R) *in* $(S, <)$ *and each ordinal number* α *such that* $b(z) < \alpha$*,* $\forall z \in L \cup R$*,* $b(\{L|R\}) \leqslant \alpha$*, where* $\{L|R\}$ *its unique member of* $m_S(L, R)$*.* □

Remark 21. Let $\mathcal{S} = (S, <, b)$ be a full class of surreal numbers.
• Condition (ii) above entails that S is a *proper* class.
• Condition (iii) above guarantees that $(S, <)$ is a η_∞-class.
• Since the order relation in On is linear (is an well-order), according the notation in condition (iii), $m_S(L, R) = \{x \in I_S(L, R) : \forall y \in S, y \in I_S(L, R) \to b(x) \leq b(y)\}$.
• By condition (iv), $b(\{\emptyset|\emptyset\}) = 0$. □

As mentioned in section 3 of chapter 4 in [All87], by results proven in Conway's book [Con01], the constructions of surreal numbers classes presented in our subsection 3.1 (by Conway cuts, by Cuesta-Dutari cuts and by the space of sign-expansions), endowed with natural "birthday" functions, are all full classes of surreal numbers.

Definition 22. *Let* $\mathcal{S} = (S, <, b)$ *and* $\mathcal{S}' = (S', <', b')$ *be full classes of surreal numbers. A surreal (mono)morphism* $f : \mathcal{S} \to \mathcal{S}'$ *is a function* $f : S \to S'$ *such that:*
(i) $\forall x, y \in S,\ x < y \iff f(x) <' f(y)$*;*
(ii) $\forall x \in S,\ b'(f(x)) = b(x)$*.* □

Remark 23. Let $\mathcal{S} = (S, <, b)$ and $\mathcal{S}' = (S', <', b')$ be full classes of surreal numbers.
• Since $<$ and $<'$ are linear order, a surreal morphism is always injective and condition (i) is equivalent to:
(i)' $\forall x, y \in S,\ x < y \longrightarrow f(x) <' f(y)$.
• Naturally, we can define a ("very-large") category whose objects are the full classes of surreal numbers and the arrows are surreal morphisms, with obvious composition and identities. Clearly, an isomorphism in such category is just a surjective morphism. □

Proposition 24. *Let* $\mathcal{S} = (S, <, b)$ *and* $\mathcal{S}' = (S', <', b')$ *be full classes of surreal numbers. Then:*
(i) There is a unique surreal (mono)morphism $f : \mathcal{S} \to \mathcal{S}'$ *and it is an isomorphism.*
(ii) For each ordinal number α*,* $b^{-1}([0, \alpha))$ *is a set. Or, equivalently,* b *is a locally small function.*
(iii) The function $(L, R) \in C_s(S, <) \overset{t}{\mapsto} \{L|R\} \in S$ *is surjective.* □

In particular, all the constructions of surreal numbers classes presented in our subsection 3.1, endowed with natural birthday functions, are canonically isomorphic, through a unique isomorphism.

In the section 4 of chapter 4 in [All87], named "Subtraction in No", we can find the following result:

Proposition 25. *Let $\mathcal{S} = (S, <, b)$ be a full class of surreal numbers. Then there is a unique function $- : S \to S$ such that:*
(i) $b(-x) = b(x), \forall x \in S$;
(ii) $-(-x) = x, \forall x \in S$;
(iii) $x < y \iff -y < -x, \forall x, y \in S$;
(iv) $-\{L|R\} = \{-R| - L\}, \forall (L, R) \in C_s(S, <)$. □

Remark 26. Let $\mathcal{S} = (S, <, b)$ be a full class of surreal numbers.
• In the presence of condition (ii), condition (iii) is equivalent to:
(iii)' $x < y \to -y < -x, \forall x, y \in S$.
• By condition (iii), condition (iv) makes sense, since $L < R \Rightarrow -R < -L$.
• By condition (iv), $-\{\emptyset|\emptyset\} = \{\emptyset|\emptyset\}$. □

We finish this Subsection registering the following useful results whose proofs can be found in [All87], pages 125, 126.

Fact 27. *Let $\mathcal{S} = (S, <, b)$ be a full class of surreal numbers. Let $A, A', B, B', \{x\}, \{x'\} \subseteq S$ be subsets such that $A < B$ and $A' < B'$ and $x = \{A|B\}, x' = \{A'|B'\}$. Then:*

(a) If A and A' are mutually cofinal and B and B' are mutually coinitial, then $\{A|B\} = \{A'|B'\}$.

(b) Suppose that (A, B) and (A', B') are timely representations of x and x' respectively, i.e $b(z) < b(x), \forall z \in A \cup B$ and $b(z') < b(x'), \forall z' \in A' \cup B'$. If $x = x'$ then A and A' are mutually cofinal and B and B' are mutually coinitial.

□

1.3.3 Ordinals in No

The results presented in this Subsection can be found in the Chapter 4 of [All87].

The ordinals can be embedded in a very natural way in the field No. The function that makes this work is recursively defined as follows:

Definition 28. $j(\alpha) = \{j[\alpha]|\emptyset\}$, $\alpha \in On$.

The following result establishes a relation between the function j and the birthday function:

Proposition 29. *[*[Gon86]*, p. 41] $b \circ j = id_{On}$*

That map j encodes completely the ordinal order into the surreal order:

Proposition 30. *$\alpha < \beta$ iff $j(\alpha) < j(\beta)$, $\forall \alpha, \beta \in On$.*

We have also that $j(0) = 0$, $j(1) = 1$. In fact, that embedding preserves also some algebraic structure. Although the sum and product of ordinals are not commutative, we have alternative definitions sum and product in On closely related to the usual operations that are commutative:

Definition 31. *If α and β are ordinals we can define the Hessemberg Sum (respesctively Hessemberg Product) of α and β as the sum (product) of the normal forms of α and β as if they are polynomials. (See* [Gon86], *p. 41)*

Fact 32. *[*[Gon86]*, p.42] The Hessemberg sum and product of ordinals are mapped by j to the surreal sum and product.*

In other words, the semi-ring $(On, \dot{+}, \dot{\times}, 0, 1)$ has an isomorphic copy in No given by the image of j

Chapter 2

Introducing Surreal Algebras

Motivated by the structure definable in the class No of all surreal numbers, we introduce in this Chapter the notion of surreal algebra (SUR-algebra) as a (higher-order) structure $\mathcal{S} = (S, <, *, -, t)$, satisfying some properties were, in particular, $<$ is an acyclic relation on S where $t : C_s(S) \to S$ is a function that gives a coherent choice of witness of η_∞ density of $(S, <)$. Every SUR-algebra turns out to be a proper class. Besides the verification that No indeed support the SUR-algebra structure, we have defined two distinguished SUR-algebras SA and ST, respectively the "free surreal algebra"' and "the free transitive surreal algebra", that will be useful in the sequel of this work. We also have introduced the notion of partial SUR-algebra (that can be a improper class) and describe some examples and constructions in the corresponding categories. We have provided, by categorical methods, some universal results that characterizes the SUR-algebras SA and ST, and also some relative versions with base ("urelements") $SA(I)$, $ST(I')$ where I, I' are partial SUR-algebra satisfying a few constraints.

2.1 Axiomatic definition

Definition 33. *A surreal algebra (or SUR-algebra) is an structure $\mathcal{S} = (S, <, -, *, t)$ where:*
- *$<$ is a binary relation in S;*
- *$* \in S$ is a distinguished element;*
- *$- : S \to S$ is a unary operation;*
- *$t : C_s(S) \to S$ is a function, where $C_s(S) = \{(A, B) \in P_s(S) \times P_s(S) : A < B\}$.*

Satisfying the following properties:

(S1) *$<$ is an acyclic relation.*

(S2) *$\forall x \in S$, $-(-x) = x$.*

(S3) $-* = *$.

(S4) $\forall a, b \in S$, $a < b$ *iff* $-b < -a$.

(S5) $\forall (A, B) \in C_s(S)$, $A < t(A, B) < B$.

(S6) $\forall (A, B) \in C_s(S)$, $-t(A, B) = t(-B, -A)$.

(S7) $* = t(\emptyset, \emptyset)$.

□

Remark 34.
• Let $(S, <)$ be the underlying relational structure of a surreal algebra $\mathcal{S}$. Then $<$ is an irreflexive relation, by condition (S1), and by (S5), $(S, <)$ is a η_∞-relational structure. As a consequence S is a *proper* class: see 7 in the Subsection 1.1.3. The other axioms establish the possibility of choice of witness for the η_∞ property satisfying additional coherent conditions.
• Note that (S3) follows from (S7) and (S6) : $-* = -t(\emptyset, \emptyset) = t(-\emptyset, -\emptyset) = t(\emptyset, \emptyset) = *$.
• Axiom (S7) establish that the SUR-algebra structure is "an extension by definitions" of a simpler (second-order) language: without a symbol for constant $*$.
• In the presence of (S2), condition (S4) is equivalent to:
(S4)' $\forall a, b \in S$, $a < b \Rightarrow -b < -a$.
• By condition (S4), condition (S6) makes sense, since $A < B \Rightarrow -B < -A$ (and if A, B are sets, then $-A$, $-B$ are sets).

□

A morphism of surreal algebras is a function that preserves all the structure on the nose. More precisely:

Definition 35. *Let $\mathcal{S} = (S, <, -, *, t)$ and $\mathcal{S}' = (S', <', -', *', t')$ be SUR-algebras. A morphism of SUR-algebras $h : \mathcal{S} \to \mathcal{S}'$ is a function $h : S \to S'$ that satisfies the conditions below:*

(Sm1) $h(*) = *'$.

(Sm2) $h(-a) = -'h(a)$, $\forall a \in S$.

(Sm3) $a < b \implies h(a) <' h(b)$, $\forall a, b \in S$.

(Sm4) $h(t(A, B)) = t'(h[A], h[B])$, $\forall (A, B) \in C_s(S)$.[1]

□

[1]Note that, by property (Sm3), $(A, B) \in C_s(S) \implies (h[A], h[B]) \in C_s(S')$, thus (Sm4) makes sense.

Definition 36. The category of SUR-algebras:

We will denote by $SUR - alg$ the ("very-large") category such that $Obj(SUR - alg)$ is the class of all SUR-algebras and $Mor(SUR - alg)$ is the class of all SUR-algebras morphisms, endowed with obvious composition and identities.

□

Remark 37.

Of course, we have the same "size issue" in the categories $ZF-alg$, of all ZF-algebras (Section 1.2, Chapter 1), and in $SUR-alg$: each object can be a (respect., is a) proper class, thus it cannot be represented in NBG background theory this "very large" category. The mathematical (pragmatical) treatment of this question, that we will adopt in the Chapters 1, 2 and 3, is to assume a stronger background theory: NBG (or ZFC) and also three Grothendieck universes $U_0 \in U_1 \in U_2$. The members of U_0 represents "the sets"; the members of U_1 represents "the classes"; the members of U_2 represents "the meta-classes". Thus a category $\mathcal{C}$ is: (i) "small", whenever $\mathcal{C} \in U_0$; (ii) "large", whenever $\mathcal{C} \in U_1 \setminus U_0$; (iii) "very large", whenever $\mathcal{C} \in U_2 \setminus U_1$.

□

2.2 Examples and constructions

2.2.1 The surreal numbers as SUR-algebras

The structure $(No, <, b)$ of full surreal numbers class, according the Definition 20 in the Subsection 1.3.2 in Chapter 1, induces a unique structure of SUR-algebra $(No, <, -, *, t)$, where:
• The function $t : C_s(No, <) \to No$ is such that $(A, B) \mapsto t(A, B) := \{A|B\}$;
• The distinguished element $* \in No$ is given by $* := \{\emptyset|\emptyset\}$;
• The function $- : No \to No$ is the unique function satisfying the conditions in Proposition 25 and Remark 26.

This SUR-algebra has two distinctive additional properties:
• t is a surjective function;
• $<$ is a strict linear order (equivalently, since $<$ is acyclic, $<$ is a total relation).

2.2.2 The free surreal algebra

We will give now a new example of surreal algebra, denoted SA[2], which is not a linear order and satisfies a nice universal property on the category of all surreal algebras (see Section 2.4). The construction, is based is based on a cumulative Conway's cuts hierarchy over a family of binary relations. [3].

We can define recursively the family of **sets** SA_α as follows:

Suppose that, for all $\beta < \alpha$, we have constructed the sets SA_β and $<_\beta$, binary relations on SA_β, and denote $SA^{(\alpha)} = \bigcup_{\beta<\alpha} SA_\beta$ and $<^{(\alpha)} = \bigcup_{\beta<\alpha} <_\beta$. Then, for α we define:

- $SA_\alpha = SA^{(\alpha)} \cup \{\langle A, B\rangle : A, B \subseteq SA^{(\alpha)} \text{ and } A <^{(\alpha)} B\}$.[4]
- $<_\alpha = <^{(\alpha)} \cup \{(a, \langle A, B\rangle), (\langle A, B\rangle, b) : \langle A, B\rangle \in SA_\alpha \setminus SA^{(\alpha)} \text{ and } a \in A, b \in B\}$.
- The class SA[5] is the union $SA := \bigcup_{\alpha\in On} SA_\alpha$.
- $< := \bigcup_{\alpha\in On} <_\alpha$ is a binary relation on SA.

Fact: Note that that:
(a) $SA^{(0)} = \emptyset$, $SA^{(1)} = SA_0 = \{\langle\emptyset, \emptyset\rangle\}$ and $SA_1 = \{\langle\emptyset, \emptyset\rangle, \langle\emptyset, \{\langle\emptyset, \emptyset\rangle\}\rangle, \langle\{\langle\emptyset, \emptyset\rangle\}, \emptyset\rangle\}$. By simplicity, we will denote $* := \langle\emptyset, \emptyset\rangle = 0$, $-1 := \langle\emptyset, \{*\}\rangle$ and $1 := \langle\{*\}, \emptyset\rangle$ thus $SA_1 = \{0, -1, 1\}$.
(b) $<_0 = \emptyset$ and $<_1 = \{(-1, 0), (0, 1)\}$.
(c) -1 and 1 are $<$-incomparable.
(d) $SA^{(\alpha)} \subseteq SA_\alpha$, $\alpha \in On$.
(e) $SA_\beta \subseteq SA_\alpha$, $\beta \leq \alpha \in On$.
(f) $SA^{(\beta)} \subseteq SA^{(\alpha)}$, $\beta \leq \alpha \in On$.
(g) $<^{(\alpha)} = <_\alpha \cap SA^{(\alpha)} \times SA^{(\alpha)}$, $\alpha \in On$ (by the definition of $<_\alpha$).
(h) $<_\beta = <^{(\alpha)} \cap SA_\beta \times SA_\beta$, $\beta < \alpha \in On$.
(i) $<_\beta = <_\alpha \cap SA_\beta \times SA_\beta$, $\beta \leq \alpha \in On$ (by items (g) and (h) above).
(j) $<_\alpha = < \cap SA_\alpha \times SA_\alpha$, $\alpha \in On$.
(k) $C_s(SA_\alpha, <_\alpha) = C_s(SA, <) \cap (P_s(SA_\alpha) \times P_s(SA_\alpha))$, $\alpha \in On$ (by item (j)).
(l) $C_s(SA^{(\alpha)}, <^{(\alpha)}) = C_s(SA, <) \cap (P_s(SA^{(\alpha)}) \times P_s(SA^{(\alpha)}))$, $\alpha \in On$.

---

[2]The "A" in SA is to put emphasis on **a**cyclic.

[3]Starting from the emptyset, and performing a cumulative construction based on Cuesta-Dutari completion of a linearly ordered set, we obtain No: see for instance [All87].

[4]The expression $\langle A, B\rangle$ is just an alternative notation for the ordered pair (A, B), used for the reader's convenience.

[5]Soon, we will see that SA is a *proper* class.

We already have defined $<$ and $*(=\langle\emptyset,\emptyset\rangle)$ in SA, thus we must define $t: C_s(SA)\to SA$ and $-: SA\to SA$ to complete the definition of the structure SA: this will be carry out by recursion on well-founded relations on SA and $C_s(SA)$[6] that will be defined below.

For each $x\in SA$, we define its rank as $r(x):=min\{\alpha\in On: x\in SA_\alpha\}$. Since for each $\beta<\alpha$, $SA_\beta\subseteq SA^{(\alpha)}\subseteq SA_\alpha$, it is clear that $r(x)=\alpha$ iff $x\in SA_\alpha\setminus SA^{(\alpha)}$.

Following Conway ([Con01], p.291), we can define for the SA setting the notions of: "old members", "made members" and "new members". More precisely, for each ordinal α:
• The set of **old** members w.r.t. α is the subset of SA of all members "born **before** day α". $O(SA,\alpha):=SA^{(\alpha)}$;
• The set of **made** members w.r.t. α is the subset of SA of all members "born **on or before** day α". $M(SA,\alpha):=SA_\alpha$;
• The set of **new** members w.r.t. α is the subset of SA of all members "born **on** day α". $N(SA,\alpha):=SA_\alpha\setminus SA^{(\alpha)}$.

We will denote $x\prec y$ in SA iff $r(x)<r(y)$ in On.

Claim 1: $\prec$ is an well-founded relation in SA.

Proof. Let $y\in SA$ and let $\alpha:=r(y)$. Given $x\in SA$, $r(x)<\alpha$ iff $x\in O(SA,\alpha)=SA^{(\alpha)}$. Therefore, the subclass $\{x\in SA: x\prec y\}$ is a subset of SA.
Now let X be a non-empty subset of SA. Then $r[X]$ is a non-empty subset of On and let $\alpha:=min(r[X])$. Consider any $a\in r^{-1}[\{\alpha\}]\cap X$, then clearly a is a $\prec$-minimal member of X. □

We have an induced "rank" on the class (of small $<$-Conway cuts) $C_s(SA)=\{(A,B)\in P_s(A)\times P_s(B): A<B\}$, $R(A,B):=min\{\alpha\in On: A\cup B\subseteq SA^{(\alpha)}\}$. We can also define a binary relation on the class $C_s(SA)$:
$(A,B)\lhd(C,D)$ in $C_s(SA)$ iff $R(A,B)<R(C,D)$ in On.

Claim 2: $\lhd$ is an well-founded relation in $C_s(SA)$.

Proof. Let $(C,D)\in C_s(SA)$ and let $\alpha:=R(C,D)$. Given $(A,B)\in C_s(SA)$, $R(A,B)<\alpha$ iff $\exists\beta<\alpha, A\cup B\subseteq O(SA,\beta)=SA^{(\beta)}$. Therefore, the subclass $\{(A,B)\in C_s(SA):$

[6]From now on, we will omit the binary relation on a class when it is clear from the setting.

$(A, B) \lhd (C, D)\}$ is a subset of $C_s(SA)$.
Now let Y be a non-empty subset of $C_s(SA)$. Then $R[Y]$ is a non-empty subset of On and let $\alpha := min(R[Y])$. Consider any $(A, B) \in R^{-1}[\{\alpha\}] \cap Y$, then clearly (A, B) is a $\lhd$-minimal member of Y. □

Let H be the (class) function $H(p, g)$ where, for each $p = (C, D) \in C_s(SA)$ and g a (set) function with domain $p^{\lhd} := \{(A, B) \in C_s(SA) : (A, B) \lhd p\}$, given by $H(p, g) := \langle C, D\rangle$ (i.e., H is just first coordinate projection). Then H is a class function and we can define by $\lhd$-recursion a unique (class) function $t : C_s(SA) \to SA$ by $t(p) = H(p, t_{\restriction p^{\lhd}})$, i.e. $t(C, D) = \langle C, D\rangle$. The range of t is included in SA: since A and B are *subsets* of SA such that $A < B$, there exists $\alpha \in On$ such that $A, B \subseteq SA^{(\alpha)}$; since $<$ is the reunion of the increasing compatible family of binary relations $\{<_\beta : \beta \in On\}$, the have that $A <^{(\alpha)} B$, thus $\langle A, B\rangle \in SA_\alpha \subseteq SA$.

Claim 3: $\forall \alpha \in On, M(SA, \alpha) = C_s(O(SA, \alpha))$. Thus $N(SA, \alpha) = C_s(SA^{(\alpha)}) \backslash SA^{(\alpha)}$.

Proof. Since $M(SA, \alpha) = O(SA, \alpha) \cup C_s(O(SA, \alpha))$, we just have to prove that, $SA^{(\alpha)} \subseteq C_s(SA^{(\alpha)})$, for each $\alpha \in On$. Suppose that $SA^{(\beta)} \subseteq C_s(SA^{(\beta)})$ for each $\beta \in On$ such that $\beta < \alpha$. By the assumption, we have $SA^{(\alpha)} = \bigcup_{\beta<\alpha} SA_\beta = \bigcup_{\beta<\alpha} C_s(SA^{(\beta)})$. Since $SA^{(\beta)} \subseteq SA^{(\alpha)}$ and $<^{(\beta)} = <^{(\alpha)} \cap (SA^{(\beta)} \times SA^{(\beta)})$[7], we have $C_s(SA^{(\beta)}) \subseteq C_s(SA^{(\alpha)})$, thus $\bigcup_{\beta<\alpha} C_s(SA^{(\beta)}) \subseteq C_s(SA^{(\alpha)})$. Summing up, we conclude that $SA^{(\alpha)} \subseteq C_s(SA^{(\alpha)})$ and the result follows by induction. □

Claim 4: $C_s(SA) = SA$ and $t : C_s(SA) \to SA$ is the identity map, thus, in particular, t is a bijection.

Proof. By items (k) and (l) in the Fact above, $C_s(SA, <) = \bigcup_{\alpha\in On} C_s(SA_\alpha, <_\alpha) = \bigcup_{\alpha\in On} C_s(SA^{(\alpha)}, <^{(\alpha)})$. By Claim 3 above, $SA_\alpha = C_s(SA^{(\alpha)}, <^{(\alpha)}), \forall \alpha \in On$, thus $\bigcup_{\alpha\in On} C_s(SA^{(\alpha)}, <^{(\alpha)}) = \bigcup_{\alpha\in On} SA_\alpha = SA$. Summing up, we obtain $C_s(SA) = SA$. Then $t : C_s(SA) \to SA$, given by $(A, B) \mapsto \langle A, B\rangle$ is the identity map. □

For each $x \in SA$, denote $(L_x, R_x) \in C_s(SA)$ the unique representation of x: in fact, $x = \langle L_x, R_x\rangle$.

Claim 5: $r \circ t = R$.

Proof. The functional equation is equivalent to:

[7] The non trivial inclusion $<^{(\beta)} \supseteq <^{(\alpha)} \cap (SA^{(\beta)} \times SA^{(\beta)})$ holds since for every pair (x, y) in the right side there are $\delta < \beta$ and $\gamma < \alpha$ (that we can assume $\gamma \geq \delta$) such that $(x, y) \in <_\gamma \cap SA_\delta \times SA_\delta = <_\delta \subseteq <_\beta$.

$\forall (A,B) \in C_s(SA), \forall \gamma \in On,\ \langle A,B\rangle \in SA_\gamma$ iff $A \cup B \subseteq SA^{(\gamma)}$.
If $(A,B) \in C_s(SA)$ and $A \cup B \subseteq SA^{(\gamma)}$ then, since $A < B$ we have $A <^{(\gamma)} B$, thus $\langle A,B\rangle \in SA_\gamma$ by the recursive definition of SA_γ. On the other hand, if $(A,B) \in C_s(SA)$ and $\langle A,B\rangle \in SA_\gamma$, then by Claim 3 above, $(A,B) \in C_s(SA,<) \cap C_s(SA^{(\gamma)}), <^{(\gamma)}) = C_s(SA^{(\gamma)}), <^{(\gamma)})$, thus $A \cup B \subseteq SA^{(\gamma)}$. □

Claim 6: Let $(A,B) \in C_s(SA)$ and $\alpha \in On$, then: $\forall a \in A, \forall b \in B,\ r(a), r(b) < \alpha$ iff $r(t(A,B)) \le \alpha$. In particular: $\forall a \in A, \forall b \in B,\ r(a), r(b) < r(t(A,B))$.

Proof. The equivalence is just a rewriting of the equivalence proved above:
$A \cup B \subseteq SA^{(\alpha)}$ iff $\langle A,B\rangle \in SA_\alpha$. □

Claim 7: $\forall x,y \in SA,\ x < y \ \Rightarrow\ r(x) \ne r(y)$. In particular, the relation $<$ in SA is irreflexive.

Proof. Suppose that there are $x,y \in SA$ such that $x < y$ and $r(x) = r(y) = \alpha \in On$. Thus $x,y \in SA_\alpha \setminus SA^{(\alpha)}$ and, since $x,y \in SA_\alpha$ and $(x,y) \in <$, we get $(x,y) \in <_\alpha \setminus <^{(\alpha)}$. Thus $(x,y) = (a, \langle L_y, R_y\rangle)$ for some $a \in L_y \subseteq SA^{(\alpha)}$ or $(x,y) = (\langle L_x, R_x\rangle, d)$ for some $d \in R_x \subseteq SA^{(\alpha)}$. In both cases we obtain $x = a \in SA^{(\alpha)}$ or $y = d \in SA^{(\alpha)}$, contradicting our hypothesis. □

Claim 8: Let $A,B \subseteq SA$ be subclasses such that $A < B$, then $r[A] \cap r[B] = \emptyset$.

Proof. Suppose that $A < B$ and that there are $a \in A$ and $b \in B$ such that $r(a) = r(b) \in r[A] \cap r[B]$. Then $a < b$ and $r(a) = r(b)$, contradicting the Claim 7 above. □

Claim 9: Let $(A,B), (C,D) \in C_s(SA)$. Then $\langle A,B\rangle < \langle C,D\rangle$ iff $\langle A,B\rangle \in C$ (then $r(\langle A,B\rangle) < r(\langle C,D\rangle))$ or $\langle C,D\rangle \in B$ (then $r(\langle C,D\rangle) < r(\langle A,B\rangle))$.

Proof. ($\Leftarrow$) If $\langle A,B\rangle = c \in C$ and $r(\langle C,D\rangle) = \alpha$, then $(c, \langle C,D\rangle) \in <_\alpha \subseteq <$, thus $\langle A,B\rangle < \langle C,D\rangle$. The other case is analogous.
($\Rightarrow$) Suppose that $\langle A,B\rangle < \langle C,D\rangle$. By Claim 7 above we have $\alpha = r(\langle A,B\rangle) \ne r(\langle C,D\rangle) = \gamma$. If $\alpha < \gamma$ we have $SA_\alpha \subseteq SA^{(\gamma)}$ and $\langle C,D\rangle \in SA_\gamma \setminus SA^{(\gamma)}$, thus $(\langle A,B\rangle, \langle C,D\rangle) \in <_\gamma \setminus <^{(\gamma)}$ and we have $\langle A,B\rangle \in C$. If $\gamma < \alpha$ we conclude, by an analogous reasoning, that $\langle C,D\rangle \in B$. □

Claim 10: Let $(A,B), (C,D) \in C_s(SA)$. Then $A < \langle C,D\rangle < B$ and $R(A,B) \le R(C,D)$ iff $A \subseteq C$ and $B \subseteq D$.

Proof. ($\Leftarrow$) Let $R(C,D) = \alpha$, then $\forall c \in C, \forall d \in D,\ (c, \langle C,D\rangle), (\langle C,D\rangle, d) \in <_\alpha \subseteq <$.

If $A \subseteq C$ and $B \subseteq D$, then $A < \langle C, D\rangle < B$ and $A \cup B \subseteq C \cup D \subseteq SA^{(\alpha)}$, i.e. $R(A, B) \leq \alpha = R(C, D)$.
($\Rightarrow$) Let $A < \langle C, D\rangle < B$ and suppose that there is $a \in A \backslash C$, then $\langle L_a, R_a\rangle = a < \langle C, D\rangle$. Since $\langle L_a, R_a\rangle \notin C$, then by Claim 9 above, we have $\langle C, D\rangle \in R_a$, thus:
$R(C, D) =_{Claim5} r(\langle C, D\rangle) <_{Claim6} r(\langle L_a, R_a\rangle) = r(a) < r(\langle A, B\rangle) = R(A, B)$. Analogously, if $A < \langle C, D\rangle < B$ and $B \setminus D \neq \emptyset$, we obtain $R(C, D) < R(A, B)$. □

Claim 11: For each $(A, B) \in C_s(SA, <)$, $A < \langle A, B\rangle < B$. In particular, $(SA, <)$ is a η_∞ *proper* class.

Proof. By Claim 7 above, $<$ is an irreflexive relation. By Claim 10 above, for each $(A, B) \in C_s(SA, <)$, $A < \langle A, B\rangle < B$, thus $(SA, <)$ is a η_∞ class. It follows from 7 in the Subsection 1.1.3, that SA is proper class. □

Claim 12: For each $(A, B) \in C_s(SA, <)$ and each $z \in SA$ such that $A < z < B$, then $r(t(A, B)) \leq r(z)$.

Proof. Suppose that the result is false and let α the least ordinal such that there are $(A, B) \in C_s(SA)$ and $z \in SA$ such that $A < z < B$, but $r(z) < r(t(A, B)) = R(A, B) = \alpha$: thus $\alpha > 0$. By a simple analysis of the cases α ordinal limit and α successor, we can see that there are $A' \subseteq A, B' \subseteq B$ such that $R(A', B') = \alpha' < \alpha$ and $A' < z < B'$, contradicting the minimality of α[8]. □

Define, by recursion on the well-ordered proper class $(On, <)$, a function $s : On \to SA$ by $s(\alpha) := \langle s[\alpha], \emptyset\rangle$, $\alpha \in On$.

Claim 13: $r \circ s = id_{On}$. In particular, the function $r : SA \to On$ is surjective and SA is a proper class.

Proof. We will prove the result by induction on the well-ordered proper class $(On, <)$. Let $\alpha \in On$ and suppose that $r(s(\beta) = \beta$, for all ordinal $\beta < \alpha$. Then:
$r(s(\alpha)) = r(\langle s[\alpha], \emptyset\rangle) = r(t(s[\alpha], \emptyset)) =_{Claim\ 5} R(s[\alpha], \emptyset) = min\{\gamma \in On : s[\alpha] \cup \emptyset \subseteq SA^{(\gamma)}\}$.
By the induction hypothesis, we have:
(IH) $s(\beta) \in SA_\beta \setminus SA^{(\beta)}$, for all ordinal $\beta < \alpha$.
Since $s(\beta) \in SA_\beta$, we have $s(\beta) \in SA^{(\alpha)}$, $\forall \beta < \alpha$. If $s[\alpha] \cup \emptyset \subseteq SA^{(\gamma)}$ for some $\gamma < \alpha$, then $s(\gamma) \in SA^{(\gamma)}$, in contradiction with (IH). Summing up, we conclude that $r(s(\alpha) = \alpha$, and the result follows by induction. □

[8] Hint: in the case $\alpha = \gamma + 1$, use Claim 7.

Claim 14: There is a unique function $- : SA \to SA$, such that:
(i) $\forall x \in SA, r(-x) = r(x)$;
(ii) $\forall x \in SA, -(-x) = x$;
(iii) $\forall x, y \in SA$, $x < y$ iff $-y < -x$;
(iv) $\forall (A, B) \in C_s(SA), -t(A, B) = t(-B, -A)$.

Proof. Let $z \in SA$ and suppose that a function $-$ is defined for all $x, y \in SA$, such that $x, y \prec z$, satisfying the conditions (i)–(iv) adequately restricted to the subset $SA^{(\alpha)}$, for $\alpha := r(z) \in On$. Then $z = \langle L_z, R_z\rangle = t(L_z, R_z) \in SA_\alpha \setminus SA^{(\alpha)}$ for a unique $(L_z, R_z) \in C_s(SA)$ (Claim 4) and α is the least $\gamma \in On$ such that $L_z \cup R_z \subseteq SA^{(\gamma)}$ (Claim 5), thus $\forall x \in R_z \cup L_z, r(x) < r(z)$. Then $-x$ is defined $\forall x \in L_z \cup R_z$, satisfying the conditions (i)–(iv) restricted to the subset $SA^{(\alpha)}$. Since $x < y$, $\forall x \in L_z \forall y \in R_z$, it holds, by condition (iii), $-y < -x$ then $-R_z < -L_z$ and since $-R_z, -L_z$ are the images of a function on sets, $(-R_z, -L_z) \in C_s(SA)$. Moreover, by condition (i), α is the least $\gamma \in On$ such that $-R_z \cup -L_z \subseteq SA^{(\gamma)}$, i.e. $t(-R_z, -L_z) = \langle -R_z, -L_z\rangle \in SA_\alpha \setminus SA^{(\alpha)}$. Define $-z := t(-R_z, -L_z)$.
Now we will prove that the conditions (i)–(iv) still holds for all members in $SA_\alpha \setminus SA^{(\alpha)}$.
(i) Let $x \in SA_\alpha$. If $x \in SA^{(\alpha)}$, this condition holds by hypothesis. If $x \in SA_\alpha \setminus SA^{(\alpha)}$, then by the recursive definition above, $-x = \langle -R_x, -L_x\rangle \in SA_\alpha \setminus SA^{(\alpha)}$, thus $r(-x) = \alpha = r(x)$. Thus (i) holds in SA_α.
(ii) Let $x \in SA_\alpha \setminus SA^{(\alpha)}$, then $-x, -(-x) \in SA_\alpha \setminus SA^{(\alpha)}$ (by the validity of condition (i) on SA_α established above). $-(-x) = -(-t(L_x, R_x)) = -t(-R_x, -L_x) = t(-(-L_x), -(-R_x)) = t(L_x, R_x) = x$, since by hypothesis the conditions (iii) and (ii) holds for members of $SA^{(\alpha)}$. Thus (ii) holds in SA_α.
(iii) We suppose that $\forall x, y \in SA^{(\alpha)}$, $x < y$ iff $-y < -x$. Let $x, y \in SA_\alpha$ such that $x < y$. If both $x, y \in SA^{(\alpha)}$ then, by hypothesis $-y < -x$. Otherwise, by Claim 7, there is exactly one between x, y that is a member of $SA_\alpha \setminus SA^{(\alpha)}$. By Claim 9: if $r(y) < r(x) = \alpha$ then $y \in R_x$; if $r(x) < r(y) = \alpha$ then $x \in L_y$. Thus: if $r(y) < r(x) = \alpha$ then $-y \in -R_x = L_{-x}$, thus $-y < -x$; if $r(x) < r(y) = \alpha$ then $-x \in -L_y = R_{-y}$, thus $-y < -x$. Then we have proved that $\forall x, y \in SA_\alpha$, $x < y \Rightarrow -y < -x$. Since the conditions (i) and (ii) have already be established on SA_α, we also have $\forall x, y \in SA_\alpha$, $-y < -x \Rightarrow x = -(-x) < -(-y) = y$.
(iv) Suppose that $t(A, B) = \langle A, B\rangle = \langle -B, -A\rangle = t(-B, -A)$ holds for all $(A, B) \in \bigcup_{\beta<\alpha} C_s(SA) \cap P_s(SA^{(\beta)}) \times P_s(SA^{(\beta)}) = \bigcup_{\beta<\alpha} C_s(SA^{(\beta)}) = \bigcup_{\beta<\alpha} SA_\beta = SA^{(\alpha)}$. We must prove that the condition still holds for all $(C, D) \in C_s(SA) \cap P_s(SA^{(\alpha)}) \times P_s(SA^{(\alpha)}) = C_s(SA^{(\alpha)} = SA_\alpha$. Let $z = (C, D) = \langle C, D\rangle = t(C, D) \in SA_\alpha \setminus SA^{(\alpha)}$. Then just by the recursive definition of $-z$, we have $-z = -t(C, D) = t(-B, -A)$, as we wish. □

Finally, we will prove that SA satisfies all the 7 axioms of SUR-algebra:

(S7) $* = t(\emptyset, \emptyset)$.
This holds by our definition of $*$.

