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Distribuições espectrais e angulares da radiação síncrotron no âmbito da teoria quântica

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Spectral and angular distributions of synchrotron radiation in quantum theory

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Resumo

Consideramos as características da radiação síncrotron (RS) no âmbito da teoria quântica. Para simplificar a descrição teórica do processo de radiação restringimos à consideração da emissão de único fóton. Para transições quânticas arbitrárias, as distribuições espectrais e angulares da potência da RS são dadas de forma analítica exata. Tratamos separadamente partículas escalares (bósons) e com spin $\hbar/2$ (elétrons). Atenção especial é dada às transições particulares, a saber, as transições ao primeiro estado excitado e estado fundamental. É mostrado que os componentes de polarização linear da radiação de elétron se trocam em relação à orientação de spin quando o elétron passa para o estado fundamental. Este fato pode ser considerado como uma comprovação analítica para a presença de π -componente da radiação quântica no plano de movimento.

Analizamos minuciosamente a radiação emitida pela partícula fracamente excitada. Várias funções são introduzidas para descrever a evolução dos perfis de distribuições angulares para sistemas de dois e três níveis. Para transições quânticas do primeiro estado excitado ao estado fundamental a análise comparativa da radiação de bósons e elétrons é realizada, e isso ajuda a estimar a influência de spin e sua direção sobre as características da RS. A radiação de elétrons não polarizados é considerada separadamente. Observando o comportamento dos ângulos efetivos, é fácil perceber a inconsistência da conclusão clássica bem conhecida sobre a concentração de radiação ultra-relativista total no plano do movimento. Mostramos que os ângulos efetivos da radiação quântica tendem aos valores finitos e não desaparecem na região ultrarelativista.

Uma revisão breve da teoria clássica inclui a introdução do conceito novo, isto é a n-parte do espectro. A fim de encontrar um análogo clássico adequado para a radiação das partículas fracamente excitadas, a ideia de reduzir o espectro clássico foi desenvolvida. Constatamos que as características da radiação calculadas para o espectro clássico reduzido permanecem em boa concordância, tanto quantitativa quanto qualitativa, com os seus análogos quânticos, pelo menos no que diz respeito aos espectros quânticos de uma ou duas harmônicas. Neste sentido, a teoria clássica do espectro reduzido pode ser chamada de representativa.

A evolução do máximo no espectro da radiação é considerada em capítulo separado. A aproximação, comumente considerada na teoria clássica para frequência crítica, é inválida quando as correções quânticas entram em cena. Mas existe uma possibilidade de encontrar as condições para o máximo transferir-se à harmônica maior do espectro quântico. É mostrado que as transferências ocorrem sucessivamente, começando com a harmônica principal no caso não relativístico, e este resultado permanece válido, independentemente de spin. Para uma partícula escalar existe um conjunto fixo dos valores críticos do campo

externo, de tal modo que a transferência do máximo da radiação entre duas harmônicas específicas pode acontecer somente quando a intensidade do campo externo é maior do que o valor crítico associado com essas harmônicas. Se essa condição não for satisfeita, a posição do máximo permanece inalterada. Verificamos que a presença de spin perturba esta condição, no caso do elétron os valores críticos da intensidade do campo dependem de número do nível inicial.

Abstract

In the framework of quantum theory the characteristics of synchrotron radiation (SR) are considered. In order to simplify theoretical description the process of radiation is restricted to single-photon emission. For arbitrary quantum transitions the spectral-angular distributions of SR power are given in exact analytical form. Scalar particles (bosons) and particles with spin $\hbar/2$ (electrons) are treated separately. Special attention is given to the particular transitions, namely, to the transitions to first excited and ground states. It is shown that the components of linear polarization of radiation from electron switch places due to the orientation of spin when the electron jumps to the ground state. This fact can be considered an analytical proof for the presence of π -component of quantum radiation in the plane of motion.

The radiation emitted from weakly excited particles is thoroughly analysed. To describe the evolution of the profiles of angular distributions various functions are introduced both for two- and three-level systems. For quantum transitions from the first excited state to the ground state the comparative analysis of radiation from bosons and electrons is performed, which helps to estimate the influence of spin and its direction on the characteristics of radiation. The radiation from unpolarized electron is considered separately. Tracking the behavior of effective angles allows to discover the inconsistency of well-known classical conclusion about the concentration of total (summed over spectrum) ultrarelativistic radiation in the plane of motion. It is shown that the effective angles of quantum radiation tend to finite values and do not vanish in ultrarelativistic region.

A brief review of classical theory includes an introduction of the new concept, n -part of spectrum. In order to find an adequate classical analogue for the radiation from weakly excited particles, the idea to reduce classical spectrum was developed. It turns out that the characteristics of radiation calculated for reduced classical spectrum stay in good quantitative and qualitative agreement with their quantum analogues, at least for single-harmonic and two-harmonic quantum spectra, and classical theory of a reduced spectrum can be claimed representational in this sense.

The evolution of maximum in radiation spectrum is considered in separate chapter. A well-known approximation obtained for critical frequency in the framework of classical theory is invalid when quantum corrections enter the picture. But there appears a possibility to find the conditions for the maximum to shift to the highest harmonic of finite quantum spectrum. It is shown that the shifts occur successively starting with primary harmonic in non-relativistic case, and this result remains valid independently of spin. For a scalar particle there exists a fixed set of numbers, which are the critical values of external field, such that the shift of radiation maximum in the spectrum of boson can only happen when the

intensity of external field is greater than certain critical value related to corresponding harmonic. If this condition is not satisfied, the position of maximum remains unchanged. It turns out that the presence of spin perturbs this picture, so that the critical values of field intensity depend on the number of initial level.

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Introduction

Synchrotron radiation (SR) is such a well-known and thoroughly studied phenomenon that it seems to be almost impossible to add anything essential in this field. Thanks to its unique properties, this radiation has a broad spectrum of scientific and technical applications (see e.g. [1–9]), and electron-storage rings serving as radiation sources are spread over the whole globe. Wide spectrum of SR [10, 11] gives an opportunity to work with frequencies required in particular experiment. One of the most important features of SR is a high degree of polarization [12–15].

Together with increasing number of practical applications, theoretical description of SR evolved significantly. It should be pointed out that experiments and theory always remained in good agreement (see e.g. [16, 17]). At present, theory of SR is a well-developed branch of theoretical physics, there are lots of publications on related topics. Apart from well-known monographs [18–31], papers and reviews [32–43], they appear often in laboratory reports and proceedings of workshops.

Shortly before the synchrotron radiation itself was physically observed for the first time, the possibility of its existence was theoretically predicted by Ivanenko and Pomeranchuk [44] and independently somewhat later by Schwinger [45]. Although these papers are considered a starting point for the theory of SR, the so-called classical description of this phenomenon is based on the results obtained by Schott in the end of nineteenth century. In [46, 47] Schott calculated the power of radiation emitted from a particle that moved along a circular trajectory. Initially, he tried to create a classical model of the stable atom and, in this regard, performed an expansion of electromagnetic field into a Fourier series and found the solutions of Maxwell-Lorentz equations for an electron moving on a circular orbit. In what follows, with the term "classical theory of SR" we refer to the well-known Schott formula obtained for the power of radiation, its derivation and consequences. Classical theory of SR is widely used in the physics of accelerators and mostly gives results which are quantitatively and qualitatively adequate with respect to this field of research.

At the same time, it is not very hard to find the features of radiation which classical theory fails to describe. For example, the discreteness of radiation influences the particle so that its trajectory widens, and these fluctuations are of a macroscopic order. An exact analytical description of such properties can be given in the framework of quantum theory of SR. Quantum approach to the characteristics of radiation is based on relativistic quantum mechanics and quantum electrodynamics, it was developed principally by Sokolov and Ternov in the 1960s [20, 39] (in this connection one should not forget about [48–51] and other important papers). By that time SR has already become a research tool for many fields of science where classical description was quite sufficient. In some works, the detailed analytical description of

radiation process itself didn't appear at all, since SR played a role of an instrument in an experiment, e.g. it was used for the studies of the properties of particular materials, etc. As a result, quantum theory has received undeservedly little attention (the reader may find some information on quantum theory and useful examples of the manifestation of quantum corrections in [52–57]).

Even now, most textbooks devoted to SR are aimed at a broad audience of the practitioners and SR users with different research interests, which do not necessarily lie in the field or imply a disciplinary background other than accelerator physics. As a result, these theoretical sources focus on classical approach to the radiation, and even though the material can be given in a very detailed form and contain useful examples, usually, it doesn't cover purely quantum processes and effects. The recoil effect, which is to be mentioned in chapter 5, can be considered a good illustration of such gap. Other example which comes readily to mind is the spin of a particle. Obviously, the effect of self-polarization of electrons and positrons in storage rings is of purely quantum nature. We do not claim a great need in exhaustively covering quantum theory of synchrotron radiation in every manual, a good overview would be quite enough to provide a better understanding. In general, many prospective users of SR would benefit greatly from having a working knowledge of particular aspects of quantum approach (some useful tips on the quantum approach to radiation may be found in [58]).

Of course, both classical and quantum theories of SR have already passed the stage of an exciting boom and can be considered as closed independent theoretical structures. Furthermore, for some values of radiation parameters these approaches remain in good agreement. Still, particular gaps can be found within any of these theories, and, sometimes, an attempt to remove them leads to the discovery of new and unexpected properties of radiation. This paper contains corresponding examples for spectral-angular distributions of SR.

In order to avoid excessively cumbersome terminology, in what follows we use a common SR jargon. Namely, the terms "classical" and "quantum" are to accompany various characteristics of SR. The reader is supposed to associate the corresponding characteristic with either classical or quantum theory of radiation. For instance, the concept "classical spectrum" should be understood as "the spectrum of radiation in classical theory", i.e. a well-known expansion of radiation emitted from a particle when its motion is periodic and considered in the framework of classical electrodynamics (see chapter 1).

Unlike classical theory, in quantum mechanics one should be very careful using a concept of trajectory. However, we can assume that the intensity of external field is an analogue of corresponding classical vector. Hereinafter we use the term "external" with respect to the magnetic field that provides the spiral (or circular) motion of the particle in order to distinguish it from the radiation fields associated with emitting charge. If in the classical theory the field provides circular motion of a charge, one bears this picture in mind for the considerations of quantum analogy. E.g., when the classical field is directed along the z -axis and the particle moves along a circular trajectory in xy -plane, (x, y, z) being a Cartesian frame, we refer to this plane as to "the plane of motion" or "the plane of orbit". In quantum theory one may choose the potentials that would provide the vector of external field intensity to be $\vec{H} = (0, 0, H)$ and continue using the term "plane of motion" for xy . The concept of radius of circular orbit can be transferred to the quantum picture as well (see [25]), but here we do not go into details about it. This short explanation is

supposed to help the reader to avoid confusion and to serve an apology for the abuse of terminology.

In the framework of quantum theory we separately consider the radiation from a particle with elementary charge e_0 and spin $\hbar/2$ and the radiation from a scalar particle, namely, the particle with e_0 and spin zero. The former is to be referred to as an *electron* while for the latter the term *boson* is used. The model of scalar particle might not appear useful from practical point of view (though in some particular practical cases spin properties can be neglected), but theoretically it is an excellent tool to either simplify the study of SR characteristics or distinguish the perturbation caused by spin properties when comparing to the characteristics of spinor particle. The calculations that we perform for a spin one half particle, in some sense, are closer to reality, since in different facilities (for example, ELSA (Germany) [59] and ALBA (Spain) [60]) polarized electrons are accelerated.

In this regard, we find it convenient to label the characteristics of radiation from boson with superscript 'b' and those of an electron - with 'e'. In some cases this labelling will be omitted for brevity, and in order to avoid confusion the reader will be notified in advance. Similarly to the terms *classical* and *quantum*, we use "bosonic" to indicate that the corresponding characteristic was considered for the radiation emitted from boson, e.g. a *bosonic curve* stands for the plot of some characteristic of radiation from scalar particle, etc.

Let W^{qu} be the power of radiation emitted from quantum particle, while W^{cl} stands for the power of radiation calculated in the framework of classical theory. The relation between W^{qu} and W^{cl} has the form [61]

$$W^{qu} = W^{cl} \left(1 - \frac{55\sqrt{3}}{16} \xi + (45 \pm 3) \xi^2 - \dots \right). \quad (1)$$

Here positive sign corresponds with an electron, whereas for a boson negative sign should be used [35]. $\xi = \gamma H/H_0$ is a fundamental quantum parameter [25, 62], γ is the relativistic factor ($1 \leq \gamma < \infty$), $H_0 = m_0^2 c^3 [|e_0| \hbar]^{-1} = 4,41 \cdot 10^{13} \text{ } Oe$ is the Schwinger field [61] (m_0 is the rest mass of the particle, c is the speed of light and e_0 stands for the charge of the particle; when these parameters refer to an electron we get the value of Schwinger field intensity). Generally, contemporary SR sources have $\xi \sim 10^{-6}$ (Table 1, an explicit list of accelerator facilities with radiation parameters and useful references can be found in [63]). As an example, the field of the superconducting dipole magnets of LHC reaches 8.3 T [64, 65] so that ξ -parameter is $\sim 10^{-5}$. It allows us to assume a possible increase of ξ for future sources. In accordance with (1), the values of $\xi \sim 10^{-2}$ already oblige the researcher to take quantum corrections into account.

Table 1: The values of ξ -parameter for contemporary SR sources

Facility	Energy, GeV	Magnetic field intensity, T	$\xi, 10^{-6}$
Canadian Light Source	2.9	1.35	1.7
Australian Synchrotron	3	1.3	1.71
SOLEIL Synchrotron	2.75	0.74	0.85
LNLS Synchrotron	1.37	1.47	1.03

In astrophysics SR is currently the main experimental source of our knowledge about physical processes in deep space [66–71]. Neutron stars and magnetars [72–76] are space objects with extremely powerful magnetic fields (see Table 2 [77, 78]), the intensities are high enough for the particles to be relativistic even at low Landau levels. For magnetars the value of parameter ξ does not allow to neglect quantum corrections (even at low levels ξ can reach $\sim 1 - 10$), and, therefore, the characteristics of radiation can only be treated adequately with the help of quantum theory.

Table 2: The magnetic field intensities of magnetars

Name	External field intensity, $10^{10} T$	Name	External field intensity, $10^{10} T$
SGR 1806-20	20	CXOU J171405.7-381031	5
1E 1841-045	7	1RXS J170849.0-400910	4.7
SGR 1900+14	7	CXOU J010043.1-721134	3.9
SGR 0526-66	5.6	1E 1048.1-5937	3.9

Radiation from relativistic particles that initially occupy low levels is of our special interest. It is exactly the case of substantial manifestation of quantum corrections. In this paper we consider weakly excited particles, namely, quantum two-level and three-level systems. That is, we consider quantum transitions from the first excited state to the ground state, as well as two possible transitions from the second excited state. For each of quantum transitions we assume the emission of a single photon though at different frequencies. Of course, one can complicate this problem by taking the possibility of many-photon emission into account. The corrections of higher orders, which are associated with the emission of several photons at the particular transition (two-photon emission, the emission of three photons, etc) are small in comparison with the first order corrections associated with the emission of a single photon. We restrict ourselves to considering simpler case, but this already allows to estimate the influence of quantum effects.

For synchrotron radiation it is already a tradition to start any manuscript with a brief or detailed review of classical theory, and even the manuals on technical applications of SR are no exception here. Partly out of respect for this tradition but, also, in order to provide necessary theoretical and conceptual backgrounds we start the analysis by focusing on the origin of Schott formula, polarization properties and basic characteristics of classical radiation. Details are omitted for a sophisticated reader not to get bored, at the same time, we intended to avoid long repeats of well-known results. However, various references serve to compensate this brevity. Also, first chapter introduces the reader to the fundamental quantitative properties of classical SR. For a radiation from classical particle the concepts of partial contribution with respect to spectral harmonics and electro-magnetic polarization are to be treated analytically and numerically, their definitions modified in order to be adapted for further study and comparative analysis. The reduced classical spectrum is to be considered, and, in this connection, we introduce the concept of n -part of spectrum in the framework of classical theory.

Second chapter is concerned with quantum theory of radiation, where we treat bosons and electrons separately and find spectral-angular distributions of the power. The radiation at arbitrary quantum transitions from n^{th} to n'^{th} excited state is to be considered. We give exact analytical expressions for spectral-angular distributions of radiation power both for total radiation and for the components of circular and linear polarization. The probabilistic approach to the definition of radiation power implies the calculation of matrix elements based on the solutions of Klein-Gordon-Fock and Dirac equations for scalar and spinor particles correspondingly. Further in this chapter we focus on particular quantum transitions, namely, on the transitions to the first excited state (when the penultimate harmonic of quantum spectrum is emitted) and the transitions to the ground state. It turned out that the radiation emitted due to these transitions has a number of interesting properties that have no classical interpretation and can only be explained in the framework of quantum theory.

Quantum and classical results obtained for particular spectral harmonics are expressed in terms of so-called Laguerre functions (the exact definition can be found in Appendix) and Bessel functions, some concepts, like effective angles of radiation, are defined implicitly (e.g., as the solutions of a system of integral equations), so we perform a great number of numerical simulations to track the behavior of SR characteristics. We hope that numerous plots ease the work for reader, since visual interpretation simplifies the details of comparative analysis.

Chapter 3 deals with quantum transitions from the first excited state. Practically, the corresponding relation between the energy of the particle $m_0 c^2 \gamma$ and the normalized magnetic field H/H_0 brings us back to magnetars with their surface field intensity sufficient to accelerate particles when n is low. In order to follow the evolution of angular profiles we introduce the effective angles of radiation as well as the deviation angles, the numerical analysis of which detects inconsistency between classical and quantum results. This chapter, again, starts with the description of boson, and then we switch to the radiation from spinor particle. Special attention is paid to unpolarized electron. The comparative analysis can be found in a separate section and reveals the influence of spin and its direction on radiation from quantum particle.

Since for $n = 1$ quantum spectrum contains a single harmonic, all the expressions have a much simpler form and can be given in terms of elementary functions or integrals of elementary functions. To estimate the significance of quantum corrections we compare new results with the predictions of classical theory (the primary harmonic of classical spectrum is considered a candidate for an adequate description of radiation emitted by the particle that initially occupied the first level and then jumps to the ground state).

Chapter 1 is a prelude to the study of spectral properties within the quantum theory, to which we turn our attention in chapter 4. Obviously, when initially the particle is in the second excited state it has two possibilities of transition. So, in this case two harmonics can be emitted and we already deal with a *spectrum*. Three-level system is the simplest example when the spectral characteristics of quantum radiation can be investigated. We perform the analysis both for bosons and electrons and, comparing the results with their classical analogues, set a question about the representativeness of classical description. Here the reduced classical spectrum (see chapter 1) enters the picture. The idea to reduce infinite classical

spectrum in order to provide a better correspondence with short quantum spectrum belongs to Sokolov, but it wasn't developed and the question about how to reduce the spectrum remained a puzzle. Intuitively, cutting the classical spectrum at the harmonic whose number corresponds with the initial quantum level of the particle seems an adequate solution. But here, again, we embark on a dangerous path. At first glance, this idea seems trivial, but the nature of classical and quantum spectrum should keep the reader from jumping to conclusions, one has to treat carefully quantitative and qualitative similarities and avoid any kind of confusion in this regard.

Last chapter focuses on tracking the evolution of maximum in quantum spectrum of SR. Almost any textbook on synchrotron radiation contains a well-known approximation for the critical frequency in high energy limit. Namely, classical theory predicts that the maximum of radiation shifts to higher frequencies as $\sim \gamma^3$ (see e.g. [13, 25, 79]) while the velocity of particle tends to reach ultrarelativistic domain. Nothing similar can we expect in quantum theory since quantum spectrum is finite and the number of emitted harmonic cannot exceed the number of initial energy level n . Still, we can set a problem of finding the conditions (if such conditions exist) for the maximum of radiation to lie at the highest harmonic of quantum spectrum. In accordance with the scheme used in previous chapters, we start with a scalar particle and then consider the maximum in the spectrum of electron. Note, that chapter 5 contains an overview of corresponding results of classical theory, which was included in order to provide the necessary basic understanding of the shift processes in radiation spectrum.

New results may serve a confirmation of how perspective and broad the theory of synchrotron radiation is, complete and well-formulated yet it continues to bring out various questions and problems, and this can be reasonably looked at as one of its greatest features.

Chapter 1

Classical Theory of Synchrotron Radiation

1.1 Spectral and Angular Distributions of Synchrotron Radiation in Classical Theory

In the framework of classical theory we consider the motion of a particle with elementary charge e_0 in a constant uniform magnetic field $\vec{H} = (0, 0, H)$. The power of radiation emitted from a classical particle can be defined as

$$W_0^{cl} = \oint_p (\vec{S} \cdot d\vec{p}), \quad (1.1)$$

see, for example, [80, p. 443]. Here $\vec{S}$ is the Poynting vector and $d\vec{p}$ is an element of a spherical surface p of radius R' , which contains the point P (the position of an observer, figure 1.1); the vector $d\vec{p}$ is directed along the normal to the surface p

$$d\vec{p} = \vec{n}' dp = \vec{n}' |\vec{R}'|^2 d\Omega', \quad \vec{n}' = \frac{\vec{R}'}{R'}, \quad (1.2)$$

where $d\Omega'$ is a solid angle associated with dp . Thus, [81, p. 67]

$$W_0^{cl} = \int (\vec{S} \cdot \vec{n}') R'^2 d\Omega'. \quad (1.3)$$

We restrict ourselves to the study of radiation in so-called "wave-zone" or "radiation zone" [80, p. 460]. That is, we consider those regions where the significance of the radiation field exceeds that of a convective field (the energy associated with velocity fields will not be taken into account). In the wave-zone the relation between the fields of a charge, $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$, is that of an electromagnetic plane wave

$$\tilde{\mathbf{E}} = (\tilde{\mathbf{H}} \times \vec{n}'), \quad \tilde{\mathbf{H}} = -(\tilde{\mathbf{E}} \times \vec{n}'), \quad (\tilde{\mathbf{E}} \cdot \vec{n}') = (\tilde{\mathbf{H}} \cdot \vec{n}') = (\tilde{\mathbf{H}} \cdot \tilde{\mathbf{E}}) = 0. \quad (1.4)$$

So, for the Poynting vector we find $\tilde{\mathbf{S}} = \frac{c}{4\pi} (\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}) = \frac{c}{4\pi} \vec{n}' |\tilde{\mathbf{E}}|^2$ and, therefore,

$$W_0^{cl} = \frac{c}{4\pi} \int |\tilde{\mathbf{E}}|^2 |\vec{R}'|^2 d\Omega'. \quad (1.5)$$

To find the power of emitted radiation we need to estimate $|\tilde{\mathbf{E}}|^2$. The field of radiation of a classical particle can be defined with the use of retarded potentials [80, p. 423] (see also [82–84])

$$\vec{A} = \frac{e_0}{c} \int \frac{\vec{v}(\tau)}{R'} \delta\left(\tau - t + \frac{R'}{c}\right) d\tau. \quad (1.6)$$

Clearly, (1.4) implies that in the wave-zone the scalar potential vanishes. We find it convenient to use figure 1.1 in order to explain the meaning of various parameters in (1.6). We assume that a charged particle moves along a circular orbit of radius R , i.e. the motion is periodic.

Figure 1.1: The motion of a classical charged particle in a constant uniform magnetic field.

Vector $\vec{r}$ defines the position of an observer P , $\vec{r}$ is constant. The distance between the current position of our particle and the observer is given by $\vec{R}'$, $\vec{R}' = \vec{r} - \vec{R}$. τ is the time in a chosen laboratory system (x, y, z) which is associated with the position $\vec{R}$, $\vec{R} = \vec{R}(\tau)$. Meanwhile, t is the moment when the observer "records" the radiation at the point P , $t = \tau + \frac{R'(\tau)}{c}$. The velocity of the particle is $\vec{v}(\tau) = \frac{d\vec{R}}{d\tau}$. The position of the classical particle in the xy -plane can be given by $\psi = \omega\tau$, $\omega = \frac{v}{R}$, $v = |\vec{v}|$. $\eta = \psi - \varphi$ is the angle between the projection of vector $\vec{r}$ and $\vec{R}(\tau)$. The radiation zone is supposed to be located at large distances from the source, so that $|\vec{r}|$ is supposed to be large enough to neglect the orders of $1/r$ higher than 1 (we do not take into account the terms with $(\frac{1}{r})^n$, $n > 1$). And since

$$R' = \sqrt{(\vec{r} - \vec{R})^2} = r \left(1 - 2 \frac{(\vec{r} \cdot \vec{R})}{r^2} + \frac{R^2}{r^2} \right)^{1/2} \approx r - R \sin \theta \cos \eta, \quad (1.7)$$

in the wave-zone $R' \sim r$. The relation (1.7) allows us to write

$$\vec{A} = \frac{e_0}{cR'} \int \vec{v} \delta\left(\tau - t + \frac{r}{c} - \frac{R \sin \theta \cos \eta}{c}\right) d\tau. \quad (1.8)$$

The motion is periodic, and we use the Fourier expansion of the δ -function

$$\delta(\tau) = \frac{\omega}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu\omega\tau} \quad (1.9)$$

to rewrite (1.8) in the form

$$\vec{A} = \frac{e_0}{cR'} \sum_{\nu=-\infty}^{\infty} \frac{1}{2\pi} e^{-i\nu\phi} \int_{-\pi}^{\pi} \vec{v} e^{i\nu(\alpha - \beta \sin \theta \sin \alpha)} d\alpha, \quad (1.10)$$

$$\phi = \omega t - \frac{\omega r}{c} - \varphi + \frac{\pi}{2}, \quad \alpha = \eta + \frac{\pi}{2}, \quad \beta = \frac{v}{c}. \quad (1.11)$$

Here we introduce the unit vectors $\vec{e}_2, \vec{e}_3, \vec{e}_{-1}, \vec{e}_1$ such that

$$\vec{e}_2 = \frac{(\vec{n}' \times \vec{j})}{\sqrt{1 - (\vec{n}' \cdot \vec{j})^2}}, \quad \vec{e}_3 = (\vec{n}' \times \vec{e}_2), \quad \vec{e}_g = \frac{\vec{e}_2 + ig\vec{e}_3}{\sqrt{2}}, \quad g = \pm 1, \quad (1.12)$$

$$(\vec{e}_2 \cdot \vec{e}_3) = (\vec{n}' \cdot \vec{e}_2) = (\vec{n}' \cdot \vec{e}_3) = 0, \quad (1.13)$$

where $(\times)$ stands for vector product, $(\cdot)$ is the scalar product, $\vec{j}$ defines the direction of the magnetic field, $\vec{j} = (0, 0, 1)$. Obviously, $\vec{e}_2$ and $\vec{e}_3$ define the linear polarization, whereas $\vec{e}_g, g = 1, -1$, stands for the circular polarization components, right and left respectively.

Although it is a standard labelling method [20,25], it is convenient to use the following table in order to avoid confusion.

Table 1.1: The components of radiation polarization

i	-1	1	0	2	3
type of polarization	left circular	right circular	total radiation	σ -component	π -component

The velocity can be given by

$$\vec{v} = v_2 \vec{e}_2 + v_3 \vec{e}_3 = v_1 \vec{e}_1 + v_{-1} \vec{e}_{-1}, \quad (1.14)$$

where the amplitudes $v_2 = (\vec{v} \cdot \vec{e}_2)$ and $v_3 = (\vec{v} \cdot \vec{e}_3)$. In the wave-zone we have

$$v_2 = v \cos \eta = v \sin \alpha, \quad v_3 = v \cos \theta \sin \eta = -v \cos \theta \cos \alpha, \quad (1.15)$$

and thus

$$A_2 = \frac{e_0 \beta}{R'} \sum_{\nu=-\infty}^{\infty} \frac{1}{2\pi} e^{-i\nu\phi} \int_{-\pi}^{\pi} e^{i\nu(\alpha - \beta \sin \theta \sin \alpha)} \sin \alpha d\alpha, \\ A_3 = -\frac{e_0 \beta}{R'} \sum_{\nu=-\infty}^{\infty} \frac{1}{2\pi} e^{-i\nu\phi} \cos \theta \int_{-\pi}^{\pi} e^{i\nu(\alpha - \beta \sin \theta \sin \alpha)} \cos \alpha d\alpha. \quad (1.16)$$

To perform the integration over α it is useful to bear in mind the integral representation of Bessel functions [85, p. 511], [86]

$$J_\nu(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(\nu\alpha - z \sin \alpha)} d\alpha, \quad (1.17)$$

as well as their properties [85, p. 513,518]

$$J_{\nu+1}(z) + J_{\nu-1}(z) = \frac{2\nu}{z} J_\nu(z), \quad J'_\nu(z) = \frac{1}{2}(J_{\nu-1}(z) - J_{\nu+1}(z)). \quad (1.18)$$

Using these expressions one can easily get

$$A_2 = \frac{e_0 \beta}{R'} \sum_{\nu=-\infty}^{\infty} e^{-i\nu\phi} i J'_\nu(\nu \beta \sin \theta), \quad A_3 = -\frac{e_0 \beta}{R'} \cot \theta \sum_{\nu=-\infty}^{\infty} e^{-i\nu\phi} J_\nu(\nu \beta \sin \theta). \quad (1.19)$$

Within the wave-zone the intensity of the radiation field can be given by

$$\tilde{\mathbf{E}} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad R' \sim r, \quad (1.20)$$

and therefore (see [87])

$$\begin{aligned} \tilde{E}_2 &= -\frac{2e_0\beta^2}{RR'} \sum_{\nu=1}^{\infty} \nu J'_\nu(\nu\beta \sin \theta) \cos \nu\phi, \\ \tilde{E}_3 &= \frac{2e_0\beta^2}{RR'} \cot \theta \sum_{\nu=1}^{\infty} \nu J_\nu(\nu\beta \sin \theta) \sin \nu\phi, \end{aligned} \quad (1.21)$$

where the derivative is taken with respect to the argument of the Bessel function and the following properties are used

$$J_\nu(\nu z) = J_{-\nu}(-\nu z), \quad J'_\nu(\nu z) = -J'_{-\nu}(-\nu z). \quad (1.22)$$

Now we use (1.21) in place of $\tilde{\mathbf{E}}$ in (1.5) and perform the integration over t , $\frac{1}{T} \int_0^T dt$ (averaging over the period of motion). Note that

$$\begin{aligned} \frac{1}{T} \int_0^T \cos \nu\phi \cos \nu'\phi dt &= \frac{1}{T} \int_0^T \sin \nu\phi \sin \nu'\phi dt = \frac{1}{2} \delta_{\nu,\nu'}, \\ \frac{1}{T} \int_0^T \sin \nu\phi \cos \nu'\phi dt &= 0. \end{aligned} \quad (1.23)$$

The radius of orbit R may be defined in terms of the energy of the moving particle (E) and the intensity of the magnetic field ($\vec{H} = H\vec{j}$) [25, 33, 87]

$$R = \frac{\beta E}{e_0 H}, \quad E = \frac{m_0 c^2}{\sqrt{1 - \beta^2}}. \quad (1.24)$$

Finally, for spectral-angular distributions of SR power we get

$$\frac{dW_i^h(\nu, \beta, \theta)}{d\Omega} = \frac{Q_0^{cl} A^{cl}(\beta)}{16} F_i^h(\nu, \beta, \theta), \quad Q_0^{cl} = \frac{e_0^4 H^2}{m_0^2 c^3}, \quad A^{cl}(\beta) = \beta^2(1 - \beta^2), \quad d\Omega = \sin \theta d\theta. \quad (1.25)$$

The explicit expressions for the functions $F_i^h(\nu, \beta, \theta)$ are known to form the Schott formula [47, p. 642], they have been cited repeatedly (see, for example, [20, 25, 87, 88])

$$\begin{aligned} F_2^h(\nu, \beta, \theta) &= 16 \nu^2 J_\nu'^2(\nu x), \quad F_3^h(\nu, \beta, \theta) = 16 \frac{\cos^2 \theta}{x^2} \nu^2 J_\nu^2(\nu x), \\ F_0^h(\nu, \beta, \theta) &= F_2^h(\nu, \beta, \theta) + F_3^h(\nu, \beta, \theta), \quad F_g^h(\nu, \beta, \theta) = \frac{F_0^h(\nu, \beta, \theta)}{2} + g \frac{\cos \theta}{x} \alpha_\nu(x), \\ \alpha_\nu(x) &= 16 \nu^2 J_\nu'(\nu x) J_\nu(\nu x), \quad x = x(\beta, \theta) = \beta \sin \theta. \end{aligned} \quad (1.26)$$

We used (1.12) in order to find the components of the circular polarization. Index h serves to distinguish spectral-angular and angular distributions in classical theory, it refers to the power of radiation associated with particular harmonic ν of classical spectrum.

Integrating (1.25) over θ from 0 to $\pi/2$, we obtain the power of radiation associated with a harmonic ν emitted into the region above the orbit of motion (upper half-plane) - $W_i^{(+h)}$ [25]. To obtain the

expression which would describe the radiation at this harmonic in the lower half-plane - $W_i^{(-)h}$ - one has to consider $[\pi/2, \pi]$ as the interval of integration in (1.25).

$$\begin{aligned}
W_i^{(+h)}(\nu, \beta) &= \frac{Q_0^{cl} A^{cl}(\beta)}{16} f_i^h(\nu, \beta), \quad f_i^h(\nu, \beta) = \int_0^{\pi/2} F_i^h(\nu, \beta, \theta) \sin \theta d\theta, \\
f_2^h(\nu, \beta) &= 8 \left[\frac{2\nu}{\beta} J'_{2\nu}(2\nu\beta) + \frac{\nu}{\beta} \int_0^{2\nu\beta} J_{2\nu}(x) dx - \frac{2\nu^2}{\beta^2} \int_0^{2\nu\beta} \frac{J_{2\nu}(x)}{x} dx \right], \\
f_3^h(\nu, \beta) &= 8 \frac{\nu^2}{\beta^2} \left[2 \int_0^{2\nu\beta} \frac{J_{2\nu}(x)}{x} dx - \frac{1}{\nu\beta} \int_0^{2\nu\beta} J_{2\nu}(x) dx \right], \\
f_0^h(\nu, \beta) &= f_2^h(\nu, \beta) + f_3^h(\nu, \beta), \quad f_g^h(\nu, \beta) = \frac{f_0^h(\nu, \beta)}{2} + g \frac{8\nu}{\beta^2} J_\nu^2(\nu\beta).
\end{aligned} \tag{1.27}$$

And summing (1.25) over the spectrum (which is over ν , $1 \leq \nu < \infty$) we get the angular distributions of the total radiated power

$$\frac{dW_i^{cl}(\beta, \theta)}{d\Omega} = \sum_{\nu=1}^{\infty} \frac{dW_i^h(\nu, \beta, \theta)}{d\Omega} = \frac{Q_0^{cl} A^{cl}(\beta)}{16} F_i(\beta, \theta), \tag{1.28}$$

$$F_i(\beta, \theta) = \sum_{\nu=1}^{\infty} F_i^h(\nu, \beta, \theta). \tag{1.29}$$

With the use of sums

$$\sum_{\nu=1}^{\infty} \nu^2 J_\nu^2(\nu z) = \frac{1}{16} \frac{z^2(4+z^2)}{(1-z^2)^{7/2}}, \quad \sum_{\nu=1}^{\infty} \nu^2 [J'_\nu(\nu z)]^2 = \frac{1}{16} \frac{(4+3z^2)}{(1-z^2)^{5/2}}, \tag{1.30}$$

known for Bessel functions [89, p. 573], simple expressions can be obtained for $F_i(\beta, \theta)$ (see e.g. [20, 25])

$$\begin{aligned}
F_2(\beta, \theta) &= \frac{4+3x^2}{(1-x^2)^{5/2}}, \quad F_3(\beta, \theta) = \frac{(4+x^2)\cos^2\theta}{(1-x^2)^{7/2}}, \\
F_0(\beta, \theta) &= F_2(\beta, \theta) + F_3(\beta, \theta), \quad F_g(\beta, \theta) = \frac{F_0(\beta, \theta)}{2} + g \frac{\cos\theta}{x} \alpha(x), \\
\alpha(x) &= \sum_{\nu=1}^{\infty} \alpha_\nu(x) = 16 \sum_{\nu=1}^{\infty} \nu^2 J'_\nu(\nu x) J_\nu(\nu x).
\end{aligned} \tag{1.31}$$

In [90, 91] circular polarization was studied in the framework of classical theory. For any possible value of parameter x a good approximation was found for the function $\alpha(x)$ (see expression (23) in [90, p. 417])

$$\begin{aligned}
\alpha(x) &\approx \frac{4x \chi_2(x)}{(1-x^2)^3}, \\
\chi_2(x) &= 1 + 2 \left(\frac{x}{2}\right)^2 - \frac{1}{2} \left(\frac{x}{2}\right)^4 + \frac{1}{9} \left(\frac{x}{2}\right)^6 - \frac{5}{288} \left(\frac{x}{2}\right)^8 + \frac{4}{225} \left(\frac{x}{2}\right)^{10} - \frac{2657}{2880} \left(\frac{x}{2}\right)^{12}.
\end{aligned} \tag{1.32}$$

We have been using this particular approximation only to perform numerical simulations, the way of derivation of (1.32) can be found in [90] and, in the framework of current thesis, we find it unnecessary to repeat the explicit derivation. We restrict ourselves to commenting that the expansion of the product of the Bessel function and its derivative was truncated, and the orders higher than $(\frac{x}{2})^{10}$ were omitted. The approximation was performed so that the values of $\chi_2(0.9)$ and $\chi_2(1)$ coincide with corresponding exact values no less than up to the orders 10^{-5} .

