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Problemas de campo forte na eletrodinâmica e teoria quântica de campos

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Strong field problems in electrodynamics and quantum field theory

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Resumo

Esta tese de doutorado é devotada a problemas de campos fortes em eletrodinâmica e teoria quântica de campos. Alguns sistemas físicos bem conhecidos são estudados sob o formalismo da eletrodinâmica quântica (QED) com campos externos e eletrodinâmica não-linear. Primeiramente estudamos propriedades estatísticas de estados quânticos de Dirac e Klein-Gordon massivos que interagem com campos elétricos dependentes do tempo que viola a estabilidade do vácuo, primeiro em termos gerais e em seguida para um campo de fundo específico. Como ponto de partida, derivamos uma expressão não-perturbativa de tais campos. Construímos operadores de densidade reduzidos para subsistemas de elétrons e pósitrons e discutimos o efeito de decoerência que pode ocorrer no curso de evolução devido à uma medição intermediária. Calculamos a perda de informação em estados em QED devido a reduções parciais e uma possível decoerência por meio da entropia de von Neumann. Em seguida consideramos um campo elétrico específico, denominado por campo T-constante, como campo de fundo forte. Este modelo exatamente solúvel nos permite calcular, explicitamente, todas as propriedades estatísticas de vários estados quânticos de campos massivos e carregados em consideração. Utilizamos uma abordagem não-perturbativa para a QED com X-degraus elétricos críticos e consideramos dois exemplos de configuração de campo de tipo exponencial (campo simétrico que varia lentamente e campo do tipo pico). Os números médios de partículas criadas por essas configurações de campo são calculados. As condições quando espaços "in" e "out" de QED com campos em consideração são unitariamente equivalentes são obtidos. Então construímos um operador de densidade geral, cuja condição inicial é o vácuo. Tal operador descreve a deformação de um estado de vácuo inicial por X-degraus elétricos críticos. Encontramos as reduções do estado deformado para subsistemas de elétrons e pósitrons e calculamos a perda de informação destas reduções. A consideração geral é ilustrada por meio de um estudo de estados de vácuo quântico entre duas placas de capacitor. Calculamos as medidas de emaranhamento destes estados reduzidos como entropias de von Neumann. Por fim, determinamos o campo de uma partícula puntiforme em movimento em eletrodinâmica local não-linear. Utilizamos como um modelo a lagrangiana de Euler-Heisenberg truncada no seu termo de ordem principal em uma expansão, em série de potências, do primeiro invariante de campo eletromagnético. Calculamos a energia total do campo produzido por uma partícula puntual e mostramos que a mesma é finita; portanto tornando sua configuração de campo como um soliton. Definimos o tensor de energia-momento finito para esta configuração e demonstramos que suas componentes satisfazem a relação mecânica padrão de uma partícula massiva livre que se move.

Palavras-chave: Eletrodinâmica quântica; Operador densidade; Entropia; Emaranhamento; Criação de partículas; Eletrodinâmica não-linear; Potenciais retardados; Carga móvel; Soliton

Abstract

This thesis is devoted to strong field problems in electrodynamics and quantum field theory. Some well known physical systems are studied in a framework of quantum electrodynamics with external field and nonlinear electrodynamics. First, the statistical properties of states of quantized charged massive Dirac and Klein-Gordon fields interacting with a time-dependent background that violates the vacuum stability, first in general terms and then for a special electromagnetic background. As a starting point, a nonperturbative expression for the density operators of such fields. The reduced density operators for electron and positron subsystems are constructed and a decoherence that may occur in course of the evolution due to an intermediate measurement is discussed. The loss of the information in QED states due to partial reductions and a possible decoherence is studied by calculating the von Neumann entropy. Next, the so-called T -constant external electric field as an external background is considered. This exactly solvable example allows the explicit calculation of all statistical properties for various quantum states of the massive charged fields under consideration. Next, a nonperturbative approach to QED with x -electric critical potential steps is used. The general consideration is illustrated by the example of so-called exponential in two different configurations (slowly varying field and sharp peak field); differential and full mean numbers of particles created by these field configurations are calculated. The conditions when in- and out- spaces of the QED under consideration are unitarily equivalent are found. Then, a general density operator with the vacuum initial condition is constructed. Such an operator describes a deformation of the initial vacuum state by x -electric critical potential steps. The reductions of the deformed state to electron and positron subsystems are found, and the loss of the information in these reductions is calculated. The general consideration is illustrated by studying the deformation of the quantum vacuum between two capacitor plates. The entanglement measures of these reduced states are calculated as von Neumann entropies. Third, the field of a moving pointlike charge is determined in nonlinear local electrodynamics. The Euler-Heisenberg Lagrangian of quantum electrodynamics truncated at the leading term of its expansion in powers of the first field invariant is used as a model Lagrangian. The total energy of the field produced by a point charge is calculated and shown to be finite; thereby making its field configuration a soliton. A finite energy-momentum vector of this field configuration is defined to demonstrate that its components satisfy the standard mechanical relation characteristic of a freely moving massive particle.

Keywords: Quantum electrodynamics; Density operator; Entropy; Entanglement; Particle creation; Nonlinear electrodynamics; Delayed potentials; Moving charge; Soliton

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Introduction

Traditionally it was believed that quantum theory is intended to describe the phenomena of the microcosm and the macroscopic field strengths are described by the classical theory. However, the development of quantum field theory led to an understanding of the fact that there are non-trivial quantum phenomena (such as vacuum polarization, the Klein paradox, the Hawking black-hole radiation, and the creation of electron-positron pairs by external fields from vacuum) that can not be described classically. Already at the beginning of creation of relativistic quantum mechanics, these effects caused a number of paradoxes, giving rise to a question of correctness of its direct application to processes with strong fields. It became clear that quantum effects can, under certain conditions, change the classical evolution of strong fields due to the backreaction. This fact was noticed by such giants of theoretical physics as Feynman and Schwinger. The case of homogeneous and constant electric fields interacting with electrons was examined by Schwinger[1], who obtained the vacuum-to-vacuum transition probability using his method of effective action. Such fields admit analytical solutions of the Dirac equation and were frequently used in various QFT calculations; see Ref. [2] for a review. It was demonstrated that electron-positron pairs must be observed in electric fields having the order of the critical field. The Feynman causal propagator approach[3] was generalized by Nikishov and Narozhny[4, 5, 6] in order to calculate pair creation and pair scattering in the zero-order approximation with respect to the radiative interaction. Depending on the structure of external backgrounds, different approaches were proposed to calculate the pair creation effect in the framework of relativistic quantum mechanics[7, 8, 9, 10].

There are two general ways to account the effects of strong field. On one hand, in quantum electrodynamics it is possible to account the effects of particle production (or, in other words, vacuum instability) nonperturbatively[12, 13, 14]. This way of consideration is possible when the exact solutions of Dirac equation in the presence of the external background are known. On the other hand, it would be, probably, very beneficial to have a classical description of strong-field problems for comparison. To achieve that, one can study self-interaction of electric fields in vacuum or taken against a very strong background. This way implies the use of the nonlinear electrodynamics, that accounts for the effects of interaction between electromagnetic fields when at least one of them is large.

The above-mentioned effect of particle creation by strong electromagnetic and gravitational fields has an essentially quantum nature. It was first considered in the framework of the relativistic quantum mechanics with understanding that all the questions can be answered only in the framework of quantum field theory (QFT). QFT with external background is to a certain extent an appropriate model for such calculations. In the framework of this model, particle creation is closely related to a violation of vacuum stability with time. Backgrounds (external fields) that violate vacuum stability are to be electriclike fields that are able to produce nonzero work when interacting with charged particles. Depending on the structure of such backgrounds, different approaches for calculating the effect were proposed and realized.

A well-known example of strong-field QFT is the so-called Furry picture of QED, in which the matter field (Dirac field) is quantized with the help of exact solutions of the Dirac equation in magnetic fields, such that the Feynman diagrams are calculated exactly with respect to the interaction with these magnetic fields[15]. The studies [12, 13, 16, 17, 18, 19, 20, 21] proposed a generalization of the Furry picture to a class of electric-like external backgrounds that violate the vacuum stability (in other words, they create electron-positron pairs from vacuum). This class of external backgrounds was restricted by fields switched on and off at the initial and final time instants, respectively. In the framework of this generalized Furry picture, many general results were obtained[22, 23, 24, 25, 26] and specific quantum effects were calculated; see the review in Ref. [14] and the more recent studies[27, 28, 29, 30].

By analogy with the original Furry picture, the new formulation was, in fact, based on the quantization of the Dirac field with the help of some adequate exact solutions of the Dirac equation with specific uniform asymptotics at $t \rightarrow \infty$. In fact, the above-mentioned external backgrounds represent the so-called t -electric potential steps (their exact definition is given below). In case the Dirac equation with a specific t -electric potential step can be solved, the general formulation[12, 13, 16, 17, 18, 19, 20, 21] allows one to calculate, nonperturbatively with respect to the external field, all the QED processes, both without photons (zero-order processes) and with an arbitrary number of photons. The cases when such solutions can be found explicitly (analytically) are called exactly solvable, e.g., see a complete collection of exact solutions of relativistic wave equations in Ref. [31]. In QED with t -electric potential steps, there exist a few exactly solvable cases that have real physical importance. Those are the Sauter-like (or adiabatic) electric field[5, 28, 32], the T -constant electric field (a uniform electric field which effectively acts during a sufficiently large but finite time interval T)[24, 25, 26, 28, 29, 30, 33, 34], the exponentially decaying electric field[35], and certain combinations of them.

However, there exist many physically interesting situations where external backgrounds formally are not switched off at the time infinity, the corresponding backgrounds formally being not t -electric potential steps. As an example, time-independent nonuniform electric fields that are concentrated in restricted space areas may be used. The latter fields represent a kind of spatial or, as they will be called conditionally, x -electric potential steps for charged particles. The x -electric potential steps can also create particles from the vacuum, the Klein paradox being closely related to this process[36, 37, 38]. Just after the original Klein's paper the problem was studied by Sauter, who considered both the Klein step[37] and a more realistic smoothed potential step, which is called the Sauter potential[38]. To avoid confusion, the Klein paradox should be distinguished from the Klein tunneling through the square barrier, e.g., see Ref. [39] and references therein. This tunneling without an exponential suppression occurs when an electron is incident on a high barrier, even when it is not high enough to create particles. Approaches elaborated for treating quantum effects in the t -electric potential steps are not directly applicable to the x -electric potential steps. Some heuristic calculations of the particle creation by x -electric potential steps in the framework of the relativistic quantum mechanics were presented by Nikishov in Refs. [6, 40] and later developed by Hansen and Ravndal in Ref. [41]. One should also mention the Damour work[42], that contributed significantly in applying semiclassical methods for treating strong field problems in astrophysics. In fact, this work presents a first step to bridge the gap between approaches to quantum effects in potential steps developed within relativistic quantum mechanics and QFT. Using Damour's approach, mean numbers of pairs created by a strong uniform electric field confined between two capacitor plates separated by a finite distance was calculated in Ref. [43]. A detailed historical review can be found in Refs. [39, 41]. Nikishov had tested his way of calculation using the special case of a constant and uniform electric field, which is possible both for the t -electric potential steps and the x -electric

potential steps, see Refs. [6, 40, 44]. At that time, however, no justification for such calculations from the QFT point of view was known. In the recent work[76], Gavrilo and Gitman succeeded to construct a consistent version of QED with the so-called x -electric critical potential steps.

At the present day, the theory of intense fields has a lot of important physical applications in astrophysics, cosmology, neutrino physics, nuclear physics, as well as in the physics of nanostructures. Recent progress in laser physics makes it possible to hope that this effect will soon be observed experimentally in laboratory conditions[45]. Due to synthesization of graphene and other nanomaterials, particle creation by external fields has now become an observable effect[46, 47]. The case of a constant uniform electric field has many similarities with the case of the de Sitter background, e.g., see Refs. [27, 48] and references therein. In particular, the particle creation effect is crucial for understanding the conductivity of graphene in the nonlinear regime[33, 49]. It turned out that, in a certain area, the physics of such structures is described by a quantum field model with unstable vacuum, where almost any electrical fields can be considered as strong fields, and methods developed in quantum electrodynamics (QED) with unstable vacuum allow nonperturbative calculations. All this makes the theory of intensive fields not only academically interesting, but also really important and relevant to condensed matter physics. Thus, it can be said that the strong field effects changed the status of quantum theory, and, along with the problems of quantum entanglement and quantum information, made it a theory that describes a real macroscopic physics with a solid experimental basis.

As was said before, the other possible way of studying the problems in question is to use nonlinear electrodynamics. It is planned to construct the field of a uniformly moving charge in nonlinear electrodynamics, or, in other words, to study a nonlinear extension of the Liénard-Wiechert potentials. The problem of the field caused by a moving charge, besides belonging to fundamental problems of electrodynamics, is also of certain practical importance when applied to charged particle beams in accelerators[50]. Its nonlinear extension may have an effect in small-impact-parameter scattering at high energies where close vicinities of the charge come into play. Nonlinear electrodynamics models as a way of describing strong field have been studied for a long time, and have attracted the attention of many researchers (see, for example, Refs. [51, 52, 53] and the references therein). Probably the most known model of nonlinear electrodynamics is Born-Infeld model[52], which first solved the problem of divergence of a field of a point charge, i.e., reproduced finiteness of the field mass of the charged particle. This property has been encouraging attempts to attribute the experimental value of the electron mass to the energy carried by its field, the electromagnetic contribution being sometimes almost exhausting[53]. However, the nonlinear model[54], which originates from truncation of quantum electrodynamics at the second power of the field invariant $\mathfrak{F}(x) = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, may be thought of as the simplest model among those that result in finiteness of the field energy of a point charge, because it admits an analytical solution to the field equations. Besides, its Lagrangian does not possess the disadvantage of being a singular function of the field. It is this quality that led to presence of a maximum value of electric field, responsible, in the end, for the finiteness of the field energy in the Born-Infeld[52], and other[51], [53] models. On the other hand, in the model of Ref. [54] the field of a point charge is still singular near the charge, but this singularity is suppressed to the extent sufficient for the convergence of the field-mass integral.

The thesis is organized as follows. In section 1, “QED with external field”, a review of a general formulation of quantum electrodynamics with external field is given. After, some more important details of QED with t -dependent external field are stressed; density operators for the system with external t -electric potential step with different initial conditions are constructed, and their reductions are calculated. Entropy of the states these density operators describe is found for different initial conditions, and T -constant electric field is considered as an example. In subsection 1.5 the results

on QED with x -dependent (non-uniform) external field are presented. First, some general theory is given and illustrated by the example of so-called peak field; in addition, the unitarity of in- and out-Fock spaces in QED with x -electric potential steps is proven. Second, deformation of initial vacuum state is considered and the density operator of the system that initially has been in a vacuum state is constructed. Its reductions are calculated and the corresponding von Neumann entropies are found. Finally, the so-called L -constant field is used to illustrate the results obtained. Section 2, “Moving charge as a soliton” is devoted to the problem of a moving charge in nonlinear electrodynamics. The minimally nonlinear model is discussed, and fields of the moving charge in this model are calculated. Then, a soliton representation for point-like charge is constructed: the energy-momentum tensor is found and the particle-like relation between momentum and the rest field mass, along with the mass finiteness is demonstrated. Appendix 1 contains some explicit calculations, demonstrating the fact that non-linear expressions for the charged particle satisfy the linear Maxwell equations with the point-like charge current. Appendix 2 contains basic elements of QED with x -electric critical potential steps that are necessary for the understanding of the subsection 1.5.

1 QED with external field

As it was mentioned in the Introduction, detailed formalism of quantum electrodynamics with external field was developed by Gitman and others; see Refs. [12, 13, 14]. But it will be useful to outline the most important points for clarity.

The Lagrangian of this theory is

$$\begin{aligned}
 L &= L_\gamma + L_e + L_{\text{int}}, \\
 L_\gamma &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \\
 L_e &= \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi, \quad \bar{\psi} = \psi^\dagger \gamma^0, \\
 L_{\text{int}} &= -j_\mu A^\mu, \quad j^\mu = e\bar{\psi}\gamma^\mu\psi,
 \end{aligned} \tag{1.1}$$

where $A^\mu(x)$ are electromagnetic field potentials and $\psi(x)$ is a spinor (Dirac) field. The corresponding equations of motion are

$$\begin{aligned}
 \partial_\nu F^{\nu\mu} - j^\mu &= \square A^\mu - \partial^\mu (\partial_\nu A^\nu) - j^\mu = 0, \\
 (i\gamma_\mu \partial^\mu - e\gamma_\mu A^\mu - m) \psi &= 0.
 \end{aligned} \tag{1.2}$$

This gauge theory has Abelian gauge transformations of the form

$$\begin{aligned}
 A^\mu &\rightarrow A'^\mu = A^\mu + \partial^\mu \xi, \\
 \psi &\rightarrow \psi' = \exp\{-ie\xi\} \psi.
 \end{aligned} \tag{1.3}$$

When quantizing theories with gauge symmetries, one encounters a number of well-known problems that can be dealt with in various ways. One method is to pass over from the original gauge-invariant Lagrangian (1.3) to another Lagrangian which, while defining a physically equivalent theory, is no longer gauge invariant. Accordingly, the following expression

$$L_\gamma = -\frac{1}{2}A^{\mu,\nu}A_{\mu,\nu}, \quad A^{\mu,\nu} = \partial^\nu A^\mu \tag{1.4}$$

taken as a Lagrangian for the electromagnetic field will be used. This differs by a four-divergence from the Lagrangian

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2, \tag{1.5}$$

which correspond to Feynman gauge. The equations of motion in the electromagnetic sector now change and become

$$\square A^\mu - j^\mu = 0, \tag{1.6}$$

whence, in view of the current conservation law $\partial_\mu j^\mu = 0$, it follows that

$$\square \partial_\mu A^\mu = 0. \tag{1.7}$$

In order to consider the Hamiltonian formalism, one defines the general momenta

$$\begin{aligned}\pi_\mu &= \frac{\partial L}{\partial \dot{A}^\mu} = -\dot{A}^\mu, \\ p_\psi &= \frac{\partial_r L}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0, \quad p_{\bar{\psi}} = \frac{\partial_r L}{\partial \dot{\bar{\psi}}} = 0.\end{aligned}\tag{1.8}$$

There are two primary second-class constraints which, in the present case, exhaust all the constraints of the theory

$$p_\psi - i\bar{\psi}\gamma^0 = 0, \quad p_{\bar{\psi}} = 0.\tag{1.9}$$

The Hamiltonian then is formed in a standard way,

$$\begin{aligned}H &= H_\gamma + H_e + H_{\text{int}}, \\ H_\gamma &= \frac{1}{2} \int (-\pi^2 + A^{\mu,i} A_{\mu,i}) d\mathbf{x}, \quad i = 0, 1, 2; \\ H_e &= \int \bar{\psi}(x) (-\gamma \bar{\nabla} + m) \psi d\mathbf{x}, \\ H_{\text{int}} &= \int j_\mu(x) A^\mu(x) d\mathbf{x}.\end{aligned}\tag{1.10}$$

The nonzero equal-time commutation relations (the constraints (1.9) having been taken into account) are

$$\begin{aligned}[A^\mu(x), \pi_\nu(y)]_- &= i\delta_\nu^\mu \delta(\mathbf{x} - \mathbf{y}), \\ [\psi(x), \bar{\psi}(y)]_+ &= \gamma^0 \delta(\mathbf{x} - \mathbf{y}), \quad x^0 = y^0.\end{aligned}\tag{1.11}$$

The field operators $\varphi(x)$ in the interaction picture are related to the corresponding operators $\varphi(\mathbf{x})$ in the Schrödinger picture in the following way:

$$\varphi(x) = e^{iH_0 t} \varphi(\mathbf{x}) e^{-iH_0 t}.\tag{1.12}$$

Taking $H_0 = H_\gamma + H_e$, one can see that in the interaction representation the operators of electromagnetic $A^\mu(x)$ and spinor $\psi(x)$ field satisfy the free equations of motion. Therefore, the operator $A^\mu(x)$ satisfies the equation

$$\square A^\mu(x) = 0.\tag{1.13}$$

Consequently, it can be represented as

$$A^\mu(x) = \sum_{\lambda=0}^3 \int [c_\lambda f_\lambda^\mu(x) + c_\lambda^\dagger f_\lambda^{\mu*}(x)] d\mathbf{k}, \quad n = (\mathbf{k}, \lambda),\tag{1.14}$$

where $f_n^\mu(x)$ is the photon wave function,

$$f_n^\mu(x) = \frac{\exp(-ikx)}{\sqrt{(2\pi)^3 2k_0}} e^\mu(\mathbf{k}, \lambda), \quad k_0 = |\mathbf{k}|;\tag{1.15}$$

$e^\mu(\mathbf{k}, \lambda)$ are four linear-independent polarization vectors, $\lambda = 0, 1, 2, 3$ subject to the relations

$$\begin{aligned}e_\mu(\mathbf{k}, \lambda) e^{\mu*}(\mathbf{k}, \lambda') &= \eta_{\lambda\lambda'}, \\ \sum_\lambda e_\mu(\mathbf{k}, \lambda) \eta_{\lambda\lambda'} e_\nu^*(\mathbf{k}, \lambda) &= \eta_{\mu\nu}.\end{aligned}\tag{1.16}$$

It follows that the orthonormality conditions for the photon wave functions are

$$\begin{aligned}(f_n, f_{n'}) &= \eta_{\lambda\lambda'} \delta(\mathbf{k} - \mathbf{k}'), \quad (f_n, f_{n'}^*) = 0, \\ (f, g) &= i \int f^*(x) \left(\overleftarrow{\partial}_0 - \overrightarrow{\partial}_0 \right) g(x) d\mathbf{x}.\end{aligned}\tag{1.17}$$

The commutation relations (1.11) for the Schrödinger operators, taking constraint (1.9) into account, produce the equal-time commutation relations for the operators $A^\mu(x)$ taken in the interaction representation

$$\begin{aligned}\left[\dot{A}_\mu(x), A_\nu(y) \right]_- &= i\eta_{\mu\nu} \delta(\mathbf{x} - \mathbf{y}), \\ [A_\mu(x), A_\nu(y)]_- &= 0, \quad x^0 = y^0,\end{aligned}\tag{1.18}$$

which imply the commutation relations for c_n^\dagger, c_n :

$$\begin{aligned}\left[c_n, c_{n'}^\dagger \right]_- &= -\eta_{\lambda\lambda'} \delta(\mathbf{k} - \mathbf{k}'), \\ [c_n, c_{n'}]_- &= [c_n^\dagger, c_{n'}^\dagger]_- = 0.\end{aligned}\tag{1.19}$$

Operators c_n are photon annihilation operators, and c_n^\dagger are photon creation operators.

Following the procedure of the second quantization for fields $A^\mu(x)$, one arrives at the representation in terms of Fock creation and annihilation operators, where

$$H_\gamma = - \sum_{\lambda=0}^3 \eta_{\lambda\lambda} \int k_0 c_n^\dagger c_n d\mathbf{k},\tag{1.20}$$

and the physical subspace of states is constrained by the condition

$$(c_{\mathbf{k}0} - c_{\mathbf{k}3}) |\Psi\rangle = 0.\tag{1.21}$$

The spinor field $\psi(x)$ satisfies the Dirac equation

$$(i\gamma^\mu \partial_\mu - m) \psi(x) = 0.\tag{1.22}$$

Let $\pm\varphi_n(x)$ be positive (+) and negative (-) energy solutions of the Dirac equation. Quantum number n includes momentum \mathbf{p} and spin s , and solutions $\pm\varphi_n(x)$ have the form

$$\begin{aligned}+\varphi_n(x) &= \sqrt{\frac{m}{(2\pi)^3 p_0}} e^{-ipx} u_s(\mathbf{p}), \\ -\varphi_n(x) &= \sqrt{\frac{m}{(2\pi)^3 p_0}} e^{ipx} v_s(\mathbf{p}), \\ p_0 &= \sqrt{\mathbf{p}^2 + m^2}, \\ (\gamma^\mu p_\mu - m) u_s(\mathbf{p}) &= (\gamma^\mu p_\mu + m) v_s(\mathbf{p}) = 0.\end{aligned}\tag{1.23}$$

Spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ are normalized as

$$\begin{aligned}u_s^\dagger(\mathbf{p}) u_{s'}(\mathbf{p}) &= v_s^\dagger(\mathbf{p}) v_{s'}(\mathbf{p}) = \delta_{ss'} \frac{p_0}{m}, \\ u_s^\dagger(\mathbf{p}) v_{s'}(\mathbf{p}) &= 0,\end{aligned}\tag{1.24}$$

and the orthonormality conditions for the solutions (1.23)

$$\int \lambda \varphi_n^\dagger(x) \lambda' \varphi_{n'}(x) d\mathbf{x} = \delta_{\lambda\lambda'} \delta_{nn'}, \quad \delta_{nn'} = \delta_{ss'} \delta(\mathbf{p} - \mathbf{p}'). \quad (1.25)$$

The completeness relation holds for functions $\varphi_n(x)$,

$$\sum_{\lambda n} \lambda \varphi_n(x) \lambda \varphi_n^\dagger(y) = \delta(\mathbf{x} - \mathbf{y}), \quad x^0 = y^0, \quad (1.26)$$

and the integration over \mathbf{p} is included in summation over quantum number n .

The spinor field $\psi(x)$ can be decomposed in the solutions (1.23) as

$$\psi(x) = \sum_n [a_n + \varphi_n(x) + b_n^+ - \varphi_n(x)]. \quad (1.27)$$

The commutation relations (1.11) for the spinor field $\psi(x)$ imply that a_n , a_n^\dagger and b_n , b_n^+ are Fermi creation and annihilation operators

$$\begin{aligned} [a_n, a_m^\dagger]_+ &= [b_n, b_m^+]_+ = \delta_{nm}, \\ [a_n, a_m]_+ &= [b_n, b_m]_+ = 0. \end{aligned} \quad (1.28)$$

The operators a_n^\dagger , a_n can be interpreted as creation and annihilation operators of electrons, and b_n^+ , b_n as creation and annihilation operators of positrons. Using these operators together with those of photons (1.19) c_n^\dagger and c_n , one can construct the Fock space, which is understood as the space of the initial and final states. The general state vector with fixed number of particles in this space is defined as

$$b_n^+ \dots a_m^\dagger \dots c_k^\dagger \dots |0\rangle, \quad (1.29)$$

where $|0\rangle$ is the corresponding vacuum vector defined by the conditions

$$c_n |0\rangle = a_n |0\rangle = b_n |0\rangle = 0. \quad (1.30)$$

The scattering operator, or S -matrix, is defined as

$$\begin{aligned} S &= \lim_{\substack{t \rightarrow \infty, \\ t' \rightarrow -\infty}} \exp\{iH_0 t\} \exp\{-iH(t-t')\} \exp\{-iH_0 t'\} \\ &= T \exp\left\{-i \int j_\mu(x) A^\mu dx\right\}, \quad j^\mu(x) = \frac{e}{2} [\bar{\psi}(x) \gamma^\mu, \psi(x)]_-, \end{aligned} \quad (1.31)$$

where the operators $A_\mu(x)$ and $\psi(x)$ relate to the interaction representation. The S -matrix elements between the states (1.29) determine the scattering probability amplitudes of particles. By reducing the S -matrix to the normal form using Wick's theorem, one can obtain an expansion of these amplitudes in a perturbative series in powers of the radiative interaction.