(S5) $\forall (A,B) \in C_s(SA),\ A < t(A,B) < B$.
This holds by the Claim 10 above.

(S6) $\forall (A,B) \in C_s(SA),\ -t(A,B) = t(-B,-A)$.
This holds by Claim 14.(iv).

(S3) $-* = *$.
Since $-* = -t(\emptyset,\emptyset) = t(-\emptyset,-\emptyset) = t(\emptyset,\emptyset) = *$.

(S2) $\forall x \in SA,\ -(-x) = x$.
This holds by Claim 14.(ii).

(S4) $\forall a, b \in SA,\ a < b$ iff $-b < -a$.
This holds by Claim 14.(iii).

(S1) $<$ is an acyclic relation.
Suppose that $<$ is not acyclic and take $x_0 < ... < x_n < x_0$ a cycle in $(SA,<)$ of minimum length $n \in \mathbb{N}$. Since $<$ is an irreflexive relation (see the Claim 7), $n > 0$. Let $\alpha = max\{r(x_i) : i \leq n\}$ and let j be the least $i \leq n$ such that $r(x_j) = \alpha$.
If $j = 0$: Since $x_0 < x_1$ and $x_n < x_0$, then by Claim 7, $r(x_1), r(x_n) < r(x_0)$. Writing $x_0 = \langle L_{x_0}, R_{x_0}\rangle$ (since, by Claim 4, $SA = C_s(SA)$), we obtain from Claim 9 that $x_n \in L_{x_0}$ and $x_1 \in R_{x_0}$. As $L_{x_0} < R_{x_0}$, we have $x_n < x_1$ and then $x_1 < ... < x_n < x_1$ is a cycle of length $n-1 < n$, a contradiction.
If $j > 0$: Then define $j^- := j-1$ and $j^+ := j+1$ (respec. $j^+ = 0$), if $j < n$ (respec. $j = n$).
Then by Claim 7, $r(x_{j^-}), r(x_{j^+}) < r(x_j)$ and by Claim 9: $x_{j^-} \in L_{x_j}$ and $x_{j^+} \in R_{x_j}$. As $L_{x_j} < R_{x_j}$, we have $x_{j^-} < x_{j^+}$ and then we can take a sub-cycle of the original one omitting x_j: this new cycle has of length $n-1 < n$, a contradiction.

2.2.3 The free transitive surreal algebra

We will give now a new example of surreal algebra, denoted ST[9], which is a strict partial order[10] that is not linear and satisfies a nice universal property on the category of all **transitive** surreal algebras (see Section 2.4). The construction is similar to the construction of SA in the previous subsection: it is based on a cumulative Conway's cuts hierarchy over a family of binary (transitive) relations.

We can define recursively the family of **sets** ST_α as follows:

Suppose that, for all $\beta < \alpha$, we have constructed the sets ST_β and $<_\beta$, binary relations on ST_β, and denote $ST^{(\alpha)} = \bigcup_{\beta<\alpha} ST_\beta$ and $<^{(\alpha)} = \bigcup_{\beta<\alpha} <_\beta$. Then, for α we define:

[9] The "T" in ST is to put emphasis on **t**ransitive.
[10] Recall that a binary relation is that is a strict partial order iff it is a transitive and acyclic relation.

- $ST_\alpha = ST^{(\alpha)} \cup \{\langle A, B\rangle : A, B \subseteq ST^{(\alpha)}$ and $A <^{(\alpha)} B\}$.
- $<_\alpha$= the transitive closure of the relation $<'_\alpha$, where $<'_\alpha := (<^{(\alpha)} \cup \{(a, \langle A, B\rangle), (\langle A, B\rangle, b) : \langle A, B\rangle \in ST_\alpha \setminus ST^{(\alpha)}$ and $a \in A, b \in B\})$.
- The (proper) class ST is the union $ST := \bigcup_{\alpha \in On} ST_\alpha$.
- $< := \bigcup_{\alpha \in On} <_\alpha$ is a binary (transitive) relation on ST.

The following result is straightforward an completely analogous to the corresponding items in the Fact in the previous Subsection on SA:

Fact 1: Note that that:
(a) $ST^{(0)} = \emptyset$, $ST^{(1)} = ST_0 = \{\langle \emptyset, \emptyset\rangle\}$. By simplicity, we will denote $0 := \langle \emptyset, \emptyset\rangle$, $1 := \langle \emptyset, \{0\}\rangle$, $-1 := \langle \{0\}, \emptyset\rangle$. Thus: $ST_0 = \{0\}$, $SA_1 = \{0, 1, -1\}$.
(b) $<_0 = \emptyset$, $<_1 = \{(-1, 0), (0, 1), (-1, 1)\}$.
(c) $-1 < 0 < 1$, $-1 < \langle\{-1\}, \{1\}\rangle < 1$, but $0, \langle\{-1\}, \{1\}\rangle$ are $<$-incomparable.
(d) $ST^{(\alpha)} \subseteq ST_\alpha$, $\alpha \in On$.
(e) $ST_\beta \subseteq ST_\alpha$, $\beta \leq \alpha \in On$.
(f) $ST^{(\beta)} \subseteq ST^{(\alpha)}$, $\beta \leq \alpha \in On$. □

Analogously to in the SA case, we can define rank functions $r : ST \to On$[11] and $R : C_s(ST) \to On$ that induces well-founded relations on ST and on $C_s(ST)$.

The results below are almost all (the exception are the items (m), (n), (o)) analogous to corresponding items in the Fact in the previous Subsection on SA. However, the techniques needed in the proofs are different than in SA case and deserve a careful presentation.

Fact 2:
(g) $<^{(\alpha)} = <_\alpha \cap SA^{(\alpha)} \times SA^{(\alpha)}$, $\alpha \in On$.
(h) $<_\beta = <^{(\alpha)} \cap SA_\beta \times SA_\beta$, $\beta < \alpha \in On$.
(i) $<_\beta = <_\alpha \cap SA_\beta \times SA_\beta$, $\beta \leq \alpha \in On$.
(j) $<_\alpha = < \cap SA_\alpha \times SA_\alpha$, $\alpha \in On$.
(k) $C_s(ST_\alpha, <_\alpha) = C_s(ST, <) \cap P_s(ST_\alpha) \times P_s(ST_\alpha)$, $\alpha \in On$.
(l) $C_s(ST^{(\alpha)}, <^{(\alpha)}) = C_s(ST, <) \cap P_s(ST^{(\alpha)}) \times P_s(ST^{(\alpha)})$, $\alpha \in On$.
(m) $\forall \alpha \in On$, $<_\alpha$ is a transitive and a acyclic relation on ST_α.
(n) $<$ is a transitive and acyclic relation (or, equivalently, it is a strict partial order) on ST.

[11] For each $x \in ST$, $r(x) = \alpha \in On$ iff $x \in ST_\alpha \setminus ST^{(\alpha)}$.

(o) Let $x, y \in ST$ and denote $\alpha := max\{r(x), r(y)\}$. Then are equivalent:
- $x < y$.
- Exists $n \in \mathbb{N}$, exists $\{z_0, \cdots, z_{n+1}\} \subseteq ST_\alpha$ such that: $x = z_0, y = z_{n+1}$; $z_j \in L_{z_{j+1}}$ or $z_{j+1} \in R_{z_j}$, for all $j \leq n$; $\{z_1, \cdots, z_n\} \subseteq ST^{(\alpha)}$.

Proof. Item (i) follows from items (g) and (h). Items (k) and (l) follows from item (j). Items (n) and (o) are direct consequences of item (m) (for the item (o) is required to perform induction in α), since $<= \bigcup_{\alpha \in On} <_\alpha$.

(g) Clearly $<^{(\alpha)} \subseteq <_\alpha \cap SA^{(\alpha)} \times SA^{(\alpha)}$. To show the converse inclusion let $x, y \in SA^{(\alpha)}$ be such that $x <_\alpha y$ and let $x = x_0 <'_\alpha \dots <'_\alpha x_n = y$ be a sequence in $(ST_\alpha, <'_\alpha)$ with the number $k = card(\{i \leqslant n : r(x_i) = \alpha\}$ being minimum. We will show that $k = 0$, thus the sequence is just $x = x_0 <^{(\alpha)} \dots <^{(\alpha)} x_n = y$ and then $x <^{(\alpha)} y$ because $<^{(\alpha)}$ is a transitive relation (since $<_\beta$, $\beta \in On$ is a transitive relation, by construction). Suppose, by absurd, that $k > 0$ and let j be the least $i \leq n$ such that $r(x_j) = \alpha$. By our hypothesis on x, y we have $0 < j < n$. Since $x_{j-1} <'_\alpha x_j <'_\alpha x_{j+1}$, we have $r(x_{j-1}), r(x_{j+1}) < r(x_j) = \alpha$ and $x_{j-1} \in L_{x_j}$, $x_{j+1} \in R_{x_j}$. As $L_{x_j} <^{(\alpha)} R_{x_j}$, we have $x_{j-1} <^{(\alpha)} x_{j+1}$ and then we can take a sub-cycle of the original one omitting x_j: this new cycle has $k - 1 < k$ members with rank α, a contradiction.

(h) We only prove the non-trivial inclusion. Let $x, y \in SA_\beta$ be such that $x <^{(\alpha)} y$. Since $<^{(\alpha)} = \bigcup_{\gamma < \alpha} <_\gamma$, let β' be the least $\gamma < \alpha$ such that $x <_{\beta'} y$. We will prove that $\beta' \leq \beta$, thus we obtain $x <_\beta y$, as we wish. Suppose, by absurd, that $\beta' > \beta$. Then $(x, y) \in <_{\beta'} \cap SA^{(\beta')} \times SA^{(\beta')}$, and by the item (g) proved above $(x, y) \in <^{(\beta')}$. Thus there is some $\gamma < \beta'$ such that $x <_\gamma y$, contradicting the minimality of β'.

(j) Let $x, y \in SA_\alpha$ be such that $x < y$. Since $<= \bigcup_{\gamma \in On} <_\gamma$, let α' be the least $\gamma \in On$ such that $x <_{\alpha'} y$. We will prove that $\alpha' \leq \alpha$, thus we obtain $x <_\alpha y$, as we wish. Suppose, by absurd, that $\alpha' > \alpha$. Then $(x, y) \in <_{\alpha'} \cap SA^{(\alpha')} \times SA^{(\alpha')}$, and by the item (g) proved above $(x, y) \in <^{(\alpha')}$. Thus there is some $\gamma < \alpha'$ such that $x <_\gamma y$, contradicting the minimality of α'.

(m) By definition of $<_\gamma$, $<_\gamma$ is a transitive relation, $\forall \gamma \in On$.
Suppose that the statement is false and let $\alpha \in On$ be the least ordinal such that $(ST_\alpha, <_\alpha)$ has some cycle. Then $\forall \beta < \alpha$, $<_\beta$ is an acyclic relation but $<_\alpha$ has some cycle (or, equivalently, $<'_\alpha$ has some cycle). Let $x_0 <'_\alpha \dots <'_\alpha x_n <'_\alpha x_0$ be a cycle in $(ST_\alpha, <'_\alpha)$ with the number $k = card(\{i \leqslant n : r(x_i) = \alpha\}$ being minimum. Note that $k > 0$, otherwise $x_0, \dots, x_n \in SA^{(\alpha)}$ and the cycle is $x_0 <^{(\alpha)} \dots <^{(\alpha)} x_n <^{(\alpha)} x_0$, thus there is a $\beta < \alpha$ and a cycle $x_0 <_\beta \dots <_\beta x_n <_\beta x_0$ in $(ST_\beta, <_\beta)$, contradicting our hypothesis. Let j be the least $i \leq n$ such that $r(x_j) = \alpha$.
<u>If $j = 0$</u>: Since $x_0 <'_\alpha x_1$ and $x_n <'_\alpha x_0$, then $r(x_1), r(x_n) < r(x_0) = \alpha$. Writing $x_0 = \langle L_{x_0}, R_{x_0} \rangle$, we have that $x_n \in L_{x_0}$ and $x_1 \in R_{x_0}$. As $L_{x_0} <^{(\alpha)} R_{x_0}$, we have $x_n <^{(\alpha)} x_1$, and then $x_1 <'_\alpha \dots <'_\alpha x_n <'_\alpha x_1$ is a cycle in $(ST_\alpha, <'_\alpha)$ with $k - 2 < k$ members with rank α, a contradiction.
<u>If $j > 0$</u>: Then define $j^- := j - 1$ and $j^+ := j + 1$ (respect. $j^+ = 0$), if $j < n$ (respect.

$j = n$).
Then $r(x_{j^-}), r(x_{j^+}) < r(x_j) = \alpha$ and: $x_{j^-} \in L_{x_j}$, $x_{j^+} \in R_{x_j}$. As $L_{x_j} <^{(\alpha)} R_{x_j}$, we have $x_{j^-} <^{(\alpha)} x_{j^+}$ and then we can take a sub-cycle of the original one omitting x_j: this new cycle has $k-1<k$ members with rank α, a contradiction. □

Since the harder part was already done, we just sketch the construction of the SUR-algebra structure $(ST, <, -, *, t)$:
• As in the SA case, from the well founded relation on $C_s(ST)$ we can define recursively a function with range ST, $t : C_s(ST) \to ST$ by $t(A, B) = \langle A, B\rangle$. We can prove, by induction, that $ST_\alpha = C_s(ST^{(\alpha)})$, $\alpha \in On$. Thus t is a bijection (is the identity function). Moreover, if $(A, B) \in C_s(ST)$, then $A < t(A, B) < B$.
• We define $* := 0 = t(\emptyset, \emptyset)$.
• As in the SA case, we can define (recursively) the function $- : ST \to ST$ by $-\langle A, B\rangle := \langle -B, -A\rangle$.

The verification of the satisfaction of the SUR-algebra axioms (S2)–(S7) are analogous as in the SA case. The satisfaction of (S1) was proved in item (m) of Fact 2 above.

2.2.4 The cut surreal algebra

In this Subsection we present a generalization of the SA, ST constructions. Given a surreal algebra S, we can define a new surreal algebra whose domain is $C_s(S)$ with the following relations and operations:

Definition 38. *Let $(S, <, -, *, t)$ be a surreal algebra. Consider the following structure in $C_s(S)$*
• $*' = (\emptyset, \emptyset)$
• $-'(A, B) = (-B, -A)$
• $(A, B) <' (C, D) \iff t(A, B) < t(C, D)$
• $t'(\Gamma, \Delta) = (t[\Gamma], t[\Delta])$, $\Gamma, \Delta \subseteq_s C_s(S)$, $\Gamma <' \Delta$

Proposition 39. *With this operations $(C_s(S), <', -', *', t')$ is a surreal algebra.*

Proof.

(S1) $<'$ is acyclic because any cycle $(A_0, B_0) <' \ldots <' (A_n, B_n)$ induces a cycle $t(A_0, B_0) < \ldots < t(A_n, B_n)$ in S, which is acyclic.

(S2) $-' -' (A, B)) = -'(-B, -A) = (- - A, - - B) = (A, B)$.

(S3) $-'*' = -'(\emptyset, \emptyset) = (-\emptyset, -\emptyset) = (\emptyset, \emptyset) = *'$.

(S4) $(A, B) <' (C, D)$ iff $t(A, B) < t(C, D)$ iff $-t(C, D) < -t(A, B)$ iff $t(-D, -C) < t(-B, -A)$ iff $(-D, -C) <' (-B, -A)$ iff $-'(C, D) <' -'(A, B)$.

(S5) Let $(\Gamma, \Delta) \in C_s(C_s(S))$. Then $\Gamma <' \Delta$ and thus $t[\Gamma] < t[\Delta]$. Since S satisfies **(S5)**, $t[\Gamma] < t(t[\Gamma], t[\Delta]) < t[\Delta]$. By the definition of $<'$, $\Gamma <' (t[\Gamma], t[\Delta]) <' \Delta$ and then $\Gamma <' t'(\Gamma, \Delta) <' \Delta$.

(S6) $-'t'(\Gamma, \Delta) = -'(t[\Gamma], t[\Delta]) = (-t[\Delta], -t[\Gamma]) = (t[-'\Delta], t[-'\Gamma]) = t'(-'\Delta, -'\Gamma)$

(S7) $t'(\emptyset, \emptyset) = (t[\emptyset], t[\emptyset]) = (\emptyset, \emptyset) = *'$

□

Some properties of the sur-algebra S are transferred to $C_s(S)$ as we can see in the above proposition:

Proposition 40.

(a) If S is transitive then $C_s(S)$ is transitive.

(b) If S is linear then $C_s(S)$ is pre-linear, i.e., denote $\sim_t$ the equivalence relation on $C_s(S)$ given by $(A, B) \sim_t (C, D)$ iff $t(A, B) = t(C, D)$. Then it holds exactly one between of the alternatives: $(A, B) <' (C, D)$; $(A, B) \sim_t (C, D)$; $(C, D) <' (A, B)$.

Proof.

(a) Suppose that we have $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in C_s(S)$ satisfying $(A_1, B_1) <' (A_2, B_2) <' (A_3, B_3)$. Then, by definition, $t(A_1, B_1) < t(A_2, B_2) < t(A_3, B_3)$. Since $<$ is transitive, we have that $t(A_1, B_1) < t(A_3, B_3)$ and then $(A_1, B_1) <' (A_3, B_3)$.

(b) Is straightforward.

□

If follows almost directly by the definition of the structure in $C_s(S)$ that:

Proposition 41. *$t : C_s(S) \to S$ is a morphism of surreal algebras.*

□

Remark 42. In the case of the three principal examples of SUR-algebras we have that $t : C_s(SA) \to SA$ and $t : C_s(ST) \to ST$ are bijections and $t : C_s(No) \to No$ is a surjection. □

Proposition 43. *If $f : S \to S'$ is a sur-algebra morphism then $C_s(f) : C_s(S) \to C_s(S') : (A, B) \mapsto (f[A], f[B])$ is a morphism of sur-algebras.*

□

Proposition 44. *C_s determines a functor from SUR to SUR, and t determines a natural transformation $t : Id_{SUR-alg} \to C_s$*

□

From a direct application of the Propositions 40 and 41, we obtain the following:

Proposition 45. *Let $\mathcal{S} = (S, <, -, *, t)$ a SUR-algebra.*

1. *If $\mathcal{S}$ is universal on the category $SUR - alg$ then the following diagram commutes:*

$$(S \stackrel{!}{\to} C_s(S) \stackrel{t}{\to} S) = (S \stackrel{id_S}{\to} S)$$

2. *If $\mathcal{S}$ is an object of the full subcategory $SUR_T - alg \hookrightarrow SUR - alg$, of all* **transitive** *SUR-algebra, and is universal on the this category $SUR_T - alg$, then the following diagram commutes:*

$$(S \stackrel{!}{\to} C_s(S) \stackrel{t}{\to} S) = (S \stackrel{id_S}{\to} S)$$

□

Remark 46. Note that: $C_s(SA) = SA$ and $C_s(ST) = ST$.

□

2.3 Partial Surreal Algebras and morphisms

In several recursive constructions, the intermediate stages plays an important role in the comprehension of the object constructed. As we have seen in the Section 2.1, all surreal algebra is a *proper class* but, on the other hand, the intermediate stages of the constructions of No, SA, ST are sets. To gain some flexibility and avoid technical difficulties, we introduce in this Section the (more general and flexible) notion of *partial* surreal algebra: every SUR-algebra is a partial SUR-algebra and this new notion can be supported on a set. Besides simple examples, that contains in particular the intermediate stages of No, SA, ST, and a relativized notion of Cut (partial) SUR-algebra, we are interest on general constructions of partial SUR-algebras: for that we will consider two kinds of morphisms between them. We will perform general constructions as products, sub partial-SUR-algebra and certain kinds of directed colimits. As an application of the latter construction, we are able to prove some universal properties satisfied by SA and ST (and natural generalizations), that justifies its names of (relatively) free SUR-algebras.

Definition 47. *A* **partial surreal algebra (pSUR-algebra)** *is a structure* $\mathcal{S} = (S, <, -, *, t)$ *where* S *is a class (proper or improper),* $* \in S$, $-$ *is an unary function in* S, $<$ *is a binary relation in* S *and* $t : C_s^t(S) \to S$ *is a* partial function, *i.e.,* $C_s^t(S) \subseteq C_s(S)$, *satisfying:*

(pS1) $<$ *is an acyclic relation.*

(pS2) $\forall x \in S,\ -(-x) = x$

(pS3) $-* = *$.

(pS4) $\forall a, b \in S,\ a < b$ *iff* $-b < -a$

(pS5) *If* $(A, B) \in C_s^t(S)$, *then* $A < t(A, B) < B$.

(pS6) *If* $(A, B) \in C_s^t(S)$, *then* $(-B, -A) \in C_s^t(S)$ *and* $-t(A, B) = t(-B, -A)$.

(pS7) $(\emptyset, \emptyset) \in C_s^t(S)$ *and* $* = t(\emptyset, \emptyset)$.

□

Note that (pS1), (pS2), (pS3) and (pS4) coincide, respectively, with the SUR-algebra axioms (S1), (S2), (S3) and (S4). The statements (pS5), (pS6) and (pS7) are relative versions of, respectively, the SUR-algebra axioms (S5), (S6) and (S7). SUR-algebras are precisely the pSUR-algebras $\mathcal{S}$ such that $C_s^t(S) = C_s(S)$.

Definition 48. *Let* $\mathcal{S} = (S, <, -, *, t)$ *and* $\mathcal{S}' = (S', <', -', *', t')$ *be partial SUR-algebras. Let* $h : S \to S'$ *be (total) function and consider the conditions below:*

(Sm1) $h(*) = *'$.

(Sm2) $h(-a) = -'h(a)$, $\forall a \in S$.

(Sm3) $a < b \implies h(a) <' h(b)$, $\forall a, b \in S$.

(pSm4) $(A, B) \in C_s^t(S) \implies (h[A], h[B]) \in C_s^{t'}(S')$ *and* $h(t(A, B)) = t'(h[A], h[B])$, $\forall (A, B) \in C_s^t(S)$.

(fpSm4) $(A, B) \in C_s(S) \implies (h[A], h[B]) \in C_s^{t'}(S')$ *and* $h(t(A, B)) = t'(h[A], h[B])$, $\forall (A, B) \in C_s^t(S)$.

We will say that $h : \mathcal{S} \to \mathcal{S}'$ *is:*
• *a* **partial SUR-algebra morphism (pSUR-morphism)** *when it satisfies: (Sm1), (Sm2), (Sm3) and (pSm4);*
• *a* <u>**full**</u> **partial SUR-algebra morphism (fpSUR-morphism)** *when it satisfies: (Sm1), (Sm2), (Sm3) and (fpSm4).*

□

Remark 49.

- Note that the property (Sm3) entails: $(A, B) \in C_s(S) \implies (h[A], h[B]) \in C_s(S')$.
- The conditions (Sm1), (Sm2) and (Sm3) are already present in the definition of SUR-algebra morphism. The property:

(Sm4)
$h(t(A, B)) = t'(h[A], h[B]), \forall (A, B) \in C_s(S)$;
completes the definition of SUR-algebra morphism.

- Every full partial SUR-algebra morphism is partial SUR-algebra morphism.
- Let S, S' be partial SUR-algebras and $h : S \to S'$ is a map. Suppose that S or S' is a SUR-algebra, then h is a pSUR morphism iff h is a fpSUR-morphism.
- If S is a partial SUR-algebra, then: $id_S : S \to S$ is a pSUR-morphism and $id_S : S \to S$ is a fpSUR-morphism iff S is a SUR-algebra.
- Let $f : S \to S'$, $f' : S' \to S''$ be pSUR morphisms:
  • Then $f' \circ f$ is a pSUR-morphism.
  • If f is fpSUR-morphism, then $f' \circ f$ is a fpSUR-morphism. In particular, the composition of fpSUR-morphisms is a fpSUR-morphism.

□

Definition 50. The category of partial SUR-algebras:

We will denote by $pSUR-alg$ *the ("very-large") category such that* $Obj(pSUR-alg)$ *is the class of all partial SUR-algebras and* $Mor(pSUR-alg)$ *is the class of all partial SUR-algebras morphisms, endowed with obvious composition and identities.*

□

Remark 51.

(a) Of course, we have in the category $pSUR-alg$ the same "size issue" present in the categories of $ZF-alg$ and $SUR-alg$: we will adopt the same "solution" explained in Remark 37. An alternative is to consider only "small" partial SUR-algebras (and obtain a "large" category –instead of very large– $pSUR_s-alg$, of all small partial SUR-algebras) since we will see that there are set-size partial SUR-algebras: we will not pursue this track because our main concern in considering partial SUR-algebras is get flexibility to make (large indexed) categorial constructions with small partial SUR-algebras to obtain a total SUR-algebra as a (co)limit process, i.e., we want $pSUR \supseteq SUR$.

(b) As we saw above that, even if the class of full morphism of partial SUR-algebras is closed under composition, it does not determines a category under composition, since it lacks the identities for the small partial SUR-algebras. However this notion will be useful to perform constructions of total SUR-algebra as colimit of a large diagram small partial SUR-algebras and fpSUR-morphisms between them (see Subsection 2.3.4).

□

52. Denote Σ-str the (very large) category such that:

(a) The objects of Σ-str are the structures $\mathcal{S} = (S, <, -, *, t)$ where S is a class, $* \in S$, $-$ is an unary function in S, $<$ is a binary relation in S and $t : D^t \to S$ is a function such that $D^t \subseteq P_s(S) \times P_s(S)$.

(b) Let $\mathcal{S} = (S, <, -, *, t)$ and $\mathcal{S}' = (S', <', -', *', t')$ be partial SUR-algebras. A Σ-morphism, $h : \mathcal{S} \to \mathcal{S}'$, is a (total) function $h : S \to S'$ satisfying the conditions below:

(Σm1) $h(*) = *'$.

(Σm2) $h(-a) = -'h(a)$, $\forall a \in S$.

(Σm3) $a < b \implies h(a) <' h(b)$, $\forall a, b \in S$.

(Σm4) $(h \times h)[D^t] \subseteq D^{t'}$ and $h(t(A, B)) = t'(h[A], h[B])$, $\forall (A, B) \in D^t$.

(c) Endowed with obvious composition and identities, Σ-str is a very large category and

$$SUR - alg \hookrightarrow pSUR - alg \hookrightarrow \Sigma - str$$

are inclusions of full subcategories.

□

2.3.1 Simple examples

In this short Subsection we just present first examples of partial SUR-algebras and its morphisms.

Example 53. Let $(G, +, -, 0, <)$ be a linearly ordered group. For each $a \in G$ such that $a \geq 0$ (respect. $a \in G \cup \{\infty\}$ such that $a > 0$) then $X_a := [-a, a] \subseteq G$ (respect. $X_a :=]-a, a[\subseteq G$), is a partial SUR-algebra, endowed with obvious definitions of $*, -, <$

and such that:
(1) $C_s^t(X_a) := \{(x^<, x^>) : x \in X_a\}$, $t(x^<, x^>) := x \in X_a$ (t is bijective);
or, alternatively,
(2) $C_s^t(X_a) := \{(L, R) \in C_s(X_a) : \exists(!)x \in X_a\ L^{\leq} = x^<,\ R^{\geq} = x^>\}$, $t(L, R) := x \in X_a$ (t is surjective).

Note that if $b \geq a$, then the inclusion $X_a \hookrightarrow X_b$ is a $pSUR$-morphism, if X_a, X_b are endowed with the second kind of t-map.

□

Another simple (and useful) class of examples are given by the ordinal steps of the recursive constructions of the SUR-algebras SA, ST and No.

Example 54. For any ordinal α we have that the Σ-structure $(SA_\alpha, <_\alpha, -_\alpha, *_\alpha, t_\alpha)$ is a partial SUR-algebra with the below definitions:

- $*_\alpha = *$
- $-_\alpha = - \restriction_{SA_\alpha}$
- $<_\alpha = < \restriction_{SA_\alpha \times SA_\alpha}$
- $C_s^t(SA_\alpha) = C_s(SA^{(\alpha)})$ and $t_\alpha = t \restriction_{C_s(SA^{(\alpha)})}$

□

Just like in the previous example, we have:

Example 55. For any ordinal α we have that the Σ-structure $(ST_\alpha, <_\alpha, -_\alpha, *_\alpha, t_\alpha)$ is a partial SUR-algebra with the below definitions:

- $*_\alpha = *$
- $-_\alpha = - \restriction_{ST_\alpha}$
- $<_\alpha = < \restriction_{ST_\alpha \times ST_\alpha}$
- $C_s^t(ST_\alpha) = C_s(ST^{(\alpha)})$ and $t_\alpha = t \restriction_{C_s(ST^{(\alpha)})}$

□

Example 56. For any given $\alpha \in On$, the Σ-structure $(No_\alpha, *_\alpha, -_\alpha, <_\alpha, t_\alpha)$ is a partial SUR-algebra with the operations defined below:

- $*_\alpha = 0$
- $-_\alpha = - \restriction_{No_\alpha}$
- $<_\alpha = < \restriction_{No_\alpha \times No_\alpha}$
- $C_s^t(No_\alpha) = C_s(No^{(\alpha)})$ and $t_\alpha = t \restriction_{C_s(No^{(\alpha)})}$

□

Remark 57.

- Note that in the three examples above $S = SA, ST, No$, the inclusion $S_\alpha \hookrightarrow S_\beta$ is a pSUR-morphism, where $\alpha \leq \beta \leq \infty$ are "extended" ordinals, with the convention $S_\infty := S$.

- We can also define partial SUR-algebras on the sets $SA^{(\alpha)}, ST^{(\alpha)}, No^{(\alpha)}$, for each $\alpha \in On \setminus \{0\}$ (this is useful!).

- Note that $i_\alpha : SA^{(\alpha)} \hookrightarrow SA_\alpha$ is a fpSUR-algebra morphism, for each $\alpha \in On \setminus \{0\}$. It can be established, by induction on $\alpha \in On \setminus \{0\}$ that for each $\gamma < \alpha$ $i_{\gamma\alpha} : SA_\gamma \hookrightarrow SA_\alpha$ is a fpSUR-morphism. An analogous situation occurs to the partial SUR-algebras $ST_\gamma \hookrightarrow ST^{(\alpha)} \hookrightarrow ST_\alpha$.

□

2.3.2 Cut partial Surreal Algebras

In this short Subsection we present an adaption/generalization of the notion of "Cut Surreal Algebra", introduced in the Subsection 2.2.4, to the realm of *partial* SUR-algebra.

Definition 58. *Let $\mathcal{S} = (S, <, -, *, t)$ be a partial SUR-algebra. The* Cut structure *of $\mathcal{S}$ is the Σ-structure $\mathcal{S}^{(t)} = (S', <', -', <', t')$, where:*

1. $S' := C_s^t(S)$
2. $*' := (\emptyset, \emptyset)$
3. $-'(A, B) := (-B, -A)$
4. $(A, B) <' (C, D) \iff t(A, B) < t(C, D)$
5. $\forall \Gamma, \Delta \subseteq C_s^t(S)$, $(\Gamma, \Delta) \in dom(t')$ *iff* $\Gamma <' \Delta$ *and* $(t[\Gamma], t[\Delta]) \in dom(t)$
6. $t' : C_s^{t'}(C_s^t(S)) \to C_s^t(S)$, $(\Gamma, \Delta) \mapsto t'(\Gamma, \Delta) := (t[\Gamma], t[\Delta])$

□

The list below is a sequence of results on Cut Partial SUR-algebras that extend the results presented in the Subsection 2.2.4 on Cut SUR-algebras: its proofs will be omitted.

Proposition 59. *Let $\mathcal{S} = (S, <, -, *, t)$ be a partial SUR-algebra. Then:*

(a) $\mathcal{S}^{(t)} = (S', <', -', <', t')$ as defined above is a partial SUR-algebra. Moreover, if $\mathcal{S}$ is a SUR-algebra, i.e. $C_s^t(S) = C_s(S)$, then $\mathcal{S}^{(t)}$ is a SUR-algebra, i.e. $C_s^{t'}(C_s^t(S)) = C_s(C_s(S))$.

(b) $t : C_s^t(S) \to S$ is a morphism of partial SUR-algebras. Moreover, if $\mathcal{S}$ is a SUR-algebra, then t is a fpSUR-algebra morphism.

□

Proposition 60. *Let $\mathcal{S} = (S, <, -, *, t)$ be a partial SUR-algebra. Then:*

(a) If S is transitive, then $C_s^t(S)$ is transitive.

(b) If S is linear, then $C_s^t(S)$ is pre-linear[12].

□

Proposition 61.

(a) If $f : \mathcal{S} \to \mathcal{S}'$ is a morphism of partial SUR-algebras then $C_s^t(f) : C_s^t(S) \to C_s^t(S')$, given by: $(A, B) \mapsto (f[A], f[B])$ is a morphism of partial SUR-algebras.

(b) The cut partial SUR-algebra construction determines a (covariant) functor C_s^t : $pSUR \to pSUR$:

$$(S \overset{f}{\to} S') \;\mapsto\; (C_s^t(S) \overset{C_s^t(f)}{\to} C_s^t(S'))$$

(c) The t-map determines a natural transformation between functors on $pSUR-alg$, $t : Id_{pSUR-alg} \to C_s^t$.

□

2.3.3 Simple constructions on pSUR

In this Section, we will verify the full subcategory $pSUR-alg \hookrightarrow \Sigma-str$ is closed under some simple categorial constructions: as (Σ-)substructure and non-empty products. We also present some results on initial objects and (weakly) terminal objects.

