Universidade de São Paulo
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Princípios de Grandes Desvios para a Condutividade Microscópica de Férmions em Cristais

Nelson Javier Buitrago Aza

Orientador: Prof. Dr. Walter Alberto de Siqueira Pedra

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Banca Examinadora:

Prof. Dr. Domingos Humberto Urbano Marchetti (IFUSP)

Prof. Dr. João Carlos Alves Barata (IFUSP)

Prof. Dr. Paulo Teotônio Sobrinho (IFUSP)

Prof. Dr. Marco Merkli (Memorial University of Newfoundland)

Prof. Dr. Pedro Lauridsen Ribeiro (UFABC)

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Large Deviation Principles for the Microscopic Conductivity of Fermions in Crystals

Nelson Javier Buitrago Aza

Supervisor: Prof. Dr. Walter Alberto de Siqueira Pedra

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the degree of Doctor of Science.

Examining Committee:

Prof. Dr. Domingos Humberto Urbano Marchetti (IFUSP)

Prof. Dr. João Carlos Alves Barata (IFUSP)

Prof. Dr. Paulo Teotônio Sobrinho (IFUSP)

Prof. Dr. Marco Merkli (Memorial University of Newfoundland)

Prof. Dr. Pedro Lauridsen Ribeiro (UFABC)

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ABSTRACT

This Thesis deals with the existence of Large Deviation Principles (LDP) in the scope of fermionic systems at equilibrium. The physical motivation beyond our studies are experimental measures of electric resistance of nanowires in silicon doped with phosphorus atoms. The latter demonstrate that quantum effects on charge transport almost disappear for nanowires of lengths larger than a few nanometers, even at very low temperature (4.2°K). In order to mathematically prove the latter, we divide our work in several steps:

1. In the first step, for non-interacting lattice fermions with disorder, we show that quantum uncertainty of microscopic electric current density around their (classical) macroscopic values is suppressed, exponentially fast with respect to the volume of the region of the lattice where an external electric field is applied. Disorder is modeled by a random external potential along with random, complex-valued, hopping amplitudes. The celebrated tight-binding Anderson model is one particular example of the general case considered here. Our mathematical analysis is based on Combes–Thomas estimates, the Akcoglu–Krengel ergodic theorem, and the large deviation formalism, in particular the Gärtner–Ellis theorem.
2. Secondly, we prove that for weakly interacting fermions on the lattice, the logarithm moment generating function $J(s), s \in \mathbb{R}$ of probability distributions associated with KMS states can be written as the limit of logarithms of Gaussian Berezin integrals. The covariances of the Gaussian integrals are shown to have a uniform determinant bound (via Hölder inequalities for Schatten norms) and to be summable in general cases of interest, including systems that are not translation invariant.
3. In the third step we analyze expansions of logarithms of Gaussian Berezin integrals, which combined with constructive methods of quantum field theory is useful to show the analyticity of $J(s)$ for s in a neighborhood of 0.

We finally discuss how to combine steps 2–3 in order to prove (mathematically speaking) for interacting fermions in equilibrium the experimental results above mentioned. In fact, the found results in this Thesis generalize previous works in the scope of LDP used to study quantum systems.

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RESUMO

Esta tese trata a existência de Princípios de Grandes Desvios (PGD), no âmbito de sistemas fermiônicos em equilíbrio. A motivação física detrás de nossos estudos são medidas experimentais de resistência elétrica de nanofios de silício dopados com átomos de fósforo. Estas medidas mostram que efeitos quânticos no transporte de carga elétrica quase desaparecem para nanofios de comprimentos maiores que alguns nanômetros, mesmo para temperaturas muito baixas (4.2°K). A fim de provar matematicamente tal efeito, dividimos nosso trabalho em diversos passos:

1. No primeiro passo, para férmions não interagentes numa rede com desordem, mostramos que a incerteza quântica da densidade da corrente elétrica microscópica, em torno de seus valores macroscópicos (clássicos), é suprimida exponencialmente rápido em relação ao volume da região da rede onde um campo elétrico externo é aplicado. A desordem é modelada como um potencial elétrico aleatório juntamente com amplitudes aleatórias de saltos com valores complexos. O célebre modelo de Anderson de *tight-binding* é um exemplo particular do caso geral considerado aqui. Nossa análise matemática é baseada em estimativas de Combes–Thomas, o Teorema Ergódico de Akcoglu–Krengel e no formalismo de Grandes Desvios, em particular o Teorema de Gärtner–Ellis.
2. Em segundo lugar, provamos que, para férmions interagindo fracamente na rede, as funções geradoras $J(s)$, $s \in \mathbb{R}$ de cumulantes de distribuições de probabilidades associadas com estados KMS pode ser escrito como o limite de logaritmos de integrais gaussianas de Berezin. Mostramos que os determinantes das covariâncias associadas às integrais gaussianas são majorados uniformemente (via desigualdades de Hölder para normas Schatten). Tais covariâncias são também somáveis, em casos gerais de interesse, incluindo assim, sistemas que não são invariantes por translação.
3. No terceiro passo, analisamos expansões de logaritmos de integrais gaussianas de Berezin, e assim combinando com métodos construtivos de teoria quântica de campos, mostramos a analiticidade de $J(s)$ para s nas vizinhanças de 0.

Finalmente, discutimos como combinar os passos 2–3, a fim de provar (matematicamente falando) os resultados experimentais mencionados acima para férmions interagindo em equilíbrio. De fato, os resultados encontrados nesta tese, generalizam trabalhos prévios no âmbito do PGD usado para o estudo de sistemas quânticos.

Numero pondere et mensura Deus omnia condidit.
Sir Isaac Newton

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I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Sir Isaac Newton

IN 1900, *P. Drude* (1863–1906) proposed the first model to describe electrical conduction in metals [AM76]. In order to maintain electrical neutrality, he assumed that atoms were composed by *positively-charged ions* and *negatively-charged free electrons*. Ions were considered to be static while the free electrons could move *freely* in the metal. Thus, at *constant* temperature $T > 0$ each one of these electrons move in straight line until it gets *scattered* by some ion in a *random* direction with a *average time* between collisions τ . Moreover, other kind of interactions, e.g., electron–electron and electron–ion, were not considered. Surprisingly, these basic assumptions lead to outstanding achievements of physical predictions when compared with experimental data and *good* descriptions of physical behaviors.

More precisely, assuming $\tau \in \mathbb{R}^+$ constant, Drude’s model (DM) is in accordance with AC, DC and *thermal* conductivity properties, e.g., the *Wiedemann–Franz law*. Recall that AC conductivities are originated by potentials $V(t) \in C_0^{(\infty)}(\mathbb{R}; \mathbb{R})$ dependent of time, and typically are described by a periodic function while in the DC case the potentials are constant. Furthermore, Wiedemann–Franz law gives a ratio of the thermal to electrical conductivity, and in 1905 *H. A. Lorentz* (1853–1928) used DM to show the linear dependency with the temperature T of this ratio.

Although DM work relatively well for AC and DC–conductivities for T constant, the coefficient of proportionality, $L \in \mathbb{R}^+$, in the Wiedemann–Franz law was around half of the actual value. Such a discrepancy was consequence of the apparent classical behavior of electrons where it was reasonable to try its statistics via *Boltzmann* distribution. At the time of the Drude–Lorentz model the *fermionic* nature of electrons it was not known, thus the now well-known *Pauli Exclusion Principle* of fermions was not taken into account in the calculations. It was not until 1933 that *A. J. W. Sommerfeld* (1868–1951) did an important adjustment of DM. He used the *Fermi–Dirac* statistics for non–interacting fermions. With this, it was significantly improved

thermal quantities, in particular the value of L . From now on, we restrict this work just to deal with electric conductivities in *conductors*, whereas *semiconductors* and *superconductors* are not taken into account. In a similar way, heat conductivities are not the purpose of this Thesis, and hence in the sequel we will not mention these.

As remarked in [AM76, Chapter 3], there are considerable problems in the Drude–Lorentz–Sommerfeld model. The usual way to tackle these is to consider the *spatial configuration* of metals as periodic arrays of atoms or *crystal lattices*. In fact, experimental evidence, via x -ray diffraction, demonstrates that metals have crystalline structure. Such a regularity is exploited to model systems in *solid state*. From a mathematical point of view, there are a few of rigorous proofs on the existence of crystals, e.g., in the *Falicov–Kimball model*, Kennedy and Lieb shown that there is crystallization for high enough temperatures while in low temperatures such a regularity is broken down [KL86]. In this work we will assume the existence of crystals by physical assumptions. Thus, we will assume that atoms or ions shape a periodic structure in the crystal and the conduction electrons can move on this [Sal13].

We now quote [Son52]

... it must be admitted that there is no entirely rigorous quantum theory of conductivity. . .

See, e.g., [KLM07, KM08, KM14, BC13, BPH14a, BPH15, BPH16, BPH14b, BP14a, BP15, Wag13, DG11] for examples of mathematically rigorous derivations of linear conductivity from first principles of quantum mechanics and thermodynamics. As we shall formulate in the following sections, the main purpose of the present work is to combine the mentioned works with *Large Deviation* (LD) techniques. This, in order to mathematically to prove an exponential suppression of the quantum uncertainty of microscopic electric current densities (around their classical, macroscopic values), a result found experimentally a few years ago [WMR⁺12, Fer12].

1.1 CLASSICAL OHM’S LAW AND BEYOND

It is widely accepted that from a classical point of view, electrons in a *piece* of a *conductor material* obey both *Newton’s Laws* and *Maxwell equations*. In practice, when a *potential difference* $V \in \mathbb{R}$ is applied between the ends of the material (commonly a wire connected to a battery), electrons accelerate and a *flow* of electrons or *current* is created. In the DC regime, G. S. Ohm (1789–1854) based in experimental outcomes established that the current $I \in \mathbb{R}$ in a test material is proportional to the potential difference $V \in \mathbb{R}$. In physics, this expression known as *Ohm’s Law* has the simple form: $V = IR$, where $R \in \mathbb{R}^+$ is the *resistance* of the test material, and in general it is non uniform on this. In circuits, a similar expression can be found for the AC case but here, the resistance is reemplaced by the *impedance* in the particular system. Moreover, in absence of magnetic fields, G. R. Kirchhoff (1824–1887) shown that at time $t \in \mathbb{R}$ and $x \in \mathcal{R}$, the *density current* $\mathbb{J}(t, x) \in \mathbb{R}^d$ and the *electric field* $E(t, x) \in \mathbb{R}^d$ have a similar relation, namely:

$\mathbb{J}(t, x) = \sigma E(t, x)$, where $\sigma \in \mathbb{R}^+$ is the *conductivity* of the material. Here, $\mathcal{R} \subset \mathbb{R}^d$ is the region where the material is embedded. When last expression is valid we say that the material is *Ohmic*. Thus, supposing that if a uniform current through a *wire* $\mathcal{W}$ of length ℓ , the resistance of the material is given by the formula:

$$(1.1.1) \quad R = \frac{\ell}{\sigma A},$$

where A is the cross-sectional of $\mathcal{W}$, and then R is linearly proportional to ℓ . Furthermore, it is well-known that the *resistivity* $\rho = 1/\sigma$ of the material depends on the temperature within which this is present. For room temperatures such a behaviour is linear while in low temperature it is not, and by the above formula the same is expected for the resistance.

As in the DM, the success of Ohm's law comes from the fact that a few assumptions are required to obtain very good descriptions when compared with experimental tests. Thus, in first instance Ohm's law is thought to describe *macroscopic* conductivities since *quantum* effects were not taken into account. For example, to obtain expression (1.1.1) it was supposed that electrons move uniformly through cross-sectional A the distance ℓ , and neither interactions between them nor *imperfections/impurities* of the material were considered.

In 1957, R. W. Landauer (1927–1999) found a theoretical expression for resistances in *quantum conductors* at low temperatures [Lan57]. Basically, he shown that the *conductance* was *quantized* in a factor $2e^2/h$, with e the electrical charge and h the *Planck's* constant. This *universality* shows an independence of the resistance with the size of the wire. Such a result is explained by interference effects in the *mesoscopic* regime [Kit04]:

... The elastic scattering length ℓ_e , is less than the sample dimensions, but the phase coherence length ℓ_ϕ , is greater. Electrons therefore propagate diffusively, but phase-coherently, through the sample...

As remarked in [Kit04], Landauer's formula was experimentally verified and is used to study quantum transport in nanostructures.

Thus, it seems that Ohm's law also fail in the *microscopic* regime. Nevertheless, recently it was experimentally proved that this is valid for *nanowires*. A detailed explanation of these results can be found, for instance, in "Ohm's Law in a Quantum World" by D. K. Ferry, and "Ohm's Law Survives to the Atomic Scale" by B. Weber et. al., published in Science Magazine at 2012 [WMR⁺12, Fer12]. Quoting [Fer12]:

... In the 1920s and 1930s, it was expected that classical behavior would operate at macroscopic scales but would break down at the microscopic scale, where it would be replaced by the new quantum mechanics. The pointlike electron motion of the classical world would be replaced by the spread out quantum waves. These quantum waves would lead to very different behavior.

... Ohm's law remains valid, even at very low temperatures, a surprising result that reveals classical behavior in the quantum regime.

D. K. Ferry, 2012

Indeed, in [WMP⁺12] they embedded phosphorus atoms in silicon crystals, creating nanowires satisfying a linear dependency w.r.t. the size of the wire where a potential difference V was applied. This discovery about Ohm's law leads to believe that electronic conduction in microscopic scales closely resembles conduction on macroscopic scales. A physical interpretation of this is that quantum coherence associated to microscopic currents decays *very fast* w.r.t. the size of the region where V is applied. Hence, it is natural to expect that in the *thermodynamic limit* microscopic *measures* converge *very fast* to their respective macroscopic values. In other words, the quantum uncertainty of microscopic electric current around their (classical) macroscopic values is suppressed, *very fast* w.r.t. the region of the lattice where V is applied. More precisely, we *informally* formulate the following:

CONJECTURE 1. Let $\mathcal{R} \subset \mathbb{R}^d$ be a finite region where we do quantum measurements of current and let $\mu_{\mathcal{R}}, \mu_{\mathbb{R}^d}$ be the corresponding measures in the microscopic and macroscopic regimes. We conjecture a fast convergence of $\mu_{\mathcal{R}}$ to $\mu_{\mathbb{R}^d}$ as $\mathcal{R}$ fills $\mathbb{R}^d$. ◇

1.2 FORMULATION OF THE PROBLEM

According to “Joule's law”, electrical resistance of a conductive material in presence of electric fields can also be quantified in terms of *heat production*. In fact, [Ara76, Ara77] Araki introduced the notion of *relative entropy* in the context of *algebraic statistical quantum mechanics* to study *equilibrium states of lattice systems*. Recently Bru and Pedra [BPH14a, BP14a, BP14b, BPH14b, BP15, BPH15, BPH16] use relative entropies to analyse the *entropy production* and hence the heat production of *infinite fermion systems*¹:

Indeed, from first principles of thermodynamics, it was studied the entropy production of a C^* -dynamical system at equilibrium, obtained from a CAR algebra, which implements the Pauli Exclusion Principle. This system represents, in a convenient form, the presence of *disorder* where the conductive material is embedded (due to: impurities, crystal lattice defects, etc.) and/or the interaction between the charge carriers and also to the presence of an external electric field $E_A(t, x)$ applied in $t \in [t_0, t_1], x \in \mathcal{R} \subset \mathbb{R}^d$. Here, $t_0, t_1 \in \mathbb{R}$ and $\mathcal{R}$ is a *convex* finite region such that in other regions and time intervals, $E_A(t, x)$ is considered to be null. $\mathbf{A}(t, x)$ is a compactly

¹For a brief historical review of the importance of thermodynamic in electrical conductivity, see [BP14b].

supported, in space and time, vector potential in such a way that we use the *Weyl gauge*. Due to the time-dependency of the electric field, the underlying physical system is non-autonomous. Note that, because the form of the electric field, the relative entropy of the system at time $t = t_1$ with respect to (w.r.t.) its initial state (at $t = t_0$) is always well defined and finite [JP03, OP04].

More precisely, assuming that the system is at equilibrium at fixed temperature $T > 0$ before the external electric field is applied, the entropy production is always a non-negative quantity. As stressed in [WS78], this positivity property is directly related to the “passivity” of the *KMS states*. Moreover, as defined in [BPH14a, BP14a], the heat production is directly proportional to the entropy production, implying its non-negativity. Since the heat production is correctly described by the Ohm’s and Joule’s Laws, the conductivity $\Xi_{\mathcal{R}}(t)$ in the region $\mathcal{R} \subset \mathbb{R}^d$ is a *positive definite function*. Recall that $\mathcal{R} \subset \mathbb{R}^d$ is a convex finite region where we apply, $E_A(t, x)$, between $t = t_1$ and $t = t_0$. In particular, in [BPH15] it was proven that imposing the *AC-condition*, namely,

$$\int_{t_0}^{t_1} E_A(s, x) ds = 0, \quad x \in \mathbb{R}^d$$

it is possible to show, for *cyclic processes* on equilibrium states, that the total heat production per unity of volume is given by

$$(1.2.1) \quad Q_{\mathcal{R}}(\mathcal{E}) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle \vec{w}, \Xi_{\mathcal{R}}(s_1 - s_2) \vec{w} \rangle \mathcal{E}_{s_2} \mathcal{E}_{s_1} ds_2 ds_1 \geq 0.$$

Here, $\vec{w} \in \mathbb{R}^d$ is a normalized vector, i.e., $\|\vec{w}\| = 1$ and $\mathcal{E} \in C_0^\infty(\mathbb{R}; \mathbb{R})$, the temporal dependency of the external electric field. At this point, by convenience, we introduce the *current density*, in direction of $\vec{w} \in \mathbb{R}^d$, $\mathbb{I}_{\mathcal{R}}^{(\mathcal{E})}$ by

$$(1.2.2) \quad \mathbb{I}_{\mathcal{R}}^{(\mathcal{E})} \doteq \sum_{k,q=1}^d w_k \int_{-\infty}^0 \{\mathcal{E}(\alpha)\}_q \{\mathcal{C}_{\mathcal{R}}(-\alpha)\}_{k,q} d\alpha,$$

where $\mathcal{C}_{\mathcal{R}} \in C^1(\mathbb{R}; \mathbb{R}^d)$ is the space-averaged transport coefficient observable (see discussions around Expression (2.3.74) for details). Note that the current density $\mathbb{I}_{\mathcal{R}}^{(\mathcal{E})}$ is an observable and hence it is natural to study the quantum fluctuations associated to this. Expression (1.2.1) combined with *Bochner’s theorem* [RS75, Sim15], implies that $\Xi_{\mathcal{R}}(t)$ is the *Fourier transform* of a positive measure $\mu_{AC, \mathcal{R}}$, denoted *in-phase microscopic AC-conductivity measure* such that

$$(1.2.3) \quad Q_{\mathcal{R}}(\mathcal{E}) = \frac{1}{2} \int_{\mathbb{R} \setminus 0} |\hat{\mathcal{E}}_\nu|^2 \mu_{AC, \mathcal{R}}(d\nu),$$

where $\hat{\mathcal{E}}$ is the Fourier transform of $\mathcal{E}$. Expression (1.2.3) describes the relation between the electric resistance and the heat production, which it is valid even for interacting systems and in presence of disorder. Note that such a conclusion corresponds to the well-known *empirical*

Joule’s law in the AC–regime. Thus, the measure $\mu_{AC,\mathcal{R}}(d\nu)$ characterizes the conductivity in the region $\mathcal{R}$, w.r.t. an angular frequency ν in the temporal part of $E_A(t, x)$, and for this reason it is named “AC”–conductivity measure. In [KLM07, KM08], A. Klein, O. Lenoble and P. Müller use a time–dependent spatially homogeneous electric field that is adiabatically switched on, in order to study for first time the concept of conductivity measure. Their scope was restricted to non–interacting fermion systems embedded at cubic crystals in presence of disorder.

Thus as Joule’s Law, in the set of works [BPH14a, BP14a, BP14b, BPH14b, BP15, BPH15, BPH16], the Ohm’s Law in quantum scales it was also proven. This is in accordance with Weber’s et al. [WMR⁺12] results, described above. Moreover, based on the second law of thermodynamics, in [BP15] the existence of an *in–phase macroscopic AC–conductivity measure* μ_{AC} was shown, according with classical electrodynamics.

