Universidade de São Paulo
Instituto de Física

**PROPRIEDADES DA LUZ EMITIDA POR UM
OSCILADOR PARAMÉTRICO ÓTICO (OPO)
EM CHIP DE SILÍCIO**

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PROPERTIES OF THE LIGHT EMITTED BY A
SILICON ON-CHIP OPTICAL PARAMETRIC
OSCILLATOR (OPO)

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Abstract

The Optical Parametric Oscillator (OPO) has been one of the most versatile source of non-classical states of light. Usual configurations of such devices are a macroscopic second order nonlinear crystals inside an optical cavity. Recently the use of silicon photonics techniques allowed the implementation of high quality factor microcavities and OPOs which include several technological advantages over usual configuration as a small size, bigger bandwidth, CMOS compatibility, facility to engineer the dispersion properties and compatibility with commercial optical fiber communications. Nevertheless the nonlinearity present within these systems is a third order nonlinearity for which theoretical calculations lack in the literature.

Here we describe theoretically the quantum properties of the light generated in an OPO with a third order nonlinearity. We showed that the effects of phase modulation (which are not present in the second order nonlinearity) and dispersion are determinant in the way that oscillation and entanglement is produced in the system. Despite of these effects, bipartite and tripartite entanglement is predicted with the use of the Schmidt modes formalism.

We also describe the system when there are more modes exited within the cavity and a frequency comb is formed. In such a situation, using again the Schmidt modes formalism, multipartite entanglement was predicted as well.

Resumo

O oscilador paramétrico ótico (OPO) tem sido uma fonte muito versátil de estados não clássicos da luz. A configuração usual destes OPOs consiste em um cristal macroscópico com não linearidade de segunda ordem no interior de uma cavidade ótica. Recentemente, devido ao desenvolvimento da fotonica de silício, foi possível a implementação de microcavidades óticas e OPOs que possuem varias vantagens sobre OPOs usuais. Não entanto a não linearidade destes sistemas é de terceira ordem.

Neste trabalho, descrevemos teoricamente as propriedades quânticas da luz gerada num OPO com não linearidade de terceira ordem. Mostra-se que os efeitos de modulação de fase (não presentes na não linearidade de segunda ordem) e a dispersão são determinantes para a geração e o emaranhamento produzido no sistema. Emaranhamento bi e tri partito foi predito teoricamente usando o formalismo de modos de Schmidt.

Também foi feita uma descrição quando mais modos da cavidade são excitados gerando um pente de frequência. Nesta situação, e utilizando novamente o formalismo de modos de Schmidt, foi predito emaranhamento multimodo destes sistemas.

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Introduction

One of the areas of the human knowledge that has more influence in our daily life is computer and information science. Almost every modern human activity uses it in some way or another, and the technologies for information processing and communication play a major role in contemporary society.

The development of computer science has been extremely fast, boosted by the invention of the transistor in the late 1950's and the development of integrated circuit technologies. Exponential growth of computational power has persisted over decades, mostly due to the miniaturization of the transistors which are based on semiconductor technology (this behavior is known as Moore's "law" which states that the number of transistors on an integrated chip doubles approximately every two years). Nowadays, transistors on chips are fabricated at a scale of around a few tens of nanometers. But if this scale is reduced even further (which is necessary to follow Moore's "law" and to increase the power of computation in the same rate as until now), classical physics ceases to be a suitable theory to describe computation processes and it is necessary to take into account quantum mechanical effects ¹ like quantum tunneling and interference effects among others.

But although quantum mechanical effects are a limitation to the fabrication of smaller standard microprocessors they are not, in principle, a limitation for the construction and the improvement of more powerful computing devices. Instead, it could be quite the opposite. Using the properties of quantum mechanics, a totally new and promising way of computation is arising. In this way, quantum properties such as the superposition of states, quantum interference and entanglement are used to develop much more efficient algorithms which can have even an exponential speed up² when compared to the best of

¹Although quantum mechanics is the underlying theory that describes the behavior of semiconductors (materials the transistors are made of), the computation itself is completely classical in the sense that the information is encoded, processed and transmitted by currents and/or voltages that can be described by the classical theory.

²This exponential speed up is characterized in terms of computational complexity, when an algorithm that solves a problem with an exponential number of steps in the size of the input is replaced with another algorithm that solves it with a polynomial number of steps.

their classical counterparts, and to transmit and share information in safer ways. A new field of investigation, called *Quantum Information* science is born.

Among the first ones to propose to use quantum systems as computers was Richard Feynman in 1982 [1]. He proposed the most straightforward application and also one of the most interesting, that was to use controllable, stable and well known quantum systems to simulate another quantum system which is not so well known. The simulation on a classical computer, should require an exponential amount of classical resources as the number of particles increases. In this field, much progress has been achieved, such as the calculation of the spectrum of the H_2 molecule using linear optics [2], simulations of quantum phase transitions using optical lattices [3], or the use of trapped ions to simulate different physical systems as in [4–7], among others.

In classical modern computation, the information is encoded in the so-called bit, which is a binary entity that takes one of two possible states that are named 0 or 1. In contrast, in the binary version of quantum computation the information is encoded in the so-called quantum bit, which is a two level quantum system labeled as $|0\rangle$ and $|1\rangle$ that, as allowed by quantum mechanics, can be in the more general superposed state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ [8]. The possibility of using quantum properties for the formulation of algorithms has been shown to potentially provide great advantage over its classical counterpart, for instance, in Shor’s algorithm [9] (experimentally implemented in 2001 [10]) for integer factorization³ or in Grover’s algorithm [11] for searching on a data base, where quantum properties provide great advantage over the most efficient classical algorithms proposed until now. Furthermore, in the area of quantum cryptography, protocols as the Quantum Key Distribution (QKD) [12, 13] also exploit quantum properties to secretly share information in a safer way.

The approach for quantum computation described above is called quantum information over discrete variables (DV) where the physical systems have a finite number d of accessible states. These systems are called *qudits* (the case $d = 2$ described before is called *qubit*). Examples of these qudits can be the polarization of photons, spin 1/2 systems as electrons, or setups where just a few internal states of atoms or molecules are available. This kind of quantum computation was the first to be developed, maybe because its direct parallel with digital classical computation predominant nowadays. The protocols and algorithms mentioned before are all over discrete variable systems.

Another approach, somehow comparable with the classical analogue computation, is the so called quantum information over continuous variables (CV) where the physical systems

³One of the most used cryptosystems, the so-called RSA, is based on the computational difficulty to perform integer factorization of big numbers. The construction of a technological practical quantum computer would mean a big security breach. But the solution for this problem could also be in the quantum domain through quantum cryptography technologies.

have an observable with continuous spectrum of states i.e., infinite dimensional systems as, for example, the position and momentum of a particle or the quadratures of a mode of the electromagnetic field. The development of CV quantum information has lagged behind the DV approach, but several advances on CV systems have been made, as for example the experimental implementation of unconditional quantum teleportation [14] and CV versions of quantum key distribution [15] among others.

In a practical implementation of a quantum computer, it may be advantageous to have different physical systems that take care of different computational tasks connected by communication channels. This is the idea of a quantum network proposed by Kimble [16], where every node of the network represents one of those physical systems, each one optimized for a particular assignment and all of them linked by quantum channels communicating information between the nodes. The nodes can be, for instance, optical setups [17] (linear and nonlinear), atomic systems [18] (cold atoms or hot vapors), nitrogen vacancy center in diamond [19, 20], superconducting devices [21], optical lattices [22], trapped ions [23], quantum dots [24], Bose-Einstein condensates [25], or any other suitable physical system for a certain task. The quantum channels can be provided by light⁴ propagating in free space or through optical fiber (which are low loss media for light).

In this area, the *Laboratório de Manipulação Coerente de Átomos e Luz* (LMCAL)⁵ has made consistent progress working in the implementation of protocols of quantum communication and quantum information, as the first generation of multicolor CV entanglement [26, 27] using parametric down conversion with an OPO built with a second order ($\chi^{(2)}$) nonlinear crystal. With that source of entangled states, the group is working to use it as a tool for quantum communication (the links on the quantum network), performing protocols as entanglement swapping or quantum state teleportation between the different frequencies in which each node of a quantum network works on. For instance, an experiment is underway at LMCAL in which an OPO is pumped with a laser at 780nm, a wavelength resonant with the D1 line of rubidium atoms, generating a signal and idler field around 1560nm which corresponds to the C band transmission window in fiber-optic communications and also compatible with silicon based on-chip micro-resonators (see Figure 1). In this way, quantum communication between two different physical systems can be performed. Light interaction with Rubidium atoms is another research line that is being investigated at LMCAL, and which I was involved with, mainly on the characterization of the coherence properties of those systems as presented in the appendix B. The study of silicon based on-chip optical microresonators is a recently opened

⁴Light is considered the best entity for communication because it is fast and has low interaction with the environment which translates in low decoherence.

⁵Brazilian name which stands for Laboratory of Coherent Manipulation for Atoms and Light

research line which is being developed at the LMCAL in collaboration with the Professor Michal Lipson at Columbia University and on which this work is focused. Silicon based

FIGURE 1: Proposal for a proof of principle for a quantum network at the LMCAL. The OPO is used as the quantum channel between rubidium atoms and silicon-based optical microcavities. At the top the configuration is such as the reflected pump is compatible with the atoms while the twins beam are compatible with the microcavities. As the strongest correlations on this system are between the twin beams, at the bottom there is another configuration where one of the twin beam is compatible with the atoms while the other one is with the micro cavities.

on-chip OPOs, that are the main object of study in this text, have several technological advantages compared with other physical systems for the implementation of quantum information devices, starting by their size, of the order of the μm . They also rely on lithography technology for fabrication, which is the method used for the construction of basically all the information technologies nowadays. That gives a great control and liberty for the design of the required geometries, and also it makes possible to fabricate optical cavities with extremely high quality factors (of the order of 10^6 and more). As mentioned before, these devices work at the conventional C band for fiber communication, making them compatible with the current technologies and also the possibility of being integrated within optical system circuits.

Within these chips, the third order non linearity ($\chi^{(3)}$) is used to generate a pair of signal and idler fields from a pump field through the process of four wave mixing (FWM), where two photons of the pump field are annihilated and, simultaneously, a pair of signal-idler photons are created. Quantum properties, (in this case, intensity squeezing) was

experimentally obtained recently [28], as a first step to demonstrate entanglement. If the pump field is strong enough, secondary cascaded FWM processes can occur, where the original signal and idler fields can act as the pump in a new FWM process. They can also combine with the pump, again through FWM, and generate several other pairs of signal-idler fields, forming a frequency comb from a single pumped mode. Experimental realizations of frequency combs on silicon based chips was also recently demonstrated [29].

This device can be a potential scalable generator of quantum cluster states (the number of excited modes can be modified by the pump power and detuning), the fundamental resource for the so-called one way quantum computation [30], a type of measurement-based computation where the computation is done by repeatedly measuring subsystems from a highly entangled state, the cluster state, instead of performing controllable unitary operations over the system, which can be, in general, difficult to carry out. In one way quantum computation, the difficulty of performing these unitary operations accurately over a physical system is translated into the difficulty to create a big enough cluster state, i.e., in the physical implementation of a scalable source of cluster states. A silicon based on-chip micro-resonator may be a solution to this problem.

The aim of this work is to study, within the framework of quantum information and quantum networks described before, the properties of light generated from a silicon based on chip OPO, focusing on the quantum properties such as entanglement and its distribution over the modes of on-chip microcavities.

This text is organized in the following way. In chapter 1 a short introduction to the concepts of quantum mechanics that will be used through the whole text is presented, such as the uncertainty principle and the quantization of the electromagnetic field. In chapter 2, selected topics on classical and quantum nonlinear optics, necessary to describe the on-chip OPO, are briefly exposed and in chapter 3 the formal concept of entanglement is introduced and certain separability criteria for continuous variables are described. Finally, in chapter 4, the description of the properties, classical and quantum, of the light generated on an OPO with $\chi^{(3)}$ media is discussed. At the beginning we focus on three mode interaction (pump, signal and idler) comparing it with the corresponding process in a $\chi^{(2)}$ OPO and afterwards we analyze the correlations generated on the multimode system (frequency comb) produced by cascaded FWM processes.

Chapter 1

Basic Topics of Quantum Mechanics

The main goal of this chapter is to introduce basic concepts of the theory of quantum mechanics as a theory that describes the physical properties of the systems in a fundamental level and with a grater accuracy than classical mechanics does. The concepts of this theory, as well as the notation that will be introduced here, will be used throughout this work, which justifies the existence of this chapter. Other explanations with more details and/or with different approaches can be found in any standard introductory book of quantum mechanics as [31–33].

1.1 Introduction

Classical mechanics describes the state of a physical system through a set of variables $\mathbf{X} = \{p_1, \dots, p_n, q_1, \dots, q_n\}$ which represents a point in the so-called phase space and where n is the dimension or the number of degrees of freedom of the system. The pair p_i, q_i describes the i th degree of freedom and are called conjugate variables of each other. The dynamics of the system is given by a function $H(\mathbf{X}, t)$ called Hamiltonian¹, through the so-called Hamilton equations

$$\begin{aligned}\frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i} \\ \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}.\end{aligned}\tag{1.1}$$

Then, given an initial condition $\mathbf{X}_0$ and the Hamiltonian of the system, we can have all the physical information of the system for all times through the integration of the

¹Generally, when written in an appropriate coordinate system, the Hamiltonian corresponds to the energy of the system.

Hamilton equations, due to the fact that every observable (physical quantity) of the system should be a function of $\mathbf{X}(t)$.

In quantum mechanics, these observables are promoted to linear hermitian operators which act on a complex vector space that also have the properties of a Hilbert space. The state of the system is given by a normalized vector $|\psi\rangle$ in this Hilbert space. The evolution of this state is ruled by the Schrödinger equation

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}(t) |\psi(t)\rangle, \quad (1.2)$$

where $\hbar$ is the reduced Planck constant and $\hat{H}(t) = \hat{H}(\hat{\mathbf{X}}, t)$ is the Hamiltonian operator. The operators that form the vector $\hat{\mathbf{X}} = \{\hat{p}_1, \dots, \hat{p}_n, \hat{q}_1, \dots, \hat{q}_n\}$ fulfill the canonical commutation relation

$$\begin{aligned} \hat{q}_i \hat{p}_j - \hat{p}_j \hat{q}_i &\equiv [\hat{q}_i, \hat{p}_j] = i\hbar \delta_{i,j} \\ [\hat{q}_i, \hat{q}_j] &= [\hat{p}_i, \hat{p}_j] = 0. \end{aligned} \quad (1.3)$$

Since the vector $|\psi\rangle$ represents the state of the system, all the physical information should be contained within it. The quantities that are experimentally accessible are the mean values of a given observable. In quantum mechanics these observables are represented by Hermitian operators and their mean value is defined as:

Let $\hat{A}$ be an Hermitian operator, its mean value at a time t over the state $|\psi(t)\rangle$ is defined as

$$\langle \hat{A} \rangle(t) = \langle \psi(t) | \hat{A} | \psi(t) \rangle \quad (1.4)$$

where $\langle \phi | \psi \rangle$ is the inner product on the Hilbert space between the vectors $|\psi\rangle$ and $|\phi\rangle$. For every measurement of the observable $\hat{A}$, the only possible outcomes are given by its eigenvalues a_n

$$\hat{A} |a_n\rangle = a_n |a_n\rangle, \quad (1.5)$$

(with $|a_n\rangle$ the corresponding eigenvector) with a probability $P_n(t) = |\langle a_n | \psi(t) \rangle|^2$. This justifies the definition (1.4) because, writing the state $|\psi\rangle$ in the basis of eigenvectors of $\hat{A}$ as $|\psi(t)\rangle = \sum_n \langle a_n | \psi \rangle |a_n\rangle$ (assuming that the set $\{|a_n\rangle\}$ forms an orthonormal basis of the Hilbert space), we have that $\langle A \rangle(t) = \sum_n a_n P_n(t)$.

In the former definitions we have been working in the so-called Schrödinger picture, where the state vector evolves over time through the Schrödinger equation (1.2), meanwhile the

operators stay fixed in time. Nevertheless there is another formalism where the vector state is fixed and are the operators which evolve in time. This formalism is called the Heisenberg picture. We can define the time evolution operator $\hat{U}(t)$, that relates the initial state $|\psi(0)\rangle$ at the time $t = 0$ with the state at an arbitrary time t , $|\psi(t)\rangle$ through

$$|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle. \quad (1.6)$$

With this definition, and the Schrödinger equation (1.2) we can deduce that the time evolution operator satisfies the equation

$$i\hbar \frac{d\hat{U}(t)}{dt} = \hat{H}(t)\hat{U}(t), \quad (1.7)$$

together with the initial condition $\hat{U}(0) = \hat{\mathbb{I}}$, with $\hat{\mathbb{I}}$ the identity operator. The solution, in the case that $[\hat{H}(t), \hat{H}(t')] = 0$, is given by ²

$$\hat{U}(t) = \exp \left[-\frac{i}{\hbar} \int_0^t \hat{H}(t') dt' \right]. \quad (1.8)$$

Using equation (1.6) on the definition of the mean value (1.4) we have that

$$\langle \hat{A} \rangle(t) = \langle \psi(0) | \hat{U}^\dagger(t) \hat{A}_S \hat{U}(t) | \psi(0) \rangle \equiv \langle \psi(0) | \hat{A}_H(t) | \psi(0) \rangle, \quad (1.9)$$

where the time dependent operator has been defined as $\hat{A}_H(t) \equiv \hat{U}^\dagger(t) \hat{A}_S \hat{U}(t)$, and where the subscripts H and S stand for the Heisenberg and Schrödinger pictures respectively. This operator fulfills the so called Heisenberg equation

$$i\hbar \frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H(t), \hat{H}(t)]. \quad (1.10)$$

In this formalism we can calculate more general mean values like, for instance, mean values of the kind $\langle \hat{A}(t) \hat{B}(t') \rangle$ with $t \neq t'$ which are related with a two time correlation of the observables $\hat{A}$ and $\hat{B}$ and can not be calculated on the Schrödinger picture.

There exists a third formalism that can be thought as intermediary between Schrödinger and Heisenberg pictures that is called the interaction or Dirac picture. In this picture we have a total Hamiltonian that can be decomposed as

$$\hat{H} = \hat{H}_0 + \hat{H}_I, \quad (1.11)$$

²It is possible to solve the equation in the case that the Hamiltonian does not commute with itself for different times, which is done using the time ordering operator. A description of this procedure can be found in several standard text books, for instance [34]

where, for the usefulness of the formalism, $\hat{H}_0$ is a well known Hamiltonian for which its eigenvectors and eigenvalues are known, and $\hat{H}_I$ is an interaction or a perturbation that, in the most general case, is difficult to diagonalize. With this decomposition it is defined a time evolution operator $\hat{U}_0(t)$ as a function of just the Hamiltonian $\hat{H}_0$ in the following way

$$\hat{U}_0(t) = \exp \left[-\frac{i}{\hbar} \int_0^t \hat{H}_0(t') dt' \right]. \quad (1.12)$$

Then, the states and operators in the Dirac picture (both time dependent) are, in terms of the operators on the Schrödinger picture,

$$|\psi_D(t)\rangle = \hat{U}_0^\dagger(t) |\psi_S(t)\rangle \quad (1.13)$$

$$\hat{A}_D(t) = \hat{U}_0^\dagger(t) \hat{A}_S \hat{U}_0(t). \quad (1.14)$$

Here the subscripts S and D stand for the Schrödinger and Dirac picture respectively. In the Dirac picture, the time evolution of the operators is ruled by the Hamiltonian $\hat{H}_{0D} = \hat{H}_0$ through the equation

$$i\hbar \frac{d}{dt} \hat{A}_D(t) = [\hat{A}_D(t), \hat{H}_0], \quad (1.15)$$

and the evolution of the states is ruled by the Hamiltonian $\hat{H}_{ID} = \hat{U}_0^\dagger(t) \hat{H}_I \hat{U}_0(t)$ by the modified Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi_D(t)\rangle = \hat{H}_{ID}(t) |\psi_D(t)\rangle. \quad (1.16)$$

The advantage of the interaction formalism is that the "trivial" evolution given by $\hat{H}_0$ is removed from the Schrödinger equation and it is left in the time evolution of the operators.

1.2 The Density Operator

Very often we find situations when it is not possible to know exactly which one is the vector state of the system, for instance, when we deal with a great amount of particles, or when we have loss of information due to that we are dealing with a quantum open system and also due to uncontrollable imperfections in the experimental preparation of the state. In those cases it is possible to know that the system is in a specific vector state $|\psi_i\rangle$ with some probability p_i . Then, whenever we want to calculate the mean value of an observable, this has to be taken into account. So, the equation (1.4) should be modified

in the following way

$$\langle \hat{A} \rangle(t) = \sum_i p_i \langle \psi_i(t) | \hat{A} | \psi_i(t) \rangle = \sum_i p_i \langle \psi_i(0) | \hat{A}(t) | \psi_i(0) \rangle. \quad (1.17)$$

Let the set $\{|n\rangle\}$ be an orthonormal basis of the Hilbert space, then the identity operator can be written as $\hat{\mathbb{I}} = \sum_n |n\rangle \langle n|$. Using this, the equation above can be written in the following way

$$\begin{aligned} \langle \hat{A} \rangle &= \sum_i p_i \langle \psi_i | \hat{A} \cdot \hat{\mathbb{I}} | \psi_i \rangle = \sum_{i,n} p_i \langle \psi_i | \hat{A} | n \rangle \langle n | \psi_i \rangle = \sum_n \langle n | \left(\sum_i p_i | \psi_i \rangle \langle \psi_i | \right) \hat{A} | n \rangle \\ &= \text{Tr}\{\hat{\rho} \hat{A}\}. \end{aligned} \quad (1.18)$$

In the former equation was defined the density operator $\hat{\rho}$ as $\hat{\rho} = \sum_i p_i | \psi_i \rangle \langle \psi_i |$, where we have that $\sum_i p_i = 1$ (all the probabilities add up to 1). The density operator is a more general way for representing the state of a system as a classical probability distribution p_i (an ensemble) of different quantum vector states.

In the Schrödinger picture, it is the density operator which evolves with time, thus, using the time evolution operator (1.6) we have

$$\hat{\rho}(t) = \sum_i p_i | \psi_i(t) \rangle \langle \psi_i(t) | = \hat{U}(t) \left(\sum_i p_i | \psi_i(0) \rangle \langle \psi_i(0) | \right) \hat{U}^\dagger(t) = \hat{U}(t) \hat{\rho}(0) \hat{U}^\dagger(t).$$

The evolution of this operator is given by the so called Liouville-von Neumann equation

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = [\hat{H}(t), \hat{\rho}(t)], \quad (1.19)$$

which can be easily deduced from equation (1.7).

Every density operator must satisfy the following properties:

- It is a positive semidefinite operator; that is, for every vector $|\phi\rangle$ we have that $\langle \phi | \hat{\rho} | \phi \rangle \geq 0$ and thus, all its eigenvalues must be equal to or greater than zero. This is necessary for being consistent with the probabilistic interpretation of the density operator (positive probabilities).
- $\hat{\rho}$ has unitary trace, i.e. $\text{Tr}\{\hat{\rho}\} = 1$. This means that the sum of all the probabilities add up one.
- $\text{Tr}\{\hat{\rho}^2\} \leq 1$ and $\text{Tr}\{\hat{\rho}^2\} = 1$ if and only if the density operator is of the form $\hat{\rho} = |\psi\rangle \langle \psi|$ (which is known as a pure state).

Given any orthonormal basis of the Hilbert space $\{|\varphi_i\rangle\}$ the more general form that the density operator can take is

$$\hat{\rho} = \sum_{i,j} p_{i,j} |\varphi_i\rangle \langle \varphi_j|. \quad (1.20)$$

In the former equation the coefficients $p_{i,i} \geq 0$ are called populations, because they represent the probability that the system is on the vector state $|\varphi_i\rangle$, and the coefficients $p_{i,j}$ $i \neq j$ (in general, complex numbers) are called coherences because they carry information about the possibility of having quantum interference between the states $|\varphi_i\rangle$ and $|\varphi_j\rangle$, i.e. that the state of the system is composed, at least in some proportion, as a coherent superposition of those states.

1.3 Uncertainty Relations And Covariance Matrix

Due to the fact that the quantum theory is an intrinsically probabilistic theory, we will find that the outcomes of different measurements of the same observables have fluctuations, i.e. deviations from the mean values, that are also known as noise (even in the hypothetical case that we could have perfect measurements devices). This noise comes from the quantum character of nature (assuming that quantum mechanics describes the nature in a fundamental way), and it is impossible to remove it from the measurements by developing more accurate experimental techniques. Not just that, as long as the theory is built on a base of non-commuting operators, some relations are imposed between the noises of different quantities which are called uncertainty relations.

The demonstration of the uncertainty relations shown here is, somehow, different from the typical demonstration that can be found on a standard introductory text book of quantum mechanics [31–33]. The aim of doing this demonstration in that way is to have an early introduction to the covariance matrix that will be very important in the next sections. This demonstration is based on the positivity of this covariance matrix.

Let $\hat{A}$ and $\hat{B}$ be two Hermitian operators which correspond to two different observables, it is possible to construct a vector with this operators as $\hat{\mathbf{X}} = \{\hat{A}, \hat{B}\}^T$ and the covariance matrix of these operators is defined as

$$\mathbb{C} = \langle \delta \hat{\mathbf{X}}_i \delta \hat{\mathbf{X}}_j \rangle = \begin{pmatrix} \langle \delta \hat{A} \delta \hat{A} \rangle & \langle \delta \hat{A} \delta \hat{B} \rangle \\ \langle \delta \hat{B} \delta \hat{A} \rangle & \langle \delta \hat{B} \delta \hat{B} \rangle \end{pmatrix} \quad (1.21)$$

where the operator $\delta \hat{Y}$, defined as $\delta \hat{Y} = \hat{Y} - \langle \hat{Y} \rangle$, is the fluctuation around the mean value. The matrix $\mathbb{C}$ is positive semidefinite, which is indicated as

$$\mathbb{C} \geq 0. \quad (1.22)$$

This property can be easily shown defining the operator $\hat{Y} = z_1 \delta \hat{A} + z_2 \delta \hat{B}$ with z_1, z_2 being any complex numbers, and by the positivity of $\hat{\rho}$ ³

$$\langle \hat{Y}^\dagger \hat{Y} \rangle \geq 0, \quad (1.23)$$

then

$$\langle \hat{Y}^\dagger \hat{Y} \rangle = \sum_{i,j} z_i^* z_j \langle \delta \hat{X}_i \delta \hat{X}_j \rangle = \mathbf{z}^\dagger \cdot \mathbb{C} \cdot \mathbf{z} \geq 0, \quad (1.24)$$

where $\mathbf{z} = \{z_1, z_2\}^T$, which means that $\mathbb{C}$ is a positive semidefinite matrix. Given that, we have that the determinant of $\mathbb{C}$ must be greater or equal to zero, which finished the demonstration in the following way [35]

$$\det \mathbb{C} = \langle \delta \hat{A}^2 \rangle \langle \delta \hat{B}^2 \rangle - \left| \langle \delta \hat{A} \delta \hat{B} \rangle \right|^2 \geq 0 \quad (1.25)$$

And writing $\delta \hat{A} \delta \hat{B} = \frac{1}{2} \left(\left\{ \delta \hat{A}, \delta \hat{B} \right\} + \left[\delta \hat{A}, \delta \hat{B} \right] \right)$ where the part with the commutator (anti-commutator) is an anti-Hermitian (Hermitian) operator and then its mean value is a pure imaginary (real) quantity. Thus $\left| \langle \delta \hat{A} \delta \hat{B} \rangle \right|^2 = \frac{1}{4} \left(\left| \langle \left\{ \delta \hat{A}, \delta \hat{B} \right\} \rangle \right|^2 + \left| \langle \left[\delta \hat{A}, \delta \hat{B} \right] \rangle \right|^2 \right)$, which leads to the Robertson-Schrödinger uncertainty relation

$$\langle \delta \hat{A}^2 \rangle \langle \delta \hat{B}^2 \rangle \geq \frac{1}{4} \left(\left| \langle \left\{ \delta \hat{A}, \delta \hat{B} \right\} \rangle \right|^2 + \left| \langle \left[\delta \hat{A}, \delta \hat{B} \right] \rangle \right|^2 \right). \quad (1.26)$$

As a particular case, when we have $\hat{A} = \hat{q}$ and $\hat{B} = \hat{p}$, due to the canonical commutation relation (1.3), the relation above take the form

$$\begin{aligned} \langle \delta \hat{q}^2 \rangle \langle \delta \hat{p}^2 \rangle &\geq \frac{1}{4} \left(\left| \langle \{ \delta \hat{q}, \delta \hat{p} \} \rangle \right|^2 + \hbar^2 \right), \\ \text{which implies } \langle \delta \hat{q}^2 \rangle \langle \delta \hat{p}^2 \rangle &\geq \frac{\hbar^2}{4}. \end{aligned} \quad (1.27)$$

This last line is known as the Heisenberg uncertainty relation. Where the anti-commutator term is not taken into account because it is state dependent.

³Note that, due to that $\hat{\rho}$ is positive semidefinite, there exists an orthonormal base $\{|\varphi_i\rangle\}$, in which it can be diagonalized as $\hat{\rho} = \sum_i p_i |\varphi_i\rangle \langle \varphi_i|$ with $p_i \geq 0$ and $\sum_i p_i = 1$. And defining $|\psi_i\rangle = \hat{Y} |\varphi_i\rangle$, for any operator $\hat{Y}$, we have that, $\langle \hat{Y}^\dagger \hat{Y} \rangle = \sum_i p_i \langle \psi_i | \psi_i \rangle \geq 0$.

For one dimensional systems the uncertainty relation (1.26) is enough for a operator $\hat{\rho}$ be a *bona fide* density operator, but for bigger dimensions, relation (1.22) with $\hat{\mathbf{X}}$ containing all the position and momentum operators is a stronger relation than (1.26).