We can define a notion of substructure in the categories $pSUR$ and Σ-str:

Definition 62. *Let $\mathcal{S} = (S, <, -, *, t)$ and $\mathcal{S}' = (S', <', -', *', t')$ be Σ-structures. $\mathcal{S}$ will be called a Σ-substructure of $\mathcal{S}$ whenever:*

[12] I.e., denote $\sim_t$ the equivalence relation on $C_s^t(S)$ given by $(A, B) \sim_t (C, D)$ iff $t(A, B) = t(C, D)$. Then it holds exactly one between of the alternatives: $(A, B) <' (C, D)$; $(A, B) \sim_t (C, D)$; $(C, D) <' (A, B)$.

(s1) $S \subseteq S'$;
(s2) $<=<'_{\restriction S\times S}$;
(s3) $- = -'_{\restriction S\times S}$;
(s4) $* = *'$;
(s5) $dom(t) = t'^{-1}[S] \cap (P_s(S) \times P_s(S)) := \{(A, B) \in dom(t') \cap (P_s(S) \times P_s(S)) : t'(A, B) \in S\} \subseteq dom(t')$ *and* $t = t'_{\restriction} : dom(t) \to S$.

□

Remark 63.

(a) The inclusion $i : S \hookrightarrow S'$ determines a Σ-morphism.

(b) By conditions (s1) and (s2) above note that $C_s(S, <) = C_s(S', <') \cap (P_s(S) \times P_s(S))$.

(c) By item (b): if $dom(t') \subseteq C_s(S', <')$, then $dom(t) \subseteq C_s(S, <)$.

(d) By the results presented in the Subsections 2.2.2 and 2.2.3, for any two extends ordinals $\alpha \leq \beta \leq \infty$ we have:
• SA_α is a Σ-substructure of SA_β.
• ST_α is a Σ-substructure of ST_β.

(e) An useful generalization of the notion of Σ-substructure is the notion of Σ-embedding: a Σ-morphism $j : \mathcal{S} \to \mathcal{S}'$ is a Σ-embedding when:
(e1) it is injective;
(e2) $\forall a, b \in S, (a < b \Leftrightarrow j(a) <' j(b))$;
(e3) $\forall (A, B) \in P_s(S) \times P_s(S), ((A, B) \in dom(t) \Leftrightarrow t'(j[A], j[B]) \in range(j))$.

(e) An inclusion $i : S \hookrightarrow S'$ determines a Σ-embedding precisely when $\mathcal{S}$ is a Σ-substructure of $\mathcal{S}'$. Note that the Σ-embeddings $j : \mathcal{S} \to \mathcal{S}'$ are precisely the Σ-morphisms described (uniquely) as $j = i \circ h$, where $i : \mathcal{S}^j \hookrightarrow \mathcal{S}'$ is a Σ-substructure inclusion and $h : \mathcal{S} \to \mathcal{S}^j$ is a Σ-isomorphism.

(f) For technical reasons, we consider an even more general notion: a Σ-morphism $j : \mathcal{S} \to \mathcal{S}'$ is a $\Sigma - quasi$-embedding whenever it satisfies the conditions (e1) and (e3) above.

□

By a straightforward verification we obtain the:

Proposition 64. *Let* $j : \mathcal{S} \to \mathcal{S}'$ *be a* Σ*-embedding of* Σ*-structures. If* $\mathcal{S}'$ *is a partial SUR-algebra, then* $\mathcal{S}$ *is a partial SUR-algebra.*

□

Definition 65. *Given a* non-empty *indexed* set *of partial* Σ*-structure* $\mathcal{S}_i = (S_i, <_i, -_i, *_i, t_i)$, $i \in I$, *we define the* Σ*-structure product* $\mathcal{S} = (S, <, -, *, t)$ *as follows:*
(a) $S = \prod_{i\in I} S_i$;
(b) $<= \{((a_i)_{i\in I}, (b_i)_{i\in I}) : a_i <_i b_i, \forall i \in I\}$;
(c) $-(a_i)_{i\in I} = (-_i a_i)_{i\in I}$;
(d) $* = (*_i)_{i\in I}$;
(e) $dom(t) = \bigcap_{i\in I}(\pi_i \times \pi_i)^{-1}[dom(t_i)] = \{((A_i)_{i\in I}, (B_i)_{i\in I}) \in P_s(S) \times P_s(S)) : (A_i, B_i) \in dom(t_i), \forall i \in I\}$ *and* $t((A_i)_{i\in I}, (B_i)_{i\in I})) = (t_i(A_i, B_i))_{i\in I}$.

□

Note that: For each $i \in I$, the projection $\pi_i : S \to S_i$ is a Σ-structure morphism.

By a straightforward verification we obtain:

Proposition 66. *Keeping the notation above.*

(a) The pair $(\mathcal{S}, (\pi)_{i\in I})$ *above defined constitutes a(the) categorial product in* Σ*-str. I.e., for each diagram* $(\mathcal{S}', (f_i)_{i\in I})$ *in* Σ*-str such that* $f_i : \mathcal{S}' \to \mathcal{S}_i$, $\forall i \in I$, *there is a unique* Σ*-morphism* $f : \mathcal{S}' \to \mathcal{S}$ *such that* $\pi_i \circ f = f_i, \forall i \in I$.

(b) Suppose that $\{\mathcal{S}_i : i \in I\} \subseteq$ *pSUR-alg. Then* $\mathcal{S} \in$ *pSUR-alg and* $(\mathcal{S}, (\pi)_{i\in I})$ *is the product in the category pSUR-alg.*

□

Proposition 67. *Let* $f : \mathcal{S} \to \mathcal{S}'$ *be a pSUR-alg morphism. If* $(S, <)$ *is strictly linearly ordered, then:*
(a) $\forall a, b \in S$, $a < b \iff f(a) <' f(b)$;
(b) f *is an injective function.*

Proof. If $a < b$, then $f(a) <' f(b)$, since f is a Σ-structure morphism. Suppose that $f(a) <' f(b)$ but $a \not< b$, then $a = b$ or $b < a$, thus $f(a) = f(b)$ or $f(b) <' f(a)$. In any case, we get a contradiction with $f(a) <' f(b)$, since $<'$ is an acyclic relation. This establishes item (a). Item (b) is similar, since $<$ satisfies trichotomy and $<'$ is acyclic.

□

The result above yields some information concerning the empty product (= terminal object) in pSUR-algebras.

Proposition 68. *If there exists a* weakly *terminal object*[13] $\mathcal{S}_1$ *in the category pSUR-alg then* $\mathcal{S}_1$ *must be a proper class.*

Proof. Suppose that $\mathcal{S}_1$ is an weakly terminal object in pSUR-alg. Since the (proper class) SUR-algebra No is strictly ordered, then by Proposition 67 above anyone of the

[13]Recall that an object in a category is weakly terminal when it is the target of *some* arrow departing from each object of the category.

existing morphisms $f : No \to \mathcal{S}_1$ is injective. Then $\mathcal{S}_1$ (and $C^t_s(\mathcal{S}_1)$) must be a proper class. $\square$

If we consider the small size version of pSUR, we can guarantee by an another application of Proposition 67, that this (large but not very-large) category does not have (weakly) terminal objects: there are small abelian linearly ordered abelian groups (or even the additive part of a ordered/real closed field) of arbitrary large cardinality, and we have seen in Example 53 how to produce small pSUR-algebras from that structures.

Concerning initial objects we have the following:

Proposition 69.

(a) Consider the Σ*-structure* $\mathcal{S}_0 = (S_0, <, -, *, t)$ *over a singleton set* $S_0 := \{*\}$*, with* $< := \emptyset$*,* $D^t = dom(t) := \emptyset$ *(thus* $\mathcal{S}_0 \notin pSUR\!-\!alg$*) and with* $- : S_0 \to S_0$ *and* $t : D^t \to S_0$ *the unique functions available. Then* $\mathcal{S}_0$ *is the (unique up to unique isomorphism) initial object in* Σ*-str.*

(b) Consider the Σ*-structure* $\mathcal{S}^p_0 = (S_0, <, -, *, t^p)$ *over a singleton set* $S_0 := \{*\}$*, with* $< := \emptyset$*,* $D^{t^p} = dom(t^p) := \{(\emptyset, \emptyset)\} \subseteq C_s(S_0, <)$ *and with* $- : S_0 \to S_0$ *and* $t^p : D^{t^p} \to S_0$ *the unique functions available. Then* $\mathcal{S}^p_0$ *is the (unique up to unique isomorphism) initial object in pSUR-alg.*

Proof.

(a) Let $\mathcal{S}'$ be a Σ-structure and let $h : \{*\} \to S'$ be the unique function such that $h(*) = *' \in S'$, then clearly h is the unique Σ-structure morphism from $\mathcal{S}_0$ into $\mathcal{S}'$: note that $(h \times h)_{\restriction} : dom(t) = \emptyset \to dom(t')$ is such that $t' \circ (h \times h)_{\restriction} = h \circ t$.

(b) It is easy to see that $\mathcal{S}^p_0$ is a partial SUR-algebra. Let $\mathcal{S}'$ be a partial SUR-algebra and let $h : \{*\} \to S'$ be the unique function such that $h(*) = *' \in S'$, then clearly h is the unique Σ-structure morphism from $\mathcal{S}_0$ into $\mathcal{S}'$: since $(\emptyset, \emptyset) \in dom(t')$, note that $(h \times h)_{\restriction} : dom(t^p) := \{(\emptyset, \emptyset)\} \to dom(t')$ is such that $t' \circ (h \times h)_{\restriction} = h \circ t$.

$\square$

2.3.4 Directed colimits of partial Surreal Algebras

One of the main general constructions in Mathematics is the colimit of an upward directed diagram. In the realm of partial SUR-algebras this turns out to be essential for the constructions of SUR-algebras and to obtain general results about them. We can recognize the utility of this process by the cumulative constructions of our main examples: No, SA, ST. Thus we will be concerned only with the colimit of **small** partial SUR-algebras, but over a possibly *large* directed diagram.

This Section is completely technical but its consequences/applications are very interesting: see the entire Section 2.4 and the Theorems at the end of Section 3.1.

Recall that:

• Given a regular "extended" cardinal κ (where "$card(X) = \infty$" means that X is a proper class[14]), a partially ordered class $(I, \leq)$ will be κ-directed, if every subclass $I' \subseteq I$ such that $card(I') < \kappa$ admits an upper bound in I.

• $pSUR_s - alg$ denotes the full subcategory of $pSUR - alg$ determined by of all *small* partial SUR-algebras and its morphisms (then $SUR - alg \cap pSUR_s - alg = \emptyset$). Analogously, we will denote Σ_s-str the full subcategory of Σ-str determined by of all *small* partial Σ-structures and its morphisms.

70. The (first-order) directed colimit construction: Let $(I, \leq)$ is a ω-directed ordered class and consider $\mathcal{D} : (I, \leq) \to \Sigma_s - str$, $(i \leq j) \mapsto (\mathcal{S}_i \stackrel{h_{ij}}{\to} \mathcal{S}_j)$ be a diagram. Define:
• $S_\infty := (\sqcup_{i \in I} S_i)/\equiv$, the set-theoretical colimit, i.e. $\equiv$ is the equivalence relation on the class $\sqcup_{i \in I} S_i$ such that $(a_i, i) \equiv (a_j, j)$ iff there is $k \geq i, j$ such that $h_{ik}(a_i) = h_{jk}(a_j) \in S_k$;
• $h_j : S_j \to S_\infty$, $a_j \mapsto [(a_j, j)]$;
• $* := [(*_i, i)]$ $(= [(*_j, j)], \forall i, j \in I)$;
• $-[(a_i, i)] = [(-_i a_i, i)]$;
• $[(a_i, i)] < [(a_j, j)]$ iff there is $k \geq i, j$ such that $h_{ik}(a_i) <_k h_{jk}(a_j) \in S_k$

□

With the construction above, it is straightforward to verify that $(S_\infty, <, -, *)$ is the colimit in the appropriate category of *first-order* (but possibly large) structures[15], with colimit co-cone $(h_j : S_j \to S_\infty)_{j \in I}$ and, if $\mathcal{D} : (I, \leq) \to pSUR_s - alg$, then the same (colimit) co-cone is in the "first-order part" of the category $pSUR - alg$, i.e., it satisfies the properties [pS1]–[pS4] presented in Definition 47. However, to "complete" the Σ-structure (respect. pSUR-algebra) we will need some extra conditions below:

Proposition 71. *Let $\mathcal{D} : (I, \leq) \to \Sigma_s - str$, $(i \leq j) \mapsto (\mathcal{S}_i \stackrel{h_{ij}}{\to} \mathcal{S}_j)$ be a diagram such that:*
(i) $(I, \leq)$ is a ω-directed ordered class and $h_{ij} : S_i \to S_j$ is a injective Σ-morphism, whenever $i \leq j$;
or;
(ii) $(I, \leq)$ is a ∞-directed ordered class (e.g. $(On, \leq)$),
then $S_\infty := (\sqcup_{i \in I} S_i)/\equiv$ is a (possibly large) partial Σ-structure and $(h_j : S_j \to S_\infty)_{j \in I}$ is a colimit cone in the category $\Sigma - str$.

[14]Recall that in NBG, all the proper classes are in bijection, by the global form of the axiom of choice.
[15]I.e., we drop the second-order part of the Σ-structure: the map $t : dom(t) \subseteq P_s(S_\infty) \times P_s(S_\infty) \to S_\infty$.

□

Proposition 72. *If $\mathcal{D} : (I, \leq) \to pSUR_s - alg$, $(i \leq j) \mapsto (\mathcal{S}_i \stackrel{h_{ij}}{\to} \mathcal{S}_j)$ is a diagram, where:*
(i) $(I, \leq)$ is a ω-directed ordered class and $h_{ij} : S_i \to S_j$ is a injective pSUR-morphism, whenever $i \leq j$;
or;
(ii) $(I, \leq)$ is a ∞-directed ordered class (e.g. $(On, \leq)$)
then S_∞ is a (possibly large) partial SUR-algebra and $(h_j : S_j \to S_\infty)_{j \in I}$ is a colimit cone in the category $pSUR - alg$.

□

Proposition 73. *The subclass of morphisms $fpSUR \subseteq pSUR$ is* closed *under directed colimits in the cases (i) and (ii) described in the Proposition above. More precisely: if $\mathcal{D} : (I, \leq) \to pSUR_s - alg$ is a directed diagram satisfying (i) or/and (ii)above and such that $h_{ij} : S_i \to S_j$ is a fpSUR-morphism, whenever $\underline{i < j}$, then the colimit co-cone $\forall j \in I$, $(h_j : S_j \to S_\infty)_{j \in I}$ is formed by fpSUR-algebra morphisms. Moreover:*
(a) If $(I, \leq)$ is ∞-directed and the transition arrows $(h_{ij})_{i \leq j}$ are injective, then S_∞ is a SUR-algebra (thus it is a proper class);
(b) If the transition arrows $(h_{ij})_{i \leq j}$ are injective (respect. $\Sigma - quasi$-embedding, Σ-embedding), then the cocone arrows $(h_j)_{j \in I}$ are injective (respect. $\Sigma - quasi$-embedding, Σ-embedding);
(c) If $t_i : C_s^{t_i}(S_i) \to S_i$ is injective (respect. surjective/bijective), $\forall i \in I$, then $t^\infty : C_s^{t^\infty}(S^\infty) \to S^\infty$ is injective (respect. surjective/bijective).

□

Example 74.

We have noted in Remark 57 that for each sequence of ordinal $\gamma < \beta < \alpha$, $i_{\gamma\beta} : SA_\gamma \hookrightarrow SA_\beta$ is fpSUR-algebra morphism. It is also a Σ-embedding. Then, for each $\alpha > 0$, $SA^{(\alpha)} \cong colim_{\gamma<\alpha} SA_\gamma$ as a pSUR-algebra and $i_\gamma^{(\alpha)} : SA_\gamma \hookrightarrow SA^{(\alpha)}$ determines a colimit co-cone of an ω-directed diagram[16] formed by fpSUR-algebras embeddings. Moreover $SA = SA_\infty \cong colim_{\gamma \in On} SA_\gamma$ and $i_\gamma^\infty : SA_\gamma \hookrightarrow SA$ determines a colimit co-cone a ∞-directed diagram over formed by fpSUR-algebras embeddings.

Analogous results holds for $ST^{(\alpha)} \cong colim_{\gamma<\alpha} ST_\gamma$, $\alpha > 0$, and $ST \cong colim_{\gamma \in On} ST_\gamma$.

□

[16]In fact it is κ directed diagram, where κ is any regular cardinal such that $\kappa \geq \alpha + \omega$.

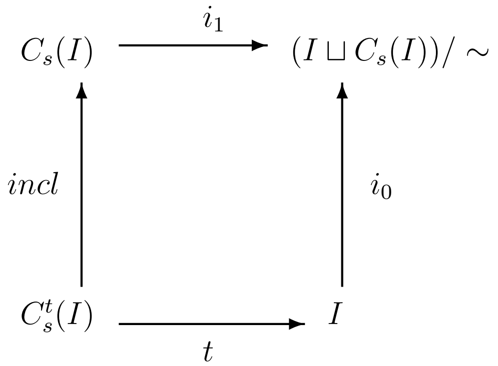
2.4 Universal Surreal Algebras

In this final Section of Chapter 2, we present some *categorical-theoretic* universal properties[17] concerning SUR-algebras and partial SUR-algebras. We will need notions, constructions and results developed in the previous Sections of this Chapter to provide, for each small *partial* SUR-algebra I, a "best" SUR-algebra over I, $SA(I)$, (respect. a "best" *transitive* SUR-algebra over I, $ST(I)$). As a consequence of this result (and its proof) we will determine the SUR-algebras SA and ST in the category of SUR by universal properties that characterizes them uniquely up to unique isomorphisms: these will justify the adopted names "$SA =$ the free surreal algebra" and "$ST =$ the free transitive surreal algebra".

We start with the following

75. Main construction: Let $\mathcal{I} = (I, <, -, *, t)$ be a partial SUR-algebra. Consider:

(a) The set-theoretical pushout diagram over $(I \stackrel{t}{\leftarrow} C_s^t(I) \stackrel{incl}{\hookrightarrow} C_s(I))$:



Note that:

- $I^+ = (I \sqcup C_s(I))/\sim$, is the vertex of the set-theoretical pushout diagram, where $\sim$ is the least equivalence relation[18] on $I \sqcup C_s(I)$ such that $(x, 0) \sim ((A, B), 1)$ iff $(A, B) \in C_s^t(I)$ and $x = t(A, B)$.
- If I is small, then I^+ is small.
- $\forall (A, B), (C, D) \in C_s(I) \setminus C_s^t(I)$, $((A, B), 1) \sim ((C, D), 1)$ iff $(A, B) = (C, D)$ (by induction on the number of steps that witness the transitive closure).
- $\forall x, y \in I$, $(x, 0) \sim (y, 0)$ iff $x = y$ (by induction on the number of steps needed in the transitive closure).
- Since $C_s^t(I) \hookrightarrow C_s(I)$ is injective function, then $i_0 : I \to (I \sqcup C_s(I))/\sim$, $x \mapsto [(x, 0)]$

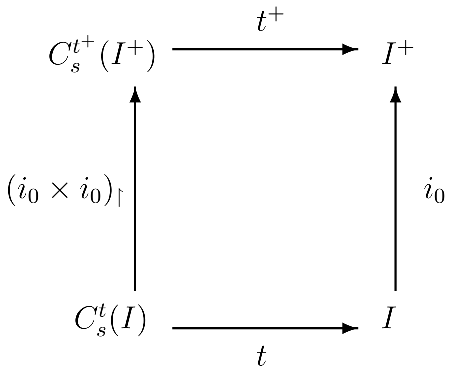
[17] An analysis of model-theoretic universal properties of the "first-order part" of (partial) SUR-algebras, and its possible connections with categorial-theoretic universality presented here, will be theme of future research, see Chapter 5 for more details.

[18] Recall that the least equivalence relation on a set X that contains $R \subseteq X \times X$ is obtained from R adding the opposite relation R^{-1} and the diagonal relation Δ_X, and then taking the transitive closure $trcl(R \cup R^{-1} \cup \Delta_X) = R^{(eq,X)}$.

is an injective function (see above) and $(i_0)^+ : (P_s(I) \times P_s(I)) \to (P_s(I^+) \times P_s(I^+))$, $(A, B) \mapsto (i_0[A], i_0[B])$ is an injective function.

(b) Let $\mathcal{I}^+ := (I^+, *^+, -^+, <^+, t^+)$ the Σ-structure defined below:
• $I^+ := (I \sqcup C_s(I))/\sim$.
• $*^+ := [(*, 0)] = [((\emptyset, \emptyset), 1)]$.
• $-^+[(x, 0)] := [(-x, 0)]$;
$-^+[((A, B), 1)] := [((-B, -A), 1)]$.
• Define $<^+$ by cases (only three):
$[(x, 0)] <^+ [(y, 0)]$ iff $x < y$;
$[(x, 0)] <^+ [((A, B), 1)]$ iff $x \in A$, whenever $(A, B) \in C_s(I) \setminus C^t_s(I)$;
$[((A, B), 1)] <^+ [(y, 0)]$ iff $y \in B$, whenever $(A, B) \in C_s(I) \setminus C^t_s(I)$.
Note that $C_s(i_0) = (i_0)^+{}_{\restriction} : C_s(I) \to C_s(I^+)$, $(A, B) \mapsto (i_0[A], i_0[B])$, is an injective function with adequate domain and codomain.
• Define $C^{t^+}_s(I^+) := range(C_s(i_0)) \subseteq C_s(I^+)$ (thus $C_s(I) \cong C^{t^+}_s(I^+)$) and $t^+ : C^{t^+}_s(I^+) \to I^+$, $(i_0[A], i_0[B]) \mapsto t^+(i_0[A], i_0[B]) := [((A, B), 1)]$.
Note that $(A, B) \in dom(t)$ iff $t^+(i_0[A], i_0[B]) \in range(i_0)$.

Thus we obtain another set-theoretical pushout diagram that is isomorphic to the previous pushout diagram:



□

We describe below the main technical result in this Section:

Lemma 76. *Let $\mathcal{I} = (I, *, -, <, t)$ be a (small) partial SUR-algebra and keep the notation in 75 above. Then*

(a) $\mathcal{I}^+ = (I^+, *^+, -^+, <^+, t^+)$ is a (small) partial SUR-algebra.

(b) $i_0 : I \to I^+$ is a Σ-embedding and full morphism of partial SUR-algebras.

(c) If $t : C^t_s(I) \to I$ is injective (respect. surjective/bijective), then $t^+ : C^{t^+}_s(I^+) \to I^+$ is injective (respect. surjective/bijective).

(d) If $\mathcal{S}' = (S', *', -', <', t')$ is a partial SUR-algebra, then for each fpSUR-algebra morphism $f : \mathcal{I} \to \mathcal{S}'$ there is a unique pSUR-algebra morphism $f^+ : \mathcal{I}^+ \to \mathcal{S}'$ such that $f^+ \circ i_0 = f$. In particular, if $\mathcal{S}'$ is a SUR-algebra, then f and f^+ are automatically fpSUR-algebras morphisms. Moreover:
• *If t' is injective, then f is a $\Sigma -$ quasi-embedding iff f^+ is a $\Sigma -$ quasi-embedding.*

Proof. Items (a), (b) and (c) are straightforward verifications. We will just sketch the proof of the universal property in item (d).

Candidate and uniqueness:
Suppose that there is a pSUR-algebra morphism $f^+ : \mathcal{I}^+ \to \mathcal{S}'$ such that $f^+ \circ i_0 = f$. Since $f : I \to S$ is a full partial SUR-algebra morphism, we have $(f \times f)_\restriction : C_s(I) \to C_s^{t'}(S')$. Then $(f^+ \times f^+)_\restriction : C_s^{t^+}(I^+) \to C_s^{t'}(S') : (\Gamma, \Delta) = (i_0[A], i_0[B]) \mapsto (f^+[\Gamma], f^+[\Delta]) = (f[A], f[B]) \in C_s^{t'}(S')$ and $f^+(t^+(\Gamma, \Delta)) = t'(f^+[\Gamma], f^+[\Delta]) = t'((f[A], f[B])) \in S'$. Since $range(i_0) \cup range(i_1) = I^+$, the function f^+ is determined by f:
• $f^+(z) = f(x) \in S'$, whenever $z = [(x, 0)] \in range(i_0)$;
• $f^+(z) = t'((f[A], f[B])) \in S'$, whenever $z = ([(A, B), 1)] \in range(i_1)$.

Existence:
Since $f : I \to S$ is a full partial SUR-algebra morphism, we have $(f \times f)_\restriction : C_s(I) \to C_s^{t'}(S')$, then the arrows

$$(C_s(I) \overset{t' \circ (f \times f)_\restriction}{\longrightarrow} S' \overset{f}{\longleftarrow} I)$$

yields a commutative co-cone over the diagram

$$(I \overset{t}{\leftarrow} C_s^t(I) \overset{incl}{\hookrightarrow} C_s(I)).$$

By the universal property of set-theoretical pushout, there is a unique function $f^+ : I^+ \to S'$ such that:
• $f^+ \circ i_0 = f$;
• $f^+ \circ i_1 = t' \circ (f \times f)_\restriction$.
Thus it remains only to check that $f^+ : I^+ \to S'$ is a pSUR-algebra morphism:
• $f^+(*^+) = f^+([*, 0]) = f(*) = *'$;
• $f^+(-^+[(x, 0)]) = f(-x) = -'f(x) = -'f^+([(x, 0)])$;
$f^+(-^+[((A, B), 1)]) = t'((f \times f)_\restriction(-B, -A)) = t'(f[-B], f[-A]) = -'t'(f[B], f[A]) = -'f^+([((A, B), 1)])$.
• If $[(x, 0)] <^+ [(y, 0)]$, then $x < y$ thus $f^+([(x, 0)]) = f(x) <' f(y) = f^+([(y, 0)])$;
If $(A, B) \in C_s(I) \setminus C_s^t(I)$:
- if $[(x, 0)] <^+ [((A, B), 1)]$, then $x \in A$ and $f(x) \in f[A]$. Since$(f[A], f[B]) \in C_s^{t'}(S')$, thus $f^+([(x, 0)]) = f(x) <' t'(f[A], f[B]) = f^+([((A, B), 1)])$;
- if $[((A, B), 1)] <^+ [(y, 0)]$, then $y \in B$ and $f(y) \in f[B]$. Since $(f[A], f[B]) \in C_s^{t'}(S')$, thus $f^+([((A, B), 1)]) = t'(f[A], f[B]) <' f(y) = f^+([(y, 0)])$.

• If $(\Gamma, \Delta) = (i_0[A], i_0[B]) \in C_s^{t^+}(I^+)$, then $(f^+[\Gamma], f^+[\Delta]) = (f[A], f[B]) \in C_s^{t'}(S')$ and $f^+(t^+(\Gamma, \Delta)) = f^+(t^+((i_0[A], i_0[B]))) = f^+([((A, B), 1)]) = t' \circ (f \times f)_{\restriction}([((A, B), 1)]) = t'(f[A], f[B]) = t'(f^+[i_0[A]], f^+[i_0[B]]) = t'(f^+[\Gamma], f^+[\Delta])$.

□

Remark 77.

In the setting above, we can interpret the Conway's notions in a very natural way:
• $Old(I) := i_0[I] \cong I$;
• $Made(I) := I^+$;
• $New(I) := I^+ \setminus i_0[I]$.

Note that if $t : C_s^t(I) \to I$ is surjective (e.g. $I = No^{(\alpha)}, SA^{(\alpha)}, ST^{(\alpha)}$, $\alpha \in On \setminus \{0\}$), then $t^+ : C_s^{t^+}(I^+) \to I^+$ and every "made member" is represented by a Conway cut in of "old members". This representation is unique, whenever $t : C_s^t(I) \to I$ is bijective (e.g., $I = SA^{(\alpha)}, ST^{(\alpha)}$, $\alpha \in On \setminus \{0\}$).

When $I = SA^{(\alpha)}, \alpha \in On \setminus \{0\}$ and $C_s^t(I) = \{(A, B) \in C_s(SA^{(\alpha)}, <_{(\alpha)}) : t(A, B) = \langle A, B \rangle \in SA^{(\alpha)}$ ($t : C_s(I) \to I$ is bijective), then $t^+ : C_s^{t^+}(I^+) \to I^+$ can be identified with the (bijective) map $C_s(SA^{(\alpha)}, <_{(\alpha)}) \to SA_\alpha$.

□

A slight modification in the construction of the Σ-structure presented in 75 above, just replacing $<^+$ by $<^+_{(tc)} := trcl(<^+)$, yields the following:

Lemma 78. *Let $\mathcal{I} = (I, *, -, <, t)$ be a (small) partial SUR-algebra and keep the notation in 75 above. Then*

(a) $\mathcal{I}^+_{(tc)} = (I^+, *^+, -^+, <^+_{(tc)}, t^+)$ is a (small) transitive partial SUR-algebra.

(b) $i_0 : I \to I^+_{(tc)}$ is a $\Sigma-$quasi-embedding (see Remark 63.(f)) and full morphism of partial SUR-algebras. Moreover, if $\mathcal{I}$ is a transitive SUR-algebra, then $i_0 : I \to I^+_{(tc)}$ is a Σ-embedding.

(c) If $t : C_s^t(I) \to I$ is injective (respect. surjective/bijective), then $t^+ : C_s^{t^+}(I^+) \to I^+$ is injective (respect. surjective/bijective).

(d) If $\mathcal{S}' = (S', *', -', <', t')$ is a partial transitive SUR-algebra, then for each fpSUR-algebra morphism $f : \mathcal{I} \to \mathcal{S}'$ there is a unique pSUR-algebra morphism $f^+ : \mathcal{I}^+ \to \mathcal{S}'$ such that $f^+ \circ i_0 = f$. In particular, if $\mathcal{S}'$ is a transitive SUR-algebra, then f and f^+ are automatically fpSUR-algebras morphisms.

□

When $I = ST^{(\alpha)}, \alpha \in On \setminus \{0\}$ and $C_s^t(I) = \{(A, B) \in C_s(ST^{(\alpha)}, <_{(\alpha)}) : t(A, B) = \langle A, B \rangle \in ST^{(\alpha)}\}$ ($t : C_s(I) \to I$ is bijective), then $t^+ : C_s^{t^+}(I^+_{(tc)}) \to I^+_{(tc)}$ can be identified with the (bijective) map $C_s(ST^{(\alpha)}, <_{(\alpha)}) \to ST_\alpha$.

Remark 79.

Note that applying the construction $(\ )^+$ to the SUR-algebra SA we obtain $(SA)^+ \cong C_s(SA) = SA$.

Applying both constructions $(\ )^+$ and $(\ )^+_{(tc)}$ to the SUR-algebra ST we obtain $(ST)^+ = (ST)^+_{(tc)} \cong C_s(ST) = ST$.

□

Now we are ready to state and prove the main result of this Section:

Theorem 80. *Let $\mathcal{I}$ be any small partial SUR-algebra. Then there exists SUR-algebras denoted by $SA(\mathcal{I})$ and $ST(\mathcal{I})$, and pSUR-morphisms $j^A_I : \mathcal{I} \to SA(\mathcal{I})$ and $j^T_I : \mathcal{I} \to ST(\mathcal{I})$ such that:*

(a)
(a1) j^A_I is a fpSUR-morphism and a Σ-embedding;
(a2) If $t : C^t_s(I) \to I$ is injective (respect. surjective/bijective), then $t^\infty : C^{t^\infty}_s(SA(I)) \to SA(I)$ is injective (respect. surjective/bijective);
(a3) $j^A_I : \mathcal{I} \to SA(\mathcal{I})$ satisfies the universal property: for each SUR-algebra $\mathcal{S}$ and each pSUR-morphism $h : \mathcal{I} \to \mathcal{S}$, there is a unique SUR-morphism $h_A : SA(\mathcal{I}) \to \mathcal{S}$ such that $h_A \circ j^A_I = h$. Moreover:
- *If t' is injective, then h is a $\Sigma-$quasi-embedding iff h_A is a $\Sigma-$quasi-embedding.*

(b)
(b1) j^T_I is a fpSUR-morphism and a $\Sigma-$quasi-embedding, that is a Σ-embedding whenever $\mathcal{I}$ is transitive;
(b2) If $t : C^t_s(I) \to I$ is injective (respect. surjective/bijective), then $t^\infty : C^{t^\infty}_s(ST(I)) \to ST(I)$ is injective (respect. surjective/bijective);
(b3) $j^T_I : \mathcal{I} \to ST(\mathcal{I})$ satisfies the universal property: for each **transitive** *SUR-algebra $\mathcal{S}$ and each pSUR-morphism $h : \mathcal{I} \to \mathcal{S}$, there is a unique SUR-morphism $h_T : ST(\mathcal{I}) \to \mathcal{S}$ such that $h_T \circ j^T_I = h$.*

Proof.

Item (a): based on Lemma 76 and Proposition 73, we can define, by transfinite recursion, a convenient *increasing* (compatible) family of diagrams $D_\alpha : [0, \alpha] \to pSUR-alg$, $\alpha \in On$, where:
(D0) $D_0(\{0\}) = I$;
(D1) For each $0 \leq \gamma < \beta < \alpha$, $D_\alpha(\gamma, \beta) = D_\beta(\gamma, \beta) : D_\beta(\gamma) \to D_\beta(\beta)$ is Σ-embedding and a fpSUR-morphism;

Just define $D_\alpha(\alpha) = (D^{(\alpha)}_\alpha)^+$, where $D^{(\alpha)}_\alpha := colim_{\beta<\alpha} D_\alpha(\beta)$ and take, for $\beta < \alpha$, $D_\alpha(\beta, \alpha) = (h^\alpha_\beta)^+ : D_\alpha(\beta) \to (colim_{\beta<\alpha} D_\alpha(\beta))^+$ be the unique pSUR-morphism –

that is automatically a fpSUR-morphism and a Σ-embedding, whenever h^α_β satisfies this conditions (see Lemma 76.(d))– such that $(h^\alpha_\beta)^+ \circ i_0 = h^\alpha_\beta$, where $i_0 : (colim_{\beta<\alpha} D_\alpha(\beta)) \to (colim_{\beta<\alpha} D_\alpha(\beta))^+$ and where $h^\alpha_\beta : D_\alpha(\beta) \to (colim_{\beta<\alpha} D_\alpha(\beta))$ is the colimit co-cone arrow: by the recursive construction and by Proposition 73 h^α_β is a fpSUR-morphism and a Σ-embedding. This completes the recursion.

Gluing this increasing family of diagrams we obtain a diagram $D_\infty : On \to pSUR-alg$.

