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Quebras de Simetria em Sistemas Aleatórios Pseudo-Hermitianos

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Symmetry Breaking in Pseudo-Hermitian Random Systems

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Resumo

Simetrias compõe parte integral da análise na Teoria das Matrizes Aleatórias (RMT). As simetrias de inversão temporal e rotacional são aspectos-chave do Ensemble Gaussiano Ortogonal (GOE), enquanto esta última é quebrada no Ensemble Gaussiano Simplético (GSE) e ambas são quebradas no Conjunto Unitário Gaussiano (GUE). Desde o final da década de 1990, o crescente interesse no campo dos sistemas quânticos $\mathcal{PT}$ -simétricos levou os pesquisadores a considerar o efeito, em matrizes aleatórias, dessa classe de simetrias, bem como simetrias pseudo-hermitianas. A principal questão a ser respondida pela pesquisa apresentada nesta tese é se a simetria $\mathcal{PT}$ ou, de forma mais geral, a pseudo-Hermiticidade implica alguma distribuição de probabilidade específica para os autovalores. Ou, em outras palavras, se há um aspecto comum transmitido por tal simetria que pode ser usada para modelar alguma classe particular de sistemas físicos.

A abordagem inicial considerada consistiu na introdução de um conjunto pseudo-hermitiano, isospectral ao conjunto β -Hermite, que apresentaria o tipo de quebra de realidade típico dos sistemas $\mathcal{PT}$ -simétricos. Nesse modelo, a primeira abordagem adotada foi a introdução de perturbações que quebraram a realidade dos espectros. Os resultados obtidos permitem concluir que a transformação em seu similar pseudo-hermitiano conduz a um sistema assintoticamente instável. Esse modelo foi estendido ao considerar um η pseudo-hermitiano não positivo, que leva a uma quebra similar na realidade dos espectros. Este caso apresenta um comportamento mais próximo do típico dos sistemas $\mathcal{PT}$ -simétricos presentes na literatura.

Um modelo denso geral baseado em projetores foi proposto, e duas realizações particulares deste modelo receberam atenção mais detalhada. O comportamento espectral também foi similar àquele típico da simetria $\mathcal{PT}$ para as duas realizações consideradas, e seus limites assintóticos foram conectados a conjuntos clássicos de teoria de matriz aleatória. Além disso, as propriedades de seus polinômios característicos médios foram obtidas e os limites assintóticos desses polinômios também foram considerados e relacionados a polinômios clássicos. O comportamento estatístico deste conjunto foi estudado e comparado com o destes polinômios.

Impor a pseudo-Hermiticidade não parece implicar qualquer distribuição particular de autovalores, sendo a característica comum a quebra da realidade dos autovalores comumente encontrados na literatura de simetria $\mathcal{PT}$. O resultado mais notável dos estudos apresentados nesta tese é o fato de que uma interação pseudo-hermitiana pode ser construída de tal forma que o comportamento espectral médio possa ser controlado calibrando-se o mecanismo de interação, bem como sua intensidade.

Palavras-chave: Simetria $\mathcal{PT}$; Teoria de Matrizes Aleatórias; Mecânica Quântica Pseudo-Hermitiana.

Abstract

The role of symmetries is an integral part of the analysis in Random Matrix Theory (RMT). Time reversal and rotational symmetries are key aspects of the Gaussian Orthogonal Ensemble (GOE), whereas the latter is broken in the Gaussian Symplectic Ensemble (GSE) and both are broken in the Gaussian Unitary Ensemble (GUE). Since the late 1990s, growing interest in the field of $\mathcal{PT}$ -symmetric quantum systems has led researchers to consider the effect, in random matrices, of this class of symmetries, as well as that of pseudo-Hermitian symmetries. The primary question to be answered by the research presented in this thesis is whether $\mathcal{PT}$ -symmetry or, more generally, pseudo-Hermiticity implies some specific probability distribution for the eigenvalues. Or, in other words, whether there is a common aspect imparted by such a symmetry which may be used to model some particular class of physical systems.

The initial approach considered consisted of introducing an pseudo-Hermitian ensemble, isospectral to the β -Hermite ensemble, which would present the type of reality-breaking typical of $\mathcal{PT}$ -symmetrical systems. In this model, the first approach taken was to introduce perturbation which broke the reality of the spectra. The results obtained allow the conclusion that the transformation into its pseudo-Hermitian similar leads into a system which is asymptotically unstable. An extension of this model was to consider a non-positive pseudo-Hermitian η , which lead to similar breaking in the reality of the spectra. This case displays behavior closer to that typical of the $\mathcal{PT}$ -symmetric systems present in the literature.

A general dense projector model was proposed, and two particular realizations of this model were given more detailed attention. The spectral behavior was also similar to that typical of $\mathcal{PT}$ -symmetry for the two realizations considered, and their asymptotic limits were shown to connect to classical ensembles of random matrix theory. Furthermore, the properties of their average characteristic polynomials were obtained and the asymptotic limits of these polynomials were also considered and were related to classical polynomials. The statistical behavior of this ensemble was studied and compared to that of these polynomials.

Imposing the pseudo-Hermitian does seem not imply any particular eigenvalue distribution, the common feature being the breaking of the reality of the eigenvalues commonly found in $\mathcal{PT}$ -symmetry literature. The most notable result of the studies presented herein is the fact that a pseudo-Hermitian interaction may be constructed such that the average spectral behavior may be controlled by calibrating the mechanism of interaction as well as its intensity.

Keywords: $\mathcal{PT}$ -Symmetry; Random Matrix Theory; Pseudo-Hermitian Quantum Mechanics.

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Chapter 1

Introduction

In this chapter, the main subject matter of this thesis is introduced. Some background regarding $\mathcal{PT}$ -symmetric quantum mechanics and Random Matrix Theory is given, and some general properties are discussed. An overview of the topics to be discussed in the following chapters is presented, and the main results are outlined.

1.1 Random Matrix Theory

In this section some concepts of Random Matrix Theory (RMT) that are fundamental to the work presented in the rest of the text are reviewed. The approach taken in presenting this section is intended to give a general sense of how matrices similar to those studied in the late 1920s by J. , Wishart [1] appeared in physics, but it is by no means meant to be an extensive review, as that is beyond the scope of this text. However, further details about random matrix theory and its role in physics, both in the established models as well as in the frontiers of research, may be found in the literature [2–4].

Firstly, let us consider a classical quantum mechanical (not necessarily finite) vector space $\mathcal{V}$ over $\mathcal{D}$ with an orthonormal basis

$$\{\Psi_\alpha\} \subset \mathcal{V}, \quad \alpha \in \mathcal{I}, \tag{1.1}$$

where $\mathcal{I} \subset \mathbb{R}$, and with a metric $d\mu$ such that

$$\int_{\mathcal{D}} \psi_{\alpha'}^\dagger \psi_{\alpha} d\mu = \delta(\alpha' - \alpha)$$

with $\delta(z)$ denoting the Dirac- δ . Let us consider a linear map acting on a subspace $S \subset \mathcal{V}$ with image $T \subset \mathcal{V}$, that is

$$\hat{O} : S \rightarrow T, \quad \phi \in S \rightarrow \hat{O}\phi \in T.$$

Then $\hat{O}$ may be represented using the basis (1.1) by the matrix elements

$$O_{\alpha', \alpha} \equiv \int_{\mathcal{D}} \psi_{\alpha'}^\dagger \hat{O} \psi_{\alpha} d\mu.$$

This construction is general and may describe in an abstract way even infinitely dimensional vector spaces. One such an example would be the Hilbert space of the solutions of the Harmonic oscillators [5], given in bra-ket notation by:

$$\{|n\rangle, n = 0, 1, \dots\}, \quad \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$

where the $\hat{a}$ and $\hat{a}^\dagger$ are the ladder operators, such that, in arbitrary units (a.u.), $\hat{x} = \frac{\hat{a}^\dagger + \hat{a}}{2}$ and $\hat{p} = \frac{\hat{a}^\dagger - \hat{a}}{2}$. Then, the matrix describing the quadratic interaction $\frac{1}{2}x^2$ in this basis is

$$\begin{aligned} \langle m | \frac{\hat{x}^2}{2} | n \rangle = & \frac{1}{8} (\sqrt{(n+2)(n+1)} \delta_{m, n+2} \\ & + (2n+1) \delta_{m, n} + \sqrt{n(n-1)} \delta_{m, n-2}) \end{aligned}$$

which is an elementary textbook result in quantum mechanics.

Another good example is the finite square well, given by a constant potential $V(x)$ acting on the real line such that

$$V(x) = \begin{cases} V_0, & x < 0, \quad x > a \\ 0, & 0 < x < a \end{cases}.$$

Figure 1.1: Numerically computed bound state energies (dashed lines) for the finite square well with $V_0 = 100$ a.u. and $a = 2.0$ a.u.

For any bound state $E < V_0$, the allowed energies are obtained by solving the Schrödinger equation

$$-\frac{1}{2} \frac{\partial^2}{\partial x^2} \psi(x) + V(x) \psi(x) = E \psi(x)$$

and imposing the boundary conditions for each region $x < 0$, $0 < x < a$ and $x > a$, as well as imposing continuity for the wave function and its derivative with respect to x . This is, again, a textbook result, and the energies E are obtained by solving the transcendental equation

$$\tan(\sqrt{2E}a) = \frac{2E - V_0}{2\sqrt{2E(V_0 - E)}}.$$

This equation has to be solved numerically for the general case, but it has a discrete spectrum. This may be seen in Fig. 1.1, where the discrete spectrum of a finite square well with $V_0 = 100$ a.u. and width $a = 2.0$ a.u. has been calculated numerically. This shows that the spectrum, for energies below the threshold of the well, is discrete.

In the region beyond that threshold, namely $E > V_0$, the solutions fall into a continuous spectra. This serves as a toy model for systems which have two different spectral regions, one that is discrete and one which is continuous.

In many applications, the interest lies in the discrete region of the spectra in which

the energies are quantized. Such a case occurs, for example, in the energy levels of heavy nuclei, where, although the many interacting particles create both continuous and discrete parts of the spectrum, some of the most experimentally interesting physics lie in the latter [2].

In his papers from the 1950s, E. P. Wigner was the first to introduce the use of random matrices in the study of the energy spectra of heavy nuclei, by proposing that the spacing between energy levels in heavy nuclei should resemble the spacings between eigenvalues of random matrices [6, 7].

In an earlier model studied by A. M. Lane, R. G. Thomas and E. P. Wigner [8] based on a square well model by H. Feschbach, C. E. Porter and V. F. Weisskopf [9], the nucleon-nucleus interaction was modeled as an independent particle interaction with a statistical approach to the behavior of the elements of the Hamiltonian. From this study, E. P. Wigner's aforementioned papers introduced a simplified model in which the interactions were modeled by an infinite bordered matrix, that is a matrix which may be written as

$$H = \mathcal{D} + \mathcal{B}$$

where $\mathcal{D}$ is an infinite diagonal matrix whose elements are all the integers and $\mathcal{B}$ is a symmetric real matrix in which M subdiagonals are non-zero with the same absolute value,

$$\mathcal{B}_{k,l} = \begin{cases} b_{k,l} & , \quad |k-l| \leq M \quad , \quad |b_{k,l}| = b \\ 0 & , \quad |k-l| > M \end{cases} .$$

The original approach was split into three cases, all of which are bound to the same asymptotic limit. Thus in this framework, Wigner was able to obtain a limiting law for the density of the eigenvalues of very large real symmetric matrices with random elements, given by

$$\sigma(x) \propto (A - x^2) \tag{1.2}$$

which became known as the Wigner semi-circle. This study was further expanded in the following decades, and further generalizations were made to incorporate non-real elements with Gaussian distributions. This was most notoriously encapsulated in what has

become known as Dyson's threefold way following the work of Freeman J. Dyson in the early 1960s [10]. This allowed the results of the real Gaussian ensemble to be extended to complex and quaternion matrices, which are matrices whose elements are elements of the complex field $\mathbb{C}$ or the quaternion non-commutative associative division algebra $\mathbb{H}$. We may denote an element from these sets as

$$x = x_0 \hat{e}_0 \in \mathbb{R} \quad , \quad x = x_0 \hat{e}_0 + x_1 \hat{e}_1 \in \mathbb{C} \quad \text{and} \quad x = x_0 \hat{e}_0 + x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3 \in \mathbb{H},$$

respectively, where $\hat{e}_0 = 1$ is the real unit and $\hat{e}_1 = i$, $\hat{e}_2 = j$, $\hat{e}_3 = k$ are the imaginary units. Furthermore, denoting a random variable x distributed in the domain $\mathcal{D}$ with probability density function F as

$$\{x \in \mathcal{D} | F(x)\}$$

and denoting the Gaussian distribution function as

$$f_{\bar{x},\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \bar{x})^2}{2\sigma^2}\right), \quad -\infty < x < +\infty.$$

then each element of the random matrix H may be defined as

$$H_{l,m}^{(\beta)} = \begin{cases} x_0 \hat{e}_0 = x_0 & , \quad \{x_0 \in \mathbb{R} | F(x) = f_{0,1}(x)\} & , \quad l = m \\ \sum_{k=0}^{\beta-1} x_k \hat{e}_k & , \quad \{x_k \in \mathbb{R} | F(x_k) = f_{0,\sigma^2}(x_k)\} & , \quad l < m \\ \overline{H_{l,m}} & , \quad l > m \end{cases} \quad (1.3)$$

where β is what is known as the Dyson β index, and corresponds to $\beta = 1, 2, 4$ for real, complex and quaternion matrices, respectively, and the $\bar{\cdot}$ denotes conjugation. These elements are all normally distributed with zero mean.

The variance σ^2 may be chosen to suit the problem at hand. For instance, it is typical to consider the classical Gaussian ensembles with $\sigma^2 = 1/2$ for each component of the off diagonal matrix elements [2], whereas for the Ginibre ensemble it is typical to consider the total variance to be $\sigma_T^2 = 1/2$ such that $\sigma^2 = 1/2\beta$.

Dyson's threefold way also connected the ensembles of Gaussian matrices each with a symmetry group. The real, complex and quaternion matrices display analogous invariance of their distributions with respect to orthogonal, unitary and symplectic group transformations, respectively. These are now more commonly referred to as the Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE) and Gaussian Symplectic Ensemble (GSE) as a result of this correspondence [2, 3].

More recently, Ioana Dumitriu and Alan Edelman utilized the method of Householder rotations to reduce the matrices of the classical Gaussian ensembles to a common tridiagonal form, parametrized by the Dyson β index [11]. This entails utilizing the property of the Householder transform which allows one to replace a number of its last elements with zeros. In other words, if we take a vector v and apply a Householder transform, $\mathbf{h}_k$, within the relevant symmetry group

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_N \end{bmatrix} \Rightarrow \mathbf{h}_k \cdot v = \begin{bmatrix} v_1 \\ \vdots \\ v_{k-1} \\ ||v_k + v_{k+1} + \dots + v_N|| \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

reducing it such that, due to the symmetry of the original matrices, we may reduce the row and column of the same index simultaneously as

$$\tilde{H}_k = \mathbf{h}_k H \mathbf{h}_k^\dagger \quad (1.4)$$

where $(.)^\dagger$ denotes the adjoint operation of the corresponding ensemble.

This allowed the matrix to be written in an isospectral tridiagonal form as

$$T_{l,m}^{(\beta)} = \begin{cases} x_l & , \quad \{x_l | F(x_l) = f_{0,1}(x_l)\} & , \quad l = m \\ \frac{1}{\sqrt{2}} y_l & , \quad \{y_l | F(y_l) = g_{\beta(N-l)}(y_l)\} & , \quad |l - k| = 1 \\ 0 & , \quad |l - k| > 1 \end{cases} \quad (1.5)$$

where the off-diagonal elements are distributed according to the χ distribution, whose probability density function is given by

$$g_\nu(y) = \frac{x^{\nu-1} \exp\left(-\frac{x^2}{2}\right)}{2^{\frac{\nu}{2}-1} \Gamma\left(\frac{\nu}{2}\right)}, \quad (1.6)$$

and their calculations were made with variance $\sigma^2 = \frac{1}{2\beta}$. The eigenvalue joint probability density function for this ensemble becomes

$$\phi_\beta(\lambda_1, \lambda_2, \dots, \lambda_N) = \frac{1}{(2\pi)^{N/2}} \prod_{m=1}^N \frac{\Gamma(1 + \frac{\beta}{2})}{(1 + \frac{\beta}{2}m)} \cdot \prod_{k < l} |\lambda_k - \lambda_l|^\beta \cdot \exp\left(-\sum_{k=1}^N \frac{\lambda_k^2}{2}\right). \quad (1.7)$$

This generalizes the classical Gaussian ensembles, in the sense that the distribution of the eigenvalues is the same as the GOE, GUE and GSE for the respective values of the Dyson β index. This became known as the β -Hermite ensemble, and eigenvalues of this ensemble may be calculated for any real value of β . Furthermore, as in the case of the Gaussian ensembles [3], the β -ensembles are connected to the Hermite polynomials through their average characteristic polynomial. The large matrix asymptotics of the eigenvalue probability density function, ρ , is the Wigner semi-circle law [2]

$$\rho(\lambda) = \frac{1}{\pi\beta} \sqrt{2N\beta - \lambda^2}. \quad (1.8)$$

which is a generalization of (1.2).

Of further particular interest to the present thesis is the idea of considering the behavior of average characteristic polynomials. This is a possible additional avenue to explore statistical properties of the ensemble [12]. To do so, let the determinantal equation for the

characteristic polynomial of a matrix $A \in M_{N \times N}(\mathbb{K})$ be

$$c(\lambda) = \det(A - \lambda \mathbf{I}_N)$$

where $\mathbb{K}$ is the set to which the elements of the matrix belong and $\mathbf{I}_N$ be the order N identity matrix. If the elements of A , $a_{i,j}$ are random variables with respective distributions $f_{i,j}(a_{i,j})$, $i, j = 1, 2, \dots, N$, then one may consider the average characteristic polynomial by performing the average over all the random variables. Namely,

$$C(\lambda) \equiv \int \int \dots \int \det(A - \lambda \mathbf{I}_N) \prod_{i,j} f_{i,j}(a_{i,j}) da_{i,j}$$

noting that for the terms of the determinant that do not include a particular $a_{i,j}$, the integral for that particular element may be calculated independently and results in unity. Correlated matrices may also be described in this fashion, although one should consider carefully the integrals over the correlated elements. Particularly for matrices which are related to their transpose or transpose conjugate in some way, one should express the diagonal opposite of the element with a δ distribution. Namely,

$$a_{j,i} = f(i, j) \delta(a_{j,i} - \bar{a}_{i,j})$$

where $f(i, j)$ is the function which connects the i -th row and j -th column element with its transpose or transpose conjugate. This polynomial, $C(\lambda)$, is the average characteristic polynomial, and we denote, compactly, the averaging over the matrix elements as

$$\langle \cdot \rangle \equiv \prod_{i,j} \int_{\mathcal{D}_{i,j}} (\cdot) f_{i,j}(a_{i,j}) da_{i,j}$$

over the domains of the respective distributions, $\{a_{i,j} \in \mathcal{D}_{i,j} | F(a_{i,j}) = f_{i,j}(a_{i,j})\}$.

Taking the particular case of the β -Hermite ensemble as an illustrative example, we

have the matrix

$$H_\beta = \begin{pmatrix} x_N & y_{N-1}/\sqrt{2} & 0 & \dots \\ y_{N-1}/\sqrt{2} & x_{N-1} & y_{N-2}/\sqrt{2} & \dots \\ 0 & y_{N-2}/\sqrt{2} & x_{N-2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where each of these elements follow the distributions of (1.5). Thus each diagonal element is uncorrelated, whereas the off-diagonal elements appearing exclusively in the lower and upper subdiagonal are identical. In other words, let the characteristic polynomial of an $N \times N$ matrix H_β be

$$p_N(\lambda) \equiv \det(H_\beta - \lambda \mathbf{I}_N) = \begin{vmatrix} x_N - \lambda & y_{N-1}/\sqrt{2} & 0 & \dots \\ y_{N-1}/\sqrt{2} & x_{N-1} - \lambda & y_{N-2}/\sqrt{2} & \dots \\ 0 & y_{N-2}/\sqrt{2} & x_{N-2} - \lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}.$$

By expanding this determinant over the first row, we obtain

$$\begin{aligned} p_N(\lambda) &= (x_N - \lambda) \begin{vmatrix} x_{N-1} - \lambda & y_{N-2}/\sqrt{2} & 0 & \dots \\ y_{N-2}/\sqrt{2} & x_{N-2} - \lambda & y_{N-3}/\sqrt{2} & \dots \\ 0 & y_{N-3}/\sqrt{2} & x_{N-3} - \lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} - \frac{y_{N-1}}{\sqrt{2}} \begin{vmatrix} y_{N-1}/\sqrt{2} & y_{N-2}/\sqrt{2} & 0 & \dots \\ 0 & x_{N-2} - \lambda & y_{N-3}/\sqrt{2} & \dots \\ 0 & y_{N-3}/\sqrt{2} & x_{N-3} - \lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} \\ &= (x_N - \lambda) \begin{vmatrix} x_{N-1} - \lambda & y_{N-2}/\sqrt{2} & 0 & \dots \\ y_{N-2}/\sqrt{2} & x_{N-2} - \lambda & y_{N-3}/\sqrt{2} & \dots \\ 0 & y_{N-3}/\sqrt{2} & x_{N-3} - \lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} - \frac{y_{N-1}^2}{2} \begin{vmatrix} x_{N-2} - \lambda & y_{N-3}/\sqrt{2} & 0 & \dots \\ y_{N-3}/\sqrt{2} & x_{N-3} - \lambda & y_{N-4}/\sqrt{2} & \dots \\ 0 & y_{N-4}/\sqrt{2} & x_{N-4} - \lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} \\ &= (x_N - \lambda) p_{N-1}(\lambda) - \frac{y_{N-1}^2}{2} p_{N-2}(\lambda), \end{aligned}$$

where p_{N-1} and p_{N-2} are the characteristic polynomials of the matrices obtained by re-

moving the first row and column of H_β and removing the first and second row of H_β , respectively. The averaging of this equation is thus straightforward [13] to calculate using the independence of distinct elements

$$\begin{aligned}
P_N(\lambda) &= \langle p_N(\lambda) \rangle = \left\langle (x_N - \lambda)p_{N-1}(\lambda) - \frac{y_{N-1}^2}{2}p_{N-2}(\lambda) \right\rangle \\
&= -\lambda \langle p_{N-1}(\lambda) \rangle - \frac{N-1}{2}\beta \langle p_{N-2}(\lambda) \rangle \\
&= -\lambda P_{N-1}(\lambda) - \frac{\beta(N-1)}{2}P_{N-2}(\lambda).
\end{aligned} \tag{1.9}$$

Noting that the Hermite polynomials, $\mathbf{H}_N$ for $N = 0, 1, 2, \dots$, follow the recurrence

$$\mathbf{H}_N(x) = 2x\mathbf{H}_{N-1}(x) - 2(N-1)\mathbf{H}_{N-2}(x)$$

we may identify through (1.9) the P_N polynomial with the corresponding Hermite polynomial as

$$P_N(\lambda) = \left(\frac{\beta}{4}\right)^{N/2} \mathbf{H}_N\left(-\frac{\lambda}{\sqrt{\beta}}\right). \tag{1.10}$$

This result, already present in the Edelman and Dumitriu article, is in accord with the expected result for the classical Gaussian cases [2, 3], and is the reason for the nomenclature of β -Hermite ensemble. It is worthy to note that the similarly named β -Laguerre and β -Jacobi ensembles also display the analogous property with respect to their namesake polynomials.

1.2 An Overview of $\mathcal{PT}$ -Symmetry and Pseudo-Hermitian Quantum Mechanics

The historical background of the study of non-Hermitian operators with $\mathcal{PT}$ symmetric Hamiltonians and the properties of pseudo-Hermitian operators is given in this section. Some of the general properties of the $\mathcal{PT}$ -symmetric Hamiltonian were studied in the paper by C. M. Bender *et al* [14], and some further generalizations, with special focus on the framework put forth by A. Mostafazadeh [19] of pseudo-Hermitian operators.

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. Then a linear operator $\mathcal{L} : \mathcal{H} \mapsto \mathcal{H}$ is called a

symmetrizable operator if there exists a Hermitian operator $\alpha : \mathcal{H} \mapsto \mathcal{H}$ such that $\alpha \neq 0$ and

$$\alpha \mathcal{L} = \mathcal{L}^\dagger \alpha.$$

J. Dieudonné considered in the early 1960s [15] the problem of the spectrum of such an operator $\mathcal{L}$ when α is a positive-definite bounded operator, which is to say that $\forall u \neq 0 \in \mathcal{H}$

$$\langle \alpha u, u \rangle_{\mathcal{H}} > 0$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the inner product of $\mathcal{H}$. These operators were then called quasi-Hermitian due to the fact that all of their eigenvalues are real, which follows from an earlier result from P. Lax [16].

Some decades later, F. G. Scholtz *et al.* [17] proposed a framework in which one could construct a consistent quantum mechanics theory for quasi-Hermitian operators. In such a framework, the construction of valid observables becomes tied to parameters related to the non-Hermitian interaction models. The ability to construct a symmetrization operator α then becomes tied to the ability to describe probabilities in the typical quantum mechanical sense. This follows from the fact that the inner product induced by a positive-definite linear operator over $\mathcal{V}$ defines a norm for that space.

In the late 1990s, C. M. Bender *et al.* [14], considered a non-Hermitian Hamiltonian which has an entirely real spectrum in some regions of its parameter space. Namely, they considered the resulting spectrum by studying the ϵ -parametrized unidimensional Hamiltonian

$$H_\epsilon = p^2 + x^2(ix)^\epsilon \tag{1.11}$$

where $\hat{x}$ is the position operator and $\hat{p}$ is the momentum operator. What they found was that for $\epsilon \geq 0$ the spectra are entirely real, even though the $\mathcal{H}$ is in general non-Hermitian

$$H_\epsilon^\dagger = [p^2 + x^2(ix)^\epsilon]^\dagger = p^2 + [(-ix)^\epsilon]^\dagger [x^2]^\dagger = p^2 + (-1)^\epsilon x^2(ix)^\epsilon.$$

Furthermore, complete non-reality of the spectrum is only observed for $\epsilon < -1$ as even for $-1 \leq \epsilon < 0$ there is a finite number of real eigenvalues. This, they surmised, was the result of the system verifying $\mathcal{PT}$ -symmetry. This means that the Hamiltonian commutes

with the $\mathcal{PT}$ operator, where

$$\mathcal{P}\psi(x) = \psi(-x)$$

and

$$\mathcal{T}\psi(x) = \bar{\psi}(x)$$

are the parity and time reversal operators, respectively. Thus, the reality of the spectrum of a given Hamiltonian would be tied to the existence of an unbroken phase of $\mathcal{PT}$ symmetry, in which the operators $\mathcal{H}$ and $\mathcal{PT}$ share a complete set of eigenfunctions. A few years later, C. M. Bender *et al* [18] introduced the $\mathcal{C}$ operator which, for the description of a positive inner product, tackles the problem of dealing with indefinite metrics arising in $\mathcal{PT}$ -symmetric systems.