1.4 Quantization of the Electromagnetic Field

We are interested in describing quantum properties of the light, therefore, in this section, a description of the canonical quantization procedure of the electromagnetic field is done.

Procedures of field quantization as the one done here can be found in any standard book of quantum electrodynamics as [36, 37] which consist in the rigorous method of canonical quantization of any field theory. This procedure consists of beginning with a classical Lagrangian formulation of the field theory, and through it, finding the pairs of canonically conjugated fields in order to formulate the corresponding Hamiltonian theory and then promoting the field to operators which fulfill and specific canonical commutation relations.

A more "phenomenological" procedure, that in general is enough for many proposes, can be found in many quantum optics text books as [38–40] among others. The aim of introducing this more formal procedure is to apply the same method when we deal with the problem of the quantization of the electromagnetic field in nonlinear media in chapter 2.

1.4.1 Basic Topics of Classical Electrodynamics and Maxwell's Equations

Quantum electrodynamics (QED), is one of the most accurate and successful physical theory developed until now. But in order to have a quantum description of the electromagnetic field, it is necessary to start with a classical description and then to proceed with the canonical quantization of the field.

In the classical theory, the electromagnetic field is described by a two real vector fields, the electric and magnetic field, that satisfy the so-called Maxwell's equations ⁴

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\rho(\mathbf{r}, t)}{\epsilon_0} \quad (1.28)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \quad (1.29)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) + \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = 0 \quad (1.30)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} = \mu_0 \mathbf{J}(\mathbf{r}, t) \quad (1.31)$$

where $\mathbf{B}(\mathbf{r}, t)$, $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{J}(\mathbf{r}, t)$ and $\rho(\mathbf{r}, t)$ are the magnetic, the electric, and the density of current fields, and the density of charge respectively, c is the velocity of light, and μ_0 and ϵ_0 are the permeability and permittivity of free space respectively.

Given a set of point like charged particles with positions $\mathbf{r}_i(t)$ and velocities $\mathbf{v}_i(t)$, the fields $\mathbf{J}(\mathbf{r}, t)$ and $\rho(\mathbf{r}, t)$ are given by

$$\mathbf{J}(\mathbf{r}, t) = \sum_i q_i \mathbf{v}_i \delta(\mathbf{r}_i(t) - \mathbf{r}); \quad \rho(\mathbf{r}, t) = \sum_i q_i \delta(\mathbf{r}_i(t) - \mathbf{r}), \quad (1.32)$$

where $\delta(x)$ is the Dirac's delta function.

The Maxwell's equations together with the equation of motion of the charges given by the Lorentz force,

$$m_i \frac{d^2 \mathbf{r}_i}{dt^2} = q_i (\mathbf{v}_i \times \mathbf{B}(\mathbf{r}_i(t), t) + \mathbf{E}(\mathbf{r}_i(t), t)), \quad (1.33)$$

describe, in principle, all the classical electrodynamics⁵.

The homogeneous Maxwell's equations (1.29) and (1.30) are trivially solved if we introduce the so-called scalar and vector potentials $A_0(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ defined by

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) \quad (1.34)$$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla A_0(\mathbf{r}, t) - \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t). \quad (1.35)$$

Then, the definitions of the Maxwell's equation with sources (1.28) and (1.31) take the form

⁴For a more extensive and complete classical description of the electromagnetic field, the reader can refer to text books like [41–43].

⁵The equation of motion for the particles written here is given by the Newton equation, which is valid only for non-relativistic particles

$$\nabla^2 A_0(\mathbf{r}, t) + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}(\mathbf{r}, t)) = -\frac{\rho(\mathbf{r}, t)}{\epsilon_0} \quad (1.36)$$

$$\left(\nabla^2 \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} \right) - \nabla \left(\nabla \cdot \mathbf{A}(\mathbf{r}, t) + \frac{1}{c^2} \frac{\partial A_0(\mathbf{r}, t)}{\partial t} \right) = -\mu_0 \mathbf{J}(\mathbf{r}, t). \quad (1.37)$$

Introducing the potentials explicitly reduces the degrees of freedom from 6 (each one of the three components of the electric and magnetic fields) to just 4 (the three components of the vector potential and the scalar potential). This shows that the electric and magnetic fields are just one physical entity, the electromagnetic field, and not two independent quantities. Physically we can see this fact in, for example, the context of special relativity, when we can have in one inertial reference frame just electric field, but when we change to another frame, we can have also magnetic field. Additionally, as we will see later, on quantum electrodynamics, the photon is defined as the quantum excitation of the whole electromagnetic field.

At the moment we will focus on the quantization of the free field, i.e. when no sources are present ($\mathbf{J}(\mathbf{r}, t) = 0$ and $\rho(\mathbf{r}, t) = 0$). In this case equations (1.36) and (1.37) or equivalently the Maxwell equations (1.28) and (1.31) can be obtained from the Euler-Lagrange equations

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial (\partial_t \Psi)} \right) + \sum_{j=1}^3 \partial_j \left(\frac{\partial \mathcal{L}}{\partial (\partial_j \Psi)} \right) - \frac{\partial \mathcal{L}}{\partial \Psi} = 0 \quad (1.38)$$

(Ψ stands for any field that the Lagrangian depends on, like A_0 or any of the components of $\mathbf{A}$.) from the following Lagrangian density

$$\mathcal{L}(A_0, \mathbf{A}, \partial_\mu A_0, \partial_\mu \mathbf{A}) = \frac{\epsilon_0}{2} (\mathbf{E}^2 - c^2 \mathbf{B}^2) \quad (1.39)$$

where the subscript μ stands for the time or one of the Cartesian component ($\mu = \{t, x, y, z\}$) and the dependence of the Lagrangian $\mathcal{L}$ with $\mathbf{A}$, A_0 and their derivatives is given through the definitions (1.34) and (1.35). For $\Psi = A_0$, equation (1.38) give us the Gauss's law (1.28) and for $\Psi = \mathbf{A}_j, j = x, y, z$, it give us the Ampere-Maxwell's law (1.31).

Having found the correct Lagrangian density that gives us the Maxwell's equations through the Euler-Lagrange equation, we can proceed to formulate a Hamiltonian theory of the electromagnetism. In order to do that we begin by finding the canonical conjugate variables of the fields A_0 and $\mathbf{A}$ by

$$\begin{aligned} \Pi_0 &= \frac{\partial \mathcal{L}}{\partial (\partial_t A_0)} = 0 \\ \boldsymbol{\Pi} &= \frac{\partial \mathcal{L}}{\partial (\partial_t \mathbf{A})} = \epsilon_0 (\partial_t \mathbf{A} + \nabla A_0) = -\epsilon_0 \mathbf{E}. \end{aligned} \quad (1.40)$$

The conjugate variable related to A_0 identically vanishes due to the fact that the time derivative $\partial_t A_0$ does not appear on the Lagrangian. This tells us that A_0 is not an independent dynamical variable. In fact, given $\mathbf{A}$ and $\partial_t \mathbf{A}$ at some instant of time, we can use equation (1.36) to solve A_0 in terms of $\mathbf{A}$. This also reduces the degrees of freedom of the electromagnetic field to the three components of the vector $\mathbf{A}$.

Having the canonical conjugated variables, the Hamiltonian density can be deduced from a Legendre transformation

$$\mathcal{H} = (\partial_t A_0) \Pi_0 + (\partial_t \mathbf{A}) \cdot \boldsymbol{\Pi} - \mathcal{L} \quad (1.41)$$

which gives as a result the following Hamiltonian density for the free electromagnetic field

$$\mathcal{H}(\mathbf{A}, \boldsymbol{\Pi}) = \frac{\epsilon_0}{2} (\mathbf{E}^2 + c^2 \mathbf{B}^2) - A_0 (\nabla \cdot \mathbf{E}) \quad (1.42)$$

so A_0 acts as a Lagrange multiplier imposing the Gauss law $\nabla \cdot \mathbf{E} = 0$ as a constraint of the theory.

It is important to realize that equations (1.34) and (1.35) do not define uniquely the potentials $\mathbf{A}(\mathbf{r}, t)$ and $A_0(\mathbf{r}, t)$. This is because, given some potentials which produce the physical electric and magnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, any other potentials related with the formers by the transformations (called gauge transformation)

$$\mathbf{A}(\mathbf{r}, t) \rightarrow \mathbf{A}(\mathbf{r}, t) + \nabla \Lambda(\mathbf{r}, t) \quad (1.43)$$

$$A_0(\mathbf{r}, t) \rightarrow A_0(\mathbf{r}, t) - \frac{\partial}{\partial t} \Lambda(\mathbf{r}, t), \quad (1.44)$$

will produce the same electromagnetic field. In these equations $\Lambda(\mathbf{r}, t)$ can be any arbitrary well behaved scalar field.

This gives us the possibility to reduce again the degrees of freedom of the field by imposing a gauge. One of the most common choices, and the one that will be used through this text unless otherwise stated, is the so called Coulomb gauge given by

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0 \quad (1.45)$$

which transforms the Maxwell's equation (1.36) and (1.37) for the free field into

$$\nabla^2 A_0(\mathbf{r}, t) = 0 \quad (1.46)$$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{A}(\mathbf{r}, t) = 0. \quad (1.47)$$

The first one of the equations above allows us to set $A_0 = 0$ and also tells us that the electric field for the free theory in the Coulomb gauge is

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t). \quad (1.48)$$

Finally, the total amount of degrees of freedom for the electromagnetic field have been reduced to just two, which physically correspond to the two independent directions of polarization of the light.

1.4.2 Modes of the Electromagnetic Field and Quantization

Having a Hamiltonian formulation of the electromagnetic theory, the quantization of the theory is straightforward. Nevertheless, in the case of the electromagnetic field, this has to be made carefully in order to be self consistent with the theory. It is necessary to take into account that, in the Coulomb gauge, both conjugate fields $\mathbf{A}$ and $\mathbf{\Pi}$ are transversal⁶. In this case, when we promote the vector field variables to field operators, the commutation relations between the canonical conjugate fields is given by

$$\begin{aligned} \left[\hat{\mathbf{A}}_i(\mathbf{r}, t), \hat{\mathbf{A}}_j(\mathbf{r}', t) \right] &= \left[\hat{\mathbf{\Pi}}_i(\mathbf{r}, t), \hat{\mathbf{\Pi}}_j(\mathbf{r}', t) \right] = 0, \\ \left[\hat{\mathbf{A}}_i(\mathbf{r}, t), \hat{\mathbf{\Pi}}_j(\mathbf{r}', t) \right] &= i\hbar \delta_{i,j}^T(\mathbf{r} - \mathbf{r}') \end{aligned} \quad (1.49)$$

where

$$\delta_{i,j}^T(\mathbf{r}) \equiv \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{r}} \left(\delta_{i,j} - \frac{k_i k_j}{k^2} \right) \quad (1.50)$$

is the so-called transversal delta function, which acts as a "projector" over the subspace of transversal vector fields⁷ and has the property that $\sum_i \partial_i \delta_{i,j}^T(\mathbf{r}) = 0$, wherefrom comes the transversality of the fields.

Another equivalent and useful way to quantize the field is in terms of the so-called modes of the electromagnetic fields. For this, we can find particular solutions for the wave equation (1.47) in the form $\mathbf{A}(\mathbf{r}, t) = \mathbf{u}_n(\mathbf{r}) e^{-i\omega_n t}$. Then, the vector field $\mathbf{u}_n(\mathbf{r})$, called mode function (profile) is found solving the vector Helmholtz equation

$$\left(\frac{\omega_n^2}{c^2} + \nabla^2 \right) \mathbf{u}_n(\mathbf{r}) = 0 \quad (1.51)$$

⁶A vector field $\mathbf{V}_T$ is said to be transversal if its divergence is equal to 0, i.e $\nabla \cdot \mathbf{V}_T = 0$. On the other hand $\mathbf{V}_L$ is called longitudinal if $\nabla \times \mathbf{V}_L = 0$. Helmholtz's theorem states that any sufficiently smooth, rapidly decaying vector field in three dimensions can be written as $\mathbf{V} = \mathbf{V}_L + \mathbf{V}_T$

⁷This, in the sense that $\mathbf{V}_{T_i}(\mathbf{r}) = \sum_j \int d^3r' \delta_{i,j}^T(\mathbf{r} - \mathbf{r}') \mathbf{V}_j(\mathbf{r}')$

together with the appropriate boundary conditions of the physical system that we are dealing with. In this case, as long as we are interested in localized modes inside a microring cavity (see figure 1.1), these cavity modes have to fulfill the continuity conditions for the magnetic and electric fields on the interface of the cavity and must vanish far away of the ring.

FIGURE 1.1: Geometry corresponding for a silicon nitride microring cavity on silicon substrate.

These mode profiles also have to fulfill the transversality condition and form a complete set of functions on this transversal space, which can be chosen to be an orthonormal set. This translates into the following

$$\sum_n \mathbf{u}_{n_i}(\mathbf{r}) \mathbf{u}_{n_j}^*(\mathbf{r}') = \delta_{i,j}^T (\mathbf{r} - \mathbf{r}') \quad (1.52)$$

$$\int \mathbf{u}_n(\mathbf{r}) \cdot \mathbf{u}_m^*(\mathbf{r}) d^3r = \delta_{n,m}$$

($\mathbf{u}_{n_i}$ $i = x, y, z$ are the Cartesian components of the vector field $\mathbf{u}_n$). Given these cavity modes, the most general solution of equation (1.47) can be written in the form

$$\mathbf{A}(\mathbf{r}, t) = \sum_n N_n (\alpha_n(0) e^{-i\omega_n t} \mathbf{u}_n(\mathbf{r}) + \alpha_n^*(0) e^{i\omega_n t} \mathbf{u}_n^*(\mathbf{r})) \quad (1.53)$$

where the variables $\alpha_n(0)$ have to be chosen in such a way that the solution satisfies the initial condition $\mathbf{A}(\mathbf{r}, 0)$. $N_n = \sqrt{\hbar/(2\epsilon_0\omega_n)}$ is a normalization constant that has been chosen in this particular way in convenience for the quantization. When working with mode profiles, the quantization of the theory corresponds to the promotion to operators

of the quantities α_n in the form $\alpha_n \rightarrow \hat{a}_n$ and $\alpha_n^* \rightarrow \hat{a}_n^\dagger$. Then, the field operators are written as

$$\begin{aligned}\hat{\mathbf{A}}(\mathbf{r}, t) &= \sum_n \sqrt{\frac{\hbar}{2\epsilon_0\omega_n}} \left(\hat{a}_n(t)\mathbf{u}_n(\mathbf{r}) + \hat{a}_n^\dagger(t)\mathbf{u}_n^*(\mathbf{r}) \right) \\ \hat{\mathbf{E}}(\mathbf{r}, t) &= -\hat{\mathbf{\Pi}}(\mathbf{r}, t)/\epsilon_0 = -i \sum_n \sqrt{\frac{\omega_n\hbar}{2\epsilon_0}} \left(\hat{a}_n(t)\mathbf{u}_n(\mathbf{r}) - \hat{a}_n^\dagger(t)\mathbf{u}_n^*(\mathbf{r}) \right),\end{aligned}\quad (1.54)$$

where the operators $\hat{a}_n(t) = \hat{a}_n(0)e^{-i\omega_n t}$ and $\hat{a}_n^\dagger(t) = \hat{a}_n^\dagger(0)e^{i\omega_n t}$ are called annihilation and creation operators of the mode n , due to that these operators are responsible for the annihilation and creation of photons on the n mode of the electromagnetic field.

The canonical commutator (1.50) in term of these operators is equivalent to

$$\left[\hat{a}_n(t), \hat{a}_m^\dagger(t) \right] = \delta_{m,n} \quad (1.55)$$

Finally, the total energy of the field can be computed as the integral over the whole space of the Hamiltonian density (1.42)

$$\hat{H} = \int \hat{\mathcal{H}}(\hat{\mathbf{A}}, \hat{\mathbf{\Pi}}) d^3r = \sum_n \hbar\omega_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right) \quad (1.56)$$

where, to deduce the equation above, relation (1.55) and properties (1.52) have been used. The operator $\hat{n}_n = \hat{a}_n^\dagger \hat{a}_n$ is known as the number operator, and its eigenstates $|n\rangle_n$ ⁸ represent states with a definite number of photons in the mode n . The ground state is a state without photons $|0\rangle_n$, and it fulfills the relation $\hat{a}_n |0\rangle_n = 0$. Equation (1.56) is nothing more than the Hamiltonian of an infinite number of independent harmonic oscillators and thus, the first term represent the energy of the excitations of these oscillators while the second term correspond with the so-called zero point energy, the energy when there are no photons on the field (vacuum), which is, indeed, a divergent term, and plays a fundamental role in some quantum phenomena such as the Casimir effect. Nevertheless, as long as we are not interested in phenomena related with this zero point energy, and due to that the energy itself have no physical meaning, but just the difference of energies, we will neglect this term and, without it, the Hamiltonian represent the energy relative to the vacuum state.

⁸Not to confuse the n inside the ket and the subscript n , the first one denotes the number of photons in the state, and the subscript is the label of the mode where the photons are.

1.4.3 Field Quadratures, Coherent States and Squeezed States

Classically, a mono-chromatic electric field, in a given point of the space, can be described through the so-called Fresnel representation (also know as phasor representation), which consists in writing the field as

$$E(t) = E_0 e^{-i\omega t} + E_0^* e^{i\omega t} \quad (1.57)$$

where $E_0 = |E_0|e^{i\phi}$ is the complex amplitude of the field at some given time t_0 , and the quantity $\varphi = \phi - \omega t_0$ is the phase of the field. This means that it can be represented as a point in a complex space as in the figure (1.2) .

FIGURE 1.2: Fresnel diagram for the classical field, where X e Y are the quadratures of the field. $E_0 = |E_0|e^{i\varphi}$ with φ the phase of the field represented as the angle between the X axis and the arrow, and with $|E_0|$ the real amplitude of the field represented as the length of the arrow.

Instead of describing the field in terms of the complex amplitude, it can be described by its real $X = E_0 + E_0^* = 2\Re(E_0)$ and imaginary $Y = -i(E_0 - E_0^*) = 2\Im(E_0)$ parts. These quantities are known as the **quadratures** of the field and, in terms of them, the field can be written as

$$E(t) = X \cos(\omega t) + Y \sin(\omega t). \quad (1.58)$$

In the case of a quantized mode of the electromagnetic field, it is possible to write the electric field as

$$\hat{E}(t) = \mathcal{E}_\omega (\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) \quad (1.59)$$

where $\mathcal{E}_\omega$ is a suitable constant with units of electric field (volts per meter) and the operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy the commutation relation (1.55). Equally, it is possible to define the quadrature operators as

$$\hat{X}_\beta = \hat{a}e^{-i\beta} + \hat{a}^\dagger e^{i\beta} \quad (1.60)$$

$$\hat{Y}_\beta = \hat{X}_{\beta+\pi/2} = -i(\hat{a}e^{-i\beta} - \hat{a}^\dagger e^{i\beta}) \quad (1.61)$$

that satisfy the commutation relation

$$[\hat{X}_\beta, \hat{Y}_\beta] = 2i. \quad (1.62)$$

These operators are a generalization of the quantum version of the quadratures defined before where the axes are rotated an angle β in relation to the previous definition. In terms of this quadratures, the field is

$$\hat{E}(t) = \mathcal{E}_\omega (\hat{X}_\beta \cos(\omega t - \beta) + \hat{Y}_\beta \sin(\omega t - \beta)) \quad (1.63)$$

These quadratures are equivalent to the position and momentum operator of the quantum harmonic oscillator (properly normalized to obtain dimensionless operators and the desired commutation relation).

Due to the fact that these quadratures do not commute with each other, it is impossible to have a quantum state with well defined values of both quadratures. As a consequence, it is not possible anymore to represent the field as a point in a complex space as before. Nevertheless, it is possible to do an equivalent representation to the Fresnel diagram for a quantum state as in the figure (1.3) where the arrow represents the mean value of the field and the shaded shape represents the uncertainty (noise) on the field quadratures.

There are two particularly interesting states of the field that are called coherent and squeezed states. These are characterized by being minimum uncertainty states, which means that they are states that satisfy the equality at the Heisenberg uncertainty relation $\langle \delta \hat{X}_\beta^2 \rangle \langle \delta \hat{Y}_\beta^2 \rangle \geq 1$ for some β .

The coherent states can be defined as eigenstates of the annihilation operator with eigenvalue α (which also works as a label for these coherent states)

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \quad (1.64)$$

or as generated from applying the so-called displacement operator $\hat{D}(\alpha)$ to the vacuum state

$$\hat{D}(\alpha) |0\rangle = |\alpha\rangle \quad (1.65)$$

$$\text{with } \hat{D}(\alpha) = \exp\{\alpha \hat{a}^\dagger - \alpha^* \hat{a}\}. \quad (1.66)$$

These states distribute the noise equally for every quadrature, i.e. $\langle \delta \hat{X}_\beta^2 \rangle = 1$ for any β , which can be represented on a Fresnel diagram as circle around the tip of the arrow as in figure 1.3.

The coherent states can be thought as the closest states to classical states because of their noise properties (minimum uncertainty and equally distributed) and their time evolution that resembles a classical harmonic oscillator [44].

FIGURE 1.3: Fresnel diagram for a quantum field, where the arrow represents the mean value of the field, the pointed circle represents the uncertainty of a coherent state which is symmetrically distributed for all the quadratures and the shaded ellipse represent a squeezed state with a reduction of noise at the amplitude quadrature and noise excess at the phase quadrature.

The squeezed states are also minimum uncertainty states, but in these states the noise of one quadrature is below the Standard Quantum Limit (SQL) defined as the noise of the vacuum state (or a coherent state), and the noise of the corresponding orthogonal quadrature has excess of noise in such a way that $\langle \delta \hat{X}_\beta^2 \rangle > 1$ and $\langle \delta \hat{Y}_\beta^2 \rangle < 1$ fulfilling that $\langle \delta \hat{X}_\beta^2 \rangle \langle \delta \hat{Y}_\beta^2 \rangle = 1$ for some β . This can be represented as an ellipse on the Fresnel diagram where the major axis represents noise excess and the minor axes noise compression (squeezing).

If we chose the quadratures in equations (1.60) and (1.61) with $\beta = \varphi$, being the phase of the mean value of the field (this mean field is defined as the expectation value of the annihilation operator $\alpha = \langle \hat{a} \rangle = |\alpha|e^{i\varphi}$, see figure 1.3). We can show that for intense fields (i.e. in which their fluctuations are much smaller than the mean value) the noise of the observable $\hat{X}_\varphi \equiv \hat{P}$ is proportional to the noise of the number operator $\hat{n}$ (which itself is proportional to the intensity of the field), and that is why $\hat{P}$ is also refereed as the intensity quadrature. On the other hand the orthogonal quadrature $\hat{Y}_\varphi = \hat{Q}$ is known as phase quadrature because the noise in this quadrature can be related with noise in the phase of the field.

As we will see later, the concept of squeezing is closely related with entanglement due to the fact that a noise reduction on a global variable (an operator acting on the whole system, like the EPR variables) of a composite system implies correlation between the subsystem that composes it and which under certain circumstances translates as entanglement (see chapter 3).

1.4.4 Quasi-monochromatic Field And Frequency Domain Operators

The electric field produced by a laser is usually described by one single frequency mode of the electromagnetic field in a coherent state, but in order to have a more accurate description of a real beam, we should describe it as composed of different frequency components around a central frequency. This central frequency is called carrier and usually contains most of the power of the beam. Frequencies around the carrier are called side bands and they cover an interval of the spectrum, the size of which is called bandwidth.

An electric field with a well defined polarization, a band width $\Delta\omega$ and a carrier frequency ω_c can be written, in accordance to equation (1.54) in the form

$$\hat{E}(t) = \sum_{\omega_n=\omega_c-\Delta\omega/2}^{\omega_c+\Delta\omega/2} \mathbf{u}_{\omega_n} \sqrt{\frac{\hbar\omega_n}{2\epsilon_0}} \left(\hat{a}_{\omega_n} e^{-i\omega_n t} + \hat{a}_{\omega_n}^\dagger e^{i\omega_n t} \right). \quad (1.67)$$

Given that all the modes have close frequencies (for instance, the frequencies within the bandwidth of a cavity mode) then we have that $\mathbf{u}_{\omega_n} \approx \mathbf{u}_{\omega_c}$ and $\omega_n \approx \omega_c$ and the equation above can be written as

$$\hat{E}(t) = \mathcal{E}_c \left(\hat{A}(t) e^{-i\omega_c t} + \hat{A}^\dagger(t) e^{i\omega_c t} \right), \quad (1.68)$$

where we have defined the quasi-monochromatic slowly varying annihilation operator $\hat{A}(t)$ as

$$\hat{A}(t) = \sum_{\Omega_n = -\Delta\omega/2}^{\Delta\omega/2} \hat{a}_{\Omega_n} e^{-i\Omega_n t}. \quad (1.69)$$

As in the single frequency case, we can define quadrature operators as

$$\begin{aligned} \hat{X}_\beta(t) &= \hat{A}(t)e^{-i\beta} + \hat{A}^\dagger(t)e^{i\beta} \\ \hat{Y}_\beta(t) &= -i \left(\hat{A}(t)e^{-i\beta} - \hat{A}^\dagger(t)e^{i\beta} \right). \end{aligned} \quad (1.70)$$

In term of this quadratures the electric field is written as

$$\hat{E}(t) = \mathcal{E}_c(\hat{X}_\beta(t) \cos(\beta - \omega_c t) - \hat{Y}_\beta(t) \sin(\beta - \omega_c t)). \quad (1.71)$$

These operators fulfill the following commutation relations [45]

$$\begin{aligned} [\hat{A}(t), \hat{A}^\dagger(t')] &= \delta(t - t'), \\ [\hat{X}_\beta(t), \hat{Y}_\beta(t')] &= 2i\delta(t - t'). \end{aligned} \quad (1.72)$$

The quantities that are going to be calculated through this work are the so called spectral density noises which are closely related with the two time correlation function. Thus, given two time dependent operators $\hat{O}(t)$ and $\hat{O}'(t)$ the two time correlation function is defined as

$$C_{O,O'}(t, t') = \langle \delta\hat{O}(t)\delta\hat{O}'(t') \rangle. \quad (1.73)$$

If these operators describe what is known in probability theory as a stationary process, the two time correlation function will depend only on the difference of times $\tau = t - t'$ ⁹. This is a totally valid assumption given that the situations addressed here are on the continuous wave regime.

The spectral density noise function is defined as the Fourier transform of the correlation function

$$S_{O,O'}(\omega) = \int_{-\infty}^{\infty} C_{O,O'}(\tau) e^{-i\omega\tau}. \quad (1.74)$$

⁹If $\hat{O} = \hat{O}'$, C is called the self correlation function and will be notated as $C_O(t, t')$, otherwise it is called cross-correlation

It is also possible to define the spectral density noise directly from the operators. The Fourier transform of the operators is given by

$$\hat{O}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{O}(t) e^{-i\omega t} dt \quad (1.75)$$

$$\hat{O}^\dagger(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{O}^\dagger(t) e^{-i\omega t} dt, \quad (1.76)$$

where from this definition it is important to point out that $[\hat{O}(\omega)]^\dagger = \hat{O}^\dagger(-\omega)$ (i.e. $\hat{O}^\dagger(\omega)$ is not the adjoint operator of $\hat{O}(\omega)$)

Then the spectral density noise is given by

$$\langle \delta \hat{O}(\omega) \delta \hat{O}'(\omega') \rangle = S_{O,O'}(\omega) \delta(\omega + \omega'). \quad (1.77)$$

The stationarity condition has as a consequence the operator $\hat{O}$ at the frequency ω is correlated with the operator $\hat{O}'(\omega')$ only at the frequency $\omega' = -\omega$. Applying these definitions to the quadratures of the electromagnetic field introduced before we have that, in the frequency domain the commutation relations (1.72) are

$$\begin{aligned} [\hat{A}(\omega), \hat{A}^\dagger(\omega')] &= \delta(\omega + \omega'), \\ [\hat{X}_\beta(\omega), \hat{Y}_\beta(\omega')] &= 2i\delta(\omega + \omega') \end{aligned} \quad (1.78)$$

In the same way the was done for the covariance matrix, it is possible to gather all the fourier transform of the quadratures in a vector $\hat{\mathbf{Z}}_\beta = \{\hat{X}_\beta, \hat{Y}_\beta\}$ and the spectral density noise matrix is defined as

$$S_{ij}(\omega) \delta(\omega + \omega') = \langle \delta \hat{Z}_i(\omega) \delta \hat{Z}_j(\omega') \rangle = \begin{pmatrix} S_{X_\beta}(\omega) & S_{X_\beta Y_\beta}(\omega) \\ S_{Y_\beta X_\beta}(\omega) & S_{Y_\beta}(\omega) \end{pmatrix} \delta(\omega + \omega'). \quad (1.79)$$

This matrix is also, as in the case of the covariance matrix (1.21), positive semidefinite. This, together with the commutation relations (1.78), allows to write an "uncertainty-like" relation

$$S_{X_\beta}(\omega) S_{Y_\beta}(\omega) \geq 1 \quad (1.80)$$

To demonstrate this inequality we use some properties deduced from the positive semidefiniteness of $S(\omega)$ such as the fact that $S_{X_\beta}(\omega)$ and $S_{Y_\beta}(\omega)$ are real and positive and also that $S_{X_\beta}(\omega) = S_{X_\beta}(-\omega)$ and $S_{Y_\beta}(\omega) = S_{Y_\beta}(-\omega)$. From $\det(S) \geq 0$ we have that

$$S_{X_\beta}(\omega)S_{Y_\beta}(\omega) \geq |S_{X_\beta Y_\beta}(\omega)|^2, \quad (1.81)$$

And using the symmetry explained before we have that

$$S_{X_\beta}(\omega)S_{Y_\beta}(\omega) \geq \frac{1}{2} (|S_{X_\beta Y_\beta}(\omega)|^2 + |S_{Y_\beta X_\beta}(-\omega)|^2), \quad (1.82)$$

Finally using the commutation relations (1.78) in the form $S_{X_\beta Y_\beta}(\omega) - S_{Y_\beta X_\beta}(-\omega) = 2i$ we have that

$$\begin{aligned} 2 (|S_{X_\beta Y_\beta}(\omega)|^2 + |S_{Y_\beta X_\beta}(-\omega)|^2) = \\ |S_{X_\beta Y_\beta}(\omega) + S_{Y_\beta X_\beta}(-\omega)|^2 + |S_{X_\beta Y_\beta}(\omega) - S_{Y_\beta X_\beta}(-\omega)|^2 = \\ |S_{X_\beta Y_\beta}(\omega) + S_{Y_\beta X_\beta}(-\omega)|^2 + 4 \end{aligned} \quad (1.83)$$

Putting everything together we demonstrate (1.80) as follow

$$\begin{aligned} S_{X_\beta}(\omega)S_{Y_\beta}(\omega) &\geq \frac{1}{4} (|S_{X_\beta Y_\beta}(\omega) + S_{Y_\beta X_\beta}(-\omega)|^2 + 4) \\ &\text{or } S_{X_\beta}(\omega)S_{Y_\beta}(\omega) \geq 1. \end{aligned} \quad (1.84)$$

It is also common to say, in this frequency domain regime, that we have squeezing of the "quadrature" $\hat{X}_\beta$ at the frequency ω if we have $S_{X_\beta}(\omega) < 1$.

