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Aplicações do Bootstrap Funcional Extremo

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Applications of the Extremal Functional Bootstrap

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Abstract

The study of conformal symmetry is motivated through an example in statistical mechanics and then rigorously developed in quantum field theories in general spatial dimensions. In particular, primary fields are introduced as the fundamental objects of such theories and then studied in the formalism of radial quantization. The implications of conformal invariance on the functional form of correlation functions are studied in detail. Conformal blocks are defined and various approaches to their analytical and numerical calculation are presented with a special emphasis on the one-dimensional case.

Building on these preliminaries, a modern formulation of the conformal bootstrap program and its various extensions are discussed. Examples are given in which bounds on the scaling dimensions in a one-dimensional theory are derived numerically. Using these results I motivate the technique of using the extremal functional bootstrap which I then develop in more detail. Many technical details are discussed and examples shown.

After a brief discussion of conformal field theories with a boundary I apply numerical methods to find constraints on the spectrum of the 3D Ising model. Another application is presented in which I study the 4-point function on the boundary of a particular theory in Anti-de-Sitter space in order to approximate the mass spectrum of the theory.

Key-Words: “Statistical Mechanics”; “Conformal Invariants”; “Phase Transition”; “Scale Invariance”; “Bootstrap”; “Critical Exponent”.

Resumo

O estudo da simetria conforme é motivado através de um exemplo em mecânica estatística e em seguida rigorosamente desenvolvido em teorias de campos quânticos em dimensões espaciais gerais. Em particular, os campos primários são introduzidos como os objetos fundamentais de tais teorias e então estudados através do formalismo de quantização radial. As implicações da invariância conforme na forma funcional das funções de correlação são estudadas em detalhe. Blocos conformes são definidos e várias abordagens para seu cálculo analítico e numérico são apresentadas com uma ênfase especial no caso unidimensional.

Com base nessas preliminares, uma formulação moderna do programa de bootstrap conforme e suas várias extensões são discutidas. Exemplos são dados em que limites nas dimensões de escala em uma teoria unidimensional são derivados numericamente. Usando esses resultados, motivei a técnica de usar o bootstrap funcional extremo, que depois desenvolvo em mais detalhes. Diversos detalhes técnicos são discutidos e exemplos são apresentados.

Após uma breve discussão das teorias de campo conformes com fronteiras, eu aplico métodos numéricos para encontrar restrições no espectro do modelo de Ising em 3D. Outra aplicação é apresentada em que eu estudo a função de 4 pontos na fronteira de uma teoria particular no espaço Anti-de-Sitter, a fim de aproximar o espectro de massa da teoria.

Palavras-Chave: “Mecânica estatística”; “Invariantes conformes”; “Transição de fase”; “Invariância de escala”; “Bootstrap”; “Expoente crítico”.

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Chapter 1

Introduction

Phase transitions are an important phenomenon in nature. They occur when the macroscopic properties of a system depend discontinuously on an external parameter. This discontinuity is generally used to define different phases of matter that behave qualitatively differently. The point in a phase diagram where this division disappears is called a critical point. At the critical point the system becomes self-similar over a big range of scales, a property that makes it both mathematically interesting as well as very difficult to study because long-range correlations become non-negligible. The system is effectively both strongly coupled and has a macroscopic number of degrees of freedom. However, taking the strong constraint implied by this scale invariance seriously, one can restrict the space of possible theories in non-trivial ways. This thesis deals with the formulation and analysis of such symmetry constraints through the lense of what is known as the *conformal bootstrap program* [1]. The present chapter aims to first illustrate the emergence of phase transitions and scale invariance through a simple example. The second chapter will introduce the mathematical framework of conformal field theory. In Ch. 3 I will discuss in detail a set of functions known as conformal blocks that will be necessary for the formulation of the conformal bootstrap in Ch. 4. Ch. 5 describes the specific numerical technique that I used for most of the bootstrap calculations in this work. Finally Ch. 6 and Ch. 7 demonstrate examples of conformal bootstrap calculations in systems that have boundaries.

1.1 Phase Transitions in Mean-Field Approximation

1.1.1 Critical Points

Although phase transitions tend to be hard to study quantitatively we can sometimes get a good qualitative picture of the physics by studying the mean-field approximation. This

following arguments closely follow the discussion in [2]. Consider an Ising-like system with partition function

$$Z = \text{Tr} \exp \left\{ \frac{\beta}{2} \sum_{r,r'} J(r-r') s(r) s(r') + \beta H \sum_r s(r) \right\}, \quad (1.1.1)$$

where $s = \pm 1$ is the local spin orientation, J a ferromagnetic interaction and H an external magnetic field. Since this problem is, in general, intractable we choose to write

$$s(r)s(r') = (M + (s(r) - M))(M + (s(r') - M)), \quad (1.1.2)$$

where we introduced the mean field M whose value is thus far unspecified. We assume the correlations of the fluctuations around the mean field to be negligible, i.e.

$$\langle (s(r) - M)(s(r') - M) \rangle \approx 0. \quad (1.1.3)$$

Here the mean-field approach is justified as long as the spatial dimension of the system is sufficiently high because in higher dimensions each lattice site will couple to enough neighbors to lock there state to the average behavior of the lattice. This way we can rewrite the partition function as

$$Z \approx \text{Tr} \exp \left\{ \frac{\beta}{2} \sum_{r,r'} J(r-r') (M(s(r) + s(r')) - M^2) + \beta H \sum_r s(r) \right\}, \quad (1.1.4)$$

which can be evaluated as

$$Z = e^{-\frac{1}{2}\beta N J M^2} (2 \cosh \beta (J M + H))^N, \quad (1.1.5)$$

where N is the number of sites (later to be taken to the thermodynamic limit $N \rightarrow \infty$) and we defined $J \equiv \sum_{r,r'} J(r-r')$. The value for M is the one that minimizes the free energy per site $f = -\frac{k_B T}{N} \ln Z$. This condition is given by

$$M = \tanh \beta (J M + H). \quad (1.1.6)$$

This equation can be most easily solved by visualizing it graphically. For $H \neq 0$ there is a unique solution and $\text{sgn}(H) = \text{sgn}(M)$ and we can think of the two cases as phases. If $H = 0$ we expect M to be small we Taylor expand the free energy and find

$$f = \text{const.} + \frac{1}{2} J (1 - \beta J) M^2 + O(M^4). \quad (1.1.7)$$

This allows us to identify a critical temperature $T_c = \frac{J}{k_B}$. For high temperatures $T > T_c$ the only minimum is $M = 0$, but at temperatures below the critical temperature there are two minima $M \neq 0$. This means the material magnetizes spontaneously. We have identified a critical point at $T = \frac{J}{k_B}$ and $H = 0$.

1.1.2 Scale Invariance at Criticality

We now want to study the correlation function of the spin variable $G(r) \equiv \langle s(r)s(0) \rangle$. Since the system is assumed to be translationally symmetric we can put one s at the origin and since there is also a $\mathbb{Z}_2$ -symmetry in the system we can assume without loss of generality that $s(0) = 1$. We then have

$$\begin{aligned} \langle s(r)s(0) \rangle &= \frac{\text{Tr } s(r) \exp \left\{ \frac{\beta}{2} \sum_{r,r'} J(r-r') s(r)s(r') + \beta H \sum_r s(r) \right\}}{Z} \\ &= k_B T \frac{\partial}{\partial h(r)} \ln Z|_{h=0}, \end{aligned} \quad (1.1.8)$$

where $h(r)$ is a local perturbation of H . Using the mean field approximation we can take this derivative explicitly to find

$$G(r) = \tanh \left(\beta \sum_{r'} J(r-r') M(r') \right) \quad (1.1.9)$$

at zero magnetic field. The mean field condition (1.1.6) is still valid, but $M(r)$ is now dependent on r because we broke translational symmetry when imposing $s(0) = 1$, so we write

$$M(r) = \tanh \left(\beta \sum_{r'} J(r-r') M(r') \right). \quad (1.1.10)$$

It follows that $G(r) = M(r)$. We are mostly interested in the behavior of $G(r)$ at large r , where we expect M to be small. So in order solve Eq. (1.1.10) we can Taylor expand in M

$$M(r) = \beta \sum_{r'} J(r-r') M(r') + \text{corrections}. \quad (1.1.11)$$

We are assuming short-range interactions for our model so the corrections with J^2 should only be relevant at short distances. We encode this by bunching all of them together as a delta function. Going into Fourier space we have

$$\tilde{M}(k) = \beta \tilde{J}(k) \tilde{M}(k) + \text{const.} \quad (1.1.12)$$

We approximate $\tilde{J}(k)$ by its lowest order terms

$$\tilde{J}(k) = J (1 - R^2 k^2) + O(k^4), \quad (1.1.13)$$

where $R^2 \equiv \frac{\sum_r r^2 J(r)}{\sum_r J(r)}$ roughly measures the range of the interaction. Remembering that $G(r) = M(r)$, we find

$$\tilde{G}(k) = \frac{\text{const.} R^{-2}}{k^2 + \xi^{-2}}, \quad (1.1.14)$$

where $\xi \equiv Rt^{-\frac{1}{2}}$ and $t \equiv \frac{T-T_c}{T_c}$. The Fourier transform can now be inverted to yield

$$G(r) \sim \frac{1}{r^{\frac{d-1}{2}}} e^{-\frac{r}{\xi}}. \quad (1.1.15)$$

This produces the well-known result that asymptotically correlation functions fall off exponentially fast. We also see that ξ can be interpreted as a correlation length. Interestingly as $T \rightarrow T_c$ the correlation length diverges $\xi \rightarrow \infty$. This means that the system really does loose its scale as claimed in the introduction. At the critical temperature we find

$$G(r) \sim \frac{1}{r^{d-2}}. \quad (1.1.16)$$

Here the value $\frac{d-2}{2}$ can be said to be the scaling dimension of the field s because it determines how this field behaves when rescaling the system. We should not expect the mean-field approach to yield reliable answers in this regime because the correlation functions are not small so the real values of the scaling dimensions will receive corrections. Similar arguments to the ones in this section can be made when boundary surfaces are present in a theory. This will be discussed in more detail in Ch. 6.

There is, however, one more interesting property to be learned from this result. The scaling behavior that we determined does not depend on the precise nature of J . We neglected many details about the short-distance behavior of the coupling but none of these details would contribute to the value $\frac{d-2}{2}$.

It turns out that this continues to be true outside of mean-field theory. This phenomenon is called *critical universality* [3] and means that many physically very distinct theories behave very similarly close to their critical points. Their behavior is determined by their properties under scaling which is given by the so-called *critical exponents* [4] which are closely related to the scaling dimensions mentioned before. One particularly striking example of criticality is the equivalence of the critical exponents of ferromagnetic samples that are described in this section and many gases at their critical point in the $p - V$ -diagram.

1.1.3 Symmetries at Criticality

As shown in the previous section, systems exhibit scale invariance at their critical point. We also assumed translation invariance which is physically reasonable in the absence of boundary effects. Rotational symmetry also holds at the critical point which is somewhat subtle considering the underlying theory will usually describe the continuum limit of a lattice theory. The point here is that the effect of the lattice becomes less pronounced at large distances because as we saw before the short distance details have no effect on the critical theory.

While these symmetries do form a closed group, in practice they often get enhanced by the so-called conformal symmetry. In 2D this can be proved using fairly general assumptions but in higher dimensions the distinction is not yet completely understood, see [5] for a review.

The approach to critical systems taken in this thesis follows the recent revival of the conformal bootstrap program. The idea is to start with the assumption of conformal symmetry instead of a specific microscopic description. We initiate the study of conformal field theory (CFT) in the next chapter.

1.2 Quantum Field Theory

1.2.1 Renormalization Group Flows

Another very important motivation for the study of CFTs comes from Quantum Field Theory (QFT). A generic QFT is usually specified by its Lagrangian density in terms of the underlying fields and their coupling constants. For example, a generic scalar field can have

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \sum_{n \in \mathbb{N}} g_n \phi^n. \quad (1.2.1)$$

In path integral quantization one calculates from this the partition function of the system [6]:

$$Z = \int [d\phi] e^{i \int d^D x \mathcal{L}}. \quad (1.2.2)$$

In order to calculate scattering amplitudes of the theory one has to study the n -point functions of the fields. When computing these, one picks up quantum corrections from loop contributions. These modify the effective values of the coupling, usually by an infinite amount. These values must then be renormalized to fit experimental values.

A physical interpretation of the somewhat mysterious renormalization prescription was found through the study of the renormalization group [7]. The idea is that any QFT is only defined up to some energy cutoff Λ , or equivalently, down to a certain microscopic size, called the ultra-violet (UV). Fluctuations in the fields can only be up to Λ . If we are interested in physical processes at lower energy then we can find a Lagrangian with a lower cutoff $\tilde{\Lambda}$, such that the predictions of both theories are identical below $\tilde{\Lambda}$. This new Lagrangian describes what is known as an *Effective Field Theory*. The way to derive effective Lagrangians from microscopic Lagrangians is to separate the fields into high energy modes and low energy modes and to integrate out the high energy modes in Eq. (1.2.2). This flow will modify the couplings in the Lagrangian and make them energy dependent $g_n(\Lambda)$. This process is called *Renormalization Group Flow* (RG flow).

1.2.2 Space of QFTS

The RG flow is irreversible [8, 9] and, thus, cannot have limit cycles. At low energies, the RG flow of a well-behaved theory must therefore converge to a fixed point, the so-called infra-red (IR). At the fixed point a change of the energy scale does not affect the couplings anymore. In other words, the theory is scale invariant. As in the case of statistical systems, this symmetry can usually be enhanced to include conformal symmetries, so the fixed points in the parameter space of QFTs are given by CFTs.

To define a QFT it is often convenient to start from a CFT and introduce some small perturbation. If the perturbation dies out along the flow, the perturbation is called *irrelevant*. If it grows macroscopic, however, it is called *relevant*. In this case the QFT will flow away from its CFT in the UV and end up in a different CFT at the IR. Thinking of QFTs as flows between CFTs makes it clear why their study is worthwhile. The most ambitious goal of the conformal bootstrap program is to completely classify all CFTs. This would teach us a lot about the structure of the parameter space of all QFTs.

1.3 Further Motivations

While I have focused on statistical mechanics and QFT as physically clear motivation for the study of CFTs there are many more subtle reasons to study them. Since many 2D CFTs have been solved through the use of algebraic methods [1] they provide examples of strongly coupled quantum field theories that we can nevertheless extract information from.

Aside from this they constitute a fundamental building block of string theory because they capture the symmetries of the world-sheet [10]. Especially in recent years they have garnered more and more attention because of the AdS/CFT correspondence, which establishes a duality between gravitational theories and CFTs, see for example [11] for an introduction.

Although all of these topics are hugely interesting they will not be the focus of my study here. First of all I will mostly avoid discussions of the very well developed field of 2D CFT and second of all I will not make any use of supersymmetry, so the theories studied here are largely unrelated to the ones in string theory.

Chapter 2

Conformal Field Theory

2.1 Preliminaries

In this section I will develop the general theory of conformal transformations and conformal invariance. This chapter mostly follows the treatment in [12].

2.1.1 Conformal invariance

Under a general coordinate transformation the metric at a point P transforms as

$$g_{\mu\nu}(\tilde{x}(P)) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x(P)). \quad (2.1.1)$$

A conformal transformation is an angle preserving coordinate transformation such that

$$g_{\mu\nu}(\tilde{x}(P)) = \Omega^2(x(P)) g_{\mu\nu}(x(P)). \quad (2.1.2)$$

The factor Ω is called the scale factor of a transformation. Notice that Poincaré transformations are the special case of $\Omega = 1$. To find the general transformations satisfying this relation we look at an infinitesimal transformation $\tilde{x}^\mu = x^\mu + \epsilon^\mu$. Using $\Omega^2 = 1 + \omega$, where ω is infinitesimal, we find

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \delta_{\mu\nu} \omega(x). \quad (2.1.3)$$

Tracing on both sides gives

$$\omega = \frac{2}{d} \partial_\mu \epsilon^\mu. \quad (2.1.4)$$

Applying a partial derivative to Eq. (2.1.3) we obtain the conformal killing equation

$$\delta_{\mu\nu} \partial_\rho \omega = \partial_\rho \partial_\nu \epsilon_\mu + \partial_\rho \partial_\mu \epsilon_\nu. \quad (2.1.5)$$

Taking cyclic permutations of the indices and then taking a linear combination of them yields

$$\delta_{\rho\mu}\partial_\nu\omega + \delta_{\nu\rho}\partial_\mu\omega - \delta_{\mu\nu}\partial_\rho\omega = 2\partial_\mu\partial_\nu\epsilon_\rho. \quad (2.1.6)$$

Contracting μ with ν we get

$$\partial^2\epsilon_\rho = \frac{2-d}{2}\partial_\rho\omega. \quad (2.1.7)$$

Applying another ∂^ρ derivative and comparing to Eq. (2.1.4) finally gives

$$(d-1)\partial^2\omega = 0. \quad (2.1.8)$$

This and Eq. (2.1.7) imply that we must consider dimensions $d \geq 3$ if we are to learn anything from this approach. If we apply ∂^ρ to Eq. (2.1.6) we get

$$(2-d)\partial_\mu\partial_\nu\omega = \delta_{\mu\nu}\partial^2\omega \quad (2.1.9)$$

so that Eq. (2.1.8) actually implies that ω is at most linear in x , which in turn implies through Eq. (2.1.6) that $\partial_\mu\partial_\nu\epsilon_\rho$ is constant. In other words, the most general allowed infinitesimal transformation is given by

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho, \quad (2.1.10)$$

where $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. We can now study the constraints on these parameters separately. There are no restrictions on a^μ because all equations involving ϵ only include its derivative. Plugging the linear part into Eq. (2.1.3) we find

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}\delta_{\mu\nu}b^\rho{}_\rho \quad (2.1.11)$$

which means that b must be the sum of an anti-symmetric tensor and a trace, i.e. $b_{\mu\nu} = \lambda\delta_{\mu\nu} + m_{\mu\nu}$, with $m_{\mu\nu} = -m_{\nu\mu}$. Now plugging the quadratic term into Eq. (2.1.6) we find

$$c_{\rho\mu\nu} = \delta_{\rho\mu}b_\nu + \delta_{\nu\rho}b_\mu - \delta_{\mu\nu}b_\rho \quad (2.1.12)$$

where we defined a new and different $b_\mu \equiv \frac{1}{d}c^\alpha{}_\alpha\mu$. This means that the quadratic part of the transformation can be entirely described by just one vector (as opposed to a rank 3 tensor). This then corresponds to the infinitesimal transformation of what is called the *special conformal transformations* or SCT

$$\epsilon_\mu = 2x_\mu(x \cdot b) - b_\mu x^2 \quad (2.1.13)$$

The first three of the infinitesimal transformations are simply exponentiated. In order to obtain the finite version of Eq. (2.1.13), assume b^μ to be finite but rescale it by some infinitesimal ϵ . This way we can realize that this corresponds to the differential equation

$$x'^\mu \equiv \frac{\tilde{x}^\mu - x^\mu}{\epsilon} = 2x^\mu(x \cdot b) - b^\mu x^2 \quad (2.1.14)$$

In order to solve this equation we make an inspired choice for a change of variables, namely $y^\mu = \frac{x^\mu}{x^2}$. Working through a few lines of algebra this magically reduces our differential equation to

$$y'^\mu = -b^\mu \quad (2.1.15)$$

which can be solved

$$y^\mu = y_0^\mu - tb^\mu \quad (2.1.16)$$

where t is really just our finite version of ϵ , that we will set equal to 1 anyway, because the whole point was to send $\epsilon b^\mu \rightarrow b^\mu$. Resubstituting x we obtain an algebraic equation that can be solved for x^μ in terms of x_0^μ and b^μ . Collecting all these results we can write down all of our finite transformations

(Translation)	$\tilde{x}^\mu = x^\mu + a^\mu$
(Scaling)	$\tilde{x}^\mu = \alpha x^\mu$
(Lorentz)	$\tilde{x}^\mu = M^\mu_\nu x^\nu$
(SCT)	$\tilde{x}^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2(b \cdot x) + b^2 x^2}$

(2.1.17)

Having derived all these formulas we should pause and wonder whether we actually found all conformal transformations. In fact we immediately see that we did not because reflections have a scale factor $\Omega^2 = 1$ but are not included in our list. The reason why we missed them is that they cannot be continuously connected to the identity. It out that conformal transformations actually allow for another type of transformation that generalizes reflections - the so-called inversion

$$\tilde{x}^\mu = \frac{x^\mu}{x^2}. \quad (2.1.18)$$

One can check that its scaling factor is given by $\Omega = x^2$. Depending on whether or not a theory is invariant under change of parity, its symmetry group may or may not include inversions.

Going back to our derivation of the finite SCT we actually have more to learn. The fact that our specific change of variables $y^\mu = \frac{x^\mu}{x^2}$ helped us solve the differential equation was no accident. People usually visualize the SCT by saying that it is an inversion, followed by a translation b , followed by another inversion. In equations, this is just

$$\frac{\tilde{x}^\mu}{\tilde{x}^2} = \frac{x^\mu}{x^2} - b^\mu \quad (2.1.19)$$

which is precisely what we found in Eq. (2.1.16). In fact, this helps us derive the scale factor of SCTs, $\Omega = x^2 \left(\frac{x}{x^2} - b \right)^2$.

There is one more formula worth mentioning because it will come in handy later. We know that distances $|x - y|$ are invariant under both translations and rotations and that they simply scale as expected under dilations. What we don't know is their behavior under SCTs. Just plugging in the transformation and crunching some algebra yields

$$|\tilde{x} - \tilde{y}| = \frac{|x - y|}{(1 - 2(b \cdot x) + b^2 x^2)^{\frac{1}{2}} (1 - 2(b \cdot y) + b^2 y^2)^{\frac{1}{2}}}. \quad (2.1.20)$$

In the special case in which $x \rightarrow y$ this looks just like the behavior that the scale factor of the SCT told us we should expect.