Just as in [BPH15], for the sake of technical simplicity and without loss of generality (w.l.o.g.), we only consider from now on an increasing sequence $\{\Lambda_L\}_{L \in \mathbb{R}_0^+}$ of boxes on the crystal lattice $\mathfrak{Q} \doteq \mathbb{Z}^d$, $d \in \mathbb{N}$, defined by

$$(1.2.4) \quad \Lambda_L \doteq \{(x_1, \dots, x_d) \in \mathfrak{Q} : |x_1|, \dots, |x_d| \leq L\} \in \mathcal{P}_f(\mathfrak{Q}),$$

for $L \in \mathbb{R}_0^+$, instead of general convex regions $\mathcal{R}$ where the electric field is non–vanishing. In (1.2.4), $\mathcal{P}_f(\mathfrak{Q}) \subset 2^{\mathfrak{Q}}$ is the set of all finite subsets of $\mathfrak{Q}$. Thus, for instance we could denote by $\{\mu_{\Lambda_L}\}_{L \in \mathbb{R}^+}$, the *finite* family of microscopic (AC) conductivity measures. Note that despite the abundance of studies regarding $\{\mu_{\Lambda_L}\}_{L \in \mathbb{R}^+}$, it is still pending a rigorous proof of its fast convergence to μ_{AC} as $L \rightarrow \infty$ [BPH15, BPH14b]. As stressing in [ABPR17], the latter is closely related to show a fast convergence of the current distribution $\{\mathfrak{m}_{\Lambda_L}\}_{L \in \mathbb{R}^+}$ associated to the current density observable (1.2.2). This is the main motivation of this Thesis, where we show an *exponential* convergence in the thermodynamic limit of the family of measures $\{\mathfrak{m}_{\Lambda_L}\}_{L \in \mathbb{R}^+}$, according with [WMR⁺12, Fer12]. In fact, from a mathematical point of view, $\{\mathfrak{m}_{\Lambda_L}\}_{L \in \mathbb{R}^+}$ quantifies the probability of deviations, due to quantum uncertainty, from the expectation value of $\mathbb{I}_{\Lambda_L}^{(\mathcal{E})}$. We notice that since $\{\mathfrak{m}_{\Lambda_L}\}_{L \in \mathbb{R}^+}$ is a family of probability measures, it is natural to associate the measurable space $(\mathbb{R}, \Sigma)$, to this family of probability measures, where Σ is the *Borel σ –algebra* of $\mathbb{R}$.

In *probability theory*, for a family of measures, a natural mathematical framework to define its “very fast” convergence, is the *Large Deviation Theory* (LDT). In such a context, when a family of measures obeys a *Large Deviation Principle* (LDP), the convergence is exponential w.r.t. a natural parameter indexing the elements of a family of measures with a well–defined *rate function* I . In the case of a family of current distribution $\{\mathfrak{m}_{\Lambda_L}\}_{L \in \mathbb{R}^+}$ and other related measures, such parameter would be related to the radius of the region where the external electric field acts.

We now reformulate Conjecture 1 as:

CONJECTURE 2. Consider an interacting fermion system at equilibrium, which is in presence of disorder and is subjected to an external electric field. Let $\{\mathfrak{m}_{\Lambda_L}\}_{L \in \mathbb{R}^+}$ be a family of current distributions. Then, the quantum uncertainty of microscopic electric current densities is suppressed, exponentially fast w.r.t. the region Λ_L , i.e., $\{\mathfrak{m}_{\Lambda_L}\}_{L \in \mathbb{R}^+}$ satisfies an LDP as $L \rightarrow \infty$. $\diamond$

1.3 RESULTS

Before presenting our results concerning to this Thesis, we briefly explain as an LDP is formulated in the fermionic setting and for general details see the subsequent chapters. Let $\mathcal{H}$ be a one-particle Hilbert space and let $\mathcal{U} \equiv \mathcal{U}(\mathcal{H})$ be the CAR C^* -algebra generated by the identity $\mathbb{1}$ and $\{a(\psi)\}_{\psi \in \mathcal{H}}$ satisfying for $\psi, \varphi \in \mathcal{H}$,

$$(1.3.1) \quad \{a(\psi), a(\varphi)\} = 0 \quad \text{and} \quad \{a(\psi), a(\varphi)^*\} = \langle \psi, \varphi \rangle_{\mathcal{H}} \mathbb{1},$$

where as is usual, for $A, B \in \mathcal{U}$, $\{A, B\} \doteq AB + BA \in \mathcal{U}$ denotes the *anticommutator* and $a(\psi)$ and $a(\psi)^*$ are respectively, the annihilation and creation operators. We notice that a combination of Riesz–Markov theorem and functional calculus permit to claim that for any self-adjoint operator $A \in \mathcal{U}$ and any state $\rho \in \mathcal{U}^*$, there is a unique probability measure $\mathfrak{m}_{\rho, A}$ on $\mathbb{R}$ such that

$$(1.3.2) \quad \mathfrak{m}_{\rho, A}(\text{spec}(A)) = 1 \quad \text{and} \quad \rho(f(A)) = \int_{\mathbb{R}} f(x) \mathfrak{m}_{\rho, A}(dx)$$

for all complex-valued continuous functions $f \in C(\mathbb{R}; \mathbb{C})$. $\mathfrak{m}_{\rho, A}$ is called the *distribution* of the observable A in the state ρ . Thus, for $s \in \mathbb{R}$, finite logarithmic moment generating functions are defined as

$$(1.3.3) \quad J_{\Lambda_L}(s) \doteq \frac{1}{|\Lambda_L|} \ln \rho(e^{s|\Lambda_L|A_L}),$$

where $\{A_L\}_{L \in \mathbb{R}} \subset \mathcal{U}$ denotes a family of self-adjoint operators. The importance of Expressions (1.3.2)–(1.3.3) lies in that we can write an LDP for $\{A_L\}_{L \in \mathbb{R}}$ for any state $\rho \in \mathcal{U}^*$. Indeed, if $A_L \in \mathcal{U}_L, L \in \mathbb{R}^+$ is an element of the local C^* -algebra defined by the sequence of boxes of (1.2.4) and I is the rate function associated to $\{A_L\}_{L \in \mathbb{R}^+} \subset \mathcal{U}$, we have for any borel subset $\mathcal{G}$ of $\mathbb{R}$,

$$(1.3.4) \quad -\inf_{x \in \mathcal{G}} I(x) \leq \liminf_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln \mathfrak{m}_{\rho, A_L}(\mathcal{G}) \leq \limsup_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln \mathfrak{m}_{\rho, A_L}(\mathcal{G}) \leq -\inf_{x \in \mathcal{G}} I(x).$$

Here, $\mathcal{G}$ is the interior of $\mathcal{G}$, while $\bar{\mathcal{G}}$ is its closure. Gärtner–Ellis Theorem provides us with sufficient conditions to verify (1.3.4). Note that a first step to obtain an LDP is showing the existence of the limit $J(s) = \lim_{L \rightarrow \infty} J_{\Lambda_L}(s)$, in such case, the rate function I is the *Legendre–Fenchel* transform of J . In addition, in the last case, it is common to obtain the *upper* bond, and we say that an upper LDP exists. Moreover, if the three inequalities in (1.3.4) hold true we will say there is a *full* LDP. However, for that purpose we need to make use of different mathematic techniques which will depend on the underlying system into consideration.

Remarks 1.3.5.

1. Usually, the self-adjoint operators $\{A_L\}_{L \in \mathbb{R}^+}$ are taken in expression (1.3.3) as an average over interactions $\{\Phi_{\Lambda_L}\}_{L \in \mathbb{R}^+} \in \mathcal{U}^+ \cap \mathcal{U}$ inside the region Λ_L . Here, $\mathcal{U}^+$ denotes the *even* elements of $\mathcal{U}$.
2. Three technical motivations for studying LDP here are: (i) to tackle disordered system, (ii) to study an LDP for self-adjoint operators $\{A_{L,t}\}_{L \in \mathbb{R}^+, t \in \mathbb{R}} \in \mathcal{U}$ evolving at time $t \in \mathbb{R}$, and (iii) to deal with commutators of interactions, i.e., $\{i[\Phi_{\Lambda_{L_1}}, \Phi_{\Lambda_{L_2}}]\}_{\Lambda_{L_1}, \Lambda_{L_2} \subset \Lambda_L}$, for $\Phi_{\Lambda_{L_1}}, \Phi_{\Lambda_{L_2}} \in \mathcal{U}^+ \cap \mathcal{U}$. In fact, a few researchers have addressed the studies of Large Deviation Theory for (i)–(iii). *

The final objective of this Thesis is to prove Conjecture 2 as general as possible. In order to tackle this we divide the present work in a series of Chapters and well-known supporting results in Appendices. Chapter 2 is devoted to introduce the mathematical concepts and notations. In the sequel we describe the main results of the works [ABPR17] and [ABPM17, ABPM16] in Chapters 3, 4 and 5 respectively. Chapter 3 refers to the existence of a Large Deviation Principle in the free fermion case whereas Chapters 4 and 5 are devoted to the existence and analyticity of the logarithmic moment generating function in the scope of weakly interacting fermion systems at equilibrium. Finally, we mention how to tackle the general prove of Conjecture 2 for the interacting case.

In the what follows we briefly describe the importance of the works [ABPR17, ABPM17, ABPM16] as well as their general results. Note that in Chapters 3 and 4 we expose some results and proofs regarding these. In contrast with [ABPM17], in Chapter 4 we describe *all* the formulation without use the *self-dual* formalism, which is presented in Appendix C.

1.3.1 ACCURACY OF CLASSICAL CONDUCTIVITY THEORY AT ATOMIC SCALES FOR FREE FERMIONS IN DISORDERED MEDIA

We model disorder in the system coming from a measure space $(\Omega, \mathfrak{A}_\Omega, \alpha_\Omega)$. Here, Ω is a set containing information on random external potential in the lattice *sites*, with values on $[-1, 1]$, and the random *hopping* in lattice *edges*, with values on the *complex* unit disc. $\mathfrak{A}_\Omega$ is the σ -algebra generated by Ω and α_Ω an *ergodic* probability measure.

For the crystal lattice $\mathfrak{Q} \doteq \mathbb{Z}^d$, with $d \in \mathbb{N}$, let $\mathcal{H} \doteq \ell^2(\mathfrak{Q})$ be the one-particle Hilbert space with canonical orthonormal basis $\{e_x\}_{x \in \mathfrak{Q}}$, such that

$$(1.3.6) \quad e_x(y) \doteq \delta_{x,y}, \quad \text{for } x, y \in \mathfrak{Q},$$

where δ is the Kronecker delta. Similar to the Anderson model, the random *tight-binding* Hamiltonian $h^{(\omega)} \in \mathcal{B}(\mathcal{H})$ is given by

$$h^{(\omega)} \doteq \Delta_{\omega, \vartheta} + \lambda \omega_1, \quad \omega \in \Omega, \lambda, \vartheta \in \mathbb{R}_0^+,$$

where $\Delta_{\omega, \vartheta} = \Delta_{\omega, \vartheta}^* \in \mathcal{B}(\mathcal{H})$ describes the *kinetic* part, with $\vartheta \in \mathbb{R}_0^+$ the strength of the hopping disorder, whereas $\lambda \omega_1 = \lambda \omega_1^* \in \mathcal{B}(\mathcal{H})$ is the external potential, with $\lambda \in \mathbb{R}_0^+$ and ω_1 the function $\omega_1: \mathfrak{L} \rightarrow [-1, 1]$ identified with the multiplication operator. Note that $h^{(\omega)}$ generates the dynamics of the system via the unitary group $\{e^{it h^{(\omega)}}\}_{t \in \mathbb{R}}$ and it is useful to bound *two-point* correlation functions via *Combes–Thomas* estimates. Indeed, if f is an analytic function on the strip $|\Im z| < \eta$ with $\eta \in \mathbb{R}^+$, and $h^{(\omega)}$ satisfies the “Combes–Thomas summability”

$$S(h^{(\omega)}, \mu) \doteq \sup_{x \in \mathfrak{L}} \sum_{y \in \mathfrak{L}} \left(e^{\mu|x-y|} - 1 \right) \left| \langle e_x, h^{(\omega)} e_y \rangle_{\mathcal{H}} \right| < \frac{\eta}{2},$$

for $\mu \in \mathbb{R}_0^+$, then

$$\left| \langle e_x, f(h^{(\omega)}) e_y \rangle_{\mathcal{H}} \right| \leq D \|f\|_{\infty} e^{-\mu|x-y|},$$

where $D \in \mathbb{R}^+$ and $x, y \in \mathfrak{L}$. In particular, we are able to bound two point correlations w.r.t. the unitary group $\{e^{it h^{(\omega)}}\}_{t \in \mathbb{R}}$

$$\left| \langle e_x, e^{it h^{(\omega)}} e_y \rangle_{\mathcal{H}} \right| \leq D e^{|t|\eta - \mu \min\left\{1, \frac{\eta}{2S(h^{(\omega)}, \mu)}\right\}|x-y|}.$$

Note that Combes–Thomas estimates are useful because in free fermion systems the KMS–equilibrium state is a “quasi–free state” $\rho^{(\omega)} \in \mathcal{U}^*$ given by

$$\rho^{(\omega)}(a^*(e_x) a(e_y)) = \left\langle e_x, \frac{1}{1 + e^{\beta h^{(\omega)}}} e_y \right\rangle_{\mathcal{H}}, \quad x, y \in \mathfrak{L}.$$

Here, $\beta > 0$ is the *inverse temperature* and $\{a(\psi)\}_{\psi \in \mathcal{H}}$ is a generator of the CAR C^* –algebra $\mathcal{U} \equiv \mathcal{U}_{\mathcal{H}}$, which is also generated by the identity $\mathbb{1}$. Recall that the free fermion dynamics is given by a strongly continuous group $\tau^{(\omega)} \doteq \{\tau_t^{(\omega)}\}_{t \in \mathbb{R}}$ of (Bogoliubov) $*$ –automorphisms of $\mathcal{U}$, such that the (KMS) quasi–free state $\rho^{(\omega)}$ is stationary w.r.t. $\tau^{(\omega)}$

$$\rho^{(\omega)} \circ \tau_t^{(\omega)} = \rho^{(\omega)}, \quad \beta \in \mathbb{R}^+, \omega \in \Omega, \lambda, \vartheta \in \mathbb{R}_0^+, t \in \mathbb{R}.$$

In a similar way, dynamics and quasi–free states can also be defined on finite regions taking the increasing sequence of boxes given by Expression (1.2.4) for $L \in \mathbb{R}_0^+$. In fact, let $\mathcal{U}_L \equiv \mathcal{U}_{\Lambda_L} \subset \mathcal{U}$ be the local sub–algebra generated by $\mathbb{1}$ and $\{a(e_x)\}_{x \in \Lambda_L}$, we denote by $\tau^{(\omega, \Lambda_L)} \doteq \{\tau_t^{(\omega, \Lambda_L)}\}_{t \in \mathbb{R}}$ and $\rho^{(\omega, \Lambda_L)}$, the strongly continuous group $*$ –automorphisms of $\mathcal{U}_L$ and the quasi–free state on $\mathcal{U}_L^*$, respectively. Note that $\tau^{(\omega, \Lambda_L)}$ converges strongly to $\tau^{(\omega)}$ and $\rho^{(\omega, \Lambda_L)}$ weakly* to $\rho^{(\omega)}$ as $L \rightarrow \infty$.

We can now formulate main results of Chapter 3, in regards to the existence of an LDP for *microscopic linear response currents* $\mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}$ depending explicitly of a normalized vector $\|\vec{w}\|_{\mathbb{R}^d} = 1$ and an electric field $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$:

Theorem 1.3.7 (Large Deviation Principle for Currents):

There is a measurable subset $\tilde{\Omega} \subset \Omega$ of full measure such that, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, the limit

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln \varrho^{(\omega)} \left(e^{|\Lambda_L| \mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}} \right)$$

exist and equals

$$J^{(\mathcal{E})} \doteq \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \mathbb{E} \left[\ln \varrho^{(\cdot)} \left(e^{|\Lambda_L| \mathbb{I}_{\Lambda_L}^{(\cdot, \mathcal{E})}} \right) \right].$$

Moreover, for any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, the map $s \mapsto J^{(s, \mathcal{E})}$ from $\mathbb{R}$ to itself is continuously differentiable and convex. Thus, $\{\mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}\}_{L \in \mathbb{R}^+}$ satisfies an LDP, in the KMS state $\varrho^{(\omega)}$, with speed $|\Lambda_L|$ and good rate function $I^{(\mathcal{E})}$ defined on $\mathbb{R}$ by

$$I^{(\mathcal{E})}(x) \doteq \sup_{s \in \mathbb{R}} \{sx - J^{(s, \mathcal{E})}\}.$$

Remarks 1.3.8.

1. The explicit definition of microscopic linear response currents depends on the *space-averaged* conductivity $\Xi_{\Lambda_L}(t)$, which are the Fourier transform of the microscopic conductive measure given by Expressions (1.2.1)–(1.2.3).
2. This result demonstrates that quantum coherence can very rapidly disappear w.r.t. growing space-scales. Thus, quantum uncertainty of microscopic electric currents is suppressed, *exponentially fast* w.r.t. the volume $|\Lambda_L| = \mathcal{O}(L^d)$ (in lattice units (l.u.), $d \in \mathbb{N}$ being the space dimension) of the region of the lattice where the electric field is applied. *

Because of the *randomness* of the model, for the proof, we invoke Akcoglu–Krengel Ergodic Theorem (AKET), which generalizes the celebrated Birkhoff additive Ergodic Theorem. This states that for the probability space $(\Omega, \mathfrak{A}_\Omega, \mathfrak{a}_\Omega)$, an *additive* process $\{\mathfrak{F}^{(\omega)}(\Lambda)\}_{\Lambda \in \mathcal{P}_f(\mathfrak{Q})}$ and any *regular* sequence $\{\Lambda^{(L)}\}_{L \in \mathbb{R}^+} \subset \mathcal{P}_f(\mathfrak{Q})$, there is a measurable subset $\tilde{\Omega} \subset \Omega$ of full measure such that, for all $\tilde{\omega} \in \tilde{\Omega}$,

$$\lim_{L \rightarrow \infty} \left\{ |\Lambda^{(L)}|^{-1} \mathfrak{F}^{(\tilde{\omega})}(\Lambda^{(L)}) \right\} = \mathbb{E} \left[\mathfrak{F}^{(\omega)}(\{0\}) \right].$$

Here, the additive process $\{\mathfrak{F}^{(\omega)}(\Lambda)\}_{\Lambda \in \mathcal{P}_f(\mathfrak{Q})}$ will depend on the finite Gärtner–Ellis generating function. The latter is nothing but the map $s \mapsto J_{\Lambda_L, \Lambda^{(L)}}^{(\omega, s, \mathcal{E})}$ from $\mathbb{R}$ to itself depending on the

Hamiltonian $h_{\Lambda^{(\vartheta)}} \in \mathcal{H}$ and the linear response operator $\mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})} \in \mathcal{U}$ as follows:

$$J_{\Lambda_L, \Lambda^{(\vartheta)}}^{(\omega, \mathcal{E})} \doteq \frac{1}{|\Lambda_L|} \ln \frac{\text{tr} \left(e^{-\beta d \langle Q, h_{\Lambda^{(\vartheta)}}^{(\omega)} Q \rangle} e^{|\Lambda_L| \mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}} \right)}{\text{tr} \left(e^{-\beta d \langle Q, h_{\Lambda^{(\vartheta)}}^{(\omega)} Q \rangle} \right)},$$

where the symbol $d \langle Q, BQ \rangle$ denotes the second quantization of the operator $B \in \mathcal{H}$. The boxes Λ_L and $\Lambda^{(\vartheta)}$ refer to the finite boxes where are defined the linear response operator and the equilibrium state respectively, these satisfy $\Lambda_L \subset \Lambda^{(\vartheta)} \in \mathcal{P}_f(\mathfrak{L})$. Thus, to apply AKET we split the box Λ_L in smaller equal boxes. Last but not least, to show that in the thermodynamic limit the *boundary interaction* between boxes is negligible we persistently use, throughout the proof, bounds of the form

$$\left| \ln \text{tr} (C e^{H_1}) - \ln \text{tr} (C e^{H_0}) \right| \leq \sup_{\alpha \in [0,1]} \sup_{u \in [-1/2, 1/2]} \left\| e^{u(\alpha H_1 + (1-\alpha)H_0)} (H_1 - H_0) e^{-u(\alpha H_1 + (1-\alpha)H_0)} \right\|_{\mathcal{U}}$$

where $H_0, H_1 \in \mathcal{U}$ are arbitrary self-adjoint elements, C is a positive element and tr is the normalized trace. The last inequality is useful for free fermion systems, because its quasi-free structure allows the right-hand side to behave as $\|H_1 - H_0\|_{\mathcal{U}}$. However, in the interacting case, such an inequality is arbitrarily bigger than $\|H_1 - H_0\|_{\mathcal{U}}$ at large volumes.

To conclude, note that, in the experimental setting of [ZDA⁺06, WMR⁺12], contacts are used to impose an electric potential difference to the nanowires. These contacts yield supplementary resistances to the systems that are well-described by Landauer's formalism [Lan57] when a *ballistic* charge transport takes place in the nanowires. In our model, the purely ballistic charge transport is reached when $\vartheta = 0$ and $\lambda \rightarrow 0^+$, as proven in [BPH14b, Theorem 4.6]. When the nanowire resistance becomes relatively small as compared to the contact resistances, then the charge transport in the nanowire is well-described by a ballistic approximation and Landauer's formalism applies, as also experimentally verified in [ZDA⁺06]. This is the reason why [WMR⁺12] reaches much smaller length scales than [ZDA⁺06]: the material used in [WMR⁺12] has a much larger linear resistivity (about 1000Ω/nm, see [WMR⁺12, Fig. 1 E]) than the one of [ZDA⁺06] (23Ω/nm, see [ZDA⁺06, discussions after Eq. (2)]).

1.3.2 GENERATING FUNCTIONS AS GAUSSIAN BEREZIN INTEGRALS AND DETERMINANT BOUNDS

Chapter 4 is devoted to the first step towards the existence of a Large Deviation Principle (LDP) for weakly *interacting* fermion systems in equilibrium. See Appendix C for the self-dual approach closer to the work [ABPM17]:

Firstly we introduce some notation:

Notation 1.3.9.

