## Universidade de São Paulo Instituto de Física

# Aspectos de laços de Wilson holográficos 

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# Aspects of holographic Wilson loops 

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A Jona, Pedro y Pepe, quienes afortunadamente no son físicos, por mantener a la distancia mis pies en la tierra por tantos años.

A Janett. A mis padres. A Laura.
"In my opinion, one who intends to write a book ought to consider carefully the subject about which he wishes to write. Nor would it be inappropriate for him to acquaint himself as far as possible with what has already been written on the subject. If on his way he should meet an individual who has dealt exhaustively and satisfactorily with one or another aspect of that subject, he would do well to rejoice as does the bridegroom's friend who stands by and rejoices greatly as he hears the bridegroom's voice. When he has done this in complete silence and with the enthusiasm of a love that ever seeks solitude, nothing more is needed; then he will carefully write his book as spontaneously as a bird sings its song, and if someone derives benefit or joy from it, so much the better."

## Resumo

Nesta tese estudamos alguns aspectos da correspondência AdS/CFT. Em particular aqueles que involvem quantidades que podem ser calculadas exatamente, como os laços de Wilson. A correspondência calibre/gravidade ou AdS/CFT nos permite interpretar duas teorias diferentes com as mesmas simetrias globais como descrições complementares da "mesma física". Os valores de expectação dos laços de Wilson em várias representações podem ser calculados em ambos lados da dualidade usando modelos de matrizes na teoria de calibre e cordas e branas no lado da gravidade. Do ponto de vista holográfico, a receita geral nos diz que devemos minimizar a folhamundo das cordas ou volumemundo das branas com limite no laço. Essas técnicas, já aplicadas aos casos das representações fundamental e (anti)simétrica do laço de Wilson, podem ser extendidas a representações mais complicadas cujo dual são branas coincidentes e geometrias "bubbling". Um fenômeno interessante também é aquele que ocorre quando consideramos dois laços, o que na teoria de gravidade se traduz como a solução conectada tipo catenoide. A existência de superficies conectadas entre dois laços de Wilson depende do valor de vários parâmetros que descrevem a geometria e posição relativa dos laços. Aqui, devido a esses parâmetros, existe uma transição de fase chamado de Gross-Ooguri, onde a solução conectada não é "energéticamente favorável" com respeito à solução desconectada, i.e. dois laços independentes. Muito mais interessante e rico é o caso de laços de Wilson na presença de defeitos, i.e. regiões de interfase devidas à presença, no caso estudado aqui, de uma D5 brana. Estudamos também o correlador entre dois laços neste caso.

Palavras chave Correspondência AdS/CFT; laços de Wilson; ação DBI nãoabeliana; transição de Gross-Ooguri.

## Abstract

In this thesis we study some aspects of the AdS/CFT correspondence. In particular those involving observables that can be computed exactly, as Wilson loops. The gauge/gravity correspondence or AdS/CFT allows us to interpret two different theories with the same global symmetries as complementary descriptions of the "same physics". The expectation values of Wilson loops in several representations can be calculated on both sides of the duality by using localization in the gauge theory and strings and branes on the gravity side. From the holographic point of view, the general recipe tells us that we have to minimize the string worldsheet or brane worldvolume with the loop as boundary. These techniques, already applied to the fundamental and (anti)symmetric reprentations of the Wilson loops, can be extended to more complicated reprentations whose duals are coincident branes and bubbling geometries. An interesting phenomenom also is that in which we consider two loops, that on the gravity side translate into the connected catenoid-like solution. The existence of connected solutions between two loops depends on the values of several parameters that describe the geometry and relative position between the loops. Here, due to these parameters, exists a transition knows as Gross-Ooguri, in which the connceted solution becomes energetically non-favorable with respect to the disconnected solution, i.e. two independent loops. Much more interesting and rich is the case of two Wilson loops in the presence of defects, i.e. interfase regions due to the presence of, in this case, a D5 brane. We also study the correlator of two Wilson loops in this case.

Keywords AdS/CFT correspondence; Wilson loops; nonabelian DBI; Gross-Ooguri phase transition.

## Contents

Acknowledgments i

Resum0 iii

Abstract iv

Index iv

1 Introduction and Overview 1

2 Basics on the AdS/CFT correspondence 7
2.1 Superstring and supergravity: quick review . . . . . . . . . . . . . . . . 7
2.2 D3-branes in type IIB, the $A d S_{5} \times S^{5}$ geometry . . . . . . . . . . . . . . 9
2.3 D3-branes in type IIB, the $\mathcal{N}=4$ theory . . . . . . . . . . . . . . . . . . 11
2.3.1 $\quad$ A short pit stop on $\mathcal{N}=4 D=4 U(N)$ gauge theory. . . . . . . 14
2.3.2 The large $N$ limit for any gauge theory . . . . . . . . . . . . . . 15
2.4 Motivating the AdS/CFT duality . . . . . . . . . . . . . . . . . . . . . . 17
2.4.1 Holographic duality . . . . . . . . . . . . . . . . . . . . . . . . 18
2.4.2 The dictionary . . . . . . . . . . . . . . . . . . . . . . . . . . . 19

3 Wilson loops, perturbative and exact results 21
3.1 (Supersymmetric) Wilson loops . . . . . . . . . . . . . . . . . . . . . . 21
3.1.1 Wilson loops in perturbation theory . . . . . . . . . . . . . . . . 24
3.1.2 Summing planar graphs . . . . . . . . . . . . . . . . . . . . . . 26
3.2 A brief introduction to matrix models and localization . . . . . . . . . . . 27
3.2.1 The saddle point method . . . . . . . . . . . . . . . . . . . . . . 30
3.2.2 Orthogonal polynomials ..... 31
3.2.3 Supersymmetric localization ..... 33
4 Holographic Wilson loops ..... 36
4.1 Boundary conditions ..... 38
4.2 Legendre transform and the elimination of the divergence ..... 40
4.3 Three examples ..... 41
5 Branes as Wilson loops ..... 46
5.1 Wilson loops in $k$-symmetric and $k$-antisymmetric representations ..... 48
5.1.1 The large $N \sim k$ limit ..... 51
$5.2 \quad D$-branes and holography ..... 52
5.2.1 $\quad \mathrm{D}_{k}$-brane and the $k$-antisymmetric representation ..... 53
5.2.2 $\mathrm{D} 3_{k}$-brane and the $k$-symmetric representation ..... 56
5.3 Beyond the leading order ..... 59
5.4 Rectangular tableau and the need of a nonabelian DBI action ..... 60
5.4.1 Vertical rectangular tableau ..... 61
5.4.2 Horizontal rectangular tableau ..... 63
5.5 NADBI for two coincident D3 branes ..... 66
5.5.1 The U(2) case ..... 70
6 Wilson loop correlators in holography ..... 77
6.1 Two connected Wilson loops with $\Delta \phi=0$ ..... 81
6.1.1 Evaluating the action ..... 86
6.2 Two connected Wilson loops with $\Delta \phi \neq 0$ ..... 90
6.3 Disconnected case with the defect ..... 98
6.3.1 Evaluating the action ..... 109
6.4 Phase transition in the presence of the defect ..... 113
7 Results and outlook ..... 121
A Symmetric/fundamental Wilson loop correlator ..... 124
B Perturbative computation ..... 132

Bibliography 135

## Chapter 1

## Introduction and Overview

String theory [1,2] is one of its most fascinating attempts to formulate a consistent theory of everything. One of the most captivating features of string theory is that it contains gravity, having a graviton as an state of its spectrum. In order to be, mathematically, consistent, string theory must be defined in ten-dimensional backgrounds making it necesssary to understand mechanisms to compactify the extra dimensions and making contact with the real world. More than forty years after its discovery, there are still many people trying to make progress in it. Not only theoretical physicists but also mathematicians and even philosophers are working hard to understand the fundamentals of this theory and to clarify some aspects of its formulation.

It was at the end of the nineties that a new avenue of research originated from string theory appeared: the gauge/gravity (also known as gauge/string, in general, or simply AdS/CFT) duality, that establishes that string theory in a certain background can be described by (or as) a gauge theory [3-7]. The main point of this duality is that there must exist one-to-one relations between observables on both sides. This is what we know as the dictionary, or more precisely a bijection, of the duality. Both sides of the duality have parameters such as coupling constant, central charge and radii of curvature which are related in simple ways. In particular, string theory is dual to a quantum field theory, but depending on which regime of parameters one considers that either the stringy behavior or the particle behavior of the theory is manifest. Also, string and quantum field theory are complementary, i.e when it is hard to work with one of them then it is easier to work with the other. This is called weak/strong coupling correspondence. In this sense we say
that the correspondence ensures that we can look at a quantum field theory from a different perspective, in which it looks like a string theory. We actually are not changing the "nature" only its description. This means that, in the strong statement, all the physics of one description can be mapped onto all the physics of the other. And, since string theory contains gravity, this means that a theory of quantum gravity could be mapped to a quantum field theory without it. It is also said that the duality is holographic: information of the bulk is encoded at its boundary. In this case, information about strings is encoded at the boundary of some space as a gauge theory [8,9]. It was first studied the duality between Type IIB superstrings in $A d S_{5} \times S_{5}$ and $\mathcal{N}=4 D=4 S U(N)$ gauge theory [10], being the simplest and best understood case until now since the gauge dual theory has the largest supersymmetry in four dimensions and it is also a conformal field theory, i.e a theory that does not depend on scales. In this case, it is assumed that the field theory "lives" on the stack of D3 branes that source the ten-dimensional background. A lot of cases were implemented thenceforth by trying to get evidence of the duality between string theory in certain backgrounds and (supersymmetric) gauge theories. A particular example is the duality between M theory/Type IIA string theory in $A d S_{4} \times S^{7} / A d S_{4} \times C P_{3}$ and $\mathcal{N}=6 D=3$ Chern-Simons theory, the most supersymmetric gauge theory in three dimensions [11]. Another case that appeared almost at the same time that the first case was the Klebanov-Witten background in which the $S^{5}$ part was replaced by a singular Calabi-Yau space $T^{11}$. The dual field theory of string in this background corresponds to $\mathcal{N}=1 S U(N) \times S U(N)$ gauge theory [12] (see also [13,14]). Other cases include brane intersections that, on the gauge side, generate new degrees of freedom, or defects that act as boundaries in the field theory [15,16] (see also [17]). Also, results in a gauge (or string) theory that inicially do not have their corresponding results in a string (or gauge) theory serve as predictions and as a guide to pursue them, e.g non-commutative gauge theories and deformed string backgrounds [18,19] (see also [20] and references therein). Lots of results have been obtained since the original statement of the conjecture, that allow us to say that the duality is indeed true, so it appears as a new striking and useful theoretical tool to make calculations and, principally, predictions. In particular, this could help to explore quantum field theory in the non-perturbative regime.

There are now techniques to test the duality exactly, i.e. to have results that can be
compared at the same regime, large $N$ and large coupling constant: localization [21-24]. In particular, a largely studied observable in gauge theory is the Wilson operator [25, 26]: a non-local gauge invariant operator that describes the path of a heavy "quark", an object in the fundamental representation, in the (supersymmetric) field theory. It is known as Wilson loop if the path is closed and circular Wilson loop if that closed path is a circle. Their expectation value in $\mathcal{N}=4$ gauge theory was computed perturbatively and exactly [27-[29]. In the gauge/gravity duality, the (fundamental) Wilson loop expectation value was first calculated by finding the area of the string worldsheet, given by the string action, in the string background whose boundary is the loop itself [30-36]. Wilson loops in higher representations, which describe systems of "quarks" or a generalized "quark", were studied by generalizing the latter idea: by placing higher dimensional objects of string theory, branes, as probes in these backgrounds with the loop as a boundary, and minimizing the action that describe them [37, 38]. As mentioned, there are methods in gauge theory that can produce exact results that include the regime in which the gauge theory is more easily described by string theory. The calculation of the expected value of Wilson loops in arbitrary representations was largely studied by using localization techniques [39-42], because it was shown that when calculating the expectation value of Wilson loops, Feynman diagrams involving loop corrections and vertices cancel each other at each order in their expansion due to supersymmetry, resulting in a single counting problem that can be expressed as a matrix model. This also worked for arbitrary representations [43-45].

If instead of the expectation value of a single Wilson loop, one wanted to compute the correlator between two loops, the relevant string worldsheet surface would be the one connecting the two different contours [46-49]. This is similar to the Plateau's problem (mentioned in [50]), in which one has to determine the shape of a thin soap film stretched between two rings lying on parallel planes. Modifying the geometry of these two rings introduces a phase structure: there are critical values for the parameters that separate the 'catenoid' (connected or continuous) solution and the 'Goldschmidt' (disconnected or discontinuous) solution [51]. When the two rings are separated beyond a certain critical value, the catenoid solution becomes unstable and breaks into the Goldschmidt one. Similarly, in the case of the Wilson loop, the string worldsheet describes a
catenoid-like solution until, at certain values of the radii of the loops and the distance, it becomes energetically unfavored with respect to the Goldschmidt-like solution. This is called a Gross-Ooguri phase transition [48, 52, 53], and can be understood as transition due to the string breaking that connects the two loops. This behavior is difficult to obtain in the field theory but it can be studied in a certain limit of Feynman diagrams called ladder/rainbows [29,54-59]. The discontinuous (broken) solution corresponds, in this case, to two minimized surfaces in $A d S$, each attached to a different loop, i.e two separated Wilson loops. These minimal surfaces are the usual onshell regularized actions with each loop as boundary.

An interesting setup that has received recent attention in AdS/CFT is the one in which a defect is introduced in the gauge theory and its corresponding string dual [15, 16, 60]. This defect is typically obtained by considering systems of intersecting branes (see also [61,62]). In particular, intersecting D3 and D5 branes along three of the four worldvolume directions allows to construct three-dimensional defects inside the $\mathcal{N}=4$ worldvolume of the $N \mathrm{D} 3$ branes. The end result is that there are two different gauge groups on each side of the defect brane. This is because $n$ D3 branes now end on the D5 defect, so on one side we have the usual $S U(N)$ and on the other side we have $S U(N-n)$. On the string theory side, the solution corresponding to a single D5 wall inside $A d S_{5} \times S^{5}$ was computed by considering that the D5 brane introduces a "magnetic" two-form flux that couples with the four-form of the D3 branes [60]. The expectation value of a single Wilson loop in this case was calculated in [63, 64]. In this work, the minimal worldsheet surface is attached to the loop and ends on the defect. The presence of the defect introduces then new boundary conditions for the string: the usual ones along the loop and the ones that ensure that the worldsheet surface end on the defect. Here, we compute the Wilson loop correlator in the presence of a $D 5$ defect from holography. We, later, study the Gross-Ooguri phase transition in this geometry, investigating in detail how this transition depends on the (numerous) parameters of the setting. We compare this case with the case without any defect and observe that the presence of the defect modifies the critical values of the parameters for the transition. In this case we have to consider the radii and separation of the loops, the relative distance to the defect and its inclination with respect to the axis that connects the loops.

Since the latter is studied by considering the connected solution of a single worldsheet attached to two loops and its transition into two separated minimal worldsheet surfaces, the gauge dual of this correlator will correspond to the expectation value of two Wilson loops in the fundamental representations. As we mentioned above, the duality was tested also for higher representations, which, on the string side, correspond to putting probe branes in the string background. Thus, for $\mathcal{N}=4$ SYM theory, it was found that a Wilson loop in the symmetric representations corresponds to a D3 probe in $\operatorname{AdS} S_{5} \times S^{5}$ [50] and that the antisymmetric representation case to a D5 probe instead, both with the loop as boundary [65]. It is still object of study to find corrections to the expectation value of the corresponding matrix model for those cases [66-70]. In general, for an arbitrarily high representation, the dual string description is in terms of bubbling geometries [7174]. Here, the string background is strongly deformed by a large number of branes in it. Even the simplest case of two probe (non-backreacting) branes, which corresponds to a rectangular representation, resulted to be very difficult to study in the gravitatinal prescription because it requires an action for more than one probe branes [75-77]. This case is going to be mentioned too.

The Wilson loop correlator in its string theory description was studied in the fundamental representation (only a string probe in the geometry). If we consider one of the string attached to the loop as before but the other one attached now to a D3 brane, this is the case of a Wilson loop correlator for mixed representations; in particular, the symmetric/fundamental correlator. The antisymmetric/fundamental case, involving a D5 brane as boundary of one end of the string was studied before [78, 79]. In general, on both the string and gauge sides, the D5 (antisymmetric) case seems to be easier to deal with.

This work is organized as follows. In chapter 2 we review th basics of the AdS/CFT correspondence. There we make brief stops about string theory and supergravity. Next, we focus on the simplest, and firstly studied, but still rich case of supergravity in $\operatorname{Ad} S_{5} \times$ $S^{5}$ and $\mathcal{N}=4 S U(N)$ gauge theory in order to motivate the correspondence. There we say some words about brane dynamics and what means to have a theory "living" on branes. Reader surely will find more complete details in the references we give. In chapter 3. and continuing with the review, we start studying Wilson loops both perturbatively and exactly by using localization methods. We give a very short mention about local-
ization, a powerful technique that precisely allows to get exact results even for higher representations. In chapter 4, we develop the idea of holographic Wilson loops, i.e. a configuration in a string background that gives the same result as in the gauge theory. We review how important are the boundary conditions we must impose and how a certain Legendre transform eliminates the divergence when calculating the area of the corresponding string worldsheet attached to the loop. In chapter 5, we continue in the gravity (string) side but now for higher representations. There we explain our attempt to extend some known results about Wilson loops in higher representations, in particular the case of "rectangular" representations both symmetric and antisymmetric without going to the socalled bubbling case. In the gravity dual these cases correspond to D-branes probing the background but equally attached to a loop as in the fundamental case, which correspond to a single string. In chapter 6, we study a very interesting phenomenom: the GrossOoguri phase transition of two connected Wilson loops into two separated ones. At the same time, we consider the case of those Wilson loops in the presence of a D5 defect. The situation becomes interesting and rich in parameters, which include the distance between loops and with respect to the defect, the ratio of the radii and inclination of the defect.

## Chapter 2

## Basics on the AdS/CFT correspondence

As usual in all works about/involving the AdS/CFT correspondence, in this first chapter, we start with a quick, and not very self-contained, review.

In 1997, J. Maldecena conjectured that the large $N$ limit of superconformal field theories with $U(N)$ gauge symmetry in $D$ dimensions is governed by supergravity (and, in the strong case, string theory) on a high dimensional $A d S$ space [8,10]. This correspondence is "holographic" in the sense that "information" of the theory within some volume can be described in terms of a different theory on its boundary (see [3]). This is the "holographic principle".

In his seminal paper [10] Maldacena focused on the case of $\mathcal{N}=4 D=4 S U(N)$ gauge theory, which is conjecturally equivalent to type IIB superstring theory on $\operatorname{AdS} S_{5} \times$ $S_{5}$. So, let us understand some details about this identification, making first some comments about the constructions of each side, and later motivating the duality.

### 2.1 Superstring and supergravity: quick review

String theory was born out of attempts to understand the strong interactions (see the classic GSW textbook [1] for a short and understandable historical introduction). But soon it became one of the most successful theoretical descriptions that include gravity in a nat-

[^0]ural way. String theory began as a theory involving one-dimensional extended bosonic objects (closed and open) which move in a 26 -dimensional spacetime. Later, bosonic strings showed a number of drawbacks; principally, its spectrum contains tachyons and no fermionic states. Supersymmetry can be implemented in three ways: the Ramond-NeveuSchwarz and the Green-Schwarz, which differ in where the supersymmetry is manifested, either on the worldsheet or on the spacetime; and also the Pure Spinor formalism by Berkovits, which mixed features of the last two descriptions. The critical dimension of spacetime in the supersymmetric string theory, which can be obtained by canceling the central charge for an anomaly-free theory (see the Polchinski textbooks, [84], for detailed calculations), is 10 .

In particular, the so-called type II superstring theory contains two Majorana-Weyl spinors and combinations of the R (fermionic) and NS (bosonic) sectors: R-R (bosonic), NS-NS (bosonic), R-NS (fermionic) and N-SR (fermionic). These sectors correspond to two ways to impose periodicity on the two-component worldsheet spinor. Since we have two fermionic degrees of freedom, our theory is $\mathcal{N}=2$ (type II). We can choose the chirality of the spinor: for the same chirality, this is known as type II A, for opposite chirality, this is known as type IIB.

For our purposes, type IIB superstring theory will be the central matter. In general, spectra of states in string theory is go like $M^{2} \sim 1 / \alpha^{\prime}$, so when $\alpha^{\prime} \rightarrow 0$, massive states become non-propagating leaving only the massless states as propagating. For type IIB we have in the RR sector (bosonic) a scalar $C_{0}$, a two-form gauge field, $C_{2}$, and a four-form, $C_{4}$. In the NS-NS sector (bosonic): we have a dilaton, $\phi$, an antisymmetric two-form, $B_{2}$; and the graviton, $g_{M N}$ (or the metric). And, in the fermionic sector we have the dilatino and the gravitino. This is the field content of type IIB supergravity, the low energy (massless) limit of type IIB superstring. In particular, as we will see below, the $C_{4}$ RR (gauge) field is sourced by three-dimensional objects, D3 branes, where open strings end. In the next section we will see that on those branes, due to the fact that the endpoints of open strings describe gauge fields, we can define gauge theories.

### 2.2 D3-branes in type IIB, the $A d S_{5} \times S^{5}$ geometry

It was pointed out by Polchinski [85] that $D$-branes are the same as $p$-branes, classical (solitonic) solutions to supergravity with mass and Ramond-Ramond charge, by calculating the tension and charges of the D-branes from the string theory and matching with the $p$-brane solutions of supergravity.

As we reviewed before, in type IIB theory there are massive charged objects which act as sources for the RR gauge fields. Specializing to D3 branes, the action in the string frame reads ${ }^{2}$

$$
\begin{equation*}
S=\frac{1}{(2 \pi)^{7} l_{s}^{8}} \int d^{10} x \sqrt{-G}\left(e^{-2 \phi}\left(R+4(\partial \phi)^{2}\right)-\frac{2}{5!} F_{(5)}^{2}\right), \tag{2.1}
\end{equation*}
$$

where $F_{(5)}$ is the field strength of the four-form potential, $C_{(4)}$ (sourced by the D 3 brane), where $F_{(5)}=d C_{(4)}$, with ${ }^{3}$

$$
\begin{equation*}
N=\int_{S^{5}} F_{(5)} \tag{2.2}
\end{equation*}
$$

units of RR charge. A general $p$-brane solution can be found in [3] (and references therein), where the mass and charge, in the $p=3$ case, are related by 4

$$
\begin{equation*}
M \geq \frac{N}{(2 \pi)^{7} g_{s} l_{s}^{4}} . \tag{2.3}
\end{equation*}
$$

The solution whose mass is at the lower bound of the last inequality is called an extremal 3 -brane. The solution - metric, dilaton and five-form - in this case is

$$
\begin{align*}
d s^{2} & =\frac{1}{\sqrt{H(r)}}\left(-d t^{2}+d \vec{x}^{2}\right)+\sqrt{H(r)}\left(d r^{2}+r^{2} d \Omega_{5}^{2}\right), \quad H(r)=1+\frac{R^{4}}{r^{4}}  \tag{2.4}\\
e^{\phi} & =g_{s}  \tag{2.5}\\
F & =(1+*) d t \wedge d x_{1} \wedge d x_{2} \wedge d x_{3} \wedge d\left(H^{-1}\right) \tag{2.6}
\end{align*}
$$

where $R^{4}=4 \pi g_{s} N l_{s}^{4} \vdash^{5}$ In the near horizon region, when $r \rightarrow 0(r \ll R)$, we can approximate

$$
\begin{equation*}
H(r) \sim \frac{R^{4}}{r^{4}} \tag{2.7}
\end{equation*}
$$

[^1]The geometry becomes

$$
\begin{equation*}
d s^{2}=\frac{r^{2}}{R^{2}} \eta_{\mu \nu} d x^{\mu} d x^{\nu}+\frac{R^{2}}{r^{2}} d r^{2}+R^{2} d \Omega_{5}^{2} \tag{2.8}
\end{equation*}
$$

which is the metric of $A d S_{5} \times S^{5}$ (see [3, 5] for references about this space and its properties). When $r$ is large ( $r \gg R$ ), the metric becomes Minkowski in ten dimensions. In this description, branes are considered to be dissolved into the closed string sector. This means that their effects remain and are taken into account in the geometry. Isometries of this background are: $S O(2,4) \equiv S U(2,2)$ of $A d S_{5}$ and $S O(6) \equiv S U(4)$ of $S^{5}$. As we will se later, these are going to be also symmetries of a gauge theory; in particular, the first one due to conformal symmetry, and the second due to R-symmetry of $\mathcal{N}=4$ super Yang-Mills in $D=4$.

We are dealing with the low energy limit of string theory, when energies are smaller than the string energy scale $1 / l_{s}$, i.e

$$
\begin{equation*}
E \ll E_{s}=\frac{1}{\sqrt{\alpha^{\prime}}} . \tag{2.9}
\end{equation*}
$$

So we can take $\alpha^{\prime} \rightarrow 0$ to take the string energy scale to infinity. An important property of the metric $\sqrt{2.4}$ is its so-called non-constant redshift factor coming from the $g_{00}$ term,

$$
\begin{equation*}
g_{00}=\left(1+\frac{R^{4}}{r^{4}}\right)^{-1 / 4} \tag{2.10}
\end{equation*}
$$

which goes to 1 at large $r$, and to $r / R$ when $r \rightarrow 0$. The energy $E_{p}$ of a particle measured by an observer at constant position $r$ differs from the energy, $E_{i}$, of the same particle as measured by an observer at infinity as

$$
\begin{equation*}
E_{i}=\left(1+\frac{R^{4}}{r^{4}}\right)^{-1 / 4} E_{p} \tag{2.11}
\end{equation*}
$$

When the particle approaches the throat of $A d S_{5} \times S^{5}$ at $r \rightarrow 0$, it appears to have lower and lower energy to the observer at infinity. This gives another notion of the low energy regime.

Then, we have to distinguish two kinds of low energy excitations (as seen from infinity):

- particles approaching the throat with any energy that will be at low-energy as viewed from infinity, and
- massless particles (gravitons) propagating in the bulk (away from $r=0$ ).

These modes decouple since the bulk massless particles have large wavelengths that do not "see" the throat, and excitations living in the throat cannot escape from it due to the gravitational potential. So, we have free bulk closed string modes (free gravity) in flat space and supergravity (and in general, superstrings) in the $\operatorname{Ad} S_{5} \times S^{5}$ region.

It is also important to mention that we can trust the supergravity solution when $R^{4} \gg$ $l_{s}^{4}$. This corresponds to say that $\alpha^{\prime} \rightarrow 0$, the supergravity limit of superstrings. On the other hand, this requirement translates into

$$
\begin{equation*}
\frac{R^{4}}{l_{s}^{4}}=4 \pi g_{s} N \gg 1 \Rightarrow g_{s} N \gg 1 . \tag{2.12}
\end{equation*}
$$

### 2.3 D3-branes in type IIB, the $\mathcal{N}=4$ theory

Since there are lots of very good reviews and books about these introductory topics, we will not enter into details, only giving some quick arguments. D-branes are objects where open strings end. This is because the endpoints of these strings can satisfy two types of boundary conditions: Dirichlet and Neumann. The first one means that the endpoints are fixed, and the second one means that they are free, and moving at speed of light. So, if we choose $p+1$ coordinates satisfying Neumann boundary conditions for $p$ spatial coordinates and time, the endpoints move freely on a $p+1$ dimensional "wall" in spacetime: the worldvolume of a D-brane, or more precisely, a $D p$-brane (see [86] for an extensive review about D-branes).

In [87] proved that $D p$-branes are actually dynamical, so we can write an action for them by minimizing their worldvolume just like we write the action for the particle by minimizing its worldline. So, if we consider that the transverse coordinates are dynamical scalars on the worldvolume, the action for a single brane will be

$$
\begin{equation*}
S[x]=-T_{p} \int d^{p+1} \xi \sqrt{-\operatorname{det} h_{\mu \nu}} \quad \text { where } \quad h_{\mu \nu}=\partial_{\mu} x^{M} \partial_{\nu} x^{N} G_{M N}(x), \tag{2.13}
\end{equation*}
$$

where $T_{p}$ is the tension of the brane

$$
\begin{equation*}
T_{D_{p}} \propto \frac{1}{g_{s} l_{s}^{p+1}}, \tag{2.14}
\end{equation*}
$$

which has units of energy per length. Since in string theory the background metric $G_{M N}$ appears together with the B-field and the dilaton at the massless level of closed strings, the action must also contain these fields ${ }^{6}$

$$
\begin{equation*}
S_{D B I}=-T_{p} \int d^{p+1} \xi e^{-\phi} \sqrt{-\operatorname{det}\left[h_{\mu \nu}+\alpha^{\prime} B_{\mu \nu}+\alpha^{\prime} F_{\mu \nu}\right]}, \quad B_{\mu \nu}=\partial_{\mu} x^{M} \partial_{\nu} x^{N} B_{M N}(x), \tag{2.15}
\end{equation*}
$$

where $\mu, \nu$ are coordinates on the worldvolume, and $M, N$ are coordinates of the target space, the spacetime. Notice that we have also included a gauge field $F_{\mu \nu}$, coming from the fact that massless open string endpoints with Neumann boundary conditions produce gauge fields living on the brane. The presence of the B-field is also required to have charge conservation, since it carries the string charge that must go somewhere when the string ends. This is the Dirac-Born-Infeld action, which at first order in $\alpha^{\prime}$ contains Maxwell theory and the action for the scalar coordinate fields. The action for the brane needs another terms, the so-called Wess-Zumino term that describes the coupling to the corresponding antisymmetric tensor field, the Ramond-Ramond field $C_{(p+1)}$, that appears in the superstring theory, and is sourced by the $D p$-brane in the same way the electromagnetic vector is sourced by the charged point particle. This terms is the generalization of the electromagnetic case in which the electromagnetic field couples to the worldline of the point particle,

$$
\begin{equation*}
S_{W Z}=q \int A \tag{2.16}
\end{equation*}
$$

to

$$
\begin{equation*}
S_{W Z}=\mu_{p} \int P\left[C_{(p+1)}\right] \tag{2.17}
\end{equation*}
$$

where $\mu_{p}$ is the brane charge, $P[\cdots]$ is the pullback on the worldvolume, and $\mu_{p} \sim T_{p}$. Moreover, one can consider more than one brane, say $N$ D-branes, and add labels to the endpoints of the open string ending on the brane, $|i\rangle$, with $i$ going from 1 to $N$, also called Chan-Patton factors. For coincident branes the endpoints have $N$ possible labels, and then the gauge fields living on this stack of branes are $U(N)$ gauge fields. This is the origin of the $U(N)$ theory "living" on $N$ D3 branes. But here a problem arises: there is no a well-defined action describing $N$ coincident branes; this is because in the nonabelian case the scalars in the adjoint representation of $U(N)$ become matrices, so their

[^2]interpretation as position of the branes will not be valid, or clear. Moreover, in order to have gauge symmetry, we need to trace the square rooted expression of the action, and produces more problems. There is until now no complete expression for the action of this non-abelian DBI action 7 but we expect that it should contain the non-abelian action for $U(N)$ (when $\alpha^{\prime} \rightarrow 0$ ) in the same way the $U(1)$ theory ${ }^{8}$ appear from the DBI action for one single brane,
\[

$$
\begin{equation*}
S_{N D 3} \sim \int d^{4} \xi \operatorname{Tr}\left(-\frac{1}{4 g_{s}} F_{\mu \nu}^{a} F^{a \mu \nu}+\cdots\right) \tag{2.18}
\end{equation*}
$$

\]

In the superstring theory, the D-branes are supersymmetric, and live in ten dimensions, so we expect that the gauge theory on them be also supersymmetric. One thing to notice is that by comparing this action with the analogous term coming from the expansion of the DBI action, one finds that

$$
\begin{equation*}
T_{p} \sim \frac{1}{g^{2}}=\frac{1}{g_{s}} . \tag{2.19}
\end{equation*}
$$

The relation $g^{2}=g_{s}$, where $g^{2}$ is the YM coupling and $g_{s}$ is the string coupling, is because we need two open strings with coupling $g$ to be able to form one closed string with coupling $g_{s}$. This result also allows us to say that $D p$-branes are indeed non-perturbative objects, since they become heavy at weak coupling. Let us focus on the theory on D3 branes whose worldvolume is $3+1$ dimensional, or simply in $D=4$. The gauge theory on them has gauge fields $A_{\mu}^{a}$, in general, where $a=1, \cdots, N^{2}$ is the adjoint index of $U(N)$; with two on-shell degrees of freedom (polarizations), and six scalars $\Phi_{I}$ corresponding to the six transverse coordinates to the D3 brane, giving a total of eight bosonic on-shell degrees of freedom. Since we are working with supersymmetric theories, we need the fermionic partners of these eight bosonic fields. A minimal spinor in four dimensions has two on-shell degrees of freedom, so we will need four of them, leading to a $\mathcal{N}=4$ gauge theory, with $U(N)$ gauge group. Also, the field content we just found has a global $S U(4)$ symmetry, the $R$-symmetry, which rotates the supercharges in the theory. Under this symmetry, the vector field is a singlet, it changes only by a phase, the spinors are in the fundamental representation and the scalar rotates in the adjoint representation. This $S U(4) \equiv S O(6)$ symmetry coincides, as we will mention and expand later, with the isometry of the $S^{5}$ part of the background of $A d S_{5} \times S^{5}$.

[^3]
### 2.3.1 A short pit stop on $\mathcal{N}=4 D=4 U(N)$ gauge theory.

We have just seen quickly how a nonabelian gauge theory emerges as the worldvolume theory of the stack of $N$ D3 branes. But it was difficult to extract the dynamics by studying the nonabelian DBI action of them. Luckily, $\mathcal{N}=4 D=4$ super Yang-Mills was already studied separately. In the context of AdS/CFT, we review [3, 5], a pair of well known and very good references.

Supersymmetric field theories are nice since they are more constrained than usual field theories. In them bosonic and fermionic corrections cancel, as we will see, and lead to finite results. Even if they are not realized in Nature, they serve as "toy models" from which we can extract analytic results that could serve as a guide to guess the behavior of more realistic theories. In supersymmetry, Poincaré and internal (gauge) symmetries are mixed in a nontrivial way by adding $\mathcal{N}$ fermionic generators resulting in a general Lie algebra called superalgebra. The $\mathcal{N}=4 D=4$ case is even more special. The Poincaré (bosonic) group of symmetry can be enlarged to include scale invariance; this, in general, allows us to expand the Poincaré group to the full conformal group (bosonic), which includes scale invariance and Poincaré. Thus, we can extend not only the Poncaré group but the full conformal group to a supergroup: the superconformal group. This is the most symmetric (constrained) group we can have and this is precisely the symmetry group of $\mathcal{N}=4 D=4$ SYM: $\operatorname{PSU}(2,2 \mid 4)$. The bosonic part of this superconformal group is $S O(2,4) \times S U(4)$. The $U(N)$ gauge symmetry rotates each field and is going to be present as a trace in front of the lagrangian and composite operators.

The action of $\mathcal{N}=4 D=4 U(N)$ gauge theory is [5]

$$
\begin{align*}
S_{\mathrm{SYM}}= & \int d^{4} x \operatorname{tr}\left\{-\frac{1}{g^{2}} F^{2}+\frac{\theta}{8 \pi^{2}} F \tilde{F}-i \bar{\lambda} \bar{\sigma} D \lambda-(D \Phi)^{2}\right. \\
& \left.+g C \lambda[\Phi, \lambda]+\bar{C} \bar{\lambda}[\Phi, \bar{\lambda}]+\frac{g^{2}}{2}[\Phi, \Phi]^{2}\right\} \tag{2.20}
\end{align*}
$$

where the trace is over gauge indices, and we have hidden the space indices. As known in supersymmetry, the scalar potential must vanish in the supersymmetric ground state, so $[\Phi, \Phi]=0$. We have two options: $\{\Phi\}=0$, or at least one $\Phi \neq 0$. The first one is called the superconformal phase and the second one is called the spontaneously broken or Coulomb phase. Explicitly, the field content is $\left(A_{\mu}, \lambda_{\alpha}^{a}, \Phi^{I}\right)$, where $\mu=0, \cdots, 3$,
$a=1, \cdots, 4, \alpha= \pm$ and $I=1, \cdots, 6$. In particular, notice the six scalars; they are precisely those matrix "coordinates" of the stack of $N \mathrm{D} 3$ branes we mentioned above. $\mathcal{N}=4$ SYM has another symmetry, most easily espressed by first combining $g$ and $\theta$ as

$$
\begin{equation*}
\tau:=\frac{\theta}{2 \pi}+\frac{4 \pi i}{g^{2}} . \tag{2.21}
\end{equation*}
$$

The theory is invariant under $S L(2, \mathbb{Z})$, i.e,

$$
\begin{equation*}
t \rightarrow \frac{a \tau+b}{c \tau+d}, \quad a d-b c=1 \quad a, b, c, d \in \mathbb{Z} \tag{2.22}
\end{equation*}
$$

### 2.3.2 The large $N$ limit for any gauge theory

One of the first hints that gauge theory could be described in terms of string theory comes from the behavior of gauge theories at large $N$ limit, where it was observed by 't Hooft that gauge theories simplify [88].

Before starting the analysis, it is important to mention that the $U(N)$ group can be written as

$$
\begin{equation*}
U(N)=S U(N) \times U(1) . \tag{2.23}
\end{equation*}
$$

Here the non-diagonal part, $S U(N)$, correspond to excitations connecting the branes; and the $U(1)$ term is related to the center of mass motion of all the branes. Then, we can consider that the stack of branes is located close the origin of space, this is the low energy limit, and then the gauge theory is actually $S U(N)$, not $U(N)$.

Instead of focusing on the $\mathcal{N}=4 S U(N)$ SYM theory, let us describe a general $S U(N)$ theory in four dimensions whose Lagrangian schematically looks like

$$
\begin{equation*}
\mathcal{L}=\frac{1}{g^{2}} \operatorname{Tr}\left(\left(\partial \Phi^{I}\right)^{2}+\left(\Phi^{I}\right)^{3}+\left(\Phi^{I}\right)^{4}\right) \tag{2.24}
\end{equation*}
$$

where $\Phi^{I}$ are fields in the adjoint representation of $S U(N)$, so

$$
\begin{equation*}
\Phi^{I}=\left(\Phi^{I a}\right)_{j}^{i}\left(T^{a}\right)_{i}^{j}, \text { with } a=1, \cdots, N^{2}-1 \text { and } i=1, \cdots, N, j=1, \cdots, \bar{N}, \tag{2.25}
\end{equation*}
$$

and $N$ and $\bar{N}$ represents fundamental and antifundmental indices, respectively. So, we can work with $\left(\Phi^{I a}\right)^{i}{ }_{j}=\Phi^{I}\left(T^{a}\right)^{i}{ }_{j}$, and the so-called double-line notation, in which the adjoint field we wrote before is represented by a direct product of a fundamental and an antifundmental field, $\left(\Phi^{I a}\right)_{j}^{i}$. The "gluon" propagator can be written as

$$
\begin{equation*}
\left\langle\Phi^{i}{ }_{j} \Phi^{k}\right\rangle \propto\left(\delta^{i}{ }_{l} \delta^{j}{ }_{k}-\frac{1}{N} \delta^{i}{ }_{j} \delta^{l}{ }_{k}\right) . \tag{2.26}
\end{equation*}
$$

At large $N$ the second term can be neglected, so the propagator for the adjoint fields is the same of the fundamental-antifundmental pair. Thus, Feynman diagrams involving $\Phi$ may be viewed as a network of double lines involving fundamental/antifundmental fields.