By simplicity we will just denote:
- $SA(I)_\alpha := D_\infty(\alpha)$, $\alpha \in On$;
- $SA(I)_\infty := colim_{\alpha\in On} SA(I)_\alpha$;
- $D_\alpha(\beta,\alpha) = j^A_{\beta,\alpha}$, for each $0 \leq \beta \leq \alpha \leq \infty$ (since the family $(D_\alpha)_\alpha$ is increasing, we just have to introduce notation for "new arrows").

Then we set: $SA(I) := SA(I)_\infty$ and $j^A_I := j^A_{0,\infty}$.

The verification that $SA(I)$ is a SUR-algebra that satisfies the property in item (a2) and that j^A_I satisfies item (a1)[19], follows the recursive construction of the diagram and from a combination of Proposition 73 and Lemma 76.

By the same Lemma and Proposition combined, it can be checked by induction that for each $\alpha \in On$, there is a unique pSUR-morphism $h_\alpha : SA(\mathcal{I})_\alpha \to \mathcal{S}$ such that $h_\alpha \circ j^A_{0,\alpha} = h$ and such that h_α is injective (respect. $\Sigma - quasi$-embedding, Σ-embedding), whenever h is injective (respect. $\Sigma - quasi$-embedding, Σ-embedding). By applying one more time the colimit construction, we can guarantee that there is a unique pSUR-morphism $h^A := h_\infty : SA(\mathcal{I})_\infty \to \mathcal{S}$ such that $h^A \circ j^A_I = h$ and that it satisfies the additional conditions.

The proof of item (b) is analogous to the proof of item (a): basically we just have to replace to use of technical Lemma 76 by other technical Lemma 78. In general, we can on guarantee that $j^T_{\beta,\alpha}$ is a Σ-embedding and a fpSUR-morphism only for $0 < \beta < \alpha \leq \infty$.

□

In particular, taking $I = \mathcal{S}_0$ as the *initial object* in pSUR-alg (see Proposition 69 in Subsection 2.3.3), we have that $SA \cong SA(I)$ and $ST \cong ST(I)$, and they satisfy corresponding universal properties:

Corollary 81.

(a) SA is universal (= initial object) over all SUR-algebras, i.e. for each SUR-algebra $\mathcal{S}$, there is a unique SUR-algebra morphism $f_S : SA \to \mathcal{S}$.

[19] In fact, $j^A_{\beta,\alpha}$ is Σ-embedding whenever $0 \leq \beta \leq \alpha \leq \infty$ and $j^A_{\beta,\alpha}$ is a fpSUR-morphism whenever $0 \leq \beta < \alpha \leq \infty$.

(b) ST is universal (= initial object) over all **transitive** *SUR-algebras, i.e. for each* **transitive** *SUR-algebra $\mathcal{S}'$, there is a unique SUR-algebra morphism $h_{\mathcal{S}'} : ST \to \mathcal{S}'$.*

Proof. Item (a): Since for each each SUR-algebra $\mathcal{S}$ there is a unique pSUR-morphism $u_S : \mathcal{S}_0 \to \mathcal{S}$ then, by Theorem 80.(a) above, $SA(\mathcal{S}_0)$ is a SUR-algebra that has the required universal property, thus we only have to guarantee that $SA \cong SA(\mathcal{S}_0)$. Taking into account the Remark 77 and the constructions performed in the proof of the item (a) in Theorem above, that we have a (large) family of compatible pSUR-isomorphisms $SA_\alpha \cong SA(I)_\alpha, \forall \alpha \in On$. Thus $SA = \bigcup_{\alpha \in On} SA_\alpha \cong colim_{\alpha \in On} SA(\mathcal{S}_0)_\alpha = SA(\mathcal{S}_0)_\infty = SA(\mathcal{S}_0)$.

For item (b) the reasoning is similar: note that $I = \mathcal{S}_0 = \{*\}$ is a transitive partial SUR-algebra to conclude that $ST(\mathcal{S}_0)$ has the required universal property and note that by the proof of item (b) in Theorem 80 above, that $ST = \bigcup_{\alpha \in On} ST_\alpha \cong colim_{\alpha \in On} ST(\mathcal{S}_0)_\alpha = ST(\mathcal{S}_0)_\infty = ST(\mathcal{S}_0)$.

□

This Corollary describes, in particular, that SA and ST are "rigid" as Σ-structures and :
• SA and $C_s(SA)$ are isomorphic SUR-algebras and the universal map $SA \to C_s(SA)$ is the unique iso from SA to $C_s(SA)$;
• ST and $C_s(ST)$ are isomorphic SUR-algebras and the universal map $ST \to C_s(ST)$ is the unique iso from ST to $C_s(ST)$.

We finish this Section with an application of the Corollary above: we obtain some non-existence results.

Corollary 82.
(i) Let $\mathcal{L}$ be a linear SUR-algebra, i.e., $<$ is a total relation (for instance take $\mathcal{L} = No$). Then there is no SUR-algebra morphism $h : \mathcal{L} \to ST$.
(ii) Let $\mathcal{T}$ be a transitive SUR-algebra, i.e., $<$ is a transitive relation (for instance take $\mathcal{T} = ST, No$). Then there is no SUR-algebra morphism $h : \mathcal{T} \to SA$.

Proof. (i) Suppose that there is a SUR-algebra morphism $h : \mathcal{L} \to ST$. Since the binary relation $<$ in L is acyclic and total, it is a strictly linear order, in particular it is transitive. Let $a, b \in L$, since L is linear, $a < b$ in $L \Leftrightarrow h(a) < h(b)$ in ST. Now, by the universal property of ST (see Theorem above) there is a unique SUR-algebra morphism $u : ST \to \mathcal{L}$ and then $h \circ u = id_{ST}$. Summing up, $h : (L, <) \to (ST, <)$ is an isomorphism of structures, thus $(ST, <)$ is a strictly ordered class, but the members of ST 0 and $\langle\{-1\}, \{1\}\rangle$ are not comparable by Fact 1.(c) in the Subsection 2.2.3, a contradiction.

(ii) Suppose that there is a SUR-algebra morphism $h : \mathcal{T} \to SA$. Since the binary relation $<$ in T is transitive, by the universal property of ST there is a (unique) SUR-algebra morphism $v : ST \to \mathcal{T}$, thus we get a SUR-algebra morphism $g = h \circ v : ST \to SA$. By the universal property of SA there is a unique SUR-algebra morphism $u : SA \to ST$ (u is a inclusion) and then $g \circ u = id_{SA}$. Thus, for each $a, b \in SA$, $a < b$ in $SA \Leftrightarrow u(a) < u(b)$ in ST, but the members of SA denoted by -1 and 1 are not related (see Fact.(c) in the Subsection 2.2.2) and $u(-1) < u(1)$ in ST (by Fact 1.(c) in the Subsection 2.2.3), a contradiction.

□

Chapter 3

SUR-algebras and ZF-algebras

In this Chapter, we will establish relations, in both directions, between certain classes of (equipped) SUR-algebras and certain classes of (equipped) ZF-algebras, that "explains" and "expands" the relations $On \underset{b}{\overset{j}{\rightleftarrows}} No$. We introduce here the following concepts: anchored SUR-algebras (Section 1), the hereditary positive subclass of an equipped SUR-algebra (Section 3), the space of signs associated to some standard ZF-algebra (Section 4).

Even if we present some general definitions, constructions and results, the main goal of this Chapter is not develop an extensive and systematic study of the introduced concepts[1], but provide means to appreciate the content of the Section 2, named "The main diagram", that summarizes the relationship between the structured classes On, V, No, SA and will provide a theoretical basis for the development of the next Chapter on "set theory based on surreal numbers".

3.1 Anchored Surreal Algebras

Motivated by the axiomatization of the class of all surreal number $(No, <, b)$, presented in the Subsection 1.3.2 of Chapter 1, where $b : No \to On$ is the "birthday function", we begin the present Section introducing a generalization: the concept of anchor on a SUR-algebra. We develop some general results on anchored SUR-algebras, provide two distinct anchors on the free SUR-algebra SA and another anchor on the free transitive SUR-algebra ST and establish characterization results of these anchored SUR-algebras in the same vein that the class of all surreal numbers can be axiomatized by its birthday function.

Definition 83. *Let $\mathcal{C} = (C, \prec, \Phi)$ be a rooted well-founded class (see 5 in the Subsection*

[1]We intend develop these themes in a future research project.

1.1.2, Chapter 1). Denote $\sqsubseteq$ *the pre-order relation on* C *associated with* $\prec$*, according 3 in the Subsection 1.1.1, Chapter 1. Let* $\mathcal{S}$ *be a SUR-algebra. An* **anchor** $\mathbf{b}$ *of* $\mathcal{S}$ *in* $\mathcal{C}$ *(notation:* $\mathbf{b} : \mathcal{S} \to \mathcal{C}$*) is a function* $\mathbf{b} : S \to C$ *that satisfies the conditions (b1)–(b5) below:*

(b1) $\mathbf{b}$ *is a surjective function.*

(b2) $\mathbf{b}$ *has small fibers (see 8 in the Subsection 1.2.1, Chapter 1).*

(b3) $\mathbf{b}(*) = \Phi$.

(b4) $\mathbf{b}(-a) = \mathbf{b}(a)$, $\forall a \in S$.

(b5) For each subsets $A, B \subseteq S$ *such that* $A < B$ *and each* $c \in C$ *such that* $\forall a \in A, \forall b \in B$, $\mathbf{b}(a)$, $\mathbf{b}(b) \prec c$*, then* $\mathbf{b}(t(A, B)) \sqsubseteq c$.

The binary relations $\prec, \sqsubseteq$ *in* C *determines binary relations* $\prec_{\mathbf{b}}, \sqsubseteq_{\mathbf{b}}$ *in* S *by:*

- $a \prec_{\mathbf{b}} b$ *iff* $\mathbf{b}(a) \prec \mathbf{b}(b)$, $\forall a, b \in S$;
- $a \sqsubseteq_{\mathbf{b}} b$ *iff* $\mathbf{b}(a) \sqsubseteq \mathbf{b}(b)$, $\forall a, b \in S$.

For each subsets $A, B \subseteq S$ *such that* $A < B$*, consider* $Mum_{\mathbf{b}}(A, B)$ *the subclass of all* $u \in S$ *such that* $A < u < B$ *and that is a* $\sqsubseteq_{\mathbf{b}}$*-mini***mum** *for that property (i.e., if* $A < z < B$*, then* $u \sqsubseteq_{\mathbf{b}} z$*).*

The anchor $\mathbf{b} : \mathcal{S} \to \mathcal{C}$ *will be called* **strict** *if it also satisfy (b6) and (b7) below:*

(b6) $\forall (A, B) \in C_s(S)$, $t(A, B) \in Mum_{\mathbf{b}}(A, B)$.

(b7) $\forall (A, B) \in C_s(S)$*, if* $s, s' \in Mum_{\mathbf{b}}(A, B)$ *are* $<$*-comparable (i.e.,* $s = s'$ *or* $s < s'$ *or* $s' < s$*), then* $s = s'$.

- *The pair* $(\mathcal{S}, \mathbf{b})$*, where* $\mathbf{b} : \mathcal{S} \to \mathcal{C}$ *is an anchor, is called an anchored SUR-algebra.*
- *If* $\mathbf{b} : \mathcal{S} \to \mathcal{C}$ *and* $\mathbf{b}' : \mathcal{S}' \to \mathcal{C}$ *are anchor over the same rooted well-founded class* $\mathcal{C}$*, a* $\mathcal{C}$*-morphism of anchored SUR-algebras* $h : (\mathcal{S}, \mathbf{b}) \to (\mathcal{S}', \mathbf{b}')$ *is a SUR-algebra morphism* $h : \mathcal{S} \to \mathcal{S}'$ *such that* $\mathbf{b}' \circ h = \mathbf{b}$.
- *We denote* $\mathcal{C} - ancSUR - alg$ *the category*[2] *naturally obtained from the setting described just above.*

□

Remark 84. Let $\mathbf{b} : \mathcal{S} \to \mathcal{C}$ be an anchor. Note that:

On sizes: Since S is a proper class (see Remark 34) then C is a proper class too: this follows from the conditions (b1) and (b2) and the equality $S = \bigcup_{c \in C} \mathbf{b}^{-1}[\{c\}]$.

[2]Apart from the usual questions concerning sizes that occurs in category-theory.

On (b2): As we saw in 8 in Subsection 1.2.1 in Chapter 1, condition (b2) is equivalent to:
(b2)' $\mathbf{b}$ is a locally small function.

On (b3): The condition $\mathbf{b}(*) = \Phi$ is inessential, since it follows from (b5) and the hypothesis that Φ is the unique root in $(C, \prec)$ (see 5, Chapter 1).

On (b4): The condition $\mathbf{b}(-a) = \mathbf{b}(a)$, $\forall a \in S$, means that "the members of S born in pairs, at the same time".

On (b5): A stronger version of condition (b5) will appear in some examples:
(b5)^{strong} For each subsets $A, B \subseteq S$ such that $A < B$, then:
for each $c \in C$: $\forall a \in A, \forall b \in B$, $\mathbf{b}(a)$, $\mathbf{b}(b) \prec c$ (in C) iff $\mathbf{b}(t(A,B)) \sqsubseteq c$;
or, equivalently:
$\mathbf{b}(t(A,B)) = min_{\sqsubseteq}\{c \in C : \forall a \in A, \forall b \in B, \mathbf{b}(a), \mathbf{b}(b) \prec c\}$.
This strong condition entails:
• $t^{-1}[\mathbf{b}^{-1}[\{\Phi\}]] = \{(\emptyset, \emptyset)\}$; moreover, if $t : C_s(S) \to S$ is a surjective function, then $\mathbf{b}^{-1}[\{\Phi\}] = \{*\}$.
• $\forall (A,B) \in C_s(S)$, $\forall a \in A, \forall b \in B$: $\mathbf{b}(a)$, $\mathbf{b}(b) \prec \mathbf{b}(t(A,B))$

On (b6), (b7):

If $\mathcal{C}$ is a rooted extensional well-founded class, note that $\forall (A,B) \in C_s(S)$, $\mathbf{b}[Mum_{\mathbf{b}}(A,B)]$ is a singleton in C, whenever $Mum_{\mathbf{b}}(A,B)$ is non-empty subclass of S.

Other versions of conditions (b6) and (b7) will appear in examples:

(b7)^{comp} Means that (b7) holds and that, moreover, every pair $s, s' \in Mum_{\mathbf{b}}(A,B)$ are $<$-comparable.
This condition and (b6) entails: $Mum_{\mathbf{b}}(A,B) = \{t(A,B)\}$.

Let $Mal_{\mathbf{b}}(A,B)$ the subclass of all $s \in S$ such that $A < s < B$ and that is $\prec_{\mathbf{b}}$-mini**mal** for that property (i.e., if $z \prec_{\mathbf{b}} s$, then $A < z < B$ doesn't hold). Since $Mum_{\mathbf{b}}(A,B) \subseteq Mal_{\mathbf{b}}(A,B)$ (because $\prec$ is irreflexive and $(S,<)$ is an η_∞ class), we define:
(b6)^{weak} $t(A,B) \in Mal_{\mathbf{b}}(A,B)$;
(b7)^{strong} if $s, s' \in Mal_{\mathbf{b}}(A,B)$ are $<$-comparable (i.e., $s = s'$ or $s < s'$ or $s' < s$), then $s = s'$.

If $\prec$ is a strict linear order relation in C, then: $\forall x, y \in S$, $x \sqsubseteq_{\mathbf{b}} y$ iff $x \not\succ_b by$. Thus $Mum_{\mathbf{b}}(A,B) = Mal_{\mathbf{b}}(A,B)$, $\forall (A,B) \in C_s(A)$ and (b6) $\Leftrightarrow$ (b6)^{weak}, (b7) $\Leftrightarrow$ (b7)^{strong}.
□

85. ***No*** **as (strict) anchored SUR-algebra:** Naturally, the SUR-algebra $(No, <, -, *, t)$ (Chapter 2, Section 2.1), endowed with the birthday function $b : No \twoheadrightarrow On$, is a strictly anchored SUR-algebra, since:
• $(On, \in, \emptyset)$ is an well-founded extensional rooted class, that is derived from the standard ZF-algebra $\mathcal{O}n$ (see Remark 17.(a) and 18);
• By a careful reading of the results presented in the Section 3.2 of Chapter 1 and the fact

that $(On, \in)$ is a strictly linearly ordered class, we can see that are fulfilled all the axioms (b1)–(b7) (and some other stronger versions). For that, note that in On: $\prec = \in = < = \subsetneq$, $\sqsubseteq =^3 \subseteq\ = \leq$.

This anchored SUR-algebra has some distinctive additional properties:
• $t : C_s(No) \to No$ is a surjective function.
• $(No, <)$ is a strictly linearly ordered class (equivalently, since $<$ is acyclic, $<$ is a total relation).
• It is anchored in $(On, \in, \emptyset)$ that is an well-ordered class.
• It satisfies $(b6)^{weak}$, $(b7)^{strong}$ and $(b7)^{comp}$.
• It does not satisfy $(b5)^{strong}$ since: $\{-1|1\} = 0$ and $\mathbf{b}(0) = 0$ and $\mathbf{b}(\pm 1) = 1$, where $0 := * = \{\emptyset|\emptyset\}$, $1 := \{\emptyset|\{0\}\}$, $-1 := \{\{0\}|\emptyset\}$.

□

Definition 86. The binary relations derived from an anchor: *Each function* $\mathbf{b} : S \to C$ *(in particular, each anchor* $\mathbf{b} : \mathcal{S} \to \mathcal{C}$*) induces the following binary relations in* S.
Let $x, y \in S$:
• $x \equiv_{\mathbf{b}} y$ *iff* $\mathbf{b}(x) = \mathbf{b}(y)$;
• $x \prec_{\mathbf{b}} y$ *iff* $\mathbf{b}(x) \prec \mathbf{b}(y)$;
• $x \sim_{\mathbf{b}} y$ *iff* $\mathbf{b}(x)^{\prec} = \mathbf{b}(y)^{\prec}$;
• $x \sqsubseteq_{\mathbf{b}} y$ *iff* $\mathbf{b}(x) \sqsubseteq \mathbf{b}(y)$;
• $x \sqsubseteq'_{\mathbf{b}} y$ *iff* $\forall s \in S, s \prec_{\mathbf{b}} x \Rightarrow s \prec_{\mathbf{b}} y$.

□

Proposition 87. *Let* $\mathbf{b} : S \to C$ *be a function. Keeping the notation in the Definition above, we have:*
(a) $\equiv_{\mathbf{b}}$ *and* $\sim_{\mathbf{b}}$ *are equivalence relations in* S *and, whenever* $\mathbf{b}$ *satisfies the condition (b2), every* $\equiv_{\mathbf{b}}$*-equivalence class is a* subset *of* S.
(b) $\sqsubseteq_{\mathbf{b}}$ *and* $\sqsubseteq'_{\mathbf{b}}$ *are pre-order relations in* S *and* $\sim_{\mathbf{b}} = (\sqsubseteq_{\mathbf{b}}) \cap (\sqsubseteq_{\mathbf{b}})^{op}$.
(c) $\equiv_{\mathbf{b}}\ \subseteq\ \sim_{\mathbf{b}}$ *and, whenever* $\mathbf{b}$ *satisfies the condition (b1), they coincide iff* $\prec$ *is an extensional relation in* C.
(d) $\sqsubseteq'_{\mathbf{b}} = \sqsubseteq_{\mathbf{b}}$, *whenever* $\mathbf{b}$ *satisfies condition (b1).*
(e) $\prec_{\mathbf{b}}$ *is an well-founded relation in* S, *whenever* $\mathbf{b}$ *satisfies the condition (b2).*
(f) $\prec_{\mathbf{b}}$ *and* $\sqsubseteq_{\mathbf{b}}$ *are compatible with the equivalence relations* $\equiv_b$ *and* $\sim_b$ *(e.g.: if* $x \prec_{\mathbf{b}} y$ *and* $y \sim_b y'$, *then* $x \prec_b y'$.*)*
(g) $x \equiv_{\mathbf{b}} y \Leftrightarrow -x \equiv_{\mathbf{b}} -y$, *whenever* $\mathbf{b}$ *satisfies condition (b4).*
(h) $x \sim_{\mathbf{b}} y \Leftrightarrow -x \sim_{\mathbf{b}} -y$, *whenever* $\mathbf{b}$ *satisfies the condition (b4).*
(i) $x \prec_{\mathbf{b}} y \Leftrightarrow -x \prec_{\mathbf{b}} -y$, *whenever* $\mathbf{b}$ *satisfies the condition (b4).*
(j) $root(S, \prec_{\mathbf{b}}) = \mathbf{b}^{-1}[\{\Phi\}]$, *whenever* $\mathbf{b}$ *satisfies the condition (b1).*
(k) $* \in root(S, \prec_{\mathbf{b}})$, *whenever* $\mathbf{b}$ *satisfies the conditions (b1) and (b3).*

[3] Since, for each ordinal α and set x, if $x \in \alpha$ then x is an ordinal.

(l) If $\mathbf{b}$ *satisfies condition (b1) then:* $\mathbf{b}$ *satisfies (b5) iff for each* $(A,B) \in C_s(S)$ *and each* $s \in S$ *such that* $\forall z \in A \cup B, z \prec_{\mathbf{b}} s$*, then* $t(A,B) \sqsubseteq_{\mathbf{b}} s$*.*

Proof. Items (a), (b) follows directly from the definitions. Items (d) and (l) are consequences of the surjectivity of the function $\mathbf{b}$ (anchor condition (b1)). Since $-(-x) = x$, in the items (g), (h), (i) it is enough to verify the implications ($\Rightarrow$): they are direct consequence of the anchor condition (b4). It remains to prove items (c), (e), (j) and (k).

(c) Clearly $\equiv_{\mathbf{b}} \subseteq \sim_{\mathbf{b}}$.
Suppose that $\prec$ is an extensional relation in C and let $x, x' \in S$, then: $\mathbf{b}(x) = \mathbf{b}(x')$ iff $\forall c \in C, (c \prec \mathbf{b}(x) \Leftrightarrow c \prec \mathbf{b}(x'))$. Thus $(x,x') \in \equiv_{\mathbf{b}}$ iff $(x,x') \in \sim_{\mathbf{b}}$.
Conversely, if $\equiv_{\mathbf{b}} = \sim_{\mathbf{b}}$, and $x, x' \in S$, then: $\forall x, x' \in S$, $\mathbf{b}(x) = \mathbf{b}(x')$ iff $\forall c \in C, (c \prec \mathbf{b}(x) \Leftrightarrow c \prec \mathbf{b}(x'))$. Since $\mathbf{b}$ is surjective, we get: $\forall a, a' \in C$, $a = a'$ iff $\forall c \in C, (c \prec a \Leftrightarrow c \prec a')$, i.e. $\prec$ is an extensional relation in C.

(e) Let $x \in S$. By definition of $\prec_{\mathbf{b}}$, for each $x \in S$ we have $x^{\prec_{\mathbf{b}}} := \{y \in S : y \prec_{\mathbf{b}} x\} = \bigcup \{\mathbf{b}^{-1}[\{c\}] : c \prec \mathbf{b}(x)\}$. Since $\prec$ is an well founded relation in C, the subclass $\{c \in C : c \prec \mathbf{b}(x)\}$ is a subset of C. Therefore, by anchor condition (b2), the subclass of S given by $x^{\prec_{\mathbf{b}}}$ is a subset of S.
Now let X be a non-empty subset of S. Then $\mathbf{b}[X]$ is a non-empty subset of C, thus we can select $c \in \mathbf{b}[X]$ a $\prec$-minimal member. Consider any $z \in \mathbf{b}^{-1}[\{c\}] \cap X$, then clearly z is a $\prec_{\mathbf{b}}$-minimal member of X.

(j) $root(S, \prec_{\mathbf{b}}) = \mathbf{b}^{-1}[\{\Phi\}]$:
($\supseteq$) Let $x \in S$ be such that $\mathbf{b}(x) = \Phi$. Then $\forall z \in S$, $z \prec_{\mathbf{b}} x$ iff $\mathbf{b}(z) \prec \mathbf{b}(x) = \Phi$. Thus $z \not\prec_{\mathbf{b}} x$, $\forall z \in S$, i.e. x is a $\prec_{\mathbf{b}}$-root.
($\subseteq$) Let $x \in S$ be such that $\forall z \in S, z \not\prec_{\mathbf{b}} x$, then $\forall z \in S$, $\mathbf{b}(z) \not\prec \mathbf{b}(x)$. Since $\mathbf{b}$ is surjective: $\forall c \in C, c \not\prec \mathbf{b}(x)$. Thus $\mathbf{b}(x) \in root(C, \prec) = \{\Phi\}$, i.e. $x \in \mathbf{b}^{-1}[\{\Phi\}]$.

(k) Since $\mathbf{b}(*) = \Phi$ (anchor condition (b3)), we have $* \in \mathbf{b}^{-1}[\{\Phi\}]$ and the result follows from item (j) above.

□

Remark 88. A SUR-algebra $\mathcal{S} = (S, <, *, -, t)$ is a higher-order structure such that the underlying first-order structure $(S, <)$ is acyclic and a η_∞-relational structure (see Remark 34). An anchor $\mathbf{b} : \mathcal{S} \to \mathcal{C}$ on $\mathcal{S}$ induces another binary relation $\prec_{\mathbf{b}}$ on S that is well-founded (thus it is acyclic, in particular) and that, under convenient additional hypothesis, can encode the anchor itself (see 116 and 117 in Section 3 below). First-order structures of the form $(S, <, \prec)$ were considered by P. Ehrlich in [Ehr01] and by J. Hamkins in [Ham13], with the development of some interesting model-theoretic aspects. Thus the concept of anchor in a SUR-algebra is a sort of balance: at one hand it generalizes the "birthday function", a seminal ingredient for the development of the theory of surreal numbers, and at the other hand is related to interesting first-order structures that have been considered by respectable specialists in foundations of mathematics.

□

Before we provide new examples of anchored SUR-algebra, we will establish some simple results on more special kinds of anchors, to indicate the strength of the constrains imposed by the anchor axioms.

Proposition 89. *Let $\mathcal{S}$ be a SUR algebra and $\mathcal{C} = (C, \prec, \Phi)$ be a rooted extensional well-founded class (see Subsection 1.1.2, Chapter 1). Let $\lhd$ be a (well-founded) relation induced by some anchor $\mathbf{b} : \mathcal{S} \to \mathcal{C}$, i.e. $\lhd = \prec_{\mathbf{b}}$, then there is only one such anchor $\mathbf{b}$.*

Proof. Let $\mathbf{b} : \mathcal{S} \to \mathcal{C}$ and $\mathbf{b}' : \mathcal{S} \to \mathcal{C}$ be anchors such that $\prec_{\mathbf{b}} = \prec_{\mathbf{b}'}$ and denote $\lhd$ this binary relation in S. By Proposition 87.(e), $\lhd$ is a well-founded relation in S.
We will prove that $\forall y \in S, \mathbf{b}(y) = \mathbf{b}'(y)$ by $\lhd$-induction. Suppose that $\mathbf{b}(x) = \mathbf{b}'(x)$ for all $x \in y^{\lhd}$. Since $\mathbf{b}$ and $\mathbf{b}'$ are surjective, we have:
$\mathbf{b}(y)^{\prec} = \{c \in C : c \prec \mathbf{b}(y)\} = \{\mathbf{b}(x) \in C : x \lhd y\} \underset{IH}{=} \{\mathbf{b}'(x) \in C : x \lhd y\} = \{c' \in C : c \prec \mathbf{b}'(y)\} = \mathbf{b}'(y)^{\prec}$.
Since $\prec$ is extensional, we have $\mathbf{b}(y) = \mathbf{b}'(y)$ and thus $\mathbf{b} = \mathbf{b}'$. □

Proposition 90. *Let $\mathbf{b} : \mathcal{S} \to \mathcal{C}$ be an anchored SUR-algebra such that:*
(T) *For each $x \in S$, there exists $(A, B) \in C_s(S)$ such that $x = t(A, B)$ and $\forall a \in A, \forall b \in B$, $\mathbf{b}(a), \mathbf{b}(b) \prec \mathbf{b}(x)$.* [4]
Then for each SUR-algebra $\mathcal{S}'$, there is at most one SUR-algebra morphism $f : \mathcal{S} \to \mathcal{S}'$.

In particular: $SUR - alg(\mathcal{S}, \mathcal{S}) = \{id_{\mathcal{S}}\}$ and then the structure of anchored SUR-algebra $(\mathcal{S}, \mathbf{b})$ is rigid, i.e., id_S is a the unique SUR-automorphism $f : \mathcal{S} \to \mathcal{S}$ such that $\mathbf{b} \circ f = \mathbf{b}$.

Proof. The proof is a simple $\prec_{\mathbf{b}}$-induction: given SUR-algebras morphisms $f, g : \mathcal{S} \to \mathcal{S}'$, suppose that $f(z) = g(z)$ for all $z \in x^{\prec_{\mathbf{b}}}$. Take any (A, B) that is $\mathbf{b}$-timely Conway-cut representation of x. Since $A \cup B \subseteq x^{\prec_{\mathbf{b}}}$, we have:

$$f(x) = f(t(A, B)) = t'(f[A], f[B]) \underset{IH}{=} t'(g[A], g[B]) = g(t(A, B)) = g(x).$$

Thus $\forall y \in S,\ f(y) = g(y)$.

□

Remark 91.

- If $(b5)^{strong}$ holds, then the condition (T) in the Proposition above is equivalent to require that $t : C_s(S) \to S$ be a surjective function.

[4] Following [All87], section 2, chapter 4, page 125, (see also Fact 27, Chapter 1) a pair (A, B) as above will be called a $\mathbf{b}$-*timely Conway cut representation of* x. Note that, in particular, $t : C_s(S) \to S$ must be surjective.

• The anchor $b : No \to On$ described in 85 satisfies the conditions in the Propositions 89 and 90 above:
- $(On, \in \emptyset)$ is an extensional rooted well founded-class;
- it satisfies condition (T) above (but does not satisfy $(b5)^{strong}$): by the Cuesta-Dutari representation of a surreal number y (see [All87], pages 125, 129) we have $y = \{L_y | R_y\}$, where $L_y = \{x \in No : x < y, b(x) \prec b(y)\}$ and $R_y = \{z \in No : y < z, b(z) \prec b(y)\}$.

• Soon we will provide anchors on the SUR-algebra SA that satisfies the conditions in the Propositions 89 and 90 above.

• For the "timely" anchored SUR-algebras $\mathcal{S}$, i.e., those satisfying the condition (T), we may wonder about the existence of some (unique) morphism with source $\mathcal{S}$ and target a SUR-algebra $\mathcal{S}'$. By (T), we could be tempted to define recursively a morphism $f : \mathcal{S} \to \mathcal{S}'$ by $f(t(A, B)) := t'(f[A], f[B])$, since $A \cup B \subseteq x^{\prec_b}$. Clearly, by the SUR-algebra properties of $\mathcal{S}'$, such rule f must preserves $*$ and $-$. But the main issue is that when such rule can really define a function. We will re-address this topic in the future (for the anchored SUR-algebra No, see [All87], pages 125, 126).

□

In the sequel, we will provide (strict) anchors structures on the SUR-algebras SA and ST (see Subsections 2.2.2 and 2.2.3, Chapter 2).

Proposition 92. Connecting SA and On:
We have a pair of functions ($On \underset{r}{\overset{s}{\rightleftarrows}} SA$), defined in the Subsection 2.2.2 of Chapter 2:
• The function $s : On \to SA$, defined by recursion on the well-ordered proper class $(On, <)$, by $s(\alpha) := \langle s[\alpha], \emptyset \rangle$, $\alpha \in On$.
• The function $r : SA \to On$, given by $r(x) = min\{\alpha \in On : x \in SA_\alpha\}$, $x \in SA$.
Then:
(i) $r \circ s = id_{On}$.
(ii) r is a strict anchor of SA, that satisfies $(b5)^{strong}$, on the rooted extensional well-founded class $(On, \in, \emptyset)$.
Moreover, $\forall x \in SA$, if $x = t(A, B)$ then: $r(x) = min_{\subseteq}\{\alpha \in On : \forall a \in L_x, \forall b \in R_x, r(a), r(b) \in \alpha\} = \bigcup_{z \in L_x \cup R_x} (r(z))^+$.

Proof. In the items below, all the "Claims" and "Facts" mentioned are in the Subsubsection 2.2.2, Chapter 2. Item (i) was established in Claim 13. We will prove item (ii):

(b1) r is surjective: since it has a section $s : On \to SA$ (i.e. $r \circ s = id_{On}$).

(b2) r is locally small (or has small fibers): since $r^{-1}[\{\alpha\}] = SA_\alpha \setminus SA^{(\alpha)} \subseteq SA_\alpha$ is a set, for each $\alpha \in On$.

(b3) $r(*) = r(\langle \emptyset, \emptyset \rangle) = 0 = \emptyset$, by Fact.(a) (b4) $r(-x) = r(x), \forall x \in SA$, by Claim 14.(i).

(b5)^{strong} For each $(A, B) \in C_s(SA)$ and each $\alpha \in On$, then:
(S) $\forall a \in A, \forall b \in B$, $r(a)$, $r(b) \prec \alpha$ (in On) iff $r(t(A, B)) \sqsubseteq \alpha$.
In On, we have the relations $\prec = \in = <$ and $\sqsubseteq = \subseteq = \leq$. Thus the expression (S) above can be described as:
(S') $\forall a \in A, \forall b \in B$, $r(a), r(b) < \alpha$ iff $r(t(A, B)) \leq \alpha$.
The validity of (S') is exactly the content of Claim 6. Equivalently, $r(t(A, B)) = min_{\leq}\{\alpha \in On : \forall a \in A, \forall b \in B, r(a), r(b) < \alpha\}$.

Let $(A, B) \in C_s(SA)$ and $Mum_r(A, B) = \{u \in SA : A < u < B, \forall z \in SA(A < z < B \Rightarrow u \sqsubseteq_r z)\}$.

Since $\prec$ in On is a strict linear order, $Mum_r(A, B) = Mal_r(A, B) = \{u \in SA : A < u < B, \forall z \in SA(A < z < B \Rightarrow z \not\prec_r u)\}$. Thus (b6) $\Leftrightarrow$ (b6)^{weak}, (b7) $\Leftrightarrow$ (b7)^{strong} (see Remark 84 above).

(b6) $t(A, B) \in Mum_r(A, B)$:
$A < t(A, B) < B$, by Claim 11;
$A < z < B \Rightarrow r(t(A, B)) \leq r(z)$, by Claim 12.

(b7) If $s, s' \in Mum_r(A, B)$ are $<$-comparable (i.e., $s = s'$ or $s < s'$ or $s' < s$), then $s = s'$.
Since $\prec = \in = <$ is an extensional relation in On, $r[Mum_r(A, B)] = \{r(t(A, B)\}$ (Remark 84). Thus, for all $s, s' \in Mum_r(A, B)$, $r(s) = r(s')$ and by Claim 7, $s \not< s'$ and $s' \not< s$.