1. $\mathcal{H}$ is a Hilbert space.
2. For $\mathcal{H}$, we associate the CAR C^* algebra $\mathcal{U} \equiv \text{CAR}(\mathcal{H})$ generated by the identity $\mathbb{1}$ and $\{a(\varphi)\}_{\varphi \in \mathcal{H}}$ satisfying: (i) for any $\varphi \mapsto a(\varphi)^*$ is linear and (ii) for $\varphi_1, \varphi_2 \in \mathcal{H}$

$$a(\varphi_1)a(\varphi_2)^* + a(\varphi_2)^*a(\varphi_1) = \langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} \mathbb{1}.$$

3. For $k \in \mathbb{N}_0$, $\mathcal{H}^{(k)}$ denotes the copy of $\mathcal{H}$ such that $\varphi^{(k)}$ is the copy of $\varphi \in \mathcal{H}$.
4. For any $n \in \mathbb{N}$

$$(1.3.10) \quad \mathbf{H}^{(n)} \doteq \bigoplus_{k=0}^{n-1} \mathcal{H}^{(k)}. \quad \clubsuit$$

Let $H \in \mathcal{U}$ be the Hamiltonian given by

$$H = d\Gamma(h) + W, \quad W = W^* \in \mathcal{U},$$

where, $h = h^* \in \mathcal{B}(\mathcal{H})$. $h = h^* \in \mathcal{B}(\mathcal{H})$ is the free part of the Hamiltonian and $d\Gamma(h) \in \mathcal{U}$ its second quantization. W is the interparticle interaction, which we only require to be finite range. Thus, in a first instance, disorder systems are included in our results at least for random, but ergodic, interactions, as described in §§1.3.1. As is usual in the fermionic context, interactions are written as even elements of $\mathcal{U}$. Hence, if K is a self-adjoint element of $\mathcal{U}$ one can deal with *Berezin (Grassmann) integrals* in order to write the quantity

$$\text{tr}(e^{-\beta H} e^{sK}), \quad \beta \in \mathbb{R}^+, s \in \mathbb{R},$$

as a limit of these, where tr is the tracial state on $\mathcal{U}$. This in turns, suitably allows to handle with *Gaussian Berezin integrals*

$$(1.3.11) \quad \frac{\text{tr}(e^{-\beta(d\Gamma(h)+W)} e^{sK})}{\text{tr}(e^{-\beta d\Gamma(h)})} = \lim_{n \rightarrow \infty} \int d\mu_{C_h^{(n)}}(\mathbf{H}^{(n_\beta)}) e^{\mathcal{W}_{W,sK}^{(n_\beta)}}.$$

Here: (i) $\mathbf{H}^{(n_\beta)}$ is the Hilbert space given by (1.3.10) with $n_\beta \doteq n + \lfloor n/\beta \rfloor$, (ii) $\int d\mu_{C_h^{(n)}}(\mathbf{H}^{(n)})$ is the Gaussian Berezin integral on some Grassmann algebra over $\mathbf{H}^{(n)}$ associated with an explicit covariance $C_h^{(n)}$ only dependent on $h, \beta \in \mathbb{R}^+$ and $n \in \mathbb{N}$ and (iii) $\mathcal{W}_{W,sK}^{(n)}$ is the result of a canonical mapping from $\mathcal{U}$ to the Grassmann algebra over $\mathbf{H}^{(n_\beta)}$ and only depends on $W, sK, \beta \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Note the Expression (1.3.11) has a similar form than the factor inside the “ln” in the finite logarithmic moment generating function

$$(1.3.12) \quad J_{\Lambda_L}(s) \doteq \frac{1}{|\Lambda_L|} \ln \frac{\text{tr}(e^{-\beta(d\Gamma(h)+W_L)} e^{sK_L})}{\text{tr}(e^{-\beta d\Gamma(h)+W_L})},$$

where $K_L \equiv K_{\Lambda_L} \in \mathcal{U}$ and $W_L \equiv W_{\Lambda_L} \in \mathcal{U}$ are the operators $K \in \mathcal{U}$ and $W \in \mathcal{U}$ restricted to the box Λ_L given by (1.2.4). In fact, Expressions similar to (1.3.11) will permit to show the analyticity around $s = 0$ of $J_{\Lambda_L}(s)$ for weakly interacting fermion systems, when the interactions involving $H, K \in \mathcal{U}$ are *finite range translation invariant*. We now present our main results w.r.t. the covariance $C_h^{(n)}$ (see notations 1.3.9):

Theorem 1.3.13 (Determinant bounds):

Let h be a self-adjoint operator on $\mathcal{H}$ and we denote by $\mathcal{H}$, the space of continuous linear functions over $\mathcal{H}$. Then, for $\beta \in \mathbb{R}^+$, $n, m, M, N \in \mathbb{N}$ with $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$, all $k_1, \dots, k_{N+M} \in \{0, \dots, n_\beta - 1\}$ ($n_\beta \doteq n + \lfloor n/\beta \rfloor$), $j_1 \dots j_{N+M} \in \{1, \dots, m\}$, $\mathfrak{M} \in \text{Mat}(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$, and $\bar{\varphi}_1, \dots, \bar{\varphi}_N \in \mathcal{H}$, $\varphi_{N+1}, \dots, \varphi_{N+M} \in \mathcal{H}$, the following bound holds true:

$$\left| \det \left[\mathfrak{M}_{j_q, j_{N+l}} \bar{\varphi}_q^{(k_q)} \left(C_h^{(n)} \varphi_{N+l}^{(k_{N+l})} \right) \right]_{q,l=1}^N \right| \leq \delta_{N,M} \prod_{q=1}^N \|\bar{\varphi}_q\|_{\mathcal{H}} \mathfrak{M}_{j_q, j_q}^{1/2} \prod_{l=1}^N \|\varphi_{N+l}\|_{\mathcal{H}} \mathfrak{M}_{j_{N+l}, j_{N+l}}^{1/2}. \quad \spadesuit$$

Remark 1.3.14. In this theorem the Kronecker delta means that if $N \neq M$ the left-hand side of the inequality is zero, see Theorem 2.2.33. ✱

Theorem 1.3.13 does not require of translation invariance, a stronger result than those found in the literature. It also works for disorder systems, thus as far as we know, generalizes the results on LDT in the scope of quantum spin and fermion systems. Additionally, the methods used to prove such Theorem replace the celebrated Gram bounds on which a great quantity of works involving fermion systems are based. Moreover, in order to tackle the problem of the existence of $J_{\Lambda_L}(s)$ as $L \rightarrow \infty$, we take in (1.3.3) the state $\rho \in \mathcal{U}^*$ as being a KMS state. Recall that the latter exists because the finite range interaction condition. However, in contrast with other results in the literature, we do not impose any condition on the inverse of temperature $\beta > 0$ nor in the spatial dimension, in such a way that uniqueness of KMS states is not necessary.

In order to state the second main result w.r.t. covariances $C_h^{(n)}$ we first introduce some notation:

Notation 1.3.15.

1. $\{\mathfrak{e}_x\}_{x \in \mathcal{X}}$ is the canonical orthonormal basis of $\mathcal{H}$, see Expression (1.3.6).
2. Gapped Hamiltonians $h \in \mathcal{B}(\mathcal{H})$ with spectrum $\text{spec}(h)$, satisfy

$$g_h \doteq \inf \{ \varepsilon > 0 : [-\varepsilon, \varepsilon] \cap \text{spec}(h) \neq \emptyset \} > 0.$$

3. For $\Lambda_{L_1}, \Lambda_{L_2} \in \mathcal{P}_f(\mathcal{Q})$ with $\Lambda_{L_1} \subset \Lambda_{L_2}$, $\beta > 0$, $n \in \mathbb{N}$, $n_\beta \doteq n + \lfloor n/\beta \rfloor$, $\mathfrak{H}^{(n_\beta)}$ defined by (1.3.10) and the

covariance C as given in Theorem 1.3.13, for $h = h_{L_2}$, the decay parameter is

$$(1.3.16) \quad \omega_h \doteq \limsup_{L_1 \rightarrow \infty} \limsup_{L_2 \rightarrow \infty} \lim_{n \rightarrow \infty} \sup_{k_1 \in \{0, \dots, n_\beta - 1\}} \sup_{x_1 \in \Lambda_{L_1}} \left\{ \beta n^{-1} \sum_{k_2=0}^{n_\beta-1} \sum_{x_2 \in \Lambda_{L_1}} \left| \left\langle e_{x_1}^{(k_1)}, C_{h_{L_2}}^{(n)} e_{x_2}^{(k_2)} \right\rangle_{\mathbb{H}^{(n_\beta)}} \right| \right\}. \quad \clubsuit$$

We finally state:

Theorem 1.3.17 (Summability of the covariance):

Fix $d \in \mathbb{N}$ and let $h \in \mathcal{B}(\mathcal{H})$ be any self-adjoint operator on $\mathcal{H}$. Then, for any $\beta \in \mathbb{R}^+$,

1. For a self-adjoint operator $h \in \mathcal{B}(\mathcal{H})$:

$$\omega_h \leq 8\mathcal{O}\left((\beta + 1)^d\right).$$

2. For gapped self-adjoint operator:

$$\omega_h \leq \mathcal{O}\left((g_h^{-1} + 1)^{d+1}\right).$$

♣

Note that by the form of the decay parameter, Expression (1.3.16), to proof Theorem (1.3.17) we use Combes–Thomas estimates described in §§1.3.1. As a consequence, item 1 is only useful for non-zero temperatures. However, in the gapped case 2, we can study systems at zero temperature. In our discussion we notice that such bounds also corresponds to translation-invariant free fermion systems. Indeed, in the same way that in Theorem 1.3.13, above theorem applies even for disorder systems. However, as mentioned, when studying analyticity around zero of $\lim_{L \rightarrow \infty} J_{\Lambda_L}(s)$ for weakly interacting fermions, we will assume that the systems are finite range translation invariants. This in turns, by Bryc’s Theorem, it will imply a central limit theorem for the family of distribution measures $\{\mathfrak{m}_{\rho, K_L}\}_{L \in \mathbb{R}_0^+}$ associated to the observables $\{K_L\}_{L \in \mathbb{R}_0^+} \in \mathcal{U}$ and the KMS-state $\rho \in \mathcal{U}^*$, see subsequent discussion to Expression (1.3.3).

Remark 1.3.18. The method we set forth in [ABPM17, ABPM16] can be used to study relative entropy densities [JOPP11, Section 2.6] and quantum normal fluctuations [Ver11, Section 6] for fermion systems on lattices. It can also be useful in the theory of “quantum hypothesis testing” via Chernoff and Hoeffding bounds. See for instance [HMO07, Section VI]. For more explanations, see [ABPM16]. ✱

1.4 ANALITICITY OF GENERATING FUNCTIONS FROM BRYDGES–KENNEDY TREE EXPANSIONS

For $s \in \mathbb{R}$ we define the map $s \mapsto J_{L_0}(s)$, as the finite logarithmic moment generating function $J_{L_0}(s) \equiv J_{\Lambda_{L_0}}(s)$ given by

$$(1.4.1) \quad J_{L_0, L_i, L_f}(s) \doteq \frac{1}{|\Lambda_{L_0}|} \left(g_{L_0, L_i, L_f}(s) - g_{L_0, L_i, L_f}(0) \right),$$

where

$$(1.4.2) \quad g_{L_0, L_i, L_f}(s) \doteq \ln \frac{\text{tr} \left(e^{-\beta(d\langle Q, h_{L_f} Q \rangle + W_{L_i})} e^{sK_{L_0}} \right)}{\text{tr} \left(e^{-\beta d\langle Q, h_{L_f} Q \rangle} \right)}.$$

Here, for $p = \{f, i, o\}$, $\mathcal{H}_{L_p} \equiv \mathcal{H}_{\Lambda_{L_p}}$ as well as $\mathcal{U}_{L_p} \equiv \mathcal{U}(\mathcal{H}_{\Lambda_{L_p}})$: (i) $h_{L_p} \in \mathcal{B}(\mathcal{H}_{L_p})$ is a self-adjoint operator on $\mathcal{H}_{L_p}$ for $p = \{f, i, o\}$. $d\langle Q, h_{L_f} Q \rangle \in \mathcal{U}^+ \cap \mathcal{U}_{L_f}$ describes the non-interaction part, (ii) $W_{L_i} \in \mathcal{U}^+ \cap \mathcal{U}_{L_i}$ describes the interparticle interactions and (iii) $K_{L_o} \in \mathcal{U}^+ \cap \mathcal{U}_{L_o}$.

By convenience we use the canonical inclusion $\mathcal{U}_{L_o} \subset \mathcal{U}_{L_i} \subset \mathcal{U}_{L_f}$. Thus, in a suitable sense we will take in (1.4.1) firstly the limit $L_f \rightarrow \infty$, secondly $L_i \rightarrow \infty$ and finally $L_o \rightarrow \infty$ to get the existence of the logarithm moment generating function $J(s)$. Thus, we use in Chapter 5 the so-called *Brydges-Kennedy tree expansion* to show, at any dimension $d \geq 1$ and any fixed inverse temperature² β , the existence of an analytic continuation to a centered disk in the complex plane of the logarithmic moment generating function

$$J(s) \doteq \lim_{L_o \rightarrow \infty} \lim_{L_i \rightarrow \infty} \lim_{L_f \rightarrow \infty} J_{L_0, L_i, L_f}(s)$$

associated with any weak* accumulation point³ of the sequence of Gibbs states of weakly interacting fermions on lattices. Our method thus applies to systems for which the uniqueness of the KMS state is not known. The convergence of the Brydges–Kennedy tree expansion in non-relativistic fermionic constructive quantum field theory is ensured at weak interaction, provided that (i) determinants arising in the expansion can be efficiently bounded and (ii) the covariance is summable.

For $\mathcal{H}$ be a finite-dimensional Hilbert space with orthonormal basis $\{\psi_i\}_{i \in I}$ and $C \in \mathcal{B}(\mathcal{H})$ an operator acting on $\mathcal{H}$ we define w.r.t. the covariance operator C , the “Grassmann–Laplace

²More precisely, β must not be small, as in most previous results for $d > 1$, but rather the *interparticle* interaction W_{L_i} has to be small enough, depending on β .

³The function J a priori depends on the weak* accumulation point of Gibbs states, fixed by the choice of a subsequence of $L_f \geq L_i \geq 0$.

operator" and the "Gaussian convolution" respectively by:

$$\Delta_C \doteq \sum_{i,j \in I} \psi_i^*(C\psi_j) \frac{\delta}{\delta\psi_j} \frac{\delta}{\delta\psi_i^*} \quad \text{and} \quad \mu_{C*} \equiv e^{\Delta_C} \doteq \mathbf{1}_{\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})} + \sum_{m=1}^{\dim \mathcal{H}} \frac{\Delta_C^m}{m!}.$$

Recall now that $\bar{\mathcal{H}}$ is the space of all continuous linear functionals on $\mathcal{H}$. Thus, the projector P_C on the Grassmann algebra $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ with range $\mathbb{C}\mathbf{1}$ is uniquely defined by the conditions $P_C(\mathbf{1}) = \mathbf{1}$, and

$$P_C(\bar{\varphi}_1 \cdots \bar{\varphi}_m \varphi_1 \cdots \varphi_n) = 0$$

for all $m, n \in \mathbb{N}_0$ with $m + n \geq 1$ and all $\bar{\varphi}_1, \dots, \bar{\varphi}_m \in \bar{\mathcal{H}}$, $\varphi_1, \dots, \varphi_n \in \mathcal{H}$. If $m = 0$ then there is no $\bar{\varphi}$ in the above equation. Mutatis mutandis for $n = 0$. Then, for any invertible operator $C \in \mathcal{B}(\mathcal{H})$ we can show the relation between Gaussian integrals and convolutions

$$P_C \circ \mu_C * (\cdot) = \int d\mu_C(\mathcal{H}) \wedge (\cdot).$$

Hence, via the exponential function in Grassmann algebras, i.e.,

$$e^W \doteq \mathbf{1} + \sum_{k=1}^{\infty} \frac{W^k}{k!}, \quad W \in \wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}}),$$

with $\wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ the positive elements of $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$, we intend to find $W_1 \in \wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ through $W_0 \in \wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ such that

$$\mu_C * e^{-W_0} \doteq e^{\Delta_C} e^{-W_0} = e^{-W_1}.$$

Then, noting that $P_C(e^{-W_1}) = e^{-P_C(W_1)}$ is true and also

$$P_C(e^{-W_1}) = P_C(\mu_C * e^{-W_0}) = \int d\mu_C(\mathcal{H}) e^{-W_0},$$

we have

$$W_1 = -\log \left(\int d\mu_C(\mathcal{H}) e^{-W_0} \right).$$

With above properties we intend to show the existence and uniqueness of $W_t \in \wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ such that

$$\mu_{tC} * e^{-W_0} = e^{-W_t}, \quad t \in [0, 1],$$

which is guaranteed by the Polchinski equation, namely,

$$(1.4.3) \quad \forall t \in [0, 1] : \partial_t W_t = \Delta_C W_t + (\nabla W_t, \nabla W_t)_C,$$

where $W_0 \in \Lambda_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ and for $A, B \in \Lambda_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ and $(\nabla A, \nabla B)_C$ is given by Expression 5.2.11.

We now define

$$(1.4.4) \quad W_t^{(1)} \doteq \mu_{tC} * W_0, \quad t \in [0, 1],$$

and consider the infinite hierarchy of coupled ODEs in $\Lambda_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ with initial values for all integers $k \geq 2$,

$$(1.4.5) \quad \forall t \in [0, 1] : \quad \partial_t W_t^{(k)} = \Delta_C W_t^{(k)} + \sum_{l, q \in \mathbb{N} : l+q=k} (\nabla W_t^{(l)}, \nabla W_t^{(q)})_C, \quad W_0^{(k)} = 0.$$

Lemma 5.2.22 asserts that the series given by

$$\tilde{W}_t \equiv \sum_{k=1}^{\infty} W_t^{(k)} \doteq \mu_{tC} * W_0 + \sum_{k=2}^{\infty} W_t^{(k)}$$

is the unique solution of the Polchinski initial value problem (1.4.3). *Brydges–Kennedy tree expansions* $W_t^{(k)}$ are nothing but well-defined objects satisfying (1.4.5). In fact, $W_t^{(k)}$ depend of W_0 and a bounded operator $\mathbf{C} \in \mathcal{B}(\mathbf{H}^{(k)})$ (see Notation 1.3.9), which has explicit dependency of the covariance C .

As mentioned, we need to impose two conditions to the interaction between particles, namely: (a) finite range and (b) translation invariant behavior. Both conditions are necessary to assure the absolute convergence of the Brydges–Kennedy series. Hence, for the covariance operator $C_{h_{L_f}}^{(n)} \in \mathcal{B}(\mathbf{H}_{L_f}^{(n_\beta)})$ provided from a self-adjoint operator h_{L_f} on $\mathcal{H}_{L_f}$ (see the beginning of this subsection) we assume that both Theorem 1.3.13 and Theorem 1.3.17 are satisfied. Brydges–Kennedy assertions state that:

For some $\epsilon > \gamma_{C_{h_{L_f}}^{(n)}}$, the covariance matrix $C_{h_{L_f}}^{(n)} \in \mathcal{B}(\mathbf{H}_{L_f}^{(n_\beta)})$ and the element

$$W_0 = \frac{\beta s}{n} \sum_{k=n}^{n_\beta-1} \chi^{(k)}(K_{L_o}) - \frac{\beta}{n} \sum_{k=0}^{n-1} \chi^{(k)}(W_{L_i}) \in \Lambda^*(\mathbf{H}_{L_f}^{(n_\beta)} \oplus \bar{\mathbf{H}}_{L_f}^{(n_\beta)}),$$

such that

$$4\epsilon^{-2}\omega_h \|W_0\|^{(4\epsilon)} < 1,$$

with $h = \lim_{L_f} h_{L_f}$ and the norm $\|\cdot\|^{(\epsilon)}$, $\epsilon \in \mathbb{R}^+$ defined over the space of *antisymmetric functions*

(see Definition 5.3.14) we have:

1. The Brydges–Kennedy tree expansion

$$W_t \doteq \sum_{k=1}^{\infty} W_t^{(k)}, \quad t \in [0, 1],$$

is absolutely convergent, uniformly for $t \in [0, 1]$, and it is the unique solution of the Polchinski initial value problem (5.2.10).