Also, it is important to understand the behavior of the coupling $g$ as we take $N$ large in the asymptotically pure $S U(N)$ Yang-Mills theory. There, the beta function is

$$
\begin{equation*}
\mu \frac{d g}{d \mu}=-\frac{11}{3} N \frac{g^{3}}{16 \pi^{2}}+\mathcal{O}\left(g^{5}\right) \tag{2.27}
\end{equation*}
$$

which is not well-behaved at large $N$. Let us rescale the coupling constant as $g \rightarrow g / \sqrt{N}$, then the las expression does not depend on $N$. More precisely, one can define the 't Hooft coupling as

$$
\begin{equation*}
\lambda:=g^{2} N . \tag{2.28}
\end{equation*}
$$

Back to our general lagrangian, this definition leads to

$$
\begin{equation*}
\mathcal{L}=\frac{N}{\lambda} \operatorname{Tr}\left(\left(\partial \Phi_{A}\right)^{2}+\Phi_{A}^{3}+\Phi_{A}^{4}\right) . \tag{2.29}
\end{equation*}
$$

Then, in the Feynman diagrams: vertices $(\mathrm{V})$ scale as $N / \lambda$, propagators $(\mathrm{E})$ as $\lambda / N$, and the loops ( F ) (sum over the indices in the trace) contributes with a factor of $N$. Hence, we can write

$$
\begin{equation*}
\operatorname{diagram}(V, E, F) \sim N^{V-E+F} \lambda^{E-V}=N^{\chi} \lambda^{E-V} \tag{2.30}
\end{equation*}
$$

where $\chi=V-E+F$ is the Euler characteristic of the surface (plane graph) corresponding to the diagram. For closed oriented surfaces, the Euler characteristic can be computed from its genus, $\tilde{g}$, as

$$
\begin{equation*}
\chi=2-2 \tilde{g} . \tag{2.31}
\end{equation*}
$$

At large $N$, and fixed $\lambda$, diagrams will be dominated by the surfaces of maximal $\chi$ (or minimal genus), which means that the dominant topology will be the sphere (or a plane). We can conclude that these diagrams, called planar diagrams, will give a contribution of order $N^{2}$. This result will be valid, in particular, for the $\mathcal{N}=4 S U(N)$ gauge theory, and also important to motivate the AdS/CFT correspondence since it is related to the expansion in $g_{s}$, the closed string coupling in string theory.

Let us say some words about our new coupling. Since we have $N$ D3 branes on top of each other, the effective loop expansion parameter for the open strings ending on them
is $g_{s} N$ rather than $g_{s}$, since each open string comes with the Chan-Paton factor $N$. Thus, expansions are valid when $g_{s} N \ll 1$. As we saw above, $g_{s}=g^{2}$, so we can conclude that in the field theory "side"

$$
\begin{equation*}
\lambda=g^{2} N \ll 1 . \tag{2.32}
\end{equation*}
$$

As we mentioned, these D3 branes were put on a flat ten-dimensional target where we have open strings, that is why we have branes, and closed strings. At low energy, only the massless states can be excited, so we can write the following schematic effective action

$$
\begin{equation*}
S=S_{\text {bulk }}+S_{\text {brane }}+S_{\text {int }} . \tag{2.33}
\end{equation*}
$$

Here $S_{\text {bulk }}$ describes the massless closed string modes, ten-dimensional supergravity, plus corrections coming from the integration of the massive modes. The brane action $S_{\text {brane }}$ when $\alpha^{\prime} \rightarrow 0$ is the $\mathcal{N}=4 U(N)$ theory we saw above, plus higher derivative corrections. $S_{\text {int }}$ describes the interaction between the brane (open string) modes and the bulk (closed string) modes. When $\alpha^{\prime} \rightarrow 0$, the bulk part reduces to free closed strings (free gravity) and interactions are suppressed.

### 2.4 Motivating the AdS/CFT duality

In this section we will present the usual two heuristic arguments suggesting that there exists a correspondence (or duality or equivalence) between string theory and gauge theory: the low energy descriptions and global symmetries.

As we said before, the dynamics of $N$ D3 branes in the low energy regime, when $\alpha^{\prime} \rightarrow 0$, can be described by the excitations of the open string endpoints on the D-branes: gauge fields, parallel to the branes, and scalar fields, transversal to them. This gives a field theory that lives on the four-dimensional worldvolume of the D3 branes: $\mathcal{N}=4$ $S U(N)$ superconformal gauge theory in four dimensions with coupling $\lambda=g^{2} N \ll 1$ (or $g_{s} N \ll 1$ ), plus decoupled massless closed string modes (supergravity) in the Minkowski bulk spacetime. On the other hand, branes can in fact be also seen as solitonic solutions of type IIB supergravity (the low energy, $\alpha^{\prime} \rightarrow 0$, limit of superstring theory) with mass $M$ and $N$ units of RR gauge field. The solution metric, in this case is $A d S_{5} \times S^{5}$ where we have type IIB supergravity modes plus decoupled supergravity in flat space. This
description is valid when $g_{s} N \gg 1$, contrary to the condition for the open string (field theory) side. These are the "two faces" of the D-branes.

$$
\mathcal{N}=4 D=4 S U(N) \text { gauge theory } \equiv \text { Type IIB superstring theory in } A d S_{5} \times S^{5}
$$

The complementarity of regimes where we can reliably perform calculations makes the AdS/CFT correspondence useful, but also hard to prove. To close this heuristic "derivation", let us notice some interesting facts coming from the symmetries of spacetime in the gravity side and the symmetries in the field theory. As we mentioned, the isometries of $A d S_{5}$ and $S^{5}$ are $S O(2,4)$ and $S O(6)$ respectively; but $S O(2,4)$ is the conformal group in four dimensions, the conformal symmetry of $\mathcal{N}=4 S U(N)$ SYM. Moreover, the $S O(6) \equiv S U(4)$ correspond to the R-symmetry of our field theory. Besides of theses symmetries, both sides have a $S L(2, \mathbb{Z})$, called S-duality.

### 2.4.1 Holographic duality

The $A d S_{5}$ metric is

$$
\begin{equation*}
d s^{2}=\alpha^{\prime}\left[\frac{U^{2}}{L^{2} / \alpha^{\prime}} \eta_{\mu \nu} d x^{\mu} d x^{\nu}+\frac{L^{2}}{\alpha^{\prime}} \frac{d U^{2}}{U^{2}}\right], \tag{2.34}
\end{equation*}
$$

here the coordinates $x^{\mu}$ may be thought of as the coordinates along the worldvolume of the brane and can be identified with the gauge theory coordinates. The coordinate $U$, together with those of $S^{5}$, describes the transverse directions to the brane. As $U \rightarrow \infty$, we approach to the so-called boundary of $\operatorname{Ad} S_{5}$. Moreover, as we know, $\mathcal{N}=4 \mathrm{SYM}$ is a CFT, so $x \rightarrow \Lambda x$ is a symmetry of the theory. At the same time, in the gravity side, this transformation is also a symmetry, the rescalings $U \rightarrow U / \Lambda$ and $U \rightarrow \Lambda x$ leave the metric invariant. Now, when $\Lambda \ll 1$, in the gauge theory, we get physics at large distances, i.e., at small energies (IR). In the gravity side, the $U$ coordinate goes to zero, the near horizon limit. Whereas, when $\Lambda \gg 1$ we have physics at short distances in the gauge theory, which means high energies (UV). In this case, $U \rightarrow \infty$, the boundary of $A d S_{5}$. Thus, $U$ can be identified with the renormalization group scale in the gauge theory

$$
\begin{equation*}
E \sim U . \tag{2.35}
\end{equation*}
$$

This allows us to say that the field theory is not defined only on the boundary of $A d S_{5}$, where we have its UV limit (and thus, the IR limit in the gravity side); the field theory
describes all the physics inside the $A d S_{5}$ bulk. Because of the energy scale in the gravity side is related directly to the radial coordinate of $A d S_{5}$, we say that different regions (slices of constant $U$ ) correspond to different energy scales in the field theory.

### 2.4.2 The dictionary

Since it was conjectured, the equivalence between two different theories, we must mention the statement in a more "precise" way by showing the explicit translation of objects of one side into its dual side. This is known as "dictionary". 9

Let us explain some details about the correspondence between operators in the gauge theory and fields in the gravity side. As a conformal theory, $\mathcal{N}=4$ SYM does not allow us to define asymptotic states, since it does not make sense to construct states separated by large distances. So, we will work instead with the collection of all local, and also the non-local, gauge invariant, operators $\mathcal{O}(x)$ that are polynomials in the fields of the theory. As we know, the physical quantities in a field theory are correlation functions of these operators. This is the way we measure in theory,

$$
\begin{equation*}
\left\langle\mathcal{O}\left(x_{1}\right) \mathcal{O}\left(x_{2}\right) \cdots \mathcal{O}\left(x_{n}\right)\right\rangle=\left.\frac{\delta^{n}}{\delta J_{1}\left(x_{1}\right) \delta J_{2}\left(x_{2}\right) \cdots \delta J_{n}\left(x_{n}\right)} \ln Z_{C F T}\right|_{J=0} \tag{2.36}
\end{equation*}
$$

where

$$
\begin{equation*}
Z_{C F T}[J]:=\left\langle\exp \left(-\int d x \mathcal{L}_{J}\right)\right\rangle, \quad \mathcal{L}_{J}=\mathcal{L}+\sum_{i} J_{i}(x) \mathcal{O}_{i}(x) . \tag{2.37}
\end{equation*}
$$

Gauge invariant operators are defined as the trace of polynomial functions of the scalar fields of the gauge theory. The conformal dimension of the operators is $\Delta$. In the gravity side, the $A d S_{5}$ space can be written in the following way

$$
\begin{equation*}
d s^{2}=R^{2} \frac{d y^{2}+d x^{2}}{y^{2}} \tag{2.38}
\end{equation*}
$$

where we set $r=R^{2} / y$. In this metric, the boundary is located at $y=0$. As we mentioned before, the boundary of $A d S_{5}$ corresponds to the UV limit of the gauge theory. By solving the Klein-Gordon equation for a single scalar field, $\phi(x, y)$, we obtain the following solution near the boundary

$$
\begin{equation*}
\phi(x, y) \approx\left\langle\mathcal{O}_{\Delta}(x)\right\rangle y^{\Delta}+J(x) y^{4-\Delta} . \tag{2.39}
\end{equation*}
$$

[^4]where
\[

$$
\begin{equation*}
\left\langle\mathcal{O}_{\Delta}(x)\right\rangle \equiv \lim _{y \rightarrow 0} y^{-\Delta} \phi(x, y), \quad \text { and } \quad J(x) \equiv \lim _{y \rightarrow 0} y^{\Delta-4} \phi(x, y) \tag{2.40}
\end{equation*}
$$

\]

With this solution we can find an interesting relation for the mass of the scalar and $\Delta$ :

$$
\begin{equation*}
m^{2} R^{2}=\Delta(\Delta-4) \tag{2.41}
\end{equation*}
$$

In order to see that the functions in front of each term in the solution for the scalar, $\phi(x, y)$, correspond precisely to the expected value of the operator $\mathcal{O}_{\Delta}(x)$ and to the source $J(x)$ on the boundary, we must establish the equivalence between partition functions in both sides (see [8, 9] for details)

$$
\begin{equation*}
Z_{C F T}[J(x)]=\left\langle\exp \left(-\int d^{4} x \mathcal{O}_{\Delta}(x) J(x)\right)\right\rangle \equiv Z_{\text {strings }}\left[\left.y^{\Delta-4} \phi(x, y)\right|_{y \rightarrow 0}\right] \tag{2.42}
\end{equation*}
$$

The complete correspondence between the representations of $S U(2,2 \mid 4)$ on both sides of the duality is given in [5].

In the next chapter we will study one of the most important gauge invariant operators: Wilson loops, or in this case supersymmetric Wilson loops. These operators can be calculated exactly for arbitrary values of $g_{s}$ and $N$, so it is possible to compare with supergravity results, and test the correspondence.

## Chapter 3

## Wilson loops, perturbative and exact results

In this chapter we study Wilson loops in the $\mathcal{N}=4 S U(N)$ gauge theory, the supersymmetric extension of the Wilson loops in gauge theory. These operators, since they can also be calculated exactly, will provide a strong verification of the AdS/CFT correspondence.

Since Wilson loops are the central topic in this work, here we will give some important and almost self-contained details about them. First, we start with the usual definitions in quantum field theory and later, we move on to the supersymmetric case which we are going to develop even more in the next chapter.

## 3.1 (Supersymmetric) Wilson loops

A Wilson loops is a non-local gauge invarian ${ }^{1}$ operator which is associated with the phase acquired by a heavy particle in the fundamental representation of the gauge group around a path $\mathcal{C}$ (see the textbook [25] for details). It is defined as

$$
\begin{equation*}
W_{\mathcal{C}}:=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \oint_{\mathcal{C}} d s A_{\mu} \frac{d x^{\mu}}{d s}\right), \tag{3.1}
\end{equation*}
$$

where $\mathcal{C}$ denotes a closed loop in spacetime parametrized as $x^{\mu}=x^{\mu}(s)$, and the trace is over the fundamental representation of the gauge group. 2 We say that this operator

[^5]is non-local because it depends on a curve, not on some particular point of spacetime. Moreover, formally the Wilson loop defined in this way is the trace of the holonomy of the gauge connection $A$ around the curve $\mathcal{C}$. If we consider the loop contour is a rectangle $\mathcal{C}=L \times T$, where $T$ (euclidean "time") is large, the expectation value of the Wilson loop contains the potential between two charges (quark-antiquark) separated by a distance $L$
\[

$$
\begin{equation*}
\left\langle W_{\mathcal{C}}\right\rangle=e^{-T V(L)} . \tag{3.2}
\end{equation*}
$$

\]

If we assume that $V(L) \propto L$ (confining phase, i.e. an increasing force against separation), the exponent in (3.2) describes the area of $\mathcal{C}$ : the area law. This result is usual in QCD, which is known to be confining [26, 89]. On the other hand, for QED ${ }^{3}$, we have $V(L) \propto$ $1 / L$, which is known as Coulomb phase (non-confining, i.e less force with separation). Theories in this case are scale invariant and, in general, conformal. Hence, the Wilson loop plays the role of order parameter.

In the supersymmetric case, and in particular, in the $\mathcal{N}=4 U(N)$ case (instead of $S U(N)$ ), as we saw before, the field content of $\mathcal{N}=4$ theory consists of the gauge field $A_{\mu}$, four Weyl fermions $\lambda_{\alpha}^{a}, \bar{\lambda}_{a}^{\dot{\alpha}}(a=1, \cdots, 4, \alpha, \dot{\alpha}=1,2)$, which leads to have sixteen supercharges, and six scalar fields $\Phi^{I}(I=1, \cdots, 6)$ being all of them in the adjoint representation of the gauge group $U(N)$. But we do not have massive quarks (particles in the fundamental representation) in $\mathcal{N}=4$, so we need to perform a setup in which something with the same behavior appears. To do this, let us consider the breaking of $U(N+1)$ to $U(N) \times U(1)$ by giving some expectation value to the scalar fields, which will parametrize a point in $S^{5}$ in the dual case as we will see in chapter 4. The phase factor associated in this case to the trajectory of this "W-boson" (vector gauge field that become massive by eating the scalar fields) gives to the (Maldacena-) Wilson loop operator the following form in Euclidean space $[27,32,-34]^{4}$,

$$
\begin{equation*}
W_{\mathcal{C}}:=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left[\oint_{\mathcal{C}} d \tau\left(i A_{\mu}(\tau) \dot{x}^{\mu}+\Phi^{i}(\tau) \dot{y}^{i}\right)\right], \tag{3.3}
\end{equation*}
$$

where $x^{\mu}(\tau)$ parametrizes the loop. Notice that this definition does not represent a pure phase as the pure gauge case. It is in Minkowski space that the definition of Wilson loop

[^6]is written as a pure phase, i.e with an $i$ in front of the integral [30, 31]
\[

$$
\begin{equation*}
W_{\mathcal{C}}:=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left[i \oint_{\mathcal{C}} d \tau\left(A_{\mu}(\tau) \dot{x}^{\mu}+\Phi^{i}(\tau) \dot{y}^{i}\right)\right] \tag{3.4}
\end{equation*}
$$

\]

These two versions of the Wilson loops are related by a Wick rotation of the six "internal" coordinates, $y^{i} \rightarrow i y^{i}$ [32].

The supersymmetry transformations in the ten-dimensional representation of the fields [36] are

$$
\begin{equation*}
\delta_{\epsilon} A_{\mu}=\bar{\Psi} \Gamma_{\mu} \epsilon, \quad \delta_{\epsilon} \Phi^{i}=\bar{\Psi} \Gamma^{i} \epsilon, \tag{3.5}
\end{equation*}
$$

where $\epsilon$ is a ten-dimensional Majorana-Weyl spinor. By requiring invariance of the Euclidean $W_{\mathcal{C}}$ (3.3), the last transformations lead to the conditions:

$$
\begin{equation*}
\left(i \Gamma_{\mu} \dot{x}^{\mu}+\Gamma^{i} \dot{y}^{i}\right) \epsilon=0 . \tag{3.6}
\end{equation*}
$$

If the last condition results to be nilpotent, then

$$
\begin{equation*}
\left(i \Gamma_{\mu} \dot{x}^{\mu}+\Gamma^{i} \dot{y}^{i}\right)^{2} \epsilon=\left(\dot{x}^{2}-\dot{y}^{2}\right) \epsilon=0 \tag{3.7}
\end{equation*}
$$

so

$$
\begin{equation*}
\dot{x}^{2}-\dot{y}^{2}=0 \tag{3.8}
\end{equation*}
$$

which is solved by making

$$
\begin{equation*}
\dot{y}^{i}(\tau)=|\dot{x}(\tau)| \theta^{i}(\tau), \tag{3.9}
\end{equation*}
$$

where $\left\{\theta^{i}\right\}$ labels a point on the unit $S^{5}$. From (3.9) we see that if we set $x_{0} \rightarrow i x_{0}$,

$$
\begin{equation*}
|\dot{x}|=i \sqrt{x_{0}^{2}-x_{1}^{2}-x_{2}^{2}-x_{3}^{2}}, \quad x_{0}^{2}-x_{1}^{2}-x_{2}^{2}-x_{3}^{2}>0 \tag{3.10}
\end{equation*}
$$

i.e. a timelike Wilson loop in Minkowski space is a total phase. Back to Euclidean space, nilpotency of the supersymmetry condition leads to $\dot{x}^{2}=\dot{y}^{2}$ which breaks half of supersymmetries to eight, so the loop will be called $1 / 2$ BPS [27, 32]. Now, in general supersymmetry will be local, since (3.9) depends on $\tau$. In order to have global supersymmetry, we must fix a point over $S^{5}$, i.e. $\theta^{i}=\theta_{0}^{i}$ and also set $|\dot{x}|$ to be a constant, so the path will be actually either a line or a circle.

Notice that, even though we wrote the supersymmetric extension of the Wilson loop given in (3.1), there are no fermionic coordinates in (3.3). The reason of not having fermionic fields in the supersymmetric Wilson loop is that they are descendants of the operator above [3, 38] (see [96], a good textbook on conformal field theory) ${ }^{5}$

[^7]In general, for arbitrary representations, the supersymmetric Wilson loop in $\mathcal{N}=4$ is

$$
\begin{equation*}
W_{\mathcal{C}}=\frac{1}{\operatorname{dim} \mathcal{R}} \operatorname{Tr}_{\mathcal{R}} \mathcal{P} \exp \oint_{\mathcal{C}} d \tau\left(i A_{\mu}(\tau) \dot{x}^{\mu}+\Phi^{i}(\tau) \dot{y}^{i}\right) \tag{3.11}
\end{equation*}
$$

which reduces to (3.3) when $\mathcal{R}=$so $\operatorname{dim}$ $\square$ $=N$.

### 3.1.1 Wilson loops in perturbation theory

Let us compute the expectation value of the Wilson loop in the fundamental representation of $S U(N)$ by expanding (3.3) as

$$
\begin{equation*}
\left\langle W_{\mathcal{C}}\right\rangle=\sum_{n=0}^{\infty} A_{n} \lambda^{n}=\sum_{n=0}^{\infty} C_{n} . \tag{3.12}
\end{equation*}
$$

Then,

$$
\begin{align*}
\left\langle W_{\mathcal{C}}\right\rangle & =1+C_{1} \\
& =1+\frac{1}{N} \int d \tau_{1} \int d \tau_{2} \operatorname{Tr}\left(-\dot{x}_{1}^{\mu} \dot{x}_{2}^{\nu}\left\langle A_{\mu}\left(x_{1}\right) A_{\nu}\left(x_{2}\right)\right\rangle+\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right| \theta_{1}^{i} \theta_{2}^{j}\left\langle\Phi^{i}\left(x_{1}\right) \Phi^{j}\left(x_{2}\right)\right\rangle\right) \\
& +\cdots . \tag{3.13}
\end{align*}
$$

By remembering the fact that the fields are in the adjoint representation, $A_{\mu}=A_{\mu}^{a} T^{a}$ and $\Phi^{i}=\Phi^{i a} T^{a}$, the second term can be written as

$$
\begin{align*}
\left\langle W_{\mathcal{C}}\right\rangle & =1+\frac{g^{2} N}{8 \pi^{2}} \int d \tau_{1} \int d \tau_{2} \frac{\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|-\dot{x}_{1} \cdot \dot{x}_{2}}{\left|x_{1}-x_{2}\right|^{2}}+\cdots \\
& =1+\frac{\lambda}{8 \pi^{2}} \int d \tau_{1} \int d \tau_{2} \frac{\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|-\dot{x}_{1} \cdot \dot{x}_{2}}{\left|x_{1}-x_{2}\right|^{2}}+\cdots \tag{3.14}
\end{align*}
$$

where we have considered the propagators in Feynman gauge ${ }^{6}$

$$
\begin{equation*}
\left\langle A_{\mu}(x) A_{\nu}(x)\right\rangle=\frac{g^{2}}{4 \pi^{2}} \frac{\eta_{\mu \nu}}{|x-y|^{2}}, \quad\left\langle\Phi^{i}(x) \Phi^{j}(x)\right\rangle=\frac{g^{2}}{4 \pi^{2}} \frac{\delta_{i j}}{|x-y|^{2}} \tag{3.15}
\end{equation*}
$$

Let us analyze the last term when $x_{1} \rightarrow x_{2}$. Notice that even though $\mathcal{N}=4$ is UV finite, we can see that there exists a UV divergence when $x_{1}=x_{2} \cdot 7$ so when $1 /\left|x_{1}-x_{2}\right|^{2}=1 / \epsilon^{2}$ $(\epsilon \rightarrow 0)$ we get [33]

$$
\begin{equation*}
\frac{\lambda}{8 \pi^{2}} \int d \tau_{1} \int d \tau_{2} \frac{\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|-\dot{x}_{1} \cdot \dot{x}_{2}}{\left|x_{1}-x_{2}\right|^{2}} \overbrace{=}^{x_{1} \rightarrow x_{2}+\epsilon} \frac{\lambda}{4 \pi^{2} \epsilon} \int d \tau_{1}\left|\dot{x}\left(\tau_{1}\right)\right|\left(1-\frac{\dot{y}^{2}}{\dot{x}^{2}}\right) . \tag{3.16}
\end{equation*}
$$

[^8]As we mentioned, supersymmetry imposes (3.8)

$$
\begin{equation*}
\dot{x}^{2}-\dot{y}^{2}=0 \tag{3.17}
\end{equation*}
$$

which leads to the cancellation of the divergence. One important thing to notice is that the cancellation does not occur between bosonic and fermionic contribution, but only bosonic parts. Then we can say that supersymmetry imposed a constraint to cancel the divergences $\sqrt[8]{ }$ Does it happen with the other terms in the expansion? It was proven in [27, 36] that UV divergences cancel each other also at order $\lambda^{2}$. This is supposed to happen at each order in the $\lambda$ expansion because of conformal symmetry. It was argued in [27] that part of the singular parts at order $\lambda^{2}$ survive to compensate loop correction to the propagators. In general, it was mentioned in [32] that at order $\lambda^{n}$, the linear divergence has the following general form

$$
\begin{equation*}
\frac{\lambda^{n}}{\epsilon} \int d \tau_{1}\left|\dot{x}\left(\tau_{1}\right)\right| G_{n}\left(\frac{\dot{y}^{2}}{\dot{x}^{2}}\right), \tag{3.18}
\end{equation*}
$$

where $G_{n}(z)$ is a polynomial, and as we saw before, $G_{n}(1)=0$. AdS/CFT will allow us to see that there is no divergence in the expansion.

On the other hand, we can calculate the expected value of the Wilson loop, in the $1 / 2$ BPS case for the line and the circle, which is related to the line by a conformal transformation.

- The straight line, which can be parametrized by

$$
\begin{equation*}
x(\tau)=(\tau, 0,0,0), \quad-\infty<\tau<\infty \tag{3.19}
\end{equation*}
$$

which leads to cancel the first contribution in $\lambda$. Then, the value of the Wilson line is

$$
\begin{equation*}
\left\langle W_{\text {line }}\right\rangle=1 . \tag{3.20}
\end{equation*}
$$

This result is also valid at strong coupling. The symmetries preserved by the line are $S O(1,2) \times S O(3)$ (translations and inversion in one dimension plus rotations in three spatial dimensions) in spacetime and $S O(5) \subset S O(6)$ (the fixed point in $S^{5}$ ) because we fixed one $\theta^{i}=\theta_{0}$.

[^9]- The circle can be parametrized by

$$
\begin{equation*}
x(\tau)=(\cos \tau, \sin \tau, 0,0), \quad 0 \leq \tau \leq 2 \pi \tag{3.21}
\end{equation*}
$$

which produces

$$
\begin{equation*}
\left\langle W_{\text {circle }}\right\rangle=1+\frac{\lambda}{16}+\cdots \tag{3.22}
\end{equation*}
$$

It is worth to notice that the line and the circle are related by a conformal transformation (inversion),

$$
\begin{equation*}
x^{\mu} \rightarrow \frac{x^{\mu}}{|x|} \tag{3.23}
\end{equation*}
$$

One could have thought that the results should be the same due to conformal symmetry. The reason of this discrepancy is that the circular Wilson loop is determined by the conformal anomaly that emerges when we perform this kind of transformations. Another reason to expect different results is the fact that the (special) conformal transformation line-circle, is not a symmetry of $\mathbb{R}^{4}$ but of $S^{4}$, since it brings a point at infinity to a point at a finite distance. It was shown in [34] the difference between the Wilson line and the Wilson loop comes from the divergence in the gauge transformation that appears in the gauge propagator under conformal transformation.

### 3.1.2 Summing planar graphs

As we have seen in (2.30), for large $N$ and fixed $\lambda$, only planar diagrams are relevant. This is a huge simplification, but actually these diagrams are composed by ladder (for the case of the circular loop) or rainbow (for the case of the line) diagrams, which are those who do not have any interaction, and also loop and vertex diagrams. It was proved in [27] that the $\lambda^{2}$ contributions of the one-loop correction and the internal three-vertex cancel each other for the circle and the line and was conjectured that the same occurs at each level in $\lambda$. In that paper it was then assumed that only ladder or rainbow diagrams are involved in calculations. Let us consider the $2 n$-th order term in the Taylor expansion of the loop in (3.12),

$$
\begin{align*}
C_{2 n}= & \frac{1}{N} \int_{0}^{2 \pi} d \tau_{1} \int_{0}^{\tau_{1}} d \tau_{2} \cdots \int_{0}^{2 n-1} d \tau_{2 n} \times \\
& \times \operatorname{Tr}\left\langle\left(i A_{1} \cdot \dot{x}_{1}+\left|\dot{x}_{1}\right| \Phi_{1} \cdot \theta\right) \cdots\left(i A_{2 n} \cdot \dot{x}_{2 n}+\left|\dot{x}_{2 n}\right| \Phi_{2 n} \cdot \theta\right)\right\rangle \tag{3.24}
\end{align*}
$$

where $A_{1}=A\left(\tau_{1}\right)$ and $\Phi_{1}=\Phi\left(\tau_{1}\right)$. The factor $1 /(2 n)$ ! cancels with the $(2 n)$ ! factor coming from the particular ordering we choose. To construct the general term, let us consider

$$
\begin{equation*}
\left\langle\left(i A_{i}^{a} \cdot \dot{x}_{i}+\left|\dot{x}_{i}\right| \Phi_{i}^{a} \cdot \theta\right)\left(i A_{j}^{b} \cdot \dot{x}_{j}+\left|\dot{x}_{j}\right| \Phi_{j}^{b} \cdot \theta\right)\right\rangle=\frac{g^{2}}{4 \pi^{2}} \frac{\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|-\dot{x}_{1} \cdot \dot{x}_{2}}{\left|x_{1}-x_{2}\right|^{2}}=\frac{g^{2}}{4 \pi^{2}} \frac{1}{2} \delta^{a b} \tag{3.25}
\end{equation*}
$$

which is constant. Wick contractions in (2.30) lead to a product of $n$ two-point functions, and then

$$
\begin{equation*}
C_{2 n}=\frac{1}{(2 n)!}\left(\frac{\lambda}{4}\right)^{n} A_{n} \tag{3.26}
\end{equation*}
$$

Then, we can calculate the number of planar diagrams with $n$ internal propagators, $A_{n}$,

$$
\begin{equation*}
A_{n}=\frac{(2 n)!}{(n+1)!n!} \tag{3.27}
\end{equation*}
$$

So we can sum all the planar ladder diagrams as [27],

$$
\begin{equation*}
\left\langle W_{\mathcal{C}}\right\rangle_{\text {ladders }}=\sum_{n=0}^{\infty} \frac{(\lambda / 4)^{n}}{(n+1)!n!}=\frac{2}{\sqrt{\lambda}} I_{1}(\sqrt{\lambda}) \tag{3.28}
\end{equation*}
$$

where $I_{1}(\sqrt{\lambda})$ is a generalized Bessel function. At large $\lambda$,

$$
\begin{equation*}
\left\langle W_{\mathcal{C}}\right\rangle_{\text {ladders }} \sim e^{\sqrt{\lambda}} . \tag{3.29}
\end{equation*}
$$

As was also mentioned in [27], the fact that (3.25) is independent of the coordinates, and that $\left\langle W_{\mathcal{C}}\right\rangle_{\text {ladders }}$ involves a sum the ladder diagrams, maps the problems to a zerodimensional matrix model. In the next section we will review the matrix model theory.

### 3.2 A brief introduction to matrix models and localization

As was mentioned above, the fact that the sum of the scalar and vector contributions is constant, and that the sum of all ladder diagrams reduces to the counting of diagrams, allowed Erickson et al in [27] to conjecture that the number of planar ladders can be calculated from the infinite $N$ limit of a matrix model. In this section, we are going to present some basic results about matrix models in order to understand the connection with large $N$ gauge theory and then to understand the Erickson-Semenoff-Zarembo idea.

At large $N$, we learned that the perturbative expansion is dominated by planar diagrams, which were represented schematically by (2.30),

$$
\begin{equation*}
\operatorname{diagram}(V, E, F) \sim N^{V-E+F}=N^{\chi} \tag{3.30}
\end{equation*}
$$

where $\chi$ is the Euler characteristic of the surface associated to the diagram. This, in turn, can be connected with the results given by the following model [23],

$$
\begin{equation*}
\frac{1}{g_{s}} S(M)=\frac{1}{2 g_{s}} \operatorname{Tr} M^{2}+\frac{1}{g_{s}} \sum_{p \geq 3} \frac{g_{p}}{p} \operatorname{Tr} M^{p} \tag{3.31}
\end{equation*}
$$

where we can assume that $g_{s}=g^{2}$ is the string (or Yang-Mills) coupling and $g_{p}$ is new coupling depending on $p$, and the field is a $N \times N$ hermitian matrix, $M$, with constant entries. We can impose also $U(N)$ symmetry, which rotates the matrices. This is a matrix model which was used to understand the internal geometry of a 2D surface that can be discretized as a sum over randomly triangulated surfaces in 2D gravity [97]. Under rescaling of the matrix field, $M \rightarrow M \sqrt{N}$, the action 3.31, can be written as

$$
\begin{equation*}
\frac{1}{g_{s}} S(M)=N\left(\frac{1}{2 g_{s}} \operatorname{Tr} M^{2}+\frac{1}{g_{s}} \sum_{p \geq 3} \frac{g_{p}}{p} \operatorname{Tr} M^{p}\right) \tag{3.32}
\end{equation*}
$$

which produces the same behavior

$$
\begin{equation*}
N^{V-E+F}=N^{\chi} . \tag{3.33}
\end{equation*}
$$

For small $g$ (or large $N$ ), i.e the planar limit, we can write the partition function for the matrix model,

$$
\begin{equation*}
Z=\frac{1}{\operatorname{vol}(U(N))} \int \mathcal{D} M e^{-\frac{1}{2 g_{s}} \operatorname{Tr} M^{2}} \tag{3.34}
\end{equation*}
$$

the Gaussian version of the model, where we have rescaled back the matrix $M \rightarrow M / \sqrt{N}$.
As usual in quantum field theory, we compute

$$
\begin{equation*}
\langle f(M)\rangle=\frac{\int \mathcal{D} M f(M) e^{-\frac{1}{2 g_{s}} \operatorname{Tr} M^{2}}}{\int \mathcal{D} M e^{-\frac{1}{2 g_{s}} \operatorname{Tr} M^{2}}} \tag{3.35}
\end{equation*}
$$

Due to the matrix indices, propagators can be represented by the double-line notation ,or "fatgraphs" (see again [23]), presented above for the large $N$ expansion in gauge theories.

The $\operatorname{vol}(U(N))$ factor is the volume factor of the "gauge" group needs after fixing a gauge. The measure $d M$ in the "path" integral is the so-called Haar measure

$$
\begin{equation*}
\mathcal{D} M=\prod_{i}^{N} d M_{i i} \prod_{i<j}^{N} d \operatorname{Re}\left(M_{i j}\right) d \operatorname{Im}\left(M_{i j}\right) \tag{3.36}
\end{equation*}
$$

Since we have gauge invariance, we can fix it to write the theory in terms of diagonal matrices

$$
\begin{equation*}
M \rightarrow U M U^{\dagger}=D \tag{3.37}
\end{equation*}
$$

where $D$ is diagonal with elements $\left\{m_{i}\right\}$. So, we now have $N$ parameters instead of $N^{2}$. We can perform the Faddeev-Popov method to compute the gauge fixed partition function. After some calculations, following [23], we can arrive to

$$
\begin{equation*}
Z=\frac{1}{N!} \frac{1}{(2 \pi)^{N}} \int \prod_{i=1}^{N} d m_{i} \Delta^{2}(m) \exp \left(-\frac{1}{2 g^{2}} \sum_{i} m_{i}^{2}\right) \tag{3.38}
\end{equation*}
$$

or

$$
\begin{equation*}
Z=\frac{1}{N!} \int \prod_{i=1}^{N}\left(\frac{d m_{i}}{2 \pi}\right) \Delta^{2}(m) \exp \left(-\frac{N}{2 \lambda} \sum_{i} m_{i}^{2}\right) \tag{3.39}
\end{equation*}
$$

where

$$
\Delta(m)=\left|\begin{array}{ccccc}
1 & m_{1} & m_{1}^{2} & \cdots & m_{1}^{N-1}  \tag{3.40}\\
1 & m_{2} & m_{2}^{2} & \cdots & m_{2}^{N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & m_{N} & m_{N}^{2} & \cdots & m_{N}^{N-1}
\end{array}\right|=\prod_{1 \leq i<j \leq N}\left|m_{i}-m_{j}\right|
$$

is called the Vandermonde derteminant; and $\lambda=g^{2} N$. But why do we worry about this? As was conjectured by Erickson, Semenoff and Zarembo in [27], the number of planar ladder diagrams with $n$ propagators can be calculated by

$$
\begin{equation*}
A_{n}=\left\langle\frac{1}{N} \operatorname{tr} M^{n}\right\rangle \tag{3.41}
\end{equation*}
$$

by following the arguments in [98,99], in which a zero-dimensional field theory, i.e. a theory without spacetime in which fields are constants, was considered. Given the form of the general term in the expansion of the Wilson loop (3.26), one can say that

$$
\begin{equation*}
\left\langle W_{\mathcal{C}}\right\rangle_{\text {ladders }}=\left\langle\frac{1}{N} \operatorname{tr} e^{M}\right\rangle \tag{3.42}
\end{equation*}
$$

which was the result that Drukker and Gross conjectured in [28], by arguing that the circular and line Wilson operators differ by the contribution of the point at infinity that
must be added under conformal transformation from the line to the circle. So, since the result only comes from a point, there is no spacetime to be considered, and as we saw above, calculations reduce to counting diagrams: a zero-dimensional field theory, i.e. a matrix model.

In the following, we shall review some details about the solution of the model, which we can be obtained with two methos:

- Saddle-point analysis, and
- Orthogonal polynomials.


### 3.2.1 The saddle point method

The partition function defined before in (3.39) can be rewritten as

$$
\begin{equation*}
Z=\frac{1}{N!} \int \prod_{i=1}^{N}\left(\frac{d m_{i}}{2 \pi}\right) e^{N^{2} S_{\mathrm{eff}}} \tag{3.43}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{\mathrm{eff}}=-\frac{1}{\lambda N} \sum_{i} \frac{m_{i}^{2}}{2}+\frac{2}{N^{2}} \sum_{i<j} \ln \left|m_{i}-m_{j}\right| . \tag{3.44}
\end{equation*}
$$

The first term in the action is a sum over the square of the eigenvalues, $\lambda_{i}$, which gives a factor of $N$. Then, it is of order $N$ which cancels the $1 / N$ factor and leaves the quadratic part being of order $\mathcal{O}(1)$. The second term of the action can be thought of as a repulsive potential for the eigenvalues, there $1 / N^{2}$ plays the role of $\hbar$. So, we can imagine that large $N$ translates into $\hbar \rightarrow 0$, the classical limit of the action: the saddle point.

Varying the action with respect to $\lambda_{i}$,

$$
\begin{equation*}
\delta S_{\mathrm{eff}}=0 \rightarrow \frac{1}{2 \lambda} V^{\prime}\left(m_{i}\right)=\frac{1}{N} \sum_{j \neq i} \frac{1}{m_{i}-m_{j}}, \quad i, j=1, \cdots, N, \tag{3.45}
\end{equation*}
$$

where $V\left(m_{i}\right)=\sum_{i} m_{i}^{2} / 2$. Also, we can define the eigenvalue distribution as

$$
\begin{equation*}
\rho(\xi)=\frac{1}{N} \sum_{j} \delta\left(\xi-m_{j}\right) \tag{3.46}
\end{equation*}
$$

From (3.44), we can define an effective potential

$$
\begin{equation*}
V_{\mathrm{eff}}=V\left(m_{i}\right)-\frac{2 \lambda}{N} \sum_{i<j} \ln \left|m_{i}-m_{j}\right|, \tag{3.47}
\end{equation*}
$$

with a minimum in $\tilde{m}_{i}$. When $\lambda$ (the 't Hooft coupling) is small, the quadratic term dominates and eigenvalues tend to be in the minimum, a single value $\tilde{m}_{i}$; when $\lambda$ starts to grow, the repulsive part of the potential dominates and separates the eigenvalues from each other over a curve $\mathcal{C}$. At large $N$, we can treat $\left\{m_{i}\right\}$ as a continuum $\xi$, so

$$
\begin{equation*}
\frac{1}{N} \sum_{i} f\left(m_{i}\right) \rightarrow \int_{\mathcal{C}} f(\xi) \rho(\xi) d \xi \Rightarrow \int_{\mathcal{C}} \rho(\xi) d \xi=1 \tag{3.48}
\end{equation*}
$$

and the action (3.44)

$$
\begin{equation*}
S_{\mathrm{eff}}=-\frac{1}{\lambda} \int \xi^{2} \rho(\xi) d \xi+2 \int \ln \left|\xi-\xi^{\prime}\right| \rho(\xi) \rho\left(\xi^{\prime}\right) d \xi d \xi^{\prime} \tag{3.49}
\end{equation*}
$$

So we can rewrite the saddle point equation (3.45) as

$$
\begin{equation*}
\frac{1}{2 \lambda} V^{\prime}(\xi)=\frac{\xi}{\lambda}=f_{\mathcal{C}} \frac{\rho\left(\xi^{\prime}\right) d \xi^{\prime}}{\xi-\xi^{\prime}} \tag{3.50}
\end{equation*}
$$

It is possible to invert this expression to find that [22]

$$
\begin{equation*}
\rho(\xi)=\frac{1}{2 \pi \lambda} \sqrt{4 \lambda-\xi^{2}} \tag{3.51}
\end{equation*}
$$

where $\xi \in[-2 \sqrt{\lambda}, 2 \sqrt{\lambda}]$, or rescaled to

$$
\begin{equation*}
\rho(x)=\frac{1}{\pi \lambda} \sqrt{\lambda-x^{2}}, \quad-\sqrt{\lambda} \leq x \leq \sqrt{\lambda} \tag{3.52}
\end{equation*}
$$

This is the Wigner semi-circle distribution. The integral in (3.49) is dominated by the $\rho$ which minimizes $S_{\text {eff }}$, and in particular it becomes zero. Thus, we can calculate

$$
\begin{equation*}
\left\langle\frac{1}{N} \operatorname{Tr} M^{n}\right\rangle=\left(\frac{1}{N} \sum_{i} m_{i}^{n}\right) \rightarrow \int d \xi \xi^{n} \rho(\xi) \tag{3.53}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\left\langle\frac{1}{N} \operatorname{Tr} e^{M}\right\rangle=\frac{2}{\sqrt{\lambda}} I_{1}(\sqrt{\lambda}) \tag{3.54}
\end{equation*}
$$

The last result coincides with (3.28). It is worth to notice that this method is valid only for large $N$ but for all $\lambda$.