Chapter 2

Introduction to Non-linear Optics

Light shows itself as one of the most useful tools in the field of quantum information and, as it travels at the fastest speed allowed by the nature, it is also a good system for carrying quantum communications tasks. It usually interacts weakly with the environment and consequently suffers a low level of decoherence, hardly achievable for others system, being this decoherence one of the major problems to be solved to technically build a quantum computer.

In order to generate entangled states, some interaction between the systems is necessary. Light, in principle, does not interact with itself as can be deduced from the linearity of the Maxwell equations (1.28-1.31), thus, to generate entangled states of light, a nonlinear medium is needed to acts as an effective interaction between the different modes of the electromagnetic field.

In this chapter we will make a brief introduction to the topic of nonlinear optics that will be useful later, starting with a brief description of the classic nonlinear optics and after that, make the quantization of the theory. In this way we will have the necessary theoretical framework to analyze the properties of the light produced inside third order nonlinear cavities. The description done here can be found in a more exhaustive way in nonlinear optics text books as [46, 47] among others.

2.1 Classical Non-Linear Optics

Classically, the interaction of light and matter is given by the source terms ρ and $\mathbf{J}$ in the Maxwell equations (1.28-1.31). Microscopically, inside a material medium these source terms can be decomposed into free charges and current densities, those who can move all over the material, and bound ones, which intrinsically "belong" to the atoms

or molecules that constitute the material and are localized around them. Therefore we can write the charge and current densities as

$$\begin{aligned}\rho(\mathbf{r}, t) &= \rho_{free}(\mathbf{r}, t) + \rho_{bound}(\mathbf{r}, t) \\ \mathbf{J}(\mathbf{r}, t) &= \mathbf{J}_{free}(\mathbf{r}, t) + \mathbf{J}_{bound}(\mathbf{r}, t).\end{aligned}\tag{2.1}$$

From this point it will be supposed that there are no free current densities present in our material since that would be the case in silicon based on-chip microcavities that we intend to describe (non magnetic material), although including it would be straightforward following the same procedure that will be done with the charge density.

Macroscopically, the bound charge density can be written as (see, for instance [41] chapter 6)

$$\rho_{bound}(\mathbf{r}, t) = \rho_{mol}(\mathbf{r}, t) - \nabla \cdot \mathbf{P}(\mathbf{r}, t) + \dots\tag{2.2}$$

where $\rho_{mol}(\mathbf{r}, t)$ is the total mean charge density averaged over a macroscopic volume compared with the molecules that constitute the material, which is zero in neutral materials. $\mathbf{P}(\mathbf{r}, t)$ is the mean macroscopic electric dipolar moment and the suspension point stands for terms related to the quadrupole, octupole and higher order electrical moments. Terms higher than the dipole will be neglected and $\rho_{mol}(\mathbf{r}, t)$ and $\rho_{free}(\mathbf{r}, t)$ will be set to zero as is the case of the systems we want to describe.

Given that, the Maxwell equations with sources inside material medium are written as

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 0\tag{2.3}$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) - \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} = \mathbf{0},\tag{2.4}$$

where the so-called displacement field $\mathbf{D}$ and the magnetizing field $\mathbf{H}$ are defined as

$$\begin{aligned}\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} &= \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}\end{aligned}\tag{2.5}$$

with $\mathbf{M}$ the magnetization field that will be also set to zero, because we will deal with non magnetic materials.

Phenomenologically, we can try to write the polarization as a power expansion of the electric field as

$$\mathbf{P} = \epsilon_0 \left(\chi^{(1)} : \mathbf{E} + \chi^{(2)} :: \mathbf{E}^2 + \chi^{(3)} ::: \mathbf{E}^3 + \dots \right) \equiv \mathbf{P}_L + \mathbf{P}_{NL}\tag{2.6}$$

Where the linear polarization corresponds to $\mathbf{P}_L = \epsilon_0 \chi^{(1)} : \mathbf{E}$ and the non-linear polarization to the quadratic and higher order terms in the electric field. The linear and nonlinear susceptibilities $\chi^{(n)}$ are, in the most general case, frequency dependent $n + 1$ rank tensor.

This expansion can be seen as the deviation of the bound charge from an harmonic motion. When bound charges experience a force produced by an electric field, the movement can be treated as an harmonic oscillator as long as the displacement is small. In this case we can show that this displacement, and thus the dipole moment is linear with the field. But if the field is strong enough it will produce a significant displacement and the movement will be no longer harmonic. We can deal with this anharmonicity as a perturbation, so the dipole moment will be written as a power expansion in the field, giving as a result a polarization of the form (2.6). The fact that the displacement has to be large enough to have a nonlinear behavior tells us that these effects appear just when intense fields are present. This is possible to obtain, for instance, with a powerful enough laser light or with the help of an optical cavity such as the case we will study, where a high quality cavity increases several times the incident field amplitude inside the cavity enhancing the nonlinear effects.

In terms of this nonlinear polarization, the wave equation that the electric field fulfills is given by

$$\nabla \times \nabla \times \mathbf{E} + \frac{n^2}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2} \quad (2.7)$$

where $n^2 = (1 + \chi^{(1)})$ is the refractive index. A useful approximation that can be applied is the so called slowly varying envelope approximation (SVEA) in which we suppose that the electric and polarization fields are of the form $\mathbf{V}(\mathbf{r}, t) = \mathbf{V}(z)e^{ikz - \omega t}$, i.e. they are waves propagating in some arbitrary z direction and the envelope $\mathbf{V}(z)$ slowly varies with z over distances compared with the wavelength $\lambda = 2\pi/k$.

$$\left| \frac{\partial^2 \mathbf{E}}{\partial z^2} \right| \ll \left| k \frac{\partial \mathbf{E}}{\partial z} \right|. \quad (2.8)$$

Within this approximation, the wave equation (2.7) takes the form

$$\frac{\partial \mathbf{E}}{\partial z} = \frac{i\omega^2 \mu_0}{2k} \mathbf{P}_{NL} e^{-ikz} \quad (2.9)$$

which is a useful equation to describe paraxial wave propagation in nonlinear media.

The first nonlinear term appearing in the expansion is the second order nonlinearity $\chi^{(2)}$ and, when present, it usually is the leading term of the nonlinear phenomenon appearing

in different systems. The phenomena produced as a consequence of the second order nonlinearity include, for instance

$\chi^{(2)}$ **effects:**

- Second harmonic generation: Generation of a wave with twice the frequency of the incident wave.
- Wave rectification: Generation of a d.c. field from an oscillating incident field.
- Parametric amplification: Interaction of a weak signal with a strong pump, generating an idler wave and the amplification of the signal.

If a material is symmetric under spatial reflection, that is, under the change $\mathbf{r} \rightarrow -\mathbf{r}$ (centrosymmetric) then the second order susceptibility identically vanishes and the non-linear leading term is the $\chi^{(3)}$ term. This is the case of the on chip microcavities we will study and the one we will focus on. Among the phenomena that the $\chi^{(3)}$ term are responsible for are

$\chi^{(3)}$ **effects:**

- d.c. induced second harmonic generation: A d.c. electric field interacts with an oscillating field generating its second harmonic.
- d.c Kerr effect: An intensity dependent refractive index due to a d.c. field.
- Third harmonic generation: The generation of a wave with three times the frequency of the incident wave
- Self-phase modulation (SPM) and cross-phase modulation (XPM): Kerr effect suffered by a wave induced by itself (SPM) or by another wave (XPM)
- Four wave mixing (FWM): Interaction (energy exchange) between four waves inside the media.

It is the FWM process the one who produces the generation of several modes in systems as on chip cavities. When starting with just a monochromatic pump, we can generate several signal and idler pairs through a cascade FWM mechanism.

2.2 Quantum Nonlinear Optics

Now, with this brief description of the classical theory of nonlinear optics, we can proceed to make the quantization of the theory. In this case we will have a different approach based on [48, 49], in which instead of defining potentials as in equations (1.34) and (1.35), we take advantage that we are in a medium with no free charges and currents. Then, defining the so-called dual potentials

$$\mathbf{D} = \nabla \times \mathbf{\Lambda} \quad \mathbf{B} = \mu_0 \left(\frac{\partial \mathbf{\Lambda}}{\partial t} + \nabla \Lambda_0 \right), \quad (2.10)$$

equations (2.3) and (2.4) are trivially satisfied (In contrast, potentials (1.34) and (1.35) trivially satisfy equations (1.29) and (1.30)). The Lagrangian density from which the Gauss and Faraday's laws can be derived is

$$\mathcal{L}(\Lambda) = \frac{1}{2} \left(\frac{1}{\mu_0} \mathbf{B}^2 - \frac{1}{\epsilon_0} \mathbf{D}^2 \right) + \frac{1}{\epsilon_0} \mathbf{D} \cdot \left(\frac{1}{2} \eta^{(1)} : \mathbf{D} + \frac{1}{3} \eta^{(2)} :: \mathbf{D}^2 + \frac{1}{4} \eta^{(3)} ::: \mathbf{D}^3 + \dots \right) \quad (2.11)$$

where here, we have written the polarization as a power series of the displacement field and not of the electric field as before

$$\mathbf{P} = \eta^{(1)} : \mathbf{D} + \eta^{(2)} :: \mathbf{D}^2 + \eta^{(3)} ::: \mathbf{D}^3 + \dots \quad (2.12)$$

From the Lagrangian density, the canonical conjugated momenta can be derived obtaining

$$\Pi_0 = \frac{\partial \mathcal{L}}{\partial (\partial_t \Lambda_0)} = 0 \quad \mathbf{\Pi} = \frac{\partial \mathcal{L}}{\partial (\partial_t \mathbf{\Lambda})} = \mathbf{B}. \quad (2.13)$$

As before, we have that the canonical conjugated momentum to Λ_0 is null, but the one associated to $\mathbf{\Pi}$ is $\mathbf{B}$ and not $\mathbf{E}$ as in the free theory, this is due to the formulation of the theory being done in terms the dual potential instead of the regular potentials.

Given the conjugated fields, the Hamiltonian density of the system is, after some manipulation and using integration by parts,

$$\begin{aligned} \mathcal{H}(\Lambda, \mathbf{\Pi}) = & \left(\frac{1}{\mu_0} \mathbf{B}^2 - \frac{1}{\epsilon_0} \mathbf{D}^2 \right) - \frac{1}{\epsilon_0} \mathbf{D} \cdot \left(\frac{1}{2} \eta^{(1)} : \mathbf{D} + \frac{1}{3} \eta^{(2)} :: \mathbf{D}^2 \right. \\ & \left. + \frac{1}{4} \eta^{(3)} ::: \mathbf{D}^3 + \dots \right) + \Lambda_0 (\nabla \cdot \mathbf{B}) \end{aligned} \quad (2.14)$$

where the last term can be dropped out given that $\nabla \cdot \mathbf{B} = 0$ (here Λ_0 acts as a Lagrange's multiplier with the Gauss' Law as a constrain of the theory) and the field $\mathbf{D}$ has to be written in terms of the variable $\mathbf{\Lambda}$ through (2.10).

Here, as in the case of the free field, equations (2.10) do not define uniquely the potentials and, again, we can choose an adequate gauge. In this case it is useful to set $\nabla \cdot \mathbf{\Lambda} = 0$ which with help of Gauss law we can choose also $\Lambda_0 = 0$ to eliminate the redundancy (given by $\Pi_0 = 0$) of the theory.

The quantization is done by promoting the vector fields $\mathbf{\Lambda}$ and $\mathbf{B}$ to operators and imposing canonical commutation relation

$$\left[\hat{\mathbf{\Lambda}}_i(\mathbf{r}, t), \hat{\Pi}_j(\mathbf{r}', t) \right] = \left[\hat{\mathbf{\Lambda}}_i(\mathbf{r}, t), \hat{\mathbf{B}}_j(\mathbf{r}', t) \right] = i\hbar \delta_{ij}^T(\mathbf{r}' - \mathbf{r}) \quad (2.15)$$

between them. The mode description is done dealing with the nonlinearity as a perturbation and assuming that it is weak enough to not to modify the mode profile appreciably. Thus, if we are dealing with cavity modes, they are found by taking into account only the linear polarization. As in the free field case, we can make the ansatz $\mathbf{\Lambda}(\mathbf{r}, t) = N_n \mathbf{u}_n(\mathbf{r}) e^{-i\omega_n t}$ with N_n a normalization constant and then, the corresponding Helmholtz equation is

$$\nabla \times \left(\beta^{(1)} \nabla \times \mathbf{u}_n(\mathbf{r}) \right) = \frac{\omega_n}{c^2} \mathbf{u}_n(\mathbf{r}) \quad (2.16)$$

where $\beta^{(1)} = (1 + \chi^{(1)})^{-1}$ is related with the refractive index of the medium. This equation defines, through the boundary condition imposed by the geometry of the cavity, the mode profiles and the oscillation frequencies. These cavity frequencies are, as long as the dispersion is not too strong, approximately separated by the same value called the free spectral range ω_{FSR} .

In terms of theses modes the fields are

$$\hat{\mathbf{\Lambda}}(\mathbf{r}, t) = \sum_n \sqrt{\frac{\hbar}{2\mu_0\omega_n}} \left(\hat{a}_n(t) \mathbf{u}_n(\mathbf{r}) + \hat{a}_n^\dagger(t) \mathbf{u}_n^*(\mathbf{r}) \right) \quad (2.17)$$

$$\hat{\mathbf{B}}(\mathbf{r}, t) = \hat{\Pi}(\mathbf{r}, t)/\epsilon_0 = i \sum_n \sqrt{\frac{\mu_0\omega_n\hbar}{2}} \left(\hat{a}_n(t) \mathbf{u}_n(\mathbf{r}) - \hat{a}_n^\dagger(t) \mathbf{u}_n^*(\mathbf{r}) \right). \quad (2.18)$$

$$\hat{\mathbf{D}}(\mathbf{r}, t) = \sum_n \sqrt{\frac{\hbar}{2\mu_0\omega_n}} \left(\hat{a}_n(t) \nabla \times \mathbf{u}_n(\mathbf{r}) + \hat{a}_n^\dagger(t) \nabla \times \mathbf{u}_n^*(\mathbf{r}) \right) \quad (2.19)$$

with

$$[\hat{a}_n, \hat{a}_n^\dagger] = 1. \quad (2.20)$$

Splitting the Hamiltonian density in a linear and a nonlinear part

$$\begin{aligned} \mathcal{H}_L &= \left(\frac{1}{\mu_0} \mathbf{B}^2 - \frac{1}{\epsilon_0} \mathbf{D}^2 \right) - \frac{1}{2\epsilon_0} \mathbf{D} \cdot \eta^{(1)} : \mathbf{D} \\ \mathcal{H}_{NL} &= -\frac{1}{\epsilon_0} \mathbf{D} \cdot \left(\frac{1}{3} \eta^{(2)} :: \mathbf{D}^2 + \frac{1}{4} \eta^{(3)} ::: \mathbf{D}^3 + \dots \right) \end{aligned}$$

we can show that the linear Hamiltonian is

$$H_L = \sum_n \hbar \omega_n \hat{a}_n^\dagger \hat{a}_n \quad (2.21)$$

which has the same form as the free field case. The subscript n that labels the mode is chosen in a way that it sorts the cavity frequencies in an increasing order.

As long as we are mostly interested in the $\chi^{(3)}$ nonlinearity we set $\chi^{(2)} = 0$ and work on the $\chi^{(3)}$ term. For a completely isotropic and homogeneous media the Hamiltonian simplifies because the susceptibility $\chi^{(3)}$ is a scalar. We also suppose that, as long as the modes to be considered are not too spectrally far away, the susceptibility $\chi^{(3)}$ is frequency independent in the region of interest. Given that, the nonlinear Hamiltonian is given by

$$\hat{H}_{NL} = -\frac{1}{4\epsilon_0} \eta^{(3)} \int_V (\hat{\mathbf{D}} \cdot \hat{\mathbf{D}})^2 d^3r \quad (2.22)$$

Then, when written in terms of mode functions (2.19) we will find terms containing products of the form $\hat{a}_i \hat{a}_j \hat{a}_k \hat{a}_l$, $\hat{a}_i^\dagger \hat{a}_j \hat{a}_k \hat{a}_l$, $\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l$, $\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k^\dagger \hat{a}_l$ and $\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l^\dagger$. The operators $\hat{a}_i(t)$ are mostly driven by a rapidly time varying component of the form $e^{-i\omega_i t}$ and, as long as we suppose that only modes with frequency close to a central pump frequency ω_p (that expand, at most, around an octave) are excited and relevant to the problem, only terms that have a small oscillation are significant on the Hamiltonian. Those terms are of the form $\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l$ with the constrain that $\omega_i + \omega_j \approx \omega_k + \omega_l$. The others terms oscillate rapidly and will not contribute significantly inasmuch as they average approximately zero on time. Then we will neglect these rapidly oscillating terms in the Hamiltonian, this approximation is known in the literature as the rotation wave approximation (RWA) and, within this approximation, the nonlinear Hamiltonian is transformed into

$$\hat{H}_{NL} = -\hbar \eta_3 \sum' \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l, \quad (2.23)$$

where the coefficient η_3 is proportional to the third order nonlinear susceptibility $\eta^{(3)}$ and to the mode overlapping $\int_u (\mathbf{u}_i^* \cdot \mathbf{u}_j^*)(\mathbf{u}_k \cdot \mathbf{u}_l) d^3r$ that we will consider it independent of i, j, k and l that is valid as long as the modes profiles are no too different. The symbol $\sum'$ stands for a sum over the indices i, j, k and l subject to the constrain $i + j = k + l$.

The Hamiltonian (2.23) will be our starting point for the study of the properties of the light generated on an on-chip silicon based OPO. First considering just the interaction between three modes of the cavity, for subsequently to introduce gradually more and more modes until built a full description of a the frequency comb generated on the cavity.

Chapter 3

Entanglement

The phenomenon of quantum entanglement is, perhaps, one of the most disturbing (in a good sense) and fascinating within the quantum theory, mostly because of its counterintuitive nature without parallel in the classical world. Entanglement boils down, qualitatively, to the fact that when we are dealing with several interacting systems, they lose their "individuality" in the sense that, differently from classical physics, a complete description of each subsystem separately can not be done without a loss of information, even if the subsystems are far away from each other and no longer interact. This implies that quantum theory is nonlocal in that sense (the description of the system, generally, can not be done "joining" the local descriptions of the individual subsystem) and just a characterization of the system as a whole can give us a complete description.

The fact that, in an entangled state, any attempt to make an individual description of the state of any subsystem leads to an information loss indicates that these subsystems are strongly correlated. Indeed, the correlations that arise from a quantum entangled state can be stronger than any other possible correlation allowed by the classical theory. These stronger correlations could make us think that in some sense, the use of quantum theory for some tasks as, for example computation, can be intrinsically more advantageous than using classical physics, although such a statement has not been proven yet.

Since the first stages of quantum theory, the founding fathers of the theory, such as Schrödinger [50] and Einstein, Podolski and Rosen in their famous paper about the EPR paradox [51], called the attention to this particular property of quantum systems. This seemed so strange that in the EPR paper the authors question the completeness of the theory. Their argument was based on the following example. Given a two particle quantum state represented by the wave function on the position basis of the form

$\psi(x_1, x_2) = C_x \delta(x_1 - x_2)$ ¹ (which is an entangled state), that represents perfectly correlated positions, we can compute the wave function in the momentum basis getting as a result $\varphi(p_1, p_2) = C_p \delta(p_1 + p_2)$ which tells us that the particles also have perfectly anti-correlated momenta. This means that when measuring the position of particle 1, immediately the position of particle 2 will be known, and vice versa. The same applies for measurements of the momentum. Measuring the momentum of particle 1, the momentum of particle 2 can be precisely deduced, and vice versa. All of this, even if the particles are far enough that they are causally disconnected and are not able to interact any more. As long as the measurements are causally disconnected, there would be no way that measurements on one particle could affect the state of the other, so performing measurements of x_1 on particle 1 and p_2 on particle 2 with arbitrary precision, we would have information, also with arbitrary precision, about the conjugated variables x_2 and p_1 . However, quantum theory itself imposes restrictions about the measurements on conjugated variables given by Heisenberg uncertainty principle (1.3), in apparent contradiction with what was exposed before. In this way of thinking the authors of the EPR paper claim that, even though quantum mechanics gives adequate results describing several phenomena, it has internal inconsistencies that make it an incomplete theory.

That statement is founded on the interpretation that was given to the previous example based on the concepts of *locality and elements of reality* (which together are called *local realism*). The locality indicates that a system located at some position can only be affected or influenced by whatever is at the same position, and that the interaction has a maximum speed of propagation, the speed of the light on vacuum. The second concept indicates that whenever an observable that describes a system in a theory has a definite value independently of whether it is measured or not, then that observable constitutes an *element of reality*.

The limitation of Einstein, Podolsky and Rosen was to suppose that any complete theory that describes nature should be local realistic.

As an answer to the problem presented by the EPR paradox, Bohr published a paper in the same year and with the same title [53], defending quantum theory by explaining that there was no paradox due to the fact that the variables $x_1 - x_2$ and $p_1 + p_2$ in fact commute, so there would be no restriction to the arbitrary accuracy of the measurement of both quantities at the same time.

Despite that, through time, quantum theory kept satisfactorily giving explanation for several phenomena previously unexplained. Given the fact that there was no way to

¹This wave function does not correspond to a physically possible state given that it is not normalizable, nevertheless it is possible to make a more accurate example in terms of the so called two-mode squeezed vacuum states which leads to the same result [52].

test the local realism hypotheses, the discussion about the completeness of the theory remained more at a philosophical level for several years until 1964, when John S. Bell formulated the famous theorem named after him [54, 55]. There he shows, in a very simple way, that any theory that satisfies the local realism hypothesis must also satisfy a set of inequalities, now called Bell inequalities. This theorem takes the discussion of the completeness of quantum mechanics out of the philosophical realm, towards the physical realm, due to the possibility of testing the violation of these inequalities experimentally. In that way, one can verify if the description of nature agrees more with a local realistic theory or with a quantum mechanical like theory.

The first experiments on Bell's theorem tests were done by Clauser *et al.* in 1978 [56] and Aspect *et al.* in 1982 [57], using the polarization of photons. A Bell like inequality, known as CHSH inequality [58], was violated. These Bell test experiments, as others made in different physical systems like ions [59], or in the domain of continuous variables using the quadratures of the electromagnetic field [60], besides confirming the validity of quantum theory, made the use of entangled states an experimental reality.

The phenomenon of entanglement, because of its counterintuitive nature, was used in the early days of quantum mechanics to try to refute the theory but nowadays this same phenomenon is being used to try to have a better understanding of the foundations of quantum mechanics, as in the analysis of quantum decoherence, the quantum-classical transition, the measurement problem and wave function collapse [61–63]. It also has interesting technological implications as in quantum computation and information, and quantum cryptography [8].

3.1 Formal Definition of Entanglement

Formally, entanglement arises in quantum systems that are described by a Hilbert space H_{sis} which is made from the tensor product of two or more subsystems

$$H_{sis} = H_1 \otimes \cdots \otimes H_N, \quad (3.1)$$

where N is the number of subsystems and $H_n, n = 1, 2, \dots, N$ is the Hilbert space of the n th subsystem. Let the set $\{|i\rangle_n\}$ be a basis on H_n , the most general pure state of the total system is given by

$$|\Psi\rangle = \sum_{i_1, i_2, \dots, i_N} c_{i_1, i_2, \dots, i_N} |i_1\rangle_1 \otimes |i_2\rangle_2 \otimes \cdots \otimes |i_N\rangle_N. \quad (3.2)$$

If this state can be written as a tensor product of N vector states, each one belonging to a different subspace, as

$$|\Psi\rangle_{sep} = |\psi_1\rangle_1 \otimes |\psi_2\rangle_2 \otimes \cdots \otimes |\psi_N\rangle_N, \quad (3.3)$$

then the state is called fully separable. An **entangled pure state** is defined as any pure state that is not fully separable, i.e. that can not be written as (3.3).

A vector state can also be partially separable if it can be written as, for example,

$$|\Psi\rangle = |\psi_1\rangle_1 \otimes \cdots \otimes |\psi_{i-1}\rangle_{i-1} \otimes |\psi_{i+1}\rangle_{i+1} \otimes \cdots \otimes |\psi_{j-i}\rangle_{j-i} \otimes |\psi_{j+1}\rangle_{j+1} \otimes \cdots \otimes |\psi_N\rangle_N \otimes |\Phi\rangle_{i,j} \quad (3.4)$$

where $|\Phi\rangle_{i,j}$ belongs to the composite Hilbert space $\mathcal{H}_i \otimes \mathcal{H}_j$ of the subsystems i and j and it is not separable in that Hilbert space. The state (3.4) is an example of an entangled state where just the subsystems i and j are entangled to each other but are separable from the rest. We can have several different ways in which a vector state can be entangled, which correspond to the number of different ways that we can group (partitions) the N subsystem, which increase rapidly with N .

Nonetheless, whenever it is not possible to describe the state of the system with a vector state, the previous definitions are not useful anymore. We can, in principle, make an analog definition in terms of the density matrix for a fully separable state as

$$\hat{\rho}_{sep} = \hat{\rho}_1 \otimes \cdots \otimes \hat{\rho}_N, \quad (3.5)$$

However, when dealing with density matrices it is necessary to be more careful. In fact the state above is just a particular case of a fully separable state called **product state**.

The most general way of having a fully separable state is by a statistical mixture of product states as

$$\hat{\rho}_{sep} = \sum_i p_i (\hat{\rho}_1^i \otimes \cdots \otimes \hat{\rho}_N^i), \quad (3.6)$$

with $p_i \geq 0$ and $\sum_i p_i = 1$. Finally, as before, an entangled state is defined as any state that is not separable. Here the entanglement structure is even more complicated because an entangled state can be the result of statistical mixture of partially separable states. For example, a system composed of three subsystems A , B and C , can have a density matrix that can be written as

$$\hat{\rho}_{sys} = p_1(\hat{\rho}^{A,B} \otimes \hat{\rho}^C) + p_2(\hat{\rho}^{A,C} \otimes \hat{\rho}^B) + p_3(\hat{\rho}^{B,C} \otimes \hat{\rho}^A), \quad (3.7)$$

which is a statistical mixture where, in the first term, systems A and B are entangled and separate from C , in the second term, systems A and C are entangled and separate from B and in the last term systems B and C are entangled and separate from A .

3.2 Separability Criteria

The fact that there is not an operational definition of an entangled state makes it a really difficult task to, whenever we have a density matrix of a system, distinguish whether it corresponds to an entangled state or not. This task is known as the separability problem and, computationally talking, grows exponentially with the number of subsystems and the dimension of these subsystems. The violation of a Bell inequality can be used as a test of entanglement, but that is a very strong criterion because there are many entangled states that do not violate any Bell inequality but however they are useful for some quantum information tasks, as for example the so-called Gaussian states in continuous variable (CV) systems. We also face the problem that several of the finite dimensional criteria are no longer applicable to continuous variable systems.

Fortunately there are other more flexible criteria for CV systems that can be used to identify entangled states as for example the positive partial transpose and criteria based on the sum of variances.

3.2.1 Positive Partial Transpose (PPT)

The PPT criterion, also known as the Peres-Horodecki criterion [64, 65], is a separability test for bipartite systems based on the concepts of positive maps and completely positive maps.

A positive map is a linear transformation $\mathcal{E} : \mathcal{H} \rightarrow \mathcal{E}(\mathcal{H}) \subset \mathcal{H}$ that preserves positivity, that is, it takes a positive operator $\hat{\rho} \in \mathcal{H}$ and transforms it into another positive operator $\mathcal{E}(\hat{\rho})$. Here $\mathcal{H}$ is the Hilbert space made of the set of all positive operators acting on $\mathbf{H}$. Formally this is written as

$$\mathcal{E} \text{ is positive iff } \forall \hat{\rho} > 0 \Rightarrow \hat{\rho}' = \mathcal{E}(\hat{\rho}) > 0. \quad (3.8)$$

We can think that this property, together with the preservation of the trace (that means conservation of probabilities), provides enough conditions to impose on a map for being a candidate to represent any physical process. However, it can be the case that trivial

extensions of this map to higher dimensional spaces do not preserve positivity. To overcome this we define completely positive maps, which are maps $\mathcal{E} : \mathcal{H}_A \rightarrow \mathcal{E}(\mathcal{H}_A) \subset \mathcal{H}_A$ such that $\mathcal{E} \otimes \mathbb{I}_B : \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow (\mathcal{E} \otimes \mathbb{I}_B)(\mathcal{H}_A \otimes \mathcal{H}_B) \subset \mathcal{H}_A \otimes \mathcal{H}_B$ is also positive, i.e.