We finish the proof showing that, for each $x \in SA$, $r(x) = min_{\subseteq}\{\alpha \in On : \forall a \in L_x, \forall b \in R_x, r(a), r(b) \in \alpha\} = \bigcup_{z \in L_x \cup R_x}(r(z))^+$:
The first equality follows from the proof above, since in On, $\leq = \subseteq$ and $< = \in$. Let $X := min_{\subseteq}\{\alpha \in On : \forall a \in L_x, \forall b \in R_x, r(a), r(b) \in \alpha\}$ and $X' := \bigcup_{z \in L_x \cup R_x}(r(z))^+ \in On$. Consider any $z \in L_x \cup R_x$, then $r(z) \in r(z)^+ \subseteq X'$, i.e. $X \subseteq X'$. On the other hand, any $z \in L_x \cup R_x$, $r(z) \in X$, then $r(z)^+ \subseteq X$, thus $X' \subseteq X$. Summing up, $X = X'$.

□

By a sequence of reasonings analogous to the above, but founded over the "Facts" and "Claims" presented in the Subsection 2.2.3 of Chapter 2, we obtain the following:

Fact 93. Connecting ST and On:
We have a pair of functions ($On \overset{s'}{\underset{r'}{\rightleftarrows}} ST$), defined in the Subsection 2.2.3 of Chapter 2:
• *The function $s' : On \to ST$, defined by recursion on the well-ordered proper class $(On, <)$, by $s'(\alpha) := \langle s'[\alpha], \emptyset \rangle$, $\alpha \in On$.*
• *The function $r' : ST \to On$, given by $r'(x) = min\{\alpha \in On : x \in ST_\alpha\}$, $x \in SA$.*
Then:
(i) $r' \circ s' = id_{On}$.
(ii) r' is a strict anchor of ST, that satisfies $(b5)^{strong}$, on the rooted extensional well-

founded class $(On, \in, \emptyset)$.
Moreover, $\forall x \in ST$*, if* $x = t(A, B)$ *then:* $r'(x) = min_{\subseteq}\{\alpha \in On : \forall a \in A, \forall b \in B, r'(a), r'(b) \in \alpha\} = \bigcup_{z \in A \cup B}(r'(z))^+$.

□

Now, we will connect the initial SUR-algebra SA and the initial ZF-algebra $\mathcal{V}$: since $\mathcal{V}$ is a standard ZF-algebra (Subsection 1.2.2, Chapter 1), $(V, \in, \emptyset)$ is an well-founded extensional rooted class.

Definition 94. *(i) Let* $j^* : V \to SA$*, defined by* $\in$*-recursion by* $j^*(X) = t(j^*[X], \emptyset)$, $X \in V$.
(ii) Consider $\prec_r$ *the well-founded relation induced by the above defined strict anchor function* $r : SA \to On$*, of* SA *on the rooted extensional well-founded class* $(On, \in, \emptyset)$*. Since* $t : C_s(SA) \to SA$ *is a bijection (is the identity function), and* $r : SA \to On$ *is a* strict *anchor, we can define a function* $b^* : SA \to V$ *by* $\prec_r$*-recursion:* $b^*(x) = b^*[L_x] \cup b^*[R_x]$*, where* $x = \langle L_x, R_x \rangle = t(L_x, R_x)$.

□

Proposition 95.
(i) $b^* \circ j^* = id_V$*, thus* b^* *is surjective and* j^* *is injective.*
(ii) Let $X, Y \in V$*. Then:* $X \in Y$ *iff* $j^*(X) < j^*(Y)$.
(iii) Let $Y \in V$ *and* $a \in SA$*. If* $a < j^*(Y)$ *and* $b^*(a) \in b^*(j^*(Y))$ *then* $a = j^*(X)$ *for a unique* $X \in Y$.

Proof. All the "Claims" mentioned below are in the Subsection 2.2.2, Chapter 2.
(i) By $\in$-induction: $b^*(j^*(Y)) \underset{def.\ j^*}{=} b^*(t(j^*[Y], \emptyset)) \underset{def.\ b^*}{=} b^*[j^*[Y]] \cup b^*[\emptyset] = \{b^*(j^*(X)) : X \in Y\} \cup \emptyset \underset{IH}{=} Y$.
(ii) If $X \in Y$, then $j^*(X) \in j^*[Y]$ and, since $j^*(Y) = \langle j^*[Y], \emptyset \rangle$, then $j^*(X) \in L_{j^*(Y)}$. Thus, by Claims 4 and 9, $j^*(X) < j^*(Y)$. Conversely, if $j^*(X) < j^*(Y)$, then since $R_{j^*(X)} = \emptyset$, we must have $j^*(X) \in L_{j^*(Y)} = j^*[Y]$. This means that there is $X' \in Y$ such that $j^*(X) = j^*(X')$. By, item (i) above, j^* is injective, then $X = X' \in Y$, as we would like.
(iii) If $a < j^*(Y)$, then by Claims 4 and 9, $a \in L_{j^*(Y)}$ or $j^*(Y) \in R_a$. In the latter case, $b^*(j^*(Y)) \in b^*[R_a] \subseteq b^*(a)$, and since $\in$ is an irreflexive relation in V, this is incompatible with the second hypothesis, $b^*(a) \in b^*(j^*(Y))$. Thus $a \in L_{j^*(Y)} = j^*[Y]$, and exists $X \in Y$ such that $a = j^*(X)$. Since j^* is injective, such X is uniquely determined.

□

Let us register the following useful:

Fact 96.

(i) For each $x \in SA$, $b^*(x) = b^*[L_x] \cup b^*[R_x] = min_{\subseteq}\{Y \in V : \forall a \in L_x, \forall b \in R_x, b^*(a), b^*(b) \in Y\} = \bigcup_{z \in L_x \cup R_x} \{b^*(z)\}$.

(ii) By Claims 4 and 9 in the Subsection 2.2.2, Chapter 2: $\forall x, y \in SA$, $x < y$ iff $x \in L_y$ (thus $b^*(x) \prec b^*(y)$) or $y \in R_x$ (thus $b^*(y) \prec b^*(x)$). In particular:

- $\forall x, y \in SA$, $x < y \Rightarrow b^*(x) \neq b^*(y)$.
- $\forall (A, B) \in C_s(SA)$, $b^*[A] \cap b^*[B] = \emptyset$.

□

Proposition 97.
(i) j^* extends s, through i. More precisely:
$(SA \stackrel{s}{\leftarrow} On) = (SA \stackrel{j^*}{\leftarrow} V \stackrel{i}{\leftarrow} On)$.
(ii) b^* lifts r, through ρ. More precisely:
$(SA \stackrel{r}{\rightarrow} On) = (SA \stackrel{b^*}{\rightarrow} V \stackrel{\rho}{\rightarrow} On)$.

Proof. (i) By $\in$-induction. Let $\alpha \in On$, then $j^*(i(\alpha)) = \langle j^*[i(\alpha)], \emptyset \rangle = t(\{j^*(Y) : Y \in V, Y \in i(\alpha)\}, \emptyset) = t(\{j^*(i(\beta)) : \beta \in On, \beta \in \alpha\}, \emptyset) \underset{IH}{=} t(\{r(\beta) : \beta \in On, \beta \in \alpha\}, \emptyset) = \langle r[\alpha], \emptyset \rangle = r(\alpha)$.
(ii) By $\prec_r$-induction. Let $x = t(A, B)$, then, by the Fact 96 just above:
$\rho(b^*(x)) = \rho(b^*(t(A, B))) = \rho(b^*[A] \cup b^*[B]) = \rho(\bigcup_{z \in A \cup B} \{b^*(z)\}) =^5 \bigcup_{z \in A \cup B} \rho(\{b^*(z)\}) =^6 \bigcup_{z \in A \cup B} (\rho(b^*(z)))^+ \underset{IH}{=} \bigcup_{z \in A \cup B} (r(z))^+ = r(t(A, B)) = r(x)$.

□

Now, we are ready to state and prove a result analogous to Proposition 92:

Proposition 98. Connecting SA and V:
We have a pair of functions $(V \underset{b^*}{\overset{j^*}{\rightleftarrows}} SA)$, defined in 94 above. Then:
(i) $b^* \circ j^* = id_V$.
(ii) b^* is an weakly[7] *strict anchor of SA, that satisfies* $(b5)^{strong}$*, on the rooted extensional well-founded class $(V, \in, \emptyset)$.*

Proof. First of all, note that in $(V, \in, \emptyset)$, $\prec = \in$ and $\sqsubseteq = \subseteq$. Item (i) was proved in the Proposition 95.(i). Now we will prove item (ii).

- From item (i), b^* is surjective, i.e. (b1) holds.
- Let $x \in V$ and denote $\rho(x) = \alpha \in On$ then, by Proposition 97.(ii):
$b^{*-1}[\{x\}] \subseteq b^{*-1}[V_{\alpha+1} \setminus V_\alpha] = b^{*-1}[\rho^{-1}[\{\alpha\}] = SA_\alpha \setminus SA^{(\alpha)}$
thus $b^{*-1}[\{x\}]$ is a set and (b2) holds.

[5] See the Subsection 1.2.1
[6] Idem.
[7] I.e., it satisfies (b7) and, instead (b6), it satisfies $(b6)^{weak}$, see Remark 84.

• The verifications of conditions (b3) and (b4) are straightforward:
(b3) $b^*(*) = b^*(t(\emptyset, \emptyset)) = b^*[\emptyset] \cup b^*[\emptyset] = \emptyset$.
(b4) Let $x \in SA$, we prove by $\prec_r$-induction that $b^*(-x) = b^*(x)$:
Since t is bijective, let $x = t(A, B)$, then $\forall z \in A \cup B$, $b^*(z) \prec b^*(x)$ and:
$b^*(-x) = b^*(-t(A, B)) = b^*(t(-B, -A)) = b^*[-B] \cup b^*[-A] \underset{IH}{=} b^*[B] \cup b^*[A] = b^*(t(A, B)) = b^*(x)$.

• Note that for each $(A, B) \in C_s(SA)$, $b^*(t(A, B)) = b^*[A] \cup b^*[B] = min_{\subseteq}\{X \in V : \forall a \in A, \forall b \in B, b^*(a), b^*(b) \in X\}$ (see Fact 96). Equivalently, for each $X \in V$, then: $\forall a \in A, \forall b \in B$, $b^*(a), b^*(b) \in X$ iff $b^*(t(A, B)) = b^*[A] \cup b^*[B] \subseteq X$. This means that b^* verifies the condition (b5)^{strong}.

• The condition (b7) is satisfied:
Let $(A, B) \in C_s(SA)$. Since $\prec = \in$ is an extensional relation in V, $b^*[Mum_{b^*}(A, B)]$ has at most one member (Remark 84). Thus, for all $s, s' \in Mum_{b^*}(A, B)$, $b^*(s) = b^*(s')$ and, by Fact 96, $s \nless s'$ and $s' \nless s$.

• Let $(A, B) \in C_s(SA)$ and let $z \in SA$ such that $z \prec_{b^*} t(A, B)$. Then $b^*(z) \in b^*[A] \cup b^*[B]$. If $b^*(z) = b^*(a)$, for some $a \in A$, then by Fact 96, it can't hold $A < z$. Likewise, if $b^*(z) = b^*(b)$, for some $b \in B$, then it can't hold $z < B$. Thus $t(A, B) \in Mal_{b^*}(A, B)$ and (b6)^{weak} holds.

□

Note this anchor (SA, b^*) satisfies the conditions (T) in the Proposition 90 (but the conclusion is not new, since we saw in Chapter 2 that SA is the initial object in the category SUR).

In the sequel, we present characterizations/axiomatizations of the anchored SUR-algebras (SA, r), (ST, r') and (SA, b^*) in the same vein as the axiomatization of the SUR-algebra (No, b) (see Subsection 1.3.2, Chapter 1).

Theorem 99. An axiomatization of $SA \xrightarrow{r} On$:
The anchored SUR-algebra (SA, r) is axiomatized by the following properties:
(rSA1) $t : C_s(SA) \to SA$ is bijective.
(rSA2) $\forall u \in SA$, if $u = t(A, B)$ then: $r(u) = min_{\subseteq}\{\alpha \in On : \forall a \in A, \forall b \in B, r(a), r(b) \in \alpha\} = \bigcup_{v \in A \cup B}(r(v))^+$.
(rSA3) $\forall u, v \in SA$, denote $u = t(L_u, R_u)$ and $v = t(L_v, R_v)$, then are equivalent:
• $u < v$;
• $u \in L_v$ or[8] $v \in R_u$.

More precisely, if $(\mathcal{S}, \mathbf{b})$, is an anchored SUR-algebra, $\mathbf{b} : \mathcal{S} \to On$, then are equivalent:
(i) $(\mathcal{S}, \mathbf{b})$ satisfies the conditions (rSA1), (rSA2), (rSA3) above.
(ii) There exists a unique isomorphism of anchored SUR-algebras $u : (SA, r) \overset{\cong}{\to} (\mathcal{S}, \mathbf{b})$.

[8] By (rSA2), this is a non-inclusive disjunction.

In particular:
• *Any anchored SUR-algebra* $(\mathcal{S}, \mathbf{b})$ *that satisfies (rSA1), (rSA2), (rSA3) must be strict and satisfies* $(b5)^{strong}$*, since* (SA, r) *is an strictly anchored SUR-algebra satisfying* $(b5)^{strong}$.
• *If* $(\mathcal{S}, \mathbf{b})$ *and* $(\mathcal{S}', \mathbf{b}')$ *are anchored SUR-algebras that satisfies (rSA1), (rSA2), (rSA3), then there is a unique isomorphism of anchored SUR-algebras* $f : (\mathcal{S}, \mathbf{b}) \overset{\cong}{\to} (\mathcal{S}', \mathbf{b}')$.

Proof. $(ii) \Rightarrow (i)$ is clear from the results on (SA, r): see Subsection 2.2.2 and Proposition 92. We will show that $(i) \Rightarrow (ii)$.

Denote $\mathcal{Z} := (Z, \bigvee, s)$ the standard ZF-algebra *On*.

Let $(\mathcal{S}, \mathbf{b})$ be anchored SUR-algebra that satisfies (rSA1), (rSA2), (rSA3). For each $z \in Z$, let $S^z := \mathbf{b}^{-1}[z^{\prec}]$. Then clearly:
• $u \in S^z$ iff $-u \in S^z$;
• $S^{\Phi} = \emptyset$;
• $z \neq \Phi$ iff $* \in S^z$;
• $t_z : C_s(S^z, <_{\restriction}) \to S^{s(z)}$ is a bijection, by (rSA1), (rSA2);
• If $y \prec z$, equivalently $y \varepsilon z$ or $y < z$, see Remark 17.(d) (in particular: $\prec$ is a transitive relation), then $y^{\prec} \subseteq z^{\prec}$ and $S^y \subseteq S^z$.

Thus, since $z < s(z)$, S^z is a partial SUR-algebra whenever $z \neq \Phi$, where $C_s^t(S^z, <) := t_z^{-1}[S^z] \subseteq t_z^{-1}[S^{s(z)}] = C_s(S^z, <)$ and $t_z \restriction : C_s^t(S^z, <) \to S^z$ is the partial structure map. Moreover, $0 < y \leq z \mapsto S^y \subseteq S^z$ is a increasing family of partial SUR-algebras and $S = \bigcup_{z \in Z} S^z$ (a directed colimit of partial SUR-algebras, see Subsection 2.3.4, Chapter 2).

Let $(\mathcal{S}, \mathbf{b})$ and $(\mathcal{S}', \mathbf{b}')$ be anchored SUR-algebras that satisfies (rSA1), (rSA2), (rSA3). We will show that there is a *unique* SUR-algebra morphism $f : S \to S'$ such that $\mathbf{b} = \mathbf{b}' \circ f$. Then, by uniqueness, the SUR-algebra morphism $f' : S' \to S$ such that $\mathbf{b}' = \mathbf{b} \circ f'$ is the inverse morphism of f, completing the proof.

Claim: For all $x \in Z$ there is a unique function $f_x : S^x \to S'^x$ that is a partial SUR-algebra morphism (whenever $x \neq \Phi$) and such that $\mathbf{b}'(f_x(u)) = \mathbf{b}(u)$, for all $u \in S^x$.

Proof: Suppose, by induction, that for all $y \prec x$ there is a unique $f_y : S^y \to S'^y$ satisfying the conditions above. Note that if there is a map $f_x : S^x \to S'^x$ satisfying the conditions, then $f_y = (f_x)_{\restriction}$, $\forall y \prec x$, by the uniqueness of f_y. Since $x^{\prec} = \bigcup_{y \varepsilon x} s(y)^{\prec}$ and $S^x = \mathbf{b}^{-1}[x^{\prec}] = \bigcup_{y \varepsilon x} S^{s(y)}$ we consider two cases:

• x is a limit: then $x^{\prec} = \bigcup_{y \varepsilon x} y^{\prec}$ and $(f_x : S^x \to S'^x) = (\bigcup_{y \varepsilon x} f_y : \bigcup_{y \varepsilon x} S^y \to \bigcup_{y \varepsilon x} S'^y)$ (it is the unique possibility, and it works).

• x is a successor: then $x = s(z), z \varepsilon x$. Note that $\bigcup_{y \varepsilon x} S^y = \bigcup_{y \varepsilon x, y \neq z} S^{s(y)}$. Let $u \in S^{s(z)} \setminus \bigcup_{y \varepsilon x} S^y$, then $\mathbf{b}(u) = z$ and $u = t(L_u, R_u)$, with $z = min_{\leq}\{w \in Z : \forall v \in L_u \cup R_u, \mathbf{b}(v) < z\}$. If there is f_x satisfies the conditions, we must have $f_x(u) = f_x(t(L_u, R_u)) = t'(f_x[L_u], f_x[R_u]) = t'(\bigcup_{y \varepsilon x} f_y[L_u], \bigcup_{y \varepsilon x} f_y[R_u])$, $z = min_{\leq}\{w \in Z : \forall v' \in$

$\bigcup_{y\varepsilon x} f_y[L_u \cup R_u], \mathbf{b}'(v') < z\}$. Thus there is at most one map $f_x : S_x \to S'_x$ satisfying the conditions. Now *define* $f_x(u) := t'(f_z[L_u], f_z[R_u])$, $u \in S_x$. Then this map satisfies all the conditions required. Indeed if $u < v \in S^{s(z)}$, then by (rSA3) $u \in L_v$ or (exclusive) $v \in R_u$, thus at most one member between u, v (they are distinct, since $<$ is irreflexive), belongs to $\mathbf{b}^{-1}[\{z\}]$. Since $L'_{f_x(u)} = f_z[L_u]$ and $R'_{f_x(u)} = f_z[R_u]$, we obtain $f_x(u) <' f_x(v)$. The other conditions on f_x are even easier to prove. □

By the Claim above, the family of functions $\{f_x : S^x \to S'^x; x \in Z\}$, is compatible (i.e., $y \leq x \Rightarrow f_y = (f_x)_\restriction$), thus $(f : S \to S') = (\bigcup_{y\in Z} f_y : \bigcup_{y\in Z} S^y \to \bigcup_{y\in Z} S'^y)$: it is the unique possibility, and it works (i.e., $f : S \to S'$ is a SUR-algebra morphism such that $\mathbf{b}' \circ f = \mathbf{b}$).

□

Theorem 100. An axiomatization of $ST \overset{r'}{\to} On$**:**
The anchored SUR-algebra (SA, r) *is axiomatized by the following properties:*
(rST1) $t : C_s(SA) \to SA$ *is bijective.*
(rST2) $\forall u \in ST$, *if* $u = t(A,B)$ *then:* $r'(u) = min_{\subseteq}\{\alpha \in On : \forall a \in A, \forall b \in B, r'(a), r'(b) \in \alpha\} = \bigcup_{v\in A\cup B}(r'(v))^+$.
(rST3) $\forall u, v \in ST$, *if* $u = t(A,B)$ *and* $v = t(C,D)$ *and denote* $\alpha := max\{r'(u), r'(v)\}$ *then are equivalent:*
- $u < v$.
- *Exists* $n \in \mathbb{N}$, *exists* $\{w_0, \cdots, w_{n+1}\}$ *where* $u = w_0$, $v = w_{n+1}$ *and such that:* $r'(w_i) < \alpha$, *whenever* $1 \leq i \leq n$; $w_j \in L_{w_{j+1}}$ *or* $w_{j+1} \in R_{w_j}$, *whenever* $0 \leq j \leq n$.

More precisely, if $(\mathcal{S}, \mathbf{b})$, *is an anchored SUR-algebra,* $\mathbf{b} : \mathcal{S} \to On$, *then are equivalent:*
(i) $(\mathcal{S}, \mathbf{b})$ *satisfies the conditions (rST1), (rST2), (rST3) above.*
(ii) There exists a unique isomorphism of anchored SUR-algebras $u : (ST, r') \overset{\cong}{\to} (\mathcal{S}, \mathbf{b})$.

In particular:
- *Any anchored SUR-algebra* $(\mathcal{S}, \mathbf{b})$ *that satisfies (rST1), (rST2), (rST3) must satisfies* $(b5)^{strong}$, *since* (ST, r') *is an anchored SUR-algebra satisfying* $(b5)^{strong}$.
- *If* $(\mathcal{S}, \mathbf{b})$ *and* $(\mathcal{S}', \mathbf{b}')$ *are anchored SUR-algebras that satisfies (rST1), (rST2), (rST3), then there is a unique isomorphism of anchored SUR-algebras* $f : (\mathcal{S}, \mathbf{b}) \overset{\cong}{\to} (\mathcal{S}', \mathbf{b}')$.

Proof. $(ii) \Rightarrow (i)$ is clear from the results on (ST, r'): see Subsection 2.2.3 and Fact 93. The proof of $(i) \Rightarrow (ii)$ follows closely the proof of the Theorem above: it can be established an analogous:
Claim: For all $x \in Z = On$ there is a unique function $f_x : S^x \to S'^x$ that is a partial SUR-algebra morphism (whenever $x \neq \Phi$) and such that $\mathbf{b}'(f_x(u)) = \mathbf{b}(u)$, for all $u \in S^x$. It is proof is established by induction in $x \in Z = On$, by considering separately the cases x limit and x successor: in the latter case, we must to adapt the corresponding part of the proof of Theorem 99, using the condition (rST3) instead (rSA3), but everything works.

By the Claim above, the family of functions $\{f_x : S^x \to S'^x; x \in Z\}$, is compatible (i.e., $y \leq x \Rightarrow f_y = (f_x)_{\restriction}$), thus $(f : S \to S') = (\bigcup_{y\in Z} f_y : \bigcup_{y\in Z} S^y \to \bigcup_{y\in Z} S'^y)$: it is the unique possibility, and it works (i.e., $f : S \to S'$ is a SUR-algebra morphism such that $\mathbf{b}' \circ f = \mathbf{b}$).

□

Theorem 101. An axiomatization of $SA \xrightarrow{b^*} V$:
The anchored SUR-algebra (SA, b^*) is axiomatized by the following properties:
(bSA1) $t : C_s(SA) \to SA$ is bijective.
(bSA2) $\forall u \in SA$, if $u = t(A, B)$ then: $b^*(u) = min_{\subseteq}\{x \in V : \forall a \in A, \forall b \in B, b^*(a), b^*(b) \in x\} = \bigcup_{v\in A\cup B}\{b^*(v)\}$.
(bSA3) $\forall u, v \in SA$, denote $u = t(L_u, R_u)$ and $v = t(L_v, R_v)$, then are equivalent:
* *$u < v$;*
* *$u \in L_v$ or[9] $v \in R_u$.*

More precisely, if $(\mathcal{S}, \mathbf{b})$, is an anchored SUR-algebra, $\mathbf{b} : \mathcal{S} \to V$, then are equivalent:
(i) $(\mathcal{S}, \mathbf{b})$ satisfies the conditions (bSA1), (bSA2), (bSA3) above.
(ii) There exists a unique isomorphism of anchored SUR-algebras $u : (SA, b^*) \xrightarrow{\cong} (\mathcal{S}, \mathbf{b})$.

In particular:
* *Any anchored SUR-algebra $(\mathcal{S}, \mathbf{b})$ that satisfies (bSA1), (bSA2), (bSA3) must be* weakly[10] *strict and satisfies $(b5)^{strong}$, since (SA, b^*) is an strictly anchored SUR-algebra satisfying $(b5)^{strong}$.*
* *If $(\mathcal{S}, \mathbf{b})$ and $(\mathcal{S}', \mathbf{b}')$ are anchored SUR-algebras that satisfies (bSA1), (bSA2), (bSA3), then there is a unique isomorphism of anchored SUR-algebras $f : (\mathcal{S}, \mathbf{b}) \xrightarrow{\cong} (\mathcal{S}', \mathbf{b}')$.*

Proof. $(ii) \Rightarrow (i)$ is clear from the results on (SA, b^*): see Subsection 2.2.2 and Proposition 98. The proof of $(i) \Rightarrow (ii)$ follows closely the proof of the Theorem 99 above.

Denote $\mathcal{Z} := (Z, \bigvee, s)$ the standard ZF-algebra V.

Let $(\mathcal{S}, \mathbf{b})$ be anchored SUR-algebra that satisfies (bSA1), (bSA2), (bSA3). For each $z \in Z$, let $S^z := \mathbf{b}^{-1}[z^{\prec}]$. Then clearly:
* $u \in S^z$ iff $-u \in S^z$;
* $S^{\Phi} = \emptyset$;
* If z is a transitive set, then: $z \neq \Phi$ iff $\emptyset \in z$ iff $* \in S^z$;
* If z is a non-empty transitive set then $(S^z, <_{\restriction}, -_{\restriction}, *, t_{\restriction})$ is a partial SUR-algebra, where $t_{\restriction} : C^t_s \to S^z$ and $C^t_s(S^z) = t^{-1}[S^z] \cap C_s(S^z, <)$;
* Since $V = \bigcup_{\alpha\in On} V_\alpha$ and each V_α is a transitive set, each $S^{V_{\alpha^+}}$ is a partial SUR-algebra and $t_{V_\alpha} : C_s(S^{V_\alpha}, <_{\restriction}) \to S^{V_{\alpha^+}}$ is a bijection, by (bSA1), (bSA2) (note that $\mathbf{b} \circ \rho$ provides an anchor on $\mathcal{S}$ that satisfies the condition on Theorem 99);
* The map $0 < \alpha \leq \beta \in On \mapsto S^{V_\alpha} \subseteq S^{V_\beta}$ is a increasing family of partial SUR-algebras

[9] By (bSA2), this is a non-inclusive disjunction.
[10] I.e., it satisfies (b7) and, instead (b6), it satisfies $(b6)^{weak}$, see Remark 84.

and $S = \bigcup_{\alpha \in On} S^{V_{\alpha^+}}$ (a directed colimit of partial SUR-algebras, see Subsection 2.3.4, Chapter 2).

It can be established the following:

Claim: For all $\alpha \in On$ there is a unique function $f_\alpha : S^{V_\alpha} \to S'^{V_\alpha}$ that is a partial SUR-algebra morphism (whenever $\alpha \neq \Phi$) and such that $\mathbf{b}'(f_\alpha(u)) = \mathbf{b}(u)$, for all $u \in S^{V_\alpha}$.
It is proof is established by induction in $\alpha \in On$, by considering separately the cases α limit and α successor: in the latter case, we must to adapt the corresponding part of the proof of Theorem 99, using the condition (bSA3) that coincides with (rSA3), but everything works.

By the Claim above, the family of functions $\{f_\alpha : S^{V_\alpha} \to S'^{V_\alpha}; \alpha \in On\}$, is compatible (i.e., $\gamma \leq \alpha \Rightarrow f_\gamma = (f_\alpha)_{\restriction}$), thus $(f : S \to S') = (\bigcup_{\alpha \in On} f_\alpha : \bigcup_{\alpha \in Z} S^\alpha \to \bigcup_{\alpha \in Z} S'^\alpha)$: it is the unique possibility, and it works (i.e., $f : S \to S'$ is a SUR-algebra morphism such that $\mathbf{b}' \circ f = \mathbf{b}$).

□

We finish this Section with some general commentaries:

Proposition 102. *Let $(\mathcal{S}, \mathbf{b}, \mathcal{C})$ be an anchored SUR-algebra satisfying $(b5)^{strong}$ and such that $\mathcal{C}$ is a rooted well-founded class underlying to a standard ZF-algebra. Then for each $(A, B) \in C_s(S)$, $\mathbf{b}(t(A, B)) = min_{\sqsubseteq}\{c \in C : \forall a \in A, \forall b \in B, \mathbf{b}(a), \mathbf{b}(b) \prec c\} = \bigvee_{z \in A \cup B} s(\mathbf{b}(z))$.*

Proof. Recall from Subsection 1.2.2, Chapter 1, that in a standard ZF-algebra $\mathcal{C}$ it holds:
- $d \prec c$ iff $s(d) \leq c$;
- $d \sqsubseteq c$ iff $d \leq c$.

Let $x := \mathbf{b}(t(A, B)) = min_{\sqsubseteq}\{c \in C : \forall a \in A, \forall b \in B, \mathbf{b}(a), \mathbf{b}(b) \prec c\}$ (by $(b5)^{strong}$) and $x' := \bigvee_{z \in A \cup B} s(\mathbf{b}(z))$.
Consider any $z \in A \cup B$, then $\mathbf{b}(z) \prec s(\mathbf{b}(z)) \leq x'$, thus $\mathbf{b}(z) \prec x'$, i.e. $x \leq x'$. On the other hand, for any $z \in A \cup B$, $\mathbf{b}(z) \prec x$, then $s(\mathbf{b}(z)) \leq x$, thus $x' \leq x$. Summing up, $x = x'$. □

Remark 103.

(i) There are many possible notions of morphism between anchored SUR-algebras over different rooted well-founded classes, $(\mathcal{S}, \mathbf{b}, \mathcal{C}) \to (\mathcal{S}', \mathbf{b}', \mathcal{C}')$, that determine categories. The simplest one is just given just by "commutative squares", i.e. pairs (f, v) where $f : \mathcal{S} \to \mathcal{S}'$ is a SUR-algebra morphism, $v : \mathcal{C} \to \mathcal{C}'$ is a morphism of rooted well founded classes (i.e., it preserves the root and the binary relation) and $v \circ \mathbf{b} = \mathbf{b}' \circ f$. Note that if $x \prec_{\mathbf{b}} y$ in S, then $f(x) \prec_{\mathbf{b}'} f(y)$ in S'. With the obvious notions of identity and composition, these data provides a category: $ancSUR{-}alg$.

Some examples of arrows: $(can, id_{On}) : (SA, r, On) \to (ST, r', On)$ and $(id_{SA}, \rho) : (SA, \mathbf{b}, V) \to (SA, r, On)$.

(ii) If $f : \mathcal{S} \to \mathcal{S}'$ is a SUR-algebra morphism with small fibers and $\mathbf{b}' : \mathcal{S}' \to \mathcal{C}'$ is an anchor in $\mathcal{S}'$, then consider $\mathcal{C}$ the substructure of $(C', \prec', \Phi')$ where $C = \mathbf{b}'[f[S]]$. Note that $\mathbf{b}'(f(*)) = \mathbf{b}'(*') = \Phi'$. Suppose that the inclusion $i : C \hookrightarrow C'$ reflects minimal elements of every non-empty subclass $K \subseteq C$. Then $\mathcal{C}$ is a rooted well-founded class, $i : \mathcal{C} \hookrightarrow \mathcal{C}'$ is a morphism of rooted well-founded class, $\mathbf{b} := \mathbf{b}' \circ f : \mathcal{S} \to \mathcal{C}$ is an anchor in $\mathcal{S}$ and $(f, i) : (\mathcal{S}, \mathbf{b}, \mathcal{C}) \to (\mathcal{S}', \mathbf{b}', \mathcal{C}')$ is a morphism of anchored SUR-algebras.

As an example consider: $(can, id_{On}) : (SA, r, On) \to (ST, r', On)$.

(iii) If $v : \mathcal{C} \to \mathcal{C}'$ is a surjective map with small fibers such that $v(\Phi) = \Phi'$ and $c \prec d$ in $C \Leftrightarrow v(c) \prec' v(d)$ in C' and $\mathbf{b} : \mathcal{S} \to \mathcal{C}$ is an anchor in $\mathcal{S}$, then $\mathbf{b}' := v \circ \mathbf{b} : \mathcal{S} \to \mathcal{C}'$ is an anchor in $\mathcal{S}$ and $(id_S, v) : (\mathcal{S}, \mathbf{b}, \mathcal{C}) \to (\mathcal{S}, \mathbf{b}', \mathcal{C}')$ is a morphism of anchored SUR-algebras.

(iv) Another notion of morphism –that is related to item (iii) above– is given by pairs (f, v) where:
- $f : \mathcal{S} \to \mathcal{S}'$ is a SUR-algebra morphism;
- $v : \mathcal{C} \to \mathcal{C}'$ is such that: $v(\Phi) = \Phi'$, $c \prec d$ in $C \Leftrightarrow v(c) \prec' v(d)$ in C' and v is surjective;
- $\forall s \in S$, $\mathbf{b}'(f(s)) \sqsubseteq' v(\mathbf{b}(s))$ (respect. and/or $v(\mathbf{b}(s)) \sqsubseteq' \mathbf{b}'(f(s))$).

As an example consider: $(\rho^*, id_{On}) : (SA, r, On) \to (No, b, On)$, see Proposition 105.(i) in the next Section.

(v) Another interesting possibilities to define morphisms (f, v) is when:
- we just consider morphisms $v : \mathcal{C} \to \mathcal{C}'$ between rooted well founded classes that are underlying to standard ZF-algebras: example, $i : On \hookrightarrow V$;

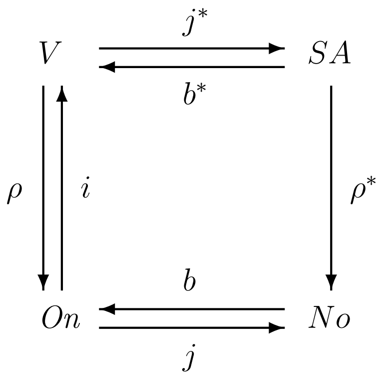
or
- we just consider morphisms $v : \mathcal{C} \to \mathcal{C}'$ of rooted well-founded classes that are underlying to ZF-algebra morphisms of standard ZF-algebras: example $\rho : V \to On$.

Concerning the second possibility: the anchor $b^* : SA \to V$ is the "best anchor on SA that satisfies $(b5)^{strong}$", since there is a unique ZF-algebra morphism $u : V \to C$, where C is a standard ZF-algebra and given $\mathbf{b} : SA \to C$ an anchor that satisfies $(b5)^{strong}$ then, by Proposition 102, $u \circ b^* = \mathbf{b}$.