### 3.2.2 Orthogonal polynomials

Another way to solve matrix models is by using orthogonal polynomials, which was used by Drukker and Gross in [28] (see also [100] for an old reference) to calculate the expectation value of the circular loop for any value of $N$, since the technique does not depend
on $N$, neither on $\lambda$. Let us consider the rescaled partition (3.39),

$$
\begin{equation*}
Z=\int \prod_{i=1}^{N} d m_{i} \Delta^{2}(m) \exp \left(-\frac{2 N}{\lambda} \sum_{i} m_{i}^{2}\right) \tag{3.55}
\end{equation*}
$$

Thus, we write,

$$
\begin{equation*}
\left\langle\frac{1}{N} \operatorname{Tr} e^{M}\right\rangle=\frac{1}{Z} \int \prod_{i=1}^{N} d m_{i} \Delta^{2}(m) \exp \left(-\frac{2 N}{\lambda} \sum_{i} \lambda_{i}^{2}\right) \frac{1}{N} \sum_{k} \exp \lambda_{k} \tag{3.56}
\end{equation*}
$$

or, rescaling again and absorbing all extra factors into $Z$,

$$
\begin{equation*}
\left\langle\frac{1}{N} \operatorname{Tr} e^{M}\right\rangle=\frac{1}{Z} \int \prod_{i=1}^{N} d m_{i} \Delta^{2}(m) \exp \left(-\sum_{i} m_{i}^{2}\right) \frac{1}{N} \sum_{k} \exp \left(\sqrt{\frac{\lambda}{2 N}} m_{k}\right) . \tag{3.57}
\end{equation*}
$$

The Vandermonde determinant in (3.40) can be written as

$$
\begin{equation*}
\Delta\left(m_{i}\right)=\prod_{i<j}\left(m_{j}-m_{i}\right)=\operatorname{det}\left[m_{i}^{j-1}\right], \quad i, j=1, \cdots, N, \tag{3.58}
\end{equation*}
$$

where the element $m_{i}^{j-1}$ represents $m_{i}$ to the $(j-1)$-th. We can also get the same result if instead we use

$$
\begin{equation*}
\Delta(m)=\operatorname{det}\left[P_{j-1}\left(m_{i}\right)\right], \tag{3.59}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{j-1}\left(m_{i}\right)=x^{j-1}+\sum_{k=0}^{j-2} a_{k} x^{k}, \quad i, j=1, \cdots, N \tag{3.60}
\end{equation*}
$$

is a general polynomial that can be chosen to our convenience. Thus,

$$
\begin{equation*}
\left\langle\frac{1}{N} \operatorname{Tr} e^{M}\right\rangle=\frac{1}{Z} \int \prod_{i=1}^{N} d m_{i}\left(\operatorname{det}\left[P_{j-1}\left(\lambda_{i}\right)\right]\right)^{2} \exp \left(-\sum_{i} m_{i}^{2}\right) \frac{1}{N} \sum_{k} \exp \left(\sqrt{\frac{\lambda}{2 N}} m_{k}\right) . \tag{3.61}
\end{equation*}
$$

The last integral contains a sum that can be extracted from the whole expression,

$$
\begin{equation*}
\left\langle\frac{1}{N} \operatorname{Tr} e^{M}\right\rangle=\frac{1}{N} \sum_{k} \frac{1}{Z} \int \prod_{i=1}^{N} d m_{i}\left(\operatorname{det}\left[P_{j-1}\left(m_{i}\right)\right]\right)^{2} \exp \left(-\sum_{i} m_{i}^{2}\right) \exp \left(\sqrt{\frac{\lambda}{2 N}} m_{k}\right) . \tag{3.62}
\end{equation*}
$$

The appropriate polynomial we can use are proportional to the Hermite polynomials, orthonormalized with respect to $d m_{i} \exp \left(-m_{i}^{2}\right)$. From the last expression $\sqrt{3.62}$ we see that we can integrate for $i \neq k$ by using orthogonality. We end up with 9

$$
\begin{equation*}
\left\langle\frac{1}{N} \operatorname{Tr} e^{M}\right\rangle=\frac{1}{N} \sum_{k} \frac{1}{Z} \int d m P_{j}(m)^{2} \exp \left[-\left(m-\sqrt{\frac{\lambda}{8 N}}\right)^{2}\right] \exp \left(\frac{\lambda}{8 N}\right) \tag{3.63}
\end{equation*}
$$

[^10]and finally,
\[

$$
\begin{equation*}
\left\langle\frac{1}{N} \operatorname{Tr} e^{M}\right\rangle=\frac{1}{N} L_{N-1}^{1}\left(-\frac{\lambda}{4 N}\right) \exp \left(\frac{\lambda}{8 N}\right) \tag{3.64}
\end{equation*}
$$

\]

where $L_{n}^{m}(x)$ is a generalized Laguerre polynomial. This method does not depend on making $N$ large, so the last result includes also non-planar corrections. For large $N$ we can prove that

$$
\begin{equation*}
\left\langle\frac{1}{N} \operatorname{Tr} e^{M}\right\rangle=\frac{2}{\sqrt{\lambda}} I_{1}(\sqrt{\lambda})+\mathcal{O}\left(\frac{1}{N^{2}}\right) . \tag{3.65}
\end{equation*}
$$

We stress that the matrix model that was considered above is quadratic in the field $M$. Erickson, Semenoff and Zarembo suggested that the Wilson loops could be calculated by a matrix model, since was the only thing that mattered at large $N$. Later, Drukker and Gross used the quadratic model to do the calculations and get an expression valid for any $N$, that reduces to the result obtained by Erickson et all at large $N$. But, they did not know if one can produce the same result at large $N$ from a non-Gaussian matrix model. They argued that a Gaussian matrix model is the correct model to be considered since the results coincide with those coming from the AdS/CFT correspondence, at large $N$ and large $\lambda$. It was Pestun in 2007 who showed that the matrix model must be Gaussian by using a technique called localization. In the next section we will review briefly his method.

### 3.2.3 Supersymmetric localization

Localization is a powerful technique that allows to calculate supersymmetric path integrals explicitly. It is based on having a continuum symmetry that acts on the space, it is possible to express integrals over that space as sums of contributions coming from points that are invariant under the symmetry. If the symmetry is the one that mixes fermionic and bosonic fields, i.e. supersymmetry, the reduction of the path integral to a finite integral over moduli space (classical values of the fields) is called supersymmetric localization. It was Pestun [39] who used the localization principle to compute path integrals involving supersymmetric operators, in particular the Wilson loop. Let us see how it works, schematically [40].

Let $Q$ be the supersymmetric transformation (fermionic), so $Q^{2}$ is a bosonic symmetry. Its effect over the action is

$$
\begin{equation*}
Q S[X]=0 \tag{3.66}
\end{equation*}
$$

Now consider the partition function corresponding to the previous action but perturbed by a $Q$-exact term

$$
\begin{equation*}
Z=\int \mathcal{D} X e^{-(S[X]+t Q V[X])}, \quad Q V[X] \geq 0 \tag{3.67}
\end{equation*}
$$

for an arbitrary real parameter $t$, and the fermionic functional $V$ such that $Q^{2} V[X]=0$. We can see that,

$$
\begin{equation*}
\frac{d Z}{d t}=-\int \mathcal{D} X(Q V[X]) e^{-(S[X]+t Q V[X])}=-\int \mathcal{D} X Q\left(V[X] e^{-(S[X]+t Q V[X])}\right)=0 \tag{3.68}
\end{equation*}
$$

since it is a total derivative. ${ }^{10}$ So $Z$ does not depend on the deformation parameter and it can be computed at, for example, $t \rightarrow \infty$ without any problem. Similary, for the Wilson loop,

$$
\begin{equation*}
\frac{d\langle W\rangle}{d t}=-\frac{1}{Z} \int \mathcal{D} X Q\left(V[X] e^{-(S[X]+t Q V[X])} W\right)=0 \tag{3.69}
\end{equation*}
$$

for supersymmetric Wilson operators, with $Q W=0$. Thus,

$$
\begin{equation*}
\langle W\rangle=\frac{1}{Z} \int \mathcal{D} X e^{-S[X]} W=\lim _{t \rightarrow \infty} \frac{1}{Z} \int \mathcal{D} X e^{-(S[X]+t Q V[X])} W \tag{3.70}
\end{equation*}
$$

The only non-vanishing contributions to the expected value are field configurations (moduli) satisfying

$$
\begin{equation*}
Q V[X]=0 \tag{3.71}
\end{equation*}
$$

because we take $t \rightarrow \infty$ for $t Q V[X]$ being finite. Then, this "localized" set of (bosonic) fields leads to a finite-dimensional integral. For a nice choice of $V[X]$ the integral can be computed by evaluating the action at $Q V=0$. In Pestun's paper, who focused on the $\mathcal{N}=4$ SYM theory on $S^{4} \cdot{ }^{[11}$ this condition leads to a quadratic constant action (see [39], and also the collection [42], for details)

$$
\begin{equation*}
S\left[\Phi_{0}=M\right] \propto \frac{1}{g^{2}} \operatorname{Tr}\left(M^{2}\right), \tag{3.72}
\end{equation*}
$$

for the unique non-zero scalar in the localized configuration, $\Phi_{0}$. And since the scalars are in the adjoint representation, they are actually matrices. So the localized theory is indeed made of constant matrices with a quadratic action. This proves that the matrix model that Erickson et al used to calculate the circular Wilson loop in $\mathcal{N}=4$ was correct.

[^11]In chapter 5 we will extend briefly the matrix model to the case of Wilson loops in higher representations. We will also try to study subleading corrections of the exact results. But before that, let us see in the next chapter how the correspondence allows us to get the same results from a configuration of strings in ten dimensions.

## Chapter 4

## Holographic Wilson loops

In this chapter let us try to reproduce the field theory results from the gravity side because this is how the duality works: comparing and matching results. In particular, for Wilson loops in $\mathcal{N}=4 D=4 S U(N)$ gauge theory at large $N{ }^{1}$ and large $\lambda$, we will understand how to obtain the expectation value of a Wilson loop in type IIB supergravity in $\operatorname{Ad} S_{5} \times$ $S^{5}$.

According to the Maldacena prescription [3.8], we can calculate the expectation value of an operator in the gauge theory at large $\lambda$ by evaluating the string action whose worldsheets in $A d S_{5} \times S^{5}$ satisfy boundary conditions [30]

$$
\begin{equation*}
\left\langle W_{\mathcal{C}}\right\rangle \equiv \int_{\mathcal{C}} \mathcal{D} X e^{-S[X]}=Z_{\text {strings }} \xrightarrow{\lambda \rightarrow \infty} \exp \left(-S_{\text {onshell }}\right) \tag{4.1}
\end{equation*}
$$

The string partition in the r.h.s. of the last equation defines a complicated two-dimensional sigma model which cannot be solved explicitly $\cdot 2$ But simplifications occur at large $\lambda$ : here the sigma model becomes weakly coupled, and the path integral is dominated by the saddle point because stringy fluctuations are suppressed (since $\alpha^{\prime} \rightarrow 0$ ). The resulting action in this limit is the Nambu-Goto action or, equivalently, the Polyakov action, whose solution describes the minimal area of the worldsheet [36] (see [34] for a complete analysis of the string action in this background),

$$
\begin{equation*}
S[X]:=S_{N G}=T \int_{\Sigma} d \tau d \sigma \sqrt{\operatorname{det} g_{a b}}, \quad g_{a b}=G_{M N} \partial_{a} Z^{M} \partial_{b} Z^{N}, \quad T=\frac{1}{2 \pi \alpha^{\prime}} . \tag{4.2}
\end{equation*}
$$

[^12]or
\[

$$
\begin{equation*}
S[X]:=S_{N G}=\frac{\sqrt{\lambda}}{2 \pi} \int_{\Sigma} d \tau d \sigma \sqrt{\operatorname{det} g_{a b}}, \quad g_{a b}=G_{M N} \partial_{a} Z^{M} \partial_{b} Z^{N} \quad \sqrt{\lambda}=\frac{L^{2}}{\alpha^{\prime}} . \tag{4.3}
\end{equation*}
$$

\]

where we simply extracted the dimensional part of the induced metric. So,

$$
\begin{equation*}
S_{\text {onshell }}=\frac{\sqrt{\lambda}}{2 \pi} A[\mathcal{C}] \tag{4.4}
\end{equation*}
$$

We can see that $\alpha^{\prime} \rightarrow 0$ means large string tension, so the string (classically imagined) does not "vibrate"; and also $\alpha^{\prime} \rightarrow 0$ means large $\lambda$. So, in this limit, the resulting action must give only the area described by the string worldsheet attached to the loop. But this dual expression is not going to give us the correct answer. If we naively compute the onshell action, i.e. the minimal area described by the string, we expect it diverges; that leads to $\left\langle W_{\mathcal{C}}\right\rangle=0$. This is because we are not considering subtle details about the boundary conditions we impose. Another way to see this is to consider the case of $N+1$ D3 branes instead. Thus, their resultant worldvolume theory is going to be $U(N+1)$. Let us separate one single brane from the stack and connect them by a string. This translates into a breaking of $U(N+1)$ to $U(N) \times U(1)$, as we said in chapter 3. This means that

$$
\Phi_{U(N+1)}^{I}=\left(\begin{array}{cc}
\Phi_{U(N)}^{I}=0 & 0  \tag{4.5}\\
0 & \Phi_{N+1}^{I}=\Phi_{0}^{I}
\end{array}\right)
$$

The string state connecting the branes will have a mass

$$
\begin{equation*}
M=\frac{1}{2 \pi \alpha^{\prime}} \ell . \tag{4.6}
\end{equation*}
$$

If we see this string connect as a straight line connecting the position of the stack at the boundary of space, with the separated brane, $\ell$ is going to be along the radial coordinate of $A d S_{5}$. So, an infinitely separated brane will produce a infinite massive string state connecting the branes. This, in turn, will produce massive fundamental states on the worldvolume theory: "quarks". In order to eliminate from the action this infinite mass, we explicitly write

$$
\begin{equation*}
\left\langle W_{\mathcal{C}}\right\rangle \sim \lim _{M \rightarrow \infty} e^{-\left(\frac{\sqrt{\lambda}}{2 \pi} A[\mathcal{C}]-M L[\mathcal{C}]\right)} \tag{4.7}
\end{equation*}
$$

where $L[\mathcal{C}]$ is the length of the loop.

### 4.1 Boundary conditions

Branes are higher-dimensional surfaces where open strings can end. If we attach a string to them we need to specify the behavior of its endpoints on that "plane": free to move on the plane and attached to it. The worldvolume of the field theory living on the brane is four-dimensional; so if we define a $1 / 2$ BPS Wilson loop there, we established that the condition on the loop variables, $\left(x^{\mu}(\tau), y^{i}(\tau)\right)$, is

$$
\begin{equation*}
\dot{x}^{2}=\dot{y}^{2} . \tag{4.8}
\end{equation*}
$$

This condition can be obtained in the gravity side by giving boundary conditions to the worldsheet of a string stretched between a probe D3 brane and the $N \mathrm{D} 3$ branes.

At large $\lambda$, the $N \mathrm{D} 3$ branes become geometry of $A d S_{5} \times S^{5}$. The string, which is massive due to the separation of the branes, resembles the W-boson and each of their endpoints behaves as "quarks". From the worldsheet point of view, the gauge field, $A_{\mu}$, generated by the endpoint of the string and the scalar field, which can be understood as transversal coordinates, couple at the boundary to the string worldsheet as [33]

$$
\begin{equation*}
\oint d \tau\left[A_{\mu} \frac{\partial X^{\mu}}{\partial \tau}+\Phi^{i} P^{i}\right]_{\sigma=0} \tag{4.9}
\end{equation*}
$$

where $A_{\mu}$ couples only to $X^{\mu}$ and the scalar fields are precisely the transversal coordinates as viewed by the worldsheet, so that they couple only to the (transversal) momentum $P^{i}$ [32,33] associated to $Y^{i}$ along $\sigma$.

Let us review this in a slightly more general way. Remember the Nambu-Goto action in (4.3)

$$
\begin{equation*}
S_{N G}=\frac{\sqrt{\lambda}}{2 \pi} \int_{\Sigma} d^{2} \sigma \sqrt{g}, \quad g_{a b}=G_{M N} \partial_{a} Z^{M} \partial_{b} Z^{N} \quad \sqrt{\lambda}=\frac{L^{2}}{\alpha^{\prime}} \tag{4.10}
\end{equation*}
$$

where $Z^{M}=\left\{X^{\mu}, Y^{i}\right\}, a, b=\tau, \sigma$ and $G_{M N}=\left\{G_{\mu \nu}, G_{i j}\right\}$. Varying the Nambu-Goto action we get

$$
\begin{align*}
\delta S_{N G} & =-\frac{\sqrt{\lambda}}{2 \pi} \int d \tau d \sigma \partial_{a}\left(\sqrt{g} g^{a b} \partial_{b} Z^{J} G_{J I} \delta Z^{I}\right) \\
& =\frac{\sqrt{\lambda}}{2 \pi} \int d \tau\left[\sqrt{g} g^{2 b} \partial_{b} Z^{J} G_{J I} \delta Z^{I}\right]_{\sigma=\sigma_{0}}^{\sigma=\sigma_{1}} \tag{4.11}
\end{align*}
$$

where we used the fact that the fields vanish at $\tau= \pm \infty$. We identify the momentum along $\sigma$ conjugate to $Z^{I}$ as

$$
\begin{equation*}
P_{I}(\tau, \sigma)=\frac{\delta S}{\delta \partial_{\sigma} Z^{I}}=\frac{\sqrt{\lambda}}{2 \pi} \sqrt{g} g^{2 b} \partial_{b} Z^{J} G_{I J} \tag{4.12}
\end{equation*}
$$

At the boundary we need that

$$
\begin{align*}
X^{\mu}\left(\tau, \sigma_{0}\right) & =x^{\mu}(\tau) \\
P_{Y}^{i}\left(\tau, \sigma_{0}\right)=\sqrt{g} g^{2 b} \partial_{b} Y^{i}\left(\tau, \sigma_{0}\right) & =\dot{y}^{i}(\tau), \tag{4.13}
\end{align*}
$$

where we assumed that at $\sigma=0$ the string ends at the boundary and the loop is parametrized by $\tau$. Four directions along the brane are fixed, so they are now Dirichlet, while for the six other transverse directions the string momentum is fixed, so we have six Neumann boundary conditions. Consequently, the Wilson loop in the worldvolume theory imposes "complementary" boundary conditions on the string worldsheet. We say complementary because a free open string on the D3 brane obeys four Neumann, since it is free to move on the brane, and six Dirichlet boundary conditions, since the endpoints have fixed transverse positions. Now, due to the loop on the stack of branes, the string endpoints cannot leave the branes and go with them in their transverse space. Another way to arrive at this is by starting from the ten-dimensional gauge theory from which we can reduce into four-dimensional gauge theory by T-duality [32]: a Wilson loop in ten-dimensions is a worldsheet disc bounded by the loop, so all boundary conditions we have are Dirichlet and without the loop the string endpoints are free to move on the space-filling D9-branes. When going to the four-dimensional gauge theory, with the loop, we perform T-duality along six directions turning Dirichlet conditions into Neumann's.

Now, let us follow [32] and see how the $1 / 2$ BPS condition of the Wilson loop we imposed in the field theory side is translated into the gravity side. Let us consider $\operatorname{AdS} S_{5} \times$ $S^{5}$,

$$
\begin{equation*}
\frac{d s^{2}}{L^{2}}=G_{\mu \nu} d X^{\mu} d X^{\nu}+G_{i j} d X^{i} d X^{j}=\frac{1}{Y^{2}}\left(\sum_{\mu=0}^{3} d X^{\mu} d X^{\mu}+\sum_{i=4}^{9} d Y^{i} d Y^{i}\right) \tag{4.14}
\end{equation*}
$$

The Hamilton-Jacobi equation for the minimal surface on a Riemannian manifold, i.e our worldsheet, is ${ }^{3}$ [32, 104]

$$
\begin{equation*}
G^{I J}\left(\frac{\delta S}{\delta Z^{I}}\right)\left(\frac{\delta S}{\delta Z^{J}}\right)=G_{I J} \partial_{1} Z^{I} \partial_{1} Z^{J} \tag{4.16}
\end{equation*}
$$

[^13]\[

$$
\begin{equation*}
h_{a b} P_{I}^{a} P_{J}^{b} G^{I J}=0 . \tag{4.15}
\end{equation*}
$$

\]

In the case of interest at $Y=0$ in $A d S_{5} \times S^{5}$, this becomes (3.8)

$$
\begin{equation*}
\dot{x}^{2}-\dot{y}^{2}=0, \tag{4.17}
\end{equation*}
$$

at the boundary, which means that the minimal surface, which is unique [33], ending at the boundary of $\operatorname{Ad} S_{5} \times S^{5}$ requires $\dot{x}^{2}=\dot{y}^{2}$, which is nothing else than the $1 / 2$ BPS condition of the Wilson loop.

Moreover, this condition allows us to reinterpret the six Neumann boundary conditions give above as Dirichlet ones [32]. By redefinition

$$
\begin{equation*}
\theta^{i}=\frac{\dot{y}^{i}}{|\dot{y}|}, \quad \sum_{i} \theta^{i} \theta^{i}=1 \tag{4.18}
\end{equation*}
$$

So $\left\{\theta^{i}\right\}$ are coordinates of the $S^{5}$, which are fixed. Hence, the supersymmetric Wilson loop lies at the boundary of $A d S_{5}$ and at a fixed point on $S^{5}$, and it is precisely its supersymmetry condition that is translated into the minimal area condition.

### 4.2 Legendre transform and the elimination of the divergence

We saw above that the area of the worldsheet with boundary in $\mathcal{C}$ is infinite because the string stretches from the boundary of $\operatorname{AdS} S_{5} \times S^{5}$. The second term in 4.7) contains the divergent part when $M \rightarrow \infty$, so we took it off by hand. We can understand this cancellation by another method. The Nambu-Goto action (4.3) depends on $X^{\mu}$ and $Y^{i}$, and it would be nice to define the area if those coordinates satisfy full Dirichlet boundary conditions. Since this is not the case, we perform a Legendre transform to put the action in terms of $X^{\mu}$ and $P^{i}$, which, as bounded dynamical variables, can be thought as having Dirichlet boundary conditions [32,33] (also see [29]). Let us vary a general string action with $Y^{i}$ to get

$$
\begin{equation*}
\delta S=\int d \tau d \sigma \partial_{a}\left(\frac{\partial \mathcal{L}}{\partial \partial_{a} Y^{i}}\right)=\oint d \tau\left[P_{i}^{\sigma} \delta Y^{i}\right]_{\sigma=\sigma_{0}}^{\sigma=\sigma_{1}} \tag{4.19}
\end{equation*}
$$

which results to be a functional of $Y^{i}$. Let us define

$$
\begin{equation*}
\tilde{S}=S-\oint d \tau\left[P_{i}^{\sigma} Y^{i}\right]_{\sigma=\sigma_{0}}^{\sigma=\sigma_{1}} \tag{4.20}
\end{equation*}
$$

such that

$$
\begin{equation*}
\delta \tilde{S}=-\oint d \tau\left[\delta P_{i}^{\sigma} Y^{i}\right]_{\sigma=\sigma_{0}}^{\sigma=\sigma_{1}} \tag{4.21}
\end{equation*}
$$

which is now a functional of the momentum $P_{i}$, not $Y^{i}$. In this case, assuming that $\sigma=\sigma_{1}$ is at the boundary,

$$
\begin{equation*}
P_{i}\left(\tau, \sigma_{0}\right)=\frac{\delta S}{\delta \partial_{\sigma} Y^{i}}=\frac{\sqrt{\lambda}}{2 \pi} \frac{\dot{y}^{i}}{y^{2}} . \tag{4.22}
\end{equation*}
$$

Then, at $y=\epsilon$, the divergent part of the Legendre transformed action is

$$
\begin{equation*}
\tilde{S}=S-\left.\oint_{\mathcal{C}} d \tau P_{i} Y^{i}\right|_{\sigma_{1}}=S-\frac{\sqrt{\lambda}}{2 \pi \epsilon} \oint_{\mathcal{C}} d \tau|\dot{y}| . \tag{4.23}
\end{equation*}
$$

The onshell action is divergent and proportional to the length of the loop, so

$$
\begin{equation*}
\tilde{S}=\frac{\sqrt{\lambda}}{2 \pi \epsilon} \int d \tau(|\dot{x}|-|\dot{y}|) . \tag{4.24}
\end{equation*}
$$

Thus, the divergence cancels when the constrain $|\dot{x}|=|\dot{y}|$ is satisfied. These results lead us to say that the minimal surface located at the boundary of $A d S_{5} \times S^{5}$ is given by the Legendre transform of the Nambu-Goto action for the bosonic string when the supersymmetric condition is satisfied.

### 4.3 Three examples

In this section we review three holographic results for $\langle W\rangle$. Henceforth, let us consider the $A d S_{5} \times S^{5}$ space as parametrized by

$$
\begin{equation*}
d s^{2}=\frac{L^{2}}{y^{2}}\left(d y^{2}+d x_{0}^{2}+d x_{1}^{2}+d x_{2}^{2}+d x_{3}^{2}\right)+L^{2}\left(d \theta^{2}+\sin ^{2} \theta d s_{S^{4}}^{2}\right), \quad L^{2}=\sqrt{\lambda} \alpha^{\prime} \tag{4.25}
\end{equation*}
$$

## Wilson line

For the line along Euclidean time, we can write the parametrization $x^{\mu}=(\tau, 0,0,0)$ and $y=\sigma$. The Nambu-Goto action reduces to

$$
\begin{equation*}
S=\frac{\sqrt{\lambda}}{2 \pi} \int_{-\infty}^{+\infty} d \tau \int_{\epsilon}^{+\infty} d \sigma \frac{1}{\sigma^{2}}=\frac{\sqrt{\lambda}}{2 \pi \epsilon} \int d \tau|\dot{x}| \quad \rightarrow M=\frac{\sqrt{\lambda}}{2 \pi \epsilon} . \tag{4.26}
\end{equation*}
$$

Notice that $M \rightarrow \infty$ as $\epsilon \rightarrow 0$. The regularized minimal action can be calculated by

$$
\begin{equation*}
\tilde{S}=\frac{\sqrt{\lambda}}{2 \pi \epsilon} \int d \tau(|\dot{x}|-|\dot{y}|)=0 \tag{4.27}
\end{equation*}
$$

Thus the expectation value of the Wilson loop is then

$$
\begin{equation*}
\langle W[C]\rangle=1, \tag{4.28}
\end{equation*}
$$

which coincides with the field theory result in (3.20) and is also valid at strong coupling.

## Circular loop

The circular case was computed in [49]. We can calculate the minimal surface by using two method: (a) solving the equation of motion and, (b) making use of conformal symmetry. We choose that the worldsheet lives at a fixed point on $S^{5}$, so the $\theta^{i}$,s are constants.

Let us see review the first method. We need a surface that ends on a circle of radius $a$ at $Y=0$, so can write the ansatz for the $\operatorname{Ad} S_{5}$ metric [36,49]

$$
\begin{align*}
X^{\mu} & =(R(\sigma) \cos \tau, R(\sigma) \sin \tau, 0,0), \quad 0 \leq \tau \leq 2 \pi \\
Y & =\sigma \tag{4.29}
\end{align*}
$$

The Nambu-Goto action becomes

$$
\begin{equation*}
S=\frac{\sqrt{\lambda}}{2 \pi} \int d \tau d \sigma \frac{1}{\sigma^{2}} \sqrt{R^{2}\left(R^{\prime 2}+1\right)} \tag{4.30}
\end{equation*}
$$

The corresponding equation of motion solves as

$$
\begin{equation*}
R(\sigma)=\sqrt{a^{2}-\sigma^{2}} \tag{4.31}
\end{equation*}
$$

which goes to zero at $\sigma=0$. We can find this as solution of the quadratic equation for the hemisphere

$$
\begin{equation*}
x_{1}^{2}+x_{2}^{2}+y^{2}=a^{2}, \quad x_{1}^{2}+x_{2}^{2}=\left(\sqrt{a^{2}-y^{2}}\right)^{2} . \tag{4.32}
\end{equation*}
$$

Back to the action,

$$
\begin{equation*}
S=\sqrt{\lambda} \int_{\epsilon}^{a} d \sigma \frac{a}{\sigma^{2}}=\sqrt{\lambda}\left(\frac{a}{\epsilon}-1\right) \tag{4.33}
\end{equation*}
$$

which clearly has a divergent part. The regularization term is

$$
\begin{equation*}
\frac{1}{2 \pi \epsilon} \oint_{\mathcal{C}} d \tau|\dot{x}|=\sqrt{\lambda} \frac{a}{\epsilon} \tag{4.34}
\end{equation*}
$$

Thus, $\tilde{S}=-\sqrt{\lambda}$, which is independent of the radius $a$ of the loop, as required by conformal invariance. Finally,

$$
\begin{equation*}
\langle W\rangle=e^{\sqrt{\lambda}} \tag{4.35}
\end{equation*}
$$

which coincides with our last result in (3.29).
Now let us analyze the result by using conformal symmetry in the Euclidean case. The parametrization of the line at the boundary, $y=0$ is $x=(\tau, 0,0,0)$, where $-\infty<\tau<$ $\infty$, then the transformation

$$
\begin{equation*}
\tilde{x}_{i}=\frac{x_{i}+b_{i} x^{2}}{1+2 b \cdot x+b^{2} x^{2}} \tag{4.36}
\end{equation*}
$$

where $b_{i}=(0, b, 0,0)$ is a constant vector in $\mathbb{R}^{4}$, maps the line into a circle with finite radius $a^{2} \rightarrow 1 / b^{2}$ (when $\tau \rightarrow \infty$ ). So $\left(\tilde{x}_{1}\right)^{2}+\left(\tilde{x}_{2}\right)^{2}=a^{2}$. If we want to move this circle into the bulk, we can do

$$
\begin{equation*}
\left(\tilde{x}_{1}\right)^{2}+\left(\tilde{x}_{2}\right)^{2}=a^{2}-y^{2}, \tag{4.37}
\end{equation*}
$$

which leads to the same parametrization in $\sqrt{4.29}$ with $R(y)=\sqrt{a^{2}-y^{2}}$. So, we wrote the same solution for the minimal surface without solving the equation of motion.

Here we saw that it is possible to map the infinite line into a circle. But, again, why does $\langle W$ (circle) $\rangle \neq\langle W$ (line $)\rangle$ ? Since expectations values are different for conformally equivalent paths, the conformal invariance has been violated. This violation comes from the fact that we have regularized the area.

The fact that for the Wilson line $\langle W\rangle=1$ independently of $\lambda$ tells us that it is protected by supersymmetry as a global BPS object [28]. In the gravity side, even though the conformal symmetry is broken, the result is as expected. It means that the cutoff did not break superconformal symmetry and the BPS nature is preserved. Now, since in the circular case the result is different as we saw also in the field theory, we say that the cutoff broke conformal invariance leading to a conformal anomaly which contributed to the $\lambda$ dependent expectation value. So quantum corrections are possible for the circular Wilson loop [29].

## Rectangular loop: quark-antiquark potential

Let us consider two parallel Wilson lines. This was studied by Maldacena [30] and by Rey and Yee [31]. In this case, let us consider a string connecting two lines, whose endpoints
will describe a "quark" and an "antiquark" in the field theory. The string state can be considered as a infinitely massive W-boson.

We choose the lines to point in the $x_{0}$ (Euclidean "time") direction, and we put the endpoints in the $x_{1}$ direction, at $-L / 2$ and $+L / 2$, so they are separated by a distance $L$. The lines are then parametrized by

$$
\begin{align*}
& x_{1}^{\mu}=(+\tau,+L / 2,0,0), \\
& x_{2}^{\mu}=(-\tau,-L / 2,0,0), \tag{4.38}
\end{align*}
$$

with $\tau \in[-T / 2,+T / 2]$. So we are interested in the minimal area spanned by the string worldsheet attached to a rectangular loop at the boundary of $A d S_{5}$ and fixed at a point in $S^{5}$.

The $A d S_{5} \times S^{5}$ metric can be written as

$$
\begin{equation*}
d s^{2}=\alpha^{\prime}\left[\frac{U^{2}}{L^{2} / \alpha^{\prime}}\left(d X_{0}^{2}+d X_{1}^{2}+d X_{2}^{2}+d X_{3}^{2}\right)+\frac{L^{2}}{\alpha^{\prime}} \frac{d U^{2}}{U^{2}}+\frac{L^{2}}{\alpha^{\prime}} d \Omega_{5}^{2}\right] \tag{4.39}
\end{equation*}
$$

as we did in subsection 2.4.1. The boundary, in this case, is at $U \rightarrow \infty$. In $A d S_{5} \times S^{5}$ we can set

$$
\begin{align*}
X^{\mu} & =(\tau, \sigma, 0,0)  \tag{4.40}\\
U & =U(\sigma) \tag{4.41}
\end{align*}
$$

with $\tau \in[-T / 2,+T / 2]$ and $\sigma \in[-L / 2,+L / 2]$. At the boundary,

$$
\begin{equation*}
U( \pm L / 2) \rightarrow \infty \tag{4.42}
\end{equation*}
$$

In the Nambu-Goto action all this becomes

$$
\begin{equation*}
S_{N G}=\frac{1}{2 \pi} T \int_{-L / 2}^{+L / 2} d \sigma \sqrt{\left(\partial_{\sigma} U\right)^{2}+U^{4} / \tilde{L}^{4}}, \quad \tilde{L}^{2}=L^{2} / \alpha^{\prime}=\sqrt{\lambda} . \tag{4.43}
\end{equation*}
$$

Since the action does not depend explicitly on $\sigma$, there is a constant of motion:

$$
\begin{equation*}
\frac{U^{4} / \tilde{L}^{4}}{\sqrt{\left(\partial_{\sigma} U\right)^{2}+U^{4} / \tilde{L}^{4}}}=\text { constant } \tag{4.44}
\end{equation*}
$$

Defining $U_{0}$ to be the minimum value of $U$, which by symmetry occurs at $\sigma=0$, we can check that

$$
\begin{equation*}
\sigma=\frac{\tilde{L}^{2}}{U_{0}} \int_{1}^{U / U_{0}} \frac{d y}{y^{2} \sqrt{y^{4}-1}}, \quad y=U / U_{0} \tag{4.45}
\end{equation*}
$$

which gives $\sigma=\sigma\left(U, U_{0}\right)$. Then, by inverting, $U=U\left(\sigma, U_{0}\right) . U_{0}$ can be determined by remembering that [30] (see also [7]),

$$
\begin{equation*}
\frac{L}{2}=\frac{\tilde{L}^{2}}{U_{0}} \int_{1}^{\infty} \frac{d y}{y^{2} \sqrt{y^{4}-1}}=\frac{\tilde{R}^{2}}{U_{0}} \frac{\sqrt{2} \pi^{3 / 2}}{\Gamma(1 / 4)^{2}} . \tag{4.46}
\end{equation*}
$$

From (4.45),

$$
\begin{equation*}
d \sigma=\tilde{L}^{2} U_{0}^{2} \frac{d U}{U^{2} \sqrt{U^{4}-U_{0}^{4}}} \Rightarrow\left(\frac{d U}{d \sigma}\right)^{2}=\frac{U^{4}}{\tilde{L}^{4} U_{0}^{4}}\left(U^{4}-U_{0}^{4}\right) . \tag{4.47}
\end{equation*}
$$

So, back to the action, this becomes

$$
\begin{equation*}
S_{N G}=2 \frac{T}{2 \pi} \int_{U_{0}}^{\infty} d U \frac{U^{2}}{\sqrt{U^{4}-U_{0}^{4}}}=\frac{T U_{0}}{\pi} \int_{1}^{\infty} d y \frac{y^{2}}{\sqrt{y^{4}-1}} . \tag{4.48}
\end{equation*}
$$

This integral does not converge, so we must consider a cutoff:

$$
\begin{equation*}
S_{N G}=\lim _{y_{\max } \rightarrow \infty} \frac{T U_{0}}{\pi} \int_{1}^{y_{\max }} d y \frac{y^{2}}{\sqrt{y^{4}-1}}=\frac{T U_{0}}{\pi}\left(\frac{U_{\max }}{U_{0}}-\frac{\sqrt{\pi} \Gamma(3 / 4)}{\Gamma(1 / 4)}\right) . \tag{4.49}
\end{equation*}
$$

We have isolated the divergence,

$$
\begin{equation*}
\frac{2 T U_{\max }}{2 \pi} \tag{4.50}
\end{equation*}
$$

in which we recognize the length of the loop, $2 T$ and the "W-boson" mass, $U_{\max } / 2 \pi$. We, them simply substract this term and write the finite regularized action as

$$
\begin{equation*}
\tilde{S}_{N G}=-T \frac{U_{0} \Gamma(3 / 4)}{\sqrt{\pi} \Gamma(1 / 4)}=-T \frac{4 \pi^{2} \sqrt{\lambda}}{\Gamma(1 / 4)^{4}} \frac{1}{L}, \tag{4.51}
\end{equation*}
$$

where we used (4.46), the definition of $U_{0}$. We see that the action has the form $\tilde{S}_{N G}=$ $-T V(L)$, where $V(L) \propto 1 / L$ due to conformal invariance, as expected. An important detail is that the latter results shows that the potential goes as $\sqrt{\lambda}$, i.e. $\tilde{S}_{N G} \propto-T \sqrt{\lambda} / L$, i.e. a polynomial in $\lambda$, as opposed to $S \propto-T \lambda / L$ which is a perturbative result (see, for example, [27, 30]). This indicates some screening of the charges, as mentioned in [30].

Until now we have reviewed how Wilson loops are useful to check the AdS/CFT correspondence, but we have focused only on the case of the fundamental representation of $U(N)$, for which we only need to consider the minimal worldsheet surface attached to that loop. We will see in the next chapter how this idea is extended to higher representations, and how the string dynamics is not very useful to describe higher representations, i.e. lots of strings or, equivalently, lots of "quarks".

## Chapter 5

## Branes as Wilson loops

We learned that on the worldvolume of $N$ coincident D3 branes lives a $U(N)$ (or $S U(N)$, see [3]) $\mathcal{N}=4$ gauge theory. A Wilson loop in the fundamental representation can be constructed from $U(N+1)$ and breaking it into $U(1) \times U(N)$ by choosing a nonzero (large) vacuum expectation for the scalar fields, which in turn fixes a point on the unit $S^{5}$. A $U(1)$ vector field appears which "eats" the scalar by Higgs mechanism to become (very) massive, and transforms in the fundamental representation of $U(N)$ [30]. Then, a Wilson loop represents the phase along a path of the corresponding massive quark in the fundamental representation of $U(N)$, which is produced by decaying of the vector field, and it was defined in (3.3). In the gravity side, we saw that massive quarks and the Wilson loop itself can be obtained by considering $N+1$ D3-branes and separating one of them, a single string links the "probe" brane to the other $N$ branes. On the worldvolume of the $N$ branes, a vector and scalar field can be defined since the endpoints of the string carry Chan-Paton factors and pull the branes towards the probe. The correspondence says that at large $N$ and large $\lambda$, the field theory results when we "measure" the Wilson loop must coincide with the evaluation of the minimal area described by the worldsheet of a single string, which in this case stretches from the boundary of the $\operatorname{AdS} S_{5} \times S^{5}$ space to some finite position. The Gaussian matrix model helped to calculate the expectation value of the Wilson loop in the field theory side and the Nambu-Goto action, which defines the area of the worldsheet, allowed to compute this expectation value in the gravity side for the case of the linear and the circular paths.

We could extend these results and expect that the correspondence works when we
consider "more quarks" in the fundamental representation of $U(N)$ (or $S U(N)$ ), arranged into a tensor product, which translates into having more strings stretching between the $N$ D3 branes and the probe D3 brane. Let us consider $n$ quarks in the fundamental representation of $U(N)$,

$$
\begin{equation*}
\underbrace{\square \otimes \square \otimes \cdots \otimes \square}_{n}=\sum_{i} \mathcal{R}_{i} . \tag{5.1}
\end{equation*}
$$

which can be also seen as a single composed "quark". These tensor product forms a reducible representation, which can be reduced into irreducible representations characterized by a partition $\mathcal{R}_{i}=\left\{n_{1}, n_{2}, \cdots, n_{N}\right\}$, where $n_{1} \geq n_{2} \geq \cdots \geq n_{N}$. and $n=n_{1}+n_{2}+\cdots+n_{N}$. As a diagram, the partition can be expressed by a Young tableau,

where we put $n_{1}$ boxes (quarks) in the first row, $n_{2}$ boxes in the second row and so on. For a given irreducible representation $\mathcal{R}$ of $U(N)$, we write the Wilson loop as we did in (3.11) [43]

$$
\begin{equation*}
\mathcal{W}_{\mathcal{R}}[\mathcal{C}]:=\frac{1}{\operatorname{dim} \mathcal{R}} \operatorname{Tr}_{\mathcal{R}} \mathcal{P} \exp \left[\oint_{\mathcal{C}} d \tau\left(i A_{\mu}(\tau) \dot{x}^{\mu}+\Phi^{i}(\tau) \dot{y}^{i}\right)\right] \tag{5.3}
\end{equation*}
$$

In this chapter we will focus on two cases: the $k$-symmetric and the $k$-antisymmetric representations,

which correspond to a single row and a single column with $k$ boxes, respectively.