Integrating (1.29) over $d\Omega$ one can obtain the expressions for total power of classical radiation (see e.g. (7) on [90, p. 415] or [20, 25])

$$W_0^{cl}(\beta) = \frac{2}{3} Q_0^{cl} A^{cl}(\beta) f^{cl}(\beta), \quad f^{cl}(\beta) = \frac{3}{8} f_0^{cl}(\beta), \quad f_0^{cl}(\beta) = f_2^{cl}(\beta) + f_3^{cl}(\beta),$$

$$f_2^{cl}(\beta) = \frac{6 + \beta^2}{3(1 - \beta^2)^2}, \quad f_3^{cl}(\beta) = \frac{2 - \beta^2}{3(1 - \beta^2)^2}, \quad f_2^{cl}(0) = 2, \quad f_3^{cl}(0) = \frac{2}{3}. \quad (1.33)$$

In order to describe the degree of polarization in the upper half-plane we introduce the functions $q_i^{cl}(\beta)$ such that

$$W_i^{(+cl)}(\beta) = \frac{1}{2} W_0^{cl}(\beta) q_i^{cl}(\beta). \quad (1.34)$$

Here $W_i^{(+cl)}(\beta)$ defines the amount of total (summed over spectrum) polarized radiation emitted above the orbit of motion. Namely, in (1.28) the integration over θ was performed from 0 to $\pi/2$. The relation between the amount of total (summed over spectrum) radiation emitted in the lower half-plane ($W_i^{(-cl)}(\beta)$) and in the upper half plane ($W_i^{(+cl)}(\beta)$) has the form

$$W_i^{(-cl)}(\beta) = W_i^{(+cl)}(\beta) \quad (i = 0, 2, 3), \quad W_g^{(-cl)}(\beta) = W_{-g}^{(+cl)}(\beta). \quad (1.35)$$

The definition of $q_i^{cl}(\beta)$ implies

$$q_0^{cl}(\beta) = q_2^{cl}(\beta) + q_3^{cl}(\beta) = q_g^{cl}(\beta) + q_{-g}^{cl}(\beta) = 1. \quad (1.36)$$

For the functions $q_i^{cl}(\beta)$ we can write explicitly [25]

$$q_2^{cl}(\beta) = \frac{f_2^{cl}(\beta)}{f_0^{cl}(\beta)} = \frac{6 + \beta^2}{8}, \quad q_3^{cl}(\beta) = \frac{f_3^{cl}(\beta)}{f_0^{cl}(\beta)} = \frac{2 - \beta^2}{8},$$

$$q_g^{cl}(\beta) = \frac{1}{2} + g \frac{f_1^{cl}(\beta)}{f_0^{cl}(\beta)} = \frac{1}{2} + g \frac{3(1 - \beta^2)^2}{2\beta^2} \sum_{\nu=1}^{\infty} \nu J_{\nu}^2(\nu\beta). \quad (1.37)$$

Further numerical simulations, again, require some simplification. Here we also use the results obtained in [90] for the characteristics of circularly polarized radiation, and for the functions $q_g^{cl}(\beta)$ we write the approximation which is valid for all possible values of x , $|x| < 1$, (see (23) in [90, p. 417])

$$q_g^{cl}(\beta) \approx \frac{1}{2} + g \frac{3}{8} \chi_1(\beta), \quad f_1^{cl}(\beta) \approx \frac{\chi_1(\beta)}{(1 - \beta^2)^2},$$

$$\chi_1(x) = 1 - \left(\frac{x}{2}\right)^2 - \frac{1}{6} \left(\frac{x}{2}\right)^4 - \frac{5}{36} \left(\frac{x}{2}\right)^6 - \frac{65}{288} \left(\frac{x}{2}\right)^8 + \frac{1397}{1080} \left(\frac{x}{2}\right)^{10}. \quad (1.38)$$

For non-relativistic limit one can easily find

$$q_2^{cl}(0) = 3/4, \quad q_3^{cl}(0) = 1/4, \quad q_g^{cl}(0) = (4 + 3g)/8. \quad (1.39)$$

The plots of $q_i^{cl}(\beta)$ and the analysis of their behavior will be given in the section 1.2.2.

In order to define the part of polarized radiation emitted into a small solid angle ($d\Omega$) near some direction (given by θ) it is convenient to introduce $p_i^{cl}(\beta, \theta)$

$$\frac{dW_i^{cl}(\beta, \theta)}{d\Omega} = W_0^{cl}(\beta) p_i^{cl}(\beta, \theta), \quad \text{i. e. } p_i^{cl}(\beta, \theta) = \frac{F_i(\beta, \theta)}{4f_0^{cl}(\beta)}. \quad (1.40)$$

Figures 1.2 - 1.5 illustrate the behavior of these functions and confirm a number of well-known (in classical SR theory) results [20, 25, 39]. For instance, at any energy in the plane of orbit ($\theta = \pi/2$) the

Figure 1.2: The plots of $p_2^{cl}(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.3: The plots of $p_3^{cl}(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.4: The plots of $p_0^{cl}(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.5: The plots of $p_1^{cl}(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

radiation is completely linearly polarized, $p_3^{cl}(\beta, \pi/2) = 0$. For the components of linear polarization, as well as for the total power of radiation, one can observe the symmetry of angular distribution profiles with respect to the plane of motion, namely, $p_i^{cl}(\beta, \pi - \theta) = p_i^{cl}(\beta, \theta)$, ($i = 0, 2, 3$). The components of circular polarization mirror each other with respect to $\pi/2$, $p_g^{cl}(\beta, \pi - \theta) = p_{-g}^{cl}(\beta, \theta)$.

It is easy to find

$$\begin{aligned}
 2p_2^{cl}(\beta, 0) &= 2p_3^{cl}(\beta, 0) = p_0^{cl}(\beta, 0) = \frac{2}{f_0^{cl}(\beta)}, \quad p_{-1}^{cl}(\beta, 0) = 0, \\
 p_2^{cl}(\beta, \pi/2) &= p_0^{cl}(\beta, \pi/2) = \frac{4 + 3\beta^2}{4(1 - \beta^2)^{5/2} f_0^{cl}(\beta)}, \quad p_2^{cl}(0, \theta) = \frac{3}{8}, \quad p_3^{cl}(0, \theta) = \frac{3 \cos^2 \theta}{8}, \\
 p_0^{cl}(0, \theta) &= \frac{3(1 + \cos^2 \theta)}{8}, \quad p_1^{cl}(0, \theta) = \frac{3(1 + \cos \theta)^2}{16}.
 \end{aligned} \tag{1.41}$$

According to their definitions, the functions $q_i^{cl}(\beta)$ can be interpreted in terms of $p_i^{cl}(\beta, \theta)$ as follows

$$q_i^{cl}(\beta) = 2 \int_0^{\pi/2} p_i^{cl}(\beta, \theta) \sin \theta d\theta. \tag{1.42}$$

The degree of polarization of radiation emitted in θ -direction can be studied with the use of the functions

$$q_i^{cl}(\beta, \theta) = \frac{p_i^{cl}(\beta, \theta)}{p_0^{cl}(\beta, \theta)} = \frac{F_i(\beta, \theta)}{F_0(\beta, \theta)}. \tag{1.43}$$

The evolution of q_i^{cl} ($i = 1, 2, 3$) with β and with θ can be observed in the figures 1.6 - 1.8. It follows from definition that $q_0^{cl}(\beta, \theta) = q_2^{cl}(\beta, \theta) + q_3^{cl}(\beta, \theta) = q_g^{cl}(\beta, \theta) + q_{-g}^{cl}(\beta, \theta) = 1$.

Figure 1.6: The plots of $q_2^{cl}(\beta, \theta)$, $\beta = 0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99, 0.999, 1$. Figure 1.7: The plots of $q_3^{cl}(\beta, \theta)$, $\beta = 0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99, 0.999, 1$. Figure 1.8: The plots of $q_1^{cl}(\beta, \theta)$, $\beta = 0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99, 0.999, 1$.

As was in the case of $p_i^{cl}(\beta, \theta)$, the behavior of functions $q_i^{cl}(\beta, \theta)$ indicates the presence of σ -component of linear polarized radiation in the plane of orbit. In fact, σ -component dominates at any energy (except for the direction of external field in ultrarelativistic domain where the distribution of radiation power between the components of linear polarization is half to half). Functions $q_2^{cl}(\beta, \theta)$ increase with θ on $[0, \pi/2]$. So, the contribution of σ -component increases as θ gets closer to the orbit. On the contrary, for every β the functions $q_3^{cl}(\beta, \theta)$ decrease on $[0, \pi/2]$ and, thus, the absence of π -component of radiation near the plane of orbit is confirmed.

It could be seen in the figures 1.6 - 1.8 that the functions $q_i^{cl}(\beta, \theta)$, as well as $p_i^{cl}(\beta, \theta)$, for $i = 2, 3$ are symmetrical with respect to $\pi/2$, $q_i^{cl}(\beta, \pi - \theta) = q_i^{cl}(\beta, \theta)$ and $q_g^{cl}(\beta, \pi - \theta) = q_{-g}^{cl}(\beta, \theta)$. Similarly to (1.36) we can find particular values of $q_i^{cl}(\beta, \theta)$

$$q_2^{cl}(\beta, 0) = q_3^{cl}(\beta, 0) = \frac{1}{2}, \quad q_g^{cl}(\beta, 0) = \frac{1+g}{2}, \quad q_2^{cl}(\beta, \frac{\pi}{2}) = 2 q_g^{cl}(\beta, \frac{\pi}{2}) = 1, \\ q_2^{cl}(0, \theta) = \frac{1}{1 + \cos^2 \theta}, \quad q_3^{cl}(0, \theta) = \frac{\cos^2 \theta}{1 + \cos^2 \theta}, \quad q_1^{cl}(0, \theta) = \frac{(1 + \cos \theta)^2}{2(1 + \cos^2 \theta)}, \quad (1.44)$$

and at $\theta = 0$ we have $2q_2^{cl}(0, 0) = 2q_3^{cl}(0, 0) = q_1^{cl}(0, 0) = 1$.

1.2 First Harmonic of Classical Spectrum

1.2.1 Polarization Properties of Radiation Associated with the First Harmonic

Let us focus on studying the radiation associated with the first (or "primary") harmonic [92]. Assuming $\nu = 1$ in (1.25) and (1.26) for the distribution of power in this case we obtain

$$F_2^h(1, \beta, \theta) = 16 J_1'^2(x), \quad F_3^h(1, \beta, \theta) = 16 \frac{\cos^2 \theta}{x^2} J_1^2(x),$$

$$F_0^h(1, \beta, \theta) = F_2^h(1, \beta, \theta) + F_3^h(1, \beta, \theta), \quad (1.45)$$

$$F_g^h(1, \beta, \theta) = \frac{F_0^h(1, \beta, \theta)}{2} + g \frac{\cos \theta}{x} \alpha_1(x), \quad \alpha_1(x) = 16 J_1'(x) J_1(x).$$

Figures 1.9 - 1.12 show the behavior of the functions $F_i^h(1, \beta, \theta)$.

Figure 1.9: The plots of $F_2^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.10: The plots of $F_3^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.11: The plots of $F_0^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.12: The plots of $F_1^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

According to (1.27), the power of radiation associated with first harmonic in upper half-plane is defined by the functions

$$f_2^h(1, \beta) = 8 \left[\frac{2}{\beta} J_2'(2\beta) + \frac{1}{\beta} \int_0^{2\beta} J_2(x) dx - \frac{2}{\beta^2} \int_0^{2\beta} \frac{J_2(x)}{x} dx \right],$$

$$f_3^h(1, \beta) = \frac{8}{\beta^2} \left[2 \int_0^{2\beta} \frac{J_2(x)}{x} dx - \frac{1}{\beta} \int_0^{2\beta} J_2(x) dx \right], \quad f_0^h(1, \beta) = f_2^h(1, \beta) + f_3^h(1, \beta), \quad (1.46)$$

$$f_1^h(1, \beta) = \frac{f_0^h(1, \beta)}{2} + \frac{8}{\beta^2} J_1^2(\beta).$$

Let us consider the analogues of $q_i^{cl}(\beta)$ "constructed" for a certain harmonic. In other words, we introduce the functions which define the degree of radiation polarization for this particular harmonic

$$W_i^{(+h)}(\nu, \beta) = \frac{1}{2} W_0^h(\nu, \beta) q_i^h(\nu, \beta), \quad W_0^h(\nu, \beta) = \int_0^\pi \frac{dW_0^h(\nu, \beta, \theta)}{d\Omega} \sin \theta d\theta. \quad (1.47)$$

With the help of (1.27), this definition can be rewritten in the form

$$q_i^h(\nu, \beta) = \frac{f_i^h(\nu, \beta)}{f_0^h(\nu, \beta)}, \quad (1.48)$$

and, therefore, for the first harmonic

$$q_i^h(1, \beta) = \frac{f_i^h(1, \beta)}{f_0^h(1, \beta)}. \quad (1.49)$$

It is easy to see that in non-relativistic limit $q_i^h(1, 0) = q_i^{cl}(0)$, ($i = 0, 2, 3, \pm 1$). Together with (1.39), this equality reaffirms a well-known conclusion of classical theory, when $\beta \rightarrow 0$ the only frequency radiated is the primary one [92, p. 4].

Similarly, for each harmonic ν we can calculate the contribution of polarized radiation, emitted into a solid angle $d\Omega$ in θ -direction, to the total power of radiation associated with this harmonic

$$\frac{dW_i^h(\nu, \beta, \theta)}{d\Omega} = W_0^h(\nu, \beta) p_i^h(\nu, \beta, \theta), \quad p_i^h(\nu, \beta, \theta) = \frac{F_i^h(\nu, \beta, \theta)}{2f_0^h(\nu, \beta)}. \quad (1.50)$$

The graphs of functions $p_i^h(1, \beta, \theta)$, are given in the figures 1.13 - 1.16. The behavior of functions $p_i^h(1, \beta, \theta)$ differs fundamentally from the behavior of $p_i^{cl}(\beta, \theta)$. Indeed, with the help of the figures 1.2 - 1.5 it was shown that for the total (summed over spectrum) radiation emitted in the direction of external magnetic field ($\theta = 0$) $p_i^{cl}(\beta, \theta)$ decreases with energy. However, as it could be seen in figures 1.13 - 1.16, for the primary harmonic we have an opposite tendency. In addition, for the first harmonic the relative amount of total (summed over linear or circular polarization components) radiation emitted in the plane of motion ($\theta = \pi/2$) decreases with β . Nevertheless, it doesn't vanish as was in the case of total classical spectrum. In non-relativistic limit $p_i^h(1, 0, \theta) = p_i^{cl}(0, \theta)$, and in the plane of orbit we have

$$p_2^h(1, \beta, \pi/2) = p_0^h(1, \beta, \pi/2) = 16J_1^2(\beta) = 2p_1^h(1, \beta, \pi/2), \quad p_3^h(1, \beta, \pi/2) = 0. \quad (1.51)$$

By analogy with (1.43) we introduce the functions which define the part of polarized radiation of a particular frequency emitted in the direction θ

$$q_i^h(\nu, \beta, \theta) = \frac{p_i^h(\nu, \beta, \theta)}{p_0^h(\nu, \beta, \theta)} = \frac{F_i^h(\nu, \beta, \theta)}{F_0^h(\nu, \beta, \theta)}. \quad (1.52)$$

The behavior of $q_i^h(1, \beta, \theta)$ ($i = 1, 2, 3$) is shown in the figures 1.17 - 1.19.

Unlike the functions $p_i^h(1, \beta, \theta)$, the behavior of $q_i^h(1, \beta, \theta)$ stays in good qualitative agreement with the results which classical theory gives for the total spectrum. Within the interval $[0, \pi/2]$, same as $q_2^{cl}(\beta, \theta)$, $q_2^h(1, \beta, \theta)$ increases, while $q_3^h(1, \beta, \theta)$ and $q_1^h(1, \beta, \theta)$ decreases similarly to $q_3^{cl}(\beta, \theta)$ and $q_1^{cl}(\beta, \theta)$. It means that the radiation of primary harmonic emitted in fixed direction has the same the polarization properties as the total (summed over spectrum) radiation in this direction. We note that in the non-relativistic limit $q_i^h(1, \beta, \theta) = q_i^{cl}(\beta, \theta)$, and at $\theta = \pi/2$ we also have $q_i^h(1, \beta, \pi/2) = q_i^{cl}(\beta, \pi/2)$.

Figure 1.13: The plots of $p_2^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.14: The plots of $p_3^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.15: The plots of $p_0^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.16: The plots of $p_1^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

1.2.2 Partial Contribution of the First Harmonic

The definition of partial contribution of spectral components to the total power was introduced in [93]. However, in [93], for each type of polarization i the portion of radiation associated with harmonic ν was considered. Here we use the term "partial contribution" for the relation between the power of radiation associated with fixed harmonic ν and with polarization type i and total power (summed over spectral and polarization components), i.e. we single out both the polarization component and the harmonic. In the framework of current research such modification is performed for the reasons of convenience.

For fixed ν , for i component of polarization, partial contribution can be defined in terms of $\tilde{q}_i^h(\nu, \beta)$

$$W_i^{(+h)}(\nu, \beta) = \frac{1}{2} W_0^{cl}(\beta) \tilde{q}_i^h(\nu, \beta), \quad \tilde{q}_i^h(\nu, \beta) = \frac{f_i^h(\nu, \beta)}{2f_0^{cl}(\beta)} = \frac{3(1 - \beta^2)^2 f_i^h(\nu, \beta)}{16}. \quad (1.53)$$

Since $\tilde{q}_i^h(\nu, \beta)$ differs from $f_i^h(\nu, \beta)$, whose plots for $\nu = 1$ are given in the figure 1.20, only by a scaling factor $3(1 - \beta^2)^2/16$, we do not find it necessary to plot these functions. It is sufficient to give the plots of $\tilde{q}_i^h(\nu, \beta)$ together with $q_i^{cl}(\beta)$ and $q_i^h(1, \beta)$ (see figures 1.21 and 1.22) in order to complete further analysis.

Obviously, there is a relation between the functions $\tilde{q}_i^h(\nu, \beta)$ and $q_i^{cl}(\beta)$

$$\sum_{\nu=1}^{\infty} \tilde{q}_i^h(\nu, \beta) = q_i^{cl}(\beta). \quad (1.54)$$

Figure 1.17: The plots of $q_2^h(1, \beta, \theta)$, $\beta = 0, 0.6, 0.8, 1$. Figure 1.18: The plots of $q_3^h(1, \beta, \theta)$, $\beta = 0, 0.6, 0.8, 1$.

Figure 1.19: The plots of $q_1^h(1, \beta, \theta)$, $\beta = 0, 0.6, 0.8, 1$.

Figure 1.20: The plots of $f_i^h(1, \beta)$. The number of curve corresponds with the number of polarization component i .

It is easy to verify that $\tilde{q}_i^h(1, 0) = q_i^h(1, 0) = q_i^{cl}(0)$.

The functions $\tilde{p}_i^h(\nu, \beta, \theta)$ define partial contribution of radiation associated with harmonic ν emitted in θ -direction

$$\frac{dW_i^h(\nu, \beta, \theta)}{d\Omega} = W_0^{cl}(\beta) \tilde{p}_i^h(\nu, \beta, \theta), \quad \tilde{p}_i^h(\nu, \beta) = \frac{F_i^h(\nu, \beta, \theta)}{4f_0^{cl}(\beta)} = \frac{3(1 - \beta^2)^2 F_i^h(\nu, \beta, \theta)}{32}. \quad (1.55)$$

These functions differ from $F_i^h(1, \beta, \theta)$, whose plots are given in figures 1.9 - 1.12, by the scaling factor $3(1 - \beta^2)^2/32$. In the non-relativistic limit they are identical to $p_i^{cl}(\beta, \theta)$ and $p_i^h(1, \beta, \theta)$ which we introduced earlier. We also give explicit expressions for $\tilde{p}_i^h(1, \beta, \theta)$ in the plane of motion

$$\begin{aligned} \tilde{p}_2^h(1, \beta, \pi/2) &= \tilde{p}_0^h(1, \beta, \pi/2) = 2\tilde{p}_1^h(1, \beta, \pi/2) = \\ &= 3J_1^2(\beta)(1 - \beta^2)^2/32, \quad \tilde{p}_3^h(1, \beta, \pi/2) = 0. \end{aligned} \quad (1.56)$$

The relation between $\tilde{p}_i^h(\nu, \beta, \theta)$ and $p_i^{cl}(\beta, \theta)$ has the following form

$$\sum_{\nu=1}^{\infty} \tilde{p}_i^h(\nu, \beta, \theta) = p_i^{cl}(\beta, \theta). \quad (1.57)$$

Also, it is convenient to introduce the functions $\tilde{q}_i^h(\nu, \beta, \theta)$

$$\tilde{q}_i^h(\nu, \beta, \theta) = \frac{\tilde{p}_i^h(\nu, \beta, \theta)}{p_0^{cl}(\beta, \theta)} = \frac{F_i^h(\nu, \beta, \theta)}{F_0(\beta, \theta)}, \quad \sum_{\nu=1}^{\infty} \tilde{q}_i^h(\nu, \beta, \theta) = q_i^{cl}(\beta, \theta). \quad (1.58)$$

Figure 1.21: The plots of $q_2^{cl}(\beta)$ (1), $q_2^h(1, \beta)$ (2) and $\tilde{q}_2^h(1, \beta)$ (3).

Figure 1.22: The plots of $q_1^{cl}(\beta)$ (1), $q_1^h(1, \beta)$ (2) and $\tilde{q}_1^h(1, \beta)$ (3).

Corresponding plots are given in the figures 1.23 - 1.26.

Figure 1.23: The plots of $\tilde{q}_2^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.24: The plots of $\tilde{q}_3^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.25: The plots of $\tilde{q}_0^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.26: The plots of $\tilde{q}_1^h(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Using the definition, in non-relativistic limit we obtain $\tilde{q}_i^h(1, \beta, \theta) = q_i^h(1, \beta, \theta) = q_i^{cl}(1, \beta, \theta)$. At $\theta = \pi/2$ we have

$$\tilde{q}_2^h(1, \beta, \frac{\pi}{2}) = \tilde{q}_0^h(1, \beta, \frac{\pi}{2}) = 2\tilde{q}_1^h(1, \beta, \frac{\pi}{2}) = \frac{16J_1'^2(\beta)(1-\beta^2)^{5/2}}{4+3\beta^2}, \quad \tilde{q}_3^h(1, \beta, \frac{\pi}{2}) = 0. \quad (1.59)$$

For $i = g, 0, 2$ the contribution of first harmonic decreases with β in the plane of orbit and in ultrarelativistic limit it vanishes for any type of polarization.

1.3 Classical Spectrum of Synchrotron Radiation and its Primary Harmonics $\nu = 1$ and $\nu = 2$

In this work we plan to consider the simplest problem of the spectral properties of SR in terms of the quantum theory, namely, we are about to study the main characteristics of radiation emitted by quantum particles in the second excited state ($n = 2$). When $n = 2$, the radiation spectrum of quantum particle consists of two harmonics ($\nu = 1$ and $\nu = 2$). Thus, there appeared an idea to compare quantum data with the characteristics of radiation associated with a reduced classical spectrum, which would contain the same number of harmonics as its quantum analogue. In this regard we study the properties of radiation associated with the first two harmonics of the classical spectrum. In the previous section 1.2 we focused on the description of radiation associated with first harmonic. Now we consider the second harmonic of classical spectrum, as well as the integral radiation of first two harmonics.

According to classical theory, the spectrum of SR has an infinite number of harmonics [46]. Let us pick out the "basis" of classical spectrum, i.e. the part of spectrum starting from $\nu = 1$ to $\nu = n$. For this picking we introduce the term *n-part of spectrum*. For example, the 2-part of spectrum includes two harmonics - the primary ($\nu = 1$) and the next one ($\nu = 2$). Let us consider particular characteristics of *n-part* of the spectrum, namely, the contribution of radiation associated with harmonic ν to the total radiation associated with *n-part* of the spectrum, angular distributions, etc.

Picking out first n harmonics of the spectrum one can calculate the degree of polarization of the radiation associated with these harmonics taken all together. We introduce the functions $\bar{q}_i^h(n, \beta)$

$$\bar{q}_i^h(n, \beta) = \frac{\sum_{\nu=1}^n f_i^h(\nu, \beta)}{\sum_{\nu=1}^n f_0^h(\nu, \beta)}, \quad \bar{q}_i^h(1, \beta) = q_i^h(1, \beta), \quad \bar{q}_0^h(n, \beta) = 1. \quad (1.60)$$

In fact, the sum $\sum_{\nu=1}^n f_i^h(\nu, \beta)$ defines the amount of radiation with polarization type i associated with first n harmonics of the spectrum. At the same time, the sum $\sum_{\nu=1}^n f_0^h(\nu, \beta)$ defines total radiation (summed over polarization components) of particular set. The plots of functions $\bar{q}_i^h(n, \beta)$ will be given in section 4.1.

In addition, apart from the degree of radiation polarization, it is interesting to consider the relation $\tilde{q}_i^h(n, \nu, \beta)$. It defines the contribution of the i -polarization component associated with harmonic ν to the total power of radiation associated with the *n-part* of spectrum

$$\tilde{q}_i^h(n, \nu, \beta) = \frac{f_i^h(\nu, \beta)}{\sum_{\nu=1}^n f_0^h(\nu, \beta)}, \quad \tilde{q}_i^h(1, 1, \beta) = \bar{q}_i^h(1, \beta) = q_i^h(1, \beta). \quad (1.61)$$

In the section 1.2.1 we introduced the functions $p_i^h(\nu, \beta, \theta)$. Figures 1.27-1.30 illustrate the plots of these functions for the second harmonic of radiation ($\nu = 2$).

Figure 1.27: The plots of $p_2^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.28: The plots of $p_3^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.29: The plots of $p_0^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.30: The plots of $p_1^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

We note that unlike the function $p_i^h(1, \beta, \theta)$ functions $p_i^h(2, \beta, \theta)$ vanish in the direction of magnetic field. This property can be observed for all $p_i^h(\nu, \beta, \theta)$ ($\nu \neq 1, i = 0, 1, 2, 3$), only the first harmonic is emitted in the direction $\theta \approx 0$. The radiation associated with other harmonics is negligible.

Figure 1.31: The plots of $p_2^h(2, 1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.32: The plots of $p_3^h(2, 1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

We define now the partial contribution of radiation with polarization type i emitted in the direction

θ to the radiation associated with n -part of the spectrum

$$p_i^h(n, \nu, \beta, \theta) = \frac{F_i^h(\nu, \beta, \theta)}{\sum_{\nu=1}^n f_0^h(\nu, \beta)}. \quad (1.62)$$

In the figures 1.31-1.34 and 1.35-1.38 we give the plots of functions $p_i^h(2, 1, \beta, \theta)$ and $p_i^h(2, 2, \beta, \theta)$ for σ - and π -components of linear polarization, total radiation and right circular polarization.

Figure 1.33: The plots of $p_0^h(2, 1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.34: The plots of $p_1^h(2, 1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.35: The plots of $p_2^h(2, 2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.36: The plots of $p_3^h(2, 2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.37: the plots of $p_0^h(2, 2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.38: The plots of $p_1^h(2, 2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

For all the types of polarization i the functions $p_i^h(2, 1, \beta, \theta)$ decrease with β , and in the direction of magnetic field in ultrarelativistic region they have a weak minima. In its turn, $p_i^h(2, 2, \beta, \theta)$ increase with energy of a particle up to a certain β near 1, almost on the whole interval $[0, \pi]$, $p_i^h(2, 2, \beta = 0, \theta) = 0$. Thus, the behavior of functions $p_i^h(2, 1, \beta, \theta)$ and $p_i^h(2, 2, \beta, \theta)$ also confirms the well-known conclusion of classical theory. Namely, in non-relativistic limit the radiation spectrum is reduced to a single harmonic (the first one) and then the maximum of radiation shifts to higher harmonics with energy.

We can define angular distributions of radiation power associated with first n harmonics of classical spectrum, i.e. the contribution of radiation with a fixed type of polarization emitted in a particular direction to the total power of radiation of n -part of the spectrum (the analogues of the functions $p_i^{cl}(\beta, \theta)$ for n -part of spectrum)

$$P_i^h(n, \beta, \theta) = \frac{\sum_{\nu=1}^n F_i^h(\nu, \beta, \theta)}{2 \sum_{\nu=1}^n f_0^h(\nu, \beta)} = \sum_{\nu=1}^n p_i^h(n, \nu, \beta, \theta). \quad (1.63)$$

Figure 1.39: The plots of $P_2^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.40: The plots of $P_3^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.41: The plots of $P_0^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.42: The plots of $P_1^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

The evolution of angular distributions for the radiation power of n -part of classical spectrum with energy is given in the figures 1.39 - 1.42.

Figure 1.43: The plots of $\tilde{P}_2^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.44: The plots of $\tilde{P}_3^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.45: The plots of $\tilde{P}_0^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 1.46: The plots of $\tilde{P}_1^h(2, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Let us consider the contribution of radiation associated with n -part of the spectrum and θ -direction to the total power emitted by classical particle (the power of unpolarized radiation associated with total classical spectrum)

$$\tilde{P}_i^h(n, \beta, \theta) = \frac{\sum_{\nu=1}^n F_i^h(\nu, \beta, \theta)}{4 f_0^{cl}(\beta)} = \sum_{\nu=1}^n \tilde{p}_i^h(\nu, \beta, \theta). \quad (1.64)$$

The plots of $\tilde{P}_i^h(n, \beta, \theta)$ are given in the figures 1.43 - 1.46.

The functions $\tilde{P}_i^h(2, \beta, \theta)$ decrease with β and $\tilde{P}_i^h(2, \beta \rightarrow 1, \theta) = 0$. The main part of radiation shifts to higher harmonics with energy, therefore, the contribution of first two harmonics loses its significance as β tends to 1.

Chapter 2

An Introduction to Quantum Theory of Synchrotron Radiation

2.1 The Power of Radiation in Quantum Theory

The energy of a relativistic particle can be given by

$$E = \frac{m_0 c^2}{\sqrt{1 - \beta^2}} = m_0 c^2 \gamma, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad (2.1)$$

where β is the analogue of the normalized classical velocity of the particle, $\beta = \vec{v}/c$ (see (1.24)). The following notations are to be used

$$E = \hbar K, \quad k_0 = \frac{m_0 c}{\hbar}, \quad \Rightarrow \quad \frac{k_0}{K} = \sqrt{1 - \beta^2}. \quad (2.2)$$

Generally, we assume that the particle moves along a spiral coiling around the direction of the magnetic field intensity $\vec{H}$. For the reasons of convenience let us set the z -axis along the vector $\vec{H}$, $\vec{H} = (0, 0, H)$, $H = |\vec{H}|$, as we did in the classical theory.

In quantum theory the power of radiation is described by

$$W_i = \frac{ce^2}{2\pi} \int d^3\kappa \delta(\kappa' - \kappa) S_i, \quad (2.3)$$

$\vec{\kappa} = \kappa \vec{\kappa}_0$ is the wave vector of emitted photon, $\vec{\kappa}_0$ is the unit vector whose direction is given by the angles θ and φ' as follows

$$\vec{\kappa}_0 = (\cos \varphi' \sin \theta, \sin \varphi' \sin \theta, \cos \theta), \quad (2.4)$$

$$\int d^3\kappa = \int_0^\pi d\theta \int_0^{2\pi} d\varphi' \int_0^\infty \kappa^2 \sin \theta d\kappa, \quad (2.5)$$

$\kappa' = K - K'$, where K defines the energy of the particle in the initial state, and K' stands for the energy of the final state. So, the integrand δ -function gives the energy conservation

$$\kappa = K - K'. \quad (2.6)$$

The well-known property of the δ -function is

$$\delta(f(x)) = \frac{\delta(x - x_s)}{|f'(x_s)|}, \quad x_s | f(x_s) = 0, \quad (2.7)$$

and $\frac{\partial}{\partial \kappa}(\kappa - (K - K')) = 1 + \frac{\partial K'}{\partial \kappa}$, so integrating over φ' , from (2.3) and (2.5) we get

$$W_i = ce^2 \int_0^\pi S_i \cdot \frac{\kappa^2 \sin \theta d\theta}{1 + \frac{\partial K'}{\partial \kappa}} . \quad (2.8)$$

It is important to note that, generally, elements S_i depend on the direction φ' , and this dependence should be taken into account while integrating over φ' . However, when the specified direction (in our case it is the direction of the external field) is perpendicular to the plane of the orbit the power of radiation is supposed not to depend on φ' . That is, all the possible directions of photon emission $0 \leq \varphi' < 2\pi$ are equiprobable for fixed θ . In this connection, without loss of generality, for the problem in question we can assume $\varphi' = \pi/2$. Similarly to the notations used in classical theory, index i denotes polarization. The elements S_i depend on κ , so that (2.8) implies $S_i = S_i|_{\kappa=K-K'}$.

The analytical definitions of S_i are given below

$$S_i = |(\langle \vec{D} \rangle \cdot \vec{e}_i)|^2 , \quad (2.9)$$

$$\langle \vec{D} \rangle = (\langle D_1 \rangle, \langle D_2 \rangle, \langle D_3 \rangle) \equiv (\langle D_x \rangle, \langle D_y \rangle, \langle D_z \rangle) . \quad (2.10)$$

Matrix elements $\langle \vec{D} \rangle$ should be chosen with respect to the type of radiating particle and have the form

$$\langle D_a \rangle = \int \psi'^+ e^{-i(\vec{\kappa} \cdot \vec{r})} D_a \psi \vec{dr} , \quad (2.11)$$

where the functions ψ and ψ' represent the initial and final state of the particle correspondingly, and are the solutions of the equations of motion, +-index denotes the hermitian conjugation. Roughly speaking, $\vec{r}$ gives the "position" of the particle, $\vec{dr} = r dr d\varphi dz$. When the particle moves along a spiral, we have

$$\vec{r} = (r \sin \varphi, r \cos \varphi, z) , \quad (2.12)$$

$$(\vec{\kappa} \cdot \vec{r})|_{\varphi'=\frac{\pi}{2}} = \kappa(r \sin \varphi \sin \theta + z \cos \theta) . \quad (2.13)$$

Vectors $\vec{e}_i$, $|\vec{e}_i| = 1$, describe the polarization of the radiation. These vectors can be introduced similarly to their analogues that describe polarization in the classical theory (1.12). We remind the reader that the unit vectors which define the polarization components should form a full orthogonal set with the unit vector collinear with the direction of the photon emission $\vec{\kappa}_0$, i.e.

$$(\vec{\kappa}_0 \cdot \vec{e}_i) = (\vec{e}_i \cdot \vec{e}_j)|_{i \neq j} = 0, \{i, j\} = \{2, 3\}, \vec{\kappa}_0 = (\vec{e}_2 \times \vec{e}_3) . \quad (2.14)$$

Right and left circular polarization can be described with the use of linear combinations as follows

$$\vec{e}_g = \frac{1}{\sqrt{2}}(\vec{e}_2 + ig\vec{e}_3), \quad g = -1, 1 . \quad (2.15)$$

so, the components of the circular polarization are perpendicular to the direction of the photon emission as well,

$$(\vec{\kappa}_0 \cdot \vec{e}_g) = 0, \quad g = -1, 1. \quad (2.16)$$

To satisfy all the requirements given above we take the unit vectors of the form

$$\vec{e}_2 = \frac{(\vec{\kappa}_0 \times \mathbf{j})}{\sqrt{1 - (\vec{\kappa}_0 \cdot \mathbf{j})^2}}, \quad \vec{e}_3 = \frac{\vec{\kappa}_0(\vec{\kappa}_0 \cdot \mathbf{j}) - \mathbf{j}}{\sqrt{1 - (\vec{\kappa}_0 \cdot \mathbf{j})^2}}, \quad (2.17)$$

to describe the linear polarizations.

The unit vector $\mathbf{j}$ should correspond to some specific direction. We assume $\mathbf{j}$ and the external field intensity to be collinear. Since the intensity of the external field is given by $\vec{H} = (0, 0, H)$, then $\mathbf{j} = \vec{H}/|H| = (0, 0, 1)$. Assuming $\varphi' = \pi/2$, with $\mathbf{j} = (0, 0, 1)$ it becomes easy to obtain

$$\vec{e}_2 = (1, 0, 0), \vec{e}_3 = (0, \cos \theta, -\sin \theta), \vec{e}_g = \frac{1}{\sqrt{2}}(1, ig \cos \theta, -ig \sin \theta). \quad (2.18)$$

Finally, substituting $\vec{e}_i$ in (2.9) with (2.18) for S_i we get

$$S_2 = |\langle D_1 \rangle|^2 = \langle D_1 \rangle \langle D_1 \rangle^+, S_3 = |\langle D_2 \rangle \cos \theta - \langle D_3 \rangle \sin \theta|^2, S_0 = S_2 + S_3, \quad (2.19)$$

$$S_g = \frac{1}{2} |\langle D_1 \rangle + ig(\langle D_2 \rangle \cos \theta - \langle D_3 \rangle \sin \theta)|^2, \quad (2.20)$$

$$S_g = \frac{S_0}{2} + \frac{ig}{2} [(\langle D_2 \rangle \cos \theta - \langle D_3 \rangle \sin \theta) \langle D_1 \rangle^+ - \langle D_1 \rangle (\langle D_2 \rangle^+ \cos \theta - \langle D_3 \rangle^+ \sin \theta)] . \quad (2.21)$$

Obviously, S_0 can be given as a sum of linear polarization components as well, i.e. $S_0 = S_1 + S_{-1}$.

By analogy with the chapter on the classical theory of SR we omit the derivation of (2.3), the starting point is the definition of the power of radiation. This derivation can be found in [20]. In order to obtain (2.3) spontaneous quantum transitions, caused by second quantized electromagnetic field, were considered. The equation of motion contained a term associated with the energy of interaction of a particle and the second quantized field of photons. The power of radiation is understood in terms of the probability of spontaneous transitions.