□

3.2 The main diagram

In this short Section we register the relations between, in one hand the SUR-algebras SA, No and at the other hand, the ZF-algebras V, On: these will be useful in the next Chapter. The following diagram of classes and functions summarizes these relations:



Where:

• The arrow $\rho : V \to On$ is the universal ZF-algebra morphism: the rank function (see Proposition 12).

• The arrow $i : On \hookrightarrow V$ is the inclusion function.

• The arrow $b : No \to On$ was described in the Subsection 1.3.2, Chapter 1.

• The arrow $j : On \to No$ was described in the Subsection 1.3.3, Chapter 1.

• $\rho^* : SA \to No$ is universal SUR-algebra arrow (see Section 2.4, Chapter 2).

• $s : On \to SA$ was defined by $\in$-recursion in 92.

• $r : SA \to V$ was defined in 92.

• $j^* : V \to SA$ was defined by $\in$-recursion in 94.(i).

• $b^* : SA \to V$ was defined by $\prec_r$-recursion in 94.(ii).

Remark 104. Concerning the maps in the diagram above:

• The top $\to$ down arrows $\rho : V \to On$ and $\rho^* : SA \to No$ are "rank" arrows in the category of ZF-algebras and of SUR-algebras.

• The pairs of horizontal arrows $On \underset{b}{\overset{j}{\rightleftarrows}} No$ and $V \underset{b^*}{\overset{j^*}{\rightleftarrows}} SA$ are pairs of heteromorphisms –see [Ell07]– i.e. they are pairs of "chimera"-arrows (they have head in a category and tail in another category): the right $\to$ left component of the pairs are anchors structures from the SUR-algebras No, SA to (standard) ZF-algebras On, V; the left $\to$ right component of the pairs are "inclusions" of the ZF-algebras On, V into the SUR-algebras No, SA.

• Some extra object, ST and extra arrows (between ST and SA, No, On) can be added to this diagram, but we will not make use of these extra information.

$\square$

Proposition 105. *We have the following relations between the maps in the diagram above:*

(a) $\rho \circ i = id_{On}$
(b) $b \circ j = id_{No}$
(c) $r \circ s = id_{On}$
(d) $b^* \circ j^* = id_V$
(e) $s = j^* \circ i$
(f) $r = \rho \circ b^*$
(g) $j = \rho^* \circ s$
(h) $\rho^* \circ j^* = j \circ \rho$
(i) $b \circ \rho^* \leq r$

Proof. Item (a) is well known and was already mentioned in the Remark 14.(ii), Chapter 1. Item (b) was established in the Subsection 1.3.3, Chapter 1. Item (c) was established in the Proposition 92.(i). Item (d) was established in the Proposition 95.(i). Items (e) and (f) were established in the Proposition 97.

(g) Let us recall the definitions: $j : On \to No$ is defined by $\in$-recursion on ordinal numbers $\alpha \in On$, $j(\alpha) = \{j[\alpha]|\emptyset\} \underset{notation}{=} t'(j[\alpha], \emptyset)$ and $j^* : V \to SA$ is defined by $\in$-recursion on $x \in V$, $j^*(x) = \langle j^*[x], \emptyset\rangle \underset{notation}{=} t(j[x], \emptyset)$.

By items (e) and (f), we must to establish the equality $j = \rho^* \circ j^* \circ i$. This follows from $\in$-induction on ordinal numbers: let $\alpha \in On$, then $j^*(i(\alpha)) = \langle\{j^*(x) : x \in V, x \in i(\alpha)\}, \emptyset\rangle = \langle\{j^*(i(\beta)) : \beta \in On, \beta \in \alpha\}, \emptyset\rangle = t(\{j^*(i(\beta)) : \beta \in On, \beta \in \alpha\}, \emptyset)$, since every member of an ordinal number is an ordinal and i preserves and reflects the binary relations $=$ and $\in$. Since ρ^* is a SUR-algebra morphism:
$\rho^*(j^*(i(\alpha))) = \rho^*(t(\{j^*(i(\beta)) : \beta \in On, \beta \in \alpha\}, \emptyset)) = t'(\rho^*[j^*[i[\{\beta \in On : \beta \in \alpha\}]]], \emptyset) \underset{IH}{=} t'(j[\{\beta \in On : \beta \in \alpha\}], \emptyset) = t'(j[\alpha], \emptyset) = j(\alpha)$.

(h) We will establish the equality $\rho^* \circ j^*(x) = j \circ \rho(x)$ by $\in$-induction on $x \in V$. Unraveling both sides of the equation we obtain:
$\rho^* \circ j^*(x) = \rho^*(t(j^*[x], \emptyset)) = t'(\rho^*[j^*[x]], \rho^*[\emptyset]) = t'(\rho^*[j^*[x]], \emptyset) \underset{IH}{=} t'(j[\rho[x]], \emptyset) = t'(\{j(\rho(y)) : y \in x\}, \emptyset)$.
$j \circ \rho(x) = t'(j[\rho(x)], \emptyset) = t'(j[\bigcup_{y\in x} \rho(y)^+], \emptyset) = t'(\{j(z) : \exists y \in x, z \in \rho(y)^+\}, \emptyset)$.
Note that $\{j(\rho(y)) : y \in x\} \subseteq \{j(z) : \exists y \in x, z \in \rho(y)^+\}$. On the other hand, if $z \in \rho(y)^+$, then $j(z) = j(\rho(y))$ or $j(z) \in j[\rho(y)]$ (and $j(z) < t'(j[\rho(y)], \emptyset) = j(\rho(y))$). Summing up, the sets of surreal numbers $\{j(\rho(y)) : y \in x\}$ and $\{j(z) : \exists y \in x, z \in \rho(y)^+\}$ are mutually cofinal, and by Fact 27.(a), $t'(\{j(\rho(y)) : y \in x\}, \emptyset) = t'(\{j(z) : \exists y \in x, z \in \rho(y)^+\}, \emptyset)$, completing the proof.

(i) We will prove by $\prec_r$ induction on $x \in SA$. Write $x = t(L_x, R_x)$. Then $r(x) = \bigcup\{r(z)^+ : z \in L_x \cup R_x\}$.
• If $\{r(z)^+ : z \in L_x \cup R_x\}$ does not has maximum. Then $\forall w \in L_x \cup R_x$, $r(x) > r^(w)^+ > r(w) \geq b \circ \rho^*(w)$, by the induction hypothesis. Thus $b \circ \rho^*(x) = b \circ \rho^*(t(L_x, R_x)) = b(t(\rho^*[L_x], \rho^*[R_x])) \leq r(x)$, by the property (b5).
• If $\{r(z)^+ : z \in L_x \cup R_x\}$ has maximum $r(z)^+$. Then $\forall w \in L_x \cup R_x$, $r(x) = r(z)^+ \geq$

$r^(w)^+ > r(w) \geq b\circ\rho^*(w)$, by the induction hypothesis. Thus $b\circ\rho^*(x) = b\circ\rho^*(t(L_x, R_x)) = b(t(\rho^*[L_x], \rho^*[R_x])) \leq r(x)$ by property $(b5)$.

□

Remark 106.

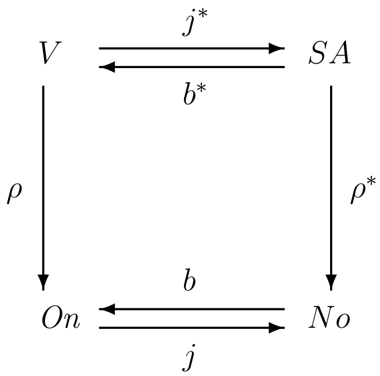
1. The free SUR-algebra SA is an expansion of the free ZF-algebra V, i.e. the class of all sets, via the map $j^* : V \to SA$, in an analogous fashion that the SUR-algebra No is an expansion of the ZF-algebra On.

2. The SUR-algebra SA is ranked on the linear SUR-algebra No of all surreal numbers in a analogous fashion that the ZF-algebra V is ranked on the well-ordered ZF-algebra On of all ordinal numbers, expanding the traditional set-theoretical link $V \overset{\rho}{\to} On$ to this new setting $SA \overset{\rho^*}{\to} No$.

3. We saw above that the pairs of horizontal arrows $On \underset{b}{\overset{j}{\rightleftarrows}} No$ and $V \underset{b^*}{\overset{j^*}{\rightleftarrows}} SA$ are pairs of heteromorphisms, connecting ZF-algebras and SUR-algebras.

4. As we already have mentioned in Remark 14.(ii), the inclusion map $i : On \hookrightarrow V$ is a section of the (unique) ZF-algebra morphism $\rho : V \to On$ (i.e. $\rho \circ i = id_{On}$), that preserves arbitrary suprema (= reunions), preserves and reflects the binary relations = and ε (= $\in$) and such that $\forall x \in V, \forall\beta \in On$, $x \in i(\beta)$ iff $x = i(\alpha)$ for some and unique $\alpha \in On$. But $i : On \hookrightarrow V$ does not preserves successors. In fact, as a consequence of the universal property of V, there is no ZF-algebra morphisms $h : On \to V$.

5. Recall that, by Corollary 82, Chapter 2 there is no SUR-algebra morphism $h : No \to SA$, but we may wonder if there is an arrow $i^* : No \to SA$ that is an "elementary" section of the (unique) SUR-algebra morphism $\rho^* : SA \to No$, i.e., $\rho^* \circ i^* = id_{No}$ and i^* preserving and reflecting every structure in sight $(=, <, *, -)$ except the t-maps. In fact, is not even clear if $\rho^* : SA \to No$ is surjective or not. The way that SA (respect. ST) lie above No, seems to be a (non-trivial) model-theoretic question that we intend to address in a future work, it seems related to some parts of the work [Ham13].



□

3.3 The Hereditary Positive members of an equipped SUR-algebra

Given an anchored SUR-algebra $(\mathcal{S}, \mathbf{b}, \mathcal{C})$ is natural to consider subclasses $P \subseteq S$ such that the restrictions of the acyclic relations $<$ and $\prec_{\mathbf{b}}$ coincide[11]. Motivated by the well-known natural "inclusion" the class On of all ordinal numbers into the class No of all surreal numbers and by our proofs that the (surjective) anchor mappings $r : SA \to On$, $b^* : SA \to V$ and $r : ST \to On$ have, in fact, "well-behaved" section mappings, we start in this Section the development of a generalization: "The Hereditary Positive members of an equipped SUR-algebra".

Two questions arise immediately:
• On the terminology: what is the meaning of being "positive" in a SUR-algebra? what are the "hereditary positives" in a SUR-algebra?
• Why (and how) the "hereditary positives" of $(\mathcal{S}, \mathbf{b}, \mathcal{C})$ are related to our original motivation: subclasses of $P \subseteq S$ such that $<_{\restriction P} = (\prec_{\mathbf{b}})_{\restriction P}$?

Obviously, in the strict *totally ordered* abelian group No there is a natural notion of positive members. On other hand, since the relation $<$ is not total in SA (respect. ST) and not every member of SA (respect. ST) are $<$-related[12] to $* = 0 = \langle \emptyset, \emptyset \rangle$, the notion of "being positive" requires a fresh conception: $u \in SA$ (respect. $u \in ST$) is positive –notation: $u \in Pos(SA)$– when there is a simple way to ensure that $u < 0$ do not occur: we just set $R_u = \emptyset$.

Concerning SA, we register the following results (that can be easily derived from the definitions and Proposition 97):

[11] In particular, $<_{\restriction P} = (\prec_{\mathbf{b}})_{\restriction P}$ is an well-founded relation in $P \subseteq S$, whenever the inclusion $P \hookrightarrow S$ reflects minimal elements.

[12] Note that $\langle A, B \rangle$ is $<$-related with 0 iff $-\langle A, B \rangle$ is $<$-related with 0.

Fact 107. *Let $u, v \in SA$. Then:*
(a) If $u \prec_{b^*} v$, then $u \prec_r v$.
(b) If $u < v \in SA$, then $u \prec_{b^*} v \Leftrightarrow u \prec_r v$.
(c) If $R_u = R_v = \emptyset$, then $u < v \implies (u \prec_{b^*} v \Leftrightarrow u \prec_r v)$.

□

Having some idea of what should be "the positives" in the main SUR-algebras of this work, we turn our attention to the "hereditary positives".

108. Concerning No:
• Every ordinal $j(\alpha)$ in No is positive, i.e. $j(\alpha) \geq 0$, and is determined by the previous ordinals in No ($j(\alpha) = \{\{j(\beta) : \beta \in \alpha\}|\emptyset\}$.
• Since $b \circ j = id_{No}$ and $\beta < \alpha \in On$ iff $j(\beta) < j(\alpha) \in No$, then $<_{\restriction j[On]} = (\prec_b)_{\restriction j[On]}$.

□

Note that in SA, since $b^* \circ j^* = id_V$ and $y \in x \in V$ iff $j^*(y) < j^*(x) \in SA$, then $<_{\restriction j^*[V]} = (\prec_{b^*})_{\restriction j^*[V]}$. The following result, concerning SA, is analogous to 108 above:

Proposition 109. *For each $u \in SA$, are equivalent:*
(a) $u \in j^*[V] \subseteq SA$.
(b) For each finite sequence $z_0, \cdots, z_n \in SA$ such that $z_0 = u$, $z_{i+1} < z_i$, $z_{i+1} \prec_{b^*} z_i$, $i < n$, then $z_j \in Pos(SA), j \leq n$.
(c) For each finite sequence $z_0, \cdots, z_n \in SA$ such that $z_0 = u$, $z_{i+1} < z_i$, $z_{i+1} \prec_r z_i$, $i < n$, then $z_j \in Pos(SA), j \leq n$.

Proof. $(a) \Leftrightarrow (b)$ follows by $\prec_{b^*}$-induction and $(b) \Leftrightarrow (c)$ follows from Fact 107 above.

□

Having experimented in particular cases the concepts of "positive" (Pos) and "hereditary positive" (HP), we pass now to a more general situation:

Definition 110.
(a) Let $\mathcal{S}$ be a SUR-algebra, define $Pos(\mathcal{S}) := \{u \in S : u = t(A, \emptyset)$, for some subset $A \subseteq S\}$.
(b) Let $(\mathcal{S}, \mathbf{b}, \mathcal{C})$ be anchored SUR-algebra, define $HP(\mathcal{S}, \mathbf{b}, \mathcal{C})$ as the range of the function $HP : C \to P_s(S)$ defined by $\prec$-recursion:
$HP(c) := \{u \in S : \mathbf{b}(u) = c, u = t(\bigcup_{d \in c^{\prec}} HP(d), \emptyset)\}$, $c \in C$.

□

Note that, for each subset $A \subseteq No$: (i) if there is $a \in A$ such that $a \geq 0$, then $\{A|\emptyset\} > 0$; (ii) otherwise, $a < 0$ for all $a \in A$ and $\{A|\emptyset\} = 0$. Thus $Pos(No) \subseteq \{u \in No : u \geq 0\}$.

111. Let $(\mathcal{S}, \mathbf{b}, \mathcal{C})$ be anchored SUR-algebra.
(a) By induction, $HP(c) \subseteq Pos(S)$ and it is a singleton whenever it is non-empty, and its unique member belongs to $\mathbf{b}^{-1}[\{c\}]$, in particular $HP(c)$ is a *subset* of S and the codomain of the function HP is adequate.
(b) If $d \prec c \in C$, then $\forall v \in HP(c) \forall w \in HP(d)$ we have $w < v$. Indeed: if $v \in HP(c)$ and $w \in HP(d)$, since $d \prec c$, we have $w < t(\bigcup_{d \in c^{\prec}} HP(d), \emptyset) = v$.
(c) *Suppose* that, for each $c \in C$, $HP(c)$ is a singleton, and denote $p(c)$ the unique member of $HP(c)$. Then we obtain a function $p : C \to S$ such that $\mathbf{b} \circ p = id_C$.
(d) Note that $* = t(\emptyset, \emptyset) \in \{u \in S : \mathbf{b}(u) = \Phi, t(\bigcup_{d \in \Phi^{\prec}} HP(d), \emptyset)\}$, thus $p(\Phi) = * \in S$.

□

Proposition 112. *Let $(\mathcal{S}, \mathbf{b}, \mathcal{C})$ be anchored SUR-algebra.*

(a) Suppose that, $HP(c) = \{p(c)\}$, for each $c \in C$. Let $c, d \in C$ and consider:
(p1) $p(d) \prec_{\mathbf{b}} p(c)$ in S
(p2) $d \prec c$ in C
(p3) $p(d) < p(c)$ in S

Then $(p1) \Leftrightarrow (p2) \Rightarrow (p3)$

(b) Suppose that $(\mathcal{S}, \mathbf{b}, \mathcal{C})$ be an anchored SUR-algebra satisfying the additional condition below:
(L) *$(\mathcal{S}, \mathbf{b}, \mathcal{C})$ be an anchored SUR-algebra such that $\mathcal{C} = (C, \prec, \Phi)$ is a rooted well-founded class where $\prec$ is a strict linear order.*
Then the items $(p1), (p2), (p3)$ are equivalent.

(c) Suppose that $(\mathcal{S}, \mathbf{b}, \mathcal{C})$ be an anchored SUR-algebra satisfying the additional condition below:
(S5) *$(\mathcal{S}, \mathbf{b}, \mathcal{C})$ be an anchored SUR-algebra satisfying $(b5)^{strong}$ and such that $\mathcal{C} = (C, \prec, \Phi)$ is a rooted well-founded class underlying to a standard ZF-algebra.*
Then $HP(c) = \{p(c)\}$, for each $c \in C$.

Proof.

(a)
$(p1) \Leftrightarrow (p2)$: By definition of $HP(d)$, $\mathbf{b}(p(d)) = d, \forall d \in C$. Thus, the equivalence follows from the definition of $\prec_{\mathbf{b}}$.
$(p2) \Rightarrow (p3)$: If $d \prec c$, then $p(d) \in \bigcup_{d \in c^{\prec}}$ thus $p(d) < t(\bigcup_{d \in c^{\prec}} HP(d), \emptyset) = p(c)$.

(b) Note that when $(C, \prec)$ is a strictly linearly ordered class then $(p3) \Rightarrow (p2)$: if $c \not\prec d$, then $c = d$ or $d \prec c$ thus $p(c) = p(d)$ or $p(d) < p(c)$ (because $(p2) \Rightarrow (p3)$) and, in any case, $p(c) \not< p(d)$, since $<$ is an acyclic relation in S.

(c) Let $(\mathcal{S}, \mathbf{b}, \mathcal{C})$ be an anchored SUR-algebra satisfying $(b5)^{strong}$ and such that $\mathcal{C}$ is a rooted well-founded class underlying to a standard ZF-algebra.

We will need the following characterization (see Proposition 102): let $(A, B) \in C_s(S)$, $\mathbf{b}(t(A,B)) = min_{\sqsubseteq}\{c \in C : \forall a \in A, \forall b \in B, \mathbf{b}(a), \mathbf{b}(b) \prec c\} = \bigvee_{z \in A \cup B} s(\mathbf{b}(z))$.

Now we will verify that $c = \mathbf{b}(t(\bigcup_{d \in c^{\prec_{\mathbf{b}}}} HP(d), \emptyset)), \forall c \in C$. Let $c \in C$ and assume, by induction, that $HP(d) = \{p(d)\}$, for all $d \prec c$. Then $d \prec c \Rightarrow d = \mathbf{b}(p(d))$ and $\mathbf{b}(t(\bigcup_{d \in c^{\prec}} HP(d), \emptyset)) \underset{IH}{=} \mathbf{b}(t(\{p(d) : d \prec c\}, \emptyset)) \underset{Claim}{=} \bigvee_{z \in p[c^{\prec}]} s(\mathbf{b}(z)) = \bigvee_{d \in c^{\prec}} s(\mathbf{b}(p(d))) \underset{IH}{=} \bigvee_{d \in c^{\prec}} s(d) = c$, where the last equality follows since $\mathcal{C}$ is obtained from a standard ZF-algebra.

□

Alternatively, we can:
• Define first a map $j : C \to S$, by $\prec$-recursion: $j(c) = t(j[c^{\prec}], \emptyset)$, $c \in C$;
• After that, consider the subclass of S given by $range(j) = j[C] \subseteq S$.

Both approaches, i.e. beginning from $HP : C \to P_s(S)$ or from $j : C \to S$, are related:

Proposition 113. *The following items are equivalent:*
(a) $\mathbf{b} \circ j = id_C$;
(b) $HP(c) \neq \emptyset$, $\forall c \in C$;
(c) $HP(c) = \{p(c)\}$, $\forall c \in C$;
(d) $j(c) = p(c)$, $\forall c \in C$.

Proof. We already have observed that $(b) \Leftrightarrow (c)$.
$(d) \Rightarrow (c)$: Since $HP(c)$ has at most one member.
$(c) \Rightarrow (d)$: Since $\{p(c)\} = HP(c) = \{u \in S : \mathbf{b}(u) = c, u = t(\bigcup_{d \in c^{\prec}} HP(d), \emptyset)\} = \{t(\{p(d) : d \in c^{\prec}\}, \emptyset)\} \cap \mathbf{b}^{-1}[\{c\}]$, we have $p(c) = t(\{p(d) : d \in c^{\prec}\}, \emptyset) \in \mathbf{b}^{-1}[\{c\}]$. Then, by $\prec$-induction, we obtain $j(c) = p(c)$, $\forall c \in C$.
$(d) \Rightarrow (a)$: As above, $j(c) = p(c) \in \mathbf{b}^{-1}[\{c\}]$, thus $\mathbf{b}(j(c)) = c, \forall c \in C$.
$(a) \Rightarrow (d)$: If $\mathbf{b}(j(c)) = c$, then $j(c) \in \mathbf{b}^{-1}[\{c\}]$, $\forall c \in C$. By induction, suppose that $j(d) = p(d)$, $\forall d \in c^{\prec} \subseteq C$. Then $HP(d) = \{j(d)\}$, $\forall d \in c^{\prec} \subseteq C$ and $HP(c) = \{u \in S : \mathbf{b}(u) = c, u = t(\bigcup_{d \in c^{\prec}} HP(d), \emptyset)\} = \{t(\{p(d) : d \in c^{\prec}\}, \emptyset)\} \cap \mathbf{b}^{-1}[\{c\}] \underset{IH}{=} \{t(\{j(d) : d \in c^{\prec}\}, \emptyset)\} \cap \mathbf{b}^{-1}[\{c\}] = \{j(c)\}$, since $j(c) \in \mathbf{b}^{-1}[\{c\}]$. Thus $p(c) = j(c)$ and the result follows by induction.

□

Combining the results above we obtain the:

Corollary 114. *For anchored SUR-algebras* $(\mathcal{S}, \mathbf{b}, \mathcal{C})$ *such that* $\mathbf{b} \circ j = id_C$ *and* ($j(d) < j(c)$ *in* $S \Rightarrow d \prec c$ *in* C)*:*

(a) The items below are equivalent:
• $j(d) \prec_{\mathbf{b}} j(c)$ *in* S
• $d \prec c$ *in* C
• $j(d) < j(c)$ *in* S

(b) For each $x \in S$ are equivalent:
• *x is hereditary positive (i.e. $x \in HP((\mathcal{S}, \mathbf{b}, \mathcal{C}))$)*
• *$x \in range(j)$*

(c) For the subclass $HP = HP((\mathcal{S}, \mathbf{b}, \mathcal{C})) \subseteq S$, we have $(C, \prec, \Phi) \underset{j}{\cong} (HP, (\prec_{\mathbf{b}})_{\restriction}, *) = (HP, <_{\restriction}, *)$.

□

Remark 115. The special cases: Note that the anchored SUR-algebras (No, b, On), $(ST, r, On), (SA, r, On), (SA, b^*, V)$ satisfy both the additional conditions in the Corollary above:
• (No, b, On) satisfies condition (L) in Proposition 112.(b), thus ($j(d) < j(c)$ in $S \;\Rightarrow\; d \prec c$ in C). Moreover, $b \circ j = Id_{On}$ (see Section 3, Chapter 1).
• (SA, b^*, V) satisfies condition (S5) in Proposition 112.(c), $b^* \circ j^* = Id_V$. Moreover, ($j^*(d) < j^*(c)$ in $SA \;\Rightarrow\; d \prec c$ in V).
• (ST, r, On) and (SA, r, On) satisfy both the conditions (L) and (S5) in Proposition 112.(b),(c), thus they satisfy both the additional conditions in the Corollary above.

This ensures that our main anchored SUR-algebras have nice notions of hereditary positive subclass: see items (a), (b), (c) in the Corollary above.

□

116. The equipped SUR-algebra induced by a anchor:

Recall that, from Definition 86, each anchor $\mathbf{b} : \mathcal{S} \to \mathcal{C}$) induces five binary relations in S: $\prec_{\mathbf{b}}, \equiv_{\mathbf{b}}, \sim_{\mathbf{b}}, \sqsubseteq_{\mathbf{b}}, \sqsubseteq'_{\mathbf{b}}$. According Proposition 87, these relations satisfy many interesting properties. The arrangement is particularly simple whenever $\prec$ is an extensional relation.

We can consider, instead of the anchor $\mathbf{b}$ on $\mathcal{S}$, equip $\mathcal{S}$ with a derived pair of binary relations $(\mathcal{S}, \prec_{\mathbf{b}}, \equiv_{\mathbf{b}})$. Note that the anchor $\mathbf{b} : S \to C$ can be essentially recovered from these data, since $S/\equiv_{\mathbf{b}} \cong C$. When $\prec$ is extensional, then the equivalence relation $\equiv_{\mathbf{b}}$ is determined by the well-founded relation $\prec_{\mathbf{b}}$ and we can consider just the structure $(\mathcal{S}, \prec_{\mathbf{b}})$.

It can be defined by $\prec_{\mathbf{b}}$-recursion a function $HP^* : S \to P_s(S)$:
$HP^*(u) := \{u' \in S : u' \equiv_{\mathbf{b}} u, u' = t(\bigcup_{w \in u^{\prec_{\mathbf{b}}}} HP^*(w), \emptyset)\}$.

Note that:
• $u_0 \equiv_{\mathbf{b}} u_1 \Rightarrow u_0^{\prec_{\mathbf{b}}} = u_1^{\prec_{\mathbf{b}}}$ and $HP^*(u_0) = HP^*(u_1)$;
• $HP^*(u) = HP(b(u))$, $\forall u \in S$ (by $\prec_{\mathbf{b}}$-induction);
• $HP^*(u)$ has at most one member.

In the process of associated a subclass of "hereditary positive members" for some anchored SUR-algebras $(\mathcal{S}, \mathbf{b})$ we consider naturally a structure given by class endowed by two binary and acyclic relations $(S, <, \prec_{\mathbf{b}})$. This seems related to the notions of

s-hierarchical structures considered by P. Ehrlich in [Ehr01].

□

To complete the "analysis $\rightleftarrows$ synthesis" process, we consider:

117. The anchor induced by an abstract equipped SUR-algebra:

We can consider equipped SUR-algebras $(\mathcal{S}, \lhd, \approx)$ where $\approx$ is an equivalence relation on S and $\lhd$ is an well-founded relation on S satisfying some additional conditions (see Proposition 87):
(1) $\lhd$ is compatible with $\approx$: i.e. $u \approx u', v \approx v', u \lhd v \Rightarrow u' \lhd v'$;
(2) Every $\approx$-equivalence class is a subset of S (in particular, there exists the quotient (proper) class $S/\approx := \{[u]_{\approx} : u \in S\}$;
(3) $root(S, \lhd) = [*] = \{u \in S : u \approx *\}$;
(4) $-u \approx u, \forall u \in S$. (5) For each $(A, B) \in C_s(S) \subseteq S$ and each $s \in S$ such that $\forall z \in A \cup B, z \lhd s$, then $t(A, B) \sqsubseteq s$, where $\sqsubseteq$ is the pre-order relation on S derived from $\lhd$.

As above, we define the notion by $\lhd$-recursion: $HP^* : S \to P_s(S)$, $HP^*(u) = \{u' \in S : u' \approx u, u' = t(\bigcup_{w \in u^{\lhd}} HP^*(w), \emptyset)\}$.

Note that: $u_0 \approx u_1 \Rightarrow u_0^{\lhd} = u_1^{\lhd}$ and $HP^*(u_0) = HP^*(u_1)$

Set $C := S/\approx$. In C define, $\Phi := [*]$ and $[u] \prec [v]$ iff $u \lhd v$

Claim 1: $(C, \prec, \Phi)$ is a rooted well founded class.
Indeed:
$[v]^{\prec} = \bigcup_{u \in v^{\lhd}} [u]$ is a subset of C and, if $A \subseteq C$ is a non-empty subclass, then $\check{A} := \bigcup_{[u] \in A} [u] \subseteq S$ is a non-empty class. Selecting any $v \in \check{A}$ a $\lhd$-minimal member of $\check{A}$, then $[v] \in A$ is a $\prec$-minimal member of A. By the condition (3) above, Φ is the only root of $(C, \prec)$.

Claim 2: $q : S \to C$, $u \mapsto [u]$ is an anchor in $\mathcal{S}$.
Indeed: (b1) obviously holds and (b2)–(b5) are obtained from (2)–(5) above.

□

118. Abstract Hereditary Positives:

We saw in Corollary 114.(c), that for "well-behaved" anchored SUR-algebras $(\mathcal{S}, \mathbf{b}, \mathcal{C})$, the subclass $HP = HP((\mathcal{S}, \mathbf{b}, \mathcal{C})) \subseteq S$, provides a perfect encoding of $\mathcal{C}$, since $(C, \prec, \Phi) \cong (HP, (\prec_{\mathbf{b}})_{\restriction}, *) = (HP, <_{\restriction}, *)$.

Alternatively, from 116 and 117 above, for an "well-behaved" equipped SUR-algebra $(\mathcal{S}, \lhd, \approx)$, we define the notion of abstract hereditary positive subclass by $\lhd$-recursion: $HP^* : S \to P_s(S)$, $HP^*(u) = \{u' \in S : u' \approx u, u' = t(\bigcup_{w \in u^{\lhd}} HP^*(w), \emptyset)\}$. Note that: $u_0 \approx u_1 \Rightarrow u_0^{\lhd} = u_1^{\lhd}$ and $HP^*(u_0) = HP^*(u_1)$.

Distinct well-behaved structures $(\lhd, \approx)$ on $\mathcal{S}$ induces distinct subclasses $HP^*(\mathcal{S}, \lhd, \approx) \subseteq S$, however $(HP^*, \lhd_{\restriction}, *) = (HP^*, <_{\restriction}, *)$, and the possible difference occurs only on the particular subclass, since they share the same root $*$ and the same binary relation ¡. The different maps $HP^* : S \to P_s(S)$ are just different retractions of the distinct inclusions $HP^*(\mathcal{S}, \lhd, \approx) \hookrightarrow S$.

We can also consider just a (nice) well-founded relation $\lhd$ on S and:
• define $HP'(\mathcal{S}, \lhd) = \bigcup\{K \in P_s(S) : i_K : K \hookrightarrow S$ reflects $\lhd$-minimal members and $< \cap (K \times K) = \lhd \cap (K \times K)\}$;
• denote $HP' := HP'(\mathcal{S}, \lhd)$.
Note that $* \in HP'$, $i : HP' \hookrightarrow S$ reflects $\lhd$-minimal members and $< \cap (HP' \times HP') = \lhd \cap (HP' \times HP')$ is a rooted well-founded class (proper or improper).
It is natural ask if there is a maximal (or even a largest) subclass $H \subseteq S$, such that $< \cap (H \times H)$ coincides with $\lhd \cap (H \times H)$, for some well-behaved structure $(\lhd, \approx)$ on $\mathcal{S}$. Such kind of extremal question is related to Remark 103.(v).

□

3.4 Spaces of Signs

In the previous Section we have introduced the (recursively-defined) subclass of *hereditary positive members* of an anchored SUR-algebra $\mathbf{b} : \mathcal{S} \to \mathcal{C}$, $HP(\mathcal{S}, \mathbf{b})$, that under convenient hypothesis encodes $\mathcal{C}$ (e.g. $HP(SA, b^*) \cong V, HP(No, b) \cong On$). Combining the work developed in the previous section with the work that we will present in this short Section, we establish relations (in both directions) between certain classes of equipped SUR-algebras and certain classes of equipped standard ZF-algebras, that "explains" and "expands" the relations $On \underset{b}{\overset{j}{\rightleftarrows}} No$ and $V \underset{b^*}{\overset{j^*}{\rightleftarrows}} SA$.

Motivated by the construction of No as a space of signs builded over the class On as a certain class of functions with domain $\in On$ and codomain as a set of ordered pairs with first component[13] $\in \{-, +\}$ (see Subsection 1.3.1.3, Chapter 1), we expand the notion of "space of signs". Roughly speaking, given an standard ZF-algebra Z (e.g., On, V) and an "operation on binary relations" op (e.g., "identity", "transitive closure") we can build a corresponding SUR-algebra *space of signs* $Sig(Z, op)$ (e.g., $Sig(On, id) \cong SA, Sig(On, trcl) \cong ST, Sig(V, id) \cong SA$) and every SUR-algebra space of signs is anchored on its underlying standard ZF-algebra, the domain of a sign function determining the anchor $d : Sig(Z, op) \to Z,\ f \mapsto d(f) = domain(f)$.

119. The space of signs of $SA \overset{r}{\to} On$**:**

[13]Given $\beta \in On$, $F : \beta \to \{-, +\}$ can be identified with $f : \beta \to \bigcup_{\alpha\in\beta}\{-, +\} \times P_s(\{-, +\}^\alpha)$, such that $f(\alpha) = (F(\alpha), F\restriction_\alpha)$, $\alpha \in \beta$.

We define, by $\in$-recursion in On a subset $Sig(On, id)(\beta) \subseteq \{f : \beta \to \bigcup_{\alpha\in\beta}\{-,+\} \times P_s(Sig(On, id)(\alpha))\}$ and a (small) partial SUR-algebra structure on $\bigcup_{\alpha\in\beta^+} Sig(On, id)(\alpha)$, $\beta \in On$, (or $\bigcup_{\alpha\in\beta} Sig(On, id)(\alpha)$, $\beta > 0$) in such a way that $Sig(On, id) := \bigcup_{\alpha\in On} Sig(On, id)(\alpha)$ (note that this reunion is disjoint) is endowed with a natural structure of anchored SUR-algebra, with the obvious anchor $d : Sig(On, id) \to On$, $f \mapsto d(f) = domain(f) \in On$. This construction $(Sig(On, id), d)$ will be a representation of (SA, r) since it satisfies the conditions of the characterization theorem 99. The steps of the $Sig(On, id)$ construction are (of course) very similar to the SA construction presented in the Subsection 2.2.2, Chapter 2, the only (irrelevant) difference is that the emphasis is on the "new" members of $Sig(On, id)$, $Sig(On, id)(\beta)$, instead of "made" members of SA, SA_β, $\beta \in On$. Thus we will just present sketches of the proofs.