Earlier that same year, A. Mostafazadeh [19] introduced the notion of a pseudo-Hermitian operator. Let $\mathcal{V}$ be a vector space. The operator $\mathcal{O} : \mathcal{V} \rightarrow \mathcal{V}$ is said to be pseudo-Hermitian if there exists a Hermitian operator $\eta : \mathcal{V} \rightarrow \mathcal{V}$ such that

$$\mathcal{O}^\dagger \eta = \eta \mathcal{O}. \quad (1.12)$$

In addition to providing this defining relation, A. Mostafazadeh showed in those papers that the eigenvalues of this class of operators are not always real but are still constrained to be complex-conjugate pairs. Furthermore, it is possible to construct a viable description of observables in the quantum-mechanical system through pseudo-Hermiticity

$$(\eta^{1/2})^2 = \eta \quad , \quad (\eta^{1/2})^\dagger = \eta^{1/2}. \quad (1.13)$$

This definition extends the $\mathcal{PT}$ -symmetry definition to an abstract class of objects by letting the non-Hermitian nature of the system to be contained in the form of the η operator as well as the Hamiltonian H .

In the case of pseudo-Hermitian systems, the breaking in the reality of the spectrum may be considered to investigate the transition from the broken and unbroken symmetry phases. The coalescence of real eigenvalues into a complex conjugate pair causes the eigenfunctions to no longer form a basis for the underlying space which, in turn, corresponds to the loss of a symmetry in the system.

1.3 Random Matrices and $\mathcal{PT}$ -Symmetry

Time reversal symmetry, by itself, was a key aspect in the analysis of the Gaussian ensembles. The Gaussian Orthogonal and Symplectic Ensembles, composed respectively of real and quaternion matrices, are invariant with respect to time-reversal, whereas the Gaussian Unitary ensemble is not symmetric with respect to time reversal [2]. On the other hand, the Orthogonal Ensemble is symmetric to rotations, whereas the Symplectic Ensemble is not. These considerations set the bounds for the use of these ensembles as models for physical systems, to the effect that this showcases the role symmetry consideration play even in classical random matrix theory.

Therefore the question of how random matrices would behave if they had not the classical Wigner symmetries, but $\mathcal{PT}$ or pseudo-Hermitian symmetries quickly became subject of research. The most notable of these initial surveys into the effect of imposing a $\mathcal{PT}$ -type symmetry in a random matrix system arguably came in the papers by Z. Ahmed and S. R. Jain [20, 21]. There they explored the properties of a pseudo-unitary ensemble of matrices, and were able to obtain properties of interest analytically to a 2×2 model. This was also later extended for full cyclic matrices [22, 23], and other researchers have since then also taken an interest in this type of problem. Some notable examples are C. Birchall and H. Schomerus who studied absorbing and amplifying resonators for coupled interacting finite systems through random matrix theory, for arbitrary matrix size [24] and E.-M. Graefe, S. Mudute-Ndumbe and M. Taylor who approached the problem of modelling $\mathcal{PT}$ -symmetric systems using split-algebraic versions of complex and quaternion numbers [25].

The present thesis describes the contributions to this line of research that were obtained during my studies. The primary question to be answered by the research herein presented is whether $\mathcal{PT}$ -symmetry or, more generally, pseudo-Hermiticity implies some specific distribution for the eigenvalues. Or, in other words, whether there is a common aspect imparted by such a symmetry which may be used to model some particular class of physical systems.

The initial approach considered consisted of introducing an pseudo-Hermitian ensemble, isospectral to the β -Hermite ensemble, which would present the type of reality-

breaking typical of $\mathcal{PT}$ -symmetrical systems. This was motivated primarily by earlier work by my supervisor [13] as well literature surrounding non-Hermitian random phenomena in condensed matter physics [26, 27]. In this model, the first approach taken was to introduce perturbation which broke the reality of the spectra. The results obtained allow the conclusion that, even though isospectral to the original β -ensemble, the transformation into its pseudo-Hermitian similar leads into a system which is asymptotically unstable. An extension of this model was to consider a non-positive pseudo-Hermitian η , which lead to similar breaking in the reality of the spectra. This case proved to be of greater potential interest as it does not require the introduction of perturbation to break the reality of the spectra, and in fact remains quite robust to perturbation even as the matrix size increases. It also displays behavior closer to that of the typical $\mathcal{PT}$ -symmetric systems present in the literature.

This led to the second model considered. In order to fully explore the implications of the results of the non-positive model of the β -ensemble, a non-positive Gaussian model, defined directly from the dense matrices of the GOE, GUE and GSE ensembles was considered. A general model was proposed, in which projectors are used to introduce interactions between subspaces of the underlying vector space. Two particular realizations of this model were considered. Namely, one in which the space is split into two contiguous sets of basis elements which are then are subject to a pseudo-Hermitian interaction and another in which the basis elements alternate between the Hermitian and pseudo-Hermitian interaction, leading to a chessboard-like interaction structure. The spectral behavior was also similar to that typical of $\mathcal{PT}$ symmetry for the two realizations considered, and their asymptotic limits were shown to connect to classical ensembles of random matrix theory. Furthermore, the properties of their average characteristic polynomials were obtained, defining a set of polynomials for the first realization and two sets of intertwined polynomials for the second realization. The asymptotic limits of these polynomials were also considered, and were also related to classical polynomials. The statistical behavior of this ensemble was studied and compared to that of these polynomials.

It is possible to ascertain through these studies that imposing the pseudo-Hermitian does seem not imply any particular effect on the eigenvalue distribution itself, having as a common feature the breaking of the reality of the eigenvalues in similar fashion to that of

$\mathcal{PT}$ -symmetry literature. However, the most notable result of the studies presented herein is the fact that a pseudo-Hermitian interaction may be constructed such that the average spectral behavior may be controlled by calibrating the mechanism of interaction as well as its intensity.

1.4 Publications

During the time through which the project of this thesis, there was a total of six publications. Those consisted of four peer-reviewed papers, one book chapter and a paper in a conference proceedings. Therefore, some results presented in this thesis previously appeared in part in those publications.

Chapter 2 is based on the following publications :

- [28] G. Marinello and M. P. Pato. “A pseudo-Hermitian β -Hermite family of matrices”. In: *Physica A: Statistical Mechanics and its Applications* 444 (Feb. 2016), pp. 1049–1061. DOI: 10.1016/j.physa.2015.10.093
- [29] G. Marinello and M. P. Pato. “Pseudo-Hermitian β -Ensembles with Complex Eigenvalues”. In: *Springer Proceedings in Physics*. Springer Nature, 2016, pp. 305–318. DOI: 10.1007/978-3-319-31356-6_20
- [30] G. Marinello and M. P. Pato. “Random non-Hermitian tight-binding models”. In: *Journal of Physics: Conference Series* 738 (Aug. 2016), p. 012040. DOI: 10.1088/1742-6596/738/1/012040

Chapters 3, 4 and 5 are based on the following publications:

- [31] G. Marinello and M. P. Pato. “Pseudo-Hermitian ensemble of random Gaussian matrices”. In: *Physical Review E* 94.1 (July 2016). DOI: 10.1103/physreve.94.012147
- [32] G. Marinello and M. P. Pato. “Pseudo-Hermitian anti-Hermitian ensemble of Gaussian matrices”. In: *Phys. Rev. E* 96 (1 July 2017), p. 012154. DOI: 10.1103/PhysRevE.96.012154. URL: <https://link.aps.org/doi/10.1103/PhysRevE.96.012154>

- [33] G Marinello and M P Pato. “Statistical behavior of the characteristic polynomials of a family of pseudo-Hermitian Gaussian matrices”. In: *Journal of Physics A: Mathematical and Theoretical* 51.37 (2018), p. 375003. URL: <http://stacks.iop.org/1751-8121/51/i=37/a=375003>

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Mathematical and Theoretical 51.37 (2018), p. 375003. URL: <http://stacks.iop.org/1751-8121/51/i=37/a=375003>.

Chapter 2

Tridiagonal Model

In this chapter, tridiagonal models of pseudo-Hermitian random matrices are considered. These models, based on the β -ensemble generalization of the classical Gaussian ensembles of square matrices, are made pseudo-Hermitian by means of two main approaches. Namely, by introducing a pseudo-Hermitian symmetry that breaks the Hermiticity of the β -ensemble by either making η asymptotically unbound or by tying it to a non-positive η .

Thus, first model studied consists of a system in which the eigenvalues move into the complex plane spontaneously as the system increases in size. In this section, the model is presented and the asymptotic properties of the norm induced by the metric operator η for this case, and also establish the connection to another previous, pseudo-Hermitian model. Numerical results are presented for the cases studied. Considering the connection of non-Hermitian tridiagonal matrices and tight-binding models [1], the meaning and implications of interpreting the β -ensemble model as a one-dimensional lattice model is considered; results for closed chains as well as a generalization of the previous model are studied numerically. The final approach taken in this context consists of further generalizing the one-dimensional lattice interpretation, and allowing for the chain to self-intersect on distinct points, leading to further structures in the eigenvalues.

Noting the properties of quasi-Hermitian operators, an additional method for breaking the reality of the spectra was introduced to the β -Ensemble. In the second model, the interactions are made such that the metric operator is no longer positive-definite, resulting in non-quasi-Hermiticity, while still maintaining the pseudo-Hermitian property. Numerical results of this ensemble are also presented, and the implications are discussed.

2.1 The Hermitian and the Quasi-Hermitian β -Hermite ensembles

Recalling from Chapter 1 that the joint density distribution of the eigenvalues for a β -Hermite ensemble matrix is given by (1.7) and its asymptotic expression, the Wigner semi-circle, is given by Eq. (1.8). In an effort to obtain a pseudo-Hermitian equivalent of the β -ensemble, in Ref. [2] these matrices were made non-Hermitian by filling the two off-diagonals with different values taken from the same distribution. It was then shown that the non-Hermitian matrices $\hat{H}_\beta$ thus constructed are pseudo-Hermitian with real eigenvalues since the matrices η and $\eta^{1/2}$ can be defined such that Eqs. (1.12) and (1.13) are satisfied.

It is straightforward to show, using the matrix $\eta^{1/2}$, that the matrix $K_\beta = \eta^{1/2} \hat{H}_\beta \eta^{-1/2}$ is Hermitian and shares with H_β the same set of eigenvalues. It is also noteworthy that the matrix η given in reference [2]

$$\eta = \text{diag} \left(1, \frac{b_{N-1}}{c_{N-1}}, \frac{b_{N-1} b_{N-2}}{c_{N-1} c_{N-2}}, \dots, \prod_{j=1}^{N-1} \frac{b_{N-j}}{c_{N-j}} \right) \quad (2.1)$$

where each b_j and c_j are sorted independently from (1.6) with parameter $\beta(N-j)$, and which obeys Eq. (1.12) fluctuates around finite values. This fact can be seen by considering the distribution of the variable b/c :

$$\begin{aligned} F_{\beta(N-j)}(z) &= \int_0^\infty \int_0^\infty f_{\beta(N-j)}(b) f_{\beta(N-j)}(c) \delta \left(z - \frac{b}{c} \right) db dc \\ &= \int_0^\infty \int_0^\infty \frac{b^{\beta(N-j)-1} \exp \left(-\frac{b^2}{2} \right)}{2^{\frac{\beta(N-j)}{2}-1} \Gamma \left(\frac{\beta(N-j)}{2} \right)} f_{\beta(N-j)}(c) \delta \left(z - \frac{b}{c} \right) db dc \\ &= \frac{z^{\beta(N-j)-1}}{2^{\beta(N-j)-2} \left[\Gamma \left(\frac{\beta(N-j)}{2} \right) \right]^2} \int_0^\infty c^{2\beta(N-j)-2} \exp \left(-\frac{(1+z^2)c^2}{2} \right) c dc \quad (2.2) \\ \{w \rightarrow \frac{(1+z^2)c^2}{2}\} &= \frac{2z^{\beta(N-j)-1}}{\left[\Gamma \left(\frac{\beta(N-j)}{2} \right) \right]^2 (1+z^2)^{\beta(N-j)}} \int_0^\infty w^{\beta(N-j)-1} \exp(-w) dw \\ &= \frac{2\Gamma(\beta(N-j))}{\left(\Gamma \left[\frac{\beta(N-j)}{2} \right] \right)^2} \frac{z^{\beta(N-j)-1}}{(z^2+1)^{\beta(N-j)}}. \end{aligned}$$

The α -th moment integral may be written, using successively the substitutions $z = \tan \phi$ and $\xi = \cos^2 \phi$, as

$$\begin{aligned} \int_0^\infty z^\alpha \frac{z^{\gamma-1}}{(z^2+1)^\gamma} dz &= \int_0^{\pi/2} (1 - \cos^2 \phi)^{\frac{\gamma+\alpha-1}{2}} (\cos^2 \phi)^{\frac{\gamma-\alpha-1}{2}} d\phi \\ &= \frac{1}{2} \int_0^1 (1-\xi)^{\frac{\gamma+\alpha}{2}-1} \xi^{\frac{\gamma-\alpha}{2}-1} d\xi = \frac{1}{2} B\left(\frac{\gamma+\alpha}{2}, \frac{\gamma-\alpha}{2}\right) \\ &= \frac{\Gamma\left(\frac{\gamma+\alpha}{2}\right) \Gamma\left(\frac{\gamma-\alpha}{2}\right)}{2\Gamma(\gamma)} \end{aligned}$$

where $B(x, y)$ is the Beta function [3]. Hence, it follows that (2.2) obeys

$$\begin{aligned} \langle z \rangle_{\beta(N-j)} &= \frac{2\Gamma(\beta(N-j)) \Gamma\left(\frac{\beta(N-j)+1}{2}\right) \Gamma\left(\frac{\beta(N-j)-1}{2}\right)}{\left[\Gamma\left(\frac{\beta(N-j)}{2}\right)\right]^2 2\Gamma(\beta(N-j))} \\ &= \frac{\Gamma\left(\frac{\beta(N-j)+1}{2}\right) \Gamma\left(\frac{\beta(N-j)-1}{2}\right)}{\Gamma^2\left[\frac{\beta(N-j)}{2}\right]}, \\ \langle z^2 \rangle_{\beta(N-j)} &= \frac{2\Gamma(\beta(N-j)) \Gamma\left(\frac{\beta(N-j)}{2} + 1\right) \Gamma\left(\frac{\beta(N-j)}{2} - 1\right)}{\left[\Gamma\left(\frac{\beta(N-j)}{2}\right)\right]^2 2\Gamma(\beta(N-j))} \\ &= \frac{\beta(N-j)}{\beta(N-j) - 2}, \\ \Rightarrow \sigma_{\beta(N-j)}^2 &= \frac{\beta(N-j)}{\beta(N-j) - 2} - \langle z \rangle_{\beta(N-j)}^2. \end{aligned} \tag{2.3}$$

These imply immediately that when $\nu \rightarrow \infty$, $\langle z \rangle_\infty \rightarrow 1$, $\langle z^2 \rangle_\infty \rightarrow 1$ and hence $\sigma_\infty^2 \rightarrow 0$.

In the case studied in Ref. [2], the fact that each pair b_j/c_j fluctuates with distribution (2.2) around unity implies that, even in the very large N asymptotic limit, the quasi-Hermitian operator T corresponding to the η matrix of Eq. (2.1) will remain bounded.

2.2 Pseudo-Hermitian β -Hermite ensemble

Here we are interested in the construction of a pseudo-Hermitian ensemble whose matrices are isospectral with those of the β -ensemble. Since all matrices similar to a matrix A share their sets of eigenvalues, this construction is not unique. The choice of interest to the present section consists in assuming real tridiagonal non-Hermitian matrices of the

form

$$G = \begin{pmatrix} a_1 & d_{N-1} & 0 & \dots & 0 & 0 & 0 \\ 1 & a_2 & d_{N-2} & \dots & 0 & 0 & 0 \\ 0 & 1 & a_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{N-1} & d_1 \\ 0 & 0 & 0 & \dots & 0 & 1 & a_N \end{pmatrix} \quad (2.4)$$

where $a_1, a_2, \dots, a_N, d_1, d_2, \dots, d_{N-1} \in \mathbb{R}$. The fact that this matrix is pseudo-Hermitian can readily be seen by defining the diagonal matrix μ with elements

$$\text{diag}(\mu) = \left(1, d_{N-1}, \dots, \prod_{k=1}^p d_{N-k}, \dots, \prod_{k=1}^{N-1} d_{N-k} \right) \quad (2.5)$$

which immediately yields

$$\mu G = \begin{pmatrix} a_1 & d_{N-1} & 0 & \dots & 0 & 0 & 0 \\ d_{N-1} & a_2 d_{N-1} & d_{N-2} d_{N-1} & \dots & 0 & 0 & 0 \\ 0 & d_{N-1} d_{N-1} & a_3 d_{N-1} d_{N-1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \prod_{k=1}^{N-2} d_{N-k} & a_{N-1} \prod_{k=1}^{N-2} d_{N-k} & \prod_{k=1}^{N-1} d_{N-k} \\ 0 & 0 & 0 & \dots & 0 & \prod_{k=1}^{N-1} d_{N-k} & a_N \end{pmatrix} = G^\dagger \mu \quad (2.6)$$

Since the elements of μ are all diagonal and positive, its square root $\mu^{1/2}$ can be constructed as

$$\text{diag}(\mu^{1/2}) = \left(1, \sqrt{d_{N-1}}, \dots, \prod_{k=1}^p \sqrt{d_{N-k}}, \dots, \prod_{k=1}^{N-1} \sqrt{d_{N-k}} \right)$$

such that the similarity transformation

$$\Omega = \mu^{1/2} G \mu^{-1/2}$$

produces a Hermitian tridiagonal matrix

$$\Omega = \begin{pmatrix} a_1 & \sqrt{d_{N-1}} & 0 & \dots & 0 & 0 & 0 \\ \sqrt{d_{N-1}} & a_2 & \sqrt{d_{N-2}} & \dots & 0 & 0 & 0 \\ 0 & \sqrt{d_{N-2}} & a_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{d_2} & a_{N-1} & \sqrt{d_1} \\ 0 & 0 & 0 & \dots & 0 & \sqrt{d_1} & a_N \end{pmatrix}. \quad (2.7)$$

Equating this Hermitian matrix with the matrix H_β , with the identification

$$F(a_k) = f_{(0,1)}(a_k), k = 1, 2, \dots, N, \quad (2.8)$$

and

$$F(d_\nu) = \chi_\nu^2/2, \quad (2.9)$$

where $\nu = \beta(N - k)$ for $k = 1, 2, \dots, N - 1$, it is found that G becomes a family of matrices G_β in which the random off-diagonal elements follow the χ_ν^2 distribution

$$\tilde{F}_\nu(x) = \frac{1}{\Gamma[\nu/2]} x^{\frac{\nu}{2}-1} \exp(-x) \quad (2.10)$$

and constitutes a pseudo-Hermitian ensemble with the same real eigenvalues of the β -Hermite ensemble.

The matrix μ is a diagonal matrix whose elements are given by the product of the subdiagonal of G_β . This means that the j -th element of μ , μ_{jj} is given by:

$$\mu_{jj} = \begin{cases} 1 & , \quad j = 1 \\ \prod_{k=1}^{j-1} d_{N-k} & , \quad 2 \leq j \leq N \end{cases}. \quad (2.11)$$

Since each of the diagonal elements d_k is an independent random variable,

$$\langle \mu_{NN} \rangle = \left\langle \prod_{k=1}^{N-1} d_{N-k} \right\rangle = \prod_{k=1}^{N-1} \langle d_{N-k} \rangle = \prod_{k=1}^{N-1} \frac{\beta(N-k)}{2} = \frac{\beta^{N-1}(N-1)!}{2^{N-1}}. \quad (2.12)$$

This implies that, as the size of the matrix increases, the norm of the average μ grows as

$\sqrt{2\pi n} \left(\frac{N-1}{2e}\right)^{N-1}$. Unlike the case of Ref. [2], in the asymptotic limit of large N , the μ_{NN} diverges. Since μ is diagonal, this also implies that this operator becomes unbounded in this asymptotic limit.

2.2.1 Complex Eigenvalues

In order to investigate the spectrum of matrices of the G_β family, a perturbation will be introduced through a sparse Hermitian matrix g . This perturbation matrix is defined such that all elements g_{ij} are zero except for the $(1, N)$ and $(N, 1)$ elements. That is

$$g_{ij} = \epsilon (\delta_{i1}\delta_{jN} + \delta_{iN}\delta_{j1}), \quad (2.13)$$

where ϵ can be very small. This matrix explicitly breaks the pseudo-Hermiticity condition and by adding the two corner elements, changing from an open linear chain to a closed circular chain. This physical interpretation is drawn from the work presented in references [1, 4] which uses a perturbation of a similar form in another context.

In Fig. 2.1, the eigenvalues of a perturbed sample matrix for $\beta = 1$ and $N = 32$ are shown as a function of the perturbation strength ϵ and in Fig. 2.2 the analogous results for the average matrix of the family are presented. For convenience of representation, the eigenvalues are rescaled by a factor of $1/R_\beta$ where $R_\beta = \sqrt{2N\beta}$ is the Wigner radius. Note that Figs. 2.1 and 2.2 have behaviors qualitatively similar not only to each other, but also to that seen in references [5–7], albeit with the inverse behavior of the parameter ϵ .

Fig. 2.3 depicts, on the other hand, the behavior of the average matrices of varying size for the same perturbation magnitude $\epsilon = 10^{-20}$. Smaller matrices remain real for this perturbation, whereas larger matrices begin to exhibit some of its eigenvalues in the complex plane. From this analysis, it may be concluded that if ϵ is kept fixed in an arbitrarily small value, by appropriately increasing the size of the matrix, part of the eigenvalues are forced to move into the complex plane. In other words, as the matrix size N increases, the magnitude of the perturbation which breaks the reality of the eigenvalues of the matrix G_β decreases. Therefore, we have a transition to complex eigenvalues as occurs in Refs. [5–9], but our notion of spontaneous break corresponds to a system which is more easily affected by perturbations as it becomes larger [10, 11].

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane.

(c) Eigenvalue behavior, projected in the $\text{Re} \times \epsilon$ plane.

(d) Eigenvalue behavior, projected in the $\text{Im} \times \epsilon$ plane.

Figure 2.1: Complex eigenvalue behavior for an $N = 32$ sample matrix, as a function of the perturbation parameter ϵ , depicted as the Z axis as well as the coloring of the eigenvalues.

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane.

(c) Eigenvalue behavior, projected in the $\text{Re} \times \epsilon$ plane.

(d) Eigenvalue behavior, projected in the $\text{Im} \times \epsilon$ plane.

Figure 2.2: Complex eigenvalue behavior for an $N = 32$ average matrix, as a function of the perturbation parameter ϵ , depicted as the Z axis as well as the coloring of the eigenvalues.

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane.

(c) Eigenvalue behavior, projected in the $\text{Re} \times N$ plane.

(d) Eigenvalue behavior, projected in the $\text{Im} \times N$ plane.

Figure 2.3: Complex eigenvalue behavior for an $N = 32$ average matrix, as a function of the matrix size, depicted as the Z axis as well as the coloring of the eigenvalues, with $\epsilon = 10^{-20}$.

To understand these results, let us consider the perturbed matrix

$$G_\beta + g = \begin{pmatrix} N(0,1) & d_{N-1} & 0 & \dots & 0 & 0 & \epsilon \\ 1 & N(0,1) & d_{N-2} & \dots & 0 & 0 & 0 \\ 0 & 1 & N(0,1) & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & N(0,1) & d_1 \\ \epsilon & 0 & 0 & \dots & 0 & 1 & N(0,1) \end{pmatrix}. \quad (2.14)$$

The characteristic polynomial $P_N(\lambda, \epsilon)$ of the matrix defined in (2.14) may be expressed exactly by performing the Taylor expansion around $\epsilon_0 = 0$ of the determinant

$$\det(G_\beta + g) = \sum_{k=0}^{\infty} \frac{d}{d\epsilon} \det(G_\beta + g) \Big|_{\epsilon=0} \frac{\epsilon^k}{k!} = \det G_\beta + \left[\det \begin{pmatrix} N(0,1) & d_{N-1} & 0 & \dots & 0 & 0 & 0 \\ 1 & N(0,1) & d_{N-2} & \dots & 0 & 0 & 0 \\ 0 & 1 & N(0,1) & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & N(0,1) & d_1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} + \det \begin{pmatrix} N(0,1) & d_{N-1} & 0 & \dots & 0 & 0 & 1 \\ 1 & N(0,1) & d_{N-2} & \dots & 0 & 0 & 0 \\ 0 & 1 & N(0,1) & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & N(0,1) & d_1 \\ 0 & 0 & 0 & \dots & 0 & 1 & N(0,1) \end{pmatrix} \right] \epsilon + \mathcal{O}(\epsilon^2)$$

such that the Laplace expansion of the determinant then yields

$$p_N(\lambda, \epsilon) = p_N(\lambda, 0) + (-1)^{N+1} (d_{N-1} d_{N-2} \dots d_1 + 1) \epsilon + \mathcal{O}(\epsilon^2), \quad (2.15)$$

showing that it is a second order polynomial in ϵ . In the limit of large N , the term $(-1)^{N+1} \epsilon$ is negligible and since the last term a second order term in ϵ , $Q_{N-2}(\lambda, 0) \frac{\epsilon^2}{2}$ is also negligible. The eigenvalues are then obtained by solving the equation

$$p_N(\lambda) = (-1)^N \epsilon d_{N-1} d_{N-2} \dots d_1. \quad (2.16)$$

In this regime of large matrices, a better insight is obtained by averaging this equation. In this case, taking the average of (2.16), we get

$$\langle p_N(\lambda) \rangle = \left(\frac{\beta}{4} \right)^{N/2} H_N \left(-\frac{\lambda}{\sqrt{\beta}} \right) = (-1)^N \epsilon \left(\frac{\beta}{2} \right)^{N-1} (N-1)! = (-1)^N \epsilon \left(\frac{\beta}{2} \right)^{N-1} (N-1)!,$$

noting the independence of each element in the right hand side. The Hermite polynomial

oscillates inside the interval containing the roots and outside it, it diverges. The amplitude of the oscillations is smaller at the region around the center region and monotonically increases as one moves to the edge. If the constant term in the rhs is smaller than the minimum amplitude at the center, then all eigenvalues are real, otherwise pairs of conjugated complex eigenvalues appears at the center. With a fixed g , the size of the right-hand side is controlled by the size N of the matrix. Therefore by increasing N there will necessarily be some value of $N = N_0$ at which the constant term gets larger than the minimum amplitude, no matter how small ϵ is. From this value on, the eigenvalues progressively move to a line in the complex plane. Vice versa, with a fixed matrix size N , starting with a very small value by increasing ϵ , from some value ϵ_0 on, pairs of conjugate eigenvalues will pop up.