The Lagrangian of QED with an external field is formally obtained from that of QED by the replacement

$$A_\mu(x) \rightarrow A_\mu(x) + A_\mu^{\text{ext}}(x), \quad (1.32)$$

where $A_\mu^{\text{ext}}(x)$ is an external electromagnetic field. The quantization procedure described above can be repeated without modification for this case also. The commutation relations (1.28) do not change. The Hamiltonian for QED with an external field differs from the Hamiltonian (1.10) by a term responsible for the interaction with the external field

$$H = H_\gamma + H_e + H_{\text{int}} + \int j^\mu(x) A_\mu^{\text{ext}}(x) d\mathbf{x}. \quad (1.33)$$

1.1 General theory: t -electric potential steps

Here we will briefly consider a special case of QFT with unstable vacuum of quantized Dirac or K-G field with time dependent electric-like background that is switched on and off at $t \rightarrow \pm\infty$. Let us repeat the most important points of QED with unstable vacuum using as an example the spinor field $\psi(x)$ interacting with an external electromagnetic field $A_\mu^{\text{ext}}(x)$. The Lagrangian of the system is

$$\mathcal{L} = \bar{\psi} (i\partial_\mu - eA_\mu^{\text{ext}}(x)) \gamma^\mu \psi - m\bar{\psi}\psi, \quad (1.1.1)$$

and the equation of motion for $\psi(x)$ is the Dirac equation in an external field

$$D(A_\mu^{\text{ext}}(x)) \psi(x) = 0, \quad D(A_\mu^{\text{ext}}(x)) = \gamma^\mu (i\partial_\mu - eA_\mu^{\text{ext}}(x)) - m. \quad (1.1.2)$$

The interaction Hamiltonian and the commutation relations for the Schrödinger operators are

$$\begin{aligned} H_e(t) &= \int \bar{\psi}(x) (-i\gamma\nabla + e\gamma^\mu A_\mu^{\text{ext}}(x) + m) \psi(x) dx \\ &= \int \psi^\dagger(\mathbf{x}) H_e(t) \psi(\mathbf{x}) d\mathbf{x}, \\ H_e(t) &= \alpha (-i\nabla - e\mathbf{A}^{\text{ext}}(x)) + eA_0^{\text{ext}}(x) + m\beta, \\ [\psi(x), \bar{\psi}(y)]_+ &= \gamma^0 \delta(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (1.1.3)$$

The time-dependence of the Hamiltonians is explicitly indicated here in order to stress the possible dependence on time of the external field. Let us turn to the scattering problem in the system under consideration. To this end, one must first construct states corresponding to particles in the initial, t_{in} , and final, t_{out} , time instants, to be understood in the final expressions as shifted into the infinitely remote past and future, respectively. The general case, characterized by the fact that the external field potentials are not switched off at t_{in} , t_{out} , will be considered here. This situation may arise even if the external field strengths disappear at t_{in} and t_{out} . For example, the electric field of the form $E_x = E_y = 0$, $E_z = E \cosh^{-2} t$ vanishes when $t \rightarrow \pm\infty$. Nevertheless, its potentials $A_0 = A_x = A_y = 0$, $A_z = -E \tanh t$, chosen in a special gauge, are different from zero and from one another in the limits $t \rightarrow \pm\infty$. Therefore, the Hamiltonians $H_e(t_{\text{in}})$ and $H_e(t_{\text{out}})$ are different. It is possible to transform them into diagonal form

$$\begin{aligned} \hat{H}_e(t_{\text{in}}) \pm \varphi_n(\mathbf{x}) &= \pm \varepsilon(t_{\text{in}}) \pm \varphi_n(\mathbf{x}), \\ \hat{H}_e(t_{\text{out}}) \pm \varphi_n(\mathbf{x}) &= \pm \varepsilon(t_{\text{out}}) \pm \varphi_n(\mathbf{x}), \end{aligned} \quad (1.1.4)$$

where $\pm \varepsilon(t_{\text{in}})$ and $\pm \varepsilon(t_{\text{out}})$ are the corresponding energies at the moments t_{in} and t_{out} . It is assumed that the sets of spinors $\{\pm \varphi_n(x)\}$ and $\{\pm \varphi_n(x)\}$ are orthonormalized and complete:

$$\begin{aligned} (\zeta \varphi_n, \zeta' \varphi_{n'}) &= (\zeta \varphi_n, \zeta' \varphi_{n'}) = \delta_{\zeta\zeta'} \delta_{nn'}, \quad \zeta, \zeta' = \pm, \\ (\varphi, \chi) &= \int \varphi^\dagger(\mathbf{x}) \chi(\mathbf{x}) d\mathbf{x}, \\ \sum_{n\zeta} \zeta \varphi_n(\mathbf{x}) \zeta \varphi_n^\dagger(\mathbf{y}) &= \sum_{n\zeta} \zeta \varphi_n(\mathbf{x}) \zeta \varphi_n^\dagger(\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (1.1.5)$$

The field $\psi(x)$ can be decomposed in bases $\{\pm\varphi_n(x)\}$ and $\{\pm\varphi_n(x)\}$ as

$$\begin{aligned}\psi(x) &= \sum_n \{a_n(t_{\text{in}}) {}^+\varphi_n(\mathbf{x}) + b_n^\dagger(t_{\text{in}}) {}^-\varphi_n(\mathbf{x})\}, \\ \psi(x) &= \sum_n \{a_n(t_{\text{out}}) {}^+\varphi_n(\mathbf{x}) + b_n^\dagger(t_{\text{out}}) {}^-\varphi_n(\mathbf{x})\}.\end{aligned}\quad (1.1.6)$$

Next, one can obtain commutation relations for creation and annihilation operators $a_n(t_{\text{in}})$, $a_n^\dagger(t_{\text{in}})$, $b_n(t_{\text{in}})$, $b_n^\dagger(t_{\text{in}})$ and $a_n(t_{\text{out}})$, $a_n^\dagger(t_{\text{out}})$, $b_n(t_{\text{out}})$, $b_n^\dagger(t_{\text{out}})$ from (1.1.3) and (1.1.5):

$$\begin{aligned}[a_n(t_{\text{in}}), a_m^\dagger(t_{\text{in}})]_+ &= [a_n(t_{\text{out}}), a_m^\dagger(t_{\text{out}})]_+ = [b_n(t_{\text{in}}), b_m^\dagger(t_{\text{in}})]_+ \\ &= [b_n(t_{\text{out}}), b_m^\dagger(t_{\text{out}})]_+ = \delta_{nm},\end{aligned}\quad (1.1.7)$$

and all the others anticommutators are zero. In terms of these operators, the Hamiltonians $H_e(t_{\text{in}})$ and $H_e(t_{\text{out}})$ are diagonal and have the form

$$\begin{aligned}\hat{H}_e(t_{\text{in}}) &= \sum_n \{ {}^+\varepsilon(t_{\text{in}}) a_n^\dagger(t_{\text{in}}) a_n(t_{\text{in}}) + | {}^-\varepsilon(t_{\text{in}}) | b_n^\dagger(t_{\text{in}}) b_n(t_{\text{in}}) \}, \\ \hat{H}_e(t_{\text{out}}) &= \sum_n \{ {}^+\varepsilon(t_{\text{out}}) a_n^\dagger(t_{\text{out}}) a_n(t_{\text{out}}) + | {}^-\varepsilon(t_{\text{out}}) | b_n^\dagger(t_{\text{out}}) b_n(t_{\text{out}}) \}.\end{aligned}\quad (1.1.8)$$

In what follows, operators of the observables in the Schrödinger representation will be denoted with the hat \hat{A} , while operators in the Heisenberg representation by the inverted hat \check{A} . In the Schrödinger picture one has: at the initial time instant t_{in} , a set of creation and annihilation operators $a_n^\dagger(t_{\text{in}})$, $a_n(t_{\text{in}})$ of electrons, as well as similar operators $b_n^\dagger(t_{\text{in}})$, $b_n(t_{\text{in}})$ of positrons, such that the corresponding vacuum at t_{in} is $|0, t_{\text{in}}\rangle$; at the final time instant t_{out} , a set of creation and annihilation operators $a_n^\dagger(t_{\text{out}})$, $a_n(t_{\text{out}})$, of electrons and similar operators $b_n^\dagger(t_{\text{out}})$, $b_n(t_{\text{out}})$ of positrons, such that the corresponding vacuum at t_{out} is $|0, t_{\text{out}}\rangle$,

$$a_n(t_{\text{in}})|0, t_{\text{in}}\rangle = b_n(t_{\text{in}})|0, t_{\text{in}}\rangle = 0, \quad a_n(t_{\text{out}})|0, t_{\text{out}}\rangle = b_n(t_{\text{out}})|0, t_{\text{out}}\rangle = 0 \quad \forall n.$$

The probability amplitude for transition from an initial to a final state $M_{\text{in} \rightarrow \text{out}}$ has the following form in the Schrödinger picture:

$$M_{\text{in} \rightarrow \text{out}} = \langle t_{\text{out}} | U(t_{\text{out}}, t_{\text{in}}) | t_{\text{in}} \rangle,$$

where $U(t, t')$ is a unitary evolution operator of the system. The density operator of an initial state $\hat{\rho}(t_{\text{in}})$ is given as an operator-valued function of the creation and annihilation operators of electrons (positrons) at the initial time instant,

$$\hat{\rho}(t_{\text{in}}) = \rho_{\text{in}}(a^\dagger(t_{\text{in}}), a(t_{\text{in}}), b^\dagger(t_{\text{in}}), b(t_{\text{in}})).$$

The mean value of a physical quantity F at the final time instant reads

$$\langle F(t_{\text{out}}) \rangle = \text{tr } \hat{\rho}(t_{\text{out}}) \hat{F}(t_{\text{out}}),\quad (1.1.9)$$

where $\hat{\rho}(t)$ is the density operator in Schrödinger representation at time instant t , and the designation tr stands for the complete trace,

$$\hat{\rho}(t_{\text{out}}) = U(t_{\text{out}}, t_{\text{in}}) \hat{\rho}(t_{\text{in}}) U^\dagger(t_{\text{out}}, t_{\text{in}}). \quad (1.1.10)$$

In order to pass to the Heisenberg picture one defines the finite-time evolution operators $\Omega_{(\pm)}$,

$$\begin{aligned}\Omega_{(+)} &= U(0, t_{\text{in}}), \quad \Omega_{(-)} = U(0, t_{\text{out}}), \quad U(t_{\text{out}}, t_{\text{in}}) = \Omega_{(-)}^\dagger \Omega_{(+)}, \\ \check{\rho} &= \hat{\rho}(0) = \Omega_{(+)} \hat{\rho}(t_{\text{in}}) \Omega_{(+)}^\dagger = \Omega_{(-)} \hat{\rho}(t_{\text{out}}) \Omega_{(-)}^\dagger,\end{aligned}\tag{1.1.11}$$

and a set of creation and annihilation operators $a_n^\dagger(\text{in})$, $a_n(\text{in})$ of in-electrons, as well as similar operators $b_n^\dagger(\text{in})$, $b_n(\text{in})$ of in-positrons, the corresponding in-vacuum $|0, \text{in}\rangle$, and a set of creation and annihilation operators $a_n^\dagger(\text{out})$, $a_n(\text{out})$, of out-electrons and similar operators $b_n^\dagger(\text{out})$, $b_n(\text{out})$ of out-positrons, and the corresponding out-vacuum $|0, \text{out}\rangle$,

$$\begin{aligned}\{a(\text{in}), \dots\} &= \Omega_{(+)} \{a(t_{\text{in}}), \dots\} \Omega_{(+)}^\dagger, \quad |0, \text{in}\rangle = \Omega_{(+)} |0, t_{\text{in}}\rangle, \\ \{a(\text{out}), \dots\} &= \Omega_{(-)} \{a(t_{\text{out}}), \dots\} \Omega_{(-)}^\dagger, \quad |0, \text{out}\rangle = \Omega_{(-)} |0, t_{\text{out}}\rangle, \\ M_{\text{in} \rightarrow \text{out}} &= \langle 0, t_{\text{out}} | \dots a(t_{\text{in}}) \Omega_{(-)}^\dagger \Omega_{(+)} a_n^\dagger(t_{\text{in}}) \dots | 0, t_{\text{in}} \rangle = \langle 0, \text{out} | \dots a(\text{out}) a_n^\dagger(\text{in}) \dots | 0, \text{in} \rangle, \\ c_v &= \langle 0, t_{\text{out}} | U(t_{\text{out}}, t_{\text{in}}) | 0, t_{\text{in}} \rangle = \langle 0, \text{out} | 0, \text{in} \rangle.\end{aligned}\tag{1.1.12}$$

The entire information concerning the processes of particle creation, annihilation and scattering is contained in the elementary probability amplitudes

$$\begin{aligned}w(+|+)_{mn} &= c_v^{-1} \langle 0, \text{out} | a_m(\text{out}) a_n^\dagger(\text{in}) | 0, \text{in} \rangle, \\ w(-|-)_{nm} &= c_v^{-1} \langle 0, \text{out} | b_m(\text{out}) b_n^\dagger(\text{in}) | 0, \text{in} \rangle, \\ w(0|-+)_{nm} &= c_v^{-1} \langle 0, \text{out} | b_n^\dagger(\text{in}) a_m^\dagger(\text{in}) | 0, \text{in} \rangle, \\ w(+ - | 0)_{mn} &= c_v^{-1} \langle 0, \text{out} | a_m(\text{out}) b_n(\text{out}) | 0, \text{in} \rangle.\end{aligned}\tag{1.1.13}$$

The amplitudes (1.1.13) can be calculated with the help of certain appropriate sets of solutions of the corresponding relativistic wave equation with an external field (Klein–Gordon, Dirac, and so on), see Refs. [12, 13, 14]. The most interesting is the case of uniform external field, that does not mix different quantum modes (solutions with the different sets of quantum numbers n). In this case the amplitudes (1.1.13) are diagonal in quantum numbers,

$$\begin{aligned}w(\zeta|\zeta)_{mn} &= \delta_{mn} w(\zeta|\zeta)_{nn}, \\ w(0|-+)_{nm} &= \delta_{mn} w(0|-+)_{nn}, \\ w(+ - | 0)_{nm} &= \delta_{mn} w(+ - | 0)_{nn}.\end{aligned}$$

The sets of in- and out-operators are related to each other by a linear canonical transformation[66], which can be written in terms of the amplitudes (1.1.13)¹:

$$\begin{aligned}a(\text{out}) &= \left[w(+|+)^\dagger \right]^{-1} a(\text{in}) - \kappa w(+ - | 0) [w(-|-)]^{-1} b^\dagger(\text{in}), \\ b^\dagger(\text{out}) &= \left[w(+|+)^\dagger \right]^{-1} w(+ - | 0)^\dagger a(\text{in}) + [w(-|-)]^{-1} b^\dagger(\text{in}),\end{aligned}\tag{1.1.14}$$

¹Condensed notations are used, for example,

$$bw(0|-+)a = \sum_{n,m} b_n w(0|-+)_{nm} a_m.$$

and by its Hermitian conjugate. As it has been demonstrated [12, 13, 14]), such a relation is given by an unitary operator V ,

$$V \{a(\text{out}), \dots\} V^\dagger = \{a(\text{in}), \dots\}, \quad |0, \text{in}\rangle = V|0, \text{out}\rangle, \quad (1.1.15)$$

which has the form $V = v_4 v_3 v_2 v_1$,

$$\begin{aligned} v_1 &= \exp \{-\kappa b(\text{out}) w(0| - +) a(\text{out})\}, \quad v_2 = \exp \{a^\dagger(\text{out}) \ln w(+|+) a(\text{out})\}, \\ v_3 &= \exp \{-\kappa b(\text{out}) \ln w(-|-) b^\dagger(\text{out})\}, \quad v_4 = \exp \{-\kappa a^\dagger(\text{out}) w(+ - |0) b^\dagger(\text{out})\}. \end{aligned} \quad (1.1.16)$$

Using this expression for V , one can find

$$c_v = \langle 0, \text{out} | V | 0, \text{out} \rangle = \exp \{-\kappa \text{tr} \ln w(-|-)\}. \quad (1.1.17)$$

1.2 Density operators

It is convenient to introduce a generating operator $\check{R}(J)$ that allows one to construct density operators $\check{\rho}$ with different initial conditions (different initial states at the initial time instant t_{in}). This generating operator has the following form [25]:

$$\begin{aligned} \check{R}(J) &= Z^{-1}(J) \underline{\check{R}}(J), \quad \text{tr} \check{R}(J) = 1, \quad Z(J) = \text{tr} \underline{\check{R}}(J), \\ \underline{\check{R}}(J) &= : \exp \left[\sum_n [a_n^\dagger(\text{in}) (J_{n,+} - 1) a_n(\text{in}) + b_n^\dagger(\text{in}) (J_{n,-} - 1) b_n(\text{in})] \right] :, \end{aligned} \quad (1.2.1)$$

where the variables $J_{n,\zeta}$ are sources for electron ($\zeta = +$) or positron ($\zeta = -$) in-operators, Z is a normalization factor (the partition function), and $: \dots :$ here and in what follows means the normal form with respect to those creation and annihilation operators that are situated inside the colon signs.

This operator can be expressed in terms of the out-operators as [25]

$$\begin{aligned} \check{R}(J) &= Z^{-1}(J) |c_v|^2 \det(1 + \kappa AB)^\kappa \underline{\check{R}}(J), \\ \underline{\check{R}}(J) &= : \exp [-a^\dagger(\text{out}) (1 - D_+) a(\text{out}) - b^\dagger(\text{out}) (1 - D_-) b(\text{out}) - a^\dagger(\text{out}) C^\dagger b^\dagger(\text{out}) - b(\text{out}) C a(\text{out})] :, \\ D_+ &= w(+|+) (1 + \kappa AB)^{-1} \mathbb{J}_+ w(+|+)^\dagger, \quad D_-^T = w(-|-)^\dagger \mathbb{J}_- (1 + \kappa BA)^{-1} w(-|-), \quad A(J) = \mathbb{J}_+ B^\dagger \mathbb{J}_-, \\ C &= w(-|-)^\dagger \mathbb{J}_- B (1 + \kappa AB)^{-1} \mathbb{J}_+ w(+|+)^\dagger + \kappa w(+ - |0)^\dagger, \quad B = \kappa w(0| - +), \quad \mathbb{J}_{mn,\zeta} = \delta_{mn} J_{n,\zeta}, \end{aligned} \quad (1.2.2)$$

where $\kappa = +1$ for Fermi case and $\kappa = -1$ for Bose case. The normalization factor Z has the form

$$Z(J) = \exp \left\{ \kappa \sum_{n,\zeta} [\ln(1 + \kappa J_{n,\zeta})] \right\} = \prod_{n,\zeta} [1 + \kappa J_{n,\zeta}]^\kappa. \quad (1.2.3)$$

In what follows, two important cases of the general density operator that correspond to the initial vacuum state and to the initial thermal state will be considered.

a) Setting $J = 0$ and using the well-known Berezin's formula for the vacuum projector [66], one obtains the density operator $\check{\rho}(0)$ that corresponds to the initial vacuum state

$$\check{\rho}(0) = : \exp \left\{ - \sum_n [a_n^\dagger(\text{in}) a_n(\text{in}) + b_n^\dagger(\text{in}) b_n(\text{in})] \right\} := |0, \text{in}\rangle \langle 0, \text{in}|. \quad (1.2.4)$$

From Eqs. (1.2.2) this operator can be presented in terms of the out-operators

$$\begin{aligned} \check{\rho}(0) = & |c_v|^2 : \exp \left\{ - \sum_n [a_n^\dagger(\text{out})a_n(\text{out}) + b_n^\dagger(\text{out})b_n(\text{out}) \right. \\ & \left. + \kappa a_n^\dagger(\text{out})w(+ - |0)_{nn} b_n^\dagger(\text{out}) + \kappa b_n(\text{out})w(+ - |0)_{nn}^\dagger a_n(\text{out}) \right\} : . \end{aligned} \quad (1.2.5)$$

The differential mean numbers $N_{n,\zeta}(0|in)$ of in-electrons and positrons in the state $\check{\rho}(0)$ are zero,

$$N_{n,+}(0|in) = \text{tr} \check{\rho}(0) a_n^\dagger(\text{in}) a_n(\text{in}) = 0, \quad N_{n,-}(0|in) = \text{tr} \check{\rho}(0) b_n^\dagger(\text{in}) b_n(\text{in}) = 0,$$

whereas differential mean numbers $N_{n,\zeta}(0|out)$ of out-electrons and positrons in the state $\check{\rho}(0)$

$$N_{n,+}(0|out) = \text{tr} \check{\rho}(0) a_n^\dagger(\text{out}) a_n(\text{out}), \quad N_{n,-}(0|out) = \text{tr} \check{\rho}(0) b_n^\dagger(\text{out}) b_n(\text{out}),$$

are equal and have the form

$$N_{n,+}(0|out) = N_{n,-}(0|out) = N_n(0|out), \quad N_n(0|out) = \frac{|w(+ - |0)_{nn}|^2}{1 + \kappa |w(+ - |0)_{nn}|^2}, \quad (1.2.6)$$

b) To obtain the density operator $\check{\rho}(\beta)$ that corresponds to the thermal initial state, one has to set $J_{n,\zeta} = J_{n,\zeta}(\beta)$,

$$J_{n,\zeta}(\beta) = e^{-E_{n,\zeta}}, \quad E_{n,\zeta} = \beta(\varepsilon_{n,\zeta} - \mu_\zeta), \quad (1.2.7)$$

where $\varepsilon_{n,\zeta}$ are energies of electrons ($\zeta = +$) or positrons ($\zeta = -$) with quantum numbers n ; μ_ζ are the corresponding chemical potentials, and $\beta = \Theta^{-1}$, where Θ is the absolute temperature [25]. It can be checked that an explicit expression for $\check{\rho}(\beta)$ in terms of the in-operators is

$$\begin{aligned} \check{\rho}(\beta) = & Z_{gr}^{-1} \exp \left[-\beta \left(\check{H} - \sum_\zeta \mu_\zeta \check{N}_\zeta \right) \right], \\ Z_{gr} = & \exp \left[\kappa \sum_{n\zeta} \ln (1 + \kappa e^{-E_{n,\zeta}}) \right]. \end{aligned} \quad (1.2.8)$$

The quantity Z_{gr} is the partition function of grand canonical ensemble, \check{H} is the Hamiltonian of the system (written in terms of in-operators),

$$\check{H} = \sum_n [a_n^\dagger(\text{in})\varepsilon_{n,+}a_n(\text{in}) + b_n^\dagger(\text{in})\varepsilon_{n,-}b_n(\text{in})],$$

and

$$\check{N}_+ = \sum_n [a_n^\dagger(\text{in})a_n(\text{in})], \quad \check{N}_- = \sum_n [b_n^\dagger(\text{in})b_n(\text{in})]$$

are operators of numbers of in-electrons and in-positrons, respectively.

Let $\check{\rho}$ be the general density matrix for an arbitrary initial state, and $N_{n,\zeta}(\cdots|in)$ be differential mean numbers of in-electrons or positrons in the state $\check{\rho}$, and $N_{n,\zeta}(\cdots|out)$ be differential mean numbers of out-electrons or positrons in the state $\check{\rho}$,

$$\begin{aligned} N_{n,+}(\cdots|in) = & \text{tr} \check{\rho} a_n^\dagger(\text{in}) a_n(\text{in}), \quad N_{n,-}(\cdots|in) = \text{tr} \check{\rho} b_n^\dagger(\text{in}) b_n(\text{in}), \\ N_{n,+}(\cdots|out) = & \text{tr} \check{\rho} a_n^\dagger(\text{out}) a_n(\text{out}), \quad N_{n,-}(\cdots|out) = \text{tr} \check{\rho} b_n^\dagger(\text{out}) b_n(\text{out}). \end{aligned} \quad (1.2.9)$$

Calculating the traces in the in-basis, one can see [25] that

$$N_{n,\zeta}(\cdots|out) = N_{n,\zeta}(\cdots|in) + N_n(0|out) \{1 - \kappa [N_{n,+}(\cdots|in) + N_{n,-}(\cdots|in)]\}. \quad (1.2.10)$$

In particular, differential mean numbers $N_{n,\zeta}(\beta|in)$ of in-electrons or positrons in the state $\check{\rho}(\beta)$ are the well-known Fermi-Dirac ($\kappa = +1$) or Bose-Einstein ($\kappa = -1$) distributions,

$$\begin{aligned} N_{n,+}(\beta|in) &= \text{tr } \check{\rho}(\beta) a_n^\dagger(in) a_n(in) = (e^{E_{n,+}} + \kappa)^{-1}, \\ N_{n,-}(\beta|in) &= \text{tr } \check{\rho}(\beta) b_n^\dagger(in) b_n(in) = (e^{E_{n,-}} + \kappa)^{-1}. \end{aligned} \quad (1.2.11)$$

Differential mean numbers $N_{n,\zeta}(\beta|out)$ of out-electrons or positrons in the state $\check{\rho}(\beta)$ follow immediately from (1.2.10).

1.2.1 Reduced density operators for electron and positron subsystems

At any fixed time instant, the complete system of quantum electrons and positrons can be conditionally divided into two subsystems: a system of electrons and a system of positrons. Let us suppose that the external electric field is switched off at some sufficiently large time instant t_2 in such a way that at $t_{out} > t_2$ no particle creation occurs and both subsystems are spatially separated. Thus, the particle creation effect by the time-dependent uniform electric field provides a real division of the complete quantum field system into the two subsystems. One can introduce the so-called reduced density operators $\check{\rho}_\pm$ of the electron subsystem and of the positron subsystem. These operators are defined as follows:

$$\begin{aligned} \check{\rho}_+ &= \text{tr}_- \check{\rho} = \sum_{M=0}^{\infty} \sum_{\{m\}} (M!)^{-1} {}_b \langle 0, out | b_{m_M}(out) \dots b_{m_1}(out) | \check{\rho} | b_{m_1}^\dagger(out) \dots b_{m_M}^\dagger(out) | 0, out \rangle_b, \\ \check{\rho}_- &= \text{tr}_+ \check{\rho} = \sum_{M=0}^{\infty} \sum_{\{m\}} (M!)^{-1} {}_a \langle 0, out | a_{m_M}(out) \dots a_{m_1}(out) | \check{\rho} | a_{m_1}^\dagger(out) \dots a_{m_M}^\dagger(out) | 0, out \rangle_a, \end{aligned} \quad (1.2.12)$$

where $\check{\rho}$ is the density operator of the complete system, $|0, out\rangle_a$ and $|0, out\rangle_b$ are electron and positron vacua, respectively, $(a_m(out)|0, out\rangle_a = 0$, $b_m(out)|0, out\rangle_b = 0$, $|0, out\rangle = |0, out\rangle_a \otimes |0, out\rangle_b$) and tr_\pm are the so-called reduced traces. Obviously, the reduced density operators $\check{\rho}_\pm$ describe mixed states.

The reduced density operators $\check{\rho}_\pm$ can be obtained from the reduced generating operators $\check{R}_\pm(J)$ which are defined as:

$$\check{R}_\pm(J) = \text{tr}_\mp \check{R}(J). \quad (1.2.13)$$

In terms of the out-operators these have the form

$$\begin{aligned} \check{R}_+(J) &= Z_+^{-1}(J) : \exp \left\{ - \sum_n a_n^\dagger(out) [1 - K_+(J)]_{nn} a_n(out) \right\} :, \\ \check{R}_-(J) &= Z_-^{-1}(J) : \exp \left\{ - \sum_n b_n^\dagger(out) [1 - K_-(J)]_{nn} b_n(out) \right\} :, \\ K_\pm(J) &= D_\pm + C^\dagger (1 + \kappa D_\mp^T)^{-\kappa} C, \\ Z_\pm^{-1}(J) &= Z^{-1}(J) |c_v|^2 \det(1 + \kappa AB)^\kappa \det(1 + \kappa D_\mp)^\kappa. \end{aligned} \quad (1.2.14)$$

The reduced generating operators $\check{R}_\pm(J)$ allow one to obtain the reduced density operators $\check{\rho}_\pm$ for different initial states of the system. Consider as before two important cases.

a) By setting $J = 0$ in (1.2.14) one obtains the reduced density operators $\check{\rho}_\zeta(0) = \check{R}_\zeta(0)$ for the both subsystems in the case when the complete system was in the vacuum state at the initial time instant. Taking into account that

$$\begin{aligned} K_\pm(0) &= |w(+ - |0)\rangle^2 = P(+ - |0)P_v^{-1}, \\ Z_\pm^{-1}(0) &= |c_v|^2 = P_v, \quad P(+ - |0) = |\langle 0, \text{out} | a_n(\text{out}) b_n(\text{out}) | 0, \text{in} \rangle|^2, \end{aligned}$$

where $P(+ - |0)$ and P_v are probabilities of pair creation and vacuum-to-vacuum transition, respectively, one obtains explicit expressions for $\check{\rho}_\zeta(0)$:

$$\begin{aligned} \check{\rho}_+(0) &= \check{R}_+(0) = |c_v|^2 : \exp \left\{ - \sum_n a_n^\dagger(\text{out}) [1 - P(+ - |0)P_v^{-1}]_{nn} a_n(\text{out}) \right\} : , \\ \check{\rho}_-(0) &= \check{R}_-(0) = |c_v|^2 : \exp \left\{ - \sum_n b_n^\dagger(\text{out}) [1 - P(+ - |0)P_v^{-1}]_{nn} b_n(\text{out}) \right\} : . \end{aligned} \quad (1.2.15)$$

It should be noted that reduced density operators (1.2.15) were for the first time obtained in Refs. [69].

b) By setting the sources J in expression (1.2.14) equal to $J_{n,\zeta}(\beta)$ according to eqs. (1.2.7), one can see that the reduced generating operators (1.2.14) become the reduced density operators $\check{\rho}_\zeta(\beta)$ of the system that was in thermal equilibrium at the initial time instant, $\check{R}_+(J) = \check{\rho}_+(\beta)$ and $\check{R}_-(J) = \check{\rho}_-(\beta)$.

1.2.2 Decoherence in course of the evolution

In the previous sections we considered the case where information loss was due to the averaging over one of the subsystems of electrons or positrons. However, information loss can also occur due to the interaction of the quantum system with classical (or semiclassical) objects, or, in other words, due to decoherence. One can imagine two possible scenarios for this: first, it can happen during intermediate measurements by a classical tool and, second, as a result of collisions of particles with some semiclassical objects (for example, impurities in graphene). For the approach under consideration, there is no difference between which of the mechanisms is implemented, so in what follows an intermediate measurement by a classical tool as a source of the decoherence will be chosen.