### 5.1 Wilson loops in $k$-symmetric and $k$-antisymmetric representations

In principle, to compute the expectation value of the Wilson loop we must calculate the path integral

$$
\begin{equation*}
\left\langle W_{\mathcal{R}}\right\rangle=\frac{\int \mathcal{D} X e^{-S[X]} W_{\mathcal{R}}}{\int \mathcal{D} X e^{-S[X]}} \tag{5.5}
\end{equation*}
$$

where $X$ denotes the fields of $U(N)$ SYM theory. As we saw before, Erickson, Semenoff and Zarembo [27], Drukker and Gross [28] proposed and proved, respectively, that the expectation value of the Wilson loop in fundamental representation can be exactly calculated by summing ladder diagrams, and that this can be represented by a matrix model. The matrix model was proved to be Gaussian by Pestun [39].

Since the matrix model is independent of the representation of $U(N)$, we consider

$$
\begin{equation*}
Z=\int \mathcal{D} M \exp \left(-\frac{2 N}{\lambda} \operatorname{Tr} M^{2}\right) \tag{5.6}
\end{equation*}
$$

where $M$ is a $N \times N$ Hermitian matrix which transforms in the $\mathcal{R}$ representation of $U(N)$ (see [43|45], two important references). In the eigenvalue basis, $M=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \cdots, \lambda_{N}\right)$, the partition function is 3.55

$$
\begin{equation*}
Z=\int \prod_{i=1}^{N} d \lambda_{i} \prod_{i<j}\left(\lambda_{i}-\lambda_{j}\right)^{2} \exp \left(-\frac{2 N}{\lambda} \sum_{i} \lambda_{i}^{2}\right) . \tag{5.7}
\end{equation*}
$$

The expectation value of the Wilson loop is (see (3.42))

$$
\begin{equation*}
\left\langle W_{\mathcal{R}}\right\rangle=\frac{1}{\operatorname{dim\mathcal {R}}}\left\langle\operatorname{Tr} e^{M}\right\rangle \tag{5.8}
\end{equation*}
$$

where we replace $\operatorname{dim} \square=N \rightarrow \operatorname{dim} \mathcal{R}$, in general. After diagonalization we can write

$$
\begin{equation*}
\frac{1}{\operatorname{dim} \mathcal{R}}\left\langle\operatorname{Tr} e^{M}\right\rangle=\frac{1}{\operatorname{dim} \mathcal{R}}\left\langle S_{\mathcal{R}}\left(e^{\lambda_{1}}, e^{\lambda_{2}}, \cdots, e^{\lambda_{N}}\right)\right\rangle, \tag{5.9}
\end{equation*}
$$

where $S_{\mathcal{R}}\left(e^{\lambda_{1}}, e^{\lambda_{2}}, \cdots, e^{\lambda_{N}}\right)$ is the Schur polynomial associated to the representation $\mathcal{R}$ [24] (see [105] for details). Thus,

$$
\begin{equation*}
\left\langle W_{\mathcal{R}}\right\rangle=\frac{1}{\operatorname{dim} \mathcal{R}} \frac{1}{Z} \int \prod_{i=1}^{N} d \lambda_{i} \prod_{i<j}\left(\lambda_{i}-\lambda_{j}\right)^{2} S_{\mathcal{R}}\left(e^{\lambda_{1}}, e^{\lambda_{2}}, \cdots, e^{\lambda_{N}}\right) \exp \left(-\frac{2 N}{\lambda} \sum_{i} \lambda_{i}^{2}\right) \tag{5.10}
\end{equation*}
$$

For the symmetric and antisymmetric cases (5.4) with $k$-boxes, the Schur polynomials are the following [106, 107]:

- $k$-symmetric

$$
\begin{equation*}
S_{S_{k}}\left(e^{\lambda_{1}}, e^{\lambda_{2}}, \cdots, e^{\lambda_{N}}\right)=h_{k}\left(e^{\lambda_{1}}, e^{\lambda_{2}}, \cdots, e^{\lambda_{N}}\right), \tag{5.11}
\end{equation*}
$$

where $h_{k}$ is the $k^{\text {th }}$ complete symmetric polynomial, the sum of all monomials of total degree $k$ in $\left\{\exp \lambda_{i}\right\}$,

$$
\begin{equation*}
h_{k}=\sum_{i=k} \exp \lambda_{i} . \tag{5.12}
\end{equation*}
$$

The generating function for $h_{k}$ is

$$
\begin{equation*}
H(t)=\sum_{k \geq 0} h_{k} t^{k}=\prod_{i=1}^{N} \frac{1}{\left(1-t e^{\lambda_{i}}\right)}=\frac{1}{\operatorname{det}\left(1-t e^{M}\right)}=F_{S}(t) \tag{5.13}
\end{equation*}
$$

which can be inverted as

$$
\begin{equation*}
h_{k}=\frac{1}{2 \pi i} \oint d t \frac{F_{S}(t)}{t^{k+1}}, \tag{5.14}
\end{equation*}
$$

where $t \in \mathbb{C}$.

- $k$-antisymmetric

$$
\begin{equation*}
S_{A_{k}}\left(e^{\lambda_{1}}, e^{\lambda_{2}}, \cdots, e^{\lambda_{N}}\right)=e_{k}\left(e^{\lambda_{1}}, e^{\lambda_{2}}, \cdots, e^{\lambda_{N}}\right), \tag{5.15}
\end{equation*}
$$

where $e_{k}$ is the $k^{\text {th }}$ elementary symmetric polynomial, the sum of all products of $k$ distinct variables $\exp \lambda_{i}$,

$$
\begin{equation*}
e_{k}=\sum_{1 \leq i_{1}<i_{2}<\cdots<i_{k} \leq N} \exp \left(\lambda_{i_{1}}+\lambda_{i_{2}}+\cdots+\lambda_{i_{k}}\right) . \tag{5.16}
\end{equation*}
$$

The generating function for $e_{k}$ is

$$
\begin{equation*}
E(t)=\sum_{k \geq 0} e_{k} t^{k}=\prod_{i=1}^{N}\left(1+t e^{\lambda_{i}}\right) . \tag{5.17}
\end{equation*}
$$

We can redefine $E(t)$ as [45]

$$
\begin{equation*}
F_{A}(t)=t^{N} E\left(\frac{1}{t}\right)=\sum_{k \geq 0} e_{k} t^{N-k}=\prod_{i=1}^{N}\left(t+e^{\lambda_{i}}\right)=\operatorname{det}\left(t+e^{M}\right), \tag{5.18}
\end{equation*}
$$

which can be inverted as

$$
\begin{equation*}
e_{k}=\frac{1}{2 \pi i} \oint d t \frac{F_{A}(t)}{t^{N-k+1}} . \tag{5.19}
\end{equation*}
$$

The strategy in [43] was to compute the expectation value of the generating function instead of the Wilson loop's

$$
\begin{equation*}
\left\langle F_{A, S}(t)\right\rangle=\frac{1}{Z} \int \prod_{i=1}^{N} d \lambda_{i} \prod_{1 \leq i<j \leq N}\left(\lambda_{i}-\lambda_{j}\right)^{2} F_{S, A}(t) \exp \left(-\frac{2 N}{\lambda} \sum_{i} \lambda_{i}^{2}\right) . \tag{5.20}
\end{equation*}
$$

The expectation value of the Wilson loop can be then calculated by

$$
\left\langle W_{S, A}\right\rangle=\frac{1}{\operatorname{dim}[S, A]} \oint \frac{d t}{2 \pi i}\left\langle F_{S, A}(t)\right\rangle \times \begin{cases}\frac{1}{t^{k+1}} & \text { for } S_{S}  \tag{5.21}\\ \frac{1}{t^{N-k+1}} & \text { for } S_{A}\end{cases}
$$

The dimensions of the rank $k$ symmetric and antisymmetric representations are, respectively,

$$
\begin{equation*}
\operatorname{dim}[S]=\frac{(N+k-1)!}{k!(N-1)!}, \quad \operatorname{dim}[A]=\frac{N!}{k!(N-k)!} . \tag{5.22}
\end{equation*}
$$

The path integral can be rewritten as

$$
\begin{equation*}
\left\langle F_{S, A}(t)\right\rangle=\frac{1}{Z} \int \prod_{i=1}^{N} d \lambda_{i} \exp \left(-S_{S, A}\left[\left\{\lambda_{i}\right\}\right]\right), \tag{5.23}
\end{equation*}
$$

where

$$
S_{S, A}=\frac{2 N}{\lambda} \sum_{i} \lambda_{i}^{2}-\sum_{i<j} \log \left(\lambda_{i}-\lambda_{j}\right)^{2}+\sum_{i=1}^{N}\left\{\begin{array}{ll}
+\log \left(1-t e^{\lambda_{i}}\right) & \text { for } S_{S}  \tag{5.24}\\
-\log \left(t+e^{\lambda_{i}}\right) & \text { for } S_{A}
\end{array} .\right.
$$

The first two terms are of order $N^{2}$ and the last one is of order $N$, so at large $N$ the first two terms will dominate. So the expectation value for the generating functions will be

$$
\begin{equation*}
\left\langle F_{S}(t)\right\rangle=\left\langle\exp \left(\operatorname{Tr} \log \left[t+e^{M}\right]\right)\right\rangle \rightarrow \exp \left(-N \int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} d x \rho(x) \log \left(1-t e^{x}\right)\right) \tag{5.25}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle F_{A}(t)\right\rangle=\left\langle\exp \left(-\operatorname{Tr} \log \left[1-t e^{M}\right]\right)\right\rangle \rightarrow \exp \left(N \int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} d x \rho(x) \log \left(t+e^{x}\right)\right) \tag{5.26}
\end{equation*}
$$

where $\rho(x)$ was defined in (3.51) in subsection 3.2.1.
At large $N$, the circular Wilson loop is given by (5.8), where

- $k$-symmetric

$$
\begin{equation*}
\left\langle\operatorname{Tr}_{S_{k}} e^{M}\right\rangle=\oint \frac{d t}{2 \pi i} \frac{1}{t^{k+1}} \exp \left(-N \int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} d x \rho(x) \log \left(1-t e^{x}\right)\right) . \tag{5.27}
\end{equation*}
$$

- $k$-antisymmetric

$$
\begin{equation*}
\left\langle\operatorname{Tr}_{A_{k}} e^{M}\right\rangle=\oint \frac{d t}{2 \pi i} \frac{1}{t^{k+1}} \exp \left(N \int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} d x \rho(x) \log \left(1+t e^{x}\right)\right) \tag{5.28}
\end{equation*}
$$

where we absorbed the factor $t^{-N}$ into the exponent and changed $t \rightarrow 1 / t$.
Therefore, we can write both results as

$$
\begin{equation*}
\left\langle\operatorname{Tr}_{S_{k}, A_{k}} e^{M}\right\rangle=\oint \frac{d t}{2 \pi i} \frac{1}{t^{k+1}} \exp \left(\mp N \int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} d x \rho(x) \log \left(1 \mp t e^{x}\right)\right) \tag{5.29}
\end{equation*}
$$

where the - sign corresponds to the symmetric representation, and the + one to the antisymmetric. Notice that this is the saddle point approximation we reviewed in subsection 3.2.1.

### 5.1.1 The large $N \sim k$ limit

We have already considered the large $N$ limit, which reduced (5.10) to computing residues in (5.29). It is interesting to take also the limit when the number of boxes, the number of "quarks", is large. Let us define,

$$
\begin{equation*}
f=\frac{k}{N} \tag{5.30}
\end{equation*}
$$

which is kept fixed, and change

$$
\begin{equation*}
t=e^{\sqrt{\lambda} z} \tag{5.31}
\end{equation*}
$$

to write (5.29) as

$$
\begin{equation*}
\left\langle\operatorname{Tr}_{S_{k}, A_{k}} e^{M}\right\rangle=\frac{\sqrt{\lambda}}{2 \pi i} \int_{C} d z \exp \left[\mp N\left(\frac{2}{\pi} \int_{-1}^{+1} d x \sqrt{1-x^{2}} \log \left(1 \mp e^{-\sqrt{\lambda}(x-z)}\right) \pm f \sqrt{\lambda} z\right)\right] \tag{5.32}
\end{equation*}
$$

Now $t=0$ is mapped to $z \rightarrow-\infty$, and $t \rightarrow \infty$ to $z \rightarrow \infty$.
In the large $N$ limit, the integral is dominated by the saddle point, whose equation is

$$
\begin{equation*}
\frac{2}{\pi} \int_{-1}^{+1} d x \frac{\sqrt{1-x^{2}}}{1 \mp e^{\sqrt{\lambda}(x-z)}} \pm f=0 \tag{5.33}
\end{equation*}
$$

which is an equation in $z$. In order to compare with the supergravity result, we need to take $\lambda \rightarrow \infty$.

Let us review the corresponding results for the symmetric and antisymmetric cases. At large $N$, 5.32) can be evaluated using the saddle point method when we choose the upper sign (see [43, 108] and also [41])

$$
\begin{equation*}
\left\langle W_{S}\right\rangle=\exp \left[2 N\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right)\right], \quad \kappa=\frac{k \sqrt{\lambda}}{4 N} . \tag{5.34}
\end{equation*}
$$

which is the expectation value of a Wilson loop in the symmetric representation.
It was computed also in [43, 108] that, in the antisymmetric representation, the Wilson loop is

$$
\begin{equation*}
\left\langle W_{A}\right\rangle=\exp \left(\frac{2 N}{3 \pi} \sqrt{\lambda} \sin ^{3} \theta_{k}\right) \tag{5.35}
\end{equation*}
$$

where $\theta_{k}$ satisfies

$$
\begin{equation*}
\pi f=\theta_{k}-\frac{1}{2} \sin 2 \theta_{k} . \tag{5.36}
\end{equation*}
$$

$1 / N$ corrections were studied in [66, 109, 110].
Let us see next how these results appear in the corresponding gravity context.

### 5.2 D-branes and holography

We saw that the expectation value of a Wilson loops in the fundamental representation of $U(N)$ (or $S U(N)$ ) corresponds to calculating the minimal area of the worldsheet of an open string ending on a D3 brane probing $N$ D3 branes. For higher representations, one could think that the expectation value of the Wilson loop should correspond to having more (fundamental) strings attached to the boundary of $A d S$. But calculating the minimal area of coincident worldsheets is a daunting task due to the complicated geometry.

It was found, for first time, in [111] that a bunch of fundamental strings ending on branes are described better by another brane! This was later confirmed in [37, 38, 50, 65]. To be precise, let us consider $k$ fundamental strings. Calculating the minimal area would involve the evaluation of the string action containing string corrections since the $k$ worldsheets could interact and develop conical singularities and branch cuts. A horrible collective worldsheet! Each string carries a unit of "electric flux" which is communicated to the brane through the $B$-field (see [86, 112] for the basics). Now, $k$ fundamental strings will insert $k$ units of electric field to the probe D3 brane. Following [111], we can consider
instead a "string-equivalent" $D$-brane carrying $k$ units of electric field on its worldvolume. The dynamics of this brane is given by the Dirac-Born-Infeld action.

In [37, 38] it was proved that $1 / 2$ BPS Wilson loops of $\mathcal{N}=4$ SYM in symmetric and antisymmetric representations of $S U(N)$ can be described by D3 and D5 branes, respectively, carrying $k$ units of fundamental string charge, which in their worldvolume action means to insert a Chern-Simons term.

As we know, a circular or straight line Wilson operator breaks one-half of the superconformal symmetries of $\mathcal{N}=4$ SYM. Just for recalling: in order for the Wilson loop to be supersymmetric, each point in the loop must preserves supersymmetry, it leads to the condition $\ddot{x}=0$. This is satisfied by the straight line loop and, by conformal transformation, by the circular loop. The line, or the circle, breaks the $S O(4,2) \times S O(6)$ (bosonic) isometry of $\mathcal{N}=4$ into $S O(2,1) \times S O(3) \times S O(5)$ : conformal transformations along the line and rotations. For higher representations, since, in spacetime, the loops is the same as in the fundamental case, the symmetries that must be preserved are still $S O(2,1) \times S O(3) \times S O(5)$ or $S U(1,1) \times S U(2) \times S O(5)$.

### 5.2.1 $\quad \mathrm{D} 5_{k}$-brane and the $k$-antisymmetric representation

Let us consider a D5 brane probing $N$ D3 branes in flat space. The configuration in flat space can be expressed as (see [37, 38] and [86])

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| D3 | X | X | X | X |  |  |  |  |  |  |
| D5 | X |  |  |  |  | X | X | X | X | X |

The effect of the D5 brane on the four-dimensional worldvolume theory of $N \mathrm{D} 3$ branes is to insert a codimension three defect. This perturbations correspond to the $(3,5)$ and $(5,3)$ string states, i.e. strings connecting the D5 with the D3. These new degrees of freedom are localized in the defect. The $(5,5)$ strings are not dynamical, but they are important to identify the D5 brane with the Wilson loop in the antisymmetric representation. As explained in [37, 38], the D5 branes inserts a defect term in the partition function of $\mathcal{N}=4$ theory that includes fermionic fields due to the endpoints of the $(3,5)$ and $(5,3)$ strings, transforming in the fundamental representation of $U(N)$. Also, since the D 5 is
charged, a Chern-Simons term must be included, which brings the electric charge of the D5 into the D3 worldvolume as well as the interaction between it and the defect fields. As better explained in the references, integrating out those defect states from the partition function has the effect of inserting into the $\mathcal{N}=4$ path integral a Wilson loop in the $k$-antisymmetric representation, for $k \leq N$. It results that the charge of the D5 we add corresponds to the number of strings endpoints or fermionic states on the D3 branes.

Here, we will follow some details of the calculations in [65] and [113] (reviewed in [41]), where it was computed and proved that indeed D5 branes can be used to compute the circular Wilson loop in the antisymmetric representation. Let us consider the $\operatorname{Ad} S_{5} \times$ $S^{5}$ space as a fibration of two-dimensional space with $A d S_{2} \times S^{2} \times S^{4}$ fiber, in which the $S U(1,1) \times S U(2) \times S O(5)$ isometry is explicit.

$$
\begin{equation*}
d s^{2}=\cosh ^{2} u d s_{A d S_{2}}^{2}+d u^{2}+\sinh ^{2} u d s_{S^{2}}^{2}+d \theta^{2}+\sin ^{2} \theta d \Omega_{4}^{2}, 0 \leq u, 0 \leq \theta \leq \pi \tag{5.37}
\end{equation*}
$$

with

$$
\begin{equation*}
d s_{A d S_{2}}^{2}=d \xi^{2}+\sinh ^{2} \xi d \psi^{2}, \quad d s_{S^{2}}^{2}=d \vartheta^{2}+\sin ^{2} \vartheta d \phi^{2}, \tag{5.38}
\end{equation*}
$$

where the worldvolume coordinates of the D5 brane are taken to be $\left\{\xi, \psi, \theta_{1}, \cdots, \theta_{4}\right\}$ (static gauge). The Wilson loop is located at $u=0, \xi \rightarrow \infty$; so $u=u(\xi) \cdot{ }^{1}$ The action for the D5 brane in this background is given by

$$
\begin{equation*}
S_{D 5}=T_{5} \int d^{6} \xi \sqrt{\operatorname{det}(G+F)}-i T_{6} \int F \wedge C_{4} . \tag{5.39}
\end{equation*}
$$

Let us consider that the D5 brane wraps $S^{4}$, so the $\theta$ angles are fixed $\theta=\theta_{k}$. The $C_{4}$ background field is

$$
\begin{equation*}
C_{4}=4\left(\frac{u}{8}-\frac{1}{32} \sinh 4 u\right) \operatorname{vol}_{A d S_{2}} \wedge \operatorname{vol}_{S^{2}}-\left(\frac{3}{2} \theta-\sin 2 \theta+\frac{1}{8} \sin 4 \theta\right) \operatorname{vol}_{S^{4}} \tag{5.40}
\end{equation*}
$$

Taking the ansatz $u=0$ and $\theta=\theta_{k}$ constant, and $F=F_{\psi \xi} d \psi \wedge d \xi$, the action for the D5 brane is

$$
\begin{align*}
S_{D 5}= & \frac{2 N}{3 \pi} \sqrt{\lambda} \int d \xi \sinh \xi \sin ^{4} \theta \sqrt{1+\frac{4 \pi^{2}}{\lambda} \frac{F_{\psi \xi}^{2}}{\sinh ^{2} \xi}} \\
& +\frac{4 i N}{3} \int d \xi F_{\psi \xi}\left(\frac{3}{2} \theta_{k}-\sin 2 \theta_{k}+\frac{1}{8} \sin 4 \theta_{k}\right), \tag{5.41}
\end{align*}
$$

[^14]where $T_{D 5}=N \sqrt{\lambda} / 8 \pi^{2}$ and $\operatorname{vol}_{S^{4}}=8 \pi^{2} / 3$. Since the worldvolume gauge field, $A_{\xi}$, does not appear explicitly in the action, its corresponding conjugate momentum $i \Pi$ is conserved (constant and equal to the electric charge),
\[

$$
\begin{equation*}
\Pi=-i \frac{1}{2 \pi} \frac{\delta \mathcal{L}}{\delta F_{\psi \xi}}=\frac{2 N}{3 \pi} \frac{E \sin ^{4} \theta_{k}}{\sqrt{1-E^{2}}}+\frac{2 N}{3 \pi}\left(\frac{3}{2} \theta_{k}-\sin 2 \theta_{k}+\frac{1}{8} \sin 4 \theta_{k}\right)=k \tag{5.42}
\end{equation*}
$$

\]

with

$$
\begin{equation*}
E=-\frac{2 \pi i}{\sqrt{\lambda}} \frac{F_{\psi \xi}}{\sinh \xi} \tag{5.43}
\end{equation*}
$$

We set $\Pi=k$ because of the fundamental string charge goes into the brane as electric charge. If $E=\cos \theta_{k}$, then $F_{\psi \xi}=i \sqrt{\lambda} \sinh \xi \cos \theta_{k} / 2 \pi$,

$$
\begin{equation*}
\theta_{k}-\sin \theta_{k} \cos \theta_{k}=\pi \frac{k}{N} \leq \pi \quad \Rightarrow \quad k \leq N \tag{5.44}
\end{equation*}
$$

The boundary contribution which takes into account the effect of the boundary of $A d S$ can be computed [32, 65, 113]. Plugging the solution back to (5.41) and adding the boundary term ${ }^{2}$ we get

$$
\begin{equation*}
S_{D 5}+S_{\mathrm{bdy}, A}=-\frac{2 N}{3 \pi} \sqrt{\lambda} \sin ^{3} \theta_{k} \tag{5.45}
\end{equation*}
$$

The expectation value of the Wilson loop in the antisymmetric representation is

$$
\begin{equation*}
\left\langle W_{A_{k}}\right\rangle=\exp \frac{2 N}{3 \pi} \sqrt{\lambda} \sin ^{3} \theta_{k} \tag{5.46}
\end{equation*}
$$

the same result obtained by Hartnoll and Kumar we wrote in (5.35). Now, when $k \ll N$ the angle $\theta_{k}$ is small, so we can approximate $\theta_{k}^{3}=3 \pi k / 2 N$ and

$$
\begin{equation*}
\left\langle W_{A_{k}}\right\rangle=\exp \frac{2 N}{3 \pi} \sqrt{\lambda} \theta_{k}^{3}=e^{k \sqrt{\lambda}} \tag{5.47}
\end{equation*}
$$

or

$$
\begin{equation*}
\left\langle W_{A_{k}}\right\rangle \sim\left\langle W_{\square}\right\rangle^{k} . \tag{5.48}
\end{equation*}
$$

Which is the expected result, representing $k$ independent fundamental strings attached to the D3 brane.

There is also an interesting detail with our result, the $k$-antisymmetric and $(N-k)$ antisymmetric representations, where $k \leq N$, are related by the change $\theta_{k} \rightarrow \pi-\theta_{k}$, which means $F_{\psi \xi} \rightarrow-F_{\psi \xi}$, the complex conjugation of the electric field. Under this change, the Wilson loop is invariant.

[^15]
### 5.2.2 $\quad \mathbf{D} 3_{k}$-brane and the $k$-symmetric representation

It was shown also in [37,38] that the configuration for a D 3 brane probing $N \mathrm{D} 3$ branes corresponding to the $k$-symmetric representation of $U(N)$ can be obtained by bosonization of the fermionic field $\chi$ in the defect theory. This partition function inserts a Wilson loop in the $k$-symmetric representation in $\mathcal{N}=4$ SYM theory, so we expect that duality allows us to compute the expected value also in the gravity side in the D3 brane description. Actually, the derivation of the symmetric representation was re-shown in [38] by following the same method in [37] for the D5 brane. $k$, again, represents the number of string ending on the D3 stack. An important difference from the D5 case is that, in this case, the electric charge is arbitrary, so $k$ can take any value, even $k>N$.

In order to see how the D3 brane description actually works, we will follow [41,50] to calculate the circular Wilson loop in the symmetric representation. The $\operatorname{AdS} S_{5} \times S^{5}$ space can be written as

$$
\begin{equation*}
d s^{2}=\frac{1}{y^{2}}\left(d y^{2}+d r_{1}^{2}+r_{1}^{2} d \psi^{2}+d r_{2}^{2}+r_{2}^{2} d \phi^{2}\right)+d \Omega_{5}^{2} \tag{5.49}
\end{equation*}
$$

The circular loop is located at $r_{1}=a, r_{2}=0$ and $z \rightarrow 0$. In order to exhibit the $S U(1,1) \times S U(2) \times S O(5)$ symmetry, we change variables,

$$
\begin{equation*}
r_{1}=\frac{a \cos \eta}{\cosh \rho-\sinh \rho \cos \theta}, \quad r_{2}=\frac{a \sinh \eta \sin \theta}{\cosh \rho-\sinh \rho \cos \theta}, \quad y=\frac{a \sin \eta}{\cosh \rho-\sinh \rho \cos \theta} \tag{5.50}
\end{equation*}
$$

such that the metric becomes

$$
\begin{equation*}
d s^{2}=\frac{1}{\sin ^{2} \eta}\left(d \eta^{2}+\cos ^{2} \eta d \psi^{2}+d \rho^{2}+\sinh ^{2} \rho\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right)\right)+d \Omega_{5}^{2} \tag{5.51}
\end{equation*}
$$

where $0 \leq \rho, 0 \leq \theta \leq \pi$ and $0 \leq \eta \leq \pi / 2$. In this metric the Wilson loop is located at $\eta=\rho=0$. We go to static gauge in which the worldvolume coordinates are $\{\psi, \rho, \theta, \phi\}$, and the brane sits at a fixed point $\left\{\Theta^{I}\right\}$ on $S^{5}$. The remaining coordinate, $\eta$, will be seen as a worldvolume scalar field $\eta=\eta(\rho)$. There is also a gauge field, which is chosen to be only electric, $F_{\psi \rho}$. The four-form potential is, in terms of the old metric (5.49),

$$
\begin{equation*}
C_{4}=\frac{r_{1} r_{2}}{y^{4}} d r_{1} \wedge d \psi \wedge d r_{2} \wedge d \phi \tag{5.52}
\end{equation*}
$$

and can be written in the new coordinates (5.50). Dynamics of the D3 brane is given by
the DBI action and the corresponding Wess-Zumino term.

$$
\begin{align*}
S_{D 3}= & 2 N \int d \rho d \theta \frac{\sin \theta \sinh ^{2} \rho}{\sin ^{4} \eta} \sqrt{\cos ^{2} \eta\left(1+\eta^{\prime 2}\right)+\left(2 \pi \alpha^{\prime}\right)^{2} \sin ^{4} \eta F_{\psi \rho}^{2}} \\
& -2 N \int d \rho d \theta \frac{\cos \eta \sin \theta \sinh ^{2} \rho}{\sin ^{4} \eta}\left(\cos \eta+\eta^{\prime} \sin \eta \frac{\sinh \rho-\cosh \rho \cos \theta}{\cosh \rho-\sinh \rho \cos \theta}\right) \tag{5.53}
\end{align*}
$$

Solutions to the equations of motion are

$$
\begin{equation*}
\sin \eta=\kappa^{-1} \sinh \rho, \quad F_{\psi \rho}=\frac{i k \lambda}{8 \pi N \sinh ^{2} \rho}, \tag{5.54}
\end{equation*}
$$

where

$$
\begin{equation*}
\Pi=-i \frac{1}{2 \pi} \frac{\delta \mathcal{L}}{\delta F_{\psi \rho}}=k=\frac{4 N \kappa}{\sqrt{\lambda}} \tag{5.55}
\end{equation*}
$$

is the constant conjugate momentum associated to $A_{\psi}(\rho)$, the electric field, i.e. the electric charge or fundamental string charge dissolved on the D3 brane. By plugging this solution back into the action we can calculate the on-shell action,

$$
\begin{equation*}
S_{D 3}^{\text {onshell }}=2 N\left(\kappa \sqrt{1+\kappa^{2}}-\sinh ^{-1} \kappa\right), \tag{5.56}
\end{equation*}
$$

which is exactly finite. Even though we found a non-divergent action and naively we computed the expectation value of the circular Wilson loop, we need to do the Legendre transform as required and hope it does not produce a divergence. There are two boundary terms

$$
\begin{equation*}
S_{\mathrm{bdy}, \eta}=\lim _{\eta_{0} \rightarrow 0}-\left.\frac{1}{2 \pi} \int d \psi \eta \frac{\delta \mathcal{L}}{\delta \eta^{\prime}}\right|_{\eta_{0}}=-\frac{4 N \kappa}{\eta_{0}} \tag{5.57}
\end{equation*}
$$

and

$$
\begin{equation*}
S_{\text {bdy }, A}=\lim _{\eta_{0} \rightarrow 0}-\frac{i}{2 \pi} \int d \psi d \rho \Pi F_{\psi \rho}=-4 N \kappa \sqrt{1+\kappa^{2}}+\frac{4 N \kappa}{\eta_{0}} . \tag{5.58}
\end{equation*}
$$

Thus, the regularized action is

$$
\begin{equation*}
S_{D 3}^{\mathrm{onshell}}+S_{\mathrm{bdy}, \eta}+S_{\mathrm{bdy}, A}=-2 N\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right) . \tag{5.59}
\end{equation*}
$$

The expectation value of the circular Wilson loop is then

$$
\begin{equation*}
\left\langle W_{S_{k}}\right\rangle=\exp 2 N\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right), \tag{5.60}
\end{equation*}
$$

which is precisely the matrix model result in (5.34). At small $k$ and $\lambda \ll N^{2}$, the last expression reduces to

$$
\begin{equation*}
\left\langle W_{S_{k}}\right\rangle \sim e^{k \sqrt{\lambda}}=\left\langle W_{\square}\right\rangle^{k} . \tag{5.61}
\end{equation*}
$$

The last result coincides with $\left\langle W_{A_{k}}\right\rangle$ at small $k$,

$$
\begin{equation*}
\lim _{k \rightarrow 1}\left\langle W_{S_{k}}\right\rangle=\lim _{k \rightarrow 1}\left\langle W_{A_{k}}\right\rangle=\left\langle W_{\square}\right\rangle^{k} . \tag{5.62}
\end{equation*}
$$

For example,

$$
\begin{equation*}
\left\langle W_{\square}\right\rangle \simeq\left\langle W_{\square}\right\rangle \simeq\left\langle W_{\square}\right\rangle\left\langle W_{\square}\right\rangle, \tag{5.63}
\end{equation*}
$$

where two fundamental strings produces a circular Wilson loop separately.
But, what happens when $k \gg N$ ? This is not allowed in the antisymmetric case but only in the symmetric. In this regime the brane we considered backreacts on the geometry and deform the $A d S_{5} \times S^{5}$ space, forming the so-called bubbling geometries (see [71-73] for details).

As we said before, to consider $k$ fundamental strings to describe the $k$-representation of the circular Wilson loop, would lead to handle complicated and singular geometries. The $k \approx N$ limit, in which we dissolve the $k$ strings worldsheets into a brane is convenient because it produces the expected AdS/CFT results, and then it encodes the interactions between the coincident strings and allows to have all non-planar contributions to the expectation value of the higher rank Wilson loop.

Two kinds of branes were considered in order to describe, in this case, circular Wilson loops in the $k$-symmetric and $k$-antisymmetric representations of $U(N)$ (with $N$ large): D3 and D5 branes with $k$ units of charge. Those brane descriptions satisfied the $S O(2,1) \times S O(3) \times S O(5)$ symmetry of the Wilson loop in the gauge theory as an isometry of their induced geometries. Their worldvolumes are: $A d S_{2} \times S^{2}$ and $A d S_{2} \times S^{4}$, for the D 3 and the D 5 . We can see that the D 3 branes is entirely embedded into $A d S_{5}$ but not the D5 which has its $S^{4}$ part inside $S^{5}$, so it preserves $S O(5)$.

It is important to mention that we have assumed some results from the string worldsheet results given in the last chapter. Remember that, in order to describe a Wilson loop at the boundary of $A d S_{5} \times S^{5}$, we minimize the string worldsheet area which, in turn, yields the BPS condition for the loop as required by supersymmetry. Here we considered a brane as the "effective" behavior of $k$ worldsheets, and assume that the minimal area condition became the onshell regularized action of the brane. It was proved in [114] that the worldvolume field on the probe brane attached to the Wilson loop at the AdS boundary are constrained by the BPS condition, as expected, since the loop does not change.

### 5.3 Beyond the leading order

The last results correspond to the $N \rightarrow \infty$ and $\lambda \rightarrow \infty$ regime. Let us go back to the result of [28], in which the circular Wilson loop in the fundamental representation of $U(N)$ was computed in the matrix model for any $N, \lambda$

$$
\begin{equation*}
\left\langle W_{\text {ladders }}\right\rangle=\left\langle\frac{1}{N} \operatorname{Tr} e^{M}\right\rangle=\frac{1}{N} L_{N-1}^{1}\left(-\frac{\lambda}{4 N}\right) \exp \left(\frac{\lambda}{8 N}\right), \tag{5.64}
\end{equation*}
$$

which can be expanded in powers of $1 / N$ and Bessel functions $I_{n}$

$$
\begin{equation*}
\left\langle W_{\text {ladders }}\right\rangle=\frac{2}{\sqrt{\lambda}} I_{1}(\sqrt{\lambda})+\frac{\lambda}{48 N^{2}} I_{2}(\sqrt{\lambda})+\frac{\lambda^{2}}{1280 N^{4}} I_{4}(\sqrt{\lambda})+\cdots . \tag{5.65}
\end{equation*}
$$

When $N, \lambda \rightarrow \infty$,

$$
\begin{equation*}
\left\langle W_{\text {ladders }}\right\rangle \approx e^{\sqrt{\lambda}} \tag{5.66}
\end{equation*}
$$

Let us consider the $k$-symmetric Wilson loop result. We saw that, at small $k$,

$$
\begin{equation*}
\left\langle W_{S_{k}}\right\rangle \approx\left\langle W_{\text {ladders }}\right\rangle^{k} \approx e^{k \sqrt{\lambda}} . \tag{5.67}
\end{equation*}
$$

So we can take (5.64), and change $\lambda \rightarrow k^{2} \lambda$

$$
\begin{equation*}
\left\langle W_{S_{k}}\right\rangle=\left\langle\frac{1}{N} \operatorname{Tr} e^{k M}\right\rangle=\frac{1}{N} L_{N-1}^{1}\left(-\frac{k^{2} \lambda}{4 N}\right) \exp \left(\frac{k^{2} \lambda}{8 N}\right) \tag{5.68}
\end{equation*}
$$

or

$$
\begin{equation*}
\left\langle W_{S_{k}}\right\rangle=\left\langle\frac{1}{N} \operatorname{Tr} e^{k M}\right\rangle=\frac{1}{N} L_{N-1}^{1}\left(-4 N \kappa^{2}\right) \exp \left(2 N \kappa^{2}\right), \tag{5.69}
\end{equation*}
$$

where $k \sqrt{\lambda}=4 N \kappa$. Laguerre polynomials satisfy the differential equation,

$$
\begin{equation*}
x L_{n}^{k}(x)^{\prime \prime}+(k+1-x) L_{n}^{k}(x)^{\prime}+n L_{n}^{k}(x)=0 \tag{5.70}
\end{equation*}
$$

which leads to an equation for $\left\langle W_{S_{k}}\right\rangle$ [50]

$$
\begin{equation*}
\left[\kappa \partial_{\kappa}^{2}+3 \partial_{\kappa}-16 N^{2} \kappa\left(1+\kappa^{2}\right)\right]\left\langle W_{S_{k}}\right\rangle=0 \tag{5.71}
\end{equation*}
$$

If we set $\left\langle W_{S_{k}}\right\rangle=\exp (-N \mathcal{F})$, then

$$
\begin{equation*}
\left(\mathcal{F}^{\prime}\right)^{2}-\frac{1}{\kappa N}\left(\kappa \mathcal{F}^{\prime \prime}+3 \mathcal{F}^{\prime}\right)-16\left(1+\kappa^{2}\right)=0 \tag{5.72}
\end{equation*}
$$

If we consider $k \sqrt{\lambda} \sim \kappa N \gg 1$ since $k \sim N \gg 1$ and $\lambda \gg 1$, 5.72 becomes

$$
\begin{equation*}
\frac{d \mathcal{F}}{d \kappa}= \pm 4 \sqrt{1+\kappa^{2}} \tag{5.73}
\end{equation*}
$$

whose solution is

$$
\begin{equation*}
\mathcal{F}=\mathcal{F}_{0} \pm 2\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right), \tag{5.74}
\end{equation*}
$$

which coincides with (5.60) when $\mathcal{F}_{0}=0$ and we choose the minus sign. We can also solve (5.72) without eliminating the $1 / \kappa N$ term [114],

$$
\begin{equation*}
\mathcal{F}=-2\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right)+\frac{1}{2 N}\left(\ln \kappa^{3} \sqrt{1+\kappa^{2}}+\ln \left(32 \pi N^{3}\right)\right)+\mathcal{O}\left(\frac{1}{N^{2}}\right) . \tag{5.75}
\end{equation*}
$$

A similar result was obtained by Faraggi et al. in [109],

$$
\begin{equation*}
\mathcal{F}=-2\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right)-\frac{1}{2 N} \ln \frac{\kappa^{3}}{\sqrt{1+\kappa^{2}}} \tag{5.76}
\end{equation*}
$$

Another result appeared in [66],

$$
\begin{equation*}
\mathcal{F}=-2\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right)+\frac{1}{2 N} \ln \frac{\kappa^{3}}{\sqrt{1+\kappa^{2}}}+\frac{1}{N} \ln 4 \sqrt{\lambda} . \tag{5.77}
\end{equation*}
$$

Notice that the second term differs by a sign and a constant of the result given in the last expression, and also by an extra term. On the gravity side [115] (see also [116, 117], it was obtained

$$
\begin{equation*}
\mathcal{F}_{D 3}=-2\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right)+\frac{1}{2 N} \ln \frac{\kappa^{3}}{\sqrt{1+\kappa^{2}}} \tag{5.78}
\end{equation*}
$$

We notice that the $1 / N$ corrections are slightly different in both sides, and even in the same side. The disagreement between results could be due to subtle effects in the matrix model, or backreaction of the probe branes in the background geometry [41].

Similiar results for the antisymmetric case, and also a recent discovery of a relation between both symmetric and antisymmetric representations are given in [67,-70, 118, 119].

### 5.4 Rectangular tableau and the need of a nonabelian DBI action

In this section we will present an original, even though unpublished, work in which we explore higher representations called rectangular that allow to extend our study to include the so called nonabelian DBI action for probe branes. Let us consider the following Young
tableaux


As said in [37, 38], $\mathcal{R}_{\text {vert }}$ is dual to a set of $\ell$ coincident D5 branes in $\operatorname{AdS} S_{5} \times S^{5}$, while $\mathcal{R}_{\text {hor }}$ corresponds to $\ell \mathrm{D} 3$ branes in $A d S_{5} \times S^{5}$. This is expected, we can think $\mathcal{R}_{\text {vert }}$ and $\mathcal{R}_{\text {hor }}$ as an arrangement of $\ell$ symmetric and antisymmetric representations, respectively; those which correspond to D3s and D5s. In this description we also need to have $\ell \ll$ $k \sim N$ in both cases in order to not have backreaction in the geometry, i.e. that the presence of the branes does not modify severely the original $\operatorname{AdS} S_{5} \times S^{5}$ background. Dynamics of these probes would then be given by the so-called Nonabelian Dirac-BornInfeld (NADBI) action [75-77, 120, 121], the generalized form of the well-known DBI action for a single brane moving on a background.