2.2.2 The pseudospectra

A direct association may be made between these effects of small perturbations in the pseudo-Hermitian system and errors introduced *ad hoc*. The sensitivity to errors in a matrix can be studied through more than a single method. In linear systems it is common [12, 13] to use the condition number defined as $\kappa(A) \equiv \|A\| \cdot \|A^{-1}\|$, typically in the 2-norm product space. However, in the determination of eigensystems this quantity does not contain much information about the error. In this case, it is more appropriate to use the eigenvalue condition number [14] defined, for a given simple eigenvalue λ , as the reciprocal of the cosine of the angle between its left and right normalized eigenvectors, namely

$$\kappa_\lambda = \left(\cos \left\langle \psi_\lambda^\dagger | \phi_\lambda \right\rangle \right)^{-1}. \quad (2.17)$$

While in the Hermitian case this quantity is a fixed value, this is not true in the non-Hermitian case. This condition number provides the estimate

$$|\lambda' - \lambda| \leq \kappa_\lambda \epsilon + O(\epsilon^2) \quad (2.18)$$

of how close an eigenvalue λ' of the matrix $A + \delta A$ is of the λ of A , where the δA has a matrix 2-norm $\|\delta A\|_2 = \epsilon$, such that if $\epsilon \rightarrow 0$, $\lambda' \rightarrow \lambda$.

Figure 2.4: Logarithm of the reciprocal of the largest condition number plotted against the logarithm of the reciprocal of the minimal perturbation ϵ which leads the eigenvalues into the complex plane of the same matrix, calculated for the average matrix of each even size from $N = 8$ to $N = 80$. Matrix sizes $10k$, $k = 1, 2, \dots, 8$ are depicted with larger markers. Matrix size $N = 32$ is singled out for the analysis of the pseudospectra.

In Fig. 2.4, the reciprocal of the minimal value of the perturbation parameter ϵ which causes the eigenvalues to begin escaping into the complex plane is plotted against the largest condition number of the same matrix, for $\tilde{G}_\beta$ matrices of size 8 to 80. It shows a direct correspondence between the minimal value and the eigenvalue condition number and also shows that such a relationship is nearly the identity function.

However, the eigenvalue condition numbers only describe the order of magnitude of a particular eigenvalue's sensitivity to perturbations. A better way to understand this transition from real to complex eigenvalues is through the investigation of the pseudospectrum [15] of a matrix of the G_β ensemble. The pseudospectra provides a way to characterize the anomalous behavior of normal matrices.

There are multiple equivalent ways by which the pseudospectrum of a linear operator can be defined. One way is indeed to study the response of a matrix to perturbations. However, here another definition is used, that which uses the resolvent $(z - A)^{-1}$ of the matrix A , where the set z satisfies

$$\|(z - A)^{-1}\|_2 \geq \frac{1}{\epsilon}, \quad (2.19)$$

with ϵ being positive and very small and $\|\cdot\|_2$ again denotes the 2-norm. This choice of norm allows us, following [15], to relate the norm of $(z - A)^{-1}$ to the minimum of the

Figure 2.5: Pseudospectra of Hermitian matrices of the β ensemble.

singular values of $z - A$.

To determine the pseudospectra a discrete grid Γ over the complex plane is created in the vicinity of the eigenvalues. Noting that $s_{\min}(\cdot)$ denotes the minimum of the singular values $\{s_1, s_2, \dots, s_N\}$, of the argument, we may obtain a contour of the ϵ -Pseudospectra by solving

$$\{z \in \Gamma \mid \|z - A\|_2 = s_{\min}(z - A) = \epsilon\}.$$

The appropriate grid can be obtained by scaling the eigenvalues by a factor of $1/\sqrt{2n\beta}$ and considering a square region in the complex plane. This method was found to be the most stable for the large sparse G_β matrices.

For a normal matrix, z is expected to be at the neighborhood of the eigenvalues for small values of ϵ and an anomalous behavior, on the other hand, occurs when even for very small ϵ , z may be far from the eigenvalues. To illustrate this, we consider the spectral behavior of both a sample matrix as well as the average matrix of the Hermitian β ensemble. The resulting pseudospectra of $N = 32$ matrices of the Hermitian β ensemble are depicted for a sample matrix in Fig. 2.5.(a) and the average matrix in Fig. 2.5.(b). Their behaviors are very similar, as in both cases the pseudospectra are the expected circles around the eigenvalues and the only marked difference is the greater symmetry of the average matrix along the real axis.

On the other hand, in Fig. 2.6 we present the pseudospectrum of the average matrix of the G_β ensemble for $N = 32$. While the spectra itself is real, the elliptical shape around the center of the spectra, shown in the figure, can immediately be related to the

Figure 2.6: Pseudospectrum of the average matrices of the pseudo-Hermitian β ensemble of Sec. 2.2. The contour lines denote $\log_{10} \epsilon$ of the ϵ pseudospectrum boundary.

behavior of the eigenvalues seen in Figs 2.1 and 2.2, where the eigenvalues emerge into the complex plane forming an elliptical shape around the origin.

It is also noteworthy that in accordance to Fig. 2.4, the smallest contour of the pseudospectra in Fig. 2.6 corresponds to perturbations of order close to 10^{-12} . Thus the escaping of eigenvalues into the complex plane may be attributed to intrinsic properties of the matrix [13]. As the matrix size increases this break into the complex plane will show up no matter how small the perturbation is. This sensitivity to perturbations parallels behavior observed previously in PT-symmetric lattices [16] for which exponentially small perturbations drove the system out of the symmetric phase.

2.2.3 Controlling the Pseudo-Hermitian Interaction

Consider now the real non-Hermitian random matrix

$$H_{\beta,\alpha} = \begin{pmatrix} a_N & b_{N-1}^{1+\alpha} & 0 & & \\ b_{N-1}^{1-\alpha} & a_{N-1} & b_{N-2}^{1+\alpha} & & \\ 0 & \vdots & \ddots & \ddots & \\ & & b_2^{1-\alpha} & a_2 & b_1^{1+\alpha} \\ & & & b_1^{1-\alpha} & a_1 \end{pmatrix}, \quad (2.20)$$

where the diagonal elements $\{a_k\}$ and subdiagonal elements $\{b_k\}$ are those of the β -Hermite ensemble, and α is a real parameter. This is a family of pseudo-Hermitian matrices with respect to the diagonal metric with elements

$$\mu = \text{diag} [1, b_{N-1}^{2\alpha}, (b_{N-1}b_{N-2})^{2\alpha}, \dots, (b_{N-1}b_{N-2}\dots b_1)^{2\alpha}], \quad (2.21)$$

as may be seen by performing the same check that was done in Eq. (2.6). Hence, there exists a similarity transformation leading back to the Hermitian β ensemble. Thus the matrices defined in this way belong to a family of isospectral matrices, parametrized by the value of α , that share their eigenvalues with a matrix from the β -Hermite ensemble. This parameter α , which may assume any real value, is symmetrical with respect to zero. That is, since the matrix is real, the matrix obtained for $-\alpha$ is exactly the transpose of the matrix obtained for α whereas $\alpha = 0$ is simply a Hermitian matrix of the β ensemble.

Being defined from the β ensemble we may identify from (2.9) that $b_k^2 = d_k$, $b_k > 0$. This implies that the elements of the subdiagonals b_k are distributed according to χ distributions (1.6) with parameter $\nu = \beta(N - k)$, for the k -th element. As such, their average value is

$$\begin{aligned} \langle b_k \rangle &= \int_0^\infty \frac{b_k^\nu \exp\left(-\frac{b_k^2}{2}\right)}{2^{\frac{\nu}{2}-1} \Gamma\left(\frac{\nu}{2}\right)} db_k = \frac{1}{2^{\frac{\nu}{2}-1} \Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty (2u_k)^{\nu/2} \exp(-u_k) \frac{1}{(2u_k)^{1/2}} du_k \\ &= \frac{2^{1/2}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty u_k^{\frac{\nu+1}{2}-1} \exp(-u_k) du_k = \sqrt{2} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} = \sqrt{2} \frac{\Gamma\left(\frac{\beta(N-k)+1}{2}\right)}{\Gamma\left(\frac{\beta(N-k)}{2}\right)} \end{aligned} \quad (2.22)$$

where the transformation $u_k = b_k^2/2$ and its associated Jacobian were used. The asymptotic limit of (2.22) when $N - k \gg 1$ may be obtained through the Stirling formula for the Γ function

$$\Gamma(x) \approx \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$$

and thus

$$\begin{aligned} \lim_{N-k \gg 1} \langle b_k \rangle &\approx \sqrt{2} \frac{\sqrt{2\pi \frac{\beta(N-k)+1}{2}} \left(\frac{\beta(N-k)+1}{2e}\right)^{\frac{\beta(N-k)+1}{2}}}{\sqrt{2\pi \frac{\beta(N-k)}{2}} \left(\frac{\beta(N-k)}{2e}\right)^{\frac{\beta(N-k)}{2}}} \\ &\approx \sqrt{\frac{\beta(N-k)+1}{\beta(N-k)}} \left(\frac{\beta(N-k)+1}{\beta(N-k)}\right)^{\beta(N-k)} \sqrt{\frac{\beta(N-k)+1}{e}} \end{aligned}$$

$$\begin{aligned}
&\approx \sqrt{1 + \frac{1}{\beta(N-k)}} \left(1 + \frac{1}{\beta(N-k)}\right)^{\beta(N-k)} \sqrt{\frac{\beta(N-k) + 1}{e}} \\
&\approx \exp(1/2) \sqrt{1 + \frac{1}{\beta(N-k)}} \sqrt{\beta(N-k) + 1}
\end{aligned}$$

which clearly diverges when $N - k$ becomes arbitrarily large. The implication of this is that for very large $H_{\beta,\alpha}$ matrices, the diagonal elements of μ from (2.21) diverge for all $\alpha \neq 0$ and, thus, the metric induced by the μ operator loses its boundedness in this asymptotic limit.

Following the previous exploration, we introduce a perturbation by a matrix whose only non-zero element has a small value ϵ located at the positions i, j and j, i :

$$g_{mn} = \epsilon (\delta_{mi}\delta_{nj} + \delta_{ni}\delta_{mj}). \quad (2.23)$$

Therefore, in this model we may control both the magnitude of pseudo-Hermitian effect through the parameter α as well as the position of the perturbation. Although tridiagonal matrices share the common characteristic of representing first order nearest neighbor processes, the choice of indexes m and n will change the precise physical picture of being represented. The general case is presented in Fig. 2.7.(a). Besides the nearest neighbor interaction, the chain becomes close enough between sites m and n in such a way that a residual interaction ϵ is observed between the sites. The case of Sec. 2.2.2, for an interaction which closes the chain, is depicted in Fig. 2.7.(b). Further consideration shall be given now, as this model is revisited in the present section to investigate the effect of varying the parameter α .

As the position may be defined arbitrarily within the matrix, calculations will be focused in a particular possible set of self-intersections. Namely, those that appear in the anti-diagonal such that $m = s, n = N - s + 1$. In Fig. 2.8 we consider, for fixed $\alpha = 1$ and $\epsilon = 10^{-20}$, the spectral behavior of such matrices, taking the average $N = 64$ matrix for our calculations. Changing the position of the perturbation exhibits remarkably similar behavior to that of changing the matrix size seen in Fig. 2.3, although matrix size must be increased for the effect in the spectra to be noticeable outside of the closing element for the same ϵ . Additionally, some eigenvalues move within the real axis, forming a secondary shape within the complex boundary of the spectrum.

Figure 2.7: Depiction of the interaction g being introduced in a tridigonal matrix model of order N in the general self-intersecting case (a) and the closed chain case (b). Sites are numbered from 1 to N , and the interactions are depicted as the connecting lines.

In order to assess the effect of the magnitude of the pseudo-Hermiticity parameter α , the spectra of sample and average matrices are presented in Figs. 2.9 and 2.10, respectively, for varying α and fixed $\epsilon = 10^{-20}$. Again, the qualitative behavior of the sample matrix resembles very closely the behavior of the average, and in both figures it is noticeable that the parameter α plays a similar role to increasing the ϵ parameter. Thus it is also possible to say that the limit for large α has similar properties to the limit of large N , as in the case of the previous section. It is, however, much more straightforward to see this if we look at the metric (2.21), where taking the limit $\alpha \rightarrow \pm\infty$ will yield a diverging metric.

In order to further explore the underlying phenomena in these matrices, it is illustrative to look at the behavior of the pseudospectrum for α values distinct from the one considered in the previous section. Thus we consider the cases of $\alpha = 0.01$, $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 2.0$ in Fig. 2.11.(a)-(d), respectively. It is noticeable that in Fig. 2.11.(a), the pseudospectrum is still very similar to that of Fig. 2.5. As the parameter increases, in Fig. 2.11.(b) the pseudospectrum has already begun to deform itself such that in Fig. 2.11.(c) it is already very similar to Fig. 2.6. In the last Fig. 2.11.(d), we can see that the pseudospectra has become much more non-normal, with ϵ -boundary values much lower than the Hermitian case. Nonetheless, in all cases of Fig. 2.11 the spectra remain the same.

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane.

(c) Eigenvalue behavior, projected in the $\text{Re} \times s$ plane.

(d) Eigenvalue behavior, projected in the $\text{Im} \times s$ plane.

Figure 2.8: Complex eigenvalue behavior for an $N = 64$ average matrix, as a function of the perturbation position s , depicted as the Z axis as well as the coloring of the eigenvalues, with $\epsilon = 10^{-20}$.

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane.

(c) Eigenvalue behavior, projected in the $\text{Re} \times \alpha$ plane.

(d) Eigenvalue behavior, projected in the $\text{Im} \times \alpha$ plane.

Figure 2.9: Complex eigenvalue behavior for an $N = 32$ sample matrix, as a function of the α parameter, depicted as the Z axis as well as the coloring of the eigenvalues, with $\epsilon = 10^{-20}$.

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane.

(c) Eigenvalue behavior, projected in the $\text{Re} \times \alpha$ plane.

(d) Eigenvalue behavior, projected in the $\text{Im} \times \alpha$ plane.

Figure 2.10: Complex eigenvalue behavior for an $N = 32$ average matrix, as a function of the α parameter, depicted as the Z axis as well as the coloring of the eigenvalues, with $\epsilon = 10^{-20}$.

Figure 2.11: Pseudospectrum of the average matrices for α parameter 0.01 (a), 0.1 (b), 0.5 (c) and 2.0 (d). The contour lines denote $\log_{10} \epsilon$ of the ϵ pseudospectrum boundary.

2.3 The non-positive-definite pseudo-Hermitian β -Hermite ensemble

The previous sections discussed cases in which the similarity operator η presented interesting spontaneous behavior when considering asymptotic limits. In the present section, we construct an ensemble whose matrices have their Hermiticity progressively broken by changing the signs of their subdiagonal elements. For a given matrix $H_{\beta,\alpha}$ as written in Eq. (2.20) with $\alpha = 0$, we construct a matrix with m changed signs denoted by

$$S_{\beta,\alpha,m} = \begin{pmatrix} a_N & -b_{N-1}^{1+\alpha} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b_{N-1}^{1-\alpha} & a_{N-1} & -b_{N-2}^{1+\alpha} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & b_{N-2}^{1-\alpha} & a_{N-2} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{N-m+1} & -b_{N-(m-1)}^{1+\alpha} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & b_{N-(m-1)}^{1-\alpha} & a_{N-m} & b_{N-m}^{1+\alpha} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_3 & b_2^{1+\alpha} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & b_2^{1-\alpha} & a_2 & b_1^{1+\alpha} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & b_1^{1-\alpha} & a_1 \end{pmatrix}. \quad (2.24)$$

As in the case of the pseudo-Hermitian ensemble described in Ref. [2], the respective matrix η may be written following (1.12) for arbitrary α ,

$$\eta_m = \text{diag} \left(1, -b_{N-1}^{2\alpha}, b_{N-1}^{2\alpha} b_{N-2}^{2\alpha}, \dots, (-1)^{m-1} \prod_{k=1}^{m-1} b_{N-k}^{2\alpha}, (-1)^m \prod_{k=1}^m b_{N-k}^{2\alpha}, \right. \\ \left. \dots, (-1)^m \prod_{k=1}^{N-2} b_{N-k}^{2\alpha}, (-1)^m \prod_{k=1}^{N-1} b_{N-k}^{2\alpha} \right), \quad (2.25)$$

which is clearly a non positive-definite operator for any integer m and positive $\{b_k\}$. In the present work, we shall focus on the special case of $\alpha = 0$, for which

$$\eta_m = \text{diag} (1, -1, 1, \dots, (-1)^{m-1}, (-1)^m, \dots, (-1)^m, (-1)^m), \quad (2.26)$$

which corresponds to the Hermitian H_β of [17] for $m = 0$.

This procedure may be generalized by introducing, instead of a sign change, a smooth transition along the real line for each element b_k in the upper subdiagonal. For the k -th

element of the subdiagonal, the subdiagonal element would become

$$\hat{S}_{\beta,\alpha,f}(\tau) = \begin{pmatrix} \ddots & & & & \\ & a_{N-(k-1)+1} & b_{N-(k-1)}^{1+\alpha} & 0 & 0 \\ & b_{N-(k-1)}^{1-\alpha} & a_{N-k+1} & f(\tau) & 0 \\ & 0 & b_{N-k}^{1-\alpha} & a_{N-(k+1)+1} & b_{N-(k+1)}^{1+\alpha} \\ & & & & \ddots \end{pmatrix} \quad (2.27)$$

where $\tau \in [0, 1]$ and $f : [0, 1] \rightarrow [-b_{N-k}^{1+\alpha}, 0) \cup (0, +b_{N-k}^{1+\alpha}] \equiv I$ such that $f(0) = +b_{N-k}^{1+\alpha}$ and $f(1) = -b_{N-k}^{1+\alpha}$. By means of this parametrization, we may perform a smooth change while ensuring that a matrix η of the kind written in (1.12) is still definable. We restrict ourselves to the case of $\alpha = 0$ in this case as well.

In order to perform numerical studies of this transformation, we partition the interval I in q discrete segments

$$(2.28)$$

and choose the discrete function f_τ such that $f_\tau = b_{N-k} - \tau\delta b_{N-k}$, $\tau = 0, 1/q, 2/q, \dots, 1$ and $\delta b_{N-k} = 2b_{N-k}$.

We shift the k -th subdiagonal into each of the f_τ and study the effect on the eigenvalues. Beginning with $\tau = 0$, once the case $\tau = 1$ is reached, we perform the same procedure in the next element. In this case, if there are m fully changed signs and the k -th subdiagonal is transformed to f_τ , the $\eta_{m,j}$ matrix becomes

$$\eta_{m,\tau} = \text{diag} \left(1, -1, 1, \dots, (-1)^{m-1}, \right. \\ \left. (-1)^m, (-1)^m \frac{z_j}{b_{N-k}}, \dots, (-1)^m \frac{z_j}{b_{N-k}} \right) \quad (2.29)$$

which is again non-positive definite.

In Fig. 2.12 the spectral behavior of a sample matrix is shown as the signs are pro-

Figure 2.12: Eigenvalue trajectories of a sample matrix as the sign changes, with the full sign changes m depicted as larger solid points. The Hermitian eigenvalues are represented by the large black points.

Figure 2.13: Eigenvalue trajectories of the average matrix as the sign changes, with the full sign changes m depicted as larger solid points. The Hermitian eigenvalues are represented by the large black points.

gressively changed from the lower subdiagonal first element to its last element, as the trajectory of the eigenvalue in the complex plane. Fig.2.12.a shows the full plot, whereas details the a smaller region of the spectrum trajectories. The trajectories show some global symmetry, but this seems to disappear locally when magnified. In Fig. 2.13 the same is shown for the average matrix. It is notable that the average matrix differs from the sample again only qualitatively, with the main difference being the presence of residual real part in the eigenvalues that move toward the imaginary axis in the sample matrix, whereas in the average matrix they finish their trajectories along said axis. The effect of this is to confer greater symmetry to the trajectories, both globally as well as locally.

Figure 2.14: Real part of the eigenvalues sample matrix (a) and the average matrix (b) as the sign changes, with the full sign changes m depicted as larger solid points. The Hermitian eigenvalues are represented by the large black points.

The behavior of the real projection of these eigenvalues is also interesting, as it allows for comparison with the continuous variation in other models seen in Figs. 2.1.(c), 2.2.(c), 2.9.(c) and 2.10.(c). In Fig. 2.14 it is possible to see such behavior as the signs are progressively changed by a factor of $z = \exp(i\delta\theta)$, where $\delta\theta = \pi/100$. Unlike the previous models, the eigenvalues with greater real part move toward the complex plane before those with smaller part, and this behavior is seen both in the sample matrix in Fig.2.14.(a) as well as in the average matrix in Fig. 2.14.(b).

This change in behavior also prompts the question of how the pseudospectra is altered by these sign changes. The effect is, however, only of displacement of the contour lines, as may be seen in Figs. 2.15 and Figs. 2.16, where the pseudo spectra for the sample and average matrices, respectively, are shown for $m = 7, 15, 23$ and 31 sign changes in the $N = 32$ case. We can see that the pseudospectral contour is tightly wound around the eigenvalues, much as would be expected of a normal matrix. Thus, unlike the previous models presented, this ensemble displays movement of eigenvalues toward the complex plane albeit without the loss of matrix normality. This can be readily related to the metric of (2.29), which despite losing positive definiteness remains bounded for all possible asymptotic limits.

Figure 2.15: Pseudospectrum of a sample matrix for varying number of sign changes of the subdiagonal of a Hermitian $N = 32$ matrix.

(a) $m = 7$ sign changes

(b) $m = 15$ sign changes

(c) $m = 23$ sign changes

(d) $m = 31$ sign changes

Figure 2.16: Pseudospectrum of the average matrix for varying number of sign changes of the subdiagonal of a Hermitian $N = 32$ matrix.

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Chapter 3

Dense Models

In this chapter, we utilize projectors to introduce a break in the Hermiticity of matrices of the classical Gaussian Orthogonal, Unitary and Symplectic ensembles. It is shown that their spectral behavior, when in a pseudo-Hermitian regime, shares common traits to other pseudo-Hermitian ensembles, but that they are ultimately a subclass of the Ginibre ensemble. Nonetheless, this family of matrices is of particular interest because many algebraic properties are thus obtained, particularly those related to its anti-Hermitian limit and those related to its average characteristic polynomials.

3.1 Projector Method

Let $\mathbb{K}$ be a set and let $M_N(\mathbb{K})$ be the set of square matrices of order N whose entries are elements of $\mathbb{K}$. In the present chapter, we shall consider the classical cases of Gaussian random matrices and thus $\mathbb{K}$ may be the real $\mathbb{R}$, complex $\mathbb{C}$ or quaternion $\mathbb{H}$ numbers. We choose a hermitian matrix $H \in M_N(\mathbb{K})$

$$H = \sum_{1 \leq i \leq j \leq N} \left[h_{i,j} |i\rangle \langle j| + h_{j,i}^\dagger |j\rangle \langle i| \right] \quad (3.1)$$

where $\cdot^\dagger$ denotes conjugation in the appropriate set. Let us also consider a set of K projectors

$$\begin{aligned} P_k, \quad k = 1, \dots, K \\ P_k |\psi\rangle \perp P_l |\psi\rangle, \quad \forall \psi, \forall k \neq l \end{aligned} \tag{3.2}$$

where each projector is defined in a set I_k of basis indexes

$$P_k = \sum_{\alpha \in I_k} |\alpha\rangle \langle \alpha|. \tag{3.3}$$

Their respective image subspaces, $U(P_k)$, are such that they split the space into complements. In other words, the direct sum of their images is

$$\bigoplus_{k=1}^K U(P_k) = U(M_N(\mathbb{K})),$$

the sum of the dimensions of each individual image subspace is equal to the total dimension

$$\sum_{k=1}^K \dim(U(P_k)) = N$$

and the kernel of each projector is the direct sum of the remaining images

$$\ker(P_k) = \bigoplus_{l \neq k} U(P_l).$$

We may define a new matrix from (3.1) in terms of (3.2) by writing

$$A_\phi(r) = \sum_{k,l} \phi_{k,l}(r) P_k H P_l \tag{3.4}$$

where each element of ϕ , $\phi_{k,l}(r)$ is a function in a given parameter r . This function works on the linear operators over the vector space as a mixing function. Our description of a particular interaction between the blocks is introduced by means of this function ϕ , and their precise nature will affect directly how the eigenvalues behave. In other words, if we consider the operator representation of a system's Hamiltonian, H , the time dependent

Schrödinger equation, in scaled units, is

$$H\Psi = i\partial_t\Psi$$

which describes the complete wave function Ψ . The time evolution of the occupation probabilities, for a given basis state Ψ , will be a function of the Hamiltonian H . The projection $P_k H P_k$ describes the transitions of H on the elements of the image subspace of P_k within that subspace. Analogously, the projection $P_k H P_{k'}$ for $k \neq k'$, describes how H affects elements of the subspace tied to $P_{k'}$ as they transition into the subspace P_k . This may be seen more promptly by writing the integral equation

$$\begin{aligned}\Psi(t) &= \sum_l c_l(t) |l\rangle = -i \int H\Psi(t)dt = -i \sum_{j,k,l} \int P_j H P_k c_l(t) |l\rangle \\ \Rightarrow \sum_l c_l(t) |l\rangle &= -i \sum_{\alpha \in I_j} \sum_{\gamma \in I_k} \int h_{\alpha,\gamma} c_\gamma(t) |\alpha\rangle dt\end{aligned}$$

which is a coupled integral equation in the coefficients of $\Psi(t)$, where the coupling between different coefficients depends explicitly on the composition of the projectors through the index sets I_k .