2.1.2 Primary Fields

Now that we have some idea of the type of transformations we want to study, let's see how they might affect fields. Classically a field assigns to every point some quantity. To label points in space we are forced to pick some coordinate system. In this coordinate system our field then maps a certain set of numbers (our coordinates) to another set of numbers (the field value). If we picked a different coordinate system the field would be unaffected but its functional form could differ. This difference comes from two reasons: plugging in the same numbers into both functions would actually evaluate the field at distinct points. The second much less obvious difference comes from the fact that even if we looked at the field representations at coordinate values corresponding to the same point, the representation of the quantities may depend on the coordinate system that we picked.

Putting this into equations reads

$$\boxed{\begin{aligned} \tilde{x} &= f(x) \\ \tilde{\phi}(\tilde{x}) &= \mathcal{F}[\phi(x)] \end{aligned}}, \quad (2.1.21)$$

where $\mathcal{F}$ depends on both the transformation f and on what type of field ϕ describes. Consider, for example, scalars, tensors and spinors for which we know this transformation for Lorentz transformations and translations. However, now the symmetry group is now enlarged and we have no clue how any given object might transform under, say a scale transformation.

This point is usually glossed over in textbooks so I want to work out an example before delving into formalism. Taking a large tank of some gas, Alice and Bob would both like to measure its density and they have agreed that they should go ahead and count the number of molecules per unit volume. They take their measurements and meet to compare the results but they look nothing alike. What went wrong? Well, it turns out that Alice is American and to her a unit volume is measured in cubic inches whereas British

Bob used cubic centimeters. Can they still compare their values? Of course they can but they have to essentially make a coordinate transformation. Let x be the number of inches something is from the origin and $\tilde{x}$ the number of centimeters it is from the origin. In the language of Eq. (2.1.24) $\tilde{x} = 2.54x$. Likewise n and $\tilde{n}$ will be the numerical values that Alice and Bob got respectively. We know that $\frac{1}{\text{in}^3} = 2.54^3 \frac{1}{\text{cm}^3}$. Therefore Alice must rescale her numbers as $\tilde{n}(\tilde{x}) = 2.54^{-3}n(x)$. This makes sense because she found way more molecules in her unit volume than Bob did in his. But if they adjust the numbers like this they see that their measurements agree after all.

Fields transforming like this will later be called *primaries*. Notice that a 3 turned up in the exponent of the rescaling and that this number came from the fact that there was a length cubed in the denominator of its units. This number is called the scaling dimension Δ and it characterizes how a primary transforms under rescalings of the coordinate system. Classically we can just count the units of the quantity but once we move to quantum fields we have to remember that they can have anomalous dimension so that the scaling dimension might not match the naïve engineering dimension.

Generalizing our discussion for all conformal transformations we can write down the definition of a primary scalar field, i.e. a field that transforms as follows under conformal transformations

$$\tilde{\phi}(\tilde{x}) = \left| \frac{\partial \tilde{x}}{\partial x} \right|^{-\frac{\Delta}{d}} \phi(x). \quad (2.1.22)$$

If the field in question also has some tensor structure then that transforms as usual. Actually as you can see by playing around with our example of the density, this transformation law even extends to non-conformal transformations but we won't be needing those.

2.1.3 Conformal Generators

In the previous section we saw how any transformation we make can affect the functional form of a field in two ways, namely through the transformation of the coordinate and the field itself. We would like to have a way to neatly package this information. We first notice that all our basic transformations in Eq. (2.1.17) are parameterized by some continuous parameters, e.g. α in the case of rescalings. So for every rescaling we then have an operator S that transforms our field for us, i.e.

$$\tilde{\phi}(x) = S_{\alpha} \phi(x). \quad (2.1.23)$$

Notice that here we evaluate the two fields at the same coordinate, not at the same point as before. The way to understand the above formula is in some *representation*, which means that we describe the action of S on ϕ by the action of a square matrix (of

possibly infinite dimension) on a vector. Representation theory is a topic way too big to be discussed here, but we will only need one result anyway: The action of a continuously parametrized transformation can be written as the exponential of some operator times the parameter, or, in formulas, $S_\alpha = \exp(-i\alpha G)$, where G is called the generator of the transformation. All we need to do then is to find all the generators of the conformal group.

We assume that we know how all the fields we are interested in transform under the transformations we considered, i.e. the functional form of Eq. (2.1.24). This is physical input, something we could not know a priori through mathematics alone. It gives information about what type of field we are dealing with and is by no means obvious.

We will assume that we are doing some transformation with infinitesimal parameters ω_a

$$\begin{aligned}\tilde{x} &\simeq x + \omega_a \frac{\partial f(x)}{\partial \omega_a}, \\ \tilde{\phi}(\tilde{x}) &\simeq \phi(x) + \omega_a \frac{\delta \mathcal{F}[\phi(x)]}{\delta \omega_a}.\end{aligned}\tag{2.1.24}$$

Replacing x by $\tilde{x}$ in the second equation produces

$$\tilde{\phi}(\tilde{x}) = \phi(\tilde{x}) - \omega_a \frac{\partial f(\tilde{x})}{\partial \omega_a} \partial \phi(\tilde{x}) + \omega_a \frac{\delta \mathcal{F}[\phi(\tilde{x})]}{\delta \omega_a},\tag{2.1.25}$$

plus corrections of order $\mathcal{O}(\omega_a^2)$. Now the equation would still be valid if we just drop the tilde on x for convenience.

To get our generators into the picture we just start from their definition and expand in ω_a again.

$$\tilde{\phi}(x) \simeq (1 - i\omega_a G_a) \phi(x).\tag{2.1.26}$$

Putting our last two equations together we finally find

$$iG_a \phi = \frac{\partial f}{\partial \omega_a} \partial \phi - \frac{\delta \mathcal{F}}{\delta \omega_a}.\tag{2.1.27}$$

We will only be considering primaries for now since it is the only field we have learned about so far. As a warm-up we start with translations. In that case we know

$$\begin{aligned}f(x^\mu) &= x^\mu + a^\mu, \\ \mathcal{F}[\phi] &= \phi,\end{aligned}\tag{2.1.28}$$

so we find the generator P of translations

$$P_\mu = -i\partial_\mu.\tag{2.1.29}$$

For rotations the transformations are given by

$$\begin{aligned}f(x^\mu) &= x^\mu + \omega_{\mu\nu}^\mu x^\nu, \\ \mathcal{F}[\phi] &= \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right) \phi,\end{aligned}\tag{2.1.30}$$

where ω is an anti-symmetric tensor and the action of the operator S is as before physical input and will depend on the field in question. This way we find the generators of rotations

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}. \quad (2.1.31)$$

Next transformation on the list is dilations. Instead of α we will now use $1 + \lambda$ though, so that $\lambda \approx 0$ corresponds to an infinitesimal transformation. To know the form of $\mathcal{F}$ we see from Eq. (2.1.22) that we have to calculate the determinant of the transformations Jacobian which gives $|\frac{\partial \tilde{x}}{\partial x}| = (1 + \lambda)^d$:

$$\begin{aligned} f(x^\mu) &= x^\mu + \lambda x^\mu, \\ \mathcal{F}[\phi] &= (1 + \lambda)^{-\Delta} \phi. \end{aligned} \quad (2.1.32)$$

We now use our generator formula again to find

$$D = -i(\Delta + x^\mu \partial_\mu). \quad (2.1.33)$$

The only missing generator is the SCT. In Sec. 2.1.1 we already derived the scale factor so we can immediately write:

$$\begin{aligned} f(x^\mu) &= \frac{x^\mu - x^2 b^\mu}{1 - 2b \cdot x + b^2 x^2}, \\ \mathcal{F}[\phi] &= (1 - 2b \cdot x + b^2 x^2)^\Delta \phi. \end{aligned} \quad (2.1.34)$$

which results in

$$K_\mu = -i(2x_\mu \Delta + 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu). \quad (2.1.35)$$

However, there is actually a piece missing. When defining primary fields we noted that tensor structures should keep their usual transformation properties under rotations. It turns out that special conformal transformations do cause local rotations so we need consider this. This requires a generalization of the global rotation considered above. We want to somehow write

$$\tilde{\phi} \propto \exp\left(-i \frac{\omega_{\mu\nu}}{2} S^{\mu\nu}\right) \phi. \quad (2.1.36)$$

Notice that we had

$$\frac{\tilde{x}_\mu}{x^\nu} = \delta_{\mu\nu} + \omega_{\mu\nu} \quad (2.1.37)$$

From this we find our ω for special conformal transformations. Plugging

$$\mathcal{F}[\phi] \propto (1 + i(x_\mu b_\nu - b_\mu x_\nu) S^{\mu\nu}) \phi \quad (2.1.38)$$

Taking the derivative with respect to b yields the missing term that has to be added to the generator¹

$$K_\mu = -i(2x_\mu \Delta + 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) - 2x^\nu S_{\mu\nu}. \quad (2.1.39)$$

With these generators we can now describe how primary fields transform.

¹There is a typo in the big yellow book by DiFrancesco in Eq. (4.31) - a missing factor of 2. This mistake has been copied into many other references and is not listed in the book's errata.

2.1.4 Conformal Algebra

Just to neatly summarize our results from the previous section

$$\begin{array}{ll}
 \text{(Scaling)} & D = \tilde{\Delta} - ix^\mu \partial_\mu \\
 \text{(Translation)} & P_\mu = -i\partial_\mu \\
 \text{(SCT)} & K_\mu = 2x_\mu \tilde{\Delta} + i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu) - 2x^\nu S_{\mu\nu} \\
 \text{(Lorentz)} & L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}
 \end{array} , \quad (2.1.40)$$

where I have followed the convention of the literature of defining $\tilde{\Delta} = i\Delta$, mostly to distinguish the operator from the eigenvalue Δ . It would be useful to have the commutation relations of these operators and we will go ahead and calculate them. The only thing we need to know first is the commutator between $\tilde{\Delta}$ and $S_{\mu\nu}$. Notice that field components that $\tilde{\Delta}$ operates on just get multiplied by their scaling dimension, not mixed with each other. In other words, $\tilde{\Delta}$ is diagonal and therefore its commutator with any operator vanishes. The full algebra is:

$$\begin{array}{ll}
 [D, P_\mu] = iP_\mu \\
 [D, K_\mu] = -iK_\mu \\
 [D, L_{\mu\nu}] = 0 \\
 [P_\mu, P_\nu] = 0 \\
 [K_\mu, K_\nu] = 0 \\
 [K_\mu, P_\nu] = 2i(\delta_{\mu\nu}D - L_{\mu\nu}) \\
 [P_\rho, L_{\mu\nu}] = i(\delta_{\mu\rho}P_\nu - \delta_{\nu\rho}P_\mu) \\
 [K_\rho, L_{\mu\nu}] = i(\delta_{\mu\rho}K_\nu - \delta_{\nu\rho}K_\mu) \\
 [L_{\mu\nu}, L_{\rho\sigma}] = i(\delta_{\nu\rho}L_{\mu\sigma} + \delta_{\mu\sigma}L_{\nu\rho} - \delta_{\mu\rho}L_{\nu\sigma} - \delta_{\nu\sigma}L_{\mu\rho})
 \end{array} \quad (2.1.41)$$

Notice how strikingly similar K and P behave.

The algebra can be rewritten in an unexpected way. If we define an antisymmetric object M in the following way

$$\begin{aligned}
 M_{-1,0} &= D, \\
 M_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu), \\
 M_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu), \\
 M_{\mu\nu} &= L_{\mu\nu},
 \end{aligned} \quad (2.1.42)$$

it turns out that all commutators are just given by some Lorentz algebra on a $(d+1) + 1$ -

dimensional space, i.e.

$$[M_{ab}, M_{cd}] = i(\eta_{bc}M_{ad} + \eta_{ad}M_{bc} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}), \quad (2.1.43)$$

where the metric is $\eta = \text{diag}(-1, 1, 1 \dots)$ and the indices run over $a = -1, 0, 1 \dots d$. Mathematically, this means that the conformal group in d -dimensional Euclidean space is isomorphic to $SO(d+1, 1)$. This motivates the idea of the embedding space formalism which will be discussed in the next section.

Before that I would just like to point out that the isomorphism that we found makes it really easy to find the quadratic Casimir of the conformal group, i.e. an object that commutes with all elements of the algebra. Simply expanding the Casimir of $SO(d+1, 1)$ gives

$$\frac{1}{2}M_{ab}M^{ab} = \frac{1}{2}L_{\mu\nu}L^{\mu\nu} - D^2 + \frac{1}{2}(P_\mu K^\mu + K_\mu P^\mu). \quad (2.1.44)$$

2.1.5 Embedding Space Formalism

Motivated by the discovery in the previous section, we would like to see if we can somehow rewrite the complicated non-linear action of the conformal generators (K_μ has x^2 terms) in terms of rotations and boosts on a higher dimensional space. This old approach, originally used in [13], will prove to be very useful in its more modern form [14], when we consider objects with spin. We are starting off with a $(d+1) + 1$ -dimensional space and need to find a d -dimensional Lorentz-invariant subspace that we can identify with our original space. One natural guess is to just consider the light-cone, i.e. $X^a X^b \eta_{ab} = 0$, which already reduces our dimension by one.

The next step is a bit more subtle. Define an equivalence relation $X \cong Y$ if $Y = \lambda X$ for some $\lambda \in \mathbb{R}$. Now quotienting this out means that all points on the same light-ray are equivalent. What will happen is that these light-rays transform under Lorentz transformations like points would under conformal transformations. To make this concrete, we need to label the light-rays by some representatives. This is best done in light-cone coordinates $X^\pm = X^{-1} \pm X^0$ where the metric is written as

$$\eta = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1^{d \times d} \end{pmatrix}. \quad (2.1.45)$$

Each ray is now labeled by the representative $(X^+, X^-, X^\mu) = (1, x^2, x^\mu)$. We want to study the action of these Lorentz transformations in this space. As a first sanity check, we see that $M_{\mu\nu}$ only affects x^μ like normal rotations, just as it should. Looking at M_{-10}

provides a non-trivial example. We know that it acts on X like a boost in the 0-direction

$$\begin{pmatrix} \tilde{X}^{-1} \\ \tilde{X}^0 \\ \tilde{X}^\mu \end{pmatrix} = \begin{pmatrix} X^{-1} \cosh \alpha + X^0 \sinh \alpha \\ X^{-1} \sinh \alpha + X^0 \cosh \alpha \\ X^\mu \end{pmatrix} \implies \begin{pmatrix} \tilde{X}^+ \\ \tilde{X}^- \\ \tilde{X}^\mu \end{pmatrix} = \begin{pmatrix} (\cosh \alpha + \sinh \alpha) X^+ \\ (\cosh \alpha - \sinh \alpha) X^- \\ X^\mu \end{pmatrix}. \quad (2.1.46)$$

We moved to a different light-ray but to determine which we have to normalize to the representative which we find to be

$$(1, (\cosh \alpha + \sinh \alpha)^{-2} x^2, (\cosh \alpha + \sinh \alpha)^{-1} x^\mu). \quad (2.1.47)$$

It came out as a rescaling of X just as we expected. Carrying out the same procedure for the generators M_{-11} and M_{01} also verifies the correct behavior. This time however, it is preferable to take the infinitesimal limit to simplify the comparison with $P_1 \pm K_1$.

The embedding space formalism only gets its real power once we also lift the definition of primary fields into the embedding space. For simplicity, I am only going to talk about primaries that are traceless symmetric tensors (which includes scalars). The fields only need to be defined on the light-cone and they should satisfy some sort of scaling law that encodes the primaries' scaling dimension. Also any component pointing along the light-ray is irrelevant because that direction gets quotiented out when going to the conformal space. With these thoughts in mind, we define the tensor field $F_{a_1 \dots a_l}(X)$ with the following properties

1. defined on the light cone $X^2 = 0$;
2. homogeneous of degree $-\Delta$, i.e. $F(\lambda X) = \lambda^{-\Delta} F(X)$ for $\lambda > 0$;
3. symmetric and traceless;
4. transverse, i.e. $X^{a_1} F_{a_1 \dots a_l}(X) = 0$.

If we want to obtain the lower dimensional object we just restrict the domain (to only the representative of each ray) and project the components down

$$f_{\mu_1 \dots \mu_l} = \frac{X^{a_1}}{x^{\mu_1}} \dots \frac{X^{a_l}}{x^{\mu_l}} F_{a_1 \dots a_l}(X). \quad (2.1.48)$$

Using the embedding space formalism allows us to do computations that are simpler than they would be in the conformal space and then we just project down our results. We will see this in action in Sec. 2.2.3.

2.2 Conformal Field Theories

2.2.1 Definition

So far we have looked a lot at conformal transformations but not really at physics that is invariant under them. We will start with classical theories, which we can define in terms of their Lagrangians $\mathcal{L}(\phi, \partial_\mu \phi)$. A field theory is invariant under a transformation like Eq. (2.1.24) if and only if

$$\mathcal{L}(\tilde{\phi}, \tilde{\partial}_\mu \tilde{\phi}) = \mathcal{L}(\phi, \partial_\mu \phi). \quad (2.2.1)$$

Note that this implies that the action is invariant as well.

It is well-known that this invariance at the classical level may break at the quantum level due to anomalies, so the definition must be modified accordingly. We will instead consider the correlation functions of the theory and see how they transform. For a theory that is quantized from an action $S[\phi]$ the n-point functions are given by

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \frac{1}{Z} \int [D\phi] \phi_1(x_1) \dots \phi_n(x_n) \exp(-iS[\phi]). \quad (2.2.2)$$

We are going to see what happens when we do a conformal transformation on these functions, assuming they are calculated for scalar primaries. Transforming the dummy variables in the integral yields the relation

$$\begin{aligned} \langle \phi_1(\tilde{x}_1) \dots \phi_n(\tilde{x}_n) \rangle &= \frac{1}{Z} \int [D\tilde{\phi}] \tilde{\phi}_1(\tilde{x}_1) \dots \tilde{\phi}_n(\tilde{x}_n) \exp(-iS[\tilde{\phi}]) \\ &= \frac{1}{Z} \int [D\phi] \left| \frac{\partial \tilde{x}}{\partial x} \right|^{-\frac{\Delta_1}{d}} \phi_1(x_1) \dots \left| \frac{\partial \tilde{x}}{\partial x} \right|^{-\frac{\Delta_n}{d}} \phi_n(x_n) \exp(-iS[\phi]) \\ &= \left| \frac{\partial \tilde{x}}{\partial x} \right|^{-\frac{\Delta_1}{d}} \dots \left| \frac{\partial \tilde{x}}{\partial x} \right|^{-\frac{\Delta_n}{d}} \langle \phi_1(x_1) \dots \phi_n(x_n) \rangle. \end{aligned} \quad (2.2.3)$$

In going from the first to the second line not only did we apply the transformation rules, but also assumed $[D\tilde{\phi}] = [D\phi]$, i.e. the absence of anomalies.

We really don't want to be assuming Lagrangian descriptions for our CFTs, so we turn the logic around and make relation (2.2.3) our definition of a CFT: a CFT is a list of primary operators whose correlation functions satisfy Eq. (2.2.3) under all conformal transformations. Unfortunately that doesn't quite cover it because we assumed scalar primaries and we haven't derived the full form of how operators with spin transform under finite transformations. On top of that, every consistent CFT also satisfies some additional constraints, but since the full list of these are not completely known we are going to work with this fairly non-rigorous definition.

2.2.2 Scalar n -point functions

Much can be learned about the functional form of correlation functions in a CFT by studying Eq. (2.2.3). It is clear right away that the n -point functions must be translation invariant. For 1-point functions that means $\langle \phi(x) \rangle = c$. Then using the scaling behavior

$$\begin{aligned} c &= \langle \phi(\lambda x) \rangle \\ &= \lambda^{-\Delta} \langle \phi(x) \rangle \\ &= \lambda^{-\Delta} c, \end{aligned} \tag{2.2.4}$$

which means that either $c = 0$ or $\Delta = 0$. Physically speaking, a non-vanishing expectation value of a dimensionful field would introduce a scale in the theory. And what about the dimensionless solution? It turns out that all n -point functions involving only $\Delta = 0$ fields are constant, which means that the operator field is constant, i.e. it is proportional to the identity.

Moving on to 2-point functions, translation invariance implies that the function must depend only on the difference $x_1 - x_2$ and then by rotation invariance only on its magnitude $|x_1 - x_2|$. Using rescaling again

$$\begin{aligned} f(\lambda|x_1 - x_2|) &= \langle \phi_1(\lambda x_1) \phi_2(\lambda x_2) \rangle \\ &= \lambda^{-(\Delta_1 + \Delta_2)} \langle \phi_1(x_1) \phi_2(x_2) \rangle \\ &= \lambda^{-(\Delta_1 + \Delta_2)} f(|x_1 - x_2|). \end{aligned} \tag{2.2.5}$$

This homogeneity condition fixes

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{c_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}}. \tag{2.2.6}$$

Further insight comes from invoking Eq. (2.1.20). Applying an SCT to the 2-point function gives

$$\begin{aligned} \langle \phi_1(\tilde{x}_1) \phi_2(\tilde{x}_2) \rangle &= \frac{c_{12}}{|\tilde{x}_1 - \tilde{x}_2|^{\Delta_1 + \Delta_2}} \\ &= ((1 - 2(b \cdot x_1) + b^2 x_1^2)(1 - 2(b \cdot x_2) + b^2 x_2^2))^{\frac{\Delta_1 + \Delta_2}{2}} \langle \phi_1(x_1) \phi_2(x_2) \rangle, \end{aligned} \tag{2.2.7}$$

whereas by Eq. (2.2.3)

$$\langle \phi_1(\tilde{x}_1) \phi_2(\tilde{x}_2) \rangle = (1 - 2(b \cdot x_1) + b^2 x_1^2)^{\Delta_1} (1 - 2(b \cdot x_2) + b^2 x_2^2)^{\Delta_2} \langle \phi_1(x_1) \phi_2(x_2) \rangle. \tag{2.2.8}$$

In order for these two to be simultaneously true we must have $\Delta_1 = \Delta_2$ or $c_{12} = 0$. In other words

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{c_{\Delta_1} \delta_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}}. \tag{2.2.9}$$

There are exactly as many constants c as there are primary fields, so all of them can be normalized to one by redefining the fields appropriately.