To keep the notation simple, we will just denote $Sig(On, id)(\beta)$ by $Sig(\beta)$, in this item.

We define $Sig(\beta)$ and $<_\beta$, $\beta \in On$, by recursion, where $<_\beta$ is a binary acyclic relation in $\bigcup_{\gamma\in\beta^+} Sig(\gamma)$.

$Sig(\beta) \subseteq \{f : \beta \to \bigcup_{\alpha\in\beta}\{-,+\} \times P_s(Sig(\alpha))\}$ is the subclass (that is a subset, by induction) satisfying the following conditions:
(s1) If $\alpha \in \beta$, then $f(\alpha) \in \{-,+\} \times P_s(Sig(\alpha))$;
(s2) $(L(f), R(f)) \in C_s(Sig^{(\beta)}) \setminus C_s(Sig^{(\gamma)})$, $\forall \gamma \in \beta$, where $L(f) := \bigcup_{\alpha\in\beta}\{\pi_2 f(\alpha) : \pi_1 f(\alpha) = +\}$, $R(f) := \bigcup_{\alpha\in\beta}\{\pi_2 f(\alpha) : \pi_1 f(\alpha) = -\}$ and $C_s(Sig^{(\delta)}) := \{(A, B) : A, B \in P_s(\bigcup_{\alpha\in\delta} Sig(\alpha)) : A <^{(\delta)} B\}$, $<^{(\delta)} := \bigcup_{\alpha\in\delta} <_\alpha$, $\delta \in On$.

Note that:
• $Sig(0) = \{f : 0 \to \bigcup_{\alpha\in 0}\{-,+\} \times P_s(Sig(\alpha))\} = \{$ the unique function $f : \emptyset \to \emptyset\} = \{*\}$.
• $f : \beta \to \bigcup_{\alpha\in\beta}\{-,+\} \times P_s(Sig(\alpha)) \in Sig(\beta)$ iff $-f : \beta \to \bigcup_{\alpha\in\beta}\{-,+\} \times P_s(Sig(\alpha)) \in Sig(\beta)$, where $\{\pi_1(-f(\alpha))\} = \{+,-\} \setminus \{\pi_1(f(\alpha))\}$, $\alpha \in \beta$.
• For each $\beta \in On$ and $f \in Sig(\beta)$, define $[f]_\epsilon := \{\alpha \in \beta : \pi_1 f(\alpha) = \epsilon \in \{-,+\}\}$, then: $[f]_- \cap [f]_+ = \emptyset$ (by s1) and $L(f) \cap R(f) = \emptyset$ (otherwise $\exists \alpha \in [f]_-, \exists \alpha' \in [f]_+, \exists g \in \pi_2 f(\alpha) \cap \pi_2 f(\alpha') \subseteq Sig(\alpha) \cap Sig(\alpha') = \emptyset$).

Define $<_\beta := <^{(\beta)} \cup \{(g, f) : f \in Sig(\beta), g \in L(f)\} \cup \{(f, h) : f \in Sig(\beta), h \in R(f)\}$. We can check, by induction, that $<_\beta$ is an acyclic relation in $\bigcup_{\gamma\in\beta^+} Sig(\gamma)$.

Let $Sig := \dot{\bigcup}_{\beta\in On} Sig(\beta)$ and $< := \bigcup_{\beta\in On} <_\beta$. Define $C_s(Sig) := \{(A, B) : A, B \in P_s(\bigcup_{\alpha\in On} Sig(\alpha)) : A < B\}$, then $C_s(Sig) = \bigcup_{\beta\in On} C_s(Sig^{(\beta)})$.

We can define $- : Sig \to Sig$, $f \in Sig(\beta) \mapsto -f \in Sig(\beta)$. Define $d : Sig \to On$, $f \mapsto d(f) = domain(f)$. Since $d^{-1}[\{\beta\}] = Sig(\beta)$, d has small fibers. It is clear that $g < h$ iff $(d(g) \in d(h)$ and $g \in L(h))$ or $(d(h) \in d(g)$ and $h \in R(g))$.

Define $u_\beta : \bigcup_{\gamma\in\beta^+} Sig(\gamma) \to C_s(Sig^{(\beta)})$, $f \mapsto (L(f), R(f))$. By induction, u_β is a bijection. Denote $t_\beta = (u_\beta)^{-1} : C_s(Sig^{(\beta)}) \to \bigcup_{\gamma\in\beta^+} Sig(\gamma)$, $\beta \in On$. The gluing of this increasing and compatible family of bijections is the bijection $t : C_s(Sig) \to Sig$.

It is a routine checking that $\mathcal{S}ig = (Sig, <, -, *, t)$ is a SUR-algebra.

Let $f \in Sig(\beta)$. By (s2) β is the least ordinal $> \alpha, \forall \alpha \in [f]_- \cup [f]_+$, thus $d(f) = \beta = \bigcup\{\alpha^+ : \alpha \in \beta\} = \bigcup\{d(g)^+ : g \in L(f) \cup R(f)\}$. The map $s : On \to Sig$, $s(\beta) \in Sig(\beta)$, $\pi_1(s(\beta)(\alpha)) = +$, $\forall \alpha \in \beta$, defines a section of d, i.e. $d \circ s = id_{On}$, thus d is surjective. Then $d : Sig \to On$ is an anchor on Sig that satisfies $(b5)^{strong}$.

Summing up, the anchored SUR-algebra $(\mathcal{S}ig, d)$ satisfies the conditions (rSA1), (rSA2), (rSA3) in the characterization Theorem 99. Thus there is a unique isomorphism of anchored SUR-algebras $(SA, r) \overset{\cong}{\to} (\mathcal{S}ig, d)$.

□

120. The space of signs of $ST \overset{r'}{\to} On$:

In the same vein of 119 above, we can define, by $\in$-recursion in On a subset $Sig(On, trcl)(\beta) \subseteq \{f : \beta \to \bigcup_{\alpha\in\beta}\{-,+\} \times P_s(Sig(On, trcl)(\alpha))\}$ and a partial SUR-algebra structure on $\bigcup_{\alpha\in\beta^+} Sig(On, trcl)(\alpha)$, $\beta \in On$, in such a way that $Sig(On, trcl) := \bigcup_{\alpha\in On} Sig(On, trcl)(\alpha)$ is endowed with a natural structure of anchored SUR-algebra, with the obvious anchor $d : Sig(On, trcl) \to On$, $f \mapsto d(f) = domain(f) \in On$.

To keep the notation simple, we will just denote $Sig(On, trcl)(\beta)$ by $Sig(\beta)$, in this item.

We define $Sig(\beta)$ and $<_\beta$, $\beta \in On$, by recursion, where $<_\beta$ is a binary acyclic relation in $\bigcup_{\gamma\in\beta^+} Sig(\gamma)$.

$Sig(\beta) \subseteq \{f : \beta \to \bigcup_{\alpha\in\beta}\{-,+\} \times P_s(Sig(\alpha))\}$ is the subclass (that is a subset, by induction) satisfying the following conditions:
(s1) If $\alpha \in \beta$, then $f(\alpha) \in \{-,+\} \times P_s(Sig(\alpha))$;
(s2) $(L(f), R(f)) \in C_s(Sig^{(\beta)}) \setminus C_s(Sig^{(\gamma)})$, $\forall \gamma \in \beta$, where $L(f) := \bigcup_{\alpha\in\beta}\{\pi_2 f(\alpha) : \pi_1 f(\alpha) = +\}$, $R(f) := \bigcup_{\alpha\in\beta}\{\pi_2 f(\alpha) : \pi_1 f(\alpha) = -\}$ and $C_s(Sig^{(\beta)}) = \{(A, B) : A, B \in P_s(\bigcup_{\alpha\in\beta} Sig(\alpha)) : A <^{(\beta)} B\}$, $<^{(\beta)} = \bigcup_{\alpha\in\beta} <_\alpha$.

Define $<_\beta$ as the transitive closure of the relation $<^{(\beta)} \cup \{(g, f) : f \in Sig(\beta), g \in L(f)\} \cup \{(f, h) : f \in Sig(\beta), h \in R(f)\}$. We can check, by induction, that $<_\beta$ is an acyclic and transitive relation in $\bigcup_{\gamma\in\beta^+} Sig(\gamma)$. Then $< := \bigcup_{\beta\in On} <_\beta$ is an acyclic and transitive relation in $Sig := \dot{\bigcup}_{\beta\in On} Sig(\beta)$.

The definitions of $*, -, t, d, s$ are similiar to the Sig^{SA} case. Then $\mathcal{S}ig = (Sig, <, -, *, t)$ is a SUR-algebra such that $t : C_s(Sig) \to Sig$ is bijective and $d : Sig \to On$ is an anchor on Sig that satisfies $(b5)^{strong}$, since for each $\beta \in On$ and $f \in Sig(\beta)$, $d(f) = \beta = \bigcup\{\alpha^+ : \alpha \in \beta\} = \bigcup\{d(g)^+ : g \in L(f) \cup R(f)\}$.

Summing up, the anchored SUR-algebra $(\mathcal{S}ig, d)$ satisfies the conditions (rST1), (rST2), (rST3) in the characterization Theorem 100. Thus there is a unique isomorphism of anchored SUR-algebras $(ST, r') \overset{\cong}{\to} (\mathcal{S}ig, d)$.

□

121. The space of signs of $SA \overset{b^*}{\to} V$:

We define, by recursion on $y \in V$, a subset $Sig(V, id)(y) \subseteq \{f : y \to \bigcup_{z \in y} \{-, +\} \times P_s(Sig(V, id)(z))\}$, and we admit that we have already defined $Sig(V, id)(z), \forall z \in V_{\rho(y)}$ and $S^{V_{\rho(y)}} := \bigcup_{z \in V_{\rho(y)}} Sig(V, id)(z)$ is a (small) partial SUR-algebra whenever $\rho(y) > 0$. Since $V = \bigcup_{\alpha \in On} V_\alpha$ (V_α is a transitive set), the class $Sig(V, id) := \bigcup_{y \in V} Sig(V, id)(y)$ (note that this reunion is disjoint) is endowed with a natural structure of anchored SUR-algebra, with the obvious anchor $d : Sig(V, id) \to V$, $f \mapsto d(f) = domain(f) \in V$. This construction $(Sig(V, id), d)$ will be a representation of (SA, b^*) since it satisfies the conditions of the characterization theorem 101. The steps of the $Sig(V, id)$ construction are (of course) very similar to the SA construction presented in the Subsection 2.2.2, Chapter 2, the only (irrelevant) difference is that the emphasis is on the "new" members of $Sig(V, id)$, $Sig(V, id)(y)$, instead of "made" members of SA, $SA_{\rho(y)}$, $y \in V$.

We will just present sketches the constructions and results. To keep the notation simple, we will just denote $Sig(V, id)(y)$ by $Sig(y)$, $y \in V$, in this item.

Let $Sig(y) \subseteq \{f : y \to \bigcup_{z \in y} \{-, +\} \times P_s(Sig(z))\}$ be the subclass (that is a subset, by induction) satisfying the following conditions:
(s1) If $x \in y$, then $f(x) \in \{-, +\} \times P_s(Sig(x))$;
(s2) $(L(f), R(f)) \in C_s(Sig^{(y)}) \backslash C_s(Sig^{(z)}), \forall z \in P_s(y) \backslash \{y\}$, where $L(f) := \bigcup_{x \in y} \{\pi_2 f(x) : \pi_1 f(x) = +\}$, $R(f) := \bigcup_{x \in y} \{\pi_2 f(x) : \pi_1 f(x) = -\}$ and $C_s(Sig^{(y)}) := \{(A, B) : A, B \in P_s(\bigcup_{x \in y} Sig(x)) : A <^{(y)} B\}$, $<^{(y)} := \bigcup_{z \in y} <_z$, $y \in V$.

Note that:
• $S^{V_\emptyset} = \emptyset$.
• $Sig(\emptyset) = \{f : \emptyset \to \bigcup_{z \in V_0} \{-, +\} \times P_s(Sig(z))\} = \{$ the unique function $f : \emptyset \to \emptyset\} = \{*\}$.
• $f : y \to \bigcup_{z \in V_{\rho(y)}} \{-, +\} \times P_s(Sig(z)) \in Sig(y)$ iff $-f : y \to \bigcup_{z \in V_{\rho(y)}} \{-, +\} \times P_s(Sig(z)) \in Sig(y)$, where $\{\pi_1(-f(x))\} = \{+, -\} \setminus \{\pi_1(f(x))\}$, $x \in y$.
• For each $y \in V$ and $f \in Sig(y)$, define $[f]_\epsilon := \{x \in y : \pi_1 f(x) = \epsilon \in \{-, +\}\}$, then: $[f]_- \cap [f]_+ = \emptyset$ (by s1) and $L(f) \cap R(f) = \emptyset$ (otherwise $\exists x \in [f]_-, \exists x' \in [f]_+, \exists g \in \pi_2 f(x) \cap \pi_2 f(x') \subseteq Sig(x) \cap Sig(x') = \emptyset$).

If $\beta \neq \emptyset$, $S^{V_\beta} := \bigcup_{z \in V_\beta} Sig(z)$ is a partial SUR-algebra:
• $<_\beta$, $\beta \in On$, is a binary acyclic relation on $S^{V_{\beta+}}$ given by $<_\beta := <^{(V_\beta)} \cup \bigcup_{\rho(y) = \beta \in On} \{(g, f) : f \in Sig(y), g \in L(f)\} \cup \{(f, h) : f \in Sig(y), h \in R(f)\}$.
• Define $u_\beta : S^{V_{\beta+}} \to C_s(Sig^{(V_\beta)})$, $f \mapsto (L(f), R(f))$. By induction, u_β is a bijection. Denote $t_\beta = (u_\beta)^{-1} : C_s(Sig^{(V_\beta)}) \to S^{V_{\beta+}}$, $\beta \in On$.

Let $Sig := \dot{\bigcup}_{y \in V} Sig(y) = \bigcup_{\beta \in On} S^{V_\beta}$ and $< := \bigcup_{\beta \in On} <_\beta$. Define $C_s(Sig) :=$

$\{(A,B) : A, B \in P_s(\bigcup_{y\in V} Sig(y)) : A < B\}$, then $C_s(Sig) = \bigcup_{y\in V} C_s(Sig^{(y)})$. The gluing of the increasing and compatible family of bijections $\{t_\beta : \beta \in On\}$ is the bijection $t : C_s(Sig) \to Sig$. We can define $- : Sig \to Sig$, $f \in Sig(y) \mapsto -f \in Sig(y)$. Define $d : Sig \to On$, $f \mapsto d(f) = domain(f)$. Since $d^{-1}(\beta) = Sig(\beta)$, d has small fibers. It is clear that $g < h$ iff $(d(g) \in d(h)$ and $g \in L(h))$ or $(d(h) \in d(g)$ and $h \in R(g))$.

It is a routine checking that $\mathcal{S}ig = (Sig, <, -, *, t)$ is a SUR-algebra.

Let $f \in Sig(y)$, $y \in V$. By (s2) y is the $\subseteq$-least $w \in V$ such that $\forall x \in [f]_- \cup [f]_+, x \in w$, thus $d(f) = y = \bigcup\{\{x\} : x \in y\} = \bigcup\{\{d(g)\} : g \in L(f) \cup R(f)\}$. The map $s : V \to Sig$, $s(y) \in Sig(y)$, $\pi_1(s(y)(x)) = +$, $\forall x \in y$, defines a section of d, i.e. $d \circ s = id_{On}$, thus d is surjective. Then $d : Sig \to V$ is an anchor on Sig that satisfies $(b5)^{strong}$.

Summing up, the anchored SUR-algebra $(\mathcal{S}ig, d)$ satisfies the conditions (bSA1), (bSA2), (bSA3) in the characterization Theorem 101. Thus there is a unique isomorphism of anchored SUR-algebras $(SA, b^*) \overset{\cong}{\to} (\mathcal{S}ig, d)$.

□

We finish this short Section with some commentaries that indicates the possible expansions of the constructions above.

Remark 122.

We have defined above anchored SUR-algebras "space of signs" $Sig(Z, op)$, where $Z(= On, V)$ is a standard ZF-algebra and $op(= id, trcl)$ is an "operation" on relations, such that $t : C_s(Sig(Z, op)) \to Sig(Z, op)$ is a bijection. Obviously such constructions cannot represent our original motivation: the anchored SUR-algebra (No, b) is such that $t : C_s(No) \to No$ is surjective, but its fibers are *proper classes*.

The process of obtaining an anchored SUR-algebras "space of signs" can be slightly generalized adding another components to the construction: a initial step (a convenient partial SUR-algebra) and a coherent specification of equivalence relations: the No situation is the motivation to consider coherent equivalence relations (see Fact 27).

If Z is a standard ZF-algebra such that $s(z) \leq s(\Phi)$ iff $z = \Phi$, then we define a "space of signs specification" as a 6-upla $\sigma = (Z, L, v, op, I, \sim)$, where:
• I is partial SUR-algebra such that $t : C_s^t(I) \to I$ is surjective;
• $L \subseteq Z$ is a well-ordered subclass that contains the root $\Phi \in Z$ and such that $v : L \to Z$ satisfies $v(\Phi) = s(\Phi)$, $\lambda' < \lambda \in L \Rightarrow v(\lambda') \varepsilon v(\lambda)$ and $\forall z \in Z, \exists \lambda \in L$, $z \varepsilon v(\lambda)$. Denote $r_v(z) = \lambda \in L$ iff $z \varepsilon v(\lambda)$ but $z \not\varepsilon v(\lambda')$, $\forall \lambda' < \lambda$.

By a convenient cumulative construction by ε-recursion (sketched below) we can obtain an anchored SUR-algebra $Sig(\sigma)$ –that from now on we just denote simply by Sig – that expands the partial SUR-algebra I, such that $t : C_s(Sig) \to Sig$ is surjective and the domain function determines the anchor $d : Sig(\sigma) \to Z$, $f \mapsto d(f) = domain(f)$:

• $Sig(\Phi) = I$.
• For each $y \in Z \setminus \{\Phi\}$, $Sig(y) \subseteq \{f : y \to \bigcup_{z\varepsilon y}\{-,+\} \times P_s(Sig(z))\}$.
• Define $\Sigma^{(\lambda)} = C_s(\bigcup_{z\varepsilon v(\lambda)} Sig(z)) = \{(A,B) : A, B \in P_s(\bigcup_{z\varepsilon v(\lambda)} Sig(z)), A <^{(\lambda)} B\}$, where $<^{(\lambda)} = op(\bigcup_{\lambda'\varepsilon\lambda} <_{\lambda'})$. Note that $\Sigma^{(\Phi)} = C_s(I)$.
• $\sim_l \subseteq \Sigma^{(\lambda)} \times \Sigma^{(\lambda)}$ is an equivalence relation compatible with $<^{(\lambda)}$ (see. for instance, Fact 27) and $\sim_\Phi$ is such that $\Sigma^{(\Phi)}/\sim_\Phi \cong I$.
• A coherent family of injective maps $j^\lambda_{z'} : Sig(z') \rightarrowtail \Sigma^{(\lambda)}/\sim_\lambda$, $r_v(z') \leq \lambda$ with disjoint images that covers $\Sigma^{(\lambda)}/\sim_\lambda$ (thus $\bigcup_{r_v(z)=\lambda} Sig(z) \cong \Sigma^{(\lambda)}/\sim_\lambda \setminus \bigcup_{r_v(z')<\lambda} j^\lambda_{z'}[Sig(z')]$ is a class of "new members" at level λ and $\Sigma^{(\lambda)}/\sim_\lambda$ represents the "made members" at level λ).
• Given a "pre-number" $(A,B) \in \Sigma^{(\lambda)} = C_s(\bigcup_{z\varepsilon v(\lambda)} Sig(z))$ define $t(A,B)$ as the unique function $f \in Sig(z')$ such that $j^\lambda_{z'}(f) = [(A,B)]_{\sim_\lambda}$.
• Define $Sig = \bigcup_{z\in Z} Sig(z) \cong colim_{\lambda\in L}\Sigma^{(\lambda)}$.

It can be defined a notion of morphism of space of signs specification (a morphism of rooted well founded classes that is compatible with additional structure): this will induce an anchored SUR-algebra morphism (see Remark 103).

□

Chapter 4

Set Theory in the free Surreal Algebra

The initial motivation for this work is to obtain an "algebraic set theory for surreals" along the lines of the Algebraic Set Theory: to establish abstract and general links between the class of all surreal numbers and a universe of "surreal sets" similar to (but expanding it) the (ZF-algebra) relations between the classes On and V, giving the first steps towards a certain kind of (alternative) "relative set theory" (see [Fre16] for another presentation of this general notion).

Noting that:
(i) the (injective) map $j : On \to No$, is a kind of "homomorphism", partially encoding the ordinal arithmetic;
(ii) the traditional set-theoretic constructions (in V) keep some relation with its (ordinal) complexity (e.g., $x \in y \ \rightarrow \ \rho(x) < \rho(y)$, $\rho(\{x\}) = \rho(P(x)) = \rho(x) + 1$, $\rho(\bigcup_{i\in I} x_i) = \bigcup_{i\in I} \rho(x_i)$);
then we wonder about the possibility of this new structured domain SA determines a category, by the encoding of arrows (and composition) as objects of SA, in an analogous fashion to the way that the class V of all sets determines a category, i.e. by the encoding of some notion of "function" as certain surreal set (i.e. an objects of SA); testing, in particular, the degree of compatibility of such constructions with the map $j^* : V \to SA$ and examining if this new expanded domain could encode homomorphically the cardinal arithmetic.

Remark that the usual set/class theories (as ZFC or NBG) have the power of "encode" (syntactically) its Model Theory: constructions of models of set theory by the Cohen forcing method or through boolean valued models method are done by a convenient encoding of the fundamental binary relations $\in$ and $=$.

Thus, for us, there are three main requirements for a theory deserves be named a "Set Theory":

(i) have the potential to define arrows (category) as entities of the theory, through a fundamental binary relation;
(ii) be the "derived set theory" of a free object in a category (like in ZF-algebra setting);
(iii) its "internal" category is a topos-like category.

We develop in this Chapter the first steps of a certain kind of set theory based (or ranked) on surreal numbers, that expands the relation between V and On.

In fact, we work out a "positive set theory" on SA ranked on No, that expands the ZF-algebra relationship $V \to On$ through the "positive" map $j^+ : V \to SA$, given by $j^+(X) = j^*(X) = \langle j^+[X], \emptyset \rangle$, $X \in V$. Thus the free/initial SUR-algebra SA supports, in many senses, an expansion of the free/initial ZF-algebra V and its underlying set theory. To accomplish this, we use the functions j^* and b^* that establishes connections between V and SA. Under logical and set-theoretical perspective, the map $j^+ : V \to SA$ preserves and reflects many constructions. On the category-theoretic perspective, the map j^+ defines a full, faithful and logical functor $j^+ : Set \to Cat^+(SA)$, from the topos Set associated to V into the topos $Cat^+(SA)$ associated to SA.

4.1 Set Theory in the Surreal Algebra SA

The aim of this Section is the introduction of certain set-theoretic-like structure on the class SA. To accomplish this we will use the functions j^* and b^* that establishes connections between V and SA.

The requirements for the set-theoretic operations are the following: for the "standard sets", i.e., the SA members in the copy $j^*[V]$ of the set-theoretic universe V, the operations are essentially the same.

All the proofs in this section are straightforward verifications and will be frequently omitted.

4.1.1 The maps $(\cdot)^p$ and $(\cdot)^n$

In order to study the maps b^* and j^* we need to introduce two functions of great importance for the constructions in SA

Definition 123. *If $A \in P_s(SA)$, we define:*
- $A^p := \langle A, \emptyset \rangle \in SA$;
- $A^n := \langle \emptyset, A \rangle \in SA$;

We have then two **bijective** *maps that associates subsets of SA to elements of SA:*
- $(\cdot)^p : P_s(SA) \to Pos(SA)$;
- $(\cdot)^n : P_s(SA) \to Neg(SA)$.

□

Remark 124. If $x \in V$ we can consider $j^+(x) := \langle j^+[x], \emptyset\rangle$ and $j^-(x) := \langle \emptyset, j^-[x]\rangle$, then $j^+(x) = j^*(x) = -j^-(x)$ (by induction on $\rho(x)$).

□

Definition 125. *Let $u = \langle L_u, R_u\rangle, v = \langle L_v, R_v\rangle \in SA$. Define:*
(a) $u \prec v$ iff $b^*(u) \in b^*(v)$.
(b) $l(u, v)$ iff $u < v$ and $u \prec v$;
(c) $r(u, v)$ iff $u < v$ and $v \prec u$.
(d) $u\varepsilon v$ iff $l(u, v)$, whenever $v \in Pos(SA)$.

□

Remark 126.

(a) By Proposition 87.(c) and 98 $\prec$ is a well founded relation in SA. We also have that $u < v$ implies $u \prec v$ or $v \prec u$ (by Fact 96).

(b) By Proposition 95 and its proof:
(i) $b^* \circ j^* = id_V$, thus b^* is surjective and j^* is injective.
(ii) Let $X, Y \in V$. Then: $X \in Y$ iff $j^*(X) < j^*(Y)$.
(iii) Let $Y \in V$ and $a \in SA$. If $a < j^*(Y)$ and $b^*(a) \in b^*(j^*(Y))$ then $a = j^*(X)$ for a unique $X \in Y$.

(c) If $v \in SA$, then $\{u \in SA : l(u, v)\}, \{w \in SA : r(v, w)\} \in P_s(SA)$.

(d) If $w = \langle W, \emptyset\rangle \in Pos(SA)$, then $w^\varepsilon = \{u \in SA : l(u, w)\} = \{u \in SA : u \in L_w\} = \{u \in SA : u \in W\} \in P_s(SA)$

□

4.1.2 Inclusion, union, intersection and power

For the definitions of inclusion, union and order we have definitions, for all SA members, that are compatible with the maps j^+ and j^-.

Definition 127. *Let $u, v \in SA$. We define the inclusion $\subseteq^*$ as the following relation: $u \subseteq^* v$ iff $L_u \subseteq L_v$ and $R_u \supseteq R_v$*

□

Definition 128. *Let $\{u_i\}_{i\in I}$ an indexed non-empty set of elements of SA.*

1. $\bigcap^*_{i\in I} u_i := \langle \bigcap_{i\in I} L_{u_i}, \bigcup_{i\in I} R_{u_i}\rangle$

2. $\bigcup_{i\in I}^{*} u_i := \langle \bigcup_{i\in I} L_{u_i}, \bigcap_{i\in I} R_{u_i} \rangle$
   (when $I = \emptyset$ *define* $\bigcup^{*} \emptyset = \langle \emptyset, \emptyset \rangle$*)*

□

The next result follows directly from the correspondent properties of $\subseteq$:

Proposition 129. $\subseteq^{*}$ *is an order relation in* SA*.*

□

In respect of the maps j^{+} and $\rho^{*} : SA \to No$ we have the following result:

Proposition 130. *Let* $X, Y \in V$ *and* $u, v \in SA$

1. $X \subseteq Y$ *iff* $j^{+}(X) \subseteq^{*} j^{+}(Y)$
2. $u \subseteq^{*} v$ *implies* $\rho^{*}(u) \leqslant \rho^{*}(v)$

□

With this definitions we have interesting properties, similar to the correspondent ones we have in set theory:

Proposition 131. *Let* $u_i \in SA$ *for* $i \in I \neq \emptyset$ *and let* $v \in SA$*. Then:*

1. $\forall i \in I,\ u_i \subseteq^{*} v \iff \bigcup_{i\in I}^{*} u_i \subseteq^{*} v$
2. $\forall i \in I,\ u_i \supseteq^{*} v \iff \bigcap_{i\in I}^{*} u_i \supseteq^{*} v$

□

Another interesting fact about j^{*} and our definitions of union and intersection is the following result:

Proposition 132. *If* $\{A_i\}_{i\in I}$ *is an indexed family of sets in* V *we have:*

1. $j^{+}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I}^{*} j^{+}(A_i)$
2. $j^{+}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I}^{*} j^{+}(A_i)$

□

Definition 133. *Let* $u = \langle U, \emptyset \rangle \in Pos(SA)$. *We define* $P^+(u) := \langle \{\langle z, \emptyset \rangle : z \subseteq U\}, \emptyset \rangle$

□

The relationship between (positive) powers and unions in SA extends the relationship between powers and unions in V.

Proposition 134. *For each* $\mathcal{C}, U \in P_s(Pos(SA))$:
$\bigcup^* \mathcal{C} \subseteq^* U^p$ *iff* $\mathcal{C}^p \subseteq^* P^+(U^p)$

□

4.1.3 Singletons, doubletons and ordered pairs

In order to define the notion ordered pairs in SA we need suitable notions of singletons and doubletons:

Definition 135. *Let* $u, v, w \in SA$.
- $\{u\}^+ := \langle \{u\}, \emptyset \rangle$
- $\{u, v\}^+ := \langle \{u, v\}, \emptyset \rangle$
- $(u, v)^+ := \{\{u\}^+, \{u, v\}^+\}^+$
- $(u, v, w)^+ := ((u, v)^+, w)^+$

Proposition 136. $(u, v)^+ = (u', v')^+ \in SA$ *iff* $u = u'$ *and* $v = v'$ *iff* $(u, v) = (u', v') \in V$

All these constructions are compatible with the set theoretic notions induced by j^*.

Proposition 137. *Let* $x, y \in V$.

1. $j^+(\emptyset) = \emptyset^+$
2. $j^+(\{x\}) = \{j^+(x)\}^+$
3. $j^+((x, y)) = (j^+(x), j^+(y))^+$

Proof.

1. $j^+(z) = \emptyset^+$ iff $z = \emptyset$
2. $j^+(z) = \{r\}^+$ iff $\exists! x \in V, z = \{x\}, j^+(x) = r$
3. $j^+(z) = (r, s)^+$ iff $\exists! x, y \in V, z = (x, y), j^+(x) = r, j^+(y) = s$

□

4.1.4 Cartesian product, relations and functions

Having the notion of (positive) ordered pair in SA, we are able to define (positive) cartesian product, relations and funcions in SA.

Definition 138.

1. *If* $w \in Pos(SA)$, $dom^+(w) := \langle \{a : \exists b, l((a,b)^+, w)\}, \emptyset \rangle \in Pos(SA)$ *and* $ran^+(w) := \langle \{b : \exists a, l((a,b)^+, W)\}, \emptyset \rangle \in Pos(SA)$.

2. *A positive (small) relation* r *is a positive* SA *member whose* ε*-members are positive ordered pairs. More precisely,* $r = R^p = \langle R, \emptyset \rangle$, *where* $R = \langle \{(a_i, b_i)^+ : i \in I\}, \emptyset \rangle \in Pos(SA)$.

3. *Positive cartesian product: if* $u, v \in Pos(SA)$, $(u \times^+ v) := \langle \{(a,b)^+ : l(a,u), l(b,v)\}, \emptyset \rangle \in Pos(SA)$.

4. *Positive composition of positive relations:* $R^p \circ^+ S^p = \langle \{(a,c)^+ : \exists b, l((a,b)^+, S^p), l((b,c)^+, R^p)\}, \emptyset \rangle \in Pos(SA)$.

5. *Positive identity relations: if* $v \in Pos(SA)$, *define* $\Delta_v^+ = \langle \{(u,u)^+ : u \varepsilon v\}, \emptyset \rangle \in Pos(SA)$.

6. *If* $u, v \in Pos(SA)$, *then a positive functional relation from* u *into* v *is a positive relation* r *such that* $dom^+(r) = u, ran^+(r) \subseteq^* v$ *and such that* $(a,b)^+, (a,b')^+ \in L_r \Rightarrow b = b'$.

□

Proposition 139.

(a) Let $u = \langle U, \emptyset \rangle, v = \langle V, \emptyset \rangle \in Pos(SA)$, *then:* $(a,b)^+ \ \varepsilon \ u \times^+ v$ *iff* $(a,b) \in L_u \times L_v = U \times V$. *Thus*

(b) If $r \in Pos(SA)$ *is a positive relation, then:* $r \subseteq^* u \times^+ v$, *where* $u = dom^+(r), v = ran^+(r)$ *and* $R := \{(a,b) : (a,b)^+ \in L_r\} \subseteq L_u \times L_v$, $L_u \subseteq dom(R)$ *and* $L_v \subseteq ran(R)$.

(c) If $u, v \in Pos(SA)$ *and* $r \in Pos(SA)$ *is a positive functional relation from* u *into* v *then* $\{(a,b) : (a,b)^+ \in L_r\}$ *is a functional relation from* L_u *into* L_v.

(d) If $x, y \in V$, *then* $j^+(x,y) = (j^+(x), j^+(y))^+$.

(e) If $X, Y \in V$, *then* $j^+(X \times Y) = j^+(X) \times^+ j^+(Y)$.

(f) If R, S *are small relations in* V, *then* $j^+(R \circ S) = j^+(R) \circ^+ j^+(S)$.

(g) If $X \in V$, *then* $j^+(\Delta_X) = \Delta^+_{j^+(X)}$.

(h) If $X, Y \in V$, *then* $r \subseteq^* j^+(X) \times^+ j^+(Y)$ *is a positive functional relation from* $j^+(X)$ *into* $j^+(Y)$ *iff* $r = j^+(R)$ *for a unique* $R \subseteq X \times Y$, *a functional relation from* X *into* Y.

Proof. We will just sketch some proofs.

Item (e): $j^+(X \times Y) = \langle j^+[X \times Y], \emptyset\rangle = \langle\{(j^+(a), j^+(b))^+ : (a,b) \in X \times Y\}, \emptyset\rangle = \langle\{(j^+(a), j^+(b))^+ : (a,b) \in X \times Y\}, \emptyset\rangle = \langle\{(j^+(a), j^+(b))^+ : (j^+(a), j^+(b)) \in j^+[X] \times j^+[Y]\}, \emptyset\rangle = j^+(X) \times^+ j^+(Y)$

Item (g): $j^+(\Delta_X) = \langle\{j^+((x,x)) : x \in X\}, \emptyset\rangle = \langle\{(j^+(x), j^+(x))^+ : x \in X\}, \emptyset\rangle = \Delta^+_{j^+(X)}$

□

4.2 Logical analysis of the functions $V \overset{j^*}{\underset{b^*}{\rightleftarrows}} SA$

Since $j^*(x) = \langle j^*[x], \emptyset\rangle \in Pos(SA)$, $x \in V$, we have the injective (class) function $j^+ : V \to Pos(SA)$. We saw in the previous Section that j^+ is "elementary", i.e., it preserves and reflects many set-theoretic constructions: $\emptyset$, singletons, (ordered) pairs, cartesian products, relations, functions, composition of relations/functions, identities, ...