In the following sections, we introduce our specific choices for mixing ϕ as well as the projector structures expressed through the P_k . These will be the basis of further in-depth discussions in the following chapters.

3.2 The Models

It is worthwhile to reintroduce the symmetry case which has been the basis for the present model. In Ref. [1] projection operators P and Q were used to write

$$A = PHP + QHQ + \xi_1 PHQ + \xi_2 QHP, \quad (3.5)$$

where

$$P = \sum_{i=1}^M |i\rangle \langle i| \quad (3.6)$$

with $Q = \mathbf{I}_N - P$ [2] and H is a Hermitian $N \times N$ random matrix taken from one of the classes of the Gaussian ensemble as defined in Sec. 1.1. The two first terms of (3.5) are Hermitian diagonal blocks while the two other terms represent the off-diagonal blocks which becomes non-Hermitian if $\xi_1 \neq \bar{\xi}_2$.

To apply this structure to the PT symmetric case, we consider parameters such that the adjoint of the operator A is

$$A^\dagger = PHP + QHQ + \bar{\xi}_1 QHP + \bar{\xi}_2 PHQ, \quad (3.7)$$

with bar indicating complex conjugation. We then write $\xi_1 = r_1 \exp(i\alpha_1)$ and $\xi_2 = r_2 \exp(i\alpha_2)$ such that

$$A = PHP + QHQ + r_1 \exp(i\alpha_1) PHQ + r_2 \exp(i\alpha_2) QHP. \quad (3.8)$$

For this operator to obey the pseudo-Hermitian condition (1.12), η must be of the form

$$\eta = zP + wQ \quad (3.9)$$

as no higher order terms are possible [2]. It is then straightforward to show that using (3.9) on (3.5) and (3.7) implies

$$\begin{aligned} \eta A &= (zP + wQ)(PHP + QHQ + \xi_1 PHQ + \xi_2 QHP) \\ &= zPHP + z\xi_1 PHQ + wQHQ + w\xi_2 QHP, \\ A^\dagger \eta &= (PHP + QHQ + \bar{\xi}_1 QHP + \bar{\xi}_2 PHQ)(zP + wQ) \\ &= zPHP + z\bar{\xi}_1 QHP + wQHQ + w\bar{\xi}_2 PHQ, \end{aligned}$$

such that imposing (1.12), with ξ_1 and ξ_2 given as in (3.8), we obtain

$$zr_1 \exp(i\alpha_1) = wr_2 \exp(-i\alpha_2) \quad \text{and} \quad zr_1 \exp(-i\alpha_1) = wr_2 \exp(i\alpha_2).$$

It should be noted that by manipulating the above equalities, such as by *e.g.* dividing the

first by the second, we further obtain

$$\exp(i2\alpha_1) = \exp(-i2\alpha_2).$$

This implies that for the pseudo-Hermitian condition to be verified, we must have that

$$r_2 w = (-1)^k r_1 z \quad , \quad \alpha_1 + \alpha_2 = k\pi \quad , \quad k = 0, 1 \quad (3.10)$$

up to a global phase of $m\pi$, for integer m . The case for $k = 0$ defines an operator η which is positive definite, as it consists of the scaling of two interacting blocks by a positive number. Thus η has a positive square root, and following Eq. (1.13), the eigenvalues of H are real [3]. In this model, we are interested in the case of $k = 1$, for which said operator is not positive-definite, and it may be written as

$$\eta = r_2 P - r_1 Q.$$

Noting that α_1 and α_2 may differ up to a phase of opposite sign, we may then express A for the case of interest, $k = 1$, as

$$A(\alpha) = PHP + QHQ + r_1 \exp(i\alpha) PHQ - r_2 \exp(-i\alpha) QHP. \quad (3.11)$$

In principle, the parameters of the above model are the integer M which defines the size of the blocks, the phase α and r_1 and r_2 which weighs the importance of the pseudo-Hermitian elements. However, defining the operator

$$\mu(\alpha) = P \exp\left(\frac{i\alpha}{2}\right) + Q \exp\left(-\frac{i\alpha}{2}\right)$$

and noting that this operator has an inverse

$$\mu(\alpha)^{-1} = \mu(-\alpha)$$

we find a similarity transformation

$$\begin{aligned}
A(\alpha)\mu(\alpha) &= (PHP + QHQ + r_1 e^{i\alpha} PHQ - r_2 e^{-i\alpha} QHP) [Pe^{i\alpha/2} + Qe^{-i\alpha/2}] \\
&= \exp\left(\frac{i\alpha}{2}\right) (PHP + r_1 PHQ) + \exp\left(-\frac{i\alpha}{2}\right) (QHQ - r_2 QHP) \\
&= [Pe^{i\alpha/2} + Qe^{-i\alpha/2}] (PHP + QHQ + r_1 PHQ - r_2 QHP) \\
&= \mu(\alpha)A(0) \\
\Rightarrow A(\alpha)\mu(\alpha) &= \mu(\alpha)A(0) \iff A(\alpha) = \mu(\alpha)A(0)\mu(\alpha)^{-1}.
\end{aligned}$$

This allows us to switch, by means of this similarity transformation, between the phased matrix $A(\alpha)$ and the unphased $A(0)$. Thus, α determines an isospectral family of matrices parametrized by this phase. On the other hand, in terms of the eigenvectors of the matrix $A(0)$ the eigenvectors for a given eigenvalue λ can be expressed for a particular value of α as

$$|\Psi_\alpha(\lambda)\rangle = \mu(\alpha) |\Psi_0(\lambda)\rangle$$

since for an eigenvector of $A(0)$ we have that

$$\begin{aligned}
A(0) |\Psi_0(\lambda)\rangle &= \lambda |\Psi_0(\lambda)\rangle \\
\iff \mu(\alpha)A(0) |\Psi_0(\lambda)\rangle &= \lambda \mu(\alpha) |\Psi_0(\lambda)\rangle \\
\iff \mu(\alpha)A(0)\mu(\alpha)^{-1} \mu(\alpha) |\Psi_0(\lambda)\rangle &= \lambda \mu(\alpha) |\Psi_0(\lambda)\rangle \\
\iff A(\alpha)\mu(\alpha) |\Psi_0(\lambda)\rangle &= \lambda \mu(\alpha) |\Psi_0(\lambda)\rangle.
\end{aligned}$$

Therefore, when α goes from zero to 2π , the matrix returns to its original value but the eigenvectors change sign. It is also interesting to observe that for $\alpha = \frac{\pi}{2}$ elements of the interacting blocks are multiplied by the same imaginary unit.

In order to further explore the model, a choice for the parameters must be made. For this purpose, we will take the case in which $\xi_1 = -\xi_2^* = r \exp(i\alpha)$, and defining

$$A_1 = PHP + QHQ = \frac{A + A^\dagger}{2}, \quad (3.12)$$

$$A_2 = e^{i\alpha} PHQ - e^{-i\alpha} QHP = \frac{A - A^\dagger}{2r} \quad (3.13)$$

we have $H = A_1 + A_2$, where $A_1^\dagger = A_1$ and $A_2^\dagger = -A_2$ are the Hermitian and anti-Hermitian parts of H , respectively. Therefore

$$\text{tr}(A_1^2 - A_2^2) = \text{tr}H^2 = \text{tr}(H^\dagger H). \quad (3.14)$$

Thus we may write the trace of $H^\dagger H$ in terms of A using the explicit forms of A_1 and A_2 , (3.12) and (3.13). To do so, we first note that

$$A_1^2 = \frac{A^2 + AA^\dagger + A^\dagger A + A^{\dagger 2}}{4}$$

and

$$A_2^2 = \frac{A^2 - AA^\dagger - A^\dagger A + A^{\dagger 2}}{4r^2}.$$

Thus

$$\text{tr}(H^\dagger H) = \text{tr} \left[\left(1 - \frac{1}{r^2}\right) \frac{A^2 + A^{\dagger 2}}{4} + \left(1 + \frac{1}{r^2}\right) \frac{AA^\dagger + A^\dagger A}{4} \right]. \quad (3.15)$$

This implies that we may obtain the density of A directly from the j.p.d.f. of H , that is

$$\begin{aligned} P(A) &= Z_N \left| \frac{\partial H}{\partial A} \right| \exp \left\{ -\frac{\beta}{2} \text{tr} [H^\dagger(A, A^\dagger) H(A, A^\dagger)] \right\} = \\ &\zeta_N(r) \exp \left\{ -\frac{\beta}{2} \text{tr} \left[\left(1 - \frac{1}{r^2}\right) \frac{AA + A^\dagger A^\dagger}{4} + \left(1 + \frac{1}{r^2}\right) \frac{AA^\dagger + A^\dagger A}{4} \right] \right\} \end{aligned} \quad (3.16)$$

since the Jacobian $\left| \frac{\partial H}{\partial A} \right|$ is linear in all elements of H and is, hence, a polynomial in r . This distribution describes a transition from a situation in which the eigenvalues lie in the real axis, for $r = 0$, into one in which the eigenvalues move into the complex plane, as r goes to 1.

The similarity of (3.16) to that of the transition from the Hermitian to the Ginibre case known in the literature [4] suggests the ansatz that the resulting distribution of eigenvalues are likely to follow an elliptic law [5–7]. This is a known property for the general case of $N \times N$ complex matrices [8, 9] for which the axis are known

$$a = \sqrt{\frac{2N}{1 + r^2}} \quad (3.17)$$

$$b = r^2 \sqrt{\frac{2N}{1+r^2}} \quad (3.18)$$

and for which the eccentricity may be calculated to give

$$\left\{ \begin{array}{ll} \epsilon = \sqrt{1-r^4} & 0 \leq r \leq 1 \\ \epsilon = \sqrt{1-\frac{1}{r^4}} & r > 1 \end{array} \right. . \quad (3.19)$$

Let us now consider the case in which both r_1 and r_2 are both equal to 1. By direct substitution in (3.16), we have that

$$P(A) = Z_N^{-1} \exp \left[-\frac{\beta}{2} \text{tr}(A^\dagger A) \right] . \quad (3.20)$$

which is the j.p.d.f. of the Ginibre ensemble. This implies that we have a particular class of the non-Hermitian Gaussian matrices, resembling those of the Ginibre ensemble, since the elements are all i.i.d. Gaussian variables, with the exception of the diagonal elements. This pseudo-Hermitian ensemble, introduced using two contiguously indexed P and Q projectors, is just a particular case which can be generalized. To see this, with $k = 1, 2, \dots, K$ let us introduce the K projectors P_K

$$P_k = \sum_{i=1}^{M_k} |i_k\rangle \langle i_k| , \quad (3.21)$$

where $\sum_{k=1}^K M_k = N$ and $\{i_k\}_{1,\dots,M_1} \cup \{i_k\}_{1,\dots,M_2} \cdots \cup \{i_k\}_{1,\dots,M_K} = \{1, 2, \dots, N\}$, such that

$$\sum_{k=1}^K P_k = 1 \quad , \quad P_k P_l = \delta_{k,l} P_k . \quad (3.22)$$

Using this set of projection operators and taking a matrix H of the Gaussian ensemble, an operator A can then be constructed as

$$A = \sum_{k=1}^K P_k H P_k + \sum_{j>i} (r^{s_{ij}} P_i H P_j + r^{s_{ij}} \cos[(i-j)\pi] P_j H P_i) , \quad (3.23)$$

where $s_{ij} = 1/2 - \cos[(j-i)\pi]/2$. The corresponding matrix A is made of diagonal Hermitian blocks that come from the first terms in the right hand side. Then, the neighbouring subdiagonal is made of pseudo-Hermitian blocks, the next of Hermitian ones, followed by pseudo-Hermitian ones, and so on. The parameter r controls the importance of the Hermitian terms in relation to the pseudo-Hermitian ones.

The metric operator is

$$\eta = \sum_{k=1}^K (-1)^{k+1} P_k \quad (3.24)$$

which may be seen by direct application of (3.24) and its inverse to (3.23)

$$\begin{aligned} \eta A \eta^{-1} &= \left(r \sum_{l=1}^K (-1)^{l+1} P_l \right) A \left(r^{-1} \sum_{p=1}^K (-1)^{p+1} P_p \right) \\ &= \sum_{l,p=1}^K (-1)^{l+p} P_l A P_p = \sum_{l,p=1}^K \left[\sum_k \delta_{l,k} \delta_{p,k} P_k H P_k \right. \\ &\quad \left. + \sum_{j>i} (-1)^{l+p} r^{s_{ij}} \{ \delta_{l,i} \delta_{p,j} P_i H P_j + \delta_{l,j} \delta_{p,i} \cos[(i-j)\pi] P_j H P_i \} \right] \\ &= A^\dagger \end{aligned} \quad (3.25)$$

since $\cos[(i-j)\pi] = (-1)^{i+j}$.

It is noteworthy that, as in the case of a single size M block, we may obtain the j.p.d.f. of entries by considering the matrices

$$\begin{aligned} A_1 &= \sum_{k=1}^K P_k H P_k + \sum_{\substack{j>i \\ \text{mod}(i-j,2)=0}} (P_i H P_j + P_j H P_i) = \frac{A + A^\dagger}{2}, \\ A_2 &= \sum_{\substack{j>i \\ \text{mod}(i-j,2)=1}} (P_i H P_j - P_j H P_i) = \frac{A - A^\dagger}{2r}. \end{aligned}$$

As in the case of two projectors, we note that

$$\text{tr}(A_1^2 - A_2^2) = \text{tr}(H^\dagger H) \quad (3.26)$$

which follows from the fact that A_1 and A_2 are again the Hermitian and anti-Hermitian parts of A , respectively. Therefore, as in the previous case, the j.p.d.f. will be given by

(3.16).

Also as it happens with the $K = 2$ case, a phase can be introduced by the unitary transformation $\mu A(\alpha) \mu^\dagger$ with

$$\mu = \sum_{k=1}^K \exp \left[(-1)^{k+1} i \frac{\alpha}{2} \right] P_k. \quad (3.27)$$

For the two cases presented in this chapter, the interaction between the blocks may be described by a ϕ , as defined in Eq. (3.4), given by

$$\phi_{k,l}(r) = \begin{cases} r & , \quad k < l \text{ and } \text{parity}(k) \neq \text{parity}(l) \\ -r & , \quad k > l \text{ and } \text{parity}(k) \neq \text{parity}(l) \\ 1 & , \quad \text{otherwise} \end{cases} \quad (3.28)$$

in both cases. In the first case, the two projectors are such that $\dim(U(P_1)) = M$ and $\dim(U(P_2)) = N - M$, leading to a block matrix

$$\left(\begin{array}{c|c} A_{M \times M} & B_{M \times N} \\ \hline -B_{M \times N}^\dagger & C_{(N-M) \times (N-M)} \end{array} \right)$$

composed of Hermitian diagonal blocks A and C of size M and $N - M$ and a pseudo-Hermitian off-diagonal block B of size $M \times (N - M)$ as well as its adjoint $B^\dagger$. In the other case there are N projectors, each with $\dim(U(P_k)) = 1$, leading to a chessboard-like structure with alternating Hermitian and pseudo-Hermitian subdiagonals

$$\left(\begin{array}{cccc} \{\mathbf{h}\} & \{\mathbf{ph}\} & \{\mathbf{h}\} & \dots \\ \{\mathbf{ph}\} & \{\mathbf{h}\} & \{\mathbf{ph}\} & \dots \\ \{\mathbf{h}\} & \{\mathbf{ph}\} & \{\mathbf{h}\} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right)$$

where $\{\mathbf{h}\}$ represents the hermitian subdiagonal elements and $\{\mathbf{ph}\}$ the pseudo-Hermitian ones, which we recall from the definition of H are numbers belonging to the respective $\mathbb{K}$ over which H was defined.

For $\phi_{k,l}(r)$ as written in Eq. (3.28), and the P_k for each case, the resulting matrix A is a pseudo-Hermitian matrix obeying the pseudo-Hermitian condition

$$A^\dagger = \eta A \eta^{-1} \quad (3.29)$$

with a common form for the η operator

$$\eta = \sum_{k=1}^K (-1)^{k-1} P_k. \quad (3.30)$$

3.3 Commutation Properties and the Anti-Hermitian Limit

One feature of the model insofar constructed through the projector method is that η is an involution, $\eta^2 = 1$. We may take advantage of this fact to obtain some tractable properties of the model. To do this let us consider a matrix which may be written as

$$A(r) = \mathcal{H} + r\mathcal{S} \quad (3.31)$$

such that $A(r)$ verifies Eq. (1.12) and where $\mathcal{H}$ is a Hermitian matrix and $\mathcal{S}$ is an anti-Hermitian matrix, with r a positive parameter. That the matrices of our considered models follow this property is an immediate consequence of applying the definition (3.28) to (3.4).

The operator η will then commute with the Hermitian part $\mathcal{H}$,

$$[\mathcal{H}, \eta] = 0 \quad (3.32)$$

and anti-commute with the anti-Hermitian part $\mathcal{S}$

$$\{\mathcal{S}, \eta\} = 0. \quad (3.33)$$

Introducing the operators P and Q that project into these subspaces, the metric can be expressed as

$$\eta = P - Q. \quad (3.34)$$

For A in the two projector case, we have that

$$A = PHP + QHQ + r(PHP - QHP), \quad (3.35)$$

The matrix defined in (3.35) verifies both (3.32) and (3.33). This can be seen by noting that, for (3.35), $\mathcal{H} = PHP + QHQ$ and $\mathcal{S} = PHQ - QHP$ and therefore

$$\begin{aligned} (P - Q)\mathcal{H} - \mathcal{H}(P - Q) &= PHP - QHQ - (PHP - QHQ) = 0 \\ (P - Q)\mathcal{S} + \mathcal{S}(P - Q) &= PHQ + QHP + (-QHP - PHQ) = 0 \end{aligned} \quad (3.36)$$

In the N projector case, the Hermitian and anti-Hermitian parts of A may be written as, respectively

$$\mathcal{H} = \sum_{\text{mod } (|i-j|, 2)=0}^N P_i H P_j, \quad \mathcal{S} = \sum_{\text{mod } (|i-j|, 2)=1} [P_i H P_j - P_j H P_i]. \quad (3.37)$$

where $\text{mod } (x, y)$ denotes the remainder of the division between integers x and y and $\delta_{i,j}$ is the Kronecker δ . This means that the Hermitian part consists of all the products $P_i H P_j$ where both i and j have the same integer parity and the anti-Hermitian part consists of all the analogous products where i and j have different parities.

The commutation and anti-commutation relations, (3.32) and (3.33), are also valid in this case. This can be seen fairly straightforwardly by first noting that

$$P\mathcal{H} = \sum_{i,j \text{ odd}} P_i H P_j = \mathcal{H}P, \quad Q\mathcal{H} = \sum_{i,j \text{ even}} P_i H P_j = \mathcal{H}Q, \quad (3.38)$$

and that

$$PS = \sum_{\substack{i \text{ odd} \\ j \text{ even}}} [P_i H P_j - P_j H P_i] = SQ \quad , \quad QS = \sum_{\substack{i \text{ even} \\ j \text{ odd}}} [P_i H P_j - P_j H P_i] = SP. \quad (3.39)$$

Eq. (3.38) implies immediately that $(P - Q)\mathcal{H} = \mathcal{H}(P - Q)$ which in turn imply (3.32), whereas (3.39) similarly implies that $(P - Q)\mathcal{S} = \mathcal{S}(Q - P)$, which in turn imply (3.33).

We note additionally that the anti-Hermitian part of the pseudo-Hermitian matrix A , as defined from (3.4) and (3.28) is obtained by taking the limit

$$\mathcal{S} = \lim_{r \rightarrow \infty} \frac{A}{r}, \quad (3.40)$$

whose joint density distribution of its matrix elements is immediately found to be given by

$$P(\mathcal{S}) = \frac{1}{W_N} \exp \left(-\frac{\beta}{2} \text{tr } \mathcal{S} \mathcal{S}^\dagger \right). \quad (3.41)$$

We remark that this should not be mistaken as being an anti-pseudo-Hermitian matrix, which is defined in the literature [10] as an operator which verifies (1.12) for an antilinear, anti-Hermitian η .

There is a noteworthy property of matrices which are both pseudo- and anti-Hermitian regarding their trace. Let S be an $N \times N$ diagonalizable pseudo-Hermitian matrix verifying

$$S^\dagger = \mu S \mu^{-1} = -S \quad (3.42)$$

and with diagonalizations

$$SU = U\Xi,$$

$$SV = V\bar{\Xi}.$$

Using the pseudo-Hermitian relation (3.42) we obtain the connection between the two above diagonalizations

$$V^{-1} = U^\dagger \mu$$

and applying this to the trace of $S^\dagger S$, we obtain, through the cyclic property of the trace

and the anti-Hermiticity of S ,

$$\text{tr}(S^\dagger S) = \text{tr}(S^\dagger S V U^\dagger \mu) = \text{tr}(U^\dagger \mu S^\dagger S V) = -\text{tr}(U^\dagger \mu S V \Xi) = -\text{tr}(U^\dagger \mu V \Xi \Xi) = \text{tr}(\Xi \Xi).$$

Therefore, for any pseudo- and anti- Hermitian diagonalizable matrix S , we have that

$$\text{tr}(S^\dagger S) = \text{tr}(\Xi \Xi) = \text{tr}(\Xi \Xi) = \sum_{k=1}^N |\xi_k|^2 \quad (3.43)$$

where Ξ is the diagonal matrix containing the eigenvalues $\{\xi_k\}_{k=1,2,\dots,N}$ of S .

The anti-commutation of $\mathcal{S}$ with the metric η defines a discrete symmetry characteristic of the presence of chirality, that is of a P -symmetry [11]. So, apart from the fact that we are dealing with conjugate pairs of imaginary eigenvalues, we should expect to recover properties of the so-called chiral ensemble. To see this, let us consider the eigenvalue equation

$$\mathcal{S}\Psi = \lambda\Psi$$

together with the equation

$$\mathcal{S}\eta\Psi = -\lambda\eta\Psi$$

obtained using the anti-commutator relation (3.33). Adding and subtracting them we obtain

$$\mathcal{S}(\mathbf{I} + P - Q)\Psi = (\mathbf{I} - P + Q)\lambda\Psi$$

$$\mathcal{S}(\mathbf{I} - P + Q)\Psi = (\mathbf{I} + P - Q)\lambda\Psi$$

which, using the complementarity of $\mathcal{P}$ and $\mathcal{Q}$, become

$$\mathcal{S}P\Psi = \lambda Q\Psi \quad , \quad \mathcal{S}Q\Psi = \lambda P\Psi.$$

Applying Q and P to these, respectively, and using their involution properties, they become

$$Q\mathcal{S}P^2\Psi = (Q\mathcal{S}P)P\Psi = \lambda Q\Psi \quad (3.44)$$

and

$$P\mathcal{S}Q^2\Psi = (P\mathcal{S}Q)Q\Psi = \lambda P\Psi. \quad (3.45)$$

Easily, these equations decouple as

$$(Q\mathcal{S}P)^\dagger (Q\mathcal{S}P) P\Psi = -\lambda^2 P\Psi \quad (3.46)$$

and

$$(P\mathcal{S}Q)^\dagger (P\mathcal{S}Q) Q\Psi = -\lambda^2 Q\Psi. \quad (3.47)$$

It is convenient to introduce matrices $\mathcal{S}^\pm$ defined as

$$S^+ = Q\mathcal{S}P \quad , \quad S^- = P\mathcal{S}Q$$

which will simplify the analysis of the problem.

For the model discussed in this chapter, in the realization with two large contiguous diagonal square blocks of size M and $N-M$, and off-diagonal blocks of size $N \times (N-M)$ and $(N-M) \times N$, $\mathcal{S}^+$ has dimension $(N-M) \times M$. Its elements may be written as

$$\mathcal{S}_{ij}^+ = (Q\mathcal{S}P)_{ij} = -H_{M+i,j}^* \quad (3.48)$$

whereas $\mathcal{S}^-$ has dimension $M \times (N-M)$ and elements

$$\mathcal{S}_{ij}^- = (P\mathcal{S}Q)_{ij} = H_{i,M+j}. \quad (3.49)$$

The above matrices generate the Wishart matrices $(\mathcal{S}^\pm)^\dagger \mathcal{S}^\pm$ with distribution [12]

$$P(\mathcal{S}^\pm) = C_M^{-1} \exp \left[-\frac{\beta}{2} \text{tr}(\mathcal{S}^\pm)^\dagger \mathcal{S}^\pm \right], \quad (3.50)$$

which share the same set of eigenvalues with the distribution of a Laguerre ensemble

$$P_\beta(y_1, \dots, y_M) = \frac{1}{C_M} \prod_{k=1}^M y_k^{\beta(N-2M+1)/2-1} e^{-\beta y_k/2} \prod_{j>i} |y_j - y_i|^\beta \quad (3.51)$$

The case of N projectors behaves differently, however. Denoting $M = [N/2]$ and

$\text{sgn}(x) = x/|x|$, we have the matrices

$$\mathcal{S}_{ij}^+ = (Q\mathcal{S}P)_{2i,2j-1} = \text{sgn}(2i - 2j + 1)H_{2i,2j-1} \quad (3.52)$$

and

$$\mathcal{S}_{ij}^- = (P\mathcal{S}Q)_{2i-i,2j} = \text{sgn}(2i - 2j - 1)H_{2i-1,2j}. \quad (3.53)$$

Their dimensions are $M \times M$ if N is even and $M \times (M+1)$ and $(M+1) \times M$, respectively, if N is odd. This means that these matrices may still be reduced to the Wishart case and (3.50) still holds. Therefore, making the substitution $y_k = \lambda_k^2$, the joint distribution of the eigenvalues of the anti-pseudo-Hermitian ensemble, is obtained as

$$P_\beta(\lambda_1, \dots, \lambda_M) = \frac{1}{Z_M} \prod_{\gamma=1}^M \exp\left(-\frac{\beta|\lambda_\gamma|^2}{2}\right) |\lambda_\gamma|^{\beta(N-2M+1)-1} \prod_{\xi > \gamma} |\lambda_\xi^2 - \lambda_\gamma^2|^\beta. \quad (3.54)$$

In (3.54), the term $N - 2M$ in the exponent is the number of zeros and M is the number of complex conjugate pairs. This corresponds to the number of differences between non-zero eigenvalues and the zeros in the former case, and the term $|\lambda|^{\beta-1}$ corresponds to the number of differences between eigenvalues and their complex conjugates. Thus the factorization of the Jacobian in the Hermitian case extends to the pseudo-Hermitian case in terms of these differences.