Next up on the list are the 3-point functions. One might be tempted to think that we can still only form pairwise differences like $|x_1 - x_3|$ etc. but what about for example $|x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3|$? These more general linear combinations of the points do not obey any simple transformation properties under SCTs like the one given by Eq. (2.1.20). Because of this I am not going to consider them.²

Assuming then that our function can only depend on $x_{ij} \equiv |x_i - x_j|$, it must be a linear combination of terms of the form $x_{12}^{-\gamma_{12}} x_{13}^{-\gamma_{13}} x_{23}^{-\gamma_{23}}$ with $\gamma_{12} + \gamma_{13} + \gamma_{23} = \Delta_1 + \Delta_2 + \Delta_3$ so that the behavior under rescaling is as it should be. Again by applying an SCT we find

$$\tilde{x}_{12}^{-\gamma_{12}} \tilde{x}_{13}^{-\gamma_{13}} \tilde{x}_{23}^{-\gamma_{23}} = \Omega_1^{\frac{\gamma_{12}+\gamma_{13}}{2}} \Omega_2^{\frac{\gamma_{12}+\gamma_{23}}{2}} \Omega_3^{\frac{\gamma_{13}+\gamma_{23}}{2}} x_{12}^{-\gamma_{12}} x_{13}^{-\gamma_{13}} x_{23}^{-\gamma_{23}}, \quad (2.2.10)$$

where $\Omega_i = (1 - 2(b \cdot x_i) + b^2 x_i^2)$. The whole 3-point function must transform as

$$\langle \phi_1(\tilde{x}_1) \phi_2(\tilde{x}_2) \phi_3(\tilde{x}_3) \rangle = \Omega_1^{\Delta_1} \Omega_2^{\Delta_2} \Omega_3^{\Delta_3} \langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle. \quad (2.2.11)$$

Since this is valid for all choices of x_i , each term in the 3-point function must individually transform like this. Solving for the γ_{ij} we actually see that only one such term exists and we can write

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle = \frac{\lambda_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}} \quad (2.2.12)$$

This time the remaining constant λ_{123} cannot be normalized away because there are no degrees of freedom left. These *structure constants* encode the dynamics of the CFT since they tell us how different fields interact with each other.

Moving on to the 4-point functions we run into our first problem pretty quickly. With four different points we can for the first time actually form invariants under the conformal group. Take the following *conformal cross-ratios*

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad (2.2.13)$$

and any function $f(u, v)$ will be invariant under all of our available transformations. We can however still extract some information about the 4-point function. Proceeding exactly as we did for the 3-point function, we can see that the proper scaling behavior forces the 4-point function to be made of terms of the form $x_{12}^{-\gamma_{12}} x_{13}^{-\gamma_{13}} x_{14}^{-\gamma_{14}} x_{23}^{-\gamma_{23}} x_{24}^{-\gamma_{24}} x_{34}^{-\gamma_{34}}$ with $\sum_{i<j} \gamma_{ij} = \sum_i \Delta_i$. Applying the SCT on the individual terms we find

$$\prod_{i<j} \tilde{x}_{ij}^{\gamma_{ij}} = \Omega_1^{\frac{\gamma_{12}+\gamma_{13}+\gamma_{14}}{2}} \Omega_2^{\frac{\gamma_{12}+\gamma_{23}+\gamma_{24}}{2}} \Omega_3^{\frac{\gamma_{13}+\gamma_{23}+\gamma_{34}}{2}} \Omega_4^{\frac{\gamma_{14}+\gamma_{24}+\gamma_{34}}{2}} \prod_{i<j} x_{ij}^{\gamma_{ij}} \quad (2.2.14)$$

²This is a subtlety that I haven't seen being discussed anywhere. It is possible that there is a rigorous argument showing that n -point functions can only depend on pairwise differences of points, but if so I am not aware of it.

and on the correlation function itself

$$\langle \phi_1(\tilde{x}_1) \phi_2(\tilde{x}_2) \phi_3(\tilde{x}_3) \phi_4(\tilde{x}_4) \rangle = \Omega_1^{\Delta_1} \Omega_2^{\Delta_2} \Omega_3^{\Delta_3} \Omega_4^{\Delta_4} \langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle. \quad (2.2.15)$$

At first sight solving these equations seems problematic because we have four linearly independent equations for six unknowns. That was of course to be expected because there are two degrees of freedom that can be absorbed in $f(u, v)$. We can fix that by arbitrarily imposing

$$\gamma_{12} + \gamma_{34} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 \quad \gamma_{14} + \gamma_{23} = \Delta_1 - \Delta_2 - \Delta_3 + \Delta_4 \quad (2.2.16)$$

This way we finally obtain the 4-point function

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = \left(\frac{x_{24}}{x_{14}} \right)^{\Delta_1 - \Delta_2} \left(\frac{x_{14}}{x_{13}} \right)^{\Delta_3 - \Delta_4} \frac{f(u, v)}{x_{12}^{\Delta_1 + \Delta_2} x_{34}^{\Delta_3 + \Delta_4}}. \quad (2.2.17)$$

It turns out that higher point functions while being much more complicated are not going to teach us anything new about the CFT in a sense to be discussed later. However the question of how to determine correlators between primaries with tensorial or spinorial structure is an interesting one that is still a topic of active research. In order to study any but the simplest situations we will need to employ the embedding space formalism.

2.2.3 Correlation functions with spin

As a warm up, I will first rederive the scalar 2-point function in the embedding space. What is the most general Lorentz scalar that can be formed out of two vectors X_1 and X_2 on the light-cone? It's just their dot product (because their norms are zero). Rescaling it conveniently we define

$$X_{ij} \equiv -2X_i \cdot X_j = x_{ij}^2 \quad (2.2.18)$$

Remembering that each field has to individually satisfy homogeneity, the 2-point function is

$$\langle \Phi_1(X_1) \Phi_2(X_2) \rangle = \frac{c \delta_{12}}{X_{12}^{\frac{1}{2}(\Delta_1 + \Delta_2)}}, \quad (2.2.19)$$

which is exactly what we found before.

Now for the correlator of two vectors $\langle V_a(X_1) V_b(X_2) \rangle$ we already know we must get some rank 2 tensor. The only such objects we can form with X_1 and X_2 are η_{ab} and $X_{1,a} X_{2,b}$. All the other terms that are possible like $X_{1,a} X_{1,b}$ etc. can be neglected because they project down to zero by Eq. (2.1.48). Taking into account the homogeneity of the fields, we almost completely fix the functional form of the 2-point function

$$\langle V_a(X_1) V_b(X_2) \rangle = \frac{\eta_{ab} + \alpha \frac{X_{1a} X_{2b}}{X_{12}}}{X_{12}^{\Delta}}. \quad (2.2.20)$$

Lastly the α can be fixed by requiring transversality $X_1^a \langle V_a(X_1) V_b(X_2) \rangle = 0 \implies \alpha =$
 2. Now we can project our expression to find

$$\langle v^\mu(x_1) v^\nu(x_2) \rangle = \frac{\delta^{\mu\nu} - 2 \frac{x_{12}^\mu x_{12}^\nu}{|x_{12}|^2}}{|x_{12}|^\Delta}. \quad (2.2.21)$$

Another correlator that is very important in the conformal bootstrap is the one between two scalars and a vector. We can build it up by recycling what we already know about the correlator of three scalars, i.e. how to get the proper scaling laws:

$$\langle \phi_1(X_1) \phi_2(X_2) V_a(X_3) \rangle = \frac{\text{const.}}{X_{12}^{\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3)} X_{13}^{\frac{1}{2}(\Delta_1 + \Delta_3 - \Delta_2)} X_{23}^{\frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1)}} \times \text{tensor}. \quad (2.2.22)$$

By tensor I mean an object that transforms correctly under Lorentz-transformation but does not transform under rescalings. In this case we need to build a scale invariant vector out of the vectors that we have. Any terms proportional to $X_{3,a}$ would disappear during the projection so the only thing we can write is the linear combination

$$\frac{(X_2 \cdot X_3) X_{1,a} - (X_1 \cdot X_3) X_{2,a}}{\sqrt{X_{12} X_{23} X_{13}}}. \quad (2.2.23)$$

The minus sign is fixed by the transversality with respect to Z . We can now do the projection which gives us

$$\langle \phi_1(x_1) \phi_2(x_2) v_\mu(x_3) \rangle = \frac{\lambda_{123}}{x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{13}^{\Delta_1 + \Delta_3 - \Delta_2} x_{23}^{\Delta_2 + \Delta_3 - \Delta_1}} R_\mu(x_1, x_2, x_3), \quad (2.2.24)$$

where we defined the quantity $R^\mu(x_1, x_2, x_3) = \frac{|x_{13}| |x_{23}|}{|x_{12}|} \left(\frac{(x_1 - x_3)^\mu}{x_{13}^2} - \frac{(x_2 - x_3)^\mu}{x_{23}^2} \right)$. It turns out that we can even generalize this to symmetric traceless tensors of arbitrary spin l by just multiplying by l copies of R_μ instead of one.

2.3 Radial Quantization

2.3.1 Foliation of Space-Time

So far we have seen how correlation functions are constrained if we impose conformal symmetry. Since we want to describe quantum field theories it is clear that the fields must be some sort of operators, and the fact that they depend on one space-time coordinate means that we are referring to local operators. But what are the states that the operators act on?

Usually when studying QFT we know our state at a specific time. We are then interested in how that state evolves in time, which we know by applying the unitary operator

$e^{-\frac{iH\Delta t}{\hbar}}$. This takes us from some time slice into another one, each of which equipped with its own Hilbert space. But it is already clear that this way of quantizing the theory is not unique. We could have chosen a different reference frame and the differing notions of simultaneity would have given us a different set of time slices. Since the physics should not depend on what foliation of space-time we choose, we are free to look for one that is advantageous for the problem at hand.

In the case of CFTs, it makes sense to choose the so-called radial quantization [15]. Here we foliate the space into concentric spheres around some arbitrary point. Going from one sphere to the next amounts to changing the scale of the system so our unitary evolution operator is $e^{-\frac{iD\Delta\lambda}{\hbar}}$, where λ gives some measure of the scale at which we are looking at the system.

On the Hilbert spaces defined in radial quantization it makes sense to choose a basis of eigenstates of D . Since $L_{\mu\nu}$ and D commute we also need to label the spin of our states

$$\begin{aligned} D|\Delta, l\rangle_{\{t\}} &= i\Delta|\Delta, l\rangle_{\{t\}}, \\ L_{\mu\nu}|\Delta, l\rangle_{\{s\}} &= (S_{\mu\nu}^{(l)})_{\{s\}}^{\{t\}}|\Delta, l\rangle_{\{t\}}, \end{aligned} \tag{2.3.1}$$

Now we assume the existence of a unique ground state $|0\rangle$ that gets annihilated by all symmetry generators.

It is of course important to know how our symmetries act on operators as well. These relations were already found in (2.1.40) when looking at the transformation of primary fields. However, some care needs to be taken. The explicit form given tells us what happens when some generator G acts only on one field or operator. But when applying G to a product of operators it will satisfy the Leibniz rule $G(\mathcal{O}_1\mathcal{O}_2\ldots) = G(\mathcal{O}_1)\mathcal{O}_2\ldots + \mathcal{O}_1G(\mathcal{O}_2\ldots)$. This means (2.1.40) actually gives the commutator $[G, \mathcal{O}(x)]$ rather than $G\mathcal{O}(x)$. Now we can define primary operators as operators that satisfy these relations, but we can actually be more economical by going to the special point $x = 0$. Here it follows that

$$\begin{aligned} [K_\mu, \mathcal{O}(0)] &= 0, \\ [P_\mu, \mathcal{O}(x)] &= -i\partial_\mu\mathcal{O}(x), \\ [D, \mathcal{O}(0)] &= i\Delta\mathcal{O}(0), \\ [L_{\mu\nu}, \mathcal{O}(0)] &= S_{\mu\nu}\mathcal{O}(0), \end{aligned} \tag{2.3.2}$$

which will be enough to define the behavior of primary fields.

2.3.2 Operator-State Correspondence

Let's see what happens when we insert a primary operator at the origin and apply it to a vacuum

$$\begin{aligned} D(\mathcal{O}(0)|0\rangle) &= [D, \mathcal{O}(0)]|0\rangle + \mathcal{O}(0)D|0\rangle \\ &= i\Delta|0\rangle + 0 \\ &= D|\Delta\rangle. \end{aligned} \tag{2.3.3}$$

So we can create any eigenstate of D by inserting the appropriate primary operator and therefore represent any state we like as a superposition of operators inserted at the origin. Surprisingly this logic can be reversed by bringing an arbitrary operator $\mathcal{O}(x)$ to the origin and then representing it as a state. This is called the *operator-state correspondence*.

2.3.3 Conjugate Operators

With all these technicalities sorted out we are almost ready to properly understand the quantum version of our correlation functions

$$\langle \phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n) \rangle = \langle 0|\mathcal{R}\{\phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)\}|0\rangle, \tag{2.3.4}$$

where $\mathcal{R}$ means radial ordering. The only ingredient that is missing is the conjugation operation $|0\rangle \rightarrow \langle 0|$ and $\phi \rightarrow \phi^\dagger$. Since we are working in Euclidean signature conjugation actually takes our operator to a different point in space as seen by the following quick argument. In Lorentzian theory we have

$$\phi(x, t) = e^{-iHt}\phi(x, 0)e^{iHt}. \tag{2.3.5}$$

Assuming $\phi(x, 0)$ is Hermitean, this implies $\phi(x, t)^\dagger = \phi(x, t)$. If we Wick rotate the equation first though we find

$$\phi(x, t_E) = e^{-Ht_E}\phi(x, 0)e^{Ht_E} \rightarrow \phi(x, t_E)^\dagger = e^{Ht}\phi(x, 0)^\dagger e^{-Ht}, \tag{2.3.6}$$

which implies $\phi(x, t_E)^\dagger = \phi(x, -t_E)$ instead. This means that the property of unitarity $\langle \Psi|\Psi \rangle \geq 0$ is here better understood as a *reflection positivity* because we evaluate the inner product of states at symmetric points about the $t_E = 0$ plane. This also illustrates that the action of conjugation on the fields is quantization dependent. Specifically we are interested in radial quantization. Intuitively we would expect that just as before we had $t_E \leftrightarrow -t_E$, in radial quantization there should be some relation $0 \leftrightarrow \infty$. One reasonable guess is that we have to act with an inversion $x^\mu \leftrightarrow \frac{x^\mu}{x^2}$. It turns out that this guess is correct and the precise form for scalar primary fields is [16]

$$\phi(x^\mu)^\dagger = \mathcal{I}\phi(x^\mu)\mathcal{I} = x^{-2\Delta}\phi\left(\frac{x^\mu}{x^2}\right), \tag{2.3.7}$$

where $\mathcal{I}$ is the inversion operator. Interestingly this leads to the relation

$$P^{\mu\dagger} = \mathcal{I}P^\mu\mathcal{I} = K^\mu, \quad (2.3.8)$$

thus confirming the finding from Sec. 2.1.1 that an SCT is equivalent to an inversion followed by a translation and another inversion.

2.3.4 Unitarity Bounds

Looking at equation (2.2.9) it is clear that for $\Delta < 0$ the correlation between two points increases with distance. Physically, this seems entirely unreasonable so we would expect physical theories to satisfy $\Delta \geq 0$. We are now in a position to prove this result rigorously. As we have seen in the previous section the Wick transform of our Euclidean CFT can only be unitary (and therefore physical) if reflection positivity is satisfied. Consider a primary state $|\Psi\rangle = \mathcal{O}_{\Delta,\ell}|0\rangle$ and act with P :

$$\begin{aligned} |P^1|\Psi\rangle|^2 &= \langle\Psi|K^1P^1|\Psi\rangle \\ &= \langle\Psi|[K^1, P^1]|\Psi\rangle + \langle\Psi|P^1K^1|\Psi\rangle \\ &= -2i\langle\Psi|D|\Psi\rangle \\ &= 2\Delta\langle\Psi|\Psi\rangle \\ &\geq 0 \end{aligned} \quad (2.3.9)$$

For scalar primary states a stronger bound can be found by looking at $|P^\mu P_\mu|\Psi\rangle|^2$ which leads to the condition $2\Delta d \cdot (4\Delta - 2(d-1))\langle\Psi|\Psi\rangle \geq 0$. This is satisfied for either $\Delta = 0$ or $\Delta \geq \frac{d-2}{2}$. It is an important sanity check that even this stronger bound does not exclude $\Delta = 0$ because the identity operator must be present in any theory. The spinorial bound can also be strengthened by considering $\langle\Psi_a|K^\mu P^\nu|\Psi_b\rangle$ which leads to the condition $\Delta \geq \ell + d - 2$ [16].

The unitarity bounds also imply that primary states (which get annihilated by K) must exist. It is simple to show that $P|\Delta\rangle \propto |\Delta+1\rangle$ and $K|\Delta\rangle \propto |\Delta-1\rangle$. Since scaling dimensions are bounded from below, this means at some point a state has to be annihilated by K which we have just been assuming up until now.

2.4 Operator Product Expansion

The operator product expansion (OPE) is the last basic element of conformal field theory we need to put into place in order to be able to think about conformal bootstrapping. While the OPE is a well-known tool in quantum field theory in general [17], it takes on

a more central role in CFT. The basic idea is to write the product of two operators at some points as the sum of operators at some other point. When thinking of the operator-state correspondence it is clear why this should work. If we think of the product of two operators acting on the vacuum state we have

$$\begin{aligned}\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle &= \sum_{\Delta,\ell} f_{\mathcal{O}_1\mathcal{O}_2\Delta\ell} C(x_1, x_2, P)|\Delta, \ell\rangle \\ &= \sum_k C_{12k}(x_{13}, x_{23}, \partial_3)\mathcal{O}_k(x_3)|0\rangle.\end{aligned}\tag{2.4.1}$$

The first step is to recognize that we just get some state which can be rewritten as the sum of primary states transported to some point as shown in the second step. Each state corresponds to an operator so in the third line the sum is rewritten over primary states. Since the point around which we quantize the theory is arbitrary, this equation does not depend on $|0\rangle$ and is in fact an operator equation. Conformal symmetry greatly restricts the precise form that the differential operators C_{ijk} can take but only one observation will matter for us. If we consider this operator equation inside a 3-point function we find

$$\begin{aligned}\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle &= \sum_k C_{12k}\langle\mathcal{O}_k\mathcal{O}_3\rangle, \\ \frac{\lambda_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{13}^{\Delta_{12}+\Delta_3}x_{23}^{-\Delta_{12}+\Delta_3}} &= C_{123}\frac{1}{x_{13}^{\Delta_1+\Delta_2}}.\end{aligned}\tag{2.4.2}$$

This means that of course C_{123} has to act precisely in such a way as to make the kinematical part of the equation hold, but, in particular, it means $C_{123} \propto \lambda_{123}$ is the only way in which dynamical information gets encoded. If we know for example that a certain 3-point function $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle$ vanishes due to some symmetry then it also means we know that $\mathcal{O}_3$ cannot appear in the OPE of $\mathcal{O}_1\mathcal{O}_2$. Another important observation is the fact that the differential operators C do not depend on Δ_1 and Δ_2 separately but only on the combination $\Delta_{12} \equiv \Delta_1 - \Delta_2$ [18]. These considerations will be very important for the bootstrap program.

Chapter 3

Conformal Blocks

In chapter 2 we learned that scalar 4-point functions are restricted to be of the form given in Eq. (2.2.17). We also noted that the dynamical information of the theory is given by its OPE coefficients which are the same as the coefficient in the 3-point functions. The goal of this chapter will be to disentangle the kinematical and dynamical contributions to the function $g(u, v)$ in the 4-point functions. This will essentially be like rewriting a wave function in terms of spherical harmonics. The basis functions we want to decompose our functions in are called *conformal blocks*. While techniques for calculating very general blocks are known, I will stick to the simplest case of blocks appearing in the 4-point function of scalars. For a review see [19].

3.1 Definition

In the conformal block expansion we want to capture the contribution that a specific operator has on the correlator in question. It is simple to show that all descendants of a certain primary have the same OPE coefficients with any other field as the primary itself. To be economical it therefore makes sense to include the contribution of an entire conformal multiplet at once and label it by its primary. To do that we define the projector

$$|\mathcal{O}| \equiv \sum_{\alpha \in \mathcal{O}, P\mathcal{O}, PP\mathcal{O} \dots} \frac{|\alpha\rangle\langle\alpha|}{\langle\alpha|\alpha\rangle}, \quad (3.1.1)$$

where $\mathcal{O}$ are primaries and we obtain the descendants via the action of P . Since $1 = \sum_{\mathcal{O}} |\mathcal{O}|$ we can insert this into a 4-point function as an identity. Picking points such that

$|x_i| > |x_j|$ where $i = 1, 2$ and $j = 3, 4$, we get

$$\begin{aligned}
\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle &= \langle 0 | \mathcal{R}\{\phi_1(x_1)\phi_2(x_2)\} \mathcal{R}\{\phi_3(x_3)\phi_4(x_4)\} | 0 \rangle \\
&= \sum_{\mathcal{O}} \langle 0 | \mathcal{R}\{\phi_1(x_1)\phi_2(x_2)\} | \mathcal{O} \rangle \langle \mathcal{O} | \mathcal{R}\{\phi_3(x_3)\phi_4(x_4)\} | 0 \rangle \\
&= \left(\frac{x_{24}}{x_{14}} \right)^{\Delta_{12}} \left(\frac{x_{14}}{x_{13}} \right)^{\Delta_{34}} \frac{f(u, v)}{x_{12}^{\Delta_1 + \Delta_2} x_{34}^{\Delta_3 + \Delta_4}}.
\end{aligned} \tag{3.1.2}$$

By multiplying everything by $\left(\frac{x_{24}}{x_{14}} \right)^{\Delta_{21}} \left(\frac{x_{14}}{x_{13}} \right)^{\Delta_{43}} x_{12}^{\Delta_1 + \Delta_2} x_{34}^{\Delta_3 + \Delta_4}$, we obtain expressions that should only depend on the conformal cross ratios u and v . In the previous section, we also saw that the OPE coefficients must enter linearly. In this case we would get a contribution of $\lambda_{12\mathcal{O}}$ from $\langle \phi_1\phi_2 | \alpha \rangle$ and $\lambda_{34\mathcal{O}}$ from $\langle \beta | \phi_3\phi_4 \rangle$. Excluding all these factors we therefore define our conformal blocks

$$\begin{aligned}
g_{\mathcal{O}}^{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(u, v) &\equiv g_{\Delta, \ell}^{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(u, v) \\
&\equiv \frac{1}{\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}} \left(\frac{x_{24}}{x_{14}} \right)^{\Delta_{21}} \left(\frac{x_{14}}{x_{13}} \right)^{\Delta_{43}} x_{12}^{\Delta_1 + \Delta_2} x_{34}^{\Delta_3 + \Delta_4} \\
&\quad \times \langle 0 | \mathcal{R}\{\phi_1(x_1)\phi_2(x_2)\} | \mathcal{O} \rangle \langle \mathcal{O} | \mathcal{R}\{\phi_3(x_3)\phi_4(x_4)\} | 0 \rangle
\end{aligned} \tag{3.1.3}$$

These blocks are independent of the specific theory because we removed all dynamical information from them. Given these conformal blocks, we could recover the 4-point function through

$$g(u, v) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} g_{\mathcal{O}}^{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(u, v). \tag{3.1.4}$$

Since, as mentioned in chapter 2, the OPE depends on Δ_1 and Δ_2 only through their differences, the same is true for the conformal blocks so that we can use the notation $g_{\mathcal{O}}^{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(u, v) \equiv g_{\mathcal{O}}^{\Delta_{12}, \Delta_{34}}(u, v)$.