2. For any $t \in [0, 1]$, $e^{-t\Delta_C} e^{-W_0} = e^{-W_t}$.
3. For any $t \in [0, 1]$,

$$|P_C W_t| \leq 2w(\mathcal{J}) \frac{\|W_0\|^{(4\epsilon)}}{1 - (4\epsilon^{-2} \|C\|_{1,\infty}) \|W_0\|^{(4\epsilon)}}$$

while

$$\left\| (\mathbf{1}_{\wedge^*(\mathbb{H} \oplus \bar{\mathbb{H}})} - P_C) W_t \right\|^{(\epsilon - \gamma_C)} \leq \frac{\|W_0\|^{(4\epsilon)}}{1 - (4\epsilon^{-2} \|C\|_{1,\infty}) \|W_0\|^{(4\epsilon)}}.$$

The main result we assert is: We finally assert:

Theorem 1.4.6:

Let $h \in \mathcal{B}(\mathcal{H})$ be any self-adjoint operator such that h and $\mathcal{H}$ are the limits of h_i and $\mathcal{H}_i$ as $L_i \rightarrow \infty$. Consider $\mathcal{W}_\lambda(s) \in \wedge^*(\mathbb{H}^{(n_\beta)} \oplus \bar{\mathbb{H}}^{(n_\beta)})$ given by

$$\mathcal{W}_\lambda(s) \doteq \frac{\beta s}{n} \sum_{k=n}^{n_\beta-1} \chi^{(k)}(K_{L_0}) - \lambda \frac{\beta}{n} \sum_{k=0}^{n-1} \chi^{(k)}(U_{L_i}) \in \wedge^*(\mathbb{H}^{(n_\beta)} \oplus \bar{\mathbb{H}}^{(n_\beta)}),$$

and $J_{L_0, L_i, L_f}(s)$ in (1.4.1). Then for $s \in \mathbb{R}$, $\beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}$ small enough, the generating function

$$\begin{aligned} J(s) &= \lim_{L_0 \rightarrow \infty} \lim_{L_i \rightarrow \infty} \lim_{L_f \rightarrow \infty} J_{L_0, L_i, L_f}(s) \\ &= \lim_{L_0 \rightarrow \infty} \frac{1}{|\Lambda_{L_0}|} \lim_{L_i \rightarrow \infty} \lim_{n \rightarrow \infty} \left(\ln \int d\mu_{C_h^{(n)}}(\mathbb{H}^{(n_\beta)}) e^{\mathcal{W}_\lambda(s)} - \ln \int d\mu_{C_h^{(n)}}(\mathbb{H}^{(n_\beta)}) e^{\mathcal{W}_\lambda(0)} \right), \end{aligned}$$

exists. Moreover, for $s \in \mathbb{C}$ in a neighborhood of zero, $J(s)$ is an analytic function such that for all $m \in \mathbb{N}_0$ we have

$$\left| \frac{d^m}{ds^m} J(s) \right| \leq m!.$$

THIS chapter is devoted to the mathematical framework of the present Thesis. Thus, we will persistently use throughout each subsequent Chapters the definitions and results introduced here. This chapter is organized as follows: (a) a first part corresponding to the interest of studying Large Deviation Theory, (b) a few results related to different algebras which are found throughout this work, (c) the well-known algebraic setting when dealing with Lattice Fermion Systems and (d) the connection between Large Deviation Theory and Lattice Fermion Systems, which it will be used to study the existence of a *Large Deviation Principle* for the different systems covered in this work.

Notation 2.0.1.

1. We denote by D any positive and finite generic constant. These constants do not need to be the same from one statement to another.
2. A norm on the generic vector space $\mathcal{X}$ is denoted by $\|\cdot\|_{\mathcal{X}}$ and the identity map of $\mathcal{X}$ by $\mathbb{1}_{\mathcal{X}}$. $\mathcal{L}(\mathcal{X})$ and $\mathcal{B}(\mathcal{X})$ denote the spaces of all linear operators and bounded linear operators on $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ respectively.
3. The identity element of any vector space $\mathcal{X}$ is denoted by $\mathbb{1}$, provided it exists. The scalar product of any Hilbert space $\mathcal{X}$ is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\text{Tr}_{\mathcal{X}}$ represents the usual trace on $\mathcal{B}(\mathcal{X})$.
4. For each $k \in \mathbb{N}_0$, denote by $\mathcal{X}^{(k)}$ a copy of some vector space $\mathcal{X}$. The corresponding copy of $\xi \in \mathcal{X}$ is denoted by $\xi^{(k)}$.
5. When we want to consider the element of $\mathcal{H}$ which is related to φ via the Riesz representation theorem, we employ the notation φ^* . Analogously, to the vector from $\mathcal{H}$ to which $\bar{\varphi}$ is identified via the Riesz representation theorem, we employ $\bar{\varphi}^*$. ♣

2.1 LARGE DEVIATION THEORY

It is well-known that physical systems can be analysed using *probability theory*. For instance, in *classical* statistics mechanics at equilibrium, when a set of particles is confined in a finite region of the space, to study its properties, physicists make use of the concepts of *ensembles*: *microcanonical*, *canonical* and *grand canonical* [Tou09, Sal10]. Thus, in the thermodynamic limit, i.e., when for example the number the particles $n \in \mathbb{N}$ or volume V grows indefinitely, it is

convenient to employ one or other to describe physical quantities of interest. In particular, we know that the entropy achieves its maximum value at equilibrium. With that in mind, we can exploit the *saddle-point method* or *Varadhan's lemma* to seek explicitly the “entropy” $I(M)$ related to some physical quantity M . Varadhan's lemma is nothing but a technology to deal with integrals of the form

$$G(f) \doteq \lim_{n \rightarrow \infty} \frac{1}{a_n} \ln \int_{\mathbb{R}} e^{a_n f(x)} m_n(dx),$$

where $\{m_n\}_{n \in \mathbb{N}}$ is the density measure of $\{X_n\}_{n \in \mathbb{N}}$, a sequence of real-valued random variables related with the probability distribution of the quantity M , e.g., mean energy per particle, magnetization, etc. Here, a_n is some positive, increasing, divergent sequence and $f \in C_b(\mathbb{R})$ is a bounded and continuous real-valued function. When last limit exists, it follows that the “entropy” $I(M)$ is related with $G(f)$ via the *Legendre–Fenchel transform*, namely,

$$G(f) = \sup_{M \in \mathcal{D}_I} \{f(M) - I(M)\},$$

where $\mathcal{D}_I$ is the domain of I , see Definition 2.1.3. For more details, see Chapter 4. Indeed, in basic models of physics, last expression is easily verified, e.g., for *non-interacting particle systems* and *2D Ising models* with periodic boundary conditions to name a few. As mentioned in Chapter 1, the main purpose of this Thesis is to show the existence of such entropies for physical quantities related to transport coefficients and weakly interacting systems. In order to tackle this, Large Deviations (LD) techniques will be employed.

From a mathematical point of view, Large Deviation Theory (LDT) can be thought as a theory which goes beyond the Law of Large Numbers (LLN) and complements the Central Limit Theorem (CLT) in the sense that the former is related to the statistics near the expected value, whereas the latter describes the frequency of large departures from this value. As already mentioned, in Physics LDT is often related to some kind of entropy and it is a good conceptual scope to formulate rigorously various problems from statistical mechanics. Large deviations can even be studied for *non-independent and identically distributed* (i.i.d.) random-variables, what is the typical situation in quantum statistical physics.

2.1.1 LARGE DEVIATION PRINCIPLE

In order to illustrate LDT we consider the following simple example:

2.1.1.1 Example. Consider n i.i.d. random real valued variables $\{X_i\}_{i=1}^n$ and let

$$S_n = \frac{1}{n} \sum_{i=1}^n X_i$$

be the mean. Assuming that the density measures m_{X_i} of the random variables $\{X_i\}_{i=1}^n$ are Gaussian, i.e.

m_{X_i} is absolutely continuous w.r.t. to the Lebesgue measure λ on $\mathbb{R}$ with

$$\frac{dm_{X_i}}{d\lambda}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \text{ for } x \in \mathbb{R} \text{ } \lambda\text{-a.e.},$$

we want to find the density measure m_{S_n} of the sum S_n . Here, m is the expectation value of $\{X_i\}_{i=1}^n$ and σ its standard deviation. It is straightforward to show that m_{S_n} is absolutely continuous w.r.t. λ and [Tou09, Kle13]

$$\frac{dm_{S_n}}{d\lambda}(S) = \sqrt{\frac{n}{2\pi\sigma^2}} e^{-n\frac{(S-m)^2}{2\sigma^2}} \text{ for } S \in \mathbb{R} \text{ } \lambda\text{-a.e.}$$

Note that if $S \neq m$, then the density $\frac{dm_{S_n}}{d\lambda}(S)$ goes to zero at large n . Therefore, the probability measure m_{S_n} gets concentrated around the mean m while n increases. Indeed, for all $S \neq m$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left[\frac{\frac{dm_{S_n}}{d\lambda}(S)}{\exp(-nI(S))} \right] = 0$$

for some fixed exponential rate $I(S) \in \mathbb{R}$. On the other hand, at $S = m$ this density increases as $n \rightarrow \infty$. The map $S \mapsto I(S)$ is called “rate function” of the family of probability measures $\{m_{S_n}\}_{n \in \mathbb{N}}$. In fact,

$$(2.1.2) \quad I(S) = \frac{(S-m)^2}{2\sigma^2} = -\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\frac{dm_{S_n}}{d\lambda}(S) \right)$$

and we notice that $I(S)$ is a convex, non-negative, lower semi-continuous function. *

The above example sheds light on important properties of the rate functions that will be essential in a more general framework LDT, namely positivity and convexity (in fact, in the example the rate function is even strictly convex and continuous). The positivity of rate functions is related to the normalization of probability measures. Basically, if I were negative somewhere, we would obtain that the total density measure of the considered family of random variables diverges, as $n \rightarrow \infty$. The convexity of rate functions plays an essential role in LDT and will be discussed later in more detail. Note, however, that rate functions are allowed to be non-convex. See [Tou09] for concrete examples.

We now give the precise formulation of the concepts of rate function and LDP heuristically discussed in above example:

2.1.3 DEFINITION (RATE FUNCTIONS). Let $(\mathcal{X}, \tau)$ be a topological space. Any lower semi-continuous function $I: \mathcal{X} \rightarrow [0, \infty]$ is called a “rate function” in that space. We note that, for all $a \in [0, \infty)$, the level set $\Psi_I(a) = \{x: I(x) \leq a\}$, is a closed subset of $\mathcal{X}$, by lower semi-continuity. I is said to be a “good rate function” if for $a \in [0, \infty)$, all the level sets $\Psi_I(a)$ are compact subsets of $\mathcal{X}$. The set of points of $(\mathcal{X}, \tau)$ for which the rate function I takes finite values is called the “effective domain of I ”. The latter is denoted by $\mathcal{D}_I \doteq \{x: I(x) < \infty\}$. ■

Based on the above definition and Example 2.1.1, we define the *cornerstone* of LDT:

2.1.4 DEFINITION (LARGE DEVIATION PRINCIPLE). Let $(\mathcal{X}, \tau)$ be a topological space and denote by $\Sigma(\tau)$ the Borel σ -algebra on $\mathcal{X}$. Let $\{\mathfrak{m}_\epsilon\}_{\epsilon>0}$ be a family of probability measures of the measurable space $(\mathcal{X}, \Sigma(\tau))$. We say that this family satisfies the “Large Deviation Principle” (LDP) with rate function I if, for all $\Gamma \in \Sigma(\tau)$ we have

$$(2.1.5) \quad -\inf_{x \in \overset{\circ}{\Gamma}} I(x) \leq \liminf_{\epsilon \rightarrow 0} \epsilon \ln \mathfrak{m}_\epsilon(\Gamma) \leq \limsup_{\epsilon \rightarrow 0} \epsilon \ln \mathfrak{m}_\epsilon(\Gamma) \leq -\inf_{x \in \bar{\Gamma}} I(x),$$

where $\overset{\circ}{\Gamma}$ and $\bar{\Gamma}$ are the interior and closure of Γ , respectively. ▀

Note that, for any interval $\Gamma \subset \mathbb{R}$, the rate function of Example 2.1.1, $I(S) = \frac{(S-m)^2}{2\sigma^2}$, satisfies

$$\inf_{S \in \overset{\circ}{\Gamma}} I(S) = \inf_{S \in \bar{\Gamma}} I(S) \doteq I_\Gamma.$$

Hence,

$$(2.1.6) \quad \lim_{\epsilon \rightarrow 0} \epsilon \ln \mathfrak{m}_\epsilon(\Gamma) = -I_\Gamma.$$

Taking the particular case of a countable family of measures $\mathfrak{m}_n$, we note that (2.1.2) is nothing but the above expression for $\epsilon = a_n^{-1}$ with $a_n \doteq n$, $n \in \mathbb{N}$. Sets Γ satisfying (2.1.6) are called “continuity sets” [DZ98]. However, although continuity sets are useful and important in several situations, as stressed by Touchette in [Tou09], there are other kinds of sets which turn out to be relevant for LDPs. In the present study, we shall use the Gärtner–Ellis Theorem discussed below. It will give the third inequality of (2.1.5) for closed sets C in an Hausdorff Topological Vector Space $\mathcal{X}$:

$$\limsup_{\epsilon \rightarrow 0} \epsilon \ln \mathfrak{m}_\epsilon(C) \leq -\inf_{x \in C} I(x).$$

In a similar way, in some special cases, it also gives the first inequality of (2.1.5) for open sets O of $\mathcal{X}$. Indeed, in general, the first inequality of (2.1.5) is only valid for the intersection of open sets with a so-called set of “exposed points”. The precise statement of the Gärtner–Ellis Theorem is given below. To define the rate function in the context of the Gärtner–Ellis Theorem we introduce the Legendre–Fenchel transform:

2.1.7 DEFINITION (LEGENDRE–FENCHEL TRANSFORM). Let $\mathcal{X}$ be an Topological Vector Space (TVS) and $\mathcal{X}^*$ its dual. Let $f: \mathcal{X} \rightarrow (-\infty, \infty]$ be a functional, its Legendre–Fenchel Transform (LFT) f^* , is the convex lower semi-continuous functional $f^*: \mathcal{X}^* \rightarrow (-\infty, \infty]$ defined for $x^* \in \mathcal{X}^*$ as

$$f^*(x^*) \doteq \sup_{x \in \mathcal{X}} \{x^*(x) - f(x)\}. \quad \text{▀}$$

In Lemma 2.1.11 we give the basic properties of LFT, in the $\mathbb{R}^d$ case. Indeed, we use convex analysis and its relation with rate functions. We aim to study the behavior of distributions of

random variables by using LDT via the Gärtner–Ellis Theorem (GET). Note, however, the range of application of LDT goes far beyond of this particular case.

Let X_ϵ be a family of random variables taking values in the TVS $\mathcal{X}$ and let m_ϵ be the distribution (probability) measure of X_ϵ . If $\mathcal{X}^*$ is the dual of $\mathcal{X}$, we define the logarithmic moment generating function $J_{m_\epsilon}: \mathcal{X}^* \rightarrow (-\infty, \infty]$ as

$$J_\epsilon(\lambda) \equiv \ln \mathbb{E}[e^{\langle \lambda, X_\epsilon \rangle}] = \ln \left(\int_{\mathcal{X}} e^{\lambda(x)} m_\epsilon(dx) \right),$$

for $x \in \mathcal{X}, \lambda \in \mathcal{X}^*$, where $\langle \lambda, x \rangle \doteq \lambda(x)$. Here, m_ϵ is the density measure of X_ϵ . From now on, we will assume that the “scaled cumulant generating function”, also called “free energy”,

$$(2.1.8) \quad \limsup_{\epsilon \rightarrow 0} \epsilon J_\epsilon \left(\frac{\lambda}{\epsilon} \right) \doteq J(\lambda)$$

exists. By Hölder’s inequality, the free energy is a convex function [Ell05], as its LFT. We give these properties in Lemma 2.1.11 for the $\mathbb{R}^d$ case.

From now on, we denote the domain of a function $f: \mathcal{X} \rightarrow (-\infty, \infty]$ by

$$\mathcal{D}_f \doteq \{x \in \mathcal{X} : f(x) < \infty\}.$$

To state Lemma 2.1.11 for $\mathbb{R}^d$, as well as the GET, we need a further notion, which is crucial to study convex functions:

2.1.9 DEFINITION (EXPPOSED POINT). Let $I(x)$ be the Legendre–Fenchel transform of $J(\lambda)$. We say that $x \in \mathcal{X}$ is an “exposed point” of I if there exists $\lambda \in \mathcal{X}^*$ and $y \neq x$ such that

$$(2.1.10) \quad I(y) - I(x) > \langle \lambda, y - x \rangle = \lambda(y - x).$$

In this case, λ is an “exposing hyperplane” for the exposed point x . We further define following set of the set of exposed points of I :

$$\mathcal{E}(J, I) \doteq \{x \in \mathcal{X} : \lambda \in \mathring{\mathcal{D}}_J \text{ exposing hyperplane for } x\}. \quad \blacksquare$$

Next Lemma gives the main property of the free energy, in the finite dimensional case:

2.1.11 LEMMA. Let X_n be a sequence of $\mathbb{R}^d$ –valued random variables, $d \in \mathbb{N}$, and, for all $\lambda \in \mathbb{R}^d$, define

$$(2.1.12) \quad J_n(\lambda) \doteq \ln \mathbb{E}[e^{\langle \lambda, X_n \rangle}] = \ln \left(\int_{\mathcal{X}} e^{\lambda(x)} m_n(dx) \right)$$

where m_n is the density measure of X_n . Suppose that the limit

$$(2.1.13) \quad \lim_{n \rightarrow \infty} \frac{1}{a_n} J_n(a_n \lambda) \doteq J(\lambda)$$

exists for some positive, increasing, divergent sequence a_n , all $\lambda \in \mathbb{R}^d$ and $0 \in \mathring{\mathcal{D}}_J$. Then:

1. $J: \mathbb{R}^d \rightarrow (-\infty, \infty]$ is convex.
2. $I: \mathbb{R}^d \rightarrow [0, \infty]$ is convex and is a good rate function, according to Definition 2.1.3.
3. If $y = \nabla J(\eta)$ for some $\eta \in \mathring{\mathcal{D}}_J$, then

$$I(y) = \langle \eta, y \rangle - J(\eta).$$

Moreover, $y \in \mathcal{E}(J, I)$ has η as exposing hyperplane. ♦

We are now in position to formulate the GET:

Theorem 2.1.14 (Gärtner–Ellis):

Let a_n be a positive, increasing, divergent sequence and suppose that the free energy J of Lemma 2.1.11 exists. Let $C \subset \mathbb{R}^d$ and $O \subset \mathbb{R}^d$ be, respectively, closed and open sets. We say that we have a large deviation upper bound for C and a lower bound for O , if

$$(2.1.15) \quad \limsup_{n \rightarrow \infty} \frac{1}{a_n} \ln m_n(C) \leq - \inf_{x \in C} I(x) \text{ and } \liminf_{n \rightarrow \infty} \frac{1}{a_n} \ln m_n(O) \geq - \inf_{x \in O \cap \mathcal{E}} I(x).$$

Here, $I(x)$ is the Legendre–Fenchel transform of $J(\lambda)$ and $\mathcal{E} \equiv \mathcal{E}(J, I)$ is the set of exposed points defined in 2.1.9.

Furthermore, if we assume that

- J is lower semi-continuous on $\mathbb{R}^d$ and differentiable on $\mathring{\mathcal{D}}_J$,
- either $\mathcal{D}_J = \mathbb{R}^d$ or $\lim_{\lambda \in \mathring{\mathcal{D}}_J, \lambda \rightarrow \partial \mathcal{D}_J} |\nabla J(\lambda)| = \infty$, where $\partial \mathcal{D}$ is the boundary of $\mathcal{D}$,

then we may replace $O \cap \mathcal{E}$ by O in the inequality for $\liminf$ in (2.1.15). ♣

Remarks 2.1.16.

- As mentioned at beginning of this section, the LFT of $J(\lambda)$, $I(\cdot)$ is called entropy associated to the sequence X_n of $\mathbb{R}^d$ -valued random variables.
- If the two additional conditions in Gärtner–Ellis Theorem above are satisfied, then a LDP holds for the good rate function $I(\cdot)$. ✱

A version of the GET for TVS has been proven by Baldi [Bal88]. In order to state it the following definition will be used:

2.1.17 DEFINITION. Let $\Sigma(\tau)$ be the Borel σ -algebra of a topological space $(\mathcal{X}, \tau)$. Suppose that all the compact subsets of $\mathcal{X}$ belong to $\Sigma(\tau)$. We say that the family of probability measures $\mathfrak{m}_\epsilon$ on $(\mathcal{X}, \Sigma(\tau))$ is “exponentially tight” if, for every $\alpha \in \mathbb{R}$, there exists a compact set $K_\alpha \subset \mathcal{X}$ such that

$$\limsup_{\epsilon \rightarrow 0} \epsilon \ln \mathfrak{m}_\epsilon(\complement K_\alpha) < -\alpha,$$

where $\complement K_\alpha$ stands for the complement of K_α . ▀

Detailed discussions on exponential tightness can be found, for instance, in [DZ98, OV05]. Observe also that, usually, the notion of exponential tightness already appears in the proof of the (finite dimensional) GET.

Theorem 2.1.18 (Baldi):

Let $\mathfrak{m}_\epsilon$ be a family of exponentially tight probability measures on $\mathcal{X}$ and $\mathbf{J}$ defined in (2.1.8).