This is in fact the case, and in order to obtain the density in terms of eigenvalues for the general matrix A with pseudo-Hermiticity operator η , we must firstly obtain an equation for the diagonalization of A [13]. We begin by writing

$$AU = U\Lambda \quad (3.55)$$

and the diagonalization in terms of the complex conjugate eigenvalues

$$AV = V\bar{\Lambda}. \quad (3.56)$$

which corresponds to switching the positions of the eigenvectors of the conjugate pairs in (3.55). Taking the adjoint of (3.55) and considering the pseudo-Hermiticity of A as in

(??), however we have that

$$U^\dagger \eta A = \bar{\Lambda} U^\dagger \eta \quad (3.57)$$

such that, multiplying Eq. (3.56) by $U^\dagger \eta$ to the left and subtracting (3.57) multiplied by V to the right we get

$$U^\dagger \eta V \bar{\Lambda} - \bar{\Lambda} U^\dagger \eta V = 0 \quad (3.58)$$

and, therefore, $U^\dagger \eta V = \mathbf{1}$ and thus the Hermiticity of η implies that $V^{-1} = (\eta U)^\dagger$ or, alternatively,

$$U^{-1} = (\eta V)^\dagger. \quad (3.59)$$

Now, to define the Jacobian of interest we must perform a change of variables that maps the elements of A into the N eigenvalues and another set of M variables which describe the remaining degrees of freedom. In other words,

$$P(\Gamma_j, p_\mu^{\{k\}}) = P(A) \det \left[\frac{\partial (a_{i,i}, a_{i,j}^{\{k\}})}{\partial (\Gamma_j, p_\mu)} \right] \quad (3.60)$$

where Γ_j are the eigenvalues, p_μ are the M additional variables needed by the constraints in the eigenfunctions and the last term denotes the Jacobian of the transformation.

To obtain this Jacobian, we must initially consider that

$$A = U \Lambda U^{-1}. \quad (3.61)$$

Therefore

$$\frac{\partial}{\partial \Gamma_j} A_{\gamma,\xi} = \frac{\partial}{\partial \Gamma_j} [U \Lambda U^{-1}]_{\gamma,\xi} = \frac{\partial}{\partial \Gamma_j} \sum_{\delta,\epsilon} U_{\gamma,\delta} \Lambda_{\delta,\epsilon} U_{\xi,\epsilon}^{-1}$$

and since Λ is a diagonal matrix containing the eigenvalues of A , and noting that the remaining terms are of U and its inverse,

$$\frac{\partial A_{\gamma,\xi}}{\partial \Gamma_j} = \delta_{j,\gamma} \delta_{\gamma,\xi}. \quad (3.62)$$

The inverse relation $U^{-1}U = \mathbf{1}$ gives us

$$\begin{aligned} \frac{\partial}{\partial p_\mu} U^{-1}U &= \left(\frac{\partial}{\partial p_\mu} U^{-1} \right) U + U^{-1} \left(\frac{\partial}{\partial p_\mu} U \right) = \mathbf{0} \\ \Rightarrow \left(\frac{\partial}{\partial p_\mu} U^{-1} \right) U &= -U^{-1} \left(\frac{\partial}{\partial p_\mu} U \right) \equiv \Theta_\mu. \end{aligned} \quad (3.63)$$

This new matrix Θ_μ has a property of interest,

$$\Theta_\mu^\dagger = U^\dagger \left(\frac{\partial}{\partial p_\mu} U^{-1} \right)^\dagger = V^{-1} \left(\frac{\partial}{\partial p_\mu} V \right) \quad (3.64)$$

which may be used in specific realizations of A to obtain further information about the change of variables.

Using the definition in the above Eq. (3.63), multiplying (3.55) from the left by U^{-1} and taking the derivative of the resulting equation by a parameter p_μ we obtain

$$U^{-1} \frac{\partial A}{\partial p_\mu} U = U^{-1} \left(\frac{\partial}{\partial p_\mu} U \right) \Lambda - \Lambda \left(\frac{\partial}{\partial p_\mu} U^{-1} \right) U = (\Theta_\mu \Lambda - \Lambda \Theta_\mu) \quad (3.65)$$

or, in component terms,

$$\left(U^{-1} \frac{\partial A}{\partial p_\mu} U \right)_{\gamma, \xi} = (\Theta_\mu)_{\gamma, \xi} (\Gamma_\xi - \Gamma_\gamma). \quad (3.66)$$

Therefore, instead of writing the Jacobian matrix directly, we follow the approach of [13] and write the auxiliary matrix

$$\Omega = \begin{pmatrix} \left[U^{-1} \frac{\partial A}{\partial \Gamma_j} U \right]_{\xi, \xi} & \left[U^{-1} \frac{\partial A}{\partial \Gamma_j} U \right]_{\gamma, \xi} \\ \left[U^{-1} \frac{\partial A}{\partial p_\mu} U \right]_{\xi, \xi} & \left[U^{-1} \frac{\partial A}{\partial p_\mu} U \right]_{\gamma, \xi} \end{pmatrix} = \begin{pmatrix} \delta_{j, \gamma} & 0 \\ 0 & [\Theta_\mu]_{\gamma, \xi} (\Gamma_\xi - \Gamma_\gamma) \end{pmatrix}. \quad (3.67)$$

The determinant of this matrix is then the Jacobian up to a function on the eigenvector parameters

$$\phi(p_\mu) \det \Omega = \det \left[\frac{\partial(a_{i,i}, a_{i,j})}{\partial(\Gamma_j, p_\mu)} \right]. \quad (3.68)$$

Thus the Jacobian factors out.

3.4 Statistics of the Average Characteristic Polynomial

Additionally, it is convenient to present here the method through which we shall consider the statistical behavior of the characteristic polynomial of a random matrix. This will be used in Chapters 4 and 5 to study the statistics of the roots of the characteristic polynomial. It is thus convenient to explore some general properties are shared by the models discussed in both of these Chapters.

Let us consider a matrix A . We call A a *sample matrix* if some or all of its entries are random variables sorted from some distribution. As an example, for a matrix of the β -ensemble of A. Edelman and I. Dumitriu used in chapter 2, those correspond only to the elements of the diagonal and first upper and lower subdiagonal which have Gaussian and χ distributions, respectively. The behavior of the first two moments of a single characteristic polynomial of a sample matrix A is then given by [14],

$$M_1(z) = \langle \det [A - z] \rangle \quad (3.69)$$

$$M_2(z, w) = \langle \det [(A - z) (A^\dagger - w)] \rangle \quad (3.70)$$

from which the variance is given by

$$\Sigma(z) = M_2(z, \bar{z}) - \overline{M_1(z)} M_1(z) = M_2(z, \bar{z}) - M_1(\bar{z}) M_1(z) \quad (3.71)$$

and the variance relative to the mean by

$$\rho(z) = \frac{M_2(z, \bar{z})}{M_1(\bar{z}) M_1(z)} - 1. \quad (3.72)$$

In the above equations, the averaging $\langle . \rangle$ is again taken over the ensemble. The determinants in Eqs. (3.69) and (3.70) may be expressed in simpler terms by considering a sample matrix A . Since the sample matrix chosen is not *a priori* diagonalizable, we must consider its Jordan canonical form

$$A = U \left[\left(\bigoplus_{k=1}^{\kappa} J_k \right) \oplus \Lambda \right] U^{-1}$$

where $J_k, k = 1, 2, \dots, \kappa$ are the Jordan blocks corresponding to the eigenvalues $\tau_1, \tau_2, \dots, \tau_\kappa$ of multiplicity $m_1, m_2, \dots, m_\kappa$, Λ is a diagonal matrix whose entries are the ν unique eigenvalues of A , $\{\alpha_l\}_{l=1,2,\dots,\nu}$ and U is the similarity transformation between A and its Jordan form, noting that $\sum_{k=1}^{\kappa} m(k) + \nu = N$.

Thus Eq. (3.69) becomes

$$\begin{aligned} M_1(z) &= \left\langle \prod_{k=1}^{\kappa} \det(z\mathbf{I}_{m(k)} - J_k) \det(z\mathbf{I}_\nu - \Lambda) \right\rangle \\ &= \left\langle \prod_{k=1}^{\kappa} (z - \tau_k)^{m(k)} \prod_{l=1}^{\nu} (z - \alpha_l) \right\rangle = \prod_{k=1}^N (z - \lambda_k) \end{aligned} \quad (3.73)$$

where in the last equality, the definition of the roots of the characteristic polynomial was used, and any potentially multiple roots are repeated with distinct indexes. Eq. (3.73) is thus straightforward to calculate from the roots, λ_k , of the average characteristic polynomial and does not require us to consider sample matrices for its calculation. On the other hand, (3.70) becomes

$$\begin{aligned} M_2(z, w) &= \langle \det(A - z) \det(A^\dagger - w) \rangle = \langle \det(A - z) \det \eta^{-1} \det(A - w) \det \eta \rangle \\ &= \left\langle \prod_{k=1}^{\kappa} (z - \tau_k)^{m(k)} \prod_{l=1}^{\nu} (z - \alpha_l) \prod_{k'=1}^{\kappa} (z - \tau_{k'})^{m(k')} \prod_{l'=1}^{\nu} (z - \alpha_{l'}) \right\rangle \\ &= \sum_{l=1, m=1}^{N, N} \left\langle f_N^{(l)} f_N^{(m)} \right\rangle (-z)^{N-l} (-w)^{N-m} \end{aligned} \quad (3.74)$$

where (1.12) was used. In the above Eq. (3.74) the coefficients cannot be obtained directly from the roots of the characteristic polynomial. Instead, they are symmetrical functions over the N eigenvalues of each sample matrix A such that

$$\begin{aligned} f_N^{(0)} &= 1, \quad f_N^{(1)} = \mu_1 + \mu_2 + \dots + \mu_N, \\ f_N^{(2)} &= \mu_1\mu_2 + \mu_1\mu_3 + \dots + \mu_{N-1}\mu_N, \quad \dots, \quad f_N^{(N)} = \mu_1\mu_2 \dots \mu_N, \end{aligned}$$

where

$$\{\mu_k\}_{k \in I} = \left\{ \underbrace{\tau_1, \tau_1, \dots, \tau_1}_{m(1) \text{ times}}, \underbrace{\tau_2, \tau_2, \dots, \tau_2}_{m(2) \text{ times}}, \dots, \underbrace{\tau_\kappa, \tau_\kappa, \dots, \tau_\kappa}_{m(\kappa) \text{ times}}, \alpha_1, \alpha_2, \dots, \alpha_\nu \right\}$$

is a re-indexing I of the eigenvalues.

In other words, each $f_n^{(l)}$ is the sum of all unique permutations of the l -uple products of the $\mu_{1,2,\dots,n}$. The coefficients of order N can be obtained from lower-order coefficients recursively by means of

$$f_N^{(l)} = f_{N-1}^{(l)} + \mu_N f_{N-1}^{(l-1)}$$

which works for all l as long as we impose the boundary conditions

$$f_N^{(0)} = 1, \quad f_N^{(l)} = 0, l > N.$$

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Chapter 4

Separated Model

In this chapter, we consider the first of the two projector models to be studied. As introduced in the previous chapter, by choosing a contiguous set of basis elements, we split the $N \times N$ matrix in a 2×2 block matrix. In this block, we introduce a pseudo-Hermitian interaction between the image subspaces associated with the projectors

$$\mathcal{P} = \sum_{k=1}^M |k\rangle \langle k|$$

and

$$\mathcal{Q} = \sum_{k=M+1}^N |k\rangle \langle k|$$

by means of an interaction described by the Hamiltonian

$$A = \mathcal{P}H\mathcal{P} + \mathcal{Q}H\mathcal{Q} + r\mathcal{P}H\mathcal{Q} - \mathcal{Q}H\mathcal{P}.$$

In this context, H is a random matrix of the Gaussian ensembles of Sec. 1.2.

We begin by presenting in Sec. 1 results regarding the spectral behavior of sample matrix so defined. This is followed in Sec. 2 by the study the behavior of the average characteristic polynomials of this ensemble and in Sec. 3 by presenting some asymptotic limits. Finally, in Sec. 4, some further numerical results are presented, exploring the statistics surrounding the average characteristic polynomial and the spectral density over the complex plane.

4.1 Spectral Behavior

We begin by considering the spectral behavior of a sample matrix of size $N = 32$ with distinct values of block size. For this matrix, we vary the interaction strength r for block sizes $M = 2, 4, 8$ and 16 . The results are presented in Figs. 4.1-4.4. Figs. 4.1.a-4.4.a show the trajectory of the eigenvalues in the complex plane as a function of the r parameter, depicted in the z axis. It may be noted that the number of eigenvalues that escape into the complex plane is twice the block size in each case, corresponding to one pair per block size. This may be seen also in Figs. 4.1.c-4.4.c and 4.1.d-4.4.d, where it is of special interest to note that in the case for $M = N/2$ all the eigenvalues leave the real axis. Fig. 4.1.b-4.4.b show similar behavior to the $\mathcal{PT}$ -symmetry breaking figure of [1] and also to the behavior of Figs. 2.1, 2.2, 2.9 and 2.10 of Chapter 2.

It is, however, important to investigate whether this is an effect of small matrices or whether it applies also to larger matrix size. In Figs. 4.5.a, 4.5.b, 4.5.c and 4.5.d, the evolution of the spectra of a matrix of size $N = 200$ for block sizes $M = 25, 50, 100$, respectively, for matrices of the $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ cases are shown. For $M = 25$, the spectra remains predominantly real, and only some of the eigenvalues move into the complex plane, although all of them still remain close to the Ginibre circular limit. When moving to $M = 50$, the number of eigenvalues in the complex plane increases in number and, simultaneously, the gaps in the real eigenvalues start to become increasingly more noticeable. In the case of $M = 75$, the number of eigenvalues in the complex plane becomes even larger, and the thinning of the eigenvalues in the real axis continue, including the edges becoming compressed. Finally, when $M = N/2 = 100$, most of the eigenvalues become complex with small fluctuations both around the real axis and the Ginibre limit.

These eigenvalues that remain in the real axis are likely to be random fluctuations. To illustrate this, in Fig. 4.6.a, we present the average fraction of real eigenvalues of a sample of 100 $N = 100$ $\mathbb{R}$ matrices for varying block size. Three different choices of the interaction strength r are considered, $r = 0.25, 1.0$ and 4.0 , and increasing the interaction strength decreases the number of real eigenvalues for the same value of the parameter r , specially for larger block sizes. Such behavior is intensified for larger matrix size, as may be seen in 4.6.b, where the same calculations are presented for a sample of 100 $N = 400$

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane, as a function of the interaction strength, depicted as the Z axis as well as the coloring of the eigenvalues.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane, with color depicting the interaction strength.

(c) Eigenvalue behavior, projected in the $\text{Re} \times s$ plane, with color depicting the interaction strength.

(d) Eigenvalue behavior, projected in the $\text{Im} \times s$ plane, with color depicting the interaction strength.

Figure 4.1: Complex eigenvalue behavior for an $N = 32$ $\mathbb{R}$ matrix with upper block size $M = 2$, as a function of the interaction strength r .

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane, as a function of the interaction strength, depicted as the Z axis as well as the coloring of the eigenvalues.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane, with color depicting the interaction strength.

(c) Eigenvalue behavior, projected in the $\text{Re} \times s$ plane, with color depicting the interaction strength.

(d) Eigenvalue behavior, projected in the $\text{Im} \times s$ plane, with color depicting the interaction strength.

Figure 4.2: Complex eigenvalue behavior for an $N = 32$ $\mathbb{R}$ matrix with upper block size $M = 4$, as a function of the interaction strength r .

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane, as a function of the interaction strength, depicted as the Z axis as well as the coloring of the eigenvalues.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane, with color depicting the interaction strength.

(c) Eigenvalue behavior, projected in the $\text{Re} \times s$ plane, with color depicting the interaction strength.

(d) Eigenvalue behavior, projected in the $\text{Im} \times s$ plane, with color depicting the interaction strength.

Figure 4.3: Complex eigenvalue behavior for an $N = 32$ $\mathbb{R}$ matrix with upper block size $M = 8$, as a function of the interaction strength r .

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane, as a function of the interaction strength, depicted as the Z axis as well as the coloring of the eigenvalues.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane, with color depicting the interaction strength.

(c) Eigenvalue behavior, projected in the $\text{Re} \times s$ plane, with color depicting the interaction strength.

(d) Eigenvalue behavior, projected in the $\text{Im} \times s$ plane, with color depicting the interaction strength.

Figure 4.4: Complex eigenvalue behavior for an $N = 32$ $\mathbb{R}$ matrix with upper block size $M = 16$, as a function of the interaction strength r .

real matrices where the $r = 1.0$ and $r = 4.0$ overlap almost entirely. This suggests that in the asymptotic limit of very large N , there is very little dependence on the parameter r with regard to how many eigenvalues become complex, and that it is rather determined mainly by the block size M .

It should be noted that the symmetry around $M = N/2$ is to be expected, as the cases of block size M and $N - M$ consist of the the same block construction, just with the order of the blocks interchanged. Thus, the fact that the elements follow the same distributions, altered only by the symmetry of their respective block, implies that the statistical behavior should be the same as is seen. Additionally, the $\mathbb{C}$ and $\mathbb{H}$ cases behave similarly to these cases, and it is also worthy to mention that this contrasts to the result in the literature [2], in which there is an asymptotic tendency for all the eigenvalues to become complex. This seems to be tied, in the present case, to the existing pseudo-Hermitian block-wise quality of the matrix.

4.2 Characteristic Polynomial

In order to consider the characteristic polynomials of the present case, let us write the two projectors as

$$\mathcal{P} = \sum_{k=1}^M |k\rangle \langle k| \quad (4.1)$$

and

$$\mathcal{Q} = \sum_{k=M+1}^N |k\rangle \langle k| \quad (4.2)$$

in the orthonormal basis of states $|k\rangle_{k=1,2,\dots,N}$. The resulting pseudo-Hermitian matrix will be denoted as

$$A^{(M,N)}(r) = \mathcal{P}H\mathcal{P} + \mathcal{Q}H\mathcal{Q} + r\mathcal{P}H\mathcal{Q} - r\mathcal{Q}H\mathcal{P} \quad (4.3)$$

and its characteristic polynomial as

$$q_{M,N}(\lambda; r) = \det(A_{M,N}(r) - \lambda\mathbb{I}). \quad (4.4)$$

(a) $M = 25$

(b) $M = 50$

(c) $M = 75$

(d) $M = 100$

Figure 4.5: Sample matrix of size $N = 200$ spectra for $M = 25$ (a), 50 (b), 75 (c) and 100 (d).

Figure 4.6: Fraction of eigenvalues which remain in the real axis as a function of the block size M .

This q polynomial will not only be different for each choice of block size M , but also will vary for each individual sample matrix. As seen in the case of Chapter 2, looking into the average behavior may provide further insights that might be otherwise obscured in a sample due to its innate random fluctuations. Therefore, to better assess the behavior of the ensemble as a whole, we focus instead on the average polynomial of this ensemble, which we shall denote as

$$Q_{M,N}(\lambda; r) = \langle q_{M,N}(\lambda; r) \rangle \quad (4.5)$$

where $\langle . \rangle$ denotes the average over the ensemble.

In order to obtain a treatable form of (4.5) we perform the Laplace expansion of the determinant (4.4) by one of its rows or columns. Choosing the first column seems to be the most convenient choice [3], such that we obtain

$$Q_{M,N}(\lambda) = \sum_{k=1}^N \left\langle (-1)^{1+k} \left(A_{k,1}^{(M,N)} - \delta_{k,1} \lambda \right) \det(C_{k,1}) \right\rangle \quad (4.6)$$

where $C_{k,1}$ is the minor matrix [4] obtained by removing row k and column 1 from $A^{(M,N)}$.

To continue, we first note the structure of the matrix

$$A^{(M,N)} - \lambda \mathbb{I} = \begin{pmatrix} D^{(1,M)} & S^{(M,N)} \\ -[S^{(M,N)}]^\dagger & D^{(M+1,N)} \end{pmatrix}$$

where $D^{(i,j)}$ is the diagonal block of the Hermitian H from row i to row j and $S^{(l,m)}$ is the upper-right off-diagonal block with l rows and m columns of H multiplied by the factor r . This suggests that in the Laplace expansion of the determinant there are three separate cases, namely: the diagonal element, occurring once per column; the $M - 1$ Hermitian elements of the column in the $D^{(1,M)}$ block; and the $N - M$ non-Hermitian elements of the column in the $-S^\dagger$ block.

The contribution of the diagonal element is

$$\left\langle A_{1,1}^{(M,N)} - \lambda \right\rangle \langle \det C_{1,1} \rangle = -\lambda Q_{M-1,N-1-(M-1)} = -\lambda Q_{M-1,N-1} \quad (4.7)$$

since the elements of the diagonal of A are all i.i.d. with zero mean. It should be noted that the minor matrix $C_{1,1}$ corresponds to the original matrix with the first row and the first column removed. It is thus a $N - 1$ matrix, with an upper hermitian block of order $M - 1$ and a lower pseudo-Hermitian block of order $N - M$, otherwise following the same structural characteristics of the original matrix. Therefore, the average behavior of the minor, $\det C_{1,1}$, is exactly the same as a sample matrix $A^{(M-1,N-1)}$.

The contribution of the Hermitian non-diagonal elements may be seen by looking e.g. at the k -th element of the chosen column

$$\left\langle \overline{A_{1,k}^{(M,N)}} \det C_{k,1} \right\rangle,$$

which requires us to expand the determinant yet again. This minor matrix may be written

as

$$C_{k,1} = \left(\begin{array}{ccc|c} x & A_{1,k}^{(M,N)} & u & S^{(M,N)} \\ G & v & D^{(M-k-1,M)} & \\ \hline & -[S^{(M,N)}]^\dagger & & D^{M+1,N} \end{array} \right)$$

where we define the row vectors

$$x = (A_{1,2}^{(M,N)} \ A_{1,3}^{(M,N)} \ \dots \ A_{1,k-1}^{(M,N)})$$

$$u = (A_{1,k+1}^{(M,N)} \ A_{1,k+2}^{(M,N)} \ \dots \ A_{1,M}^{(M,N)})$$

and the column vector

$$v = \begin{pmatrix} A_{3,k}^{(M,N)} \\ A_{4,k}^{(M,N)} \\ \vdots \\ A_{k-1,k}^{(M,N)} \\ A_{k,k}^{(M,N)} - \lambda \\ \overline{A_{k,k+1}^{(M,N)}} \\ \vdots \\ \overline{A_{k,M}^{(M,N)}} \end{pmatrix}.$$

where $\bar{x}$ denotes the complex conjugate of x .

Because the elements of each row of $A^{(M,N)}$ are i.i.d., $\langle \overline{A_{1,k}} v_j \rangle = 0$ for all $j = 1, \dots, M-2$. Therefore, by expanding the determinant in the only element with non-zero contribution, $A_{1,k}^{(M,N)}$, the contribution becomes

$$(-1)^k \langle |A_{1,k}^{(M,N)}|^2 \rangle \langle \det C_{k,1;1,k} \rangle$$

where $C_{i,j;l,m}$ is the second-order minor matrix of $A^{(M,N)}$, obtained by removing both the i -th row and j -th column as well as the l -th row and m -th column. Since the distribution of the elements of $H^{(M,N)}$ has known total variance σ^2 , the only remaining step is to note that, as in the diagonal case, the determinant of this second-order minor behaves analogously to the minor obtained in diagonal case. In this last case, however, leading to the characteristic polynomial of a matrix $A^{(M-2,N-2)}$, again following the same characteris-

tics of the original matrix. Therefore

$$\begin{aligned} \left\langle \overline{\left(A_{1,k}^{(M,N)}\right)} \det C_{k,1} \right\rangle &= (-1)^k \langle |H_{1,k}|^2 \rangle \langle \det C_{k,1} \rangle \\ &= (-1)^k \sigma^2 Q_{M-2,N-2}, \end{aligned}$$

where it should be noted that due to the removal of one row and column, the exponent of the (-1) term of the second expansion is decreased by one as the k -th row of the original matrix becomes the $(k-1)$ -th row of the reduced matrix. In addition, the same reasoning is valid for all the remaining Hermitian block column elements, which implies that the total contribution of these $M-1$ elements, noting the sign of the determinant expansion, becomes

$$\sum_{\text{Hermitian}} = -(M-1)\sigma^2 Q_{M-2,N-2}. \quad (4.8)$$

The contribution of the non-Hermitian elements, on the other hand may be seen by looking at an element of the chosen column lying within this block, e.g. the k -th element, with $k \geq 1$,

$$\left\langle \overline{\left(-A_{1,M+k}^{(M,N)}\right)} \det C_{M+k,1} \right\rangle,$$

which, as in the case of the Hermitian contributions, requires us to expand the determinant another time. This minor may be written as

$$C_{M+k,1} = \begin{pmatrix} a & A_{1,M+k}^{(M,N)} & b \\ D^{(2,M+k-1)} & c & Z \\ -Z^\dagger & d & D^{(M+k,N)} \end{pmatrix}$$

where we introduce the row vectors

$$a = \left(A_{1,2}^{(M,N)} \ A_{1,3}^{(M,N)} \ \dots \ A_{1,M}^{(M,N)} \ A_{1,M+1}^{(M,N)} \ \dots \ A_{M+k-1,M}^{(M,N)} \right) \quad ,$$

$$b = \left(A_{1,M+k+1}^{(M,N)} \ A_{1,M+k+2}^{(M,N)} \ \dots \ A_{1,N}^{(M,N)} \right),$$

the column vectors

$$c = \begin{pmatrix} A_{2,M+k}^{(M,N)} \\ A_{3,M+k}^{(M,N)} \\ \vdots \\ A_{M+k-1,M+k}^{(M,N)} \end{pmatrix}, \quad d = \begin{pmatrix} -\overline{A_{M+k,M+k+1}^{(M,N)}} \\ -\overline{A_{M+k,M+k+2}^{(M,N)}} \\ \vdots \\ -\overline{A_{M+k,N}^{(M,N)}} \end{pmatrix},$$

and the matrix

$$Z = \begin{pmatrix} A_{2,M+k+1}^{(M,N)} & A_{2,M+k+2}^{(M,N)} & \dots & A_{2,N}^{(M,N)} \\ A_{3,M+k+1}^{(M,N)} & A_{3,M+k+2}^{(M,N)} & \dots & A_{3,N}^{(M,N)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M+k-1,M+k+1}^{(M,N)} & A_{M+k-1,M+k+2}^{(M,N)} & \dots & A_{M+k-1,N}^{(M,N)} \end{pmatrix}.$$

Again, the successive expansion of the determinant, and taking into account the independence of the elements of each row in the first row, of this minor implies that the contribution becomes

$$\begin{aligned} & \left\langle -\overline{A_{1,M+k}^{(M,N)}} \det C_{M+k,1} \right\rangle = \\ & -(-1)^{(M+k)} \left\langle \left| A_{1,M+k}^{(M,N)} \right|^2 \right\rangle \left\langle \det \begin{pmatrix} D^{(2,M+k-1)} & Z \\ -Z^\dagger & D^{(M+k+1,N)} \end{pmatrix} \right\rangle = \\ & -(-1)^{(M+k)} r^2 \left\langle \left| H_{1,M+k}^{(M,N)} \right|^2 \right\rangle \langle \det C_{M+k,1;1,M+k} \rangle \\ & = -(-1)^{(M+k)} r^2 \sigma^2 Q_{M-1,N-2}. \end{aligned}$$

where we note that, in this case, the average determinant remaining is the polynomial corresponding to that of the ensemble with two rows and columns removed, yielding a total matrix size of $N - 2$, and with only one of those rows and columns having been removed from the $M \times M$ block, leading to a $(M - 1) \times (M - 1)$ block instead. We also note that, again, the further removal of a row implies that there is one less row and column in the expansion, and consequentially the exponent of the sign is reduced accordingly.