3.2 Conformal Casimir

The definition given in Eq. (3.1.3) is clearly very unwieldy and not too useful because it makes reference to a specific theory. We do however know that the left-hand side must be conformally invariant and, as we will see, it must also be an eigenfunction of a certain differential operator. Using this information will enable us to actually calculate the blocks in general. The general procedure and results are derived in [18, 20, 21].

First we must realize that the quadratic Casimir from Eq. (2.1.44) acts on the projector $|\mathcal{O}\rangle$ like it would on a state $|\mathcal{O}\rangle$:

$$\begin{aligned}
\mathcal{C}|\mathcal{O}\rangle &= |\mathcal{O}\rangle \mathcal{C} = \lambda_{\Delta, \ell} |\mathcal{O}\rangle, \\
\lambda_{\Delta, \ell} &= \Delta(\Delta - d) + \ell(\ell + d - 2).
\end{aligned} \tag{3.2.1}$$

The Casimir is constructed from operators M_{ab} which act on any particular field as a differential operator. For example, $[M_{-1,\mu}, \mathcal{O}(x_1)] = -i \frac{\partial}{\partial x_1^\mu} \mathcal{O}(x_1)$. The Casimir would therefore act on two fields like

$$\begin{aligned} \mathcal{C}\phi_1(x_1)\phi_2(x_2)|0\rangle &= \frac{1}{2}M^{ab}M_{ab}\phi_1(x_1)\phi_2(x_2)|0\rangle \\ &= \frac{1}{2}M^{ab}[M_{ab}, \phi_1(x_1)]\phi_2(x_2)|0\rangle + \frac{1}{2}M^{ab}\phi_1(x_1)[M_{ab}, \phi_2(x_2)]|0\rangle \\ &= \frac{1}{2}(\mathcal{M}_1^{ab} + \mathcal{M}_2^{ab})(\mathcal{M}_{ab,1} + \mathcal{M}_{ab,2})\phi_1(x_1)\phi_2(x_2)|0\rangle, \end{aligned} \quad (3.2.2)$$

where $\mathcal{M}_{ab,i}$ indicates the differential operator you get from acting on $\phi_i(x_i)$ with M_{ab} . Now by inserting the conformal Casimir into $\langle 0|\mathcal{R}\{\phi_1(x_1)\phi_2(x_2)\}|\mathcal{O}|\mathcal{R}\{\phi_3(x_3)\phi_4(x_4)\}|0\rangle$ we can find on the one hand

$$\begin{aligned} \langle \phi_1(x_1)\phi_2(x_2)|\mathcal{O}|\mathcal{C}\phi_3(x_3)\phi_4(x_4)\rangle &= \frac{1}{2}(\mathcal{M}_3^{ab} + \mathcal{M}_4^{ab})(\mathcal{M}_{ab,3} + \mathcal{M}_{ab,4}) \\ &\quad \times \langle \phi_1(x_1)\phi_2(x_2)|\mathcal{O}|\phi_3(x_3)\phi_4(x_4)\rangle, \end{aligned} \quad (3.2.3)$$

and on the other hand

$$\langle \phi_1(x_1)\phi_2(x_2)|\mathcal{O}|\mathcal{C}\phi_3(x_3)\phi_4(x_4)\rangle = \lambda_{\Delta,\ell}\langle \phi_1(x_1)\phi_2(x_2)|\mathcal{O}|\phi_3(x_3)\phi_4(x_4)\rangle. \quad (3.2.4)$$

Comparing these equations yields a differential equation

$$\begin{aligned} \frac{1}{2}(\mathcal{M}_3 + \mathcal{M}_4)^{ab}(\mathcal{M}_3 + \mathcal{M}_4)_{ab} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{12}} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{34}} \frac{g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u,v)}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \\ = \lambda_{\Delta,\ell} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{12}} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{34}} \frac{g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u,v)}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}}. \end{aligned} \quad (3.2.5)$$

This can be expanded and rewritten in terms of conformal cross ratios. The equation then takes the form [21]

$$\mathcal{D}g_{\Delta,\ell}(u,v) = \lambda_{\Delta,\ell}g_{\Delta,\ell}(u,v), \quad (3.2.6)$$

for

$$\begin{aligned} \mathcal{D} &= (1-u-v)\frac{\partial}{\partial v} \left(v\frac{\partial}{\partial v} + a+b \right) + u\frac{\partial}{\partial u} \left(2u\frac{\partial}{\partial u} - d \right) \\ &\quad - (1+u-v) \left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + a \right) \left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + b \right), \end{aligned} \quad (3.2.7)$$

where $a = -\frac{1}{2}\Delta_{12}$ and $b = \frac{1}{2}\Delta_{34}$. The exact solution to this equation in the case of identical scalars is known in even dimensions. These solutions are usually expressed in terms of the coordinates z and $\bar{z}$ defined by

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \quad (3.2.8)$$

Figure 3.1: Any non-degenerate arrangement of four points can be mapped into this configuration without loss of generality.

These variables have a simple geometric interpretation if we switch to a specific coordinates system. Four arbitrary points can be mapped to the arrangement in Fig. 3.1 as follows: using conformal invariance send point x_4 to infinity; then translate the system to center on x_1 ; now rotate and rescale the system such that $x_3^i = \delta_0^i$. Since x_1 , x_3 and x_4 now lie on a line there is still enough freedom to rotate x_2 into a plane. This suggests defining $z = x_2^0 + ix_2^1$ and $\bar{z} = x_2^0 - ix_2^1$ which does in fact satisfy Eq. (3.2.8) in this arrangement.

These coordinates are, of course, also motivated by the special case of 2 dimensions where their introduction allows us to write the algebra of global symmetries as a direct product and enables the use of the powerful tools of complex analysis. I will not make use of any such methods in this work.

3.3 Conformal Blocks in 1D

To illustrate how equation Eq. (3.2.6) can be used to actually compute a conformal block, I want to present the general argument of [22], modified to apply to the case of one dimension, which was not discussed in that paper. In the special case of one dimension all points must lie on a line so instead of the z and $\bar{z}$ coordinates the 1D blocks can only depend on z . We want to look at the expression

$$\lim_{R \rightarrow \infty} \langle \phi(R) \phi(1) \phi(z) \phi(0) \rangle = \lim_{R \rightarrow \infty} \frac{1}{R^{2\Delta_\phi} z^{2\Delta_\phi}} g(u, v), \quad (3.3.1)$$

with $0 < z < 1$ to maintain radial ordering. Inserting the identity operator in the $|\Delta\rangle$ basis between z and 1 and applying the states at R and 0 to the outer vacuum states gives

$$g = \sum_{\Delta} z^{\Delta} \langle \phi | \phi(1) | \Delta \rangle \langle \Delta | \phi(1) | \phi \rangle. \quad (3.3.2)$$

The z^{Δ} -term comes from scaling $\phi(z)$ from z to 1 using $e^{-iD\lambda}$. In higher dimension $\langle \Delta | \phi(1) | \phi \rangle$ could be a function of the angle $\arg z$ and have its own spin structure, but in 1D it has to be a scalar that depends only on Δ .

Collecting the contributions of all descendants we can therefore write

$$g_{\Delta}(z) = \sum_{n=0}^{\infty} A_n z^{\Delta+n}. \quad (3.3.3)$$

We can now rewrite the differential operator (3.2.7) in z -coordinates and for the case of 1D.

$$\begin{aligned} \mathcal{D} &= \mathcal{D}_0 + \mathcal{D}_1, \\ \mathcal{D}_0 &= z^2 \partial_z^2, \\ \mathcal{D}_1 &= -z (z^2 \partial_z^2 + z \partial_z). \end{aligned} \quad (3.3.4)$$

The operator is separated such that $\mathcal{D}_0$ preserves the degree of the monomials in Eq. (3.3.3) and $\mathcal{D}_1$ increases it by 1. Plugging these expressions into Eq. (3.2.6) gives

$$\sum_{n=0}^{\infty} A_n (\lambda_{\Delta+n} z^{\Delta+n} - (\Delta+n)^2 z^{\Delta+n+1}) = \sum_{n=0}^{\infty} A_n \lambda_{\Delta} z^{\Delta+n}. \quad (3.3.5)$$

Rearranging the terms and comparing equal powers of z gives the recursion relation

$$A_{n+1} = \frac{(\Delta+n)^2}{(\Delta+n+1)(\Delta+n) - \Delta(\Delta-1)} A_n. \quad (3.3.6)$$

This equation has a closed form solution so that we can write

$$\begin{aligned} g_{\Delta}(z) &= \sum_{n=0}^{\infty} \frac{(\Delta)_n (\Delta)_n}{(2\Delta)_n} \frac{z^{\Delta+n}}{n!} \\ &= z^{\Delta} {}_2F_1(\Delta, \Delta, 2\Delta; z), \end{aligned} \quad (3.3.7)$$

where $(x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)}$ is the rising factorial, in the bootstrap literature often called the Pochhammer symbol, and ${}_2F_1$ is a hypergeometric function.

3.4 Radial Coordinates

While the z coordinates are useful to study the properties of the conformal blocks and their derivatives, numerically much better results can be achieved when using the radial coordinates from [22]. This improves the rate of convergence both for the individual blocks

Figure 3.2: Radial coordinates provide a more symmetric analogue of z -coordinates.

when calculated as a series expansion, as well as the convergence of the 4-point function OPE [23]. This will be very important later when we consider bootstrap applications. Radial coordinates are defined by

$$\rho = \frac{z}{(1 + \sqrt{1 - z})^2} \quad \Longleftrightarrow \quad z = \frac{4\rho}{(1 + \rho)^2}, \quad (3.4.1)$$

as well as $\eta = \cos \arg \rho$. These coordinates also have a geometric interpretation if one thinks of moving the points into a particular configuration as shown in Fig. 3.2.

Taking as an example the conformal block worked out in the previous section we can see the improved convergence in action. Using radial coordinates and using some hypergeometric identities we can rewrite the conformal block as

$$g_{\Delta}(\rho) = \frac{(4r)^{\Delta}}{\sqrt{1 - r^2}} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, \Delta + \frac{1}{2}; \frac{r^2}{r^2 - 1} \right). \quad (3.4.2)$$

In conformal bootstrap applications people are usually interested in what is called the crossing symmetric point $z = \bar{z} = \frac{1}{2}$. This corresponds to $\frac{r^2}{r^2 - 1} \approx -0.03$ so a lot fewer terms are needed in the expansion in order to get good convergence.

3.5 Recursion Relation

The analytical expressions for conformal blocks are of limited utility. Firstly they are not available for all possible blocks but, more importantly, they cannot be evaluated very efficiently, especially once we try to calculate arbitrary derivatives. Furthermore, the most

efficient formulations of the numerical bootstrap only work with the specific approximation we are about to derive and would not work with the closed form expressions. The approximation scheme we are interested in comes from a type of recursion relation that was introduced in [24] and is often called the *rational representation* of the conformal blocks. The special case of two-dimensional CFTs was already studied in [25]. It can best be understood in terms of the general representation theoretic arguments given in [26]. I will present the same argument here, adapted to the special case of 1D. This is not strictly necessary because we already have a closed form solution of the conformal block and could derive the rational approximation directly from it. It is still instructive to derive it from first principles because the recursion relation is indispensable in higher dimensions. Also, on the way we will find the 1D mixed conformal blocks with little extra effort.

The basic idea is as follows. A conformal block is given as a sum of the contribution to the 4-point function over all descendants in a conformal multiplet $\mathcal{H}_\mathcal{O}$ labeled by a primary with dimension Δ_A^* :

$$\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}G_{\Delta,\ell} = \sum_{\alpha \in \mathcal{H}_\mathcal{O}} \frac{\langle \phi_1|\phi_2|\alpha \rangle \langle \alpha|\phi_3|\phi_4 \rangle}{\langle \alpha|\alpha \rangle}. \quad (3.5.1)$$

If the norm $\langle \alpha|\alpha \rangle$ of some descendant in that multiplet with dimension $\Delta_A = \Delta_A^* + n_A$ vanishes, then the conformal block must diverge. Assuming the conformal blocks are analytic functions in Δ , the vanishing of the norm means that nearby conformal blocks can be approximated by

$$G_\Delta \xrightarrow{\Delta \rightarrow \Delta_A^*} \frac{R_A}{\Delta - \Delta_A^*} G_{\Delta_A, \ell_A}. \quad (3.5.2)$$

The right-hand is also a conformal block because it still needs to satisfy the Casimir equation (3.2.6). The zeros could be of higher order which does in fact present a problem in even dimensions. For 1D we will see that this is not the case. The residue R_A consists of three parts

$$R_A = M_A^{(L)} Q_A M_A^{(R)}, \quad (3.5.3)$$

where Q_A comes from the inverse of the norm of the vanishing intermediate state $|\alpha\rangle$ after we peel off the divergence and $M_A^{(L)}$ and $M_A^{(R)}$ correspond to $\langle \phi_1|\phi_2|\alpha \rangle$ and $\langle \alpha|\phi_3|\phi_4 \rangle$, respectively.

As seen in the previous section, there is a factor of $(4r)^\Delta$ in the conformal blocks which turns out to be generic. This prefactor should be taken out in order to meaningfully study the large Δ limit of the conformal blocks. Using all the facts mentioned so far, we can finally state the recursion relation

$$\begin{aligned} g_{\Delta,\ell}(r,\eta) &= (4r)^\Delta h_{\Delta,\ell}(r,\eta), \\ h_{\Delta,\ell}(r,\eta) &= h_{\infty,\ell}(r,\eta) + \sum_A \frac{R_A}{\Delta - \Delta_A^*} (4r)^{n_A} h_{\Delta_A, \ell_A}(r,\eta). \end{aligned} \quad (3.5.4)$$

This relation is useful because plugging in an expression on the right-hand side that is accurate up to some order $O(r^k)$ gives an expression on the left-hand side that is accurate to at least one order higher, because $n_A \geq 1$. Of course, in order to apply this formula, we need to find all primary descendants and their corresponding residues as well as the large Δ limit $h_{\infty, \ell}$. Since we are primarily interested in 1D CFT, all spin indices ℓ become unnecessary and $\eta = 1$ is a constant. This will simplify the analysis a lot as there will only be one type of primary descendant as opposed to the three found in higher dimensions.

3.5.1 Primary Descendants

We need to find states of zero norm. Since primary states are normalized, those null states must necessarily be descendants, but they will also be annihilated by K so they are called primary descendants. In general, there are lots of combinations of P^μ that might produce primary descendants but in 1D only P^n is possible. Let $|\Delta\rangle$ be a primary state. Then

$$\begin{aligned}
 KP^n|\Delta\rangle &= (P^n K + [K, P^n])|\Delta\rangle \\
 &= \sum_{j=1}^n P^{n-j} [K, P] P^{j-1} |\Delta\rangle \\
 &= \sum_{j=1}^n P^{n-j} 2i D P^{j-1} |\Delta\rangle \\
 &= 2i \sum_{j=1}^n P^{n-1} i(\Delta + j - 1) |\Delta\rangle \\
 &= -2n \left(\Delta + \frac{1}{2}(n-1) \right) P^{n-1} |\Delta\rangle.
 \end{aligned} \tag{3.5.5}$$

The case of $n = 0$ is of course trivially zero because we assumed a primary state. The non-trivial relation states that we can have a primary descendant if $\Delta = \Delta_n^* = \frac{1-n}{2}$ for $n = 1, 2, 3, \dots$. Note that all of these correspond to non-unitary operator dimensions, just as we should have been expecting, because this is precisely the type of anomaly that the unitarity bounds avoid.

The norm of these states can be computed in the same fashion

$$\begin{aligned}
 N_n &= \langle \Delta | K^n P^n | \Delta \rangle \\
 &= -2n \left(\Delta + \frac{1}{2}(n-1) \right) \langle \Delta | K^{n-1} P^{n-1} | \Delta \rangle \\
 &= -n (2\Delta + n - 1) N_{n-1}.
 \end{aligned} \tag{3.5.6}$$

This recursion relation is readily solved using $N_0 = 1$ and yields

$$N_n = (-1)^n n! (2\Delta)_n. \tag{3.5.7}$$

3.5.2 Residues

To calculate the residues R_A all that is missing are the coefficients $M_A^{(L)}$ and $M_A^{(R)}$ of the 3-point functions. As we know from Eq. (2.2.12), the 3-point function is given by

$$\langle \mathcal{O}_\Delta(x_0) \phi_3(x_3) \phi_4(x_4) \rangle = \frac{\lambda_{34} \mathcal{O}}{x_{03}^{\Delta+\Delta_{34}} x_{04}^{\Delta-\Delta_{34}} x_{34}^{\Delta_3+\Delta_4-\Delta}}. \quad (3.5.8)$$

As discussed in Sec. 3.3, ϕ_3 gets transported to the insertion point of $|\mathcal{O}\rangle\langle\mathcal{O}|$ so the $x_{03}^{-(\Delta+\Delta_{34})}$ factor will dominate. Therefore, inserting $P\mathcal{O}$ instead of $\mathcal{O}$ is just like taking the derivative

$$-i\partial_0 x_{03}^{-(\Delta+\Delta_{34})} = i(\Delta + \Delta_{34}) x_{03}^{-(\Delta+\Delta_{34}+1)}. \quad (3.5.9)$$

This proportionality factor is exactly what we were looking for. Multiple derivatives are also easy to work out, so that we have

$$M_n^{(R)} = i^n (\Delta + \Delta_{34})_n \quad M_n^{(L)} = (-i)^n (\Delta - \Delta_{12})_n. \quad (3.5.10)$$

We can now finally write

$$R_n = \frac{(\Delta - \Delta_{12})_n (\Delta + \Delta_{34})_n}{2(2\Delta)_{n-1} n!}. \quad (3.5.11)$$

These have to be evaluated as $\Delta \rightarrow \Delta_A^*$. Notice that for the special case of identical scalars half of these residues vanish and we recover precisely the residues we would expect from our closed form solution. Having noticed this, we can also make an educated guess about the mixed conformal block

$$\begin{aligned} g_{\Delta}^{\Delta_{12}, \Delta_{34}}(z) &= \sum_{n=0}^{\infty} \frac{(\Delta - \Delta_{12})_n (\Delta + \Delta_{34})_n}{(2\Delta)_n} \frac{z^{\Delta+n}}{n!} \\ &= z^{\Delta} {}_2F_1(\Delta - \Delta_{12}, \Delta + \Delta_{34}, 2\Delta; z), \end{aligned} \quad (3.5.12)$$

which is, in fact, correct [21].

3.5.3 Large Δ Limit

The last piece needed in order to make sense of the recursion relation (3.5.4) is the behavior at large Δ . This time we need to use the Casimir equation (3.2.6) again but it will simplify greatly because the second order pieces cancel exactly at large Δ . Rewriting the differential operator in radial coordinates and remembering to strip off the $(4r)^\Delta$ prefactor we get

$$\frac{d}{dr} h_\infty(r) = \frac{4(\Delta_{12} - \Delta_{34})(1 - r^2)^2 - 4r + r(1 + r^2)(2 + (1 - r^2))}{(1 - r^2)^3} h_\infty(r). \quad (3.5.13)$$

Assuming $g \xrightarrow{r \rightarrow 0} 1$ the solution is given by

$$h_\infty(r) = \frac{1}{\sqrt{1 - r^2}} \left(\frac{1 - r}{1 + r} \right)^{\Delta_{12} - \Delta_{34}}. \quad (3.5.14)$$

3.5.4 Recursion Example

To illustrate how the recursion relation can be used to evaluate the conformal blocks I explicitly work out the first terms for the identical scalar case and compare them to the exact solution. To start the recursion we expand $h_\infty(r)$ around $r = 0$. To second order this just gives $h_\infty(r) \approx 1$. Assuming $h_\Delta = 1$ to lowest order and plugging this into the recursion relation (3.5.4) gives

$$\begin{aligned} h_\Delta &= 1 + \frac{r^2}{2} - \frac{1}{64(\Delta + \frac{1}{2})}(4r)^2 + O(r^4) \\ &= 1 + \frac{\Delta}{2\Delta + 1}r^2 + O(r^4), \end{aligned} \quad (3.5.15)$$

and the next order

$$\begin{aligned} h_\Delta &= 1 + \frac{r^2}{2} + \frac{3}{8}r^4 - \frac{1}{64(\Delta + \frac{1}{2})}(4r)^2 \left(1 + \frac{3}{8}r^2\right) - \frac{9}{16(2\Delta + 3)}r^4 + O(r^6) \\ &= 1 + \frac{\Delta}{2\Delta + 1}r^2 + \frac{3\Delta(\Delta + 1)}{2(2\Delta + 1)(2\Delta + 3)}r^4 + O(r^6). \end{aligned} \quad (3.5.16)$$

Since $r^2 \approx 0.03$ the convergence is relatively fast. This equation leads to two different expressions for the conformal blocks and their z -derivatives around the crossing symmetric point, both of which will be very important for the numerical bootstrap applications presented in Ch. 6 and Ch. 7. The first approximation comes from taking all the wanted derivatives and evaluating them at $r = r(z = 1/2)$. Resumming the residues of each pole then gives the expression

$$g_\Delta^{(m)} = (4\rho)^\Delta \left(\sum_{j=0}^{\infty} \frac{R_j^{(m)}}{\Delta + j + \frac{1}{2}} + \sum_{j=0}^{\infty} \Delta^j P_j^{(m)} \right). \quad (3.5.17)$$

The polynomial part comes from picking up a factor of Δ each time we take the derivative of $(4r)^\Delta$. This approximation is useful because once the matrices $R_j^{(m)}$ and $P_j^{(m)}$ are precomputed up to some cutoff, the conformal block derivatives can be evaluated very efficiently. This is discussed in detail in App. A.