$\mathcal{E}$ is completely positive iff

$$\forall \hat{\rho} \in \mathcal{H}_A \otimes \mathcal{H}_B, \hat{\rho} > 0, \Rightarrow \hat{\rho}' = (\mathcal{E} \otimes \mathbb{I}_B)(\hat{\rho}) > 0. \quad (3.9)$$

The PPT criterion uses the fact that, if restricted to separable states, positive and completely positive maps are equivalent. In the PPT criterion the positive, but not completely positive map used to test entanglement on a state is the transposition map $\mathcal{T}$. It is trivially positive because the eigenvalues of a matrix and its transpose are the same. But when applied to one subsystem of a composite system, which is called partial transposition, the result is a positive operator if the initial state is separable, but may not preserve positivity if the initial state is entangled.

Thus, let $\hat{\rho}_{AB}^{sep} = \sum_i p_i(\hat{\rho}_A^i \otimes \hat{\rho}_B^i)$ be a separable state composed of two parties A and B . We have that the operator $\hat{\sigma}_{AB}$ given by

$$\hat{\sigma}_{AB} = (\mathbb{I}_A \otimes \mathcal{T}_B)\hat{\rho}_{AB}^{sep} = (\mathbb{I}_A \otimes \mathcal{T}_B) \sum_i p_i(\hat{\rho}_A^i \otimes \hat{\rho}_B^i) = \sum_i p_i(\hat{\rho}_A^i \otimes \mathcal{T}(\hat{\rho}_B^i)) \quad (3.10)$$

can be also a candidate to represent another quantum state, i.e. it is also a positive operator. This because the states $\mathcal{T}(\hat{\rho}_B^i)$ are positive provided that $\hat{\rho}_B^i$ are also positive. Therefore, if the partial transposition map $\mathbb{I}_A \otimes \mathcal{T}_B$ takes a general state $\hat{\rho}_{AB}$ out of the positive definite operators set, it means that the state $\hat{\rho}_{AB}$ is an entangled state. Any other positive but not completely positive map can be used in the same way to detect entanglement.

It has been proved that the PPT criterion is necessary and sufficient for bipartite systems of the kind (2×2) and (2×3) (the notation here is *(System A dimension $\times$ System B dimension)*) [65]. But in higher dimensions there exist entangled states that have a positive partial transposition, called bound entanglement, so the criteria is no longer necessary.

Now, in the case of continuous variables (CV), the process of testing positivity is, in most of the cases, not feasible because the density operator is an infinite dimensional matrix. Nonetheless it is possible to adapt this criterion to be easily applied in CV systems. In order to do that, we will look at how the covariance matrix is affected by the partial transposition map, using for that the Wigner function [66].

The Wigner function is one of the most used quasi-probability functions in quantum mechanics; it is a representation of the density operator in a phase space. With this

function, we can calculate the expectation value of any operator in a "classical" way and it is defined as a function of the density operator as

$$W_\rho(p, q) = \frac{1}{\pi\hbar} \int \langle q+x | \hat{\rho} | q-x \rangle e^{2ixp/\hbar} dx \quad (3.11)$$

where $|x\rangle$ is an eigenstate of the position operator $\hat{x}$ (or a quadrature in the case of the electromagnetic field), $\hat{x}|x\rangle = x|x\rangle$.

The Wigner function associated with the transposed density operator correspond to a specular reflection of the p coordinate in the phase space on the Wigner function of the untransposed density operator as²

$$\begin{aligned} W_{\rho^T}(p, q) &= \frac{1}{\pi\hbar} \int \langle q+x | \hat{\rho}^T | q-x \rangle e^{2ixp/\hbar} dx \\ &= \frac{1}{\pi\hbar} \int \langle q-x | \hat{\rho} | q+x \rangle e^{2ixp/\hbar} dx \\ &= \frac{1}{\pi\hbar} \int \langle q+x | \hat{\rho} | q-x \rangle e^{-2ixp/\hbar} dx \\ &= W_\rho(-p, q). \end{aligned} \quad (3.12)$$

In the case of a bipartite system it is convenient to define the vectors

$$\boldsymbol{\xi} = \{p_1, q_1, p_2, q_2\}; \quad \hat{\boldsymbol{\xi}} = \{\hat{p}_1, \hat{q}_1, \hat{p}_2, \hat{q}_2\} \quad (3.13)$$

where the first vector contains the *c-numbers* associated to the operators which form the second vector. Following the same procedure as before the effect of the partial transposition of the density operator in the bipartite Wigner function is given by

$$\hat{\rho} \rightarrow W(\boldsymbol{\xi}) \quad (3.14)$$

$$\hat{\rho}^{T_2} \rightarrow W(\boldsymbol{\Lambda} \cdot \boldsymbol{\xi}) = W(p_1, q_1, -p_2, q_2) \quad (3.15)$$

where the matrix $\boldsymbol{\Lambda}$ is defined as $\boldsymbol{\Lambda} = \text{diag}(1, 1, -1, 1)$ and the superscript (T_2) means the partial transposition of the second subsystem.

Defining the symmetric covariance matrix in a similar way as (1.21) we have

$$V_{ij} = \frac{1}{2} \left\langle \{ \delta \hat{\xi}_i, \delta \hat{\xi}_j \} \right\rangle = \int \delta \xi_i \delta \xi_j W(\boldsymbol{\xi}) d^4 \xi. \quad (3.16)$$

²Due to the fact that the transposition operation depends on the basis in which it is done, if the Wigner function is written in terms of the matrix elements in the momentum space, the transposition operation would correspond to a specular reflection of the q coordinate, instead of the p coordinate.

This symmetric covariance matrix, as a consequence of the positivity of the density operator, is also positive semidefinite and, unlike the covariance matrix defined at (1.21), it is real and not complex.

When applying the partial transposition to the density operator of the bipartite system, the symmetric covariance matrix is transformed as

$$\hat{\rho} \rightarrow \mathbb{V} \implies \hat{\rho}^{T_2} \rightarrow \tilde{\mathbb{V}} = \mathbf{\Lambda} \cdot \mathbb{V} \cdot \mathbf{\Lambda}. \quad (3.17)$$

The separability criteria is now translated into testing if this new symmetric covariance matrix $\tilde{\mathbb{V}}$ correspond to a covariance matrix produced by a quantum state, i.e. a positive semidefinite density operator. In addition to $\tilde{\mathbb{V}}$ being positive, it also has to fulfill the following relation.

Starting from the covariance matrix $\mathbb{C}$ defined in (1.21) we have that, given that it is also positive, the matrix $\mathbb{V}$ of a quantum state must satisfy

$$C_{ij} = \langle \delta \hat{\xi}_i \delta \hat{\xi}_j \rangle = \frac{1}{2} \langle \{ \delta \hat{\xi}_i, \delta \hat{\xi}_j \} \rangle + \frac{1}{2} \langle [\delta \hat{\xi}_i, \delta \hat{\xi}_j] \rangle = V_{ij} + i\Omega_{ij} \geq 0 \quad (3.18)$$

where $\mathbf{\Omega}$ comes from the commutation relations (1.3) as (the equation above is nothing more than the uncertainty principle explained at Section 1.3)

$$[\hat{\xi}_i, \hat{\xi}_j] = 2i\Omega_{ij} \quad \mathbf{\Omega} = \begin{pmatrix} \mathbb{J} & 0 \\ 0 & \mathbb{J} \end{pmatrix} \quad \mathbb{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.19)$$

and therefore any separable state must fulfill, according to the PPT criterion that

$$\tilde{\mathbb{V}} + i\mathbf{\Omega} \geq 0 \quad (3.20)$$

Any violation of the relation above means that the initial density matrix corresponds to an entangled state. Because in this case we have infinite dimensional systems, the violation of the PPT criteria is not a necessary condition for entanglement. However, if the quantum state is describe by a Gaussian Wigner function then the PPT becomes a necessary condition for $1 \times N$ systems³ [66]. This is related with the fact that for CV Gaussian states, the Wigner function is fully described by the covariance matrix, so an infinite number of parameter is not needed in this case to characterize the state.

³this notation means that we are testing entanglement between two subsystems in which one of them is composed by one mode of the field while the second one is composed by N modes

3.2.2 Sum of Variances

Another separability criteria frequently used, and which doesn't need to calculate/measure all the covariance matrix elements, is that initially proposed by Duan *et al.* [67] and that is closely related to the EPR variables (the total momentum and the relative position).

The uncertainty principle imposes a limit for the variances of these variables that any quantum state has to fulfill, but it is possible to derive another, more flexible, inequality (sometimes called DGCZ⁴ inequality), which must be fulfilled by any separable state. Thus only entangled states have strong enough correlations for violating this limit and to approach to the limit imposed by the uncertainty principle.

Here it will be shown the demonstration made in [68] which is a generalization of the original DGCZ inequality [67] for any observable and not only for the EPR variables, and it is done basically by the same procedure.

Let $\hat{r}_i, \hat{s}_i, i = 1, 2$ be two observables, each one corresponding to the subsystem 1 or 2 respectively, that satisfy the commutation relations

$$[\hat{r}_i, \hat{s}_i] = i\hat{C}_i \quad (3.21)$$

where $\hat{C}_i$ is an hermitian operator of the subsystem i . With those operators it is possible to define global observables of the total system as

$$\hat{u} = a_1\hat{r}_1 + a_2\hat{r}_2 \quad (3.22)$$

$$\hat{v} = b_1\hat{s}_1 + b_2\hat{s}_2 \quad (3.23)$$

with a_i, b_i being real parameters. The variances of these global operators must fulfill the uncertainty relation (equation 1.26 without the anti-commutator term) for any physical quantum state $\hat{\rho}$

$$\langle \delta \hat{u}^2 \rangle \langle \delta \hat{v}^2 \rangle \geq \frac{1}{4} \left| a_1 b_1 \langle \hat{C}_1 \rangle + a_2 b_2 \langle \hat{C}_2 \rangle \right|^2. \quad (3.24)$$

It is possible to rewrite this uncertainty relation in terms of the sum of the variances rather than the product. For this, it is used the fact that the function $f(x) = \gamma_1 x + \gamma_2/x$, $\gamma_1, \gamma_2 > 0$ has as minimum value $f_{min} = 2\sqrt{\gamma_1 \gamma_2}$, thus

⁴from the authors Duan, Giedke, Cirac and Zoller

$$\begin{aligned}
\langle \delta \hat{u}^2 \rangle + \langle \delta \hat{v}^2 \rangle &\geq \langle \delta \hat{u}^2 \rangle + \frac{1}{4 \langle \delta \hat{u}^2 \rangle} \left| a_1 b_1 \langle \hat{C}_1 \rangle + a_2 b_2 \langle \hat{C}_2 \rangle \right|^2 \\
\langle \delta \hat{u}^2 \rangle + \langle \delta \hat{v}^2 \rangle &\geq \left| a_1 b_1 \langle \hat{C}_1 \rangle + a_2 b_2 \langle \hat{C}_2 \rangle \right|.
\end{aligned} \tag{3.25}$$

Any bipartite separable state according to (3.6) is given by

$$\hat{\rho}_{sep} = \sum_k p_k (\hat{\rho}_1^k \otimes \hat{\rho}_2^k), \tag{3.26}$$

in this separable state, the variance of the observable $\hat{u}$ is

$$\begin{aligned}
\langle \delta \hat{u}^2 \rangle_{sep} &= \langle \hat{u}^2 \rangle_{sep} - \langle \hat{u} \rangle_{sep}^2 \\
&= \sum_k p_k (a_1^2 \langle \hat{r}_1^2 \rangle_k + a_2^2 \langle \hat{r}_2^2 \rangle_k + 2a_1 a_2 \langle \hat{r}_1 \rangle_k \langle \hat{r}_2 \rangle_k) - \langle \hat{u} \rangle_{sep}^2 \\
&= \sum_k p_k (a_1^2 \langle (\delta \hat{r}_1^{(k)})^2 \rangle_k + a_2^2 \langle (\delta \hat{r}_2^{(k)})^2 \rangle_k) + \sum_k p_k \langle \hat{u} \rangle_k^2 - \left(\sum_k p_k \langle \hat{u} \rangle_k \right)^2
\end{aligned} \tag{3.27}$$

where the notation $\delta \hat{r}_i^{(k)} = \hat{r}_i - \langle \hat{r}_i \rangle_k$ and $\langle \cdots \rangle_k$ stands for the expectation value over the product state $\hat{\rho}_1^k \otimes \hat{\rho}_2^k$. The Cauchy-Schwartz inequality $(\sum_k p_k) \left(\sum_k p_k \langle \hat{u} \rangle_k^2 \right) \geq (\sum_k p_k \langle \hat{u} \rangle_k)^2$, taking into account the fact that $(\sum_k p_k) = 1$, tell us that the last two terms of the previous equation form a positive quantity so

$$\begin{aligned}
\sum_k p_k (a_1^2 \langle (\delta \hat{r}_1^{(k)})^2 \rangle_k + a_2^2 \langle (\delta \hat{r}_2^{(k)})^2 \rangle_k) + \sum_k p_k \langle \hat{u} \rangle_k^2 - \left(\sum_k p_k \langle \hat{u} \rangle_k \right)^2 &\geq \\
&\sum_k p_k (a_1^2 \langle (\delta \hat{r}_1^{(k)})^2 \rangle_k + a_2^2 \langle (\delta \hat{r}_2^{(k)})^2 \rangle_k).
\end{aligned} \tag{3.28}$$

Putting this result together with an equivalent inequality for the observable $\langle \delta \hat{v}^2 \rangle_{sep}$ we have

$$\begin{aligned}
\langle \delta \hat{u}^2 \rangle_{sep} + \langle \delta \hat{v}^2 \rangle_{sep} &\geq \sum_k p_k \left(a_1^2 \langle (\delta \hat{r}_1^{(k)})^2 \rangle_k + b_1^2 \langle (\delta \hat{s}_1^{(k)})^2 \rangle_k + a_2^2 \langle (\delta \hat{r}_2^{(k)})^2 \rangle_k + b_2^2 \langle (\delta \hat{s}_2^{(k)})^2 \rangle_k \right).
\end{aligned} \tag{3.29}$$

Finally, with help of the uncertainty relation for the operator r_i and s_i in the sum form $a_1^2 \langle (\delta \hat{r}_i^{(k)})^2 \rangle_k + b_1^2 \langle (\delta \hat{s}_i^{(k)})^2 \rangle_k \geq |a_i b_i \langle \hat{C}_i \rangle_k|$ it is possible to deduce that

$$\langle \delta \hat{u}^2 \rangle_{sep} + \langle \delta \hat{v}^2 \rangle_{sep} \geq \sum_k p_k \left(|a_1 b_1 \langle \hat{C}_1 \rangle_k| + |a_2 b_2 \langle \hat{C}_2 \rangle_k| \right). \tag{3.30}$$

Also the triangle inequality tell us that $\sum_k p_k \left| \langle \hat{C}_i \rangle_k \right| \geq \left| \sum_k p_k \langle \hat{C}_i \rangle_k \right| = \left| \langle \hat{C}_i \rangle \right|$, and finally we have that any separable state must fulfill

$$\langle \delta \hat{u}^2 \rangle_{sep} + \langle \delta \hat{v}^2 \rangle_{sep} \geq \left| a_1 b_1 \langle \hat{C}_1 \rangle \right| + \left| a_2 b_2 \langle \hat{C}_2 \rangle \right|. \quad (3.31)$$

The inequality above imposes a stronger restriction on the separable states than the one imposed for the uncertainty principle due to the fact that $\left| a_1 b_1 \langle \hat{C}_1 \rangle \right| + \left| a_2 b_2 \langle \hat{C}_2 \rangle \right| \geq \left| a_1 b_1 \langle \hat{C}_1 \rangle + a_2 b_2 \langle \hat{C}_2 \rangle \right|$. As an example we can go back to the original EPR variables $\hat{u} = \hat{x}_1 - \hat{x}_2$ and $\hat{v} = \hat{p}_1 + \hat{p}_2$. In this case the right hand side of (3.25) is equal to zero because the EPR variables commute between each other, but for a separable state the variances of the EPR variables can not be arbitrary small, due to the fact that the right hand side of the separability inequality (3.31) is equal to 2. Therefore, only entangled states have strong enough correlations between the positions and momenta such that they can violate the separability inequality and approach the saturation of the uncertainty inequality.

Given that we are interested mostly on the spectral properties of the fields through the noise spectral densities defined in section (1.4.4). We would like to have an equivalent of (3.31) valid in the frequency domain.

$$S_u(\omega) + S_v(\omega) \geq \left| a_1 b_1 S_{[r_1, s_1]}(\omega) \right| + \left| a_2 b_2 S_{[r_2, s_2]}(\omega) \right| \quad (3.32)$$

where we have defined $S_{[r_i, s_i]}(\omega)$ as $S_{[r_i, s_i]}(\omega) \delta(\omega + \omega') = \langle [\hat{r}_i(\omega), \hat{s}_i(\omega')] \rangle$ and $\hat{u}$ and $\hat{v}$ defined as

$$\begin{aligned} \hat{u}(\omega) &= a_1(\omega) \hat{r}_1(\omega) + a_2(\omega) \hat{r}_2(\omega) \\ \hat{v}(\omega) &= b_1(\omega) \hat{s}_1(\omega) + b_2(\omega) \hat{s}_2(\omega) \end{aligned}$$

with $a_i(\omega)^* = a_i(-\omega)$ (and the same for b_i) which implies that $\hat{u}(t)$ and $\hat{v}(t)$ are self-adjoint operators.

Frequency domain inequalities as (3.32) are widely used although, up to my knowledge, no formal proof of it can be found in the literature⁵.

From the definition of noise spectral density (1.77) we have that

$$\begin{aligned} S_u(\omega) \delta(\omega + \omega') &= \langle \delta \hat{u}(\omega) \delta \hat{u}(\omega') \rangle = \langle \hat{u}(\omega) \hat{u}(\omega') \rangle - \langle \hat{u}(\omega) \rangle \langle \hat{u}(\omega') \rangle \\ &= a_1(\omega) a_1(\omega') \langle \hat{r}_1(\omega) \hat{r}_1(\omega') \rangle + a_2(\omega) a_2(\omega') \langle \hat{r}_2(\omega) \hat{r}_2(\omega') \rangle + \\ &\quad a_1(\omega) a_2(\omega') \langle \hat{r}_1(\omega) \hat{r}_2(\omega') \rangle + a_2(\omega) a_1(\omega') \langle \hat{r}_2(\omega) \hat{r}_1(\omega') \rangle - \langle \hat{u}(\omega) \rangle \langle \hat{u}(\omega') \rangle. \end{aligned}$$

⁵In [52], published on 2005, the authors point out the same fact.

As before, for a separable state, the system is described by a classical statistical mixture of uncorrelated systems, i.e. the average of product of two operators belonging to different subsystems have the form $\langle \hat{r}_i \hat{r}_j \rangle = \sum_k p_k \langle \hat{r}_i \rangle_k \langle \hat{r}_j \rangle_k$ $i \neq j$ where $\langle \dots \rangle_k$ is the expectation value on the k -th realization of the ensemble and p_k is the probability of having such realization and therefore $\sum_k p_k = 1$. With the definition $\delta^k \hat{r} = \hat{r} - \langle \hat{r} \rangle_k$ we have that

$$\begin{aligned}
\langle \delta \hat{u}(\omega) \delta \hat{u}(\omega') \rangle &= S_u(\omega) \delta(\omega + \omega') \\
&= \sum_k p_k [a_1(\omega) a_1(\omega') \langle \hat{r}_1(\omega) \hat{r}_1(\omega') \rangle_k + a_2(\omega) a_2(\omega') \langle \hat{r}_2(\omega) \hat{r}_2(\omega') \rangle_k + \\
&\quad a_1(\omega) a_2(\omega') \langle \hat{r}_1(\omega) \rangle_k \langle \hat{r}_2(\omega') \rangle_k + a_2(\omega) a_1(\omega') \langle \hat{r}_2(\omega) \rangle_k \langle \hat{r}_1(\omega') \rangle_k] - \langle \hat{u}(\omega) \rangle \langle \hat{u}(\omega') \rangle. \\
&= \sum_k p_k [a_1(\omega) a_1(\omega') \langle \delta^k \hat{r}_1(\omega) \delta^k \hat{r}_1(\omega') \rangle_k + a_2(\omega) a_2(\omega') \langle \delta^k \hat{r}_2(\omega) \delta^k \hat{r}_2(\omega') \rangle_k + \\
&\quad a_1(\omega) a_1(\omega') \langle \hat{r}_1(\omega) \rangle_k \langle \hat{r}_1(\omega') \rangle_k + a_2(\omega) a_2(\omega') \langle \hat{r}_2(\omega) \rangle_k \langle \hat{r}_2(\omega') \rangle_k + \\
&\quad a_1(\omega) a_2(\omega') \langle \hat{r}_1(\omega) \rangle_k \langle \hat{r}_2(\omega') \rangle_k + a_2(\omega) a_1(\omega') \langle \hat{r}_2(\omega) \rangle_k \langle \hat{r}_1(\omega') \rangle_k] - \langle \hat{u}(\omega) \rangle \langle \hat{u}(\omega') \rangle. \\
&= \sum_k p_k \left(a_1(\omega) a_1(\omega') \langle \delta \hat{r}_1^{(k)}(\omega) \delta \hat{r}_1^{(k)}(\omega') \rangle_k + a_2(\omega) a_2(\omega') \langle \delta \hat{r}_2^{(k)}(\omega) \delta \hat{r}_2^{(k)}(\omega') \rangle_k \right) \\
&\quad + \sum_k p_k \langle \hat{u}(\omega) \rangle_k \langle \hat{u}(\omega') \rangle_k - \left(\sum_k p_k \langle \hat{u}(\omega) \rangle_k \right) \left(\sum_k p_k \langle \hat{u}(\omega') \rangle_k \right) \\
&= \sum_k p_k \left(|a_1(\omega)|^2 S_{r_1}^{(k)}(\omega) + |a_2(\omega)|^2 S_{r_2}^{(k)}(\omega) \right) \delta(\omega + \omega') \\
&\quad + \sum_k p_k \langle \hat{u}(\omega) \rangle_k \langle \hat{u}(\omega') \rangle_k - \left(\sum_k p_k \langle \hat{u}(\omega) \rangle_k \right) \left(\sum_k p_k \langle \hat{u}(\omega') \rangle_k \right). \tag{3.33}
\end{aligned}$$

Now, we define the quantity

$$A_k = \int_{\omega_0 - \epsilon_1}^{\omega_0 + \epsilon_2} \langle \hat{u}(\omega) \rangle_k d\omega \tag{3.34}$$

with ω_0 and $\epsilon_1, \epsilon_2 > 0$ are arbitrary. Thus, starting from the trivial inequality

$$\sum_{km} p_k p_m |A_k - A_m|^2 \geq 0 \tag{3.35}$$

after some simple algebraic manipulation and using that the operator $\hat{u}(t)$ is hermitian and therefore $[\hat{u}(\omega)]^\dagger = \hat{u}(-\omega)$, it is easy to show that

$$\begin{aligned}
\sum_{km} p_k p_m |A_k - A_m|^2 &= 2 \int_{\omega_0 - \epsilon_1}^{\omega_0 + \epsilon_2} d\omega \int_{\omega_0 - \epsilon_1}^{\omega_0 + \epsilon_2} d\omega' \left[\sum_k p_k \langle \hat{u}(\omega) \rangle_k \langle \hat{u}(-\omega') \rangle_k \right. \\
&\quad \left. - \left(\sum_k p_k \langle \hat{u}(\omega) \rangle_k \right) \left(\sum_k p_k \langle \hat{u}(-\omega') \rangle_k \right) \right]. \tag{3.36}
\end{aligned}$$

From (3.33), the term inside square brackets is of the form $f(\omega) \delta(\omega - \omega')$ with

$$f(\omega) = S_u(\omega) - \sum_k p_k \left(|a_1(\omega)|^2 S_{r_1}^{(k)}(\omega) + |a_2(\omega)|^2 S_{r_2}^{(k)}(\omega) \right). \quad (3.37)$$

Integrating over the Dirac's delta function we have

$$\sum_{km} p_k p_m |A_k - A_m|^2 = 2 \int_{\omega_0 - \epsilon_1}^{\omega_0 + \epsilon_2} f(\omega) d\omega \geq 0 \Rightarrow f(\omega) \geq 0. \quad (3.38)$$

Given that ω_0 , ϵ_1 and ϵ_2 are arbitrary quantities. From the definition of $f(\omega)$ we have that

$$S_u(\omega) \geq \sum_k p_k \left(|a_1(\omega)|^2 S_{r_1}^{(k)}(\omega) + |a_2(\omega)|^2 S_{r_2}^{(k)}(\omega) \right). \quad (3.39)$$

From here, the procedure is analog of the one used before. Putting the inequality above together with the equivalent one for the quantity $S_v(\omega)$ and using the uncertainty like inequality on the frequency domain (1.80) on the form $S_X(\omega) + S_Y(\omega) \geq |S_{[X,Y]}(\omega)|$ we have that

$$\begin{aligned} S_u(\omega) + S_v(\omega) &\geq \sum_k p_k \left(|a_1(\omega)|^2 S_{r_1}^{(k)}(\omega) + |b_1(\omega)|^2 S_{s_1}^{(k)}(\omega) + |a_2(\omega)|^2 S_{r_2}^{(k)}(\omega) + |b_2(\omega)|^2 S_{s_2}^{(k)}(\omega) \right) \\ &\geq \sum_k p_k \left(\left| a_1(\omega) b_1(\omega) S_{[r_1, s_1]}^{(k)}(\omega) \right| + \left| a_2(\omega) b_2(\omega) S_{[r_2, s_2]}^{(k)}(\omega) \right| \right) \\ &\geq \left| \sum_k p_k a_1(\omega) b_1(\omega) S_{[r_1, s_1]}^{(k)}(\omega) \right| + \left| \sum_k p_k a_2(\omega) b_2(\omega) S_{[r_2, s_2]}^{(k)}(\omega) \right| \\ &= \left| a_1(\omega) b_1(\omega) S_{[r_1, s_1]}(\omega) \right| + \left| a_2(\omega) b_2(\omega) S_{[r_2, s_2]}(\omega) \right| \end{aligned} \quad (3.40)$$

which demonstrates equation (3.32).

The concepts introduced in the text until here will allow us to make a model and to study the desired properties of the $\chi^{(3)}$ based OPO in the next chapter.

Chapter 4

Optical Parametric Oscillator: $\chi^{(2)}$ vs $\chi^{(3)}$

In this chapter we will use the theory presented in the previous chapters in order to make a description of the quantum state of the light generated in a silicon based micro resonator. The approach we will take is to begin with the most simple case making all suitable approximations, then gradually rise the level of complexity to have a more general view of the properties of the light that can be generated on silicon based nonlinear microcavity. At first, we will study the system of just three modes generated with the process of four wave mixing (FWM) and compare it with the process that occurs in a $\chi^{(2)}$ medium, which is the most common type of nonlinearity used for the generation of non-classical states of light. The description of the light generated in an $\chi^{(2)}$ OPO can be found in several places including some of the theses produced at the LMCAL as for example [69] and will not be introduced here on details, just some result will be.

Although the main objective is to describe an on chip microcavity, the theory developed here is applicable to any kind of OPO based on a $\chi^{(3)}$ process as long as the assumptions that will be done are still fulfilled.

Hamiltonian First, let us consider a lossless ring cavity with an, also lossless (losses will be introduced later), non-linear medium (NLM) of length l inside it (see fig (4.1)). This medium can have either a $\chi^{(2)}$ or $\chi^{(3)}$ non-linearity. The cavity is pumped with a continuous wave (CW) field of frequency ν_p and generates two other light beams called signal and idler of frequencies ν_s and ν_i respectively.

In this part, the description of the OPO is made with propagating waves and not with cavity modes.

FIGURE 4.1: OPO schematics consisting in a ring cavity with a nonlinear medium of length l inside. M_c stands for the coupling mirror, $\hat{A}_{in}$ are the incoming fields to the cavity. Inside the cavity the fields propagate in the clockwise direction. $\hat{A}_1$ are the intracavity fields just before the nonlinear medium and after M_c , $\hat{A}_2$ are the intracavity fields after completing a round trip inside the cavity and just before M_c and l in the length of the nonlinear medium.

If the NLM medium has a $\chi^{(2)}$ nonlinearity, the interaction Hamiltonian is given by

$$\hat{H}_{int}^{(2)} = \hbar \zeta^{(2)} \left(\hat{a}_p^\dagger \hat{a}_s \hat{a}_i e^{-i\Delta k_2 z} + h.c. \right), \quad (4.1)$$

where " $h.c$ " stands for the hermitian conjugate, $\hat{a}_j$, $j = p, s, i$ are the annihilation operators of the pump, signal and idler propagating modes, $\Delta k_2 = k_p - k_s - k_i$ is the phase mismatch for the $\chi^{(2)}$ process, $\zeta^{(2)}$ is proportional to the second order nonlinear coefficient, and the mode frequencies are related by the conservation of energy $\nu_p = \nu_s + \nu_i$. The phase mismatch is not, in general, null because here the wave number $k_j = k(\nu_j) = \nu_j n(\nu_j)/c$ is given by the refractive index $n(\nu)$ which in the case of a dispersive material it is frequency dependent. This Hamiltonian describes the process of parametric down conversion (PDC) where one photon from the pump is annihilated and a pair of "twins" photons, called signal and idler, are simultaneously generated.