1. If $C \in \mathcal{X}$ is a closed set then

$$\limsup_{\epsilon \rightarrow 0} \epsilon \ln \mathfrak{m}_\epsilon(C) \leq -\inf_{x \in C} \mathbf{I}(x).$$

2. Define the following set of exposed points of $\mathbf{I}$

$$\begin{aligned} \mathcal{F}(\mathbf{J}, \mathbf{I}) &\doteq \{x \in \mathcal{X} : \lambda \in \mathcal{X}^* \text{ exposing hyperplane for } x \text{ with } \mathbf{J}(\alpha\lambda) < \infty \text{ for} \\ &\text{some } \alpha > 1 \text{ and such that the limit } \lim_{\epsilon \rightarrow 0} \epsilon \mathbf{J}_{\mathfrak{m}_\epsilon}\left(\frac{\lambda}{\epsilon}\right) \text{ exists.}\} \end{aligned}$$

If $O \in \mathcal{X}$ is an open set then

$$\liminf_{\epsilon \rightarrow 0} \epsilon \ln \mathfrak{m}_\epsilon(O) \geq -\inf_{x \in O \cap \mathcal{F}(\mathbf{J}, \mathbf{I})} \mathbf{I}(x).$$

3. Furthermore, if for every open set $O \in \mathcal{X}$,

$$\inf_{x \in O \cap \mathcal{F}(\mathbf{J}, \mathbf{I})} \mathbf{I}(x) = \inf_{x \in O} \mathbf{I}(x)$$

it follows that $\mathfrak{m}_\epsilon$ satisfies the LDP with a good rate function $\mathbf{I}$. ♣

As we have stated before, LDP goes beyond of LLN. In fact, *strong* LLN (SLLN) is a straight consequence of LDP, namely, for a sequence i.i.d. real-valued random variables $\{X_n\}_{n \in \mathbb{N}}$, $X_\infty \in \mathbb{R}$, SLLN claims that

$$\frac{\mathcal{S}_n}{n} \xrightarrow{\text{a.s.}} \mathbb{E}(X_i),$$

where $\mathcal{S}_n = \sum_{i=1}^n X_i$ and as is usual $\mathbb{E}(\cdot)$ denotes the *expectation value* and “a.s.” the *almost surely* convergence. Indeed, consider the sequence i.i.d. real-valued random variables $\{X_n\}_{n \in \mathbb{N}}$, $X_\infty \in \mathbb{R}$

$\mathbb{R}$. We replace X_i by $X_i - \mathbb{E}(X_i)$, such that we can suppose $\mathbb{E}(X_i) = 0$ with $i \in \mathbb{N}$ or $i = \infty$. We now invoke [Eil05, Theorem II.3.3], which tells us that if an LDP exists, then for the closure of $\Gamma \in \mathbb{R}$ there is a $N \doteq N(\bar{\Gamma}) > 0$ such that $m_n(\bar{\Gamma}) \leq e^{-a_n N}$, with the sequence a_n as given in Lemma 2.1.11. Thus, for any $\epsilon > 0, \bar{\Gamma}$, the closed set $\{z \in \mathbb{R} : |z| \geq \epsilon\}$, and $a_n = n \in \mathbb{N}$, it follows that

$$m_n(\bar{\Gamma}) = \mathbb{P}\left(\left|\frac{\mathcal{S}_n}{n}\right| \geq \epsilon\right) \leq e^{-nN}.$$

Since $\sum_{n \in \mathbb{N}} e^{-nN}$ converges, we use *Borel–Cantelli Theorem* and then

$$\frac{\mathcal{S}_n}{n} \xrightarrow{\text{a.s.}} 0 \Rightarrow \frac{\mathcal{S}_n}{n} \xrightarrow{\text{a.s.}} \mathbb{E}(X_i).$$

For more details see [Kle13] and [Sim15, Part 1] as well as [Eil05] for the $\mathbb{R}^d$ case.

On the other hand, we also claimed that LDT complements the CLT. Bryc noticed that, under certain assumptions, the CLT is a consequence of the LDT [Bry93]: If for $d = 1$, (2.1.13) exists and admits an analytic continuation to a neighborhood of $z = 0 \in \mathbb{C}$, the (rescaled) family of measures $m_{\frac{1}{a_n} X_n}$ converge, in the weak* sense, to the normal distribution $\mathcal{N}_{0, \sigma^2}$ with

$$\sigma^2 = \frac{d^2}{dz^2} G(z) \Big|_{z=0},$$

where $G(\cdot)$ is the free energy given in (2.1.13). Bryc stated the result for real-valued random variables and Jaksic et al. [JOPP11, Appendix A–Theorem A.8], generalized this for finite dimension $d \geq 1$. They explore Vitali’s convergence theorem, cf. [JOPP11, Appendix B–Theorem B.1], such that when $n \rightarrow \infty$, all the derivatives of $\frac{1}{a_n} G_n(a_n z)$ converge uniformly to the derivatives of $G(z)$. In particular, the *usual* CLT follows for $a_n = n$.

Theorem 2.1.19 (Bryc):

Fix $d \in \mathbb{N}, \epsilon \in \mathbb{R}^+$. Let $z_\epsilon \doteq \{z_i\}_{i=1}^d$ be a set of complex numbers such that $(z_1, \dots, z_d) \in \mathbb{C}^d$ with $\max_i |z_i| < \epsilon$. In addition, let m_n be the density measure of X_n , a sequence of $\mathbb{R}^d$ -valued random variables. Suppose that for all $z_\epsilon \in \mathbb{R}^d$, the free energy (2.1.13) of Lemma 2.1.11 exists and has an analytic continuation to z_ϵ . Then for the expectation value v , the family of measures m_n of $\sqrt{a_n}(X_n - v)$ converges in distribution to the normal distribution $\mathcal{N}_{0, \sigma^2}$ with variance $\sigma^2 = \frac{\partial^2}{\partial z_i \partial z_j} G(z) \Big|_{z=0}$. $\clubsuit$

2.2 SOME ALGEBRAS

This section is devoted to introduce concepts and definitions of different algebras used through this work.

2.2.1 C^* -ALGEBRAS

C^* -algebras are a special case of Banach algebras with the structure of bounded operators acting on a Hilbert space [BR03a]. Let $\mathcal{U}$ be a vector space in $\mathbb{K}$ representing the field of real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. We define a C^* -algebra in a constructive way as follows:

- $\mathcal{U}$ is an “algebra” if it is endowed by an associative and distributive multiplication, i.e., if for $A, B \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{K}$, we have

$$\begin{aligned} A(BC) &= A(BC), \\ A(B + C) &= AB + AC, \\ \alpha\beta(AB) &= (\alpha A)(\beta B). \end{aligned}$$

- For $A \in \mathcal{U}$, the map $A \rightarrow A^*$ with $A^* \in \mathcal{U}$ is called an “involution” of the algebra, if it satisfies

$$\begin{aligned} A^{**} &= A, \\ (AB)^* &= B^*A^*, \\ (\alpha A + \beta B)^* &= \bar{\alpha}A^* + \bar{\beta}B^*. \end{aligned}$$

Here, for $z \in \mathbb{K}$, $\bar{z} \in \mathbb{K}$ is its complex conjugate. The involution map is also called “adjoint operation”.

- Algebras with involution are called “ $*$ -algebras”.
- The subset $\mathcal{U}' \in \mathcal{U}$ is called “self-adjoint” if $A \in \mathcal{U}'$ implies that $A^* \in \mathcal{U}'$.
- $\mathcal{U}$ is “normed” if for $A \in \mathcal{U}$ we associate the real number $\|A\|$, called “norm” of A , which satisfies

$$\begin{aligned} \|A\| &\geq 0, \text{ with } \|A\| = 0 \text{ iff } A = 0 \\ \|\alpha A\| &= |\alpha| \|A\|, \\ \|A + B\| &\leq \|A\| + \|B\|, \\ \|AB\| &\leq \|A\| \|B\|. \end{aligned}$$

- If $\mathcal{U}$ is a normed space, we say that it is a “Banach algebra” if $\mathcal{U}$ is complete w.r.t. the norm $\|\cdot\|$.
- If the Banach algebra $\mathcal{U}$ has involution and if $\|A\| = \|A^*\|$, for $A \in \mathcal{U}$, then $\mathcal{U}$ is a “Banach $*$ -algebra”.

Remarks 2.2.1.

- $\mathbb{1}$ is the identity of $\mathcal{U}$ if this has identity.
- An element x in a $*$ -algebra $\mathcal{U}$ is said to be isometric or an isometry if $x^*x = \mathbb{1}$. ✱

We finally define:

2.2.2 DEFINITION (C*-ALGEBRAS). A C^* -algebra is a Banach $*$ -algebra that satisfies $\|A^*A\| = \|A\|^2$, for all $A \in \mathcal{U}$. ■

For the present work it is interesting the following one:

2.2.3 Example (C*-algebra of bounded operators). Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The set of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ is a C^* -algebra with identity. In this case, the operator norm is defined by

$$\|A\|_{\mathcal{B}(\mathcal{H})} \doteq \sup_{\varphi \in \mathcal{H}, \|\varphi\|=1} \|A\varphi\|_{\mathcal{H}}.$$

To prove that in fact it is a C^* -algebra it is enough to note that

1.

$$\|A^*A\|_{\mathcal{B}(\mathcal{H})} \leq \|A^*\|_{\mathcal{B}(\mathcal{H})} \|A\|_{\mathcal{B}(\mathcal{H})} = \|A\|_{\mathcal{B}(\mathcal{H})}^2,$$

because Banach $*$ -algebra properties.

2. By inner-norm properties of $\mathcal{H}$ one has

$$\begin{aligned} \|A\|_{\mathcal{B}(\mathcal{H})}^2 &= \left(\sup_{\varphi \in \mathcal{H}, \|\varphi\|=1} \|A\varphi\|_{\mathcal{H}} \right)^2 = \sup_{\varphi \in \mathcal{H}, \|\varphi\|=1} \|A\varphi\|_{\mathcal{H}}^2 \\ &= \sup_{\varphi \in \mathcal{H}, \|\varphi\|=1} \langle A\varphi, A\varphi \rangle_{\mathcal{H}} = \sup_{\varphi \in \mathcal{H}, \|\varphi\|=1} \langle \varphi, A^*A\varphi \rangle_{\mathcal{H}} \\ &\leq \|\varphi\| \|A^*A\|_{\mathcal{B}(\mathcal{H})} = \|A^*A\|_{\mathcal{B}(\mathcal{H})}. \end{aligned}$$

The conclusion follows. ✱

2.2.2 CAR ALGEBRAS

Due to the *Pauli's exclusion principle* in the fermionic setting, it is usual to implement *Canonical Anti-commutation Relations* (CAR) algebra to tackle this, which as is usual is defined as follows:

2.2.4 DEFINITION (CAR-ALGEBRAS). Let $\mathcal{H}$ be a (*one-particle*) finite Hilbert space. A CAR algebra $\mathcal{U} \equiv \text{CAR}(\mathcal{H})$ is a C^* -algebra generated by the identity $\mathbb{1}_{\mathcal{U}}$ and $\{a(\psi)\}_{\psi \in \mathcal{H}}$ satisfying for all $\psi, \varphi \in \mathcal{H}$,

$$(2.2.5) \quad \{a(\psi), a(\varphi)\} = 0, \quad \{a(\psi), a(\varphi)^*\} = \langle \psi, \varphi \rangle_{\mathcal{H}} \mathbb{1}_{\mathcal{U}},$$

where $\{A, B\} \doteq AB + BA \in \mathcal{U}$ is the *anti-commutator* between the elements $A \in \mathcal{U}$ and $B \in \mathcal{U}$. ■

The involution element of $a(\psi)$, $\psi \in \mathcal{H}$, namely $a(\psi)^*$, exists due to the definition of C^* -algebra, see Definition 2.2.2. Note that the maps $\mathcal{H} \rightarrow \mathcal{U}$, $\varphi \mapsto a(\varphi)$ and $\varphi \mapsto a(\varphi)^*$ are antilinear and linear respectively. Moreover, we also note that $\{a(\psi), a(\varphi)\} = 0$ at (2.2.5), which for $\psi = \varphi$ it is equivalent to $a(\psi)^*a(\psi)^* = 0$. This is the well-known Pauli's exclusion principle, that states that it is impossible the creation of several fermions in the same state. Finally, for all $\psi \in \mathcal{H}$,

$$(2.2.6) \quad \|a(\psi)\|_{\mathcal{U}} \leq \|\psi\|_{\mathcal{H}}.$$

Remark 2.2.7. Usually, for $\psi \in \mathcal{H}$, the operator $a(\psi)$ is called the annihilation operator whereas its involution $a(\psi)^*$ is the creation operator. Through this Thesis, we use both $a(\psi)^*$ and $a^*(\psi)$, to denote the involution of $a(\psi)$, see §2.3–2.3.1 *

2.2.3 GRASSMANN ALGEBRAS AND BEREZIN CALCULUS

Let $\mathcal{X}$ be a topological vector space, with dual space $\mathcal{X}^*$, i.e., the space of all continuous linear functionals on $\mathcal{X}$. We define *Grassmann algebras* as follows:

1. For every $n \in \mathbb{N}$ and $x_1^*, \dots, x_n^* \in \mathcal{X}^*$, we define the completely antisymmetric n -linear form $x_1^* \wedge \dots \wedge x_n^*$ from $\mathcal{X}^n$ to $\mathbb{C}$ by

$$(2.2.8) \quad x_1^* \wedge \dots \wedge x_n^*(y_1, \dots, y_n) \doteq \det\left((x_k^*(y_l))_{k,l=1}^n\right), \quad y_1, \dots, y_n \in \mathcal{X}.$$

In particular, for any permutation π of $n \in \mathbb{N}$ elements with sign $(-1)^\pi$,

$$(2.2.9) \quad x_1^* \wedge \dots \wedge x_n^* \doteq (-1)^\pi x_{\pi(1)}^* \wedge \dots \wedge x_{\pi(n)}^*, \quad x_1^*, \dots, x_n^* \in \mathcal{X}^*.$$

2. Let $\wedge^0 \mathcal{X} \doteq \mathbb{C}$, while, for all $n \in \mathbb{N}$, we use the definition

$$(2.2.10) \quad \wedge^n \mathcal{X} \doteq \text{lin}\{x_1^* \wedge \dots \wedge x_n^* : x_1^*, \dots, x_n^* \in \mathcal{X}^*\}.$$

We then define the vector space

$$(2.2.11) \quad \wedge^* \mathcal{X} \doteq \bigoplus_{n=0}^{\infty} \wedge^n \mathcal{X}.$$

Recall that the infinite direct sum of a family $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ of vector spaces, like in the above definition, is the subspace of the product space $\prod_{n=0}^{\infty} \mathcal{X}_n$, the elements of which are sequences that eventually vanish.

3. For $n, m \in \mathbb{N}_0$, $\xi \in \wedge^n \mathcal{X}$ and $\zeta \in \wedge^m \mathcal{X}$, their *exterior product* $\xi \wedge \zeta \in \wedge^{n+m} \mathcal{X}$ is defined

by

$$(2.2.12) \quad \xi \wedge \zeta(x_1, \dots, x_{n+m}) \doteq \frac{1}{n!m!} \sum_{\pi \in \mathcal{S}_{n+m}} (-1)^\pi \xi(x_{\pi(1)}, \dots, x_{\pi(n)}) \zeta(x_{\pi(n+1)}, \dots, x_{\pi(n+m)}) ,$$

where $\mathcal{S}_N$ is the set of all permutations of $N \in \mathbb{N}$ elements. This prescription uniquely defines an associative product on $\wedge^* \mathcal{X}$, which is consistent with the definition of the n -linear form (2.2.8):

$$x_1^* \wedge (x_2^* \wedge \dots \wedge x_n^*) = x_1^* \wedge \dots \wedge x_n^* , \quad n \in \mathbb{N}, x_1^*, \dots, x_n^* \in \mathcal{X}^* .$$

Compare (2.2.12) with (2.2.8), using the Leibniz formula to compute the determinant.

2.2.13 DEFINITION (GRASSMANN ALGEBRA). The Grassmann algebra on a topological vector space $\mathcal{X}$ is the (associative and distributive) algebra $(\wedge^* \mathcal{X}, +, \wedge)$. $\blacksquare$

Notation 2.2.14.

When there is no risk of ambiguity, we use $x_1^* \wedge \dots \wedge x_n^* \equiv x_1^* \cdots x_n^*$ to denote exterior products. $\clubsuit$

For $n \in \mathbb{N}_0$, $\wedge^{*n} \mathcal{X}$ is precisely the subspace of elements of degree n of the *graded* algebra $\wedge^* \mathcal{X}$. The unit of the Grassmann algebra $\wedge^* \mathcal{X}$ is denoted by

$$\mathbb{1} \doteq 1 \in \wedge^{*0} \mathcal{X} \subset \wedge^* \mathcal{X}$$

and $[\xi]_0$ stands for the zero-degree component of any element ξ of $\wedge^* \mathcal{X}$. For $n = 0$, the linear forms on $\mathcal{X}$ are the complex numbers. The subspace of $\wedge^* \mathcal{X}$ generated by monomials $\bar{x}_1 \wedge \dots \wedge \bar{x}_n$ of even order $n \in 2\mathbb{N}_0$ forms a *commutative* subalgebra, the “even subalgebra” of $\wedge^* \mathcal{X}$, which will be denoted by $\wedge_+^* \mathcal{X}$ in the sequel. Explicitly:

$$(2.2.15) \quad \wedge_+^* \mathcal{X} \doteq \bigoplus_{n=0}^{\infty} \wedge^{*2n} \mathcal{X} .$$

The exponential function in Grassmann algebra is important in the sequel. It is defined, as is usual, by

$$(2.2.16) \quad e^\xi \doteq \mathbb{1} + \sum_{k=1}^{\infty} \frac{\xi^k}{k!} , \quad \xi \in \wedge^* \mathcal{X} .$$

This function is well-defined because, by definition of $\wedge^* \mathcal{X}$, $\xi \in \wedge^* \mathcal{X}$ is contained in some finite-dimensional subalgebra of $\wedge^* \mathcal{X}$. Indeed, by finite dimensionality, for any norm on the space $\wedge^* \mathcal{X}$ and any $\xi \in \wedge^* \mathcal{X}$, there is a constant $D_\xi < \infty$ such that

$$(2.2.17) \quad \|\xi^k\|_{\wedge^* \mathcal{X}} \leq (D_\xi)^k , \quad k \in \mathbb{N} .$$

Hence, the series defining e^ξ is absolutely convergent in the normed space $\wedge^* \mathcal{X}$.