Consider the case when the unitary evolution of the system is interrupted by a single intermediate measurement. The external field starts to act at the time instant t_{in} , the system is evolving in a unitary way from t_{in} to t_1 , during the time interval T_1 , then at t_1 a decoherence takes place, and then again the unitary evolution proceeds from t_1 to t_{out} during the time interval T_2 . In this case, if one considers the Heisenberg picture, the out-set of creation and annihilation operators for electrons and positrons of interval T_1 is the in-set of interval T_2 .

Suppose that during the time interval T_1 the system is described by density operator $\check{\rho}(0)$, i.e. the system is in the vacuum state at initial time instant t_{in} . Differential mean numbers of electrons and positrons at time instant t_1 are the numbers of electrons/positrons created by external field from vacuum $N_n(0|\text{out})$ (1.2.6). It is known from general theory that the electrons and positrons created in pairs by external field are entangled.

During time interval T_2 the system is described by a density operator which is denoted as $\check{\rho}'$. The latter in terms of in-set of creation-annihilation operators for electrons and positrons must describe

the system without quantum correlations between the electrons and positrons created (i.e. new “initial” state of the system in time interval T_2 is the state without any entanglement).

Such an operator can be obtained by using von Neumann’s reduction principle ([70]). Let a system be in a pure state which is described by a state vector $|\psi\rangle$, or equivalently by a density operator $\hat{\rho}$ that is in such a case the projector, $\hat{\rho} = \hat{P}_\psi = |\psi\rangle\langle\psi|$. And let \hat{R} be a self-adjoint observable of the system. In the simplest case, when this observable has a nondegenerate discrete spectrum the following spectral decomposition holds $\hat{R} = \sum_\alpha r_\alpha P_{\varphi_\alpha}$, where r_α are possible eigenvalues of the observable, and P_{φ_α} are projectors on to the corresponding eigenvectors $|\varphi_\alpha\rangle$, $\hat{P}_{\varphi_\alpha} = |\varphi_\alpha\rangle\langle\varphi_\alpha|$. When measuring the observable \hat{R} , one obtains the eigenvalues r_α with the probabilities $|\langle\varphi_\alpha|\psi\rangle|^2 = \langle\varphi_\alpha|\hat{P}_\psi|\varphi_\alpha\rangle = \langle\varphi_\alpha|\hat{\rho}|\varphi_\alpha\rangle$, and just after the measurement the state vector $|\psi\rangle$ is reduced to the vector $|\varphi_\alpha\rangle$, or the density operator $\hat{\rho}$ is reduced to the operator $\hat{\rho}' = \hat{P}_{\varphi_\alpha}$. A more general case, where the system is in a mixed state, is described by the density operator $\hat{\rho}$ with a simple discrete spectrum, $\hat{\rho} = \sum_n \lambda_n P_{\psi_n}$, $P_{\psi_n} = |\psi_n\rangle\langle\psi_n|$, λ_n being statistical weights of corresponding states P_{ψ_n} , and \hat{R} being the above mentioned observable. Then the measurement is presented as follows. The eigenvalues r_α are measured with the following probabilities

$$\sum_n \lambda_n |\langle\varphi_\alpha|\psi_n\rangle|^2 = \langle\varphi_\alpha|\hat{\rho}|\varphi_\alpha\rangle,$$

and just after the measurement the density operator $\hat{\rho}$ is reduced to the operator $\hat{\rho}'$,

$$\hat{\rho}' = \sum_\alpha \langle\varphi_\alpha|\hat{\rho}|\varphi_\alpha\rangle \hat{P}_{\varphi_\alpha}.$$

The same, obviously, can be done in Heisenberg picture.

The density operator $\check{\rho}(0)$ is

$$\check{\rho}(0) = |0, \text{in}\rangle\langle 0, \text{in}|. \quad (1.2.16)$$

The in-vacuum $|0, \text{in}\rangle$ is connected to the out-vacuum $|0, \text{out}\rangle$ by relation (1.1.15). Then density operator $\check{\rho}(0)$ can be presented as

$$\check{\rho}(0) = V|0, \text{out}\rangle\langle 0, \text{out}|V^\dagger. \quad (1.2.17)$$

The case of uniform external field is considered here. This field does not mix different quantum modes. Thus, amplitudes (1.1.13) are diagonal. Thus, it is possible to factorize V defined by (1.1.15) as:

$$V = \prod_n V_n, \quad V_n = v_{4n} v_{3n} v_{2n} v_{1n}, \quad (1.2.18)$$

where

$$\begin{aligned} v_{1n} &= \exp \{ -\kappa b_n(\text{out}) w (0| - +)_{nn} a_n(\text{out}) \}, \quad v_{2n} = \exp \{ a_n^\dagger(\text{out}) [\ln w (+|+)]_{nn} a_n(\text{out}) \}, \\ v_{3n} &= \exp \{ -\kappa b_n(\text{out}) [\ln w (-|-)]_{nn} b_n^\dagger(\text{out}) \}, \quad v_{4n} = \exp \{ -\kappa a_n^\dagger(\text{out}) w (+ - |0)_{nn} b_n^\dagger(\text{out}) \}. \end{aligned}$$

Making use of the explicit form of V_n and definition (1.1.15) one can write

$$|0, \text{in}\rangle = c_v \prod_n \sum_{m=0} \frac{(-1)^m}{m!} [\kappa w (+ - |0)_{nn} a_n^\dagger(\text{out}) b_n^\dagger(\text{out})]^m |0, \text{out}\rangle, \quad (1.2.19)$$

and then one can easily calculate c_v ,

$$c_v = \prod_n [w (-|-)_{nn}]^{-\kappa}. \quad (1.2.20)$$

Thus, density operators $\check{\rho}(0)$ (1.2.17) can be represented as

$$\begin{aligned} \check{\rho}(0) &= |c_v|^2 \prod_n \left(\sum_{m=0}^{\infty} \frac{[-\kappa w (+ - |0)_{nn} a_n^\dagger(\text{out}) b_n^\dagger(\text{out})]^m}{m!} \right) \\ &\times \check{P}_0 \prod_{n'} \left(\sum_{m'=0}^{\infty} \frac{[-\kappa w (+ - |0)_{n'n'}^\dagger b_{n'}(\text{out}) a_{n'}(\text{out})]^{m'}}{m'!} \right), \end{aligned} \quad (1.2.21)$$

where $\check{P}_0 = |0, \text{out}\rangle\langle 0, \text{out}|$.

1.2.3 Measurement of differential mean numbers in the system

Suppose that one is going to measure the physical quantity, which is the number of particles, in the state $\check{\rho}(0)$ of the system under consideration. The operator corresponding to such physical quantity is $\check{N}(\text{out})$,

$$\check{N}(\text{out}) = \sum_{n, \zeta} \check{N}_{n, \zeta}(\text{out}) = \sum_n [a_n^\dagger(\text{out}) a_n(\text{out}) + b_n^\dagger(\text{out}) b_n(\text{out})]. \quad (1.2.22)$$

Its eigenstates are mutually orthonormal vectors of the following form

$$\begin{aligned} |s, \text{out}\rangle &= |\{i, l\}_{LP}, \text{out}\rangle_a \otimes |\{j, k\}_{KQ}, \text{out}\rangle_b, \\ |\{i, l\}_{LP}, \text{out}\rangle_a &= \frac{[a_{i_1}^\dagger(\text{out})]^{l_1}}{\sqrt{l_1!}} \dots \frac{[a_{i_P}^\dagger(\text{out})]^{l_P}}{\sqrt{l_P!}} |0, \text{out}\rangle_a, \\ |\{j, k\}_{KQ}, \text{out}\rangle_b &= \frac{[b_{j_1}^\dagger(\text{out})]^{k_1}}{\sqrt{k_1!}} \dots \frac{[b_{j_Q}^\dagger(\text{out})]^{k_Q}}{\sqrt{k_Q!}} |0, \text{out}\rangle_b, \\ L &= 0, 1, 2, \dots, \quad P = 1, 2, \dots, L, \quad i = i_1, \dots, i_P, \quad l_1 + l_2 + \dots + l_P = L, \\ K &= 0, 1, 2, \dots, \quad Q = 1, 2, \dots, K, \quad j = j_1, \dots, j_Q, \quad k_1 + k_2 + \dots + k_Q = K, \end{aligned}$$

such that its eigenvalues are

$$\hat{N}(\text{out})|s, \text{out}\rangle = (L + K)|s, \text{out}\rangle,$$

where s is the full set of quantum numbers $K, L, \{i\}, \{j\}, P, Q$, and $|\{i, l\}_{LP}, \text{out}\rangle_a$ is a state with L electrons distributed in P groups i_1, \dots, i_P , with l_1 electrons in the group i_1 , l_2 electrons in the group i_2 , and so on. Analogously, $|\{j, k\}_{KQ}, \text{out}\rangle_b$ is a state with K positrons distributed in Q groups j_1, \dots, j_Q , with k_1 positrons in group j_1 , k_2 positrons in group j_2 , and so on.

According to von Neumann [70], the density operator $\check{\rho}(0)$ is after such measurement reduced to the operator $\check{\rho}_N$ of the form

$$\check{\rho}_N = \sum_s \langle s, \text{out} | \check{\rho}(0) | s, \text{out} \rangle \check{P}_s, \quad \check{P}_s = |s, \text{out}\rangle\langle s, \text{out}|. \quad (1.2.23)$$

Due to the structure of the density operator $\check{\rho}(0)$ given by Eq. (1.2.21), the weights $\langle s, \text{out} | \check{\rho}(0) | s, \text{out} \rangle$

are non-zero only when the states $|s, \text{out}\rangle$ are states with integer number of pairs. Thus, one obtains

$$\begin{aligned}\check{\rho}_N &= |c_v|^2 \sum_f W_f \check{P}_f, \quad \sum_f = \sum_{M=0}^{\infty} \sum_{Z=1}^M \sum_{\{m,n\}}, \quad \check{P}_f = |f, \text{out}\rangle \langle f, \text{out}|, \\ W_f &= |w(+ - |0\rangle_{n_1 n_1})|^{2m_1} \dots |w(+ - |0\rangle_{n_Z n_Z})|^{2m_Z}, \quad m_1 + m_2 + \dots + m_Z = M, \\ |f, \text{out}\rangle &= \frac{[a_{n_1}^\dagger(\text{out})b_{n_1}^\dagger(\text{out})]^{m_1}}{m_1!} \dots \frac{[a_{n_Z}^\dagger(\text{out})b_{n_Z}^\dagger(\text{out})]^{m_Z}}{m_Z!} |0, \text{out}\rangle,\end{aligned}\tag{1.2.24}$$

where f is a complete set of quantum numbers $M, Z, \{m\}, \{n\}$, and $|f, \text{out}\rangle$ is a state with total number of pairs M distributed in Z groups, m_1 pairs being in the group n_1 , m_2 pairs being in the group n_2 and so on. Unlike (1.2.21), the latter expression contains only terms diagonal in f . Thus, the measurement destroy nondiagonal terms of density operator (1.2.21).

Let us now calculate the reduced (in the sense of sect. 1.2.1) operators $[\check{\rho}_N]_\zeta$:

$$\begin{aligned}[\check{\rho}_N]_\zeta &= \text{tr}_{-\zeta} \check{\rho}_N = |c_v|^2 \sum_f W_f \text{tr}_{-\zeta} \check{P}_f, \\ \text{tr}_{-} \check{P}_f &= \frac{[a_{n_1}^\dagger(\text{out})]^{m_1}}{\sqrt{m_1!}} \dots \frac{[a_{n_Z}^\dagger(\text{out})]^{m_Z}}{\sqrt{m_Z!}} |0, \text{out}\rangle_{aa} \langle 0, \text{out}| \frac{[a_{n_Z}(\text{out})]^{m_Z}}{\sqrt{m_Z!}} \dots \frac{[a_{n_1}(\text{out})]^{m_1}}{\sqrt{m_1!}}, \\ \text{tr}_{+} P_f &= \frac{[b_{n_1}^\dagger(\text{out})]^{m_1}}{\sqrt{m_1!}} \dots \frac{[b_{n_Z}^\dagger(\text{out})]^{m_Z}}{\sqrt{m_Z!}} |0, \text{out}\rangle_{bb} \langle 0, \text{out}| \frac{[b_{n_Z}(\text{out})]^{m_Z}}{\sqrt{m_Z!}} \dots \frac{[b_{n_1}(\text{out})]^{m_1}}{\sqrt{m_1!}}.\end{aligned}\tag{1.2.25}$$

On the other hand, one can calculate the reduced density operators $\check{\rho}_\zeta(0)$ by taking reduced traces (1.2.12) of operator (1.2.21) to verify that they have exactly the same form,

$$[\check{\rho}_N]_\zeta = \check{\rho}_\zeta(0).$$

1.2.4 Measurements of differential mean numbers in the subsystems

Suppose now that one measures the number of either electrons or positrons. The corresponding operators of these physical quantities are

$$\begin{aligned}\check{N}_+(\text{out}) &= \sum_n \check{N}_{n,+}(\text{out}) = \sum_n a_n^\dagger(\text{out}) a_n(\text{out}), \\ \check{N}_-(\text{out}) &= \sum_n \check{N}_{n,-}(\text{out}) = \sum_n b_n^\dagger(\text{out}) b_n(\text{out}).\end{aligned}\tag{1.2.26}$$

The spectra of operators (1.2.26) are

$$\begin{aligned}|s_+, \text{out}\rangle_a &= |\{i, l\}_{LP}, \text{out}\rangle_a = \frac{[a_{i_1}^\dagger(\text{out})]^{l_1}}{\sqrt{l_1!}} \dots \frac{[a_{i_P}^\dagger(\text{out})]^{l_P}}{\sqrt{l_P!}} |0, \text{out}\rangle_a, \\ L &= 0, 1, 2, \dots, \quad P = 1, 2, \dots, L, \quad i = i_1, \dots, i_P, \quad l_1 + l_2 + \dots + l_P = L, \\ |s_-, \text{out}\rangle_b &= |\{j, k\}_{KQ}, \text{out}\rangle_b = \frac{[b_{j_1}^\dagger(\text{out})]^{k_1}}{\sqrt{k_1!}} \dots \frac{[b_{j_Q}^\dagger(\text{out})]^{k_Q}}{\sqrt{k_Q!}} |0, \text{out}\rangle_b, \\ K &= 0, 1, 2, \dots, \quad Q = 1, 2, \dots, K, \quad j = j_1, \dots, j_Q, \quad k_1 + k_2 + \dots + k_Q = K, \\ \hat{N}_+(\text{out})|s_+, \text{out}\rangle_a &= L|s_+, \text{out}\rangle_a, \quad \hat{N}_-(\text{out})|s_-, \text{out}\rangle_b = K|s_-, \text{out}\rangle_b.\end{aligned}$$

The states $|\{i, l\}_{LP}, \text{out}\rangle_a$ and $|\{j, k\}_{KQ}, \text{out}\rangle_b$ are defined in the same way as in the previous subsection.

The density operators after such measurements, which are denoted by $\check{\rho}_{N_+}$ and $\check{\rho}_{N_-}$, respectively, have the form

$$\begin{aligned} \check{\rho}_{N_+} &= \sum_{s_+} {}_a \langle s_+, \text{out} | \check{\rho}(0) | s_+, \text{out} \rangle_a P_{s_+}, \quad P_{s_+} = |s_+, \text{out}\rangle_a \langle s_+, \text{out}|, \quad \sum_{s_+} = \sum_{L=0}^{\infty} \sum_{P=1}^L \sum_{\{i,l\}}, \\ \check{\rho}_{N_-} &= \sum_{s_-} {}_b \langle s_-, \text{out} | \check{\rho}(0) | s_-, \text{out} \rangle_b P_{s_-}, \quad P_{s_-} = |s_-, \text{out}\rangle_b \langle s_-, \text{out}|, \quad \sum_{s_-} = \sum_{K=0}^{\infty} \sum_{Q=1}^K \sum_{\{j,k\}}. \end{aligned} \quad (1.2.27)$$

Let us now calculate the quantities ${}_a \langle s_+, \text{out} | \check{\rho}(0) | s_+, \text{out} \rangle_a P_{s_+}$ and ${}_b \langle s_-, \text{out} | \check{\rho}(0) | s_-, \text{out} \rangle_b P_{s_-}$. Due to the structure of $\check{\rho}(0)$ they are equal and have the form

$$\begin{aligned} &{}_a \langle s_+, \text{out} | \check{\rho}(0) | s_+, \text{out} \rangle_a P_{s_+} = {}_b \langle s_-, \text{out} | \check{\rho}(0) | s_-, \text{out} \rangle_b P_{s_-} \\ &= |c_v|^2 \frac{\left[-\kappa w (+ - |0\rangle_{i_1 i_1} a_{i_1}^\dagger(\text{out}) b_{i_1}^\dagger(\text{out}) \right]^{l_1}}{l_1!} \dots \frac{\left[-\kappa w (+ - |0\rangle_{i_p i_p} a_{i_p}^\dagger(\text{out}) b_{i_p}^\dagger(\text{out}) \right]^{l_p}}{l_p!} |0, \text{out}\rangle \\ &\times \langle 0, \text{out} | \frac{\left[-\kappa w (+ - |0\rangle_{i_p i_p}^\dagger b_{i_p}(\text{out}) a_{i_p}(\text{out}) \right]^{l_p}}{l_p!} \dots \frac{\left[-\kappa w (+ - |0\rangle_{i_1 i_1}^\dagger b_{i_1}(\text{out}) a_{i_1}(\text{out}) \right]^{l_1}}{l_1!}. \end{aligned} \quad (1.2.28)$$

It is not difficult to see that density operators $\check{\rho}_{N_+}$ and $\check{\rho}_{N_-}$ have exactly the same form as (1.2.24), namely, they are sums over all possible projectors on to states with integer amount of pairs:

$$\check{\rho}_{N_+} = \check{\rho}_{N_-} = \check{\rho}_N. \quad (1.2.29)$$

Thus, it should be stressed that measurements of N , N_+ and N_- produce the same reductions. The reduced density operators $[\check{\rho}_{N_+}]_\zeta = \text{tr}_{-\zeta} \check{\rho}_{N_+}$ and $[\check{\rho}_{N_-}]_\zeta = \text{tr}_{-\zeta} \check{\rho}_{N_-}$ are equal to reduced density operators $\check{\rho}_\zeta(0)$ given in (1.2.15).

1.3 Entropy and entanglement of electron and positron subsystems

As was already said in the Introduction, the measure of the information loss in a quantum state $\check{\rho}$ can be identified with the entropy of such a state, namely with the von Neumann information entropy S [70],

$$S(\check{\rho}) = -k_B \text{tr} \check{\rho} \ln \check{\rho}. \quad (1.3.1)$$

Let $\hat{\rho}(t_{\text{in}}) = \check{\rho}(\beta)$, where $\check{\rho}(\beta)$ is given by (1.2.8), then

$$S(\check{\rho}(\beta)) = k_B \left[\ln Z_{gr} + \sum_{n\zeta} E_{n,\zeta} N_{n,\zeta}(\beta | \text{in}) \right], \quad (1.3.2)$$

The corresponding differential mean numbers $N_{n,\zeta}(\beta | \text{in})$ are Fermi-Dirac or Bose-Einstein distributions, given by (1.2.11). The entropy (1.5.63) can be written in terms of the Bose (Fermi) occupation number alone, if one takes into account that

$$e^{-E_{n,\zeta}} = \frac{N_{n,\zeta}(\beta | \text{in})}{1 - \kappa N_{n,\zeta}(\beta | \text{in})}. \quad (1.3.3)$$

Then

$$S(\check{\rho}(\beta)) = -k_B \sum_{n\zeta} \{ \kappa [1 - \kappa N_{n,\zeta}(\beta|\text{in})] \ln [1 - \kappa N_{n,\zeta}(\beta|\text{in})] + N_{n,\zeta}(\beta|\text{in}) \ln N_{n,\zeta}(\beta|\text{in}) \}. \quad (1.3.4)$$

This expression has a form similar to expressions for entropy of the grand canonical ensemble for Fermi- and Bose-particles[71].

Especially interesting information is obtained by calculating the von Neumann information entropy of reduced density operators of both the electron and positron subsystems $S(\hat{\rho}_\pm)$,

$$S(\hat{\rho}_\pm) = -k_B \text{tr}_\pm(\hat{\rho}_\pm \ln \hat{\rho}_\pm). \quad (1.3.5)$$

According to the general theory they coincide $S(\hat{\rho}_+) = S(\hat{\rho}_-)$ and can be treated as a measure of the quantum entanglement of these subsystems.

It is also known that one can recognize entanglement by evaluating the so-called Schmidt measure, which is the trace of the squared reduced density operators[74]

$$\tilde{S}(\hat{\rho}_\pm) = -\text{tr} [(\hat{\rho}_\pm)^2]. \quad (1.3.6)$$

Let us calculate von Neumann entropy for both the electron and positron subsystems in the two important cases of the vacuum initial state and the thermal initial state, that are described by the reduced density operators $\check{\rho}_\zeta(0)$ and $\check{\rho}_\zeta(\beta)$.

1.3.1 Vacuum initial state

The entropy for the reduced density operator of the system with initial vacuum state has the form

$$S(\check{\rho}_\zeta(0)) = -k_B \text{tr}_\zeta(\check{\rho}_\zeta(0) \ln \check{\rho}_\zeta(0)). \quad (1.3.7)$$

The terms $\ln \check{\rho}_\zeta(0)$ at the right-hand side of (1.3.7) can be written as

$$\begin{aligned} \ln \check{\rho}_+(0) &= \ln \left[|c_v|^2 : \exp \left\{ - \sum_n a_n^\dagger(\text{out}) (1 - P(+ - |0)P_v^{-1})_{nn} a_n(\text{out}) \right\} : \right], \\ \ln \check{\rho}_-(0) &= \ln \left[|c_v|^2 : \exp \left\{ - \sum_n b_n^\dagger(\text{out}) (1 - P(+ - |0)P_v^{-1})_{nn} b_n(\text{out}) \right\} : \right]. \end{aligned} \quad (1.3.8)$$

Transforming the normal-form exponents into ordinary exponents with the help of the relation given in Ref. [25], and recalling that $|c_v|^2 = P_v$, one obtains

$$\begin{aligned} \ln \check{\rho}_+(0) &= \ln P_v + \sum_n a_n^\dagger(\text{out}) \ln [P(+ - |0)P_v^{-1}]_{nn} a_n(\text{out}), \\ \ln \check{\rho}_-(0) &= \ln P_v + \sum_n b_n^\dagger(\text{out}) \ln [P(+ - |0)P_v^{-1}]_{nn} b_n(\text{out}). \end{aligned} \quad (1.3.9)$$

Taking into account that the matrices $P(+ - |0)P_v^{-1}$ are diagonal, one can rewrite (1.3.7) as

$$\begin{aligned} S(\check{\rho}_+(0)) &= -k_B \left\{ \ln P_v + \sum_n \text{tr}_+ (\check{\rho}_+(0) a_n^\dagger(\text{out}) a_n(\text{out})) \ln [P(+ - |0)P_v^{-1}]_{nn} \right\}, \\ S(\check{\rho}_-(0)) &= -k_B \left\{ \ln P_v + \sum_n \text{tr}_- (\check{\rho}_-(0) b_n^\dagger(\text{out}) b_n(\text{out})) \ln [P(+ - |0)P_v^{-1}]_{nn} \right\}, \end{aligned} \quad (1.3.10)$$

where $\text{tr}_+ \check{\rho}_+(0) a_n^\dagger(\text{out}) a_n(\text{out}) = N_n(0|\text{out})$ and $\text{tr}_- \check{\rho}_-(0) b_n^\dagger(\text{out}) b_n(\text{out}) = N_n(0|\text{out})$ are differential mean numbers of out-electrons and out-positrons, respectively. They obviously coincide. Thus, the entropy takes the form

$$S(\check{\rho}_\zeta(0)) = -k_B \left\{ \ln P_v + \sum_n N_n(0|\text{out}) [\ln P(+ - |0) P_v^{-1}]_{nn} \right\}. \quad (1.3.11)$$

One can use the pair creation probability and the vacuum-to-vacuum probability written in terms of differential mean numbers (see, for example, [28])

$$P(- + |0)_{n,n'} = \delta_{n,n'} \frac{P_v N_n(0|\text{out})}{1 - \kappa N_n(0|\text{out})}, \quad P_v = \exp \left\{ \kappa \sum_n \ln [1 - \kappa N_n(0|\text{out})] \right\}, \quad (1.3.12)$$

to obtain

$$\begin{aligned} S(\check{\rho}_\zeta(0)) &= \sum_n S(\check{\rho}_{n,\zeta}(0)), \\ S(\check{\rho}_{n,\zeta}(0)) &= -k_B [\kappa (1 - \kappa N_n(0|\text{out})) \ln (1 - \kappa N_n(0|\text{out})) + N_n(0|\text{out}) \ln N_n(0|\text{out})]. \end{aligned} \quad (1.3.13)$$

Let us consider the Schmidt entanglement measure (1.3.6),

$$\tilde{S}(\check{\rho}_\zeta(0)) = -\text{tr} [\check{\rho}_\zeta(0)]^2. \quad (1.3.14)$$

Here

$$\begin{aligned} [\check{\rho}_+(0)]^2 &= P_v^2 \left\{ : \exp \left[- \sum_n a_n^\dagger(\text{out}) (1 - P(+ - |0) P_v^{-1})_{nn} a_n(\text{out}) \right] : \right\}^2, \\ [\check{\rho}_-(0)]^2 &= P_v^2 \left\{ : \exp \left[- \sum_n b_n^\dagger(\text{out}) (1 - P(+ - |0) P_v^{-1})_{nn} b_n(\text{out}) \right] : \right\}^2. \end{aligned} \quad (1.3.15)$$

Using the following relation (here D and \tilde{D} are some matrices)

$$: \exp [-a^\dagger(\text{out}) D a(\text{out})] : : \exp [-a^\dagger(\text{out}) \tilde{D} a(\text{out})] := : \exp [-a^\dagger(\text{out}) (D + \tilde{D} - D\tilde{D}) a(\text{out})] : , \quad (1.3.16)$$

one obtains

$$\begin{aligned} [\check{\rho}_+(0)]^2 &= P_v^2 : \exp \left\{ \sum_n a_n^\dagger(\text{out}) \left[(P(+ - |0) P_v^{-1})^2 - 1 \right]_{nn} a_n(\text{out}) \right\} : , \\ [\check{\rho}_-(0)]^2 &= P_v^2 : \exp \left\{ \sum_n b_n^\dagger(\text{out}) \left[(P(+ - |0) P_v^{-1})^2 - 1 \right]_{nn} b_n(\text{out}) \right\} : . \end{aligned} \quad (1.3.17)$$

Calculating the traces in (1.3.14) with the account of (1.3.12), one finally obtains

$$\begin{aligned} \tilde{S}(\check{\rho}_\zeta(0)) &= -P_v^2 \det \left[1 + \kappa (P(+ - |0) P_v^{-1})^2 \right]^\kappa \\ &= - \prod_n [1 - 2\kappa N_n(0|\text{out}) + (1 + \kappa) (N_n(0|\text{out}))^2]^\kappa. \end{aligned} \quad (1.3.18)$$

1.3.2 Thermal initial state

The entropy for the operators $\check{\rho}_\zeta(\beta)$, which describe the system that has been in thermal equilibrium at the initial time instant, has the form

$$S(\check{\rho}_{\beta,\zeta}) = -k_B \text{tr}_\zeta \check{\rho}_\zeta(\beta) \ln \check{\rho}_\zeta(\beta). \quad (1.3.19)$$

Transforming expressions $\ln \check{\rho}_\zeta(\beta)$ as

$$\begin{aligned} \ln \check{\rho}_+(\beta) &= \ln Z_\zeta(J_\beta) + \sum_n a_n^\dagger(\text{out}) \ln [K_+(J_\beta)]_{nn} a_n(\text{out}), \\ \ln \check{\rho}_-(\beta) &= \ln Z_\zeta(J_\beta) + \sum_n b_n^\dagger(\text{out}) \ln [K_-(J_\beta)]_{nn} b_n(\text{out}), \end{aligned} \quad (1.3.20)$$

one can write

$$S(\check{\rho}_{\beta,\zeta}) = k_B \left\{ \ln Z_\zeta(J_\beta) - \sum_n N_{n,\zeta}(\beta|\text{out}) [\ln K_\zeta(J_\beta)]_{nn} \right\}, \quad (1.3.21)$$

where $N_{n,\zeta}(\beta|\text{out})$ are given by (1.2.10) with $N_{n,\zeta}(\cdots|\text{in}) = N_{n,\zeta}(\beta|\text{in})$. One can express diagonal elements of $K_\zeta(J_\beta)$ in terms of the corresponding occupation numbers $N_{n,\zeta}(\beta|\text{out})$

$$[K_\zeta(J_\beta)]_{nn} = \frac{N_{n,\zeta}(\beta|\text{out})}{1 - \kappa N_{n,\zeta}(\beta|\text{out})}, \quad (1.3.22)$$

and do the same to $Z_\zeta(J_\beta)$ by means of the normalization condition ($\text{tr}_\zeta \check{\rho}_{\beta,\zeta} = 1$)

$$Z_\zeta(J_\beta) = \exp \left\{ -\kappa \sum_n \ln [1 - \kappa N_{n,\zeta}(\beta|\text{out})] \right\}, \quad (1.3.23)$$

to rewrite expression (1.3.21) for the entropy in the form

$$\begin{aligned} S(\check{\rho}_\zeta(\beta)) &= -k_B \sum_n S(\check{\rho}_{n,\zeta}(\beta)), \quad S(\check{\rho}_{n,\zeta}(\beta)) \\ &= -k_B \{ \kappa [1 - \kappa N_{n,\zeta}(\beta|\text{out})] \ln [1 - \kappa N_{n,\zeta}(\beta|\text{out})] + N_{n,\zeta}(\beta|\text{out}) \ln N_{n,\zeta}(\beta|\text{out}) \}. \end{aligned} \quad (1.3.24)$$

Considering expressions (1.3.24), (1.3.13), and (1.3.4) one can see that they all have similar forms.