2.2 Synchrotron Radiation

We focus on the motion of charged particles in external magnetic field and separately consider spinor and scalar particles. We have mentioned that the external field is supposed to be directed along z -axis, so that for the 4-vector-potential we can write

$$A^\mu = (\phi, \vec{A}), \phi = 0, A_x = -\frac{y}{2}H, A_y = \frac{x}{2}H, A_z = 0. \quad (2.22)$$

Assuming the particle to have a negative elementary charge $e_0 = -|e_0|$, we introduce the generalized momenta

$$\hat{\mathcal{P}} = \hat{\mathbf{p}} + \frac{|e_0|}{c} \vec{A}, \hat{\mathbf{p}} = -i\hbar \vec{\nabla}. \quad (2.23)$$

For both scalar and spinor particles we consider stationary states

$$\Psi = e^{-i\frac{E}{\hbar}t} \psi(\vec{r}). \quad (2.24)$$

To simplify further calculations it is convenient to introduce the notations

$$\bar{\gamma} = \frac{|e_0|H}{2c\hbar} = \frac{1}{2} b \frac{m_0^2 c^2}{\hbar^2}, b = \frac{H}{H_0}, H_0 = \frac{m_0^2 c^3}{|e_0| \hbar}. \quad (2.25)$$

2.2.1 Electron

The Motion of an Electron in a Uniform Magnetic Field

In the framework of relativistic quantum mechanics, the motion of an electron in a magnetic field is described by the Dirac equation (see the original works [94,95] and [96, p. 257] or textbooks, e.g. [20,97,98])

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi , \quad (2.26)$$

where the Hamiltonian $\hat{H}$, with respect to (2.22) - zero scalar potential, has the form

$$\hat{H} = c (\vec{\alpha} \cdot \hat{\mathcal{P}}) + m_0 c^2 \rho_3 , \quad (2.27)$$

$\vec{\alpha}$, $\vec{\rho}$ are the Dirac matrices [98, p. 103], [96] (together with these matrices we define $\vec{\sigma}$ which we will use later)

$$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \quad \vec{\rho} = (\rho_1, \rho_2, \rho_3), \quad \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) , \quad (2.28)$$

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma'_j \\ \sigma'_j & \mathbf{0} \end{pmatrix}, \quad \sigma_j = \begin{pmatrix} \sigma'_j & \mathbf{0} \\ \mathbf{0} & \sigma'_j \end{pmatrix}, \quad j = 1, 2, 3 , \quad (2.29)$$

$$\rho_1 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} \mathbf{0} & -i\mathbf{I} \\ i\mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}, \quad (2.30)$$

$\mathbf{0}$, $\mathbf{I}$ are zero- and unit- $[2 \times 2]$ -matrices correspondingly, and σ' are $[2 \times 2]$ Dirac matrices

$$\sigma'_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma'_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma'_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.31)$$

The solution of (2.26) is a 4-component spinor Ψ , and, taking (2.24) into account we can write (though we give a step-by-step solution of the equations below, the reader may find useful to follow [99] or [100–103])

$$\Psi = e^{-i \frac{E}{\hbar} t} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \begin{pmatrix} E - m_0 c^2 & 0 & -c\hat{\mathcal{P}}_z & -c(\hat{\mathcal{P}}_x - i\hat{\mathcal{P}}_y) \\ 0 & E - m_0 c^2 & -c(\hat{\mathcal{P}}_x + i\hat{\mathcal{P}}_y) & c\hat{\mathcal{P}}_z \\ -c\hat{\mathcal{P}}_z & -c(\hat{\mathcal{P}}_x - i\hat{\mathcal{P}}_y) & E + m_0 c^2 & 0 \\ -c(\hat{\mathcal{P}}_x + i\hat{\mathcal{P}}_y) & c\hat{\mathcal{P}}_z & 0 & E + m_0 c^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = 0 . \quad (2.32)$$

It follows from the symmetry of the problem that both the z -component of the generalized momentum $\hat{\mathcal{P}}_z$ and the z -component of the total angular momentum $\hat{\mathcal{M}}_z$ are integrals of motion. The total angular momentum is the sum of orbital angular momentum $\hat{M}$ and spin angular momentum $\hat{S}$,

$$\hat{\mathcal{M}}_j = \hat{M}_j + \hat{S}_j, \quad \hat{M}_j = (\vec{r} \times \mathbf{p})_j, \quad \hat{S}_j = \frac{\hbar}{2} \sigma_j . \quad (2.33)$$

In this connection it is convenient to use cylindrical coordinate system and write the expressions in terms of (r, φ, z)

$$\hat{M}_z \equiv \hat{M}_3 = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar \frac{\partial}{\partial \varphi}, \quad \hat{\mathcal{M}}_z \equiv \hat{\mathcal{M}}_3 = \hbar \left(-i \frac{\partial}{\partial \varphi} + \frac{1}{2} \sigma_3 \right), \quad (2.34)$$

the wave-function in (2.32) is an eigenfunction of both $\hat{\mathcal{P}}_z$ and $\hat{\mathcal{M}}_z$

$$\hat{\mathcal{P}}_z \Psi = \hbar k_3 \Psi, \quad \hat{\mathcal{M}}_z \Psi = \hbar(l - 1/2) \Psi . \quad (2.35)$$

The first equation of (2.35) allows us to write

$$\psi_\lambda(\vec{r}) = \frac{e^{ik_3 z}}{\sqrt{L}} \psi_\lambda(r, \varphi), \quad \lambda = 1, 2, 3, 4. \quad (2.36)$$

The second equation of the pair (2.35) together with (2.32) gives

$$\begin{cases} (-i\frac{\partial}{\partial\varphi} + \frac{1}{2})\psi_{1,3} = (l - \frac{1}{2})\psi_{1,3}, \\ (-i\frac{\partial}{\partial\varphi} - \frac{1}{2})\psi_{2,4} = (l - \frac{1}{2})\psi_{2,4}; \end{cases} \Rightarrow \begin{cases} \psi_{1,3}(\vec{r}) = \frac{e^{ik_3 z}}{\sqrt{L}} \frac{e^{i(l-1)\varphi}}{\sqrt{2\pi}} \psi_{1,3}(r), \\ \psi_{2,4}(\vec{r}) = \frac{e^{ik_3 z}}{\sqrt{L}} \frac{e^{il\varphi}}{\sqrt{2\pi}} \psi_{2,4}(r). \end{cases} \quad (2.37)$$

Using the notation (2.25) it isn't hard to obtain

$$\begin{aligned} \hat{\mathcal{P}}_x - i\hat{\mathcal{P}}_y &= -\hbar e^{-i\varphi} \left(\frac{1}{r} \frac{\partial}{\partial\varphi} + i\frac{\partial}{\partial r} + i\bar{\gamma}r \right), \\ \hat{\mathcal{P}}_x + i\hat{\mathcal{P}}_y &= \hbar e^{i\varphi} \left(\frac{1}{r} \frac{\partial}{\partial\varphi} - i\frac{\partial}{\partial r} + i\bar{\gamma}r \right). \end{aligned} \quad (2.38)$$

Now we use (2.38) and the wave-functions (2.37) which, together with (2.35), allow us to write (2.32) in the following form

$$\begin{cases} (K - k_0)\psi_1 - k_3\psi_3 + i\left(\frac{l}{r} + \frac{\partial}{\partial r} + \bar{\gamma}r\right)\psi_4 = 0, \\ (K - k_0)\psi_2 + k_3\psi_4 - i\left(\frac{l-1}{r} - \frac{\partial}{\partial r} + \bar{\gamma}r\right)\psi_3 = 0, \\ (K + k_0)\psi_3 - k_3\psi_1 + i\left(\frac{l}{r} + \frac{\partial}{\partial r} + \bar{\gamma}r\right)\psi_2 = 0, \\ (K + k_0)\psi_4 + k_3\psi_2 - i\left(\frac{l-1}{r} - \frac{\partial}{\partial r} + \bar{\gamma}r\right)\psi_1 = 0. \end{cases} \quad (2.39)$$

Applying successively the operators $i(\frac{l}{r} \pm \frac{\partial}{\partial r} + \bar{\gamma}r)$, it is easy to transform these equations to the form

$$\begin{aligned} \{K^2 - k_3^2 - k_0^2 - (\frac{l}{r} + \frac{\partial}{\partial r} + \bar{\gamma}r)(\frac{l-1}{r} - \frac{\partial}{\partial r} + \bar{\gamma}r)\}\psi_{1,3}(r) &= 0, \\ \{K^2 - k_3^2 - k_0^2 - (\frac{l-1}{r} - \frac{\partial}{\partial r} + \bar{\gamma}r)(\frac{l}{r} + \frac{\partial}{\partial r} + \bar{\gamma}r)\}\psi_{2,4}(r) &= 0, \end{aligned} \quad (2.40)$$

so, that the resulting system is

$$\begin{aligned} \{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(l-1)^2}{r^2} - \bar{\gamma}^2 r^2 + K^2 - k_0^2 - k_3^2 - 2\bar{\gamma}l\}\psi_{1,3}(r) &= 0, \\ \{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{l^2}{r^2} - \bar{\gamma}^2 r^2 + K^2 - k_0^2 - k_3^2 - 2\bar{\gamma}(l-1)\}\psi_{2,4}(r) &= 0. \end{aligned} \quad (2.41)$$

Each of these equations is (A.1), and for the first pair of equations

$$A = (l-1)^2, \quad B = \bar{\gamma}^2, \quad C = K^2 - k_0^2 - k_3^2 - 2\bar{\gamma}l, \quad (2.42)$$

whereas for the second pair

$$A = l^2, \quad B = \bar{\gamma}^2, \quad C = K^2 - k_0^2 - k_3^2 - 2\bar{\gamma}(l-1). \quad (2.43)$$

We put

$$s = \frac{K^2 - k_0^2 - k_3^2}{4\bar{\gamma}} - l, \quad s = 0, 1, 2, \dots, \quad n = s + l, \quad n = 0, 1, 2, \dots, \quad (2.44)$$

and obtain discrete energy levels (2.2)

$$K \equiv K_n = \sqrt{k_0^2 + k_3^2 + 4\bar{\gamma}n}, \quad (2.45)$$

and (see (A.14)) the wave-functions can be written in the form

$$\psi_{1,3}(r) = \sqrt{2\bar{\gamma}} c_{1,3} I_{n-1,s}(\bar{\gamma}r^2), \quad \psi_{2,4}(r) = \sqrt{2\bar{\gamma}} i c_{2,4} I_{n,s}(\bar{\gamma}r^2), \quad (2.46)$$

$$\psi(\vec{r}) = \sqrt{\frac{\bar{\gamma}}{\pi L}} e^{ik_3 z} e^{i(l-1)\varphi} \begin{pmatrix} c_1 I_{n-1,s}(\bar{\gamma}r^2) \\ i c_2 e^{i\varphi} I_{n,s}(\bar{\gamma}r^2) \\ c_3 I_{n-1,s}(\bar{\gamma}r^2) \\ i c_4 e^{i\varphi} I_{n,s}(\bar{\gamma}r^2) \end{pmatrix} \quad (2.47)$$

where c_λ ($\lambda = 1, 2, 3, 4$) are to be defined later. We perform "partial" normalization, so that with the use of (A.16) (which lead to the inclusion of the factor $\sqrt{\frac{\bar{\gamma}}{\pi L}}$), the normalization condition

$$\int \psi^+(\vec{r})\psi(\vec{r})d\vec{r} = 1 , \quad (2.48)$$

transforms to a condition for coefficients c_λ

$$\sum_{\lambda=1}^4 c_\lambda^\dagger c_\lambda = 1 . \quad (2.49)$$

With the use of (2.38) and (A.17) one can calculate

$$\begin{aligned} c_{1,3}(\hat{P}_x - i\hat{P}_y)\psi_{2,4}(\vec{r}) &= \hbar\sqrt{4\bar{\gamma}n} c_{2,4}\psi_{1,3}(\vec{r}) , \\ c_{2,4}(\hat{P}_x + i\hat{P}_y)\psi_{1,3}(\vec{r}) &= \hbar\sqrt{4\bar{\gamma}n} c_{1,3}\psi_{2,4}(\vec{r}) . \end{aligned} \quad (2.50)$$

Substituting (2.46) together with the results (2.50) and (2.32) and excluding the factor $c\hbar$, with (2.2) taken into account, we get the system of equations for c_λ

$$\begin{pmatrix} K - k_0 & 0 & -k_3 & -\sqrt{4\bar{\gamma}n} \\ 0 & K - k_0 & -\sqrt{4\bar{\gamma}n} & k_3 \\ -k_3 & -\sqrt{4\bar{\gamma}n} & K + k_0 & 0 \\ -\sqrt{4\bar{\gamma}n} & k_3 & 0 & K + k_0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 . \quad (2.51)$$

From (2.45) follows that the rank of the matrix of system (2.51) equals 2. For instance, the third (3) and forth (4) equations can be given as linear combinations of (1) and (2)

$$(3) = \frac{-k_3 \cdot (1) - (2)\sqrt{4\bar{\gamma}n}}{K - k_0} , \quad (4) = \frac{k_3 \cdot (2) - (1)\sqrt{4\bar{\gamma}n}}{K - k_0} . \quad (2.52)$$

Thus, the system (2.51) implies the possibility to write $c_{1,2}$ in terms of $c_{3,4}$

$$c_1 = \frac{k_3 c_3 + \sqrt{4\bar{\gamma}n} c_4}{K - k_0} , \quad c_2 = \frac{\sqrt{4\bar{\gamma}n} c_3 - k_3 c_4}{K - k_0} , \quad (2.53)$$

and taking into account the normalization (2.49), we get

$$c_1 = \frac{k_3 A + \sqrt{4\bar{\gamma}n} B}{\sqrt{2K(K - k_0)}} , \quad c_2 = \frac{\sqrt{4\bar{\gamma}n} A - k_3 B}{\sqrt{2K(K - k_0)}} , \quad (2.54)$$

$$c_3 = A\sqrt{\frac{1}{2}\left(1 - \frac{k_0}{K}\right)} , \quad c_4 = B\sqrt{\frac{1}{2}\left(1 - \frac{k_0}{K}\right)} , \quad (2.55)$$

where A and B undergo

$$A^+ A + B^+ B = 1 . \quad (2.56)$$

To describe the properties of spin one can use either vector-operator of polarization or tensor-operator (see [20], the original papers [104–106], and also [107]). Though z -projection of vector-operator $\hat{\pi}_z$ commutes with Hamiltonian

$$\hat{\pi}_z = \rho_3 \sigma_z + \rho_1 \frac{\hat{P}_z}{m_0 c} , \quad [\hat{\pi}_z, \hat{H}] = 0 , \quad (2.57)$$

the use of $\hat{\pi}_z$ can hardly be convenient for real analysis since in the relativistic case it transforms to $\sim \hat{P}_z$ and, therefore, does not contain spinor matrix.

So, for our study it will be better and more fruitful to consider the tensor-operator of polarization. Its component associated with the magnetic field direction

$$\hat{\Pi}_z \equiv \hat{\Pi}_3 = \frac{\hbar}{2} \left(\sigma_3 + \rho_2 \frac{(\vec{\sigma} \times \hat{\mathcal{P}})_3}{m_0 c} \right) \quad (2.58)$$

is an integral of motion, $[\hat{\Pi}_z, \hat{H}] = 0$. It means that $\psi(\vec{r})$ is an eigenfunction of $\hat{\Pi}_z$

$$\hat{\Pi}_z \psi = \zeta \Lambda \psi, \quad \zeta = \pm 1, \quad (2.59)$$

ζ describes the direction of spin with respect to the magnetic field $\vec{H}$: $\zeta = 1$ when spin is collinear with the vector of field intensity, $\zeta = -1$ when the spin of the particle has an opposite direction relative to $\vec{H}$.

The equation (2.59) can be written explicitly

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{\hat{\mathcal{P}}_x - i\hat{\mathcal{P}}_y}{m_0 c} \\ 0 & -1 & \frac{\hat{\mathcal{P}}_x + i\hat{\mathcal{P}}_y}{m_0 c} & 0 \\ 0 & -\frac{\hat{\mathcal{P}}_x - i\hat{\mathcal{P}}_y}{m_0 c} & 1 & 0 \\ \frac{\hat{\mathcal{P}}_x + i\hat{\mathcal{P}}_y}{m_0 c} & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \zeta \Lambda \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \quad (2.60)$$

Using the system (2.32) and the notations (2.2) it is easy to rewrite (2.60) in the form

$$\begin{pmatrix} K - \zeta \Lambda_0 & 0 & -k_3 & 0 \\ 0 & K + \zeta \Lambda_0 & 0 & k_3 \\ -k_3 & 0 & K + \zeta \Lambda_0 & 0 \\ 0 & k_3 & 0 & K - \zeta \Lambda_0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0, \quad \Lambda_0 = \Lambda k_0. \quad (2.61)$$

For the consistency of this system the determinant should satisfy

$$\det \begin{vmatrix} K - \zeta \Lambda_0 & 0 & -k_3 & 0 \\ 0 & K + \zeta \Lambda_0 & 0 & k_3 \\ -k_3 & 0 & K + \zeta \Lambda_0 & 0 \\ 0 & k_3 & 0 & K - \zeta \Lambda_0 \end{vmatrix} = [K^2 - \Lambda_0^2 - k_3^2]^2 = 0, \Rightarrow \Lambda_0 = \sqrt{K^2 - k_3^2}. \quad (2.62)$$

Substituting the coefficients $c_{1,3}$ ($c_{2,4}$) in the form (2.54) - (2.55) in the first (second) equation of system (2.61), with the use of

$$k_3 = \sqrt{(K - \zeta \Lambda_0)(K + \zeta \Lambda_0)}, \quad \sqrt{4\gamma n} = \sqrt{(\Lambda_0 - \zeta k_0)(\Lambda_0 + \zeta k_0)}, \quad (2.63)$$

we get the relation between A and B

$$A \sqrt{(K + \zeta \Lambda_0)(\Lambda_0 - \zeta k_0)} = \zeta B \sqrt{(K - \zeta \Lambda_0)(\Lambda_0 + \zeta k_0)}, \quad (2.64)$$

which, together with normalization (2.56), allows us to write

$$A = \sqrt{\frac{(1 + \zeta \frac{k_0}{\Lambda_0})(1 - \zeta \frac{\Lambda_0}{K})}{2(1 - \frac{k_0}{K})}}, \quad B = \zeta \sqrt{\frac{(1 - \zeta \frac{k_0}{\Lambda_0})(1 + \zeta \frac{\Lambda_0}{K})}{2(1 - \frac{k_0}{K})}}. \quad (2.65)$$

The following equality can be checked easily

$$1 \pm \zeta \sqrt{1 - \frac{k_3^2}{K^2}} = \frac{1}{2} \left(\sqrt{1 + \frac{k_3}{K}} \pm \zeta \sqrt{1 - \frac{k_3}{K}} \right)^2. \quad (2.66)$$

Finally, substituting (2.65) into (2.54) - (2.55) with (2.66) taken into account for the coefficient c_λ we obtain

$$\begin{aligned} c_1 &= \frac{1}{2} \sqrt{\frac{1}{2} \left(1 + \zeta \frac{k_0}{\Lambda_0} \right)} \left[\sqrt{1 + \frac{k_3}{K}} + \zeta \sqrt{1 - \frac{k_3}{K}} \right] , \\ c_2 &= -\frac{\zeta}{2} \sqrt{\frac{1}{2} \left(1 - \zeta \frac{k_0}{\Lambda_0} \right)} \left[\sqrt{1 + \frac{k_3}{K}} - \zeta \sqrt{1 - \frac{k_3}{K}} \right] , \\ c_3 &= \frac{1}{2} \sqrt{\frac{1}{2} \left(1 + \zeta \frac{k_0}{\Lambda_0} \right)} \left[\sqrt{1 + \frac{k_3}{K}} - \zeta \sqrt{1 - \frac{k_3}{K}} \right] , \\ c_4 &= \frac{\zeta}{2} \sqrt{\frac{1}{2} \left(1 - \zeta \frac{k_0}{\Lambda_0} \right)} \left[\sqrt{1 + \frac{k_3}{K}} + \zeta \sqrt{1 - \frac{k_3}{K}} \right] . \end{aligned} \quad (2.67)$$

$$(2.68)$$

So, the expressions (2.47) and (2.67) define the state of a Dirac particle in a constant uniform magnetic field. Further study of SR characteristics implies the use of $\psi(\vec{r})$ in (2.11).

Spectral-Angular Distributions of Radiation Emitted from an Electron

For an electron, the elements $\vec{D}$ in (2.11) are Dirac matrices $\vec{\alpha}$, given by (2.29).

It is easy to obtain

$$\begin{aligned} \psi'^+(\vec{r})\alpha_1\psi(\vec{r}) &= \frac{i\bar{\gamma}}{\pi L} e^{i(k_3-k'_3)z} e^{i(l-l')\varphi} [(c_1'^+c_4 + c_3'^+c_2)I_{n'-1,s'}I_{n,s}e^{i\varphi} - \\ &\quad (c_2'^+c_3 + c_4'^+c_1)I_{n',s'}I_{n-1,s}e^{-i\varphi}] , \\ \psi'^+(\vec{r})\alpha_2\psi(\vec{r}) &= \frac{\bar{\gamma}}{\pi L} e^{i(k_3-k'_3)z} e^{i(l-l')\varphi} [(c_1'^+c_4 + c_3'^+c_2)I_{n'-1,s'}I_{n,s}e^{i\varphi} + \\ &\quad (c_2'^+c_3 + c_4'^+c_1)I_{n',s'}I_{n-1,s}e^{-i\varphi}] , \\ \psi'^+(\vec{r})\alpha_3\psi(\vec{r}) &= \frac{\bar{\gamma}}{\pi L} e^{i(k_3-k'_3)z} e^{i(l-l')\varphi} [(c_1'^+c_3 + c_3'^+c_1)I_{n'-1,s'}I_{n,s}e^{i\varphi} - \\ &\quad (c_2'^+c_4 + c_4'^+c_2)I_{n',s'}I_{n-1,s}e^{-i\varphi}] . \end{aligned} \quad (2.69)$$

The argument of the Laguerre functions in (2.69) is $\bar{\gamma}r^2$, we omit it to simplify the formulae.

Integrating these expressions in order to obtain the matrix elements $\langle \alpha_a \rangle$ (see (2.11)) we use an integral representations of the Bessel functions (1.17) as well as the value of the integral [20]

$$\int_0^\infty I_{n',s'}(x)I_{n,s}(x)J_{l-l'}(\sqrt{qx})dx = I_{n,n'}\left(\frac{q}{4}\right)I_{s,s'}\left(\frac{q}{4}\right) , \quad n-s-l = n'-s'-l' . \quad (2.70)$$

Finally, for the matrix elements of the Dirac matrices we get [108]

$$\begin{aligned} \langle \alpha_1 \rangle &= iI_{s,s'}(x)[(c_1'^+c_4 + c_3'^+c_2)I_{n,n'-1}(x) - (c_2'^+c_3 + c_4'^+c_1)I_{n-1,n'}(x)]\delta_{k'_3,k_3-\kappa\cos\theta} , \\ \langle \alpha_2 \rangle &= I_{s,s'}(x)[(c_1'^+c_4 + c_3'^+c_2)I_{n,n'-1}(x) + (c_2'^+c_3 + c_4'^+c_1)I_{n-1,n'}(x)]\delta_{k'_3,k_3-\kappa\cos\theta} , \\ \langle \alpha_3 \rangle &= I_{s,s'}(x)[(c_1'^+c_3 + c_3'^+c_1)I_{n-1,n'-1}(x) - (c_2'^+c_4 + c_4'^+c_2)I_{n,n'}(x)]\delta_{k'_3,k_3-\kappa\cos\theta} , \end{aligned} \quad (2.71)$$

where we used the notation

$$x = \frac{\kappa^2 \sin^2 \theta}{4\bar{\gamma}} . \quad (2.72)$$

In fact, the expressions (2.71) imply the conservation of the z -component of momentum

$$k'_3 = k_3 - \kappa \cos \theta . \quad (2.73)$$

This equality together with energy conservation (2.6) and (2.2) gives the relation between κ and K

$$\kappa = \frac{K}{\sin^2 \theta} \left[1 - \frac{k_3}{K} \cos \theta + \sqrt{\left(1 - \frac{k_3}{K} \cos \theta\right)^2 - \left(\beta^2 - \frac{k_3^2}{K^2}\right) \frac{\nu}{n} \sin^2 \theta} \right], \quad \nu = n - n'. \quad (2.74)$$

Initially we assumed a spiral motion of a particle, i.e. a non-zero component of the initial momentum along the direction of the magnetic field. But it all can easily be transformed and adapted to describe a motion on a circular orbit if we assume $k_3 = 0$. In this case the dependence (2.74) has the form

$$\kappa = \frac{K}{\sin^2 \theta} \left(1 - \sqrt{1 - \beta^2 \frac{\nu}{n} \sin^2 \theta} \right). \quad (2.75)$$

The energy of the emitted photon is $\hbar\omega = c\kappa$, and from (2.75) with (2.1) and (2.2) we get

$$\hbar\omega = \frac{\nu}{n} \frac{m_0 c^2 \gamma \beta^2}{1 + \sqrt{1 - \frac{\nu}{n} \beta^2 \sin^2 \theta}}, \quad (2.76)$$

i.e. the frequency of the emitted photon. Moreover, at $k_3 = 0$ from (2.45), (2.1) and (2.25) follows the relation between the external field intensity, the energy of the electron and the number of the initial level n

$$\gamma^2 = 1 + 2nb, \quad (\xi = b \sqrt{1 + 2nb}). \quad (2.77)$$

Let us introduce the notations

$$q = 1 - \frac{\nu}{n} \beta^2, \quad p = 1 - \frac{\nu}{n} \beta^2 \sin^2 \theta, \quad 0 < q \leq p \leq 1, \quad (2.78)$$

so that we can rewrite (2.72) as

$$x^e \equiv x^e(n, \nu, \beta, \theta) = \nu \frac{1 - \sqrt{1 - \frac{\nu}{n} \beta^2 \sin^2 \theta}}{1 + \sqrt{1 - \frac{\nu}{n} \beta^2 \sin^2 \theta}} = \nu \frac{1 - \sqrt{p}}{1 + \sqrt{p}}, \quad 0 < x^e \leq x_0^e < \nu, \quad (2.79)$$

where

$$x_0^e \equiv x_0^e(n, \nu, \beta) = x^e(n, \nu, \beta, \frac{\pi}{2}) = \nu \frac{1 - \sqrt{1 - \frac{\nu}{n} \beta^2}}{1 + \sqrt{1 - \frac{\nu}{n} \beta^2}} = \nu \frac{1 - \sqrt{q}}{1 + \sqrt{q}}. \quad (2.80)$$

Further in this section we omit the upper index e , we use x and x_0 for x^e and x_0^e correspondingly.

It isn't hard to obtain inverse expressions, i.e. q and p can be defined in terms of x and x_0

$$q = \left[\frac{\nu - x_0}{\nu + x_0} \right]^2, \quad p = \left[\frac{\nu - x}{\nu + x} \right]^2. \quad (2.81)$$

We are about to use the expressions which could easily be obtained

$$\sin^2 \theta = \frac{x}{x_0} \frac{(\nu + x_0)^2}{(\nu + x)^2} = \frac{1 - p}{1 - q}, \quad \cos^2 \theta = \frac{(x_0 - x)(\nu^2 - x_0 x)}{x_0(\nu + x)^2} = \frac{p - q}{1 - q}. \quad (2.82)$$

To calculate the matrix elements $\langle \alpha_i \rangle$ one has to define the energy of a photon as well as the energy (K' , see (2.6)) and the momentum (k'_3 , see (2.73)) of the final state of the electron in terms of q , p , x_0 and x

$$\kappa = K \frac{2x_0(\nu + x)}{(\nu + x_0)^2} = K \frac{(q + \sqrt{p})}{(1 + \sqrt{p})}, \quad K' = K \frac{(\nu^2 - 2x_0 x + x_0^2)}{(\nu + x_0)^2} = K \frac{(q + \sqrt{p})}{(1 + \sqrt{p})}, \quad (2.83)$$

$$k'_3 = - \frac{2K \sqrt{x_0(x_0 - x)(\nu^2 - x_0 x)}}{(\nu + x_0)^2} = - \frac{K \sqrt{(p - q)(1 - q)}}{1 + \sqrt{p}}. \quad (2.84)$$

From (2.67) for $k_3 = 0$ (ζ' is the spin in the final state) follows

$$\begin{aligned} c_1 &= \frac{1}{2} \sqrt{\frac{1}{2}(1 + \zeta \sqrt{1 - \beta^2})} (1 + \zeta), \\ c_2 &= -\frac{\zeta}{2} \sqrt{\frac{1}{2}(1 - \zeta \sqrt{1 - \beta^2})} (1 - \zeta), \\ c_3 &= \frac{1}{2} \sqrt{\frac{1}{2}(1 + \zeta \sqrt{1 - \beta^2})} (1 - \zeta), \\ c_4 &= \frac{\zeta}{2} \sqrt{\frac{1}{2}(1 - \zeta \sqrt{1 - \beta^2})} (1 + \zeta), \end{aligned} \quad (2.85)$$

$$\begin{aligned} c'_1 &= \frac{1}{2} \sqrt{\frac{1}{2} \left(1 + \zeta' \sqrt{\frac{1 - \beta^2}{q}} \right)} \left(\sqrt{1 + \frac{k'_3}{K'}} + \zeta' \sqrt{1 - \frac{k'_3}{K'}} \right), \\ c'_2 &= -\frac{\zeta'}{2} \sqrt{\frac{1}{2} \left(1 - \zeta' \sqrt{\frac{1 - \beta^2}{q}} \right)} \left(\sqrt{1 + \frac{k'_3}{K'}} - \zeta' \sqrt{1 - \frac{k'_3}{K'}} \right), \\ c'_3 &= \frac{1}{2} \sqrt{\frac{1}{2} \left(1 + \zeta' \sqrt{\frac{1 - \beta^2}{q}} \right)} \left(\sqrt{1 + \frac{k'_3}{K'}} - \zeta' \sqrt{1 - \frac{k'_3}{K'}} \right), \\ c'_4 &= \frac{\zeta'}{2} \sqrt{\frac{1}{2} \left(1 - \zeta' \sqrt{\frac{1 - \beta^2}{q}} \right)} \left(\sqrt{1 + \frac{k'_3}{K'}} + \zeta' \sqrt{1 - \frac{k'_3}{K'}} \right), \end{aligned} \quad (2.86)$$

where, as follows from (2.83) and (2.84),

$$\frac{k'_3}{K'} = -\frac{2\sqrt{x_0(x_0 - x)(\nu^2 - x_0x)}}{\nu^2 - 2x_0x + x_0^2} = -\frac{\sqrt{(p - q)(1 - q)}}{q + \sqrt{p}}, \quad \frac{k_0}{\Lambda'_0} = \sqrt{\frac{1 - \beta^2}{q}}. \quad (2.87)$$

Here we give some in-between calculations

$$\begin{aligned} c'^+_1 c_4 + c'^+_3 c_2 &= \frac{1}{4} \mathcal{F}(-\zeta, \zeta') \left(\sqrt{1 + \frac{k'_3}{K'}} + \zeta \zeta' \sqrt{1 - \frac{k'_3}{K'}} \right), \\ c'^+_2 c_3 + c'^+_4 c_1 &= \frac{1}{4} \mathcal{F}(\zeta, -\zeta') \left(\sqrt{1 - \frac{k'_3}{K'}} + \zeta \zeta' \sqrt{1 + \frac{k'_3}{K'}} \right), \\ c'^+_1 c_3 + c'^+_3 c_1 &= \frac{1}{4} \mathcal{F}(\zeta, \zeta') \left(\sqrt{1 + \frac{k'_3}{K'}} - \zeta \zeta' \sqrt{1 - \frac{k'_3}{K'}} \right), \\ c'^+_2 c_4 + c'^+_4 c_2 &= \frac{1}{4} \mathcal{F}(-\zeta, -\zeta') \left(\sqrt{1 - \frac{k'_3}{K'}} - \zeta \zeta' \sqrt{1 + \frac{k'_3}{K'}} \right), \end{aligned} \quad (2.88)$$

where for convenience we introduced the function $\mathcal{F}(\zeta, \zeta')$ of the form

$$\mathcal{F}(\zeta, \zeta') = \sqrt{\left(1 + \zeta \sqrt{1 - \beta^2} \right) \left(1 + \zeta' \sqrt{\frac{1 - \beta^2}{q}} \right)}. \quad (2.89)$$

Thus, summing over s' , $0 \leq s' < \infty$, and taking into account the properties (A.18) of the Laguerre functions, from (2.19) we obtain

$$\begin{aligned} \sum_{s'} S_2 &= \frac{1}{8} \left(1 + \zeta \zeta' \sqrt{1 - \frac{k'^2_3}{K'^2}} \right) [\mathcal{F}(-\zeta, \zeta') I_{n, n'-1}(x) - \zeta \zeta' \mathcal{F}(\zeta, -\zeta') I_{n-1, n'}(x)]^2, \\ \sum_{s'} S_3 &= \frac{1}{16} \left\{ \left(\sqrt{1 + \frac{k'_3}{K'}} + \zeta \zeta' \sqrt{1 - \frac{k'_3}{K'}} \right) [\mathcal{F}(-\zeta, \zeta') I_{n, n'-1}(x) + \zeta \zeta' \mathcal{F}(\zeta, -\zeta') I_{n-1, n'}(x)] \cos \theta - \right. \\ &\quad \left. - \left(\sqrt{1 + \frac{k'_3}{K'}} - \zeta \zeta' \sqrt{1 - \frac{k'_3}{K'}} \right) [\mathcal{F}(\zeta, \zeta') I_{n-1, n'-1}(x) + \zeta \zeta' \mathcal{F}(-\zeta, -\zeta') I_{n, n'}(x)] \sin \theta \right\}^2. \end{aligned} \quad (2.90)$$

Finally, for the spectral-angular distributions of the radiation power emitted by a spinor particle and its polarization components, in terms of x_0 and x we get

$$\frac{dW_i}{d\Omega} = \frac{1}{8} \frac{Q_0 A(\beta)(\nu + x)^3}{(2n)^3 x_0 (\nu - x)} \mathcal{W}_i(\zeta, \zeta', n, \nu, \beta, \theta), \quad A(\beta) = \frac{\beta^6}{1 - \beta^2} = \frac{(\gamma^2 - 1)^3}{\gamma^4}, \quad (2.91)$$

$$\begin{aligned} \mathcal{W}_2(\zeta, \zeta', n, \nu, \beta, \theta) &= [d(\zeta, \zeta')]^2 [\mathcal{F}(-\zeta, \zeta') I_{n, n'-1}(x) - \zeta \zeta' \mathcal{F}(\zeta, -\zeta') I_{n-1, n'}(x)]^2, \\ \mathcal{W}_3(\zeta, \zeta', n, \nu, \beta, \theta) &= \{d(\zeta, \zeta') [\mathcal{F}(-\zeta, \zeta') I_{n, n'-1}(x) + \zeta \zeta' \mathcal{F}(\zeta, -\zeta') I_{n-1, n'}(x)] \cos \theta + \\ &\quad + d(\zeta, -\zeta') [\mathcal{F}(\zeta, \zeta') I_{n-1, n'-1}(x) + \zeta \zeta' \mathcal{F}(-\zeta, -\zeta') I_{n, n'}(x)] \sin \theta\}^2, \end{aligned} \quad (2.92)$$

$$\mathcal{W}_0(\zeta, \zeta', n, \nu, \beta, \theta) = \mathcal{W}_2(\zeta, \zeta', n, \nu, \beta, \theta) + \mathcal{W}_3(\zeta, \zeta', n, \nu, \beta, \theta) \quad (2.93)$$

$$\mathcal{W}_g(\zeta, \zeta', n, \nu, \beta, \theta) = \mathcal{W}_0 + g\mathcal{V}, \quad (2.94)$$

$$\begin{aligned} \mathcal{V} &= d(\zeta, \zeta') \{d(\zeta, \zeta') [\mathcal{F}(-\zeta, \zeta') I_{n, n'-1} + \zeta \zeta' \mathcal{F}(\zeta, -\zeta') I_{n-1, n'}] \cos \theta + \\ &\quad + d(\zeta, -\zeta') [\mathcal{F}(\zeta, \zeta') I_{n-1, n'-1}(x) + \zeta \zeta' \mathcal{F}(-\zeta, -\zeta') I_{n, n'}(x)] \sin \theta\} \times \\ &\quad \times \{\mathcal{F}(-\zeta, \zeta') I_{n, n'-1} - \zeta \zeta' \mathcal{F}(\zeta, -\zeta') I_{n-1, n'}\}. \end{aligned} \quad (2.95)$$

These expressions can be rewritten in terms of q and p

$$\frac{dW_i}{d\Omega} = \frac{Q_0(1-q)^2(1+\sqrt{q})(\sqrt{p}+\sqrt{q})}{8(1-\beta^2)\sqrt{p}(1+\sqrt{p})^3} \overline{\mathcal{W}}_i(\zeta, \zeta', n, \nu, \beta, \theta), \quad (2.96)$$

$$\begin{aligned} \overline{\mathcal{W}}_2(\zeta, \zeta', n, \nu, \beta, \theta) &= [\bar{d}(\zeta, \zeta')]^2 [\mathcal{F}(-\zeta, \zeta') I_{n, n'-1}(x) - \zeta \zeta' \mathcal{F}(\zeta, -\zeta') I_{n-1, n'}(x)]^2, \\ \overline{\mathcal{W}}_3(\zeta, \zeta', n, \nu, \beta, \theta) &= \{\bar{d}(\zeta, \zeta') [\mathcal{F}(-\zeta, \zeta') I_{n, n'-1}(x) + \zeta \zeta' \mathcal{F}(\zeta, -\zeta') I_{n-1, n'}(x)] \cos \theta + \\ &\quad + \bar{d}(\zeta, -\zeta') [\mathcal{F}(\zeta, \zeta') I_{n-1, n'-1}(x) + \zeta \zeta' \mathcal{F}(-\zeta, -\zeta') I_{n, n'}(x)] \sin \theta\}^2, \end{aligned} \quad (2.97)$$

$$\overline{\mathcal{W}}_0(\zeta, \zeta', n, \nu, \beta, \theta) = \overline{\mathcal{W}}_2(\zeta, \zeta', n, \nu, \beta, \theta) + \overline{\mathcal{W}}_3(\zeta, \zeta', n, \nu, \beta, \theta) \quad (2.98)$$

$$\overline{\mathcal{W}}_g(\zeta, \zeta', n, \nu, \beta, \theta) = \overline{\mathcal{W}}_0 + g\overline{\mathcal{V}}, \quad (2.99)$$

$$\begin{aligned} \overline{\mathcal{V}} &= \bar{d}(\zeta, \zeta') \{\bar{d}(\zeta, \zeta') [\mathcal{F}(-\zeta, \zeta') I_{n, n'-1} + \zeta \zeta' \mathcal{F}(\zeta, -\zeta') I_{n-1, n'}] \cos \theta + \\ &\quad + \bar{d}(\zeta, -\zeta') [\mathcal{F}(\zeta, \zeta') I_{n-1, n'-1}(x) + \zeta \zeta' \mathcal{F}(-\zeta, -\zeta') I_{n, n'}(x)] \sin \theta\} \times \\ &\quad \times \{\mathcal{F}(-\zeta, \zeta') I_{n, n'-1} - \zeta \zeta' \mathcal{F}(\zeta, -\zeta') I_{n-1, n'}\}. \end{aligned} \quad (2.100)$$

$$d(\zeta, \zeta') = \begin{cases} \sqrt{x_0(x_0 - x)}, & \zeta' = -\zeta, \\ \sqrt{\nu^2 - x_0 x}, & \zeta' = \zeta, \end{cases} \quad \bar{d}(\zeta, \zeta') = \begin{cases} \sqrt{\frac{(1-\sqrt{q})(\sqrt{p}-\sqrt{q})}{(1+\sqrt{q})(\sqrt{p}+\sqrt{q})}}, & \zeta' = -\zeta, \\ 1, & \zeta' = \zeta. \end{cases} \quad (2.101)$$

Obviously, we can distinguish two types of quantum transitions, namely, those when spin direction is conserved ($\zeta' = \zeta$) and the spin-flip transitions ($\zeta' = -\zeta$). It is convenient to introduce the index k to perform a thorough study of each of these transitions. $k = 1$ stands for the transitions when no spin-flip occur and $k = 2$ for spin-flip transitions. So, we can introduce $W_i^{(k)}(\zeta, n, \nu, \beta)$, the spectral-angular distribution of the emitted power with respect to the initial spin and the type of transition

$$\begin{aligned} W_i^{(1)}(\zeta, n, \nu, \beta) &= W_i(\zeta, \zeta, n, \nu, \beta), \quad W_i^{(2)}(\zeta, n, \nu, \beta) = W_i(\zeta, -\zeta, n, \nu, \beta), \\ \sum_{\zeta'=\pm\zeta} W_i(\zeta, \zeta', n, \nu, \beta) &= W_i^{(1)}(\zeta, n, \nu, \beta) + W_i^{(2)}(\zeta, n, \nu, \beta). \end{aligned} \quad (2.102)$$