From now on, we will replace $\mathcal{X}$ by a finite Hilbert space $\mathcal{H}$. Thus, $\wedge^* \mathcal{H}$ is the Grassmann algebra associated to $\mathcal{H}$ as defined in §2.2. We denote by $\bar{\mathcal{H}}$, the space of all continuous linear functionals on $\mathcal{H}$. Berezin derivatives act as follows on elements of $(\mathcal{H} \oplus \bar{\mathcal{H}})^*$

$$(2.2.18) \quad \frac{\delta}{\delta \varphi}(\mathbb{1}) = \frac{\delta}{\delta \bar{\varphi}}(\mathbb{1}) = \frac{\delta}{\delta \varphi}(\bar{\varphi}) = \frac{\delta}{\delta \bar{\varphi}}(\varphi) = 0$$

for all $\varphi \in \mathcal{H}$ and

$$(2.2.19) \quad \frac{\delta}{\delta \varphi_1} \varphi_2 = \langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} \mathbb{1}, \quad \frac{\delta}{\delta \bar{\varphi}_1}(\bar{\varphi}_2) = \langle \bar{\varphi}_2, \bar{\varphi}_1^* \rangle_{\bar{\mathcal{H}}} \mathbb{1}$$

for all $\varphi_1, \varphi_2 \in \mathcal{H}$, see Notation 2.0.1–5. Berezin derivatives anticommute with one another:

$$(2.2.20) \quad \frac{\delta}{\delta \varphi_1} \frac{\delta}{\delta \varphi_2} = -\frac{\delta}{\delta \varphi_2} \frac{\delta}{\delta \varphi_1}, \quad \frac{\delta}{\delta \bar{\varphi}_1} \frac{\delta}{\delta \bar{\varphi}_2} = -\frac{\delta}{\delta \bar{\varphi}_2} \frac{\delta}{\delta \bar{\varphi}_1}, \quad \frac{\delta}{\delta \varphi_1} \frac{\delta}{\delta \bar{\varphi}_2} = -\frac{\delta}{\delta \bar{\varphi}_2} \frac{\delta}{\delta \varphi_1}$$

for all $\varphi_1, \varphi_2 \in \mathcal{H}$ and all $\bar{\varphi}_1, \bar{\varphi}_2 \in \bar{\mathcal{H}}$. Note that $\delta/\delta \varphi$ is an annihilation operator on $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$, viewed as the fermionic Fock space $\mathcal{F}((\mathcal{H} \oplus \bar{\mathcal{H}})^*)$. See [ABPM17, §6.4]. Furthermore, Grassmann derivatives are so-called “derivations of degree one”, i.e., for all $\xi_1 \in \wedge^{*n}(\mathcal{H} \oplus \bar{\mathcal{H}})$, $n \in \mathbb{N}$, all $\xi_2 \in \wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ and any $\varphi \in \mathcal{H}$, $\bar{\varphi} \in \bar{\mathcal{H}}$,

$$(2.2.21) \quad \frac{\delta}{\delta \varphi}(\xi_1 \xi_2) = \left(\frac{\delta}{\delta \varphi} \xi_1 \right) \xi_2 + (-1)^n \xi_1 \frac{\delta}{\delta \varphi} \xi_2,$$

$$(2.2.22) \quad \frac{\delta}{\delta \bar{\varphi}}(\xi_1 \xi_2) = \left(\frac{\delta}{\delta \bar{\varphi}} \xi_1 \right) \xi_2 + (-1)^n \xi_1 \frac{\delta}{\delta \bar{\varphi}} \xi_2$$

Recall that, for each $k \in \mathbb{N}_0$, $\mathcal{H}^{(k)}$ denotes a copy $\mathcal{H}$ and the corresponding copy of $\xi \in \mathcal{H}$ is written as $\xi^{(k)}$, see Notation 2.0.1. For any $K \subset \{0, \dots, N\}$ with $N \in \mathbb{N}_0$, we define

$$(2.2.23) \quad \mathbf{H}^{(K)} \oplus \bar{\mathbf{H}}^{(K)} \doteq \bigoplus_{k \in K} (\mathcal{H}^{(k)} \oplus \bar{\mathcal{H}}^{(k)}),$$

and for any $n \in \mathbb{N}$, let

$$(2.2.24) \quad \mathbf{H}^{(n)} \oplus \bar{\mathbf{H}}^{(n)} \doteq \bigoplus_{k=0}^{n-1} (\mathcal{H}^{(k)} \oplus \bar{\mathcal{H}}^{(k)}).$$

The corresponding scalar product is the usual one for direct sums of Hilbert spaces. In particular, for any orthonormal basis (ONB) $\{\psi_i\}_{i \in I}$ of $\mathcal{H}$, the union $\bigcup_{k=0}^{n-1} \{\psi_i^{(k)}\}_{i \in I}$ is an ONB of $\mathbf{H}^{(n)}$. Therefore,

we identify

$$\wedge^* \left(\mathbb{H}^{(K)} \oplus \bar{\mathbb{H}}^{(K)} \right)$$

with the Grassmann subalgebra of

$$\wedge^* \left(\mathbb{H}^{(N)} \oplus \bar{\mathbb{H}}^{(N)} \right)$$

generated by

$$\bigcup_{k \in K} \{ \bar{\varphi}^{(k)} \mid \bar{\varphi} \in \bar{\mathcal{H}} \} \cup \{ \varphi^{(k)} \mid \varphi \in \mathcal{H} \}.$$

We also identify the Grassmann subalgebra $\wedge^* (\mathcal{H}^{(0)} \oplus \bar{\mathcal{H}}^{(0)})$ with the Grassmann algebra $\wedge^* (\mathcal{H} \oplus \bar{\mathcal{H}})$, i.e.,

$$(2.2.25) \quad \wedge^* (\mathcal{H}^{(0)} \oplus \bar{\mathcal{H}}^{(0)}) \equiv \wedge^* (\mathcal{H} \oplus \bar{\mathcal{H}}).$$

We are now in a position to define the so-called *Berezin integral* [Ber66, Section I.3]:

2.2.26 DEFINITION (BEREZIN INTEGRAL). Let $\{\psi_i\}_{i \in J}$ be any ONB of $\mathcal{H}$. For all $k \in \{0, \dots, N\}$, we define the linear map

$$\int d(\mathbb{H}^{(k)}) : \wedge^* (\mathbb{H}^{(N)} \oplus \bar{\mathbb{H}}^{(N)}) \rightarrow \wedge^* (\mathbb{H}^{\{0, \dots, N\} \setminus K} \oplus \bar{\mathbb{H}}^{\{0, \dots, N\} \setminus K})$$

by

$$\int d(\mathbb{H}^{(k)}) \doteq \prod_{i \in J} \left(\frac{\delta}{\delta \psi_i^{(k)}} \frac{\delta}{\delta \psi_i^{*(k)}} \right). \quad \blacksquare$$

For $N = 0$, the Berezin integral defines a linear form from $\wedge^* (\mathcal{H}^{(0)} \oplus \bar{\mathcal{H}}^{(0)}) \equiv \wedge^* (\mathcal{H} \oplus \bar{\mathcal{H}})$ to $\mathbb{C}1 \equiv \mathbb{C}$. Recall that usual integrals can be seen as linear forms on an algebra of continuous functions. This partially justifies the use of the term “integral” for the map $\int d(\mathcal{H}^{(k)} \oplus \bar{\mathcal{H}}^{(k)})$. Note that the Berezin integral does not depend on the particular choice of the ONB $\{\psi_i\}_{i \in J}$ of $\mathcal{H}$: Considering the case $N = 0$ without loss of generality, observe that, for all $i \in J$, the basis transformation

$$\tilde{\psi}_i = \sum_{j \in J} U_{ij} \psi_j$$

for any unitary transformation U on $\mathcal{H}$, yields to

$$\frac{\delta}{\delta(U\psi_i)} = \sum_{j \in I} \langle U\psi_i, \psi_j \rangle_{\mathcal{H}} \frac{\delta}{\delta\psi_j}.$$

Finally, since Berezin derivatives anticommute with each other, one has

$$\prod_{i \in I} \left(\frac{\delta}{\delta(U\psi_i)} \frac{\delta}{\delta(U\psi_i^*)} \right) = \prod_{j \in I} \left(\frac{\delta}{\delta(\psi_j)} \frac{\delta}{\delta(\psi_j^*)} \right).$$

One can define a “circle” product $\circ$ and an involution $*$ on $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ in such a way that $\mathcal{B}(\wedge^* \mathcal{H})$ and $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ are isomorphic $*$ -algebras. To this end, for all $i, j, k, l \in \mathbb{N}_0$ we define first the following isomorphisms of linear spaces

$$(2.2.27) \quad \kappa_{(i,j)}^{(k,l)}: \wedge^*(\mathcal{H}^{(i)} \oplus \mathcal{H}^{*(j)}) \rightarrow \wedge^*(\mathcal{H}^{(k)} \oplus \mathcal{H}^{*(l)})$$

the unique isomorphism of linear spaces such that $\kappa_{(i,j)}^{(k,l)}(z\mathbf{1}) = z\mathbf{1}$ for $z \in \mathbb{C}$ and, for any $m, n \in \mathbb{N}_0$ so that $m + n \geq 1$, and all $\varphi_1, \dots, \varphi_{m+n} \in \mathcal{H}$,

$$(2.2.28) \quad \kappa_{(i,j)}^{(k,l)}(\bar{\varphi}_1^{(i)} \cdots \bar{\varphi}_m^{(i)} \varphi_{m+1}^{(j)} \cdots \varphi_{m+n}^{(j)}) = \bar{\varphi}_1^{(k)} \cdots \bar{\varphi}_m^{(k)} \varphi_{m+1}^{(l)} \cdots \varphi_{m+n}^{(l)}$$

with $\varphi_1 \wedge \varphi_2 \equiv \varphi_1 \varphi_2$ (see Notation 2.0.1–4 and 2.2.14). Here, if $m = 0$, then there is no $\bar{\varphi}$ in the above equation. Mutatis mutandis for $n = 0$.

By using Notation 2.0.1–4, we define bilinear elements of Grassmann algebra and compare with Definition 2.3.15:

2.2.29 DEFINITION (BILINEAR ELEMENTS OF GRASSMANN ALGEBRAS). Let $\mathcal{H}$ be a finite Hilbert space with ONB $\{\psi_j\}_{j \in I}$.

1. Fix an ONB $\{\psi_i\}_{i \in I}$ of $\mathcal{H}$ and define, for all $H \in \mathcal{B}(\mathcal{H})$, a bilinear element of the Grassmann algebra $\wedge^* \mathcal{H}$ by

$$(2.2.30) \quad \langle \mathcal{H}, H\mathcal{H} \rangle \doteq 2 \sum_{i,j \in I} \langle \psi_i, H\psi_j \rangle_{\mathcal{H}} \psi_j^* \wedge \psi_i.$$

2. Given $k, l \in \mathbb{N}_0$,

$$\langle \mathcal{H}^{(k)}, \mathcal{H}^{(l)} \rangle \doteq \sum_{j \in I} \psi_j^{*(k)} \wedge \psi_j^{(l)}.$$

□

Note that this definition *does not depend* on the particular choice of the ONB. We also notice that

$\langle \mathcal{H}^{(k)}, \mathcal{H}^{(l)} \rangle^m = 0$ whenever $m > \dim \mathcal{H}$. On the other hand, elementary computations yield

$$(2.2.31) \quad \int d(\mathcal{H}) e^{\langle \mathcal{H}, H \mathcal{H} \rangle} = \det(H).$$

Gaussian Berezin integrals are then defined as follows:

2.2.32 DEFINITION (GAUSSIAN BEREZIN INTEGRALS). Let $C \in \mathcal{B}(\mathcal{H})$ be any invertible self-dual operator. The Gaussian Berezin integral with covariance $C \in \mathcal{B}(\mathcal{H})$ is the linear map $\int d\mu_C(\mathcal{H})$ from $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ to $\mathbb{C}\mathbb{1}$ defined by

$$\int d\mu_C(\mathcal{H}) \xi \doteq \det(C) \int d(\mathcal{H}) e^{\langle \mathcal{H}, C^{-1} \mathcal{H} \rangle} \wedge \xi, \quad \xi \in \wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}}). \quad \blacksquare$$

C in above definition is also named the “covariance” of the integral. We can prove:

Theorem 2.2.33 (Properties of Gaussian Berezin integrals):

Let $m, n \in \mathbb{N}_0$ and $\bar{\varphi}_1, \dots, \bar{\varphi}_m \in \bar{\mathcal{H}}, \varphi_1, \dots, \varphi_n \in \mathcal{H}$ then

$$\int d\mu_C(\mathcal{H}) \mathbb{1} = 1 \quad \text{and} \quad \int d\mu_C(\mathcal{H}) \bar{\varphi}_1 \cdots \bar{\varphi}_m \varphi_1 \cdots \varphi_n = \delta_{m,n} \det[\bar{\varphi}_k(C\varphi_l)]_{k,l=1}^m \mathbb{1}. \quad \clubsuit$$

PROOF. See [ABPM17, Proposition 3.7 and Corollary 3.8].

End

By using the objects defined by (2.2.16), Definition 2.2.29 and (2.2.27)–(2.2.28), as well as Berezin integrals (Definition 2.2.26), we introduce a new product on $\wedge^* \mathcal{H}$ depending on $\mathcal{H}$ [Ped05]:

2.2.34 DEFINITION (CIRCLE PRODUCTS). For any $\xi_0, \xi_1 \in \wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$, we define their circle product $\xi_0 \circ \xi_1 \in \mathcal{H}$ by

$$\xi_0 \circ \xi_1 \doteq (-1)^{\dim \mathcal{H}} \int d(\mathcal{H}^{(1)}) \chi_{(0,0)}^{(0,1)}(\xi_0) \chi_{(0,0)}^{(1,0)}(\xi_1) e^{-\langle \mathcal{H}^{(0)}, \mathcal{H}^{(0)} \rangle} e^{\langle \mathcal{H}^{(0)}, \mathcal{H}^{(1)} \rangle} e^{-\langle \mathcal{H}^{(1)}, \mathcal{H}^{(1)} \rangle} e^{\langle \mathcal{H}^{(1)}, \mathcal{H}^{(0)} \rangle} \quad \blacksquare$$

It directly follows from this definition that the circle product is associative and distributive on the space $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ with the same unit $\mathbb{1}$ as with the exterior product $\wedge$:

2.2.35 LEMMA (ELEMENTARY PROPERTIES OF THE CIRCLE PRODUCT). *The circle product $\circ$ has the following properties:*

1. *Associativity: For all $N \in \mathbb{N}$, $N \geq 2$, and $\xi_0, \dots, \xi_{N-1} \in \wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$,*

$$\xi_0 \circ \dots \circ \xi_{N-1} = (-1)^{(N-1) \dim \mathcal{H}} \left(\prod_{k=1}^{N-1} \int d(\mathcal{H}^{(k)}) \right) \left(\prod_{k=0}^{N-1} \mathcal{X}_{(0,0)}^{(k, k+1 \bmod N)}(\xi_k) \right) \prod_{k=0}^{N-1} e^{-\langle \mathcal{H}^{(k)}, \mathcal{H}^{(k)} \rangle} e^{\langle \mathcal{H}^{(k)}, \mathcal{H}^{(k+1 \bmod N)} \rangle}.$$

2. *Distributivity over the addition: For $n_1, n_2 \in \mathbb{N}$, $z_1, \dots, z_{n_1+n_2} \in \mathbb{C}$, and $\xi_1, \dots, \xi_{n_1+n_2} \in \wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$,*

$$\left(\sum_{j=1}^{n_1} z_j \xi_j \right) \circ \left(\sum_{j=1}^{n_2} z_{j+n_1} \xi_{j+n_1} \right) = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} (z_{j_1} z_{j_2+n_1}) \xi_{j_1} \circ \xi_{j_2+n_1}.$$

3. *The unit of the circle product coincides with the unit $\mathbb{1}$ of the exterior product $\wedge$:*

$$z\mathbb{1} \circ \xi = \xi \circ z\mathbb{1} = z\xi, \quad z \in \mathbb{C}, \xi \in \wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}}).$$

4. *Clustering property: For any subspace $\mathcal{Y} \subset \mathcal{H}$ with orthogonal complement $\mathcal{Y}^\perp$,*

$$\xi_0 \circ \xi_1 = \xi_0 \wedge \xi_1, \quad \xi_0 \in \wedge^* \mathcal{Y}, \xi_1 \in \wedge^* \mathcal{Y}^\perp.$$

5. *For any $\varphi_1, \varphi_2 \in \mathcal{H}$ the Canonical Anti-commutation Relations (CAR) are satisfied:*

$$\varphi_1 \circ \varphi_2 = \varphi_1 \wedge \varphi_2 + \bar{\varphi}_2(\varphi_1) \mathbb{1}. \quad \blacklozenge$$

PROOF. For the proof see [ABPM17, Lemma 4.3.32] and [BP].

End

Remark 2.2.36. By Lemma 2.2.35, for $m, n \in \mathbb{N}_0$ so that $m+n \geq 1$, and all $\varphi_1, \dots, \varphi_{m+n} \in \mathcal{H}$,

$$(\bar{\varphi}_1) \circ \dots \circ (\bar{\varphi}_m) \circ \varphi_{m+1} \circ \dots \circ \varphi_{m+n} = (\bar{\varphi}_1) \wedge \dots \wedge (\bar{\varphi}_m) \wedge \varphi_{m+1} \wedge \dots \wedge \varphi_{m+n}.$$

If $m=0$, then there is no “ $\bar{\varphi}$ ” in the above equation. Mutatis mutandis for $n=0$. *

2.2.37 DEFINITION (INVOLUTION ON $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$). The antilinear map $\xi \mapsto \xi^*$ from $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ to itself is uniquely defined for any $\mathcal{H}$ by the conditions $1^* = 1$ and

$$(\bar{\varphi}_1 \circ \dots \circ \bar{\varphi}_m \circ \varphi_{m+1} \circ \dots \circ \varphi_{m+n})^* = \varphi_{m+n}^* \circ \dots \circ \varphi_{m+1}^* \circ \bar{\varphi}_m^* \circ \dots \circ \bar{\varphi}_1^*,$$

for any $\bar{\varphi}_1, \dots, \bar{\varphi}_m \in \bar{\mathcal{H}}$ and $\varphi_{m+1} \circ \dots \circ \varphi_{m+n} \in \mathcal{H}$ with $n, m \in \mathbb{N}_0$ such that $n+m \geq 1$. □

We can show that the Grassmann algebra $(\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}}), +, \wedge)$ is a $*$ -algebra (see discussions around Definition 2.2.2):

2.2.38 LEMMA (GRASSMANN ALGEBRA AS $*$ -ALGEBRA). $(\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}}), +, \wedge)$ equipped with the involution $*$ defined from Definition 2.2.37 is a $*$ -algebra, i.e.,

$$(2.2.39) \quad (\xi_0 \wedge \xi_1)^* = \xi_1^* \wedge \xi_0^*, \quad \xi_0, \xi_1 \in \wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}}). \quad \blacklozenge$$

PROOF. It suffices to show (2.2.39) for monomials ξ_0, ξ_1 in $\varphi \in \mathcal{H}$ and $\bar{\varphi} \in \tilde{\mathcal{H}}$, by linearity and antilinearity. This special case can be shown with straightforward computations. See for instance Remark 2.2.36. End

The algebra $(\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}}), +, \circ)$ endowed with this involution shares, by construction, all properties CAR algebra except for the norm:

Theorem 2.2.40 (From Grassmann algebra to CAR algebra):

Let $\mathcal{H}$ be a finite (one-particle) Hilbert space, then $(\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}}), +, \circ, *)$ is a $*$ -algebra generated by $\mathbb{1}$ and the family $\{\varphi^*\}_{\varphi \in \mathcal{H}}$ of elements satisfying 2.2.4. ⚡

PROOF. By 2.2.37, the map $\varphi \mapsto \varphi^*$ is antilinear and its involution is linear. Take $\varphi_1, \varphi_2 \in \mathcal{H}$, and from Lemma 2.2.35, in particular Assertion 5 and elementary computations we have

$$\varphi_1^* \circ \varphi_2 + \varphi_2 \circ \varphi_1^* = \bar{\varphi}_1 \circ \varphi_2 + \varphi_2 \circ \bar{\varphi}_1 = \langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} \mathbb{1}, \quad \varphi_1, \varphi_2 \in \mathcal{H}. \quad \text{End}$$

For $\mathcal{H}$ finite dimensional, recall that $\dim \mathcal{U} = 2^{2\dim \mathcal{H}} = \dim (\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}}))$, see discussions around Expression (2.3.13). Then, there is a canonical $*$ -isomorphism between the CAR C^* -algebra $\mathcal{U} \equiv \mathcal{U}(\mathcal{H})$ and $\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$:

2.2.41 DEFINITION (CANONICAL ISOMORPHISM OF $*$ -ALGEBRA). For the finite Hilbert space $\mathcal{H}$ we define the canonical isomorphism

$$\kappa: (\mathcal{U}, +, \cdot, *) \rightarrow (\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}}), +, \circ, *)$$

via the conditions $\kappa(z\mathbb{1}) = z\mathbb{1}$ and $\kappa(a(\varphi)) = \varphi^*$ for all $\varphi \in \mathcal{H}$. ▀

From definition 2.2.41 we note that for any $n, m \in \mathbb{N}_0$ with $m+n \geq 1$ and $\bar{\varphi}_1, \dots, \bar{\varphi}_m \in \tilde{\mathcal{H}}$ and $\varphi_{m+1} \circ \dots \circ \varphi_{m+n} \in \mathcal{H}$ we have that the isomorphism κ acts as:

$$\kappa(a(\bar{\varphi}_1) \cdots a(\bar{\varphi}_m) a(\varphi_{m+1}) \cdots a(\varphi_{m+n})) = \varphi_1 \cdots \varphi_m \bar{\varphi}_{m+1} \cdots \bar{\varphi}_{m+n},$$

where

$$\varphi_1 \cdots \varphi_m \bar{\varphi}_{m+1} \cdots \bar{\varphi}_{m+n} \doteq \varphi_1 \wedge \cdots \wedge \varphi_m \wedge \bar{\varphi}_{m+1} \wedge \cdots \wedge \bar{\varphi}_{m+n}.$$

Above, if $m = 0$, then there is neither “ $a(\bar{\varphi})$ ” nor “ φ ” in the equations. Mutatis mutandis for $n = 0$.

To make $(\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}}), +, \circ, *)$ a CAR (C^* –) algebra, it suffices to equip this $*$ –algebra with the following norm:

$$(2.2.42) \quad \|\xi\|_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})} \doteq \|\kappa^{-1}(\xi)\|_{\mathcal{U}}, \quad \xi \in \wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}}).$$

See Definition C.1.1 and Theorem 2.2.40. In this case, κ is an isometry, see Remark 2.2.1.

2.3 LATTICE FERMION SYSTEMS

In classical and quantum systems, it is often to deal with problems such that the number of particles changes in time and this quantity in general is non constant. This could be explained because the system into consideration interacts with its environment. In order to tackle these kind of systems, we usually employ the *second quantization* method, which allows to study successfully *many–body bosonic* and *fermionic* systems. In particular, it is well–known that in the fermionic context, such a depiction gives profound consequences. In fact, we will see that using Fock spaces ideas we can construct the algebraic structure to describe *Lattice Fermion Systems* using CAR C^* –algebras defined in previous section. Thus, in contrast to [ABPM17, ABPM16] we do not use self–dual CAR algebras.