Next, let us find the Schmidt measure for subsystems of positrons and electrons for the system with a thermal state at the initial time instant; subsystems of such state are described by the reduced density operator $\check{\rho}_\zeta(\beta)$. The entanglement measure of electron and positron subsystem is given by

$$\tilde{S}(\check{\rho}_\zeta(\beta)) = -\text{tr} [\check{\rho}_\zeta(\beta)]^2, \quad (1.3.25)$$

where the squares of the operators $\check{\rho}_\zeta(\beta)$ are

$$\begin{aligned} [\check{\rho}_+(\beta)]^2 &= Z_+^{-2}(J_\beta) : \exp \left[\sum_n a_n^\dagger(\text{out}) [K_+^2(J_\beta) - 1]_{nn} a_n(\text{out}) \right] : , \\ [\check{\rho}_-(\beta)]^2 &= Z_-^{-2}(J_\beta) : \exp \left[\sum_n b_n^\dagger(\text{out}) [K_-^2(J_\beta) - 1]_{nn} b_n(\text{out}) \right] . \end{aligned} \quad (1.3.26)$$

As a result, we get

$$\tilde{S}(\check{\rho}_\zeta(\beta)) = - \prod_n \{ 1 - 2\kappa N_{n,\zeta}(\beta|\text{out}) + (1 + \kappa) [N_{n,\zeta}(\beta|\text{out})]^2 \}^\kappa. \quad (1.3.27)$$

1.3.3 Entropy of measurement-reduced density operators

The entropy of a density operator $\check{\rho}_N$ (1.5.49) has the form

$$S(\check{\rho}_N) = -k_B \text{tr} \check{\rho}_N \ln \check{\rho}_N. \quad (1.3.28)$$

Representation (1.2.4) allows one to factorize the complete vacuum into the product of single-mode vacua,

$$\begin{aligned} |0, \text{out}\rangle\langle 0, \text{out}| &= \prod_n |0, \text{out}\rangle_{nn}\langle 0, \text{out}|, \\ a_n(\text{out})|0, \text{out}\rangle_n &= 0, \quad b_n(\text{out})|0, \text{out}\rangle_n = 0. \end{aligned} \quad (1.3.29)$$

Using this fact and the representation for $|c_v|^2$ from (1.2.20), one can rewrite density operator (1.5.49) as a product of single-mode density operators:

$$\begin{aligned} \check{\rho}_N &= \prod_n \check{\rho}_{N,n}, \quad \text{tr} \check{\rho}_{N,n} = 1, \quad \check{\rho}_{N,n} = |c_v|_n^2 \sum_{f=0} W_{f,n} |f, \text{out}\rangle_{nn}\langle f, \text{out}|, \\ |c_v|_n^2 &= |w(-|-)_{nn}|^{-2\kappa}, \quad W_{f,n} = |w(+ - |0)_{nn}|^{2f}, \quad |f, \text{out}\rangle_n = \frac{[a_n^\dagger(\text{out})b_n^\dagger(\text{out})]^f}{f!} |0, \text{out}\rangle_n. \end{aligned} \quad (1.3.30)$$

Quantities $|c_v|_n^2$ and $|w(+ - |0)_{nn}|^2$ can be expressed via differential numbers $N_n(0|\text{out})$ as

$$|c_v|_n^2 = (1 - \kappa N_n(0|\text{out}))^\kappa, \quad |w(+ - |0)_{nn}|^2 = \frac{N_n(0|\text{out})}{1 - \kappa N_n(0|\text{out})}. \quad (1.3.31)$$

Due to expression (1.3.30), entropy (1.3.28) can be written as

$$S(\check{\rho}_N) = -k_B \sum_n \text{tr} \check{\rho}_{N,n} \ln \check{\rho}_{N,n}. \quad (1.3.32)$$

In order to calculate the trace of the operator $\check{\rho}_{N,n} \ln \check{\rho}_{N,n}$, one can use the formal decomposition

$$\check{\rho}_{N,n} \ln \check{\rho}_{N,n} = \check{\rho}_{N,n} \sum_{k=1}^{\infty} k^{-1} (\check{\rho}_{N,n} - 1)^k = \sum_{k=1}^{\infty} k^{-1} \sum_{l=0}^k C_k^l (\check{\rho}_{N,n})^{l+1} (-1)^{k-l}, \quad (1.3.33)$$

where C_k^l are binomial coefficients. Due to the orthonormality of the states $|f, \text{out}\rangle_n$ the density operators $(\check{\rho}_{N,n})^{l+1}$ have the form

$$(\check{\rho}_{N,n})^{l+1} = |c_v|_n^{2(l+1)} \sum_{f=0}^{\infty} (W_{f,n})^{l+1} |f, \text{out}\rangle_{nn}\langle f, \text{out}|. \quad (1.3.34)$$

Substituting (1.3.34) into (1.3.33), one obtains

$$\check{\rho}_{N,n} \ln \check{\rho}_{N,n} = |c_v|_n^2 \sum_{f=0}^{\infty} W_{f,n} \ln (|c_v|_n^2 W_{f,n}) |f, \text{out}\rangle_{nn}\langle f, \text{out}|. \quad (1.3.35)$$

Then

$$\text{tr} \check{\rho}_{N,n} \ln \check{\rho}_{N,n} = N_n(0|\text{out}) \ln N_n(0|\text{out}) + \kappa [1 - \kappa N_n(0|\text{out})] \ln [1 - \kappa N_n(0|\text{out})]. \quad (1.3.36)$$

Thus, the entropy of density operator (1.5.49) reads

$$S(\check{\rho}_N) = -k_B \sum_n \{ \kappa [1 - \kappa N_n(0|out)] \ln [1 - \kappa N_n(0|out)] + N_n(0|out) \ln N_n(0|out) \}. \quad (1.3.37)$$

The result has the same form as the entropy $S(\check{\rho}_\zeta(0))$ given by (1.3.13). Thus, it can be said that measurement of N , N_+ or N_- leads to the same information loss as the reduction over one of subsystems of electrons or positrons.

It was shown in the sect. 1.2.2 that reduction of the density operator $\check{\rho}_N$ over electrons and positrons transforms it in $[\check{\rho}_N]_\zeta = \check{\rho}_\zeta(0)$. This means that if one calculates entropy of density operator $[\check{\rho}_N]_\zeta$, one obtains the same expression (1.3.37) again. The conditional entropy [57] $S_{cond} = S(\check{\rho}_N) - S([\check{\rho}_N]_\zeta)$, which is used as a measure of correlations between subsystems, is zero. This fact means that all quantum correlations between the electrons and positrons are lost due to decoherence, and there is no entanglement left after the measurement.

1.4 T-constant external electric field

To illustrate some of the above general formulas, let's consider the so-called T -constant electric field as an external background. Such a field acts only during a finite time T and it is constant within this time interval. Using this field allows one to avoid troubles with the definition of in- and out-states inherent to external fields non-switched at $t \rightarrow \pm\infty$. Another important point is that this field produces a finite work in a finite space volume. Let us consider $d = (D + 1)$ -dimensional space, then the T -constant electric field \mathbf{E} is acting during the time interval $T = t_{out} - t_{in}$,

$$\mathbf{E} = (0, E(t), 0, \dots, 0), \quad E(t) = \begin{cases} 0, & -\infty < t \leq t_{in} \\ E > 0, & t_{in} < t < t_{out} \\ 0, & t_{out} \leq t < \infty \end{cases}, \quad (1.4.1)$$

Processes of pair creation in such field were studied in Refs. [72, 73, 28, 75]. Similar to these works, sufficiently long time T of the field action will be considered.

Since there is no particle production after the time instant t_{out} , differential mean numbers of particles $N_{n,\zeta}(\dots|out)$ created in a given state $n = \mathbf{p}, r$ (\mathbf{p} is a D -dimensional vector of momentum and r is spin) depend only on the time interval. Electric field acting during the sufficiently long time T creates a considerable number of pairs only in a finite region in the momentum space. Since it is supposed that $T \gg \max\{1, E_c/E\}$, one needs to consider only the range

$$|p_\perp| \leq \sqrt{eE} \left[\sqrt{eET} \right]^{1/2}, \quad -T/2 \leq p_1/eE \leq T/2 \quad (1.4.2)$$

in the momentum space, see [28] for details. Note that for the case $d = 2$ there are no transversal components of momentum.

1.4.1 Vacuum initial state

First let us consider the case when the system initially was in the vacuum state. For this case differential mean numbers in the momentum range (1.4.2) are

$$N_n(0|out) = e^{-\pi\lambda}, \quad \lambda = (p_\perp^2 + m^2)/eE. \quad (1.4.3)$$

They have the same form as in the case of the constant uniform electric field [69, 65], and are the same for bosons and fermions. Entropy (1.3.13) is expressed in terms of $N_n(0|\text{out})$ and does not depend on the spin quantum number r , thus, the summation over the latter results in the factor $\gamma_{(d)} = 2^{\lfloor \frac{d}{2} \rfloor - 1}$.

First, let's consider the Dirac case ($\kappa = +1$):

$$S(\check{\rho}_{n,\zeta}(0)) = -k_B [(1 - N_n(0|\text{out})) \ln(1 - N_n(0|\text{out})) + N_n(0|\text{out}) \ln N_n(0|\text{out})]. \quad (1.4.4)$$

For the case of the T -constant electric field the mean number of particles created $N_n(0|\text{out})$ can vary only within the range $(0, 1)$ and depends only on the strength of the external field. Expression (1.4.4) is symmetric with respect to $N_{n,\zeta}(0|\text{out})$. It reaches maximum at $N_n(0|\text{out}) = 1/2$ and turns to zero at $N_n(0|\text{out}) = 1$ and $N_n(0|\text{out}) = 0$. This fact can be interpreted as follows. In the case of $N_n(0|\text{out}) = 0$ there are no particles created by the external field and the initial vacuum state in the mode remains unchanged. The case $N_n(0|\text{out}) = 1$ corresponds to situation when a particle is created with certainty. The maximum of (1.4.4), corresponding to $N_n(0|\text{out}) = 1/2$, is associated with the state with the maximum amount of uncertainty.

Representing the logarithm in the first term of expression (1.4.4) as the Taylor series in powers of $N_n(0|\text{out})$, one sees that $S(\check{\rho}_{n,\zeta}(0))$ is proportional to $N_n(0|\text{out})$. The latter plays the role of the cut-off parameter for the integral over p_\perp [28]. Thus, the summation over the quantum numbers can be reduced to an integration over momenta that satisfy restrictions (1.4.2),

$$\sum_n \rightarrow \frac{\gamma_{(d)} V}{(2\pi)^{d-1}} \int d\mathbf{p},$$

where V is the D -dimensional spatial volume. The mean numbers (1.4.3) do not depend on the longitudinal component of momentum. Outside of the range (1.4.2), the contribution to the integral is very small, and this allows us to extend the integration limits of p_\perp to infinity. Integration over p_\perp can be performed using the Taylor series. The result of the integration is

$$S(\check{\rho}_\zeta(0)) = \gamma_{(d)} k_B \frac{(eE)^{\frac{d}{2}} TV}{(2\pi)^{d-1}} A_{Dirac}(d, E_c/E), \quad (1.4.5)$$

where the factor TV can be considered as d -dimensional volume. To get finite and correct expressions, one should use the volume normalization. The factor $A_{Dirac}(d, E_c/E)$ has the form

$$A_{Dirac}(d, E_c/E) = \left[\sum_{l=1}^{\infty} l^{-d/2} \exp[-\pi l E_c/E] - \sum_{l=1}^{\infty} l^{-1} (l+1)^{\frac{2-d}{2}} \exp[-\pi (l+1) E_c/E] + \left(\pi \frac{E_c}{E} + \frac{d-2}{2} \right) \exp(-\pi E_c/E) \right].$$

It is possible to estimate the entropy in strong $E_c/E \ll 1$, critical $E_c/E = 1$, and weak $E_c/E \gg 1$ field limits. For example, for a strong field with $d = 4$ one has $A_{Dirac}(4, 0) = \pi^2/6$, for the critical field, one has $A_{Dirac}(4, 1) \approx 0, 22$. In the case of a weak field the entropy has a small value of the order of $(\pi E_c/E) \exp[-\pi E_c/E]$ for any d . For $d = 3$ the following estimations hold $A_{Dirac}(3, 0) \approx 0, 93$, $A_{Dirac}(3, 1) \approx 0, 2$; for $d = 2$ the factor $A(2, 0)$ is a value of order of 1, and $A(2, 1) = e^{-\pi}$.

Let us consider the K-G case ($\kappa = -1$). The entropy is given by

$$S(\check{\rho}_{n,\zeta}(0)) = k_B \{ [1 + N_n(0|\text{out})] \ln [1 + N_n(0|\text{out})] - N_n(0|\text{out}) \ln N_n(0|\text{out}) \}. \quad (1.4.6)$$

Expression (1.4.6) just increases with $N_n(0|\text{out})$. After a summation over the quantum numbers, entropy (1.3.11) takes the form

$$S(\check{\rho}_\zeta(0)) = k_B \frac{(eE)^{\frac{d}{2}} TV}{(2\pi)^{d-1}} A_{K-G}(d, E_c/E), \quad (1.4.7)$$

where

$$A_{K-G}(d, E_c/E) = \left[\sum_{l=1}^{\infty} l^{-d/2} (-1)^{l-1} \exp[-\pi l E_c/E] + \sum_{l=1}^{\infty} l^{-1} (l+1)^{\frac{2-d}{2}} (-1)^{l-1} \exp[-\pi(l+1)E_c/E] + \left(\pi \frac{E_c}{E} + \frac{d-2}{2} \right) \exp(-\pi E_c/E) \right].$$

Estimates for different field strengths are the following: $A_{K-G}(4, 0) \approx 2, 21$, $A_{K-G}(4, 1) \approx 0, 22$; $A_{K-G}(3, 0) \approx 1, 78$, $A_{K-G}(3, 1) \approx 0, 2$; $A_{K-G}(2, 0) \approx 1$, $A_{K-G}(2, 1) \approx e^{-\pi}$. In the case of weak field the entropy is a small value of the order of $(\pi E_c/E) \exp[-\pi E_c/E]$ for any d again.

It was mentioned before that the entropy of density operator $\check{\rho}_N$, given by (1.3.37), is exactly of the same form as the entropy of $\check{\rho}_\zeta(0)$, given by (1.3.13), hence all the considerations for the case with the intermediate measurement are the same.

1.4.2 Mixed initial state

It should be noted that entropy (1.3.24) of the system that has been in thermal equilibrium at the initial time instant is expressed in terms of differential mean numbers $N_{n,\zeta}(\beta|\text{out})$ (1.2.10) of particles created by the external field, whereas initial differential numbers of particles are

$$N_{n,\zeta}(\beta|\text{in}) = [\exp \beta(\varepsilon_n - \mu) + \kappa]^{-1}, \quad \varepsilon_n = \sqrt{m^2 + p_\perp^2 + (p_1 + eET/2)^2}. \quad (1.4.8)$$

Let us discuss two cases, the first one being the case of low temperature

$$\beta(\varepsilon_\perp - \mu) \gg 1, \quad \varepsilon_\perp = \sqrt{m^2 + p_\perp^2},$$

when all the energies of the particles created with a given p_\perp are considerably higher than the temperature, and the second being the case of high temperature $\beta eET \ll 1$, when all the energies of the created particles are much lower than the temperature. It is assumed for simplicity that $eET \gg \mu$, and that T is sufficiently large to provide $(eET)^2 \gg m^2 + p_\perp^2$.

In the low temperature case, the number of created particles does not depend on the longitudinal momenta:

$$N_{n,\zeta}(\beta|\text{in}) \approx \exp(-\beta\varepsilon_n) \rightarrow 0, \quad N_{n,\zeta}(\beta|\text{out}) \rightarrow N_{n,\zeta}(0|\text{out}).$$

In this limit entropy $S(\check{\rho}_{n,\zeta}(\beta))$ tends to that of the zero temperature case (initial vacuum state). Then integration over transversal momenta can be done exactly as in the initial vacuum case.

Formal calculations of $N_{n,\zeta}(\beta|\text{out})$ and of the entropy in the case of high temperature, $\beta eET \ll 1$, are also quite simple. However, it was shown in Ref. [75] that in the Dirac case under such a condition the current density is much greater than the current density of particles created from vacuum, due to the work of the external field performed over the particles (which was denoted as $\text{Re}\langle j_\mu(t) \rangle_\theta^c$ in [75]) already existing in the initial state. Therefore in such a case the particle creation effect may be disregarded.

1.5 General theory: x -electric potential steps

The general theory of QED with x -electric potential steps was formulated in Ref. [76]. Unlike the case of QED with a t -dependent potential steps, the general theory is formulated in Heisenberg picture from the beginning. It should also be noted that only the fermion case will be considered in this section.

It was shown that in the presence of x -electric potential steps the quantized Dirac field can be described in terms of in and out electrons and positrons. Such particles are characterized by quantum numbers n that can be divided in five ranges Ω_i , $i = 1, \dots, 5$. The corresponding quantum numbers are denoted by n_i , so that $n_i \in \Omega_i$. The manifold of all the quantum numbers n is denoted by Ω , so that $\Omega = \Omega_1 \cup \dots \cup \Omega_5$. Basic elements of QED with unstable vacuum in the presence of x -electric critical potential steps are given in Appendix 2.

The linear canonical transformation between the in and out sets of creation and annihilation operators in the Klein zone (a and b operators are related to electrons and positrons, respectively) can be written in the following form

$$\begin{aligned} {}^-a_n(\text{in}) &= w_n(+|+)^{-1} [{}^+a_n(\text{out}) + w_n(+|-|0) {}^+b_n^\dagger(\text{out})], \\ {}^-b_n^\dagger(\text{in}) &= w_n(-|-)^{-1} [{}^+b_n^\dagger(\text{out}) - w_n(+|-|0) {}^+a_n(\text{out})], \end{aligned} \quad (1.5.1)$$

where

$$\begin{aligned} w(+|+)_{n'n} &= c_v^{-1} \langle 0, \text{out} | {}^+a_{n'}(\text{out}) {}^-a_n^\dagger(\text{in}) | 0, \text{in} \rangle, \\ w(-|-)_{n'n} &= c_v^{-1} \langle 0, \text{out} | {}^+b_{n'}(\text{out}) {}^-b_n^\dagger(\text{in}) | 0, \text{in} \rangle, \end{aligned} \quad (1.5.2)$$

are relative scattering amplitudes of electrons and positrons, and

$$\begin{aligned} w(+|-|0)_{n'n} &= c_v^{-1} \langle 0, \text{out} | {}^+a_{n'}(\text{out}) {}^+b_n(\text{out}) | 0, \text{in} \rangle, \\ w(0|-|+)_{nn'} &= c_v^{-1} \langle 0, \text{out} | {}^-b_n^\dagger(\text{in}) {}^-a_{n'}^\dagger(\text{in}) | 0, \text{in} \rangle. \end{aligned} \quad (1.5.3)$$

are relative amplitudes for pair creation and pair annihilation, and

$$c_v = c_v^{(K)} = \prod_n w_n(-|-)^{-1}. \quad (1.5.4)$$

All the amplitudes can be expressed via the coefficients $g(\zeta|\zeta')$ which, in turn, are calculated via corresponding solutions of the Dirac equation with x -electric potential steps.

1.5.1 Peak x -dependent electric field

To illustrate the general formulation, let us consider the example of so-called x -dependent peak electric field. We consider an external electromagnetic field, placed in ($d = D + 1$)-dimensional Minkovsky space, parametrized by the coordinates $X = (X^\mu, \mu = 0, 1, \dots, D) = (t, \mathbf{r})$, $X^0 = t$, $\mathbf{r} = (x, \mathbf{r}_\perp)$, $\mathbf{r}_\perp = (X^2, \dots, X^D)$. The potentials for the external electromagnetic field are chosen as $A^\mu(X) = \delta_0^\mu A_\mu(x)$,

$$A^\mu(X) = (A^0 = A_0(x), A^k = 0, k = 1, \dots, D), \quad (1.5.5)$$

which corresponds to vanishing magnetic field and to the electric field

$$\mathbf{E}(X) = \mathbf{E}(x) = (E_x(x), 0, \dots, 0), \quad E_x(t) = -A_0'(x) = E(x). \quad (1.5.6)$$

The electric field (1.5.6) is directed along the x -axis and is homogeneous. The main properties common to any x -electric potential steps are

$$A_0(x) \xrightarrow{x \rightarrow \pm\infty} A_0(\pm\infty), \quad E(x) \xrightarrow{|x| \rightarrow \infty} 0, \quad (1.5.7)$$

where $A_0(\pm\infty)$ are some constant quantities, and the derivative of the potential $A'_0(x)$ does not change its sign for any $x \in \mathbb{R}$. For definiteness, we suppose that

$$A'_0(x) \leq 0 \implies \begin{cases} E(x) = -A'_0(x) \geq 0 \\ A_0(-\infty) > A_0(+\infty) \end{cases}. \quad (1.5.8)$$

The electric charge of the electron is $q = -e$, $e > 0$. The potential energy of electron in this field is $U(x) = -eA_0(x)$, and the magnitude of the corresponding x -potential steps is

$$\mathbb{U} = U_R - U_L > 0, \quad U_R = -eA_0(+\infty), \quad U_L = -eA_0(-\infty). \quad (1.5.9)$$

We are interested in the critical steps, for which $\mathbb{U} > 2m$ and the vacuum is unstable in the Klein zone.

The system under consideration consists of a Dirac field $\psi(X)$ interacting with the so-called peak electric field. This electric field is composed of independent parts, wherein for each one the Dirac equation is exactly solvable. The field in consideration grows exponentially from the minus infinity $x = -\infty$, reaches a maximal amplitude E at $x = 0$ and decreases exponentially to the infinity $x = +\infty$. Its maximum $E > 0$ occurring at a very sharp point, say at $x = 0$, such that the limit

$$\lim_{x \rightarrow -0} E'(x) \neq \lim_{x \rightarrow +0} E'(x), \quad (1.5.10)$$

is not defined. The latter property implies that a peak at $x = 0$ is present. We label the exponentially increasing interval by $S_L = (-\infty, 0]$ and the exponentially decreasing interval by $S_R = (0, +\infty)$, where the field and its x -electric potential step are

$$E(x) = E \begin{cases} e^{k_1 x}, & x \in S_L \\ e^{-k_2 x}, & x \in S_R \end{cases}, \quad A_0(x) = E \begin{cases} k_1^{-1} (-e^{k_1 x} + 1), & x \in S_L \\ k_2^{-1} (e^{-k_2 x} - 1), & x \in S_R \end{cases}, \quad eE > 0, \quad k_1, k_2 > 0. \quad (1.5.11)$$

It should be noted that, for example, the so-called exponentially decaying electric field, given by the potential

$$A_0^{\text{ed}}(x) = E \begin{cases} 0, & x \in S_L \\ k_2^{-1} (e^{-k_2 x} - 1), & x \in S_R \end{cases}, \quad (1.5.12)$$

can be considered as a particular case of the peak field, when the peak field switches abruptly on at $x = 0$, that is, k_1 is sufficiently large, $k_1 \rightarrow \infty$.

1.5.2 Dirac equation with x -electric potential step

The Dirac equation for the system under consideration has the following form:

$$i\partial_0\psi(X) = \hat{H}\psi(X), \quad \hat{H} = \gamma^0 (-i\gamma^j \partial_j + m) + U(x), \quad j = 1, \dots, D, \quad (1.5.13)$$

where the Dirac field $\psi(X)$ is a $2^{[d/2]}$ -component spinor (where $[d/2]$ is integer part of $d/2$) in d dimensions, \hat{H} is the one-particle Hamiltonian, γ^μ are $2^{[d/2]} \times 2^{[d/2]}$ gamma-matrices in d dimensions (see, for example, Ref ([77]):

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \text{diag}(\underbrace{1, -1, \dots, -1}_d), \quad \mu, \nu = 0, 1, \dots, D. \quad (1.5.14)$$

Due to the configuration of the field (1.5.11), the structure of Dirac spinor $\psi(X)$ in directions X^0 and X^2, \dots, X^D is a simple plane wave, so we consider stationary solutions of the Dirac equation (1.5.13) of the form

$$\begin{aligned}\psi_n(X) &= \exp[-ip_0t + i\mathbf{p}_\perp \mathbf{r}_\perp] \psi_n(x), \quad n = (p_0, \mathbf{p}_\perp, \sigma), \\ \psi_n(x) &= \{\gamma^0 [p_0 - U(x)] + i\gamma^1 \partial_x - \gamma^\perp \mathbf{p}_\perp + m\} \phi_n(x), \\ \mathbf{p}_\perp &= (p^2, \dots, p^D), \quad \gamma^\perp = (\gamma^2, \dots, \gamma^D),\end{aligned}\tag{1.5.15}$$

where $\psi_n(x)$ and $\phi_n(x)$ are spinors that depend on x alone. These spinors are stationary states with the fixed energy p_0 and with definite transversal momenta \mathbf{p}_\perp (the index \perp stands for the spatial components perpendicular to the electric field, e.g., $\mathbf{p}_\perp = (p^2, \dots, p^D)$). Substituting Eq. (1.5.15) into Eq. (1.5.13), we obtain the second-order differential equation for $\phi_n(x)$:

$$\left(\hat{p}_x^2 - i\gamma^0 \gamma^1 \frac{\partial U(x)}{\partial x} - [p_0 - U(x)]^2 + \mathbf{p}_\perp^2 + m^2 \right) \phi_n(x) = 0, \quad \hat{p}_x = -i\partial_x.\tag{1.5.16}$$

Spinning variables are separated by substitution

$$\phi_n(x) = \phi_n^{(\chi)}(x) = \varphi_n(x) v_{\chi, \sigma},\tag{1.5.17}$$

where $v_{s, \sigma}$ is the set of constant orthonormalized spinors with $\chi = \pm 1$, $\sigma = (\sigma_1, \dots, \sigma_{[d/2]-1})$, $\sigma_j = \pm 1$, satisfying the following relations

$$\gamma^0 \gamma^1 v_{\chi, \sigma} = \chi v_{\chi, \sigma}, \quad v_{\chi, \sigma}^\dagger v_{\chi', \sigma'} = \delta_{\chi, \chi'} \delta_{\sigma, \sigma'}.\tag{1.5.18}$$

The quantum numbers s and σ describe a spin polarization and provide parametrization of the solutions. In d dimensions there are exist only $J_{(d)} = 2^{[d/2]-1}$ different spin states. It was shown in Ref. [78] that it is sufficient to work with only one value of χ . The scalar functions $\varphi_n(x)$ have to obey the second order differential equations

$$\begin{aligned}\left(-\partial_x^2 - i\chi e E e^{k_1 x} - \left[p_0 + \frac{eE}{k_1} (-e^{k_1 x} + 1) \right]^2 + \pi_\perp^2 \right) \varphi_n(x) &= 0, \quad x \in S_L, \\ \left(-\partial_x^2 - i\chi e E e^{-k_2 x} - \left[p_0 + \frac{eE}{k_2} (e^{-k_2 x} - 1) \right]^2 + \pi_\perp^2 \right) \varphi_n(x) &= 0, \quad x \in S_R,\end{aligned}\tag{1.5.19}$$

where $j = 1$ for $x \in S_L$, and $j = 2$ for $x \in S_R$, and $\pi_\perp^2 = p_\perp^2 + m^2$. In each interval we introduce new variables η_j

$$\eta_j = i h_j e^{(-1)^{j+1} k_j x}, \quad h_j = \frac{2eE}{k_j^2},\tag{1.5.20}$$

and represent the scalar functions $\varphi_n(x)$ as

$$\begin{aligned}\varphi_n^j(x) &= e^{-\eta_j/2} \eta_j^{\nu_j} \rho_j(x), \quad \nu_1 = i \frac{p^L}{k_1}, \quad p^L = \sqrt{\pi_1^2 - \pi_\perp^2}, \\ \nu_2 &= i \frac{p^R}{k_2}, \quad p^R = \sqrt{\pi_2^2 - \pi_\perp^2}, \quad \pi_j = p_0 + (-1)^{j+1} \frac{eE}{k_j}.\end{aligned}\tag{1.5.21}$$

The functions $\rho_j(x)$ satisfy confluent hypergeometrical equation

$$\left(\eta_j \frac{\partial^2}{\partial \eta_j^2} + [c_j - \eta_j] \frac{\partial}{\partial \eta_j} - a_j \right) \rho_j(x) = 0,$$

$$c_j = 2\nu_j + 1, \quad a_j = \frac{1}{2} (1 - \chi) + (-1)^j \frac{i\pi_j}{k_j} + \nu_j. \quad (1.5.22)$$

A fundamental set of solutions for the equation is composed by two linearly independent confluent hypergeometric functions:

$$\Phi(a_j, c_j; \eta_j) \quad \text{and} \quad \eta_j^{1-c_j} e^{\eta_j} \Phi(1 - a_j, 2 - c_j; \eta_j),$$

where functions Φ are defined as

$$\Phi(a, c; \eta) = 1 + \frac{a \eta}{c \cdot 1!} + \frac{a(a+1) \eta^2}{c(c+1) \cdot 2!} + \dots \quad (1.5.23)$$

The general solution in both intervals S_L and S_R can be presented as their linear combination

$$\begin{aligned} \varphi_n^j(x) &= A_2^j y_1^j(\eta_j) + A_1^j y_2^j(\eta_j), \\ y_1^j &= e^{-\eta_j/2} \eta_j^{\nu_j} \Phi(a_j, c_j; \eta_j), \\ y_2^j &= e^{\eta_j/2} \eta_j^{-\nu_j} \Phi(1 - a_j, 2 - c_j; -\eta_j), \end{aligned} \quad (1.5.24)$$

where coefficients A_1^j, A_2^j must be defined by boundary conditions.