Let us have a taste of how these rectangular tableaux become complicated.

### 5.4.1 Vertical rectangular tableau

A vertical rectangular tableau should correspond to a group of D5 probes in $\operatorname{Ad} S_{5} \times S^{5}$. From (5.10), we see that the Wilson loop insertion is represented by the Schur function

$$
\begin{equation*}
\left\langle W_{\mathcal{R}}\right\rangle=\frac{1}{\operatorname{dim} \mathcal{R}} \frac{1}{Z} \int \prod_{i=1}^{N} d \lambda_{i} \prod_{i<j}\left(\lambda_{i}-\lambda_{j}\right)^{2} S_{\mathcal{R}}\left(e^{\lambda_{1}}, e^{\lambda_{2}}, \cdots, e^{\lambda_{N}}\right) \exp \left(-\frac{2 N}{\lambda} \sum_{i} \lambda_{i}^{2}\right), \tag{5.80}
\end{equation*}
$$

and in the case of the vertical (antisymmetric) tableau, we can express the Schur function as the determinant of a $N \times N$ composed by elementary symmetric polynomials (5.15) [105],

$$
S_{\mathcal{R}}=\left|e_{\lambda_{i}-i+j}\right|=\left|\begin{array}{cccc}
e_{\lambda_{1}} & e_{\lambda_{1}+1} & \cdots & e_{\lambda_{\ell}+\ell-1}  \tag{5.81}\\
e_{\lambda_{2}-1} & e_{\lambda_{2}} & \cdots & \\
\vdots & & \ddots & \\
e_{\lambda_{\ell}-\ell+1} & & & e_{\lambda_{\ell}}
\end{array}\right|
$$

where $i=1, \cdots, N$ and $\lambda_{i}$ are eigenvalues. This is the so-called second Jacobi-Trudi identity [106]. Now, if $\lambda_{\ell+1}=\cdots=\lambda_{\lambda_{N}}=0$, the determinant simplifies to the determinant of a $\ell \times \ell$ matrix. Let us work with the simplest case, $\ell=2$, thus

$$
\begin{equation*}
S_{\mathcal{R}}=e_{k}^{2}-e_{k+1} e_{k-1} . \tag{5.82}
\end{equation*}
$$

So, we need to compute $\left\langle e_{k+1} e_{k-1}\right\rangle$. By using the generating functions of the elementary symmetric polynomials we have

$$
\begin{equation*}
\left\langle e_{m} e_{n}\right\rangle=\frac{1}{(2 \pi i)^{2}} \oint \frac{d t}{t^{N-m+1}} \oint \frac{d s}{s^{N-n+1}}\left\langle F_{A}(t) F_{A}(s)\right\rangle . \tag{5.83}
\end{equation*}
$$

The expectation values can be calculated by

$$
\begin{equation*}
\left\langle F_{A}(t) F_{A}(s)\right\rangle=\frac{1}{Z} \int \prod_{i=1}^{N} d \lambda_{i} \exp \left(-S_{A}(t, s)\right), \tag{5.84}
\end{equation*}
$$

with

$$
\begin{equation*}
S_{A}(t, s)=\frac{2 N}{\lambda} \sum_{i} \lambda_{i}^{2}-\sum_{i<j} \log \left(\lambda_{i}-\lambda_{j}\right)^{2}-\sum_{i=1}^{N} \log \left(t+e^{\lambda_{i}}\right)-\sum_{i=1}^{N} \log \left(s+e^{\lambda_{i}}\right) . \tag{5.85}
\end{equation*}
$$

The first two terms dominate since they are of order $N^{2}$, the Gaussian model, while the last two enter as insertions since they are of order $N$. Therefore, in the large $N$ limit the eigenvalue distribution is governed by the Wigner semi-circle law, thus

$$
\begin{equation*}
\left\langle F_{A}(t) F_{A}(s)\right\rangle \rightarrow \exp \left(N \int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} d x \rho(x)\left(\log \left(t+e^{x}\right)+\log \left(s+e^{x}\right)\right)\right)=\left\langle F_{A}(t)\right\rangle\left\langle F_{A}(s)\right\rangle . \tag{5.86}
\end{equation*}
$$

It means that

$$
\begin{equation*}
\left\langle e_{m} e_{n}\right\rangle=\left\langle e_{m}\right\rangle\left\langle e_{n}\right\rangle . \tag{5.87}
\end{equation*}
$$

Since

$$
\begin{equation*}
\left\langle W_{A_{k}}\right\rangle=\frac{1}{\operatorname{dim} A_{k}}\left\langle e_{k}\right\rangle, \tag{5.88}
\end{equation*}
$$

we can write

$$
\begin{equation*}
\left\langle W_{\mathcal{R}_{\text {vert }}}\right\rangle_{\ell=2}=\frac{1}{\operatorname{dim} \mathcal{R}_{\text {vert }}}\left(\left(\operatorname{dim} A_{k}\left\langle W_{A_{k}}\right\rangle\right)^{2}-\operatorname{dim} A_{k+1} \operatorname{dim} A_{k-1}\left\langle W_{A_{k+1}}\right\rangle\left\langle W_{A_{k-1}}\right\rangle\right), \tag{5.89}
\end{equation*}
$$

which is a determinant that can be generalized for any $\ell$ as

$$
\begin{equation*}
\left\langle W_{\mathcal{R}_{\text {vert }}}\right\rangle_{\ell}=\frac{1}{\operatorname{dim} \mathcal{R}_{\text {vert }}}\left|\operatorname{dim} A_{k-i+j}\left\langle W_{A_{k-i+j}}\right\rangle\right|, \tag{5.90}
\end{equation*}
$$

with $\ell \ll k \sim N . \operatorname{dim} \mathcal{R}_{\text {vert }}$ of the vertical tableau (5.79) can be computed by induction from the $\operatorname{dim} A_{k}$, a single column tableau, and it is given by

$$
\begin{equation*}
\operatorname{dim} \mathcal{R}_{\mathrm{vert}}(\ell, k)=\prod_{i=1}^{\ell} \frac{(N-1+i)!(\ell-i)!}{(N-k-1+1)!(k+\ell-i)!}, \tag{5.91}
\end{equation*}
$$

where $\ell \geq 1$, and for $\ell=1$ we see that $\operatorname{dim} \mathcal{R}_{\text {vert }}(1, k)=\operatorname{dim} A_{k}=\frac{N!}{k!(N-k)!}$, as expected. For $\ell=2$, we get

$$
\begin{equation*}
\operatorname{dim} \mathcal{R}_{\text {vert }}(2, k)=\frac{N!(N+1)!}{(N-k)!(k+1)!(N-k+1)!k!} . \tag{5.92}
\end{equation*}
$$

In the gravity side, as we learned, we have to consider two D5 branes as probes in the $A d S_{5} \times S^{5}$ geometry. Also, as we mentioned above, higher representations in the gauge theory side requires more than one branes whose dynamics is given by the not welldefined nonabelian DBI action. Unfortunately, we have not developed computations in the gravity side in this case, but we developed some ideas for the horizontal Young tableau.

### 5.4.2 Horizontal rectangular tableau

In the case of the horizontal (symmetric) tableau, which corresponds to a D3 branes probing $A d S_{5} \times S^{5}$, we can express the Schur function in (5.80) as the determinant of a $N \times N$, composed by complete (homogeneous) symmetric polynomials,

$$
S_{\mathcal{R}}=\left|h_{\lambda_{i}-i+j}\right|=\left|\begin{array}{cccc}
h_{\lambda_{1}} & h_{\lambda_{1}+1} & \cdots & h_{\lambda_{\ell}+\ell-1}  \tag{5.93}\\
h_{\lambda_{2}-1} & h_{\lambda_{2}} & \cdots & \\
\vdots & & \ddots & \\
h_{\lambda_{\ell}-\ell+1} & & & h_{\lambda_{\ell}}
\end{array}\right|
$$

where $i=1, \cdots, N$ and $\lambda_{i}$ are eigenvalues. This is the so-called first Jacobi-Trudi identity [106]. Now, if $\lambda_{\ell+1}=\cdots=\lambda_{\lambda_{N}}=0$, the determinant simplifies to the determinant of a $\ell \times \ell$ matrix. Let us work with the simplest case, $\ell=2$ with $\lambda_{i}=k$, thus

$$
\begin{equation*}
S_{\mathcal{R}}=h_{k}^{2}-h_{k+1} h_{k-1} . \tag{5.94}
\end{equation*}
$$

So again, we need to compute $\left\langle h_{k+1} h_{k-1}\right\rangle$. In the same way we did it before,

$$
\begin{equation*}
\left\langle h_{m} h_{n}\right\rangle=\frac{1}{(2 \pi i)^{2}} \oint \frac{d t}{t^{m+1}} \oint \frac{d s}{s^{n+1}}\left\langle F_{S}(t) F_{S}(s)\right\rangle . \tag{5.95}
\end{equation*}
$$

The expectation values can be calculated by

$$
\begin{equation*}
\left\langle F_{S}(t) F_{S}(s)\right\rangle=\frac{1}{Z} \int \prod_{i=1}^{N} d \lambda_{i} \exp \left(-S_{S}(t, s)\right) \tag{5.96}
\end{equation*}
$$

with

$$
\begin{equation*}
S_{S}(t, s)=\frac{2 N}{\lambda} \sum_{i} \lambda_{i}^{2}-\sum_{i<j} \log \left(\lambda_{i}-\lambda_{j}\right)^{2}+\sum_{i=1}^{N} \log \left(1-t e^{\lambda_{i}}\right)+\sum_{i=1}^{N} \log \left(1-s e^{\lambda_{i}}\right) . \tag{5.97}
\end{equation*}
$$

In the large $N$ limit, again,

$$
\begin{equation*}
\left\langle F_{S}(t) F_{S}(s)\right\rangle \rightarrow \exp \left(-N \int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} d x \rho(x)\left(\log \left(1-t e^{x}\right)+\log \left(1-s e^{x}\right)\right)\right)=\left\langle F_{S}(t)\right\rangle\left\langle F_{S}(s)\right\rangle \tag{5.98}
\end{equation*}
$$

then,

$$
\begin{equation*}
\left\langle h_{m} h_{n}\right\rangle=\left\langle h_{m}\right\rangle\left\langle h_{n}\right\rangle . \tag{5.99}
\end{equation*}
$$

From

$$
\begin{equation*}
\left\langle W_{S_{k}}\right\rangle=\frac{1}{\operatorname{dim} S_{k}}\left\langle h_{k}\right\rangle, \tag{5.100}
\end{equation*}
$$

we conclude that

$$
\begin{equation*}
\left\langle W_{\mathcal{R}_{\text {hor }}}\right\rangle_{\ell=2}=\frac{1}{\operatorname{dim} \mathcal{R}_{\text {hor }}}\left(\left(\operatorname{dim} S_{k}\left\langle W_{S_{k}}\right\rangle\right)^{2}-\operatorname{dim} S_{k+1} \operatorname{dim} S_{k-1}\left\langle W_{S_{k+1}}\right\rangle\left\langle W_{S_{k-1}}\right\rangle\right), \tag{5.101}
\end{equation*}
$$

which is a determinant that can be generalized, again, for any $\ell$ as

$$
\begin{equation*}
\left\langle W_{\mathcal{R}_{\text {hor }}}\right\rangle_{\ell}=\frac{1}{\operatorname{dim} \mathcal{R}_{\text {hor }}}\left|\operatorname{dim} S_{k-i+j}\left\langle W_{S_{k-i+j}}\right\rangle\right|, \tag{5.102}
\end{equation*}
$$

with $\ell \ll k \sim N . \operatorname{dim} \mathcal{R}_{\text {hor }}$ of the horizontal tableau (5.79) can be computed by induction from the $\operatorname{dim} S_{k}$, a single row tableau, giving

$$
\begin{equation*}
\operatorname{dim} \mathcal{R}_{\text {hor }}=\prod_{i=1}^{\ell} \frac{(N+k-i)!(\ell-i)!}{(N-i)!(k+\ell-i)!}, \tag{5.103}
\end{equation*}
$$

where $\ell \geq 1$, and for $\ell=1$ we see that $\operatorname{dim} \mathcal{R}_{\text {hor }}=\operatorname{dim} S_{k}=\frac{(N+k-1)!}{k!(N-1)!}$, again, as expected. For $\ell=2$, we obtain

$$
\begin{equation*}
\operatorname{dim} \mathcal{R}_{\mathrm{vert}}=\frac{(N+k-1)!(N+k-2)!}{(N-1)!(k+1)!(N-2)!k!} . \tag{5.104}
\end{equation*}
$$

We will do some further computations in this case. Let us remember that

$$
\begin{equation*}
\left\langle W_{S_{k}}\right\rangle=\exp 2 N\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right), \quad \kappa=\frac{k \sqrt{\lambda}}{4 N}, \tag{5.105}
\end{equation*}
$$

which should be obtained in the gravity side by considering D3 branes in the $\operatorname{Ad} S_{5} \times S^{5}$ background. Moreover, we saw in (5.78) that the first corrected action, and the expectation value is

$$
\begin{equation*}
\left\langle W_{S_{\kappa}}\right\rangle=\sqrt{\frac{\kappa^{3}}{\sqrt{1+\kappa^{2}}}} \exp \left(2 N\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right)\right)=\sqrt{\frac{\kappa^{3}}{\sqrt{1+\kappa^{2}}}}\left\langle W_{S_{k}}\right\rangle_{N \rightarrow \infty} . \tag{5.106}
\end{equation*}
$$

The factor in front of $\left\langle W_{S_{k}}\right\rangle_{N \rightarrow \infty}$ in the last result contains the $1 / N$ correction. Since $k \sim N(5.102)$ would vanish if $N \rightarrow \infty$, so we need to consider corrections and then (5.106). Since

$$
\begin{equation*}
\frac{(k \pm 1) \sqrt{\lambda}}{4 N}=\kappa \pm \frac{\sqrt{\lambda}}{4 N} \rightarrow \kappa(1 \pm \epsilon), \quad \epsilon \sim \frac{1}{N} \ll 1, \tag{5.107}
\end{equation*}
$$

the nonabelian nature can be extracted from $1 / N$ corrections. The expectation value we want to compute is

$$
\begin{equation*}
\left\langle W_{\mathcal{R}_{\text {hor }}}\right\rangle_{\ell=2}=\frac{1}{\operatorname{dim} \mathcal{R}_{\text {hor }}}\left|\operatorname{dim} S_{k+i-j}\left\langle W_{\kappa+(i-j) \epsilon \epsilon}\right\rangle\right| . \tag{5.108}
\end{equation*}
$$

From 5.101), by using $\operatorname{dim} S_{k}=\frac{(N+k-1)!}{k!(N-1)!}$, we can see that

$$
\begin{equation*}
\operatorname{dim} S_{k+1} \operatorname{dim} S_{k-1}=\left(\operatorname{dim} S_{k}\right)^{2} \frac{(N+k) k}{(N+k-1)(k+1)} . \tag{5.109}
\end{equation*}
$$

If we fix $k / N=q$, we can expand the last result for large $k$, and get

$$
\begin{equation*}
\operatorname{dim} S_{k+1} \operatorname{dim} S_{k-1}=\left(\operatorname{dim} S_{k}\right)^{2}\left(1-\frac{1}{k(1+q)}\right) . \tag{5.110}
\end{equation*}
$$

So we can see that there is an effect of order $1 / k$ only looking at the dimension of the representations. We can compute $\left\langle W_{\mathcal{R}_{\text {nor }}}\right\rangle_{\ell=2}$, by inserting the different dimensions and the result (5.106), and get

$$
\begin{equation*}
\left\langle W_{\mathcal{R}_{\text {hor }}}\right\rangle_{\ell=2}=\left(\left\langle W_{S_{k}}\right\rangle^{N \rightarrow \infty}\right)^{2}\left(1+u(\kappa)+\frac{1}{k} v(\kappa)\right), \tag{5.111}
\end{equation*}
$$

where $u(\kappa)$ and $v(\kappa)$ are functions to be determined and compared with the gravity result. ${ }^{3}$ We can write the last result as

$$
\begin{equation*}
\left\langle W_{\mathcal{R}_{\text {hor }}}\right\rangle_{\ell=2}=\exp \left(4 N\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right)\right)\left(1+u(\kappa)+\frac{1}{k} v(\kappa)\right) . \tag{5.112}
\end{equation*}
$$

[^16]So, the expectation value of the Wilson loop in the horizontal rectangular representation contains two copies of the Drukker-Fiol result for the single row plus corrections coming from the nonabelian nature of $U(2)$, in other words, from the fact that the two D 3 branes coincide and interact. We will see below how this result could help to make predictions for the nonabelian generalization of the DBI action, which is up to now not well-defined.

### 5.5 NADBI for two coincident D3 branes

As we already mentioned, $\ell$ coincident branes are needed to describe rectangular representations of $U(N)$. Let us discuss some details about the possible generalization of the DBI action. For more than one branes, the gauge group of the theory on their worldvolume is $U(\ell)$, and the adjoint fields become $\ell \times \ell$ matrices, so the interpretation of the scalar fields as positions of the brane is not valid anymore. However, there is a way out: even though the scalars are matrices, positions of the brane could be given by eigenvalues of the matrix scalar field.


Figure 5.1: Scheme of two probe D3 branes corresponding to a Wilson loop in rectangular horizontal representation.

Let us start by explaining the case of the DBI action for one single brane, a D3. The action for a $D_{p}$-brane is

$$
\begin{equation*}
S_{D B I}=-T_{p} \int d^{p+1} \xi e^{-\phi} \sqrt{-\operatorname{det}\left[P[G+B]_{a b}+2 \pi \alpha^{\prime} F_{a b}\right]} . \tag{5.113}
\end{equation*}
$$

Interactions with the background gauge fields are given by the Wess-Zumino term

$$
\begin{equation*}
S_{W Z}=\mu_{p} \int P\left[\sum C^{(n)} e^{B}\right] e^{2 \pi \alpha^{\prime} F}=\mu_{p} \int P\left[C^{(n)} \wedge(1+B+\cdots)\right]\left(1+2 \pi \alpha^{\prime} F+\cdots\right) . \tag{5.114}
\end{equation*}
$$

Here $C^{(n)}$ represents the $n$-form RR potential, $B$ is the NSNS two-form potential, $F=d A$ where $A_{a}$ is a $U(1)$ field, and $P[\cdots]$ denotes the pullback of the bulk spacetime fields to the $(p+1)$-dimensional worldvolume of the $D_{p}$-brane,

$$
\begin{equation*}
P[E]_{a b}=E_{\mu \nu} \frac{\partial x^{\mu}}{\partial \sigma^{a}} \frac{\partial x^{\nu}}{\partial \sigma^{b}}=E_{a b}+2 \pi \alpha^{\prime} E_{i a} \partial_{b} \Phi^{i}+2 \pi \alpha^{\prime} E_{i b} \partial_{a} \Phi^{i}+\left(2 \pi \alpha^{\prime}\right)^{2} E_{i j} \partial_{a} \Phi^{i} \partial_{b} \Phi^{j} . \tag{5.115}
\end{equation*}
$$

The brane charge is defined as the charge under $C^{(p+1)}$, in which $\mu_{p}= \pm T_{p}$. In general, the WZ term allows us to couple the $D_{p}$-brane to another RR potentials with lower form degree than $p+1$. The last action, in the same way the Nambu-Goto for the fundamental string, represents the proper volume swept out by the $D_{p}$-brane. If we go to the static gauge $\sigma^{a}=x^{a}$, with $a=0,1, \cdots, p$, the other coordinates, $x^{i}$, with $i=p+1, \cdots, 9$, correspond to scalar fields in the worldvolume theory

$$
\begin{equation*}
x^{i}(\sigma)=2 \pi \alpha^{\prime} \Phi^{i}(\sigma) . \tag{5.116}
\end{equation*}
$$

The square root in the DBI action resums an infinite series of stringy corrections coming from the fact that this is an effective action which arises through integrating out massive modes in string theory.

Now let us consider $\ell$ parallel branes. Strings stretching between them become massless when branes approach each other, then $A_{a}=A_{a}^{(n)} T^{n}$ where $\left\{T^{n}\right\}$ are $\ell^{2}$ Hermitian generators of $U(\ell)$ and $n=1, \cdots, \ell^{2}-1$, and $U(1)$ becomes $U(\ell)$. The field strength is now

$$
\begin{equation*}
F_{a b}=\partial_{a} A_{b}-\partial_{b} A_{a}+i\left[A_{a}, A_{b}\right] . \tag{5.117}
\end{equation*}
$$

The scalars $\Phi^{i}$ also transform in the adjoint of $U(\ell)$, with

$$
\begin{equation*}
D_{a} \Phi^{i}=\partial_{a} \Phi^{i}+i\left[A_{a}, \Phi^{i}\right] . \tag{5.118}
\end{equation*}
$$

Since scalars are now matrices, as we mentioned above, they cannot represent positions of the branes in the transverse space.

It was suggested in [123], that for general non-commuting scalars, positions are given by eigenvalues. It was proposed by Myers in [75] (and reviewed in [76, 77]) by using T-duality on the DBI action of a $D 9$-brane that the action for $\ell D_{p}$-branes is

$$
\begin{equation*}
S_{D B I}=-T_{D p} \int d^{p+1} \sigma \operatorname{STr}^{(g)}\left(e^{-\phi} \sqrt{\operatorname{det} M_{a b} \operatorname{det} Q_{j}^{i}}\right), \tag{5.119}
\end{equation*}
$$

where the superscript $(g)$ means trace over the gauge group, and

$$
\begin{equation*}
M_{a b}=P\left[E_{a b}+E_{a i}\left(Q^{-1}-\delta\right)^{i j} E_{j b}\right]+\lambda F_{a b}, \tag{5.120}
\end{equation*}
$$

and

$$
\begin{equation*}
Q_{j}^{i}=\delta_{j}^{i}+i \lambda\left[\Phi^{i}, \Phi^{k}\right] E_{k j}, \quad \lambda=2 \pi \alpha^{\prime}, \tag{5.121}
\end{equation*}
$$

with

$$
\begin{equation*}
E_{M N}=G_{M N}+B_{M N}, \tag{5.122}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{a b}=\partial_{a} A_{b}-\partial_{b} A_{a}+i\left[A_{a}, A_{b}\right] . \tag{5.123}
\end{equation*}
$$

To incorporate the nonabelian gauge symmetry of the action, Tseytlin [120, 121], suggested the symmetrized trace "STr" in front of the action (see some general properties in [121]). This trace is necessary to remove the ambiguity in the ordering of the products of the terms inside the action. The standard prescription instructs us to expand the square root in powers of $\lambda$ and take the trace over all possible orderings because it is not clear how to take this symmetrized trace without expanding the square root:

$$
\begin{equation*}
\operatorname{STr}^{(g)}\left(T^{n_{1}} T^{n_{2}} \cdots T^{n_{K}}\right)=\frac{1}{K!} \operatorname{Tr}\left(T^{n_{1}} T^{n_{2}} \cdots T^{n_{K}}+\text { permutations }\right) . \tag{5.124}
\end{equation*}
$$

The WZ term [75, 77]

$$
\begin{equation*}
S_{W Z}=T_{D 3} \int \operatorname{STr}^{(g)}\left(P\left[e^{i\left(2 \pi \alpha^{\prime}\right) i_{\Phi} i_{\Phi}} C^{(4)}\right] \wedge e^{\left(2 \pi \alpha^{\prime}\right) F}\right) \tag{5.125}
\end{equation*}
$$

will be a generalization of the WZ term in (5.114), where $i_{\Phi}$ is an interior product which reduces the form degree by -1 . The nonabelian pullback is defined as

$$
\begin{equation*}
P[E]_{a b}=E_{a b}+\lambda E_{i a} D_{b} \Phi^{i}+\lambda E_{i b} D_{a} \Phi^{i}+\lambda^{2} E_{i j} D_{a} \Phi^{i} D_{b} \Phi^{j} . \tag{5.126}
\end{equation*}
$$

The metric can be written as

$$
\begin{equation*}
\frac{d s^{2}}{L^{2}}=G_{a b} d x^{a} d x^{b}+G_{i j} d x^{i} d x^{j}, \quad G_{a i}=0 \tag{5.127}
\end{equation*}
$$

Let us take $p=3$ and $B_{M N}=\phi=0$. The non-abelian pullback is [77] (some terms vanish because $G_{i a}=0$ )

$$
\begin{equation*}
P\left[G_{a b}\right]=G_{a b}+\lambda^{2} G_{i j} D_{a} \Phi^{i} D_{b} \Phi^{j}, \quad x^{i}=\lambda \Phi^{i} \tag{5.128}
\end{equation*}
$$

and

$$
\begin{equation*}
P\left[G_{a i}\left(Q^{-1}-\delta\right)^{i j} G_{j b}\right]=\lambda^{2} G_{k i}\left(Q^{-1}-\delta\right)^{i j} G_{j l} D_{a} \Phi^{k} D_{b} \Phi^{l} \tag{5.129}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{a} \Phi^{i}=\partial_{a} \Phi^{i}+i\left[A_{a}, \Phi^{i}\right] \tag{5.130}
\end{equation*}
$$

is necessary to preserve gauge symmetry. Then,

$$
\begin{gather*}
S_{D B I}=-T_{D 3} \int d^{4} \sigma \operatorname{STr}^{(g)}\left(\sqrt{\operatorname{det} M_{a b}} \sqrt{\operatorname{det} Q_{j}^{i}}\right), \quad T_{D 3}=N / 2 \pi^{2} L^{4},  \tag{5.131}\\
M_{a b}=G_{a b}+\lambda^{2}\left(G_{i j}+G_{i k}\left(Q^{-1}-\delta\right)^{k l} G_{l j}\right) D_{a} \Phi^{i} D_{b} \Phi^{j}+\lambda F_{a b} . \tag{5.132}
\end{gather*}
$$

Since

$$
\begin{equation*}
\left(Q^{-1}-\delta\right)^{i j}=-i \lambda\left[\Phi^{i}, \Phi^{j}\right], \lambda \ll 1, \tag{5.133}
\end{equation*}
$$

then

$$
\begin{equation*}
M_{a b}=G_{a b}+\lambda^{2}\left(G_{i j}-i \lambda\left[\Phi^{k}, \Phi^{l}\right] G_{k i} G_{l j}\right) D_{a} \Phi^{i} D_{b} \Phi^{j}+\lambda F_{a b}, \tag{5.134}
\end{equation*}
$$

or

$$
\begin{equation*}
M_{a b}=G_{a b}+\lambda F_{a b}+\lambda^{2} D_{a} \Phi^{i} D_{b} \Phi^{j} G_{i j}-i \lambda^{3}\left[\Phi^{k}, \Phi^{l}\right] D_{a} \Phi^{i} D_{b} \Phi^{j} G_{k i} G_{l j} \tag{5.135}
\end{equation*}
$$

The background fields are functionals of the nonabelian scalars. After fixing the static gauge (5.127), the metric $G_{a b}$ in the D -brane action would be given by a nonabelian Taylor expansion [75-77]

$$
\begin{align*}
G_{a b} & =\left.\exp \left[\lambda \Phi^{i} \partial_{x^{i}}\right] G_{a b}\left(x^{a}, x^{i}\right)\right|_{x^{i}=0} \\
& =G_{a b}\left(x^{a}, 0\right)+\lambda \Phi^{i} \partial_{x^{i}} G_{a b}\left(x^{a}, 0\right)+\lambda^{2} \Phi^{i} \Phi^{j} \partial_{x^{i}} \partial_{x^{j}} G_{a b}\left(x^{a}, 0\right)+\cdots \tag{5.136}
\end{align*}
$$

### 5.5.1 The $\mathbf{U}(2)$ case

In this section we explore further the NADBI and try to make comparisons with the matrix models results. Let us consider $\ell=2$, then the gauge field on the D 3 brane probe worldvolume is $U(2)$; the gauge field is $A_{a}=A_{a}^{(n)} T^{n}$ with $a=0,1,2,3$ and $(n)=0,1,2,3$ counts the generators of $U(2)$ (the $2 \times 2$ identity and the $S U(2)$ generators). The scalar fields are $\Phi^{i}=\Phi^{i(n)} T^{n}$ with $i=1, \cdots, 6$. Notice now that we actually have "too many" scalars because they are matrices, they do not longer describe positions.

Let us explore the case of two probe D3 branes in $A d S_{5} \times S^{5}$. The metric is

$$
\begin{equation*}
\frac{d s^{2}}{L^{2}}=\frac{1}{\sin ^{2} \eta}\left(d \eta^{2}+\cos ^{2} \eta d \psi^{2}+d \rho^{2}+\sinh ^{2} \rho\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right)\right) \tag{5.137}
\end{equation*}
$$

where $\rho \in[0, \infty), \theta \in[0, \pi), 0 \leq \psi, \phi \leq 2 \pi$ and $\eta \in[0, \pi / 2]$. In this coordinate system the boundary is located at $\eta=0$ (and also at $\rho \rightarrow \infty$ ). The circular loop is located at $\eta=\rho=0$. We take $\{\psi, \rho, \theta, \phi\}$ as the brane world-volume coordinates. The embedding function (transverse coordinate) is $\eta=\eta(\rho)$ [50]. In the nonabelian extension of the DBI action, we must promote the transverse coordinate to a transverse scalar field

$$
\begin{equation*}
\eta(\rho) \rightarrow \lambda \Phi(\rho) \tag{5.138}
\end{equation*}
$$

where $\Phi(\rho)$ is a $2 \times 2$ matrix. The other five scalars are related to the $S^{5}$ part of the background, but we do not need to worry about them if we conveniently fix a point on $S^{5}$ to be the north pole of the sphere, so we set them to zero. With only one scalar field $\Phi$ in our problem, the non-abelian DBI action will simplify drastically because all commutators $\left[\Phi^{i}, \Phi^{j}\right]$ vanish.

With these simplifications the non-abelian DBI action becomes

$$
\begin{equation*}
S_{D B I}=-T_{D 3} \int d^{4} \sigma \operatorname{STr}^{(g)} \sqrt{\operatorname{det} M_{a b}} \tag{5.139}
\end{equation*}
$$

where

$$
\begin{equation*}
M_{a b}=G_{a b}+\lambda F_{a b}+\lambda^{2} D_{a} \Phi^{i} D_{b} \Phi^{j} G_{i j} . \tag{5.140}
\end{equation*}
$$

Note that the components of the pullback are now $2 \times 2$ matrices. In particular, the metric components are now viewed as functionals of the scalars

$$
\begin{equation*}
G_{M N}(x) \rightarrow G_{M N}\left[\Phi^{i}\right] . \tag{5.141}
\end{equation*}
$$

In principle, this transition should be done using the so called non-abelian Taylor expansion (5.136), but we will follow an alternative approach using the properties of the Pauli matrices, as we will describe below. After doing that, we need to evaluate the determinant in the DBI action (5.139). For our embedding it has the form

$$
\begin{equation*}
\operatorname{det} M_{\mu \nu}=\left(M_{\psi \psi} M_{\rho \rho}-M_{\psi \rho} M_{\rho \psi}\right) M_{\theta \theta} M_{\phi \phi}, \tag{5.142}
\end{equation*}
$$

where

$$
\begin{align*}
M_{\psi \psi} & =G_{\psi \psi}-\lambda^{2}\left[A_{\psi}, \Phi\right]^{2} G_{\eta \eta}, \\
M_{\psi \rho} & =-\lambda A_{\psi}^{\prime}+i \lambda^{2}\left[A_{\psi}, \Phi\right] \Phi^{\prime} G_{\eta \eta} \\
M_{\rho \psi} & =\lambda A_{\psi}^{\prime}+i \lambda^{2}\left[A_{\psi}, \Phi\right] \Phi^{\prime} G_{\eta \eta}, \\
M_{\rho \rho} & =G_{\rho \rho}+\lambda^{2}\left(\Phi^{\prime}\right)^{2} G_{\eta \eta}, \\
M_{\theta \theta} & =G_{\theta \theta} \\
M_{\phi \phi} & =G_{\phi \phi} \tag{5.143}
\end{align*}
$$

After promoting the embedding function to a matrix $(\eta \rightarrow \lambda \Phi)$ we also need to promote $G_{M N}$ to matrices, that now are functionals of $\Phi$. We take the Pauli matrices and the identity as the generators of $U(2): T^{a}=\{1 / 2 \mathbb{I}, 1 / 2 \vec{\sigma}\}$. So the scalar field has the following form

$$
\Phi_{a} T^{a}=\frac{1}{2}\left(\begin{array}{cc}
\Phi_{0}+\Phi_{3} & \Phi_{1}-i \Phi_{2}  \tag{5.144}\\
\Phi_{1}+i \Phi_{2} & \Phi_{0}-\Phi_{3}
\end{array}\right),
$$

whose eigenvalues (which describe the positions of the branes [123]) are

$$
\begin{equation*}
\lambda_{1,2}=\left\{\Phi_{0}-|\vec{\Phi}|, \Phi_{0}+|\vec{\Phi}|\right\}, \quad|\vec{\Phi}|=\sqrt{\Phi_{1}^{2}+\Phi_{2}^{2}+\Phi_{3}^{2}} \tag{5.145}
\end{equation*}
$$

For coincident branes $\lambda_{1}=\lambda_{2}$, so $|\vec{\Phi}|=0$, then $\Phi=\frac{1}{2} \Phi_{0} \mathbb{I} .{ }^{4}$
In the metric background (5.137) we have sines and cosines. Writing them as exponentials and using the properties of the Pauli matrices we obtain

$$
\begin{align*}
\sin (\lambda \Phi) & =a_{1} \mathbb{I}+a_{2}(\hat{\Phi} \cdot \vec{\sigma}) \\
\cos (\lambda \Phi) & =b_{1} \mathbb{I}+b_{2}(\hat{\Phi} \cdot \vec{\sigma}), \tag{5.146}
\end{align*}
$$

[^17]where
\[

\left\{$$
\begin{array}{l}
a_{1}=\sin \left(\frac{\lambda}{2} \Phi_{0}\right) \cos \left(\frac{\lambda}{2}|\vec{\Phi}|\right)  \tag{5.147}\\
a_{2}=\cos \left(\frac{\lambda}{2} \Phi_{0}\right) \sin \left(\frac{\lambda}{2}|\vec{\Phi}|\right)
\end{array}
$$,\left\{$$
\begin{array}{l}
b_{1}=\cos \left(\frac{\lambda}{2} \Phi_{0}\right) \cos \left(\frac{\lambda}{2}|\vec{\Phi}|\right) \\
b_{2}=-\sin \left(\frac{\lambda}{2} \Phi_{0}\right) \sin \left(\frac{\lambda}{2}|\vec{\Phi}|\right)
\end{array}
$$ \quad, \hat{\Phi}=\frac{\vec{\Phi}}{|\vec{\Phi}|}\right.\right.
\]

With the above expressions we can compute the metric components. Explicitly, we get

$$
\begin{align*}
\frac{G_{\eta \eta}[\Phi]}{L^{2}} & =\frac{1}{\sin ^{2} \lambda \Phi}=\frac{2-2 \cos \lambda \Phi_{0} \cos \lambda|\vec{\Phi}|}{\left(\cos \lambda \Phi_{0}-\cos \lambda|\vec{\Phi}|\right)^{2}} \mathbb{I}-\frac{2 \sin \lambda \Phi_{0} \sin \lambda|\vec{\Phi}|}{\left(\cos \lambda \Phi_{0}-\cos \lambda|\vec{\Phi}|\right)^{2}}(\hat{\Phi} \cdot \vec{\sigma}), \\
\frac{G_{\psi \psi}[\Phi]}{L^{2}} & =\frac{\cos ^{2} \lambda \Phi}{\sin ^{2} \lambda \Phi}=\frac{2-\cos 2 \lambda \Phi_{0}-\cos 2 \lambda|\vec{\Phi}|}{2\left(\cos \lambda \Phi_{0}-\cos \lambda|\vec{\Phi}|\right)^{2}} \mathbb{I}-\frac{2 \sin \lambda \Phi_{0} \sin \lambda|\vec{\Phi}|}{\left(\cos \lambda \Phi_{0}-\cos \lambda|\vec{\Phi}|\right)^{2}}(\hat{\Phi} \cdot \vec{\sigma}) \tag{5.148}
\end{align*}
$$

The other metric components are proportional to $G_{\eta \eta}$, namely

$$
\begin{equation*}
G_{\rho \rho}=G_{\eta \eta}, \quad G_{\theta \theta}=\sinh ^{2} \rho G_{\eta \eta}, \quad G_{\phi \phi}=\sinh ^{2} \rho \sin ^{2} \theta G_{\eta \eta} \tag{5.149}
\end{equation*}
$$

Note that all metric components commute with each other and with $\Phi$. We saw that for coincident branes we can take $\Phi=\frac{1}{2} \Phi_{0} \mathbb{I}$, and then $\eta=\frac{\lambda}{2} \Phi_{0} \mathbb{I}$. With this choice, everything that appears in the DBI action commutes, thus we do not need to worry about any ambiguity and the symmetrized trace is simply a regular trace. Let us choose $A_{\psi}(\rho)=$ $A_{\psi}^{(0)}(\rho) \mathbb{I}+\vec{A}_{\psi}(\rho) \cdot \vec{\sigma}$ just like in [50]. Then, the determinant 5.142 is

$$
\begin{equation*}
\sqrt{\operatorname{det} M_{a b}}=\sqrt{A \mathbb{I}+B\left(\overrightarrow{A_{\psi}^{\prime}} \cdot \vec{\sigma}\right)} \frac{\sinh ^{2} \rho \sin \theta}{\sin ^{2} \frac{\lambda \Phi_{0}}{2}} \tag{5.150}
\end{equation*}
$$

Since the Pauli matrices do not appear as products, there is only one term, so we do not need to symmetrize the square root:

$$
\begin{equation*}
\operatorname{STr} \sqrt{\operatorname{det} M_{a b}}=\operatorname{Tr} \sqrt{\operatorname{det} M_{a b}} \tag{5.151}
\end{equation*}
$$

The formula of the trace of the determinant of a $2 \times 2$ matrix implies that

$$
\begin{equation*}
\operatorname{Tr} \sqrt{\operatorname{det} M_{a b}}=\sqrt{2} \frac{\sinh ^{2} \rho \sin \theta}{\sin ^{2} \eta} \sqrt{A+\sqrt{A^{2}-B^{2}\left|\vec{F}_{\rho \psi}\right|^{2}}} \tag{5.152}
\end{equation*}
$$

with

$$
\begin{align*}
& A=\frac{\cos ^{2} \eta\left(1+\eta^{\prime 2}\right)}{\sin ^{4} \eta}+\lambda^{2}\left(F_{\rho \psi}^{(0) 2}+\left|\vec{F}_{\rho \psi}\right|^{2}\right)  \tag{5.153}\\
& B=\lambda^{2} F_{\rho \psi}^{(0)}{ }^{2} \tag{5.154}
\end{align*}
$$

Back to the DBI action, we write

$$
\begin{equation*}
S_{D B I}=-\sqrt{2} T_{D 3} \int d^{4} \sigma \frac{\sinh ^{2} \rho \sin \theta}{\sin ^{2} \eta} \sqrt{A+\sqrt{A^{2}-B^{2}\left|\vec{F}_{\rho \psi}\right|^{2}}} \tag{5.155}
\end{equation*}
$$

The WZ part is the one given in Drukker and Fiol [50], but multiplied by 2 due to the matrix nature of the variable $\eta$

$$
\begin{equation*}
S_{W Z}=-4 N \int d \rho d \theta \frac{\cos \eta \sin \theta \sinh ^{2} \rho}{\sin ^{4} \eta}\left(\cos \eta+\eta^{\prime} \sin \eta \frac{\sinh \rho-\cosh \rho \cos \theta}{\cosh \rho-\sinh \rho \cos \theta}\right) \tag{5.156}
\end{equation*}
$$

It is important to notice that when $\vec{F}_{\rho \psi}=0$, we get the abelian action of Drukker and Fiol multiplied by 2, as expected. So, the nonabelian nature of our system is contained in the off-diagonal terms of $F_{\rho \psi}$. Since making them to vanish breaks the $U(2)$ symmetry to $U(1) \times U(1)$, they are related to the off-diagonal terms of $\Phi, \vec{\Phi}$, by T-duality [75].

Since we have only considered that the gauge field on the two D3 brane system get nonabelian effects, leaving the scalar field duplicated due to the matrix nature of the field, we can take the result in [50] and deform it to include this effect.