2.3.1 FOCK SPACES AND SECOND QUANTIZATION

Let $\mathcal{F}(\mathcal{H})$ be an *appropriate* Hilbert space and $\mathcal{P}_n$ the set of particles $\{p_i\}_{i=1}^n$ that changes at the time. $\mathcal{F}(\mathcal{H})$ is called *Fock space*, where $\mathcal{H}$ is a (one–particle) Hilbert space. In quantum mechanics, typically $\mathcal{H}$ is separable. In fact, it is usual to deal with the Hilbert space of *square–integrable functions* $\mathcal{H} \doteq L^2(\mathcal{R})$. $\psi \in \mathcal{H}$ is a *normalized wave function* and $\mathcal{R} \subset \mathbb{R}^d$ is the region where the set of particles $\mathcal{P}_n$ are confined to move, such that the coordinate of the particle p_i is $x_i \in \mathbb{R}$ for $i = \{1, \dots, n\}$. Consider the n tensor product space

$$(2.3.1) \quad \mathcal{H}^n \doteq \mathcal{H} \otimes \dots \otimes \mathcal{H},$$

where $\mathcal{H}^0 \doteq \mathbb{C}$. Hence, for n *state vectors* of a single particle $f_1, \dots, f_n \in \mathcal{H}$, the vector $f_1 \otimes \dots \otimes f_n \in \mathcal{H}^n$ is associated to the state of the particle 1 in the state f_1 , the particle 2 in the state f_2 , and so on [AJP06]. We formally define the Fock space as

$$(2.3.2) \quad \mathcal{F}(\mathcal{H}) \doteq \bigoplus_{n \geq 0} \mathcal{H}^n.$$

Thus for an element $\Psi \in \mathcal{F}(\mathcal{H})$, Ψ is the sequence of functions $\{\psi_n\}_{n \geq 0}$ such that $\psi_0 \in \mathbb{C}$ and $\psi_n \in \mathcal{H}^n$ for $n \geq 1$. Explicitly Ψ is

$$(2.3.3) \quad \Psi \doteq \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \dots\},$$

where for $n \geq 0$, $\psi_n \doteq f_1 \otimes \dots \otimes f_n \in \mathcal{H}^n$ and

$$(2.3.4) \quad (f_1 \otimes \dots \otimes f_n)(x_1, \dots, x_n) \doteq f(x_1) \dots f(x_n),$$

with $x_1, \dots, x_n \in \mathcal{R}$. It is defined the inner product on $\mathcal{F}(\mathcal{H})$ as

$$\langle \Psi, \Phi \rangle \doteq \sum_{n \geq 0} \langle \psi_n, \phi_n \rangle_{\mathcal{H}^n}.$$

for $\Psi, \Phi \in \mathcal{F}(\mathcal{H})$. Note that for $\mathcal{H} = L^2(\mathcal{R})$, $\mathcal{R} \subset \mathbb{R}^d$ the norm is given by

$$\|\Psi\|^2 \doteq |\psi_0|^2 + \sum_{n \geq 1} \int_{\mathcal{R}^n} |\psi_n(x_1, \dots, x_n)|^2 \mu(dx_1) \dots \mu(dx_n) < \infty.$$

Regarding to the bosonic and fermionic statistics nature of particles, we need to consider two Fock spaces, namely *Bosonic* and *Fermionic Fock spaces*. $\mathcal{F}_{\pm}$ will denote the symmetric–bosonic (anti–symmetric–fermionic) Fock space, for bosons (fermions) and the $+$ ($-$) sign is used to represent that the associated wave functions are symmetric (anti–symmetric) by interchanging each pair of particles $p_i, p_j \in \mathcal{P}_n$ with $i \neq j$. In order to describe interchange of particles we introduce the operator $P_{\pm} \in \mathcal{B}(\mathcal{H}^n)$ of *symmetrization (anti–symmetrization)* on $\mathcal{F}(\mathcal{H})$ such that for $f_1, \dots, f_n \in \mathcal{H}$

$$(2.3.5) \quad P_{\pm}(f_1 \otimes \dots \otimes f_n) = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \varepsilon_{\pi} f_{\pi(1)} \otimes \dots \otimes f_{\pi(n)},$$

where $\mathcal{S}_n$ denotes the set of all permutations of $n \in \mathbb{N}$ elements, with $\pi \in \mathcal{S}_n$ a permutation, $\varepsilon_{\pi} \equiv 1$ for bosons and for fermions ε_{π} is $+1$ (-1) if the permutation is even (odd). Note that $P_{\pm}$ is an *orthogonal projection*, i.e., $P_{\pm}^2 = P_{\pm}$ and $P_{\pm} = P_{\pm}^*$. We now define the n –linear form $f_1 \wedge \dots \wedge f_n \in \wedge^n \mathcal{H}$ as

$$(2.3.6) \quad f_1 \wedge \dots \wedge f_n \doteq \frac{1}{\sqrt{n!}} \sum_{\pi \in \mathcal{S}_n} \varepsilon_{\pi} f_{\pi(1)} \otimes \dots \otimes f_{\pi(n)},$$

where for $n = 0$, $\wedge^0 \mathcal{H} \doteq \mathbb{C}$ and for $n \geq 1$

$$\wedge^n \mathcal{H} \doteq \text{lin}\{f_1 \wedge \dots \wedge f_n : f_1, \dots, f_n \in \mathcal{H}\}.$$

We are able to write using (2.3.4) that

$$(2.3.7) \quad (f_1 \wedge \cdots \wedge f_n)(x_1, \dots, x_n) = \frac{1}{\sqrt{n!}} \sum_{\pi \in S_n} \varepsilon_\pi f_{\pi(1)}(x_1) \cdots f_{\pi(n)}(x_n)$$

We finally define

$$\mathcal{F}_\pm(\mathcal{H}) \equiv P_\pm \mathcal{F}(\mathcal{H}) \doteq \bigoplus_{n \geq 0} P_\pm \mathcal{H}^n = \bigoplus_{n \geq 0} \wedge^n \mathcal{H},$$

such that in terms of linear forms, Fock spaces are denoted as

$$(2.3.8) \quad \wedge_\pm \mathcal{H} \doteq \mathcal{F}_\pm(\mathcal{H}),$$

which is the Fock space associated to the (one-particle) Hilbert space $\mathcal{H}$. In Chapter 4 we present a similar construction of *Grassmann algebras*, used in the fermionic case.

FERMIONIC SYSTEMS AND THE *wedge product* $\wedge$

Let $\mathcal{H}$ be a separable Hilbert space with a basis $\{e_1, \dots, e_{\dim \mathcal{H}}\}$. We consider the *wedge product* $\wedge$ presented in (2.3.6)–(2.3.7) for the fermion case [JOPP11]. Straightforward calculations show that $\{e_{j_1} \wedge \cdots \wedge e_{j_n} | 1 \leq j_1 < \cdots < j_n \leq \dim \mathcal{H}\}$ is a basis of the wedge product $\wedge^n \mathcal{H}$ for $n \leq \dim \mathcal{H}$ with $\dim \wedge^n \mathcal{H} = \binom{\dim \mathcal{H}}{n}$ and that terms of the form $e_{j_1} \wedge \cdots \wedge e_{j_n}$ for $n > \dim \mathcal{H}$ are $0 \in \wedge^n \mathcal{H}$. It follows that the inner product in $\mathcal{F}_-(\mathcal{H})$ is given by

$$\langle \Psi, \Phi \rangle = \sum_{n=0}^{\dim \mathcal{H}} \langle \psi_n, \phi_n \rangle_{\mathcal{H}^n}$$

and $\dim \mathcal{F}_-(\mathcal{H}) = \sum_{n=0}^{\dim \mathcal{H}} \binom{\dim \mathcal{H}}{n} = 2^{\dim \mathcal{H}}$.

Let $A \in \mathcal{L}(\mathcal{H})$ be a linear operator and consider $f_1, \dots, f_n \in \mathcal{H}$. For $n \geq 1$, we define the linear operators $\langle Q, AQ \rangle_n, d \langle Q, AQ \rangle_n \in \mathcal{L}(\wedge^n \mathcal{H})$ on $\wedge^n \mathcal{H}$ such that

$$\begin{aligned} \langle Q, AQ \rangle_n (f_1 \wedge \cdots \wedge f_n) &\doteq A f_1 \wedge \cdots \wedge A f_n, \text{ and} \\ d \langle Q, AQ \rangle_n (f_1 \wedge \cdots \wedge f_n) &\doteq A f_1 \wedge \cdots \wedge f_n + \cdots + f_1 \wedge \cdots \wedge A f_n, \end{aligned}$$

where for $n = 0$, $\langle Q, AQ \rangle_0(A) = \mathbb{1}$ and $d \langle Q, AQ \rangle_0(A) = 0$. We have the following:

2.3.9 PROPOSITION. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then, for $n \in \mathbb{N}$*

$$(i) \quad \langle Q, A^* Q \rangle_n = \langle Q, AQ \rangle_n^*,$$

$$(ii) \quad d \langle Q, A^* Q \rangle_n = d \langle Q, AQ \rangle_n^*,$$

$$(iii) \quad \langle Q, ABQ \rangle_n = \langle Q, AQ \rangle_n \langle Q, BQ \rangle_n,$$

$$(iv) \quad d \langle Q, (A + \lambda B)Q \rangle_n = d \langle Q, AQ \rangle_n + \lambda d \langle Q, BQ \rangle_n,$$

$$(v) \quad d \langle Q, AQ \rangle_n = \frac{d}{dt} \langle Q, e^{tA} Q \rangle_n \Big|_{t=0},$$

$$(vi) \quad \langle Q, e^{tA} Q \rangle_n = e^{d \langle Q, AQ \rangle_n}.$$

□

PROOF. For the sake of completeness we present some proofs of the assertions, namely (iii), (v) and (vi). Indeed, for (iii) we note that $\langle Q, AQ \rangle_n \langle Q, BQ \rangle_n (f_1 \wedge \cdots \wedge f_n)$ is

$$\langle Q, AQ \rangle_n (Bf_1 \wedge \cdots \wedge Bf_n) = ABf_1 \wedge \cdots \wedge ABf_n = \langle Q, ABQ \rangle_n (f_1 \wedge \cdots \wedge f_n),$$

and then $\langle Q, ABQ \rangle_n = \langle Q, AQ \rangle_n \langle Q, BQ \rangle_n$. Secondly, we write $\frac{d}{dt} \langle Q, e^{tA} Q \rangle_n (f_1 \wedge \cdots \wedge f_n) \Big|_{t=0}$ as

$$\begin{aligned} \frac{d}{dt} \langle Q, e^{tA} Q \rangle_n (f_1 \wedge \cdots \wedge f_n) \Big|_{t=0} &= \frac{d}{dt} (e^{tA} f_1 \wedge \cdots \wedge e^{tA} f_n) \Big|_{t=0} \\ &= (Ae^{tA} f_1 \wedge \cdots \wedge e^{tA} f_n + \cdots + e^{tA} f_1 \wedge \cdots \wedge Ae^{tA} f_n) \Big|_{t=0} \\ &= Af_1 \wedge \cdots \wedge f_n + \cdots + f_1 \wedge \cdots \wedge Af_n \\ &= d \langle Q, AQ \rangle_n (f_1 \wedge \cdots \wedge f_n), \end{aligned}$$

and $d \langle Q, AQ \rangle_n = \frac{d}{dt} \langle Q, e^{tA} Q \rangle_n \Big|_{t=0}$ follows. Finally, to show (vi), we use as induction hypothesis (v) in order to have that the m^{th} -derivative of $\langle Q, e^{tA} Q \rangle_n$ at $t = 0$ is

$$(2.3.10) \quad \frac{d^m}{dt^m} \langle Q, e^{tA} Q \rangle_n \Big|_{t=0} = (d \langle Q, AQ \rangle_n)^m.$$

Due to $A \in \mathcal{B}(\mathcal{H})$, we can use Measurable Functional Calculus as well as Taylor's expansion around $t = 0$ of the function $t \mapsto e^{tA}$ for $a \in \mathbb{R}$:

$$\langle Q, e^{tA} Q \rangle_n (f_1 \wedge \cdots \wedge f_n) = \left(\mathbb{1} + \sum_{m=1}^{\infty} \frac{t^m}{m!} \frac{d^m}{dt^m} \langle Q, e^{tA} Q \rangle_n \Big|_{t=0} \right) (f_1 \wedge \cdots \wedge f_n).$$

With (2.3.10) we have

$$\langle Q, e^{tA} Q \rangle_n (f_1 \wedge \cdots \wedge f_n) = \sum_{m=0}^{\infty} \frac{t^m}{m!} (d \langle Q, AQ \rangle_n)^m (f_1 \wedge \cdots \wedge f_n) = e^{td \langle Q, AQ \rangle_n} (f_1 \wedge \cdots \wedge f_n),$$

which concludes the proof. End

We now define for $A \in \mathcal{B}(\mathcal{H})$, the elements $\langle Q, AQ \rangle, d \langle Q, AQ \rangle \in \mathcal{B}(\wedge^* \mathcal{H})$ as

$$\langle Q, AQ \rangle \doteq \bigoplus_{n=0}^{\dim \mathcal{H}} \langle Q, AQ \rangle_n, \quad d \langle Q, AQ \rangle \doteq \bigoplus_{n=0}^{\dim \mathcal{H}} d \langle Q, AQ \rangle_n.$$

With Proposition (2.3.9) we can show:

2.3.11 COROLLARY. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then, for $n \in \mathbb{N}$

- (i) $\langle Q, A^*Q \rangle = \langle Q, AQ \rangle^*$,
- (ii) $d \langle Q, A^*Q \rangle = d \langle Q, AQ \rangle^*$,
- (iii) $\langle Q, ABQ \rangle = \langle Q, AQ \rangle \langle Q, BQ \rangle$,
- (iv) $d \langle Q, (A + \lambda B)Q \rangle = d \langle Q, AQ \rangle + \lambda d \langle Q, BQ \rangle$,
- (v) $d \langle Q, AQ \rangle = \left. \frac{d}{dt} \langle Q, e^{tA}Q \rangle \right|_{t=0}$,
- (vi) $\langle Q, e^A Q \rangle = e^{d \langle Q, AQ \rangle}$. ◻

In particular, if A is invertible, $\langle Q, AQ \rangle^{-1} = \langle Q, A^{-1}Q \rangle$. For $A = \mathbb{1}$

$$d \langle Q, \mathbb{1}Q \rangle = \bigoplus_{n=0}^{\dim \mathcal{H}} d \langle Q, \mathbb{1}Q \rangle_n = \bigoplus_{n=0}^{\dim \mathcal{H}} n,$$

and then $d \langle Q, \mathbb{1}Q \rangle$ is the observable that measure the number of particles in the system, and it is named the “number operator” N . In Chapter 3 we will use *second quantization operators* to analyse the free-fermion case such as explained in Chapter 1. These will have similar properties to those presented above for $\langle Q, AQ \rangle$ and $d \langle Q, AQ \rangle$ (see Definition 2.3.15 below).

ANNIHILATION AND CREATION OPERATORS IN BOSONIC AND FERMIONIC SYSTEMS

We now introduce “annihilation” and “creation operators” for bosons and fermion systems.

Fix $f, f_1, \dots, f_n \in \mathcal{H}$ and define $a(f), a^*(f) \in \mathcal{F}(\mathcal{H})$ as

$$\begin{aligned} a(f)(f_1 \otimes \dots \otimes f_n) &= \sqrt{n} \langle f, f_1 \rangle f_2 \otimes \dots \otimes f_n \\ a^*(f)(f_1 \otimes \dots \otimes f_n) &= \sqrt{n+1} f \otimes f_1 \otimes \dots \otimes f_n, \end{aligned}$$

i.e., $a(\cdot)$ maps $\mathcal{H}^n$ into $\mathcal{H}^{n-1}$ and $a^*(\cdot)$ maps $\mathcal{H}^n$ into $\mathcal{H}^{n+1}$. We now define $\Omega \in \mathcal{F}(\mathcal{H})$ as the *vacuum* vector such that $[\Omega]_0 \doteq 1 \in \mathcal{H}^0$ and $[\Omega]_n \doteq 0 \in \mathcal{H}^n$ for $n \geq 1$. Hence, $a(f)\Omega = 0$ and $a^*(f)\Omega = f$ for all $f \in \mathcal{H}$ and indeed physically vacuum vector is associated with the state $(1, 0, 0, \dots)$ without particles. Moreover, straightforward calculations show that $a(f)^* = a^*(f)$ and

$$a_{\pm}(f) = a(f)P_{\pm} \text{ and } a_{\pm}^*(f) = P_{\pm}a^*(f),$$

where $P_{\pm}$ is the symmetrization (anti-symmetrization) operator. For details see [AJP06, Pages 189–190] and [BR03b]. We now consider $\Psi \in \mathcal{F}_{\pm}(\mathcal{H})$ given by Expression (2.3.3), namely,

$$\Psi \doteq \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \dots\},$$

where $x_1, \dots, x_n \in \mathcal{R} \subset \mathbb{R}^d$, $\psi_0 \in \mathbb{C}$ and $\psi_n \in \mathcal{H}^n$ for $n \geq 1$. Then, we have for $x, x_1, \dots, x_n \in \mathcal{R}$ that

$$\begin{aligned} (a_{\pm}(f)\psi)_n(x_1, \dots, x_n) &= \sqrt{n+1} \sum_{x \in \mathcal{R}} \overline{f(x)} \psi_{n+1}(x, x_1, \dots, x_n) \\ (a_{\pm}^*(f)\psi)_n(x_1, \dots, x_n) &= \frac{1}{\sqrt{n}} \sum_{k=1}^n (\pm 1)^{k-1} f(x_k) \psi_{n-1}(x_1, \dots, \check{x}_k, \dots, x_n), \end{aligned}$$

where the symbol $\check{}$ means that the corresponding coordinate x_k was omitted. Hence, $a_{\pm}(\cdot)$ and $a_{\pm}^*(\cdot)$ are antilinear and linear operators respectively. Finally, we define $a_{\pm}(x)$ and $a_{\pm}^*(x)$ such that

$$\begin{aligned} (a_{\pm}(x)\psi)_n(x_1, \dots, x_n) &= \sqrt{n+1} \psi_{n+1}(x, x_1, \dots, x_n) \\ (a_{\pm}^*(x)\psi)_n(x_1, \dots, x_n) &= \frac{1}{\sqrt{n}} \sum_{k=1}^n (\pm 1)^{k-1} \delta(x - x_k) \psi_{n-1}(x_1, \dots, \check{x}_k, \dots, x_n), \end{aligned}$$

to write

$$(2.3.12) \quad a_{\pm}(f) = \sum_{x \in \mathcal{R}} \overline{f(x)} a_{\pm}(x) \quad \text{and} \quad a_{\pm}^*(f) = \sum_{x \in \mathcal{R}} f(x) a_{\pm}^*(x).$$

Since this Thesis is devoted to fermions systems we will pay attention in the properties of the underlying systems. However, we also can derive properties in the scope of the bosonic case. For instance, we can deduce the *Canonical Commutation Relations* (CCR) as follows: take the n -linear form $f_1 \wedge \dots \wedge f_n$ and apply $a_+(g)a_+^*(f)$ and $a_+^*(f)a_+(g)$ on the left side, where $f_1, \dots, f_n, f, g \in \mathcal{H}$, see Expression (2.3.7). By combining $a_+(g) = a(g)P_+$, $a_+^*(f) = P_+a^*(f)$ and $P_+^2 = P_+$, see (2.3.5), straightforward calculations show that

$$\begin{aligned} a_+(g)a_+^*(f)(f_1 \wedge \dots \wedge f_n) &= \sum_{k=1}^{n+1} \langle g, f_k \rangle P_+ f_1 \wedge \dots \wedge \check{f}_k \wedge \dots \wedge f_{n+1}, \quad \text{and} \\ a_+^*(f)a_+(g)(f_1 \wedge \dots \wedge f_n) &= \sum_{k=1}^n \langle g, f_k \rangle P_+ f_1 \wedge \dots \wedge \check{f}_k \wedge \dots \wedge f_{n+1}, \end{aligned}$$

where the symbol $\check{}$ indicates that the corresponding f_k was omitted and $f_{n+1} \doteq f$. As is usual $[A, B] \doteq AB - BA \in \mathcal{X}$ denotes the commutator between the elements $A \in \mathcal{X}$ and $B \in \mathcal{X}$. Then, subtracting last expressions we get

$$[a_+(g), a_+^*(f)] = \langle g, f \rangle \mathbb{1}_{\mathcal{F}_+(\mathcal{H})}.$$

In a similar way we can show that

$$[a_+(g), a_+(f)] = 0 \quad \text{and} \quad [a_+^*(g), a_+^*(f)] = 0.$$

Note that $a_+^*(\cdot)$ is an unbounded operator: Let $\mathfrak{f}_n \doteq f \otimes \cdots \otimes f \in \mathcal{H}^n$, $f \in \mathcal{H}$ with $\|f\| = 1$. Since $a_+^*(f)\mathfrak{f}_n = \sqrt{n+1}\mathfrak{f}_{n+1}$ we find that $\|a_+^*(f)\mathfrak{f}_n\| = \sqrt{n+1}$, and we get that for $n \rightarrow \infty$ indeed the norm is unbounded. For other basic remarks and comments see for example [BR03b, AJP06], and for recent rigorous results found by Lewin and collaborators see [Lew15] and references therein.