This Hamiltonian leads to the followings propagation equations inside the NLM

$$\begin{aligned} \frac{d\hat{a}_p}{dz} &= -i\zeta^{(2)} \hat{a}_s \hat{a}_i e^{-i\Delta k_2 z} \\ \frac{d\hat{a}_s}{dz} &= -i\zeta^{(2)} \hat{a}_p \hat{a}_i^\dagger e^{i\Delta k_2 z} \\ \frac{d\hat{a}_i^\dagger}{dz} &= i\zeta^{(2)} \hat{a}_p^\dagger \hat{a}_s e^{-i\Delta k_2 z}. \end{aligned} \quad (4.2)$$

On the other hand, if we have a $\chi^{(3)}$ NLM the interaction Hamiltonian inside it takes the following more complicated form

$$\begin{aligned} \hat{H}_{int}^{(3)} = & -\hbar\zeta^{(3)} \left[\frac{1}{2} \left(\hat{a}_p^\dagger \hat{a}_p^\dagger \hat{a}_p \hat{a}_p + \hat{a}_s^\dagger \hat{a}_s^\dagger \hat{a}_s \hat{a}_s + \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \right) + \right. \\ & \left. 2 \left(\hat{a}_p^\dagger \hat{a}_s^\dagger \hat{a}_p \hat{a}_s + \hat{a}_p^\dagger \hat{a}_i^\dagger \hat{a}_p \hat{a}_i + \hat{a}_s^\dagger \hat{a}_i^\dagger \hat{a}_s \hat{a}_i \right) + \left(\hat{a}_s^\dagger \hat{a}_i^\dagger \hat{a}_p \hat{a}_p e^{-i\Delta k_3 z} + \hat{a}_p^\dagger \hat{a}_p^\dagger \hat{a}_s \hat{a}_i e^{i\Delta k_3 z} \right) \right]. \end{aligned} \quad (4.3)$$

The first term in parenthesis is the so called self-phase modulation (SPM) term, the second one is the cross-phase modulation (XPM) and the last one is the FWM term. The phase modulation terms commute with the number operators $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$, $j = p, s, i$ so they do not affect the intensity of the fields, meanwhile the FWM term is the responsible for the energy interchange between the three fields.

The phase mismatch in this case is given by $\Delta k_3 = 2k_p - k_s - k_i$, $\zeta^{(3)}$ is proportional to the third order nonlinear coefficient $\chi^{(3)}$ and here energy conservation implies the relation between the fields frequencies $2\nu_p = \nu_s + \nu_i$.

In this case the propagation equations are

$$\begin{aligned} \frac{d\hat{a}_p}{dz} &= i\zeta^{(3)} \left(\hat{a}_p^\dagger \hat{a}_p + 2\hat{a}_s^\dagger \hat{a}_s + 2\hat{a}_i^\dagger \hat{a}_i \right) \hat{a}_p + i\zeta^{(3)} \hat{a}_p^\dagger \hat{a}_s \hat{a}_i e^{i\Delta k_3 z} \\ \frac{d\hat{a}_s}{dz} &= i\zeta^{(3)} \left(2\hat{a}_p^\dagger \hat{a}_p + \hat{a}_s^\dagger \hat{a}_s + 2\hat{a}_i^\dagger \hat{a}_i \right) \hat{a}_s + i\zeta^{(3)} \hat{a}_p^2 \hat{a}_i^\dagger e^{-i\Delta k_3 z} \\ \frac{d\hat{a}_i}{dz} &= i\zeta^{(3)} \left(2\hat{a}_p^\dagger \hat{a}_p + 2\hat{a}_s^\dagger \hat{a}_s + \hat{a}_i^\dagger \hat{a}_i \right) \hat{a}_i + i\zeta^{(3)} \hat{a}_p^2 \hat{a}_s^\dagger e^{-i\Delta k_3 z}. \end{aligned} \quad (4.4)$$

Propagation equations (4.2) and (4.4) are the quantum version of the equation (2.9) in the situation when we have only three different frequencies and the nonlinear polarization $\mathbf{P}_{NL}$ contains just either the $\chi^{(2)}$ or the $\chi^{(3)}$ term respectively.

4.1 No Pump Depletion

In this section we will start by restricting ourselves to the classical scenario, so we make the substitution $\hat{a}_j \rightarrow \alpha_j$ and $\hat{a}_j^\dagger \rightarrow \alpha_j^*$, $j = p, s, i$ on the propagation equations (4.2) and (4.4). We will also consider the case where the pump amplitude is constant, i.e. it has no depletion. In this situation, the pump field is much stronger than the signal and idler fields and its depletion is negligible. Formally we neglect terms of second order in the signal-idler amplitudes on the propagation equations, and finally we end up with a set of linear equations for the propagation of the signal and idler modes.

Within the approximations stated above the propagation equations for the $\chi^{(2)}$ process are the following

$$\begin{aligned}\frac{d\alpha_p}{dz} &\approx 0 \quad \rightarrow \alpha_p(z) = \alpha_p(0) \\ \frac{d\alpha_s}{dz} &\approx -i\zeta^{(2)}\alpha_p(0)\alpha_i^*e^{i\Delta k_2 z} \\ \frac{d\alpha_i^*}{dz} &\approx i\zeta^{(2)}\alpha_p(0)^*\alpha_s e^{-i\Delta k_2 z},\end{aligned}\tag{4.5}$$

and for the $\chi^{(3)}$ case

$$\begin{aligned}\frac{d\alpha_p}{dz} &\approx i\zeta^{(3)}|\alpha_p|^2\alpha_p \quad \rightarrow \alpha_p(z) = \alpha_p(0)e^{i\zeta^{(3)}P_p z}; \quad P_p = |\alpha_p(0)|^2 \\ \frac{d\alpha_s}{dz} &\approx i2\zeta^{(3)}|\alpha_p|^2\alpha_s + i\zeta^{(3)}\alpha_p(0)^2e^{i(2\zeta^{(3)}P_p - \Delta k_3)z}\alpha_i^* \\ \frac{d\alpha_i^*}{dz} &\approx -i2\zeta^{(3)}|\alpha_p|^2\alpha_i - i\zeta^{(3)}\alpha_p(0)^*2e^{-i(2\zeta^{(3)}P_p - \Delta k_3)z}\alpha_s.\end{aligned}\tag{4.6}$$

As we can see again, the main differences between the $\chi^{(2)}$ and $\chi^{(3)}$ rise from the effect of the phase modulation term. The pump amplitude¹ remains constant while it propagates through a $\chi^{(2)}$ NLM, but it acquires a phase on the $\chi^{(3)}$ NLM due to the SPM (Kerr effect). The effect of XPM is present on the propagation of the signal and idler beam through the first term of the equations above and also affects the FWM process given that it is phase dependent (the phase determines the direction of the energy flow).

Both, $\chi^{(2)}$ and $\chi^{(3)}$ systems of equations have the same form (linear set of equation on the signal and idler amplitudes) and, therefore, the same solution which can be analytically written as

$$\begin{aligned}\alpha_s(z) &= \mu\alpha_s(0) + \nu\alpha_i^*(0) \\ \alpha_i(z) &= \mu\alpha_i(0) + \nu\alpha_s^*(0)\end{aligned}\tag{4.7}$$

where the coefficients μ and ν are, of course, different for the $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearity. In the $\chi^{(2)}$ process, their explicit form are

$$\begin{aligned}\mu_2 = \mu_2(z) &= \left(\cosh(g_2 z) - \frac{i\Delta k_2}{2g_2} \sinh(g_2 z) \right) e^{i\frac{\Delta k_2 z}{2}} \\ \nu_2 = \nu_2(z) &= \frac{-i\zeta^{(2)}\alpha_p(0)}{g_2} \sinh(g_2 z) e^{i\frac{\Delta k_2 z}{2}} \\ \text{with } g_2 &= \sqrt{(\zeta^{(2)})^2 P_p - \left(\frac{\Delta k_2}{2}\right)^2}\end{aligned}\tag{4.8}$$

¹Remember that these are slowly varying amplitudes, so the fast evolution of the phase in the form kz was removed

and in the $\chi^{(3)}$ case

$$\begin{aligned}\mu_3 = \mu_3(z) &= \left(\cosh(g_3 z) + \frac{i\kappa}{2g_3} \sinh(g_3 z) \right) e^{i\left(\zeta^{(3)} P_p + \frac{\Delta k_3}{2}\right)z} \\ \nu_3 = \nu_3(z) &= \frac{i\zeta^{(3)} \alpha_p(0)^2}{g_3} \sinh(g_3 z) e^{i\left(\zeta^{(3)} P_p + \frac{\Delta k_3}{2}\right)z} \\ \text{with } g_3 &= \sqrt{\left(\zeta^{(3)} P_p\right)^2 - \left(\frac{\kappa}{2}\right)^2} \text{ and } \kappa = 2\zeta^{(3)} P_p - \Delta k_3\end{aligned}\quad (4.9)$$

It is possible to show explicitly that the coefficients μ and ν fulfill the following relation

$$|\mu|^2 - |\nu|^2 = 1 \quad (4.10)$$

as it should be as long as transformation (4.7) is a symplectic transformation².

The coefficient $g_i, i = 2, 3$ is sometimes called the gain rate of the process, since we have an exponential amplification through the propagation inside the NLM (given by the sinh and cosh terms) and g_i is the exponential growth rate constant. The behavior of this constant is different depending on whether we have a $\chi^{(2)}$ or a $\chi^{(3)}$ process. For instance, at a given pump power in order to maximize the gain of the system you have to get a null phase mismatch $\Delta k_2 = 0$ in a $\chi^{(2)}$ case, but in a $\chi^{(3)}$ situation you will have the biggest growth at a null 'non linear' phase mismatch $\kappa = 2\zeta^{(3)} P_p - \Delta k_3 = 0$.

4.1.1 Input-Output relation

As we have a linear input-output relation between the incoming signal and idler fields at the entrance and exit of the NLM we can construct a vector $\mathbf{A} = \{\alpha_s, \alpha_s^*, \alpha_i, \alpha_i^*\}^T$ and write the relation between the fields at the entrance of the NLM, $\mathbf{A}(0)$ and at the exit, $\mathbf{A}(l)$ in a matrix form as

$$\mathbf{A}(l) = \mathbb{S}(l) \cdot \mathbf{A}(0); \quad \mathbb{S}(l) = \begin{pmatrix} \mu(l) & 0 & 0 & \nu(l) \\ 0 & \mu^*(l) & \nu^*(l) & 0 \\ 0 & \nu(l) & \mu(l) & 0 \\ \nu^*(l) & 0 & 0 & \mu^*(l) \end{pmatrix} \quad (4.11)$$

We can also write the relations between the fields at the coupling mirror M_c (see fig (4.1)) which connects the intracavity fields with the input and output fields

²A symplectic transformation preserves commutation relations in the quantum case or Poisson's brackets in the classical one

$$\begin{aligned} \mathbf{A}_1 &= r\mathbf{A}_2 + t\mathbf{A}_{in} \\ \mathbf{A}_{out} &= -r\mathbf{A}_{in} + t\mathbf{A}_2 \end{aligned} \quad r^2 + t^2 = 1 \quad (4.12)$$

while the propagation through the rest of the cavity (from the exit of the NLM to just before M_c) is given by

$$\mathbf{A}_2 = \exp(i\Phi)\mathbf{A}(l) \quad \Phi = \text{diag}\{\phi_s, -\phi_s, \phi_i, -\phi_i\} \quad (4.13)$$

with ϕ_s and ϕ_i being the phase gained for the signal and idler field respectively. Combining these last three equations we can write an input-output relation (identifying $\mathbf{A}_1 \rightarrow \mathbf{A}(0)$)

$$\mathbf{A}_{out} = \mathbb{R} \cdot \mathbf{A}_{in} \quad \mathbb{R} = (\mathbb{S} \exp(i\Phi) - r\mathbb{I})(\mathbb{I} - r\mathbb{S} \exp(i\Phi))^{-1} \quad (4.14)$$

It is relevant to say that this linear input-output relation limits our solution to the regime of amplification. The oscillation (i.e. non-zero amplitude for signal and idler, without being seeded, above some pump power (*threshold*)) is not present in this solution. This is a consequence of the undepleted pump assumption that transforms the nonlinear interaction into a linear relation between signal and idler. Nevertheless, this solution is also useful for the calculations of the fluctuations under this threshold as we will see below.

4.1.2 Quantum fluctuations

All the results that were obtained under the assumptions of an undepleted pump until here, are also valid for the quantum operators $\hat{a}_s$ and $\hat{a}_i$ because in the case of an undepleted classical pump, the evolution equations are linear in the signal and idler amplitudes and no ordering problem will rise when we change these amplitudes for their corresponding bosonic operators. This allows us to easily calculate the quantum fluctuations of the field as follows³

$$\mathbb{V}^a = \langle \delta \hat{\mathbf{A}} \delta \hat{\mathbf{A}}^{\dagger T} \rangle = \begin{pmatrix} \langle \delta \hat{a}_s \delta \hat{a}_s^\dagger \rangle & \langle \delta \hat{a}_s \delta \hat{a}_s \rangle & \langle \delta \hat{a}_s \delta \hat{a}_i^\dagger \rangle & \langle \delta \hat{a}_s \delta \hat{a}_i \rangle \\ \langle \delta \hat{a}_s^\dagger \delta \hat{a}_s^\dagger \rangle & \langle \delta \hat{a}_s^\dagger \delta \hat{a}_s \rangle & \langle \delta \hat{a}_s^\dagger \delta \hat{a}_i^\dagger \rangle & \langle \delta \hat{a}_s^\dagger \delta \hat{a}_i \rangle \\ \langle \delta \hat{a}_i \delta \hat{a}_s^\dagger \rangle & \langle \delta \hat{a}_i \delta \hat{a}_s \rangle & \langle \delta \hat{a}_i \delta \hat{a}_i^\dagger \rangle & \langle \delta \hat{a}_i \delta \hat{a}_i \rangle \\ \langle \delta \hat{a}_s \delta \hat{a}_i^\dagger \rangle & \langle \delta \hat{a}_i^\dagger \delta \hat{a}_s \rangle & \langle \delta \hat{a}_i^\dagger \delta \hat{a}_i^\dagger \rangle & \langle \delta \hat{a}_i^\dagger \delta \hat{a}_i \rangle \end{pmatrix}, \quad (4.15)$$

³In this work, the symbol † applied to a vector of operators means the vector of the hermitian adjoint of the operators without transposition, mean while when it is applied to a matrix or vector of *c-numbers* it means the conjugate transpose

where we have used the fluctuation operator vector $\delta\hat{\mathbf{A}} = \{\delta\hat{a}_s, \delta\hat{a}_s^\dagger, \delta\hat{a}_i, \delta\hat{a}_i^\dagger\}^T$, and $\mathbb{V}$ is the covariance matrix (in this case, written in terms of the annihilation and creator operators instead of the quadratures as in sections 1.3 and 3.2.1). Then the covariance matrix of the output field in terms of the input covariance matrix is

$$\mathbb{V}_{out}^a = \mathbb{R} \cdot \mathbb{V}_{in}^a \cdot \mathbb{R}^\dagger. \quad (4.16)$$

We can also write any of the equations above in terms of the quadratures of the field defined as

$$\begin{aligned} \hat{x}(\varphi) &= \frac{1}{\sqrt{2}} \left(\hat{a}e^{-i\varphi} + \hat{a}^\dagger e^{i\varphi} \right) \\ \hat{y}(\varphi) &= \hat{x}\left(\varphi + \frac{\pi}{2}\right) = \frac{-i}{\sqrt{2}} \left(\hat{a}e^{-i\varphi} - \hat{a}^\dagger e^{i\varphi} \right) \\ [\hat{x}(\varphi), \hat{y}(\varphi)] &= i. \end{aligned} \quad (4.17)$$

Given that definition, we construct another vector $\hat{\mathbf{X}} = \{\hat{x}_s, \hat{y}_s, \hat{x}_i, \hat{y}_i\}^T$ which is related with the vector $\hat{\mathbf{A}} = \{\hat{a}_s, \hat{a}_s^\dagger, \hat{a}_i, \hat{a}_i^\dagger\}^T$ by a linear transformation

$$\hat{\mathbf{X}} = \mathbb{U}_x^a \cdot \hat{\mathbf{A}} \quad (4.18)$$

$$\mathbb{U}_x^a = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi_s} & e^{i\varphi_s} & 0 & 0 \\ -ie^{-i\varphi_s} & ie^{i\varphi_s} & 0 & 0 \\ 0 & 0 & e^{-i\varphi_i} & e^{i\varphi_i} \\ 0 & 0 & -ie^{-i\varphi_i} & ie^{i\varphi_i} \end{pmatrix}. \quad (4.19)$$

In this new representation the equation (4.11) with $\varphi_s = \varphi_i = 0$ is written as

$$\begin{aligned} \hat{\mathbf{X}}_{out} &= \mathbb{R}_x \cdot \hat{\mathbf{X}}_{in} \quad \mathbb{R}_x = (\mathbb{S}_x - r\mathbb{I})(\mathbb{I} - r\mathbb{S}_x)^{-1} \\ \hat{\mathbf{X}}(l) &= \mathbb{S}_x(l) \cdot \hat{\mathbf{X}}(0) \\ \mathbb{S}_x(l) &= \begin{pmatrix} \Re(\mu) & -\Im(\mu) & \Re(\nu) & \Im(\nu) \\ \Im(\mu) & \Re(\mu) & \Im(\nu) & -\Re(\nu) \\ \Re(\nu) & \Im(\nu) & \Re(\mu) & -\Im(\mu) \\ \Im(\nu) & -\Re(\nu) & \Im(\mu) & \Re(\mu) \end{pmatrix} \end{aligned} \quad (4.20)$$

Where $\Re$ and $\Im$ stand for the real and imaginary part respectively. Another interesting representation is by means of the sum and subtraction subspaces defined as

$$\begin{aligned}\hat{x}_{\pm} &= \frac{1}{\sqrt{2}} (\hat{x}_s \pm \hat{x}_i) \\ \hat{y}_{\pm} &= \frac{1}{\sqrt{2}} (\hat{y}_s \pm \hat{y}_i) \\ \hat{\mathbf{X}}_{pm} &= \{\hat{x}_+, \hat{y}_+, \hat{x}_-, \hat{y}_-\}^T = \mathbb{U}_{pm}^x \cdot \hat{\mathbf{X}}\end{aligned}$$

with

$$\mathbb{U}_{pm}^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad (4.21)$$

in which (4.11) takes the form

$$\begin{aligned}\hat{\mathbf{X}}_{pm}(l) &= \mathbb{S}_{pm}(l) \cdot \hat{\mathbf{X}}_{pm}(0) \\ \mathbb{S}_{pm}(l) &= \begin{pmatrix} \Re(\mu + \nu) & \Im(\nu - \mu) & 0 & 0 \\ \Im(\mu + \nu) & \Re(\mu - \nu) & 0 & 0 \\ 0 & 0 & \Re(\mu - \nu) & -\Im(\mu + \nu) \\ 0 & 0 & \Im(\mu - \nu) & \Re(\mu + \nu) \end{pmatrix}. \end{aligned} \quad (4.22)$$

This representation is particularly interesting because in this sum-subtraction basis, the modes are decoupled from each other.

The covariance matrices in those representations are given by a similarity transformation of (4.16) as

$$\begin{aligned}\mathbb{V}_{out}^x &= \mathbb{U}_x^a \cdot \mathbb{V}_{out}^a \cdot \mathbb{U}_x^{a\dagger} \\ \mathbb{V}_{out}^{pm} &= \mathbb{U}_{pm}^x \cdot \mathbb{V}_{out}^x \cdot \mathbb{U}_{pm}^{x\dagger}\end{aligned} \quad (4.23)$$

Next, with all the analytical formulas developed above, we proceed to analyze the squeezing generated for the systems described within this approach. We will analyze the same set of parameters used in [70] so we will set $\Delta k_2 = \Delta k_3 = 3.9718m^{-1}$, $l = 0.0018m$ and $r = 65/66$ which were taken from real experimental setups. As we can see from equations (4.8-4.9) the output fields and, as a consequence, the covariance matrix depend explicitly of the incoming pump phase through the term $\alpha_p(0) = A_p(0)e^{i\theta_p(0)}$ on the ν coefficient. Thus, we will take two different values for this parameter, the same one used in [70] of $\theta_p = 1.59\pi$ in order to compare, and we will also use $\theta_p = \theta_{opt}$ defined as the optimal value of the pump phase to obtain a maximum squeezing value.

Figure 4.2 shows plots of the quadrature fluctuations of the sum and difference modes in the case of a $\chi^{(2)}$ process as a function of the pump amplitude. At the left side there are the fluctuations for $\theta_p = 1.59\pi$. For the fluctuations of the quadratures $\hat{Y}_+$ and $\hat{X}_-$ we have that $\Delta^2 \hat{Y}_+ = \Delta^2 \hat{X}_-$ ⁴ and they show squeezing at almost all the values of the pump power except for values around $\chi^{(2)} A_p \approx 8.48195$ which is the value that is identified as the "threshold" in [70]⁵. For the $\hat{X}_+$ and $\hat{Y}_-$ quadratures have that $\Delta^2 \hat{X}_+ = \Delta^2 \hat{Y}_-$ and they show noise excess for all values of A_p . We also see that the fluctuations of all the four quadratures diverge at the "threshold" value and go to the shot noise for $A_p = 0$ and $A_p \rightarrow \infty$. At the right side of figure 4.2 we have $\theta_p = \theta_{opt} = 1.499\pi$, the value of the pump phase that give us the best squeezing. In this case, as opposed to the former case, we have squeezing for all the values of A_p even getting closed to total squeezing ($\Delta^2 \hat{X}_+ = \Delta^2 \hat{Y}_- \approx 0$) at $\chi^{(2)} A_p \approx 8.48149$ really close to the "threshold"⁶, and in this case we have a minimal uncertainty states i.e. $\Delta^2 \hat{X}_+ \Delta^2 \hat{Y}_+ = \Delta^2 \hat{X}_- \Delta^2 \hat{Y}_- = 1/4$.

FIGURE 4.2: Noises of the difference and the sum of the signal and idler phase $\hat{Y}$ and $\hat{X}$ quadratures in a $\chi^{(2)}$ process for two different values of θ_p . At the left side we have the same value of [70] of $\theta_p = 1.59\pi$ and at the right side we have $\theta_p = \theta_{opt} = 1.499\pi$. The shot noise is at $1/2$ and is shown with an horizontal line, the vertical line correspond to the threshold power.

Figure 4.3 shows the fluctuations of the sum and subtraction quadratures for the $\chi^{(3)}$ process. In the case of $\theta_p = 1.59\pi$ (left side) we got a maximum noise reduction of 0.386823 for $\chi^{(3)} P_p \approx 2.99295$ and at some point the quadratures $\hat{Y}_+$ and $\hat{X}_-$ start to have more noise than the $\hat{X}_+$ and $\hat{Y}_-$ quadratures which at some point even start to get a slight squeezing at some point. When we optimize the pump phase (right side of figure 4.3) we get a maximum squeezing of 0.33332 for $\chi^{(3)} P_p \approx 4.89719$ but in this case the quadratures $\hat{Y}_+$ and $\hat{X}_-$ are squeezed for all pump power values and we have again a minimum uncertainty state.

⁴Sometimes we will use the standard notation $\Delta^2 \hat{X}$ for the variance $\langle \delta \hat{X}^2 \rangle$

⁵This "threshold" will also be defined at the next section

⁶Strictly speaking, the matrix to invert at the definition of $\mathbb{R}$ in (4.14) is singular at the threshold thus, the value of the covariance matrix at this point just make sense as a limit. And actually I can not say if these two values (the threshold and the value of the pump which maximizes the squeezing) are really different or this is just a numerical error

FIGURE 4.3: Noises of the difference and the sum of the signal and idler $\hat{Y}$ and $\hat{X}$ quadratures in a $\chi^{(3)}$ process for two different values of θ_p . $\theta_p = 1.59\pi$ (left) and $\theta_p = \theta_{opt} = 1.49803\pi$ (right). The shot noise is at $1/2$ and is shown with an horizontal line, the vertical line correspond to the threshold power.

4.1.3 Schmidt Decomposition

Following references [70, 71] we can define other two rotated modes as

$$\begin{aligned}\hat{\mathbf{A}}' &= e^{i\frac{\theta_\mu - \theta_\nu}{2}} \hat{\mathbf{A}} \\ \hat{\mathbf{A}}'' &= e^{-i\frac{\theta_\mu + \theta_\nu}{2}} \hat{\mathbf{A}}.\end{aligned}\quad (4.24)$$

where θ_μ and θ_ν are the phases of the μ and ν coefficients respectively, i.e. $\mu = |\mu|e^{i\theta_\mu}$ ($\nu = |\nu|e^{i\theta_\nu}$). In this new rotated basis we have that equation (4.7) takes the following form

$$\begin{aligned}\hat{a}_s''(z) &= |\mu|\hat{a}_s'(0) + |\nu|\hat{a}_i'^\dagger(0) \\ \hat{a}_i''(z) &= |\mu|\hat{a}_i'(0) + |\nu|\hat{a}_s'^\dagger(0).\end{aligned}\quad (4.25)$$

This transformation is done because in this basis, equation (4.22) for the sum and subtraction modes takes a very simple and diagonal form

$$\begin{aligned}\hat{\mathbf{X}}_\pm''(l) &= \mathbb{S}_\pm^{rot}(l) \cdot \hat{\mathbf{X}}_\pm'(0) \\ \mathbb{S}_\pm^{rot}(l) &= \begin{pmatrix} |\mu| + |\nu| & 0 & 0 & 0 \\ 0 & |\mu| - |\nu| & 0 & 0 \\ 0 & 0 & |\mu| - |\nu| & 0 \\ 0 & 0 & 0 & |\mu| + |\nu| \end{pmatrix},\end{aligned}\quad (4.26)$$

which leads to an also diagonal form for the output covariance matrix

$$\mathbb{V}_{out}^{\pm rot} = \mathbb{R}^{rot} \cdot \mathbb{V}_{in}^{\pm} \cdot \mathbb{R}^{rot\dagger} = \begin{pmatrix} -\frac{r+|\mu|+|\nu|}{r(|\mu|+|\nu|)-1} & 0 & 0 & 0 \\ 0 & \frac{r-|\mu|+|\nu|}{r|\mu|-r|\nu|-1} & 0 & 0 \\ 0 & 0 & \frac{r-|\mu|+|\nu|}{r|\mu|-r|\nu|-1} & 0 \\ 0 & 0 & 0 & -\frac{r+|\mu|+|\nu|}{r(|\mu|+|\nu|)-1} \end{pmatrix}$$

where $\mathbb{R}^{rot} = (\mathbb{S}_{\pm}^{rot} - r\mathbb{I}) (\mathbb{I} - r\mathbb{S}_{\pm}^{rot})^{-1}$. (4.27)

These expressions for the covariance and $\mathbb{S}$ matrix are actually the Schmidt decomposition of the these matrices. Formally the Schmidt, or singular value decomposition, is a sort of more general diagonalization factorization for any matrix. So given any matrix $\mathbb{A}$ its Schmidt decompositions is given by $\mathbb{A} = \mathbb{U} \cdot \mathbb{D} \cdot \mathbb{V}^\dagger$ where $\mathbb{V}$ and $\mathbb{U}$ are unitary matrices and $\mathbb{D}$ is a diagonal matrix with non negative entries called singular values. In the case of physical linear transformations, that are represented by symplectic matrices as the ones defined above by $\mathbb{S}$, the singular value decomposition exposes explicitly the fact that any linear transformation as the one defined by (4.5) and (4.6) are equivalent to a set of independent squeezing processes [72, 73]

$$\hat{a}_m(z) = \delta_m(z) \hat{a}_m(0) + \kappa_m(z) \hat{a}_m^\dagger(0) \quad (4.28)$$

with $\delta_m(z)$ and $\kappa_m(z)$ being singular values of the symplectic matrix representing the process, and where m enumerates the new modes defined by the Schmidt decomposition.

This Schmidt decomposition is also related with a variational problem of quadratic forms and therefore it gives us the basis in which we obtain the minimum and maximum noise at the covariance matrix and the maximum and minimum gain at the $\mathbb{S}$ matrix. It is in this sense that the "threshold" condition is defined at [70], as the point where the parametric gain G equals the losses

$$G_{max} r^2 = 1 \quad (4.29)$$

Where the maximum gain, which is given by $\mathbb{S}_{\pm}^{rot}$, is $G_{max} = (|\mu| + |\nu|)^2$. Then, for a given set of parameters (non linear coefficient, phase mismatch and length l of the NLM) the equations above give us the value of the "threshold" pump power.

Like the covariance matrix depends only on the absolute value of the coefficients μ and ν , the noises do not depend of the pump phase θ_p . Figure 4.4 shows us the noises of the sum and subtraction of quadratures for the $\chi^{(2)}$ process at the left, and for the $\chi^{(3)}$ to the right. Both have essentially the same behavior of the $\chi^{(2)}$ noise for the optimized pump phase (right side of figure 4.2) which tells us that, for a second order nonlinearity, the effect of the rotation to an optimal detected quadrature can be emulated by adjusting

the pump phase, in contrast with the third order nonlinearity when you can not reach maximum squeezing by adjusting the pump phase (Figure 4.3) but you obtain total noise reduction by rotating your detected quadratures as dictated in Schmidt transformation (4.24).

FIGURE 4.4: Noises of the difference and the sum of the signal and idler $\hat{Y}^{rot}$ and $\hat{X}^{rot}$ quadratures in a $\chi^{(2)}$ (left) and $\chi^{(3)}$ (right) process. The shot noise is at 1/2 and is shown with an horizontal line, the vertical line correspond to the threshold power.

The angle of this rotation is given by the phases of the μ and ν coefficients. For the ν coefficient the value of the phase can be easily extracted by its analytical expression (4.8) and (4.9) as $\theta_{\nu_2} = -\pi/2 + \theta_p + \Delta k_2 l/2$ and $\theta_{\nu_3} = \pi/2 + 2\theta_p + \Delta k_3 l/2 + \zeta^{(3)} P_p$. The phase of the μ coefficient doesn't have such a nice analytical form and a plot of this phase is shown at figure 4.5 for the same parameters used before and for a huge range of pumps powers. As we can see at the plot, the contribution of μ in the rotation in the case of $\chi^{(2)}$ is really small compared with the value of for a $\chi^{(3)}$, directly influenced by the XPM of the pump.

FIGURE 4.5: θ_μ for the $\chi^{(2)}$ (left) and $\chi^{(3)}$ (right) process.

4.2 $\chi^{(2)}$ With Pump Depletion

As was pointed out before, the undepleted pump approximation just allows us to describe the process of parametric amplification and not the parametric oscillation. The next step in order to have a more complete model for our system is to include the depletion of the pump in our calculations.