The ansatz that includes the solution of [50] is

$$
\begin{align*}
\eta(\rho) & =\sin ^{-1}\left(\kappa^{-1} \sinh \rho\right) \\
F_{\rho \psi}^{(0)} & =\frac{i \kappa(1+\epsilon) \sqrt{\lambda}}{2 \pi \sinh ^{2} \rho} \\
\left|\vec{F}_{\rho \psi}\right| & =\epsilon \frac{\kappa \sqrt{\lambda}}{2 \pi \sinh ^{2} \rho}, \tag{5.157}
\end{align*}
$$

where $\epsilon \ll 1$ is a parameter that allows us to insert the nonabelian nature. We can see from the action that there are four conserved quantities associated to the gauge fields charges. In the abelian case, where we only have $F_{\rho \psi}^{(0)}$, the conserved momentum $k_{0}$ corresponds to the fundamental string charge on the brane. In this case, we have two branes so we expect to have two conserved quantities equal to the fundamental string charges on each brane, but we have four momenta. After inserting (5.157) into (5.155) and (5.156), we get

$$
\begin{align*}
k^{(0)} & =\frac{8 N \kappa}{\sqrt{\lambda}}+\frac{8 N \epsilon}{\sqrt{\lambda}} \frac{\left(1+\kappa^{2}\right)}{\kappa} \\
|\vec{k}| & =\frac{8 N \epsilon}{\sqrt{\lambda}} \frac{\left(1+\kappa^{2}\right)}{\kappa}+\frac{24 N \epsilon^{2}\left(1+\kappa^{2}\right)}{\kappa^{3} \sqrt{\lambda}}, \tag{5.158}
\end{align*}
$$

where $|\vec{k}|=\left(k^{(1)}\right)^{2}+\left(k^{(2)}\right)^{2}+\left(k^{(3)}\right)^{2}$, and $\epsilon^{2}=\epsilon_{1}^{2}+\epsilon_{2}^{2}+\epsilon_{3}^{2}$. If we write the $2 \times 2$ momentum matrix as $k=k^{(0)} \mathbb{I}+\vec{k} \cdot \vec{\sigma}$, its eigenvalues are

$$
\begin{align*}
& \lambda_{1}=\frac{8 N \kappa}{\sqrt{\lambda}}-\frac{24 N \epsilon^{2}\left(1+\kappa^{2}\right)}{\kappa^{3} \sqrt{\lambda}}=2 k+\cdots,  \tag{5.159}\\
& \lambda_{2}=\frac{8 N \kappa}{\sqrt{\lambda}}+\frac{24 N \epsilon^{2}\left(1+\kappa^{2}\right)}{\kappa^{3} \sqrt{\lambda}}+\frac{16 N \epsilon\left(1+\kappa^{2}\right)}{\kappa \sqrt{\lambda}}=2 k+\cdots . \tag{5.160}
\end{align*}
$$

In the same way, the conserved momentum in the abelian case is equal to the number of the strings (and then the fundamental string charge) attached to the brane, this two eigenvalues must correspond to the number of strings attached to each branes. Thus, we can expect that for $U(\ell)$ for any $\ell$, the $\ell \times \ell$ matrix momentum with $\ell$ eigenvalues which can be interpreted as the number of strings attached to each brane. Back to our problem, we are studying the rectangular horizontal tableau, so we impose that $\lambda_{1}=\lambda_{2}$, which leads to a form for $\epsilon$,

$$
\begin{equation*}
\epsilon=-\frac{\kappa^{2}}{3} \ll 1 . \tag{5.161}
\end{equation*}
$$

So, we expect $\kappa$ to be small. This fact could lead to think that $\kappa N$ is not large anymore, which we considered before when we included a subleading term in (5.75). This particular value of $\epsilon$, leads to write

$$
\begin{equation*}
\lambda_{1}=\lambda_{2}=\frac{8 N \kappa}{\sqrt{\lambda}}-\frac{8 N \kappa\left(1+\kappa^{2}\right)}{3 \sqrt{\lambda}}=2 k-\cdots, \tag{5.162}
\end{equation*}
$$

which should be the number of strings attached to each brane. On the other hand, if we compute the equations of motion and plug our ansatz in them, by expanding in $\epsilon$ up to order $\epsilon^{4}$ we can "solve" the equations for $\epsilon$, and get

$$
\begin{equation*}
\epsilon=-\frac{4 \kappa^{2}}{15} \ll 1 . \tag{5.163}
\end{equation*}
$$

Which is slightly different than our $\epsilon$ for equal momenta, but not too much. We can see that $\epsilon$ in ".161 "solves" the equation of motion up to order $\kappa^{8}$, so we can close the equations of motion and have equal momenta. Now, we need to add the corresponding boundary terms in the same way it was done in [32,50]. In this case we have five boundary terms, one associated with $\eta$ and the others corresponding to the conserved charges

$$
\begin{equation*}
\Pi_{\eta}=\frac{1}{2 \pi} \frac{\delta \mathcal{L}}{\delta \eta}, \quad i \Pi^{(0)}=\frac{1}{2 \pi} \frac{\delta \mathcal{L}}{\delta F_{\rho \psi}^{(0)}}=k^{(0)} \text { and } \Pi^{(i)}=\frac{1}{2 \pi} \frac{\delta \mathcal{L}}{\delta F_{\rho \psi}^{(i)}}=k^{(i)} . \tag{5.164}
\end{equation*}
$$

If we take the ansatz and plug it back into the action $S=S_{D B I}+S_{W Z}$, we get the onshell action

$$
\begin{equation*}
S_{D 3}^{\text {onshell }}=4 N\left(\kappa \sqrt{1+\kappa^{2}}-\sinh ^{-1} \kappa\right)+8 N \epsilon \kappa \sqrt{1+\kappa^{2}}+\frac{64 N^{2} \epsilon^{2} \kappa}{\eta_{0}} . \tag{5.165}
\end{equation*}
$$

Notice how we have nonabelian effects containing an IR divergence. The boundary terms are,

$$
\begin{equation*}
S_{\mathrm{bdy}, \eta}=-\left.\lim _{\eta_{0} \rightarrow 0} \int d \psi \eta \Pi_{\eta}\right|_{\eta_{0}}=-\frac{8 N \kappa}{\eta_{0}}-\frac{8 N \epsilon}{\kappa \eta_{0}}, \tag{5.166}
\end{equation*}
$$

and

$$
\begin{align*}
S_{\mathrm{bdy}, A^{(0)}}= & -\left.\lim _{\eta_{0} \rightarrow 0} \int d \psi d \rho i \Pi^{(0)} F_{\psi \rho}^{(0)}\right|_{\eta_{0}}=-8 N \kappa \sqrt{1+\kappa^{2}}+\frac{8 N \kappa}{\eta_{0}}-\frac{8 N \epsilon \sqrt{1+\kappa^{2}}}{\kappa}+\frac{8 N \epsilon}{\kappa \eta_{0}}, \\
& \sum_{i} S_{\mathrm{bdy}, A^{(i)}}=-\left.\lim _{\eta_{0} \rightarrow 0} \int d \psi d \rho \Pi^{(i)} F_{\psi \rho}^{(i)}\right|_{\eta_{0}}=\frac{8 N \epsilon^{2}}{\kappa}+\frac{8 N \epsilon^{2}}{\kappa \eta_{0}} . \tag{5.167}
\end{align*}
$$

Then, the regularized worldvolume is

$$
\begin{equation*}
S_{\mathrm{tot}}=S_{D 3}^{\mathrm{onshell}}+S_{\mathrm{bdy}, \eta}+S_{\mathrm{bdy}, A^{(0)}}+\sum_{i} S_{\mathrm{bdy}, A^{(i)}}, \tag{5.169}
\end{equation*}
$$

thus,
$S_{\text {tot }}=-4 N\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right)+8 N \epsilon \kappa \sqrt{1+\kappa^{2}}-\frac{8 N \epsilon \sqrt{1+\kappa^{2}}}{\kappa}+\frac{8 N \epsilon^{2}}{\kappa}+\frac{8 N \epsilon^{2}}{\kappa \eta_{0}}$.

Notice that the first term in the last result corresponds to $2 S_{\text {Drukker-Fiol }}$ plus corrections and an unwanted still divergent term ${ }^{5}$. We can write the last result as in (5.112)

$$
\begin{equation*}
\left\langle W_{\mathcal{R}_{\text {hor }}}\right\rangle_{\ell=2}=\exp \left(4 N\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right)\right)\left(1+8 N \epsilon \kappa+\frac{8 N \epsilon \sqrt{1+\kappa^{2}}}{\kappa}\right) \tag{5.171}
\end{equation*}
$$

In this chapter we tried to use the DBI action written by Myers in its exact form to try to match with the gauge theory result in (5.112). Both results are quite new, and in principle could be different because we are working with a deformation of the Drukker-Fiol result and, in general, with expansions.

It is important to mention that we worked with the compact form of the NADBI action proposed by Myers, which is known to not capture the full IR physics (small $\alpha^{\prime}$ ). Let

[^18]us say some words about it. In the same way the abelian action can be expanded to see how it contains the abelian Maxwell theory plus nonlinear corrections, we can expand the nonabelian action 5.113) and 5.114 and get terms over all orders of $F_{a b}, D_{a} \Phi^{i}$ and $\left[\Phi^{i}, \Phi^{j}\right]$, but this expansion actually matches with string theory only up to fourth order in $F$. At sixth order, symmetrized trace will lead to produce commutators between $F$ 's [77, 120, 121], they in turn will introduce derivatives of $F$ leading to an ambiguity. Reviews about higher order corrections can be found in [124-127].

It would be interesting to get to the bottom of these ambiguities and produce a successfull match between these quantities on the gauge theory side and on the gravity side.

## Chapter 6

## Wilson loop correlator in holography: defects and phase transition(s)

If instead of the expectation value of a single Wilson loop, one wanted to compute the correlator between two loops, the relevant string surface would be the one connecting the two different contours. This is similar to a well-known (solved) problem called the Plateau's problem, in which one has to determine the shape of a thin soap film stretched between two rings lying on parallel planes (see [51] and referenced textbooks therein). Modifying the geometry (radii and distance) of these two rings introduces a phase diagram: there are critical values for the parameters that allow to go from the "catenoid" solution to the so-called "Goldschmidt" solution, which corresponds to two separated "domes". When the two rings are separated beyond a certain critical value, the catenoid solution becomes unstable and breaks into the Goldschmidt one. Similarly, in the case of the Wilson loop, the string world-sheet describes a catenoid-like solution until, at certain values of the radii of the loops and the distance, it becomes energetically unfavored with respect to the Goldschmidt-like solution. This transition is called a Gross-Ooguri phase transition [48]. The discontinuous solution corresponds in this case to two minimal surfaces in $A d S$, each attached to a different loop. These minimal surfaces are the onshell regularized actions with each loop as boundary.

The expectation value of a Wilson loop in the fundamental representation was reviewed in chapter 3. By looking at the result from localization, (3.64), we see that it does not depend neither on the radius of the loop nor on its distance with respect to the origin. This
can be seen also from the definition itself in (3.11), if we now consider

$$
\begin{equation*}
x^{\mu}=(R \cos s, R \sin s, L, 0), \tag{6.1}
\end{equation*}
$$

the expectation value of the Wilson loop does not change.
If we consider now two disconnected (separated) and independent loops, say at $x=L$ and $x=L+h$, the total action is trivially given by twice the value above, because the result does not depend on the position in $x$ ( $x_{3}$ in the four-dimensional space)

$$
\begin{equation*}
S_{0}=-2 \sqrt{\lambda} \tag{6.2}
\end{equation*}
$$

As we will see, this is not the case when a defect is introduced. The position of the loops with respect to the defect is going to add a parameter to the system and make things more interesting.

An interesting setup that has received some recent attention in holography [15, 16] is the one in which a defect is introduced in the gauge theory. This defect is typically obtained by considering systems of intersecting branes [15]. In particular, intersecting D3 and D5 branes along the $\left\{x_{0}, x_{1}, x_{2}\right\}$-directions allow to construct three-dimensional defects inside the four-dimensional world-volume of the $N \mathrm{D} 3$ branes corresponding to the $\mathcal{N}=4$ SYM directions. The end result is that there are two different gauge groups on each side of the defect brane (see [15] and [63, 128] for recent reviews). This is because $n$ D3 branes now end on the D5 defect, so on one side we have the usual $\operatorname{SU}(N)$ and on the other side we have $S U(N-n)$.


Figure 6.1: Scheme of the construction of the DCFT due to the intersection of a D5 branes with $N$ D3 branes. For our study we will focus only on the $S U(N)$ side $\left(x_{3}>0\right)$.

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| N D3 | $\times$ | $\times$ | $\times$ | $\times$ |  |  |  |  |  |  |
| D5 | $\times$ | $\times$ | $\times$ |  | $\times$ | $\times$ | $\times$ |  |  |  |

Table 6.1: Brane configuration of the D3-D5 system.

In the string theory side, the solution corresponding to a single D5 wall inside $A d S_{5} \times$ $S^{5}$ was obtained [60] by considering that the D5 brane introduces a "magnetic" two-form flux that couples with the four-form of the D3 branes and integrates on a $S^{2} \subset S^{5}$. The expectation value of a single Wilson loop in this case was calculated [63]. In this case, the minimal world-sheet surface is attached to the loop and ends on the defect. The presence of the defect introduces new boundary conditions for the string: the usual ones along the loop and the ones that ensure that the world-sheet surface ends on the defect.


Figure 6.2: Scheme of the solution of a D5 (probe) defect in $A d S_{5} \times S^{5}$.

In this chapter, we compute the Wilson loop correlator in the presence of a D5 defect from holography. We study the Gross-Ooguri phase transition in this geometry, investigating in detail how this transition depends on the (numerous) parameters of the setting. We compare this case with the case without any defect present and observe that the defect allows the connected surface to survive at larger values of the distance. The Gross-Ooguri phase transition is still present but now it depends on more parameters, those coming from the defect.

In section 6.1 we review the Wilson loop correlator, in which two Wilson loops in the fundamental representation are connected by a catenoid-like solution. We extend the analysis of [129] by considering different radii and plot the behavior of the connected solution, and its limiting parameters and the phase transition to two disconnected Wilson loops without defect. In Appendix A we study the holographic symmetric/fundamental Wilson loop which correspond to the solution connecting a a D3 brane and a loop. This follows the study in [78] in which the antisymmetric/fundamental correlator was studied. In section 6.2 we review the connected fundamental/fundamental correlator when polar angles in $S^{5}$ are considered. In section 6.3 we review the case of Wilson loops ending on a D5 brane defect as studied in [63, 64]. We will consider two loops ending on the defect as the disconnected phase instead of the two Wilson loops without the defect. In
section 6.4 we study the phase transition from the connected catenoid-like solution to the disconnected loops ending on the defect and discover that the presence of the defect allows a longer connected phase of the loop at certain values of the parameters.

### 6.1 Two connected Wilson loops with $\Delta \phi=0$

The connected correlator of two concentric (fundamental) Wilson loops, with radii $R_{1}$ and $R_{2}$ and separated by a distance $h$, was initially studied in [52, 53, 79, 130] with no separation in $S^{5}$, and later reviewed in [58] where it was considered also a separation in $S^{5}$. It is given by the action of a string attached to the two circles $C_{1}$ and $C_{2}$ at the boundary of $A d S_{5}$

$$
\begin{equation*}
\left\langle W\left(C_{1}\right) W\left(C_{2}\right)\right\rangle_{\text {conn }}=\exp \left(-S_{\text {conn }}\right) \tag{6.3}
\end{equation*}
$$

The Nambu-Goto action that describes the world-sheet area is [53]

$$
\begin{equation*}
S_{\mathrm{conn}}=\frac{\sqrt{\lambda}}{2 \pi} \int d \tau d \sigma \sqrt{\operatorname{det} g} \tag{6.4}
\end{equation*}
$$

with the background metric

$$
\begin{equation*}
d s^{2}=\frac{1}{y^{2}}\left(d y^{2}+d x_{0}^{2}+d r^{2}+r^{2} d \varphi^{2}+d x^{2}\right)+d \phi^{2} \tag{6.5}
\end{equation*}
$$

and the world-sheet coordinates $\tau=\varphi \in[0,2 \pi]$ and $\sigma=x$, so that ${ }^{1}$

$$
\begin{equation*}
S_{\mathrm{conn}}=\sqrt{\lambda} \int_{L}^{L+h} d x \frac{r}{y^{2}} \sqrt{1+r^{\prime 2}+y^{\prime 2}} \tag{6.6}
\end{equation*}
$$

In this ansatz, the coordinates $r$ and $y$ have been taken to depend on $x$ (the prime denotes a derivative w.r.t. $x$ ). The corresponding equations of motion are

$$
\begin{equation*}
r^{\prime \prime}-\frac{r}{k^{2} y^{4}}=0, \quad y^{\prime \prime}+\frac{2 r^{2}}{k^{2} y^{5}}=0 \tag{6.7}
\end{equation*}
$$

where

$$
\begin{equation*}
k=\frac{r}{y^{2}} \frac{1}{\sqrt{1+r^{\prime 2}+y^{\prime 2}}} \tag{6.8}
\end{equation*}
$$

is a constant of motion coming from the independence of the Lagrangian on $x \|^{2}$ Then, the action becomes

$$
\begin{equation*}
S_{\mathrm{conn}}=\frac{\sqrt{\lambda}}{k} \int_{L}^{L+h} d x \frac{r^{2}}{y^{4}} \tag{6.9}
\end{equation*}
$$

[^19]The boundary conditions we require are

$$
\begin{gather*}
y(L)=0=y(L+h),  \tag{6.10}\\
r(L)=R_{2}, \quad r(L+h)=R_{1} . \tag{6.11}
\end{gather*}
$$

Notice that these expression are the sames as in [52, 53, 79, 129] and also in [58]. These conditions come from the configuration of the loops: the first loop, with radius $R_{2}$, is located at $x=L$; and the second loop, with radius $R_{1}$, is located at $x=L+h$. Both loops are located at the boundary of $A d S$, so we require that at $x=L$ and $x=L+h$, $y=0$. By combining the equations of motion (6.7) and (6.8), we get

$$
\begin{equation*}
r^{2}+y^{2}+(x+\tilde{c})^{2}=a^{2}, \quad \tilde{c}=c-L \tag{6.12}
\end{equation*}
$$

where the constants $a$ and $c$ are determined by the boundary conditions

$$
\begin{equation*}
c=\frac{R_{2}^{2}-R_{1}^{2}}{2 h}-\frac{h}{2}, \quad a^{2}=c^{2}+R_{2}^{2} . \tag{6.13}
\end{equation*}
$$

Here we have introduced a shifted constant $\tilde{c}$ to define $c$ in the same way as [52,53]. The reparametrization

$$
\begin{equation*}
r=\sqrt{a^{2}-(x+\tilde{c})^{2}} \cos \theta(x), \quad y=\sqrt{a^{2}-(x+\tilde{c})^{2}} \sin \theta(x) \tag{6.14}
\end{equation*}
$$

allows us to write (6.8) as

$$
\begin{equation*}
\frac{d \theta}{d x}= \pm \frac{a}{a^{2}-(x+\tilde{c})^{2}} \frac{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta}}{k a \sin ^{2} \theta} \tag{6.15}
\end{equation*}
$$

where $\theta(x) \in[0, \pi / 2]$ satisfies the boundary conditions (see 6.10p)

$$
\begin{equation*}
\theta(L)=0=\theta(L+h) . \tag{6.16}
\end{equation*}
$$

The function $\theta$ grows (plus sign in 6.15) up to some maximum $\theta_{0}$ in $x=x_{0}$, for $L \leq$ $x<x_{0}$, and then decreases (minus sign in (6.15) for $x_{0}<x \leq L+h$. Thus, the integrals to consider are

$$
\begin{equation*}
\int_{\theta(L)=0}^{\theta\left(x_{0}\right)=\theta_{0}} d \theta \frac{k a \sin ^{2} \theta}{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta}}=+\int_{L}^{x_{0}} \frac{a}{a^{2}-(x+\tilde{c})^{2}} \tag{6.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\theta\left(x_{0}\right)=\theta_{0}}^{\theta(L+h)=0} d \theta \frac{k a \sin ^{2} \theta}{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta}}=-\int_{x_{0}}^{L+h} \frac{a}{a^{2}-(x+\tilde{c})^{2}} . \tag{6.18}
\end{equation*}
$$

The left hand side of these integrals is the same so we add them. The maximum of $\theta, \theta_{0}$, occurs when $\theta^{\prime}\left(x_{0}\right)=0$ in 6.15. We can write then

$$
\begin{equation*}
\theta_{0}(k a)=\arccos \left(\frac{\sqrt{4 k^{2} a^{2}+1}-1}{2 k a}\right), \tag{6.19}
\end{equation*}
$$

where $\theta\left(x_{0}\right)=\theta_{0}$. Since the angular integral is the same on both sides we define

$$
\begin{equation*}
\mathcal{F}(k a)=\int_{0}^{\theta_{0}} d \theta \frac{k a \sin ^{2} \theta}{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta}}=\frac{1}{4} \ln \frac{a+h+c}{a-h-c}-\frac{a+c}{a-c}, \tag{6.20}
\end{equation*}
$$

where we have integrated separately $L \leq x \leq x_{0}$ and $x_{0} \leq x \leq L+h$. Notice that $k a$ becomes an important geometric parameter to control the behavior of the connected solution.

The right hand side of 6.20 is independent of $L$, because of the choice of constants in (6.13) that allows us to write $\mathcal{F}$ just like in [53]. The integral in (6.20) can be written as in [129]

$$
\begin{equation*}
\mathcal{F}(k a)=\int_{0}^{\theta_{0}} d \theta \frac{k a \sin ^{2} \theta}{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta}}=\frac{1}{2} \int_{t_{0}}^{1} d t \frac{\sqrt{1-t}}{\sqrt{-t\left(t-\beta_{+}\right)\left(t-\beta_{-}\right)}}, \tag{6.21}
\end{equation*}
$$

where we defined the change of variable $t=\cos ^{2} \theta$, and

$$
\begin{equation*}
\beta_{ \pm}=\frac{\left(1+2 k^{2} a^{2}\right) \pm \sqrt{1+4 k^{2} a^{2}}}{2 k^{2} a^{2}}=1+\frac{1}{2 k^{2} a^{2}} \pm \frac{\sqrt{1+4 k^{2} a^{2}}}{2 k^{2} a^{2}} \tag{6.22}
\end{equation*}
$$

where $\beta_{+}=1 / \beta_{-}$. Also, notice that

$$
\begin{equation*}
t_{0}=\cos ^{2} \theta_{0}=\left(\frac{\sqrt{4 k^{2} a^{2}+1}-1}{2 k a}\right)^{2}=1+\frac{1}{2 k^{2} a^{2}}-\frac{\sqrt{1+4 k^{2} a^{2}}}{2 k^{2} a^{2}}=\beta_{-}, \tag{6.23}
\end{equation*}
$$

so (6.21) becomes

$$
\begin{equation*}
\mathcal{F}(k a)=\frac{1}{2} \int_{\beta_{-}}^{1} d t \frac{\sqrt{1-t}}{\sqrt{-t\left(t-\beta_{+}\right)\left(t-\beta_{-}\right)}} . \tag{6.24}
\end{equation*}
$$

(6.24) can be written in terms of elliptic integrals (see 255.21 in [131]; see also [102]). Thus, ${ }^{3}$

$$
\begin{equation*}
\mathcal{F}(k a)=\frac{\beta_{+}-1}{\sqrt{\beta_{+}-\beta_{-}}}\left(\Pi\left(\frac{1-\beta_{-}}{\beta_{+}-\beta_{-}}, \frac{\left(1-\beta_{-}\right) \beta_{+}}{\left(\beta_{+}-\beta_{-}\right)}\right)-K\left(\frac{\left(1-\beta_{-}\right) \beta_{+}}{\left(\beta_{+}-\beta_{-}\right)}\right)\right) . \tag{6.25}
\end{equation*}
$$

From the right hand side of (6.20) we have

$$
\begin{equation*}
\mathcal{F}\left(R_{1}, R_{2}, h\right)=\frac{1}{4} \ln \frac{a+h+c}{a-h-c}-\frac{a+c}{a-c}, \tag{6.26}
\end{equation*}
$$

[^20]so, by replacing $c$ and $a$ from (6.13), we get
\[

$$
\begin{equation*}
\mathcal{F}\left(R_{1}, R_{2}, h\right)=\frac{1}{2} \ln \left(\frac{R_{1}^{2}+R_{2}^{2}+h^{2}+\sqrt{\left(R_{2}^{2}-R_{1}^{2}\right)^{2}+h^{4}+2 h^{2}\left(R_{1}^{2}+R_{2}^{2}\right)}}{2 R_{1} R_{2}}\right), \tag{6.27}
\end{equation*}
$$

\]

or

$$
\begin{equation*}
h=R_{2} \sqrt{2 \alpha\left(1+2 \sinh ^{2} \mathcal{F}\right)-\alpha^{2}-1}, \tag{6.28}
\end{equation*}
$$

where we defined $R_{1}=\alpha R_{2}$ and $a$ was defined in (6.13). From (6.28) we see that there is a maximum value of $h$,

$$
\begin{equation*}
h_{\max }=R_{2} \sinh 2 \mathcal{F}, \tag{6.29}
\end{equation*}
$$

when

$$
\begin{equation*}
\alpha=1+2 \sinh ^{2} \mathcal{F} \geq 1 \tag{6.30}
\end{equation*}
$$

Notice that $\mathcal{F}$ was defined in two ways: in (6.25) and (6.27), in terms of $k a$ and $R_{1}, R_{2}$ and $h$, respectively. But we know that $a$ is defined also in terms of $R_{1}, R_{2}$ and $h$ 6.13, or in terms of $\alpha$ and $h$ if we fix $R_{2}=1$. In principle, $k$, as defined in 6.8), could take any value, but now we know it is connected to the geometry by $\mathcal{F}$, it is useful to solve and plot (6.28)

$$
\begin{equation*}
h=R_{2} \sqrt{2 \alpha\left(1+2 \sinh ^{2} \mathcal{F}\left(k, \alpha, R_{2}, h\right)\right)-\alpha^{2}-1} \tag{6.31}
\end{equation*}
$$

to show how the geometry, i.e $\alpha$ and $h$, imposes values on $k$ or viceversa.
Notice, from 6.28), or 6.31, that $h$ is obviously invariant under the exchange of $R_{1} \rightarrow R_{2}$, i.e. $\alpha \rightarrow 1 / \alpha$. We understand this "symmetry" geometrically as the fact that the connected solution looks the same under exchange of radii.

In Figure 6.3 we show how the integration parameter controls the allowed values of $h$. For $R_{1}=R_{2}$ we reproduce the plot in [129] (black curve), where, in general they study the finite temperature case (see also [132]). There we see that the maximum value of the distance between the loops, $h_{\text {max }}$, changes if we now change the ratio between the radii, $\alpha=R_{1} / R_{2}$, around the values that gave $\alpha=1$. Although in Figure 6.3 we considered five pairs of values for the radii, only three of them appears. This is due to the obvious symmetry under $\alpha \rightarrow 1 / \alpha$ we saw before in (6.31). We also see that there is an explicit maximum for $k$ except for equal radii, $\alpha=1$. This is interesting because we see how restricted is the geometry of the solution even in this simple case. This maximum
becomes infinite as $\alpha \rightarrow 1$, so, the $\alpha=1$ curve becomes a limit for the allowed values of $k$ and $h$.


Figure 6.3: Behavior of the integration parameter $k$ with the geometric parameters, $R_{1}, R_{2}$ and $h$. We see that, for a given pair of radii, there is an exchange symmetry due to the form of the connected solution. Also, for each pair of radii, there is a maximum value of $h$ and $k$.

From Figure 6.3 we see that only for $\alpha \approx 1$, we can set $k \rightarrow \infty$. In Figure 6.4 we see how $h_{\max }$ changes as we increase the radii with $\alpha=1$. This is quite obvious because the larger the radii the larger the allowed distance between the circles for the connected solution. We could vary the radii around $\alpha=1$, for each pair of them, to see how every curve acts as a limit for the allowed values of $k$ and $h$.


Figure 6.4: Behavior of the integration parameter $k$ with the geometric parameters, $R_{1}, R_{2}$ and $h$. In this case we show the $k-h$ curve for increasing $R_{1}=R_{2}$ values.

In general, we see that $k$ and $h$ depict an allowed region, outside of which the connected solution ceases to exist. This same behavior will be also clear for the action, as we will see below.

### 6.1.1 Evaluating the action

From (6.14) and (6.15), the action (6.9) becomes

$$
\begin{equation*}
S_{\mathrm{conn}}=\frac{\sqrt{\lambda}}{k a} \int_{L}^{L+h} d x \frac{a}{a^{2}-(x+c)^{2}} \frac{\cos ^{2} \theta}{\sin ^{4} \theta}=2 \sqrt{\lambda} \int_{0}^{\theta_{0}} d \theta \frac{\cot ^{2} \theta}{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta}}, \tag{6.32}
\end{equation*}
$$

where the factor of 2 comes from including the two branches of $\theta$.
This action is actually divergent. From (6.10) and (6.14), we see in (6.16) that we must regularize now for $\theta$ at $x=L$ and $x=L+h$. Thus, from (6.14), we introduce the regulator $\epsilon$ as

$$
\begin{align*}
\theta(L) & =\arctan \frac{\epsilon}{R_{1}} \approx \frac{\epsilon}{R_{1}}, \\
\theta(L+h) & =\arctan \frac{\epsilon}{R_{2}} \approx \frac{\epsilon}{R_{2}} . \tag{6.33}
\end{align*}
$$

Because of this, the action (6.32) must be written as

$$
\begin{equation*}
S_{\mathrm{conn}}=\sqrt{\lambda} \int_{\epsilon / R_{1}}^{\theta_{0}} d \theta \frac{\cot ^{2} \theta}{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta}}+\sqrt{\lambda} \int_{\epsilon / R_{2}}^{\theta_{0}} d \theta \frac{\cot ^{2} \theta}{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta}} \tag{6.34}
\end{equation*}
$$

Let us work first with a generic radius, $R$, to understand how to regularize the action:

$$
\begin{equation*}
S_{\mathrm{conn}}=\sqrt{\lambda} \int_{\epsilon / R}^{\theta_{0}} \frac{\cot ^{2} \theta}{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta}} \tag{6.35}
\end{equation*}
$$

Under the redefinition $t=\cos ^{2} \theta$, 6.35 becomes

$$
\begin{equation*}
S_{\mathrm{conn}}=\frac{\sqrt{\lambda}}{2 k a} \int_{\cos ^{2} \epsilon / R}^{\beta_{-}} d t \frac{\sqrt{-t}}{(t-1) \sqrt{\left(\beta_{-}-t\right)(t-1)\left(t-\beta_{+}\right)}}=\frac{\sqrt{\lambda}}{2 k a} I(k a), \tag{6.36}
\end{equation*}
$$

which can expressed as in 255.04 of [131]

$$
\begin{equation*}
I(k a)=\left(\frac{A-B}{B-C}\right) \frac{g}{w^{\prime 2}}\left(w^{\prime 2} u_{1}-E\left(\varphi, w^{2}\right)+\operatorname{dn} u_{1} \operatorname{tn} u_{1}\right) \tag{6.37}
\end{equation*}
$$

with

$$
\begin{equation*}
D=\beta_{+}, \quad C=1, \quad y=\cos \frac{\epsilon}{R}, \quad B=\beta_{-}, \quad A=0 . \tag{6.38}
\end{equation*}
$$

With these values

$$
\begin{align*}
\varphi & =\arcsin \sqrt{\frac{\cos ^{2} \epsilon / R-\beta_{-}}{\left(1-\beta_{-}\right) \cos ^{2} \epsilon / R}}=\arcsin \left(1+\frac{\beta_{-}}{1-\beta_{-}} \frac{\epsilon^{2}}{R^{2}}\right) \\
w^{2} & =1-w^{\prime 2}=\frac{\left(1-\beta_{-}\right) \beta_{+}}{\left(\beta_{+}-\beta_{-}\right)}, \quad g=\frac{2}{\sqrt{\beta_{+}-\beta_{-}}} \tag{6.39}
\end{align*}
$$

where $\beta_{ \pm}$were defined in 6.22. If $\epsilon=0$, then $\varphi=\pi / 2$ and $u_{1}=F\left(\pi / 2, w^{2}\right)=$ $K\left(w^{2}\right)=K$. Since this is not the case, we expand around $u=K$ instead.

If we naively evaluate (6.37) at $u_{1}=K\left(w^{2}\right)$ we see that it diverges, the divergence coming from $\operatorname{tn} u=\operatorname{sn} u / \mathrm{cn} u$. Expanding dn $u \operatorname{tn} u \approx-1 /(u-K)$. We need to know $u-K$ in terms of $\epsilon / R$, so we expand $\operatorname{sn} u$ around $u=K$ and compare with the expansion in (6.39)

$$
\begin{equation*}
\operatorname{sn} u \approx 1-\frac{1}{2} w^{\prime 2}(u-K)^{2}=\sin \varphi \approx 1+\frac{\beta_{-}}{1-\beta_{-}} \frac{\epsilon^{2}}{R^{2}}, \tag{6.40}
\end{equation*}
$$

from which we discover that

$$
\begin{equation*}
(u-K) \approx \frac{1}{w^{\prime}} \sqrt{\frac{\beta_{-}}{1-\beta_{-}}} \frac{\epsilon}{R} . \tag{6.41}
\end{equation*}
$$

Replacing the divergent term by its approximation when $\epsilon \rightarrow 0$ we get

$$
\begin{equation*}
S_{\mathrm{conn}}=2 \sqrt{\lambda} \frac{R}{\epsilon}+\left(1+4 k^{2} a^{2}\right)^{1 / 4}\left(\left(1-w^{2}\right) K\left(w^{2}\right)-E\left(w^{2}\right)\right) . \tag{6.42}
\end{equation*}
$$

Now applying (6.42) to the case of different radii, we get

$$
\begin{equation*}
S_{\mathrm{conn}}=2 \sqrt{\lambda} \frac{R_{1}+R_{2}}{\epsilon}+2\left(1+4 k^{2} a^{2}\right)^{1 / 4}\left(\left(1-w^{2}\right) K\left(w^{2}\right)-E\left(w^{2}\right)\right) . \tag{6.43}
\end{equation*}
$$

The regularized action is then

$$
\begin{equation*}
S_{\mathrm{conn}}=2 \sqrt{\lambda}\left(1+4 k^{2} a^{2}\right)^{1 / 4}\left(\left(1-w^{2}\right) K\left(w^{2}\right)-E\left(w^{2}\right)\right) . \tag{6.44}
\end{equation*}
$$

Notice that when $k a=0, \beta_{-}=0$ then $w=1$. Since $E(1)=1$,

$$
\begin{equation*}
S_{\mathrm{conn}}=-2 \sqrt{\lambda}, \tag{6.45}
\end{equation*}
$$

which corresponds precisely to the case of two disconnected loops without defect. We can see this behavior in Figure 6.5. The action in (6.44) does not depend explicitly on the radii, $R_{1}$ and $R_{2}$, and the distance between loops, $h$, but on $k a$. These geometric parameters are encoded in $\mathcal{F}$ by (6.27).

In Figure 6.5, we plot the connected action $S_{\text {conn }}$ as a function of $h$ and see that there are several maxima corresponding to the same behavior of the $\mathcal{F}$ function with $k a$. We see three regions: the region below the dotted line, i.e the stable disconnected solution; the region above the dotted line but below the cusp, i.e the metastable solution, in which the disconnected (without defect in this case) solution is favored; and the unstable solution that touches the disconnected solution from above when $k a \rightarrow 0$ and goes to the cusp from the left. This cusp corresponds to the maximum value for the distance between the loops, $h_{\max }$, beyond which the connected solution does not exist. From Figure 6.5 we also see the well-known Gross-Ooguri phase transition, in which the disconnected solution becomes energentically favored with respect to the connected solution. As the figure shows, it happens at a distance, $h_{0}<h_{\max }$, so before the connected solution ceases to exist.

We also notice that the $\alpha=1$ curve becomes a limiting curve for the pair $\left(R_{1}, R_{2}\right)$. As we saw before, in 6.28) and Figure 6.3, the symmetry under the exchange of radii $\left(R_{1}, R_{2}\right) \rightarrow\left(R_{2}, R_{1}\right)$, which geometrically was understood due to the form of the connected solution, leads to the curves with $\alpha$ and $1 / \alpha$ being the same. This means that the connected solution preserves the area under this exchange of radii.

From Figure 6.6 we see the same behavior as in Figure 6.4. if we increase the radii, in particular with $\alpha=1$, the connected stable distance, $h_{0}$, also increases.


Figure 6.5: Connected solution parametrized by $k a$. For different values of the radii and $\alpha$, we see that there is a two-branch structure in the connected action due to the behavior of $h$ with $k$. We see also the usual (Gross-Ooguri) phase transition to the disconnected action, $S_{0} / \sqrt{\lambda}=-2$, described in [48,51,54], at some value $h_{0}<h_{\text {max }}$.


Figure 6.6: Connected solution parametrized by $k a$. For increasing values of the radii, in particular with $\alpha=1$, the distance for the Gross-Ooguri phase transition also increases.

### 6.2 Two connected Wilson loops with $\Delta \phi \neq 0$

Let us consider now the same case as in section 6.1 but now with a separation in $S^{5}$ as was considered recently in [58]. The action now is

$$
\begin{equation*}
S_{\mathrm{conn}}=\sqrt{\lambda} \int_{L}^{L+h} d x \frac{r}{y^{2}} \sqrt{1+r^{\prime 2}+y^{\prime 2}+y^{2} \phi^{\prime 2}} \tag{6.46}
\end{equation*}
$$

The corresponding equations of motion are now

$$
\begin{equation*}
r^{\prime \prime}-\frac{r}{k^{2} y^{4}}=0, \quad y^{\prime \prime}+\frac{2 r^{2}}{k^{2} y^{5}}-\frac{\ell^{2}}{k^{2} y^{3}}=0, \tag{6.47}
\end{equation*}
$$

where now we have two constants of motion:

$$
\begin{equation*}
k=\frac{r}{y^{2} \sqrt{1+r^{\prime 2}+y^{\prime 2}+y^{2} \phi^{\prime 2}}}, \quad \ell=\frac{r \phi^{\prime}}{\sqrt{1+r^{\prime 2}+y^{\prime 2}+y^{2} \phi^{\prime 2}}} \tag{6.48}
\end{equation*}
$$

are constants of motion coming from the independence of the Lagrangian on $x$ and $\phi \square^{4}$ Then, the action becomes the same as (6.9). The boundary conditions we require are the same as 6.10 and 6.11 . These conditions come from the configuration of the loops: the

[^21]first loop, with radius $R_{2}$, is located at $x=L$; and the second loop, with radius $R_{1}$, is located at $x=L+h$. Both loops are located at the boundary of $A d S$, so we require that at $x=L$ and $x=L+h, y=0$. By combining the equations of motion (6.47) and (6.48), we get, again,
\[

$$
\begin{equation*}
r^{2}+y^{2}+(x+\tilde{c})^{2}=a^{2}, \quad \tilde{c}=c-L, \tag{6.49}
\end{equation*}
$$

\]

where the constants $a$ and $c$ are determined by the boundary conditions

$$
\begin{equation*}
c=\frac{R_{2}^{2}-R_{1}^{2}}{2 h}-\frac{h}{2}, \quad a^{2}=c^{2}+R_{2}^{2} . \tag{6.50}
\end{equation*}
$$

Here we have introduced a shifted constant $\tilde{c}$ as in section 6.1. The reparametrization

$$
\begin{equation*}
r=\sqrt{a^{2}-(x+\tilde{c})^{2}} \cos \theta(x), \quad y=\sqrt{a^{2}-(x+\tilde{c})^{2}} \sin \theta(x) \tag{6.51}
\end{equation*}
$$

allows us to write (6.48) as

$$
\begin{equation*}
\frac{d \theta}{d x}= \pm \frac{a}{a^{2}-(x+\tilde{c})^{2}} \frac{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta-\ell^{2} \sin ^{2} \theta}}{k a \sin ^{2} \theta} \tag{6.52}
\end{equation*}
$$

where $\theta(x) \in[0, \pi / 2]$ satisfies the boundary conditions (see 6.10))

$$
\begin{equation*}
\theta(L)=0=\theta(L+h) . \tag{6.53}
\end{equation*}
$$

The maximum value of $\theta, \theta_{0}$, can be written as

$$
\begin{equation*}
s=\sin ^{2} \theta_{0}=\frac{\sqrt{4 k^{2} a^{2}+\left(1+\ell^{2}\right)^{2}}-\left(1+\ell^{2}\right)}{2 k^{2} a^{2}} \tag{6.54}
\end{equation*}
$$

So $0 \leq s \leq 1$. This produces a similar angular integral as in (6.55)

$$
\begin{equation*}
\int_{0}^{\theta_{0}} d \theta \frac{k a \sin ^{2} \theta}{\sqrt{\cos ^{2} \theta-k^{2} a^{2} \sin ^{4} \theta-\ell^{2} \sin ^{2} \theta}}=\frac{1}{4} \ln \frac{a+h+c}{a-h-c}-\frac{a+c}{a-c} . \tag{6.55}
\end{equation*}
$$

The 1.h.s of (6.55) can be written as

$$
\begin{equation*}
\mathcal{F}(s, t)=\int_{0}^{1} d z \frac{\sqrt{s t} z^{2}}{\sqrt{\left(1-z^{2}\right)\left(1-s z^{2}\right)\left(1+t z^{2}\right)}}, \quad t=k^{2} a^{2} s^{2} . \tag{6.56}
\end{equation*}
$$

where $z=\sin \theta / \sqrt{s}$. The internal separation (see Figure 6.7) ca be written as

$$
\begin{equation*}
\gamma=\Delta \phi=\int_{L}^{L+h} d x \phi^{\prime}=\int_{L}^{L+h} d x \frac{\ell}{k} \frac{1}{\left(a^{2}-(x+c-L)^{2}\right) \sin ^{2} \theta} \tag{6.57}
\end{equation*}
$$

or

$$
\begin{equation*}
\gamma(s, t)=\int_{0}^{1} d z \frac{2 \sqrt{1-s-t}}{\sqrt{\left(1-z^{2}\right)\left(1-s z^{2}\right)\left(1+t z^{2}\right)}} . \tag{6.58}
\end{equation*}
$$

In the same way, (6.9) becomes

$$
\begin{equation*}
S_{\mathrm{conn}}=2 \sqrt{\lambda} \int_{0}^{1} d z \frac{\sqrt{1-s z^{2}}}{z^{2} \sqrt{s\left(1-z^{2}\right)\left(1+t z^{2}\right)}} . \tag{6.59}
\end{equation*}
$$



Figure 6.7: Internal separation of the loops in $S^{5}$. In section 6.4 we will set $\gamma=\chi_{1}-\chi_{2}$, where $\chi_{1,2}$ are polar angles on the $S^{5}$.