The Wronskian of the $y_{1,2}^j(\eta_j)$ functions is

$$W = y_1^j \frac{d}{d\eta_j} y_2^j - y_2^j \frac{d}{d\eta_j} y_1^j = \frac{1 - c_j}{\eta_j}. \quad (1.5.25)$$

It can be seen from Eq. (1.5.11) that there is no electric field at the minus infinity $x \rightarrow -\infty$ and at the infinity $x \rightarrow +\infty$. These solutions have the form (1.5.15) with the functions $\varphi_n(x)$ denoted as ${}_\zeta\varphi_n(x)$ and ${}^\zeta\varphi_n(x)$, respectively. At these regions, the exact solutions of Dirac equation represent free particles,

$${}_\zeta\varphi_n(x) = {}_\zeta\mathcal{N} e^{i\zeta p^L x} \quad \text{if } x \rightarrow -\infty, \quad {}^\zeta\varphi_n(x) = {}^\zeta\mathcal{N} e^{i\zeta p^R x} \quad \text{if } x \rightarrow +\infty, \quad \zeta = \pm. \quad (1.5.26)$$

The solutions ${}_\zeta\varphi_n(x)$ and ${}^\zeta\varphi_n(x)$ asymptotically describe particles with given real momenta ζp^L and ζp^R along the axis x . The factors ${}_\zeta\mathcal{N}$ and ${}^\zeta\mathcal{N}$ are normalization constants with respect to the inner product (B.1) on x -constant hyperplane:

$$\begin{aligned} {}_\zeta\mathcal{N} &= {}_\zeta C Y, \quad {}^\zeta\mathcal{N} = {}^\zeta C Y, \quad Y = (V_\perp T)^{-1/2}, \\ {}_\zeta C &= [2\zeta p^L |(\pi_1 - \chi \zeta p^L)|]^{-1/2}, \quad {}^\zeta C = [2\zeta p^R |(\pi_2 - \chi \zeta p^R)|]^{-1/2}. \end{aligned} \quad (1.5.27)$$

By virtue of these properties, electron (positron) states can be selected as follows:

$$\begin{aligned} +\varphi_n(x) &= +\mathcal{N} \exp(-i\pi\nu_1/2) y_1^1, \quad -\varphi_n(x) = -\mathcal{N} \exp(i\pi\nu_1/2) y_2^1, \quad x \in S_L; \\ +\varphi_n(x) &= +\mathcal{N} \exp(i\pi\nu_2/2) y_2^2, \quad -\varphi_n(x) = -\mathcal{N} \exp(-i\pi\nu_2/2) y_1^2, \quad x \in S_R. \end{aligned} \quad (1.5.28)$$

Taking into account the complete set of exact solutions (1.5.24) and mutual decompositions (B.2), the functions ${}_{-}\varphi_n(x)$ and ${}_{+}\varphi_n(x)$ can be presented in the form

$${}_{+}\varphi_n(x) = \begin{cases} \eta_L [{}_{+}\varphi_n(x) g({}_{+}|{}_{+}) - {}_{-}\varphi_n(x) g({}_{-}|{}_{+})], & x \in S_L \\ {}_{+}\mathcal{N} \exp(i\pi\nu_2/2) y_2^2, & x \in S_R \end{cases}, \quad (1.5.29)$$

$${}_{-}\varphi_n(x) = \begin{cases} -\mathcal{N} \exp(i\pi\nu_1/2) y_1^1, & x \in S_L \\ \eta_R [{}_{+}\varphi_n(x) g({}_{+}|{}_{-}) - {}_{-}\varphi_n(x) g({}_{-}|{}_{-})], & x \in S_R \end{cases}, \quad (1.5.30)$$

for the whole axis x .

The functions ${}_{-}\varphi_n(x)$ and ${}_{+}\varphi_n(x)$ and their derivatives satisfy the following gluing conditions:

$${}_{-}\varphi_n(x)|_{x=-0} = {}_{+}\varphi_n(x)|_{x=+0}, \quad \partial_x {}_{-}\varphi_n(x)|_{x=-0} = \partial_x {}_{+}\varphi_n(x)|_{x=+0}. \quad (1.5.31)$$

Using Eq. (1.5.31) and the Wronskian (1.5.25), one can find each coefficient $g(\zeta|\zeta')$ and $g(\zeta|\zeta')$ in Eqs. (1.5.29) and (1.5.30). For example, applying these conditions to the set (1.5.29), the coefficient $g({}_{-}|{}_{+})$ takes the form

$$g({}_{-}|{}_{+}) = C\Delta, \quad C = \frac{-\eta_L}{2} \sqrt{\frac{|\pi_1 + \chi p^L|}{p^L p^R |\pi_2 - \chi p^R|}} \exp\left[\frac{i\pi}{2}(\nu_2 - \nu_1)\right],$$

$$\Delta = \left[k_1 h_1 y_2^2 \frac{\partial}{\partial \eta_1} y_1^1 + k_2 h_2 y_1^1 \frac{\partial}{\partial \eta_2} y_2^2 \right] \Big|_{x=0}. \quad (1.5.32)$$

The same can be done to Eq. (1.5.30) to obtain

$$g({}_{+}|{}_{-}) = C'\Delta', \quad C' = \frac{\eta_R}{2} \sqrt{\frac{|\pi_2 - \chi p^R|}{p^R p^L |\pi_1 + \chi p^L|}} \exp\left[-\frac{i\pi}{2}(\nu_2 - \nu_1)\right],$$

$$\Delta' = \left[k_1 h_1 y_1^2 \frac{\partial}{\partial \eta_1} y_2^2 + k_2 h_2 y_2^2 \frac{\partial}{\partial \eta_2} y_1^1 \right] \Big|_{x=0}. \quad (1.5.33)$$

One can easily verify that the symmetry under a simultaneous change $k_1 \Leftrightarrow k_2$ and $\pi_1 \Leftrightarrow -\pi_2$ holds,

$$g({}_{+}|{}_{-}) \Leftrightarrow -\eta_L \eta_R g({}_{-}|{}_{+}). \quad (1.5.34)$$

Using $g({}_{-}|{}_{+})$ given by Eq. (1.5.32), we find that the differential mean number of Dirac particles created is

$$N_n^{\text{cr}} = N_{n,-}^{\text{cr}} = N_{n,+}^{\text{cr}} = \langle 0, \text{in} | {}_{+}a_n^\dagger(\text{out}) {}_{+}a_n(\text{out}) | 0, \text{in} \rangle = |g({}_{-}|{}_{+})|^{-2} = |C\Delta|^{-2}. \quad (1.5.35)$$

It is clear that N_n^{cr} is a function of modulus squared of transversal momentum, \mathbf{p}_\perp^2 . It follows from Eq. (1.5.35) that N_n^{cr} is invariant under the simultaneous change $k_1 \Leftrightarrow k_2$ and $\pi_1 \Leftrightarrow -\pi_2$. Then if $k_1 = k_2$, N_n^{cr} appears to be an even function of p_0 too.

As follows from Eqs. (1.5.32), (1.5.33), if either p^R or p^L tends to zero, one of the following limits holds true:

$$|g({}_{-}|{}_{+})|^{-2} \sim p^R \rightarrow 0, \quad |g({}_{+}|{}_{-})|^{-2} \sim p^L \rightarrow 0, \quad \forall \lambda \neq 0. \quad (1.5.36)$$

These properties are essential for the justification of in- and out-particle interpretation in the general construction [76].

1.5.3 Scattering and creation of particles

We know that in the ranges Ω_i , $i = 1, 2, 4, 5$ the partial vacua are stable. Let us start with discussion of formulas obtained for these ranges. In the range Ω_2 and Ω_4 , a particle is subjected to the total reflection. In the adjacent ranges, Ω_1 and Ω_5 , a particle can be reflected and transmitted.

As an example, in the range Ω_1 one can calculate total \tilde{R} and relative R amplitudes of an electron reflection, and total \tilde{T} and relative T amplitudes of an electron transmission, which can be presented as the following matrix elements

$$\begin{aligned}
R_{+,n} &= \tilde{R}_{+,n} c_v^{-1} & \tilde{R}_{+,n} &= \langle 0, \text{out} | -a_n(\text{out}) + a_n^\dagger(\text{in}) | 0, \text{in} \rangle, \\
T_{+,n} &= \tilde{T}_{+,n} c_v^{-1} & \tilde{T}_{+,n} &= \langle 0, \text{out} | +a_n(\text{out}) + a_n^\dagger(\text{in}) | 0, \text{in} \rangle, \\
R_{-,n} &= \tilde{R}_{-,n} c_v^{-1} & \tilde{R}_{-,n} &= \langle 0, \text{out} | +a_n(\text{out}) - a_n^\dagger(\text{in}) | 0, \text{in} \rangle, \\
T_{-,n} &= \tilde{T}_{-,n} c_v^{-1} & \tilde{T}_{-,n} &= \langle 0, \text{out} | -a_n(\text{out}) - a_n^\dagger(\text{in}) | 0, \text{in} \rangle,
\end{aligned} \tag{1.5.37}$$

where the vacuum state vectors in corresponding Fock space $|0, \text{in}\rangle$, $|0, \text{out}\rangle$ and the vacuum-to-vacuum transition amplitude c_v are defined in Appendix 2.4. The relative reflection $|R_{\zeta,n}|^2$ and transition $|T_{\zeta,n}|^2$ amplitudes

$$|T_{\zeta,n}|^2 = 1 - |R_{\zeta,n}|^2, \quad |R_{\zeta,n}|^2 = \left[1 + |g(-|^\pm)|^{-2} \right]^{-1}, \quad \zeta = \pm. \tag{1.5.38}$$

Similar expressions can be derived for positron amplitudes in the range Ω_5 . In particular, relation (1.5.38) holds true literally for the positrons in the range Ω_5 . It is clear that $|R_{\zeta,n}|^2 \leq 1$. This result may be interpreted as QFT justification of the rules of time-independent potential scattering theory in the ranges Ω_1 and Ω_5 .

The limits (1.5.36) imply that

- (i) $|g(-|^\pm)|^{-2} \rightarrow 0$ in the range Ω_1 if n tends to the boundary with the range Ω_2 ($p^R \rightarrow 0$);
- (ii) $|g(-|^\pm)|^{-2} \rightarrow 0$ in the range Ω_5 if n tends to the boundary with the range Ω_4 ($p^L \rightarrow 0$).

Thus, in these two cases the relative probabilities of reflection $|R_{\zeta,n}|^2$ tend to unity; i.e. they are continuous functions of the quantum numbers n on the boundaries. It can be also seen that $|R_{\zeta,n}|^2 \rightarrow 0$ as $p_0 \rightarrow \pm\infty$.

1.5.4 Differential and total quantities

The differential number of pairs N_n^{cr} created from the vacuum is given by Eq. (1.5.35). The total number of pairs N^{cr} created by the field under consideration can be calculated by summation over all possible quantum numbers in Klein zone. Calculating this number in the fermionic case, one has to sum the corresponding differential mean numbers N_n^{cr} over the spin projections and over the transversal momenta \mathbf{p}_\perp and and energy p_0 . Since the N_n^{cr} do not depend on the spin polarization parameters σ , the sum over the spin projections produces only the factor $J_{(d)} = 2^{\lfloor \frac{d}{2} \rfloor - 1}$, where $[\dots]$ means the integer part. The sum over the momenta and the energy transforms into an integral in the following way:

$$N^{\text{cr}} = \sum_{n \in \Omega_3} N_n^{\text{cr}} = \sum_{\mathbf{p}_\perp, p_0 \in \Omega_3} \sum_{\sigma} N_n^{\text{cr}} \rightarrow \frac{V_\perp T J_{(d)}}{(2\pi)^{d-1}} \int_{\Omega_3} dp_0 d\mathbf{p}_\perp N_n^{\text{cr}}, \tag{1.5.39}$$

where V_\perp is the spatial volume of the $(d-1)$ dimensional hypersurface orthogonal to the electric field direction, x , and T is the time duration of the electric field.

1.5.5 Slowly varying field

The inverse parameters k_1^{-1} , k_2^{-1} represent scales of time duration for increasing and decreasing phases of the electric field. In particular, we have a slowly varying field at small values of both $k_1, k_2 \rightarrow 0$, obeying the conditions

$$\min(h_1, h_2) \gg \max(1, m^2/eE). \quad (1.5.40)$$

This case can be considered as a two-parameter regularization for a constant electric field.

Let us analyze how the numbers N_n^{cr} depend on the parameters p_0 and π_\perp . It can be seen from semiclassical analysis that N_n^{cr} is exponentially small in the range of very large π_\perp ,

$$\pi_\perp \gtrsim \min(eEk_1^{-1}, eEk_2^{-1}).$$

Then the range of fixed π_\perp is of interest, and in the following we assume that condition holds true, where for the case under consideration any given number K_\perp satisfies the inequality

$$\min(h_1, h_2) \gg K_\perp^2 \gg \max(1, m^2/eE). \quad (1.5.41)$$

In what follows, we additionally assume that

$$\lambda < K_\perp^2, \quad \lambda = \frac{\pi_\perp^2}{eE}. \quad (1.5.42)$$

By virtue of symmetry properties of N_n^{cr} discussed above, one can only consider p_0 either positive or negative. Then in Klein zone Ω_3 , which is, for the field (1.5.11), defined by two inequalities[76]

$$U_L + \pi_\perp \leq p_0 \leq U_R - \pi_\perp, \quad U_L = -\frac{eE}{k_1}, \quad U_R = \frac{eE}{k_2}, \quad (1.5.43)$$

Let us, for example, consider the interval $-\frac{eE}{k_1} + \pi_\perp \leq p_0 \leq 0$. In this case π_2 is negative and large, while π_1 is positive:

$$-\pi_\perp + \frac{eE}{k_2} + \frac{eE}{k_1} \geq -\pi_2 \geq eE/k_2, \quad \pi_\perp \leq \pi_1 \leq \frac{eE}{k_1}. \quad (1.5.44)$$

It is convenient to rewrite the total range for π_1 as

$$h_1 \geq \frac{2\pi_1}{k_1} > \frac{2\pi_\perp}{k_1}. \quad (1.5.45)$$

It can be divided in three subranges

$$\begin{aligned} \text{(a)} \quad & h_1 \geq 2\pi_1/k_1 > h_1 \left[1 - \left(\sqrt{h_1 g_2} \right)^{-1} \right], \\ \text{(b)} \quad & h_1 \left[1 - \left(\sqrt{h_1 g_2} \right)^{-1} \right] \geq 2\pi_1/k_1 > h_1 (1 - \varepsilon), \\ \text{(c)} \quad & h_1 (1 - \varepsilon) \geq 2\pi_1/k_1 > h_1/g_1, \\ \text{(d)} \quad & h_1/g_1 \geq 2\pi_1/k_1 \geq \frac{2\pi_\perp}{k_1}, \end{aligned} \quad (1.5.46)$$

where g_1 , g_2 , and ε are any given numbers satisfying the condition $g_1 \gg 1$, $g_2 \gg 1$, and $(\sqrt{h_1 g_2})^{-1} \ll \varepsilon \ll 1$. Note that $\tau_1 = ih_1/c_1 \approx \frac{h_1 k_1}{2|\pi_1|}$ in the subranges (a), (b), and (c) and $\tau_2 = -ih_2/(2 - c_2) \approx \frac{h_2 k_2}{2|\pi_2|}$ in the whole range (1.5.45).

In these subranges we have for $|\tau_2|$

$$\begin{aligned}
\text{(a)} \quad & 1 \leq \tau_2^{-1} < \left[1 + \left(\sqrt{h_2 g_2} \right)^{-1} \right], \\
\text{(b)} \quad & \left[1 + \left(\sqrt{h_2 g_2} \right)^{-1} \right] < \tau_2^{-1} < (1 + \varepsilon k_2/k_1), \\
\text{(c)} \quad & (1 + \varepsilon k_2/k_1) < \tau_2^{-1} < [1 + (1 - 1/g_1) k_2/k_1], \\
\text{(d)} \quad & [1 + (1 - 1/g_1) k_2/k_1] < \tau_2^{-1} \lesssim (1 + k_2/k_1).
\end{aligned} \tag{1.5.47}$$

We see that $\tau_1 - 1 \rightarrow 0$ and $\tau_2 - 1 \rightarrow 0$ in the range (a), while $|\tau_1 - 1| \sim 1$ in the range (c), and $|\tau_2 - 1| \sim 1$ in the ranges (c) and (d). In the range (b) these quantities vary from their values in the ranges (a) and (c). In the range (a) we can use the asymptotic expression of the confluent hypergeometric function given by Eq. (C.1) in Appendix 3. Using Eqs. (C.6) and (C.7) obtained in Appendix 3, we finally find the leading term as

$$N_n^{\text{cr}} = e^{-\pi\lambda} [1 + O(|\mathcal{Z}_1|)], \tag{1.5.48}$$

where $\max|\mathcal{Z}_1| \lesssim g_2^{-1}$.

In the range (c), we use the asymptotic expression of the confluent hypergeometric function given by Eq. (C.9) in Appendix 3. Then we find that

$$N_n^{\text{cr}} = e^{-\pi\lambda} [1 + O(|\mathcal{Z}_1|)^{-1} + O(|\mathcal{Z}_2|)^{-1}], \tag{1.5.49}$$

where $\max|\mathcal{Z}_1|^{-1} \lesssim \sqrt{g_1/h_1}$ and $\max|\mathcal{Z}_2|^{-1} \lesssim g_2^{-1}$.

Using the asymptotic expression Eq. (C.1) and taking into account Eq. (1.5.48) and (1.5.49), we can estimate that $N_n^{\text{cr}} \sim e^{-\pi\lambda}$ in the range (b). In the range (d), the confluent hypergeometric function $\Phi(1 - a_2, 2 - c_2; -ih_2)$ is approximated by Eq. (C.8) and the function $\Phi(a_1, c_1; ih_1)$ is approximated by Eq. (C.10) given in Appendix 2.4. In this range the differential mean numbers in the leading-order approximation are

$$N_n^{\text{cr}} \approx \sinh(2\pi p_L/k_1) \exp\left[-\frac{\pi}{k_1}(\pi_1 - p_L)\right] \sinh[\pi(\pi_1 + p_L)/k_1]^{-1}. \tag{1.5.50}$$

It is clear that N_n^{cr} given by Eqs. (1.5.50) tends to Eq. (1.5.49), $N_n^{\text{cr}} \rightarrow e^{-\pi\lambda}$, when $\pi_1 \gg \pi_\perp$. Consequently, the forms (1.5.50) are valid in the whole range (1.2.20). We find that significant value of N_n^{cr} is in the range $\pi_\perp < \pi_1 \leq eE/k_1$ and it has the form

$$N_n^{\text{cr}} \approx \exp\left[-\frac{2\pi}{k_1}(\pi_1 - p_L)\right]. \tag{1.5.51}$$

Considering positive $p_0 > 0$, we can take into account that exact N_n^{cr} is invariant under the simultaneous exchange $k_1 \leftrightarrow k_2$ and $\pi_1 \leftrightarrow -\pi_2$. In this case π_1 is positive and large, $\pi_1 > eE/k_1$, while π_2 is negative,

$$-h_2 \leq 2\pi_2/k_2 \leq -\frac{2\pi_\perp}{k_2}. \tag{1.5.52}$$

In this range, similarly to the case of the negative p_0 , the differential mean numbers in the leading-order approximation are

$$N_n^{\text{cr}} \approx \sinh(2\pi p_R/k_2) \exp\left[-\frac{\pi}{k_2}(|\pi_2| - p_R)\right] \sinh[\pi(p_R - \pi_2)/k_2]^{-1}. \tag{1.5.53}$$

Assuming $m/k_2 \gg 1$, we find that significant value of N_n^{cr} is in the range $-eE/k_2 < \pi_2 < -\pi_\perp$ and it has a form

$$N_n^{\text{cr}} \approx \exp \left[-\frac{2\pi}{k_2} (|\pi_2| - p_R) \right]. \quad (1.5.54)$$

Consequently, the quantity N_n^{cr} is almost constant over the wide range of energies p_0 for any given λ satisfying Eq. (1.5.42).

The analysis presented above reveals that the dominant contributions for particle creation by a slowly varying field occurs in the ranges of large kinetic momenta, whose differential quantities have the asymptotic forms (1.5.51) for $p_0 < 0$ and (1.5.54) for $p_0 > 0$. Therefore, one may represent the total number N^{cr} as

$$\begin{aligned} N^{\text{cr}} &= V_\perp T n^{\text{cr}}, \quad n^{\text{cr}} = \frac{J^{(d)}}{(2\pi)^{d-1}} \int_{\sqrt{\lambda} < K_\perp} dp_0 d\mathbf{p}_\perp I_{\mathbf{p}_\perp}, \quad I_{\mathbf{p}_\perp} = I_{\mathbf{p}_\perp}^{(1)} + I_{\mathbf{p}_\perp}^{(2)}, \\ I_{\mathbf{p}_\perp}^{(1)} &= \int_{-eE/k_1 + \pi_\perp}^0 dp_0 N_n^{\text{cr}} \approx \int_{\pi_\perp}^{eE/k_1} d\pi_1 \exp \left[-\frac{2\pi}{k_1} (\pi_1 - p_L) \right], \\ I_{\mathbf{p}_\perp}^{(2)} &= \int_0^{eE/k_2 - \pi_\perp} dp_0 N_n^{\text{cr}} \approx \int_{\pi_\perp}^{eE/k_2} d|\pi_2| \exp \left[-\frac{2\pi}{k_2} (|\pi_2| - p_R) \right]. \end{aligned} \quad (1.5.55)$$

Using the change of the variables

$$s = \frac{2}{k_1 \lambda} (\pi_1 - p_L),$$

and neglecting exponentially small contributions, we represent the quantity $I_{\mathbf{p}_\perp}^{(1)}$ as

$$I_{\mathbf{p}_\perp}^{(1)} \approx \int_1^\infty \frac{ds}{s} p_L e^{-\pi \lambda s}. \quad (1.5.56)$$

Similarly, using the change of variables

$$s = \frac{2}{k_2 \lambda} (|\pi_2| - p_R),$$

we represent the quantity $I_{\mathbf{p}_\perp}^{(2)}$ as

$$I_{\mathbf{p}_\perp}^{(2)} \approx \int_1^\infty \frac{ds}{s} p_R e^{-\pi \lambda s}. \quad (1.5.57)$$

The leading contributions for both integrals (1.5.56) and (1.5.57) come from the range near $s \rightarrow 1$, where p_L and p_R are approximately given by,

$$p_L \approx \frac{eE}{sk_1}, \quad p_R \approx \frac{eE}{sk_2}.$$

Consequently the leading term in $I_{\mathbf{p}_\perp}$ (1.5.55) takes the following final form:

$$I_{\mathbf{p}_\perp} \approx \left(\frac{eE}{k_1} + \frac{eE}{k_2} \right) \int_1^\infty \frac{ds}{s^2} e^{-\pi \lambda s} = eE \left(\frac{1}{k_1} + \frac{1}{k_2} \right) e^{-\pi \lambda} G(1, \pi \lambda), \quad (1.5.58)$$

where

$$G(\alpha, x) = \int_1^\infty \frac{ds}{s^{\alpha+1}} e^{-x(s-1)} = e^x x^\alpha \Gamma(-\alpha, x), \quad (1.5.59)$$

and $\Gamma(-\alpha, x)$ is the incomplete gamma function. Neglecting the exponentially small contribution, one can represent the integral over \mathbf{p}_\perp in Eq. (1.5.55) (where $I_{\mathbf{p}_\perp}$ is given by Eq. (1.5.58)) as

$$\int_{\sqrt{\lambda} < K_\perp} d\mathbf{p}_\perp I_{\mathbf{p}_\perp} \approx \int_{\sqrt{\lambda} < \infty} d\mathbf{p}_\perp I_{\mathbf{p}_\perp}.$$

Then calculating the Gaussian integral,

$$\int d\mathbf{p}_\perp \exp\left(-\pi s \frac{\mathbf{p}_\perp^2}{eE}\right) = \left(\frac{eE}{s}\right)^{d/2-1}, \quad (1.5.60)$$

we find

$$n^{\text{cr}} = r^{\text{cr}} \left(\frac{1}{k_1} + \frac{1}{k_2}\right) G\left(\frac{d}{2}, \pi \frac{m^2}{eE}\right), \quad r^{\text{cr}} = \frac{J_{(d)}(eE)^{d/2}}{(2\pi)^{d-1}} \exp\left\{-\pi \frac{m^2}{eE}\right\}. \quad (1.5.61)$$

1.5.6 Short pulse field

Choosing certain parameters of the peak field, one can obtain electric fields that exist only in a small area in a vicinity of the origin $x = 0$. The latter fields switch on and/or switch off ‘‘abruptly’’ near the point $x = 0$. Let us consider large parameters $k_1, k_2 \rightarrow \infty$ with a fixed ratio k_1/k_2 . The corresponding asymptotic potentials, $U(+\infty) = eEk_2^{-1}$ and $U(-\infty) = -eEk_1^{-1}$ define finite increments of the energy ΔU_1 and ΔU_2 for increasing and decreasing parts, respectively, (from Eq. (1.5.11))

$$\Delta U_1 = U(0) - U(-\infty) = eEk_1^{-1}, \quad \Delta U_2 = U(+\infty) - U(0) = eEk_2^{-1}. \quad (1.5.62)$$

Such a case corresponds to a very short pulse of the electric field. At the same time this configuration imitates the Klein step, which is an x -electric rectangular step (see Ref. [78]); and coincides with it as $k_1, k_2 \rightarrow \infty$. Thus, these field configurations can be considered as regularizations of rectangular step. We assume that sufficiently large k_1 and k_2 satisfy the following inequalities:

$$\Delta U_1/k_1 \ll 1, \quad \Delta U_2/k_2 \ll 1, \quad \max(p_L/k_1, p_R/k_2) \ll 1 \quad (1.5.63)$$

for any given π_\perp and $\pi_{1,2} = p_0 - U(\mp\infty)$. In this case the confluent hypergeometric function can be approximated by the first two terms in Eq. (1.5.23), which are $\Phi(a, c; \eta)$, $c_{1,2} \approx 1$, and $a_{1,2}$ are small values of order $(p_L - \pi_1)/k_1$ and p_R/k_2 , correspondingly. Then we obtain the result

$$N_n^{\text{cr}} = \frac{|\pi_1 + p^L| (\Delta U_1 + \Delta U_2 + p_R - p_L)^2}{4p_L p_R |\pi_2 - p^R|} \quad (1.5.64)$$

which does not, in fact, depend on $k_{1,2}$.