The second important expression simply implies that all of the denominators in Eq. (3.5.17) are positive for unitary dimensions Δ . This means we can pull out their product as a necessarily positive prefactor and write

$$g_\Delta^{(m)} = \chi(\Delta) Q_m(\Delta), \quad (3.5.18)$$

where Q_m are polynomials in Δ and $\chi > 0$. This expression is important for the reformulation of the conformal bootstrap as a semi-definite program, as I will discuss in Ch. 4.

Chapter 4

Conformal Bootstrap

After having sorted out all the preliminaries of conformal field theory and developed the tools to compute the conformal blocks and their derivatives, we are finally in a position to understand the conformal bootstrap. The basic idea is to take some 4-point function of a CFT and expand it in terms of conformal blocks in two different channels. Equating these two expansions then gives non-trivial crossing equations that any consistent CFT has to satisfy. These equations have been known since the 1970s due to the groundwork of [27–33]. However, aside from the case of 2D CFT, little progress was made on analyzing these complicated equations until Dolan and Osborn found the solutions for the external scalar conformal blocks [18, 20], after which the authors of [34] demonstrated in this ground breaking paper how numerical techniques could be used to derive bounds on scaling dimensions in 4D.

Since then there has been a number of generalizations and technical improvements of these original ideas. In the next sections, I will try to present a relatively modern account of the conformal bootstrap with a special focus on how the 1D case differs from the well studied situation in higher dimensions.

4.1 Crossing Equation

Consider the scalar 4-point function $\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle$. It is given by Eq. (2.2.17) that I repeat here for convenience

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = \left(\frac{x_{24}}{x_{14}} \right)^{\Delta_{12}} \left(\frac{x_{14}}{x_{13}} \right)^{\Delta_{34}} \frac{g(u, v)}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}}. \quad (4.1.1)$$

It should really be identical to any rearrangement of the fields within the correlator.¹ In particular, consider the correlator with 1 $\leftrightarrow$ 3 exchanged

$$\langle \phi_3(x_3) \phi_2(x_2) \phi_1(x_1) \phi_4(x_4) \rangle = \left(\frac{x_{24}}{x_{34}} \right)^{-\Delta_{23}} \left(\frac{x_{34}}{x_{13}} \right)^{\Delta_{14}} \frac{g(u, v)}{x_{23}^{\Delta_2 + \Delta_3} x_{14}^{\Delta_1 + \Delta_4}}. \quad (4.1.2)$$

Notice that $u \leftrightarrow v$ under this exchange. By equating these two expressions and expanding $g(u, v)$ into conformal blocks we get the equation

$$v^{\frac{\Delta_2 + \Delta_3}{2}} \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} \lambda_{34\mathcal{O}} g_{\Delta, \ell}^{\Delta_{12}, \Delta_{34}}(u, v) = u^{\frac{\Delta_1 + \Delta_2}{2}} \sum_{\mathcal{O}} \lambda_{32\mathcal{O}} \lambda_{14\mathcal{O}} g_{\Delta, \ell}^{\Delta_{32}, \Delta_{14}}(v, u). \quad (4.1.3)$$

It has become customary to rewrite this equation by symmetrizing/anti-symmetrizing it by adding/subtracting the equation one would obtain by having $\phi_1(x_3)$ and $\phi_3(x_1)$ in the original correlator. This leads to the form

$$0 = \sum_{\mathcal{O}} [\lambda_{12\mathcal{O}} \lambda_{34\mathcal{O}} F_{\mp, \Delta, \ell}^{12, 34}(u, v) \pm \lambda_{32\mathcal{O}} \lambda_{14\mathcal{O}} F_{\mp, \Delta, \ell}^{32, 14}(u, v)], \quad (4.1.4)$$

where

$$F_{\mp, \Delta, \ell}^{ij, kl}(u, v) \equiv v^{\frac{\Delta_k + \Delta_j}{2}} g_{\Delta, \ell}^{\Delta_{ij}, \Delta_{kl}}(u, v) \mp u^{\frac{\Delta_k + \Delta_j}{2}} g_{\Delta, \ell}^{\Delta_{ij}, \Delta_{kl}}(v, u) \quad (4.1.5)$$

are called the *convolved conformal blocks*.

In principle, more equations can be derived by considering other exchanges, e.g. 2 $\leftrightarrow$ 3 but these turn out to only give trivial relations.

4.2 Linear Programming

Eq. (4.1.4) is valid for any consistent CFT and therefore we have an infinite number of functional equations constraining the space of CFTs. Extracting information from these equations, however, is very difficult. In 2008 the authors of [34] finally found a way to do this rigorously for the case of unitary CFTs using linear programming. The logic goes as follows.

Consider the special case of four identical scalar fields ϕ . Then Eq. (4.1.4) reads

$$0 = 1 + \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 F_{-, \Delta, \ell}^{\phi\phi, \phi\phi}(u, v). \quad (4.2.1)$$

The antisymmetrized equation is identically zero and the 1 appears because the identity operator always appears with $\lambda = 1$. By abuse of notation, I assume that the identity

¹This is only true as long as an origin for quantization can be found such that radial ordering is satisfied. The case of 1D is special and is discussed in Section 4.7, but in higher dimensions such a quantization can always be found.

does not appear anymore in the sum. For unitary theories and Hermitean fields the OPE coefficients λ are real, so we know that all coefficients in Eq. (4.2.1) are non-negative. We can think of $F_{-\Delta,\ell}^{\phi\phi,\phi\phi}(u,v)$ as vectors in some infinite-dimensional vector space. If we make some assumptions about what operators appear in the sum and then manage to find a functional Λ from the dual space such that $\Lambda(1) > 0$ and $\Lambda\left(F_{-\Delta,\ell}^{\phi\phi,\phi\phi}\right) \geq 0$ then the sum in Eq. (4.2.1) cannot possibly add up to zero. This means that the assumptions that we made on the spectrum are not consistent and such a CFT cannot exist.

In order to make this more concrete and to explicitly describe such a Λ , it is necessary to choose a basis for the vector space of functions in which $F_{-\Delta,\ell}^{\phi\phi,\phi\phi}$ lives. It has become customary to pick the Taylor expansion coefficients around the so called crossing symmetric point $z = \bar{z} = \frac{1}{2}$ as was mentioned in Section 3.4. This is a good choice because it is a countable basis and because we get particularly favorable convergence properties [23, 35]. To make it a bit more straightforward to write the derivatives of the complex numbers z and $\bar{z}$, it is useful to instead take the derivatives in the coordinates² $a \equiv z + \bar{z}$ and $b = (z - \bar{z})^2$. This way we can write

$$\Lambda = \sum_{n,m=0}^{\infty} c_{n,m} \partial_a^n \delta\left(a - \frac{1}{2}\right) \partial_b^m \delta(b). \quad (4.2.2)$$

If $c_{n,m} = 0$ for n and m above some cutoff nothing changes about the statement that a specific CFT spectrum is ruled out so we can actually truncate the space of functionals to some finite dimensional subspace without a loss in rigor. This way, finding a suitable Λ can be implemented numerically.

A linear program has the standard form

$$\begin{aligned} & \min_x c^T x, \\ & \text{such that : } Ax \leq b, \\ & x \geq 0. \end{aligned} \quad (4.2.3)$$

Here the expression that is to be minimized is called the *objective function*. The above form is general and every other linear program can be reduced to it, including our system of inequalities

$$\begin{aligned} & \max_{\Lambda} \Lambda F(0), \\ & \text{such that : } \Lambda F(\Delta) \geq 0, \quad \forall \Delta \geq \Delta_{gap}. \end{aligned} \quad (4.2.4)$$

If the maximum is positive then the spectrum is ruled out. To speed up the computation it is possible to introduce the extra condition $\Lambda F(0) = 1$ which eliminates the need for the optimization and only looks for a feasible solution.

²In 1D this is, of course, unnecessary. Since z is already real, we can simply write $\Lambda = \sum_{n=0}^{\infty} c_n \delta^{(n)}\left(z - \frac{1}{2}\right)$.

4.2.1 Example in 1D

To illustrate these points, I work out a basic example in 1D effectively mimicking what [34] did in 4D. Specifically, I want to study the simplest case of four identical scalars ϕ and derive a bound on the maximal scaling dimension of the first operator in the OPE of $\phi \times \phi$. Following Eq. (4.1.4), we have

$$0 = \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 F_{-\Delta}^{\phi\phi,\phi\phi}(z). \quad (4.2.5)$$

The symmetrized equation does not appear because it is identically zero. The convolved conformal block is

$$F_{-\Delta}(z) \equiv F_{-\Delta,0}^{\phi\phi,\phi\phi}(z) = (1-z)^{2\Delta_\phi} z^\Delta {}_2F_1(\Delta, \Delta, 2\Delta; z) - z^{2\Delta_\phi} (1-z)^\Delta {}_2F_1(\Delta, \Delta, 2\Delta; 1-z). \quad (4.2.6)$$

Now write this as a vector equation in the Taylor basis around $z = \frac{1}{2}$ and truncate to a finite number of derivatives $F^{(m)}(\Delta) \equiv \frac{\partial^m}{\partial z^m} F_{-\Delta}(z)$. We only need to keep the odd derivatives because $F_{-\Delta}(z - \frac{1}{2}) = -F_{-\Delta}(\frac{1}{2} - z)$ so even derivatives vanish. Now for a truncated space of given dimension N , we could make some assumption on the spectrum and see if it is possible to find a linear functional that rigorously excludes any spectrum satisfying this assumption. For example, choosing $N = 4$, $\Delta_\phi = 1$ and $\Delta_{gap} = 4$ we want to find a functional Λ such that

$$\Lambda \cdot \begin{pmatrix} F^{(1)}(0) \\ F^{(3)}(0) \\ F^{(5)}(0) \\ F^{(7)}(0) \end{pmatrix} = 1, \quad \Lambda \cdot \begin{pmatrix} F^{(1)}(\Delta) \\ F^{(3)}(\Delta) \\ F^{(5)}(\Delta) \\ F^{(7)}(\Delta) \end{pmatrix} \geq 0, \quad \forall \Delta \geq \Delta_{gap} = 4. \quad (4.2.7)$$

To implement this numerically, the easiest way is to discretize Δ up to some large cutoff as well as the large Δ asymptotic value of $F^{(m)}$. This is now a linear program that can be implemented in, for example, Mathematica. Such a code is shown below.

```
g[dd_, z_] :=
  z^dd * Hypergeometric2F1[dd, dd, 2*dd, z];
d = 1.;
NN = 4;
Fm[dd_] :=
  Table[D[(1 - z)^(2*d)*g[dd, z] -
    z^(2*d)*g[dd, 1 - z], {z, n}] /. z -> .5,
    {n, 1, 2*NN, 2}];
DelMin = 4;

DeltaVec = Array[Fm[#^2] &, 200, {Sqrt[DelMin], Sqrt[100]};
DeltaVec = Append[DeltaVec, Fm[0]];
```

Figure 4.1: Extremal functional for $\Delta_\phi = 1$ and $N = 4$. The value $\Lambda \cdot F$ crosses from negative to positive at $\Delta = 3.5$ and then has a double zero at $\Delta = 7.2$. This observation will be useful in Sec. 4.3.

```
TargetVec = Table[{0, 1}, {n, 1, 200}];
TargetVec = Append[TargetVec, {1, 0}];
BoundVec = Table[{-Infinity, Infinity}, {n, 1, NN}];

lambda = LinearProgramming[Fm[0], DeltaVec, TargetVec, BoundVec]
```

A feasible solution to this problem, i.e. a not necessarily optimal functional that satisfies the constraints, is $\Lambda = (-0.500, 0.114, 0, -2.00 \cdot 10^{-11})$. This means that the assumed spectrum is ruled out. By varying Δ_{gap} we can find the optimal gap derivable from our chosen truncation, which gives us a completely rigorous, albeit suboptimal, bound. In this case our bound would be $\Delta_{gap} \approx 3.51$. The corresponding functional is called the extremal functional. Its action on F is shown in Fig. 4.1. As we increase N , the bound monotonically improves, as shown in Fig. 4.2.

This example illustrates the simplest way to actually calculate bootstrap bounds numerically, but it is not the most efficient way of doing so. Specifically we are trying to express highly non-linear inequalities in terms of many linear inequalities, effectively approximating a high-dimensional shape by a finite number of intersecting hyperplanes. Instead, it will be more efficient to find a way to encode non-linear constraints more directly, which will be explored in Sec. 4.5. This will also partially resolve the problem of how to choose the discretization.

Figure 4.2: The maximal gap Δ_{gap} for $\Delta_\phi = 1$ as a function of $1/N$ for $N = 2, 4, 6 \dots 60$. Extrapolating to $N \rightarrow \infty$ suggests that the optimal bound is $\Delta_{gap} = 3$.

4.3 Extremal Functional Method

An interesting alternative to numerically solving Eq. (4.2.4) is the extremal functional method introduced in [36] and further developed in [37]. A key observation in the previous example was that, in the extremal case, $\Lambda F(\Delta)$ crosses from negative to positive at the maximal gap and then has a discrete number of zeros after that. An example for $N = 4$ is shown in Fig. 4.1. As we increase N , more zeros appear, Δ_{gap} increases and $\Lambda F(0)$ moves closer to zero so that as $N \rightarrow \infty$ we are actually recovering a unique solution to the crossing equation by reading off the zeros. The idea of the extremal functional bootstrap is to turn this logic around. Since we are most interested in the extremal functionals rather than the more general class, we can try to construct it directly from its zeros.

This idea is made rigorous by the Karush-Kuhn-Tucker conditions [38–40] which are a set of necessary conditions on the optimality of semi-infinite linear programs. Solving these equations directly allows for highly efficient computation of optimal bootstrap bounds. This formulation can also be extended into the so-called *flow method*. All of these points will be discussed further in Ch. 5.

4.4 Global Symmetries

So far I have only discussed crossing equations that come from 4-point functions of scalars. If these satisfy some global symmetry, however, we are free to choose their

indices separately to obtain equations that are in principle different. To find out how many equations can be extracted from this, we look at the general representations that can appear and then compare index structures, each of which gives its own equation. As an illustrative example I will follow the discussion in [24].

Consider the 4-point function of scalars ϕ_i in a global $O(N)$ vector multiplet. When writing it in terms of conformal blocks it will be necessary to know what types of operators can appear in the OPE of $\phi_i \times \phi_j$. We know from representation theory that this can be decomposed into traces S , symmetric traceless operators T and anti-symmetric operators A :

$$\phi_i \times \phi_j \sim \sum_{\mathcal{O} \in S} \delta_{ij} \mathcal{O} + \sum_{\mathcal{O} \in T} \mathcal{O}_{(ij)} + \sum_{\mathcal{O} \in A} \mathcal{O}_{[ij]}. \quad (4.4.1)$$

The $\sim$ symbol signifies that these equations should be understood inside of correlators. By considering the 3-point functions we get from contracting both sides with one of these representations we can gain further information. Take for example an anti-symmetric operator $\mathcal{O}_{[ij]}$ with spin ℓ . The 3-point function was given in Eq. (2.2.24) from which we can find the following symmetry property

$$\begin{aligned} \langle \phi_i(x) \phi_j(-x) \mathcal{O}_{[ij]}(0) \rangle &= (-1)^\ell \langle \phi_i(-x) \phi_j(x) \mathcal{O}_{[ij]}(0) \rangle \\ &= (-1)^\ell \langle \phi_j(x) \phi_i(-x) \mathcal{O}_{[ij]}(0) \rangle \\ &= -(-1)^\ell \langle \phi_i(x) \phi_j(-x) \mathcal{O}_{[ij]}(0) \rangle. \end{aligned} \quad (4.4.2)$$

From this it follows that ℓ must be odd. In particular for the special case of 1D it implies that anti-symmetric operators cannot appear in the OPE. Similar arguments reveal that operators in S and T must have even spin.

Now we write out the possible tensorial structures of the 4-point function and find the crossing equations

$$\begin{aligned} 0 &= (\delta_{ij} \delta_{kl} \pm \delta_{il} \delta_{jk}) \sum_{\mathcal{O} \in S^+} \lambda_{\phi\phi\mathcal{O}}^2 F_{\mp, \Delta, \ell}^{\phi\phi, \phi\phi} \\ &+ \left(\left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{N} \delta_{ij} \delta_{kl} \right) \pm \left(\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl} - \frac{2}{N} \delta_{il} \delta_{jk} \right) \right) \sum_{\mathcal{O} \in T^+} \lambda_{\phi\phi\mathcal{O}}^2 F_{\mp, \Delta, \ell}^{\phi\phi, \phi\phi} \\ &+ ((\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \pm (\delta_{ik} \delta_{jl} - \delta_{ij} \delta_{kl})) \sum_{\mathcal{O} \in A^-} \lambda_{\phi\phi\mathcal{O}}^2 F_{\mp, \Delta, \ell}^{\phi\phi, \phi\phi}. \end{aligned} \quad (4.4.3)$$

The exponents in S^+ , T^+ and A^- refer to whether the spins appearing in the sum are even or odd. These equations can be collected and written as on vectorial equation

$$0 = \sum_{\mathcal{O} \in S^+} \lambda_{\phi\phi\mathcal{O}} V_{S, \Delta, \ell} + \sum_{\mathcal{O} \in T^+} \lambda_{\phi\phi\mathcal{O}} V_{T, \Delta, \ell} + \sum_{\mathcal{O} \in A^-} \lambda_{\phi\phi\mathcal{O}} V_{A, \Delta, \ell}, \quad (4.4.4)$$

where

$$V_{S,\Delta,\ell} = \begin{pmatrix} 0 \\ F_{-,\Delta,\ell} \\ F_{+,\Delta,\ell} \end{pmatrix}, \quad V_{T,\Delta,\ell} = \begin{pmatrix} F_{-,\Delta,\ell} \\ (1 - \frac{2}{N}) F_{-,\Delta,\ell} \\ -(1 + \frac{2}{N}) F_{+,\Delta,\ell} \end{pmatrix}, \quad V_{A,\Delta,\ell} = \begin{pmatrix} F_{-,\Delta,\ell} \\ -F_{-,\Delta,\ell} \\ F_{+,\Delta,\ell} \end{pmatrix}. \quad (4.4.5)$$

Having more equations does not automatically give tighter bounds but, since now the operators of different representations are separated, we can impose individual restrictions on their respective spectra and therefore obtain more informative bounds.

4.5 Semi-Definite Programming

As mentioned in Sec. 4.2.1, there is a better way to encode the geometry of the inequalities given by crossing than to just discretize to some arbitrary subset. The authors of [24] describe how to approximate the semi-infinite linear program in Eq. (4.2.4) as a finite semi-definite program. In order to do this we need Eq. (3.5.18):

$$g_{\Delta}^{(m)} = \chi(\Delta) Q_m(\Delta). \quad (4.5.1)$$

I will stick to this expression which is strictly speaking only valid for 1D but its generalization to higher dimensions is relatively straightforward, the most important difference being that there is a spin index for which a cutoff must be chosen.

Using this expression it is also clear that the convolved conformal blocks can be written in the same way

$$F_{\Delta}^{(m)} = \chi(\Delta) P_m(\Delta), \quad (4.5.2)$$

where P_m are again polynomials. Now the condition of $\Lambda \cdot F(\Delta)$, $\forall \Delta \geq \Delta_{min}$, can be formulated as

$$\sum_m a_m P^{(m)}(\Delta_{min} + x) \geq 0, \quad \forall x \geq 0. \quad (4.5.3)$$

A well-known theorem [24, 41] states that the positivity of a polynomial on the positive real axis

$$P(x) \geq 0, \quad \forall x \geq 0 \quad (4.5.4)$$

is equivalent to $P(x)$ being expressible as a sum

$$P(x) = a(x) + xb(x), \quad (4.5.5)$$

where a and b are sums of squares of polynomials. Defining the vector $[x]_d \equiv (1, x, \dots, x^d)^T$ and the matrix $Q_d(x) \equiv [x]_d [x]_d^T$ it can be shown that any degree $2d$ sum of squares can

be written as $\text{Tr}(AQ_d(x))$ where $A \succeq 0$ is positive semi-definite. Therefore Eq. (4.5.4) is equivalent to

$$P(x) = \text{Tr}(AQ_{d_1}(x)) + x \text{Tr}(BQ_{d_2}(x)), \quad A, B \succeq 0. \quad (4.5.6)$$

This means that bootstrap problems can be formulated as finding a linear functional $\vec{a}^T$ such that $\sum_m a_m P^{(m)}(\Delta_{\min} + x)$ can be written in the form of Eq. (4.5.6). This is a semi-definite program or, more specifically, a polynomial matrix problem whose solution is discussed in detail in [42]. The software package *Semi-Definite Program solver for the conformal Bootstrap* (SDPB) that accompanies that paper is the go-to solver in the literature.