Under the following change of variable:

$$
\begin{equation*}
\frac{z^{2}}{1-z^{2}}=\frac{1}{1+t} \frac{u^{2}}{1-u^{2}}, \tag{6.60}
\end{equation*}
$$

(6.56) becomes

$$
\begin{equation*}
\mathcal{F}(s, t)=\sqrt{\frac{s}{t}} \frac{1}{\sqrt{1+t}} \int_{0}^{1} d u \frac{\alpha^{2} u^{2}}{\sqrt{\left(1-u^{2}\right)\left(1-r^{2} u^{2}\right)\left(1-\alpha^{2} u^{2}\right)}}, \tag{6.61}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha^{2}=\frac{t}{1+t}, \quad r^{2}=\frac{s+t}{1+t} . \tag{6.62}
\end{equation*}
$$

This integral can be written in terms of elliptic integrals directly as (see definitions 110.02 and 400.01 in [131])

$$
\begin{equation*}
\mathcal{F}(s, t)=\sqrt{\frac{s}{t}} \frac{1}{\sqrt{1+t}}\left(-K\left(\frac{s+t}{1+t}\right)+\Pi\left(\frac{t}{1+t}, \frac{s+t}{1+t}\right)\right) . \tag{6.63}
\end{equation*}
$$

This is equivalent (see 117.03 in [131]) to

$$
\begin{equation*}
\mathcal{F}(s, t)=\sqrt{\frac{t}{s}} \frac{1}{\sqrt{1+t}}\left(K\left(\frac{s+t}{1+t}\right)-(1-s) \Pi\left(s, \frac{s+t}{1+t}\right)\right) \tag{6.64}
\end{equation*}
$$

given in [58]. Under (6.60) the integral in (6.58) can be written as

$$
\begin{equation*}
\gamma(s, t)=\frac{2 \sqrt{1-s-t}}{\sqrt{1+t}} \int_{0}^{1} d u \frac{1}{\sqrt{\left(1-u^{2}\right)\left(1-r^{2} u^{2}\right)}}, \tag{6.65}
\end{equation*}
$$

which corresponds to the definition 110.02 in [131], so

$$
\begin{equation*}
\gamma(s, t)=\frac{2 \sqrt{1-s-t}}{\sqrt{1+t}} K\left(\frac{s+t}{1+t}\right) . \tag{6.66}
\end{equation*}
$$

This implies that $s+t \leq 1$. The action 6.59 becomes

$$
\begin{equation*}
S_{\mathrm{conn}}=\frac{2 \sqrt{\lambda}}{\sqrt{s}} \frac{1}{\sqrt{1+t}} \int_{\epsilon}^{1} d u \frac{\sqrt{1-\alpha^{2} u^{2}}}{u^{2} \sqrt{1-u^{2}}}=\frac{2 \sqrt{\lambda}}{\sqrt{s}} \frac{1}{\sqrt{1+t}} I(s, t), \tag{6.67}
\end{equation*}
$$

where we have considered the regulator $\epsilon \rightarrow 0$. This integral in 6.67) can be written as 220.12 of [131]

$$
\begin{equation*}
I(s, t)=\alpha \int_{\epsilon}^{1} d u \frac{\sqrt{p^{2}-u^{2}}}{u^{2} \sqrt{1-u^{2}}}=\alpha \frac{g}{q^{2}}\left(\left(1-q^{2}\right) u_{1}-E\left(\varphi, q^{2}\right)+\operatorname{dn} u_{1} \operatorname{tn} u_{1}\right), \tag{6.68}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha^{2}=\frac{s+t}{1+t}, \quad p^{2}=\frac{1}{\alpha^{2}}, \quad q^{2}=\frac{1}{p^{2}}, \quad g=\frac{1}{q}, \quad \sin \varphi=\sqrt{\frac{p^{2}\left(1-\epsilon^{2}\right)}{p^{2}-\epsilon^{2}}} \approx 1-\frac{1}{2}\left(1-q^{2}\right) \epsilon^{2} . \tag{6.69}
\end{equation*}
$$

This means that $\varphi \approx \pi / 2$, thus $u_{1} \approx K\left(q^{2}\right)$. Around $u_{1} \approx K$,

$$
\begin{equation*}
\mathrm{sn} \approx 1-\frac{1}{2}\left(1-q^{2}\right)(u-K)^{2} . \tag{6.70}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\operatorname{dn} u_{1} \operatorname{tn} u_{1} \approx-\frac{1}{u-K}=-\frac{1}{\epsilon} . \tag{6.71}
\end{equation*}
$$

This is the divergent part of the action because the other terms were not divergent for $u_{1} \rightarrow K$. The regularized integral now is

$$
\begin{equation*}
I^{\mathrm{reg}}(s, t)=(1-s) K\left(q^{2}\right)-(1+t) E\left(q^{2}\right) . \tag{6.72}
\end{equation*}
$$

Finally,

$$
\begin{equation*}
S_{\mathrm{conn}}^{\mathrm{reg}}=\frac{2 \sqrt{\lambda}}{\sqrt{s}} \frac{1}{\sqrt{1+t}}\left((1-s) K\left(\frac{s+t}{1+t}\right)-(1+t) E\left(\frac{s+t}{1+t}\right)\right) \tag{6.73}
\end{equation*}
$$

just like in [58]. From (6.66] we see that we can go to the $\Delta \phi=0$ case in section 6.1 when $s+t=1$ which precisely corresponds to $\ell=0$.

Let us now analyse how the parameters $s$ and $t$ control the physical parameters (distance between loops and internal separation) and the action. In Figure 6.8, we plot curves for constant $\gamma$ and $h / R$, in this for $R_{1}=R_{2}=R$. We also plotted the $S_{\text {conn }} / \sqrt{\lambda}=-2$
curve (dashed black line) that separates the connected and disconnected regions (see Figure 2 in [58]). For different values of $\gamma$ and $h / R$ we can see, just like in [58], that there are intersections between $\gamma$ and $h / R$; this indicates that for a fixed value of $\gamma$, two values of $h / R$ are possible; but as $h / R$ grows intersections coalesce (see for example $h / R \approx 0.80$ for $\gamma \approx \pi / 2$ in Figure 6.8. For $h / R \gtrsim 0.8$, for example $h / R \approx 0.9$, there is no intersection so there is no connected solution. It is important to notice that, in general, when we have these two intersections it is the one above the dotted black line that matters. This corresponds to the dominant solution whose action is below $S_{\mathrm{conn}} / \sqrt{\lambda}=-2$. Notice also that for $\gamma \approx \pi$ we can have a connected solution only for $h / R \rightarrow 0$, and for $\gamma \approx 0$ we are limited by $h / R \approx 1$.


Figure 6.8: s-t curves for different fixed values of $\gamma$ and $h / R$. The black dotted line separates the connected region from the disconnected one, representing the GO transition limit.

In Figure 6.9 we see how the action behaves as we increase the internal separation $\gamma$. In particular, when $R_{1}=R_{2}$ we see that as we increase $\gamma$ the GO phase transition occurs at lower distance. Notice there that when set $\gamma=0$ (as studied in section 6.1), the transition distance $h / R$ goes to its maximum value, in this case, $h / R \approx 0.91$ (see [29, 52, 129]). Conversely, if we go to $\gamma=\pi$ we see that the critical distance decreases and goes to $h / R=0$, i.e. the action is larger. We also see that the action has two branches, i.e. at
fixed angle and distance there are two values for the action, two contributions; one of them can be called dominant or leading, and the other sub-dominant.


Figure 6.9: Behavior of the action for differente values of the internal separation $\gamma$ with $R_{1}=R_{2}$. As we decrese $\gamma$ we reach the maximum value of the critical distance.

How does the action behave for $\alpha \leq 1$ ? If now the radii are different we expect the allowed distance before the GO transition to decrease. Moreover, as we saw in Figure 6.9 the critical distance also decreases as we increase $\gamma$. In Figure 6.10 we show precisely how $h_{c}$ behaves with $\alpha=R_{1} / R_{2}$, in particular with $R_{2}=1$.

In Figure 6.11 we plot the behavior of the action when $\gamma=0$ but for $\alpha \leq 1$. As mentioned before, for different radii the connected solution breaks before the equal radii case, and also becomes energentically unfavorable with respect to the disconnected loops.


Figure 6.10: Behavior of the critical distance $h_{c}$ with $\alpha=R_{1}$ when $R_{2}=1$. Notice the highest value of $h_{c} \approx 0.91$ when $\alpha=1$.


Figure 6.11: Behavior of the action for $\gamma=0$ with $R_{2}=1$, so $\alpha=R_{1}$. Notice how the critical distance for the GO transition decreases as we decrease $\alpha$, and that for $\alpha=1$ we again get $h_{c} / R \approx 0.91$. This plot is similar to the one in Figure 6.5.

As reviewed in this section, there is an interesting behavior when we connect two Wilson loops: at certain distance the correlator "prefers" (energetically) to separate into two "domes". This always happens before the critical distance in which the correlator itself breaks and ceases to exist. It was argued in [48,49,52] that in the metastable region of the distance, between $h_{c}$ and the maximum value when it breaks, the radius of the connecting cylinder (i.e. the correlator) becomes of the order of the string length. After that, when we keep separating, quantum fluctuations of the surfaces start to support the worldsheet against the total collapse, so the two loops would be "connected" by a thin tube of a string scale. Thus, the correlator does not completely vanish but is mediated by the supergraviton exchange between the loops. Here lies the importance of the study of the holographic Wilson loops correlators: correlators in the strong coupling regime could give hints of glueballs in QCD (see [133,134], and also [26]). Correlator of Wilson loops
where also studied perturbatively in [55, 56] and in the matrix model in [63, 122].


Figure 6.12: Scheme of the GO phase transition. The minimal worldsheet area connecting the loops becomes larger that the sum of the minimal area of two domes. The connected solution becomes energetically unfavorable.

### 6.3 Disconnected case with the defect

As we saw before, the usual phase transition, without the defect, occurs when the action for the connected solution becomes, at some values of the parameters, equal or greater than the action corresponding to two separated loops, so that it becomes favoured with respect to the connected solution, $S_{\text {conn }} \geq S_{0}$. As we saw from Figure 6.5, this transition occurs before the connected action becomes unstable: $h_{0}<h_{\max }$. The case we want to study here is how the transition behaves when we add a defect D5 brane, i.e. a wall that divides the space into two parts, as described in [60].

Let us consider the Wilson loop in the presence of a D5 defect wrapping an $A d S_{4} \times S^{2}$ subspace of $\operatorname{Ad} S_{5} \times S^{5}$ [63]. In this case, only one endpoint of the string describing the Wilson loop lies at the boundary of $A d S_{5}$, so one set of conditions are the same as the case with no defect. The other set of conditions come from the fact that the string does go into $A d S_{5}$ and ends on the defect.

The AdS space metric is given in 6.5). The $S^{5}$ is

$$
\begin{equation*}
d s_{S^{5}}^{2}=d \theta^{2}+\sin ^{2} \theta d s_{S^{2}}^{2}+\cos ^{2} \theta d s_{\tilde{S}^{2}}^{2} \tag{6.74}
\end{equation*}
$$

where $S^{2}$ and $\tilde{S}^{2}$ are two 2-spheres: $S^{2}$ is wrapped by D5 and $\tilde{S}^{2}$ is its complementary in $S^{5}$. The solution of the probe D5 in $A d S_{5}$ was found in [60] ${ }^{5}$

$$
\begin{equation*}
y=\frac{1}{\kappa} x, \quad \mathcal{F}=-\kappa \operatorname{vol} S^{2}, \quad \theta=\frac{\pi}{2} \tag{6.75}
\end{equation*}
$$

where $\kappa$ represents the inclination of the D5 in the $x-y$ plane. Also, $\kappa$ is associated to the number of D 3 branes ending on the D5, $n$, by

$$
\begin{equation*}
\kappa=\frac{\pi n}{\sqrt{\lambda}} \tag{6.76}
\end{equation*}
$$

A fundamental string stretching from the boundary of $\operatorname{AdS} S_{5}, y=0$, to the D5 is described by the Polyakov action

$$
\begin{equation*}
S_{\mathrm{disc}}=\frac{\sqrt{\lambda}}{4 \pi} \int d \tau d \sigma \frac{1}{y^{2}}\left(y^{\prime 2}+r^{\prime 2}+r^{2}+x^{\prime 2}+y^{2} \theta^{\prime 2}\right) \tag{6.77}
\end{equation*}
$$

where the ansatz

$$
\begin{equation*}
\phi=\tau, \quad y=y(\sigma), \quad r=r(\sigma), \quad x=x(\sigma), \quad \theta=\theta(\sigma) \tag{6.78}
\end{equation*}
$$

was used. Notice that now we have not considered $x=\sigma$ because of (6.75). The equations of motion are

$$
\begin{equation*}
x^{\prime}=-c y^{2}, \quad \theta^{\prime}=m, \tag{6.79}
\end{equation*}
$$

where $c \geq 0$ and $m \geq 0$ are constants of motion, and the minus sign in front of the equation of $x$ is because of the expected behavior of the solution in the $x-y$ plane. Also, we take $c>0$ because we want that $x$ decreases with $\sigma$. The equations of motion for $y$ and $r$ are

$$
\begin{equation*}
y y^{\prime \prime}+r^{\prime 2}+r^{2}-y^{\prime 2}+c^{2} y^{4}=0, \quad y r^{\prime \prime}-2 r^{\prime} y^{\prime}-y r=0 . \tag{6.80}
\end{equation*}
$$

The Virasoro constraint is

$$
\begin{equation*}
y^{\prime 2}+r^{\prime 2}+c^{2} y^{4}+y^{2} m^{2}-r^{2}=0 \tag{6.81}
\end{equation*}
$$

[^22]In order for the string to end on the D5, we impose that [60]

$$
\begin{array}{ll}
C_{1} \equiv y^{\prime}(\tilde{\sigma})+\kappa x(\tilde{\sigma})=0, & C_{3} \equiv r^{\prime}(\tilde{\sigma})=0, \\
C_{2} \equiv y(\tilde{\sigma})-\frac{1}{\kappa} x(\tilde{\sigma})=0, & \theta(\tilde{\sigma})=\frac{\pi}{2}, \tag{6.82}
\end{array}
$$

where $\tilde{\sigma}$ is maximum value of the $\sigma$ coordinate. Let us suppose that the loop is located at $x=L$, and has radius $R$. At $\sigma=0$ the string endpoint is at the boundary of $A d S_{5}$, so

$$
\begin{array}{ll}
y(0)=0, & r(0)=R, \\
x(0)=L, & \theta(0)=\chi .
\end{array}
$$

The equation for $\theta$ is easy to solve:

$$
\begin{equation*}
\theta(\sigma)=m \sigma+\chi . \tag{6.84}
\end{equation*}
$$

From (6.82) we can determine the "size" of the string:

$$
\begin{equation*}
\tilde{\sigma}=\frac{1}{m}\left(\frac{\pi}{2}-\chi\right) . \tag{6.85}
\end{equation*}
$$

If we define (see [64], where these calculations were done)

$$
\begin{equation*}
g(\sigma)=\frac{r(\sigma)}{y(\sigma)}, \quad h(\sigma)=\frac{1}{y(\sigma)} \tag{6.86}
\end{equation*}
$$

Combining (6.80) and 6.81) gives two separated equations: one only for $g(\sigma)$,

$$
\begin{equation*}
\frac{g^{\prime \prime}(\sigma)}{g(\sigma)}=1-m^{2}+2 g^{2}(\sigma) \tag{6.87}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{h^{\prime \prime}(\sigma)}{h(\sigma)}=2 g^{2}(\sigma)-m^{2} \tag{6.88}
\end{equation*}
$$

(6.87) can be easily solved to get (see [135])

$$
\begin{equation*}
g(\sigma)=\sqrt{\frac{m^{2}-1}{k^{2}+1}} \mathrm{~ns}\left(\sqrt{\frac{m^{2}-1}{k^{2}+1}} \sigma, k^{2}\right), \tag{6.89}
\end{equation*}
$$

where $k^{2}$ is the elliptic modulus of the Jacobi elliptic function. Near $\sigma \rightarrow 0$ we see that $g(\sigma) \rightarrow 1 / \sigma$. This can be also inferred from the definition 6.86). Notice that

$$
\begin{equation*}
g(0)=\frac{r(0)}{y(0)} \rightarrow \infty \tag{6.90}
\end{equation*}
$$

which is in agreement with (6.83). Let us follow [64] and write the first equation in 6.87) as

$$
\begin{equation*}
g^{\prime 2}(\sigma)+\left(m^{2}-1\right) g^{2}(\sigma)-g^{4}(\sigma)=-\varepsilon_{0}-m^{2}, \tag{6.91}
\end{equation*}
$$

since it does not depend explicitly on $\sigma$. Here, $-\varepsilon_{0}-m^{2}$ is a convenient integration constant. Moreover, by considering that $h(\sigma)$ has the following form

$$
\begin{equation*}
h(\sigma)=\sqrt{1+g^{2}(\sigma)} z(\sigma), \tag{6.92}
\end{equation*}
$$

we can rewrite, with help of (6.87) and (6.91), (6.88) as

$$
\begin{equation*}
\left(g^{2}(\sigma)+1\right)\left(2 g(\sigma) g^{\prime}(\sigma) z^{\prime}(\sigma)+\left(g^{2}(\sigma)+1\right) z^{\prime \prime}(\sigma)\right)-\varepsilon_{0} z(\sigma)=0 \tag{6.93}
\end{equation*}
$$

If we multiply 6.93) by $2 z^{\prime}(\sigma)$ and integrate in $\sigma$, we get

$$
\begin{equation*}
\left(g^{2}(\sigma)+1\right)^{2} z^{\prime 2}(\sigma)-\varepsilon_{0} z^{2}(\sigma)+\zeta_{0}=0 \tag{6.94}
\end{equation*}
$$

where $\zeta_{0}$ is an integration constant that can be fixed by rewriting the Virasoro constraint 6.81) in terms of $g(z)$ and $z(\sigma)$, and with 6.91). The result is

$$
\begin{equation*}
\left(g^{2}(\sigma)+1\right)^{2} z^{\prime 2}(\sigma)-\varepsilon_{0} z^{2}(\sigma)+c^{2}=0 \tag{6.95}
\end{equation*}
$$

from which $\zeta_{0}=c^{2}$. From 6.95 we also see that $\varepsilon_{0} \geq 0$. The last equation can be solved by

$$
\begin{equation*}
z(\sigma)=\frac{c}{\sqrt{\varepsilon_{0}}} \cosh (v(\sigma)-\eta) \tag{6.96}
\end{equation*}
$$

if we define

$$
\begin{equation*}
v^{\prime}(\sigma)=\frac{\sqrt{\varepsilon_{0}}}{1+g^{2}(\sigma)} \tag{6.97}
\end{equation*}
$$

with $v(0)=0$ and $\eta$ being an integration constant. From we can write that

$$
\begin{align*}
& y(\sigma)=\frac{\sqrt{\varepsilon_{0}}}{c} \frac{1}{\sqrt{1+g^{2}(\sigma)}} \operatorname{sech}(v(\sigma)-\eta) \\
& r(\sigma)=\frac{\sqrt{\varepsilon_{0}}}{c} \frac{g(\sigma)}{\sqrt{1+g^{2}(\sigma)}} \operatorname{sech}(v(\sigma)-\eta) \tag{6.98}
\end{align*}
$$

From (6.79),

$$
\begin{equation*}
x^{\prime}=-\frac{\sqrt{\varepsilon_{0}}}{c} v^{\prime}(\sigma) \operatorname{sech}^{2}(v(\sigma)-\eta), \tag{6.99}
\end{equation*}
$$

which can be integrated:

$$
\begin{equation*}
x(\sigma)=x_{0}-\frac{\sqrt{\varepsilon_{0}}}{c} \tanh (v(\sigma)-\eta) . \tag{6.100}
\end{equation*}
$$

Let us impose the boundary conditions 6.83. $r(0)=R$ becomes

$$
\begin{equation*}
R=\frac{\sqrt{\varepsilon_{0}}}{c} \operatorname{sech} \eta \quad \Rightarrow \quad c=\frac{\sqrt{\varepsilon_{0}}}{R} \operatorname{sech} \eta . \tag{6.101}
\end{equation*}
$$

$x(0)=L$, on the other hand, becomes

$$
\begin{equation*}
L=x_{0}+R \sinh \eta \quad \Rightarrow \quad x_{0}=L-R \sinh \eta . \tag{6.102}
\end{equation*}
$$

A clever way to impose the boundary conditions on the D5 brane 6.82 is to impose a combination of them, since $C_{i}=0$ then $\sum_{i} A_{i} C_{i}=0$, [64]. With 6.98), the combination:

$$
\begin{equation*}
\kappa c C_{1}+\frac{1}{y(\tilde{\sigma})} C_{2}+\frac{r(\tilde{\sigma})}{y^{2}(\tilde{\sigma})} C_{3}=0 \quad \Rightarrow \quad-c x_{0}=0 \tag{6.103}
\end{equation*}
$$

This helps to find that

$$
\begin{equation*}
\frac{L}{R}=\sinh \eta \tag{6.104}
\end{equation*}
$$

now in terms of $L$ and $R$, geometric parameters. From $C_{3}=0$ we have,

$$
\begin{equation*}
\sqrt{\varepsilon_{0}} g(\tilde{\sigma}) \tanh (\eta-v(\tilde{\sigma}))+g^{\prime}(\tilde{\sigma})=0 \quad \Rightarrow \quad \eta=v(\tilde{\sigma})+\tanh ^{-1}\left(-\frac{1}{\sqrt{\varepsilon_{0}}} \frac{g^{\prime}(\tilde{\sigma})}{g(\tilde{\sigma})}\right) \tag{6.105}
\end{equation*}
$$

The last result allows to eliminate $v(\tilde{\sigma})-\eta$ from $C_{1}=0$ and $C_{2}=0$ in 6.82 with 6.98. Both boundary conditions lead to the same equation for $\kappa$, the flux,

$$
\begin{equation*}
\kappa=-\frac{\sqrt{1+g^{2}(\tilde{\sigma})} g^{\prime}(\tilde{\sigma})}{\sqrt{\varepsilon_{0} g^{2}(\tilde{\sigma})-g^{\prime 2}(\tilde{\sigma})}}, \tag{6.106}
\end{equation*}
$$

where $\kappa \geq 0$ since it is the flux due to the D5 defect. We can use 6.91) to eliminate $g^{\prime 2}(\sigma)$ in the denominator of 6.106 to get

$$
\begin{equation*}
\kappa=-\frac{g^{\prime}(\tilde{\sigma})}{\sqrt{m^{2}+\varepsilon_{0}-g^{2}(\tilde{\sigma})}} . \tag{6.107}
\end{equation*}
$$

Let us write (6.98) and (6.100) by using (6.101),

$$
\begin{align*}
y & =R \cosh \eta \frac{1}{\sqrt{1+g^{2}(\sigma)}} \operatorname{sech}(v(\sigma)-\eta), \\
r & =R \cosh \eta \frac{g(\sigma)}{\sqrt{1+g^{2}(\sigma)}} \operatorname{sech}(v(\sigma)-\eta), \\
x & =-R \cosh \eta \tanh (v(\sigma)-\eta) \tag{6.108}
\end{align*}
$$

Now, since $v(\sigma)$ is given by 6.97), we need an explicit form for $g(\sigma)$. We can write (6.91) as

$$
\begin{equation*}
g^{\prime 2}(\sigma)=\left(g^{2}-\alpha_{+}\right)\left(g^{2}-\alpha_{-}\right), \tag{6.109}
\end{equation*}
$$

with

$$
\begin{equation*}
\alpha_{ \pm}=\frac{1}{2}\left(m^{2}-1 \pm \sqrt{\left(m^{2}+1\right)^{2}+4 \varepsilon_{0}}\right) \tag{6.110}
\end{equation*}
$$

We can write (6.109) as

$$
\begin{equation*}
\pm \int_{\infty}^{g} \frac{d g}{\sqrt{\left(g^{2}-\alpha_{+}\right)\left(g^{2}-\alpha_{-}\right)}}=\sigma \tag{6.111}
\end{equation*}
$$

from which we choose the minus sign in order to have $\sigma \geq 0$. This integral can we solved in terms of Jacobi elliptic function ns (see 130.10 in [131]). As a result we get

$$
\begin{equation*}
g(\sigma)=\sqrt{\alpha_{+}} \mathrm{ns}\left(\sqrt{\alpha_{+}} \sigma, \frac{\alpha_{-}}{\alpha_{+}}\right) \tag{6.112}
\end{equation*}
$$

If we compare the last result with (6.89),

$$
\begin{equation*}
k^{2}=\frac{\alpha_{-}}{\alpha_{+}}=\frac{m^{2}-1-\sqrt{\left(m^{2}+1\right)^{2}+4 \varepsilon_{0}}}{m^{2}-1+\sqrt{\left(m^{2}+1\right)^{2}+4 \varepsilon_{0}}} \tag{6.113}
\end{equation*}
$$

since $k<1$. This allows to write

$$
\begin{equation*}
\varepsilon_{0}=-\frac{\left(k^{2}+m^{2}\right)\left(m^{2} k^{2}+1\right)}{\left(k^{2}+1\right)^{2}} \tag{6.114}
\end{equation*}
$$

We can also write $c^{2}$ from 6.101 in terms of $k^{2}$,

$$
\begin{equation*}
c^{2}=-\frac{\left(k^{2}+m^{2}\right)\left(m^{2} k^{2}+1\right)}{\left(k^{2}+1\right)^{2}\left(L^{2}+R^{2}\right)} \tag{6.115}
\end{equation*}
$$

We need $\varepsilon_{0} \geq 0$, and also that $\alpha_{+} \geq 0$ to have $g(\sigma)$ real. This leads to ${ }^{6}$

$$
\begin{align*}
& \text { region }(A) \quad-1 \leq k^{2} \leq 0 \quad m^{2} \geq-\frac{1}{k^{2}}  \tag{6.116}\\
& \text { region }(B) \quad-\infty<k^{2} \leq-1 \quad m^{2} \leq-\frac{1}{k^{2}}
\end{align*}
$$

[^23]

Figure 6.13: Allowed values of $k^{2}$ and $m^{2}$ for $\varepsilon_{0} \geq 0$. Only those values of $k^{2}$ and $m^{2}$ that leads to $g(\sigma)$ be real are selected. They depict two different regions: A and B. The black curve represents $m^{2}=-1 / k^{2}($ or $c=0)$.

From 6.115 we see that $c=q^{7}$ at the intersection of both regions,

$$
\begin{equation*}
m^{2}=-\frac{1}{k^{2}} . \tag{6.117}
\end{equation*}
$$

We can write (6.112) as

$$
\begin{equation*}
g(\sigma)=\sqrt{\frac{m^{2}-1}{k^{2}+1}} \mathrm{~ns}\left(\sqrt{\frac{m^{2}-1}{k^{2}+1}} \sigma, k^{2}\right) . \tag{6.118}
\end{equation*}
$$

as expected.
Since $\kappa \geq 0$, from 6.107) we need that $g^{\prime}(\tilde{\sigma}) \leq 0$ and

$$
\begin{equation*}
g^{2}(\tilde{\sigma}) \leq-k^{2} \frac{\left(m^{2}-1\right)^{2}}{\left(k^{2}+1\right)^{2}} \tag{6.119}
\end{equation*}
$$

From $g^{\prime}(\tilde{\sigma}) \leq 0$ we deduce that

$$
\begin{equation*}
0 \leq \tilde{\sigma} \leq \tilde{\sigma}_{\max } \quad \text { with } \quad \tilde{\sigma}_{\max }=\sqrt{\frac{k^{2}+1}{m^{2}-1}} K\left(k^{2}\right) \tag{6.120}
\end{equation*}
$$

[^24]From 6.85) we find

$$
\begin{equation*}
x \leq \frac{K\left(k^{2}\right)}{\left(\frac{\pi}{2}-\chi\right)} \tag{6.121}
\end{equation*}
$$

where it was defined

$$
\begin{equation*}
x=\sqrt{\frac{m^{2}-1}{m^{2}\left(k^{2}+1\right)}} . \tag{6.122}
\end{equation*}
$$

From 6.116 we get the same range for $x$ for both regions, so

$$
\begin{equation*}
1 \leq x \leq \frac{K\left(k^{2}\right)}{\left(\frac{\pi}{2}-\chi\right)} \tag{6.123}
\end{equation*}
$$

As said in [64], in region A, by construction, $x$ is always less or equal to $1 / \sqrt{k^{2}+1}$. Then, from 6.121, we have

$$
\begin{equation*}
1 \leq x \leq \operatorname{Min}\left(\frac{1}{\sqrt{k^{2}+1}}, \frac{K\left(k^{2}\right)}{\left(\frac{\pi}{2}-\chi\right)}\right) . \tag{6.124}
\end{equation*}
$$

It is necessary that the admisible interval of $x$ is not empty. If $k^{2}=\tan \alpha, \alpha \in\left[-\frac{\pi}{2}, 0\right]$. This is shown in Figure 6.14.


Figure 6.14: This plot shows the allowed values of $\chi$ and $\alpha$. There also appear separated regions A and B. The black curve corresponds to $\chi=\frac{\pi}{2}-K\left(k^{2}\right)$ (or $x=1$, so $m^{2}=-1 / k^{2}$, then $c=0$ ). Also, $\chi=0$ is allowed only when $k^{2}=0$ (and then $m^{2} \rightarrow \infty$ ) in region A. This leads to $\tilde{\sigma} \rightarrow 0$ (no solution).

Moreover, (6.119) can be written as

$$
\begin{equation*}
\operatorname{cn}^{2}\left(x\left(\frac{\pi}{2}-\chi\right), k^{2}\right) \leq-\frac{1}{k^{2}}\left(1-\frac{1}{x^{2}}\right) . \tag{6.125}
\end{equation*}
$$

The r.h.s of (6.125) takes values between 0 and 1 in both regions A and B. Let us consider the lower bound of the last relation,

$$
\begin{equation*}
\mathrm{cn}^{2}\left(x_{0}\left(\frac{\pi}{2}-\chi\right), k^{2}\right)=-\frac{1}{k^{2}}\left(1-\frac{1}{x_{0}^{2}}\right) . \tag{6.126}
\end{equation*}
$$

Since $x_{0} \geq 1$, this reduces the interval of $x$ to be

$$
\begin{align*}
& \operatorname{region}(A) \quad 1 \leq x_{0} \leq x \leq \operatorname{Min}\left(\frac{1}{\sqrt{k^{2}+1}}, \frac{K\left(k^{2}\right)}{\left(\frac{\pi}{2}-\chi\right)}\right)  \tag{6.127}\\
& \operatorname{region}(B) \quad 1 \leq x_{0} \leq x \leq \frac{K\left(k^{2}\right)}{\left(\frac{\pi}{2}-\chi\right)}
\end{align*}
$$

On the other hand, (6.107) can be written, by using (6.118), as

$$
\begin{equation*}
\kappa^{2}=\frac{x^{2} \mathrm{cn}^{2}\left(w, k^{2}\right) \operatorname{dn}^{2}\left(w, k^{2}\right)}{\left(k^{2} x^{2} \operatorname{cn}^{2}\left(w, k^{2}\right)+x^{2}-1\right) \operatorname{sn}^{2}\left(w, k^{2}\right)}, \tag{6.128}
\end{equation*}
$$

where $w=x(\pi / 2-\chi)=\sqrt{n} \tilde{\sigma}$. It is possible to see that when $w=K\left(k^{2}\right)$, so $\tilde{x}=$ $K\left(k^{2}\right) /(\pi / 2-\chi)$, then $\kappa=0$. If $\chi \rightarrow \pi / 2, \tilde{x}$ could be greater than $1 / \sqrt{k^{2}+1}$ and then go out the allowed values in region A. In general, we cannot find an $x$ inside A that allows small $\kappa$ for $\chi \rightarrow \pi / 2$.


Figure 6.15: Values of $x$ that lie outside region A (green) when $K\left(k^{2}\right) /(\pi / 2-\chi) \geq$ $1 / \sqrt{k^{2}+1}$. Critical values $\left(\alpha_{c}, \chi_{c}\right)$ when $\kappa=0$ (blue). The region A (blue) enlarges as we increase $\kappa$. The blue curve represents $x_{c}=1 / \sqrt{1+k_{c}^{2}}$.

In Figure 6.15, the blue curve has $m^{2} \rightarrow \infty$; since it must converge to the same point $(0,0)$, we can say that the blue curve also represents $c=0$. The $\chi=\pi / 2$ line corresponds to $k^{2} \rightarrow-\infty$, this leads to $c^{2} \rightarrow-m^{2}$; since $c^{2}$ cannot be negative, $m^{2}=0$. Thus, the total allowed region, both A and B, is surrounded by $c=0$ (the case solved in [63]).

As said in [64], we can fix $\chi$ and $\kappa$ and study how the region A changes as we increase $k^{2}$. Let $x_{c}=1 / \sqrt{k_{c}^{2}+1}$ be the critical value of $x$ that solves 6.128 in A. For $k^{2} \geq k_{c}^{2}$ there is no solution for $\kappa$. From 6.128,

$$
\begin{equation*}
\operatorname{sn}\left(\frac{(\pi / 2-\chi)}{\sqrt{k^{2}+1}}, k^{2}\right) \geq \frac{\sqrt{2}}{\sqrt{k^{2}+1+\sqrt{\left(k^{2}-1\right)^{2}-4 k^{2} \kappa^{2}}}} \tag{6.129}
\end{equation*}
$$

or

$$
\begin{equation*}
\chi \leq \frac{\pi}{2}-\sqrt{k^{2}+1} \mathrm{sn}^{-1}\left(\frac{\sqrt{2}}{\sqrt{k^{2}+1+\sqrt{\left(k^{2}-1\right)^{2}-4 k^{2} \kappa^{2}}}}, k^{2}\right) . \tag{6.130}
\end{equation*}
$$

These values ensure that the l.h.s of 6.128) is less than a fixed $\kappa^{2}$. Also, from 6.130, we can find, at fixed $\kappa$ and $\chi$, the critical value of $k^{2}$ (in region A) above which there is no solution. It was shown in [64] that as we increase $\kappa^{2}$ the allowed values of $k^{2}$ also increase (see also Figure 6.16). In Figure 6.15 this can be seen as the expansion of the region A (blue).


Figure 6.16: Expansion of the allowed region of parameters as we increase $\kappa$. Notice how $k_{c}^{2}$ goes to zero as we enlarge region A. Thus, when $\kappa \rightarrow \infty, k^{2}$ can be zero at any value of $\chi \in\left[0, \frac{\pi}{2}\right]$. Each curve for fixed $\kappa$ is given by the equality in 6.130

So, for larger values of $\kappa$ we can access larger values in region A. When $\kappa \rightarrow \infty$, region A enlarges to its maximum. In general, $\chi$ and $\kappa$ control the limits of $k^{2}$.

### 6.3.1 Evaluating the action

Back to the action (6.77), we can write it onshell as

$$
\begin{equation*}
S_{\mathrm{disc}}=\sqrt{\lambda} \int_{0}^{\tilde{\sigma}} d \sigma \frac{r^{2}}{y^{2}} . \tag{6.131}
\end{equation*}
$$

From (6.108) we get

$$
\begin{equation*}
S_{\mathrm{disc}}=\sqrt{\lambda} \int_{0}^{\tilde{\sigma}} d \sigma g^{2}(\sigma)=\sqrt{\lambda} \int_{0}^{\tilde{\sigma}} d \sigma n \mathrm{~ns}^{2}\left(\sqrt{n} \sigma, k^{2}\right), \tag{6.132}
\end{equation*}
$$

where $n=\frac{m^{2}-1}{k^{2}+1}$. This integral can be solved in terms of elliptic function (see 362.01 in [131]),

$$
\begin{equation*}
S_{\mathrm{disc}}=\left.\sqrt{\lambda n}\left(\sqrt{n} \sigma-E\left(\operatorname{am}(\sqrt{n} \sigma), k^{2}\right)-\operatorname{dn}\left(\sqrt{n} \sigma, k^{2}\right) \operatorname{cs}\left(\sqrt{n} \sigma, k^{2}\right)\right)\right|_{\sigma=0} ^{\sigma=\tilde{\sigma}} \tag{6.133}
\end{equation*}
$$

The last result is divergent at $\sigma=0$. We also assume that $\tilde{\sigma}>0$, so $k^{2}<k_{c}^{2}$ and the action is non-vanishing. As usual, let us introduce the regulator $\sigma=\epsilon$ with $\epsilon \rightarrow 0$. With this, the evaluated action becomes

$$
\begin{equation*}
S_{\mathrm{disc}}=\sqrt{\lambda n}\left(w-E\left(\operatorname{am}(w), k^{2}\right)-\operatorname{dn}\left(w, k^{2}\right) \operatorname{cs}\left(w, k^{2}\right)\right)+\frac{\sqrt{\lambda}}{\epsilon}, \tag{6.134}
\end{equation*}
$$

which, without the $\mathcal{O}(1 / \epsilon)$ term, corresponds to the onshell action.
On the other hand, if we derivate 6.105) w.r.t $\tilde{\sigma}$, then combine with 6.97) and finally integrate, we get an expression corresponding to the definition of the incomplete elliptic integral of the third kind (see 400.01 [131]). Then, we find

$$
\begin{equation*}
\eta=\sqrt{-(1+n)\left(\frac{1}{n}+k^{2}\right)} \Pi\left(\operatorname{am}\left(w, k^{2}\right),-k^{2} n, k^{2}\right) \tag{6.135}
\end{equation*}
$$

where $w=x(\pi / 2-\chi)=\sqrt{n} \tilde{\sigma}$. Thus,

$$
\begin{equation*}
\frac{L}{R}=\sinh \eta \tag{6.136}
\end{equation*}
$$

so $\eta \geq 0$. From 6.105 we can say that at $k_{c}^{2}$, for which $m^{2} \rightarrow \infty$ and then $\tilde{\sigma} \rightarrow 0, \eta$, and thus $L / R$, goes to zero. This gives an interpretation to the critical value $k_{c}^{2}$ : the value for which the Wilson loop touches the defect [64].


Figure 6.17: Behavior of $L / R$ with $k^{2}$ for $\chi=\pi / 2-K(-0.95) \approx 0.25$. Notice that as $k^{2} \rightarrow k_{c}^{2}$, the critical value for each $\kappa^{2}, L / R \rightarrow 0$, as expected.

From Figure 6.17, we can see that $L / R$ is not always monotonic. There are certain values of $\kappa^{2}$ in which $L / R$ has a maximum [64]. This leads to having two values of $k^{2}$ for one $L / R$. In Figure 6.18 and Figure 6.19 we can see, for fixed values of $\chi$, that an increasing flux also increases the critical distance $L_{c} / R$ for the Gross-Ooguri-like transition and the maximum value of $L / R$ for the solution to exist. We also see that decreasing the angle also decreases the critical value of $L_{c} / R$.

When the transition occurs the minimal area of the string worldsheet attached to the boundary and to the defect becomes larger than the minimal area of the string worldsheet ending only on the boundary and forming a dome surface because. This configuration is energentically preferred with respect to the one ending on the D5 (see Figure 6.20).