1.5.7 Unitarity in QED with x -electric potential steps

An important question is whether in- and out- spaces in the QED under consideration are unitarily equivalent? The answer is positive if the linear canonical transformation (1.5.1) (together with its adjoint transformation) is proper one. In the latter case there exists a unitary operator V , such that

$$\begin{aligned} V(a(\text{out}), a^\dagger(\text{out}), b(\text{out}), b^\dagger(\text{out})) V^\dagger &= (a(\text{in}), a^\dagger(\text{in}), b(\text{in}), b^\dagger(\text{in})), \\ |0, \text{in}\rangle &= V |0, \text{out}\rangle, \quad V^\dagger = V^{-1}. \end{aligned} \quad (1.5.65)$$

Let us denote all the out operators via α and all the in operators via β . Then the linear uniform canonical transformation between these operators can be written as (only Fermi case is considered here)

$$\beta = \Phi\alpha + \Psi\alpha^+, \quad \Phi\Phi^+ + \Psi\Psi^+ = 1, \quad \Phi\Psi^T + \Psi\Phi^T = 0. \quad (1.5.66)$$

According to Refs. [79, 80], transformation (1.5.66) is proper one if Ψ is a Hilbert-Schmidt operator, i.e., $\sum_{m,n} |\Psi_{mn}|^2 < \infty$. It is easily to see that Hilbert-Schmidt criterion for the transformation (1.5.1) reads

$$\sum_n \left[\left| \frac{w_n(+ - |0)}{w_n(+|+)} \right|^2 + \left| \frac{w_n(+ - |0)}{w_n(-|-)} \right|^2 \right] < \infty. \quad (1.5.67)$$

As it was shown in Ref. [76],

$$\left| \frac{w_n(+ - |0)}{w_n(+|+)} \right|^2 = N_n^a, \quad \left| \frac{w_n(+ - |0)}{w_n(-|-)} \right|^2 = N_n^b, \quad (1.5.68)$$

where N_n^a and N_n^b are differential mean numbers of electrons and positrons created from the vacuum by the potential step. Then, the left-hand side of Eq. (1.5.67) is the total number N of particles created from the vacuum, such that unitarity condition can be written as

$$\sum_n (N_n^a + N_n^b) = N < \infty. \quad (1.5.69)$$

Note that in- and out-spaces of the scalar QED in the presence of critical potential steps are unitarily equivalent under the same condition.

For realistic external field limited in space and time this condition is obviously satisfied.

Inequality (1.5.67) derived for QED with x -electric potential steps can be considered as one more confirmation of the consistency of the latter theory and correct interpretation of in and out particles there. One should note that qualitatively similar result was established in Ref. [13] for QED with time-dependent electric potential steps.

1.5.8 Deformation of initial vacuum state

In this section the deformation of initial vacuum state under the action of a x -electric potential step is studied. In the Heisenberg picture, the density operator of the system whose initial state is the vacuum, is given by equation

$$\hat{\rho} = |0, \text{in}\rangle\langle 0, \text{in}|. \quad (1.5.70)$$

The in and out Fock spaces are related by the unitary operator V , see (1.5.65). Then

$$\hat{\rho} = V|0, \text{out}\rangle\langle 0, \text{out}|V^\dagger. \quad (1.5.71)$$

In QED with x -electric potential steps the operator V was constructed in [76]. Since it can be factorized, the density operator (1.5.71) can be factorized as well,

$$\begin{aligned} V &= \prod_{i=1}^5 V^{(i)}, \quad |0, \text{in}\rangle^{(i)} = V^{(i)}|0, \text{out}\rangle^{(i)}, \\ \hat{\rho} &= \prod_{i=1}^5 V^{(i)}|0, \text{out}\rangle^{(i)}\langle 0, \text{out}|V^{(i)\dagger}. \end{aligned} \quad (1.5.72)$$

Due to the specific structure of the operator $V^{(i)}$, $i = 1, 2, 4, 5$, one has

$$V^{(i)}|0, \text{out}\rangle^{(i)} \langle 0, \text{out}|V^{(i)\dagger} = |0, \text{out}\rangle^{(i)} \langle 0, \text{out}| = |0, \text{in}\rangle^{(i)} \langle 0, \text{in}|, \quad i = 1, 2, 4, 5.$$

The latter relation has clear physical meaning – vacuum states in the ranges Ω_1 , Ω_2 , Ω_4 , and Ω_5 do not change with time. There is no particle creation there. Let us use the following notation

$$\begin{aligned} P' &= \prod_{i=1,2,4,5} |0, \text{out}\rangle^{(i)} \langle 0, \text{out}| = \prod_{i=1,2,4,5} |0, \text{in}\rangle^{(i)} \langle 0, \text{in}|, \\ \hat{\rho}_K &= V^{(K)} P_K V^{(K)\dagger}, \quad P_K = |0, \text{out}\rangle^{(K)} \langle 0, \text{out}|, \end{aligned} \quad (1.5.73)$$

then

$$\hat{\rho} = P' \hat{\rho}_K. \quad (1.5.74)$$

Using the following explicit form of the operator $V^{(K)} = V^{(3)}$ derived in Ref. [76],

$$\begin{aligned} V^{(K)} &= \exp \left[- \sum_{n \in \Omega_K} {}^+ a_n^\dagger(\text{out}) w_n (+ - | 0) {}^+ b_n^\dagger(\text{out}) \right] \\ &\times \exp \left[- \sum_{n \in \Omega_K} {}^+ b_n(\text{out}) \ln w_n (- | -) {}^+ b_n^\dagger(\text{out}) \right] \\ &\times \exp \left[\sum_{n \in \Omega_K} {}^+ a_n^\dagger(\text{out}) \ln w_n (+ | +) {}^+ a_n(\text{out}) \right] \\ &\times \exp \left[- \sum_{n \in \Omega_K} {}^+ b_n(\text{out}) w_n (0 | - +) {}^+ a_n(\text{out}) \right], \end{aligned}$$

one can derive two alternative expressions for the density operator $\hat{\rho}_K$.

The first one is a normal form exponential with respect to the out operators (denoted by $: \dots :$),

$$\begin{aligned} \hat{\rho}_K |c_v|^{-2} &= : \exp \left\{ - \sum_{n \in \Omega_K} [{}^+ a_n^\dagger(\text{out}) {}^+ a_n(\text{out}) + {}^+ b_n^\dagger(\text{out}) {}^+ b_n(\text{out}) \right. \\ &+ \left. {}^+ a_n^\dagger(\text{out}) w_n (+ - | 0) {}^+ b_n^\dagger(\text{out}) + {}^+ b_n(\text{out}) w_n (+ - | 0)^* {}^+ a_n(\text{out}) \right\} :. \end{aligned} \quad (1.5.75)$$

Representation (1.5.75) can be derived in the following way: Using (1.5.73) and the explicit form of $V^{(K)}$, one can write

$$\begin{aligned} \hat{\rho}_K |c_v|^{-2} &= \exp \left[- \sum_{n \in \Omega_K} {}^+ a_n^\dagger(\text{out}) w_n (+ - | 0) {}^+ b_n^\dagger(\text{out}) \right] \\ P_K \exp \left[- \sum_{n \in \Omega_K} {}^+ b_n(\text{out}) w_n (+ - | 0)^* {}^+ a_n(\text{out}) \right]. \end{aligned} \quad (1.5.76)$$

Making use of well-known Berezin representation [79] for a projection operator P_K on the vacuum state,

$$P_K = : \exp \left\{ - \sum_{n \in \Omega_K} [{}^+ a_n^\dagger(\text{out}) {}^+ a_n(\text{out}) + {}^+ b_n^\dagger(\text{out}) {}^+ b_n(\text{out}) \right\} : \quad (1.5.77)$$

and taking into account that the left and the right exponents in Eq. (1.5.76) are already normal ordered, one easily obtains representation (1.5.75).

The second representation reads:

$$\begin{aligned} \hat{\rho}_K |c_v|^{-2} &= \prod_{n \in \Omega_K} [1 - {}^+ a_n^\dagger(\text{out}) w_n (+ - |0) {}^+ b_n^\dagger(\text{out})] \\ &\times P_{K,n} [1 - {}^+ b_n(\text{out}) w_n (+ - |0)^* {}^+ a_n(\text{out})], \\ P_{K,n} &= |0, \text{out}\rangle_n^{(K)} \langle 0, \text{out}|_n^{(K)}. \end{aligned} \quad (1.5.78)$$

Representation (1.5.78) can be derived as follows. Using the fact that operators with different quantum numbers n commute, and using the relation, see, e.g., Ref. [25],

$$\exp [a^\dagger D a] = : \exp [a^\dagger (e^D - 1) a] : , \quad (1.5.79)$$

to transform exponents from $V^{(K)}$, one expands then the obtained expressions in power series. Since the out operators in $V^{(K)}$ are Fermi type, these series are reduced to finite term expressions. Their actions on the vacuum $|0, \text{out}\rangle^{(K)}$ can be easily calculated, and one arrives at Eq. (1.5.78).

Finally, consider the structure of the $|0, \text{in}\rangle$ state in terms of out operators. First of all, one has to use the fact that the state vector under discussion is factorized,

$$\begin{aligned} |0, \text{in}\rangle &= V |0, \text{out}\rangle = |0, \text{in}\rangle' |0, \text{in}\rangle^{(K)}, \\ |0, \text{in}\rangle' &= \prod_{i=1,2,4,5} |0, \text{in}\rangle^{(i)}, \quad |0, \text{in}\rangle^{(K)} = V^{(K)} |0, \text{out}\rangle^{(K)}. \end{aligned} \quad (1.5.80)$$

Then using the explicit form $V^{(K)}$, one obtains

$$|0, \text{in}\rangle^{(K)} = c_v \prod_{n \in \Omega_K} [1 - {}^+ a_n^\dagger(\text{out}) w_n (+ - |0) {}^+ b_n^\dagger(\text{out})] |0, \text{out}\rangle^{(K)}. \quad (1.5.81)$$

In each fixed mode $n \in \Omega_K$, the state vector $|0, \text{in}\rangle$ is a linear superposition of two terms – the vacuum vector in this mode and a state with an electron-positron pair.

1.5.9 Reductions to electron and positron subsystems

It should be stressed that the system under consideration can be considered as a composed from a subsystem of electrons and a subsystem of positrons. One can introduce the so-called two reduced density operators: $\hat{\rho}_+$ of the electron subsystem and $\hat{\rho}_-$ of the positron subsystem, averaging complete density operator (1.5.70) over all possible positron states or over all possible electron states, respectively,

$$\begin{aligned} \hat{\rho}_+ &= \text{tr}_- \hat{\rho} = \sum_{i=3}^5 \sum_M \sum_{\{m\} \in \Omega_i} \sum_b^{(i)} \langle M, \text{out} | \hat{\rho} | M, \text{out} \rangle_b^{(i)}, \\ \hat{\rho}_- &= \text{tr}_+ \hat{\rho} = \sum_{i=1}^3 \sum_M \sum_{\{m\} \in \Omega_i} \sum_a^{(i)} \langle M, \text{out} | \hat{\rho} | M, \text{out} \rangle_a^{(i)}, \\ |M, \text{out}\rangle_b^{(i)} &= (M!)^{-1/2} b_{m_1}^\dagger(\text{out}) \dots b_{m_M}^\dagger(\text{out}) |0, \text{out}\rangle_b^{(i)}, \\ |M, \text{out}\rangle_a^{(i)} &= (M!)^{-1/2} a_{m_1}^\dagger(\text{out}) \dots a_{m_M}^\dagger(\text{out}) |0, \text{out}\rangle_a^{(i)}. \end{aligned} \quad (1.5.82)$$

Vectors $|0, \text{out}\rangle_a^{(i)}$ and $|0, \text{out}\rangle_b^{(i)}$ are the electron and positron vacua in the Ω_i range, defined by

$$a_n^{(i)}(\text{out})|0, \text{out}\rangle_a^{(i)} = 0, \quad b_n^{(i)}(\text{out})|0, \text{out}\rangle_b^{(i)} = 0, \quad (1.5.83)$$

where $a_n^{(i)}(\text{out})$ and $b_n^{(i)}(\text{out})$ are corresponding annihilation operators of electrons and positrons in this range, respectively. Of course, these electron and positron vacua can be factorized in quantum modes, as was mentioned already above. One can see that

$$\begin{aligned} |0, \text{out}\rangle^{(1,2)} &= |0, \text{out}\rangle_a^{(1,2)} = \prod_{n \in \Omega_{1,2}} |0, \text{out}\rangle_{n,a}^{(1,2)}, \\ |0, \text{out}\rangle^{(4,5)} &= |0, \text{out}\rangle_b^{(4,5)} = \prod_{n \in \Omega_{4,5}} |0, \text{out}\rangle_{n,b}^{(4,5)}, \\ |0, \text{out}\rangle^{(3)} &= |0, \text{out}\rangle^{(K)} = |0, \text{out}\rangle_a^{(K)} \otimes |0, \text{out}\rangle_b^{(K)}, \\ |0, \text{out}\rangle_a^{(K)} &= \prod_{n \in \Omega_K} |0, \text{out}\rangle_{n,a}^{(K)}, \quad |0, \text{out}\rangle_b^{(K)} = \prod_{n \in \Omega_K} |0, \text{out}\rangle_{n,b}^{(K)}. \end{aligned} \quad (1.5.84)$$

Using Eq. (1.5.74) and representation (1.5.78) for $\hat{\rho}_K$, it is easy to calculate traces in Eqs. (1.5.82), and to obtain thus explicit forms of the reduced operators $\hat{\rho}_\pm$:

$$\begin{aligned} \hat{\rho}_+ |c_v|^{-2} &= \prod_{i=1,2} |0, \text{out}\rangle^{(i)} \langle 0, \text{out}| \\ &\otimes \prod_{n \in \Omega_K} [P_{K,a,n} + |w_n (+ - |0)\rangle|^{2+} a_n^\dagger(\text{out}) P_{K,a,n} + a_n(\text{out})], \\ \hat{\rho}_- |c_v|^{-2} &= \prod_{i=4,5} |0, \text{out}\rangle^{(i)} \langle 0, \text{out}| \\ &\otimes \prod_{n \in \Omega_K} [P_{K,b,n} + |w_n (+ - |0)\rangle|^2 + b_n^\dagger(\text{out}) P_{K,b,n} + b_n(\text{out})], \\ P_{K,a,n} &= |0, \text{out}\rangle_{n,a}^{(K)} \langle 0, \text{out}|, \quad P_{K,b,n} = |0, \text{out}\rangle_{n,b}^{(K)} \langle 0, \text{out}|. \end{aligned} \quad (1.5.85)$$

One can also consider a reduction of density operator (1.5.74), which occurs due to measurement of a physical quantity by some classical tool, or, in other words, due to decoherence. Suppose that the number of particles $N(\text{out})$ in the state $\hat{\rho}$ of the system under consideration is measured. The operator corresponding to this physical quantity is $\hat{N}(\text{out}) = \sum_{i=1}^5 \hat{N}_i(\text{out})$, where

$$\begin{aligned} \hat{N}_1(\text{out}) &= \sum_{n \in \Omega_1} [+ a_n^\dagger(\text{out}) + a_n(\text{out}) + - a_n^\dagger(\text{out}) - a_n(\text{out})], \\ \hat{N}_2(\text{out}) &= \sum_{n \in \Omega_2} a_n^\dagger a_n, \quad \hat{N}_4(\text{out}) = \sum_{n \in \Omega_4} b_n^\dagger b_n, \\ \hat{N}_3(\text{out}) &= \sum_{n \in \Omega_K} [+ a_n^\dagger(\text{out}) + a_n^\dagger(\text{out}) + + b_n^\dagger(\text{out}) + b_n(\text{out})], \\ \hat{N}_5(\text{out}) &= \sum_{n \in \Omega_5} [+ b_n^\dagger(\text{out}) + b_n(\text{out}) + - b_n^\dagger(\text{out}) - b_n(\text{out})]. \end{aligned} \quad (1.5.86)$$

According to von Neumann [70], the density operator $\hat{\rho}$ after such a measurement is reduced to the operator $\hat{\rho}_N$ of a form

$$\hat{\rho}_N = \sum_s \langle s, \text{out} | \hat{\rho} | s, \text{out} \rangle \hat{P}_s, \quad \hat{P}_s = |s, \text{out}\rangle \langle s, \text{out}|, \quad (1.5.87)$$

where $|s, \text{out}\rangle$ are eigenstates of the operator $\hat{N}(\text{out})$ with the eigenvalues s that represent the total number of electrons and positrons in the state $|s, \text{out}\rangle$,

$$\begin{aligned} \hat{N}(\text{out})|s, \text{out}\rangle &= s|s, \text{out}\rangle, \\ |s, \text{out}\rangle &= \prod_{n \in \Omega_1} [{}^+a_n^\dagger(\text{out})]^{l_{n,1}} [{}^-a_n^\dagger(\text{out})]^{k_{n,1}} \prod_{n \in \Omega_2} (a_n^\dagger)^{l_{n,2}} \prod_{n \in \Omega_4} (b_n^\dagger)^{l_{n,4}} \\ &\times \prod_{n \in \Omega_5} [{}^+b_n^\dagger(\text{out})]^{l_{n,5}} [{}^-b_n^\dagger(\text{out})]^{k_{n,5}} \prod_{n \in \Omega_K} [{}^+a_n^\dagger(\text{out})]^{l_{n,3}} [{}^+b_n^\dagger(\text{out})]^{k_{n,3}} |0, \text{out}\rangle, \\ s &= \sum_{n \in \Omega_1} (l_{n,1} + k_{n,1}) + \sum_{n \in \Omega_2} (l_{n,2}) + \sum_{n \in \Omega_4} (l_{n,4}) + \sum_{n \in \Omega_5} (l_{n,5} + k_{n,5}) + \sum_{n \in \Omega_K} (l_{n,3} + k_{n,3}). \end{aligned}$$

Note that $l_{n,i}, k_{n,i} = (0, 1)$, due to the fact that here one deals with fermions.

Due to the structure of the operator $\hat{\rho}$, the weights $\langle s, \text{out} | \hat{\rho} | s, \text{out} \rangle$ are nonzero only for pure states $|s, \text{out}\rangle$ with an integer number of pairs in Ω_K (since the initial state of the system was a vacuum, and there is no particle creation outside of the Klein zone). Thus, the operator $\hat{\rho}_N$ takes the form

$$\hat{\rho}_N |c_v|^{-2} = P' \prod_{n \in \Omega_K} [P_{K,n} + |w_n(+ - |0)|^2 {}^+a_n^\dagger(\text{out}) {}^+b_n^\dagger(\text{out}) P_{K,n} + b_n(\text{out}) {}^+a_n(\text{out})], \quad (1.5.88)$$

where operators $P_{K,n}$ and P' were defined in the previous section, see Eq. (1.5.78). Note that the measurement destroys nondiagonal terms of the density operator (1.5.78).

Since the operator V is unitary and the initial state of the system under consideration is a pure state (the vacuum state) the density operator (1.5.74) describes a pure state as well. Therefore its von Neumann entropy is zero. However, the reduced density operators $\hat{\rho}_\pm$ (1.2.17) describe already mixed states and their entropies $S(\hat{\rho}_\pm)$ are not zero,

$$S(\hat{\rho}_\pm) = -k_B \text{tr} \hat{\rho}_\pm \ln \hat{\rho}_\pm. \quad (1.5.89)$$

It is known that this entropy can be treated as a measure of the quantum entanglement of the electron and positron subsystems and can be treated as the measure of the information loss.

Using the normalization condition for the reduced density operators, $\text{tr} \hat{\rho}_\pm = 1$, the relation (1.5.79), definitions for differential mean numbers of particles N_n^a and antiparticles N_n^b created from vacuum

$$N_n^a = \text{tr} \hat{\rho}_+ a_n^\dagger(\text{out}) a_n(\text{out}), \quad N_n^b = \text{tr} \hat{\rho}_- b_n^\dagger(\text{out}) b_n(\text{out}), \quad (1.5.90)$$

and the fact that

$$N_n^a = N_n^b = N_n^{\text{cr}}, \quad |w_n(+ - |0)|^2 = N_n^{\text{cr}} (1 - N_n^{\text{cr}})^{-1}, \quad (1.5.91)$$

one can calculate traces in Eqs. (1.5.89) and rewrite RHS in these equations as

$$S(\hat{\rho}_\pm) = \sum_{n \in \Omega_K} S_n, \quad S_n = -k_B [(1 - N_n^{\text{cr}}) \ln (1 - N_n^{\text{cr}}) + N_n^{\text{cr}} \ln N_n^{\text{cr}}]. \quad (1.5.92)$$

The von Neumann-reduced density operator (1.5.88) also describes mixed state; making use of the fact that the pure states $|0, \text{out}\rangle_n^{(K)}$ and ${}^+a_n^\dagger(\text{out}) {}^+b_n^\dagger(\text{out}) |0, \text{out}\rangle_n^{(K)}$ are orthogonal and normalized, it is not difficult to show that the von Neumann entropy $S(\hat{\rho}_N)$ of the mixed state (1.5.88) coincides with the entropies $S(\hat{\rho}_\pm)$ of the reduced density operators $\hat{\rho}_\pm$.

The differential mean number of fermions created N_n^{cr} can vary only within the range $(0, 1)$. The partial entropy S_n for given n in Eq. (1.5.92) is symmetric with respect to value of N_n^{cr} . It reaches

maximum at $N_n^{\text{cr}} = 1/2$ and turns to zero at $N_n^{\text{cr}} = 1$ and $N_n^{\text{cr}} = 0$. This fact can be interpreted as follows. In the case of $N_n^{\text{cr}} = 0$ there are no particles created by the external field and the initial vacuum state in the mode remains unchanged. The case $N_n^{\text{cr}} = 1$ corresponds to the situation when a particle is created with certainty. The maximum of S_n , corresponding to $N_n^{\text{cr}} = 1/2$, is associated with the state with the maximum amount of uncertainty.

1.5.10 Deformation of the quantum vacuum between two capacitor plates

It is possible to illustrate the general consideration considering the deformation of the quantum vacuum between two infinite capacitor plates separated by a finite distance L . Some aspects of particle creation by the constant electric field between such plates (this field is also called L -constant electric field) were studied in Ref. [78]. The latter field is a particular case of x -electric potential step. Thus, let's consider the L -constant electric field in $d = D + 1$ dimensions. The field $\mathbf{E}(x) = (E^i, i = 1, \dots, D)$, $E^1 = E_x(x)$, $E^{2, \dots, D} = 0$ is chosen,

$$E_x(x) = \begin{cases} 0, & x \in (-\infty, -L/2] \\ E = \text{const} > 0, & x \in (-L/2, L/2) \\ 0, & x \in [L/2, \infty) \end{cases} .$$

The potential energy of an electron in the L -electric field under consideration is

$$U(x) = \begin{cases} U_L = -eEL/2, & x \in (-\infty, -L/2] \\ eEx, & x \in (-L/2, L/2) \\ U_R = eEL/2, & x \in [L/2, \infty) \end{cases} . \quad (1.5.93)$$

The magnitude of the corresponding x -electric is $\mathbb{U} = eEL$. The most interesting are the critical steps, for which

$$\mathbb{U} = eEL > 2m \quad (1.5.94)$$

and the vacuum is unstable in the Klein zone.

Consider a particular case with a sufficiently large length L between the capacitor plates,

$$\sqrt{eEL} \gg \max\{1, E_c/E\} . \quad (1.5.95)$$

Here $E_c = m^2/e$ is the critical Schwinger field. In what follows it will be conditionally called as large work approximation. Such kind of x -electric step represents a regularization for a constant uniform electric field and is suitable for imitating a small-gradient field.

It was shown in Ref. [78] that the main particle production occurs in an inner subrange $\tilde{\Omega}_K$ of the Klein zone, $\tilde{\Omega}_K \subset \Omega_K$,

$$\begin{aligned} \tilde{\Omega}_K : & |p_0|/\sqrt{eE} < \sqrt{eEL}/2 - K, \quad \lambda < K_\perp^2, \\ \lambda = & \frac{\mathbf{p}_\perp^2 + m^2}{eE}, \quad \sqrt{eEL} \gg K \gg K_\perp^2 \gg \max\{1, E_c/E\}. \end{aligned} \quad (1.5.96)$$

where K and K_\perp are any given positive numbers satisfying the condition (1.5.96).

The differential number of particles with quantum numbers $n \in \tilde{\Omega}_K$ created from the vacuum reads

$$\begin{aligned} N_n^{\text{cr}} &= e^{-\pi\lambda} [1 + O(|\xi_1|^{-3}) + O(|\xi_2|^{-3})], \\ \xi_1 &= \frac{-eEL/2 - p_0}{\sqrt{eE}}, \quad \xi_2 = \frac{eEL/2 - p_0}{\sqrt{eE}}. \end{aligned} \quad (1.5.97)$$

Recall that, in fact, the quantum numbers n that label electron and positron states in general formulas gather several quantum numbers,

$$n = (p_0, \mathbf{p}_\perp, \sigma), \quad \mathbf{p}_\perp = (p_2, \dots, p_D), \quad (1.5.98)$$

where for an electron p_0 is its energy, for a positron $-p_0$ is its energy, and for an electron \mathbf{p}_\perp denotes its transversal components of the momentum, whereas for a positron $-\mathbf{p}_\perp$ denotes its transversal components of the momentum. For an electron σ is its spin polarization and for a positron $-\sigma$ is its spin polarization. Note that the electron and positron in a pair created by an external field have the same quantum numbers n .

The quantity (1.5.97) is almost constant over a wide range of energy p_0 for any given $\lambda < K_\perp^2$, for these quantum numbers one can assume $N_n^{\text{cr}} \approx e^{-\pi\lambda}$. In the limiting case of the large work approximation, $\sqrt{eEL} \rightarrow \infty$, one obtains the well-known result for particle creation by a constant uniform electric field $N_n^{\text{cr}} = e^{-\pi\lambda}$, see Refs. [4, 6, 40].

In the approximation under the consideration, the total number of particles created from the vacuum is given by a sum (integral) over $n \in \tilde{\Omega}_K$,

$$N^{\text{cr}} = \sum_{n \in \Omega_K} N_n^{\text{cr}} \approx \sum_{\mathbf{p}_\perp, p_0 \in \tilde{\Omega}_K} \sum_{\sigma} N_n^{\text{cr}} = \frac{J_{(d)} TV_\perp}{(2\pi)^{d-1}} \int_{\tilde{\Omega}_K} dp_0 d\mathbf{p}_\perp N_n^{\text{cr}}. \quad (1.5.99)$$

where $J_{(d)} = 2^{[d/2]-1}$ is a spin summation factor, V_\perp is the $(d-2)$ -dimensional spatial volume in hypersurface orthogonal to the electric field direction and T is the time duration of the electric field. The integration over p_0 results in

$$N^{\text{cr}} = \frac{J_{(d)} TV_\perp LeE}{(2\pi)^{d-1}} \int_{\tilde{\Omega}_K} d\mathbf{p}_\perp e^{-\pi\lambda}. \quad (1.5.100)$$

Integrating Eq. (1.5.100) over p_\perp , one obtains that the total number of created from the vacuum particles in the large work approximation has the form

$$N^{\text{cr}} = \frac{J_{(d)} TV (eE)^{d/2}}{(2\pi)^{d-1}} \exp\left(-\pi \frac{E_c}{E}\right), \quad (1.5.101)$$

where $V = LV_\perp$ is the volume inside of the capacitor (the volume occupied by the electric field).

It is obvious that $N^{\text{cr}} < \infty$, when the values V and T are finite or, in other words, when regularization of the finite volume and finite time of the field action is used. Looking on the condition (1.5.69), it can be seen that the x -electric potential step, which represents the electric field inside of the capacitor, does not violate the unitarity in QED.