For the column elements of the non-Hermitian block, the alternating signal is again canceled, and hence the contribution due to the $N - M$ elements becomes

$$\sum_{\text{non-Hermitian}} = (N - M) \sigma^2 r^2 Q_{M-1,N-2} \quad (4.9)$$

The recurrence relation then becomes

$$\begin{aligned} Q_{M,N}(\lambda; r) = & -\lambda Q_{M-1,N-1} - \sigma^2(M-1)Q_{M-2,N-2} \\ & + \sigma^2 r^2(N-M)Q_{M-1,N-2}. \end{aligned} \quad (4.10)$$

The above recurrence relation (4.10) is valid for all M, N if we add the additional boundary conditions:

$$\begin{aligned} Q_{M,N}(\lambda; r) &= 0, \quad M < 0 \\ Q_{0,N}(\lambda; r) &= \left(\frac{\sigma}{\sqrt{2}}\right)^n \mathbf{H}_N\left(-\frac{\lambda}{\sqrt{2}\sigma^2}\right) \end{aligned} \quad (4.11)$$

where $\mathbf{H}_k$ is the Hermite polynomial in a rescaled variable. The case $M = 0$ corresponds to that in which Q becomes the N -th order polynomial of a matrix of the corresponding Gaussian ensemble [5]. The first $Q_{M,N}$ polynomials are listed in Table 4.1 up to $N = 5$. It should be noted that the symmetry seen there, namely that, for blocks sizes M and its complementary with regard to N , $N - M$,

$$Q_{M,N}(\lambda; r) = Q_{N-M,N}(\lambda; r)$$

is a result of the fact that the position of the order M block does not alter the behavior of the average polynomial. Expanding the determinant of the order $N - M$ upper block case on the last row will allow us to follow the analogous argument presented in this section, leading to the same recurrence for the average polynomial in that case. For this reason, in the numerical results we shall restrict our calculations to $M \leq N/2$, as the remaining cases, $M > N/2$ are identical to their symmetrical with respect to $N/2$.

4.3 Asymptotics of the Q Polynomials

We may also consider certain asymptotics of interest for the polynomial obtained in the previous section. We firstly consider $r \rightarrow 0$, for which the recursion relation (4.10) becomes

$$Q_{M,N}(\lambda; 0) = -\lambda Q_{M-1,N-1}(\lambda; 0) - \sigma^2(M-1)Q_{M-2,N-2}(\lambda; 0). \quad (4.12)$$

M	N	$Q_{M,N}$	M	N	$Q_{M,N}$
0	0	1	0	4	$\lambda^4 - 6\sigma^2\lambda^2 + 3\sigma^4$
0	1	$-\lambda$	1	4	$\lambda^4 + 3(r^2 - 1)\sigma^2\lambda^2 - 3r^2\sigma^4$
1	1	$-\lambda$	2	4	$\lambda^4 + 2(2r^2 - 1)\sigma^2\lambda^2 + (2r^4 + 1)\sigma^4$
0	2	$\lambda^2 - \sigma^2$	3	4	$\lambda^4 + 3(r^2 - 1)\sigma^2\lambda^2 - 3r^2\sigma^4$
1	2	$\lambda^2 + r^2\sigma^2$	4	4	$\lambda^4 - 6\sigma^2\lambda^2 + 3\sigma^4$
2	2	$\lambda^2 - \sigma^2$	0	5	$-\lambda^5 + 10\sigma^2\lambda^3 - 15\sigma^4\lambda$
0	3	$-\lambda^3 + 3\sigma^2\lambda$	1	5	$-\lambda^5 - 2(2r^2 - 3)\sigma^2\lambda^3 - 3(1 - 4r^2)\sigma^4\lambda$
1	3	$-\lambda^3 + (1 - 2r^2)\sigma^2\lambda$	2	5	$-\lambda^5 - 2(3r^2 - 2)\sigma^2\lambda^4 - 3(2r^4 - 2r^2 + 1)\sigma^4\lambda$
2	3	$-\lambda^3 + (1 - 2r^2)\sigma^2\lambda$	3	5	$-\lambda^5 - 2(3r^2 - 2)\sigma^2\lambda^4 - 3(2r^4 - 2r^2 + 1)\sigma^4\lambda$
3	3	$-\lambda^3 + 3\sigma^2\lambda$	4	5	$-\lambda^5 - 2(2r^2 - 3)\sigma^2\lambda^3 - 3(1 - 4r^2)\sigma^4\lambda$
			5	5	$-\lambda^5 + 10\sigma^2\lambda^3 - 15\sigma^4\lambda$

Table 4.1: Q Polynomials.

The recursion relation of the Hermite polynomials

$$\mathbf{H}_N(x) = 2x\mathbf{H}_{N-1}(x) - 2(N-1)\mathbf{H}_{N-2}(x) \quad (4.13)$$

may be used alongside the scaling of (4.11) to obtain the recursion for the $M = 0$ case.

$$Q_{0,N}(\lambda; r) = -\lambda Q_{0,N-1}(\lambda; r) - (N-1)\sigma^2 Q_{0,N-2}(\lambda; r). \quad (4.14)$$

From (4.10) and (4.14) we can obtain, through induction, the property that

$$Q_{M,N}(\lambda; 0) = Q_{0,M}(\lambda; 0)Q_{0,N-M}(\lambda; 0). \quad (4.15)$$

To do so, we first note that the first two cases are

$$\begin{aligned} Q_{0,0}(\lambda; 0) &= 1 \\ Q_{1,1}(\lambda; 0) &= -\lambda \end{aligned} \quad (4.16)$$

where the first follows from the boundary conditions (4.11) and the second from the recursion (4.12). To prove the case of higher N , we assume (4.15) to be true for M, N and $M - 1, N - 1$.

$$Q_{M,N}(\lambda; 0) = Q_{0,M}(\lambda; 0)Q_{0,N-M}(\lambda; 0)$$

from which, applying (4.12) and (4.14) implies that

$$\begin{aligned} Q_{M+1,N+1}(\lambda; 0) &= -\lambda Q_{0,M}(\lambda; 0)Q_{0,N-M}(\lambda; 0) - \sigma^2 M Q_{0,M-1}(\lambda; 0)Q_{0,N-M}(\lambda; 0) \\ &= Q_{0,M+1}(\lambda; 0)Q_{0,N-M}(\lambda; 0). \end{aligned}$$

By induction and considering the first two cases, (4.15) holds for all $M, N \geq 0$.

On the other hand, for the $r \rightarrow \infty$ limit we must firstly consider a scaling of Eq. (4.10) given by

$$\xi = \frac{\lambda}{\sigma r}, \quad \tilde{Q}_{M,N}(\xi) = \frac{1}{(\sigma r)^N} Q_{M,N}(\lambda(\xi); r)$$

such that (4.10) may be written as

$$\tilde{Q}_{M,N}(\xi) = -\xi \tilde{Q}_{M-1,N-1}(\xi) - \frac{M-1}{r^2} \tilde{Q}_{M-2,N-2}(\xi) + (N-M) \tilde{Q}_{M-1,N-2}(\xi)$$

which becomes

$$\begin{aligned} \tilde{Q}_{M,N}(\xi) &= -\xi \tilde{Q}_{M-1,N-1}(\xi) + (N-M) \tilde{Q}_{M-1,N-2}(\xi) + \mathcal{O}\left(\frac{1}{r^2}\right) \\ &= -\xi \tilde{Q}_{M-1,N-1}(\xi) + (N-M) \tilde{Q}_{M-1,N-2}(\xi), \quad r \rightarrow \infty \end{aligned} \tag{4.17}$$

in the asymptotic limit.

In order to go beyond Eq. (4.17), it is convenient to note that the matrix $A - \lambda \mathbf{1}$ in this limit has the block structure

$$A(r) - \lambda \mathbf{1}_N = r \begin{pmatrix} \frac{1}{r} D^{(1,M)} - \frac{\lambda}{r} \mathbf{1}_M & H^{(M,N)} \\ -H^{\dagger(M,N)} & \frac{1}{r} D^{(M+1,N)} - \frac{\lambda}{r} \mathbf{1}_{N-M} \end{pmatrix} \stackrel{r \gg 1}{\cong} \begin{pmatrix} -\lambda \mathbf{1}_M & r H^{(M,N)} \\ -r H^{\dagger(M,N)} & -\lambda \mathbf{1}_{N-M} \end{pmatrix}$$

where $H^{(M,N)}$ is the upper-right block of M rows and $N - M$ columns of the Hermitian matrix H from which A is created. It is straightforward to show from elementary determinantal properties that

$$\det [A(r) - \lambda \mathbf{1}]$$

has the eigenvalue zero with multiplicity $N - 2M$ if $M \leq N/2$ or $2M - N$ otherwise. Let us take the case in which $M \leq N/2$, for which we write

$$\tilde{Q}_{M,N}(\xi) = (-\xi)^{N-2M} \omega_M^N(-\xi^2)$$

which, when substituted into (4.17), yields

$$\omega_M^N(-\xi^2) = \xi^2 \omega_{M-1}^{N-1}(-\xi^2) + (N - M) \omega_{M-1}^{N-2}(-\xi^2). \quad (4.18)$$

Eq. (4.18) suggests a form similar to that of the recursion of the Laguerre polynomials [6],

$$n \mathbf{L}_n^{(\alpha)}(-x) = x \mathbf{L}_{n-1}^{(\alpha+1)}(-x) + (n + \alpha) \mathbf{L}_{n-1}^{(\alpha)}(-x) \quad (4.19)$$

from which we identify that

$$\omega_M^N(-\xi^2) = M! \cdot \mathbf{L}_M^{(N-2M)}(-\xi^2)$$

Following the same procedure for the $M \geq N/2$, writing

$$\tilde{Q}_{M,N}(\xi) = (-\xi)^{2M-N} \omega_M^N(-\xi^2)$$

and noting another Laguerre polynomial recurrence [6]

$$\mathbf{L}_n^{\alpha+1}(x) = \mathbf{L}_n^{\alpha}(x) + \mathbf{L}_{n-1}^{\alpha+1}(x)$$

we obtain, similarly to the previous case,

$$\omega_M^N(-\xi^2) = (N - M)! \cdot \mathbf{L}_{N-M}^{(2M-N)}(-\xi^2).$$

Thus, the $Q_{M,N}$ polynomials become, in the $r \gg 1$ limit,

$$Q_{M,N}(\lambda, r \gg 1) = \begin{cases} M! \left(-\frac{\lambda}{\sigma r}\right)^{N-2M} L_M^{(N-2M)}\left(-\frac{\lambda^2}{r^2 \sigma^2}\right) & , \quad M \leq N/2 \\ (N-M)! \left(-\frac{\lambda}{\sigma r}\right)^{2M-N} L_{N-M}^{(2M-N)}\left(-\frac{\lambda^2}{r^2 \sigma^2}\right) & , \quad M \geq N/2 \end{cases} \quad (4.20)$$

which with the exception of the degeneracy in $\lambda = 0$ are degree $2M$ and $2(N-M)$ polynomials in λ . Since the roots of the associated Laguerre polynomial L_n^α are all positive for non-negative α [6], it follows that the non-zero roots of $Q_{M,N}(\lambda; r \gg 1)$ are

$$\lambda_k = \pm i r \sigma l_k, \quad \begin{cases} k = 1, 2, \dots, M, & M \leq N/2 \\ k = 1, 2, \dots, N-M, & M \geq N/2 \end{cases} \quad (4.21)$$

where the l_k are the roots of the Laguerre polynomial of the respective degree.

4.4 Numerical Statistics

For the realization of the dense model of the present chapter, the system may be described in terms of three main parameters. Namely, the total size of the matrix, N , the size of the block, M , and the parameter r . Beyond this, it is also possible to sort the off-diagonal elements from either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ with different variance σ^2 . We shall thus fix the total variance at $\sigma^2 = \frac{1}{2}$, implying that each independent component of the off-diagonal elements has variance $\frac{1}{2\beta}$, where β is the relevant Dyson index for the set chosen.

The first set of figures, Figs. 4.7-4.9, present the results of the eigenvalue probability density in the complex plane in the region around the roots of the average characteristic polynomial Q . The calculations were done for real matrices of total size $N = 20$, all using a sample of 10^6 matrices. Therein, the darker curves represent higher concentration of eigenvalues, whereas the dashed black line represents the radius $\sqrt{N/2}$ of the asymptotic circular law of the Ginibre ensemble. Note that we have taken a smaller N as compared to the previous section because the calculation of the average characteristic polynomial becomes exponentially more expensive and thus, in order to be able to compare the results

with those of sample matrices, the smaller size was chosen.

In Fig. 4.7a, we consider the case in which $r = 1$ and the block size is $M = 2$, for which the density is displayed including the real eigenvalues, scaled to its square root so as to improve the visibility of the lower-density regions. The fact that the real eigenvalues are concentrated in a line implies that their density is unidimensional, whereas the remaining eigenvalues are distributed on a plane, thus making it harder to discern the behavior of the complex eigenvalues when those real eigenvalues are present.

In order to better appreciate the density of the eigenvalues outside of the real axis, Fig. 4.7b presents the density excluding those eigenvalues. We note that the density of the eigenvalues appears to increase in the vicinity of the complex roots of the average characteristic polynomial $Q_{2,20}(z; r)$, such that the contours seem to follow the shape of those roots.

Equivalently, Fig. 4.8 presents the analogous results for the same value of the parameter r , but with block size $M = 5$. In this case, the density including the real eigenvalues is displayed in the box in the upper right corner. We note that the increase in complex eigenvalues causes the higher-density clouds to spread further from the imaginary axis. Despite this difference, the general behavior is similar. Similar results may be seen in Fig. 4.9 for the case $M = 8$, although the density of the complex plane increases significantly, as well as the roots of the polynomial $Q_{10,20}(z; r)$ moving entirely to the complex plane. A particular characteristic of the $M = 5$ case, shared with all choices of odd M , is that one root pair of the average characteristic polynomial is always located in the imaginary axis, but for all choices of M the pairs are symmetrical with respect to the imaginary axis, appearing in mirrored pairs.

We consider the cases of complex and quaternionic matrices, respectively, in the following Figs. 4.10 and 4.11. The behavior of these cases is qualitatively similar to that seen in Fig. 4.9, although, as the Dyson parameter of the respective set increases, the eigenvalues seem to concentrate more densely in the complex plane, which is seen by the successive rounding of the corners towards the center of the outer contours.

Finally, the parameter r is varied in the real case, below and above $r = 1$, in Figs. 4.12 and 4.13, respectively. The behavior for $r = 0.5$ is qualitatively similar to that of $r = 1$, although the spread of the imaginary part of the eigenvalues decreases significantly

in the former case when compared to the latter. For $r = 1.5$, on the other hand, there are significant differences in the behavior of the spectral density. The appearance of roots of the polynomial $Q_{10,20}$ in the imaginary axis is accompanied by a distinctive change in the shape of the contours of the density distribution.

It is also noteworthy that in almost all the above cases, the exception being the last, the roots of the polynomial and a large part of the higher density dark cloud remain within the region of the asymptotic limit. Fig. 4.14 illustrates the behavior for a sample of 8×10^5 sample matrices of order $N = 80$, block size $M = 40$, parameter $r = 1.0$ and $\sigma^2 = 0.5$. In Fig. 4.14a we see that from a qualitative point of view, the case $N = 20$ of Fig. 4.9 behaves similarly to larger N , even when we exclude the real eigenvalues. The profiles seen in 4.14b and 4.14c show how abrupt the drop in density is, near the edge of the circular limit.

In order to gauge the fluctuations of the eigenvalues around the roots of the polynomial, the variance and relative variance of the characteristic polynomial was calculated for each set of parameters chosen for a set of sample matrices. In Figs. 4.15-4.19, we present these results, accompanied in each case by the respective roots of the $Q_{20,M}$.

In Fig. 4.15 we present the relative standard deviation calculated from the coefficients obtained from a sample of 10^6 $\mathbb{R}$ matrices with total size $N = 20$, block size $M = 8$ and block interaction parameter $r = 1.0$. In Fig. 4.15 we present the result of that variance in the neighborhood of the roots of the polynomial. Near the roots, the relative variance becomes highly concentrated around the roots of the polynomial. The same behavior is seen in Figs. 4.16 and 4.17, where the block sizes have been reduced to $M = 4$ and $M = 2$, respectively. The marked difference in these later cases is, however, that the presence of real roots of the respective $Q_{M,N}(z; r)$ polynomial changes the overall shape of the contour lines.

In Figs. 4.18 and 4.19 we present the same results for block size $M = 8$ and $r = 0.5$ and $r = 1.5$, respectively. It is notable that the behavior of the $r < 1$ case remains qualitatively the same, although the roots, as well as the low variance region, are distributed in a much narrower region than in the $r = 1.0$ case. For the $r > 1$ case, however, there is a significant qualitative change in the behavior of the variance, as the behavior of both the roots and the variance display the effects of the presence of pure imaginary roots.

Figure 4.7: Density of eigenvalues on the complex plane for 10^6 $N = 20$ sample matrices of the separated $\mathbb{R}$ case, with $r = 1.0$, $\sigma = 0.5$ and $M = 2$. The left graph shows the distribution including the real eigenvalues and is scaled to the square root of the density, whereas the right graph shows the unscaled density excluding the real eigenvalues. The dashed white line represents the Ginibre circular limit. The red plus signs denote the roots of $Q_{2,20}(z; r)$.

Figure 4.8: Density plot under the same conditions of Fig. 4.7, except for the block size $M = 5$ and the red plus signs denote the roots of $Q_{5,20}(z; r)$.

Figure 4.9: Density plot under the same conditions of Fig. 4.7, except for the block size $M = 10$ and the red plus signs denote the roots of $Q_{10,20}(z; r)$.

Figure 4.10: Density plot under the same conditions of Fig. 4.7 of the $\mathbb{C}$ case, with block size $M = 10$ and the red plus signs denote the roots of $Q_{10,20}(z; r)$.

Figure 4.11: Density plot under the same conditions of Fig. 4.7 of the $\mathbb{H}$ case, with block size $M = 10$ and the red plus signs denote the roots of $Q_{10,20}(z; r)$.

Figure 4.12: Density plot under the same conditions of Fig. 4.7 of the $\mathbb{R}$ case, with block size $M = 10$, $r = 0.5$ and the red plus signs denote the roots of $Q_{10,20}(z; r)$.

Figure 4.13: Density plot under the same conditions of Fig. 4.7 of the $\mathbb{R}$ case, with block size $M = 10$, $r = 1.5$ and the red plus signs denote the roots of $Q_{10,20}(z; r)$.

The above figures represent studies of the statistical behavior of a very large number of matrices, contrasted to the average behavior of the characteristic polynomial. However, it is revealing to also consider the behavior of the eigenvalues of a single matrix when subject to the variations of the parameter r and the block size M . To do so, we consider a size $N = 20$ sample matrix of the $\mathbb{R}$ case with standard deviation $\sigma^2 = 0.5$, for which we apply (4.3) and vary the parameter r such that the eigenvalues move from the real axis into the complex plane. The real eigenvalues are denoted by black lines, whereas the lighter copper-colored lines denote the real part of the complex conjugate pairs that move into the complex plane. Figs. 4.20 -4.22 present the results for upper block size $M = 2, 5$ and 10 respectively. Figs. 4.20a-4.22a present the behavior of the roots of a sample matrix, whereas the corresponding Figs. 4.20b-4.22b present the equivalent results for the corresponding behavior of the roots of $Q_{2,20}$, $Q_{5,20}$ and $Q_{10,20}$, respectively. The eigenvalues may be seen moving from their starting positions and, as the parameter increases break into the complex plane and eventually approach their asymptotic limits, in a manner consistent with $\mathcal{PT}$ -symmetric breaks in spectra [1, 7] as well as to what was seen in the model of Chapter 2; the case in Sec. 2.3 being of particular interest due to the similarity both in spectral behavior and η matrix.

The roots of the corresponding average characteristic polynomials seem to indicate that the behavior of the spectral density is compatible with the expected behavior for the

(a) Excluding real eigenvalues.

(b) Excluding real eigenvalues, profile in the real axis.

(c) Excluding real eigenvalues, profile in the imaginary axis.

Figure 4.14: Density of eigenvalues on the complex plane for 8×10^5 $N = 80$ sample matrices of the separated $\mathbb{R}$ case, with $r = 1.0$, $\sigma = 0.5$ and $M = 40$. In (a), it is shown the distribution including the real eigenvalues and is scaled to the square root of the density, with the density including the real eigenvalues depicted in the upper-right panel. The profile of the density is shown in regards to the real axis (b) and the imaginary axis (c).

Figure 4.15: Relative standard deviation of the characteristic polynomials averaged over 10^6 $N = 20$ sample matrices of the separated case, with $r = 1.0$, $\sigma = 0.5$ and $M = 10$. The red plus signs denote the roots of $Q_{10,20}(z; r)$.

Figure 4.16: Relative standard deviation plot under the same conditions of Fig. 4.15, except for $M = 5$. The red plus signs denote the roots of $Q_{5,20}(z; r)$.

Figure 4.17: Relative standard deviation plot under the same conditions of Fig. 4.15, except for $M = 2$. The red plus signs denote the roots of $Q_{2,20}(z; r)$.

Figure 4.18: Relative standard deviation plot under the same conditions of Fig. 4.15, except for $r = 0.5$. The red plus signs denote the roots of $Q_{10,20}(z; r)$.

Figure 4.19: Relative standard deviation plot under the same conditions of Fig. 4.15, except for $r = 1.5$. The red plus signs denote the roots of $Q_{10,20}(z; r)$.

ensemble average. We note that in the sample matrix some pairs of eigenvalues occasionally spontaneously return to the real axis, as pictured in Fig. 4.22, where three instances of this type of behavior may be seen. This return is, however, short lived, and an increase in the interaction strength between the blocks causes them to return to the complex plane toward their limiting positions. This behavior is a result of the intrinsic random fluctuations of the matrix, as such behavior is not echoed in the roots of the corresponding $Q_{N,M}$ polynomials.

For the $M = N/2 = 10$ case, we additionally notice that roots of the average characteristic polynomial present a significant increase in the effect of switching on the pseudo-Hermitian interaction by making r non-zero, albeit very small. This may better be seen in the box in Fig. 4.22b, where a range of lower values for r is plotted and note that, for very low values of r the pseudo-Hermitian interaction breaks the degeneracy and, thus, moves the roots into the complex plane. A tolerance of $\tau = 10^{-7}$ was allowed for the relative magnitude of the imaginary part of the roots. This becomes necessary due to the degeneracy of the roots of the average polynomial when $r \rightarrow 0$.

(a) Eigenvalues

(b) Root behavior

Figure 4.20: Eigenvalue behavior of a sample matrix (a) and characteristic polynomial root behavior (b) for $N = 20$ and variance $\sigma^2 = 0.5$ of the separated case for varying parameter r with upper block size $M = 2$.

(a) Eigenvalues

(b) Root behavior

Figure 4.21: Eigenvalue behavior of a sample matrix (a) and characteristic polynomial root behavior (b) for $N = 20$ and variance $\sigma^2 = 0.5$ of the separated case for varying parameter r with upper block size $M = 5$.

(a) Eigenvalues

(b) Root behavior

Figure 4.22: Eigenvalue behavior of a sample matrix (a) and characteristic polynomial root behavior (b) for $N = 20$ and variance $\sigma^2 = 0.5$ of the separated case for varying parameter r with upper block size $M = 10$.

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Chapter 5

Intertwined Model

In the present chapter, the second of the Gaussian models will be discussed. In this case, the choice of projector operators is such that the resulting pseudo-Hermitian matrix alternates its subdiagonals as hermitian and pseudo-Hermitian, starting as Hermitian in the diagonal, moving on to pseudo-Hermitian in the first subdiagonal, then becoming Hermitian again in the following subdiagonal and so on. The structure of this chapter follows that of the previous Chapter 4, and we draw analogies from the separated case presented therein to analyse the results of the present intertwined case. Thus, the first results presented pertain to the spectral behavior of matrices within this model. These are followed by the study of the average characteristic polynomials in this case, which is then followed by certain asymptotics of interest. Finally, some further numerical results are presented.

5.1 Spectral Behavior

The behavior of the eigenvalues in the complex plane may be investigated by looking at the evolution of the eigenvalues of a sample matrix as we change the interaction strength r . Thus, Fig. 5.1 depicts such behavior for a sample matrix of size $N = 32$ with total off-diagonal element variance $\sigma^2 = 1/2$. Unlike the case studied in the previous chapter, the only parameter to be considered is the interaction strength between the blocks, and it may be seen that the effect of the introduction of the interaction between the blocks yields results similar to those of the $M = N/2$ model of Chapter 4 depicted in Fig. 4.4, and, as

in that case, results are also similar to Figs. 2.1, 2.2, 2.9 and 2.10 of Chapter 2.