In this section the usual theory of the OPO is developed in order to have a standard to compare with. The approach of [74, 75] is followed, so a low gain approximation is made to integrate equations (4.2) through the non-linear medium, taking only first order terms in the non-linear coefficient $\chi^{(2)}$ giving as a result

$$\begin{aligned}\hat{a}_p(l) &= \hat{a}_p(0) + (\chi_{eff}^{(2)})^* \hat{a}_s(0) \hat{a}_i(0) \\ \hat{a}_s(l) &= \hat{a}_s(0) - \chi_{eff}^{(2)} \hat{a}_p(0) \hat{a}_i(0)^\dagger \\ \hat{a}_i(l) &= \hat{a}_i(0) - \chi_{eff}^{(2)} \hat{a}_p(0) \hat{a}_s(0)^\dagger\end{aligned}\tag{4.30}$$

where

$$\chi_{eff}^{(2)} = i\zeta^{(2)} l e^{i\Delta k_2 l/2} \frac{\sin(\Delta k_2 l/2)}{\Delta k_2 l/2}\tag{4.31}$$

After a round trip through the cavity, on the steady state situation, the field amplitudes should be the same due to the continuity of the fields. Then the intra-cavity signal and idler fields satisfy the following equations

$$\begin{aligned}(\bar{\Gamma} - i\bar{\Delta}_s) \hat{a}_s &= -(\chi_{eff}^{(2)}) \hat{a}_p \hat{a}_i^\dagger + t \hat{a}_s^{in} \\ (\bar{\Gamma} - i\bar{\Delta}_i) \hat{a}_i &= -(\chi_{eff}^{(2)}) \hat{a}_p \hat{a}_s^\dagger + t \hat{a}_i^{in},\end{aligned}\tag{4.32}$$

For the pump we will explore two possibilities. The first, when the pump is resonant (or quasi-resonant) with the cavity, and the second when the cavity is transparent for the pump. The first situation is known in the literature as Double Resonant OPO (DROPO), the second one as Triple Resonant OPO (TROPO). Thus the pump field satisfies the following equation

$$(\bar{\Gamma}_p - i\bar{\Delta}_p) \hat{a}_p = (\chi_{eff}^{(2)})^* \hat{a}_s \hat{a}_i + t_p \hat{a}_p^{in}\tag{4.33}$$

in the case of a TROPO ($\hat{a}_p = \text{intracavity pump}$), or

$$\hat{a}_p = \hat{a}_p^{in} + (\chi_{eff}^{(2)})^* \hat{a}_s \hat{a}_i$$

for the DROPO case ($\hat{a}_p = \text{output pump}$),

when the identification $\hat{a}_j(0) \rightarrow \hat{a}_j$; $j = p, s, i$ was made and where $\bar{\Delta}_j \ll 2\pi$; $j = p, s, i$ is the phase gained for the fields in a round trip through the cavity. In these equations a high reflectivity approximation was done. Writing the reflectivity coefficients of the coupling mirror M_c as $r_j = 1 - \bar{\Gamma}_j$ $j = p, s, i$, and given that $r_j^2 + t_j^2 = 1$, the transmission coefficient, in the case of high reflectivity ; $\bar{\Gamma}_j \ll 1$, is given by $t_j \approx \sqrt{2\bar{\Gamma}_j}$. We also have assumed that $r_i = r_s = r$.

The solution for the previous equations can be obtained through the following procedure. First, we define the fluctuation operators as

$$\delta\hat{a} = \hat{a} - \alpha \quad \text{with} \quad \alpha = Ae^{i\theta} = \langle \hat{a} \rangle. \quad (4.34)$$

Then, writing the equations (4.32) and (4.33) in terms of these fluctuations up to first order, and then taking the average we arrive to set of equations for the c numbers α_j $j = s, i, p$

$$(\bar{\Gamma} - i\bar{\Delta}_s) \alpha_s = -(\chi_{eff}^{(2)}) \alpha_p \alpha_i^* + t \alpha_s^{in} \quad (4.35)$$

$$(\bar{\Gamma} - i\bar{\Delta}_i) \alpha_i = -(\chi_{eff}^{(2)}) \alpha_p \alpha_s^* + t \alpha_i^{in},$$

and for the TROPO case

$$(\bar{\Gamma}_p - i\bar{\Delta}_p) \alpha_p = (\chi_{eff}^{(2)})^* \alpha_s \alpha_i + t_p \alpha_p^{in}$$

or for the DROPO case

$$\alpha_p = \alpha_p^{in} + (\chi_{eff}^{(2)})^* \alpha_s \alpha_i.$$

If the only non zero input field is the pump, i.e. $\alpha_s^{in} = \alpha_i^{in} = 0$ the equations above are the standard *OPO* equations. In this case there are two different behaviors of the solution. One is the trivial solution in which the signal and idler fields are zero, and the whole system behave as either an empty cavity for the pump (if we are dealing with a TROPO), or it is transparent to the pump (if we are dealing with a DROPO). This is the under threshold solution and it coincides with undepleted pump approximation studied before.

The non zero signal-idler solution is way more interesting, in this case taking the complex conjugate of one of eqns. (4.35) and multiplying it by the other one we have from the imaginary and real part that

$$\begin{aligned} \bar{\Delta}_s &= \bar{\Delta}_i \equiv \bar{\Delta} \\ A_p^2 &= \frac{\bar{\Gamma}^2 + \bar{\Delta}^2}{|\chi_{eff}^{(2)}|^2} \end{aligned} \quad (4.36)$$

which means that the intra-cavity pump (in the case of the cavity resonant with pump) or the pump at the output of the crystal (in the case of the cavity transparent to the pump) has a fixed amplitude which is independent of the incoming pump value. That means that any power difference between the incoming pump (taking account for the beam splitter transformation in the case of the pump resonant with the cavity) and intra-cavity pump is transformed into signal and idler power. Taking the absolute value of any of the equations on (4.32) and using the result above, we have that $A_s = A_i \equiv A$, which puts on evidence the fact that the signal and idler photons are created in pairs. With this result and putting equations. (4.33) in (4.32) we can solve the intra-cavity field A

$$\frac{\bar{\Gamma} A^2}{2A_{th}^2} = \frac{\bar{\Delta} \bar{\Delta}_p}{\bar{\Gamma} \bar{\Gamma}_p} - 1 \pm \sqrt{\frac{|\alpha_p^{in}|^2}{A_{th}^2} - \left(\frac{\bar{\Delta}_p}{\bar{\Gamma}_p} + \frac{\bar{\Delta}}{\bar{\Gamma}} \right)^2} \quad \text{with} \quad A_{th}^2 = \frac{\bar{\Gamma}^2 \bar{\Gamma}_p}{2|\chi_{eff}^{(2)}|^2}$$

for the cavity resonant with the pump (TROPO) or

$$\frac{\bar{\Gamma} A^2}{2A_{th}^2} = -1 + \sqrt{\frac{|\alpha_p^{in}|^2}{A_{th}^2} - \left(\frac{\bar{\Delta}}{\bar{\Gamma}} \right)^2} \quad \text{with} \quad A_{th}^2 = \frac{\bar{\Gamma}^2}{|\chi_{eff}^{(2)}|^2}$$

for the cavity transparent to the pump (DROPO) (4.37)

where $|\alpha_{th}|^2$ is the threshold power at the exactly resonance, i.e. the minimum value of the incoming power $|\alpha_p^{in}|^2$ at $\bar{\Delta}_p = \bar{\Delta} = 0$ for which a non zero solution for the signal and idler exist. In the TROPO case, the threshold power is significantly reduced due to the effect of the increase of the field intensity given by the cavity. This is explicitly shown by the factor $\bar{\Gamma}_p \ll 1$ multiplying on the definition of A_{th}^2 in the TROPO case.

Equations (4.35) do not provide values for the individual phases of the fields but give us values for the phases $\psi = \theta_\chi + \theta_p - \theta_s - \theta_i$ (θ_χ is the phase of $\chi_{eff}^{(2)}$) and $\phi = \theta_p^{in} - \theta_p$ which are given by

$$e^{i\psi} = \frac{(-\bar{\Gamma} + i\bar{\Delta})}{\sqrt{\bar{\Gamma}^2 + \bar{\Delta}^2}}$$

$$e^{i\phi} = \frac{\left[\bar{\Gamma}_p (\bar{\Gamma}^2 + \bar{\Delta}^2) + \bar{\Gamma} |\chi_{eff}^{(2)}|^2 A^2 \right] + i \left[\bar{\Delta}_p (\bar{\Gamma}^2 + \bar{\Delta}^2) - \bar{\Delta} |\chi_{eff}^{(2)}|^2 A^2 \right]}{|\chi_{eff}^{(2)}| |\alpha_p^{in}| \sqrt{2\bar{\Gamma}_p (\bar{\Gamma}^2 + \bar{\Delta}^2)}}$$

for the cavity resonant with the pump

$$e^{i\phi} = \frac{\left(\bar{\Gamma}^2 + \bar{\Delta}^2 + \bar{\Gamma} |\chi_{eff}^{(2)}|^2 A^2 \right) - i \bar{\Delta} |\chi_{eff}^{(2)}|^2 A^2}{|\chi_{eff}^{(2)}| |\alpha_p^{in}| \sqrt{\bar{\Gamma}^2 + \bar{\Delta}^2}}$$

for the cavity transparent for the pump (4.38)

4.2.1 Quantum Fluctuations

The quantum fluctuations we will calculate by taken other approach. Instead of treating the fields as propagating waves going around the cavity, now we will consider the field in its decomposition as normal modes of the cavity. Each mode is characterized by its *eigenfrequency* ω_n and its field profile $(\mathbf{E}_n(\mathbf{r}), \mathbf{B}_n(\mathbf{r}))$ which satisfies the boundary conditions imposed by the geometry of the cavity.

In this other approach we can introduce in a more direct way the influence of the analysis frequency because we will end up with a set of time evolution equations for each mode, instead of propagation equations. As long as we have dispersion we can not guarantee that we can find three modes of the cavity that satisfy the frequency matching conditions, so in general $\omega_p - \omega_s - \omega_i = D_2$.

Now the Hamiltonian is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} \quad (4.39)$$

where

$$\hat{H}_0 = \hbar \sum_j \omega_j \hat{a}_j^\dagger \hat{a}_j, \quad (4.40)$$

and

$$\hat{H}_{int}^{(2)} = \hbar \eta^{(2)} \left(\hat{a}_p^\dagger \hat{a}_s \hat{a}_i + h.c. \right), \quad (4.41)$$

Here $\eta^{(2)}$ is proportional to the second order non-linear coefficient $\chi^{(2)}$ and to the overlap between the modes profiles.

Starting from this interaction Hamiltonian, including the effect of the coupling with some external modes which act as input-output coupling of the cavity and also including other non-useful losses as absorption, scattering, etc.. (known as spurious losses), we can derive the Heisenberg-Langevin equations [76] for the pump, signal and idler modes.

$$\begin{aligned} \frac{d\hat{a}_p}{dt} &= -(\Gamma_p + i\Delta_p) \hat{a}_p - i\eta^{(2)} \hat{a}_s \hat{a}_i + \sqrt{2\gamma_p} \hat{a}_p^{in} + \sqrt{2\mu_p} \hat{a}_p^{loss} \\ \frac{d\hat{a}_s}{dt} &= -(\Gamma_s + i\Delta_s) \hat{a}_s - i\eta^{(2)} \hat{a}_p \hat{a}_i^\dagger + \sqrt{2\gamma_s} \hat{a}_s^{in} + \sqrt{2\mu_s} \hat{a}_s^{loss} \\ \frac{d\hat{a}_i}{dt} &= -(\Gamma_i + i\Delta_i) \hat{a}_i - i\eta^{(2)} \hat{a}_p \hat{a}_s^\dagger + \sqrt{2\gamma_i} \hat{a}_i^{in} + \sqrt{2\mu_i} \hat{a}_i^{loss} \end{aligned} \quad (4.42)$$

Here the losses are included through the term $\sqrt{2\mu_j} \hat{a}_j^{loss}$ (spurious losses), again the term $\sqrt{2\gamma_j} \hat{a}_j^{in}$ (coupling losses) represent the input and detected mode coupled to the

intra cavity mode $j = p, s, i$ and $\Gamma_j = \gamma_j + \mu_j$ (Total intra-cavity losses). On these time evolution equations, the quantities Γ_j are the decay rate (measured in unities of inverse of time) of the field amplitude inside the cavity, meanwhile in the previous analysis of section (4.2), $\bar{\Gamma}$ stands for the fraction of the intensity which leaks outside the cavity per round trip (dimensionless quantity). They are related by $\bar{\Gamma} = t_{rt}\Gamma$ where t_{rt} stands for the round trip time (the time which a photon takes to make one round trip inside the cavity).

The operators $\hat{a}_j^{loss}$ and $\hat{a}_j^{in}$ are delta correlated and have the following properties

$$\langle \hat{a}_j^{loss}(t) \rangle = \langle \hat{a}_s^{in}(t) \rangle = \langle \hat{a}_i^{in}(t) \rangle = 0 \quad (4.43)$$

$$\langle \hat{a}_p^{in}(t) \rangle = \alpha_{in} = \sqrt{\frac{P_{in}}{\hbar\omega_0}} \quad (4.44)$$

$$\langle \hat{a}_j^{loss}(t) \hat{a}_j^{\dagger loss}(t') \rangle = \langle \hat{a}_j^{in}(t) \hat{a}_j^{\dagger in}(t') \rangle = \delta(t - t'), \quad (4.45)$$

where P_{in} is the power of the input field. We have chosen α_{in} real, which means that the phases of the other fields are given in relation with the input pump field phase.

In the equations (4.42) we have made the replacement⁷ $\hat{a}_j \rightarrow \hat{a}_j e^{-i\nu_j}$; $j = p, s, i$. This means that the annihilation operators here are slowly varying operators given that we have *removed* the rapidly oscillating component of the carrier. We also have that these frequencies satisfy $\nu_p - \nu_s - \nu_i = 0$ and $\Delta_j \equiv \omega_j - \nu_j$ is the detuning with of the carrier in relation with the cavity resonance.

Using the phase matching condition, we have a relation between the detuning of the fields

$$\Delta_p - \Delta_s - \Delta_i = D_2 \quad (4.46)$$

In the same way as the quantities Γ_s , the variables Δ_j here are related to $\bar{\Delta}_j$ in section (4.2) through the round trip time as $\bar{\Delta} = t_{rt}\Delta$.

Writing equations (4.42) in terms of the fluctuations (4.34) up to first order, taking the average and setting $d\alpha/dt = 0$ we obtain again the equations of the classical steady state of the field which are exactly the same as (4.35) with $\chi_{eff}^{(2)} \rightarrow i\eta^{(2)}$, and therefore, the solution is of the same form of (4.37)⁸. Given that solutions we can write the linearized equations for the fluctuations of the field in a matrix form. So, if we define a vector containing the fluctuations of the fields as

⁷Is important to point out that here ν is the actual frequency of field carrier and ω is the central frequency of the cavity mode being exited, and also that ν_p is also the frequency of the incoming pump field

⁸Again, making the assumption $\Gamma_s = \Gamma_i = \Gamma$

$$\delta\hat{\mathbf{A}} = \{\delta\hat{a}_p e^{-i\theta_p}, \delta\hat{a}_p^\dagger e^{i\theta_p}, \delta\hat{a}_s e^{-i\theta_s}, \delta\hat{a}_s^\dagger e^{-i\theta_s}, \delta\hat{a}_i e^{-i\theta_i}, \delta\hat{a}_i^\dagger e^{-i\theta_i}\}^T, \quad (4.47)$$

i.e. $\delta\mathbf{A}$ is a vector of the fluctuations of the field amplitude with the phase reference given by the mean field phase. Then, these fluctuations follow the next evolution equation

$$\begin{aligned} \frac{d\delta\hat{\mathbf{A}}}{dt} &= -\mathbb{M}_a \cdot \delta\hat{\mathbf{A}} + \mathbb{T}_a \cdot \mathbb{P}_{\theta_p} \cdot \delta\hat{\mathbf{A}}^{in} + \mathbb{T}_a^{loss} \cdot \delta\hat{\mathbf{A}}^{loss} \quad (4.48) \\ \mathbb{M}_a &= \begin{pmatrix} \Gamma_p + i\Delta_p & 0 & \beta(\Gamma - i\Delta) & 0 & \beta(\Gamma - i\Delta) & 0 \\ 0 & -i\Delta_p + \Gamma_p & 0 & \beta(i\Delta + \Gamma) & 0 & \beta(i\Delta + \Gamma) \\ -\beta(i\Delta + \Gamma) & 0 & \Gamma + i\Delta & 0 & 0 & -i\Delta - \Gamma \\ 0 & -\beta(-i\Delta + \Gamma) & 0 & -i\Delta + \Gamma & i\Delta - \Gamma & 0 \\ -\beta(i\Delta + \Gamma) & 0 & 0 & -i\Delta - \Gamma & \Gamma + i\Delta & 0 \\ 0 & -\beta(-i\Delta + \Gamma) & i\Delta - \Gamma & 0 & 0 & -i\Delta + \Gamma \end{pmatrix} \\ \mathbb{T}_a &= \text{Diag}\{\sqrt{2\gamma_p}, \sqrt{2\gamma_p}, \sqrt{2\gamma}, \sqrt{2\gamma}, \sqrt{2\gamma}, \sqrt{2\gamma}\} \\ \mathbb{T}_a^{loss} &= \text{Diag}\{\sqrt{2\mu_p}, \sqrt{2\mu_p}, \sqrt{2\mu}, \sqrt{2\mu}, \sqrt{2\mu}, \sqrt{2\mu}\} \\ \mathbb{P}_\phi &= \text{Diag}\{e^{-i\phi}, e^{i\phi}, 1, 1, 1, 1\} \end{aligned}$$

with $\beta = A/A_p$ and where we have supposed $\gamma_s = \gamma_i = \gamma$ and $\mu_s = \mu_i = \mu$. As the state of the losses modes $\hat{\mathbf{A}}^{loss}$ is vacuum, its phase is arbitrary and for simplicity we choose it as the same as the intracavity field. This equation, together with the equation that relates the output field with the input and intracavity fields

$$\hat{\mathbf{A}}^{out} = -\mathbb{P}_{\theta_p^{out}} \hat{\mathbf{A}}^{in} + \mathbb{P}_{\theta_p^{out} - \theta_p} \mathbb{T}_a \cdot \hat{\mathbf{A}} \quad (4.49)$$

allows us to have an expression for the detected fluctuation outside the cavity as

$$\begin{aligned} \delta\mathbf{A}^{out}(\omega) &= \left[-\mathbb{P}_{\theta_p^{out}} + \mathbb{P}_{\theta_p^{out} - \theta_p} \cdot \mathbb{T}_a (\mathbb{M}_a + i\omega\mathbb{I})^{-1} \mathbb{P}_{\theta_p} \cdot \mathbb{T}_a \right] \cdot \delta\mathbf{A}^{in} \\ &\quad + \mathbb{P}_{\theta_p^{out} - \theta_p} (\mathbb{M}_a + i\omega\mathbb{I})^{-1} \mathbb{T}_a^{loss} \cdot \delta\mathbf{A}^{loss} \end{aligned} \quad (4.50)$$

and its corresponding noise spectral density matrix $\mathbb{V}_a(\omega)$ ⁹.

$$\begin{aligned} \mathbb{V}^a(\omega) \delta(\omega - \omega') &= \left\langle \delta\hat{\mathbf{A}}(\omega) \delta\hat{\mathbf{A}}^{\dagger T}(-\omega') \right\rangle \\ \mathbb{V}^a(\omega) &= \mathbb{P}_{\theta_p^{out}} \left[\mathbb{P}_{-\theta_p} \mathbb{T}_a (\mathbb{M}_a + i\omega\mathbb{I})^{-1} \mathbb{P}_{\theta_p} \mathbb{T}_a^{in} - \mathbb{I} \right] \mathbb{V}^{in}(\omega) \left[\mathbb{P}_{-\theta_p} \mathbb{T}_a (\mathbb{M}_a - i\omega\mathbb{I})^{-1} \mathbb{P}_{\theta_p} \mathbb{T}_a^{in} - \mathbb{I} \right]^\dagger \mathbb{P}_{-\theta_p^{out}} \\ &\quad + \left[\mathbb{T}_a (\mathbb{M}_a + i\omega\mathbb{I})^{-1} \mathbb{T}_a^{loss} \right] \cdot \left[\mathbb{T}_a (\mathbb{M}_a - i\omega\mathbb{I})^{-1} \mathbb{T}_a^{loss} \right]^\dagger \end{aligned} \quad (4.51)$$

In the equation above, the first term of $\mathbb{V}^a(\omega)$ corresponds to a transformation of the incoming noise spectral density matrix $\mathbb{V}^{in}(\omega)$, while the second term corresponds to the inserted noise by the losses of the system.

⁹Here ω is the analysis frequency which is define relative to the carrier

We can also rewrite this equations, as we did before, in terms of the sum and subtraction of the quadratures defined in (4.21)

$$\hat{\mathbf{Y}} = \{\hat{p}_p, \hat{q}_p, \hat{p}_+, \hat{q}_+, \hat{p}_-, \hat{q}_-\}^T. \quad (4.52)$$

In this particular case, the quadratures of the fields are the so called phase and intensity quadratures. This is because the angle used in (4.17) is the phase of the mean field $\varphi_j = \theta_j$ $j = p, s, i$ (see section 1.4.3).

So fluctuations in the $\hat{x}(\theta) \equiv \hat{p}$ quadrature are approximately proportional to intensity fluctuations and fluctuations on $\hat{y}(\theta) \equiv \hat{q}$ can be associated with phase fluctuations¹⁰. In this phase and intensity sum/subtraction representation the previous equation takes the form

$$\begin{aligned} \frac{d\delta\mathbf{Y}}{dt} &= -\mathbb{M}_Y \cdot \delta\mathbf{Y} + \mathbb{T}_Y^{in} \cdot \delta\mathbf{Y}^{in} + \mathbb{T}_Y^{loss} \cdot \delta\mathbf{Y}^{loss} \quad (4.53) \\ \mathbb{M}_Y &= \begin{pmatrix} \Gamma_p & -\Delta_p & \sqrt{2}\beta\Gamma & \sqrt{2}\beta\Delta & 0 & 0 \\ \Delta_p & \Gamma_p & -\sqrt{2}\beta\Delta & \sqrt{2}\beta\Gamma & 0 & 0 \\ -\sqrt{2}\beta\Gamma & \sqrt{2}\beta\Delta & 0 & -2\Delta & 0 & 0 \\ -\sqrt{2}\beta\Delta & -\sqrt{2}\beta\Gamma & 0 & 2\Gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\Gamma & 0 \\ 0 & 0 & 0 & 0 & 2\Delta & 0 \end{pmatrix} \\ \mathbb{T}_Y^{in} &= \begin{pmatrix} \cos(\phi)\sqrt{2\gamma_p} & -\sin(\phi)\sqrt{2\gamma_p} & 0 & 0 & 0 & 0 \\ \sin(\phi)\sqrt{2\gamma_p} & \cos(\phi)\sqrt{2\gamma_p} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2\gamma} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2\gamma} \end{pmatrix} \quad (4.54) \end{aligned}$$

with $\mathbb{T}_Y^{loss}$ defined in the same way as $\mathbb{T}_Y^{in}$ but with the substitution $\gamma \rightarrow \mu$. If we restrict ourselves only to the sum and subtraction modes, i.e. we suppose that the pump is a classical noiseless field ($\delta\hat{a}_p \rightarrow 0$) and we have no spurious losses ($\mu = 0$). We can start analyzing just the signal-idler correlations. In this subspace $\{\hat{p}_+, \hat{q}_+, \hat{p}_-, \hat{q}_-\}$, the sum and the subtraction fluctuations are completely decoupled and the noise spectral density matrix takes the form

¹⁰This, in the case of intense fields where the mean is much greater than the fluctuations i.e. $|\langle\hat{a}\rangle| \gg |\langle\delta\hat{a}^2\rangle|$

$$\mathbb{V}(\omega) = \begin{pmatrix} -\frac{2\Delta^2}{4\Gamma^2+\omega^2} + \frac{1}{2} + \frac{2(\Gamma^2+\Delta^2)}{\omega^2} & \frac{2\Gamma\Delta}{4\Gamma^2+\omega^2} & 0 & 0 \\ \frac{2\Gamma\Delta}{4\Gamma^2+\omega^2} & \frac{\omega^2}{8\Gamma^2+2\omega^2} & 0 & 0 \\ 0 & 0 & \frac{\omega^2}{8\Gamma^2+2\omega^2} & -\frac{2\Gamma\Delta}{4\Gamma^2+\omega^2} \\ 0 & 0 & -\frac{2\Gamma\Delta}{4\Gamma^2+\omega^2} & -\frac{2\Delta^2}{4\Gamma^2+\omega^2} + \frac{1}{2} + \frac{2(\Gamma^2+\Delta^2)}{\omega^2} \end{pmatrix} \quad (4.55)$$

and in the case of resonance ($\Delta = 0$) we have a simply diagonal form

$$\mathbb{V}(\omega) = \begin{pmatrix} \frac{4\Gamma^2+\omega^2}{2\omega^2} & 0 & 0 & 0 \\ 0 & \frac{\omega^2}{2(4\Gamma^2+\omega^2)} & 0 & 0 \\ 0 & 0 & \frac{\omega^2}{2(4\Gamma^2+\omega^2)} & 0 \\ 0 & 0 & 0 & \frac{4\Gamma^2+\omega^2}{2\omega^2} \end{pmatrix} \quad (4.56)$$

which is a "symmetric" two mode minimal uncertainty state. Due to that the noise spectral density matrix is diagonal in this case, that means that this particular base is also the Schmidt decomposition of the system and no change of basis is needed to measure the best squeezing possible.

Figure 4.6 shows the noises of the sum and subtraction quadratures as a function of the analysis frequency ω/Γ (normalized by cavity bandwidth) in the case of a noiseless classical pump. The quadratures $\hat{p}_-$ and $\hat{q}_+$ have perfect squeezing at zero analysis frequency meanwhile their conjugate quadratures ($\hat{p}_+$ and $\hat{q}_-$) noises go to infinity as imposed by the uncertainty principle. For increasing analysis frequency the squeezing and excess of noise reduce until both approach to the shot noise as $\omega \rightarrow \infty$. Also worth to point out that this noises are independent of the power of the pump power as long as its values stays above the threshold power.

When we include the pump noise, we see that the subtraction mode is still decoupled but the sum couples to the pump. At the exact resonance ($\Delta = \Delta_p = 0$) we have that $e^{i\phi} = 1$ and then, the noise spectral density matrix of the quadratures $\{\hat{p}_p, \hat{q}_p, \hat{p}_+, \hat{q}_+\}$ is given by

$$\mathbb{V}(\omega = 0) = \begin{pmatrix} \frac{\Gamma_p}{\beta^2\Gamma} + \frac{1}{2} & 0 & -\frac{\Gamma_p^{3/2}}{\sqrt{2}\beta^3\Gamma^{3/2}} & 0 \\ 0 & \frac{\Gamma^2\beta^4+\Gamma_p^2}{2(\Gamma\beta^2+\Gamma_p)^2} & 0 & \frac{\beta\sqrt{\Gamma}\Gamma_p^{3/2}}{\sqrt{2}(\Gamma\beta^2+\Gamma_p)^2} \\ -\frac{\Gamma_p^{3/2}}{\sqrt{2}\beta^3\Gamma^{3/2}} & 0 & \frac{1}{2}\left(\frac{\Gamma_p^2}{\beta^4\Gamma^2} + 1\right) & 0 \\ 0 & \frac{\beta\sqrt{\Gamma}\Gamma_p^{3/2}}{\sqrt{2}(\Gamma\beta^2+\Gamma_p)^2} & 0 & \frac{\beta^2\Gamma(\Gamma\beta^2+2\Gamma_p)}{2(\Gamma\beta^2+\Gamma_p)^2} \end{pmatrix} \quad (4.57)$$

FIGURE 4.6: Noises of the difference and the sum of the signal and idler $\hat{q}$ and $\hat{p}$ quadratures in a $\chi^{(2)}$ process including pump depletion of the pump field and working above the threshold and at exactly the resonance ($\Delta = 0$). The shot noise is at $1/2$ and is shown with an horizontal line.

If we apply the Schmidt decomposition to this system ¹¹ we will, indeed, apply a kind of rotation to the output quadratures as follow

$$\hat{\mathbf{X}}^{rot} = \begin{pmatrix} \hat{p}_1^{rot} \\ \hat{q}_1^{rot} \\ \hat{p}_2^{rot} \\ \hat{q}_2^{rot} \end{pmatrix} = \begin{pmatrix} -\sin(\vartheta) & 0 & \cos(\vartheta) & 0 \\ 0 & -\sin(\vartheta) & 0 & \cos(\vartheta) \\ \cos(\vartheta) & 0 & \sin(\vartheta) & 0 \\ 0 & \cos(\vartheta) & 0 & \sin(\vartheta) \end{pmatrix} \begin{pmatrix} \hat{p}_p \\ \hat{q}_p \\ \hat{p}_+ \\ \hat{q}_+ \end{pmatrix}$$

$$\cos(\vartheta) = \sqrt{\frac{\Gamma_p}{2\Gamma\beta^2 + \Gamma_p}} \quad (4.58)$$

which leads to a noise spectral density matrix of the diagonal form

$$\mathbb{V}(\omega = 0) = \begin{pmatrix} \frac{\beta^4 \Gamma^2}{2(\Gamma\beta^2 + \Gamma_p)^2} & 0 & 0 & 0 \\ 0 & \frac{(\Gamma\beta^2 + \Gamma_p)^2}{2\beta^4 \Gamma^2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad (4.59)$$

Figure 4.7 shows the fluctuations of the pump and sum mode amplitude and phase quadratures and also the fluctuations of the Schmidt modes defined above. At the left, the noises of the pump and sum modes for zero analysis frequency are shown, where we can see squeezing for the phases ($\hat{q}$) of both modes and consequently, noise excess for the amplitudes ($\hat{p}$) for all values of the input pump power (exactly at threshold, $P_{in} = P_{th}$, we have total squeezing for the sum mode phase and the pump phase goes to the shot

¹¹As the noise spectral matrix is a positive definite matrix, then the Schmidt deposition is equivalent to a standard diagonalization

noise), and all of them approaching to the shot noise as $P_{in} \rightarrow \infty$. At the right side of figure 4.7 the noises of the Schmidt quadratures defined above are shown, which are also minimal uncertainty states. For the rotated mode labeled as 2 both quadrature noises are at the shot noise level, the $\hat{q}_1^{rot}$ quadrature have perfect squeezing at the threshold, $\hat{p}_1^{rot} \rightarrow \infty$ and again both quadratures approach to the shot noise for increasing P_{in} . At the lower part it is shown that the Schmidt mode 1 gives actually the minimum and maximum noise possible of the system.