Let us estimate the information loss of the reduced states of the deformed vacuum, which can be calculated as entropies (1.5.92) of these states,. Using the same summation rule as in (1.5.99), one can write

$$S(\hat{\rho}_\pm) = -k_B \frac{J_{(d)} TV_\perp}{(2\pi)^{d-1}} \int_{\Omega_K} dp_0 d\mathbf{p}_\perp [N_n^{\text{cr}} \ln N_n^{\text{cr}} + (1 - N_n^{\text{cr}}) \ln(1 - N_n^{\text{cr}})]. \quad (1.5.102)$$

For Fermi particles under the consideration, $N_n^{\text{cr}} \leq 1$. This allows us to expand the logarithm in the rhs of Eq. (1.5.102) in powers of N_n^{cr} . Thus, the term $(1 - N_n^{\text{cr}}) \ln(1 - N_n^{\text{cr}})$ is represented as follows:

$$(1 - N_n^{\text{cr}}) \ln(1 - N_n^{\text{cr}}) = - (1 - N_n^{\text{cr}}) \sum_{l=1}^{\infty} l^{-1} (N_n^{\text{cr}})^l. \quad (1.5.103)$$

Using (1.5.103) in Eq. (1.5.102), one obtains the following intermediate result

$$S(\hat{\rho}_{\pm}) = k_B \frac{J_{(d)} TV_{\perp}}{(2\pi)^{d-1}} \int_{\Omega_K} dp_0 d\mathbf{p}_{\perp} \left[-N_n^{\text{cr}} \ln N_n^{\text{cr}} + (1 - N_n^{\text{cr}}) \sum_{l=1}^{\infty} l^{-1} (N_n^{\text{cr}})^l \right]. \quad (1.5.104)$$

As it has been mentioned before, the considerable amount of particles is created only in the subrange $\tilde{\Omega}_K \in \Omega_K$, where terms proportional to $|\xi_{1,2}|^{-3}$ are small and can be neglected, allowing to use the leading-order approximation $N_n^{\text{cr}} \approx e^{-\pi\lambda}$ in the rhs of Eq. (1.5.104). Then,

$$\begin{aligned} S(\hat{\rho}_{\pm}) &\approx k_B \frac{J_{(d)} TV eE}{(2\pi)^{d-1}} \int_{\tilde{\Omega}_K} d\mathbf{p}_{\perp} \left[\pi\lambda e^{-\pi\lambda} + (1 - e^{-\pi\lambda}) \sum_{l=1}^{\infty} l^{-1} e^{-\pi\lambda l} \right] \quad \text{if } d > 2; \\ S(\hat{\rho}_{\pm}) &\approx k_B \frac{TV eE}{2\pi} A(2, E_c/E) \quad \text{if } d = 2, \\ A(2, E_c/E) &= \{ \pi E_c/E \exp(-\pi E_c/E) - [1 - \exp(-\pi E_c/E)] \ln [1 - \exp(-\pi E_c/E)] \}. \end{aligned} \quad (1.5.105)$$

In the dimensions $d > 2$ the integration over the transversal components of the momentum can be easily performed. Outside of the subrange $\tilde{\Omega}_K$, the integrand is very small, so that one can extend the integration limits of p_{\perp} to the infinity. Thus,

$$S(\hat{\rho}_{\pm}) \approx k_B \frac{J_{(d)} TV (eE)^{d/2}}{(2\pi)^{d-1}} A(d, E_c/E) \quad \text{if } d > 2, \quad (1.5.106)$$

where the factor $A(d, E_c/E)$ has the form

$$\begin{aligned} A(d, E_c/E) &= (\pi E_c/E + d/2 - 1) \exp(-\pi E_c/E) \\ &+ \sum_{l=1}^{\infty} [l^{-d/2} - l^{-1}(l+1)^{(2-d)/2} \exp(-\pi E_c/E)] \exp(-\pi l E_c/E). \end{aligned} \quad (1.5.107)$$

For example, estimations of this factor for strong field $E_c/E \ll 1$ and critical field $E_c/E = 1$ with $d = 4, 3$ are $A(4, 0) = \pi^2/6$, $A(4, 1) \approx 0, 22$; $A(3, 0) \approx 0, 93$, and $A(3, 1) \approx 0, 20$. In the case of a weak field, $E_c/E \gg 1$, the entropy is exponentially small for any d ,

$$A(d, E_c/E) \approx (\pi E_c/E + d/2) \exp(-\pi E_c/E).$$

One can note, that the large work approximation (1.5.106) obtained for $S(\hat{\rho}_{\pm})$ in the case of the x -electric step under consideration coincides with the same approximation for $S(\hat{\rho}_{\pm})$ in the case of the t -electric step with an uniform electric field that is acting during a finite time interval T (the so called T -constant field) obtained in Ref. [26]. This observation confirms the fact that the T -constant and L -constant fields produce equal physical effects in the large work approximation (or as $T \rightarrow \infty$ and $L \rightarrow \infty$), such that it is possible to consider these fields as regularizations of a constant uniform electric field given by two distinct gauge conditions for electromagnetic potentials. Obviously, exact expressions for the entropies $S(\hat{\rho}_{\pm})$ differ in the general case.

Moving charge as a soliton

The problem of a moving point-like charge is one of the fundamental problems of classical electrodynamics. It is well known that Maxwell electrodynamics when applied to point-like objects is inconsistent: for example, field mass of a point-like charge in linear theory is infinite. Many nonlinear models, in particular, Born-Infeld model[52] resolve this problem; however, due to the nonlinearity it is very difficult to solve corresponding field equations (even in the absence of a charged matter). Minimally nonlinear model used in Ref. [54], on the other hand, is exactly solvable. The convergence of the field-mass integral, provided by this model, allows one to construct a soliton-like representation for the moving charge.

2.1 Minimally nonlinear model

The Lagrangian of a minimally nonlinear electrodynamics is defined as

$$L(x) = -\mathfrak{F}(x) + \frac{\sigma}{2} (\mathfrak{F}(x))^2, \quad (2.1.1)$$

where $\mathfrak{F}(x)$ makes up (with the reversed sign) the Lagrangian density of the standard linear Maxwell electrodynamics, while the second term in (2.1.1) is the quartic in the field-strength addition to it. The field-strength tensor is related to the four-vector potential $A^\mu(x)$ as $F^{\mu\nu} = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$, where $\partial^\mu = \frac{\partial}{\partial x_\mu}$. Hence, the first pair of the Maxwell equations, $[\nabla \times \mathbf{B}] = 0$ and $[\nabla \times \mathbf{E}] + \partial^0 \mathbf{B} = \mathbf{0}$, with the electric and magnetic field strengths $E_i = F^{i0}$ and $B_i = \epsilon_{ijk} F^{jk}$, remains standard. The self-coupling constant σ is presumably small enough. It has the dimensionality of inverse fourth power of mass². The four-dimensional scalar product is $(ux) = u_0 x_0 - (\mathbf{u} \cdot \mathbf{x})$, $x^2 = x_0^2 - |\mathbf{x}|^2$. The Lorentz–Heaviside system of units will be used in this section. If the quartic correction in (2.1.1) is understood as the lowest term in the expansion of the Heisenberg–Euler Lagrangian in powers of \mathfrak{F} , with $\mathfrak{G} = 0$, the value [85] of the photon selfcoupling constant is $\sigma = e^4 / (45\pi^2 m^4)$, where e and m are electron charge and mass, respectively.

The action correspondent to the Lagrangian (2.1.1) includes the selfinteraction of the electromagnetic field and interaction with a current j^ν , produced by a moving charge. It is

$$S[A] = -\frac{1}{c} \int \left(\frac{1}{c} A_\mu j^\mu - L(x) \right) d^4x.$$

The second pair of Maxwell equations should be calculated in assumption that the trajectory of the charge is fixed, and that its variation, and, thereby, the variation of the current, are zero. Thus, the

²Here and in what follows we use standard convention about summation over repeated indices; Roman letters i, j, k run from 1 to 3, while Greek letters μ, ν, λ run from 0 to 3. The three-dimensional vectors are boldfaced. Their scalar and vector products are defined, respectively, as $(\mathbf{D} \cdot \mathbf{C}) = D_i C_i$, $[\mathbf{D} \times \mathbf{C}]_i = \epsilon_{ijk} D_j C_k$.

variation of the action takes the form

$$\begin{aligned}\delta S &= -\frac{1}{c} \int \left[\frac{1}{c} j^\mu \delta A_\mu + (1 - \sigma \mathfrak{F}(x)) \delta \mathfrak{F}(x) \right] d\Omega, \\ \delta \mathfrak{F}(x) &= \frac{1}{2} F^{\mu\nu} \delta F_{\mu\nu} = -\frac{1}{2} F^{\mu\nu} \left(\frac{\delta \partial A_\mu}{\partial x^\nu} - \frac{\delta \partial A_\nu}{\partial x^\mu} \right) = -F^{\mu\nu} \frac{\partial}{\partial x^\nu} \delta A_\mu.\end{aligned}\quad (2.1.2)$$

After integrating by parts and taking into account that there is no field at infinities the latter equation takes the form

$$\delta S = -\frac{1}{c} \int \left[\frac{1}{c} j^\mu + \frac{\partial}{\partial x^\nu} [(1 - \sigma \mathfrak{F}(x)) F^{\mu\nu}] \right] \delta A_\mu d\Omega \quad (2.1.3)$$

As always, variations should vanish for any δA_μ , and one comes to the nonlinear field equation

$$\frac{\partial}{\partial x^\nu} [(1 - \sigma \mathfrak{F}(x)) F^{\mu\nu}] = -\frac{1}{c} j^\mu. \quad (2.1.4)$$

2.2 Fields of a moving charge

2.2.1 Current

Here and in what follows only the currents j^μ that correspond to a pointlike or any other charge, which is at rest in the origin in a certain Lorentz reference frame (the rest frame) and is distributed in a spherically symmetric and time-independent way in that frame, will be used. The charge moves as a whole with the speed $-\mathbf{v}$, $v < 1$, along axis 1, $v_i = \delta_{i1}v$, in the inertial frame that moves relative to the rest frame with the 4-velocity u_μ , which is the unit four-vector, $u^2 = 1$, with the components $u_0 = \gamma$, $u_i = \gamma \frac{v_i}{c}$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. The four-current is

$$j_\mu = u_\mu \rho(x), \quad (2.2.1)$$

where $\rho(x)$ is the Lorentz scalar defined as the time-independent charge density in its rest frame. This implies that under Lorentz boosts and spacial rotations only the argument of ρ is transformed.

To make sure of the validity of the four-vector representation (2.2.1) let us first define the current in the rest frame as $j'_0(x') = \rho(\mathbf{x}')$, $\mathbf{j}'(x') = 0$. All quantities relating to that frame are marked with primes throughout. Applying the Lorentz transformation to this current

$$j_0 = \gamma \left(j'_0 + \frac{v}{c} j'_1 \right), \quad j_1 = \gamma \left(j'_1 + \frac{v}{c} j'_0 \right), \quad j_2 = j'_2, \quad j_3 = j'_3, \quad (2.2.2)$$

with the account of the vanishing of its spatial component in the rest frame $j'_1 = 0$, and applying the inverse transformation to the coordinates

$$x'_0 = \gamma \left(x_0 - \frac{v}{c} x_1 \right), \quad x'_1 = \gamma \left(x_1 - \frac{v}{c} x_0 \right), \quad x'_2 = x_2, \quad x'_3 = x_3. \quad (2.2.3)$$

one gets for the moving charge (2.2.2) $j^0(x) = \gamma \rho((x_1 - \frac{v}{c} x_0) \gamma, x_2, x_3)$, $\mathbf{j}(x) = \frac{\mathbf{v}}{c} \gamma \rho((x_1 - \frac{v}{c} x_0) \gamma, x_2, x_3)$. This agrees with Eq. (2.2.1). For the pointlike charge e one has

$$\begin{aligned}j'^0(x') &= \rho(\mathbf{x}') = e \delta^3(\mathbf{x}'), & \mathbf{j}'(x') &= 0 \\ j^0(x) &= \gamma e \delta \left(\left(x_1 - \frac{v}{c} x_0 \right) \gamma \right) \delta(x_2) \delta(x_3) = e \delta \left(x_1 - \frac{v}{c} x_0 \right) \delta(x_2) \delta(x_3) = \gamma e \delta^3(\mathbf{x}'). \\ \mathbf{j}(x) &= \frac{\mathbf{v}}{c} \gamma e \delta \left(\left(x_1 - \frac{v}{c} x_0 \right) \gamma \right) \delta(x_2) \delta(x_3) = \frac{\mathbf{v}}{c} e \delta \left(x_1 - \frac{v}{c} x_0 \right) \delta(x_2) \delta(x_3) = \frac{\mathbf{v}}{c} \gamma e \delta^3(\mathbf{x}').\end{aligned}\quad (2.2.4)$$

Note that the final expressions for the current components differ by the Lorentz-factor γ from the corresponding expressions in Ref. [82], because the charge density $\rho(x)$ is defined there as the zero-component of the four-current, and not as a Lorentz scalar, like here. The both ways are equivalent, of course.

It will be shown later that the current (2.2.4) is reproduced as the right-hand side of the field equation (2.1.4) for a moving point charge in nonlinear, as well as linear, electrodynamics, and also in the weak continuity equation (2.3.6) for the energy-momentum tensor.

2.2.2 Covariant ansatz

Once the motion of the charge is given, there are no other vectors in the problem besides u_μ and the four-coordinate of the observation point x_μ , which is the argument of the differential equations (2.1.4). Therefore, the potential produced by the charge may only have the following representation

$$A^\mu = u^\mu f_1(xu, x^2) + x^\mu f_2(xu, x^2), \quad (2.2.5)$$

where f_1 and f_2 are functions of the indicated scalars. Then the field-strength tensor resulting from this potential is

$$F_{\mu\nu} = (u_\mu x_\nu - u_\nu x_\mu) f(xu, x^2) \quad (2.2.6)$$

with

$$f(xu, x^2) = \frac{\partial f_2}{\partial(xu)} - 2 \frac{\partial f_1}{\partial(x^2)}. \quad (2.2.7)$$

The requirement that in the rest frame the (electric) field F_{0i} be independent of time x_0 implies that f should be a function of the combination

$$W^2 = (xu)^2 - x^2 \quad (2.2.8)$$

of its argument. So, by definition, the invariant W is the distance from the observation point to the charge in the rest frame of the latter $W = |\mathbf{x}'| = r$. (In the rest frame it is evident that Eq. (2.2.8) defines a positive quantity $W = |\mathbf{x}'| = r$. As long as W^2 is a Lorentz invariant it remains positive in any frame.) Therefore, in what follows the representation below will be used

$$f(xu, x^2) = g(W^2). \quad (2.2.9)$$

It can be directly verified that the field tensor resulting from (2.2.6) and (2.2.9)

$$F_{\mu\nu} = (u_\mu x_\nu - u_\nu x_\mu) g(W^2) \quad (2.2.10)$$

satisfies the first pair of the Maxwell equations (the Bianchi identities)

$$\epsilon^{\mu\nu\lambda\rho} \frac{\partial F_{\nu\lambda}}{\partial x^\rho} = 0, \quad \epsilon^{0123} = 1. \quad (2.2.11)$$

However, it is easier to see this if one notes that by choosing $f_2 = 0$ in (2.2.5) and taking f_1 as

$$f_1(W^2) = \frac{1}{2} \int g(W^2) dW^2$$

one can determine the vector-potential generating the field (2.2.10), (2.2.9)

$$A_\mu = u_\mu f_1(W^2)$$

thus guaranteeing the fulfillment of (2.2.11). In the rest frame this potential is the 3-scalar $A'_0 = f_1(\mathbf{x}'^2)$, $\mathbf{A}' = 0$. Setting $f_2 = 0$ is not the only way to fix the vector potential generating Eq. (2.2.10). Another choice of the potential admitted within the gauge arbitrariness may be, for instance,

$$A_\mu = (u_\mu(ux) - x_\mu)(ux)g(W^2).$$

This potential satisfies the Lorentz-invariant gauge condition $(Au) = 0$.

We will focus on finding a solution to the second pair of the nonlinear Maxwell equations (2.1.4) using the form (2.2.10) as an ansatz.

2.2.3 Linear limit

Let us first note that the covariant extension of the Coulomb field produced by a moving point charge e , which is at rest in the origin in the rest frame, i.e., by the one, whose world line passes through the point $\tilde{x}_0 = \tilde{x}_i = 0$, is:

$$F_{\mu\nu}^{\text{lin}} = \frac{(u_\mu x_\nu - u_\nu x_\mu)}{W} \frac{e}{4\pi W^2}. \quad (2.2.12)$$

This corresponds to setting

$$g^{\text{lin}}(W^2) = \frac{e}{4\pi W^3} \quad (2.2.13)$$

in (2.2.10) in the linear limit. In the rest frame $u_i = 0$, $u_0 = 1$, one has $W = |\mathbf{x}'| = r$, and then the nonvanishing components of Eq.(2.2.12) constitute the Coulomb electric field of a point-like charge:

$$F_{0i}^{\text{lin}} = \frac{x'_i}{r} \frac{e}{4\pi r^2}. \quad (2.2.14)$$

Expression (2.2.12) satisfies, in the moving frame, the linear Maxwell equation

$$\partial^\mu F_{\mu\nu}^{\text{lin}} = \frac{1}{c} j_\nu \quad (2.2.15)$$

with the current (2.2.4) of a point charge. This rather evident statement is explicitly demonstrated in Appendix 1.

In order to reproduce the standard form of the electric and magnetic field components produced by a charge moving with the time-independent speed $\dot{u}_\mu = 0$ with its worldline $\tilde{\mathbf{x}} = \frac{\mathbf{v}}{c}\tilde{x}_0$ passing through the origin $\tilde{\mathbf{x}} = 0$ at zero time-moment $\tilde{x}_0 = 0$, written in Ref. [82] in the case of linear Maxwell electrodynamics as

$$\mathbf{E}^{\text{lin}} = \left(1 - \frac{v^2}{c^2}\right) \frac{e(\mathbf{x} - \frac{\mathbf{v}}{c}x_0)}{4\pi R^{*3}}, \quad \mathbf{B}^{\text{lin}} = [\mathbf{v} \times \mathbf{E}^{\text{lin}}], \quad (2.2.16)$$

it is sufficient to note that R^* defined in [82] (when \mathbf{v} has only the first component, $v_i = \delta_{i1}v$) as

$$R^{*2} = \left(x_1 - \frac{v}{c}x_0\right)^2 + (x_2^2 + x_3^2) \left(1 - \frac{v^2}{c^2}\right) \quad (2.2.17)$$

is related to the Lorentz scalar W (2.2.8) in the following way $R^{*2} = \left(1 - \frac{v^2}{c^2}\right) W^2$ (Note that R' of Ref. [82] is just our invariant W). With this substitution Eq. (2.2.16) is just what follows for the electric and magnetic components of (2.2.12).

The electric and magnetic fields (2.2.16) or (2.2.12) can be written also as functions of the difference $\Delta x_\mu = x_\mu - \tilde{x}_\mu$ between the coordinate of the observation point x_μ and that of the charge \tilde{x}_μ as follows

$$\begin{aligned} E_i^{\text{lin}}(\Delta x, \tilde{x}) &= F^{\text{lin}0i} = \frac{e}{4\pi W^2} \frac{(u^0 \Delta x^i - u^i \Delta x^0)}{W}, \\ B_i^{\text{lin}}(\Delta x, \tilde{x}) &= \epsilon_{ijk} F_{jk}^{\text{lin}} = \epsilon_{ijk} \frac{e}{4\pi W^2} \frac{(u_j \Delta x_k - u_k \Delta x_j)}{W} \end{aligned} \quad (2.2.18)$$

with

$$W^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} \left[\left(\Delta x_1 - \frac{v}{c} \Delta x_0\right)^2 + ((\Delta x_2)^2 + (\Delta x_3)^2) \left(1 - \frac{v^2}{c^2}\right) \right] \quad (2.2.19)$$

As long as the trajectory is fixed, neither (2.2.16), nor (2.2.18) contain the variable \tilde{x}_μ explicitly.

In order to write the fields in the Liénard-Wiechert form it is necessary to use the light-cone condition $(\Delta x)^2 = 0$, i.e. $(\Delta x_2)^2 + (\Delta x_3)^2 = (\Delta x_0)^2 - (\Delta x_1)^2$ in (2.2.18), which tells us that the charge and the observation point must be separated by a light-like interval for the field produced by the charge be nonzero in the observation point. Imposing the light-cone condition turns (2.2.19) into

$$W^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} \left(\Delta x_0 - \frac{v}{c} \Delta x_1\right)^2,$$

and (2.2.18) becomes Eqs. (63,8), (63,9) of Ref. [82] with $\dot{\mathbf{v}} = 0$ after the identification $R = \Delta x_0$, $R_i = \Delta x_i$.

In this connection, the following remark is in order. In linear electrodynamics the influence of a charge propagates with the speed of light in the vacuum c , and hence the observation point must be separated from the four-position of the source by a light-like interval, $(\Delta x)^2 = 0$, other space-time points carry no field produced by the source at this position. This is not the case in nonlinear electrodynamics. It is well known already in QED that the nonlinearity leads[86] to nontrivial dielectric permeability and magnetic susceptibility of the vacuum in an external field, thereby to deviation of the speed of propagation from that of light. The role of the external field is in our case played by the field produced by the charge itself, the propagation speed depending, as a matter of fact, on its intensity. For this reason it will be not relevant to impose the light-cone condition onto the nonlinear solution of the next section for getting an analog of the Liénard-Wiechert potential.

2.2.4 Solution to nonlinear field equations

From (2.1.4) one can conclude that its solution may be related to $F^{\text{lin}\mu\nu}$ given by (2.2.12) as

$$(1 - \sigma \mathfrak{F}(x)) F^{\mu\nu} = F^{\text{lin}\mu\nu}, \quad (2.2.20)$$

since (2.2.12) satisfies the linear Maxwell equation (2.2.15) as demonstrated in Appendix 1. Then, using the ansatz (2.2.10) and taking into account that $u^2 = 1$, one gets

$$\begin{aligned} (1 - \sigma \mathfrak{F}(x)) g &= \frac{e}{4\pi W^3}, \\ \mathfrak{F}(x) &= \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} W^2 g^2. \end{aligned} \quad (2.2.21)$$

This is a cubic equation for $g(W^2)$:

$$g^3 + \frac{2g}{\sigma W^2} - \frac{2e}{4\pi\sigma W^5} = 0. \quad (2.2.22)$$

The discriminant of this equation is positive

$$Q = \left(\frac{2}{3\sigma W^2}\right)^3 + \left(\frac{e}{4\pi\sigma W^5}\right)^2 > 0, \quad (2.2.23)$$

because $W^2 > 0$, as argued before. The only real Cardano solution for this case is

$$g = \sqrt[3]{\frac{e}{4\pi\sigma W^5} + \sqrt{Q}} + \sqrt[3]{\frac{e}{4\pi\sigma W^5} - \sqrt{Q}}. \quad (2.2.24)$$

Its asymptotic behavior near the charge

$$g \sim \left(\frac{2e}{4\pi\sigma W^5}\right)^{1/3} \text{ as } W \rightarrow 0, \quad (2.2.25)$$

will provide convergence for the field mass integral (2.4.2) in subsection "Finite field mass", 2.4. In the rest frame, when substituted into (2.2.10) with $u_0 = 1$, $u_i = 0$, $W = r$ this reproduces the result of Ref. [54]. In the limit of vanishing nonlinearity $\sigma \rightarrow 0$ Eq. (2.2.24) becomes $g = \frac{e}{4\pi W^3}$, therefore the solution Eq. (2.2.10) turns into the known expression (2.2.12) for the field of a moving charge in the linear Maxwell electrodynamics, $F_{\mu\nu} = F_{\mu\nu}^{\text{lin}}$.

Finally, from (2.2.10) for the point charge moving along the trajectory $\tilde{\mathbf{x}} = \frac{v}{c}\tilde{x}_0$ with a constant speed along axis 1, expressions for the electric and magnetic fields $E_i = F^{0i}$ and $B_i = \epsilon_{ijk}F_{jk}$ as functions of the observation coordinates are

$$\mathbf{E} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \left(\mathbf{x} - \frac{\mathbf{v}}{c}x_0\right)g(W^2), \quad \mathbf{B} = [\mathbf{v} \times \mathbf{E}]. \quad (2.2.26)$$

In terms of the distance from the charge $\Delta x_\mu = x_\mu - \tilde{x}_\mu$ these are

$$\begin{aligned} \mathbf{E} &= \left(1 - \frac{v^2}{c^2}\right)^{-1/2} g(W^2) \left(\Delta\mathbf{x} - \frac{\mathbf{v}}{c}\Delta x_0\right), \\ B_1 &= 0, \quad B_2 = -2\frac{v}{c} \left(1 - \frac{v^2}{c^2}\right)^{-1/2} g(W^2)\Delta x_3, \quad B_3(\Delta x) = 2\frac{v}{c} \left(1 - \frac{v^2}{c^2}\right)^{-1/2} g(W^2)\Delta x_2. \end{aligned} \quad (2.2.27)$$

where W^2 is the same as (2.2.19), and $g(W^2)$ is the solution (2.2.24). This is the nonlinear generalization of Eqs. (2.2.16) and (2.2.18).

2.3 Solitonic representation of the point-charge field

2.3.1 Energy-momentum tensor

In this section $c = 1$ is set. The Noether current corresponding to space-time translations

$$T^{\mu\nu} = \frac{\partial A^\lambda}{\partial x_\mu} \frac{\partial L(x)}{\partial (\partial A^\lambda / \partial x_\nu)} - g^{\mu\nu} L(x),$$

calculated using the Lagrange density (2.1.1) with the account of the field equations (2.1.4) is written as

$$T^{\mu\nu} = -[1 - \sigma\mathfrak{F}(x)] F^{\mu\lambda} F_{\lambda}^{\nu} - \frac{\partial}{\partial x_{\lambda}} [A^{\mu} (1 - \sigma\mathfrak{F}(x)) F_{\lambda}^{\nu}] - g^{\mu\nu} L(x). \quad (2.3.1)$$

This is the energy-momentum tensor of the electromagnetic field. The requirement that it should be gauge-invariant makes us omit the potential-dependent second term (that is a full derivative) to be left with the expression, symmetrical under the permutation of the indices:

$$T^{\mu\nu} = -[1 - \sigma\mathfrak{F}(x)] F^{\mu\lambda} F_{\lambda}^{\nu} - g^{\mu\nu} L(x). \quad (2.3.2)$$

The energy T^{00} and the Pointing vector T^{i0} densities have the form

$$\begin{aligned} T^{00} &= -F^{0\lambda} F_{\lambda}^0 (1 - \sigma\mathfrak{F}(x)) + \mathfrak{F}(x) - \frac{\sigma}{2} (\mathfrak{F}(x))^2, \\ T^{i0} &= -F^{i\lambda} F_{\lambda}^0 (1 - \sigma\mathfrak{F}(x)). \end{aligned} \quad (2.3.3)$$

Contravariant components of the tensor (2.3.2) are

$$T_v^{\mu} = -[1 - \sigma\mathfrak{F}(x)] F^{\mu\lambda} F_{v\lambda} - \delta_v^{\mu} L(x). \quad (2.3.4)$$

Its trace is different from zero

$$T_i^i = 2\sigma (\mathfrak{F}(x))^2. \quad (2.3.5)$$

in the nonlinear case $\sigma \neq 0$.

2.3.2 Weak continuity and energy-momentum conservation

The above construction of the energy-momentum tensor does not provide the fulfillment of its continuity, since the current is meant to be supported by outer forces and therefore the energy-momentum non-conservation should be admitted. Instead of the continuity equation an equation which may be referred to as the "partial conservation of the Noether current" is formally obtained in the form

$$\frac{\partial}{\partial x^{\mu}} T_{\nu}^{\mu} = -F_{\nu\lambda} j^{\lambda}. \quad (2.3.6)$$

To derive this relation, equation of motion (2.1.4) was used together with the Bianchi identities $\frac{\partial F_{\lambda\rho}}{\partial x^{\nu}} = -\frac{\partial F_{\rho\nu}}{\partial x^{\lambda}} - \frac{\partial F_{\nu\lambda}}{\partial x^{\rho}}$. This relation is not owing to the nonlinearity and retains its form in the standard linear theory as well. The right-hand side in (2.3.6) is not vanishing. However the continuity of the energy-momentum tensor holds in a weak form

$$u^{\nu} \frac{\partial}{\partial x^{\mu}} T_{\nu}^{\mu} = 0. \quad (2.3.7)$$

To see this, note that the 4-current of a point charge moving with the constant speed is parallel to its 4-velocity vector. Then the right-hand side of (2.3.6) disappears when contracted with the 4-velocity, $u^{\nu} F_{\nu\lambda} j^{\lambda} \sim u^{\nu} F_{\nu\lambda} u^{\lambda} = 0$. The property of weak continuity (2.3.7) will be sufficient for establishing the conservation of the energy-momentum vector.

Let us define the latter by the integral over a space-like hyperplane that is orthogonal to the four-velocity and crosses the time axis at $x_0 = s\sqrt{1 - v^2}$

$$P^{\mu} = \int u^{\nu} T_{\nu}^{\mu} \delta(ux - s) d^4x. \quad (2.3.8)$$

Via the Gauss theorem it follows from the vanishing of the 4-divergence (2.3.7) that P^μ is independent of s , and thereby of the time of observation x_0 , because the hyperplane can be shifted as a whole along the vector u^ν without affecting the value of the integral, since the fields (2.2.10) decrease at space-time infinity no less fast as in the linear electrodynamics, the nonlinearity fades away far from the charge, where its fields are weak. On the contrary, one cannot change to a space-like hyperplane inclined differently than in (2.3.8) in the definition of the energy-momentum 4-vector due to the lack of the continuity law analogous to (2.3.7) with the unit vector u^ν other than the 4-velocity.

When the field $F_{\nu\lambda}$, on which the energy-momentum tensor T_ν^μ depends, is that of a uniformly moving (or resting) point charge, the integral in (2.3.8) usually diverges and hence makes no sense. This is not the case in the nonlinear theory under consideration here, as will be shown in the next subsection. Therefore, the energy-momentum vector (2.3.8) can be taken seriously.

Bearing in mind that s is a Lorentz scalar (moreover, set equal to zero in what follows), and that T_ν^μ (2.3.2) is a tensor, the integral (2.3.8) does define a Minkowski vector. It can only be directed along u^ν , since this is the only external vector in the integrand of (2.3.8). Hence

$$P^\mu = u^\mu M_f, \quad (2.3.9)$$

where M_f is the field mass.