4.6 Mixed Correlators

In the previous section I discussed how semi-definite programming can be used to better encode the constraints given by the crossing equations. In this section we will see that semi-definite programming is crucial for the study of mixed correlators. The crux of the issue with using linear programming on the general equation (4.1.4) is that it relies on the squared OPE-coefficients being non-negative.

$$\begin{aligned} 0 &= \sum_{\mathcal{O}} [\lambda_{12\mathcal{O}} \lambda_{34\mathcal{O}} F_{\mp, \Delta, \ell}^{12,34}(u, v) \pm \lambda_{32\mathcal{O}} \lambda_{14\mathcal{O}} F_{\mp, \Delta, \ell}^{32,14}(u, v)] \\ &= \sum_{\mathcal{O} \in S} \begin{pmatrix} \lambda_{12\mathcal{O}} & \lambda_{34\mathcal{O}} \end{pmatrix} \mathcal{F}_{\mp, \Delta, \ell}^{12,34} \begin{pmatrix} \lambda_{12\mathcal{O}} \\ \lambda_{34\mathcal{O}} \end{pmatrix} \pm \begin{pmatrix} \lambda_{14\mathcal{O}} & \lambda_{32\mathcal{O}} \end{pmatrix} \mathcal{F}_{\mp, \Delta, \ell}^{32,14} \begin{pmatrix} \lambda_{14\mathcal{O}} \\ \lambda_{32\mathcal{O}} \end{pmatrix}, \end{aligned} \quad (4.6.1)$$

where $\mathcal{F}_{\mp, \Delta, \ell}^{12,34} = \begin{pmatrix} 0 & \frac{1}{2} F_{\mp, \Delta, \ell}^{12,34}(u, v) \\ \frac{1}{2} F_{\mp, \Delta, \ell}^{12,34}(u, v) & 0 \end{pmatrix}$. If we can find a non-trivial functional such that the matrices $\Lambda \cdot \mathcal{F}$ are positive semi-definite for all Δ in the assumed spectrum then the theory is again ruled out. Unfortunately, since these matrices are symmetric and traceless they will always have eigenvalues of both positive and negative sign. In order to actually extract information from this equation it needs to be supplemented by the crossing equations of identical operators.

4.7 Caveats in 1D

So far I have treated the one-dimensional bootstrap as a special case of the higher-dimensional formulation. However, this is not strictly true and I have pointed out several things that change, e.g. the lack of an angular variable and spin. Since from here on out I will mostly deal with applications in 1D I will briefly discuss some related caveats.

First of all, how do we radially quantize a theory without mention of an angular variable? The radial variable $r \in \mathbb{R}^+$ only maps to the positive real numbers, so it appears that we are losing all points with $x < 0$. In order to avoid this, we must choose the origin of our quantization such that $x > 0$ for all operator insertions. When looking at a 4-point function

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle, \quad x_1 < x_2 < x_3 < x_4, \quad (4.7.1)$$

we can now write out the first channel expansion where we merge 1 with 2 and 3 with 4 while still respecting radial ordering. In higher dimensions we could now swap some operators through the use of conformal transformations without passing through any other operator, which we certainly cannot do in 1D. To still be able to write crossing equations we have to topologically connect $r = 0$ with $r = \infty$. This precisely captures the fact that we throw away the negative real line in radial quantization. Now we can expand in a second channel by bringing operator 4 to a point with $x < 0$ and then choosing a new origin for our quantization. In this channel we merge 4 with 1 and 2 with 3. This happens to be exactly the crossing relation that we have discussed in this chapter. The other crossing relations that are somewhat trivial in higher dimensions do not actually appear at all in 1D. A brief discussion of this point can be found in [43].

Another subtlety coming from the inability to smoothly exchange operators is that the type of argument in Sec 4.4 that excluded certain representations from the OPE of two operators does not hold in 1D. This can be somewhat intuitive because that argument related internal symmetries to spin properties and spin is not present in 1D.

Chapter 5

Extremal Functional Bootstrap

5.1 Extremality Equations

After only briefly mentioning the extremal functional bootstrap in Sec. 4.3, I want to go into more detail about it, as it will be the main tool I use in the applications. I will mostly follow the development in [37]. I will start from the crossing equations and rewrite them in a way that treats gap maximization and OPE maximization on equal footing.

For the entire chapter I am going to assume that we are dealing with identical scalars in a 1D CFT. It should, in principle, be possible to adapt all of these arguments to both mixed correlators and higher dimensions. Since this introduces several technical obstructions that have not yet been overcome in practice, I will stick to the simplest case. The crossing equation can be written as

$$\sum_{\mathcal{O} \in S} \lambda_{\mathcal{O}}^2 F_{\mathcal{O}} = T \equiv -F_0, \quad (5.1.1)$$

where S is the set of all allowed operators in the spectrum. As usual, the contribution of the identity operator can be pulled out because we know that its associated OPE coefficient is always equal to one. I previously mentioned how gap maximization can be achieved by ruling out several spectra and bisect an optimal bound. A more direct way of formulating this can be achieved by introducing, for a given truncation, the auxiliary vectors

$$W^{(i)} = \left(W_1^{(i)}, \dots, W_N^{(i)} \right), \quad W_j^{(i)} = \delta_{ij} |T_i|, \quad (5.1.2)$$

which allows for the crossing equation to be written as an optimization problem

$$\begin{aligned} & \max_{\mu, \lambda^2} \left(\sum_{i=1}^N -\mu_i \right), \\ & \sum_{i=1}^N \mu_i W^{(i)} + \sum_{\mathcal{O} \in S} \lambda_{\mathcal{O}}^2 F_{\mathcal{O}} = T, \\ & \lambda^2, \mu \geq 0. \end{aligned} \quad (5.1.3)$$

Since this is a linear program it has a dual formulation

$$\begin{aligned} \min_{\Lambda \neq 0} \Lambda \cdot T, \\ \forall \mathcal{O} \in S \quad \Lambda \cdot F_{\mathcal{O}} \geq 0, \\ \forall i=1, \dots, N \quad \Lambda \cdot W^{(i)} \geq -1. \end{aligned} \quad (5.1.4)$$

At the optimal solution, both primal and dual objective functions must agree so that we have $\Lambda \cdot T = -\sum_i \mu_i$. Thus, an optimum of zero means that we have found a solution to crossing that does not include auxiliary vectors.

We will now see that OPE maximization problems take the exact same form. Their primal formulation requires that we introduce a cost function $c_{\mathcal{O}}$ that tells us what linear combination of squared OPE coefficients we are trying to optimize. In practice, this will be zero for almost all operators, usually even for all but one. The OPE maximization problem is then

$$\begin{aligned} \max_{\lambda^2} \left(\sum_{\mathcal{O} \in S} \lambda_{\mathcal{O}}^2 c_{\mathcal{O}} \right), \\ \sum_{\mathcal{O} \in S} \lambda_{\mathcal{O}}^2 F_{\mathcal{O}} = T. \end{aligned} \quad (5.1.5)$$

The corresponding dual formulation is

$$\begin{aligned} \min_{\Lambda \neq 0} \Lambda \cdot T, \\ \forall \mathcal{O} \in S \quad \Lambda \cdot F_{\mathcal{O}} \geq c_{\mathcal{O}}. \end{aligned} \quad (5.1.6)$$

Evidently gap maximization and OPE maximization can be formulated in a unified way by introducing auxiliary vectors and a cost function. Eq. (5.1.6) is a linear semi-infinite problem, meaning there are an infinite number of finite-dimensional constraints. From the literature on such problems we can learn that there are the so-called Karush-Kuhn-Tucker (KKT) conditions [44–46]. These are necessary conditions stating that any optimal solution to our linear problem will have a set of $K \leq N$ vectors $v_j \equiv F_{\mathcal{O}_j}$ and associated coefficients a_j such that

$$\text{(Crossing)} \quad \sum_{j=1}^K a_j v_j = T \quad (5.1.7)$$

$$\text{(Saturation)} \quad \Lambda \cdot v_j = c_{\mathcal{O}_j} \quad (5.1.8)$$

$$\text{(Tangency)} \quad \Lambda \cdot \nabla_{\Delta} F(\Delta_j) = \begin{cases} 0 & \text{if } v_j \notin \partial S \\ (n_j^+)_{\Delta} & \text{if } v_j \in \partial S \end{cases} \quad (5.1.9)$$

The first equation is just the truncated crossing equation. The second equation shows that the vectors in the extremal spectrum must all saturate the bounds in Eq. (5.1.6). Lastly, the third condition should be understood by thinking of how S is parametrized in terms of Δ . Whenever Δ lies right on the edge of the allowed spectrum then $\Lambda \cdot F_\Delta$ should be positive inside the region S and negative just outside of S . This fixes the direction of the derivative.

5.1.1 Geometric Interpretation

The last few points are most easily understood by thinking of the bootstrap problem geometrically as first introduced in [34] and later extended in [37]. If we think of each convolved conformal block in Eq. (5.1.1) as a vector in an N -dimensional space, then any non-negative linear combination of such vectors will be inside of a convex cone. The crossing equation can be satisfied if and only if T lies within that cone. A linear functional Λ that satisfies $\Lambda \cdot < 0$ and $\forall_{\mathcal{O} \in S} \Lambda \cdot F_{\mathcal{O}} \geq 0$ acts as a plane passing through the origin that separates T and the cone and thus rules out a spectrum S . Since the cone is self-similar along some dimension, we can drop this dimension by just looking at one slice by fixing one dimension. This slice will now be the convex hull of all F in S . We can easily visualize this in the case of $N = 3$ as shown in Fig. 5.1. The case depicted is precisely the extremal case in which T lies directly on the dividing plane Λ . We can verify that the KKT conditions do, in fact, hold. We have $K = 2 < N = 3$ vectors $v_1 = F(\Delta_{\text{gap}})$ and $v_2 = F(\Delta_2)$ that satisfy $a_1 v_1 + a_2 v_2 = T$ for some coefficients a_i . Both vectors lie within the dividing plane as well, so saturation is satisfied. Lastly, tangency means that at Δ_2 the vector $F(\Delta)$ runs parallel to the plane Λ whereas at Δ_{gap} it does not, because Δ_{gap} marks the boundary of the set S .

In this as well as higher dimensional cases, it is possible to determine the functional Λ by knowing the vectors v_i . Therefore, the bootstrap program can be carried out by solving the KKT conditions directly and then verifying that the corresponding Λ is indeed a valid functional. This is somewhat similar to the ideas of the so-called Gliozzi bootstrap [47] where we also try to solve a truncated bootstrap equation. The difference is that in this case we obtain a rigorous bound at every step of the process.

5.2 Flow Method

Another implication of the geometrical picture is the intuition that as we vary Δ_{ext} and with it T , we must be varying both Δ_{gap} and Δ_2 continuously. This allows us to estimate the solution of a different bootstrap problem from a previous one by linearizing the KKT

Figure 5.1: Example of what $F(\Delta)$ could look like for an $N = 3$ truncation. On the left $\Delta \rightarrow \infty$ so imposing a gap means cutting off part of the curve on the right. This figure was directly take from [37].

crossing condition and solving for the linear increment. This is called the flow method and it allows for big computational speedups.

Rather than treating the fully general case, I will show the example of a gap maximization and later explain modifications to the framework as needed. The KKT crossing condition is

$$\sum_{i=1}^K a_i v_i = T. \quad (5.2.1)$$

Suppose we have found a valid solution $\vec{\Delta} = (\Delta_1, \dots, \Delta_K)$ and $\vec{a} = (a_1, \dots, a_K)$ that satisfies crossing. If we now vary Δ_{gap} the equation changes as follows

$$\begin{aligned} \delta T &= \sum_{i=1}^K v_i \delta a_i + a_i \left. \frac{\partial}{\partial \Delta} v(\Delta) \right|_{\Delta=\Delta_i} \delta \Delta_i \\ &= \begin{pmatrix} \uparrow & & \uparrow & \uparrow & \uparrow \\ a_1 \partial_{\Delta} v_1 & \dots & a_K \partial_{\Delta} v_K & v_1 & \dots & v_K \\ \downarrow & & \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} \delta \vec{\Delta} \\ \delta \vec{a} \end{pmatrix} \\ &\equiv J \begin{pmatrix} \delta \vec{\Delta} \\ \delta \vec{a} \end{pmatrix} \end{aligned} \quad (5.2.2)$$

with an appropriately defined J . The arrows indicate that each column should be thought of as a vector. Now we can find the variation in our solution by inverting J :

$$\begin{pmatrix} \delta \vec{\Delta} \\ \delta \vec{a} \end{pmatrix} = J^{-1} \delta T. \quad (5.2.3)$$

5.3 Error Correction

One of the most useful features of extremal flows is that they are self-correcting. To make the notation a little more concise, define $x \equiv (\vec{\Delta}, \vec{a})$. Assuming we have an approximate solution x to crossing, we can define the error function

$$f \equiv \sum_{i=1}^K a_i v_i - T. \quad (5.3.1)$$

Linearizing the variation in x while leaving Δ_{ext} fixed we get

$$\delta f = J \delta x. \quad (5.3.2)$$

We can now improve our solution to $\tilde{x} = x - J^{-1}f$. This turns out to just be the Newton method. Iteratively, the error can be made arbitrarily small, so long as we start from a sufficiently good solution. However, for large N , "sufficiently good" can (and in practice does) require solutions accurate to within one part in 10^{20} or more. To find these approximate solutions in the first place we can use upgrading as I explain next.

5.4 Upgrading

Finding an approximate solution to crossing for small values of N is not too hard. I have already demonstrated just how to do that in Sec. 4.3. We can even get away with just guessing an answer if we start with $N = 2$. Following the example in [37], we could arbitrarily set $\Delta_{\text{gap}} = 0.3$ and try $\Delta_1 = 2$ and $a_1 = 1$. Now applying the error correction procedure outlined in the previous section, these will converge to the values¹ $\Delta_1 = 1.874$ and $a_1 = 0.576$. Now in order to find a stronger bound, we want to upgrade our solution by increasing N . When increasing to $N = 3$ we already see a problem in that it seems like we now have three equations but four unknowns, because another operator would come with its own dimension and coefficient. In practice, what happens is that the next operator comes in at $\Delta = \infty$ so there really only is the coefficient. To avoid these complications entirely we can simply skip the odd values of N . For the lowest dimensional insertions it is sufficient to just make very crude guesses, but after that it is possible and, in fact, necessary to extrapolate where the next new insertion should lie based on the previous data. Using this approach I upgraded up to $N = 80$ using 1000 bits of precision. Samples of the code I wrote for upgrading are given in App. B. This then yields the rigorous bound $\Delta_1 \leq 1.6056$. This comes quite close to what is known to be the optimal bound of $\Delta_1 \leq 1.6$, a result that was proved in [48, 49].

¹The original paper [37] contains a typo, stating a wrong value for a_1 .

Figure 5.2: The maximal gap as a function of the external dimension is shown. Inspired by the analytical solution $\Delta_1 = 2\Delta_{\text{ext}} + 1$ in [48, 49], it makes sense to plot $\frac{\Delta_{\text{gap}} - 1}{\Delta_{\text{ext}}}$. We see that for small values of Δ_{ext} we come very close to the optimal bound but that the quality of the numerical bound degrades as $\Delta_{\text{ext}} \rightarrow \infty$. The results in [37] show that for fixed N the bound converges $\frac{\Delta_{\text{gap}} - 1}{\Delta_{\text{ext}}} \xrightarrow{\Delta_{\text{ext}} \rightarrow \infty} 3$.

5.5 Gap Flow

If we apply the concept of flow explained in Sec. 5.2 to our upgraded solution, then we can efficiently derive bounds on the maximal gap as a function of Δ_{ext} . The result of this is shown in Fig. 5.2. In the usual setting of linear or semi-definite programming this would involve bisecting a solution for each value of Δ_{ext} . Instead, we can just initially use the linearized flow and then error-correct the solution. For sufficiently small perturbations in Δ_{ext} , this works but then after only a few of these linearized steps we can again start to fit a polynomial extrapolation on the solutions along the flow. These eventually allow for fairly large steps without compromising convergence and, thus, offer another significant speedup. For example, the last step size of the graph shown in Fig. 5.2 was $\delta\Delta_{\text{ext}} = 10$.

5.6 OPE Maximization

We can also use the flow method for OPE maximization by flowing the gap maximization problem into an OPE maximization problem. Specifically we are interested in the maximum allowed value for the OPE coefficient of external operator with itself, i.e. in the expression $\phi \times \phi \sim \lambda\phi + \dots$. If we ask that question exactly at the critical value of Δ_{gap} then the answer will be zero, because this is precisely what rules out higher values of the gap, the fact that at larger gaps the squared OPE coefficient becomes negative. So instead of treating Δ_{gap} like the unknown we will now fix it as an external variable and decrease

Figure 5.3: Upper bound on $a_0 \equiv \lambda_{\phi\phi\phi}^2$ for $\Delta_{\text{ext}} \equiv \Delta_\phi = 1$ as a function of Δ_{gap} . Truncation is at $N = 36$.

it while still solving the crossing condition. The truncated crossing equation now takes the form

$$T = a_0 v(\Delta_{\text{ext}}) + a_1 v(\Delta_{\text{gap}}) + \sum_{i=2}^K a_i v(\Delta_i). \quad (5.6.1)$$

We can again take the variation, but this time with respect to Δ_{gap} to get

$$\begin{aligned} -a_1 \delta v(\Delta_{\text{gap}}) &= v(\Delta_{\text{ext}}) \delta a_0 + v(\Delta_{\text{gap}}) \delta a_1 + \sum_{i=2}^K (a_i \partial_{\Delta} v|_{\Delta=\Delta_i} \delta \Delta_i + v(\Delta_i) \delta a_i) \\ &= \begin{pmatrix} \uparrow & & \uparrow & \uparrow & \uparrow \\ a_2 \partial_{\Delta} v_2 & \dots & a_K \partial_{\Delta} v_K & v_0 & \dots & v_K \\ \downarrow & & \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} \delta \Delta_2 \\ \vdots \\ \delta \Delta_K \\ \delta a_0 \\ \delta a_1 \\ \vdots \\ \delta a_K \end{pmatrix} \\ &\equiv J \cdot \delta x, \end{aligned} \quad (5.6.2)$$

again with appropriate definitions for x and J . We can apply the same technique as when flowing the maximal gap of initially estimating the linearized change $\delta x = -J^{-1} a_1 \delta v(\Delta_{\text{gap}})$ and later using an extrapolation method. Using these ideas I performed a relatively low precision flow at $\Delta_{\text{ext}} = 1$, the result of which is shown in Fig. 5.3.

The flow in the figure occurs from right to left, so we see that we do indeed start of at initially small values of a_0 that increase as we decrease Δ_{gap} . Interestingly, we observe a singularity that can be quite difficult to cross using flows, even though it is possible

to derive bounds on the other side. At the singularity we cannot make any statements about the value of the OPE coefficient. It turns out that this singularity has a physical interpretation in terms of particle resonances in a dual massive QFT in AdS, an idea that is explored thoroughly in [50–52]. This duality will be explored further in Ch. 7.

Chapter 6

Boundary Bootstrap

In this chapter I want to illustrate how the extremal functional bootstrap can be applied in the case of what is known as the *boundary bootstrap*. Specifically, I will reproduce Fig. 7 in [53], so in this chapter I will always have the 3D Ising model in mind.

6.1 Boundaries in Systems at Criticality

As discussed in Ch. 1 the correlation length of a system diverges near criticality. The large-scale geometry of the material thus becomes important. The simplest boundary one can introduce into a system is a straight $(d - 1)$ -dimensional hypersurface that splits the space in half. An intuition about the relevant physics can be gained by considering this problem in the mean field approximation. Again, my treatment is largely based on [2].

6.1.1 Mean Field for Boundary Phase Transitions

We start off with the same Ising-like Hamiltonian as in Sec. 1.1. Introducing the mean field M , neglecting correlations of fluctuations around it and then minimizing the free energy per site again leads to the same condition on M

$$M(r) = \tanh \left(\beta \sum_{r'} J(r, r') M(r') \right), \quad (6.1.1)$$

Now the mean field has to be treated as spatially varying because the boundary breaks the translational invariance of the theory. A phase transition can still be detected when $\beta \sum_{r'} J(r, r') = 1$. However, since at the boundary there are fewer neighbors to interact with, we generally expect the boundary to become ordered at a lower temperature than the bulk. This type of transition is called the *ordinary transition* and we wish to study its effect on the boundary. At this point, the mean field should depend only on the distance

from the boundary. Since at the bulk critical temperature the boundary is still disordered we expect M to be small close to the boundary and we can Taylor expand it in the distance z from the surface

$$M(r') = M(z) + \frac{\partial M(z)}{\partial z}(r' - z) + \frac{1}{2} \frac{\partial^2 M(z)}{\partial z^2}(r' - z)^2 + \dots, \quad (6.1.2)$$

which we wish to study inside of the expression $\sum_{r'} J(z, r')M(r')$. We obtain

$$\begin{aligned} \sum_{r'} J(z, r')M(r') &= \sum_{r'} J(z, r')M(z) + \frac{\partial M(z)}{\partial z} \sum_{r'} (r' - z)J(z, r') \\ &\quad + \frac{1}{2} \frac{\partial^2 M(z)}{\partial z^2} \sum_{r'} (r' - z)^2 J(z, r') + \dots \\ &= J(z)M(z) + \frac{1}{2} J R^2 \frac{\partial^2 M(z)}{\partial z^2} + \dots, \end{aligned} \quad (6.1.3)$$

where $\sum_{r'} (r' - z)J(z, r') = 0$ because of the residual symmetries. The variables R^2 and J are defined as in Sec. 1.1. We need to assume some functional form for $J(z)$ so we will approximate it to be identical to the bulk coupling up to very short-range corrections:

$$J(z) = J \left(1 - \frac{R^2}{2\lambda} \delta(z) \right), \quad (6.1.4)$$

with the minus sign and $\lambda > 0$ capturing the intuition that the boundary interaction is in general weaker than the bulk interaction. Plugging this into the inverted mean-field condition we get

$$\begin{aligned} \frac{R^2}{2} \frac{\partial^2 M(z)}{\partial z^2} + M(z) - \frac{R^2}{2\lambda} \delta(z) M(z) &= \frac{1}{\beta J} \tanh^{-1} M(z) \\ &= \frac{1}{\beta J} \left(M + \frac{1}{3} M^3 + \dots \right). \end{aligned} \quad (6.1.5)$$

Remembering that $T_c = \frac{J}{k_B}$ and $t = \frac{T - T_c}{T_c}$, we can write this as a differential equation

$$\frac{1}{2} R^2 \frac{\partial^2 M}{\partial z^2} = tM + \frac{1}{3} M^3 - \frac{R^2}{2\lambda} \delta(z) M. \quad (6.1.6)$$

We need two boundary conditions to fix this equation. It makes sense to set $M = 0$ for $z < 0$. The other boundary condition follows from this by integrating from $z = -\infty$ to $z = 0^+$ which leaves us with $M'(0) = \frac{1}{\lambda} M(0)$. This gives an interpretation to λ as being an extrapolation length of M , a distance scale at which the mean field would continuously go to zero if it were continued below $z < 0$. To lift the discontinuity caused by the delta we will therefore replace our first boundary condition by $M = 0$ for $z = -\lambda$.