5.2 Characteristic Polynomial

It is noteworthy that, in this case, the projectors may be written as

$$\mathcal{P} = \sum_{k=1}^{\lceil N/2 \rceil} |2k-1\rangle \langle 2k-1| \quad (5.1)$$

and

$$\mathcal{Q} = \sum_{k=1}^{\lfloor N/2 \rfloor} |2k\rangle \langle 2k|. \quad (5.2)$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions, respectively. The average characteristic polynomial may be deduced following a similar reasoning to that of the separated case. Namely, consider the determinant

$$P_N(\lambda; r) = \langle \det(A - \lambda \mathbb{I}) \rangle \equiv \langle D \rangle \quad (5.3)$$

by means of the Laplace expansion on the first row

$$P_N(\lambda; r) = \sum_{k=1}^N \langle (-1)^{N+k} (A_{1,k} - \delta_{1,k} \lambda) \det(D_{1,k}) \rangle \quad (5.4)$$

where $D_{l,k}$ is the matrix minor resulting from the exclusion of row l and column k from D . Since, however, the elements of the matrix are i.i.d. random variables with zero mean, only some of the averages including the minors are non-zero, analogously to the case in the preceding Chapter. The contribution from the diagonal may be derived by looking at the first term of the sum

$$\left\langle (A_{1,1} - \lambda) \begin{vmatrix} A_{2,2} - \lambda & r A_{2,3} & \dots & \phi_{2,N} A_{2,N} \\ -r \overline{A_{2,3}} & A_{3,3} - \lambda & \dots & A_{3,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N,2} \overline{A_{2,N}} & \phi_{N,3} \overline{A_{3,N}} & \dots & A_{N,N} - \lambda \end{vmatrix} \right\rangle =$$

(a) Eigenvalue behavior in the complex plane, depicted as the $\text{Re} \times \text{Im}$ plane, as a function of the interaction strength, depicted as the Z axis as well as the coloring of the eigenvalues.

(b) Eigenvalue behavior, projected in the $\text{Re} \times \text{Im}$ plane, with color depicting the interaction strength.

(c) Eigenvalue behavior, projected in the $\text{Re} \times s$ plane, with color depicting the interaction strength.

(d) Eigenvalue behavior, projected in the $\text{Im} \times s$ plane, with color depicting the interaction strength.

Figure 5.1: Complex eigenvalue behavior for an $N = 32$ intertwined matrix, as a function of the interaction strength r .

$$\begin{aligned}
&= \langle A_{1,1} - \lambda \rangle \left\langle \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & \dots & \phi_{2,N}A_{2,N} \\ -r\overline{A_{2,3}} & A_{3,3}-\lambda & \dots & A_{3,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N,2}\overline{A_{2,N}} & \phi_{N,3}\overline{A_{3,N}} & \dots & A_{N,N}-\lambda \end{vmatrix} \right\rangle \\
&= -\lambda P_{N-1}(\lambda; r).
\end{aligned} \tag{5.5}$$

In the last equality, it is noteworthy that performing the average of the characteristic polynomial as in (5.5), the sub-matrix of the upper diagonal $(N-1) \times (N-1)$ block of an order N sample matrix is indistinguishable from an order $N-1$ sample matrix. Therefore, the average polynomial given by the determinant in Eq. (5.5) coincides with the $N-1$ average polynomial, analogously to what occurred in the separated case.

The contribution of the off-diagonal elements, on the other hand, may be seen by writing explicitly e.g. the first term of the sum

$$\left\langle \phi_{1,2}A_{1,2} \begin{vmatrix} -r\overline{A_{1,2}} & A_{2,3} & \dots & \phi_{2,N}A_{2,N} \\ A_{1,3} & A_{3,3}-\lambda & \dots & A_{3,N} \\ -rA_{1,4} & -r\overline{A_{3,4}} & \dots & A_{4,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N,1}\overline{A_{1,N}} & \phi_{N,3}\overline{A_{3,N}} & \dots & A_{N,N}-\lambda \end{vmatrix} \right\rangle \tag{5.6}$$

and may be further expanded by the first row [1], noting that

$$\begin{aligned}
\langle \phi_{1,2}A_{1,2}A_{2,3} \rangle &= \langle \phi_{1,2}A_{1,2}A_{2,4} \rangle = \dots \\
&= \langle \phi_{1,2}\phi_{2,N}A_{1,2}A_{2,N} \rangle = 0
\end{aligned} \tag{5.7}$$

since the variables are i.i.d. with zero mean. The only element which is not i.i.d, $A_{1,2}$, only appears in the first element of that expansion, for which using the symmetry $\phi_{k,l} = -\phi_{l,k}$ for different parities of k, l we obtain

$$\langle \phi_{2,1}\phi_{1,2}\overline{A_{1,2}}A_{1,2} \rangle = -\phi_{1,2}^2\sigma^2 \tag{5.8}$$

where σ^2 is the total variance of the off-diagonal elements. Therefore, (5.6) becomes

$$\begin{aligned}
& -\phi_{1,2}^2 \sigma^2 \left\langle \begin{vmatrix} A_{3,3}-\lambda & rA_{3,4} & \dots & \phi_{3,N}A_{3,N} \\ -r\overline{A_{3,4}} & A_{4,4}-\lambda & \dots & \phi_{4,N}A_{4,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N,3}\overline{A_{3,N}} & \phi_{N,4}\overline{A_{4,N}} & \dots & A_{N,N}-\lambda \end{vmatrix} \right\rangle \\
& = -\phi_{1,2}^2 \sigma^2 P_{N-2}(\lambda).
\end{aligned} \tag{5.9}$$

In the last equality the determinant being averaged is equivalent to a sample matrix of the ensemble for order $N - 2$, similarly to what occurred previously above, albeit with the simultaneous exclusion of two row and column pairs from the original matrix. This procedure is the same for all the even elements of the expansion, leading to a total contribution of

$$-\left\lfloor \frac{N}{2} \right\rfloor r^2 \sigma^2 P_{N-2}(\lambda). \tag{5.10}$$

The second non-diagonal term of the sum gives

$$\left\langle \phi_{1,3} A_{1,3} \begin{vmatrix} -r\overline{A_{1,2}} & A_{2,2}-\lambda & A_{2,4} & \dots & \phi_{2,N}A_{2,N} \\ \overline{A_{1,3}} & -r\overline{A_{2,3}} & rA_{3,4} & \dots & A_{3,N} \\ -r\overline{A_{1,4}} & \overline{A_{2,4}} & A_{4,4}-\lambda & \dots & A_{4,N} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \phi_{N,1}\overline{A_{1,N}} & \phi_{N,3}\overline{A_{3,N}} & \phi_{N,4}\overline{A_{4,N}} & \dots & A_{N,N}-\lambda \end{vmatrix} \right\rangle \tag{5.11}$$

which yields, by expanding on the first element of the second row of the new determinant,

$$\begin{aligned}
\langle \phi_{1,3} A_{1,3} \phi_{3,2} \overline{A_{2,3}} \rangle &= \langle \phi_{1,3} A_{1,3} \phi_{3,4} A_{3,4} \rangle = \dots \\
&= \langle \phi_{1,3} A_{1,3} \phi_{3,N} A_{3,N} \rangle = 0
\end{aligned} \tag{5.12}$$

and

$$\langle \phi_{1,3} \phi_{3,1} \overline{A_{1,3}} A_{1,3} \rangle = \phi_{1,3}^2 \sigma^2. \tag{5.13}$$

where the symmetry $\phi_{k,l} = \phi_{l,k}$ for k, l with the same parity was used.

Similarly, Eq. (5.11) becomes

$$\begin{aligned} \phi_{1,3}^2 \sigma^2 \left\langle \begin{vmatrix} A_{2,2}-\lambda & A_{2,4} & rA_{2,5} & \dots & \phi_{2,N}A_{2,N} \\ \overline{A_{2,4}} & A_{4,4}-\lambda & rA_{4,5} & \dots & \phi_{4,N}A_{4,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{N,2}\overline{A_{2,N}} & \phi_{N,4}\overline{A_{4,N}} & \phi_{N,5}\overline{A_{5,N}} & \dots & A_{N,N}-\lambda \end{vmatrix} \right\rangle \\ = \phi_{1,3}^2 \sigma^2 R_{N-2}(\lambda). \end{aligned} \quad (5.14)$$

in which the R_k are average characteristic polynomials of the matrix within the determinant.

All the non-diagonal odd contributions to the sum will yield an equivalent determinant, albeit with both the first and $(2k+1)$ -th rows and columns removed instead of the first and third. This may be seen by representing the determinant in terms of the Hermiticity $\tilde{h}$ or pseudo-Hermiticity $\tilde{p}$ of the leading coefficient of the column

$$\begin{aligned} \langle [d_2 \quad \tilde{h}_4 \quad \tilde{p}_5 \quad \tilde{h}_6 \quad \tilde{p}_7 \quad \dots] \rangle &= \langle [d_2 \quad \tilde{p}_3 \quad \tilde{h}_4 \quad \tilde{h}_6 \quad \tilde{p}_7 \quad \dots] \rangle = \dots \\ &= \begin{cases} \langle [d_2 \quad \tilde{p}_3 \quad \tilde{h}_4 \quad \dots \quad \tilde{p}_{N-3} \quad \tilde{h}_{N-2} \quad \tilde{h}_N] \rangle & , \text{ odd } N \\ \langle [d_2 \quad \tilde{p}_3 \quad \tilde{h}_4 \quad \dots \quad \tilde{h}_{N-3} \quad \tilde{p}_{N-2} \quad \tilde{h}_{N-1}] \rangle & , \text{ even } N \end{cases} \end{aligned} \quad (5.15)$$

and, noting that swapping both a row and a column of a matrix does not alter its determinant, they can all be rearranged to have the same form as in (5.14) for the relevant parity of N . This means that their averages will yield the same result and hence, the total contribution of the odd elements of (5.4) is

$$\lfloor \frac{N-1}{2} \rfloor \sigma^2 R_{N-2}(\lambda). \quad (5.16)$$

Therefore, for the off-diagonal terms of (5.4)

$$\langle (-1)^{N+k} A_{N,k} \det(D_{N,k}) \rangle = \begin{cases} -\sigma^2 P_{N-2}(\lambda; r) & , \quad k \text{ odd} \\ +r^2 \sigma^2 R_{N-2}(\lambda; r) & , \quad k \text{ even} \end{cases}. \quad (5.17)$$

The recurrence for the R_k is obtained through the use of the same argument presented

above, albeit noting the structure of Eq. (5.15). The argument will not be presented here in full as it is very similar to the above, and yields a diagonal contribution of

$$\begin{cases} -\lambda P_{N-1}(\lambda; r) & , \quad N \text{ odd} \\ -\lambda R_{N-1}(\lambda; r) & , \quad N \text{ even} \end{cases} \quad (5.18)$$

and off-diagonal contributions of

$$- \lfloor \frac{N}{2} \rfloor r^2 \sigma^2 P_{N-2}(\lambda; r) + \lfloor \frac{N-1}{2} \rfloor \sigma^2 R_{N-2}(\lambda; r). \quad (5.19)$$

It is nonetheless illustrative to provide an example of these determinantal expansions, so as to aid in the visualization of the above property. For this purpose, we may consider the case $N = 5$ which, although small in statistical terms, display the essential recurring properties. The expansion of a given sample determinant is

$$\begin{aligned} \Delta_5 &= \begin{vmatrix} A_{1,1}-\lambda & rA_{1,2} & A_{1,3} & rA_{1,4} & A_{1,5} \\ -r\bar{A}_{1,2} & A_{2,2}-\lambda & rA_{2,3} & A_{2,4} & rA_{2,5} \\ \bar{A}_{1,3} & -r\bar{A}_{2,3} & A_{3,3}-\lambda & rA_{3,4} & A_{3,5} \\ -r\bar{A}_{1,4} & \bar{A}_{2,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda & rA_{4,5} \\ \bar{A}_{1,5} & -r\bar{A}_{2,5} & \bar{A}_{3,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} = (A_{1,1} - \lambda) \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & A_{2,4} & rA_{2,5} \\ -r\bar{A}_{2,3} & A_{3,3}-\lambda & rA_{3,4} & A_{3,5} \\ \bar{A}_{2,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda & rA_{4,5} \\ -r\bar{A}_{2,5} & \bar{A}_{3,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \\ &\quad - rA_{1,2} \begin{vmatrix} -r\bar{A}_{1,2} & rA_{2,3} & A_{2,4} & rA_{2,5} \\ \bar{A}_{1,3} & A_{3,3}-\lambda & rA_{3,4} & A_{3,5} \\ -r\bar{A}_{1,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda & rA_{4,5} \\ \bar{A}_{1,5} & \bar{A}_{3,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} + A_{1,3} \begin{vmatrix} -r\bar{A}_{1,2} & A_{2,2}-\lambda & A_{2,4} & rA_{2,5} \\ \bar{A}_{1,3} & -r\bar{A}_{2,3} & rA_{3,4} & A_{3,5} \\ -r\bar{A}_{1,4} & \bar{A}_{2,4} & A_{4,4}-\lambda & rA_{4,5} \\ \bar{A}_{1,5} & -r\bar{A}_{2,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \\ &\quad - rA_{1,4} \begin{vmatrix} -r\bar{A}_{1,2} & A_{2,2}-\lambda & rA_{2,3} & rA_{2,5} \\ \bar{A}_{1,3} & -r\bar{A}_{2,3} & A_{3,3}-\lambda & A_{3,5} \\ -r\bar{A}_{1,4} & \bar{A}_{2,4} & -r\bar{A}_{3,4} & rA_{4,5} \\ \bar{A}_{1,5} & -r\bar{A}_{2,5} & \bar{A}_{3,5} & A_{5,5}-\lambda \end{vmatrix} + A_{1,5} \begin{vmatrix} -r\bar{A}_{1,2} & A_{2,2}-\lambda & rA_{2,3} & A_{2,4} \\ \bar{A}_{1,3} & -r\bar{A}_{2,3} & A_{3,3}-\lambda & rA_{3,4} \\ -r\bar{A}_{1,4} & \bar{A}_{2,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda \\ \bar{A}_{1,5} & -r\bar{A}_{2,5} & \bar{A}_{3,5} & -r\bar{A}_{4,5} \end{vmatrix} \\ &\equiv (A_{1,1} - \lambda)\Delta_{1,1} - rA_{1,2}\Delta_{1,2} + A_{1,3}\Delta_{1,3} - rA_{1,4}\Delta_{1,4} + A_{1,5}\Delta_{1,5} \end{aligned}$$

where the $\Delta_{1,k}$ are the minor determinants outlined above. When averaged, the i.i.d. nature of the elements causes this determinant to become

$$\langle \Delta \rangle = \langle A_{1,1} - \lambda \rangle \langle \Delta_{1,1} \rangle - r \langle A_{1,2} \Delta_{1,2} \rangle + \langle A_{1,3} \Delta_{1,3} \rangle - r \langle A_{1,4} \Delta_{1,4} \rangle + \langle A_{1,5} \Delta_{1,5} \rangle$$

The first term is straightforward to calculate, as the diagonal elements are unique in the

matrix, and thus

$$\langle A_{1,1} - \lambda \rangle \langle \Delta_{1,1} \rangle = (-\lambda) \left\langle \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & A_{2,4} & rA_{2,5} \\ -r\bar{A}_{2,3} & A_{3,3}-\lambda & rA_{3,4} & A_{3,5} \\ \bar{A}_{2,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda & rA_{4,5} \\ -r\bar{A}_{2,5} & \bar{A}_{3,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle = -\lambda P_4(\lambda; r).$$

For the second term, we must expand the determinant yet again

$$\begin{aligned} \langle A_{1,2} \Delta_{1,2} \rangle &= \left\langle A_{1,2} \begin{vmatrix} -r\bar{A}_{1,2} & rA_{2,3} & A_{2,4} & rA_{2,5} \\ \bar{A}_{1,3} & A_{3,3}-\lambda & rA_{3,4} & A_{3,5} \\ -r\bar{A}_{1,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda & rA_{4,5} \\ \bar{A}_{1,5} & \bar{A}_{3,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle \\ &= -r \left\langle |A_{1,2}|^2 \begin{vmatrix} A_{3,3}-\lambda & rA_{3,4} & A_{3,5} \\ -r\bar{A}_{3,4} & A_{4,4}-\lambda & rA_{4,5} \\ \bar{A}_{3,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle - \left\langle A_{1,2} \bar{A}_{1,3} \begin{vmatrix} rA_{2,3} & A_{2,4} & rA_{2,5} \\ -r\bar{A}_{3,4} & A_{4,4}-\lambda & rA_{4,5} \\ \bar{A}_{3,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle \\ &\quad - r \left\langle A_{1,2} \bar{A}_{1,4} \begin{vmatrix} rA_{2,3} & A_{2,4} & rA_{2,5} \\ A_{3,3}-\lambda & rA_{3,4} & A_{3,5} \\ \bar{A}_{3,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle - \left\langle A_{1,2} \bar{A}_{1,5} \begin{vmatrix} rA_{2,3} & A_{2,4} & rA_{2,5} \\ A_{3,3}-\lambda & rA_{3,4} & A_{3,5} \\ -r\bar{A}_{3,4} & A_{4,4}-\lambda & rA_{4,5} \end{vmatrix} \right\rangle \\ &= -r\sigma^2 P_3(\lambda; r) \end{aligned}$$

the last equality of which only the first term is non-zero, as we may again invoke that the elements are i.i.d. The fourth term behaves similarly,

$$\begin{aligned} \langle A_{1,4} \Delta_{1,4} \rangle &= \left\langle A_{1,4} \begin{vmatrix} -r\bar{A}_{1,2} & A_{2,2}-\lambda & rA_{2,3} & rA_{2,5} \\ \bar{A}_{1,3} & -r\bar{A}_{2,3} & A_{3,3}-\lambda & A_{3,5} \\ -r\bar{A}_{1,4} & \bar{A}_{2,4} & -r\bar{A}_{3,4} & rA_{4,5} \\ \bar{A}_{1,5} & -r\bar{A}_{2,5} & \bar{A}_{3,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle \\ &= -r \left\langle \bar{A}_{1,2} A_{1,4} \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & rA_{2,5} \\ -r\bar{A}_{2,3} & A_{3,3}-\lambda & A_{3,5} \\ \bar{A}_{2,4} & -r\bar{A}_{3,4} & rA_{4,5} \\ -r\bar{A}_{2,5} & \bar{A}_{3,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle - \left\langle \bar{A}_{1,3} A_{1,4} \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & rA_{2,5} \\ \bar{A}_{2,4} & -r\bar{A}_{3,4} & rA_{4,5} \\ -r\bar{A}_{2,5} & \bar{A}_{3,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle \\ &\quad - r \left\langle |A_{1,4}|^2 \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & rA_{2,5} \\ -r\bar{A}_{2,3} & A_{3,3}-\lambda & A_{3,5} \\ -r\bar{A}_{2,5} & \bar{A}_{3,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle - \left\langle A_{1,4} \bar{A}_{1,5} \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & rA_{2,5} \\ -r\bar{A}_{2,3} & A_{3,3}-\lambda & A_{3,5} \\ \bar{A}_{2,4} & -r\bar{A}_{3,4} & rA_{4,5} \end{vmatrix} \right\rangle \\ &= -r\sigma^2 P_3(\lambda; r). \end{aligned}$$

For the third and fifth term, which yield the contributions of the R polynomials, we

have respectively

$$\begin{aligned}
\langle A_{1,3} \Delta_{1,3} \rangle &= \left\langle A_{1,3} \begin{vmatrix} -r\bar{A}_{1,2} & A_{2,2}-\lambda & A_{2,4} & rA_{2,5} \\ \bar{A}_{1,3} & -r\bar{A}_{2,3} & rA_{3,4} & A_{3,5} \\ -r\bar{A}_{1,4} & \bar{A}_{2,4} & A_{4,4}-\lambda & rA_{4,5} \\ \bar{A}_{1,5} & -r\bar{A}_{2,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle \\
&= -r \left\langle \bar{A}_{1,2} A_{1,3} \begin{vmatrix} -r\bar{A}_{2,3} & rA_{3,4} & A_{3,5} \\ \bar{A}_{2,4} & A_{4,4}-\lambda & rA_{4,5} \\ -r\bar{A}_{2,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle - \left\langle |A_{1,3}|^2 \begin{vmatrix} A_{2,2}-\lambda & A_{2,4} & rA_{2,5} \\ \bar{A}_{2,4} & A_{4,4}-\lambda & rA_{4,5} \\ -r\bar{A}_{2,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle \\
&\quad - r \left\langle A_{1,3} \bar{A}_{1,4} \begin{vmatrix} A_{2,2}-\lambda & A_{2,4} & rA_{2,5} \\ -r\bar{A}_{2,3} & rA_{3,4} & A_{3,5} \\ -r\bar{A}_{2,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle - \left\langle A_{1,3} \bar{A}_{1,5} \begin{vmatrix} A_{2,2}-\lambda & A_{2,4} & rA_{2,5} \\ -r\bar{A}_{2,3} & rA_{3,4} & A_{3,5} \\ \bar{A}_{2,4} & A_{4,4}-\lambda & rA_{4,5} \end{vmatrix} \right\rangle \\
&= -\sigma^2 \left\langle \begin{vmatrix} A_{2,2}-\lambda & A_{2,4} & rA_{2,5} \\ \bar{A}_{2,4} & A_{4,4}-\lambda & rA_{4,5} \\ -r\bar{A}_{2,5} & -r\bar{A}_{4,5} & A_{5,5}-\lambda \end{vmatrix} \right\rangle = -\sigma^2 R_3(\lambda; r)
\end{aligned}$$

and

$$\begin{aligned}
\langle A_{1,5} \Delta_{1,5} \rangle &= \left\langle A_{1,5} \begin{vmatrix} -r\bar{A}_{1,2} & A_{2,2}-\lambda & rA_{2,3} & A_{2,4} \\ \bar{A}_{1,3} & -r\bar{A}_{2,3} & A_{3,3}-\lambda & rA_{3,4} \\ -r\bar{A}_{1,4} & \bar{A}_{2,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda \\ \bar{A}_{1,5} & -r\bar{A}_{2,5} & \bar{A}_{3,5} & -r\bar{A}_{4,5} \end{vmatrix} \right\rangle \\
&= -r \left\langle \bar{A}_{1,2} A_{1,5} \begin{vmatrix} -r\bar{A}_{2,3} & A_{3,3}-\lambda & rA_{3,4} \\ \bar{A}_{2,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda \\ -r\bar{A}_{2,5} & \bar{A}_{3,5} & -r\bar{A}_{4,5} \end{vmatrix} \right\rangle - \left\langle \bar{A}_{1,3} A_{1,5} \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & A_{2,4} \\ \bar{A}_{2,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda \\ -r\bar{A}_{2,5} & \bar{A}_{3,5} & -r\bar{A}_{4,5} \end{vmatrix} \right\rangle \\
&\quad - r \left\langle \bar{A}_{1,4} A_{1,5} \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & A_{2,4} \\ -r\bar{A}_{2,3} & A_{3,3}-\lambda & rA_{3,4} \\ -r\bar{A}_{2,5} & \bar{A}_{3,5} & -r\bar{A}_{4,5} \end{vmatrix} \right\rangle - \left\langle |A_{1,5}|^2 \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & A_{2,4} \\ -r\bar{A}_{2,3} & A_{3,3}-\lambda & rA_{3,4} \\ \bar{A}_{2,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda \end{vmatrix} \right\rangle \\
&= -\sigma^2 \left\langle \begin{vmatrix} A_{2,2}-\lambda & rA_{2,3} & A_{2,4} \\ -r\bar{A}_{2,3} & A_{3,3}-\lambda & rA_{3,4} \\ \bar{A}_{2,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda \end{vmatrix} \right\rangle = +\sigma^2 \left\langle \begin{vmatrix} A_{2,2}-\lambda & A_{2,4} & rA_{2,3} \\ -r\bar{A}_{2,3} & rA_{3,4} & A_{3,3}-\lambda \\ \bar{A}_{2,4} & A_{4,4}-\lambda & -r\bar{A}_{3,4} \end{vmatrix} \right\rangle \\
&= -\sigma^2 \left\langle \begin{vmatrix} A_{2,2}-\lambda & A_{2,4} & rA_{2,3} \\ \bar{A}_{2,4} & A_{4,4}-\lambda & -r\bar{A}_{3,4} \\ -r\bar{A}_{2,3} & rA_{3,4} & A_{3,3}-\lambda \end{vmatrix} \right\rangle = -\sigma^2 R_3(\lambda; r),
\end{aligned}$$

noting the use of the property of row and column swap of determinants in the last equalities. It is important to note that for each sample matrix the contributions from each of the even elements may be distinct from one another, as is the case for the odd elements. However, when considering the average over the ensemble, their average behavior is indistinguishable, as these elements are i.i.d. Therefore, we arrive at the relation for the $N = 5$ case

$$P_5(\lambda; r) = -\lambda P_4(\lambda; r) + 2r^2 \sigma^2 P_3(\lambda; r) - 2\sigma^2 R_3(\lambda; r).$$

It is fairly straightforward to notice that the case for other N is quite similar. For example,

by removing one row and one column to reduce to the $N = 4$ case, we similarly obtain

$$\tilde{\Delta}_4 = \begin{vmatrix} A_{1,1}-\lambda & rA_{1,2} & A_{1,3} & rA_{1,4} \\ -r\bar{A}_{1,2} & A_{2,2}-\lambda & rA_{2,3} & A_{2,4} \\ \bar{A}_{1,3} & -r\bar{A}_{2,3} & A_{3,3}-\lambda & rA_{3,4} \\ -r\bar{A}_{1,4} & \bar{A}_{2,4} & -r\bar{A}_{3,4} & A_{4,4}-\lambda \end{vmatrix} \equiv (A_{1,1} - \lambda)\tilde{\Delta}_{1,1} - rA_{1,2}\tilde{\Delta}_{1,2} + A_{1,3}\tilde{\Delta}_{1,3} - rA_{1,4}\tilde{\Delta}_{1,4}$$

for which the same procedure outlined in the $N = 5$ example may be followed to obtain

$$P_4(\lambda; r) = -\lambda P_3(\lambda; r) + 2r^2\sigma^2 P_2(\lambda; r) - \sigma^2 R_2(\lambda; r)$$

and so on for other N .