For the transitions without spin-flip we can list the exact analytical forms of distributions with respect to polarization i (here we omit the argument x of Laguerre functions)

$$\begin{aligned}\mathcal{W}_2^{(1)}(\zeta, n, \nu, \beta, \theta) &= (\nu^2 - x_0 x) [\mathcal{F}(-\zeta, \zeta) I_{n, n'-1} - \mathcal{F}(\zeta, -\zeta) I_{n-1, n'}]^2, \\ \overline{\mathcal{W}}_2^{(1)}(\zeta, n, \nu, \beta, \theta) &= [\mathcal{F}(-\zeta, \zeta) I_{n, n'-1} - \mathcal{F}(\zeta, -\zeta) I_{n-1, n'}]^2,\end{aligned}\quad (2.103)$$

$$\begin{aligned}\mathcal{W}_3^{(1)}(\zeta, n, \nu, \beta, \theta) &= \left\{ \sqrt{\nu^2 - x_0 x} \cos \theta [\mathcal{F}(-\zeta, \zeta) I_{n, n'-1} + \mathcal{F}(\zeta, -\zeta) I_{n-1, n'}] + \right. \\ &\quad \left. + \sqrt{x_0(x_0 - x)} \sin \theta [\mathcal{F}(\zeta, \zeta) I_{n-1, n'-1} + \mathcal{F}(-\zeta, -\zeta) I_{n, n'}] \right\}^2, \\ \overline{\mathcal{W}}_3^{(1)}(\zeta, n, \nu, \beta, \theta) &= \left\{ \cos \theta [\mathcal{F}(-\zeta, \zeta) I_{n, n'-1} + \mathcal{F}(\zeta, -\zeta) I_{n-1, n'}] + \right. \\ &\quad \left. + \left[\frac{(1 - \sqrt{q})(\sqrt{p} - \sqrt{q})}{(1 + \sqrt{q})(\sqrt{p} + \sqrt{q})} \right]^{1/2} \sin \theta [\mathcal{F}(\zeta, \zeta) I_{n-1, n'-1} + \mathcal{F}(-\zeta, -\zeta) I_{n, n'}] \right\}^2,\end{aligned}\quad (2.104)$$

$$\begin{aligned}\mathcal{V}^{(1)}(\zeta, n, \nu, \beta, \theta) &= \left\{ (\nu^2 - x_0 x) \cos \theta [\mathcal{F}(-\zeta, \zeta) I_{n, n'-1} + \mathcal{F}(\zeta, -\zeta) I_{n-1, n'}] + \sqrt{x_0(x_0 - x)(\nu^2 - x_0 x)} \times \right. \\ &\quad \left. \times \sin \theta [\mathcal{F}(\zeta, \zeta) I_{n-1, n'-1} + \mathcal{F}(-\zeta, -\zeta) I_{n, n'}] \right\} \{ \mathcal{F}(-\zeta, \zeta) I_{n, n'-1} - \mathcal{F}(\zeta, -\zeta) I_{n-1, n'} \}, \\ \overline{\mathcal{V}}^{(1)}(\zeta, n, \nu, \beta, \theta) &= \left\{ \cos \theta [\mathcal{F}(-\zeta, \zeta) I_{n, n'-1} + \mathcal{F}(\zeta, -\zeta) I_{n-1, n'}] + \left[\frac{(1 - \sqrt{q})(\sqrt{p} - \sqrt{q})}{(1 + \sqrt{q})(\sqrt{p} + \sqrt{q})} \right]^{1/2} \times \right. \\ &\quad \left. \times \sin \theta [\mathcal{F}(\zeta, \zeta) I_{n-1, n'-1} + \mathcal{F}(-\zeta, -\zeta) I_{n, n'}] \right\} \{ \mathcal{F}(-\zeta, \zeta) I_{n, n'-1} - \mathcal{F}(\zeta, -\zeta) I_{n-1, n'} \}.\end{aligned}$$

For spin-flip transitions we get

$$\begin{aligned}\mathcal{W}_2^{(2)}(\zeta, n, \nu, \beta, \theta) &= x_0(x_0 - x) [\mathcal{F}(-\zeta, -\zeta) I_{n, n'-1} + \mathcal{F}(\zeta, \zeta) I_{n-1, n'}]^2, \\ \overline{\mathcal{W}}_2^{(2)}(\zeta, n, \nu, \beta, \theta) &= \frac{(1 - \sqrt{q})(\sqrt{p} - \sqrt{q})}{(1 + \sqrt{q})(\sqrt{p} + \sqrt{q})} [\mathcal{F}(-\zeta, -\zeta) I_{n, n'-1} + \mathcal{F}(\zeta, \zeta) I_{n-1, n'}]^2,\end{aligned}\quad (2.105)$$

$$\begin{aligned}\mathcal{W}_3^{(2)}(\zeta, n, \nu, \beta, \theta) &= \left\{ \sqrt{x_0(x_0 - x)} \cos \theta [\mathcal{F}(-\zeta, -\zeta) I_{n, n'-1} - \mathcal{F}(\zeta, \zeta) I_{n-1, n'}] + \right. \\ &\quad \left. + \sqrt{\nu^2 - x_0 x} \sin \theta [\mathcal{F}(\zeta, -\zeta) I_{n-1, n'-1} - \mathcal{F}(-\zeta, \zeta) I_{n, n'}] \right\}^2, \\ \overline{\mathcal{W}}_3^{(2)}(\zeta, n, \nu, \beta, \theta) &= \left\{ \left[\frac{(1 - \sqrt{q})(\sqrt{p} - \sqrt{q})}{(1 + \sqrt{q})(\sqrt{p} + \sqrt{q})} \right]^{1/2} \cos \theta [\mathcal{F}(-\zeta, -\zeta) I_{n, n'-1} - \mathcal{F}(\zeta, \zeta) I_{n-1, n'}] + \right. \\ &\quad \left. + \sin \theta [\mathcal{F}(\zeta, -\zeta) I_{n-1, n'-1} - \mathcal{F}(-\zeta, \zeta) I_{n, n'}] \right\}^2,\end{aligned}$$

$$\begin{aligned}\mathcal{V}^{(2)}(\zeta, n, \nu, \beta, \theta) &= \left\{ x_0(x_0 - x) \cos \theta [\mathcal{F}(-\zeta, -\zeta) I_{n, n'-1} - \mathcal{F}(\zeta, \zeta) I_{n-1, n'}] + \sqrt{x_0(x_0 - x)} \times \right. \\ &\quad \left. \times \sqrt{\nu^2 - x_0 x} \sin \theta [\mathcal{F}(\zeta, -\zeta) I_{n-1, n'-1} - \mathcal{F}(\zeta, \zeta) I_{n, n'}] \right\} \{ \mathcal{F}(-\zeta, -\zeta) I_{n, n'-1} + \mathcal{F}(\zeta, \zeta) I_{n-1, n'} \}, \\ \overline{\mathcal{V}}^{(2)}(\zeta, n, \nu, \beta, \theta) &= \left\{ \frac{(1 - \sqrt{q})(\sqrt{p} - \sqrt{q})}{(1 + \sqrt{q})(\sqrt{p} + \sqrt{q})} \cos \theta [\mathcal{F}(-\zeta, -\zeta) I_{n, n'-1} - \mathcal{F}(\zeta, \zeta) I_{n-1, n'}] + \right. \\ &\quad \left. + \left[\frac{(1 - \sqrt{q})(\sqrt{p} - \sqrt{q})}{(1 + \sqrt{q})(\sqrt{p} + \sqrt{q})} \right]^{1/2} \sin \theta [\mathcal{F}(\zeta, -\zeta) I_{n-1, n'-1} - \mathcal{F}(-\zeta, \zeta) I_{n, n'}] \right\} \times \\ &\quad \times [\mathcal{F}(-\zeta, -\zeta) I_{n, n'-1} + \mathcal{F}(\zeta, \zeta) I_{n-1, n'}].\end{aligned}$$

2.2.2 Boson

Scalar Particle in a Constant Magnetic Field

Let us consider the radiation emitted by a scalar particle of elementary charge e_0 . Relativistic motion of a boson can be described by the Klein-Gordon-Fock equation (the original papers [109–111] are in

German and the reader may use [98, p. 65] and [20,97] in order to study the Klein-Gordon-Fock equation), which, for the field (2.22), has the form

$$\left\{ \frac{1}{\hbar^2} \hat{\mathcal{P}} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + k_0^2 \right\} \Psi = 0. \quad (2.106)$$

As was in the case of a spinor particle, we search for a stationary state of the form (2.24), which is the eigenvector of both the operator of the z -component of the momentum $\hat{\mathcal{P}}_z$ and the angular momentum operator $\hat{M}_z$, i.e. by analogy with (2.35) we write

$$\hat{\mathcal{P}}_z \Psi = \hbar k_3 \Psi, \quad \hat{M}_z \Psi = -i\hbar \frac{\partial}{\partial \varphi} \Psi. \quad (2.107)$$

Thus, similarly to (2.37), the function $\psi(\vec{r})$ of (2.24) can be given in the form

$$\psi(\vec{r}) = \sqrt{\frac{\bar{\gamma}}{\pi L}} e^{ik_3 z} e^{il\varphi} \psi(r). \quad (2.108)$$

And the equation (2.106) in terms of (r, φ, z) has the form

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{l^2}{r^2} - \bar{\gamma}^2 r^2 + K^2 - k_0^2 - k_3^2 - 2\bar{\gamma}l \right\} \psi(r) = 0, \quad (2.109)$$

which resembles the equations of the system (2.41). This equation is nothing other than (A.1) with

$$A = l^2, \quad B = \bar{\gamma}^2, \quad C = K^2 - k_0^2 - k_3^2 - 2\bar{\gamma}l, \quad (2.110)$$

and its solutions are the Laguerre functions

$$\psi(r) = \tilde{c} I_{n,s}(\bar{\gamma} r^2). \quad (2.111)$$

The normalization condition (2.48) together with (A.16) give

$$2\bar{\gamma} |\tilde{c}|^2 \int_0^\infty \psi^+(r) \psi(r) r dr = |\tilde{c}|^2 = 1. \quad (2.112)$$

And, finally, we obtain

$$\psi(\vec{r}) = \sqrt{\frac{\bar{\gamma}}{\pi L}} e^{ik_3 z} e^{il\varphi} I_{n,s}(\gamma r^2). \quad (2.113)$$

Indexes s and n are defined with the help of (A.9) and (A.12), therefore

$$s = \frac{K^2 - k_0^2 - k_3^2}{4\bar{\gamma}} - l - \frac{1}{2}, \quad n + \frac{1}{2} = \frac{K^2 - k_0^2 - k_3^2}{4\bar{\gamma}}, \quad n = 0, 1, 2, \dots, \quad (2.114)$$

while the discrete energy levels are given by

$$K_n = \sqrt{k_0^2 + k_3^2 + 4\bar{\gamma}(n + 1/2)}. \quad (2.115)$$

We note, that the semi-classical approach to Klein-Gordon-Fock and Dirac equations can be found in [112, p. 300].

Spectral-Angular Distributions of the Radiation emitted from a Scalar Particle

For a boson in place of the elements $\vec{D}$ of (2.11) one is supposed to use $[\hbar\sqrt{KK'}]^{-1} \hat{\mathcal{P}}$, where K and K' are the energies of initial and final states of the particle. For the eigenfunctions (2.113) it is easy to

estimate

$$\begin{aligned}\hat{\mathcal{P}}_x\psi(\vec{r}) &= \frac{i\hbar}{2} \sqrt{\frac{\bar{\gamma}}{\pi L}} e^{ik_3 z} e^{il\varphi} \sqrt{4\bar{\gamma}} \left\{ \sqrt{n+1} e^{i\varphi} I_{n+1,s}(\bar{\gamma}r^2) - \sqrt{n} e^{-i\varphi} I_{n-1,s}(\bar{\gamma}r^2) \right\}, \\ \hat{\mathcal{P}}_y\psi(\vec{r}) &= \frac{\hbar}{2} \sqrt{\frac{\bar{\gamma}}{\pi L}} e^{ik_3 z} e^{il\varphi} \sqrt{4\bar{\gamma}} \left\{ \sqrt{n+1} e^{i\varphi} I_{n+1,s}(\bar{\gamma}r^2) + \sqrt{n} e^{-i\varphi} I_{n-1,s}(\bar{\gamma}r^2) \right\},\end{aligned}\quad (2.116)$$

and, by analogy with (2.69), to obtain

$$\begin{aligned}\psi'^+(\vec{r})\hat{\mathcal{P}}_x\psi(\vec{r}) &= \frac{i\hbar}{2} \frac{\bar{\gamma}}{\pi L} e^{i(k_3-k'_3)z} e^{i(l-l')\varphi} \sqrt{4\bar{\gamma}} I_{n',s'}(\bar{\gamma}r^2) \times \\ &\quad \times \left\{ \sqrt{n+1} e^{i\varphi} I_{n+1,s}(\bar{\gamma}r^2) - \sqrt{n} e^{-i\varphi} I_{n-1,s}(\bar{\gamma}r^2) \right\}, \\ \psi'^+(\vec{r})\hat{\mathcal{P}}_y\psi(\vec{r}) &= \frac{\hbar}{2} \frac{\bar{\gamma}}{\pi L} e^{i(k_3-k'_3)z} e^{i(l-l')\varphi} \sqrt{4\bar{\gamma}} I_{n',s'}(\bar{\gamma}r^2) \times \\ &\quad \times \left\{ \sqrt{n+1} e^{i\varphi} I_{n+1,s}(\bar{\gamma}r^2) + \sqrt{n} e^{-i\varphi} I_{n-1,s}(\bar{\gamma}r^2) \right\}, \\ \psi'^+(\vec{r})\hat{\mathcal{P}}_z\psi(\vec{r}) &= \hbar k_3 \frac{\bar{\gamma}}{\pi L} e^{i(k_3-k'_3)z} e^{i(l-l')\varphi} I_{n',s'}(\bar{\gamma}r^2) I_{n,s}(\bar{\gamma}r^2).\end{aligned}\quad (2.117)$$

Considering (1.17) and (2.70) for the matrix elements we get

$$\begin{aligned}\frac{\langle \hat{\mathcal{P}}_1 \rangle}{\hbar\sqrt{KK'}} &= i\sqrt{\frac{\bar{\gamma}}{KK'}} \left\{ \sqrt{n+1} I_{n+1,n'}(x) - \sqrt{n} I_{n-1,n'}(x) \right\} I_{s,s'}(x) \delta_{k'_3, k_3 - \kappa \cos \theta}, \\ \frac{\langle \hat{\mathcal{P}}_2 \rangle}{\hbar\sqrt{KK'}} &= \sqrt{\frac{\bar{\gamma}}{KK'}} \left\{ \sqrt{n+1} I_{n+1,n'}(x) + \sqrt{n} I_{n-1,n'}(x) \right\} I_{s,s'}(x) \delta_{k'_3, k_3 - \kappa \cos \theta}, \\ \frac{\langle \hat{\mathcal{P}}_3 \rangle}{\hbar\sqrt{KK'}} &= \frac{k_3}{\sqrt{KK'}} I_{n,n'}(x) I_{s,s'}(x) \delta_{k'_3, k_3 - \kappa \cos \theta},\end{aligned}\quad (2.118)$$

where x can be calculated with (2.72). (2.73) and (2.74) remain valid, though for a boson n should be replaced with $n + 1/2$. Thus, for $k_3 = 0$, a photon that was emitted from a boson is characterized by

$$\kappa = \frac{K}{\sin^2 \theta} \left(1 - \sqrt{1 - \frac{2\nu}{2n+1} \beta^2 \sin^2 \theta} \right), \quad \hbar\omega = \frac{2\nu}{(2n+1)} \frac{m_0 c^2 \gamma \beta^2}{\left(1 + \sqrt{1 - \frac{2\nu}{2n+1} \beta^2 \sin^2 \theta} \right)}. \quad (2.119)$$

The relation between γ and n (2.77) is also modified for a boson

$$\gamma^2 = 1 + (2n+1)b, \quad (\xi = b\sqrt{1 + (2n+1)b}). \quad (2.120)$$

Similarly to (2.78) we use the notations

$$\bar{q} = 1 - \frac{2\nu}{2n+1} \beta^2, \quad \bar{p} = 1 - \frac{2\nu}{2n+1} \beta^2 \sin^2 \theta, \quad (2.121)$$

and, obviously, the inequality $0 < \bar{q} \leq \bar{p} \leq 1$ remains valid. For a boson we introduce

$$x^b \equiv x^b(n, \nu, \beta, \theta) = \nu \frac{1 - \sqrt{1 - \frac{2\nu}{2n+1} \beta^2 \sin^2 \theta}}{1 + \sqrt{1 - \frac{2\nu}{2n+1} \beta^2 \sin^2 \theta}} = \nu \frac{1 - \sqrt{\bar{p}}}{1 + \sqrt{\bar{p}}}, \quad 0 < x^b \leq x_0^b < \nu, \quad (2.122)$$

where

$$x_0^b \equiv x_0^b(n, \nu, \beta) = x^b(n, \nu, \beta, \frac{\pi}{2}) = \nu \frac{1 - \sqrt{1 - \frac{2\nu}{2n+1} \beta^2}}{1 + \sqrt{1 - \frac{2\nu}{2n+1} \beta^2}} = \nu \frac{1 - \sqrt{\bar{q}}}{1 + \sqrt{\bar{q}}}. \quad (2.123)$$

Similarly to the case of an electron, in the section where the motion of a boson is described we omit the index b , though the notations x and x_0 are supposed to be considered with respect to (2.122) and (2.123).

The relations (2.83) remain valid with respect to the substitutions $q \rightarrow \bar{q}$ and $p \rightarrow \bar{p}$. For a scalar particle it is easy to obtain

$$\frac{\bar{\gamma}}{KK'} = \frac{\beta^2(\nu + x_0)^2}{2(2n+1)(\nu^2 - 2x_0x + x_0^2)} = \frac{\beta^2(1 + \sqrt{\bar{p}})}{2(2n+1)(\sqrt{\bar{q}} + \sqrt{\bar{p}})} , \quad (2.124)$$

$$\frac{ce_0^2\kappa^2}{1 + \frac{\partial K'}{\partial \kappa}} \frac{\bar{\gamma}}{KK'} = \frac{Q_0\beta^6}{(1 - \beta^2)} \frac{(\nu + x)^3}{2(2n+1)^3(\nu - x)} = \frac{Q_0\beta^2(1 - \bar{q})^2}{(1 - \beta^2)2(2n+1)\sqrt{\bar{p}}(1 + \sqrt{\bar{p}})^2} . \quad (2.125)$$

Performing a summation over s' with the use of the properties (A.17) we find

$$\begin{aligned} \sum_{s'} S_2 &= \frac{\bar{\gamma}}{KK'} [2\sqrt{x} I'_{n,n'}(x)]^2, \\ \sum_{s'} S_3 &= \frac{\bar{\gamma}}{KK'} \left[\frac{(\nu + x)}{\sqrt{x}} I_{n,n'}(x) \right]^2 \cos^2 \theta, \\ \sum_{s'} S_g &= \frac{1}{2} \sum_{s'} S_0 + g \cdot 2(x + \nu) I_{n,n'}(x) I'_{n,n'}(x) \cos \theta , \end{aligned} \quad (2.126)$$

and for the spectral-angular distributions of power

$$\frac{dW_i^b}{d\Omega} = \frac{Q_0 A(\beta)(\nu + x)^3}{2(2n+1)^3(\nu - x)} \mathcal{W}_i^b(n, \nu, \beta, \theta) = \frac{Q_0 \beta^2(1 - \bar{q})^2}{2(2n+1)\sqrt{\bar{p}}(1 + \sqrt{\bar{p}})^2} \mathcal{W}_i^b(n, \nu, \beta, \theta), \quad (2.127)$$

$$\mathcal{W}_2^b(n, \nu, \beta, \theta) = [2\sqrt{x} I'_{n,n'}(x)]^2, \quad \mathcal{W}_3^b(n, \nu, \beta, \theta) = \left[\frac{(\nu + x)}{\sqrt{x}} I_{n,n'}(x) \right]^2 \cos^2 \theta, \quad (2.128)$$

$$\mathcal{W}_0^b(n, \nu, \beta, \theta) = \mathcal{W}_2^b + \mathcal{W}_3^b, \quad \mathcal{W}_g^b = \frac{\mathcal{W}_0^b}{2} + g\mathcal{V}^b, \quad \mathcal{V}^b = 2(\nu + x) I_{n,n'}(x) I'_{n,n'}(x) . \quad (2.129)$$

2.3 Spontaneous Transitions to the First Excited State and to the Ground State

To simplify the analysis of the influence of quantum corrections on radiation, we distinguish two particular types of quantum transitions, namely,

$$\begin{aligned} &\text{transitions to the first excited state, } n \rightarrow 1, \text{ i.e. } n' = 1, \nu = n - 1, \\ &\text{transitions to the ground state, } n \rightarrow 0, \text{ i.e. } n' = 0, \nu = n, \end{aligned} \quad (2.130)$$

and consider them separately. It is convenient to rewrite the expressions (2.91) - (2.101) and (2.127) - (2.129) in the following form

$$\begin{aligned} \frac{dW_i^b(n, \nu, \beta, \theta)}{d\Omega} &= \frac{Q_0 A(\beta)}{2(2n+1)^3} F_i^b(n, \nu, \beta, \theta), \quad \frac{dW_i^{(k)}(\zeta, n, \nu, \beta, \theta)}{d\Omega} = \frac{Q_0 A(\beta)}{2(2n)^3} F_i^{(k)}(\zeta, n, \nu, \beta, \theta), \\ F_i^e(\zeta, n, \nu, \beta, \theta) &= F_i^{(1)} + F_i^{(2)}, \quad \bar{F}_i^e(\zeta, n, \nu, \beta, \theta) = \frac{1}{2}(F_i^{(1)} + F_i^{(2)}), \end{aligned} \quad (2.131)$$

and to introduce the function

$$B(n, \nu, \chi) = \frac{(\nu + \chi)^3}{(\nu - \chi)} \chi^{\nu-1} e^{-\chi} , \quad (2.132)$$

which allow us to write

$$F_i^b(n, \nu, \beta, \theta) = \frac{B(n, \nu, x^b)}{n!} \mathfrak{F}_i^b(n, \nu, \beta, \theta) , \quad F_i^{(k)}(\zeta, n, \nu, \beta, \theta) = \frac{B(n, \nu, x^e)}{n!} \mathfrak{F}_i^{(k)}(\zeta, n, \nu, \beta, \theta) . \quad (2.133)$$

The definition (2.132) serves to simplify the cumbersome formulae used for the functions F_i . The notations x^e , x_0^e (see (2.79) and (2.80)) and x^b , x_0^b (see (2.122) and (2.123)) are to be transformed for particular transitions as follows, for a boson

$$\begin{aligned} x^b(n, n, \beta, \theta) &\equiv \bar{x}, \quad x_0^b(n, n, \beta) \equiv \bar{x}_0, \\ x^b(n, n-1, \beta, \theta) &\equiv \bar{y}, \quad x_0^b(n, n-1, \beta) \equiv \bar{y}_0, \end{aligned} \quad (2.134)$$

and for an electron

$$\begin{aligned} x^e(n, n, \beta, \theta) &\equiv \tilde{x}, \quad x_0^e(n, n, \beta) \equiv \tilde{x}_0, \\ x^e(n, n-1, \beta, \theta) &\equiv \tilde{y}, \quad x_0^e(n, n-1, \beta) \equiv \tilde{y}_0. \end{aligned} \quad (2.135)$$

In these terms it is easy to obtain, for the *transitions to the first excited state*

for a *boson*

$$F_i^b(n, n-1, \beta, \theta) = \frac{B(n, n-1, \bar{y})}{n!} \mathfrak{F}_i^b(n, n-1, \beta, \theta), \quad (2.136)$$

$$\mathfrak{F}_2^b(n, n-1, \beta, \theta) = [n(n-1) - (2n+1)\bar{y} + \bar{y}^2]^2, \quad (2.137)$$

$$\mathfrak{F}_3^b(n, n-1, \beta, \theta) = (n - \bar{y})^2(n-1 + \bar{y})^2 \cos^2 \theta, \quad (2.138)$$

$$\mathfrak{F}_0^b(n, n-1, \beta, \theta) = \mathfrak{F}_2^b(n, n-1, \beta, \theta) + \mathfrak{F}_3^b(n, n-1, \beta, \theta), \quad (2.139)$$

$$\mathfrak{F}_g^b(n, n-1, \beta, \theta) = \frac{\mathfrak{F}_0}{2} + g(n - \bar{y})(n-1 + \bar{y})[n(n-1) - (2n+1)\bar{y} + \bar{y}^2] \cos \theta. \quad (2.140)$$

for an *electron*

$$F_i^{(k)}(\zeta, n, n-1, \beta, \theta) = \frac{B(n, n-1, \tilde{y})}{n!} \mathfrak{F}_i^{(k)}(\zeta, n, n-1, \beta, \theta),$$

$$\mathfrak{F}_i^{(k)}(\zeta, n, n-1, \beta, \theta) = \frac{\mathfrak{f}_i^{(k)}(\zeta, n, n-1, \beta, \theta)}{2((n-1)^2 - \tilde{y}_0^2)}, \quad (2.141)$$

- when the direction of the spin does not change

$$\begin{aligned} \mathfrak{f}_2^{(1)}(\zeta, n, n-1, \beta, \theta) &= ((n-1)^2 - \tilde{y}_0 \tilde{y})[(2(n+1) - \tilde{y}_0)M(n, \tilde{y}) - \\ &\quad (n-1)^2 \overline{M}(n, \tilde{y}) - \zeta(n-1)D(n, \tilde{y}_0) \overline{\overline{M}}(n, \tilde{y})], \end{aligned} \quad (2.142)$$

$$\begin{aligned} \mathfrak{f}_3^{(1)}(\zeta, n, n-1, \beta, \theta) &= (n-1)^2 \frac{(\tilde{y}_0 - \tilde{y})}{\tilde{y}_0} [(n-1)^2 \overline{M}(n, \tilde{y}) - \\ &\quad M(n, \tilde{y}) \tilde{y}_0 - \zeta(n-1)D(n, \tilde{y}_0) \overline{\overline{M}}(n, \tilde{y})], \end{aligned} \quad (2.143)$$

$$\mathfrak{f}_0^{(1)}(\zeta, n, n-1, \beta, \theta) = \mathfrak{f}_2^{(1)}(\zeta, n, n-1, \beta, \theta) + \mathfrak{f}_3^{(1)}(\zeta, n, n-1, \beta, \theta), \quad (2.144)$$

- spin-flip transitions

$$\begin{aligned} \mathfrak{f}_2^{(2)}(\zeta, n, n-1, \beta, \theta) &= (\tilde{y}_0 - \tilde{y})[((n-1)^2 - 2(n+1)\tilde{y}_0)M(n, \tilde{y}) + \\ &\quad \tilde{y}_0(n-1)^2 \overline{M}(n, \tilde{y}) + \zeta(n-1)^2 D(n, \tilde{y}_0) \overline{\overline{M}}(n, \tilde{y})], \end{aligned} \quad (2.145)$$

$$\mathfrak{f}_3^{(2)}(\zeta, n, n-1, \beta, \theta) = \tilde{y}_0((n-1)^2 - \tilde{y}_0 \tilde{y})[M(n, \tilde{y}) - \overline{M}(n, \tilde{y})\tilde{y}_0 + \zeta D(n, \tilde{y}_0)\overline{\overline{M}}(n, \tilde{y})] ,$$

$$\mathfrak{f}_0^{(2)}(\zeta, n, n-1, \beta, \theta) = \mathfrak{f}_2^{(2)}(\zeta, n, n-1, \beta, \theta) + \mathfrak{f}^{(2)}(\zeta, n, n-1, \beta, \theta) . \quad (2.146)$$

Here we used the notations

$$M(n, z) = n(n-1)^2 - 2n(n-1)z + (n+1)z^2, \quad \overline{M}(n, z) = n(n+1) - 2nz + z^2, \quad (2.147)$$

$$\overline{\overline{M}}(n, z) = n(n-1) - 2nz + z^2, \quad \overline{M}(n, z) + \overline{\overline{M}}(n, z) = 2(n-z)^2 ,$$

$$D(n, z) = \sqrt{(n-1)^2 + z^2 - 2(n+1)z} . \quad (2.148)$$

Now, let us turn our attention to **quantum transitions to the ground state**. For $n \rightarrow 0$ we have found

for a **boson**

$$F_i^b(n, n, \beta, \theta) = \frac{B(n, n, \bar{x})}{n!} \mathfrak{F}_i^b(n, n, \beta, \theta) , \quad (2.149)$$

$$\mathfrak{F}_2^b(n, n, \beta, \theta) = (n - \bar{x})^2 , \quad \mathfrak{F}_3^b(n, n, \beta, \theta) = (n + \bar{x})^2 \cos^2 \theta , \quad (2.150)$$

$$\mathfrak{F}_0^b(n, n, \beta, \theta) = \mathfrak{F}_2^b(n, n, \beta, \theta) + \mathfrak{F}_3^b(n, n, \beta, \theta), \quad (2.151)$$

$$\mathfrak{F}_g^b(n, n, \beta, \theta) = \frac{\mathfrak{F}_0}{2} + g(n^2 - \bar{x}^2) \cos \theta. \quad (2.152)$$

for an **electron**

$$F_i^{(k)}(\zeta, n, n, \beta, \theta) \equiv F_i^{(k)}(n, n, \beta, \theta) = \frac{B(n, n, \tilde{x})}{n!} \mathfrak{F}_i^{(k)}(n, n, \beta, \theta) , \quad (2.153)$$

$$\mathfrak{F}_i^{(k)}(n, n, \beta, \theta) = \frac{d^{(k)}(n, \beta)}{n + \tilde{x}_0} \mathfrak{f}_i^{(k)}(n, n, \beta, \theta) , \quad (2.154)$$

$$d^{(k)}(n, \beta) = \begin{cases} n , & k = 1, \\ \tilde{x}_0 , & k = 2 , \end{cases} \quad (2.155)$$

$$\mathfrak{f}_2^{(1)}(n, n, \beta, \theta) \equiv \mathfrak{f}_2(-1, n, n, \beta, \theta) = n^2 - \tilde{x}_0 \tilde{x} , \quad (2.156)$$

$$\mathfrak{f}_3^{(1)}(n, n, \beta, \theta) \equiv \mathfrak{f}_3(-1, n, n, \beta, \theta) = \frac{n^2(n + \tilde{x})^2}{(n^2 - \tilde{x}_0 \tilde{x})} \cos^2 \theta , \quad (2.157)$$

$$\mathfrak{f}_2^{(2)}(n, n, \beta, \theta) \equiv \mathfrak{f}_2(1, n, n, \beta, \theta) = \mathfrak{f}_3^{(1)}(n, n, \beta, \theta) , \quad (2.158)$$

$$\mathfrak{f}_3^{(2)}(n, n, \beta, \theta) \equiv \mathfrak{f}_3(1, n, n, \beta, \theta) = \mathfrak{f}_2^{(1)}(n, n, \beta, \theta) , \quad (2.159)$$

$$\mathfrak{f}_0^{(k)}(n, n, \beta, \theta) \equiv \mathfrak{f}_0(\zeta, n, n, \beta, \theta) = \mathfrak{f}_2^{(k)}(n, n, \beta, \theta) + \mathfrak{f}_3^{(k)}(n, n, \beta, \theta) . \quad (2.160)$$

The specific notations that we used for an electron in this case reflect the fact that in the ground state the spin of the electron is strictly opposite to the external field ($\zeta' = -1$) (see [113, p. 76] or any other textbook on quantum mechanics). Thus, if the initial spin is collinear with $\vec{H}$, then the transition to $n' = 0$ is a spin-flip transition. So, to avoid tautology, we can either use the value of ζ ($\zeta = 1$) as a parameter and get rid of the index (k) ($k = 2$) or remove ζ and denote the transition with ($k = 2$). Same thing occurs for the initial spin antiparallel to external field, so that we either use $\zeta = -1$ or $k = 1$.

Here we focus on some peculiar features of radiation emitted at $n \rightarrow 0$ and $n \rightarrow 1$. From the mathematical point of view, further conclusions are nothing more than a number of trivial consequences of (2.136) - (2.160). However, they uncover new intriguing properties of radiation that could not have been predicted intuitively.

Let us first note that in the ultrarelativistic limit ($\beta \rightarrow 1$) the factor $D(n, \tilde{y}_0)$ vanishes

$$\tilde{y}_0|_{\beta \rightarrow 1} = x_0^e(n, n-1, \beta \rightarrow 1) = (n+1) - 2\sqrt{n}, \Rightarrow D(n, \tilde{y}_0)|_{\beta \rightarrow 1} = 0. \quad (2.161)$$

This fact indicates that, in case of transitions to the first excited state, one can not but distinguish whether it was spin-flip or not. The distribution of radiation power doesn't give exact information on the direction of initial and final spin (ζ and ζ' correspondingly), yet it allows us to understand if $\zeta = \zeta'$ or $\zeta = -\zeta'$. In terms of F^e -functions we have

$$F_i^{(k)}(\zeta, n, n-1, \beta \rightarrow 1, \theta) = F_i^{(k)}(-\zeta, n, n-1, \beta \rightarrow 1, \theta). \quad (2.162)$$

The presence of the factor $(\tilde{y}_0 - \tilde{y})$ in (2.145) and $\tilde{y}_0$ in (2.146) explains a vanishing probability of spin-flip transitions in the non-relativistic case. It is easy to see that

$$x_0^e(n, \nu, \beta \rightarrow 0) = 0, \quad x^e(n, \nu, \beta \rightarrow 0, \theta) = 0, \quad (2.163)$$

so, the spin-flip transitions give zero contribution to the power of radiation at $\beta \rightarrow 0$ (i.e. $f_i^{(2)}(\zeta, n, n-1, \beta \approx 0, \theta) \approx 0$, $i = 0, 2, 3$).

As for the transitions with no spin-flip, for non-relativistic quantum particles one can observe a simple relation between the power distributions associated with opposite spin directions, namely

$$f_i^{(1)}(\zeta = -1, n, n-1, \beta \approx 0, \theta) = n \cdot f_i^{(1)}(\zeta = 1, n, n-1, \beta \approx 0, \theta), \quad i = 0, 2, 3. \quad (2.164)$$

It means that the probability of spin direction conservation during a quantum transition is n times greater for the case when the spin is antiparallel to the external field.

The remarkable feature of the spontaneous transitions to the ground state involves polarization. Relations (2.156) - (2.160) show that the angular distribution of the σ -component of the radiation emitted by an electron whose spin is antiparallel to the direction of the external field intensity coincides (up to a scaling factor $\tilde{x}_0/n$) with the distribution of the π -component of an electron, whose spin is parallel to $\vec{H}$. The distribution function $f_3^{(1)}$, in its turn, copies the behavior of the σ -component of an electron with $\zeta = 1$. Thus, we can assume that angular distributions of σ - and π - components switch places due to the orientation of the spin ζ .

We emphasize that in the classical theory of SR the contribution of the π -component to the total power vanishes for $\theta = \pi/2$, i.e. the σ -component is the only to be emitted in the plane of the orbit of motion. However, in [52] it was shown that the spin-flip transitions perturb the classical result, so, that one can detect a nonzero contribution of the π -component at $\theta = \pi/2$. In practice, the presence of a π -component can serve as a tool to detect the orientation of the initial spin of radiating particle. Still, in [52] the switch between the components of linear polarization was not specified. Thus, in addition to giving an exact analytical proof for the presence of π -component in the plane of motion, we established

the physical cause of this phenomenon. For the particular case of a one-harmonic spectrum, this proof can be found in our recent papers [114,115].

The factor $d^{(k)}(n, \beta)$ determines the probability of a quantum transition with respect to the initial spin. Obviously, the transitions without spin-flip (when the initial spin is parallel to the external field) are $n/\tilde{x}_0$ times more likely to occur, except for the ultrarelativistic case, where two possible types of transitions become equiprobable.

Speaking on ultrarelativistic case, not only the dependence on the initial spin direction vanishes but also the angular distributions of the components of linear polarization become equal. Namely, from (2.156) - (2.160) at $\beta \rightarrow 1$ it follows

$$f_2^{(k)}(n, n, \beta \rightarrow 1, \theta) = f_3^{(k)}(n, n, \beta \rightarrow 1, \theta) = n(n - \tilde{x})|_{\beta \rightarrow 1}, \quad (2.165)$$

$$\tilde{x}|_{\beta \rightarrow 1} = x^e(n, n, \beta \rightarrow 1, \theta) \approx n \frac{1 - |\cos \theta|}{1 + |\cos \theta|}, \quad k = 1, 2, \quad (2.166)$$

and since the factor $d^{(2)}(n, \beta \rightarrow 1) \approx n$ near $\theta = \pi/2$ we have

$$F_i^{(k)}(\zeta_1, n, n, \beta \rightarrow 1, \theta \approx \pi/2) = F_{i'}^{(k')}(\zeta_2, n, n, \beta \rightarrow 1, \theta \approx \pi/2), \quad (2.167)$$

$$\{i, i'\} = 2, 3, \quad \{k, k'\} = 1, 2, \quad \{\zeta_1, \zeta_2\} = \pm 1, \quad (2.168)$$

so, in the plane of motion, all the $n \rightarrow 0$ transitions become equiprobable independently on (linear) polarization and initial spin.

Chapter 3

Single-harmonic Spectrum of Synchrotron Radiation in Quantum Theory

In the framework of quantum theory we study the distributions of radiation emitted by a relativistic particle at the transition $1 \rightarrow 0$. Obviously, if the initial energy level of the particle in question is $n = 1$, i.e. before the emission of a photon the particle was in the first excited state, then the spectrum of the radiation contains a single harmonic $\nu = n = 1$. It is clear, that the relations (2.149) - (2.160) can be significantly simplified when $n = 1$, so that it becomes (technically) easier to follow the influence of quantum effects on radiation.

In the previous chapter one can find the relations (2.77) and (2.120) between the energy of the quantum particle, the number of the initial state and the intensity of the external field. When $n = 1$, these relations have the form $\gamma^2 = 1 + 2b$ and $\gamma^2 = 1 + 3b$ for electron and boson respectively. In other words, to provide a relativistic motion the extremely high values of external field intensity $H \sim H_0$ are required. Contemporary accelerator techniques though very well developed yet give no chance to reach such intensities, and, thus, the analysis of the particular radiation characteristics for $n = 1$ has a tutorial nature with respect to accelerator physics. In this sense, the two-level system can be considered a tool to study peculiar properties of quantum SR and to qualitatively estimate the influence of quantum effects on radiation. Particular results obtained for a single-harmonic spectrum can further be generalized to the case of arbitrary n , and special attention should be given to qualitative description of angular distributions.

As for the practical use, we turn our attention back to Magnetars, where even at $n = 1$ the parameter ξ can be of the order $\sim 10^2 - 10^3$, $\xi \gg 1$. So, the particles in the first excited state are already relativistic, and the results obtained for low n can be adequately applied to the radiation processes in deep space. We emphasize here that the contribution of quantum corrections is more noticeable the lower the level n , and the case $n = 1$ is a purely quantum system with respect to radiation.