The solution to this differential equation must approach the correct scaling behavior near the critical point as $z \rightarrow \infty$. Just below the critical temperature we must have

$$M = |t|^{-\frac{1}{2}} f \left(\frac{z + \lambda}{R|t|^{-\frac{1}{2}}} \right), \quad (6.1.7)$$

Figure 6.1: Phase diagram of Ising-like models with surface. There is a critical temperature at which the bulk orders. If the surface coupling J_s is stronger than the bulk coupling J_b the surface remains ordered at a higher temperature than the bulk.

where the function f satisfies $f(0) = 0$ and $f(\infty) = \text{const.}$ The temperature dependence in the denominator of the function argument comes from plugging in the temperature dependence of the correlation length which is the only length scale available to the system. Since f solves a non-singular differential equation, we expect it to be an analytic function. Near the boundary that leads us to expect $M \sim |t|^1$. In other words: we have different scaling exponents at the boundary than in the bulk. Of course the particular value extracted here was again derived through mean-field arguments and will not be accurate quantitatively but the general proposition that surface exponents are different from bulk exponents is true in a wide array of circumstances. In CFT terms this means that we have to define different fields at the boundary than in the bulk as well.

6.1.2 Phase diagram

As pointed out in the previous section in most cases we expect the bulk ordering to occur at higher temperatures than the surface transition because the boundary interaction is weaker. We can, however, imagine situations where local interactions at the boundary get enhanced to the point of raising the critical temperature of the boundary above the critical temperature of the bulk. In this case the bulk transition is called the *extraordinary transi-*

tion. Then there is also the critical case between the two when both transition occur at the same temperature. This is called the *special transition*. These are displayed in Fig. 6.1.

The extraordinary transition is particularly interesting because it is very stable. This is to say that at the bulk critical point any small perturbation of the surface operators must preserve the fact that the surface is already ordered. Mathematically this gets expressed by the statement that the boundary theory should not have any relevant operators. We can infer even more. The fact that energy cannot be transported across the boundary gives the condition $\partial_z T|_{z=0} = 0$. But we know that a conserved quantity is protected, so we can assert that at the extraordinary transition the lowest dimensional boundary operator (after the identity) has $\Delta_T = 3$. Having this much information makes it possible to bootstrap information about the bulk spectrum.

6.2 Boundary CFT

In this section we want to study the effects on 2-point functions that come from restricting $\mathbb{R}^d$ to $\mathbb{R}_+ \times \mathbb{R}^{d-1}$. If the bulk system is near criticality then its conformal symmetry also gets broken down from $SO(d+1, 1)$ to $SO(d, 1)$. This means, that the usual logic of restricting n -point functions via Ward identities does not carry through like it did in Ch. 2. Instead, there can be an explicit dependence on the coordinate perpendicular to the boundary. This was studied in detail in [54] so I will just state the results that pertain to our discussion. The 1-point function no longer vanishes but takes the form

$$\langle \phi(x_\mu) \rangle = \frac{a_\phi}{(2y)^\Delta}, \quad (6.2.1)$$

where I introduce the notation $x_\mu = (y, \vec{x})$ with $y \geq 0$ referring to the perpendicular direction. The 2-point functions also change significantly because it is now possible to form a conformal invariant from just two positions

$$\xi = \frac{(\vec{x}_1 - \vec{x}_2)^2}{4y_1 y_2}, \quad (6.2.2)$$

which tells us that the 2-point function will only be determined up to a function of ξ . The precise form is given by

$$\langle \phi_1(x_{1\mu}) \phi_2(x_{2\mu}) \rangle = \frac{1}{(2y_1)^{\Delta_1} (2y_2)^{\Delta_2}} \xi^{-(\Delta_1 + \Delta_2)/2} G(\xi). \quad (6.2.3)$$

I restricted my attention to scalar operators but very similar formulas can be derived using the embedding formalism. To carry out the bootstrap program for boundaries we now try to write the 2-point functions as two distinct channel decompositions. This was done

in [53]. The first channel comes from the bulk OPE which in the case of two identical scalars takes the form

$$\phi(x_1)\phi(x_2) = \frac{1}{(x_1 - x_2)^{2\Delta}} + \sum_k \lambda_k C[x_1 - x_2, \partial_{x_2}] \mathcal{O}_k(x_2). \quad (6.2.4)$$

Plugging this into the 2-point function and multiplying out the denominator gives

$$G(\xi) = 1 + \sum_k \lambda_k a_k f_{\text{bulk}}(\Delta_k, \xi), \quad (6.2.5)$$

where we defined the bulk conformal blocks

$$f_{\text{bulk}}(\Delta_k, \xi) = C[x_1 - x_2, \partial_{x_2}] \frac{1}{y_2^{\Delta_k}}. \quad (6.2.6)$$

At first glance it appears that there is no second way to expand a 2-point function but in this case there is, because we can couple with the boundary operators. As has been known in the literature for a long time, there is a way to write bulk operators as a linear combination of boundary operators in a *boundary operator expansion* (BOE) [cardy1991bulk, 54, 55]:

$$\mathcal{O}(x) = \frac{a_{\mathcal{O}}}{(2y)^\Delta} + \sum_l \mu_l D[y, \partial_{\vec{x}}] \hat{\mathcal{O}}_l(\vec{x}), \quad (6.2.7)$$

where the hat indicates a boundary operator and D just like C before is a differential operator that is fixed by the symmetries. Plugging the BOE into the 2-point function and exploiting that the boundary operators satisfy full conformal symmetry, we get the second equation

$$G(\xi) = \xi^\Delta \left(a_{\mathcal{O}}^2 + \sum_l \mu_l^2 f_{\text{bdy}}(\Delta_l, \xi) \right), \quad (6.2.8)$$

where now the boundary conformal blocks can be determined from

$$D[y_1, \partial_{\vec{x}_1}] D[y_2, \partial_{\vec{x}_2}] \frac{1}{|\vec{x}_1 - \vec{x}_2|^{2\Delta}}. \quad (6.2.9)$$

Both types of conformal blocks were calculated in [54] and are given by

$$f_{\text{bulk}}(\Delta, \xi) = \xi^{\frac{\Delta}{2}} {}_2F_1 \left(\frac{\Delta}{2}, \frac{\Delta}{2}, \Delta + 1 - \frac{d}{2}; -\xi \right), \quad (6.2.10)$$

$$f_{\text{bdy}}(\Delta, \xi) = \xi^{-\Delta} {}_2F_1 \left(\Delta, \Delta + 1 - \frac{d}{2}, 2\Delta + 2 - d; -\frac{1}{\xi} \right). \quad (6.2.11)$$

Since these functions are given by hypergeometric functions they can be expanded as series just as described in App. A which allows for an equally efficient evaluation.

By comparing the two expansions of the 2-point function then gives us a valid crossing equation

$$1 + \sum_k \lambda_k a_k f_{\text{bulk}}(\Delta_k, \xi) = \xi^\Delta \left(a_{\mathcal{O}}^2 + \sum_l \mu_l^2 f_{\text{bdy}}(\Delta_l, \xi) \right). \quad (6.2.12)$$

The major obstruction to treating this equation using the same machinery of linear programming is that we do not know if the coefficients appearing in the sums are necessarily non-negative. This is why more recent works set up the crossing equations for different kinds of defects but don't actually solve these constraints [56, 57]. While in principle non-negative constraints could potentially be approximately solved by what is called the method of determinants or Gliozzi method [47, 58–60], the method is far from well-understood and in its current form not capable of generating reliable results. However, in the special case of a planar boundary in the 3D Ising model, the authors of [53] found a number of heuristic arguments that indicate that the coefficients in the sum are non-negative in this particular model. I will therefore adopt this as an assumption in order to be able to numerically analyze the crossing equation.

6.3 Bootstrap of Extraordinary Transition

Following the discussion in [61] as well as the argument in section 6.1.2 we can see that in the extraordinary transition we expect the first boundary operator after the identity to have dimension d . Using this information as input we will then try to infer bounds on the bulk theory that will be valid even in the absence of the boundary.

6.3.1 Numerical Bootstrap

We can compactly write the crossing equation as

$$1 = \sum_{\Delta} p_{\Delta} F_{\Delta}(\xi), \quad (6.3.1)$$

where

$$p_{\Delta} \equiv (\lambda_k a_k, a_{\mathcal{O}}^2, \mu_l^2), \quad (6.3.2)$$

$$F_{\Delta}(\xi) \equiv (-f_{\text{bulk}}(\Delta, \xi), \xi^{\Delta_{\text{ext}}}, \xi^{\Delta_{\text{ext}}} f_{\text{bdy}}(\Delta, \xi)). \quad (6.3.3)$$

The assumptions on the spectrum for fixed Δ_{ext} are

$$\begin{aligned} \Delta_{\text{bulk}} &\geq \Delta_{\text{min}}, \\ \Delta_{\text{bdy}}^{(1)} &= 0, \\ \Delta_{\text{bdy}}^{(2)} &= 3. \end{aligned} \quad (6.3.4)$$

The crossing equations can now be truncated to a certain number of derivatives and then we can use linear programming to bisect the maximal Δ_{min} for different Δ_{ext} as was done in [53]. Instead, I want to show what this would look like when using an extremal flow.

Figure 6.2: a) Action of extremal functional on f_{bdy} . The functional is encountered at approximately $\Delta_{\min} = 2.3$. The value of $\Lambda \cdot f_{\text{bdy}}$ crosses from negative to positive precisely at $\Delta = 3$ which is expected because that precisely saturates the bound that was imposed. b) Action of extremal functional on f_{bulk} . The crossover occurs at $\Delta_{\min}$ but then we see a second order zero at $\Delta \approx 6.6$.

It is certainly harder than in the previously considered cases because this time it will be necessary to choose from two different types of operators when upgrading the truncation. Because of this, I first did a low-dimensional linear programming determination of $\Delta_{\min}$ with $\Delta_{\text{ext}} = 0.6$ and $N = 8$, the result of which is shown in Figure 6.2.

Taking these results, i.e. the location of the zeros in the bulk channel, we can try to directly solve a truncated version of the crossing equation via the Newton method:

$$\vec{T} = a_{\text{ext}} \vec{v}_{\text{ext}} + \sum_{i=0}^{n_{\text{bulk}}} a_{\text{bulk}}^{(i)} \vec{F}_{i,\text{bulk}} \left(\Delta_{\text{bulk}}^{(i)} \right) + a_{\text{bdy}}^{(0)} \vec{F}_{i,\text{bdy}} (3) + \sum_{i=1}^{n_{\text{bdy}}} a_{\text{bdy}}^{(i)} \vec{F}_{i,\text{bdy}} \left(\Delta_{\text{bdy}}^{(i)} \right). \quad (6.3.5)$$

The vector signifies that we are truncating the derivatives $\left. \frac{\partial^n}{\partial \xi^n} \right|_{\xi \rightarrow 1}$ to the first N elements. The vector $\vec{T} = (1, 0, \dots, 0)^T$ comes from the contribution of the identity and $\vec{v}_{\text{ext}}$ gives the contribution from $\xi^{\Delta_{\text{ext}}}$.

When using this equation as input and then upgrading the truncation as described in Section 5.4 we must now make guesses about where to insert which type of operator. I observed that there was no discernible pattern of doing so. Worse yet, the increment by which to increase N from one step to the next seems to be unpredictable as well. When choosing the wrong increment and/or wrong guess on the spectrum the algorithm would diverge. In principle, it should be possible to overcome this issue by interfacing the extremal bootstrap with SDP by numerically reading off the double zeros and using that as the initial guess. Since this is outside of the scope of this work, I had to check new spectra for convergence by hand up until $N = 40$ which then imposes a rigorous (under assumption of non-negativity), non-optimal bound of $\Delta_{\text{bulk}} \geq 1.68$. Since we are ultimately interested in finding bounds on the Ising model, as in the famous paper [62],

Figure 6.3: a) The upper bound on $\Delta_{\text{bulk}}^{(1)}$ as a function of Δ_{ext} . The blue line is found using $N = 40$ and starting the flow from $\Delta_{\text{ext}} = 0.6$ while the red line is obtained from $N = 28$ and flowing from $\Delta_{\text{ext}} = 0.5$. Crucially the slopes are the same confirming the absence of a kink that was noted in [53]. The jump comes from the fact that N is different for both sides. b) The dimension $\Delta_n = \Delta_{\text{bdy}}^{(7)}$ of the highest dimensional boundary operator in the extremal spectrum. As it diverges its associated coefficient decays to zero.

I varied the external dimension using the extremal flow, i.e. by linearizing the variation in Eq. (6.3.5) and then approximating the next solution. This works well only up to $\Delta_{\text{ext}} = 0.52$ at which point the flow method stops working, no matter how small of a step size is chosen. The reason for that is a singularity we encounter in the extremal functional as is illustrated in Figure 6.3. Being unable to cross this point we must instead resort to upgrading a new spectrum from the other side of the singularity. I started from $\Delta_{\text{ext}} = 0.5$, but was unable to go beyond $N = 28$ due to the limitations I described before.

6.3.2 Discussion

I have reproduced one of the results of [53] at higher numerical precision by using the extremal functional bootstrap. However, no kink like the one in [62] or [24] was found in the bounds. Such kinks are known to lie at points where the spectrum becomes sparse [36], which generally indicates the location of a physically relevant theory. The authors of that paper offered two explanations for the absence of the kink. The first explanation, namely that the numerics were insufficient appears very unlikely, given that my computations were done at nearly twice the number of derivatives. This gives more credence to their other suggestion, that an unknown solution to crossing could lie above the 3D CFT bound, i.e. a BCFT that violates the CFT bounds.

One very interesting point to note is that the singularity occurs close to $\Delta_{\text{ext}} = 0.518$ where the Ising model lies and where the kink is found when using the 3D equations [62]. In other words, in the boundary approach we do not discover a discontinuity in the bound,

but a discontinuity in the functional that derives it. It is certainly possible that similar phenomena hold in other situations, where a hidden kink could be discovered by using the extremal functional bootstrap rather than the standard approach. To study this question in more detail a rigorous implementation of the extremal bootstrap in higher dimensions is needed, something that should be considered in future work.

One natural question is whether it is possible to extrapolate an asymptotic value $N \rightarrow \infty$ for Δ_{ext} at which the divergence occurs. My low-dimensional results indicate that this can not be done, as this value fluctuates around $\Delta_{\text{bdy}} = 0.518$ seemingly at random as we vary N . I suspect that this is related to the hard-to-predict spectrum when upgrading.

Chapter 7

AdS Bootstrap

In this chapter I want to present a more elaborate example of 1D conformal bootstrap. The motivation for the problem formulation comes, however, from 2D field theory. The basic idea is to consider a massive QFT with a boundary and study its bulk properties by exploiting its conformal symmetry on the boundary.

7.1 Scaling Lee-Yang model

It is a well-known fact that in 2D CFT a special class of theories can be solved exactly by exploiting the infinite number of local symmetries [1]. These models, known as minimal models, can be indexed by two integer parameters and have a finite number of primary operators. Their central charge is given by

$$c_{p,q} = 1 - 6 \frac{(p-q)^2}{pq}, \quad (7.1.1)$$

with $p, q \geq 2$ being coprime integers. The operator dimensions can be found through

$$h_{r,s} = \frac{(pr - qs)^2 - (p - q)^2}{4pq}, \quad (7.1.2)$$

with the integers r and s satisfying $1 \leq r \leq q - 1$ and $1 \leq s \leq p - 1$. In the language that I used so far that means $\Delta = 2h$.

The *Lee-Yang model* [63] corresponding to the $(5, 2)$ -minimal model is one of the simplest examples one can study. We can calculate the central charge $c = -\frac{22}{5}$ and the dimensions of the primary representations $\Delta_{\mathbb{1}} = 0$ and $\Delta_{\phi} = -\frac{2}{5}$. The only non-trivial fusion rule is very simple:

$$\phi \times \phi \sim \mathbb{1} + \phi. \quad (7.1.3)$$

Clearly it is non-unitary since both the central charge and the scaling dimension of ϕ are negative.

Since we are interested in studying a massive theory we will introduce a relevant deformation of ϕ which will lead us to the *scaling Lee-Yang model* (SLYM) [64]. The action is written as

$$\mathcal{A}_{\text{SLYM}} = \mathcal{A}_{\mathcal{M}_{(5,2)}} + ig \int \phi(x) dx^2. \quad (7.1.4)$$

The coupling constant must have units $g \sim (\text{mass})^{\frac{12}{5}}$. In [65] it was shown that this theory is integrable so it can be studied using techniques such as thermodynamic Bethe ansatz to determine the mass spectrum and the scattering matrix. However, this is beyond the scope of this thesis and the numerical value is not actually important here so I will simply quote the result from [66] stating that there is only one massive particle with a mass of

$$m \approx 2.6 \cdot g^{\frac{5}{12}}. \quad (7.1.5)$$

This justifies the convention of including an imaginary factor in the deformation and tells us that the coupling g is real.

7.2 SLYM in AdS

Because of its simplicity the SLYM has been studied in the presence of different boundary conditions [64, 67]. Let us first think of the UV-limit, i.e. the Lee-Yang model. As mentioned in the previous chapter, introducing a straight boundary into a CFT breaks the symmetry down to the conformal symmetry of one fewer dimension. In the case of 2D CFT this means going from $SL(2, \mathbb{C})$ to $SL(2, \mathbb{R})$. A straight boundary at $x = 0$ can be mapped onto AdS by the coordinate transformation

$$dx^2 + dy^2 \rightarrow L^2 \frac{dx^2 + dy^2}{x^2} \quad (7.2.1)$$

through application of the residual symmetry. This does, in fact, preserve the required symmetries in the y -direction and maintains the boundary at $x \rightarrow 0$.

This motivates the study of the previously introduced scaling Lee-Yang model in AdS. This theory can not simply be mapped to a theory in flat space with a boundary, though, because it is not a conformal theory to begin with. However, since we are in AdS the boundary fields do still satisfy an $SL(2, \mathbb{R})$ -symmetry even after the deformation is turned on.

There are two independent dimensionful parameters in the theory. The AdS radius L and the mass m of the particle coming from the deformation ϕ . This means there is only one dimensionless parameter mL . We want to determine the mass spectrum of the theory as a function of this parameter. By identifying bulk states with boundary operators as was done in [68] this can be achieved by studying the 4-point function of identical boundary operators φ , whose dimension Δ_φ is controlled by the dimensionless parameter mL .

7.2.1 UV-Limit

First of all we know what this function looks like in the UV-limit where $mL \rightarrow 0$, so a theory without deformation. By the identification $m^2 L^2 = \Delta(\Delta - 2)$ this implies $\Delta_\varphi = 2$. In this case we are basically dealing with normal boundary conformal field theory as discussed in for example [12] and we can use known results. The 4-point function takes the exact form

$$\langle \varphi(0)\varphi(z)\varphi(1)\varphi(\infty) \rangle = \frac{1}{z^4} + 1 - \frac{8}{cz(1-z)} + \frac{8}{cz^2(1-z)^2} + \frac{1}{z^4(1-z)^4}. \quad (7.2.2)$$

This function can be decomposed into 1D conformal blocks to yield the precise spectrum and the squared coupling constants. In particular, it reveals that the spectrum in the OPE includes only operators with dimension $\Delta_n = 2n$ for $n \in \mathbb{N}_0$. The first few squared structure constants $c_n \equiv \lambda_{\varphi\varphi 2n}$ are

$$\begin{aligned} c_1 &= -\frac{20}{11}, \\ c_2 &= 0, \\ c_3 &= \frac{310}{77}, \\ c_4 &= \frac{1025}{429}, \\ c_5 &= \frac{1550}{2057}, \\ &\vdots \end{aligned} \quad (7.2.3)$$

The expected non-unitarity only appears at precisely the self coupling $\lambda_{\varphi\varphi\varphi}$. All other coefficients respect reality and thus non-negativity of their squares.

7.2.2 IR-Limit

On the other hand, we want to analyze the IR-limit, which corresponds to $mL \rightarrow \infty$ and thus $\Delta_\varphi \rightarrow \infty$. In this limit L becomes much larger than the Compton wavelength of the particle. Therefore, the space looks flat and the boundary effects should become negligible. This means we come back to the known SLYM. As mentioned, this model is integrable so we can write out its S-matrix exactly. This computation was carried out in [65] and can be written as

$$S(\theta) = f_1(\theta)f_2(\theta), \quad (7.2.4)$$

where θ is the particle's rapidity and f_n is a CDD-factor [69] given by

$$f_n(\theta) = \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi n}{6}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi n}{6}\right)}. \quad (7.2.5)$$

Figure 7.1: The analitcal structure of $S(s)$. There are branch cuts at $s > 4m^2$ and at $s < 0$ and first order poles at $s = m^2$ and at $s = 3m^2$.