Thus finally combining Eqs. (5.5), (5.17), (5.18) and (5.19) the average characteristic polynomial of the pseudo-Hermitian chessboard matrix is therefore given by the recurrence relations

$$P_N(\lambda; r) = \begin{cases} 0 & , N < 0 \\ 1 & , N = 0 \\ -\lambda P_{N-1}(\lambda; r) + \lfloor \frac{N}{2} \rfloor r^2 \sigma^2 P_{N-2}(\lambda; r) & , N \geq 1 \\ -\lfloor \frac{N-1}{2} \rfloor \sigma^2 R_{N-2}(\lambda; r) & \end{cases} \quad (5.20)$$

$$R_N(\lambda; r) = \begin{cases} 0 & , N < 0 \\ 1 & , N = 0 \\ -\lambda P_{N-1}(\lambda; r) & , N = 2k+1, k \in \{0, 1, 2, \dots\} \\ -\lfloor \frac{N}{2} \rfloor \sigma^2 P_{N-2}(\lambda; r) + \lfloor \frac{N-1}{2} \rfloor r^2 \sigma^2 R_{N-2}(\lambda; r) & \\ -\lambda R_{N-1}(\lambda; r) & , N = 2k, k \in \{1, 2, 3, \dots\} \\ -\lfloor \frac{N}{2} \rfloor \sigma^2 P_{N-2}(\lambda; r) + \lfloor \frac{N-1}{2} \rfloor r^2 \sigma^2 R_{N-2}(\lambda; r) & \end{cases} \quad (5.21)$$

From the above equations (5.20) and (5.21), we may obtain additional properties. For the first, we consider initially that since $P_0(\lambda; r) = R_0(\lambda; r) = 1$, the recursion relations yield that $P_1(\lambda; r) = R_1(\lambda; r) = -\lambda$. If we then assume that $P_{2(k-1)+1}(\lambda; r) = R_{2(k-1)+1}(\lambda; r)$ recursions for $N = 2k + 1, k \geq 0$, become

$$P_{2k+1}(\lambda; r) = -\lambda P_{2k} + k(r^2 - 1)\sigma^2 P_{2(k-1)+1} = R_{2k+1}(\lambda; r)$$

and therefore by induction $P_{2k+1}(\lambda; r) = R_{2k+1}(\lambda; r)$ for all $k \in \{0, 1, 2, \dots\}$. In other words, the odd P polynomials are equal to the odd R polynomials.

Furthermore, if we consider this recurrence relation applied outside of the conditions of the pseudo-Hermitian model of the present paper to an imaginary $r^2 = -1$ and $\sigma^2 = \frac{1}{2}$,

the P and Q of same order coincide, and are equal to a scaling of the Hermite polynomials

$$P_n \left(\lambda; \pm i, \sigma^2 = \frac{1}{2} \right) = R_n \left(\lambda; \pm i, \sigma^2 = \frac{1}{2} \right) \equiv T_n(\lambda) = 2^{-n} \mathbf{H}_n(-\lambda). \quad (5.22)$$

This property may be obtained by noting that, in both cases, the recurrence relation reduces to

$$T_n(\lambda) = -\lambda T_{n-1}(\lambda) - \frac{n-1}{2} T_{n-2}(\lambda) \quad (5.23)$$

for all cases of parity both in P and R . The above Eq. 5.23 is simply a rescaling of the respective Hermite polynomial recurrence by a factor of $\frac{1}{2}$, leading to Eq. (5.22). By performing this continuation into the imaginary axis, particularly for $r = qi$, with $q \in [0, 1]$, we recover the model which motivated the present study [2], which described partial quantum number conservation.

The first few P and R polynomials are listed in Tables 5.1 and 5.2, respectively.

N	P_N
0	1
1	$-\lambda$
2	$\lambda^2 + r^2 \sigma^2$
3	$-\lambda^3 + (1 - 2r^2) \lambda \sigma^2$
4	$\lambda^4 + 2(2r^2 - 1) \sigma^2 \lambda^2 + (2r^4 + 1) \sigma^4$
5	$-\lambda^5 - 2(3r^2 - 2) \sigma^2 \lambda^3 - 3(2r^4 - 2r^2 + 1) \sigma^4 \lambda$
6	$\lambda^6 + 3(3r^2 - 2) \sigma^2 \lambda^4 + 9(2r^4 - 2r^2 + 1) \sigma^4 \lambda^2 + 3r^2(2r^4 + 3) \sigma^6$
7	$-\lambda^7 + 3(3 - 4r^2) \sigma^2 \lambda^5$ $-3(12r^4 - 16r^2 + 7) \sigma^4 \lambda^3 + 3(-8r^6 + 12r^4 - 12r^2 + 3) \sigma^6 \lambda$
8	$\lambda^8 + 4(4r^2 - 3) \sigma^2 \lambda^6 + 6(12r^4 - 16r^2 + 7) \sigma^4 \lambda^4$ $+12(8r^6 - 12r^4 + 12r^2 - 3) \sigma^6 \lambda^2 + 3(8(r^4 + 3)r^4 + 3) \sigma^8$

Table 5.1: P Polynomials.

N	R_N
0	1
1	$-\lambda$
2	$\lambda^2 - \sigma^2$
3	$-\lambda^3 + (1 - 2r^2) \lambda \sigma^2$
4	$\lambda^4 + 3(r^2 - 1) \sigma^2 \lambda^2 - 3r^2 \sigma^4$
5	$-\lambda^5 - 2(3r^2 - 2) \sigma^2 \lambda^3 - 3(2r^4 - 2r^2 + 1) \sigma^4 \lambda$
6	$\lambda^6 + (8r^2 - 7) \sigma^2 \lambda^4 + 3(4r^4 - 8r^2 + 3) \sigma^4 \lambda^2 - 3(4r^4 + 1) \sigma^6$
7	$-\lambda^7 + 3(3 - 4r^2) \sigma^2 \lambda^5$ $-3(12r^4 - 16r^2 + 7) \sigma^4 \lambda^3 + 3(-8r^6 + 12r^4 - 12r^2 + 3) \sigma^6 \lambda$
8	$\lambda^8 + (15r^2 - 13) \sigma^2 \lambda^6$ $+15(r^2(3 - 2r^2)^2 - 3) \sigma^6 \lambda^2 + 15(4r^4 - 7r^2 + 3) \sigma^4 \lambda^4 - 15r^2(4r^4 + 3) \sigma^8$

Table 5.2: R Polynomials.

5.3 Asymptotics of the P and R polynomials

The same asymptotics calculated in the two projector case of Chapter 4 may also be considered here. We again begin with the limit $r \rightarrow 0$, in which the recurrences (5.20) and (5.21) become

$$\begin{cases} P_{2k}(\lambda; 0) = -\lambda P_{2k-1}(\lambda; 0) - \sigma^2(k-1)R_{2(k-1)}(\lambda; 0), \\ P_{2k+1}(\lambda; 0) = -\lambda P_{2k}(\lambda; 0) - \sigma^2 k P_{2k-1}(\lambda; 0), \\ R_{2k}(\lambda; 0) = -\lambda P_{2k-1}(\lambda; 0) - \sigma^2 k P_{2(k-1)}(\lambda; 0). \end{cases} \quad (5.24)$$

All three equations bear some degree of similarity to the recurrence (4.13) of the Hermite polynomials, with the addition to the coupling between the P_n and the R_{2k} . Let us then first define

$$\xi = -\frac{\lambda}{\sqrt{2}\sigma}, \quad u(\xi) = \left(\frac{\sigma}{\sqrt{2}}\right)^{-2k} P_{2k}(\xi; 0),$$

$$v(\xi) = \left(\frac{\sigma}{\sqrt{2}}\right)^{-(2k+1)} P_{2k+1}(\xi; 0) \quad \text{and} \quad w(\xi) = \left(\frac{\sigma}{\sqrt{2}}\right)^{-2k} R_{2k}(\xi; 0).$$

Then Eq. (5.24) becomes

$$\begin{cases} u_k(\xi) = 2\xi v_{k-1}(\xi) - 2(k-1)w_{k-1}(\xi) \\ v_k(\xi) = 2\xi u_k(\xi) - 2kv_{k-1}(\xi) \\ w_k(\xi) = 2\xi v_{k-1}(\xi) - 2ku_{k-1}(\xi) \end{cases} \quad (5.25)$$

These equations are verified by

$$u_k(\xi) = \mathbf{H}_k(\xi)\mathbf{H}_k(\xi), \quad v_k(\xi) = \mathbf{H}_k(\xi)\mathbf{H}_{k+1}(\xi), \quad w_k(\xi) = \mathbf{H}_{k+1}(\xi)\mathbf{H}_{k-1}(\xi),$$

such that the average characteristic polynomials P become, in this limit

$$\begin{cases} P_{2k}(\lambda; 0) = \left(\frac{\sigma}{\sqrt{2}}\right)^{2k} \mathbf{H}_k\left(-\frac{\lambda}{\sqrt{2}\sigma}\right) \mathbf{H}_k\left(-\frac{\lambda}{\sqrt{2}\sigma}\right), \\ P_{2k+1}(\lambda; 0) = \left(\frac{\sigma}{\sqrt{2}}\right)^{2k+1} \mathbf{H}_{k+1}\left(-\frac{\lambda}{\sqrt{2}\sigma}\right) \mathbf{H}_k\left(-\frac{\lambda}{\sqrt{2}\sigma}\right) \end{cases} \quad (5.26)$$

which coincides with the $M = \lfloor N/2 \rfloor$ case of the separated case.

The limit of $r \rightarrow \infty$ in the intertwined case may also be carried out analytically from Eq.(5.20) by the same method used in the separated case. To do so, we consider the scalings

$$\xi = \frac{\lambda}{\sigma r}, \quad \tilde{P}_n(\xi) = \frac{1}{(\sigma r)^n} P_n(\lambda(\xi); r), \quad \tilde{R}_n(\xi) = \frac{1}{(\sigma r)^n} R_n(\lambda(\xi); r)$$

such that Eq. (5.20) may be rewritten as

$$\tilde{P}_n(\xi) = -\xi \tilde{P}_{n-1}(\xi) + \lfloor \frac{n}{2} \rfloor \tilde{P}_{n-2}(\xi) - \lfloor \frac{n-1}{2} \rfloor \frac{1}{r^2} \tilde{R}_{n-2}(\xi)$$

which, in the asymptotic limit of $r \rightarrow \infty$, becomes dependent only in the $\tilde{P}_k$

$$\begin{aligned} \tilde{P}_n(\xi) &= -\xi \tilde{P}_{n-1}(\xi) + \lfloor \frac{n}{2} \rfloor \tilde{P}_{n-2}(\xi) + \mathcal{O}\left(\frac{1}{r^2}\right) \\ &= -\xi \tilde{P}_{n-1}(\xi) + \lfloor \frac{n}{2} \rfloor \tilde{P}_{n-2}(\xi), \quad r \rightarrow \infty \end{aligned} \quad (5.27)$$

Noting that the above Eq. (5.27) is different for odd and even n , we define

$$\rho_k(\xi) = \tilde{P}_{2k}(\xi), \quad \zeta_k(\xi) = \frac{1}{\xi} \tilde{P}_{2k+1}(\xi) \quad (5.28)$$

which are polynomials of order $2k$ in ξ , and have only even exponents. Therefore, (5.27) splits into two different equations depending on the parity of n , yielding

$$-(-\xi^2)\zeta_{k-1}(\xi) = \rho_k(\xi) - k\rho_{k-1}(\xi) \quad (5.29)$$

from the even case and

$$\zeta_k(\xi) = -\rho_k(\xi) + k\zeta_{k-1}(\xi) \quad (5.30)$$

from the odd case. These recursion relations are very similar to those of the Laguerre polynomials [3] and it can readily be shown that the above Eqs. (5.29) and (5.30) are verified by

$$\rho_k(\xi) = k! \cdot \mathbf{L}_k^{(0)}(-\xi^2), \quad \zeta_k(\xi) = -k! \cdot \mathbf{L}_k^{(1)}(-\xi^2), \quad (5.31)$$

where $\mathbf{L}_k^{(\alpha)}$ is the associated Laguerre polynomial of order k and parameter α . This leads us to conclude that

$$\begin{aligned} P_{2k}(\lambda; r \gg 1) &= \frac{k!}{(\sigma r)^{2k}} \cdot \mathbf{L}_k^{(0)}\left(-\frac{\lambda^2}{r^2 \sigma^2}\right) \\ P_{2k+1}(\lambda; r \gg 1) &= -\frac{k!}{(\sigma r)^{2k+1}} \cdot \lambda \cdot \mathbf{L}_k^{(1)}\left(-\frac{\lambda^2}{r^2 \sigma^2}\right), \end{aligned} \quad (5.32)$$

which can be shown to work for $k = 0$ in both cases.

Note, in particular, that the roots of the associated Laguerre polynomials with positive parameter α are all real and positive [3]. This implies that the λ of the asymptotic P are all complex conjugate pairs of the corresponding roots of the Laguerre polynomial, with an additional zero root in the odd case.

5.4 Numerical Statistics

For the case of intertwined blocks, we have two main parameters: the total size of the matrix and the parameter r . Although it is also possible to sort the elements of the matrix

from real, complex or quaternion distributions, the results are qualitatively similar, as in the separated case presented in Chapter 4. For this reason, Figs. 5.2, 5.3 and 5.4 depict the results of the eigenvalue density calculated for 10^6 sample matrices of size $N = 20$ with $r = 1$, $r = 0.5$ and $r = 1.5$, respectively. The roots of the polynomials P_{20} and R_{20} for the respective r are also represented. The dashed black line again represents the radius $\sqrt{N/2}$ of the asymptotic circular law of the Ginibre ensemble. In all three cases, we see that the eigenvalue density again follows the roots of the polynomial and, as in the separated case, increasing the parameter r above 1 causes significant changes in the symmetry of the spectrum.

Here, we again consider the variance in the neighborhood of the roots of the average characteristic polynomial. In Figs. 5.5-5.7 we present density plots of the standard deviation and relative standard deviation for the cases of $N = 20$ intertwined $\mathbb{R}$ matrices with varying interaction parameters. In Fig. 5.5, the r parameter is set to 1.0, and the results are qualitatively similar to those seen in the previous case in Fig. 4.15, namely, that the variance peaks as we approach the roots of the polynomial. Similarly, Figs. 5.6 and 5.7 for $r = 0.5$ and $r = 1.5$ respectively present similar behavior to 4.18 and 4.19, respectively, with the higher density regions again accompanying the shape of the roots of the average characteristic polynomial $P_{20}(z; r)$. It is interesting to note that, in this case, the roots of the Q_{20} seem consistently located in regions of low relative variance, except in the case of $r > 1$, for which the imaginary roots of $P_{20}(z; r)$ and $Q_{20}(z; r)$ begin to approach one another.

As in the separated case, it is interesting to look into the behavior of a sample matrix when the interacting parameter r is varied. Starting from the same Hermitian matrix used to obtain the results in Figs. 4.20a-4.22a and applying (4.3) for the projectors (5.1) and (5.2), we vary the interaction strength r in the same fashion that was done in the separated case, and the resulting behavior for both eigenvalues and roots of the average polynomial is presented in Fig. 5.8. It is notable that, although the behavior of the sample matrix is qualitatively similar to that of 4.22, the quantitative results are different. This is due to fact that, despite being obtained the same original Hermitian sample matrix, the polynomials of specific samples of each ensemble do not necessarily follow the behavior of their respective ensemble average. Therefore, even for large r , the same sample Hermitian ma-

Figure 5.2: Density of eigenvalues on the complex plane for 10^6 $N = 16$ sample matrices of the intertwined $\mathbb{R}$ case, with $r = 1.0$, $\sigma = 0.5$ excluding the real eigenvalues, with the case including the real eigenvalues depicted in the upper-right panel. The red plus signs denote the roots of P_{20} and the blue crosses denote the roots of R_{20} .

Figure 5.3: Density plot under the same conditions of Fig. 5.2, except for $r = 0.5$. The red plus signs denote the roots of P_{20} and the blue crosses denote the roots of R_{20} .

Figure 5.4: Density plot under the same conditions of Fig. 5.2, except for $r = 1.5$. The red plus signs denote the roots of P_{20} and the blue crosses denote the roots of R_{20} .

Figure 5.5: Relative standard deviation of the characteristic polynomials averaged over 10^6 $N = 16$ sample matrices of the intertwined case, with $r = 1.0$, $\sigma = 0.5$. The red plus signs denote the roots of P_{16} and the blue crosses denote the roots of R_{20} .

Figure 5.6: Relative standard deviation plot under the same conditions of Fig. 5.5, except for $r = 0.5$. The red plus signs denote the roots of P_{20} and the blue crosses denote the roots of R_{20} .

Figure 5.7: Variance plot under the same conditions of Fig. 5.5, except for $r = 1.5$. The red plus signs denote the roots of P_{20} and the blue crosses denote the roots of R_{16} .

Figure 5.8: Eigenvalue behavior of a sample matrix (a) and characteristic polynomial root behavior (b) for $N = 20$ of the intertwined case, with variance $\sigma^2 = 0.5$.

trix will produce different eigenvalues in each ensemble, even in the cases in which the average asymptotics coincide, such as in Figs. 4.22.a and 5.8.b, where the large asymptotic limits display similar qualitative and quantitative behavior, and are similarly greatly affected by the interaction being switched on.

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Chapter 6

Discussion and Final Remarks

In the present chapter, the main results of this thesis will be discussed. The relationship between the observed behaviors within the distinct models will be discussed, as will be the implication of this connection. Additionally, some final remarks regarding open questions and future problems will be presented.

6.1 Tridiagonal Model

The model introduced in Chapter 2 was based on the initial approach introduced in Ref. [1] as a path to introduce pseudo-Hermiticity into the β -Hermite ensemble [2, 3]. The studies conducted as part of this thesis have led to two separate cases of reality-breaking behavior.

The first case of this behavior occurs when the sensitivity of the system to perturbations becomes increasingly large. In Figs. 2.1 and 2.2 we may observe the first case in which this happens, as well as noting the distinctive relation of this figure to the illustration of the $\mathcal{PT}$ -symmetric cubic oscillator [4]. Namely, one should note that the eigenvalues of the cubic oscillator move into the complex plane successively from larger moduli pairs, whereas when the perturbation is introduced in that first realization of the tridiagonal model the pairs move starting from lower moduli. In fact, when considering increasing matrix size and fixed perturbation as in Fig. 2.3, the same behavior is observed. Changing the position of the perturbation may soften the effect of the perturbation, as seen in Fig. 2.8, but the movement into the complex plane is nonetheless similar despite the rise

of new real eigenvalues. This is again the case when modulating the interaction strength as in Figs. 2.9 and 2.10. We may tie this to the presence of the unbounded η operator and the related non-normality of the matrices, which is illustrated by the analysis of the condition number in Fig. 2.4 as well as the pseudospectral behavior in Figs. 2.6, 2.11, the latter particularly may be contrasted to Fig. 2.5.

For the second case, the positive-definiteness of the η operator is broken. As illustrated in Figs. 2.12 and 2.13, this causes a movement of the eigenvalues from the real axis towards the complex plane, similar to the previous case in which boundedness was lost. However, the marked distinction may be seen particularly well by comparing Fig. 2.14 with 2.1, 2.2, 2.9 and 2.10. The former behave much more like the cubic harmonic oscillator than the latter, having the eigenvalues of larger moduli being prone to become complex before those of smaller moduli. At the same time, Figs. 2.15 and 2.16 show that the eigenvalues are well defined with respect to perturbations, leading to the conclusion that this realization of the model displays characteristics that are of particular interest to represent $\mathcal{PT}$ symmetric interactions through a pseudo-Hermitian representation.

6.2 Dense Models

In Chapter 3, the next model considered was introduced. In that model, rather than considering a sparse tridiagonal matrix to introduce pseudo-Hermiticity a fully Gaussian $N \times N$ matrix model was introduced. Starting from the projector method of Ref. [5], pseudo-Hermiticity was introduced by splitting the vector space through projection operators, and subsequently mixing those split subspaces by means of a non-positive-definite pseudo-Hermitian interaction. This method was applied to matrices defined from the classical Gaussian matrices of Wigner, namely the orthogonal, unitary and symplectic ensembles of real, complex and quaternion matrices, respectively.

Two particular realizations of this method were presented and discussed. Some shared properties of these models were discussed, namely their commutation and anti-commutation relations with the associated η operator. This served to compute the anti-Hermitian limit for these models, which in both cases was connected to the Chiral ensemble of Random Matrix Theory (RMT). The spectral behavior of these two cases were discussed in further

detail each in its own Chapter.

Chapter 4 delved deeper into the case of two contiguous projectors. Spectral properties were found to be within the asymptotic limit of the Ginibre ensemble. However, some special characteristics were also seen. Namely, the spectrum was shown to deviate within that limit as a function of both the interaction parameter, r , and the $\mathcal{P}$ projector image dimension, M . Thus, the pseudo-Hermitian interaction appears to potentially allow for the control of the spectral behavior within the transition from real to purely imaginary eigenvalues. Furthermore, it was shown that it is possible to obtain for this case a recurrence relation for the average characteristic polynomials, denoted by $Q_{M,N}$. The numerical results pertaining to the statistical behavior of the spectra around these roots corroborate that, while not representative the eigenvalues of a particular sample matrix, are a good path for comparison to the global behavior of the ensemble as a whole.

The second case, consisting of two parity-wise split projectors, was discussed in Chapter 5. Again the spectral behavior was subject to variation as a function of the interaction strength. In this case, however, the projectors are fixed, as there is only one parity-wise split for the basis. Furthermore, the observed spectral behavior was very similar to that of $M = \lfloor N/2 \rfloor$ of the previous case. The average characteristic polynomials were also determined in this case, although it was shown that, unlike the separated case, two sets of interlaced polynomials are obtained, denoted by P_N and R_N . Thus, the recurrence relation for the polynomial of an $N \times N$ realization of this case is a coupling of the recursions of these two polynomial families. Again, the statistical behavior of the eigenvalues show that these polynomials are good predictors of the global behavior of the eigenvalues of the ensemble, even if also failing to describe individual sample matrices as in the previous case.

Furthermore, the common aspect of these two cases is that the observed spectral curves, seen in Figs. 4.1-4.4 and 5.1 are similar to the case to which the tridiagonal non-positive model was also compared, namely that of Ref. [4]. This may be seen more clearly also in Figs. 4.20-4.22 and 5.8, where the complex eigenvalues were compared to the average root behavior of the respective polynomial. This corroborates the idea that these non-positive models may be a promising path to introduce $\mathcal{PT}$ -symmetric characteristics into RMT models.

6.3 Model Comparison

The two models presented in this thesis, namely the tridiagonal sparse model and the full matrix dense model, are both rooted ultimately in the same cases of the Wigner Gaussian ensemble. In the case of the tridiagonal model, the starting point of the work presented is the β -Hermite generalization of the tridiagonal reduction of these models. Therein, it was possible to introduce a break in the reality of the spectrum by two different means. The first was breaking the boundedness of η and the second was introducing a non-positive η . On the other hand, for the dense model the break in reality was introduced only through the non-positive η , motivated in fact by the previous results in the tridiagonal model.

Although the behavior of the spectral samples in the unbound case was qualitatively analogous to the non-positive cases, these non-positive cases of both models were not only similar to one another but also to the standard behavior in the literature of $\mathcal{PT}$ -symmetry. The tridiagonal case has the advantage of being extendable for arbitrary β and thus allows for a parameter that the dense case does not. Particularly, the interpretation of β as an inverse temperature, as the typical notation in statistical mechanics, bears some potential for further study. The dense model, on the other hand, has the advantage of allowing the representation of various interactions between parts of the whole system by choosing appropriate projectors and interaction functions ϕ , despite being limited with respect to the thermodynamic interpretation of β .

The break in reality is clear in both cases, even if they come to be by distinct mechanisms. This break in reality indicates the transition between broken and unbroken symmetry phases, as the eigenvectors no longer form a basis when the real eigenvalues coalesce. The precise nature of the symmetry being broken is an interesting question, which may in turn provide further insight into the nature of the behavior observed, and it remains unexplored.

6.4 Open Questions

During the period in which the results presented herein were obtained, other questions arose which, due to time constraints, were not tackled directly. Some of those are inter-

esting problems in their own right and, for this reason, will be presented here.

6.4.1 Spacing Distributions

An important problem in RMT is the study of the distribution of the spacing between eigenvalues. This is closely tied to the original study by Wigner in the context of nuclear resonance models. Thus, it is very common to perform studies of this kind in models derived from classical RMT. For the studies presented in this thesis, however, this is a non-trivial problem. Studying the spacing between eigenvalues requires a systematic approach to perform what is known as unfolding of the eigenvalues, which uniformizes the eigenvalue distribution. Thus, understanding how the spacing is affected by the symmetry breaks in both models is still a problem to be studied.

6.4.2 Exceptional Point Behavior

In $\mathcal{PT}$ -symmetric systems research, it is typical to consider the behavior of the eigenstates around the points in which the real eigenvalues branch into the complex plane. These points, known in the $\mathcal{PT}$ -symmetry literature as exceptional points, may change the behavior of the eigenstates when one considers certain trajectories through the parameter space. It is thus possible to consider the statistical properties of the eigenvalues and eigenvectors around the exceptional point for the random models presented here. It should be noted, however, that random matrices are typically averse to degeneracies, and the presence of level repulsion means that classical random matrices are never, up to a nil-measure set, degenerate. Thus, even through the introduction of pseudo-Hermitian interactions presented in this thesis, it may be that the exceptional point behavior is reducible to the 2×2 toy model typically alluded to in the literature [6].

An interesting point to note is that when introducing exceptional points in a random system, there must be a suppression of the level-repulsion typical of RMT matrices. Thus, this line of inquiry is very likely to be connected to the study of the effects of the spacing between real eigenvalues.

6.4.3 Light Propagation in non-Hermitian Lattices

One problem which is currently actively researched is that of light beams propagated in non-Hermitian Lattices. Some results in the literature suggest that this allows for the modeling of effective unidirectional invisibility. The tridiagonal model presented in this thesis may be a good candidate for such a study, particularly with appropriate choices of interactions for the impurities. Although some time was dedicated to this problem, ultimately it still remains to be explored in full.

6.4.4 Laguerre and Jacobi Matrices

All of the cases considered within this thesis has as their starting point the Hermite, or Wigner, case of classical RMT. It may be possible to introduce similar pseudo-Hermitian interactions in the Laguerre, or Wishart, case, which have great importance in the study of certain statistical systems. The Laguerre ensemble is a particularly interesting case, as the eigenvalues are all positive. Thus, it may be possible to produce a model whose behavior more closely resembles that of the cubic harmonic oscillator.

It should be noted that some exploratory studies made during the development of this thesis suggest that the Laguerre ensemble is very robust with respect to its underlying symmetry. Thus, the introduction of an interaction that breaks the reality may not be as straightforward as the Hermite case.

Furthermore, the Jacobi, or MANOVA, case is also of significance in statistical modeling. Since it may be defined in similar fashion to the Hermite and Laguerre cases, it may also be an avenue of interest, although it is less common for this ensemble to appear in the context of physical research than the two former ensembles.

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