Figure 6.18: Behavior of $S_{\text {disc }}$ with $L / R$ for different values of the flux $\kappa^{2}$ at $\chi \approx 0.25$. We can see that the distance for the GO-like transition, when the curves cross $S_{0}$, the action in 6.2 , increases as we increase the flux $\kappa$.


Figure 6.19: Behavior of $S_{\text {disc }}$ with $L / R$ for different values of the flux $\kappa^{2}$ at $\chi \approx 0.12$. We can see here that for a lower value of $\chi$, all critical distances for the GO-like transition decrease. Also, notice that for this choice of $\chi$ there is only branch for the action.


Figure 6.20: Behavior of $S_{\text {disc }}$, the action of the worldsheet attached to the D5 defect and its comparison with the "dome". This is the GO-like transition studied in [64]. In this case $\chi \leq \pi / 2$ because the D5 defect is located at $\chi=\pi / 2$ in $S^{5}$.

### 6.4 Phase transition in the presence of the defect

In this section we will explore the case in which we have two Wilson loops separated by a distance along $x_{3}$ in the presence of a D5 defect. The correlator of these loops will be the same as in section 6.2, in which we reviewed the correlator of two Wilson loops with radii $R_{1}$ and $R_{2}$, separated by a distance $h$ along $x_{3}$ and by an internal angle $\gamma$. At some critical value of the distance the area of the worldsheet stretching between the loops becomes greater than the total area of two independent loops forming the usual dome in $\operatorname{Ad} S_{5}$, and thus a phase transition occurs. This critical distance is controlled by the internal angle and the radii as can be seen in Figure 6.9 and Figure 6.11. Also, the connected correlator has a maximum distance before ceasing to exist and, as we saw, this happens always before they are becoming energentically unfavored with respect to the domes. In the presence
of a D5 defect we must take care now if the corresponding minimal area connecting the loops becomes larger than the total area of the two loops. These loops could either both end on the defect or only one of them or both form domes. This behavior is now controlled by more parameters: the radii $R_{1}$ and $R_{2}$, the distance with respect to the defect, $L$, the relative distance between the loops, $h$, the internal angles $\chi_{1}$ and $\chi_{2}$, and the flux due to the defect, $\kappa$.

As a starting situation we consider that the two loops have equal radii $R_{1}=R_{2}=R$; and that the first one is placed at $L / R$ with respect to the defect and the second one is at $(L+h) / R$. With this we fix the minimal string worldsheet of the first loop and then the total action will depend only on $h / R$. For the second loop we have,

$$
\begin{equation*}
\frac{h}{R}\left(\chi_{1}, \chi_{2}, \kappa\right)=-\frac{L}{R}\left(\chi_{2}, \kappa\right)+\sinh \eta\left(\chi_{1}, \kappa\right), \tag{6.137}
\end{equation*}
$$

and the total action is

$$
\begin{equation*}
S_{\text {disc }}\left(\frac{h}{R}, \gamma, \kappa\right)=S_{\text {disc }}^{(1)}\left(\frac{L+h}{R}, \chi_{1}, \kappa\right)+S_{\text {disc }}^{(2)}\left(\frac{L}{R}, \chi_{2}, \kappa\right), \tag{6.138}
\end{equation*}
$$

where the upper indices correspond to the actions of first and second loops, respectively, and with $S_{\text {disc }}^{(2)}\left(\frac{L}{R}, \chi_{2}, \kappa\right)$ fixed at certain value of $L / R$ for which the loop is connected to the defect. We also took $R_{1}=R_{2}=R$ in order to control the transition only with $L / R$, $h / R, \kappa$ and $\gamma=\chi_{1}-\chi_{2}$.

We will fix the first loop at a distance $L / R$ for which its action $S_{\text {disc }}^{(2)}<-\sqrt{\lambda}$, and leave the relative distance $h / R$ free in $S_{\text {disc }}^{(1)}$.

In Figure 6.21, we see how the action corresponding to two Wilson loops behaves in terms of their relative separation $h / R$. There we have fixed the first loop to be always connected to the defect so we can see at which value of $h / R$ the second loop forms a dome.

As we learned from Figure 6.9 and Figure 6.11, the minimal worldsheet connecting two Wilson loops has maximum values for $\gamma \rightarrow 0$ (coincident loops in $S^{5}$ ) and $\alpha \rightarrow 1$ (equal radii). In Figure 6.21, we chose $\chi_{1} \approx 0.25, R_{1}=R_{2}=R$, and plotted the total disconnected action for three different values of parameters for the fixed loop.


Figure 6.21: Behavior of two separated Wilson loops in the presence of a D5 defect with $\kappa^{2}=2$ with $\chi_{1} \approx 0.25$. We have fixed the second loop for different values of parameters. Remember the GO-like transition occurs at $-1+S_{\text {disc }}^{(2)}$, so $h_{c} / R \approx 0.05-0.1$, so most of $S_{\text {disc }}$ correspond to one loop attached to the D5 and the other one not attached. For lower values of $\kappa^{2}$ the maximal value of $h / R$ decreases.

In Figure 6.21 we considered two cases in which the second loop is attached to the D5 brane and one in which it is not (green). For the first two cases we considered also different internal angles, one of them being equal to our choice of $\chi_{1}$, so $\gamma=0$. Notice that for $h / R \rightarrow 0$ we reduce to

$$
\begin{equation*}
S_{\mathrm{disc}}(0, \gamma, \kappa)=S_{\mathrm{disc}}^{(1)}\left(\frac{L}{R}, \chi_{1}, \kappa\right)+S_{\mathrm{disc}}^{(2)}\left(\frac{L}{R}, \chi_{2}, \kappa\right) \tag{6.139}
\end{equation*}
$$

in which for $\chi_{1}=\chi_{2} \approx 0.25$ we reduce to $2 S_{\mathrm{disc}}^{(2)} \sqrt{\lambda} \approx-3.2$. In general, we show a
scheme in Figure 6.22.


Figure 6.22: Two separated loops with the same radii. The first loop keeps fixed on the D5. Each loop can take different positions in $S^{5}$. We must take care on the second loop since its distance $(L+h) / R$ could allow the GO-like transition: $S_{\text {disc }}\left(h_{c} / R, \gamma, \kappa\right)=-\sqrt{\lambda}+S_{\text {disc }}^{(2)}\left(L / R, \chi_{2}, \kappa\right)$.

In Figure 6.23 and Figure 6.24, we see for $\chi_{1}=\chi_{2} \approx 0.25$, the comparison between the minimal string worldsheet connecting two loops of equal radii separated by a distance $h$ (red) compared with the total action of two independent loops where one of them is fixed to have $S_{\text {disc }}^{(2)} / \sqrt{\lambda}=-2$ and $S_{\text {disc }}^{(2)} / \sqrt{\lambda}=-3$, respectively, so it ends on the D5 in three cases: $\kappa^{2}=4,6,10$. When total action $S_{\text {disc }} / \sqrt{\lambda}=-3$ and $S_{\text {disc }} / \sqrt{\lambda}=-4$, the first loop forms a dome since its action $S_{\text {disc }}^{(1)} / \sqrt{\lambda}=-1$. This is the GO-like transition [64]. As we increase the flux the GO transition occurs at lower distances because the inclination of the defect increases. For lower values of the flux the connected solution does not see the defect and the GO transition occurs at $h / R \approx 0.9$ as usual.


Figure 6.23: Behavior of $S_{\text {conn }}$ compared to $S_{\text {disc }}$ for $\gamma=0$ in $S^{5}$ and $S_{\text {disc }}^{(2)} / \sqrt{\lambda}=-2$ for different values of $\kappa$. GO transition from connected to attached-dome configuration.


Figure 6.24: Behavior of $S_{\text {conn }}$ compared to $S_{\text {disc }}$ for $\gamma=0$ in $S^{5}$ and $S_{\text {disc }}^{(2)} / \sqrt{\lambda}=-3$ for different values of $\kappa$. GO transition from connected to attached-attached configuration.


Figure 6.25: General configuration of the two loops and transitions. The connected solution can break into the disconnected one (GO phase transition). The disconnected solution has three possible configurations.


Figure 6.26: Does it make any difference if we choose two different positions in $S^{5}$ but in such a way $\gamma$ does not change?


Figure 6.27: GO phase transition for two cases with the same separation in $S^{5}, \gamma \approx 0.13$ and $S_{\text {disc }}^{(2)} / \sqrt{\lambda}=-3$.

In Figure 6.27, we see, for $\kappa^{2}=6$, that if the two loops are not placed in the same positions in $S^{5}$ but their separation is the same the critical distance for the GO transition increases for the loops placed at lower polar positions in $S^{5}$.

There are now several parameters we could manipulate in order to control the transitions: $L / R, h / R, \chi$ and $\kappa$. And, if we choose different radii, also $R_{1} / R_{2}$ will be a parameter. We could choose the first loop ending on the defect, either near the the distance for which the minimal surface becomes a dome or not. If the second loop "sees" the defect or not depends on this choice. Also, different internal angles would lead to move the critical values for the transitions. The general setup is given in Figure 6.25,

## Chapter 7

## Results and outlook

There are now techniques that allow to test the AdS/CFT conjecture exactly, i.e to have results that can be compared on the weak/strong coupling on the opposite sides of the duality. In particular, a largely studied observable in gauge theory is the Wilson operator. In this case, by using localization, on the gauge side and probe branes on the gravity side, exact results were obtained. The strong coupling expansion of these results can then be successfully compared with a dual gravity computation based on minimal string surfaces in AdS, attached to the loop on the AdS boundary. If instead of the expectation value of a single Wilson loop, one wanted to compute the correlator between two loops holographically, the relevant string worldsheet surface would be the one connecting the two different contours. Modifying the geometry of these two rings introduces a phase diagram in which there are critical values for the parameters that separate the catenoid solution and the Goldschmidt solution. When the two rings are separated beyond a certain critical value, the catenoid solution becomes unstable and breaks into the Goldschmidt one. Similarly, in the case of the Wilson loop, the string worldsheet describes a catenoid-like solution until, at certain values of the radii of the loops and the distance, it becomes energetically unfavored with respect to the Goldschmidt-like solution. This is called a Gross-Ooguri (GO) phase transition. The discontinuous (broken catenoid) solution corresponds, in this case, to two minimized "dome" surfaces in $A d S$, each attached to a different loop, i.e two separated Wilson loops. These minimal surfaces are the usual onshell regularized actions with each loop as boundary. On the gauge theory side, the correlator of two Wilson loops is studied and it is found that there exist an analogous GO transition. An interesting setup
that was the focus of this work is the one in which a defect is introduced in the gauge theory. This defect is typically obtained by considering systems of intersecting branes. In particular, intersecting D3 and D5 branes along three of the four worldvolume directions allows to construct three-dimensional defects inside the $\mathcal{N}=4$ worldvolume of the $N$ D3 branes. The end result is that there are two different gauge groups on each side of the defect brane. This is because $k$ D3 branes now end on the D5 defect, so on one side we have the usual $S U(N)$ and on the other side we have $S U(N-k)$. On the string theory side, the solution corresponding to a single D5 wall inside $A d S_{5} \times S^{5}$ is computed by considering that the D5 brane is along three directions in $A d S_{5}$ and two in $S^{5}$. This reduces the allowed symmetries in both parts; in particular, the conformal symmetry of $A d S_{5}$ and the isometries of $S^{5}$. The minimal worldsheet surface is attached to the loop and ends on the defect. The presence of the defect introduces then new boundary conditions for the string: the usual ones along the loop and the ones that ensure that the worldsheet surface end on the defect. But it could happen that the string worldsheet does not touch the defect, or energetically prefers not to end on it and form a dome instead. Thus, this situation is very rich in parameters. This work is recent and ongoing. We studied the Wilson loop correlator in the presence of a D5 defect on the string theory side and the Gross-Ooguri phase transition in this geometry. We investigated in detail how this transition depends on the parameters of the setting: radii and separation of the loops, distance with respect to the defect and inclination of it, and we compared this case with the case without defect and observed that the defect modifies the GO transition region.

Since the latter was solved by considering the connected solution of a single worldsheet attached to two loops, and its transition into two separated minimal worldsheet surfaces; the gauge dual of this correlator will correspond to the expectation value of two Wilson loops in the fundamental representations. The work in the gauge theory side was already started for the correlator without the defect of two concentric loops. Also, Wilson loops in defect gauge theory were also computed and are still studied. We leave as a future work to study also correlators in defect field theory by using well-defined techniques in field theory such as perturbative expansion and localization. As we mentioned above, the duality was tested also for higher representations, which on the string side, correspond to putting probe branes in the string background. Thus, for the $\mathcal{N}=4$ SYM
theory, it was found that, in particular, a Wilson loop in the symmetric representations corresponds to a D3 probe in $A d S_{5} \times S^{5}$; and that the antisymmetric representation case to a D5 probe instead, both with the loop as boundary. In general, for an arbitrary high representation, the dual string description is in terms of bubbling geometries. Here, the string background is strongly deformed by the high number of branes in it. So, it is also interesting to extend the study of correlators of fundamental Wilson loops in defect field theory to corrrelators of symmetric/antisymmetric Wilson loops now in the presence of defects. If we consider one of the string attached to the loop as before but the other one attached now to a D3 brane, this is the case of a Wilson loop correlator for mixed representations; in particular, we setup the symmetric/fundamental correlator in Appendix A. The antisymmetric/fundamental case, involving a D5 brane as boundary of one end of the string was studied before.

Another interesting route to follow is the finite temperature case. So , there are many routes to follow, firstly, to find the corresponding solution of the finite temperature case in the presence of a defect, and later to consider the correlator of two Wilson loops in that case. This should reduce to the known results at zero temperature and, at some values of the paremeters, to the zero temperature case without defect. Another route we want to follow is to consider adding parameters that deform the string background without losing the solvability of the model. These directly come from the so-called integrable deformations in which we can consider to put Wilson loops, i.e. to try to find the minimal surface of the string worldsheet [136, 137]. These deformed string backgrounds correspond to non-commutative quantum field theories which are also largely studied. Independently of the AdS/CFT correspondence, Wilson loops in non-commutative field theories were studied since the beginning of the AdS/CFT times [18, 19, 138], and recently in [139]. Thus, we are also interested in studying the correlator of Wilson loops to make progress and predictions that will be comparable with their corresponding string dual. These studies can help to deepen and enrich our understanding of AdS/CFT correspondence.

## Appendix A

## Symmetric/fundamental Wilson loop correlator

In this section we study the correlator between two Wilson loops, one in the fundamental and the other in the symmetric representation ${ }^{1}$ We already know that the fundamental representation corresponds to taking a string and that the symmetric representation corresponds to taking a D3 brane. We discover that, after a suitable choice of coordinates, the presence of the D3 brane introduces new boundary conditions parameterized by $\kappa$. We study how the Gross-Ooguri phase transition behaves in this case, and how it reduces to the fundamental/fundamental case.

[^25]

Figure A.1: Schematic behavior of the connected solution for the holographic symmetric/fundamental Wilson loop correlator. Since the D3 brane is much heavier that the fundamental string, the D3 brane is not deformed and only sets new boundary conditions.

The Drukker-Fiol solution was found by using

$$
\begin{equation*}
\frac{d s^{2}}{L^{2}}=\frac{1}{\sin ^{2} \eta}\left(d \eta^{2}+\cos ^{2} \eta d \psi^{2}+d \rho^{2}+\sinh ^{2} \rho\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right)\right) \tag{A.1}
\end{equation*}
$$

as the $A d S_{5}$ metric. Here $0 \leq \rho, 0 \leq \theta \leq \pi$ and $0 \leq \eta \leq \pi / 2$. The solution for a D3 brane is given by

$$
\begin{equation*}
\sinh \rho=\kappa \sin \eta, \quad F_{\psi \rho}=\frac{i \kappa \sqrt{\lambda}}{2 \pi \sinh ^{2} \rho}, \quad \kappa=\frac{k \sqrt{\lambda}}{4 N} . \tag{A.2}
\end{equation*}
$$

The solution is attached to the circle on the AdS boundary at $\rho=0$, and corresponds to a Wilson loop in the symmetric representation.

The loop at the boundary appears from the original form of the AdS metric,

$$
\begin{equation*}
\frac{d s^{2}}{L^{2}}=\frac{1}{y^{2}}\left(d y^{2}+d r_{1}^{2}+r_{1}^{2} d \psi^{2}+d r_{2}^{2}+r_{2}^{2} d \phi^{2}\right) \tag{A.3}
\end{equation*}
$$

with

$$
\begin{equation*}
r_{1}=\frac{R \cos \eta}{\cosh \rho-\sinh \rho \cos \theta}, \quad r_{2}=\frac{R \sinh \rho \sin \theta}{\cosh \rho-\sinh \rho \cos \theta}, \quad y=\frac{R \sin \eta}{\cosh \rho-\sinh \rho \cos \theta} . \tag{A.4}
\end{equation*}
$$

with $x_{3}=r_{2} \cos \phi$ and $x_{4}=r_{2} \sin \phi$. The original condition for the loop is $r_{1}=R$ with $y=0$. Thus,

$$
\begin{equation*}
r_{1}^{2}+y^{2}=\frac{R^{2}}{(\cosh \rho-\sinh \rho \cos \theta)^{2}} \quad \Rightarrow \quad r_{1}=R \quad \text { if } \quad \rho=0 \tag{A.5}
\end{equation*}
$$

## APPENDIX A. SYMMETRIC/FUNDAMENTAL WILSON LOOP CORRELATOR126

Now let us consider a string worldsheet whose first boundary is attached to the D3 brane and its second boundary is attached to the loop on the AdS boundary. Let the worldsheet coordinates be $(\sigma, \psi)$. Let $\sigma=\sigma_{0}$ be the boundary on the brane, whereas $\sigma=\sigma_{1}$ be the boundary at the loop with $\sigma_{0}<\sigma_{1}$.

Let us consider the ansatz

$$
\begin{equation*}
\eta=\eta(\sigma), \quad \rho=\rho(\sigma), \quad \theta=0 . \tag{A.6}
\end{equation*}
$$

This ansatz makes $r_{2}=0$, so $x_{3}, x_{4}=0$, which does not allow to study the correlator of Wilson loops separated in $x_{3}$. We could choose instead,

$$
\begin{equation*}
\eta=\eta(\sigma), \quad \rho=\rho(\sigma), \quad \theta=\pi / 2, \quad \phi=0, \tag{A.7}
\end{equation*}
$$

which implies $x_{4}=0$ and

$$
\begin{equation*}
x_{3}(\sigma)=r_{2}=R \tanh \rho(\sigma) \tag{A.8}
\end{equation*}
$$

and

$$
\begin{equation*}
r_{1}(\sigma)=\frac{R \cos \eta(\sigma)}{\cosh \rho(\sigma)}, \quad y(\sigma)=\frac{R \sin \eta(\sigma)}{\cosh \rho(\sigma)} \tag{A.9}
\end{equation*}
$$

The boundary conditions are:

- $\sigma=\sigma_{0}$ (string attached at the boundary loop)

$$
\begin{equation*}
\rho\left(\sigma_{0}\right)=\eta\left(\sigma_{0}\right)=0 \quad \Rightarrow \quad r_{1}=R_{1}=R, \quad y=0 \quad x_{3}=0 ; \tag{A.10}
\end{equation*}
$$

- $\sigma=\sigma_{1}$ (string attached to the D3 brane + ansatz)

$$
\begin{align*}
& \rho\left(\sigma_{1}\right)=\rho_{0}=\sinh ^{-1}\left(\kappa \sin \eta\left(\sigma_{1}\right)\right)=\sinh ^{-1}\left(\kappa \sin \eta_{0}\right) \\
& x_{3}\left(\sigma_{1}\right)=h_{0}-R \tanh \rho_{0}=h\left(\rho_{0}\right) \\
& r_{1}\left(\sigma_{1}\right)=\frac{R \cos \eta_{0}}{\cosh \rho_{0}}=\frac{R \sqrt{\kappa^{2}-\sinh ^{2} \rho_{0}}}{\kappa \cosh \rho_{0}}=R_{2}\left(\rho_{0}\right) \\
& y\left(\sigma_{1}\right)=\frac{R \sin \eta_{0}}{\cosh \rho_{0}}=\frac{R \tanh \rho_{0}}{\kappa}=\frac{h\left(\rho_{0}\right)}{\kappa} . \tag{A.11}
\end{align*}
$$



Figure A.2: Scheme of the connected solution between a loop attached to the boundary of AdS and a D3 probe brane in AdS. Here we see how the presence of the D3 brane (dual to a Wilson loop in the symmetric representation) introduces a new parameter $\rho_{0}=\sinh ^{-1}\left(\kappa \sin \eta_{0}\right)$, which controls the distance $h$ in $x_{3}$ between the loops.

There are some special values for $\rho_{0}$ :

- $\rho_{0}=0$. From A.11) we have

$$
\begin{equation*}
x_{3}\left(\sigma_{1}\right)=h_{0}, \quad r_{1}\left(\sigma_{1}\right)=R ; \tag{A.12}
\end{equation*}
$$

- $\rho_{0}=\sinh ^{-1} \kappa$. From A.11 we have

$$
\begin{equation*}
x_{3}\left(\sigma_{1}\right)=h_{0}-R \frac{\kappa}{1+\kappa^{2}}, \quad r_{1}\left(\sigma_{1}\right)=0 . \tag{A.13}
\end{equation*}
$$

The string worldsheet action can be written as the Nambu-Goto action

$$
\begin{equation*}
S_{\mathrm{NG}}=\sqrt{\lambda} \int_{\sigma_{0}}^{\sigma_{1}} d \sigma \frac{\cos \eta}{\sin ^{2} \eta} \sqrt{\eta^{\prime 2}+\rho^{\prime 2}} \tag{A.14}
\end{equation*}
$$

with the integration constant

$$
\begin{equation*}
\frac{\cos \eta}{\sin ^{2} \eta} \frac{\rho^{\prime}}{\sqrt{\eta^{\prime 2}+\rho^{\prime 2}}}=m \tag{A.15}
\end{equation*}
$$

The action (A.14) is actually equivalent, for our ansatz (A.7), to

$$
\begin{equation*}
S_{\mathrm{NG}}=\sqrt{\lambda} \int_{\sigma_{0}}^{\sigma_{1}} d \sigma \frac{r}{y^{2}} \sqrt{x_{3}^{\prime 2}+y^{\prime 2}+r_{1}^{\prime 2}} \tag{A.16}
\end{equation*}
$$

## APPENDIX A. SYMMETRIC/FUNDAMENTAL WILSON LOOP CORRELATOR128

This is the usual action in AdS as studied in [52]. Let us choose now $\sigma=x_{3}=x$ and rename $r_{1}=r$, thus the action becomes

$$
\begin{equation*}
S_{\mathrm{conn}}=\sqrt{\lambda} \int_{0}^{h=R \tanh \left(\kappa \sin \eta_{0}\right)} d x \frac{r}{y^{2}} \sqrt{1+y^{\prime 2}+r^{\prime 2}} \tag{A.17}
\end{equation*}
$$

There is also a boundary term due to the gauge field at $\sigma=\sigma_{1}$ (see [78])
$S_{\mathrm{bdy}, 1}=i \int_{\sigma=\sigma_{1}} A+$ constant $=i A_{\psi}\left(\sigma_{0}\right)+$ constant $=\sqrt{\lambda}\left(\sqrt{1+\kappa^{2}}-\frac{\sqrt{1+\kappa^{2} \sin ^{2} \eta_{0}}}{\sin \eta_{0}}\right)$.

The constant is chosen so that this boundary term vanishes at $\eta_{0}=\pi / 2$ where the boundary of the worldsheet shrinks to a point (and $\rho_{0}=\sinh ^{-1} \kappa$ ).

$$
\text { If } \rho_{0} \rightarrow 0 \quad S_{\mathrm{bdy}, 1}=\sqrt{\lambda} \sqrt{1+\kappa^{2}}-\sqrt{\lambda} \frac{\kappa}{\rho_{0}} .
$$

Thus, let us just take the expression for the (usual) connected correlator between two Wilson loops, with radii $R_{1}=R$ and $R_{2}=R_{2}\left(\rho_{0}\right)$, separated by a distance $h=h\left(\rho_{0}\right)$ in $x_{3}$ of AdS:

$$
\begin{equation*}
S_{\mathrm{conn}}=2 \sqrt{\lambda} \frac{R+R_{2}\left(\rho_{0}\right)}{\epsilon\left(\rho_{0}\right)}+2\left(1+4 \tau^{2}\right)^{1 / 4}\left(\left(1-w^{2}\right) K\left(w^{2}\right)-E\left(w^{2}\right)\right) . \tag{A.20}
\end{equation*}
$$

Here $w=w(\tau)$, where $\tau=m a$ is a geometrical parameter, with

$$
\begin{equation*}
m=\frac{r}{y^{2}} \frac{1}{\sqrt{1+r^{\prime 2}+y^{\prime 2}}}, \tag{A.21}
\end{equation*}
$$

and

$$
\begin{equation*}
a^{2}=\left(\frac{R_{2}^{2}-R_{1}^{2}}{2 h}-\frac{h}{2}\right)^{2}+R_{2}^{2} \tag{A.22}
\end{equation*}
$$

We know that the divergent part of A.20 comes from evaluating the action with a cutoff in $y$ at the two sides of the connected solution which end at $y=0$. So we can translate this term to the coordinates we are using. From A.10) and A.11) with $\rho_{0} \rightarrow 0$

$$
\begin{equation*}
S_{\mathrm{conn}}=4 \sqrt{\lambda} \frac{\kappa}{\rho_{0}}+2\left(1+4 \tau^{2}\right)^{1 / 4}\left(\left(1-w^{2}\right) K\left(w^{2}\right)-E\left(w^{2}\right)\right) \tag{A.23}
\end{equation*}
$$

where $w=w(\tau)$. If we add A.19) with $\rho_{0} \rightarrow 0$ and, we get

$$
\begin{equation*}
S_{\text {conn }}^{\text {onshell }}=3 \sqrt{\lambda} \frac{\kappa}{\rho_{0}}+2 \sqrt{\lambda}\left(1+4 \tau^{2}\right)^{1 / 4}\left(\left(1-w^{2}\right) K\left(w^{2}\right)-E\left(w^{2}\right)\right)+\sqrt{\lambda} \sqrt{1+\kappa^{2}} \tag{A.24}
\end{equation*}
$$

APPENDIX A. SYMMETRIC/FUNDAMENTAL WILSON LOOP CORRELATOR129
We still have a divergent term that we must totally eliminate by a suitable Legendre transform. Let us take instead the finite part of the action as being regularized

$$
\begin{equation*}
S_{\mathrm{conn}}^{\mathrm{reg}}=2 \sqrt{\lambda}\left(1+4 \tau^{2}\right)^{1 / 4}\left(\left(1-w^{2}\right) K\left(w^{2}\right)-E\left(w^{2}\right)\right)+\sqrt{\lambda} \sqrt{1+\kappa^{2}} \tag{A.25}
\end{equation*}
$$

The parameter $\tau=m a$, where $a=a\left(\rho_{0}\right)$, parametrizes $\mathcal{F}=\mathcal{F}(\tau)$. But now we have

$$
\begin{equation*}
h_{0}-R \tanh \rho_{0}=R_{2} \sqrt{2 \alpha\left(1+2 \sinh ^{2} \mathcal{F}(\tau)\right)-\alpha^{2}-1}, \tag{A.26}
\end{equation*}
$$

where $\alpha=R / R_{2}$. In this case,

$$
\begin{equation*}
\alpha=\frac{\kappa \cosh \rho_{0}}{\sqrt{\kappa^{2}-\sinh ^{2} \rho_{0}}} \tag{A.27}
\end{equation*}
$$

which becomes 1 when $\rho_{0}=0$. Moreover, in this case, we also have

$$
\begin{equation*}
h_{0}=2 \sinh \mathcal{F}(m a), \quad a^{2}=1+\frac{h_{0}}{4} . \tag{A.28}
\end{equation*}
$$



Figure A.3: Behavior of the integration constant, $m$, with $h_{0}$, the distance between the loop and the base of the brane. Here we set $R=1$.

Notice, from Figure A.3, that, in the presence of the D3-brane, the original separation, $h_{0}$, between the loop and the base of the brane is modified by the presence of the new parameter, $\rho_{0}$. We know that when $\sinh \rho_{0}=\kappa$, the second loop shrinks at the top of the D3 brane. Notice also that when $\sinh \rho_{0}=0$, both with $\kappa=1$ and $\kappa=10$, we recover the behavior of $h_{0}$ in the case of two loops with equal radius (blue line).

Let us rewrite A.26, as

$$
\begin{equation*}
h_{0}=\tanh \rho_{0}+R_{2} \sqrt{2 \alpha\left(1+2 \sinh ^{2} \mathcal{F}(\tau)\right)-\alpha^{2}-1} \tag{A.29}
\end{equation*}
$$

where $R_{2}=R_{2}\left(\rho_{0}\right)$ and $\tau=\tau\left(\rho_{0}\right)$. So, $\rho_{0}$ indeed introduces a new parameter.
We expect that $\rho_{0}$ also modifies the behavior of the action, and then the phase transition. This appendix is a preliminary computation that study the holographic symmet-

APPENDIX A. SYMMETRIC/FUNDAMENTAL WILSON LOOP CORRELATOR131
ric/fundamental Wilson loop correlator. Since it is a preliminary computation, next step will be to do plots for the action of the connected correlator and to compare with the disconnected case, which in this case will be

$$
\begin{equation*}
S_{\text {disc }}=-2 N\left(\kappa \sqrt{1+\kappa^{2}}+\sinh ^{-1} \kappa\right)-\sqrt{\lambda} . \tag{A.30}
\end{equation*}
$$

## Appendix B

## Perturbative computation of the Wilson loop correlator in DCFT

In this appendix we start computing the correlator between two Wilson loops in defect conformal field theory (DCFT). As said in [60, 140, 141], the one-point functions of gauge invariant operators can be non-vanishing in the presence of a defect. In this case, three of the six scalars acquire a non-vanishing vev. This is called fuzzy-funnel solution,

$$
\begin{equation*}
\left\langle\Phi^{i}\right\rangle=-\frac{1}{x_{3}} t^{i} \oplus 0_{(N-n) \times(N-n)}, \quad x_{3}>0, \tag{B.1}
\end{equation*}
$$

where $t^{i}, i=4,5,6$ are generators of the $n$-dimensional irreducible representation of $S U(2)$, i.e. $n \times n$ matrices satisfying the $S U(2)$ algebra.

Let us consider the Wilson loop operator as defined in (3.3) and the connected correlator of two circular Wilson loops [58],

$$
\begin{equation*}
\left\langle W\left(\mathcal{C}_{1}, \mathcal{C}_{2}\right)\right\rangle_{c}=\left\langle W\left(\mathcal{C}_{1}\right) W\left(\mathcal{C}_{2}\right)\right\rangle-\left\langle W\left(\mathcal{C}_{1}\right)\right\rangle\left\langle W\left(\mathcal{C}_{2}\right)\right\rangle, \tag{B.2}
\end{equation*}
$$

where $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ are concentric circles of radii $R_{1}$ and $R_{2}$, respectively, separated by a distance $h$ along $x_{3}$. In addition, they are oppositelly oriented in space and separated in $S^{5}$,

$$
\begin{array}{ll}
\mathcal{C}_{1}: x^{\mu}(\tau)=\left(R_{1} \cos \tau,+R_{1} \sin \tau, L, 0\right), & n^{i}(\tau)=\left(0,0, \sin \chi_{1}, 0,0, \cos \chi_{1}\right), \\
\mathcal{C}_{2}: y^{\mu}(\tau)=\left(R_{2} \cos \tau,-R_{2} \sin \tau, L+h, 0\right), & n^{i}(\tau)=\left(0,0, \sin \chi_{2}, 0,0, \cos \chi_{2}\right), \tag{B.3}
\end{array}
$$

where we have shifted the parametrization to include the parameter $L$, the distance with respect to the defect, and $\chi_{1,2} \in\left[0, \frac{\pi}{2}\right]$. With this parametrization we get $\square^{1}$
$W\left(\mathcal{C}_{1}\right)=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(R_{1} \int_{0}^{2 \pi} d \tau\left[-i A_{1} \sin \tau+i A_{2} \cos \tau-\sin \chi_{1} \Phi_{3}^{(1)}-\cos \chi_{1} \Phi_{6}\right]\right)$,
$W\left(\mathcal{C}_{2}\right)=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(R_{2} \int_{0}^{2 \pi} d \tau\left[-i A_{1} \sin \tau-i A_{2} \cos \tau-\sin \chi_{2} \Phi_{3}^{(2)}-\cos \chi_{2} \Phi_{6}\right]\right)$.

We can study the leading order in $\lambda=g^{2} N \ll 1$ by starting with the Feynman gauge

$$
\begin{equation*}
\left\langle\Phi_{i}^{a} \Phi_{j}^{b}\right\rangle=\frac{\lambda}{4 \pi^{2} N} \frac{\delta^{a b} \delta_{i j}}{(x-y)^{2}}, \quad\left\langle A_{\mu}^{a} A_{\nu}^{b}\right\rangle=\frac{\lambda}{4 \pi^{2} N} \frac{\delta^{a b} \delta_{\mu \nu}}{(x-y)^{2}} \tag{B.5}
\end{equation*}
$$

The second term of (B.2] was calculated in [58] for each factor, but we must worry about the two different positions of the loops, $L$ and $L+h$.

We define the classical propagator between points on different loops as

$$
\begin{equation*}
U^{\mathrm{cl}}(\alpha, \beta)=\mathcal{P} \exp \int_{\alpha}^{\beta} d \tau \mathcal{A}^{\mathrm{cl}}(\tau), \quad \mathcal{A}^{\mathrm{cl}}(\tau)=-\sin \chi\left\langle\Phi_{3}\right\rangle_{\mathrm{cl}} \tag{B.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\left\langle\Phi_{3}\right\rangle_{\mathrm{cl}}^{(1)}=\frac{\sin \chi_{1}}{L+h} t_{3}, \quad\left\langle\Phi_{3}\right\rangle_{\mathrm{cl}}^{(2)}=\frac{\sin \chi_{2}}{L} t_{3}, \tag{B.7}
\end{equation*}
$$

then

$$
\begin{equation*}
U^{\mathrm{cl}}(\alpha, \beta)=\exp \left(\left[\frac{\beta \sin \chi_{1}}{L+h}-\frac{\alpha \sin \chi_{2}}{L}\right] t_{3}\right) . \tag{B.8}
\end{equation*}
$$



Figure B.1: Planar diagrams, rainbow (same circle) and ladder (circle to circle), for the case without defect [58].

[^26]

Figure B.2: Scheme of the 1-loop ladder contributions to the Wilson loop correlator in DCFT.

In Figure B.1 we recall the the planar diagrams for the case without defect. The rainbow propagator is a constant, and the ladder can be computed without considering the vev of $\Phi_{3}$,

$$
\begin{equation*}
\left\langle W\left(\mathcal{C}_{1}, \mathcal{C}_{2}\right)\right\rangle_{c}^{(0)}=\frac{\lambda}{8 \pi^{2}} \int_{0}^{2 \pi} d \alpha \int_{0}^{2 \pi} d \beta \frac{\cos (\alpha-\beta)+\cos \left(\chi_{1}-\chi_{2}\right)}{\cosh w-\cos (\alpha-\beta)} \tag{B.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\cosh w=\frac{R_{1}^{2}+R_{2}^{2}+h^{2}}{2 R_{1} R_{2}} \tag{B.10}
\end{equation*}
$$

In Figure B. 2 we see the contributions at 1-loop to ladder diagram, the tadpole and lollipop diagrams (see [140]) due to the defect. These contribute at the same order in $\lambda 2^{2}$ with

$$
\begin{equation*}
\left\langle W\left(\mathcal{C}_{1}, \mathcal{C}_{2}\right)\right\rangle_{c}^{\text {defect }}=\int_{0}^{2 \pi} d \alpha \int_{\alpha}^{2 \pi} d \beta\left\langle U^{\mathrm{cl}}(0, \alpha) \mathcal{A}(\alpha) U^{\mathrm{cl}}(\alpha, \beta) \mathcal{A}(\beta) U^{\mathrm{cl}}(\beta, 2 \pi)\right\rangle \tag{B.11}
\end{equation*}
$$

corresponding to the tadpole diagram.
We leave the continuation of this study for a future work. This will involve results given in [58, 59, 63] and will be similar to the ones in [128].

[^27]
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[^0]:    ${ }^{1}$ One of the most known reviews about AdS/CFT is [3]( the so-called MAGOO review). Some others reviews are, for example, [2]4]; also textbooks [6|7] include the basics and recent applications. Even more, philosophical (purely conceptual) works were written, for example, [80-83].

[^1]:    ${ }^{2}$ Some constants could be different depending on which review or conventions one is using. As usual, you can always "absorbe" them into the solution fields.
    ${ }^{3}$ This is just a generalized form of electromagnetism.
    ${ }^{4}$ This is a relation between mass and charge.
    ${ }^{5}$ Some authors use $L$ instead of $R$ as the $A d S$ radius.

[^2]:    ${ }^{6}$ Up to some conventions in front of the fields.

[^3]:    ${ }^{7}$ There is a well studied and detailed attempt for this by Myers [75-77].
    ${ }^{8}$ This is the usual electromagnetism theory.

[^4]:    ${ }^{9}$ Since we are dealing with a conjectured equivalence the word dictionary is also not exact.

[^5]:    ${ }^{1}$ We need to close the path and take the trace in order to have gauge invariance
    ${ }^{2}$ In general this trace is taken over some representation.

[^6]:    ${ }^{3}$ There exists confining for QED in the strong coupling regime. See for example [90, 91].
    ${ }^{4}$ The case of the pure gauge Wilson loop (with no scalar coupling), and its dual, was studied in [92-95].

[^7]:    ${ }^{5}$ See [35] for a complete definition of the supersymmetric Wilson loop.

[^8]:    ${ }^{6}$ These propagator actually contain also a $\mathcal{O}(1 / N)$ term that vanishes at large $N$.
    ${ }^{7}$ Just like other composite operators in $\mathcal{N}=4$.

[^9]:    ${ }^{8}$ This happens only for smooth loops, cusped loops does not show cancellation due to supersymmetry.

[^10]:    ${ }^{9}$ See a detailed computation in [27], and also in the master's thesis 101]. A good recommendation for integration is [102].

[^11]:    ${ }^{10}$ Since supersymmetry transformation can be thought as translations in the superspace, $Q$ can be written as a differential operator.
    ${ }^{11}$ Localization can be extended to non-compact spaces as well (see for example $103 \mid$ )

[^12]:    ${ }^{1}$ Remember that in the large $N$ regime $U(N)$ and $S U(N)$ can be harmessly exchanged.
    ${ }^{2}$ Integrability techniques allows us to say that the model is indeed fully solvable without solving it.

[^13]:    ${ }^{3}$ This equation comes from the fact that $\mathcal{H}=0$ for reparametrization invariant metric like ours. We can find an equivalent result by starting from the Polyakov action instead of the Nambu-Goto one. In this case $\mathcal{H}=0$ becomes

[^14]:    ${ }^{1}$ Notice that $\xi$ is the radial coordinate of the internal $A d S_{2}$ space.

[^15]:    ${ }^{2}$ This boundary term must be at $\xi$, since this is the radial coordinate.

[^16]:    ${ }^{3}$ We have not obtained a reliable result for these functions because we need to take better care of the expansions we perform, but we expect that the behavior to be the one we wrote above. A result appeared recently [122] where an approximate expression for the expansion of the factor in front of the squared expression of the horizontal rectangular Wilson loop was obtained.

[^17]:    ${ }^{4}$ This choice allows to have only derivatives of $A$ in the action, so a conserved momentum.

[^18]:    ${ }^{5}$ It was remarked that a better understanding of this divergences is needed in order to explain if those cancel each other or if it is due to our ansatz that they appear. This is open and ongoing.

[^19]:    ${ }^{1}$ Notice we have not considered the internal angle in $S^{5}$. This case was considered in [58].
    ${ }^{2} k$ is $K_{x}$ in [58].

[^20]:    ${ }^{3}$ This expression is the same as the one given in eq. (21) in [58].

[^21]:    ${ }^{4} k$ and $\ell$ are $K_{x}$ and $K_{\phi}$ in [58].

[^22]:    ${ }^{5}$ Here we set $x_{3}=x$ for simplicity.

[^23]:    ${ }^{6} \varepsilon_{0} \geq 0$ actually produces four regions of parameters but we choose only those that lead to real $g(\sigma)$.

[^24]:    ${ }^{7}$ This case was solved in [63].

[^25]:    ${ }^{1}$ The antisymmetric/fundamental case was studied in [78] and the antisymmetric/antisymmetric case in [79].

[^26]:    ${ }^{1}$ We chose a minus sign in front of the term involving the scalars as in [63].

[^27]:    ${ }^{2}$ Lollipop contribution results to vanish. [141]