3.1 The Characteristics of Radiation

3.1.1 Boson

For a scalar particle we use the definitions (2.134) and (2.135), though throughout this chapter we bear in mind that $n = 1$ and assume

$$x^b(1, 1, \beta, \theta) = \bar{x}, \quad x_0^b(1, 1, \beta) = \bar{x}_0. \quad (3.1)$$

Clearly,

$$\bar{x}_0 = \frac{\sqrt{3} - \sqrt{3 - 2\beta^2}}{\sqrt{3} + \sqrt{3 - 2\beta^2}}, \quad \bar{x} = \frac{\sqrt{3} - \sqrt{3 - 2\beta^2 \sin^2 \theta}}{\sqrt{3} + \sqrt{3 - 2\beta^2 \sin^2 \theta}}, \quad (3.2)$$

$$0 \leq \bar{x}_0 < 2 - \sqrt{3} \approx 0.26794919, \quad 0 \leq \bar{x} \leq \bar{x}_0.$$

The power of radiation emitted from a boson while it jumps from the first excited state to the ground state is based on the functions

$$\begin{aligned} \mathfrak{F}_2^b(1, 1, \beta, \theta) &= (1 - \bar{x})^2, \quad \mathfrak{F}_3^b(1, 1, \beta, \theta) = (1 + \bar{x})^2 \cos^2 \theta, \\ \mathfrak{F}_0^b(1, 1, \beta, \theta) &= \mathfrak{F}_2^b(1, 1, \beta, \theta) + \mathfrak{F}_3^b(1, 1, \beta, \theta), \quad \mathfrak{F}_g^b(1, 1, \beta, \theta) = \frac{\mathfrak{F}_0^b}{2} + g(1 - \bar{x}^2) \cos \theta, \end{aligned} \quad (3.3)$$

so that for the angular distributions one can write [115]

$$\frac{dW_i^b}{d\Omega} = \frac{Q_0 A(\beta)}{54} (1 + \bar{x})^3 e^{-\bar{x}} \varphi_i^b(\beta, \theta), \quad (3.4)$$

$$\varphi_2^b(\beta, \theta) = 1 - \bar{x}, \quad \varphi_3^b(\beta, \theta) = \frac{(1 + \bar{x})^2 \cos^2 \theta}{1 - \bar{x}}, \quad (3.5)$$

$$\varphi_0^b(\beta, \theta) = \varphi_2^b(\beta, \theta) + \varphi_3^b(\beta, \theta) = \varphi_{-1}^b(\beta, \theta) + \varphi_1^b(\beta, \theta), \quad \varphi_g^b(\beta, \theta) = \frac{1}{2} \varphi_0^b(\beta, \theta) + g(1 + \bar{x}) \cos \theta,$$

Integrating (3.4) over θ within $[0, \pi]$, for the total power of unpolarized radiation we get

$$W_0^b(1, 1, \beta) = \frac{4Q_0 A(\beta)}{81} f^b(\beta), \quad f^b(\beta) = \frac{3(1 + \bar{x}_0)^2}{8} f_0^b(\bar{x}_0), \quad f_0^b(x) = f_2^b(x) + f_3^b(x). \quad (3.6)$$

The functions $f_2^b(x)$ and $f_3^b(x)$ cannot be expressed in terms of elementary functions and have the form

$$f_2^b(x) = \int_0^1 \frac{(1 + xy)(1 - xy)^2}{\sqrt{(1 - y)(1 - x^2 y)}} e^{-xy} dy, \quad (3.7)$$

$$f_3^b(x) = \int_0^1 (1 + xy) \sqrt{(1 - y)(1 - x^2 y)} e^{-xy} dy, \quad f_2^b(0) = 2, \quad f_3^b(0) = \frac{2}{3}.$$

In order to obtain the power of radiation emitted in the upper half-plane, one should integrate the distributions $dW_i/d\Omega$ over θ , $0 \leq \theta < \pi/2$. In the framework of quantum theory we introduce the functions $q_i^b(\beta)$ by analogy with the classical functions $q_i^{cl}(\beta)$ which we defined in section 1.1

$$W_i^{(+b)}(1, 1, \beta) = \frac{1}{2} W_0^b(1, 1, \beta) q_i^b(\beta), \quad q_2^b(\beta) + q_3^b(\beta) = q_{-g}^b(\beta) + q_g^b(\beta) = 1. \quad (3.8)$$

Obviously, the relations $q_i^b(\beta)$ can be presented in terms of $f_i^b(\bar{x}_0)$

$$q_i^b(\beta) = \frac{f_i^b(\bar{x}_0)}{f_0^b(\bar{x}_0)}, \quad (i = 2, 3), \quad q_0^b(\beta) = 1, \quad q_g^b(\beta) = \frac{1}{2} + g \frac{f_1^b(\bar{x}_0)}{f_0^b(\bar{x}_0)}. \quad (3.9)$$

Unlike $f_i^b(\bar{x}_0)$ ($i = 2, 3$), the function $f_1^b(\bar{x}_0)$ is elementary

$$f_1^b(x) = \int_0^1 (1 - x^2 y^2) e^{-xy} dy = \frac{(1+x)^2 e^{-x} - 1}{x}, \quad f_1^b(0) = 1.$$

In the non-relativistic approximation ($\beta \approx 0$) the values of the functions $q_i^b(\beta)$ coincide with their classical analogues (1.39). Moreover, the relation between the power of radiation emitted in the upper and lower half-plane is similar to (1.35)

$$W_i^{(-)b}(1, 1, \beta) = W_i^{(+)b}(1, 1, \beta) \quad (i = 0, 2, 3), \quad W_g^{(-)b}(1, 1, \beta) = W_{-g}^{(+)b}(1, 1, \beta). \quad (3.10)$$

The plots of the functions $q_i^b(\beta)$ for $i = 1, 2$ are given in figures 3.13 and 3.14 and their description can be found in section 3.1.2.

As it was in classical theory, one can consider the contribution of the i -component of the polarized radiation emitted into a solid angle $d\Omega$ near the direction θ . Namely, we introduce the functions

$$\frac{dW_i^b(1, 1, \beta, \theta)}{d\Omega} = W_0^b(1, 1, \beta) p_i^b(\beta, \theta), \quad \text{i.e. } p_i^b(\beta, \theta) = \frac{(1 + \bar{x})^3 e^{-\bar{x}} \varphi_i^b(\beta, \theta)}{(1 + \bar{x}_0)^2 f_0^b(\bar{x})}, \quad (3.11)$$

which, same as their classical analogues, either have the property of symmetry ($i = 0, 2, 3$) or antisymmetry ($i = g = \pm 1$) with respect to $\pi/2$

$$p_i^b(\beta, \theta) = p_i^b(\beta, \pi - \theta) \quad (i = 0, 2, 3), \quad p_g^b(\beta, \theta) = p_{-g}^b(\beta, \pi - \theta).$$

It follows from the definition that the functions $q_i^b(\beta)$ can be expressed in terms of $p_i^b(\beta, \theta)$

$$q_i^b(\beta) = 2 \int_0^{\frac{\pi}{2}} p_i^b(\beta, \theta) d\Omega. \quad (3.12)$$

The plots of $p_i^b(\beta, \theta)$ for $i = 0, 1, 2, 3$ are given in figures 3.1 - 3.4.

Here we list some particular values of the functions $p_i^b(\beta, \theta)$

$$p_0^b(\beta, 0) = p_1^b(\beta, 0) = 2p_2^b(\beta, 0) = 2, \quad p_3^b(\beta, 0) = \frac{2}{(1 + \bar{x}_0)^2 f_0^b(\bar{x}_0)}, \quad p_{-1}^b(\beta, 0) = 0, \\ p_0^b\left(\beta, \frac{\pi}{2}\right) = p_2^b\left(\beta, \frac{\pi}{2}\right) = 2p_g^b\left(\beta, \frac{\pi}{2}\right) = \frac{1 - \bar{x}_0^2}{e^{\bar{x}_0} f_0^b(\bar{x}_0)}, \quad p_3^b\left(\beta, \frac{\pi}{2}\right) = 0. \quad (3.13)$$

Following the sequence of concepts given in the chapter 1, right after $p_i^b(\beta, \theta)$ we introduce the functions $q_i^b(\beta, \theta)$

$$q_i^b(\beta, \theta) = \frac{p_i^b(\beta, \theta)}{p_0^b(\beta, \theta)} = \frac{\varphi_i^b(\beta, \theta)}{\varphi_0^b(\beta, \theta)}, \quad (3.14)$$

which, similarly to $q_i^{cl}(\beta, \theta)$, have the properties

$$q_2^b(\beta, \theta) + q_3^b(\beta, \theta) = q_g^b(\beta, \theta) + q_{-g}^b(\beta, \theta) = q_0^b(\beta, \theta) = 1. \quad (3.15)$$

These functions show the evolution of the polarization rate with β and θ . Since we have (3.15), we can restrict ourselves to studying the behavior of $q_2^b(\beta, \theta)$ and $q_1^b(\beta, \theta)$. In figures 3.5 and 3.6 the dependence of q_i^b ($i = 1, 2$) on β and θ is given.

It is easy to calculate the values of $q_i^b(\beta, \theta)$ for fixed β and θ

$$q_2^b(\beta, 0) = \frac{1}{2}, \quad q_g^b(\beta, 0) = \frac{1+g}{2}, \quad q_2^b\left(\beta, \frac{\pi}{2}\right) = 1, \quad q_g^b\left(\beta, \frac{\pi}{2}\right) = \frac{1}{2}. \quad (3.16)$$

Figure 3.1: The plots of $p_2^b(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 3.2: The plots of $p_3^b(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 3.3: The plots of $p_0^b(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 3.4: The plots of $p_1^b(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

3.1.2 Electron

Polarized Electron

The distributions of radiation emitted by an electron for $n = 1$ transform to

$$\mathfrak{F}_i^{(k)}(1, 1, \beta, \theta) = \frac{d^{(k)}(1, \beta)}{1 + \tilde{x}_0} \mathfrak{f}_i^{(k)}(1, 1, \beta, \theta), \quad (3.17)$$

$$d^{(k)}(1, \beta) = \begin{cases} 1, & k = 1, \\ \tilde{x}_0, & k = 2, \end{cases} \quad (3.18)$$

$$\begin{aligned} \mathfrak{f}_2^{(1)}(1, 1, \beta, \theta) &= \mathfrak{f}_2(-1, 1, 1, \beta, \theta) = 1 - \tilde{x}_0 \tilde{x}, \\ \mathfrak{f}_3^{(1)}(1, 1, \beta, \theta) &= \mathfrak{f}_3(-1, 1, 1, \beta, \theta) = \frac{(1 + \tilde{x})^2}{(1 - \tilde{x}_0 \tilde{x})} \cos^2 \theta, \\ \mathfrak{f}_0(1)(1, 1, \beta, \theta) &= \mathfrak{f}_0(-1, 1, 1, \beta, \theta) = \mathfrak{f}_2(-1, 1, 1, \beta, \theta) + \mathfrak{f}_3(-1, 1, 1, \beta, \theta), \end{aligned} \quad (3.19)$$

Similarly to the previous section, here we simplify the notations and use the variables $\tilde{x}_0$ and $\tilde{x}$ in order to define

$$x^e(1, 1, \beta, \theta) \equiv \tilde{x}, \quad x_0^e(1, 1, \beta) \equiv \tilde{x}_0, \quad (3.20)$$

$$\tilde{x}_0 = \frac{1 - \sqrt{1 - \beta^2}}{1 + \sqrt{1 - \beta^2}} = \frac{\gamma - 1}{\gamma + 1}, \quad \tilde{x} = \frac{1 - \sqrt{1 - \beta^2 \sin^2 \theta}}{1 + \sqrt{1 - \beta^2 \sin^2 \theta}}, \quad (3.21)$$

$$0 \leq \tilde{x}_0(\beta) < 1, \quad 0 \leq \tilde{x}(\beta, \theta) \leq \tilde{x}_0(\beta).$$

Figure 3.5: The plots of $q_2^b(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$. Figure 3.6: The plots of $q_1^b(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

So, for the angular distributions of the radiation power emitted by an electron we get

$$\frac{dW_i^e}{d\Omega} = \frac{dW_i^{(k)}}{d\Omega} = \frac{Q_0 A(\beta)}{16(1 + \tilde{x}_0)} d^{(k)}(1, \beta) \frac{(1 + \tilde{x})^3 e^{-\tilde{x}}}{1 - \tilde{x}} \varphi_i^{(k)}(\beta, \theta), \quad (3.22)$$

where, similarly to (3.1.2), there are two options to include the dependence on spin

$$\varphi_i^{(k)}(\beta, \theta) \equiv \varphi_i^e(\zeta, \beta, \theta), \quad k = 1 \ (k = 2) \text{ implies } \zeta = -1 \ (\zeta = 1), \quad (3.23)$$

$$\begin{aligned} \varphi_2^e(-1, \beta, \theta) &= \varphi_3^e(1, \beta, \theta) = 1 - \tilde{x}_0 \tilde{x}, \\ \varphi_2^e(1, \beta, \theta) &= \varphi_3^e(-1, \beta, \theta) = \frac{(1 + \tilde{x})^2 \cos^2 \theta}{1 - \tilde{x}_0 \tilde{x}}, \\ \varphi_0^e(\zeta, \beta, \theta) &= \varphi_0^e(\beta, \theta) = \varphi_2^e(-1, \beta, \theta) + \varphi_3^e(-1, \beta, \theta) = \varphi_2^e(1, \beta, \theta) + \varphi_3^e(1, \beta, \theta), \\ \varphi_g^e(\zeta, \beta, \theta) &= \varphi_g^e(\beta, \theta) = \frac{\varphi_0^e(\beta, \theta)}{2} + g(1 + \tilde{x}) \cos \theta. \end{aligned} \quad (3.24)$$

Now, we can find the power of the radiation emitted in the upper half-plane ($W_i^{(+e)}$) integrating (3.22) over θ within $[0, \pi/2]$ and the total power (to obtain the total power integration should be performed within the interval $0 \leq \theta \leq \pi$)

$$W_0^e(\zeta, 1, 1, \beta) = \frac{Q_0 A(\beta)}{6} d^{(k)}(1, \beta) f^e(\beta), \quad f^e(\beta) = \frac{3(1 + \tilde{x}_0)}{8} f_0^e(\tilde{x}_0), \quad (3.25)$$

$$f_0^e(x) = f_2^e(x) + f_3^e(x).$$

The functions $f_2^e(x)$ and $f_3^e(x)$ can not be expressed in terms of elementary functions and can be written as the integrals

$$\begin{aligned} f_2^e(x) &= \int_0^1 (1 + xy) \exp(-xy) \sqrt{\frac{1 - x^2 y}{1 - y}} dy, \\ f_3^e(x) &= \int_0^1 (1 + xy) \exp(-xy) \sqrt{\frac{1 - y}{1 - x^2 y}} dy. \end{aligned}$$

We remind here of the "switch" between the components of linear polarization, which implies the non-vanishing π -component in the plane of motion ($\theta = \pi/2$). The presence of the factor $d^{(k)}(1, \beta)$ in (3.22) indicates that the spin-flip transitions are $\tilde{x}_0$ times less probable than the transitions without spin-flip.

Since

$$\tilde{x}_0 \approx \frac{1}{4} \beta^2 \ll 1, \text{ at } \beta \ll 1, \quad \tilde{x}_0 \approx 1, \text{ at } 1 - \beta^2 \ll 1, \quad (3.26)$$

in the non-relativistic approximation the only electron that radiates is the one with initial spin antiparallel to the external field, while the electron with $\zeta = 1$ does not emit, i.e. remains quasi-stable. In the relativistic case ($\gamma \gg 1$) the dependence of SR power on the spin orientation vanishes. However, the switch between σ - and π - components of the linear polarization can be observed at any energy.

As was for a scalar particle, for an electron the power of radiation emitted in the upper half-plane $W_i^{(+)\epsilon}(\zeta, 1, 1, \beta)$ can be given in terms of q^ϵ -functions

$$W_i^{(+)\epsilon}(\zeta, 1, 1, \beta) = \frac{1}{2} W_0^\epsilon(\zeta, 1, 1, \beta) q_i^\epsilon(\zeta, \beta), \quad q_g^\epsilon(\zeta, \beta) \equiv q_g^\epsilon(\beta), \quad (3.27)$$

$$q_2^\epsilon(\zeta, \beta) = q_3^\epsilon(-\zeta, \beta), \quad q_2^\epsilon(\zeta, \beta) + q_3^\epsilon(\zeta, \beta) = q_{-g}^\epsilon(\beta) + q_g^\epsilon(\beta) = q_0^\epsilon(\zeta, \beta) = 1. \quad (3.28)$$

It is not hard to verify

$$q_1^\epsilon(\beta) = \frac{1}{2} + \frac{f_1^\epsilon(\tilde{x}_0)}{f_0^\epsilon(\tilde{x}_0)}, \quad q_2^\epsilon(\zeta, \beta) = \frac{1 + \zeta - 2\zeta q_2^\epsilon(\beta)}{2}, \quad q_2^\epsilon(\beta) = \frac{f_2^\epsilon(\tilde{x}_0)}{f_0^\epsilon(\tilde{x}_0)}, \quad (3.29)$$

The function $f_1^\epsilon(x)$ is elementary, as was $f_1^b(x)$

$$f_1^\epsilon(x) = \frac{2 - (2 + x) \exp(-x)}{x}, \quad 0 \leq x \leq 1.$$

For the functions $f_i^\epsilon(x)$ one can find approximations near the points $x = 0, 1$

$$f_1^\epsilon(x) \approx 1 - \frac{x^2}{6}, \quad f_2^\epsilon(x) \approx 2 \left(1 - \frac{13}{15} x^2 \right), \quad f_3^\epsilon(x) \approx \frac{2}{3} \left(1 + \frac{11}{35} x^2 \right), \quad x \ll 1.$$

$$f_1^\epsilon(x) \approx 2 - \frac{3}{e} + \left(2 - \frac{5}{e} \right) (1 - x), \quad f_2^\epsilon(x) \approx 2 - \frac{3}{e} - \frac{4}{e} (1 - x) \ln(1 - x),$$

$$f_3^\epsilon(x) \approx 2 - \frac{3}{e} + \frac{2}{e} (1 - x) \ln(1 - x), \quad 0 < 1 - x \ll 1.$$

In its turn, in the non-relativistic limit the functions $q_i^\epsilon(\beta)$ tend to the same values as $q_i^b(\beta)$ ($q_i^{cl}(\beta)$), and in the ultrarelativistic limit

$$q_2^\epsilon(\beta \rightarrow 1) \approx \frac{1}{2}, \quad q_g^\epsilon(\beta \rightarrow) \approx \frac{1 + g}{2}. \quad (3.30)$$

Together with the plots of $q_i^b(\beta)$, the curves $q_i^\epsilon(-1, \beta)$, are given in figures 3.13, 3.14 (the number of the curve is 2).

For an electron, the relation (1.35) between the power of radiation emitted above and below the orbit remains valid.

To analyse the behavior of the angular distributions of the radiation emitted by a spin one-half particle one can introduce the functions $p_i^\epsilon(\zeta, \beta, \theta)$, which are similar to $p_i^{cl}(\beta, \theta)$ and $p_i^b(\beta, \theta)$

$$\frac{dW_i^\epsilon(\zeta, \beta, \theta)}{d\Omega} = W_0^\epsilon(\zeta, \beta) p_i^\epsilon(\zeta, \beta, \theta), \quad p_i^\epsilon(\zeta, \beta, \theta) = \frac{(1 + \tilde{x})^3 e^{-\tilde{x}} \varphi_i^\epsilon(\zeta, \beta, \theta)}{(1 - \tilde{x})(1 + \tilde{x}_0)^2 f_0^\epsilon(\tilde{x}_0)}, \quad (3.31)$$

$$p_2^\epsilon(\zeta, \beta, \theta) = p_3^\epsilon(-\zeta, \beta, \theta), \quad p_0^\epsilon(\zeta, \beta, \theta) \equiv p_0^\epsilon(\beta, \theta), \quad p_g^\epsilon(\zeta, \beta, \theta) \equiv p_g^\epsilon(\beta, \theta),$$

$$p_2^\epsilon(-1, \beta, \theta) = p_3^\epsilon(1, \beta, \theta) = p_2^\epsilon(\beta, \theta), \quad p_3^\epsilon(-1, \beta, \theta) = p_2^\epsilon(1, \beta, \theta) = p_3^\epsilon(\beta, \theta), \quad (3.32)$$

$$p_i^\epsilon(\zeta, \beta, \theta) = p_i^\epsilon(\zeta, \beta, \pi - \theta), \quad (i = 0, 2, 3), \quad p_g^\epsilon(\beta, \theta) = p_{-g}^\epsilon(\beta, \pi - \theta).$$

With (3.32) taken into account, we can restrict ourselves to the analysis of the evolution of $p_i^\epsilon(-1, \beta, \theta)$ with β and θ . The plots of $p_i^\epsilon(-1, \beta, \theta)$ are given in the figures 3.7 - 3.10 for $i = 0, 1, 2, 3$.

Figure 3.7: The plots of $p_2^e(-1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 0.95, 0.99, 0.999, 1$.

Figure 3.8: The plots of $p_3^e(-1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 0.95, 0.99, 0.999, 1$.

Figure 3.9: The plots of $p_0^e(-1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 0.95, 0.99, 0.999, 1$.

Figure 3.10: The plots of $p_1^e(-1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 0.95, 0.99, 0.999, 1$.

We can calculate particular values of the functions $p_i^e(\beta, \theta)$

$$p_0^e(\beta, 0) = p_1^e(\beta, 0) = 2p_2^e(\zeta, \beta, 0) = \frac{2}{(1 + \tilde{x}_0)^2 f_0^e(\tilde{x}_0)}, \quad p_{-1}^e(\beta, 0) = 0,$$

$$p_0^e\left(\beta, \frac{\pi}{2}\right) = p_2^e\left(\beta, \frac{\pi}{2}\right) = 2p_g^e\left(\beta, \frac{\pi}{2}\right) = \frac{(1 + \tilde{x}_0)^2}{e^{\tilde{x}_0} f_0^e(\tilde{x}_0)}, \quad p_3^e\left(\beta, \frac{\pi}{2}\right) = 0. \quad (3.33)$$

By analogy with (1.43) and (3.15), for an electron we introduce $q_i^e(\zeta, \beta, \theta)$, which defines the polarization degree of radiation at fixed ζ , β and θ

$$q_i^e(\zeta, \beta, \theta) = \frac{p_i^e(\zeta, \beta, \theta)}{p_0^e(\zeta, \beta, \theta)} = \frac{\varphi_i^e(\zeta, \beta, \theta)}{\varphi_0^e(\beta, \theta)}, \quad q_g^e(\zeta, \beta, \theta) \equiv q_g^e(\beta, \theta), \quad (3.34)$$

$$q_2^e(-1, \beta, \theta) \equiv q_2^e(\beta, \theta), \quad q_3^e(-1, \beta, \theta) \equiv q_3^e(\beta, \theta), \quad q_2^e(\zeta, \beta, \theta) = q_3^e(-\zeta, \beta, \theta),$$

$$q_2^e(\zeta, \beta, \theta) + q_3^e(\zeta, \beta, \theta) = q_g^e(\beta, \theta) + q_{-g}^e(\beta, \theta) = q_0^e(\zeta, \beta, \theta) = 1, \quad (3.35)$$

$$q_i^e(\zeta, \beta, \theta) = q_i^e(\zeta, \beta, \pi - \theta) \quad (i = 2, 3), \quad q_g^e(\beta, \theta) = q_{-g}^e(\beta, \pi - \theta).$$

The plots of the functions $q_2^e(-1, \beta, \theta)$ and $q_1^e(-1, \beta, \theta)$ are given in figures 3.11 - 3.12.

It is easy to calculate the values of $q_2^e(\beta, \theta)$ for particular points

$$q_2^e(\beta, 0) = \frac{1}{2}, \quad q_g^e(\beta, 0) = \frac{1+g}{2}, \quad q_2^e\left(\beta, \frac{\pi}{2}\right) = 1, \quad q_g^e\left(\beta, \frac{\pi}{2}\right) = \frac{1}{2}. \quad (3.36)$$

Figure 3.11: The plots of $q_2^e(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 0.95, 0.99, 0.999, 1$.

Figure 3.12: The plots of $q_1^e(\beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 0.95, 0.99, 0.999, 1$.

Unpolarized electron

Obviously, the power of radiation emitted by an unpolarized electron is half the sum of the power radiated by transversally polarized electrons with spin parallel and anti-parallel to the external magnetic field

$$\frac{d\overline{W}_i^e(n, \nu, \beta)}{d\Omega} = \frac{Q_0 A(\beta)}{4(2n)^3} (F_i^e(\zeta, n, \nu, \beta, \theta) + F_i^e(-\zeta, n, \nu, \beta, \theta)), \quad (3.37)$$

$$\overline{W}_i^e(n = 1, \nu = 1, \beta) = \frac{W_i^e(1, 1, 1, \beta) + W_i^e(-1, 1, 1, \beta)}{2} = \frac{(1 + \tilde{x}_0)}{2} W_i^e(-1, 1, 1, \beta).$$

The latest can be rewritten in the form (3.25), factor $(1 + \tilde{x}_0)/2$ included to the definition of the function $\overline{f}_e(\beta)$

$$\overline{W}_0^e(1, 1, \beta) = \frac{Q_0 A(\beta)}{6} \overline{f}^e(\beta), \quad \overline{f}^e(\beta) = \frac{(1 + \tilde{x}_0)}{2} f^e(\beta). \quad (3.38)$$

The amount of radiation emitted by an unpolarized electron in the upper (lower) half-plane can be expressed in terms of $W_i^{(+e)}(\pm\zeta, \beta)$ ($W_i^{(-e)}(\pm\zeta, \beta)$).

The relation between the power of radiation emitted by an electron with $\zeta = -1$ and an electron with $\zeta = 1$ was obtained in (3.28). It is not hard to show that the contributions of i - components of polarization ($i = 2, 3, 1, -1$) for unpolarized electron $\overline{q}_i^e(\beta)$ are related to the functions $q_i^e(-1, \beta)$ as follows

$$\begin{aligned} \overline{q}_2^e(\beta) &= \frac{\tilde{x}_0 + (1 - \tilde{x}_0)q_2^e(-1, \beta)}{1 + \tilde{x}_0}, \quad \overline{q}_3^e(\beta) = \frac{\tilde{x}_0 + (1 - \tilde{x}_0)q_3^e(-1, \beta)}{1 + \tilde{x}_0} = \\ &= 1 - \overline{q}_2^e(-1, \beta), \quad \overline{q}_g^e(\beta) = q_g^e(\beta). \end{aligned}$$

The plots of these functions for σ -component of linear polarization and for the right circular polarization are given in figures 3.13 and 3.14.

By analogy with (3.31) we introduce the functions $\overline{p}_i^e(\beta, \theta)$, which describe the part of polarized radiation emitted by an unpolarized electron into a solid angle $d\Omega$. And, obviously, there is a relation between these functions and $p_i^e(\beta, \theta)$

$$\overline{p}_2^e(\beta, \theta) = \frac{\tilde{x}_0 p_3^e(\beta, \theta) + p_2^e(\beta, \theta)}{1 + \tilde{x}_0}, \quad \overline{p}_3^e(\beta, \theta) = \frac{\tilde{x}_0 p_2^e(\beta, \theta) + p_3^e(\beta, \theta)}{1 + \tilde{x}_0},$$

$$\overline{p}_{0, \pm 1}^e(\beta, \theta) = p_{0, \pm 1}^e(\beta, \theta).$$

Figure 3.13: Plots of $q_2^b(\beta)$ (1), $q_2^e(-1, \beta)$ (2), $\bar{q}_2^e(\beta)$ (3), $q_2^{cl}(\beta)$ (4) and $q_2^h(1, \beta)$ (5).

Figure 3.14: Plots of $q_1^b(\beta)$ (1), $q_1^e(-1, \beta)$ (2), $q_1^{cl}(\beta)$ (3) and $q_1^h(1, \beta)$ (4).

Figure 3.15: The plots of $\bar{p}_2^e(\beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 3.16: The plots of $\bar{p}_3^e(\beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 0.95, 0.99, 0.999, 1$.

The plots of $\bar{p}_2^e(\beta, \theta)$ and $\bar{p}_3^e(\beta, \theta)$ are given in figures 3.15 and 3.16. Figure 3.15 illustrates that the contribution of the σ -component exceeds $\bar{p}_2^e(\beta = 0, \theta) = 3/8$ only at $\beta \approx 1$ near the plane of motion ($\theta \approx \pi/2$).

The functions $\bar{q}_i^e(\beta, \theta)$, that characterize the ratio of polarization for θ -direction at fixed β for an unpolarized electron, are given by

$$\bar{q}_2^e(\beta, \theta) = \frac{\tilde{x}_0 + (1 - \tilde{x}_0)q_2^e(\beta, \theta)}{1 + \tilde{x}_0}, \quad \bar{q}_3^e(\beta, \theta) = \frac{\tilde{x}_0 + (1 - \tilde{x}_0)q_3^e(\beta, \theta)}{1 + \tilde{x}_0} =$$

$$= 1 - \bar{q}_2^e(\beta, \theta), \quad \bar{q}_g^e(\beta, \theta) = q_g^e(\beta, \theta), \quad \bar{q}_0^e(\beta, \theta) = q_0^e(\beta, \theta) = 1.$$

It is sufficient to consider the behavior of $\bar{q}_2^e(\beta, \theta)$, its plots shown in figure 3.17.

We remind the reader that in the previous section we established the property of "switch" between the components of linear polarization of radiation emitted by an electron. Moreover, the contribution of σ -component at $\theta \approx \pi/2$ does not vanish for $\zeta = -1$. Therefore, in contrast to the predictions of the classical theory which worked well for the case of spinless particle, here the π -component of radiation remains nonzero near the plane of motion, except for the non-relativistic case $\beta \approx 0$.

Figure 3.17: Plots of $\overline{q}_2^e(\beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 0.95, 0.99, 0.999, 1$.

3.1.3 Comparative Analysis of the Angular Distributions of Radiation from Bosons and Electrons

To find how the spin itself and its directions influence the power of emitted radiation we introduce the functions $\varkappa(\zeta, \beta)$ and $\bar{\varkappa}(\beta)$. These functions describe the relation between the power radiated from a spinor particle (3.25, 3.37) and the power emitted by a boson (3.6)

$$\varkappa(\zeta, \beta) = \frac{W_0^e(\zeta, 1, 1, \beta)}{W_0^b(1, 1, \beta)}, \quad \bar{\varkappa} = \frac{\overline{W}_i^e(1, 1, \beta)}{W_0^b(1, 1, \beta)} = \frac{(1 + \tilde{x}_0)}{2} \varkappa(-1, \beta), \quad (3.39)$$

$$\varkappa(-1, \beta) = \frac{27}{8} \frac{f^e(\beta)}{f^b(\beta)}, \quad \bar{\varkappa}(\beta) = \frac{\bar{f}^e(\beta)}{f^b(\beta)}, \quad (3.40)$$

$$\varkappa(1, \beta) = \tilde{x}_0 \cdot \varkappa(-1, \beta) \leq \bar{\varkappa}(\beta) \leq \varkappa(-1, \beta).$$

The plots of the functions $f^b(\beta)$, $f^e(\beta)$ and $\bar{f}^e(\beta)$ are given in figures 3.18 - 3.19. Unlike $f^b(\beta)$ and $\bar{f}^e(\beta)$, $f^e(\beta)$ has a small maximum near $\beta = 1$. It means that for a polarized electron there is a certain energy value ($\beta = 0.98950989$, i.e. $\gamma = 6.9220849959$), which provides the maximum of emitted radiation. The functions $f^b(\beta)$ and $\bar{f}^e(\beta)$ increase monotonically with β , i.e. the radiation emitted from a boson and an unpolarized electron reaches its maximum when $\beta \rightarrow 1$, and this result corresponds with the conclusions known in classical theory.

Figure 3.18: The plots of $f^b(\beta)$ (1), $f^e(\beta)$ (2) and $\bar{f}^e(\beta)$ (3).

Figure 3.19: The plots of $f^b(\gamma)$ (1), $f^e(\gamma)$ (2) and $\bar{f}^e(\gamma)$ (3).

We have estimated numerically the values of $\varkappa$ -functions. Table 3.1 shows that $3.375 \leq \varkappa(-1, \beta) < 3.931168771$, and for the functions $\varkappa(1, \beta)$ we have $\varkappa(1, \beta) < 1$ at $\beta < \beta_0 \approx 0.8199913$ ($\gamma < \gamma_0 \approx 1.7471034$) and $\varkappa(1, \beta) > 1$ at $\beta > \beta_0$ ($\gamma > \gamma_0$). Finally, the functions $\bar{\varkappa}(\beta)$, as well as $\varkappa(-1, \beta)$, are greater than 1 within the entire interval $0 \leq \beta < 1$, namely, $1.6875 \leq \bar{\varkappa}(\beta) < 3.717$. These functions increase monotonically with β .

So, an electron with spin antiparallel to the external field radiates almost four times more than a boson at any energy. The unpolarized electron radiates ~ 2 times more than a spinless particle in the non-relativistic region, and in the relativistic area, similarly to the electron with $\zeta = -1$, its radiation is about $\sim 3 - 4$ times greater than that of a boson. If the initial spin direction coincides with the direction of external field, such electron radiates more than boson only within the relativistic region. So, for electrons the radiated power depends on spin significantly in weakly-relativistic region, but this dependence vanishes at extra-high energies.

Table 3.1: The influence of spin properties on radiation at $n = 1$

β	$f^b(\beta)$	$f^e(\beta)$	$\bar{f}^e(\beta)$	$\varkappa(-1, \beta)$	$\varkappa(1, \beta)$	$\bar{\varkappa}(\beta)$
0	1	1	0.5	3.3750	0	1.68750
0.1	1.001670487	1.002509868	0.502514377	3.37783	0.00849	1.69316
0.2	1.006728309	1.010160192	0.510234511	3.38651	0.03456	1.71053
0.3	1.015316560	1.023331235	0.523727265	3.40164	0.08019	1.74091
0.4	1.027687865	1.042723077	0.544072445	3.42438	0.14917	1.78677
0.5	1.044228543	1.069479439	0.573132304	3.45661	0.24817	1.85239
0.6	1.065499537	1.105426141	0.614125634	3.50147	0.38905	1.94526
0.7	1.092304653	1.153545206	0.672957454	3.56422	0.59438	2.07930
0.8	1.125807520	1.218988119	0.761867575	3.65434	0.91359	2.28396
0.9	1.167744113	1.311256203	0.913201081	3.78978	1.48887	2.63932
1	1.220847177	1.344542515	1.344542515	3.71695	3.71695	3.71695

3.2 The Difference Between Classical and Quantum Results

3.2.1 Linear and Circular Polarization

Figures 3.13 and 3.14 in section 3.1.2 serve to illustrate how does the contribution of i -polarization component to the total power of radiation change with energy. The plots were given for boson, for an electron with spin antiparallel to external field, for an unpolarized electron, for total classical spectrum and for the first harmonic of classical spectrum. Let us now discuss the behavior of corresponding functions and their peculiarities.

First of all, we note that the functions $q_2^b(\beta)$, $q_2^e(-1, \beta)$, $\bar{q}_2^e(\beta)$, introduced in the framework of quantum theory of SR, decrease with β in contrast to $q_2^{cl}(\beta)$, which monotonically increases with energy. That is, unlike in classical theory, for the radiation emitted at quantum transition from the first excited

state, the degree of linear polarization decreases. Moreover, in the case of electron, in ultrarelativistic limit, each component of the linear polarization is exactly a half of total power (3.30) (according to the classical theory, in ultrarelativistic limit the radiation is mostly linearly polarized and the σ -component is absolutely dominant).

Secondly, for an electron (either polarized or unpolarized) the degree of right circular polarization increases with β (which, again, contradicts the results known in classical theory). Besides, as it was shown in (3.30), $q_1^e(-1, \beta \rightarrow 1) = 1$. It means that, according to quantum theory, in ultrarelativistic limit the radiation emitted by electron is completely circularly polarized. Namely, in the upper half-plane right circular polarization is the only to be emitted (the contribution of left circular polarization component is negligibly small compared to that of the right circular polarization), whereas the lower half-plane contains left-polarized radiation only.

Finally, let us consider the curves $q_2^h(1, \beta)$ (plot 5 in the figure 3.13) and $q_1^h(1, \beta)$ (curve 4 in the figure 3.14) that correspond to the first harmonic of classical spectrum. Unlike the functions that describe the total classical spectrum, the behavior of these curves stays in good agreement with the results we have obtained with the use of quantum approach. Namely, $q_2^h(1, \beta)$ monotonically decreases and $q_1^h(1, \beta)$ monotonically increases with β . Moreover, both for $i = 1$ and $i = 2$, $q_i^h(1, \beta)$ is very close to the bosonic curve (curve 1 in 3.13, 3.14). So, the functions that correspond to the first harmonic of the classical spectrum are very much alike their quantum analogues created for the case $n = 1$. Compared to the characteristics of total (summed over spectrum) classical radiation, this set of functions seem to be a worthier contender for the description of radiation emitted at the transitions from low levels. It turned out not to be the only manifestation of such feature, and later we give more examples of how the reduction of classical spectrum makes the classical SR characteristics more representative with respect to short quantum spectrum.

As for the angular distributions p_i , we turn our attention to their behavior at $\beta \rightarrow 1$ and $\theta \rightarrow \pi/2$. The functions $p_i^h(\beta, \theta)$ have a continuous derivative with respect to β , which can be calculated in the point $\beta = 1$. The angular distributions of an electron $p_i^e(\beta, \theta)$ tend to finite values as $\beta \rightarrow 1$

$$p_i^e(\beta \rightarrow 1, \theta) |_{\theta \neq \frac{\pi}{2}} = \tilde{p}_i^e(\theta), \quad \tilde{p}_2^e(\theta) = \tilde{p}_3^e(\theta) = \frac{1}{2} \tilde{p}_0^e(\theta) = \frac{\Theta(\theta)}{2e-3},$$

$$\tilde{p}_g^e(\theta) = \frac{\Theta(\theta)}{2e-3} \left(1 + g \frac{\cos \theta}{|\cos \theta|} \right), \quad \Theta(\theta) = \frac{1}{(1 + |\cos \theta|)^3} \exp \left(\frac{2|\cos \theta|}{1 + |\cos \theta|} \right). \quad (3.41)$$

The function $\Theta(\theta)$ is continuous at all θ , and its derivative at $\theta = \frac{\pi}{2}$ has a finite break. More specifically, we get

$$\tilde{p}_2^e\left(\frac{\pi}{2}\right) = \tilde{p}_3^e\left(\frac{\pi}{2}\right) = \frac{1}{2} \tilde{p}_0^e\left(\frac{\pi}{2}\right) = \frac{1}{2e-3}, \quad (3.42)$$

whereas from (3.33) follows

$$p_2^e\left(\beta \rightarrow 1, \frac{\pi}{2}\right) = p_0^e\left(\beta \rightarrow 1, \frac{\pi}{2}\right) = \frac{2}{2e-3}, \quad p_3^e\left(\beta \rightarrow 1, \frac{\pi}{2}\right) = 0, \quad (3.43)$$

which reflects the ambiguity of double limit $\beta \rightarrow 1, \theta \rightarrow \pi/2$.

We remind here, that classical theory implies similar ambiguity. Namely, in ultrarelativistic limit $p_i^{cl}(\beta \rightarrow 1, \theta) = 0$, whereas if we change the order and start with $\theta \approx \pi/2$ then letting $\beta \rightarrow 1$, $p_i^{cl}(\beta, \theta)$

will increase infinitely. Thus, the characters of angular distributions of the power emitted by an electron and by a classical particle coincide at $\beta \rightarrow 1$, $\theta \rightarrow \pi/2$, yet we emphasize that the functions $p_i^e(\beta \rightarrow 1, \frac{\pi}{2})$ are finite.

Now let us consider the angular distribution with respect to polarization types. We perform the comparative analysis between classical particles, bosons and electrons for each particular type of polarization. It could be seen in figures 3.1, 3.7 and 3.15 that the functions $p_2^b(\beta, \theta)$, $p_2^e(\beta, \theta)$ and $\bar{p}_2^e(\beta, \theta)$ increase monotonically with θ on $[0, \pi/2]$. This result stays in a qualitative agreement with the classical theory, the functions $p_2^{cl}(\beta, \theta)$ increase with energy as well and reach their maximum in the plane of motion, $\theta \approx \pi/2$. σ -component of linear polarization of the first harmonic of the classical spectrum has an opposite behavior, namely, the functions $p_2^h(1, \beta, \theta)$ monotonically decrease at $0 \leq \theta \leq \pi/2$. Figure 3.1 illustrates explicitly that for a boson at each fixed θ the functions $p_2^b(\beta, \theta)$ decrease monotonically with β . Such behavior is not consistent with the behavior of the curves plotted for the electron (either polarized or unpolarized). Moreover, it differs from the predictions of classical theory, which implies the existence of some border θ , where the functions p_2 change their behavior and start increasing with β . For an unpolarized electron this border value of θ lies close to $\pi/2$. Thus, despite the presence of spin properties, which are of purely quantum nature, the structure of angular distributions associated with σ -component of radiation emitted by an electron, unlike a boson, stays in reasonably good qualitative agreement with that of classical particle.