FIGURE 4.7: Noises of the difference and the sum of the signal and idler phases and intensities quadratures and the pump in a $\chi^{(2)}$ process including pump depletion of the pump field, working above the threshold at exactly resonance and at zero analysis frequency. The shot noise is at $1/2$ and is shown with an horizontal line.

As it was pointed out before, the Schmidt decomposition is also related with a minimization problem, that is the property that will be used to find separability inequalities in a convenient way in order to help us to detect entanglement.

4.3 $\chi^{(3)}$ With Pump Depletion

In this section we proceed to make a similar treatment done before but this time for a NLM with a third order nonlinearity. In this case, as before, the Hamiltonian is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} \quad (4.60)$$

where

$$\hat{H}_0 = \hbar \sum_j \omega_j \hat{a}_j^\dagger \hat{a}_j, \quad (4.61)$$

is the free Hamiltonian, and the nonlinear interaction Hamiltonian is

$$\begin{aligned} \hat{H}_{int}^{(3)} = -\hbar \eta^{(3)} & \left[\frac{1}{2} \left(\hat{a}_p^\dagger \hat{a}_p^\dagger \hat{a}_p \hat{a}_p + \hat{a}_s^\dagger \hat{a}_s^\dagger \hat{a}_s \hat{a}_s + \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \right) + \right. \\ & \left. 2 \left(\hat{a}_p^\dagger \hat{a}_s^\dagger \hat{a}_p \hat{a}_s + \hat{a}_p^\dagger \hat{a}_i^\dagger \hat{a}_p \hat{a}_i + \hat{a}_s^\dagger \hat{a}_i^\dagger \hat{a}_s \hat{a}_i \right) + \left(\hat{a}_s^\dagger \hat{a}_i^\dagger \hat{a}_p \hat{a}_p + \hat{a}_p^\dagger \hat{a}_p^\dagger \hat{a}_s \hat{a}_i \right) \right] \end{aligned} \quad (4.62)$$

Here $\eta^{(3)}$ is proportional to the third order nonlinearity and to the mode profile overlap. As before, the first and second terms of the *r.h.s.* (right hand side) of the last equation are respectively the SPM and XPM terms, and the last is the FWM term.

This Hamiltonian leads to the following Heisenberg-Langevin equations of motion for the annihilation operators

$$\begin{aligned} \frac{d\hat{a}_p}{dt} &= -(\Gamma_p + i\Delta_p) \hat{a}_p + i\eta^{(3)} \left[\left(\hat{a}_p^\dagger \hat{a}_p + 2\hat{a}_s^\dagger \hat{a}_s + 2\hat{a}_i^\dagger \hat{a}_i \right) \hat{a}_p + 2\hat{a}_p^\dagger \hat{a}_s \hat{a}_i \right] + \sqrt{2\gamma_p} \hat{a}_p^{in} + \sqrt{2\mu_p} \hat{a}_p^{loss} \\ \frac{d\hat{a}_s}{dt} &= -(\Gamma_s + i\Delta_s) \hat{a}_s + i\eta^{(3)} \left[\left(2\hat{a}_p^\dagger \hat{a}_p + \hat{a}_s^\dagger \hat{a}_s + 2\hat{a}_i^\dagger \hat{a}_i \right) \hat{a}_s + \hat{a}_p^2 \hat{a}_i^\dagger \right] + \sqrt{2\gamma_s} \hat{a}_s^{in} + \sqrt{2\mu_s} \hat{a}_s^{loss} \\ \frac{d\hat{a}_i}{dt} &= -(\Gamma_i + i\Delta_i) \hat{a}_i + i\eta^{(3)} \left[\left(2\hat{a}_p^\dagger \hat{a}_p + 2\hat{a}_s^\dagger \hat{a}_s + \hat{a}_i^\dagger \hat{a}_i \right) \hat{a}_i + \hat{a}_p^2 \hat{a}_s^\dagger \right] + \sqrt{2\gamma_i} \hat{a}_i^{in} + \sqrt{2\mu_i} \hat{a}_i^{loss} \end{aligned} \quad (4.63)$$

These equations, as before, are written in terms of slowly varying operators, i.e. that we have made the replacement $\hat{a}_j \rightarrow \hat{a}_j e^{-i\nu_j}$; $j = p, s, i$. In the $\chi^{(3)}$ process the carrier frequencies are related through the equation $2\nu_p - \nu_s - \nu_i = 0$. As is the $\chi^{(2)}$ situation, it is not possible, in general, to find three cavity modes such that their resonant frequencies satisfy $2\omega_p - \omega_s - \omega_i = 0$. This is because the dispersion inside the cavity, produced by the material dispersion and the geometry of the cavity makes that the eigenfrequencies of the modes are not evenly spaced, so in general we have $2\omega_p - \omega_s - \omega_i = D_3$. Then, given that $\Delta_j \equiv \omega_j - \nu_j$ is the detuning of the field with relation to the cavity excited mode resonant frequency, the relation between the detuning of the three fields is given by

$$2\Delta_p - \Delta_s - \Delta_i = D_3. \quad (4.64)$$

Using the same procedure as before, we write these equation in terms of the fluctuation (4.34) up to first order, take the average to end up with a set of equations for the stationary mean field.

$$\begin{aligned}
& -(\Gamma_p + i\Delta_p) \alpha_p + i\eta^{(3)} [(|\alpha_p|^2 + 2|\alpha_s|^2 + 2|\alpha_i|^2) \alpha_p + 2\alpha_p^* \alpha_s \alpha_i] + \sqrt{2\gamma_p} \alpha_p^{in} = 0 \\
& -(\Gamma_s + i\Delta_s) \alpha_s + i\eta^{(3)} [(2|\alpha_p|^2 + |\alpha_s|^2 + 2|\alpha_i|^2) \alpha_s + \alpha_p^2 \alpha_i^*] + \sqrt{2\gamma_s} \alpha_s^{in} = 0 \\
& -(\Gamma_i + i\Delta_i) \alpha_i + i\eta^{(3)} [(2|\alpha_p|^2 + 2|\alpha_s|^2 + |\alpha_i|^2) \alpha_i + \alpha_p^2 \alpha_s^*] + \sqrt{2\gamma_i} \alpha_i^{in} = 0 \quad (4.65)
\end{aligned}$$

Following [77], in order to solve the equations above we split them in their real and imaginary parts as follows

$$\begin{aligned}
& -\Gamma_p A_p - 2\eta^{(3)} A_p A_s A_i \sin(\phi) + \sqrt{2\gamma_p} A^{in} \cos(\psi) = 0 \\
& -\Delta_p A_p + \eta^{(3)} (A_p^2 + 2A_s^2 + 2A_i^2) A_p + 2\eta^{(3)} A_p A_s A_i \cos(\phi) + \sqrt{2\gamma_p} A^{in} \sin(\psi) = 0 \\
& -\Gamma_s A_s + \eta^{(3)} A_p^2 A_i \sin(\phi) = 0 \\
& -\Delta_s A_s + \eta^{(3)} (2A_p^2 + A_s^2 + 2A_i^2) A_s + \eta^{(3)} A_p^2 A_i \cos(\phi) = 0 \\
& -\Gamma_i A_i + \eta^{(3)} A_p^2 A_s \sin(\phi) = 0 \\
& -\Delta_i A_i + \eta^{(3)} (2A_p^2 + 2A_s^2 + A_i^2) A_i + \eta^{(3)} A_p^2 A_s \cos(\phi) = 0 \quad (4.66)
\end{aligned}$$

where we only have an incoming field for the pump mode, i.e. $\alpha_s^{in} = \alpha_i^{in} = 0$ and $\alpha_p^{in} = A^{in} e^{i\theta_{in}}$. The quantity A^{in} is related with the incoming pump power by $A^{in} = \sqrt{P^{in}/\hbar\nu_p}$. As in the $\chi^{(2)}$ process, these equations do not give us the phases of the pump, signal and idler individually but give us value of the phases $\phi = \theta_s + \theta_i - 2\theta_p$ and $\psi = \theta_{in} - \theta_p$. The third and fifth equations of (4.66) tell us that, as long as we have that $\Gamma_s = \Gamma_i = \Gamma$, we have that the signal and idler amplitudes are equal, i.e. $A_s = A_i = A$, and with that result, the fourth and sixth equations impose that $\Delta_s = \Delta_i = \Delta$.

We can also work with normalized variables to make a little bit easier to handle the equations and reduce the number of parameters. Therefore we make the following substitutions

$$A \rightarrow \sqrt{\frac{\Gamma}{\eta^{(3)}}} A; \quad A_p \rightarrow \sqrt{\frac{\Gamma}{\eta^{(3)}}} A_p; \quad D_3 \rightarrow \frac{D_3}{\Gamma}; \quad \Delta_p \rightarrow \frac{\Delta_p}{\Gamma}; \quad A^{in} \rightarrow F\Gamma \sqrt{\frac{\Gamma}{2\gamma_p \eta^{(3)}}} \quad (4.67)$$

In terms of these new variables we can mix the first and the second equation on (4.66) and also the third with the fourth in order to eliminate the dependence on the ψ and ϕ variables and end up with a couple of equations for the amplitudes A and A_p only in terms of the experimental adjustable parameters of the system, the pump power (proportional to F^2) and its detuning Δ_p

$$A_p^4 = 1 + \left(\Delta_p - \frac{D_3}{2} - 2A_p^2 - 3A^2 \right)^2$$

$$F^2 = A_p^2 \left\{ \left(\frac{\Gamma_p}{\Gamma} + 2\frac{A^2}{A_p^2} \right)^2 + \left[\Delta_p - A_p^2 - \frac{2A^2}{A_p^2} \left(\Delta_p - \frac{D_3}{2} - 3A^2 \right) \right]^2 \right\}. \quad (4.68)$$

Once A and A_p are calculated from the equations above, the phases ϕ and ψ can be determined by the equations

$$\sin(\phi) = \frac{1}{A_p^2}; \quad \cos(\phi) = \frac{1}{A_p^2} \left(\Delta_p - \frac{D_3}{2} - 3A^2 - 2A_p^2 \right)$$

$$\sin(\psi) = \frac{A_p}{F} \left[\Delta_p - A_p^2 - \frac{2A^2}{A_p^2} \left(-3A^2 - \frac{D_3}{2} + \Delta_p \right) \right]; \quad \cos(\psi) = \frac{A_p}{F} \left(\frac{\Gamma_p}{\Gamma} + \frac{2A^2}{A_p^2} \right) \quad (4.69)$$

The numerical solutions for the above equations, in the case of perfect phase matching ($D_3 = 0$) and for $\Gamma_p = \Gamma$, are shown at the top of figure 4.8 for different values of pump detuning Δ_p . At the bottom of figure 4.8 we have numerical solutions of equations (4.69) for a resonant pump ($\Delta_p = 0$) for different values of the D_3 parameter. To the left we have the intracavity pump power A_p^2 and at the right the signal and idler power A^2 .

FIGURE 4.8: Intracavity pump (left) and signal-idler (right) amplitudes for several values of pump detuning Δ_p at perfect phase matching $D_3 = 0$ (top) and at resonance with the pump, $\Delta_p = 0$ for several values of phase matching D_3 (bottom). The dashed line represent unstable solutions

The effect of the phase modulation is totally evident here. For instance, the intracavity pump does not have a constant value above threshold like in the $\chi^{(2)}$ scenario (equation

4.36) and is also a function of the incoming pump. We can see that we have not oscillation at the exact resonance in a perfect phase matching, instead we have to pump the resonator with some detuning (even bigger than the cavity bandwidth) in order to have oscillation. This is because when we increase the pump power in order to trigger the oscillation, the phase modulation shifts the mode frequencies taking them out of resonance, so, pumping with some detuning will compensate this effect. In the case of perfect resonance this compensation is done by the phase mismatch. This effect is shown at figure 4.9 where we plotted the total nonlinear detuning of the pump $\Delta_p^{NL} = A_p^2 + 4A^2 - \Delta_p$ and the total nonlinear detuning for the signal and idler fields $\Delta^{NL} = 2A_p^2 + 3A^2 - \Delta_p + D_3/2$ for the situation where $\Delta_p = 0$. Figure 4.8 shows that the signal and idler intensities are not a monotonically increasing functions of the incoming pump but they attain a maximum value around $\Delta^{NL} \approx 0$. On the other hand the intracavity pump intensity starts to decrease at the moment that the oscillation is achieved as a consequence of an increasing Δ_p^{NL} but at the moment that it stops to increase and start to decrease the intracavity pump can grow as the incoming pump also grows.

FIGURE 4.9: Non linear detuning for the signal, idler and pump.

4.3.1 Stability Of the Solutions and Oscillation Threshold

The oscillation threshold in the $\chi^{(2)}$ case was analytically found in equations (4.37) as the minimum incoming pump power for which we can find a non zero solution for the signal and idler field. In the $\chi^{(3)}$ case, although we can not get an analytical solution for the amplitudes of the signal and idler, we can have a sense of a threshold by looking for the stability of the "trivial" $A = 0$ solution. The analysis of the stability is done in a standard way. We make a small perturbation to the steady state solution and then we look if those perturbations grow on time, in which case the solution is unstable. On the other hand, if those perturbations decay on time, then the solution is stable. The perturbation is formally introduced with the substitution $\alpha_j(t) \rightarrow \bar{\alpha}_j + \delta\alpha_j(t)$ $j = s, i$ where $\bar{\alpha}_j = 0$ (below threshold) and as long as the perturbation can be as small as we

want, we will take only up to first order on those perturbations¹². Then, their time evolution is given by

$$\frac{d}{dt} \begin{pmatrix} \delta\alpha_s \\ \delta\alpha_i^* \end{pmatrix} = \underbrace{\begin{pmatrix} -1 - i\Delta + 2iA_p^2 & i\alpha_p^2 \\ -i\alpha_p^2 & -1 + i\Delta - 2iA_p^2 \end{pmatrix}}_{\mathbb{M}} \cdot \begin{pmatrix} \delta\alpha_s \\ \delta\alpha_i^* \end{pmatrix}, \quad (4.70)$$

where the intra cavity pump A_p is given by the solution of the under threshold equation

$$F^2 = A_p^2 \left[\frac{\Gamma_p^2}{\Gamma^2} + (\Delta_p - A_p^2)^2 \right]. \quad (4.71)$$

The dependence of A_p as a function of the parameters Δ_p and F is non trivial as we can see in figure 4.10. In the left side we have A_p^2/F^2 as a function of the detuning Δ_p . We can see that for small values F the curve has the usual Lorentzian shape of a cavity resonance, but when F increases the peak of the curve gets tilted to the right, attaining the maximum value at $\Delta_p = F^2$. When F achieve the critical value of $F_{crit} = (64/27)^{1/4} \approx 1.24$ we have three solutions of equation (4.71) from which one of them is unstable as pointed by the dashed line.

At the right side of figure 4.10 we have A_p^2 as a function of the normalized incoming pump power F^2 where for $\Delta_p < \sqrt{3}$ there is a single valued solution of (4.71) but for $\Delta_p > \sqrt{3}$ we have two stable solutions and one unstable for the intracavity pump intensity A_p^2 . In both cases this bistability causes an hysteresis effect when the values of the parameter are scanned in way one or in the opposite.

¹²This procedure is similar to the calculation of the quantum fluctuations done before, nevertheless, here we are dealing with classical perturbations of the steady state solution, that is why they can be as small as we want. The perturbation $\delta\alpha$ should not be confused with the quantum fluctuation $\delta\hat{a}$

FIGURE 4.10: (Left) Intracavity pump intensity A_p^2 as a function of the detuning for two values of F , $F = 0.2 < F_{crit}$ and $F = 2.7 > F_{crit}$. (Right) Intracavity pump intensity A_p^2 as a function of the incoming pump power F^2 for two values of $\Delta_p = 0 < \sqrt{3}$ and $\Delta_p = 2.5 > \sqrt{3}$. The dashed line stand for the unstable solution.

In order to have a below threshold stable solution, we need that the real part of the all the eigenvalues of $\mathbb{M}$ in equation (4.70) have to be negative. Those eigenvalues are given by

$$\lambda = -1 \pm \sqrt{A_p^4 - (\Delta - 2A_p^2)^2}, \quad (4.72)$$

thus, the stability condition as a function of the experimentally accessible parameter Δ_p is given by

$$A_p^4 - (\Delta_p - D_3/2 - 2A_p^2)^2 < 1, \quad (4.73)$$

and in any situation where this inequality is not satisfied we will have a growth of the signal and idler field. This process some times is called modulation instability.

FIGURE 4.11: Plot of the relation $A_p^4 - (\Delta_p - D_3/2 - 2A_p^2)^2 = 1$ for several values of the phase mismatch D_3 . The upper regions bounded by every curve are the zones where the below threshold solution is unstable and the signal and idler fields are present.

The same procedure can be used to find the stability condition for the above threshold solution. Again with the substitution $\alpha_i(t) \rightarrow \bar{\alpha}_i + \delta\alpha_i(t)$ where $\bar{\alpha}_i$ is the solution of the steady state equations (4.65) and $\delta\alpha_i$ is a (classical) perturbation. Defining $\delta\alpha = \{\delta\alpha_p, \delta\alpha_p^*, \delta\alpha_s, \delta\alpha_s^*, \delta\alpha_i, \delta\alpha_i^*\}$, doing the substitution shown before in the evolution equation and taking just first order term on the perturbation we end up with a equation in the form

$$\frac{d\delta\alpha}{dt} = \mathbb{M} \cdot \delta\alpha \quad (4.74)$$

where here $\mathbb{M}$ is a matrix which is a function of the steady state solution $\bar{\alpha}_i$. In the perfect phase match condition (plotted at the top of figure 4.8) the unstable solution is shown as a dashed line, showing that from the triple solution found in this case, just one of them is stable. On the other hand, at the resonant condition (plotted at the bottom of figure 4.8) the only solution is always stable.

4.3.2 Quantum Fluctuations

For the calculation of the quantum fluctuations we proceed in the same way as in the previous analysis for the $\chi^{(2)}$ process. We made the same linearization process as before, writing equation (4.63) in terms of the quantum fluctuation $\delta\hat{a}_i = \hat{a}_i - \bar{\alpha}_i$ and keeping just first order terms in these fluctuations. Defining, as before, the vector of fluctuations as

$$\delta\hat{\mathbf{A}} = \{\delta\hat{a}_p e^{-i\theta_p}, \delta\hat{a}_p^\dagger e^{i\theta_p}, \delta\hat{a}_s e^{-i\theta_s}, \delta\hat{a}_s^\dagger e^{i\theta_s}, \delta\hat{a}_i e^{-i\theta_i}, \delta\hat{a}_i^\dagger e^{i\theta_i}\}^T, \quad (4.75)$$

where θ_i is the phase of the main field $\langle\hat{a}_i\rangle = \alpha_i = A_i e^{i\theta_i}$. The time evolution of these fluctuations are given by

$$\frac{d\delta\hat{\mathbf{A}}}{d\tau} = -\mathbb{M}_a \cdot \delta\hat{\mathbf{A}} + \frac{\mathbb{T}_a^{in}}{\Gamma} \cdot \delta\hat{\mathbf{A}}^{in} + \frac{\mathbb{T}_a^{loss}}{\Gamma} \cdot \delta\hat{\mathbf{A}}^{loss} \quad (4.76)$$

$$\mathbb{T}_a^{in} = \text{Diag}\{\sqrt{2\gamma_p} e^{i\psi}, \sqrt{2\gamma_p} e^{-i\psi}, \sqrt{2\gamma}, \sqrt{2\gamma}, \sqrt{2\gamma}, \sqrt{2\gamma}\}, \quad (4.77)$$

with $\mathbb{T}_a^{loss}$ defined in the same way of $\mathbb{T}_a^{in}$ but changing γ by μ and $\tau = \Gamma t$ is the normalized time.

4.3.2.1 Two Mode Correlations

As in the $\chi^{(2)}$ scenario we can define the sum and subtraction subspace as in equation (4.52). In this stage we will consider the pump again as a classical noiseless field (mathematically done making $\delta\hat{a}_p \rightarrow 0$) and study just the correlations between the signal and idler fields. We can, as before, write the fluctuation equations in the sum/subtraction basis, so, given the vector $\delta\hat{\mathbf{Y}} = \{\delta\hat{x}_+, \delta\hat{y}_+, \delta\hat{x}_-, \delta\hat{y}_-\}^T$, it fulfills the equation

$$\frac{d\delta\hat{\mathbf{Y}}}{d\tau} = \mathbb{M}_Y \cdot \delta\hat{\mathbf{Y}} + \frac{\mathbb{T}_Y^{in}}{\Gamma} \cdot \delta\hat{\mathbf{Y}}^{in} + \frac{\mathbb{T}_Y^{loss}}{\Gamma} \cdot \delta\hat{\mathbf{Y}}^{loss} \quad (4.78)$$

with

$$\mathbb{M}_Y = \begin{pmatrix} \mathbb{M}_+ & \mathbf{0} \\ \mathbf{0} & \mathbb{M}_- \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 4A_p^2 + 6A^2 + D_3 - 2\Delta_p & 0 & 0 \\ -6A^2 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -4A_p^2 - 4A^2 - D_3 + 2\Delta_p & 0 \end{pmatrix}. \quad (4.79)$$

We can see again that the sum and subtraction subspaces are decoupled from each other as in the $\chi^{(2)}$ process. This is because the signal and idler photons are produced in pairs (twin photons) in both processes. But here we have that the noise spectral density matrix can not be made diagonal so easily as before. Then, the best squeezing will be on a rotated basis given by the Schmidt decomposition of the system. As long as the subspaces are decoupled, the rotation done for the Schmidt decomposition is different and independent for the sum and subtraction quadratures.

These new rotated quadratures are given by

$$\begin{pmatrix} y_{\pm}^{rot} \\ x_{\pm}^{rot} \end{pmatrix} = \begin{pmatrix} \cos(\theta_{\pm}) & \sin(\theta_{\pm}) \\ -\sin(\theta_{\pm}) & \cos(\theta_{\pm}) \end{pmatrix} \begin{pmatrix} y_{\pm} \\ x_{\pm} \end{pmatrix} \quad (4.80)$$

where the angles $\theta_{\pm}$ are chosen in such a way that the noise spectral density matrix is diagonal.

Figures (4.13-4.12) and (4.15-4.14) show, comparatively, the noises of the subtraction and sum quadratures respectively with and without rotation as a function of the incoming pump power. Here, real experimental parameters from a silicon based microresonator were taken. $\omega/\Gamma = 1.5 \times 10^{-2}$ and $\gamma/\Gamma = 0.55$ which correspond to an analysis frequency of 3 MHz, a bandwidth of 200 MHz, a intrinsic and loaded quality factor of 10^6 and 450.000 respective at $\lambda_p = 1561nm$. Plots for several values of phase mismatch for at pump resonance $\Delta_p = 0$ were done.

For the subtraction subspace we have a power independent squeezing on for the x_- quadrature (the same situation present on the $\chi^{(2)}$ case) and a large noise excess for the its conjugate quadrature y_- . For the rotated quadratures, we have a slightly better squeezing for the x_-^{rot} quadrature and a slightly larger noise excess for y_-^{rot} (because the noise excess is very high, the difference between y_-^{rot} and y_- is not appreciable on the plot). The small difference between the noise of the rotated and not rotated quadrature can be explained easily looking at the rotation angle at figure 4.16, where we see that

is a very small angle, $|\theta_-/\pi| < 10^{-5}$ in every situation, so both set quadratures are practically the same.

FIGURE 4.12: Noise for the rotated y_-^{rot} (black) and original y_- (red) as a function of the incoming power quadratures with $\Delta_p = 0$ for several values of phase matching D_3 . The SQL is at $1/2$

FIGURE 4.13: Noise for the rotated x_-^{rot} (black) and original x_- (red) as a function of the incoming power quadratures with $\Delta_p = 0$ for several values of phase matching D_3 . The SQL is at $1/2$

FIGURE 4.14: Noise for the rotated x_+^{rot} (black) and original x_+ (red) quadratures as a function of the incoming power with $\Delta_p = 0$ for several values of phase matching D_3 . The SQL is at $1/2$

FIGURE 4.15: Noise for the rotated x_+^{rot} (black) and original x_+ (red) quadratures as a function of the incoming power with $\Delta_p = 0$ for several values of phase matching D_3 . The SQL is at $1/2$

For the sum subspace the effect of the rotation is more dramatic. Here we have noise excess for both quadratures y_+ and x_+ but for the rotated quadratures we have way less noise for the y_+^{rot} quadrature, enough to have a good squeezing over all the values of pump power F^2 , and the x_+^{rot} quadrature just get slightly noisier than x_+ . Looking at the rotation angle at figure (4.17) it is evident the much bigger rotation suffered by the plus subspace compared with the subtraction one.

FIGURE 4.16: Rotation angle for the subtraction subspace as a function of the incoming power with $\Delta_p = 0$ for several values of phase matching D_3 .

The dramatic effect of the quadrature rotation on the noises of the sum subspace has a direct repercussion on the detection of entanglement. For this we have that the Duan inequalities for the rotated and original quadratures have different forms in accordance with equation (3.31) and the rotation transformation (4.80). For the system to be separable it has to fulfill

$$\Delta^2 \hat{x}_- + \Delta^2 \hat{y}_+ - 1 \geq 0 \quad (4.81)$$

$$\Delta^2 \hat{x}_-^{rot} + \Delta^2 \hat{y}_+^{rot} - |C| \geq 0 \quad (4.82)$$

$$\text{with } C = \cos(\theta_+ - \theta_-)$$

In figures (4.18) and (4.19) we can see the LHS of the inequalities above. For the original quadratures (4.18), just in a small region for high pump power, the Duan inequality detects entanglement. This region is exactly the place where the y_+ quadrature has a small noise excess (See figure 4.14). Now, in the case of the rotated quadratures, both y_+^{rot} and x_-^{rot} are squeezed thus the entanglement is detected over almost all the range

FIGURE 4.17: Rotation angle for the sum subspace as a function of the incoming power with $\Delta_p = 0$ for several values of phase matching D_3 .

FIGURE 4.18: Detection of entanglement between signal and idler field using the Duan inequality (4.81) for $\Delta_p = 0$ and several values of phase matching D_3 . Values below 0 represent detected entanglement.

FIGURE 4.19: Detection of entanglement between signal and idler field using the modified Duan inequality (4.82) for $\Delta_p = 0$ and several values of phase matching D_3 . Values below 0 represent detected entanglement.

of input pump power, just in the regions where the rotation angle θ_+ is close to $-\pi/2$ the inequality is not violated, this is because when $\theta_+ \approx -\pi/2$ we have that $x_+^{\text{rot}} \approx y_+$ which commutes with y_- and therefore the quantity C in equation (4.82) is close zero $C \approx 0$ and the inequality is always fulfilled. We can see this situation as if just intensity information is taken into account which make impossible any entanglement detection.

Looking at these results we have shown how the effects of phase modulation on the third order nonlinearities affect the noise ellipse of the field generated on an OPO generating a quadrature rotation and play a main role in the features of the generated light. Although the process of parametric down conversion ($\chi^{(2)}$) is somehow similar to the FWM (this is the main reason why we always compare these two processes) the phase modulation effect is not present on the $\chi^{(2)}$ system and in order to detect better quantum properties (as squeezing and entanglement) it is necessary to perform a quadrature rotation on $\chi^{(3)}$ systems.

4.3.2.2 Three Modes Correlations

The next step to fully characterize the three mode system is to include the correspondent quantum fluctuation to pump field and then analyze the entanglement properties of this

three quantum field system. On the linearized equations for the quantum fluctuations for the three modes we have that on the sum/subtraction basis just the sum is coupled to the pump.

Defining the vector $\delta\hat{\mathbf{X}} = \{\delta\hat{x}_p, \delta\hat{y}_p, \delta\hat{x}_+, \delta\hat{y}_+, \delta\hat{x}_-, \delta\hat{y}_-\}^T$, its linearized equation of motion is given by

$$\frac{d\delta\hat{\mathbf{X}}}{d\tau} = \mathbb{M} \cdot \delta\mathbf{X} + \frac{\mathbb{T}^{in}}{\Gamma} \cdot \delta\mathbf{X}^{in} + \frac{\mathbb{T}^{loss}}{\Gamma} \cdot \delta\mathbf{X}^{loss} \quad (4.83)$$

where the matrix $\mathbb{M}$ is block diagonal of the form $\mathbb{M} = \text{diag}\{\mathbb{M}_+, \mathbb{M}_-\}$, with $\mathbb{M}_+$ the coupling matrix between the pump and the sum modes and $\mathbb{M}_-$ is the same matrix as in equation (4.79). The explicit form of the matrix $\mathbb{M}$ is on the appendix A. From the equation above we can calculate the noise spectral density matrix using equation (4.51) and from it calculate the squeezing and test the entanglement of the system.

Figures 4.20 and 4.21 show the noises of the $\hat{x}$ and $\hat{y}$ quadratures of the pump for the same parameters as before. Showing that just for higher values of the D_3 parameter (in absolute value) and only for a few specific values of pump power the $\hat{x}$ quadrature has a small level of squeezing meanwhile the $\hat{y}$ just possesses excess of noise at all the parameter values shown.