2.4 Finite field mass

It follows from (2.3.8) that

$$M_f = P^\nu u_\nu = \int u_\nu T^{\mu\nu} u_\mu \delta(ux) d^4x.$$

From (2.3.2) and (2.2.12) one calculates

$$\begin{aligned} M_f &= \int [-[1 - \sigma \mathfrak{F}(x)] u_\mu F^{\mu\lambda} F_\lambda^\nu u_\nu - L] \delta(ux) d^4x = \\ &= \int \left[\left(1 - \sigma \frac{W^2 g^2}{2}\right) W^2 g^2 + \frac{W^2 g^2}{2} + \frac{\sigma W^4 g^4}{8} \right] \delta(ux) d^4x. \end{aligned} \quad (2.4.1)$$

Thanks to the delta-function one can set $W^2 = -x^2$ in the integrand, the argument of the function $g(W^2)$ included. For any function $\Phi(x^2)$, provided the integrals below converge, the following chain relation holds

$$\int \Phi(x^2) \delta(ux) d^4x = \frac{1}{u_0} \int \Phi\left(-x_1^2 \frac{u_0^2 - u_1^2}{u_0^2} - x_2^2 - x_3^2\right) dx_1 dx_2 dx_3.$$

The above expression is integrated over x_0 with the help of the delta-function. Performing the change of the variable $x_1 \frac{\sqrt{u_0^2 - u_1^2}}{u_0} = x_1 \sqrt{1 - v^2} = x_1'$ and omitting the prime afterwards one finds that this integral is equal to

$$\int \Phi(-\mathbf{x}^2) d^3x.$$

Applying this result to (2.4.1) one gets for the field mass the integral over 3-space

$$M_f = \int \left[\frac{\mathbf{x}^2 g(\mathbf{x}^2)^2}{2} + \frac{3}{8} \sigma \mathbf{x}^4 g(\mathbf{x}^2)^4 \right] d^3x.$$

Note that the function $g(\mathbf{x}^2)$ involved here is just (2.2.24) taken in the rest frame.

With the help of Eqs.(2.2.22) the second term can be expressed as

$$\frac{3}{8}\sigma\mathbf{x}^4 g(\mathbf{x}^2)^4 = \frac{3}{4} \left(-\mathbf{x}^2 g(\mathbf{x}^2)^2 + \frac{eg(\mathbf{x}^2)}{4\pi|\mathbf{x}|} \right).$$

Then the expression for the mass can be rewritten as

$$M_f = \int \left[-\frac{\mathbf{x}^2 g(\mathbf{x}^2)^2}{4} + \frac{3}{4} \frac{eg(\mathbf{x}^2)}{4\pi|\mathbf{x}|} \right] d^3x. \quad (2.4.2)$$

Therefore, to calculate the mass M_f , one has to calculate two integrals. The first one is

$$\begin{aligned} \int \mathbf{x}^2 g(\mathbf{x}^2)^2 d^3x &= |e|^{\frac{3}{2}} \left(\frac{3}{2\sigma(4\pi)^2} \right)^{\frac{1}{4}} \frac{3}{2} I_1, \\ I_1 &= \int_0^\infty y^{\frac{2}{3}} \left(\sqrt[3]{\sqrt{1+y^4}+1} - \sqrt[3]{\sqrt{1+y^4}-1} \right)^2 dy = 0.885, \end{aligned} \quad (2.4.3)$$

and the second one is

$$\begin{aligned} e \int \frac{g(\mathbf{x}^2)}{4\pi|\mathbf{x}|} d^3x &= |e|^{\frac{3}{2}} \left(\frac{3}{2\sigma(4\pi)^2} \right)^{\frac{1}{4}} I_2, \\ I_2 &= \int_0^\infty y^{-\frac{2}{3}} \left(\sqrt[3]{\sqrt{1+y^4}+1} - \sqrt[3]{\sqrt{1+y^4}-1} \right)^2 dy = 3.984. \end{aligned} \quad (2.4.4)$$

The final result is

$$M_f = |e|^{\frac{3}{2}} \left(\frac{3}{2\sigma(4\pi)^2} \right)^{\frac{1}{4}} \frac{1}{4} \left(3I_2 - \frac{3}{2}I_1 \right) = 2.65|e|^{\frac{3}{2}} \left(\frac{3}{2\sigma(4\pi)^2} \right)^{\frac{1}{4}} < \infty,$$

and it can be seen that this is the same value as the full electrostatic energy of a stationary pointlike particle found in [54].

Conclusion

In the framework of QED with unstable vacuum in the presence of t-electric potential steps the following problems were studied. First, using a general nonperturbative expression for the density operators (of quantized Dirac or KG fields), their specific forms corresponding to different initial conditions were derived. Applying a reduction procedure to these specific density operators, we constructed mixed states of both electron and positron subsystems. Calculating the entropy of such states, we obtained the loss of information due to the reduction and, at the same time, the entanglement of electron and positron subsystems. We paid attention to the fact that any measurement in the system under consideration implies a decoherence and the corresponding modifications of the complete and the reduced density operators. We studied the results of such a decoherence and we related to it the loss of information by calculating the information entropy. To illustrate some of the obtained general results, we considered the slowly varying T-constant electric field as an external background. The following conclusions are derived. The entropy of any subsystem (of electrons or positrons) with the vacuum as the initial state is proportional to the factor $(eE)d/2$ and to the number of spin degrees of freedom $J(d)$. It grows linearly with the time of the field action T. The above behavior remains in the thermal case at low temperatures; in fact, here the entropy does not depend on the temperature.

In the framework of QED with non-uniform external fields the exponential external field in two different configurations (slowly varying symmetric field and peak field) was considered; the differential and full mean numbers of particles created from vacuum were calculated. An important question about unitarily equivalence of in and out spaces was studied in the framework of QED with x-electric potential steps. It was shown that the in- and out-Fock spaces are unitarily equivalent, if the linear canonical transformation (1.5.1) (together with its adjoint transformation) that connects in- and out-sets of operators of creation and annihilation of particles, is proper one. It was demonstrated that in the scope of the QED in the presence of critical potential steps the transformation is a proper one if the total number of particles created from vacuum is finite; for any realistic field limited in space and time this condition is obviously satisfied. This fact can be considered as one more confirmation of the consistency of the theory constructed and correct interpretation of in- and out- particles in said theory. The next step taken was a construction of a general density operator with the vacuum initial condition. Such an operator describes a deformation of the initial vacuum state by x-electric critical potential steps. Using the Bogolubov transformation, we managed to derive two different explicit expressions for this operator in terms of the out-operators (normal-form exponent and the expression in terms of vector states). Of course, both representations are equivalent. The reductions of the deformed state to electron and positron subsystems were constructed, and the loss of the information in these reductions was calculated as the corresponding von Neumann entropies. It is known that this entropy can also be treated as the entanglement measure of these reduced matrices. Therefore, the entanglement measure of the electron and positron subsystems was obtained. The general consideration was illustrated by studying the deformation of the quantum vacuum between

two capacitor plates, that is, the effect that so-called L-constant field has on initial vacuum state. It was shown that in this case entropy is proportional to the factor $(eE)d/2$, to the number of spin degrees of freedom $J(d)$, and the volume inside the capacitor $V = V_{\perp}L$ (volume occupied by electric field). It can be noted that in the leading order the expressions for the case of x-electric step under consideration and T-constant field coincide.

As it was said before, the other way to consider strong field apart from QED is the nonlinear local electrodynamics. As a model Lagrangian for the latter we used the one whose nonlinearity is the Euler-Heisenberg Lagrangian of quantum electrodynamics truncated at the leading term of its expansion in powers of the first field invariant. It was shown that the total energy of the field produced by a point charge is finite in that model; thereby its field configuration is a soliton. We defined a finite energy-momentum vector of this field configuration to demonstrate that its components satisfy the standard mechanical relation characteristic of a free moving massive particle.

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Appendix 1

In this appendix it is demonstrated that Eq. (2.2.12) satisfies linear equation (2.2.15) with the point-like charge current (2.2.4). The superscript "lin" is omitted within this Appendix. It is convenient to rewrite (2.2.12) in terms of coordinates of the coordinate system K' , which moves along the axis x_1 with constant velocity v (rest frame of the moving particle). The corresponding Lorentz boost is

$$x_0 = \gamma \left(x'_0 + \frac{v}{c} x'_1 \right), \quad x_1 = \gamma \left(x'_1 + \frac{v}{c} x'_0 \right), \quad x'_2 = x_2, \quad x'_3 = x_3,$$

and the inverse transformation is given by (2.2.3). The Lorentz transformation reduces the scalar W^2 to the form

$$\begin{aligned} W^2 &= (xu)^2 - x^2 = \\ &= x_1'^2 + x_2'^2 + x_3'^2 = r'^2. \end{aligned} \tag{A.1}$$

The relations

$$\begin{aligned} \left. \frac{\partial x'_0}{\partial x_0} \right|_x &= \gamma \left. \frac{\partial (x_0 - \frac{v}{c} x_1)}{\partial x_0} \right|_x = \gamma, \\ \left. \frac{\partial x'_1}{\partial x_0} \right|_x &= \gamma \left. \frac{\partial (x_1 - \frac{v}{c} x_0)}{\partial x_0} \right|_x = -\gamma \frac{v}{c}, \\ \left. \frac{\partial x'_0}{\partial x_1} \right|_x &= \gamma \left. \frac{\partial (x_0 - \frac{v}{c} x_1)}{\partial x_1} \right|_x = -\gamma \frac{v}{c}, \\ \left. \frac{\partial x'_1}{\partial x_1} \right|_x &= \gamma \left. \frac{\partial (x_1 - \frac{v}{c} x_0)}{\partial x_1} \right|_x = \gamma, \end{aligned} \tag{A.2}$$

where $|_x$ designates derivative calculated at constant x , follow from (2.2.3). Then

$$\begin{aligned} \left. \frac{\partial}{\partial x_0} \right|_x &= \left(\gamma \left. \frac{\partial}{\partial x'_0} \right|_{x'} - \gamma \frac{v}{c} \left. \frac{\partial}{\partial x'_1} \right|_{x'} \right), \\ \left. \frac{\partial}{\partial x_1} \right|_x &= \left(-\gamma \frac{v}{c} \left. \frac{\partial}{\partial x'_0} \right|_{x'} + \gamma \left. \frac{\partial}{\partial x'_1} \right|_{x'} \right). \end{aligned} \tag{A.3}$$

Now, with the account of (A.1)-(A.3), one can calculate divergency of $F_{\mu\nu}$ exploiting the covariant

form (2.2.12). The zeroth and first components are:

$$\begin{aligned}
\partial^\mu F_{\mu 0} &= \frac{e}{4\pi} \left[\frac{\partial}{\partial x_1} \left(\frac{-\gamma (x_1 - \frac{v}{c} x_0)}{W^3} \right) + \frac{\partial}{\partial x_2} \left(\frac{-\gamma x_2}{W^3} \right) + \frac{\partial}{\partial x_3} \left(\frac{-\gamma x_3}{W^3} \right) \right] = \\
&= \frac{e\gamma}{4\pi} \left[\frac{\partial}{\partial x'_i} \left(\frac{-x'_i}{r'^3} \right) \right], \\
\partial^\mu F_{\mu 1} &= \frac{e}{4\pi} \left[\frac{\partial}{\partial x_0} \left(\frac{\gamma (x_1 - \frac{v}{c} x_0)}{W^3} \right) + \frac{\partial}{\partial x_2} \left(\frac{-\gamma \frac{v}{c} x_2}{W^3} \right) + \frac{\partial}{\partial x_3} \left(\frac{-\gamma \frac{v}{c} x_3}{W^3} \right) \right] = \\
&= \frac{v e \gamma}{c 4\pi} \left[\frac{\partial}{\partial x'_i} \left(\frac{-x'_i}{r'^3} \right) \right], \tag{A.4}
\end{aligned}$$

The second and third components are

$$\partial^\mu F_{\mu 2,3} = \frac{e\gamma x_{2,3}}{4\pi} \left[\frac{\partial}{\partial x_0} \left(\frac{1}{W^3} \right) + \frac{v}{c} \frac{\partial}{\partial x_1} \left(\frac{1}{W^3} \right) \right] = \frac{e\gamma x'_{2,3}}{4\pi} \left[\frac{\partial}{\partial x_0} + \frac{v}{c} \frac{\partial}{\partial x_1} \right] \left(\frac{1}{r'^3} \right).$$

This is zero, because

$$\left[\frac{\partial}{\partial x_0} + \frac{v}{c} \frac{\partial}{\partial x_1} \right] r'^{-3} = \left[-\gamma \frac{v}{c} \frac{\partial}{\partial x'_1} + \gamma \frac{v}{c} \frac{\partial}{\partial x'_1} \right] r'^{-3} \equiv 0, \quad \left[\frac{\partial r'^{-3}}{\partial x'_0} \right]_{x'=const} = 0.$$

Referring to the linear Maxwell equation in the rest frame

$$\frac{\partial}{\partial x'_i} \left(\frac{-x'_i}{r'^3} \right) = 4\pi \delta^3(\mathbf{x}'), \tag{A.5}$$

one can write

$$\partial^\mu F_{\mu\nu} = 4\pi \left(e\gamma \delta^3(\mathbf{x}'), \frac{v}{c} e\gamma \delta^3(\mathbf{x}'), 0, 0 \right).$$

This coincides with the current (2.2.4).

Appendix 2: Basic elements of QED with x -electric critical potential steps

Solutions of the Dirac equation with x -electric potential steps, ${}_{\zeta}\psi_n(X)$ and ${}^{\zeta}\psi_n(X)$, $\zeta = \pm$, can be subjected to the following orthonormality conditions on x -constant hyperplane

$$\begin{aligned} ({}_{\zeta}\psi_n, {}_{\zeta'}\psi_{n'})_x &= \zeta\eta_L\delta_{\zeta,\zeta'}\delta_{n,n'}, \quad \eta_L = \text{sgn } \pi_0(L), \quad \pi_0(L) = p_0 - U_L, \\ ({}^{\zeta}\psi_n, {}^{\zeta'}\psi_{n'})_x &= \zeta\eta_R\delta_{\zeta,\zeta'}\delta_{n,n'}, \quad \eta_R = \text{sgn } \pi_0(R), \quad \pi_0(R) = p_0 - U_R, \\ (\psi, \psi')_x &= \int \psi^\dagger(X)\gamma^0\gamma^1\psi'(X) dt d\mathbf{r}_\perp. \end{aligned} \quad (\text{B.1})$$

Coefficients $g({}_{\zeta}|\zeta') = ({}_{\zeta}\psi_n, {}^{\zeta'}\psi_n)_x$ define mutual decompositions of these solutions

$$\begin{aligned} \eta_L {}_{\zeta}\psi_n(X) &= {}_+\psi_n(X)g({}_+|\zeta) - {}_-\psi_n(X)g({}_-|\zeta), \\ \eta_R {}^{\zeta}\psi_n(X) &= {}^+\psi_n(X)g({}^+|\zeta) - {}^-\psi_n(X)g({}^-|\zeta). \end{aligned} \quad (\text{B.2})$$

Unitary relations for the decomposition coefficients are

$$g({}_+|\zeta')^* g({}_+|\zeta) - g({}_-|\zeta')^* g({}_-|\zeta) = \zeta\eta_L\eta_R\delta_{\zeta,\zeta'}. \quad (\text{B.3})$$

To extract results of the one-particle scattering theory, all the constituent quantities, such as reflection and transmission coefficients etc., have to be represented with the help of the g 's, that is, the matrix elements of current in x -direction.

We find the following linear independent pairs for each n by calculating scalar products on t -constant hyperplane:

$$\begin{aligned} ({}_{\zeta}\psi_n, {}_{\zeta}\psi_{n'}) &= ({}^{\zeta}\psi_n, {}^{\zeta}\psi_{n'}) = \delta_{\sigma,\sigma'}\delta(p_0 - p'_0)\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)\mathcal{M}_n, \\ \mathcal{M}_n &= |g({}_+|^+)|^2, \quad n \in \Omega_1 \cup \Omega_5; \quad \mathcal{M}_3 = |g({}_+|^+)|^2, \quad n \in \Omega_3; \\ ({}_{\zeta}\psi_n, {}^{-\zeta}\psi_n) &= 0, \quad n \in \Omega_1 \cup \Omega_5, \quad {}_{\zeta}\psi_n \text{ and } {}^{-\zeta}\psi_n \text{ independent,} \\ ({}_{\zeta}\psi_n, {}^{\zeta}\psi_n) &= 0, \quad n \in \Omega_3, \quad {}_{\zeta}\psi_n \text{ and } {}^{\zeta}\psi_n \text{ independent.} \end{aligned} \quad (\text{B.4})$$

Then we identify:

$$\begin{aligned} \text{in - solutions} &: {}_+\psi_{n_1}, {}_-\psi_{n_1}; \quad {}_-\psi_{n_5}, {}_+\psi_{n_5}; \quad {}_-\psi_{n_3}, {}_-\psi_{n_3}, \\ \text{out - solutions} &: {}_-\psi_{n_1}, {}_+\psi_{n_1}; \quad {}_+\psi_{n_5}, {}_-\psi_{n_5}; \quad {}_+\psi_{n_3}, {}_+\psi_{n_3}. \end{aligned} \quad (\text{B.5})$$

We can decompose the Heisenberg operator $\hat{\Psi}(X)$ as

$$\begin{aligned}
\Psi(X) &\implies \hat{\Psi}(X), \quad \left[\hat{\Psi}(X), \hat{\Psi}(X') \right]_+ \Big|_{t=t'} = 0, \\
\left[\hat{\Psi}(X), \hat{\Psi}(X')^\dagger \right]_+ \Big|_{t=t'} &= \delta(\mathbf{r} - \mathbf{r}'); \quad \hat{\Psi}(X) = \sum_{i=1}^5 \hat{\Psi}_i(X), \\
\hat{\Psi}_1(X) &= \sum_{n \in \Omega_1} \mathcal{M}_n^{-1/2} \left[{}^+a_n(\text{in}) {}^+\psi_n(X) + {}^-a_n(\text{in}) {}^-\psi_n(X) \right] \\
&= \sum_{n \in \Omega_1} \mathcal{M}_n^{-1/2} \left[{}^+a_n(\text{out}) {}^+\psi_n(X) + {}^-a_n(\text{out}) {}^-\psi_n(X) \right], \\
\hat{\Psi}_2(X) &= \sum_{n \in \Omega_2} \mathcal{M}_n^{-1/2} a_n \psi_n(X), \quad \hat{\Psi}_4(X) = \sum_{n \in \Omega_4} \mathcal{M}_n^{-1/2} b_n^\dagger \psi_n(X), \\
\hat{\Psi}_3(X) &= \sum_{n \in \Omega_3} \mathcal{M}_n^{-1/2} \left[{}^-a_n(\text{in}) {}^-\psi_n(X) + {}^-b_n^\dagger(\text{in}) {}^-\psi_n(X) \right] \\
&= \sum_{n \in \Omega_3} \mathcal{M}_n^{-1/2} \left[{}^+a_n(\text{out}) {}^+\psi_n(X) + {}^+b_n^\dagger(\text{out}) {}^+\psi_n(X) \right], \\
\hat{\Psi}_5(X) &= \sum_{n \in \Omega_5} \mathcal{M}_n^{-1/2} \left[{}^+b_n^\dagger(\text{in}) {}^+\psi_n(X) + {}^-b_n^\dagger(\text{in}) {}^-\psi_n(X) \right] \\
&= \sum_{n \in \Omega_5} \mathcal{M}_n^{-1/2} \left[{}^+b_n^\dagger(\text{out}) {}^+\psi_n(X) + {}^-b_n^\dagger(\text{out}) {}^-\psi_n(X) \right], \tag{B.6}
\end{aligned}$$

where all a and b are Fermi annihilation operators and all a^\dagger and b^\dagger are Fermi creation operators.

We denote the quantum numbers in corresponding zone by n_i , so that $n_i \in \Omega_i$. The manifold of all the quantum numbers n is denoted by Ω , so that $\Omega = \Omega_1 \cup \dots \cup \Omega_5$. The in and out vacua can be factorized

$$|0, \text{in}\rangle = \prod_{i=1}^5 |0, \text{in}\rangle^{(i)}, \quad |0, \text{out}\rangle = \prod_{i=1}^5 |0, \text{out}\rangle^{(i)}, \tag{B.7}$$

where $|0, \text{in}\rangle^{(i)}$ and $|0, \text{out}\rangle^{(i)}$ are the partial vacua in the ranges Ω_i . Note that in each range Ω_i it is also possible to factorize vacuum vectors in modes with fixed quantum number n so that

$$|0, \text{in}\rangle^{(i)} = \prod_{n \in \Omega_i} |0, \text{in}\rangle_n^{(i)}, \quad |0, \text{out}\rangle^{(i)} = \prod_{n \in \Omega_i} |0, \text{out}\rangle_n^{(i)}. \tag{B.8}$$

It was shown that all in and out vacua, except the vacua in the range Ω_3 (in the so-called Klein zone) coincide,

$$|0, \text{out}\rangle^{(i)} = |0, \text{in}\rangle^{(i)}, \quad i = 1, 2, 4, 5, \quad |0, \text{out}\rangle^{(3)} \neq |0, \text{in}\rangle^{(3)}. \tag{B.9}$$

The vacuum-to-vacuum transition amplitude $c_v = \langle 0, \text{out} | 0, \text{in} \rangle$ coincides (due to Eq. (B.9)) with the vacuum-to-vacuum transition amplitude $c_v^{(3)}$ in the Klein zone,

$$c_v = \langle 0, \text{out} | 0, \text{in} \rangle = c_v^{(3)} = {}^{(3)}\langle 0, \text{out} | 0, \text{in} \rangle^{(3)}. \tag{B.10}$$

Appendix 3: Some asymptotic expansions

The asymptotic expression of the confluent hypergeometric function for large η and c with fixed a and $\tau = \eta/c \sim 1$ is given by Eq. (13.8.4) in [89] as

$$\begin{aligned}\Phi(a, c; \eta) &\simeq c^{a/2} e^{\mathcal{Z}^2/4} F(a, c; \tau), \quad \mathcal{Z} = -(\tau - 1) \mathcal{W}(\tau) \sqrt{c}, \\ F(a, c; \tau) &= \tau \mathcal{W}^{1-a} D_{-a}(\mathcal{Z}) + \mathcal{R} D_{1-a}(\mathcal{Z}), \\ \mathcal{R} &= (\mathcal{W}^a - \tau \mathcal{W}^{1-a}) / \mathcal{Z}, \quad \mathcal{W}(\tau) = [2(\tau - 1 - \ln \tau) / (\tau - 1)^2]^{1/2}\end{aligned}\tag{C.1}$$

where $D_{-a}(\mathcal{Z})$ is the Weber parabolic cylinder function (WPCF) [88]. Using Eq. (C.1) we present the functions y_2^2 , y_1^1 and their derivatives at $x = 0$ as

$$\begin{aligned}y_1^1|_{x=0} &\simeq e^{-ih_1/2} (ih_1)^{\nu_1} c_1^{a_1/2} e^{\mathcal{Z}_1^2/4} F(a_1, c_1; \tau_1), \\ \mathcal{Z}_1 &= -(\tau_1 - 1) \mathcal{W}(\tau_1) \sqrt{c_1}, \quad \tau_1 = ih_1/c_1, \\ \frac{\partial y_1^1}{\partial \eta_1} \Big|_{x=0} &\simeq e^{-ih_1/2} (ih_1)^{\nu_1} c_1^{a_1/2} e^{\mathcal{Z}_1^2/4} \left[-\frac{1}{2ih_1} + \frac{1}{c_1} \frac{\partial}{\partial \tau_1} \right] F(a_1, c_1; \tau_1); \\ y_2^2|_{x=0} &\simeq e^{ih_2/2} (ih_2)^{-\nu_2} (2 - c_2)^{(1-a_2)/2} e^{\mathcal{Z}_2^2/4} F(1 - a_2, 2 - c_2; \tau_2), \\ \mathcal{Z}_2 &= -(\tau_2 - 1) \mathcal{W}(\tau_2) \sqrt{2 - c_2}, \quad \tau_2 = -ih_2 / (2 - c_2), \\ \frac{\partial y_2^2}{\partial \eta_2} \Big|_{x=0} &\simeq e^{ih_2/2} (ih_2)^{-\nu_2} c_2^{(1-a_2)/2} e^{\mathcal{Z}_2^2/4} \left[-\frac{1}{2ih_2} - \frac{1}{2 - c_2} \frac{\partial}{\partial \tau_2} \right] F(1 - a_2, 2 - c_2; \tau_2).\end{aligned}\tag{C.2}$$

Assuming $\tau - 1 \rightarrow 0$, one has

$$\begin{aligned}\mathcal{W}^{1-a} &\approx 1 + \frac{a-1}{3} (\tau - 1), \quad \mathcal{R} \approx \frac{2(a+1)}{3\sqrt{c}}, \quad \mathcal{Z} \approx -(\tau - 1) \sqrt{c}, \\ \frac{\partial F(a, c; \tau)}{\partial \tau} &\approx \frac{2+a}{3} D_{-a}(\mathcal{Z}) + \frac{\partial D_{-a}(\mathcal{Z})}{\partial \tau} + \mathcal{R} \frac{\partial D_{1-a}(\mathcal{Z})}{\partial \tau}.\end{aligned}$$

Expanding WPCFs near $\mathcal{Z} = 0$, in the leading approximation at $\mathcal{Z} \rightarrow 0$ one obtains that

$$\begin{aligned}\frac{\partial F(a, c; \tau)}{\partial \tau} &\approx -\sqrt{\eta} D'_{-a}(0) + O(\eta), \\ F(a, c; \tau) &\approx D_{-a}(0) + O(c^{-1/2}),\end{aligned}\tag{C.3}$$

and

$$D_{-a}(0) = \frac{2^{-a/2} \sqrt{\pi}}{\Gamma(\frac{a+1}{2})}, \quad D'_{-a}(0) = \frac{2^{(1-a)/2} \sqrt{\pi}}{\Gamma(\frac{a}{2})},\tag{C.4}$$

where $\Gamma(z)$ is the Euler gamma function. We find under condition (1.5.40) that

$$\begin{aligned} p^{\text{L,R}} &\approx |\pi_{1,2}| (1 - \lambda/h_{1,2}), \quad a_{1,2} \approx (1 - \chi)/2 - i\lambda/2, \\ c_1 &\approx 1 - i \left(\lambda - \frac{2|\pi_1|}{k_1} \right), \quad 2 - c_2 \approx 1 + i \left(\lambda - \frac{2|\pi_2|}{k_2} \right), \\ \tau_1 - 1 &\approx -\frac{1}{h_1} \left(i - \lambda + \frac{2p_0}{k_1} \right), \quad \tau_2 - 1 \approx \frac{1}{h_2} \left(i + \lambda + \frac{2p_0}{k_2} \right). \end{aligned} \quad (\text{C.5})$$

Using Eqs. (C.2), (C.3), and (C.5) we represent Eq. (1.5.35) in the form

$$\begin{aligned} N_n^{\text{cr}} &= e^{-\pi\lambda/2} \left[|\delta_0|^{-2} + O\left(h_1^{-1/2}\right) + O\left(h_2^{-1/2}\right) \right], \\ \delta_0 &= e^{i\pi/4} D_{-a_1}(0) D'_{a_2-1}(0) - e^{-i\pi/4} D'_{-a_1}(0) D_{a_2-1}(0). \end{aligned} \quad (\text{C.6})$$

Assuming $\chi = 1$ and using the relations of the Euler gamma function we find that

$$\delta_0 = \exp\left(i\frac{3\pi}{4} - i\frac{\pi\chi}{2}\right) e^{\pi\lambda/4}. \quad (\text{C.7})$$

Assuming $|\tau - 1| \sim 1$, one can use the asymptotic expansions of WPCFs in Eq. (C.1), e.g., see [88, 89]. Note that $\arg(\mathcal{Z}) \approx \frac{1}{2} \arg(c)$ if $1 - \tau > 0$. Then one finds that

$$\Phi(a, c; \eta) = (1 - \tau)^{-a} [1 + O(|\mathcal{Z}|^{-1})] \quad \text{if } 1 - \tau > 0. \quad (\text{C.8})$$

In the case of $1 - \tau < 0$, one has

$$\arg(\mathcal{Z}) \approx \begin{cases} \frac{1}{2} \arg(c) + \pi & \text{if } \arg(c) < 0 \\ \frac{1}{2} \arg(c) - \pi & \text{if } \arg(c) > 0 \end{cases}.$$

Then one obtains finally that

$$\Phi(a, c; \eta) = \begin{cases} (\tau - 1)^{-a} e^{-i\pi a} [1 + O(|\mathcal{Z}|^{-1})] & \text{if } \arg(c) < 0 \\ (\tau - 1)^{-a} e^{i\pi a} [1 + O(|\mathcal{Z}|^{-1})] & \text{if } \arg(c) > 0 \end{cases}. \quad (\text{C.9})$$

The asymptotic expression of the confluent hypergeometric function $\Phi(a, c; \pm ih)$ for large real h with fixed a and c is given by Eq. (6.13.1(2)) in [88] as

$$\Phi(a, c; \pm ih) = \frac{\Gamma(c)}{\Gamma(c-a)} e^{\pm i\pi a/2} h^{-a} + \frac{\Gamma(c)}{\Gamma(a)} e^{\pm ih} (e^{\pm i\pi/2} h)^{a-c}. \quad (\text{C.10})$$

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