The evolution of the functions $p_3^b(\beta, \theta)$, $p_3^e(\beta, \theta)$ and $\bar{p}_3^e(\beta, \theta)$ is shown in figures 3.2, 3.8 and 3.16. The function $p_3^b(\beta, \theta)$ decreases monotonically with θ within the interval $[0, \pi/2]$ at any fixed β and vanishes at $\theta = \pi/2$. The later is consistent with a well-known conclusion of classical theory. Namely, no π -component of linear polarization is emitted in the plane of the orbit of a classical particle whatever the energy. Concerning the evolution of partial contributions, similarly to the case $i = 2$, classical theory stays representative with respect to spinor particle if all the harmonics of classical spectrum are taken into account. Indeed, unlike the bosonic curves, the functions $p_3^e(\beta, \theta)$ and $\bar{p}_3^e(\beta, \theta)$ monotonically decrease with θ , ($0 \leq \theta \leq \pi/2$). However, at $\beta_3^e = \sqrt{3}/2$ and $\bar{\beta}_3^e = \sqrt{6}/3$, these functions lose their monotony so that a maximum can be observed at $\theta_3^{(max)}$ and $\bar{\theta}_3^{(max)}$. Due to the fact that the components of linearly polarized radiation emitted by electron switch places with spin, for an unpolarized electron the contribution of π -component of radiation doesn't vanish in the plane of motion at any β except for $\beta = 0$. Figure 3.16 reflects this fact, at $\theta = \pi/2$ the functions $\bar{p}_3^e(\beta, \theta)$ reach finite values. It should be pointed out that the curves $p_3^b(\beta, \theta)$ evolve quite similarly to $p_3^h(1, \beta, \theta)$, though $p_3^h(1, \beta, \theta)$ decreases monotonically within $[0, \pi/2]$ both with θ and with energy.

Concerning the sum over polarization components, a number of interesting peculiarities can be observed. The functions $p_0^b(\beta, \theta)$ at any fixed β decrease monotonically with θ within $[0, \pi/2]$ (figure 3.3). And, as was for $i = 3$, their behavior stays in good agreement with the results obtained in the framework of classical theory for the first harmonic of radiation spectrum (figure 1.15 demonstrates that $p_0^h(1, \beta, \theta)$ decreases monotonically with θ , $0 \leq \theta \leq \pi/2$, at any β). The essential difference in the behavior of functions $p_0^b(\beta, \theta)$ and $p_0^h(1, \beta, \theta)$ is that for a boson the contribution of radiation emitted in the direction of magnetic field ($\theta = 0$) decreases with energy, whereas for the first harmonic this contribution increases.

However, near the plane of motion both $p_0^b(\beta, \theta)$ and $p_0^h(1, \beta, \theta)$ decrease with β . For the case of total radiation, the angular distributions, set for electron (figure 3.9), are qualitatively closer to the curves plotted for the sum over classical harmonics (figure 1.4). The functions $p_0^{cl}(\beta, \theta)$ at small β decrease monotonically with θ ($0 \leq \theta \leq \pi/2$), but they change their behavior with energy and turn to be monotonically increasing. For an electron there is a value $\beta = 1/\sqrt{2}$ such that $p_0^e(\beta, \theta)$ change their character, namely, lose their monotony. $\theta_0^{max}(\beta)$ describes the evolution of position of maxima of these functions.

As for the right circular polarization, classical theory, again, gives a more adequate description for the behavior of electron, not boson. The functions $p_1^e(\beta, \theta)$ (figure 3.10), as well as $p_1^{cl}(\beta, \theta)$ (figure 1.5), are monotonically decreasing on $[0, \pi]$ at small β . However, similarly to $p_0^e(\beta, \theta)$, at $\beta = 1/\sqrt{2}$ their monotony vanishes and a maximum enters the picture, $\theta_1^{max}(\beta)$. For a spinor particle, the important feature of the angular distribution of right circular polarization is that in ultrarelativistic limit ($\beta \rightarrow 1$) the functions $p_1^e(\beta, \theta)$ vanish in $\pi/2 \leq \theta \leq \pi$. Thus, in this case, there is no right circular polarization in the lower half-plane. Due to the symmetry of circular polarization components one can claim that in the upper half-plane left-polarized radiation vanishes at $\beta \rightarrow 1$, and its concentration is expected to be found in the lower half-plane. When β is fixed the function $p_1^b(\beta, \theta)$, similarly to $p_1^h(1, \beta, \theta)$, decreases with θ within $[0, \pi]$. Moreover, as was for $i = 0, 2, 3$, unlike $p_1^h(1, \beta, 0)$, $p_1^b(\beta, 0)$ decrease with β .

Figure 3.20 illustrates the behavior of the functions $\theta_i^{(max)}(\beta)$ for $i = 0, 1, 3$ and $\bar{\theta}_3^{(max)}(\beta)$. We note, that differentiating the functions $p_i^e(\beta, \theta)$ and $\bar{p}_3^e(\beta, \theta)$ over θ and introducing the parameter $a = 1 - \tilde{x}$, where $\tilde{x}$ is defined by (3.21), it is easy to get the following parametrical dependence of $\theta_i^{(max)}$ and $\bar{\theta}_3^{(max)}$ on β

$$\begin{cases} \beta^2 = \frac{2(2-a^3)}{(2-a)(2+a+a^2)}, \\ \cos^2 \theta_0^{(max)} = \frac{a^4}{(2-a^3)(2-a)}; \end{cases} \quad \begin{cases} \beta^2 = \frac{4(2-a^3)^2}{(2-a)^2(2+a+a^2)(2+a-a^3)}, \\ \cos^2 \theta_1^{(max)} = \frac{a}{(2-a^3)}; \end{cases}$$

$$\begin{cases} \beta^2 = \frac{4(2-a^3)(2+a^2)}{(2-a)^2(2+a+a^2)^2}, \\ \cos^2 \theta_3^{(max)} = \frac{a^2(1+a+a^2)}{(2-a^3)(2+a^2)}; \end{cases} \quad \begin{cases} \beta^2 = \frac{4(2-a^3)}{(a^2-3a^3+8)}, \\ \cos^2 \bar{\theta}_3^{(max)} = \frac{a^2(1+a^2-a^3)}{(2-a^3)(2-a)^2}. \end{cases}$$

At $\beta \rightarrow 1$ ($\gamma \rightarrow \infty$), all the angles $\theta_i^{(max)}(\beta)$ tend to $\pi/2$ as

$$\theta_0^{max}(\beta) \approx \frac{\pi}{2} - \frac{2}{\gamma^2}, \quad \theta_1^{max}(\beta) \approx \frac{\pi}{2} - (2\gamma^2)^{-1/3}, \quad \theta_3^{max}(\beta) \approx \frac{\pi}{2} - \frac{1}{\sqrt{\gamma}}. \quad (3.44)$$

Classical theory predicts the concentration of radiation in the plane of motion at $\beta \rightarrow 1$. The necessary (but not sufficient) condition for such concentration is the tendency $\theta_i^{(max)}(\beta \rightarrow 1) \rightarrow \pi/2$. And even though for an electron (and for σ -component of radiation emitted by boson) this condition is satisfied, the behavior of $p_i^b(\beta, \theta)$ and $p_i^e(\beta, \theta)$ indicates that no significant concentration of radiated power near the plane of orbit occur. Thus, we have a contradiction between the conclusions of classical and quantum theories.

The figure 3.21 shows the plots of functions $p_i^{e(max)}(\beta)$ ($i = 0, 1, 3$) and $\bar{p}_3^{e(max)}(\beta)$, defined as $p_i^{e(max)}(\beta) \equiv p_i^e(\beta, \theta_i^{(max)}(\beta))$, $\bar{p}_3^{e(max)}(\beta) \equiv \bar{p}_3^e(\beta, \bar{\theta}_3^{(max)}(\beta))$. The functions $p_3^{e(max)}(\beta)$ and $\bar{p}_3^{e(max)}(\beta)$ have identical behavior and are numerically close to each other within the entire interval $0 \leq \beta < 1$. From figure 3.21 follows that the minimal value of $\bar{p}_3^{e(max)}(\beta)$ exceeds the minimum of the functions $p_3^{e(max)}(\beta)$, which is clear from the definition of these functions.

Figure 3.20: The plots of $\theta_0^{(max)}(\beta)$ (0), $\theta_1^{(max)}(\beta)$ (1), $\theta_3^{(max)}(\beta)$ (3a) and $\bar{\theta}_3^{(max)}(\beta)$ (3b).

Figure 3.21: The plots of $p_0^{(max)}(\beta)$ (0), $p_1^{(max)}(\beta)$ (1), $p_3^{(max)}(\beta)$ (3a) and $\bar{p}_3^{(max)}(\beta)$ (3b).

Finally, let us consider the plots of the functions that define the degree of radiation polarization for a fixed direction θ , i.e. $q_i(\beta, \theta)$. The results which we have obtained numerically for boson and electron stay in good qualitative agreement with classical theory. The difference between the polarization degree of radiation emitted by scalar and spinor particle is not noticeable but at high energies, $\beta \rightarrow 1$ (see figures 3.5, 3.6 and 3.11, 3.12). Though in this case, both the first harmonic singled out of classical spectrum and the sum over classical frequencies stay representative with respect to quantum data, still one can claim that the former is more consistent with boson, whereas an electron can be represented better with total classical spectrum. The essential difference between the behavior of $q_2^e(\beta, \theta)$ and σ -component of total (summed over spectrum) classical radiation lies in the fact that in ultrarelativistic limit $q_2^{cl}(\beta, \theta)$ monotonically increases with θ within $[0, \pi/2]$. As was predicted by classical theory, only right circularly polarized radiation is emitted in the direction of external field while the left-polarized radiation vanishes near $\theta = 0$. In the plane of motion ($\theta = \pi/2$) the radiation is absolutely linearly polarized. We emphasize here certain peculiarities of the functions $\bar{q}_2^e(\beta, \theta)$, introduced for unpolarized electron (figure 3.17). Unlike $q_2^{cl}(\beta, \theta)$, $q_2^h(\beta, \theta)$, $q_2^b(\beta, \theta)$ and $q_2^e(\beta, \theta)$, in the plane of orbit the values of these functions depend on β . Similarly to $q_2^{cl}(\beta, \theta)$, $q_2^h(\beta, \theta)$, $q_2^b(\beta, \theta)$ and $q_2^e(\beta, \theta)$, $\bar{q}_2^e(\beta, \theta)$ increases monotonically within $[0, \pi/2]$ at any β , while $\bar{q}_2^e(\beta, \pi/2)$ decreases monotonically with β so that $\bar{q}_2^e(\beta \rightarrow 1, \pi/2) = 1/2$.

3.2.2 Effective Angles of Radiation

In this section we intend to study the behavior of effective angles [116–118] and deviation angles of radiation emitted from bosons and electrons. For each type of polarization and for total (summed over polarization components) radiation we compare quantum and classical effective angles.

We start with the definition of an effective angle. In [116–118] an effective angle was defined as an opening angle into which the main (the half, to be precise) part of SR power is emitted. Analytically the effective angle of i -polarized radiation can be given by $\Delta_i(\beta) = \theta_i^{(2)} - \theta_i^{(1)}$, where $\theta_i^{(2)}$ and $\theta_i^{(1)}$ are the

solutions of the following system

$$\begin{cases} \int_{\theta_i^{(1)}}^{\theta_i^{(2)}} \Phi_i(\beta, \theta) \sin \theta d\theta = \frac{1}{2} \int_0^\pi \Phi_i(\beta, \theta) \sin \theta d\theta, \\ \Phi_i(\beta, \theta_i^{(1)}) \sin \theta_i^{(1)} = \Phi_i(\beta, \theta_i^{(2)}) \sin \theta_i^{(2)}, \end{cases} \quad (3.45)$$

Here the functions $\Phi_i(\beta, \theta)$ are the angular distributions of radiation power.

Figure 3.22: The definition of effective angle.

Figure 3.23: Plots of functions $\Delta_2^b(\beta)$ (1), $\Delta_2^e(\beta)$ (2), $\overline{\Delta}_2^e(\beta)$ (3), $\Delta_2^{cl}(\beta)$ (4) and $\Delta_2^h(1, \beta)$ (5).

To give a better but less formal explanation of the concept of effective angle it is convenient to use the figure 3.22. The dark blue curve shows the evolution of radiation distribution $\Phi_i(\beta, \theta)$ with θ at fixed energy. Our problem implies the symmetry of angular distributions with respect to $\theta = \pi/2$, so that we can restrict ourselves to the consideration of either upper or lower half-plane. The power of radiation emitted into an angle $\Delta/2$ is represented by an area between the dashed lines and is exactly a half of the total power emitted in the upper half-plane (the area under the blue curve between 0 and $\pi/2$) or in the lower half-plane (dashed lines between $\theta = \pi/2$ and $\theta = \pi$). The second equation of the pair (3.45) specifies the position of the dashed lines, whose height is supposed to be equal. We note that in [116–118] the deviation angle is defined as $\alpha_i = (\theta_i^{(1)} + \theta_i^{(2)})/2$. Here we define the deviation angle as $\theta^{(2)}$ (see figure 3.22) [119, 120]. So, for i -type of polarization the effective angle is $\Delta_i(\beta) = 2(\theta_i^{(2)} - \theta_i^{(1)})$ and due to symmetry the integration in (3.45) is performed within $0 \leq \theta < \pi/2$.

Figure 3.23 illustrates the evolution of $\Delta_2^{b,e}(\beta)$, $\overline{\Delta}_2^e(\beta)$, $\Delta_2^{cl}(\beta)$, $\Delta_2^h(1, \beta)$ for $i = 2$, i.e. for σ -component of linearly polarized radiation.

The angular distributions of σ -component of radiation emitted by boson, electron (either polarized or unpolarized) and classical particle increase monotonically within $0 \leq \theta \leq \pi/2$ at any β . Therefore, $\theta_2^{b(2)} = \theta_2^{e(2)} = \overline{\theta}_2^{e(2)} = \theta_2^{cl(2)} = \pi/2$ and the system of equations (3.45) is automatically reduced to a single equation that defines $\theta_2^{(1)}(\beta)$ ($\Delta_2(\beta) = 2(\pi/2 - \theta_2^{(1)}(\beta))$). For the first harmonic of classical spectrum the equality $\theta_2^{h(2)} = \pi/2$ holds only up to a certain $\beta = 0.834016$. For $\beta > 0.834016$ one should already solve the system (3.45).

In the figures 3.24 and 3.25 the behavior of the functions $\Delta_2(\beta)$ near $\beta = 1$ is shown. The functions $\Delta_2^b(\beta)$ and $\Delta_2^e(\beta)$ have minima at $\beta_{min}^b = 0.73222$ ($\gamma_{min}^b = 4.350303$) and $\beta_{min}^e = 0.996100$ ($\gamma_{min}^e = 11.333901$) correspondingly. We note here that, unlike $\Delta_2^e(\beta)$, the function $\overline{\Delta}_2^e(\beta)$, that characterizes σ -component of radiation emitted by an unpolarized electron, decreases monotonically with β .

Figure 3.24: The plots of $\Delta_2^b(\beta)$ near $\beta = 1$.

Figure 3.25: The plots of $\Delta_2^e(\beta)$ near $\beta = 1$.

The values of effective angles associated with σ -component of radiation emitted by a scalar particle, by polarized and unpolarized electron, as well as those calculated in terms of classical theory for total spectrum and its first harmonic are given in the table 3.2.

Table 3.2: Effective angles of σ -component of linear polarization

β	$\Delta_2^b(\beta)$	$\Delta_2^e(\beta)$	$\overline{\Delta}_2^e(\beta)$	$\Delta_2^{cl}(\beta)$	$\Delta_2^h(1, \beta)$
0	1.04719755	1.04719755	1.04719755	1.04719755	1.04719755
0.1	1.04671656	1.04502569	1.04574972	1.03783150	1.04936853
0.2	1.04527546	1.03842669	1.04135160	1.00992532	1.05595289
0.3	1.04288146	1.02713622	1.03383051	0.96405908	1.06716867
0.4	1.03955286	1.01066063	1.02286551	0.90118299	1.08339148
0.5	1.03533157	0.98817211	1.00792286	0.82249231	1.10517201
0.6	1.03030950	0.95828724	0.98812178	0.72906609	1.13325784
0.7	1.02468586	0.91855653	0.96193215	0.62105987	1.16861372
0.8	1.01890174	0.86405627	0.92636476	0.49581522	1.21242733
0.9	1.01399476	0.78195676	0.87395567	0.34128290	1.25797175
1	1.01271240	0.75523025	0.75523025	0	1.27873950

The behavior of effective angle of π -component of radiation emitted by electron (polarized and unpolarized) generally coincides with the predictions of classical theory, though $\Delta_3^{cl}(\beta)$ reaches its maximum at relatively low energies. Nevertheless, as in the case of σ -component, no concentration of radiation can be found in the plane of motion (figure 3.26).

Figure 3.26: The plot of $\Delta_3^b(\beta)$ (1), $\Delta_3^e(\beta)$ (2), $\bar{\Delta}_3^e(\beta)$ (3), $\Delta_3^{cl}(\beta)$ (4) and $\Delta_3^h(\beta)$ (5).

Figure 3.27: The plot of $\theta_3^{b(2)}(\beta)$ (1), $\theta_3^{e(2)}(\beta)$ (2), $\bar{\theta}_3^{e(2)}(\beta)$ (3), $\theta_3^{cl(2)}(\beta)$ (4) and $\theta_3^{h(2)}(\beta)$ (5).

The curves drawn for polarized and unpolarized electron have their maxima at $\beta_{max}^e = 0.877238$ ($\gamma_{min}^b = 2.083095$) and $\bar{\beta}_{max}^e = 0.816024$ ($\bar{\gamma}_{max}^e = 1.730050$), while the maximum of classical curve lies in $\beta_{max}^{cl} = 0.472502$ ($\gamma_{max}^{cl} = 1.134649$).

Figure 3.26 shows that the plot of $\Delta_3^e(\beta)$ remains quite close to the bosonic curve up to a certain $\beta \approx 0.9$. Concerning unpolarized electron, one can notice significant difference between the function $\bar{\Delta}_3^e(\beta)$ and each of $\Delta_3^b(\beta)$, $\Delta_3^h(1, \beta)$, $\Delta_3^{cl}(\beta)$ even in a weakly relativistic region.

Table 3.3 gives the values of effective angles associated with π -component of radiation emitted by boson, electron (polarized and unpolarized) and well-known classical data.

Table 3.3: Effective angles of π - component of linear polarization

β	$\Delta_3^b(\beta)$	$\Delta_3^e(\beta)$	$\bar{\Delta}_3^e(\beta)$	$\Delta_3^{cl}(\beta)$	$\Delta_3^h(1, \beta)$
0	0.912345443	0.912345443	0.91234544	0.912345443	0.912345443
0.1	0.912817075	0.913053297	0.91666759	0.914441359	0.912203737
0.2	0.914232766	0.915182776	0.92963771	0.920389571	0.911775759
0.3	0.916594245	0.918749192	0.95122507	0.928958228	0.911052957
0.4	0.919901941	0.923767964	0.98121272	0.937293699	0.910021172
0.5	0.924150411	0.930230466	1.01883333	0.939468017	0.908660796
0.6	0.929319703	0.938040964	1.06191652	0.924338018	0.906947002
0.7	0.935359035	0.946835453	1.10476932	0.873701561	0.904850079
0.8	0.942155116	0.955356155	1.13223841	0.762002654	0.902335878
0.9	0.949467816	0.958309277	1.09877335	0.553426248	0.899366405
1	0.956794421	0.755230254	0.75523025	1.56451314	0.895900557

The deviation angle of π -component of radiation decreases with β for the first harmonic of classical spectrum. For an electron and for the total spectrum in classical theory it increases and turns to $\pi/2$ while $\beta \rightarrow 1$ (see figure 3.27).

The deviation angle of an unpolarized electron $\bar{\theta}_3^{(2)}$ reaches $\pi/2$ at lower energy and the system (3.45) is reduced to a single equation. The boundary value of β , at which the deviation angle of the unpolarized

electron turns to $\pi/2$ was calculated numerically $\bar{\beta}^e = 0.961084$ ($\bar{\gamma}^e = 3.619832$).

The evolution of effective angles of total ($i = 0$) radiation with β is shown in figure 3.28.

Figure 3.28: The plots of $\Delta_0^b(\beta)$ (1), $\Delta_0^e(\beta)$ (2), $\bar{\Delta}_0^e(\beta)$ (3), $\Delta_0^{cl}(\beta)$ (4) and $\Delta_0^h(\beta)$ (5).

Figure 3.29: the plots of $\theta_0^{b(2)}(\beta)$ (1), $\theta_0^{e(2)}(\beta)$ (2), $\bar{\theta}_0^{e(2)}(\beta)$ (3), $\theta_0^{cl(2)}(\beta)$ (4) and $\theta_0^{h(2)}(\beta)$ (5).

It was shown earlier that the effective angle of σ -component of radiation emitted by boson decreases, while the effective angle of π -component increases. The relation between σ and π - components of radiation associated with the first harmonic of classical spectrum has an opposite character. Thus, the sum over polarization components implies the compensation of corresponding contributions, so that the curves $\Delta_0^b(\beta)$ and $\Delta_0^h(1, \beta)$ have numerical similarity. The values of $\Delta_0(\beta)$ are given in table 3.4.

Table 3.4: Effective angles of unpolarized ($i = 0$) radiation

β	$\Delta_0^b(\beta)$	$\Delta_0^e(\beta)$	$\Delta_0^{cl}(\beta)$	$\Delta_0^h(1, \beta)$
0	1.2515401	1.2515401	1.2515401	1.2515401
0.1	1.2509756	1.2501954	1.2462733	1.2523647
0.2	1.2492616	1.2459689	1.2266085	1.2545667
0.3	1.2463334	1.2382460	1.1802285	1.2573647
0.4	1.2420730	1.2258651	1.1043601	1.2595634
0.5	1.2362890	1.2068655	1.0068511	1.2596911
0.6	1.2286797	1.1780102	0.8884075	1.2561598
0.7	1.2187614	1.1346922	0.7497707	1.2474305
0.8	1.2057219	1.0744155	0.5897613	1.2321764
0.9	1.1880935	0.9840616	0.3977306	1.2094401
1	1.1629058	0.7552303	0	1.1787761

For the total radiation the deviation angle calculated in terms of classical theory turns to $\pi/2$ at $\beta^{cl} = 0.295166$ ($\gamma^{cl} = 1.046632$), whereas the deviation angle of an electron reaches this value only at $\beta^e = 0.650672$ ($\gamma^e = 1.316900$) (see figure 3.29).

The functions $\Delta_1^b(\beta)$, $\Delta_1^e(\beta)$, $\Delta_1^h(\beta)$ introduced to describe the effective angles of the right circular polarization decrease monotonically within the whole interval of energies (see figure 3.30).

Figure 3.30: The plots of $\Delta_1^b(\beta)$ (1), $\Delta_1^e(\beta)$ (2), $\overline{\Delta}_1^e(\beta)$ (3), $\Delta_1^{cl}(\beta)$ (4) and $\Delta_1^h(\beta)$ (5).

Figure 3.31: The plots of $\theta_1^{b(2)}(\beta)$ (1), $\theta_1^{e(2)}(\beta)$ (2), $\overline{\theta}_1^{e(2)}(\beta)$ (3), $\theta_1^{cl(2)}(\beta)$ (4) and $\theta_1^{h(2)}(\beta)$ (5).

Classical theory also predicts the presence of weak maximum (the function $\Delta_1^{cl}(\beta)$). As was in the case of total radiation, the functions related to boson and first harmonic of classical spectrum almost overlap. The values of effective angles associated with right circular polarization calculated for boson, electron, total classical spectrum and its first harmonic are given in table 3.5 .

Table 3.5: Effective angles of right circular polarization component

β	$\Delta_1^b(\beta)$	$\Delta_1^e(\beta)$	$\Delta_1^{cl}(\beta)$	$\Delta_1^h(1, \beta)$
0	0.68235110	0.68235110	0.68235110	0.68235110
0.1	0.68183070	0.68211447	0.68267763	0.68173886
0.2	0.68024895	0.68134489	0.68278942	0.67986545
0.3	0.67754211	0.67985021	0.67996689	0.67662192
0.4	0.67359611	0.67726098	0.66953970	0.67183112
0.5	0.66823289	0.67293336	0.64548102	0.66525640
0.6	0.66118603	0.66574534	0.60199670	0.65661768
0.7	0.65205722	0.65362870	0.53485074	0.64561776
0.8	0.64023462	0.63226890	0.44040494	0.63198095
0.9	0.62472429	0.58984152	0.30906879	0.61550382
1	0.60374876	0.37761513	0	0.59611401

The deviation angles of boson and electron increase monotonically with β . However, the classical curve has a weak maximum near $\beta \approx 1$ (see figure 3.31). The function $\theta_1^{h(2)}(1, \beta)$ decreases monotonically with β .

We note that for an electron (either polarized or unpolarized) the following equality holds $\Delta_2^e(\beta \rightarrow 1) = \Delta_3^e(\beta \rightarrow 1) = \Delta_0^e(\beta \rightarrow 1) = 2\Delta_1^e(\beta \rightarrow 1) = 0.755230254$, and $\theta_i^{e(2)}(\beta \rightarrow 1) = \pi/2$, and the equations for $\theta_i^{e(1)}$ at $\beta = 1$ transform to

$$\left(\frac{y+3}{y+1}\right) \exp\left(\frac{y-1}{y+1}\right) = \frac{3}{2e} + 1, \quad y = \cos \theta_i^{e(1)}. \quad (3.46)$$

According to the classical theory, the effective angles of radiation vanish in the ultrarelativistic limit. It means that the radiation concentrates near the plane of motion when $\beta \rightarrow 1$. The functions $\Delta_i^b(\beta)$,

$\Delta_i^e(\beta)$ and $\overline{\Delta}_i^e(\beta)$ tend to certain finite values at $\beta \rightarrow 1$, and these values differ from 0. That is, the quantum radiation does not concentrate near $\theta \approx \pi/2$ at extremely high energies. Moreover, according to quantum theory there is a deconcentration of π -component of radiation, namely, the functions $\Delta_3^b(\beta)$, $\Delta_3^e(\beta)$ are increasing with β within $0 \leq \beta < 1$.

Chapter 4

Quantum Two-harmonic Spectrum

In the previous section we studied the properties of SR of the two-level system. Obviously, when $n = 1$ only one harmonic $\nu = n = 1$ can be emitted, and, therefore, there is no sense in setting a problem about spectral characteristics of radiation. The simplest example which allows to formulate such problem is a quantum three-level system ($n = 2$). If, initially, quantum particle occupies the second level, $n = 2$, we already deal with two possible transitions $n = 2 \rightarrow n' = 1$ ($\nu = 1$) and $n = 2 \rightarrow n' = 0$ ($\nu = 2$), so, the quantum spectrum contains two harmonics. Though technically a single harmonic (the case of $n = 1$) forms a spectrum, in what follows we use the concept of "the shortest quantum spectrum" to refer to the transitions in a three-level system. As we intended to avoid any confusion, intuitively, this choice of terminology appeared reasonable: in this work the analysis of spectral properties implies the study of the relations between the power of radiation associated with particular spectral harmonics.

The properties of radiation emitted from a spinless particle when it jumps from $n = 2$ to lower levels are defined by the functions $F_i^b(2, \nu, \beta, \theta)$. The total power of SR emitted during these transitions and its polarization components have the form

$$W_i^b(n, \beta) = \sum_{\nu=1}^n W_i^b(n, \nu, \beta), \quad W_i^b(n, \nu, \beta) = \frac{Q_0 A(\beta)}{2(2n+1)^3} \int_0^\pi F_i^b(n, \nu, \beta, \theta) \sin \theta d\theta, \quad (4.1)$$

$$W_i^b(2, \beta) = W_i^b(2, 1, \beta) + W_i^b(2, 2, \beta). \quad (4.2)$$

The plots of $W_i^b(2, \beta)$ for $i = 0, 1, 2, 3$ are shown in figure 4.1. The scaling factor $Q_0 A(\beta)[2(2n+1)^3]^{-1}|_{n=2} = Q_0 A(\beta)/250$ was omitted.

All the functions $F_i^b(\beta)$ are continuous and finite on $0 \leq \beta \leq 1$ (including the boundary points). These functions monotonically increase with β .

For an electron we consider the functions $W_i^e(\zeta, 2, \beta)$ defined as

$$W_i^e(\zeta, n, \beta) = \sum_{\nu=1}^n W_i^e(\zeta, n, \nu, \beta), \quad W_i^e(\zeta, n, \nu, \beta) = \frac{Q_0 A(\beta)}{2(2n)^3} \int_0^\pi F_i^e(\zeta, n, \nu, \beta, \theta) \sin \theta d\theta. \quad (4.3)$$

Omitting the scaling factor $Q_0 A(\beta)[2(2n)^3]^{-1}|_{n=2} = Q_0 A(\beta)/128$, the evolution of $W_i^e(\zeta, 2, \beta)$ with β can be tracked with the help of figures 4.2 and 4.3 for $\zeta = 1$ and $\zeta = -1$ respectively.

Figure 4.1: The plots of $250 W_i^b(2, \beta)[Q_0 A(\beta)]^{-1}$. The number of the curve corresponds with the number of polarization component i .

Figure 4.2: The plots of $128 W_i^e(1, 2, \beta)[Q_0 A(\beta)]^{-1}$. The number of curve corresponds with the number of polarization component i .

Figure 4.3: The plots of $128 W_i^e(-1, 2, \beta)[Q_0 A(\beta)]^{-1}$. The number of curve corresponds with the number of polarization component i .

Similarly to their analogues created for a boson, the functions $Q_0 A(\beta) W_i^e(\zeta, 2, \beta)/128$ monotonically increase with β independently of the initial spin ζ and the type of polarization i (except for the ultrarelativistic region where σ -component has a weak maximum when $\zeta = -1$).

4.1 Polarization Properties of Radiation

In the framework of quantum theory, analogously to $\vec{q}_i^h(n, \beta)$ and $q_i^{cl}(\beta)$ (see 1.60, 1.34), one can introduce the concept of polarization degree for a single harmonic and for the total radiation when $n = 2$

$$q_i^b(\beta) = \frac{W_i^b(2, \beta)}{W_0^b(2, \beta)}, \quad q_i^b(\nu, \beta) = \frac{W_i^b(2, \nu, \beta)}{W_0^b(2, \nu, \beta)}. \quad (4.4)$$

Similarly to classical theory, from the definition of $q_i^b(\beta)$ and $q_i^b(\nu, \beta)$ it follows that

$$q_0^b(\nu, \beta) = q_0^b(\beta) = 1, \quad q_3^b(\nu, \beta) = 1 - q_2^b(\nu, \beta), \quad q_3^b(\beta) = 1 - q_2^b(\beta), \quad (4.5)$$

$$q_{-1}^b(\nu, \beta) = 1 - q_1^b(\nu, \beta), \quad q_{-1}^b(\beta) = 1 - q_1^b(\beta).$$

The plots of $q_i^b(\beta)$ and $q_i^b(\nu, \beta)$ for $i = 1, 2$ are given in [121] and in the figures 4.4 - 4.5 .

Figure 4.4: The plots of $q_2^b(2, \beta)$ and $q_2^h(2, \beta)$ (a), $q_2^b(1, \beta)$ and $q_2^h(1, \beta)$ (b), $q_2^b(\beta)$ and $\bar{q}_2^h(2, \beta)$ (c), $q_2^{cl}(\beta)$ (d). Dashed curves refer to classical theory.

Figure 4.5: The plots of $q_1^b(2, \beta)$ and $q_1^h(2, \beta)$ (a), $q_1^b(1, \beta)$ and $q_1^h(1, \beta)$ (b), $q_1^b(\beta)$ and $\bar{q}_1^h(2, \beta)$ (c), $q_1^{cl}(\beta)$ (d). Dashed curves refer to classical theory.

The functions $q_2^b(\beta)$, $q_2^b(1, \beta)$ and $q_2^b(2, \beta)$ decrease with β , which means that according to quantum theory the degree of the $(i = 2)$ -linear polarization decreases with energy both for total radiation and for its spectral components. The functions $q_1^b(\beta)$, $q_1^b(1, \beta)$ and $q_1^b(2, \beta)$ monotonically increase on $0 \leq \beta \leq 1$, which indicates that the degree of right circular polarization grows as β tends to the ultrarelativistic value. As it was shown in section 1.1, $q_2^{cl}(\beta)$ increases monotonically with β , whereas $q_1^{cl}(\beta)$ decreases. Thus, classical theory of the total spectrum does not give an adequate description for the polarization degree of the weakly-excited particles. However, the results obtained in the framework of the quantum theory stay in reasonably good qualitative agreement with classical data that we obtained for separate spectral harmonics (curves a, b and c in the figures 4.4-4.5). The functions $\bar{q}_2^h(2, \beta)$, $q_2^h(2, \beta)$ and $q_2^h(1, \beta)$ decrease with β , whereas $\bar{q}_1^h(2, \beta)$, $q_1^h(2, \beta)$ and $q_1^h(1, \beta)$ increase. Within the whole interval $0 \leq \beta < 1$ the plots of classical functions stay very close to their quantum analogues. In the non-relativistic limit we have a complete agreement between the results of classical and quantum theories.

In section 1.2.2 we defined partial contributions $\tilde{q}_i^h(\nu, \beta)$ and $\tilde{q}_i^h(n, \nu, \beta)$ in the framework of classical theory. Similar functions can be introduced in quantum theory of SR emitted by a spinless particle when $n = 2$

$$\tilde{q}_i^b(\nu, \beta) = \frac{W_i^b(2, \nu, \beta)}{W_0^b(2, \beta)}, \quad (i = 0, 1, 2, 3). \quad (4.6)$$

The plots of functions $\tilde{q}_i^b(\nu, \beta)$ are given in the figures 4.6-4.9. For partial contributions we observe the qualitative accordance between classical and quantum theories, namely, the partial contribution of first harmonic decreases with β and the partial contribution of second harmonic increases with energy. At certain β (that depends on the type of polarization) the second harmonic becomes dominating. It means that the maximum in the radiation spectrum shifts to the second harmonic with energy, which also corresponds with the conclusions of classical theory. The essential difference is that when β reaches high values, $\beta \rightarrow 1$, the functions $\tilde{q}_i^b(\nu, \beta)$ tend to finite nonzero values and do not vanish as it happens in the case of $\tilde{q}_i^h(\nu, \beta)$. Moreover, we note that there is a numerical similarity between $\tilde{q}_i^b(\nu, \beta)$ and $\tilde{q}_i^h(2, \nu, \beta)$. It could be seen in the figures that the corresponding curves plotted for different types of polarization are located closely to each other within the whole interval $0 \leq \beta < 1$, and even as $\beta \rightarrow 1$ their values remain similar. Thus, unlike the results that were obtained for total classical spectrum, the data we've

got for its n -part stays in good agreement with quantum theory at any energy of emitting particle.

Figure 4.6: The plots of $\tilde{q}_2^b(1, \beta)$ (1a), $\tilde{q}_2^h(2, 1, \beta)$ (1b), $\tilde{q}_2^h(1, \beta)$ (1c) and $\tilde{q}_2^b(2, \beta)$ (2a), $\tilde{q}_2^h(2, 2, \beta)$ (2b), $\tilde{q}_2^h(2, \beta)$ (2c).

Figure 4.7: The plots of $\tilde{q}_3^b(1, \beta)$ (1a), $\tilde{q}_3^h(2, 1, \beta)$ (1b), $\tilde{q}_3^h(1, \beta)$ (1c) and $\tilde{q}_3^b(2, \beta)$ (2a), $\tilde{q}_3^h(2, 2, \beta)$ (2b), $\tilde{q}_3^h(2, \beta)$ (2c).

Figure 4.8: The plots of $\tilde{q}_0^b(1, \beta)$ (1a), $\tilde{q}_0^h(2, 1, \beta)$ (1b), $\tilde{q}_0^h(1, \beta)$ (1c) and $\tilde{q}_0^b(2, \beta)$ (2a), $\tilde{q}_0^h(2, 2, \beta)$ (2b), $\tilde{q}_0^h(2, \beta)$ (2c).

Figure 4.9: The plots of $\tilde{q}_1^b(1, \beta)$ (1a), $\tilde{q}_1^h(2, 1, \beta)$ (1b), $\tilde{q}_1^h(1, \beta)$ (1c) and $\tilde{q}_1^b(2, \beta)$ (2a), $\tilde{q}_1^h(2, 2, \beta)$ (2b), $\tilde{q}_1^h(2, \beta)$ (2c).

Similarly to (4.6) for an electron we introduce

$$\tilde{q}_i^e(\zeta, \nu, \beta) = \frac{\overline{W}_i^e(\zeta, 2, \nu, \beta)}{\sum_{\nu=1,2} \overline{W}_0^e(\zeta, 2, \nu, \beta)}, \quad \overline{W}_i^e(\zeta, n, \nu, \beta) = \frac{Q_0 A(\beta)}{2(2n)^3} \int_0^\pi \bar{F}_i^e(\zeta, n, \nu, \beta, \theta) \sin \theta d\theta, \quad (4.7)$$

- the functions, that allow to estimate how significant the contribution of particular harmonic ν and polarization type i is with respect to the total (summed over polarization components and spectrum) radiation. Though it might appear a pure tautology, we find it convenient to list the explicit definitions of $\tilde{q}_i^e(\zeta, \nu, \beta)$

$$\begin{aligned} \tilde{q}_i^e(-1, 1, \beta) &= \frac{W_i^{(1)}(-1, 2, 1, \beta) + W_i^{(2)}(-1, 2, 1, \beta)}{W_0^{(1)}(2, 2, \beta) + W_0^{(1)}(-1, 2, 1, \beta) + W_0^{(2)}(-1, 2, 1, \beta)}, \\ \tilde{q}_i^e(1, 1, \beta) &= \frac{W_i^{(1)}(1, 2, 1, \beta) + W_i^{(2)}(1, 2, 1, \beta)}{W_0^{(1)}(2, 2, \beta) + W_0^{(1)}(1, 2, 1, \beta) + W_0^{(2)}(1, 2, 1, \beta)}, \\ \tilde{q}_i^e(-1, 2, \beta) &= \frac{W_i^{(1)}(2, 2, \beta)}{W_0^{(1)}(2, 2, \beta) + W_0^{(1)}(-1, 2, 1, \beta) + W_0^{(2)}(-1, 2, 1, \beta)}, \\ \tilde{q}_i^e(1, 2, \beta) &= \frac{W_i^{(2)}(2, 2, \beta)}{W_0^{(2)}(2, 2, \beta) + W_0^{(1)}(1, 2, 1, \beta) + W_0^{(2)}(1, 2, 1, \beta)}, \end{aligned} \quad (4.8)$$

where

$$W_i^{(k)}(\zeta, n, \nu, \beta) = \frac{Q_0 A(\beta)}{2(2n)^3} \int_0^\pi F_i^{(k)}(\zeta, n, \nu, \beta, \theta) \sin \theta. \quad (4.9)$$

Figure 4.10: The plots of $\tilde{q}_2^e(1, 1, \beta)$ (1a), $\tilde{q}_2^e(1, 2, \beta)$ (2a), $\tilde{q}_2^e(-1, 1, \beta)$ (1b), $\tilde{q}_2^e(-1, 2, \beta)$ (2b), $\tilde{q}_2^h(2, 1, \beta)$ (1c) and $\tilde{q}_2^h(2, 2, \beta)$ (2c).