FIGURE 4.20: Noise of the pump $\hat{x}$ quadrature

FIGURE 4.21: Noise of the pump $\hat{y}$ quadrature

As we pointed out before, the sum quadrature noise is coupled with pump. Therefore, we can observe in figures 4.22 and 4.23 how the noise in the sum mode is modified by the presence of the noisy pump. There we can see that on most of the situations the correction made for the pump reduces the noise of the $\hat{y}$ quadrature at in some situations even achieving squeezing for high pump power. As a consequence, the $\hat{x}$ quadrature has a noise increase for high values of pump powers with respect to the noise calculated without the quantum noise of the pump.

Finally we can analyze the entanglement between the three modes. For this, we use the van Loock-Furusawa criteria [78] which is a generalization of the DGCZ criteria [67] explained before in section 3.2.2 for multipartite entanglement.

As before, here we make use of the Schmidt modes of the system, given that those modes provide us the maximum squeezing possible in the system and that way would help us to detect entanglement. For this, we define vector $\hat{\xi}$ containing the quadratures of the Schmidt modes of the sum and pump subspace on the following way

$$\begin{aligned}
 \hat{\xi} &= \{\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \hat{\xi}_4\}^T \\
 \hat{\mathbf{X}}_+ &= \{\hat{x}_p, \hat{y}_p, \hat{x}_+, \hat{y}_+\}^T \\
 \hat{\xi} &= \mathbb{U} \cdot \hat{\mathbf{X}}_+
 \end{aligned} \tag{4.84}$$

FIGURE 4.22: Noise of sum $\hat{x}$ quadrature with (red) and without (black) a noisy pump FIGURE 4.23: Noise of sum $\hat{y}$ quadrature with (red) and without (black) a noisy pump

where $\mathbb{U}$ is the Schmidt transformation (in this case, an unitary matrix) relating the Schmidt modes with the original modes. As long as the subtraction fluctuations are not affected by the pump noise, their corresponding Schmidt modes are the same as in equation (4.80). Besides having a higher dimension, the matrix $\mathbb{U}$ can also be complex and can not be seen as a geometrical rotation on a 4-dimensional space as in the previous case. In terms of the Schmidt modes with lower noise the van Look-Furusawa inequalities take the form

$$\begin{aligned}\Delta^2 \xi_i + \Delta^2 \xi_j - S_I^{i,j} &\geq 0 \\ \Delta^2 x_-^{rot} + \Delta^2 \xi_j - P_I^j &\geq 0\end{aligned}\quad (4.85)$$

where $S_I^{i,j}$ and P_I^j are positive separability constants which depend on the quadratures ξ_i and x_-^{rot} used, the bipartition I in which the inseparability is being tested and are functions of the matrices in (4.80) and (4.84). Since the system is symmetric on the interchange of the signal and idler field, the separability constants are the same for the bipartitions $I = \{s|i, p\}$ and $I = \{i|s, p\}$.

FIGURE 4.24: Inseparability test for partition (p)(s,i)

Using all the Schmidt modes, we can deduce several van Look-Furusawa-like inequalities as the relations (4.85) but just those with less noise are useful for detecting entanglement, that in our case are the quadratures x_-^{rot} , ξ_3 and ξ_4 . Figures 4.24 and 4.25 show the *l.h.s.* of the two inequalities of the form (4.85) that are violated for the Schmidt modes. The violation of the inequality (negative values) plotted on figures 4.24 tell us that we have entanglement between the system formed by the pump mode on one side and the signal

FIGURE 4.25: Inseparability test for partitions (s)(p,i)=(i)(p,s)

and idler modes on the other side. The violation of the inequality plotted on figures 4.25 shows us that we can not separate the signal from the pump-idler modes and also that the idler can not be separated from the pump-signal modes, given the symmetry of the signal and idler modes in the system.

The systematic approach of the Schmidt modes allowed us to show ,for some range of parameters, the theoretical prediction of the presence of tripartite entanglement due to the simultaneous violation of the separability inequalities, being this, up to our knowledge, the first prediction of this kind of entanglement on this kind of systems.

4.3.2.3 Multimode Correlations: Cascade FWM and Frequency Combs

Once the first three modes of the field were generated on a primary FWM process, they can interact between them through a cascaded secondary FWM process and generate a second pair of signal and idler field (see figure 4.26). This process can repeat several times until the formation of the so called frequency comb where the different modes of the cavity will be interacting through the Hamiltonian (2.23) induced by $\chi^{(3)}$ non linear medium.

$$\hat{H}_{NL} = -\hbar\eta^{(3)} \sum_{i,j,k,l=-n}^n \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \delta_{i-j+k-l} \quad (4.86)$$

where δ_i is the Kronecker delta function $\delta_0 = 1$ and $\delta_{i \neq 0} = 0$.

From Hamiltonian above, we can deduce the Heisenberg-Langevin equation of motion for the annihilation operators forming a frequency comb composed by $2n + 1$ exited cavity modes as

$$\frac{d\hat{a}_i}{dt} = -(\Gamma + i\Delta_i)\hat{a}_i + i\eta^{(3)} \sum_{j,k,l=-n}^n \hat{a}_j^\dagger \hat{a}_l \hat{a}_k \delta_{i+j-k-l} + \sqrt{2\gamma}\hat{a}_i^{in} + \sqrt{2\mu}\hat{a}_i^{loss} \quad (4.87)$$

with $i = -n, \dots, n$, that is, we are taking into account $2n + 1$ modes and where we have supposed that all the modes belonging to frequency comb have the same coupling and spurious losses. The summation above already includes all the self and cross phase modulation terms as well as the FWM terms.

FIGURE 4.26: The primary FWM is generated by the annihilation of two photons of the frequency ω_0 and the generation of one photon of frequency ω_1 and another of frequency ω_{-1} . Given these three mode oscillation on the cavity secondary cascaded FWM processes take place and generate additional modes as for instance the annihilation of photons ω_0 and ω_{-1} with the generation of photons of frequency ω_1 and ω_{-2} .

Following the same philosophy that we have taken until now, we will refine our model by adding more and more modes of the field. The procedure to be followed is the same as before. First we solve the problem for the main field, and using that solution we calculate the fluctuations of the field that give us the quantum properties of the system.

The equations of motion for the mean field are given by

$$\frac{d\alpha_i}{d\tau} = -(1 + i\Delta_i)\alpha_i + i \sum_{j,k,l=-n}^n \alpha_j^* \alpha_l \alpha_k \delta_{i+j-k-l} + F\delta_i \quad (4.88)$$

where the normalization defined on equations (4.67) was used. In the equation above we have set the mode $i = 0$ as the mode being externally pumped. We also will write the resonant frequencies of the modes as Taylor expansion up to second order around the pump mode as $\omega_j = \omega_0 + \bar{d}_2 j + \bar{d}_3 j^2/2$ and we will fix the oscillation frequency

of the carries as $\nu_j = \nu_0 + \bar{d}_2 j$, with ν_0 the frequency of the external pump. With these definitions, the detuning in the equations above is given by $\Delta_j = (\omega_j - \nu_j)/\Gamma = \Delta_0 - d_3 j^2/2$ with $d_3 = \bar{d}_3/\Gamma$.

When we add just another pair of modes, the complexity of the system increases a lot, from just one FWM term in the case of three interacting modes to seven FWM terms in the case of five modes. Therefore, it was not possible to find an analytical closed form for the steady state solution as in (4.68) and (4.69) when more than three modes are taken into account. Instead of that, the approach followed here is to start with some random initial condition and evolve on time equations (4.88) numerically a long period of time until arrive to the steady state solution, if possible.

At the top figure 4.27 we plot the power ($P_i = |\alpha_i|^2$) of the intracavity fields for different number of modes added to the model. It is important to point out that there is not another threshold for the generation of the modes $\alpha_i, |i| \geq 2$. Once the first pair of modes $i = \pm 1$ appears, all the other modes begin to oscillate as well. Those additional modes are less relevant just above threshold and also for high values of F^2 close to the point where the oscillation ceases. We also can verify that the power of the modes is completely symmetric with respect of the pump mode, that is, $P_i = P_{-i}$. This is a consequence from the fact that the evolution equations (4.87) are invariant under the transformation $i \rightarrow -i$.

Given the complexity of this new situation, it can be the possibility that, for some values of the system parameters, there is not a steady state solution as can be seen at the bottom part of figure 4.27. Instead, we can have oscillating solution where the energy inside the cavity keeps flowing between the different excited modes in a periodic way and where the symmetry pointed out before is broken, that is $P_i \neq P_{-i}$. We show an example of this behavior in figure 4.28 where we have plotted the time evolution for the same parameters as in bottom plot of 4.27 for $F^2 = 17.64$ (marked by the vertical line in 4.27). Although the power of the fields is periodic in time, this is not, in general, true for the complex amplitudes $\alpha_j(t)$ of the field which solution is of the form $\alpha_j(t) = e^{i\omega_d j t} \alpha_j^{per}(t)$ where $\alpha_j^{per}(t) = \alpha_j^{per}(t + T)$ is periodic with period T and ω_d is a slow oscillation "drift" that, in general, is not commensurable with T . This means that the carrier's frequency of the field $\alpha_j(t)$ has its detuning increased or decreased by the value $j\omega_d$.

In figure 4.28 we have plotted the intracavity power $P_j(t) = |\alpha_j(t)|^2$ at the left, as a function of time, and to the right we have parametric plots of the real versus the imaginary part of $\alpha_j(t)$ and $\alpha_j^{per}(t)$, where we can clearly see the asymmetry on the modes i and $-i$ which is not present on the steady state. The sign of the frequency ω_d depends with the initial conditions, thus for a particular set of parameter values with leads to an oscillatory behavior we will have two possible solutions, one with $+\omega_d$ and

FIGURE 4.27: Intracavity power ($P_i = |\alpha_i|^2$) of the cavity modes when we gradually add more modes to the model. (Top-Left) $\Delta_0 = 0$ and $D_3 = -4.5$. (Top-Right) $\Delta_0 = 0$ and $D_3 = -7$. (Bottom) Intracavity power for $N = 2n + 1 = 9$ modes included on the model with $\Delta_0 = 1.6$ and $D_3 = -4.5$. In this situation there are some values of pump power F^2 where no steady state was found.

FIGURE 4.28: Oscillatory solution for $n = 4$, which mean that 9 modes are taken into account in the model. At the left we have the intracavity power $P(t) = |\alpha(t)|^2$ as a function of time where in orange we have P_{+j} and in blue P_{-j} at the right of each one of these plots we have parametric plots of the real part versus the imaginary part of $\alpha_j(t)$ and $\alpha_j^{per}(t)$ with the time as a parameter.

another with $-\omega_d$. In one case spacing between the carrier's frequencies of the comb is increased and in the other is decreased by ω_d .

This kind of solution can be related with the so called breather soliton [79] which is a phenomena where a $\chi^{(3)}$ nonlinear cavity pumped by a continuous wave, a cavity soliton is created which has not a fixed amplitude and width but oscillates slowly on time.

4.4 Multimode Covariance Matrix, Schmidt Modes and Entanglement

Given that we are interested on the spectral characteristics of the fluctuations is it easier to analyze just the situations where we have a stationary solution for the classical fields and we will restrict ourself to that situations in this section. Here we will generalize the procedure done before for the case on an arbitrary number of modes and then we will show explicitly the noise properties and entanglement of the system for different number of modes.

First, we define a vector containing all the $4n + 2$ quadratures of the $2n + 1$ modes of system as $\hat{\mathbf{X}} = \{\hat{x}_{-n}, \hat{y}_{-n}, \dots, \hat{x}_n, \hat{y}_n\}$. With it, we can construct the linearized equation of motion for the fluctuations of this vector from equation (4.87) in the same way as was done until now.

$$\frac{d\delta\hat{\mathbf{X}}}{d\tau} = \mathbb{M} \cdot \delta\mathbf{X} + \frac{\mathbb{T}^{in}}{\Gamma} \cdot \delta\mathbf{X}^{in} + \frac{\mathbb{T}^{loss}}{\Gamma} \cdot \delta\mathbf{X}^{loss}$$

Where here $\mathbb{M}$ is a $(4n + 2) \times (4n + 2)$ matrix which is a function of the mean value amplitudes calculated before. Given that, we can use equation (4.51) to construct the noise spectral density matrix.

The Schmidt modes $\hat{\xi} = \{\hat{\xi}_1, \hat{\xi}_2, \dots, \hat{\xi}_{4n+2}\}$ of the system are defined through the transformation

$$\hat{\xi} = \mathbb{U} \cdot \hat{\mathbf{X}} \quad (4.89)$$

such that the matrix $\mathbb{U}$ is unitary and diagonalize the noise spectral density matrix. We can label the operators in $\hat{\xi}$ such that $\hat{\xi}_1$ has the bigger noise and $\hat{\xi}_{4n+2}$ the smaller.

From the noises of this Schmidt modes we can write several separability inequalities from each bipartition $I = \{I_1|I_2\}$ as

$$\Delta^2\xi_i + \Delta^2\xi_j - S_I^{i,j} \geq 0 \quad (4.90)$$

which have to be fulfilled for all the states that are separable in the bipartition I .

As before the term $S_I^{i,j}$ is a positive quantity which depends on the specific bipartition I taken into account and on the Schmidt modes ξ_i and ξ_j used to test the separability.

When we have more than three parties, there are more ways to partition the system than just by bipartitions but if the proof inseparability for all the possible bipartition of the

system, separability for any other kind of will be denied. The bipartition $I = \{I_1|I_2\}$ is defined by splitting the set $I_T = \{-n, \dots, n\}$ into two disjoint subsets I_1 and I_2 , that is, in such a way that $I_1 \cup I_2 = I_T$ and $I_1 \cap I_2 = \emptyset$. The set I_T has $2n + 1$ elements and therefore exists $2^{2n} - 1$ different bipartitions. Given that in our model the evolution equations (4.87) are symmetric under the inversion of the indices $i \rightarrow -i$ the total amount of non equivalent bipartitions are $2^{2n-1} + 2^{n-1} - 1$.

The separability constant can be written explicitly in terms of the components of the matrix $\mathbb{U}$ as

$$S_I^{i,j} = \left| \sum_{k,l \in I_1^{index}} U_{i,k} \Omega_{k,l} U_{j,l}^* \right| + \left| \sum_{k,l \in I_2^{index}} U_{i,k} \Omega_{k,l} U_{j,l}^* \right| \quad (4.91)$$

where I_1^{index} contains the indices of all the quadratures $\hat{X}_i$ that belong to the part I_1 (and equivalently for I_2^{index}). The quantities $\Omega_{i,j}$ are defined by the commutator or the quadratures on $\hat{\mathbf{X}}$ as $i\Omega_{i,j} = [\hat{X}_i, \hat{X}_j]$.

Due to, if the system is separable for a given bipartition I , inequality (4.90) has to be fulfilled for any pair of operators $\{\xi_i, \xi_j\}$ we can test inseparability in that particular bipartition by taken the minimum over all the combinations of the indices $\{i, j\}$, that is

$$\min_{i,j} [\Delta^2 \xi_i + \Delta^2 \xi_j - S_I^{i,j}] \geq 0 \quad (4.92)$$

has to be fulfilled if the system is separable in the bipartition I .

Finally if we took the maximum over all bipartitions I from the above equation we have that whenever we violate the inequality

$$\max_I \left[\min_{i,j} [\Delta^2 \xi_i + \Delta^2 \xi_j - S_I^{i,j}] \right] \geq 0 \quad (4.93)$$

we have full $(2n + 1)$ -partite entanglement of the system.

Given all these definitions above, we can start to apply them to test whether or not entanglement can be detected in our system always remembering the big limitation of this technique, which is the fact that on the case that the separability inequality is fulfilled does not certainly mean that we have a separable system. After the tripartite system studied in section 4.3.2.2 we can investigate the behavior of the five-partite system ($n = 2$). In this case, plotting the *l.h.s* of equation (4.93) we can find that there exist regions where full five-partite entanglement is detected on the system proved by the negative values in figure 4.29. The parameters used here were the same used through

this work except for the losses which were set at $\gamma/\Gamma = 0.99$ for values less than 0.97 no five-partite entanglement was detected.

FIGURE 4.29: Plot of the *l.h.s.* of the separability inequality (4.93) for $\Delta_0 = 0$ and several values of d_3 . Violation of this inequality, shown here as negative values on the plot, proofs full five-partite entanglement on the system

If we add more modes we are not able anymore to detect full $(2n+1)$ -partite entanglement with this procedure anymore as can be seen in figure 4.30 where the *l.h.s.* (4.93) is plotted for $n = 3$ (total number of modes $N = 2n + 1 = 7$). The fact that full seven-partite entanglement was not detected does not mean that there is not present another type of entanglement, for that we can look for all the 35 non-equivalent bipartitions for $n = 3$. Actually from all those 35 bipartition just the one where the modes which are furthest from the pump are isolated from the rest had no region of inseparability detected, that is, the bipartition $I = \{-3, 3 | -2, -1, 0, 1, 2\}$ is the only one that always fulfilled inequality (4.92) for all the parameter values where it was calculated. That can led us to think that the modes $i = \pm 3$ are so weak that the interaction inside the system is negligible, but nevertheless the interaction within the system is enough to generate quantum correlations between those two modes as can be seen from the violation of inequality (4.92) for the bipartition $I = \{-3 | -2, -1, 0, 1, 2, 3\}$ (and the equivalent one $I = \{3 | -3, -2, -1, 0, 1, 2\}$), specially for high values of F^2 . (See figures 4.30)

If we keep adding more modes, for instance if we set $n = 4$ ($N = 2n + 1 = 9$ modes), the behavior is similar than in the previous case. It was not found any parameter values where we detect full 9-partite entanglement. From all the 135 non equivalent bipartition, just the bipartitions $I = \{-4, 4 | -3, -2, -1, 0, 1, 2, 3\}$, $I = \{-3, 3 | -4, -2, -1, 0, 1, 2, 4\}$ and $I = \{-4, -3, 3, 4 | -2, -1, 0, 1, 2\}$, that is, the ones where the modes with $|i| \geq 3$ are separated from the rest, fulfilled the separability inequality (4.92), but again, correlations

FIGURE 4.30: (Left) *l.h.s.* of the seven-partite separability inequality (4.93) for $\Delta_0 = 0$ and different values of d_3 (Right) *l.h.s.* of separability inequality (4.92), demonstrating entanglement between the modes $i = \pm 3$ (regions of negative values).

between those external modes exist as can be seen from the violation of separability inequalities in the case when the modes $+i$ and $-i$ with $|i| \geq 3$ are separated from each other in a particular bipartition (See figure 4.31)

FIGURE 4.31: (Left) *l.h.s.* of the nine-partite separability inequality (4.93) for $\Delta_0 = 0$ and different values of d_3 (Right) *l.h.s.* of separability inequality (4.92), demonstrating entanglement between the modes $i = \pm 3, \pm 4$ (regions of negative values).

The analysis before showed us that there are several forms of quantum correlations between the different oscillating modes within a frequency combs generated on a $\chi^{(3)}$

non-linear OPO. Even that, for $n \geq 3$ no $(2n + 1)$ -partite entanglement was detected, this analysis also open the possibility for on-chip silicon based optical microresonators to became a versatile scalable source of entangled states.

Conclusions and Perspectives

Through this work, the conditions and the features of the entanglement between idler and signal generated in a $\chi^{(3)}$ OPO were studied, comparing them with the properties of the idler-signal entanglement in a $\chi^{(2)}$ OPO, for which the group has great expertise. The effects of phase modulation, that are not present in the $\chi^{(2)}$ OPO, are determinant for the oscillation and entanglement of the signal-idler fields. For the oscillation, we showed that the cavity has to be pumped highly detuned or it needs to have some dispersion in order to compensate the phase modulation effects, keeping the fields in an effective resonance and achieving oscillation. In the case of entanglement, we apply the method of the Schmidt decomposition showing that, differently from the $\chi^{(2)}$ case, the strongest correlations of the systems are not between phase and amplitude but it is necessary to perform a quadrature rotation to observe stronger correlations and, as a consequence, be more sensitive to the signal-idler entanglement.

We went further and included the quantum character of the pump in our model. Following the same systematic procedure of the Schmidt modes we constructed several van Look-Furusawa like inequality that allowed us to detect tripartite entanglement within the system. This is, up to our knowledge, the first theoretical prediction of tripartite entanglement above threshold in a $\chi^{(3)}$ nonlinearity OPO.

In order to include the cascaded FWM effects, we added more and more modes to interact in our model which describe a frequency comb. In the situation of five interacting modes we were able to detect five-partite entanglement, but when we added another two modes this kind of full entanglement among all the interacting modes in the system. Nevertheless quantum correlations between the those external modes were also predicted analyzing separability among all the possible bipartitions of the system.

The most immediate continuation of this work is the experimental confirmation of this theory, specially on the context of integrated silicon photonics which was one of the principal motivations of this work and which is feasible with the current technologies.

Further interesting continuations to this work can include the analysis of this system in the context of side band modes, where associate two quantum modes to each resonance cavity mode, one for upper and other for the lower side band, and in that way we can have a richer entanglement structure that can be analyzed. Given that the $\chi^{(3)}$ nonlinearity is intrinsically a non-Gaussian operation it would be interesting to study which kind of non-Gaussian state can be generated in this kind of system as long as non-Gaussian states are important in the context of universal quantum computation.

Appendix A

Some matrix definitions

The matrix $\mathbb{M}$ in the equation (4.83) explicitly is given by

$$\mathbb{M} = \begin{pmatrix} \frac{-\frac{2B^2}{A^2} - 1}{-3A^4 + \Delta A^2 + B^2(6B^2 + d - 2\Delta)} & \frac{\frac{2B^2}{A^2} - 1}{A^4 + (8B^2 - \Delta)A^2 + B^2(6B^2 + d - 2\Delta)} & -\frac{2\sqrt{2}B}{A} & -\frac{\sqrt{2}B(4A^2 + 6B^2 + d - 2\Delta)}{A} & 0 & 0 \\ \frac{\frac{2\sqrt{2}B}{A}}{\sqrt{2}B(6B^2 + d - 2\Delta)} & -\frac{\sqrt{2}B(4A^2 + 6B^2 + d - 2\Delta)}{A} & \frac{\sqrt{2}B(6B^2 + d - 2\Delta)}{A} & -\frac{2\sqrt{2}B}{A} & 0 & 0 \\ -\frac{\sqrt{2}B(6B^2 + d - 2\Delta)}{A} & -\frac{\sqrt{2}B(4A^2 + 6B^2 + d - 2\Delta)}{A} & 0 & 4A^2 + 6B^2 + d - 2\Delta & 0 & 0 \\ 0 & \frac{2\sqrt{2}B}{A} & -6B^2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4A^2 - 4B^2 - d + 2\Delta & 0 \end{pmatrix} \quad (\text{A.1})$$

Appendix B

Dynamics of Intensity Noise Correlation on a ^{87}Rb Atomic Cold Cloud

This Appendix correspond to another work carried out on the LMCAL which I was involved a significant time through my PhD .

As pointed out in the introduction, the research at the LMCAL on the field of quantum optics, applied to quantum information is done through severals fronts as OPOs built with crystals ($\chi^{(2)}$) on a macroscopic cavities [26, 27], microscopic on silicon based ($\chi^{(3)}$) chip OPO as described through this text and also with atomic media.

Coherent atom-light interactions are very interesting in the quantum computation science, allowing implementations of quantum information processing and quantum memories, which are fundamental pieces of the quantum information framework. Atomic media is also useful in the generation itself of non classical states of light as have been historically shown with the first production of squeezed states of light [80] in 1985 and recently with the direct generation of entangled states via FWM in hot vapor [81].

Coherent atom-light interaction as the electromagnetically induced transparency (EIT) have been studied for several years and have been used to characterize the atomic media, in our case, through noise intensity correlation spectroscopy, where two phase noisy lasers on a Λ EIT configuration interact with ^{87}Rb cloud of atoms trapped on a magneto-optical trap (MOT). Through the interaction with the atoms, the phase noise is transform into intensity noise which correlation on the frequency domain is measured, this method allows power broadening free measurements of the decoherence between the ground state levels

of the Λ system [82], which is directly related with the storage time of quantum memories. Theoretically, this correlation is given by

$$C(\omega) = \frac{S_{12}(\omega)}{\sqrt{S_{11}(\omega)S_{22}(\omega)}} \quad (\text{B.1})$$

$$S_{ij}(\omega)\delta(\omega - \omega') = \langle \delta I_i(\omega)\delta I_j(\omega') \rangle, \quad (\text{B.2})$$

here, the average $\langle \dots \rangle$ is a classical average made over an many realizations of the stochastic phase of the incoming field, given that the outcoming intensity is a function of this phase.

An heuristic model of a closely related quantity, the $g^{(2)}(0)$ function, defined as

$$g^{(2)}(0) = \frac{\langle \delta I_1(t)\delta I_2(t) \rangle}{\sqrt{\langle \delta I_1(t)^2 \rangle \langle \delta I_2(t)^2 \rangle}} \quad (\text{B.3})$$

was developed at [83] where, this function was directly related with the dispersive and absorptive properties of the media. This approach, give an direct and intuitive physical insight of the behavior of the quantity $g^{(2)}(0)$ as a competition of different atomic properties. However, this $g^{(2)}(0)$ have no information of the side bands (ω) which are relevant, specially when the correlation is measure at analysis frequencies outside of the EIT window. In [84] a theoretical model for the frequency domain correlation function $C(\omega)$ was developed, but although it agrees with the experiments and take account for the side bands, it is relies on numerical methods that lack for any physical insight in a similar way as the heuristic model.

In order to solve this problem we developed a perturbative model that is an intermediate model between the heuristic model at [83] and the numerical model [84], which allows us to have a physical insight of the correlation function, also in terms of competition of dispersive and absorptive terms but this time in the frequency domain, although this term have to be calculated numerically in order to extract some useful information from them. This model give as a result a paper that was published at the journal Physical Review A [85].

In figure (B.1) we can see the $g^{(2)}(0)$ function and the correlation $C(\omega)$ for severals analysis frequencies as a function of the two photon detuning δ , where we can clearly observe the effect of the side bands for analysis frequencies of 4 and 5 MHz at the correlation and that is not present in the $g^{(2)}(0)$ function. This effect is already shown at the first order solution, but as long as the lasers have a high level of noise, it is necessary to carry the calculation up to severals order more, to have a good agreement with the

FIGURE B.1: Correlation spectra $C(\omega)$ and $g^{(2)}(0)$ function at first order γ (a) and to seventh order γ^7 (b) for several values of analysis frequencies $\omega' = \omega/2\pi$ in MHz, with γ the bandwidth of the phase fluctuations. The vertical dotted lines show the EIT window

experiment (our perturbative parameter is not small enough, so the first order expansion does not give us a good accuracy).

At the moment that the measurements of the noise intensity correlation spectra were taken, an interesting feature appeared. The shape of the correlation spectra curve changed depending on the direction of in which the scan was done (see B.2). That feature was not explained by our theoretical model that was used before. To address that problem, we investigated several different phenomena that we could have ignored and that possibly would explain that feature. Experimentally we noted that the dynamics toward the steady state of the system was unusually long (see B.2), about two orders of magnitude longer than predicted by our model. With that in mind the first approach was to include in our model phenomena such as coherent population oscillation (CPO) and recoiled induced resonances (RIR). It was reported that those phenomena present a sub-natural linewidth structures of the order of KHz, which corresponded to the same order of magnitude for the transient time observed. When the calculations were done, we realized that our system had not the characteristic for those phenomena to be present, like an decay of the ground state levels (for example, due to the transit time through the interaction region in the case of vapor cells) in the case of the CPO or lower temperatures to the case of the RIR.

Another experimental feature was that the effect depended strongly with the geometry of the system (see B.3) where the effect basically disappeared on the contra-propagating

FIGURE B.2: Time evolution of the correlation for two different values of the two photon detuning (left), and correlation spectra (right) when the scan is done from positive to negative detuning (red-negative scan) and when is done from negative to positive detuning (black-positive scan)

where both of the fields involve at the EIT process (EIT fields on the graphic) propagate once towards the other.

Also was noted that the re pump field played an important role as well in the phenomena, this re-pump field takes the population from the cooling transition ($F = 2 \rightarrow F' = 3$) to the EIT transition ($F = 1 \rightarrow F' = 1$) (see figure B.3) and whenever it is closer to resonance, faster is the dynamics and less prominent is the difference between positive and negative scan on the correlation spectrum.

FIGURE B.3: Geometry of the experiment (left) and structure level used at the ^{87}Rb

The solution to this problem was into taking account to the radiation pressure felt by the cloud due to the high intensities necessary to achieve anti-correlation on the spectrum produce a strong enough force to move considerably the atoms and induce an effectively dynamical on one photon detuning in accordance to Doppler effect. On figure (B.4) the experimental results are contrasted with the theoretical model showing a good agreement in relation with the time scales of the dynamics and the dynamical behavior.

FIGURE B.4: Dynamics of the intensity correlation for several two photon detuning, theory at the left and experimental results at the right

The theoretical model reproduces the difference on the dynamics between the red detune case, the on resonance case and the blue detuned case, showing in the red detuned case that at some specific instant of time the atoms go through the one photon resonance which can be seen at the dip on the correlation curve. At resonant we are at EIT condition, so the atoms do not absorb light and therefore, they do not undergo any force giving as result a really fast evolution. Experimentally it is not so easy to be at the exact resonance of the system, and given that the central peak in figure (B.2) is too narrow, the values of the correlations on the resonance case at figure (B.4) are not the same, nevertheless, the fast dynamics stay present as long as we are inside the EIT window. Finally for the blue detuned case, the one photon resonance feature is not present anymore given that the Doppler effect in our case always produces an effective blue detuning for the atoms. Most of the characteristics of this phenomena can be explained by a three level system consisting in the two EIT fields coupled to the atom in a Λ configuration (see figure B.4) but, as mentioned before, the re pump field has a significant effect on the dynamics. This can also be taken into account by developing a five level model which includes the re pump, and as shown in (B.4) is more significant for the red detuned case.

This second part of the dynamics will get as a result two papers which right now are on process of redaction.

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