On the other hand, recall that $b^*(u) = b^*[L_u] \cup b^*[R_u]$, for each $u = \langle L_u, R_u\rangle$ in SA. We will denote $b^+ = b^*_{\restriction} : Pos(SA) \to V$. b^+ preserves some set theoretical constructions:

140. $b^* : SA \to V$ **and set-theoretical constructions:**

(a)

- $0 = \langle\emptyset, \emptyset\rangle \in Pos(SA)$ and $b^+(0) = b^*(\langle\emptyset, \emptyset\rangle) = b^*[\emptyset] \cup b^*[\emptyset] = \emptyset$
- If $x \in Pos(SA)$, then $\{x, y\}^+ \in Pos(SA)$ and $b^+(\{x, y\}^+) = b^*(\langle\{x, y\}, \emptyset\rangle) = b^*[\{x, y\}] \cup b^*[\emptyset] = \{b^*(x), b^*(y)\} = \{b^+(x), b^+(y)\}$
- If $x, y \in Pos(SA)$, then $(x, y)^+ \in Pos(SA)$ and $b^+((x, y)^+) = b^+(\{\{x\}^+, \{x, y\}^+\}^+) = \{b^*(\{x\}^+), b^*(\{x, y\}^+)\} \cup b^*[\emptyset] = \{\{b^*(x)\}, \{b^*(x), b^*(y)\}\} = (b^+(x), b^+(y))$

(b) If $u, v \in Pos(SA)$

- if $u \subseteq^* v$, then $b^+(u) \subseteq b^+(v)$
- $b^+(dom^+(u)) = dom(b^+(u))$
- $b^+(ran^+(u)) = ran(b^+(u))$
- $b^+(u \times^+ v) = b^+(u) \times b^+(v)$

(c) If r is a positive relation in SA, $r \subseteq^* u \times^+ v$. Then

- $b^+(r) \subseteq b^+(u) \times b^+(v)$ is a relation
- $b^+(r \circ^+ s) \subseteq b^+(r) \circ b^+(s)$

□

141.

(I) In order to study more closely the relations between V and SA we recall the Definition 125: let $u = \langle L_u, R_u \rangle, v = \langle L_v, R_v \rangle \in SA$:
- $u \prec v$ iff $b^*(u) \in b^*(v)$.
- $l(u, v)$ iff $u < v$ and $u \prec v$;
- $r(u, v)$ iff $u < v$ and $v \prec u$.
- $u \varepsilon v$ iff $l(u, v)$, whenever $v \in Pos(SA)$.

We also define two languages:
- $L_{Sur} = \{l(-,-), r(-,-)\}$
- $L^+_{Sur} = \{l(-,-)\}$

(II) We have a "left" interpretation $(-)^\sharp : For(L_{ZF}) \to For(L_{Sur})$ that is defined by:
- atomic: $(v_1 \in v_2) \mapsto l(v_1, v_2)$;
- $\neg, \wedge, \vee, \to, \exists, \forall$ (and $=$): defined in the obvious way.

We also have a "Reversing" interpretation: $(-)^\flat : Form(L^+_{Sur}) \to Form(L_{ZF})$:
- atomic: $l(v_1, v_2) \mapsto (v_1 \in v_2)$;
- $\neg, \wedge, \vee, \to, \exists, \forall$ (and $=$): defined in the obvious way.

(III) The (positive) set theory in SA "extends" the set theory in V in the following sense:

"Axioms" (= properties of SA, $Pos(SA)$)

Extensionality:

$$Ext^*(s, s') := \forall u \forall v((l(u, s) \leftrightarrow l(u, s')) \wedge (r(s, v) \leftrightarrow r(s', v)) \to s = s'$$

$$Ext^+(s, s') := \forall u(l(u, s) \leftrightarrow l(u, s')) \to s = s'$$

$$SA \vDash \forall s, \forall s' Ext^*(s, s')$$

$$SA \vDash \forall^{pos} s, \forall^{pos} s' Ext^+(s, s')$$

$$SA \vDash \forall x \forall x'(Ext_{ZF}(x, x'))^\sharp \leftrightarrow Ext^+(j^+(x), j^+(x'))$$

$$V \vDash \forall^{pos} s \forall^{pos} s'((Ext^+(s, s'))^\flat \to Ext_{ZF}(b^+(s), b^+(s'))$$

(IV) We have convenient (positive) versions of [emptyset], [comprehension], [regularity],[pair], [reunion], [parts],..

(V) Metatheorem: $V \models \psi(\bar{y}|\bar{a})$ iff $SA \models \psi^{\sharp}(\bar{y}|j(\bar{a}))$

□

Remark 142.

As in the algebraic set theory (see Section 2, Chapter 1 or [JM95]), we can try to describe the "surreal" set theory as the theory of derived from its the initial object SA.

Thus another possibility is consider a more comprehensive fragment of the language of the initial (equipped) SUR-algebra SA, given by $<, -, *, \prec$, where $\prec = \prec_{b^*}$[1]

We can define a **ternary** incidence relation in SA: for each $a, u, b \in SA$, $m(a, u, b)$ iff $a \in L_u$ and $b \in R_u$ (iff $l(a, u)$ and $r(u, b)$). Thus $m(a, u, b) \Rightarrow a < b, a \prec u, b \prec u$.

This seems to have good potential (that we intend to explore in the future). For instance:

• With this new language we can formulate a new and general "extensionality principle" for general members of SA:
$Ext^{SA}(u, v) : (\forall a \forall b(m(a, u, b) \Leftrightarrow m(a, v, b)) \Leftrightarrow u = v)$, for each $u, v \in SA$.
The "extensionality principle" is the formula $\forall u \forall v(Ext^{SA}(u, v))$.
This encodes $u = v$ iff $L_u = L_v$ and $R_u = R_v$.

• We have the empty surreal set axiom/property:
$Empty^{SA}(u) : (\exists u \forall a \forall b(\neg m(a, u, b))$
The "empty surreal set axiom/property" is the formula $\exists u(Empty^{SA}(u))$.
This encodes the SA member $* = \langle \emptyset, \emptyset \rangle$.

□

4.3 $Cat^{+}(SA)$ and the functor induced by $j^{+} : V \to SA$

In this Section, we describe a category of positive SA members and positive arrows, $Cat^{+}(SA)$, and establish functorial relationship between this category and the category $Cat(V) = Set$ of of all sets and functions: this turns out to be induced by the "extension" map $j^{+} : V \to SA$. The main categorial-theoretic references for this Section are [BW85] and [Bor94].

Definition 143. *We denote $Cat^{+}(SA)$ the category whose objects are all the positive objects of SA, i.e. $Obj(Cat^{+}(SA)) = Pos(SA)$ and such that, for each $u, v \in Pos(SA)$, $Cat^{+}(SA)(u, v) = \{(u, \phi, v)^{+} : \phi : u \to^{+} v$ is a positive function from u into $v\}$[2]. Since*

[1]Since $b^* : SA \to V$ is the $((b5)^{strong})$ anchor from the initial SUR-algebra SA on the initial (standard) ZF-algebra V.

[2]Notation: $(a, b, c)^{+} := ((a, b)^{+}, c^{+})^{+}$.

the composition of positive functions is a associative and the identity positive function $id_u^+ : u \to^+ u$ is an identity for that composition, this indeed turns out to be a category.

□

Note that this category $Cat^+(SA)$ is large (i.e. its class of objects and class of arrows are proper classes) but is *locally small*: if $A, B \in P_s(SA)$, let α the least ordinal such that $A, B \subseteq SA^{(\alpha)}$, then $A^p = \langle A, \emptyset \rangle, B^p = \langle B, \emptyset \rangle \in Pos_\alpha(SA) = Pos(SA) \cap SA_\alpha$ and then $Cat^+(SA)(A^p, B^p) \subseteq Pos_{\alpha+9}(SA) \subseteq SA_{\alpha+9}$, thus it is a set.

144. The functor induced by $j^+ : V \to Pos(SA)$:

By the properties established in Section 1 (see Proposition 139), the map $j^+ : V \to Pos(SA)$ determines a functor $J^+ = (J_0^+, J_1^+) : Cat(V) \to Cat^+(SA)$, given by:
• If $X \in V$, then $J_0^+(X) := j^+(X) \in Pos(SA)$;
• If $f = (X, F, Y) : X \to Y$ is a function from X into Y, then $J_1^+(f) = (j^+(X), \Phi, j^+(Y)) : J_0^+(X) \to^+ J_0^+(Y)$ is a positive function from $j^+(X)$ into $j^+(Y)$ determined by the positive functional relation Φ from $j^+(X)$ into $j^+(Y)$: $\Phi = \langle \{(j^+(a), j^+(b))^+ : (a,b) \in F\}, \emptyset \rangle$. Note that L_Φ is a set.

By the Proposition 139, the functor $J^+ : Cat(V) \to Cat^+(SA)$ is injective on objects, full and faithful (thus it reflects isomorphisms).

□

The functor $J^+ : Cat(V) \to Cat^+(SA)$ will help us to establish categorial constructions and properties of $Cat^+(SA)$. Below we construct categorial products in $Cat^+(SA)$.

Proposition 145. *Let $D : \mathcal{I} \to Cat^+(SA)$ be a diagram (= functor) over a small discrete category $\mathcal{I}$ ($\equiv$ a set I). Then D has admits a cone limit in $Cat^+(SA)$. I.e., given a function $i \in I \overset{D}{\mapsto} D(i) = u_i = \langle U_i, \emptyset \rangle \in Pos(SA)$, then we can define an object $u \in Pos(SA)$, denoted by $\prod_{i\in I}^+ u_i$ and a family of positive functions $\pi_k^+ : \prod_{i\in I}^+ u_i \to^+ u_k$, $k \in I$, such that for each $w = \langle W, \emptyset \rangle \in Pos(SA)$ and a family of positive functions $\phi_k^+ : w \to^+ u_k$, $k \in I$, there is a unique positive function $\phi : w \to^+ \prod_{i\in I}^+ u_i$ such that $\pi_k^+ \circ^+ \phi = \phi_k$, $\forall k \in I$.*

Proof. We start with a set-theoretical construction in $Pos(SA)$. Define $\prod_{i\in I}^+ u_i := \langle P, \emptyset \rangle$, where:
$P := \{\sigma \mid \sigma = (j^+(I), \Sigma, \bigcup_{i\in I}^+ u_i)^+$ is a positive function $j^+(I) \to^+ \bigcup_{i\in I}^+ u_i$ such that $\sigma(j^+(i)) \in U_i$, $i \in I\}$.
Note that:
• $\Sigma \subseteq^* j^+(I) \times^+ (\bigcup_{i\in I}^+ u_i)$;
• $L_{j^+(I)} = j^+[I] = \{j^+(i) : i \in I\}$ and $L_{\bigcup_{i\in I}^+ u_i} = \bigcup_{i\in I} U_i$;

• $(a,b)^{+} \in L_{\Sigma} \Rightarrow (a,b) \in j^{+}[I] \times \bigcup_{i\in I} U_i$;
• P is a set: indeed $P = L_{\prod^{+}_{i\in I} u_i} \cong \prod_{j^{+}(i)\in j^{+}[I]} U_i$.

For each $k \in I$, define $\pi^{+}_k = (\prod^{+}_{i\in I} u_i, \Pi^{+}_k, u_k)^{+} : \prod^{+}_{i\in I} u_i \to^{+} u_k$ by $\pi^{+}_k(\sigma) = \sigma(j^{+}(k)) \in U_k = L_{u_k}$, $\sigma \in P$. This defines a positive function from $\prod^{+}_{i\in I} u_i$ into u_k. Note that, through the bijection $P = L_{\prod^{+}_{i\in I} u_i} \cong \prod_{j^{+}(i)\in j^{+}[I]} U_i$, we correspond $\pi^{+}_k : \prod^{+}_{i\in I} u_i \to^{+} u_k$ with $proj_k : \prod_{j^{+}(i)\in j^{+}[I]} U_i \to U_k$.

Now let $w = \langle W, \emptyset\rangle \in Pos(SA)$ and a family of positive functions $\{\phi^{+}_k : w \to^{+} u_k : k \in I\}$. We have to prove that there is a unique positive function $\phi : w \to^{+} \prod^{+}_{i\in I} u_i$ such that $\pi^{+}_k \circ^{+} \phi = \phi^{+}_k$, $\forall k \in I$.

Candidate and uniqueness: Suppose that $\phi : w \to^{+} \prod^{+}_{i\in I} u_i$ is a positive function such that $\pi^{+}_k \circ^{+} \phi = \phi^{+}_k$, $\forall k \in I$.
Then for each $x \in W = L_w$, $\phi^{+}_k(x) = (\pi^{+}_k \circ^{+} \phi)(x) = \pi^{+}_k(\phi(x)) = (\phi(x))(j^{+}(k))$, $\forall k \in I$. I.e., since $\phi(x)$ is a positive function $j^{+}(I) \to^{+} \bigcup^{+}_{i\in I} u_i$, $\phi(x)$ is uniquely determined by the commutativity condition.

Existence: Consider the family of functions $f_k : W \to U_k$, $k \in I$, such that for each $x \in W = L_w$, $f_k(x) := \phi^{+}_k(x) \in L_{u_k} = U_k$. Denote $f : W \to \prod_{j^{+}(i)\in j^{+}[I]} U_i$ the unique function such that $(f(x))(j^{+}(k)) = f_k(x) = \phi^{+}_k(x), k \in I, x \in W$. Define $\phi : w \to^{+} \prod^{+}_{i\in I} u_i$ by: $x \in W = L_w \overset{\phi}{\mapsto} \phi(x) : j^{+}(I) \to^{+} \bigcup^{+}_{i\in I} u_i$, where $j^{+}(k) \in j^{+}[I] = L_{j^{+}(I)} \overset{\phi(x)}{\mapsto} (\phi(x))(j^{+}(k)) := (f(x))(j^{+}(k)) = f_k(x) = \phi^{+}_k(x) \in U_k = L_{u_k}$. Then $\pi^{+}_k \circ^{+} \phi = \phi_k$, $\forall k \in I$.

□

We saw in Section 1 that $j^{+}(A\times B) = j^{+}(A) \times^{+} j^{+}(B)$: this suggests that the functor $J^{+} : Cat(V) \to Cat^{+}(SA)$ preserves arbitrary products in general. Instead of proceed into long and direct calculations (in the same vein of the previous result), we establish below general results that will provide to us a full description of the category $Cat^{+}(SA)$.

We begin with the following:

146. The forgetful functor $\Lambda : Cat^{+}(SA) \to Cat(V)$:

By the properties established in Section 1 (see Proposition 139), we can define two maps, Λ_0, Λ_1, between proper classes, given by:
• If $u = U^{p} = \langle U, \emptyset\rangle \in Pos(SA)$, then $\Lambda_0(u) := L_u = U \in V$;
• If $\phi = (u, \Phi, w)^{+} : u \to^{+} w$ is a positive function from u into v, then $\Lambda_1(\phi) = (U, F, W) : \Lambda_0(u) \to \Lambda_0(w)$ is a function from U into W determined by the functional relation F from U into W: $F = \{(a,b) : (a,b)^{+} \in \text{Ł}_{\Phi}\} = \{(a,b) : (a,b)^{+}\varepsilon\Phi\}$. Note that F is a set.

It is easy to see that this data determines a functor $\Lambda = (\Lambda_0, \Lambda_1) : Cat^{+}(SA) \to Cat(V)$. Moreover Λ is injective on objects, full and faithful (thus it reflects isomorphisms).

$\square$

147. Connecting J^+ and Λ:

We present two "natural" ways to connect $Cat(V) \underset{\Lambda}{\overset{J^+}{\rightleftarrows}} Cat^+(SA)$:

(a) For each $x \in V$, denote $\eta_x : x \to \Lambda(J^+(x))$ the function given by: $y \in x \mapsto j^+(y) \in j^+[x] = \Lambda(J^+(x))$. Since j^+ is injective, η_x is a bijection. We will denote $\eta := (\eta_x)_{x \in V}$.

(b) For each $u \in Pos(SA)$, denote $\theta_u : u \to^+ J^+(\Lambda(u))$ the positive function given by: $v \varepsilon u \mapsto j^+(v) \varepsilon j^+(U) = J^+(\Lambda(u))$. Since j^+ is injective and $v \varepsilon u = \langle U, \emptyset \rangle \Leftrightarrow v \in L_u = U$, then θ_u is a positive bijection[3]. We will denote $\tau_u = (\theta_u)^{-1} : J^+(\Lambda(u)) \to^+ u$, the inverse bijection. We will denote $\tau := (\tau_u)_{u \in Pos(SA)}$.
Note that, in particular, every object in $Cat^+(SA)$ is positively isomorphic to a standard set positive object: i.e. for each $u \in Pos(SA)$ there are $x \in V$ and a positive isomorphism $J^+(x) \overset{\cong}{\to}^+ u$.

$\square$

Now we are ready to state and prove the:

Theorem 148. *Consider the following data: $(J^+, \Lambda, \eta, \tau)$. Then:*

(a) $\eta := (\eta_x)_{x \in V}$ determines an invertible natural transformation $\eta : Id_{Cat(V)} \overset{\cong}{\to} \Lambda \circ J^+$;
$\tau := (\tau_u)_{u \in Pos(SA)}$ determines an invertible natural transformation $\tau : J^+ \circ \Lambda \overset{\cong}{\to} Id_{Cat^+(SA)}$.

(b) $(J^+, \Lambda, \eta, \tau)$ is an equivalence of adjunction with unity η and co-unity τ.

Proof.

We will just proof that:
(i) For each $x \in V$, the function $\eta_x : x \to \Lambda(J^+(x))$ is a universal function from the objects x into the functor Λ, i.e., for each $u \in Pos(SA)$ and each function $f : x \to \Lambda(u)$, there is a unique positive function $f^\sharp : J^+(x) \to^+ u$ such that $f = \Lambda(f^\sharp) \circ \eta_x$;
(ii) For each $u \in Pos(SA)$, $\tau_u = (id_{\Lambda(u)})^\sharp : J^+(\Lambda(u)) \to^+ u$.

Then, by well-known category-theoretic results:
• η and τ are (invertible) natural transformations;
• $(J^+, \Lambda, \eta, \tau)$ is an adjunction with unity η and co-unity τ.
Thus we obtain items (a) and (b).

[3]Clearly, the positive bijections coincides with the isomorphisms in the category $Cat^+(SA)$.

(i) Let $x \in V$, $u \in Pos(SA)$ and $f : x \to \Lambda(u)$ be a function:
Candidate and uniqueness: Suppose that $\phi = (J^+(x), \Phi, u) : J^+(x) \to^+ u$ is a positive function such that $f = \Lambda(\phi) \circ \eta_x$. Then, for each $y \in x$, $(j^+(y), b)^+ \varepsilon \Phi$ iff $b = \phi(j^+(y)) = (\Lambda(\phi))(j^+(y)) = (\Lambda(\phi) \circ \eta_x)(y) = f(y)$. Thus the positive functional relation from $J^+(x)$ into u, Φ, is uniquely determined by the condition.
Existence: Define $f^\sharp := (J^+(x), \Phi, u)$ where $\Phi := \langle \{(j^+(y), f(y))^+ : y \in x\}, \emptyset \rangle$. Then Φ is a positive functional relation such that $dom^+(\Phi) = j^+(x)$ and $ran^+(\Phi) \subseteq^* u$. If $y \in x$, then $f^\sharp(j^+(y)) = f(y)$, thus $f = \Lambda(f^\sharp) \circ \eta_x$, establishing (i).

(ii) Let $u \in Pos(SA)$:
Then by the proof of (i) just above, $(id_{\Lambda(u)})^\sharp = (J^+(\Lambda(u)), \Phi, u)$, where $\Phi = \langle \{(j^+(v), id_{L_u}(v))^+ : v \in L_u\}, \emptyset \rangle = \langle \{(j^+(v), v)^+ : v \in L_u\}, \emptyset \rangle$: this is precisely the positive functional relation from $J^+(\Lambda(u))$ into u that determines τ_u, see 147.(b) above, thus $\tau_u = (id_{\Lambda(u)})^\sharp$, establishing (ii).

□

Corollary 149. *As a consequence of the previously established equivalence of categories* $Cat(V) \underset{\Lambda}{\overset{J^+}{\rightleftarrows}} Cat^+(SA)$ *we obtain:*

- *The functors $J^+ : Cat(V) \to Cat^+(SA)$ and $\Lambda : Cat^+(SA) \to Cat(V)$, preserve and reflect limits and colimits.*

- *Since $Set = Cat(V)$ is a (boolean) Grothendieck topos (in particular, it is a complete and cocomplete category), then $Cat^+(SA)$ is a (boolean) Grothendieck topos.*

- *Since $\{1\}$ is a generator subclass in the category $Cat(V)$, then $\{J^+(1)\}$ is a generator subclass in the category $Cat^+(SA)$, i.e. for each pair of distinct parallel positive functions $\phi, \psi : u \to^+ v$, there is a positive function $\theta : J^+(1) \to^+ u$ such that $\phi \circ^+ \theta \neq \psi \circ^+ \theta$. Thus, as in Set, a positive arrow in $Cat^+(SA)$ is an monomorphism iff it is a positive injection.*

- *Since $\{2\}$ is a cogenerator subclass in the category $Cat(V)$, then $\{J^+(2)\}$ is a cogenerator subclass in the category $Cat^+(SA)$, i.e. for each pair of distinct parallel positive functions $\phi, \psi : u \to^+ v$, there is a positive function $\theta : v \to^+ J^+(2)$ such that $\theta \circ^+ \phi \neq \theta \circ^+ \psi$. Thus, as in Set, a positive arrow in $Cat^+(SA)$ is an epimorphism iff it is a positive surjection.*

- *The arithmetic of cardinalities (i.e., of the equivalence classes of objects under isomorphisms) in $Cat^+(SA)$ coincides with the usual arithmetic of cardinalities is Set.*

- *Since $T : 1 \to 2$, $T(0) = 1$ is the (unique up to unique isomorphism) subobject classifier in $Cat(V)$, then $J^+(T) : J^+(1) \to^+ J^+(2)$ is the subobject classifier in $Cat^+(SA)$.*

- *J^+ and λ are logical functors (i.e. they preserves finite limits, exponentiation and subobject classifiers).*

□

We finish this Chapter with the following:

Remark 150.

The results in Theorem 148 and Corollary 149 above suggest that consider only the positive members of SA and develop set-theoretical notions and define a category through the definition of some notion of "function" as certain members of $Pos(SA)$, is "too good" to create new phenomena. Obviously the same situation is obtained when we depart from the class $Neg(SA)$ of all negative members of SA and the negative functions between them ($Cat^-(SA) \overset{J^-}{\simeq} Cat(V) \overset{J^+}{\simeq} Cat^+(SA)$). Thus we need more general classes of objects and functions as certain members of SA to obtain proper expansions of $Set = Cat(V)$.

The inclusion relation in SA defined by $\langle A, B\rangle \subseteq^* \langle A', B'\rangle$ iff $A \subseteq A'$ and $B \supseteq B'$ suggests consider a new category, denoted $Cat(SA)$, whose objects are **all** members of SA and whose arrows are pairs of ordinary functions – $f = (f_l, f_r) : u \to^* v$, where $L_u \overset{f_l}{\to} L_v$ and $R_u \overset{f_r}{\leftarrow} R_v$, $u, v \in SA$ – endowed with coordinatewise composition and identities: thus if $\langle A, B\rangle \subseteq^* \langle A', B'\rangle$, then there is a "inclusion" morphism $i = (i_l, i_r) : \langle A, B\rangle \to^* \langle A', B'\rangle$. In this way, obviously $Cat(SA) \rightarrowtail Set \times Set^{op}$ is identified with a full subcategory of the product category. Under the categorial perspective, this eventually provides a nice "set-theory", but it lacks a stronger connection with the global arrangement of members of SA as the free SUR-algebra.

A more natural, and potentially useful approach, is try to develop natural set-theoretical definitions dealing directly with the fundamental "incidence" relation between members of SA: this is a **ternary** relation in SA given by $m(a, u, b)$ iff $a \in L_u$ and $b \in R_u$ (iff $l(a, u)$ and $r(u, b)$), $a, u, b \in SA$. This seems to keep good potential: for instance, we can consider general immediate successor and immediate predecessor of any member of SA: if $u = \langle L_u, R_u\rangle \in SA$, then $suc(u) := \langle L_u \cup \{u\}, R_u\rangle$ and $pred(u) := \langle L_u, \{u\} \cup R_u\rangle$.

Of course, there is no problem in the form of the (faithful) encoding of sets ($x \in V$) as "standard members" of SA, i.e as hereditary positive members of SA ($J(x) = \langle\{J(y) : y \in x\}, \emptyset\rangle$), the point is how to present a more general but still (set-theoretical) notion of "function" between general members of SA that faithfully encodes the "standard functions" between standard members, but possibly allows non-standard functions between some standard members of SA.

The task of define a category, $Cat(SA)$, in this expanded setting (i.e. whose class of objects is SA) through the definition of some notion of "function" as certain members of SA, and still obtaining nice categorial-theoretical properties (topos?), seems to be an interesting challenge that we will intend to address in the future. We must also be able to define a functor $J : Cat(V) \to Cat(SA)$ that is injective on objects and arrows (to represent the expansion process), that will preserves/reflect nice categorial notions (logical functor?) but that **is not** an equivalence of categories.

$\square$

Chapter 5

Conclusions and future works

The present thesis is essentially a collection of elementary results where we develop, from scratch, a new (we hope!) and complementary aspect of the Surreal Number Theory.

There is much work to be done: it is clear for us that we just gave the first steps in the Surreal Algebras Theory and in Set theory based on Surreal Algebras.

In the sequel, we briefly present a (non-exhaustive) list of questions that have occurred to us during the elaboration of this thesis, that we can not be able to deal in the present work by lack of time and/or of skills, but that we intend to address in the future.

Questions on Chapter 2:

We have described some general constructions in categories of partial SUR-algebra (with at least 2 kinds of morphisms): initial object, non-empty products, substructures and some kinds of directed inductive (co)limits. There are other general constructions available in these categories like quotients and coproducts? A preliminary analysis was made and indicates that the characterizations of the conditions where such constructions exists is a non trivial task.

A specific construction like the (functor) cut surreal for SUR-algebras and its partial version turns out to be very useful to the development of the results of the (partial) SUR-algebra theory: the situation is, in some sense, parallel to the specific construction of rings of fractions construction in Commutative Algebra and Algebraic Geometry. There are other natural and nice specific constructions of (partial) SUR-algebra that, at least, provide new classes of examples?

We have provided, by categorial methods, some universal results that characterizes the SUR-algebras SA and ST, and also some relative versions with base (”urelements”) $SA(I)$, $ST(I')$ where I, I' are partial SUR-algebra satisfying a few constraints. There is an analog result satisfied by the SUR-algebra No? There are some natural expansions of No by convenient I'' are partial SUR-algebra, $No(I'')$, that also satisfies a universal property that characterizes it up to a unique isomorphism?

Questions on Chapter 3:

We have provided axiomatizations of SA (respec. ST) as an anchored SUR-algebra onto On and V (respec. onto On), in analogous fashion to the axiomatization of No through its "birthday" function $b : No \twoheadrightarrow On$. It is possible provide anchors, onto convenient rooted well-founded classes, of the relative versions $SA(I), ST(I')$ such that they provide axiomatizations of these relative constructions?

We have seen that the canonical SUR-algebra morphism $SA \to ST$ is injective, in particular it has small fibers. On the other hand, the canonical ZF-algebra morphism $\rho : V \to On$ (the rank function) has small *non-empty* fibers, in particular it is surjective and $On \hookrightarrow V$ is a section of ρ. It is natural to ask if the canonical SUR-algebra morphism $SA \to No$ share some of these properties of ρ: it has small fibers? are all its fibers non-empty (= surjectivity)? it has a section $(No, <) \to (SA, <)$? the space of signs representation of SA and No can aid us to answer some of these questions? The same kind of questions can also be posed for the canonical SUR-algebra morphism $ST \to No$.

In the process of associated a subclass of "hereditary positive members" for some anchored SUR-algebras $(\mathcal{S}, \mathbf{b})$ we consider naturally a structure given by class endowed by two binary and acyclic relations $(S, <, \prec_{\mathbf{b}})$. This seems related to the notions of s-hierarchical structures considered by P. Ehrlich in [Ehr01]: they are some algebraic structures (group, field, etc) defined over a lexicographically ordered binary tree, $(S, <, \prec_s)$. Can be illuminating to establish (and explore) a precise relation between both notions.

There is some clues obtained from some of our proofs (see also [All87] and [Ehr01]) that could be useful expand/adapt the concept of anchor for partial SUR-algebras and its relatives morphisms.

We have established links, in both directions, between SUR-algebras and ZF-algebras: we have anchored some SUR-algebras S onto standard ZF-algebras (its hereditary positive part, $HP(S)$) and for some standard ZF-algebras Z we have constructed a SUR-algebra (its space of signs, $Sig(Z)$). Moreover, the pair of functions $On \underset{b}{\overset{j}{\rightleftarrows}} No$ and $V \underset{b^*}{\overset{j^*}{\rightleftarrows}} SA$ are "chimera"-morphisms: they have head in a category and tail in another category. Notions of morphisms like that are considered in [Ell07], under the name of "heteromorphisms": ideally they occur when it is available of a pair of adjoint functors between the categories in sight. It is natural question determine subcategories of (anchored) SUR-algebras and ZF-algebras such that the mappings $S \mapsto HP(S)$ and $Z \mapsto Sig(Z)$ can be extended to adjoint pair of functors - this could be useful for obtain general results of transfer/preservation of properties from one category to another. Moreover, it will be interesting to study the relations of (relatively) free SUR-algebras and (relatively) free ZF-algebras whenever they belong to such nice subcategories of SUR and ZF.

Questions on Chapter 4:

We saw that the free/initial SUR-algebra SA is, in many senses, an expansion of the free/initial ZF-algebra V ans its underlying set theory. Relative constructions are available for SUR-algebras and for ZF-algebras (see [JM95]). In particular, it can be interesting examine possible natural expansions of set theories:
(i) with urelements B, $V(B)$, to some convenient relatively free SUR-algebra $SA(\hat{B})$;
(ii) obtained from the free transitive SUR-algebra $ST \to No$

We have developed a "positive set theory" on SA ranked on No, that expands the (ZF-algebra) relation $V \to On$ through the "positive" map $j^+ : V \to SA$, $j^+(X) = \langle j^+[X], \emptyset \rangle$, $X \in V$. Naturally we can obtain a "mirrored" set theory in SA, developed from the "negative" map $j^- : V \to SA$, $j^-(X) = \langle \emptyset, j^-[X] \rangle$, $X \in V$. There are other natural and interesting "mixed" set theories available from the free SUR-algebra support SA?

A combination of the tree lines of research above mentioned can be a interesting ("second-order") task: it will be a line of development of general relative set theories that are base independent.

Unexplored possibilities:

There exists at least two major aspects of the theory of SUR-algebras that we have not addressed in this work:
• the analysis of its model-theoretic aspects;
• the consideration of possible applications of SUR-algebras into "traditional" set/class theory, to answer specific questions on ZFC/NBG theories.

It is worthy to note that two lines of research can present interesting cross feedings, as the considerations below will indicate.

First of all, we recall that:
- rational number line $(\mathbb{Q}, <)$ is a (or "the") countable dense totally ordered set without endpoints;
- a dense totally ordered set without endpoints is a η_α-set if and only if it is $\aleph_\alpha$-saturated structure, $\alpha \in On$;
- the the surreal number line, $(No, <)$, is for proper class linear orders what the rational number line $(\mathbb{Q}, <)$ is for the countable linear orders. In fact, $(No, <)$ is a proper class Fraïssé limit of the class of all finite linear orders. The surreal numbers are set-

homogeneous and universal for all proper class linear orders.
- the relational structure $(S, <)$ underlying a SUR-algebra $\mathcal{S}$ is acyclic and η_∞, a natural generalization of the properties above mentioned.

We consider below two remarkable instances of model-theoretic properties applied to set theory that, we believe, could be related to our setting:

(I) J. Hamkins have defined in [Ham13] the notion of "hypnagogic digraph", $(Hg, \rightharpoonup)$, an acyclic digraph graded on $(No, <)$[1]. The hypnagogic digraph is a proper-class analogue the countable random $\mathbb{Q}$-graded digraph: it is the Fraïssé limit of the class of all finite No-graded digraphs. It is simply the On-saturated No-graded class digraph, making it set-homogeneous and universal for all class acyclic digraphs.
Hamkins have applied this structure, and some relativized versions, to prove interesting results concerning models of ZF set theory. For instance:
• every countable model of set theory $(M, \in^M)$, is isomorphic to a submodel of its own constructible universe $(L^M, \in^M)$;
• the class of countable models of ZFC is linearly pre-ordered by the elementary embedding relation.

As a part of a program of model theoretic studies of (relatively free) SUR-algebras, seems natural to determine (and explore) a precise relation between the No-ranked relational classes $(Hg, \rightharpoonup)$ and $(SA, <)$ (or $(ST, <)$). And what about the relativized versions of Hg and SA (or ST)? This kind of question is very natural as part of an interesting general investigating program relating Model Theory and Category Theory: in one hand we have the model-theoretic universality (from inside or above) of Hg and, on the other hand, we have the category-theoretic universality of (relatively) free constructions (to outside or below) of SA and ST.

Can we construct new models of ZF(C) by establishing relations
$[Cat(SA)] \rightsquigarrow [Hamkins\ digraph\ models]$ (and/or some variants)
in a way in some sense analogous to the relation:
$[sheaves\ over\ boolean\ algebras] \rightsquigarrow [Cohen\ forcing\ models]$?

(II) J. Hirschfeld have provided in [Hir75] a list of axioms - that include axioms for $\in$-acyclicness and for $\in$-density - that describes the model companion of ZF set theory. He emphasizes in the page 369:
"*...This model companion, however, resembles more a theory of order (Theorem 3) than a set theory, and therefore, while supplying an interesting example for model theory it does not shed any new light on set theory...*"

We can wonder about the possible relations of our SUR-set theories and model theoretic (Robinson) forcing(s). This is a natural question since the models of the model companion of ZF have a "nice" relational structure and the model theoretic forcing can

[1] I.e., it is given a "rank" function $v : Hg \to No$ such that: for each $x, y \in Hg$, if $x \rightharpoonup y$, then $v(x) < v(y)$.

provide the description of model companion/completion of a first order theory. Considerations involving large infinitary languages are also been in sight, since SUR-algebras are η_∞ acyclic relational classes.

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