Figure 4.11: The plots of $\tilde{q}_0^e(1, 1, \beta)$ (1a), $\tilde{q}_0^e(1, 2, \beta)$ (2a), $\tilde{q}_0^e(-1, 1, \beta)$ (1b), $\tilde{q}_0^e(-1, 2, \beta)$ (2b), $\tilde{q}_0^h(2, 1, \beta)$ (1c) and $\tilde{q}_0^h(2, 2, \beta)$ (2c).

Figures 4.10 - 4.11 show the evolution of $\tilde{q}_2^e(\pm 1, \nu, \beta)$ and $\tilde{q}_0^e(\pm 1, \nu, \beta)$ with β . Dashed lines illustrate the results predicted by the classical theory. We emphasize here that the reduced classical spectrum was considered due to our experience in finding adequate classical analogues of $\tilde{q}_i^b(\nu, \beta)$. As one can see in figures 4.10 - 4.11, for an electron with $\zeta = -1$ the shift of radiation maximum between spectral harmonics (the intersection of blue curves in figures 4.10 - 4.11) can be very precisely described numerically with the help of a corresponding classical value (the intersection of dashed blue curves). However, when the initial spin is collinear to the vector of the field intensity, this shift can only occur in relativistic domain (the intersection between black curves lies close to $\beta = 1$, especially in the case of σ -component). So, we can claim classical theory of reduced spectrum to give adequate results that have a reasonable similarity with the data obtained for an electron with $\zeta = -1$.

Now let us turn our attention to the polarization properties of the particular harmonics in the quantum spectrum of an electron. In this regard we introduce the functions

$$q_i^e(\zeta, \nu, \beta) = \frac{\overline{W}_i^e(\zeta, 2, \nu, \beta)}{\overline{W}_0^e(\zeta, 2, \nu, \beta)}, \quad (4.10)$$

which can be written explicitly as

$$q_i^e(\pm 1, 1, \beta) = \frac{W_i^{(1)}(\pm 1, 2, 1, \beta) + W_i^{(2)}(\pm 1, 2, 1, \beta)}{W_0^{(1)}(\pm 1, 2, 1, \beta) + W_0^{(2)}(\pm 1, 2, 1, \beta)},$$

$$q_i^e(-1, 2, \beta) = \frac{W_i^{(1)}(2, 2, \beta)}{W_0^{(1)}(2, 2, \beta)}, \quad q_i^e(1, 2, \beta) = \frac{W_i^{(2)}(2, 2, \beta)}{W_0^{(2)}(2, 2, \beta)}, \quad (4.11)$$

$$W_i^{(k)}(n, n, \beta) = \frac{Q_0 A(\beta)}{2(2n)^3} \int_0^\pi F_i^{(k)}(n, n, \beta, \theta) \sin \theta d\theta. \quad (4.12)$$

Obviously, $q_0^e(\zeta, \nu, \beta) = 1$ and, therefore, $q_3^e(\zeta, \nu, \beta) = 1 - q_2^e(\zeta, \nu, \beta)$. Figures 4.12 - 4.13 illustrate the behavior of these functions for $i = 1, 3$. The number of curve corresponds with the number of harmonic

ν , label 'a' is for $\zeta = 1$ and 'b' is for $\zeta = -1$. Independently of the initial direction of the spin (when $n = 2$), $q_2^e(\zeta, 1, \beta)$ decreases with energy, whereas $q_3^e(\zeta, 1, \beta)$ increases. It means that the contribution of the σ component of radiation emitted during the transition $2 \rightarrow 1$ is less significant at high energies, though the difference is not that big. When $\nu = 2$, i.e. during the transition to the ground state $2 \rightarrow 0$, the behavior of $q_i^e(\zeta, 2, \beta)$ changes dramatically with ζ , $q_2^e(-1, 2, \beta)$ decreases with energy while $q_2^e(1, 2, \beta)$ increases, and in the ultrarelativistic limit these functions tend to the same value. Here the reader is supposed to recall the property we established for the transitions to the ground state, namely, in the ultrarelativistic limit the spin-flip transitions and those without spin-flip become equiprobable. As for the difference in their behavior at $\beta < 1$, reference should be made to the property of switching between the polarization components. Qualitatively the behavior of $q_i^e(\zeta, \nu, \beta)$ resembles that of their analogues which were introduced for a boson and, thus, it also stays in good (qualitative) agreement with the behavior of their classical analogues (created for a reduced classical spectrum). The exception is $q_i^e(1, 2, \beta)$ which is associated with spin-flip transition to the ground state.

Figure 4.12: The plots of $q_2^e(1, 1, \beta)$ (1a), $q_2^e(-1, 1, \beta)$ (1b), $q_2^e(1, 2, \beta)$ (2a), $q_2^e(-1, 2, \beta)$ (2b).

Figure 4.13: The plots of $q_3^e(1, 1, \beta)$ (1a), $q_3^e(-1, 1, \beta)$ (1b), $q_3^e(1, 2, \beta)$ (2a), $q_3^e(-1, 2, \beta)$ (2b).

4.2 Spectral-Angular Distributions of Power

By analogy with $p_i^{cl}(\beta, \theta)$, $p_i^h(\nu, \beta, \theta)$, $p_i^h(n, \nu, \beta, \theta)$, $P_i^h(n, \beta, \theta)$ and $\tilde{P}_i^h(n, \beta, \theta)$ which define different types of spectral-angular distributions in classical theory, in quantum theory of a boson in the state with $n = 2$ one can introduce the functions $P_i^b(\nu, \beta, \theta)$ that describe angular distributions of radiation associated with harmonic ν and polarization type i . Also, we can introduce the functions $P_i^b(\beta, \theta)$ in order to track the behavior of angular distribution of the total (summed over the spectrum, but not over the polarization components) power emitted by a scalar particle

$$P_i^b(\nu, \beta, \theta) = \frac{Q_0 A(\beta)}{2(2n+1)^3} \frac{F_i^b(2, \nu, \beta, \theta)}{W_0^b(2, \beta)}, \quad P_i^b(\beta, \theta) = P_i^b(1, \beta, \theta) + P_i^b(2, \beta, \theta), \quad (i = 0, 1, 2, 3). \quad (4.13)$$

The evolution of angular distributions $P_i^b(1, \beta, \theta)$, $P_i^b(2, \beta, \theta)$ and $P_i^b(\beta, \theta)$ is shown in the figures 4.14 - 4.17, 4.18 - 4.21 and 4.22 - 4.25.

Let us consider figures 4.14-4.17 together with 1.31-1.34. The behavior of functions $P_i^b(1, \beta, \theta)$ stays in good qualitative accordance with $p_i^h(2, 1, \beta, \theta)$ for all the types of polarization i , and in non-relativistic limit $P_i^b(1, \beta \rightarrow 0, \theta) = p_i^h(2, 1, \beta \rightarrow 0, \theta)$. Small difference between the values of P_i^b and p_i^h can not be

Figure 4.14: The plots of $P_2^b(1, \beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.15: The plots of $P_3^b(1, \beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.16: The plots of $P_0^b(1, \beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.17: The plots of $P_1^b(1, \beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

observed but in the ultrarelativistic region. As it was already mentioned, in the magnetic field direction ($\theta = 0$) p_i^b , as the functions of β , have weak maxima at β close to 1. It is not the case for the functions P_i^b , at any fixed θ $P_i^b(1, \beta, \theta)$ ($i = 0, 1, 2, 3$) decreases monotonically with energy. The functions $P_i^b(1, \beta, \theta)$ ($i = 0, 2, 3$), in their turn, monotonically decrease within $0 \leq \theta < \pi/2$ at any fixed β , and the function $P_1^b(1, \beta, \theta)$ decreases within $0 \leq \theta \leq \pi$. Thus, $P_i^b(1, \beta, \theta)$ ($i = 0, 2, 3$) reach their minimal values in the plane of motion ($\theta = \pi/2$) in ultrarelativistic limit, whereas $P_1^b(1, \beta, \theta)$ tend to their maxima at $\beta \rightarrow 1$ as θ reaches the direction of external field ($\theta = \pi$). This feature remains for the functions $p_i^b(2, 1, \beta, \theta)$. Note, that in the plane of motion we have $P_3^b(2, \beta, \pi/2) = P_3^b(1, \beta, \pi/2) = 0$, π -component of linear polarization vanishes and there is only σ -component left when $\theta = \pi/2$. It means that in the plane of motion the radiation emitted by a boson is completely linearly polarized.

The evolution of $P_i^b(2, \beta, \theta)$ (figures 4.18-4.21) resembles much the behavior of $p_i^b(2, 2, \beta, \theta)$ (figures 1.35-1.38). The exception is that near the plane of the orbit $P_i^b(2, \beta, \theta)$, $i = 0, 1, 2$, increase monotonically, and the functions $p_i^b(2, 2, \beta, \theta)$ reach maximal values near $\theta = \pi/2$. In the plane of the orbit of motion one observes the absence of the π -linear polarization component at any β , namely, $P_3^b(2, \beta, \pi/2) = p_3^b(2, 2, \beta, \pi/2) = 0$. Moreover, as the classical theory predicted, $P_i^b(2, \beta \rightarrow 0, \theta) \approx 0$, i.e. in the non-relativistic domain the amount of radiation associated with the second harmonic ($\nu = 2$) is negligibly small comparing to the radiation of the first harmonic ($\nu = 1$). The functions $P_2^b(2, \beta, \theta)$ for all β are

Figure 4.18: The plots of $P_2^h(2, \beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.19: The plots of $P_3^h(2, \beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.20: The plots of $P_0^b(2, \beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.21: The plots of $P_1^b(2, \beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

monotonically increasing with θ within $0 \leq \theta < \pi/2$ (unlike $p_2^h(2, 2, \beta, \theta)$, which lose their monotony at a certain β). The functions $P_3^b(2, \beta, \theta)$, as functions of θ , have maxima at all energies (similarly to $p_3^h(2, 2, \beta, \theta)$). $P_0^b(2, \beta, \theta)$ increase monotonically within $0 \leq \theta < \pi/2$, but, similarly to $p_0^h(2, 2, \beta, \theta)$, lose their monotony at some β . The character of the functions $P_1^b(2, \beta, \theta)$ qualitatively resembles that of $p_1^h(2, 2, \beta, \theta)$, the maxima of $P_1^b(2, \beta, \theta)$ tends to $\pi/2$ with β , whereas the maxima of $p_1^h(2, 2, \beta, \theta)$, on the contrary, moves away from the plane of motion.

The behavior of $P_i^b(\beta, \theta)$ (figures 4.22-4.25) to some extent corresponds with that of their classical analogues $p_i^{cl}(\beta, \theta)$ and $P_i^h(2, \beta, \theta)$ (figures 1.2-1.5 and 1.39-1.42 correspondingly). The functions $P_2^b(\beta, \theta)$, same as $p_2^{cl}(\beta, \theta)$, increase monotonically on $0 \leq \theta < \pi/2$ independently of β , and decrease monotonically with β in the direction of the magnetic field intensity ($\theta = 0$). As the functions of β , they increase with energy in the plane of the orbit. The behavior of $P_2^h(2, \beta, \theta)$ (figure 1.39) is different, namely, at some β these functions lose their monotony and have maxima in the plane of the orbit as the functions of β . The functions $P_3^b(\beta, \theta)$ decrease monotonically on $0 \leq \theta < \pi/2$, and then reach maximal values with energy. Qualitatively their behavior corresponds with $p_3^{cl}(\beta, \theta)$ and $P_3^h(2, \beta, \theta)$, whereas particular numerical characteristics of the special points do not coincide with their classical analogues. The latter remains valid for the functions $P_1^b(\beta, \theta)$, $p_1^{cl}(\beta, \theta)$ and $P_1^h(2, \beta, \theta)$. As for the case of the total (summed over the polarization components) radiation, the behavior of functions $P_0^b(\beta, \theta)$ is closer to $P_0^h(2, \beta, \theta)$ than to $p_0^{cl}(\beta, \theta)$, namely, $P_0^b(\beta, \theta)$ decrease with θ down to some point β and then, as was for $P_0^h(2, \beta, \theta)$,

Figure 4.22: The plots of $P_2^b(\beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.23: The plots of $P_3^b(\beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.24: The plots of $P_0^b(\beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.25: The plots of $P_1^b(\beta, \theta)$ for $\beta = 0, 0.1, \dots, 0.9, 1$.

the character of the evolution changes, the points of maximum other than $\pi/2$ enter the picture (the classical curves $p_0^{cl}(\beta, \theta)$ reach their maxima exactly at $\theta = \pi/2$ for every β higher than a certain value). In the plane of motion both $P_0^b(\beta, \theta)$ and $P_0^h(2, \beta, \theta)$ have the maxima as the functions of β . However, the maximum of $P_0^b(\beta, \pi/2)$ stays very close to $\beta = 1$, which is not the case for $P_0^h(2, \beta, \theta)$. In addition, $P_0^h(2, \beta \rightarrow 0, \theta = \pi/2) < P_0^h(2, \beta \rightarrow 1, \theta = \pi/2)$, and an opposite equality holds for the functions $P_0^b(\beta, \theta)$.

Thus, we can emphasize that qualitatively the character of the angular distributions of the power emitted by a boson which initially occupied $n = 2$ has certain similarities with classical angular distributions. However, there is a principal difference, namely, for the classical angular distributions no finite limit at $\beta \rightarrow 1$ can be found, while for a scalar quantum particle all the distributions stay finite. One can also claim that the classical characteristics of the spectral-angular distributions introduced for the n -part of the spectrum (when $n = 2$) basically stay in better qualitative and quantitative agreement with the results of the quantum theory of a boson than the characteristics of the total classical spectrum.

We also intend to consider the spectral-angular distributions of the radiation emitted by electron. By analogy with (4.13) we introduce

$$P_i^e(\zeta, \beta, \theta) = \frac{Q_0 A(\beta)}{128} \frac{\sum_{\nu=1,2} \bar{F}_i^e(\zeta, 2, \nu, \beta, \theta)}{\sum_{\nu=1,2} \bar{W}_0^e(\zeta, 2, \nu, \beta)}, \quad \frac{d\bar{W}_i^e}{d\Omega} = P_i^e(\zeta, \nu, \beta, \theta) \cdot \sum_{\nu=1,2} \bar{W}^e(\zeta, 2, \nu, \beta). \quad (4.14)$$

The explicit expressions for $P_i^e(\zeta, \beta, \theta)$ are listed below

$$P_i^e(-1, \beta, \theta) = \frac{Q_0 A(\beta)}{128} \frac{[F_i^{(1)}(-1, 2, 1, \beta, \theta) + F_i^{(2)}(-1, 2, 1, \beta, \theta) + F_i^{(1)}(2, 2, \beta, \theta)]}{[W_0^{(1)}(-1, 2, 1, \beta) + W_0^{(2)}(-1, 2, 1, \beta) + W_0^{(1)}(2, 2, \beta)]},$$

$$P_i^e(1, \beta, \theta) = \frac{Q_0 A(\beta)}{128} \frac{[F_i^{(1)}(1, 2, 1, \beta, \theta) + F_i^{(2)}(1, 2, 1, \beta, \theta) + F_i^{(2)}(2, 2, \beta, \theta)]}{[W_0^{(1)}(1, 2, 1, \beta) + W_0^{(2)}(1, 2, 1, \beta) + W_0^{(2)}(2, 2, \beta)]}, \quad (4.15)$$

The behavior of $P_i^e(1, \beta, \theta)$ resembles the character of evolution of the bosonic functions $P_i^b(\beta, \theta)$ with θ when β is not too high. However, for $\beta \rightarrow 1$ this similarity disappears. If the initial spin is antiparallel to the external field, one can observe a significant analogy between $P_i^e(-1, \beta, \theta)$ and $P_i^b(\beta, \theta)$. This resemblance is both quantitative and qualitative, except for the ultrarelativistic limit. At high β the radiation concentrates near the plane of motion, $P_i^e(\zeta, \beta \rightarrow 1, \theta)$ reach maximal values near $\pi/2$, but this sort of concentration differs significantly from the classical concentration. Namely, the effective angles of radiation remain finite and do not vanish at $\beta \rightarrow 1$. Figures 4.29 and 4.33 illustrate that independently of the ζ -direction right circularly polarized radiation can not be found but in the upper half-plane when $\beta \rightarrow 1$. Obviously, extremely high values of energy β guarantee the lack of the left-polarized radiation in the upper half-plane. This peculiarity might turn essential when studying particular processes of radiation in deep space.

Figure 4.26: The plots of $P_2^e(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.28: The plots of $P_0^e(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.27: The plots of $P_3^e(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.29: The plots of $P_1^e(1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 0.99999$.

Figure 4.30: The plots of $P_2^e(-1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.31: The plots of $P_3^e(-1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.32: The plots of $P_0^e(-1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 1$.

Figure 4.33: The plots of $P_1^e(-1, \beta, \theta)$, $\beta = 0, 0.1, \dots, 0.9, 0.99999$.

4.3 Effective Angles and Deviation Angles of Radiation

Let us consider the effective angles of the radiation emitted by boson during the transitions from $n = 2$. With the notations given in sections 1.2, 1.2.2 we study the functions $\Delta_i^b(2, \beta)$ (the effective angles of the total radiation emitted by a boson) and $\Delta_i^b(2, \nu, \beta)$ (the effective angles of the radiation associated with particular harmonic ν). The results of numerical calculations that we performed for the effective angles of a boson are given in figures 4.34 - 4.37. The dashed curves refer to the classical analogues of $\Delta_i^b(2, \nu, \beta)$, namely $\Delta_i^b(2, 1, \beta)$ corresponds with $\Delta_i^h(1, \beta)$, $\Delta_i^b(2, 2, \beta)$ should be compared with $\Delta_i^h(2, \beta)$. Finally, $\Delta_i^b(2, \beta)$ refers to the function $\tilde{\Delta}_i^h(2, \beta)$, defined by (3.45), where in place of the functions $\Phi_i(\beta, \theta)$ we put the sum over two first harmonics of the classical spectrum of the form $F_i^h(1, \beta, \theta) + F_i^h(2, \beta, \theta)$.

Figure 4.34 illustrates the behavior of the effective angles associated with the σ -component of radiation. We see, that the character of classical and quantum curves for $\nu = 1$ is similar, namely, in this case the effective angles of radiation increase monotonically on $0 \leq \beta \leq 1$. The curve that corresponds with the total (summed over the spectrum) radiation in quantum theory resembles the classical curve which characterizes the radiation of two first harmonics of the classical spectrum. The functions $\tilde{\Delta}_2^h(2, \beta)$ and $\Delta_2^b(2, \beta)$ decrease with β and at $\beta = 1$ reach certain finite values. For the σ -component of linear polarization the most significant difference can be observed for the functions $\Delta_2^b(2, 2, \beta)$ and $\Delta_2^h(2, \beta)$. $\Delta_2^h(2, \beta)$ increases monotonically, and $\Delta_2^b(2, 2, \beta)$ decreases on $0 \leq \beta < 1$.

Figure 4.34: Plots of functions $\Delta_2^b(2, 2, \beta)$ and $\Delta_2^b(2, \beta)$ (1), $\Delta_2^b(2, 1, \beta)$ and $\Delta_2^b(1, \beta)$ (2), $\Delta_2^b(2, \beta)$ and $\tilde{\Delta}_2^b(2, \beta)$ (3), $\Delta_2^{cl}(\beta)$ (4).

Figure 4.35: Plots of functions $\Delta_3^b(2, 2, \beta)$ and $\Delta_3^b(2, \beta)$ (1), $\Delta_3^b(2, 1, \beta)$ and $\Delta_3^b(1, \beta)$ (2), $\Delta_3^b(2, \beta)$ and $\tilde{\Delta}_3^b(2, \beta)$ (3), $\Delta_3^{cl}(\beta)$ (4).

Figure 4.36: Plots of functions $\Delta_0^b(2, 2, \beta)$ and $\Delta_0^b(2, \beta)$ (1), $\Delta_0^b(2, 1, \beta)$ and $\Delta_0^b(1, \beta)$ (2), $\Delta_0^b(2, \beta)$ and $\tilde{\Delta}_0^b(2, \beta)$ (3), $\Delta_0^{cl}(\beta)$ (4).

Figure 4.37: Plots of functions $\Delta_1^b(2, 2, \beta)$ and $\Delta_1^b(2, \beta)$ (1), $\Delta_1^b(2, 1, \beta)$ and $\Delta_1^b(1, \beta)$ (2), $\Delta_1^b(2, \beta)$ and $\tilde{\Delta}_1^b(2, \beta)$ (3), $\Delta_1^{cl}(\beta)$ (4).

For the π -component, on the contrary, the difference can be observed between the functions $\Delta_3^b(2, 1, \beta)$ and $\Delta_3^h(1, \beta)$. The character of the evolution of effective angles of the π -component of radiation emitted from boson has a qualitative accordance with the case of the total classical spectrum. The curves $\Delta_3^b(2, \beta)$, $\tilde{\Delta}_3^h(2, \beta)$ and $\Delta_3^{cl}(\beta)$ have maxima, while $\Delta_3^{cl}(\beta) \rightarrow 0$ at $\beta \rightarrow 1$. Other functions tend to certain finite non-zero values. At the same time, there is quantitative and qualitative agreement between the functions $\Delta_3^b(2, \beta)$ and $\tilde{\Delta}_3^h(2, \beta)$, such that in the ultrarelativistic limit these functions are located close to each other.

In the case of total radiation, classical and quantum curves that refer to the first harmonic ($\nu = 1$) are very close to each other on the whole interval $0 \leq \beta < 1$. However, for other functions no such similarity can be observed.

The best correspondence between the results of classical and quantum theory can be observed for the right circular polarization ($i = 1$) (figure 4.37). All the functions, namely, $\Delta_1^b(2, 1, \beta)$, $\Delta_1^b(2, 2, \beta)$, $\Delta_1^b(2, \beta)$, $\Delta_1^h(1, \beta)$, $\Delta_1^h(2, \beta)$, $\tilde{\Delta}_1^h(2, \beta)$ and $\Delta_1^{cl}(\beta)$ are monotonically increasing within $0 \leq \beta < 1$, and both when $\nu = 1$ or for the sum over $\nu = 1, 2$ classical and quantum effective angles have similar values, even in the ultrarelativistic limit.

Thus, both for the transitions from $n = 1$ and from $n = 2$, a well-known classical affirmation about the radiation concentration in the plane of motion at $\beta \rightarrow 1$ was disproved. In the framework of quantum

theory, all the effective angles at $\beta \rightarrow 1$ tend to certain finite non-zero values as the functions of energy.

The evolution of deviation angles with energy is given in figures 4.38 - 4.40. The functions $\theta_2^{b(2)}(2, 2, \beta)$, $\theta_2^{b(2)}(2, 1, \beta)$ and $\theta_2^{b(2)}(2, \beta)$ which define the deviation angles related to σ -component of radiation are equal to $\pi/2$ within the whole region $0 \leq \beta < 1$. In other words, according to the quantum theory the main part of power associated with the σ -component of the linearly polarized radiation is emitted near the plane of motion at any energy. So, for the total spectrum of radiation the concentration in the plane of orbit increases which is not the case when $\nu = 1$ (figure 4.34).

It is shown in figure 4.38 that the functions $\theta_3^{b(2)}(2, 2, \beta)$ and $\theta_3^{b(2)}(2, 1, \beta)$ increase monotonically with β on the interval $0 \leq \beta < 1$, which is not consistent with the results of the classical theory, $\theta_3^{h(2)}(2, \beta)$ and $\theta_3^{h(2)}(1, \beta)$ decrease monotonically. Though the values of function $\tilde{\theta}_3^{h(2)}(2, \beta)$ that characterizes a deviation angle of the π -component of radiation associated with first two harmonics of the classical spectrum differ completely from the values of $\theta_3^{b(2)}(2, \beta)$ when $\beta \rightarrow 1$, similarly to $\theta_3^{b(2)}(2, \beta)$, $\tilde{\theta}_3^{h(2)}(2, \beta)$ increases within $0 \leq \beta < 1$. However, none of the quantum curves tends to $\pi/2$ in the ultrarelativistic limit as the π -component of total classical spectrum (the radiation concentrates in the plane of orbit). Thus, in the case of the π -component one can claim the results obtained for 2-part of classical spectrum qualitatively consistent with the quantum theory, whereas the total classical spectrum does not seem representative with respect to the boson.

Figure 4.38: The plots of $\theta_3^{b(2)}(2, 2, \beta)$ and $\theta_3^{h(2)}(2, \beta)$ (1), $\theta_3^{b(2)}(2, 1, \beta)$ and $\theta_3^{h(2)}(1, \beta)$ (2), $\theta_3^{b(2)}(2, \beta)$ and $\tilde{\theta}_3^{h(2)}(2, \beta)$ (3), $\theta_3^{cl(2)}(\beta)$ (4).

Figure 4.39: The plots of $\theta_0^{b(2)}(2, 2, \beta)$ and $\theta_0^{h(2)}(2, \beta)$ (1), $\theta_0^{b(2)}(2, 1, \beta)$ and $\theta_0^{h(2)}(1, \beta)$ (2), $\theta_0^{b(2)}(2, \beta)$ and $\tilde{\theta}_0^{h(2)}(2, \beta)$ (3), $\theta_0^{cl(2)}(\beta)$ (4).

Figure 4.40: The plots of $\theta_1^{b(2)}(2, 2, \beta)$ and $\theta_1^{h(2)}(2, \beta)$ (1), $\theta_1^{b(2)}(2, 1, \beta)$ and $\theta_1^{h(2)}(1, \beta)$ (2), $\theta_1^{b(2)}(2, \beta)$ and $\tilde{\theta}_1^{h(2)}(2, \beta)$ (3), $\theta_1^{cl(2)}(\beta)$ (4).

The plots of the deviation angles of the total (summed over the polarizations) radiation are of particular interest (see figure 4.39). The functions associated with the first harmonic of radiation in classical and quantum theory decrease monotonically on $0 \leq \beta < 1$ (in figure 4.39, the curves 2, dashed and solid lines respectively). However, the difference between these curves grows with β . As for the deviation angles associated with the second harmonic, according to quantum theory we have $\theta_0^{b(2)}(2, 2, \beta) = \pi/2$ at any β of $[0, 1]$. Classical theory, in its turn, gives a similar result up to a certain $\tilde{\beta} \approx 0.61815955$ ($\gamma \approx 1.27218$). Within $\tilde{\beta} \leq \beta < 1$ the function $\theta_0^{h(2)}(2, \beta)$ decreases. An unusual behavior can be observed for the deviation angle associated with the total quantum spectrum, namely, for the function $\theta_0^{b(2)}(2, \beta)$ which increases on $0 \leq \beta < \bar{\beta} \approx 0.6199$ ($\gamma \approx 1.2744$), then at $\bar{\beta} \leq \beta \leq \bar{\bar{\beta}} \approx 0.80816399$ ($\gamma \approx 1.69792$) turns to $\pi/2$, and decreases at $\bar{\bar{\beta}} \leq \beta < 1$. It means that at certain energies $\bar{\beta} \leq \beta \leq \bar{\bar{\beta}}$ the main part of power is emitted in the directions close to the plane of orbit. However, the later is not the case neither for the non-relativistic, nor for the ultrarelativistic domain.

Only for the right circular polarization one can claim that the classical theory of the total spectrum is representative with respect to quantum approach. The curve 4 and the quantum curve 3 (solid line 3 in 4.40) increase within $0 \leq \beta < 1$. Nevertheless, the function $\theta_1^{cl(2)}(\beta)$ has a weak maximum at β close to 1, which is not the case for the function $\theta_1^{b(2)}(2, \beta)$. For separate harmonics in the case of $i = 1$ one can also observe a similarity between the behavior of the curves $\theta_1^{b(2)}(2, 2, \beta)$ and $\theta_1^{h(2)}(2, \beta)$, $\theta_1^{b(2)}(2, 1, \beta)$ and $\theta_1^{h(2)}(1, \beta)$ decrease monotonically on the interval $0 \leq \beta < 1$.

Concerning the effective angles of an electron, the direction of spin plays an important role, especially for the transitions to the first excited state, $2 \rightarrow 1$. Figures 4.41 - 4.44 allow one to see that both $\Delta_2^{(k)}(2, \beta)$, $k = 1, 2$, decrease with β , similarly to $\Delta_2^b(2, 2, \beta)$ (except for the ultrarelativistic case where $\Delta_2^{(1)}(2, \beta)$ has a minimum, so that its behavior qualitatively resembles that of $\Delta_2^e(\beta)$). The functions $\Delta_2^{(k)}(\zeta, 1, \beta)$ increase with β . Due to a switch between the polarization components with respect to the direction of spin, $\Delta_0^{(1)}(2, \beta) = \Delta_0^{(2)}(2, \beta)$ which could be seen in the figure 4.43.

Considering the effective angles, the difference between scalar and spinor particle manifests itself very clearly when the particle jumps from $\{2, \zeta = -1\}$ to $\{1, \zeta = 1\}$. This transition is supposed to be accompanied by a significant energy loss due to the particular switch of spin direction.

Figure 4.41: The plots of $\Delta_2^{(1)}(1, 1, \beta)$ (1^+a), $\Delta_2^{(1)}(-1, 1, \beta)$ (1^-a), $\Delta_2^{(2)}(1, 1, \beta)$ (1^+b), $\Delta_2^{(2)}(-1, 1, \beta)$ (1^-b), $\Delta_2^{(1)}(2, \beta)$ ($2a$), $\Delta_2^{(2)}(2, \beta)$ ($2b$).

Figure 4.42: The plots of $\Delta_3^{(1)}(-1, 1, \beta)$ (1^-a), $\Delta_3^{(2)}(1, 1, \beta)$ (1^+b), $\Delta_3^{(2)}(-1, 1, \beta)$ (1^-b), $\Delta_3^{(1)}(2, \beta)$ ($2a$), $\Delta_3^{(2)}(2, \beta)$ ($2b$).

Figures 4.41 - 4.44 show that, similarly to the case of a boson, the effective angles of a spinor particle do not tend to zero at $\beta \rightarrow 1$. So, the radiation emitted by an ultrarelativistic electron at whatever transition from $n = 2$ is not concentrated near the plane $\theta = \pi/2$.

Figure 4.43: The plots of $\Delta_0^{(1)}(1, 1, \beta)$ (1^+a), $\Delta_0^{(1)}(-1, 1, \beta)$ (1^-a), $\Delta_0^{(2)}(1, 1, \beta)$ (1^+b), $\Delta_0^{(2)}(-1, 1, \beta)$ (1^-b), $\Delta_0^{(1)}(2, \beta)$ ($2a$).

Figure 4.44: The plots of $\Delta_1^{(1)}(1, 1, \beta)$ (1^+a), $\Delta_1^{(1)}(-1, 1, \beta)$ (1^-a), $\Delta_1^{(2)}(1, 1, \beta)$ (1^+b), $\Delta_1^{(2)}(-1, 1, \beta)$ (1^-b), $\Delta_1^{(1)}(2, \beta)$ ($2a$).

Chapter 5

The Evolution of Maximum in the Quantum Spectrum of the Radiation

It is well-known that SR was accidentally discovered by Floyd Haber, an assistant in the laboratory of H. C. Pollock. Adjusting the accelerator, Haber has noticed a weak light [12, 122–124]. These observation confirmed both the existence of SR, predicted by Ivanenko and Pomeranchuk in [44], and the conclusion that the critical frequency of the radiation from an electron (i.e. the frequency associated with the radiation maximum) changes with energy [10, 11, 125], so that it can switch to the visible part of the spectrum [126]. In this chapter we consider the influence of the quantum corrections on this conclusion.

5.1 The Evolution of Maximum in Classical Spectrum

As a short introduction to the essential part of this chapter, we start with a brief review of the classical picture and consider the evolution of radiation maximum in the classical spectrum. The classical spectral-angular distributions of SR are to be defined by (1.26), and these expressions allow us to consider two important cases, namely, non-relativistic and ultrarelativistic limits.

When the velocity of radiating particle is much smaller than c and, therefore, the argument of Bessel functions in (1.26) is small, to calculate an approximation for spectral-angular distributions one might use a well-known approximation [127, p. 365]

$$J_\nu(z)|_{z \rightarrow 0} = \frac{z^\nu}{2^\nu \nu!}, \Rightarrow J'_\nu(z) = \frac{z^{\nu-1}}{2^\nu (\nu-1)!}. \quad (5.1)$$

Applying (5.1), for the components of linear and circular polarization we obtain

$$\begin{aligned} F_2^h(\nu, \beta \approx 0, \theta) &= \frac{\nu^{2\nu} \beta^{2(\nu-1)} [\sin \theta]^{2(\nu-1)}}{2^{2\nu} [(\nu-1)!]^2}, \quad F_3^h(\nu, \beta \approx 0, \theta) = \frac{\nu^{2\nu} \beta^{2(\nu-1)} [\sin \theta]^{2(\nu-1)}}{2^{2\nu} [(\nu-1)!]^2} \cos^2 \theta, \\ F_0^h(\nu, \beta \approx 0, \theta) &= F_2^h + F_3^h, \quad F_g^h(\nu, \beta \approx 0, \theta) = \frac{F_0^h}{2} + g \frac{\nu^{2\nu} \beta^{2(\nu-1)} [\sin \theta]^{2(\nu-1)}}{2^{2\nu} [(\nu-1)!]^2} \cos \theta. \end{aligned} \quad (5.2)$$

We then integrate over θ to obtain the spectral distributions (the fact that $A^{cl}(\beta) \rightarrow \beta^2$ at small β should

be taken into account when considering $W_i^h(\nu, \beta)$

$$\begin{aligned} f_2^h(\nu, \beta \approx 0) &= \frac{\beta^{2(\nu-1)} \nu^{2\nu+1}}{(2\nu)!} , \quad f_3^h(\nu, \beta \approx 0) = \frac{\beta^{2(\nu-1)} \nu^{2\nu+1}}{(2\nu+1)!} , \\ f_0^h(\nu, \beta, \approx 0) &= \frac{2\nu^{2\nu+1}(\nu+1)}{(2\nu+1)!} \beta^{2(\nu-1)} . \end{aligned} \quad (5.3)$$

Here the following integral was used (3.621 in [128, p. 412])

$$\int_0^\pi [\sin \theta]^{2m+1} d\theta = 2 \frac{(2m)!!}{(2m+1)!!} . \quad (5.4)$$

The part that specifies the components of circular polarization vanishes while integrating over the θ -angle and thus we can restrict ourselves to the consideration of the total radiation and its linear polarization.

It is clear from (5.3) that the amount of the radiated power decreases with ν . Indeed, the presence of the factor $\beta^{2(\nu-1)}$ for non-relativistic velocities $\beta \approx 0$ implies the diminution of $W_i^h(\nu, \beta \approx 0)$ and its tendency to reach zero-value at high harmonics. So, when β is small we are allowed to conclude that the maximum of the radiation stays at the first harmonic of the classical spectrum.

Now let us turn our attention to the evolution, or, in other words, to the shifts of radiation maximum between the harmonics of classical spectrum when the energies are neither very low nor very high. Namely, we intend to show whether and when does the position of maximum switch from one harmonic to another and what is the order of this switching. The relation $f_i^h(\nu, \beta)/f_i^h(\nu-1, \beta)$ between the power, radiated at the harmonic ν and the power radiated at the harmonic $\nu-1$ can serve perfectly for obtaining this information. Clearly, when the equality

$$K_0^h(\nu, \beta) \equiv \frac{f_0^h(\nu, \beta)}{f_0^h(\nu-1, \beta)} = 1 \quad (5.5)$$

holds, the power radiated at ν is the same as the power radiated at $\nu-1$ and, therefore, this is exactly the moment when the shift occurs. The plots of $K_i^h(\nu, \beta)$ functions are given in figures 5.1 - 5.3.

Figure 5.1: The plots of $K_2^h(\nu, \beta)$

Figure 5.2: The plots of $K_3^h(\nu, \beta)$

It is clear from these figures that the functions $K_i^h(\nu, \beta)$ increase monotonically with β . Moreover, near the dashed line that denotes unity, we see that the higher the harmonic is, the higher is the energy value, i.e. β , at which $K_i^h(\nu, \beta) = 1$. It means that the shifts of maximum to the higher harmonics are subsequent; if $\beta_{\nu_k, i}$ is such value of energy that $K_i^h(\nu_k, \beta_{\nu_k, i}) = 1$, then for $\nu_1 < \nu_2$ we have $\beta_{\nu_1, i} < \beta_{\nu_2, i}$.

As the final step of reviewing particular facts from the classical theory of SR, we consider the ultra-relativistic limit, i.e. the case of high energies and, therefore, of $\beta \rightarrow 1$. When the particle turns to be relativistic and $1 - \beta^2 \sim 0$, a well-known fact of the classical SR theory - the fact that the radiation concentrates in a small angle and one is allowed to assume $\theta \sim \pi/2$ - helps to significantly simplify the general formulae. In this connection we can use the following asymptotic expressions [14, 45]

$$J_\nu(\nu x) \approx \frac{1}{\pi\sqrt{3}} \sqrt{1-x^2} K_{1/3} \left(\frac{\nu}{3} (1-x)^{3/2} \right), \quad J'_\nu(\nu x) \approx \frac{1}{\pi\sqrt{3}} (1-x^2) K_{2/3} \left(\frac{\nu}{3} (1-x)^{3/2} \right), \quad (5.6)$$

where $K_\alpha(z)$ are modified Bessel functions of the second kind that satisfy

$$\begin{aligned} -2K'_\alpha(z) &= K_{\alpha+1}(z) + K_{\alpha-1}(z), \\ K'_\alpha(z) &= -\frac{\alpha}{z} K_\alpha(z) - K_{1-\alpha}(z). \end{aligned} \quad (5.7)$$

To obtain spectral distributions we are to perform integration over θ as previously, and here are the integrals that would ease our work

$$\begin{aligned} \int_0^\pi \epsilon^2 K_{2/3}^2 \left(\frac{\nu}{3} \epsilon^{3/2} \right) \sin \theta d\theta &= \frac{\pi\epsilon_0}{\nu\sqrt{3}} \left[\int_y^\infty K_{2/3}(x) dx + 3K_{2/3} \left(\frac{2\nu\epsilon_0^{3/2}}{3} \right) \right], \\ \int_0^\pi \epsilon \cos^2 \theta K_{1/3} \left(\frac{\nu\epsilon^{3/2}}{3} \right) \sin \theta d\theta &= \frac{\pi\epsilon_0}{\nu\sqrt{3}} \int_y^\infty K_{2/3}(x) \frac{dx}{x}. \end{aligned} \quad (5.8)$$

Here the following notations are used

$$\epsilon = 1 - x^2, \quad y = \frac{2\nu}{3} \epsilon_0^{3/2}, \quad \epsilon_0 = \epsilon(\beta, \pi/2) = 1 - \beta^2 = \frac{1}{\gamma^2}. \quad (5.9)$$

With the help of (5.6), (5.7) and (5.8) it becomes easy to calculate the spectral distributions of radiation emitted by an ultrarelativistic particle. We note here, that unlike in section 1.1 the integration over θ is performed within the region $[0, \pi]$

$$\begin{aligned} f_2^h(\nu, \beta) &= \frac{\nu\epsilon_0}{2\sqrt{3}\pi} \left[\int_y^\infty K_{5/3}(x) dx + K_{2/3}(y) \right], \\ f_3^h(\nu, \beta) &= \frac{\nu\epsilon_0}{2\sqrt{3}\pi} \left[\int_y^\infty K_{5/3}(x) dx - K_{2/3}(y) \right]. \end{aligned} \quad (5.10)$$

In the case of the circular polarization the additional term that specifies its distribution is canceled due to the integrand being an odd function. So, similarly to the case $\beta \approx 0$, we continue the consideration of the total radiation and its linear polarizations.