It satisfies $f_n(\theta)f_n(-\theta) = 1$ and $f_n(i\pi - \theta) = f_n(\theta)$ as required. Starting from the previous observation that the theory breaks unitarity only in its self-coupling it would be interesting to find an analogue of this statement in the S-matrix. This is, in fact, possible through the tools developed in [70]. Without making the discussion very technical, the arguments in that paper can be summarized as follows: assume that the S-matrix is an analytical function of the Mandelstam variables up to poles and branch cuts. In 2D we only need the energy s . There is a branch cut at $s > 4m^2$ because at this point the spectrum becomes continuous and because of the symmetry $S(s) = S(4m^2 - s)$ there is another one at $s < 0$. If there is no particle with $m_1^2 < 4m^2$ the only pole will be due to self-interaction and therefore at $s = m^2$ and again symmetrically at $3m^2$. This is shown in Fig. 7.1. Unitarity now becomes the statement

$$|S(s)| \leq 1, \quad s \geq 4m^2, \quad (7.2.6)$$

around the branch cut plus the reality of the residue of all the poles. These residues give a measure of coupling strength. This must physically be bounded because arbitrarily large couplings should give rise to new bound states and, thus, new poles. This can also be encoded mathematically as described in detail in [70].

Now we can tentatively formulate the minimal breaking of unitarity as having a purely imaginary residue at $s = m^2$ while still satisfying unitarity around the branch cut. This will provide a bound on the self-coupling $|g_{\phi\phi\phi}|$. It turns out that the unique S-matrix that saturates this bound is, when transformed to depend on θ , identical to (7.2.4). Away from the IR-limit, the S-matrix has no meaning anymore, but we might conjecture that the SLYM in AdS continues to satisfy the condition of maximally breaking unitarity in the self-coupling while preserving unitarity in all other couplings. This is now a problem that can be studied using the 1D conformal bootstrap.

Figure 7.2: Linear functionals with different truncations $N = 3$ (blue) and $N = 5$ (orange). Non-negativity was imposed for $\Delta \geq 4$ but both functionals satisfy this for lower values of Δ as well.

7.3 Bootstrap Formulation

At this point the question to be studied is the spectrum in the OPE of a 4-point function of identical scalars with scaling dimension Δ_φ such that the lowest coupling $\lambda_{\varphi\varphi\varphi}^2$ is minimized and all higher coefficients are non-negative. The condition of having no other bound states below $4m^2$ is analogous to the spectrum having a gap between at least Δ_φ and $2\Delta_\varphi$. We can check the conjecture that this spectrum corresponds to the SLYM in AdS at zero coupling, i.e. $\Delta_\varphi = 2$. The crossing equation is given by

$$0 = F_0 + c_1 F_{\Delta_\varphi} + \sum_{\mathcal{O}} c_{\mathcal{O} \in S} F_{\Delta \mathcal{O}}. \quad (7.3.1)$$

Now we can look for a functional Λ such that $\Lambda \cdot F_2 = -1$ while $\Lambda \cdot F_\Delta \geq 0, \forall \Delta \geq 2\Delta_\varphi$. If we can find such a functional then it follows that

$$|c_1| \leq -\Lambda \cdot F_0. \quad (7.3.2)$$

Consequently, maximizing $\Lambda \cdot F_0$ gives the optimal bound. Before implementing this in the framework of the extremal functional method, it is again useful to visualize these low-dimensional functionals as found by simple linear programming. The result of this computation is shown in Fig. 7.2.

In order to construct the extremal functional we need to look at the zeros of this functional. The first feature that stands out is that functional crosses the Δ -axis not at the imposed gap but before. This means that the gap is actually larger than required and

Figure 7.3: The scaling dimensions of the lowest operators that appear in the spectrum as a function of the truncation.

anticipates the result in Eq. (7.2.3) that $c_2 = 0$, thus giving a first confirmation of the conjecture.

7.4 Extremal Functional Implementation

The previous result makes it reasonable that the extremal functional should be of odd dimensionality. One degree of freedom fixes the value c_1 and then there are pairs of variable Δ_n and c_n for $n \geq 3$. Reading off the zeros and c_1 can be error-corrected to very high precision as described in chapter 5 and then be upgraded in increments of $\Delta N = 2$. Running this process up to $N = 99$ we can track the convergence of the scaling dimensions in the spectrum. This is shown in Fig. 7.3.

Although the convergence towards the conjectured spectrum (7.2.3) looks convincing, the speed of the convergence is quite slow. For example, at $N = 99$ we have $\Delta_2 = 6.086$. However, a very precise, though non-rigorous, value can be extrapolated by fitting the convergence as a function of $1/N$ by a high-degree polynomial, as exemplified in [68]. The results for the first few operators are

$\Delta_1 = 2$	$c_1 = -1.8182 \pm 0.0002$
$\Delta_3 = 6.0000 \pm 0.0002$	$c_3 = 4.0260 \pm 0.0004$
$\Delta_4 = 7.9999 \pm 0.0003$	$c_4 = 2.3893 \pm 0.0002$
$\Delta_5 = 10.0000 \pm 0.0004$	$c_5 = 0.7535 \pm 0.0003$
$\vdots$	$\vdots$

Figure 7.4: Flowing the external dimension Δ_φ and extrapolating the result at $N = 79$ gives the deviation of the spectrum from the free theory. In the plot $n = 1$ corresponds to the bottom curve and increases as we count up the curves. As explained in [50] the shown quantity is directly related to the phase shift in $2 \rightarrow 2$ collisions in the bulk. As expected, the high energy states get the lowest corrections.

These numbers were generated through a polynomial extrapolation of order $p = 25$. I estimated the error bars by generating four different extrapolations and rounding up the largest pair-wise deviations. The four extrapolations were $(N = 99, p = 25)$, $(N = 99, p = 10)$, $(N = 49, p = 25)$ and $(N = 49, p = 10)$. While this error estimation generously overestimates the actual error, as evidenced by the high accuracy of the spectrum, it is important to keep in mind that it is completely non-rigorous. However, since in this case the correct data is known, we have some confidence that this scheme is justified.

Since all of the expected values in both the scaling dimensions and the couplings lie within the confidence intervals our previous conjecture seems to be confirmed - at least in the UV. To check it in general it would be necessary to carry out conformal perturbation calculations near Δ_φ . If these results corroborate the conjecture for some intermediate values between the UV and the IR, it is then possible to use conformal bootstrap to solve the entire RG-flow numerically, presumably at much higher precision than other methods. The result of such an intermediate computation is shown in Fig. 7.4. Here the flow method comes in especially handy because we can simply flow in the external dimension Δ_φ as explained in Sec. 5.5.

The plot only goes up to a fairly small Δ_φ because as we increase it, the precision of the method decreases. This problem is well-known in the bootstrap community and can be dealt with by greatly increasing the dimensionality of the truncation. At the time of

writing, I am working on implementing these numerics in order to extract the behavior at large Δ_φ .

Chapter 8

Conclusion

In this work I have given an outline of the conformal bootstrap program in general and the method of extremal functionals in particular. Starting from statistical field theory I motivated the study of conformal invariance and I developed the framework both in theory as well as in practice, discussing numerical details as needed. In particular, I went into great detail in talking about the evaluation of conformal blocks, showcasing 1D calculations as a simple example.

I then discussed the conformal bootstrap in its modern formulation, showing important ways in which it has been generalized and extended in recent years. In particular, I put special emphasis on the technique of extremal functionals.

In the last two chapters I presented two example applications of these techniques in two very different contexts, highlighting the versatility of these methods. In both of these examples the extremal functional bootstrap conferred some benefit either through enabling additional insight or by allowing for more efficient computations.

One important extension is, of course, to expand the applicability of the extremal functional method to higher dimensions. The particularly tricky part is the fact that upgrading is much more difficult if there are several types of operators. This problem was discussed in Ch. 6. In order to circumvent this problem, it would be useful to create an interface between SDPB and the extremal functional method. Only recently has such a tool been developed [71] and it would be interesting for a future project to use this interface in order to speed up calculations in higher dimensions.

Appendices

Appendix A

Conformal Block Evaluation

As shown in Eq. (3.5.17) the conformal block derivatives at the crossing symmetric point can be written as

$$g_{\Delta}^{(m)} = (4\rho)^{\Delta} \left(\sum_{j=0}^{\infty} \frac{R_j^{(m)}}{\Delta + j + \frac{1}{2}} + \sum_{j=0}^{\infty} \Delta^j P_j^{(m)} \right). \quad (\text{A.1})$$

Of course we are specifically interested in calculating the convolved conformal blocks which are given as

$$F_{\Delta_{\phi}, \Delta}(z) = z^{2\Delta_{\phi}} g_{\Delta}(1-z) - (1-z)^{2\Delta_{\phi}} g_{\Delta}(z). \quad (\text{A.2})$$

At $z = \frac{1}{2}$ all even derivatives will vanish so from now it is assumed that m is odd.

$$\begin{aligned} F_{\Delta_{\phi}, \Delta}^{(m)} &\equiv \left. \frac{d^m F_{\Delta_{\phi}, \Delta}}{dz^m} \right|_{z=\frac{1}{2}} \\ &= 2 \sum_{k=0}^m \binom{m}{k} (2\Delta_{\phi} - m + k + 1)_{m-k} \left(\frac{1}{2} \right)^{2\Delta_{\phi} - m + k} (-1)^k g_{\Delta}^{(k)} \\ &= 2^{m-2\Delta_{\phi}+1} \sum_{k=0}^m \underbrace{\binom{m}{k} (2\Delta_{\phi} - m + k + 1)_{m-k} (-2)^{-k}}_{\equiv C_{m,k}(\Delta_{\phi})} g_{\Delta}^{(k)}, \end{aligned} \quad (\text{A.3})$$

where it will be very useful to think of all the coefficients in front of $g_{\Delta}^{(k)}$ as a matrix valued function $C_{m,k}(\Delta_{\phi})$ of the external dimension.

We also find the expression of the target vector just by taking derivatives

$$\begin{aligned} T_{\Delta_{\phi}}^{(m)} &\equiv \left. \frac{d^m}{dz^m} ((1-z)^{2\Delta_{\phi}} - z^{2\Delta_{\phi}}) \right|_{z=\frac{1}{2}} \\ &= -2^{m-2\Delta_{\phi}+1} (2\Delta_{\phi} - m + 1)_m. \end{aligned} \quad (\text{A.4})$$

Since the prefactor $2^{m-2\Delta_{\phi}+1}$ appears in both Eq. (A.3) and Eq. (A.4) we can shave off some extra computation time by just redefining both expressions without the prefactor.

Returning to finding a calculable expression of the convolved conformal blocks we plug our expansion of the conformal blocks from Eq. (3.5.17) into Eq. (A.3) to find

$$\begin{aligned}
 F_{\Delta_\phi, \Delta}^{(m)} &= \sum_{k=0}^m C_{m,k}(\Delta_\phi) g_{\Delta}^{(k)} \\
 &= (4\rho)^\Delta \sum_{k=0}^m C_{m,k}(\Delta_\phi) \left(\sum_{j=0}^{\infty} \frac{R_j^{(k)}}{\Delta + j + \frac{1}{2}} + \sum_{j=0}^{\infty} \Delta^j P_j^{(k)} \right) \\
 &= (4\rho)^\Delta \left(\sum_{j=0}^{\infty} \frac{\tilde{R}_j^{(k)}}{\Delta + j + \frac{1}{2}} + \sum_{j=0}^{\infty} \Delta^j \tilde{P}_j^{(k)} \right),
 \end{aligned} \tag{A.5}$$

where the convolved coefficients $\tilde{R}_j^{(k)} = \sum_{k=0}^m C_{m,k}(\Delta_\phi) R_j^{(k)}$ and $\tilde{P}_j^{(k)} = \sum_{k=0}^m C_{m,k}(\Delta_\phi) P_j^{(k)}$ are calculated just by a matrix multiplication. By noting that the prefactor $(4\rho)^\Delta$ is always positive we can even absorb it into the OPE coefficients.

Altogether we find a really efficient method of calculating the convolved conformal blocks for a list of dimensions $\vec{\Delta} = (\Delta_1, \Delta_2, \dots, \Delta_n)^T$ at fixed Δ_ϕ . First we truncate the sums in Eq. (A.5) to finite orders r and p respectively. Now we collect all odd derivative convolved coefficients into matrices

$$\mathcal{P} = \begin{pmatrix} \tilde{P}_1^{(1)} & \tilde{P}_2^{(1)} & \dots & \tilde{P}_p^{(1)} \\ \tilde{P}_1^{(3)} & \tilde{P}_2^{(3)} & \dots & \tilde{P}_p^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{P}_1^{(2N-1)} & \tilde{P}_2^{(2N-1)} & \dots & \tilde{P}_p^{(2N-1)} \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} \tilde{R}_1^{(1)} & \tilde{R}_2^{(1)} & \dots & \tilde{R}_r^{(1)} \\ \tilde{R}_1^{(3)} & \tilde{R}_2^{(3)} & \dots & \tilde{R}_r^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{R}_1^{(2N-1)} & \tilde{R}_2^{(2N-1)} & \dots & \tilde{R}_r^{(2N-1)} \end{pmatrix}. \tag{A.6}$$

If we now build our list of Deltas $\vec{\Delta}$ into the matrices

$$\mathcal{D}_P = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \Delta_1 & \Delta_2 & \dots & \Delta_n \\ \Delta_1^2 & \Delta_2^2 & \dots & \Delta_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_1^{p-1} & \Delta_2^{p-1} & \dots & \Delta_n^{p-1} \end{pmatrix}, \quad \mathcal{D}_R = \begin{pmatrix} \frac{1}{\Delta_1 + \frac{1}{2}} & \frac{1}{\Delta_2 + \frac{1}{2}} & \dots & \frac{1}{\Delta_n + \frac{1}{2}} \\ \frac{1}{\Delta_1 + \frac{3}{2}} & \frac{1}{\Delta_2 + \frac{3}{2}} & \dots & \frac{1}{\Delta_n + \frac{3}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\Delta_1 + r + \frac{1}{2}} & \frac{1}{\Delta_2 + r + \frac{1}{2}} & \dots & \frac{1}{\Delta_n + r + \frac{1}{2}} \end{pmatrix} \tag{A.7}$$

we get the simple formula

$$\mathcal{F} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ F_{\Delta_\phi, \Delta_1} & F_{\Delta_\phi, \Delta_2} & \dots & F_{\Delta_\phi, \Delta_n} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} = \mathcal{P} \cdot \mathcal{D}_P + \mathcal{R} \cdot \mathcal{D}_R, \tag{A.8}$$

where the arrows indicate that each $F_{\Delta\phi,\Delta n} = (F_{\Delta\phi,\Delta n}^{(1)}, F_{\Delta\phi,\Delta n}^{(3)}, \dots, F_{\Delta\phi,\Delta n}^{(2N-1)})^T$ is a vector.

Since the calculation is based entirely on matrix operations and does not rely on the evaluation of any transcendental functions it is numerically very fast. Unfortunately many bootstrap problems rely on the near cancellation of large numbers which forces us to use arbitrary precision arithmetic. Because these are not hardware accelerated by the BLAS library [72, 73] this corresponds to a slowdown of several orders of magnitude in all calculations.

Appendix B

Code Examples

In this appendix I give some examples of the codes used to generate the results in this paper. The language I used is Julia [74] first and foremost because it has arbitrary precision numerics built into it through the *BigFloat* data type. Julia has also been used in previous bootstrap studies [75].

I assume that the conformal block tables are available to the program at a high precision. They were previously calculated and stored in a text file.

```
# Set precision of BigFloat in bits
const PRECISION = 1000
setprecision(PRECISION)

# Max truncation is given by (ORDER-1)/2 because
# only odd convolved blocks are non-zero
const ORDER = 201

# Number of Poles to be kept improves the
# accuracy of the conformal block approximation
const POLES = 100

# Parsing residues and polynomials
# text file is generated via Mathematica
f = open("residues.txt")
residues = Matrix{BigFloat}(ORDER, POLES)
for i in 1:POLES
    ln = readline(f)
    a = split(ln)
    for j in 1:ORDER
        residues[j,i] = BigFloat(String(a[j]))
    end
end
close(f)
```



```

f = open("polynomials.txt")
polynomials = Matrix{BigFloat}(ORDER, ORDER)
for i in 1:ORDER
    ln = readline(f)
    a = split(ln)
    for j in 1:i
        polynomials[i,j] = BigFloat(String(a[j]))
    end
end
close(f)

```

The next snippet of code defines a set of functions that calculate $C_{m,k}(\Delta_\phi)$ in Eq. (A.3) and $T_{\Delta_\phi}^{(m)}$ in Eq. (A.4). Notice that the convolved blocks are computed at fixed truncation. This is so that during upgrading they do not get recomputed unnecessarily. There is a trade-off, however, because this means that every matrix operation involves some index selection while not working at the maximal truncation allowed for by the order of the approximation.

```

# Evaluates the pochhammer symbol
# The data type Number accepts the BigFloat type as well
function pochhammer(x::Number, n::Integer)
    if n>0
        return (x+n-1)*pochhammer(x, n-1)
    elseif n==0 return 1
    else return pochhammer(x, n+1)/(x+n)
    end
end

# calculates the vector T at external dimension d and truncation N
function calcv(d::Number, N::Integer)
    mvec = (collect(1:2:ORDER))
    return (BigFloat(2).^mvec.*pochhammer.(2*d.-mvec+1, mvec))[1:N]
end

# calculates the coefficients in C at external dimension d
function calccoeff(d::Number, m::Integer, k::Integer)
    return -(-BigFloat(1))^k*BigFloat(2)^(m-k)
        *pochhammer(2*d-m+k+1, m-k)*binomial(big(m), k);
end

# calculates the convolved conformal blocks
function convolveblocks(d::Number)
    C = calccoeff.(d, (collect(1:2:ORDER)), range(0, ORDER)')
    return C*residues, C*polynomials
end

```

Next I actually compute the convolved conformal blocks globally and then use them to define $F_{\Delta_\phi, \Delta}^{(m)}$ and $\partial_\Delta F_{\Delta_\phi, \Delta}^{(m)}$. The matrices are defined globally to save the overhead of passing them to the functions. This gets modified in cases where Δ_ϕ changes.

```

deltaext = BigFloat("1.")
convolvedRes, convolvedPol = convolveblocks(deltaext)

function calcF(Delta::Array{BigFloat,1}, N::Integer)
    Delta1 = BigFloat(1)./(Delta' .+(2*BigFloat.(1:POLES)-1)/2)
    Delta2 = Delta'.^(0:(ORDER-1))
    return convolvedPol[1:N,:]*Delta2 + convolvedRes[1:N,:]*Delta1
end

function calcMD(Delta::Array{BigFloat,1}, N::Integer)
    Delta1 = BigFloat(-1)./(Delta' .+(2*BigFloat.(1:POLES)-1)/2).^2
    Delta2 = (Delta'.^(-1:ORDER-2)).*(0:(ORDER-1))
    Delta2[1,:] = 0
    return convolvedPol[1:N,:]*Delta2 + convolvedRes[1:N,:]*Delta1
end

```

Using all the functions defined so far I write functions for the Newton step and for the entire Newton method. For damping values greater than 1 the convergence slows down but the stability of the method increases.

Since the coefficients are in general all positive and will span a big range of values, convergence can be improved slightly by using $\exp(a)$ instead of a as coefficients. This must be remembered in the end when reading off the correct values (as well as the subtlety mentioned below Eq. (A.5).

```

function refinesolutionstep(x::Array{BigFloat,1}, damping=BigFloat(1))
    N = length(x)
    Delta = x[1:Integer(N/2)]
    a = x[Integer(N/2)+1:N]
    F = calcF(Delta, N)
    MD = calcMD(Delta, N)
    v = calcv(deltaext, N)

    f = F*exp.(a) - v
    J = hcat(MD.*exp.(a)', F.*exp.(a)')

    return x - J\f/damping, norm(f)
end

function refinesolution(x::Array{BigFloat,1},
    damping = BigFloat(1), maxiteration=200)

```

```

error = 1
for i in 1:maxiteration
    if (error < BigFloat("1e-100"))
        break
    end
    x, error = refinesolutionstep(x, damping)
end
return x, error
end

```

Next I define functions that upgrade previous solutions. The first function essentially just guesses the solution at a higher truncation very crudely, so it can be used even when no previous data is available. The second function actually extrapolates the next solution by a linear regression of previous values. This is necessary for higher-dimensional truncations.

```

function naiveupgrade(x::Array{BigFloat,1})
    N2 = Integer(length(x)/2)
    splice!(x, N2 + 1, [x[N2]+3.2, x[N2+1]])
    push!(x, x[2*N2+1]-5)

    return refinesolution(x)
end

function estimatenext(xdata::Array{Array{BigFloat,1},1})
    N = size(xdata)[1]
    deltatrend = [[xdata[i][i-j] for i in (1+j):N for j in 0:(N-2)]
    atrend = [[xdata[i][2*i-j] for i in (1+j):N for j in 0:(N-2)]

    deltareg = [linreg(j:N, deltatrend[j]) for j in 1:(N-1)]
    areg = [linreg(j:N, atrend[j]) for j in 1:(N-1)]

    deltaestimate = [deltareg[i][1]+(N+1)*deltareg[i][2]
        for i in 1:(N-1)]
    aestimate = [areg[i][1]+(N+1)*areg[i][2] for i in 1:(N-1)]

    return vcat(xdata[end][1:2], deltaestimate[end:-1:1],
        xdata[end][(N+1):(N+2)], aestimate[end:-1:1])
end

```

Finally I make an initial guess for a solution and upgrade it to $N = 60$.

```

# Initial guess
x = BigFloat.(["4", "0"])

# The data structure that will hold all intermediate solutions
xdata = Array{Array{BigFloat,1},1}()

```

```
x, error = refinesolution(x)
push!(xdata, copy(x))

# There is no data to be extrapolated yet so I use naiveupgrade
for i in 1:4
    x, error = naiveupgrade(x)
    push!(xdata, copy(x))
end
for i in 12:2:36
    x, error = refinesolution(estimatenext(xdata))
    push!(xdata, copy(x))
end

# To avoid divergences I throttle down the convergence rate
for i in 38:2:60
    x, error = refinesolution(estimatenext(xdata), BigFloat("5"), 20)
    x, error = refinesolution(x)
    push!(xdata, copy(x))
end
```

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