Obviously, to obtain the total power of radiation one is supposed to sum over all the harmonics ν . We know, that at high β the contribution of the radiation power of a single harmonic ν is higher than the one corresponding to $\nu - 1$. So, the main part of radiation is concentrated at very high ($\nu \gg 1$) harmonics when $\beta \rightarrow 1$. Since $d\nu = 1 \ll \nu$, the summation over ν can be substituted by the integration

$$\sum_{\nu=1}^\infty W_i^h(\nu, \beta \rightarrow 1) = \int W_i^h(\nu, \beta \rightarrow 1) d\nu. \quad (5.11)$$

Finally, after including the substitution $\nu \rightarrow y$ ($\nu d\nu = \frac{9\gamma^6}{4} y dy$), for the spectral distribution of the radiation power at high energies we get

$$\begin{aligned}\frac{dW_2}{dy} &= Q_0^{cl} \gamma^2 \frac{3\sqrt{3}}{8\pi} y \left[\int_y^\infty K_{5/3}(x) dx + K_{2/3}(y) \right] \equiv Q_0^{cl} \gamma^2 \frac{2}{3} Y_2(y), \\ \frac{dW_3}{dy} &= Q_0^{cl} \gamma^2 \frac{3\sqrt{3}}{8\pi} y \left[\int_y^\infty K_{5/3}(x) dx - K_{2/3}(y) \right] \equiv Q_0^{cl} \gamma^2 \frac{2}{3} Y_3(y), \\ \frac{dW_0}{dy} &= Q_0^{cl} \gamma^2 \frac{3\sqrt{3}}{4\pi} y \int_y^\infty K_{5/3}(x) dx \equiv Q_0^{cl} \gamma^2 \frac{2}{3} Y_0(y).\end{aligned}\tag{5.12}$$

Now, let us consider the behavior of the Y_i -functions. As one can see in figure 5.4, all the three $Y_i(y)$ reach their maximum values at small y and then monotonically decrease with y (this conclusion is a basic thing in classical theory, similar figures can be found in SR textbooks e.g. [18, 25], see also [14, 16, 79]). Their maximal values are put into table

	$i = 2$	$i = 3$	$i = 0$
y_i^{max}	0.441847	0.222498	0.285812
Y_i^{max}	0.464228	0.105187	0.569393
ν_i^{max}	$0.66277\gamma^3$	$0.333747\gamma^3$	$0.428718\gamma^3$

It should be emphasized that the number of the harmonic associated with the radiation maximum changes with energy as $\sim \gamma^3$ [13], i.e. as the cube of relativistic factor. It can increase unlimitedly since the classical spectrum is infinite.

Figure 5.3: The plots of $K_0^h(\nu, \beta)$

Figure 5.4: The plots of $Y_i(y)$ for the total radiation and the linear components.

So, this was the revision of the known results obtained in the framework of classical SR theory. Before we switch to the quantum description of the SR maximum and its evolution, let us generalize the reviewed conclusions. When a classical particle moves at a non-relativistic velocity, the maximal amount of radiation power emitted by this particle corresponds with the first harmonic of its radiation spectrum. Increasing the energy, the radiation maximum shifts successively to higher harmonics and in the ultrarelativistic limit ($\gamma \gg 1$) the number of classical harmonic associated with the maximal emission of SR power increases as γ^3 .

5.2 Quantum Approach to the Spectral Distributions of Synchrotron Radiation

Now we intend to track the position of the radiation maximum in the quantum spectrum for both scalar and spinor particles. It is clear, that the methods given in the previous section can not serve well within the quantum approach and can only be applied to the quantum non-relativistic radiation (where the quantum theory stays in total agreement with the classical one and predicts the absolute dominance of the first harmonic $\tilde{q}_0^b(1, \beta \rightarrow 0) = 1$, $\tilde{q}_0^e(\pm 1, 1, \beta \rightarrow 0) = 1$). Indeed, the quantum spectrum is discrete and finite at any β , whereas the classical spectrum is infinite and, moreover, at high energies can be treated as continuous (e.g. in (5.11) we used this particular property). In order to better understand the behavior of maximum, it is convenient to start with a simple case of the short quantum spectrum where only two harmonics are involved.

One can see the curve (1a) meeting (2a) at some point (see figure 4.8 in section 4.1). This intersection corresponds with the shift of radiation maximum from the first harmonic to the second. The value of energy, at which this shift could occur, can be found as the solution of $F_0^b(2, 1, \beta) = F_0^b(2, 2, \beta)$, which simply follows from the definition of $F_i^b(n, \nu, \beta)$. The value $\beta^b \approx 0.846181$ ($\gamma = 1.87654$) satisfies this equation. Figures 4.11 and 4.10 show, that electrons have similar features. Both $\tilde{q}_0^e(-1, 1, \beta)$ and $\tilde{q}_0^e(1, 1, \beta)$ are decreasing with β , which means that the dominance of the first harmonic vanishes at some point. Namely, at $\beta^e \approx 0.901862329$ ($\gamma = 2.31469$) for $\zeta = 1$ and at $\beta^e \approx 0.755693695$ ($\gamma = 1.52689$) for $\zeta = -1$, which are the roots of $F_0^e(1, 2, 1, \beta) = F_0^{(2)}(1, 2, 2, \beta)$ and $F_0^e(-1, 2, 1, \beta) = F_0^{(1)}(-1, 2, 2, \beta)$, the maximum of radiation shifts to the second (last, $\nu = n = 2$) harmonic of the quantum spectrum. Though it is not noticeable but at the low levels, one can see $\beta^e(\zeta = -1) < \beta^e(\zeta = 1)$. It means that an electron with the initial spin antiparallel to the external field switches its preferences to complete "outpour" at lower energy. It is also clear, that $\beta^e(\zeta = -1) < \beta^b < \beta^e(\zeta = 1)$. According to the classical theory, the shift of radiation maximum between the first two harmonics occurs at $\beta^{cl} \approx 0.76354$ ($\gamma = 1.54856$), which can be called representative with respect to the electron of spin $\zeta = 1$. So, for both bosons and electrons, the maximum of the non-relativistic radiation lies at the first harmonic of the spectrum, and increasing the energy it shifts to the second harmonic. It allows us to assume that at $n > 2$ the maximum in quantum radiation spectrum sequently shifts to the higher harmonics with energy, starting at the primary harmonic when $\beta \approx 0$.

However, as was emphasized previously, the essential difference between classical and quantum spectrum is that the former remains finite, $\nu \leq n$. Thus, increasing the energy, the radiation maximum stays at $\nu^{max} \leq n$ and even in ultrarelativistic case we never get anything similar to the classical approximation $\nu^{max} \sim \gamma^3$. But the maximum sequently switches to higher harmonics with energy, and, therefore, there is a value of energy such that the shift occurs between $\nu = n - 1$ and $\nu = n$. So, in the framework of quantum theory, it makes sense to raise a question about the conditions that would provide a possibility of such shift. In other words, the following problem can be formulated: we are to find the conditions (if any) under which $\nu^{max} = n$, i.e. the maximum of radiation lies at the highest harmonic of the quantum spectrum.

5.2.1 Position of the Radiation Maximum in the Quantum Spectrum of a Scalar Particle

As previously, it is convenient to begin by considering the spectrum of a scalar particle. By analogy with the definition (5.5) given in the preceding section in the framework of classical approach, we introduce the functions $K^b(n, \beta)$, which, at fixed n , are the function of β

$$K^b(n, \beta) = \frac{F_0^b(n, n, \beta)}{F_0^b(n, n-1, \beta)}. \quad (5.13)$$

The functions $F_0^b(n, n, \beta)$ and $F_0^b(n, n-1, \beta)$ are defined by the expressions (2.136) and (2.149), i.e. describe the power of radiation emitted at the transition $n \rightarrow 0$ and at $n \rightarrow 1$ respectively, and, thus, $K^b(n, \beta)$ serves to compare the radiated power. It is easy to see that

$$K(n, \beta = 0) = 0,$$

which indicates the following fact - at any n there is an interval of (small) energies, where the SR power at the penultimate harmonic exceeds the power emitted at the highest harmonic, and the spectral maximum does not shift to $\nu = n$. The definition of the functions $K^b(n, \beta)$ allows us to conclude that, when $K^b(n, \beta) > 1$, the maximum of radiation stays at highest harmonic. Otherwise it does not jump to $\nu = n$. So, $K^b(n, \beta) > 1$ represents a necessary condition for the radiation maximum to lie at $\nu = n$. Finally, the equation

$$K^b(n, \beta) = 1 \quad (5.14)$$

fixes "the moment" of the shift between two highest harmonics. The solution of $K^b(n, \beta) = 1$ for fixed n is $\beta = \beta_n$, or $\gamma_n^2 = (1 - \beta_n^2)^{-1/2}$. Note, that in the ultrarelativistic limit $\beta \rightarrow 1$, and the maximum of radiation is supposed to lie at the highest harmonic of the spectrum, so that $K^b(n, 1) > 1$.

We have found β_n (γ_n^2) for different values of n numerically, and the results are given in table 5.1 (the calculations were carried out up to $n_{max} = 10^5$). The plots of the functions $K^b(n, \beta)$ can be found

Table 5.1: The roots of $K^b(n, \beta) = 1$

n	β_n	$K^b(n, 1)$	γ_n^2
5	0.9483854	1.4035	9.9438060
10	0.9755129	1.34048	20.6720474
50	0.9953967	1.28903	108.8686036
100	0.9977333	1.28027	220.833406
500	0.9995563	1.27005	1127.12206
1000	0.9997793	1.26785	2265.99764
5000	0.9999561	1.26503	11411.26374
10000	0.9999781	1.26438	22862.03008
50000	0.9999956	1.26352	114576.7518
100000	0.9999978	1.26332	229280.8677

in figure 5.6, where the solutions of (5.14) are the points of intersection of the curves $K^b(n, \beta)$ with the dashed line.

Figure 5.5: The plots of $K^b(n, \beta)$ at different n .

Figure 5.6: The plots of $K(n, \beta)$ at different n for $\beta \rightarrow 1$.

Thus, for each fixed n there exists an energy value (β_n), which corresponds with the shift of the spectral maximum to the highest harmonic $\nu = n$.

Right after $K^b(n, \beta)$ one can introduce the functions $K_1^b(n, \beta)$

$$K_1^b(n, \beta) = \frac{F_0^b(n, n-1, \beta)}{F_0^b(n, n-2, \beta)}, \quad K_1^b(n, \beta = 0) = 0. \quad (5.15)$$

We perform the numerical analysis of the behavior of the functions $K_1^b(n, \beta)$, and, similarly to the case of $K^b(n, \beta)$, consider the equation $K_1^b(n, \beta) = 1$. In other words, one can find the energy values at which the maximum in radiation spectrum shifts to $\nu = n - 1$ starting at some lower harmonic.

Figure 5.7: The plots of $K_1^b(n, \beta)$ at different n .

Let us turn our attention to the expression (2.120) that follows from the Klein-Gordon-Fock equation and describes the energy levels. Clearly, for high values of n it can be rewritten as

$$\gamma^2 \approx 2nb, \quad n \gg 1, \quad (5.16)$$

so, for a fixed b it defines the function $\gamma^2(n)$, and it gives the function $\gamma^2(b)$ when the number n is fixed (when n is fixed, the energy of a particle can be increased with the intensity of the magnetic field). Obviously, both of these functions are linear.

When solving the equation (5.14) for a series of fixed values of n and, thus, obtaining the roots (n, β_n) or, put in a more convenient form, (n, γ_n^2) , we intend to find an approximation for $\gamma_n^2(n)$. We emphasize here, that (5.14) has been considered for whatever b , i.e. for each n the value of b satisfying (5.14) could have been different, $b_n = b(n)$. So, according to (5.16), the general form of the dependence of γ_n^2 on n

is supposed to be of the form $\gamma_n^2 = 2nb_n$ (as one can see, the value of the magnetic field intensity, that provides the shift of maximum from $\nu = n - 1$ to $\nu = n$ depends on the initial energy level). Figure (5.8) shows the way the roots of (5.14) are located at the lines $\gamma^2(b)$ defined by (5.16) for fixed n .

Figure 5.8: Lines formed by the roots $(n_a, \gamma_{n_a}^2)$.

Figure 5.9: The roots of $K^b(n, \beta) = 1$, $K_1^b(n, \beta) = 1$.

It is easy to see, that the points (n, γ_n^2) form a straight vertical line. Same thing happens with the shift between $\nu = n - 2$ and $\nu = n - 1$. It means that b_n is constant, it does not depend on n . With the help of figure 5.9 it is not hard to convince oneself that the points (n, γ_n^2) represent an approximate linear dependence of γ_n^2 on n . So, for large n one can assume

$$\begin{aligned}\gamma_n^2 &= 2b_0n, b_0 = \text{const}_0, \\ \gamma_n^2 &= 2b_1n, b_1 = \text{const}_1,\end{aligned}\tag{5.17}$$

where the first expression refers to the roots of $K^b(n, \beta) = 1$, and the second - to the roots of $K_1^b(n, \beta) = 1$. We have found the approximate values of the factors b_0 and b_1 , $b_0 \approx 1.146128793$ and $b_1 \approx 0.820093109$.

As b_0 and b_1 do not depend on n , we drew some conclusions about the shifts between quantum harmonics. If the normalized intensity of external magnetic field satisfies the inequality $b < b_0$, then, whatever the initial state of the particle, the SR maximum stays at the harmonic $\nu_{max} \leq n - 1$. If the intensity satisfies $b \geq b_0$, then the spectral maximum of SR (independently of the number of the initial state) stays at the highest possible (last) harmonic $\nu_{max} = n$.

If the field intensity b exceeds the value b_1 , but does not reach b_0 , i.e. $b_1 \leq b < b_0$, then the radiation maximum stays at the harmonic $\nu = n - 1$. When $b < b_1$ the maximum of radiation lies lower than $\nu = n - 1$, and the shift from $\nu = n - 2$ to $\nu = n - 1$ occurs at $b = b_1$.

Finally, it could be hypothesized that there exists an ordered set of values $b_0 > b_1 > b_2 > \dots > b_s$, such that if $b < b_s$, then, whatever the initial state $n > s$ of the particle, the maximum of SR lies at the harmonic $\nu_{max} \leq n - s$, and it shifts to the higher harmonics with n (but the inequality $\nu_{max} \geq n - s$ remains valid). If $b_m < b < b_{m-1}$ ($1 \leq m \leq s$), then, whatever the initial state $n > m$ of the particle, the spectral maximum of SR stays at the harmonic $\nu_{max} = n - m$. If $b \geq b_0$, then for any n we have $\nu_{max} = n$.

5.2.2 The Role of Spin in the Quantum Picture

For an electron, we introduce the functions

$$K^e(\zeta, n, \beta) = \frac{F_0^e(\zeta, n, \beta)}{\bar{F}_0^e(\zeta, n, n-1, \beta)}, \quad (5.18)$$

which are very much alike $K^b(n, \beta)$. We consider their dependence on the initial spin, averaging over the possible directions of the spin in the first excited state. As one can observe in the figures 5.10 - 5.11, the dependence of $K^e(\zeta, n, \beta)$ on β is (qualitatively) quite similar to the behavior of corresponding bosonic functions. Furthermore, the difference between $K^e(-1, n, \beta)$ and $K^e(1, n, \beta)$ is noticeable only at small values of β at the low levels, and it decreases with n as $\beta \rightarrow 1$.

Figure 5.10: The plots of functions $K^e(1, n, \beta)$ at $n = 5, 10, 22, 50, 100, 300$.

Figure 5.11: The plots of functions $K^e(-1, n, \beta)$ at $n = 5, 10, 22, 50, 100, 300$.

Now, in accordance with our plan that was used for a boson, the equation

$$K^e(\zeta, n, \beta) = 1 \quad (5.19)$$

is to be studied. Its roots, β_n and γ_n^2 , were calculated numerically for both $\zeta = 1$ and $\zeta = -1$ and can be found in table 5.2.

Since we focus on the study of the case of large n such as $n \sim 10^4 - 10^6$, without loss of generality we can restrict ourselves to the analysis of $K^e(1, n, \beta)$.

Same as Klein-Gordon-Fock equation, the Dirac equation leads to the relation (2.77) between γ^2 , n and b . For $n \gg 1$ it transforms to (5.16). Plotting the lines $\gamma^2(b)$ for fixed values of n and then nesting the roots of (5.19) of the form (n, γ_n^2) , we detect an essential difference between our predictions (made in agreement with the results obtained for a scalar particle) and the new results, see figure 5.12. Roughly speaking, this is exactly when the properties of spin enter the picture.

Table 5.2: The roots of $K^e(\zeta, n, \beta) = 1$

n	$\beta_n, \zeta = 1$	$\gamma_n^2, \zeta = 1$	$\beta_n, \zeta = -1$	$\gamma_n^2, \zeta = -1$
50	0.956036	11.9810	0.957358	11.6287
100	0.973598	19.6819	0.974265	19.1915
500	0.991702	61.1482	0.991789	61.1482
1000	0.994893	98.1484	0.994927	98.8196
5000	0.998312	296.446	0.998316	297.149
10000	0.998946	474.725	0.998948	475.435
50000	0.999644	1406.70	0.999645	1407.42
100000	0.999777	2241.26	0.999777	2241.98
600000	0.999933	7446.51	0.999933	7447.23
1000000	0.999952	10479.9	0.999952	10480.6

Figure 5.12: The roots of $K^e(1, n, \beta) = 1$ at various n . Figure 5.13: The points (n, γ_n^2) and corresponding approximation.

It is clear from figure 5.13 that the dependence of γ_n on n is not linear, and, therefore, the value of magnetic field intensity, sufficient to provide the shift of maximum, changes with n . This means that for an electron there is no fixed ordered set of critical values of the field intensity that correspond with the shifts of radiation maximum. We have found an approximation for $\gamma_n^2(n)$

$$\gamma_n^2(n) \approx k \cdot n^m, \quad (5.20)$$

where the approximate values of k and m for $\zeta = \pm 1$ are

$$\begin{aligned} k &= 1.0012284, \quad m = 0.6699919 \quad \text{for } \zeta = 1, \\ k &= 1.0028489, \quad m = 0.6698787 \quad \text{for } \zeta = -1. \end{aligned} \quad (5.21)$$

So, as was predicted by the classical theory, the maximum of radiation in the quantum spectrum shifts successively to the higher harmonics, starting with the primary harmonic in non-relativistic case. This result remains valid independently of the presence of spin or its direction.

In this section we have shown that the presence of the spin perturbs the condition of shifts. For an electron, no linear dependence between γ^2 and n can be observed, and a parabolic approximation with

the power of n close to $2/3$ was found both for the initial spin parallel and antiparallel to the magnetic field intensity.

5.2.3 The Evolution of Maximum with Respect to Polarization

Let us now consider the linear polarization and study the evolution of radiation maximum for the σ - and π -components separately. Similarly to the definitions (5.13) and (5.18), we introduce the functions

$$K_i^b(n, \beta) = \frac{F_i^b(n, n, \beta)}{F_i^b(n, n-1, \beta)}, \quad K_i^e(\zeta, n, \beta) = \frac{F_i^e(\zeta, n, n, \beta)}{F_i^e(\zeta, n, n-1, \beta)}, \quad i = 2, 3. \quad (5.22)$$

We start with focusing on the polarization properties of a boson. The plots of K_i^b are given in figures 5.14 and 5.15 from which it is clear that π -component requires a higher energy (β) for the maximum of the polarized radiation to shift from $\nu_{max} = n-1$ to $\nu_{max} = n$.

Figure 5.14: The functions $K_2^b(n, \beta)$ at various n .

Figure 5.15: The functions $K_3^b(n, \beta)$ at various n .

As was performed for the total radiation in the previous sections, here we can consider the condition of shift, $K_i^b(n, \beta) = 1$. The roots of this equation (in the form (n, γ_n^2)) are given in the figure 5.16

Figure 5.16: The roots of $K_i^b(n, \beta) = 1$ in the form $\gamma_n^2(n)$, $i = 0, 2, 3$.

and so are the approximations for $\gamma^2(n)$. For both components of the linear polarization this approximation is no different from a linear dependence, as was in the case of the total SR power. The coefficients b_i were estimated numerically

$$b_2 \approx 1.0546132, \quad b_3 \approx 1.86984386, \quad b_2 < b_0 < b_3. \quad (5.23)$$

So, the intensity of the magnetic field sufficient to provide the shift of maximum in the spectrum of the π -component of radiation is approximately twice that of the σ -component.

For a boson, the maximal value of n up to which we succeeded to perform the calculations was $n \sim 10^5$, the corresponding value of b_0 was ~ 1 ((5.17) and (5.23)), thus, in this case $\gamma \sim 10^2$ and $\xi \sim 10^2$. So, according to (1), only quantum theory can give valid descriptions for the peculiarities of the shift to $\nu_{max} = n$. Since the intensity of the magnetic field in a modern synchrotron is much smaller than b_0 or b_1 (the normalized field intensity in a contemporary source is $\sim 10^{-9} - 10^{-8}$), the maximum of radiation stays at some lower harmonics, $\nu_{max} \ll n$. For large β it means that the energy of the emitted photon is much smaller than the energy of the radiating particle. Obviously, the recoil effect here can be neglected and classical theory can give a reasonably good quantitative description of the properties of SR (ξ is small). However, for the space objects such as the neutron stars and Magnetars, b varies from $\sim 10^{-3}$ to $\sim 10^2$, i.e. can exceed the critical value b_0 , and, therefore, it becomes possible for radiation maximum to lie at the highest harmonic. We emphasize here, that the parameter ξ varies from $\sim 10^{-1}$ to $\sim 10^4$, so that one can no longer neglect quantum effects.

Figures 5.17 - 5.20 illustrate the behavior of functions $K_i^e(\zeta, n, \beta)$.

Figure 5.17: The plots of $K_2^e(-1, n, \beta)$ at various n .

Figure 5.18: The plots of $K_3^e(-1, n, \beta)$ at various n .

Figure 5.19: The plots of $K_2^e(1, n, \beta)$ at various n .

Figure 5.20: The plots of $K_3^e(1, n, \beta)$ at various n .

For each fixed n the energy value β , required for the shift of maximum to occur in the spectrum of the σ -component of radiation emitted by an electron with $\zeta = 1$, exceeds that of an electron with $\zeta = -1$. The maximum in the spectrum of the π -component demonstrates opposite tendency, namely, the electron with its initial spin opposite to the external field requires higher energies for the maximum in its radiation spectrum to stay at $\nu = n$. It makes sense due to the fact of polarization components switching with respect to the spin direction at the transitions to the ground state.

The roots of $K_i^e(\zeta, n, \beta) = 1$ for $i = 2, 3$, $\zeta = -1, 1$ can be found in figures 5.21 and 5.22 for small

($< 10^3$) and large ($< 10^6$) n respectively.

Figure 5.21: The roots of $K_i^e(\zeta, n, \beta)$, $i = 2, 3$, $\zeta = \pm 1$, $n < 10^3$.

Figure 5.22: The roots of $K_3^e(1, n, \beta)$ $n < 10^6$.

The difference between the evolution of maximum in the spectrum of the σ - and π -components at $\zeta = \pm 1$ is not noticeable but for small n . For higher numbers of the initial state, the roots of $K_i^e(\zeta, n, \beta) = 1$ and corresponding approximations hardly allow to distinguish the type of polarization as well as the direction of the initial spin. Actually, that was the reason for us not to give but a single plot of $\gamma^2(n)$ for $i = 3$ and $\zeta = 1$ (figure 5.22). To confirm quantitative similarity, we list the approximation coefficients obtained for the coefficients k_i and m_i (as in (5.20) and (5.21))

$$\begin{aligned}
 k_2 &= 1.002787538, \quad m_2 = 0.6698829555, \text{ for } \zeta = -1, \\
 k_2 &= 1.068424627, \quad m_2 = 0.6656028008, \text{ for } \zeta = 1, \\
 k_3 &= 1.034731557, \quad m_3 = 0.6678621590, \text{ for } \zeta = -1, \\
 k_3 &= 1.000923820, \quad m_3 = 0.6700142626, \text{ for } \zeta = 1.
 \end{aligned} \tag{5.24}$$

As was for the total power of radiation, k_i tends to 1 whereas $m_i \rightarrow \frac{2}{3}$.

Conclusion

In the framework of classical theory of SR we studied the radiation associated with two first harmonics of the spectrum. For the total classical spectrum and for its two-harmonic part we analysed radiation characteristics, such as polarization properties, partial contributions of separate harmonics, angular profiles of the distributions of the radiation power.

In the quantum theory, we focused on studying the radiation emitted from weakly excited particles. For both spinor and scalar particles we considered a two-level system and analysed the properties of radiation for the transitions from the first excited to the ground state. In order to formulate and solve the problem about the spectral properties of radiation we considered a three-level system. We studied the characteristics of radiation similar to those that were introduced for a two-harmonic part of classical spectrum, and we performed a detailed comparison of classical and quantum results.

We tracked the evolution of maximum in the quantum spectrum of radiation. The influence of spin on the shift of the position of maximum was treated separately. Special attention was paid to the shifts of maximum with respect to the components of linear polarization.

Numerical and analytical calculations that we performed allowed to establish new and sometimes even unexpected properties of SR, as well as to gather a number of nontrivial assumptions.

Usually, as the comparative analysis between classical and quantum results is performed, the values of radiation parameters which would provide a good similarity between these results are of particular interest. Namely, the question of the representativeness of the classical theory is under investigation. Our research contains an illustration of the fact that the classical characteristics of SR have a better correspondence with their quantum analogues when the reduced classical spectrum is considered. More specifically, if we cut off the harmonics with the numbers greater than n , where n corresponds with the initial energy level in the quantum theory, the representativeness of the classical theory increases significantly.

Along with the conclusions about the relations between the results obtained within classical and quantum approach, it is important to emphasize the properties of the radiation associated with purely quantum effects, which were discovered within our study. We have found an exact analytical proof of the fact that among quantum transitions to the ground state those without spin flip are more likely to occur. Although intuitively this conclusion seems obvious, due to an attempt to prove it analytically we established that the components of linear polarization switch places with respect to the spin orientation. Certain symmetry, though not quite obvious, of the distributions of different components of linear polarization shows up for the transitions to the first excited state as well.

The property of switching places, found for a transversally polarized electron, serves as a proof for the presence of the π -component of radiation in the plane of orbit of an unpolarized electron. We remind the reader that the classical theory predicted that in the plane of motion SR is absolutely linearly polarized, only σ -component is emitted. Besides, as far as classical approach is concerned, this well-known result was claimed to be valid at any energy. Using the methods of quantum theory a zero-contribution of the π -component can be obtained only when $\beta \rightarrow 1$. Moreover, one can distinguish another peculiarity of radiation from an unpolarized electron, namely, whatever the direction of emission the maximum of angular distribution of the σ -component is reached in non-relativistic domain. The only exception is the region near the plane of motion where ultrarelativistic radiation has a notable peak. Neither classical nor quantum theory of a polarized electron do not allow to get any similar conclusion.

As for the effective angles of radiation, one cannot rely on the representativeness of classical theory, which turns out to depend significantly on the type of polarization. For the σ -component of radiation the behavior of effective angles of the total (summed over the spectrum) classical radiation is consistent with that of an unpolarized electron.

Essential is the conclusion that we established for the components of circular polarization of radiation emitted from polarized and unpolarized electrons. Numerical analysis of the distribution profiles confirmed the analytical conclusion about the displacement of the right- and left- components with respect to the plane of orbit in the ultrarelativistic domain. When the velocity of the emitting particle is very close to that of light c , the right-circularly polarized radiation concentrates above the plane of motion. Since the problem is symmetrical, it is clear that the left-circularly polarized radiation is emitted below the plane of orbit when $\beta \rightarrow 1$. Although classical theory also assumes the dominance of the right-circular polarization component in the upper half-plane, it predicts a finite non-zero value for corresponding power below the plane of motion even when β reaches 1.

Using the methods of the quantum theory in order to calculate the power of radiation emitted from weakly excited particles, we established that the effective angles do not vanish in the ultrarelativistic limit. Thus, a common classical conclusion about the concentration of radiation in the plane of motion at $\beta \rightarrow 1$ was disproved. For π -component of radiation there is a tendency to deconcentration as the velocity reaches the ultrarelativistic value.

We considered the evolution of the radiation maximum in the quantum spectrum and found the conditions for it to lie at the highest harmonic of this spectrum. For a scalar particle this condition refers to the intensity of the external field, and the maximum can shift to higher harmonics only if this intensity is higher than a certain critical value. Critical values of field intensity corresponding to different harmonics of the spectrum form a sequence, so that the shifts of maximum are successive as in the classical theory. We established that if the intensity of the external field does not reach a critical value then the position of maximum remains unchanged, no matter what values do the other parameters of radiation have. It turned out that the presence of spin perturbs the result obtained for a scalar particle. The spin influences significantly on the evolution of maximum. When we take spin-effects into account, the position of maximum is defined not only by the intensity of the external field but also by the number of the initial energy level. No linear dependence between the energy squared and the number of initial

state can be observed.

In order to study the properties of SR we developed simple mathematical methods, which allowed to obtain exact analytical and numerical solutions of the problems that were formulated. The properties of radiation that were detected in the framework of current research are supposed to be taken into account at astrophysical observations of cosmic SR from Magnetars, neutron stars, where the intensities of the magnetic field can be of the order of the Schwinger field or even higher. Below we list principal results obtained in the framework of our investigation.

1. In the framework of classical theory of SR we introduced the concept of n -part of the spectrum. For the particular sector of infinite spectrum we studied the structure of the angular profiles of radiation. It was found that, mostly, the results obtained for the reduced classical spectrum stay in better accordance with quantum theory than the characteristics of the total spectrum.
2. We established that the linear polarization components of a transversally polarized electron *switch places* with respect to spin orientation and, therefore, for radiation from unpolarized electron, we proved the emission of the π -component in the plane of orbit.
3. With the methods of the quantum theory we demonstrated that for spinor particles one can observe the concentration of the right circular polarization and the absence of the left circular polarization in the upper half-plane in the ultrarelativistic limit. Due to the symmetry, left circularly polarized radiation concentrates below the plane of motion.
4. For weakly excited particles we disproved the well-known conclusion about the concentration of the total (summed over the spectrum) radiation in the plane of orbit for extremely high velocities. It was shown that the effective angles of radiation from spinless and spinor particle tend to finite non-zero values in the ultrarelativistic domain. This conclusion stays valid both for the total quantum radiation and for its polarization components.
5. We studied the properties of the radiation in quantum three-level systems and performed a quantitative and a qualitative comparison between the classical and quantum characteristics of a short (reduced) spectrum. With the help of the numerical simulations we made an assumption about the representativeness of the classical theory.
6. For a scalar particle we found the condition of the shift of maximum to the highest harmonic of the quantum spectrum. To make this shift possible the external field intensity is to be higher or equal to a critical value (which we estimated numerically). Further more, we studied the influence of spin on the evolution of maximum. For an electron, the critical value of the external field depends on the number of the initial state.

Appendix A

Laguerre Functions

A.1 Equations of Motion and the Laguerre Polynomials

Let us consider an equation of the form [127, p. 781]

$$y''(x) + \frac{1}{x} y'(x) - \left(\frac{A}{x^2} + Bx^2 - C \right) y(x) = 0 . \quad (\text{A.1})$$

Now we change the variable $x \rightarrow t = bx^2$, where b is a constant value which, for reasons of convenience, is supposed to be defined later.

Obviously,

$$\frac{d}{dx} = 2\sqrt{bt} \frac{d}{dt} , \quad \frac{d^2}{dx^2} = 2b \left(\frac{d}{dt} + 2t \frac{d^2}{dt^2} \right) , \quad (\text{A.2})$$

so that the equation (A.1) transforms to

$$t\ddot{y}(t) + \dot{y}(t) - \left(\frac{A}{4t} + \frac{B}{4b^2} t - \frac{C}{4b} \right) y(t) = 0, \quad \dot{y} = \frac{dy}{dt} . \quad (\text{A.3})$$

We assume that the solution of (A.3) has the form

$$y(t) = e^{-\alpha t} t^\beta u(t), \quad (\text{A.4})$$

α, β are constant values. Substituting (A.4) in (A.3) we get

$$t\ddot{u}(t) + (2\beta - 2\alpha t + 1)\dot{u}(t) + \left[\left(\beta^2 - \frac{A}{4} \right) \frac{1}{t} + \left(\alpha^2 - \frac{B}{4b^2} \right) t - 2\alpha\beta - \alpha + \frac{C}{4b} \right] u(t) = 0. \quad (\text{A.5})$$

If we choose α, β and b such that

$$\alpha = \frac{1}{2}, \quad \beta = \frac{\sqrt{A}}{2}, \quad b = \sqrt{B} , \quad (\text{A.6})$$

from (A.5) we easily obtain

$$t\ddot{u}(t) + (\sqrt{A} - t + 1)\dot{u}(t) + \left(\frac{C}{4\sqrt{B}} - \frac{\sqrt{A}}{2} - \frac{1}{2} \right) u(t) = 0 . \quad (\text{A.7})$$

This equation is nothing other than the Laguerre equation [129, p. 284]

$$t\ddot{u} + (a + 1 - t)\dot{u} + mu = 0, \quad u = u(t), \quad m = 0, 1, 2, \dots , \quad (\text{A.8})$$

$$a = \sqrt{A}, \quad m = \left(\frac{C}{4\sqrt{B}} - \frac{\sqrt{A}}{2} - \frac{1}{2} \right) , \quad (\text{A.9})$$

and its solutions are the generalized Laguerre polynomials (see [130, p. 108] or [131])

$$L_m^a(t) = \frac{1}{m!} t^{-a} e^t \frac{d^m}{dt^m} (e^{-t} t^{m+a}) . \quad (\text{A.10})$$

Finally, taking (A.4) and (A.6) into account we find the solution of (A.1)

$$y(x) = \tilde{C} e^{-\frac{(\sqrt{B}x^2)}{2}} (\sqrt{B}x^2)^{\frac{(p-m)}{2}} L_m^{p-m}(\sqrt{B}x^2), \quad (\text{A.11})$$

$$p = a + m = \frac{C}{4\sqrt{B}} + \frac{\sqrt{A}}{2} - \frac{1}{2}, \quad p = 0, 1, 2, \dots, \quad (\text{A.12})$$

$\tilde{C}$ is some constant value. Since (A.1) doesn't give any restrictions for $\tilde{C}$, one can choose $\tilde{C} = \sqrt{\frac{m!}{p!}} \tilde{c}$ and, thus, introduce so-called Laguerre functions $I_{p,m}(t)$

$$I_{p,m}(t) = \sqrt{\frac{m!}{p!}} e^{-t/2} t^{(p-m)/2} L_m^{p-m}(t), \quad (\text{A.13})$$

we can write the solutions in the form

$$y(x) = \tilde{c} I_{p,m}(\sqrt{B}x^2), \quad (\text{A.14})$$

where m and p are defined by the expressions (A.9) and (A.12).

A.1.1 Particular Properties of the Laguerre Functions

One of the features of generalized Laguerre polynomials is their orthogonality with respect to $t^a e^{-t}$ [132, p. 726]

$$\int_0^\infty t^a e^{-t} L_m^a(t) L_{m'}^a(t) dt = \frac{\Gamma(m+a+1)}{m!} \delta_{m,m'}. \quad (\text{A.15})$$

So, for Laguerre functions (A.13) at $p-m = p'-m'$ we can get

$$\int_0^\infty I_{p',m'}(\sqrt{B}x^2) I_{p,m}(\sqrt{B}x^2) x dx = \frac{1}{2\sqrt{B}} \delta_{m,m'}. \quad (\text{A.16})$$

Furthermore, with the use of (A.10) right from (A.13) one can obtain

$$\begin{aligned} I_{p+1,m}(t) &= \sqrt{\frac{t}{p+1}} \left[\frac{t+p-m}{2t} I_{p,m}(t) - I'_{p,m}(t) \right], \\ I_{p-1,m}(t) &= \sqrt{\frac{t}{p}} \left[\frac{t+p-m}{2t} I_{p,m}(t) + I'_{p,m}(t) \right]. \end{aligned} \quad (\text{A.17})$$

Also, Laguerre functions obey the following equality [25, p. 134]

$$\sum_{m=0}^\infty I_{p,m}(t) I_{p',m}(t) = \delta_{p',p}. \quad (\text{A.18})$$

Appendix B

Effective angles

All the numerical simulations in this work have been performed with the use of Wolfram Mathematica and Maple packages. Here we give an example of the code used to calculate the effective angle and the deviation angle.

```
xb[b_,  $\theta$ _] := (Sqrt[3] - Sqrt[3 - 2*b^2*(Sin[ $\theta$ ])^2]) /  
  (Sqrt[3] + Sqrt[3 - 2*b^2*(Sin[ $\theta$ ])^2])  
x0b[b_] := (Sqrt[3] - Sqrt[3 - 2*b^2]) / (Sqrt[3] + Sqrt[3 - 2*b^2])  
Ab[b_,  $\theta$ _] := Exp[-xb[b,  $\theta$ ]] * (1 + xb[b,  $\theta$ ])^3  
Fb3[b_,  $\theta$ _] := Ab[b,  $\theta$ ] * (1 + xb[b,  $\theta$ ])^2 * (Cos[ $\theta$ ])^2 / (1 - xb[b,  $\theta$ ])  
myIntb[b_?NumberQ, t1b_?NumberQ, t2b_?NumberQ] :=  
  NIntegrate[Fb3[b, t] * Sin[t], {t, t1b, t2b}, WorkingPrecision -> 20]  
impfunc1b[b_?NumberQ] :=  
  t1b /. FindRoot[{myIntb[b, t1b, t2b] == myIntb[b, 0, N[Pi, 25] / 2] / 2,  
    Fb3[b, t1b] * Sin[t1b] == Fb3[b, t2b] * Sin[t2b]},  
    {{t1b, N[Pi, 25] / 8}, {t2b, N[Pi, 25] / 4}}, WorkingPrecision -> 20]  
impfunc2b[b_?NumberQ] :=  
  t2b /. FindRoot[{myIntb[b, t1b, t2b] == myIntb[b, 0, N[Pi, 25] / 2] / 2,  
    Fb3[b, t1b] * Sin[t1b] == Fb3[b, t2b] * Sin[t2b]},  
    {{t1b, N[Pi, 25] / 8}, {t2b, N[Pi, 25] / 4}}, WorkingPrecision -> 20]  
 $\Delta b[b_] := 2 * (impfunc2b[b] - impfunc1b[b])$   
  
Table[{N[b, 2], N[impfunc2b[b], 8]}, {b, 0, 1, 1/100}]  
  
ListLinePlot[Table[{N[b, 2], N[impfunc2b[b], 8]}, {b, 0, 1, 1/100}],  
  PlotRange -> All, Frame -> True, FrameLabel -> { $\beta$ }, GridLines -> Automatic,  
  GridLinesStyle -> Directive[LightGray], PlotStyle -> {Thick},  
  LabelStyle -> Directive[Bold], DataRange -> {0, 1}, ImageSize -> 500]  
  
Table[{b, N[ $\Delta b[b]$ , 8]}, {b, 0., 1., 1/100}]  
  
ListLinePlot[Table[{b,  $\Delta b[b]$ }, {b, 0., 1., 1/100}], PlotRange -> All,  
  Frame -> True, FrameLabel -> { $\beta$ }, GridLines -> Automatic,  
  GridLinesStyle -> Directive[LightGray], PlotStyle -> {Thick},  
  LabelStyle -> Directive[Bold], DataRange -> {0, 1}, ImageSize -> 500]
```


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