CANONICAL ANTI-COMMUTATION RELATIONS

Since in this work we are dealing with fermions, to lighten notations, we will omit any mention of the negative sign – for $a(f)$, $a^*(f)$ and $\wedge \mathcal{H} \equiv \mathcal{F}(\mathcal{H})$. Using a similar procedure to the bosonic case, described in the previous subsection, we also can obtain the *Canonical Anti-Commutation Relations* (CAR), which implement the Pauli exclusion principle. However, due to that we are interested in a more general framework, in order to characterize their abstract properties, we present CAR as follows:

As in section 2.2, let $\mathcal{H}$ be a Hilbert space and $\mathcal{U} \equiv \text{CAR}(\mathcal{H})$ be the (possible infinite dimensional) CAR C^* -algebra generated the identity $\mathbb{1}$ and $\{a(\psi)\}_{\psi \in \mathcal{H}}$ satisfying for all $\psi, \varphi \in \mathcal{H}$ Expressions (2.2.5), i.e.,

$$(2.3.13) \quad \{a(\psi), a(\varphi)\} = 0, \quad \{a(\psi), a(\varphi)^*\} = \langle \psi, \varphi \rangle_{\mathcal{H}} \mathbb{1}.$$

See discussions around this Expression and Definition 2.2.4. If $\mathcal{H}$ is finite dimensional, it can be shown that exists an isomorphism π between $\mathcal{U}$ and $\text{Mat}(2^{\dim \mathcal{H}}, \mathbb{C})$, the C^* -algebra of $2^{\dim \mathcal{H}} \times 2^{\dim \mathcal{H}}$ square matrices with complex entries [BR03b, Theorem 5.2.5]. This fact is used in the Work [ABPM17] (see Appendix C) to show that the self-dual CAR C^* -algebra $\mathbb{U} \equiv \text{sCAR}(\mathfrak{H}, \mathfrak{A})$ is isomorphic to the Grassmann algebra $\wedge^* \mathfrak{H}$. Here, $\mathfrak{H} \doteq \mathcal{H} \oplus \mathcal{H}^*$ and $\mathfrak{A}$ is an *antiunitary involution* on $\mathfrak{H}$ and $\mathfrak{H}^*$ is the dual space of $\mathfrak{H}$, see Definition C.1.1. The fermionic Fock space $\wedge^* \mathfrak{H}$ will fit more natural than $\wedge \mathfrak{H}$ in the Grassmann algebra approach of Chapter 4, see Expression (2.3.8). In particular, there exists an isomorphism $\bar{\kappa}: \mathcal{U} \rightarrow \mathcal{B}(\wedge \mathcal{H})$, such that for all $A \in \mathcal{U}$,

$$\text{tr}(A) = \frac{1}{\dim \wedge \mathcal{H}} \text{Tr}_{\wedge \mathcal{H}}(\bar{\kappa}(A)),$$

where $\text{Tr}_{\wedge \mathcal{H}}$ is the usual trace of linear operators in the (finite dimensional) Hilbert space $\wedge \mathcal{H}$:

$$\text{Tr}(A) \doteq \sum_{n=1}^{\dim \wedge \mathcal{H}} \langle e_n, A e_n \rangle.$$

with $\{e_1, \dots, e_{\dim \wedge \mathcal{H}}\}$ an *Orthonormal Basis* (ONB) of $\wedge \mathcal{H}$. Explicitly for the Fermion Fock space $\mathcal{F}(\mathcal{H})$

$$(2.3.14) \quad \text{tr}(A) = 2^{-\dim \mathcal{H}} \text{Tr}_{\mathcal{F}(\mathcal{H})}(\bar{\kappa}(A)).$$

We now define [JOPP11]:

2.3.15 DEFINITION (SECOND QUANTIZATION OPERATOR). Let $\mathcal{H}$ be a finite Hilbert space with orthonormal basis $\{\psi_j\}_{j \in J}$ and let $A \in \mathcal{B}(\mathcal{H})$ be a operator whose range is finite dimensional. We define the operators $\langle Q, AQ \rangle, d \langle Q, AQ \rangle \in \mathcal{U}$ as:

$$\langle Q, e^A Q \rangle \doteq e^{d \langle Q, AQ \rangle} \quad \text{with} \quad d \langle Q, AQ \rangle \doteq \sum_{i,j \in J} \langle \psi_j, A \psi_i \rangle_{\mathcal{H}} a(\psi_j)^* a(\psi_i). \quad \blacksquare$$

This definition reminiscents the elements defined previously to Corollary 2.3.11. In fact, assertions of such a Corollary are satisfied by above definition. Definition of $d \langle Q, AQ \rangle \in \mathcal{U}$ is nothing but the second quantization of the operator $A \in \mathcal{B}(\mathcal{H})$, which it will be used to tackle non-interacting fermion systems in Chapter 3. We will see that in the scope of interacting fermions other approach will be required. From definition note that for $\{\psi_j\}_{j \in J}$

$$\langle Q, AQ \rangle a(\psi_j)^* = a(A \psi_j)^* \langle Q, AQ \rangle \quad \text{and} \quad \langle Q, A^* Q \rangle a(\psi_j) = a(\psi_j) \langle Q, A^* Q \rangle,$$

and then for $\varphi \in \mathcal{H}$

$$(2.3.16) \quad e^{d \langle Q, AQ \rangle} a(\varphi)^* e^{-d \langle Q, AQ \rangle} = a(e^A \varphi)^* \quad \text{and} \quad e^{d \langle Q, AQ \rangle} a(\varphi) e^{-d \langle Q, AQ \rangle} = a(e^{-A^*} \varphi).$$

In particular, if $U \in \mathcal{B}(\mathcal{H})$ is unitary

$$(2.3.17) \quad \langle Q, UQ \rangle a(\varphi)^\# \langle Q, U^* Q \rangle = a(U\varphi)^\#,$$

where the symbol $\#$ stands for either creation or annihilation element of the C^* -algebra $\mathcal{U}$. In regard to commutation relations we are able to show that

$$[d \langle Q, AQ \rangle, a(\varphi)^*] = a(A\varphi)^* \quad \text{and} \quad [d \langle Q, AQ \rangle, a(\varphi)] = -a(A\varphi),$$

whereas if $B \in \mathcal{B}(\mathcal{H})$ we have

$$(2.3.18) \quad [d \langle Q, AQ \rangle, d \langle Q, BQ \rangle] = d \langle Q, [A, B]Q \rangle.$$

Moreover, note

$$(2.3.19) \quad d \langle Q, AQ \rangle^* = d \langle Q, A^* Q \rangle$$

so that

$$(2.3.20) \quad \Im \{d \langle Q, AQ \rangle\} = d \langle Q, \Im \{A\} Q \rangle$$

Finally, if A self-adjoint,

$$i[d \langle Q, AQ \rangle, a(\varphi)^\#] = a(iA\varphi)^\#.$$

2.3.2 ALGEBRAIC FORMULATION OF LATTICE FERMION SYSTEMS

We will consider without loss of generality (w.l.o.g.) the lattice cubic one since it is technically easier, but this choice is not necessary for our proofs. Thus, for $d \in \mathbb{N}$, the d -dimensional lattice $\mathcal{Q} \doteq \mathbb{Z}^d$ will represent the cubic crystal and $\mathcal{P}_f(\mathcal{Q}) \subset 2^{\mathcal{Q}}$ the set of all *finite* subsets of $\mathcal{Q}$. Note that we also can consider the set of *Spins* S , such that we define $\mathcal{Q} \doteq S \times \mathbb{Z}^d$, but since our results do not depend on spin it will be not used this definition. Moreover, because we are working with fermions, w.l.o.g. these can be treated as *negatively* charged particles. In fact, the cases of particles with spin and/or positively charged particles can be treated by exactly the same methods.

As is usual, $\mathcal{H} \equiv \ell^2(\mathcal{Q})$ will denote the (one-particle) Hilbert space on $\mathcal{Q}$ with ONB $\{e_x\}_{x \in \mathcal{Q}}$. In order to define the thermodynamic limit, we introduce for $L \in \mathbb{R}_0^+$ the increasing sequence

$$(2.3.21) \quad \Lambda_L \doteq \{(x_1, \dots, x_d) \in \mathcal{Q} : |x_1|, \dots, |x_d| \leq L\} \in \mathcal{P}_f(\mathcal{Q}).$$

of side length $\ell = 2[L] + 1$. Note that such a sequence is a “Van Hove net”, i.e., the volume of the boundaries¹ $\partial\Lambda_L \subset \Lambda_L \in \mathcal{P}_f(\mathcal{Q})$ must be negligible w.r.t. the volume of Λ_L for L large enough: $\lim_{L \rightarrow \infty} \{|\partial\Lambda_L|/|\Lambda_L|\} = 0$.

The CAR C^* -algebra is $\mathcal{U} \equiv \mathcal{U}(\ell^2(\mathcal{Q}))$ with C^* -subalgebra $\mathcal{U}_\Lambda \equiv \mathcal{U}(\ell^2(\Lambda))$, for $\Lambda \in \mathcal{P}_f(\mathcal{Q})$. The latter is generated by the identity $\mathbb{1}$ and $\{a(e_x)\}_{x \in \Lambda}$ satisfying

$$(2.3.22) \quad \{a(e_x), a(e_{x'})\} = 0, \quad \{a(e_x), a(e_{x'})^*\} = \delta_{x,x'} \mathbb{1}.$$

$\mathcal{U}_\Lambda$ is called the *local fermion algebras* of the lattice $\mathcal{Q}$. Indeed, in quantum statistical mechanics $a(e_x)^*$ and $a(e_x)$ are interpreted, respectively, as the creation and annihilation of a fermion at the position $x \in \mathcal{Q}$ of the lattice according to Pauli exclusion principle. Note that for any $\Lambda \subset \Lambda' \in \mathcal{P}_f(\mathcal{Q})$ with $A \in \mathcal{U}_\Lambda$, we can associate the element $A' \in \mathcal{U}_{\Lambda'}$ such that $A' : A \mapsto A \otimes \mathbb{1}_{\mathcal{U}_{\Lambda' \setminus \Lambda}}$. This shows that $\mathcal{U}_\Lambda$ is a subalgebra of $\mathcal{U}_{\Lambda'}$ [AJP06, pag. 137]. C^* -algebras constructed like the previous one are called *uniformly hyperfinite* or *UHF*, see [EK98, §2.7]. Recall that physically $\Lambda \subset \Lambda' \in \mathcal{P}_f(\mathcal{Q})$ means that the system described by $\mathcal{U}_\Lambda$ is embedded in that described by $\mathcal{U}_{\Lambda'}$. Thus, the C^* -algebra $\mathcal{U}$ is well defined via $\mathcal{U}_\Lambda$. More precisely, $\mathcal{U}$ is the inductive limit of local algebras $\{\mathcal{U}_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathcal{Q})}$ and it is known as *fermion algebra*. In fact, $\mathcal{U}$ generally is taken as the

¹By fixing $m \geq 1$, the boundary $\partial\Lambda$ of any $\Lambda \subset \Gamma$ is defined by $\partial\Lambda \doteq \{x \in \Lambda : \exists y \in \Gamma \setminus \Lambda \text{ with } d(x, y) \leq m\}$, where $d(x, y) : \mathcal{Q} \times \mathcal{Q} \rightarrow [0, \infty)$ is the usual euclidean distance between x, y in the lattice $\mathcal{Q}$ [BP13].

completion of $\mathcal{U}_0$, being this

$$(2.3.23) \quad \mathcal{U}_0 \doteq \bigcup_{\Lambda \in \mathcal{P}_t(\mathfrak{V})} \mathcal{U}_\Lambda,$$

or also through Expression (2.3.21)

$$(2.3.24) \quad \mathcal{U}_0 \doteq \bigcup_{L \in \mathbb{R}_0^+} \mathcal{U}_{\Lambda_L}.$$

We now note that for $\psi \in \mathcal{H}$ the annihilation and creation operators $a(\psi), a(\psi)^* \in \mathcal{U}$ are written using Expression (2.3.12) as

$$(2.3.25) \quad a(\psi) \doteq \sum_{x \in \mathfrak{V}} \overline{\psi(x)} a(e_x), \quad a(\psi)^* \doteq \sum_{x \in \mathfrak{V}} \psi(x) a(e_x)^*,$$

For $\theta \in \mathbb{R} \setminus (2\pi\mathbb{Z})$ we define the unique *Bogoliubov* $*$ -automorphism σ_θ on $\mathcal{U}$ as

$$(2.3.26) \quad \sigma_\theta(a(e_x)) \doteq a(e^{i\theta} e_x) = e^{-i\theta} a(e_x),$$

since a is an antilinear map. Elements $A, B, C \in \mathcal{U}$ such that $\sigma_\pi(A) = A, \sigma_\pi(B) = -B$ and $\sigma_\theta(C) = C$, with $\theta \in [0, 2\pi)$ are called *even*, *odd* and *gauge invariant* respectively. The sets

$$\mathcal{U}^+ \doteq \{A \in \mathcal{U} | \sigma_\pi(A) = A\} \quad \text{and} \quad \mathcal{U}^- \doteq \{B \in \mathcal{U} | \sigma_\pi(B) = -B\} \subset \mathcal{U}$$

are called the *even* and *odd parts* of $\mathcal{U}$. By continuity of σ_θ , $\mathcal{U}^+$ is closed and hence a C^* -algebra, called *sub-algebra of even*. Using the same argument, the set

$$\mathcal{U}^\circ \doteq \bigcap_{\theta \in \mathbb{R} \setminus (2\pi\mathbb{Z})} \{A \in \mathcal{U} | \sigma_\theta(A) = A\} \subset \mathcal{U}^+$$

of all gauge invariant elements is also a C^* -algebra called *fermion observable algebra* [BP13].

2.3.3 STATES AND INTERACTIONS

The linear functional $\rho \in \mathcal{U}^*$ is a state if it is positive and normalized, i.e., if for all $A \in \mathcal{U}$, $\rho(A^*A) \geq 0$ and $\rho(\mathbb{1}) = 1$. Note that any ρ is continuous and *Hermitian*, i.e., for all $A \in \mathcal{U}$, $\rho(A^*) = \overline{\rho(A)}$. $\rho \in \mathcal{U}^*$ is said to be “faithful” if $A = 0$ whenever $A \geq 0$ and $\rho(A) = 0$. We denote by $E \subset \mathcal{U}^*$ the set of all states on $\mathcal{U}$. Since $\mathcal{U}$ has identity, by [BR03a, Theorem 2.3.15], we note that E is a *weak*-compact* convex set, such that its *extremal* points are the pure states (see Definition A.2.16). For any $\Lambda \in \mathcal{P}_t(\mathfrak{V})$, ρ_Λ and E_Λ denote the restriction of any $\rho \in E$ on the local sub-algebra $\mathcal{U}_\Lambda$ and the set of all states ρ_Λ on $\mathcal{U}_\Lambda$, respectively. States ρ of $\mathcal{U}$, with $\rho \circ \sigma_\pi = \rho$ are called “even states” of the C^* -algebra $\mathcal{U}$, see Expression (2.3.26).

One important class of faithful states on $\mathcal{U}$ are the called “quasi-free” states. Quasi-free states on $\mathcal{U}$ are those states that satisfy for all $m, n \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_{m+n} \in \mathcal{H}$:

$$\rho(a(\varphi_1)^* \cdots a(\varphi_m)^* a(\varphi_{m+n}) \cdots a(\varphi_{m+n})) = \det [\rho(a(\varphi_k)^* a(\varphi_{N+l}))]_{k,l=1}^n \delta_{m,n}$$

The operator $S_\rho \in \mathcal{B}(\mathcal{H})$ defined by

$$\langle \varphi_2, S_\rho \varphi_1 \rangle_{\mathcal{H}} = \rho(a(\varphi_1)^* a(\varphi_2)), \quad \varphi_1, \varphi_2 \in \mathcal{H},$$

is named the “symbol” of the quasi-free state ρ . Note that, by positivity and normalization of states, it follows that symbols are positive operators with spectrum $\text{spec}(S)$ lying on the unit interval $[0, 1]$. Conversely, any such positive operator $0 \leq S \leq \mathbb{1}$ (with the association $\mathbb{1} \equiv \mathbb{1}_{\mathcal{H}}$) on $\mathcal{H}$ uniquely defines a quasi-free state ρ_S on $\mathcal{U}$ such that

$$\rho_S(a(\varphi_1)^* a(\varphi_2)) = \langle \varphi_2, S \varphi_1 \rangle_{\mathcal{H}}, \quad \varphi_1, \varphi_2 \in \mathcal{H}.$$

Of course, the state ρ_S depends not only on S , but also on the choice of generators $\{a(\varphi)\}_{\varphi \in \mathcal{H}}$ of $\mathcal{U}$. If $\text{spec}(S) \subset (0, 1)$ we define

$$h_S \doteq \beta^{-1} \log(S^{-1} - \mathbb{1}),$$

where $\beta \in (0, \infty)$ is a fixed parameter physically interpreted as being the inverse temperature. The selfadjoint operator $h_S \in \mathcal{B}(\mathcal{H})$ is named the “one-particle Hamiltonian” associated to the symbol S at inverse temperature β . In other words the *symbol* S is seen, in this case, as the Fermi–Dirac distribution, at inverse temperature β , associated to the (bounded) Hamiltonian h_S :

$$(2.3.27) \quad S = \frac{\mathbb{1}}{\mathbb{1} + e^{\beta h_S}},$$

so that

$$(2.3.28) \quad \rho_S(a(\varphi_1)^* a(\varphi_2)) = \left\langle \varphi_2, \frac{\mathbb{1}}{\mathbb{1} + e^{\beta h_S}} \varphi_1 \right\rangle_{\mathcal{H}}, \quad \varphi_1, \varphi_2 \in \mathcal{H}.$$

One important special case of quasi-free state of $\mathcal{U}$ is the so-called “trace state” or “chaotic state”, denoted by $\text{tr} \in \mathcal{U}^*$ and defined by the condition

$$(2.3.29) \quad S_{\text{tr}} = \frac{1}{2} \mathbb{1}.$$

Equivalently, $h_{S_{\text{tr}}} = 0$. This particular name comes from the fact that physically it corresponds to the state of *maximal entropy* which occurs at infinite temperature [AJPP06]. See Appendices

[A–B](#) for more details of states, in particular in [Appendix B](#) we define the *KMS states*, which are useful to study physical systems at equilibrium.

Remark 2.3.30 (Entropy). If $\mathcal{H}$ is a finite Hilbert space, with $\mathcal{U}$ a CAR C^* -algebra associated to $\mathcal{H}$, and $A \in \mathcal{B}(\mathcal{H})$ is a positive self-adjoint operator we define the *von Neumann entropic functional* as

$$(2.3.31) \quad S(A) = -\text{Tr}(A \ln(A)),$$

where $0 \ln 0 \doteq 0$. If $B \in \mathcal{B}(\mathcal{H})$ is another self-adjoint operator with $\ker B \subset \ker A$, the *relative entropic functional* is given by

$$(2.3.32) \quad S(A, B) = \text{Tr}(A(\ln(A) - \ln(B))).$$

If $\rho, \omega \in E \subset \mathcal{U}^*$ we also can define the above entropic functionals as

$$(2.3.33a) \quad S(\rho) = -\text{Tr}(\rho \ln \rho), \quad \text{von Neumann entropy,}$$

$$(2.3.33b) \quad S(\rho, \omega) = \text{Tr}(\rho(\ln \rho - \ln \omega)), \quad \text{relative entropy.}$$

von Neumann entropy, in a sense, measures the amount of *randomness* carried by the state [\[BP13\]](#). On the other hand, relative entropy is related to the *entropy production*, which is the rate of change of the relative production with respect to the reference state ρ [\[AJPP06\]](#). *

A “ $*$ -automorphism” $\chi: \mathcal{U} \rightarrow \mathcal{U}$ is an $*$ -morphism of $\mathcal{U}$ into itself and $\ker \chi = \{0\}$, see [Definition A.2.12](#). Thus, we define the family of “spatial translations” $\{\chi_x\}_{x \in \mathfrak{L}}$ as those $*$ -automorphisms of $\mathcal{U}$ that satisfy:

$$(2.3.34) \quad \chi_x(a(e_y)) \doteq a(e_{x+y}), \quad y \in \mathfrak{L}.$$

Note that for $x \in \mathfrak{L}$ the $*$ -automorphism χ_x is unique. On the other hand, we say that the state $\rho \in \mathcal{U}^*$ is “translation invariant” if for all $x \in \mathfrak{L}$, $\rho \circ \chi_x = \rho$. In the sequel E_1 will denote the set of all translation invariant states of $\mathcal{U}$. The elements of E_1 turn out to be all even states. We define:

2.3.35 DEFINITION (ERGODIC STATES). We say that a state $\rho \in \mathcal{U}^*$ is *ergodic* if, for all $A \in \mathcal{U}$, $\rho(M_L^A)$ converges, as $L \rightarrow \infty$, and

$$\lim_{L \rightarrow \infty} \rho \left(\left(M_L^A - \rho(M_L^A) \mathbb{1} \right)^* \left(M_L^A - \rho(M_L^A) \mathbb{1} \right) \right) = 0,$$

where for the increasing sequence boxes Λ_L [\(2.3.21\)](#), $M_L^A \in \mathcal{U}$ is the *empirical mean*

$$(2.3.36) \quad M_L^A \doteq \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \chi_x(A). \quad \blacksquare$$

Existence of such ergodic states is well-known for $\rho \in E_1$. Thus, for $\rho \in E_1$, $\rho(M_L^A) = \rho(A)$ for all $A \in \mathcal{U}$. For E_1 , ergodic states are exactly the extreme point of the convex set E_1 of translation invariant states on $\mathcal{U}$ [BP13, §1.2 and Chapter 4]. By the weak*-compactness of E_1 and the *Krein–Milman theorem*, the set of ergodic states is not empty and generates E_1 as its closed convex hull. In addition, for any $M = M^* \in \mathcal{U}$, the non-negative number

$$\rho((M - \rho(M)\mathbb{1})^2) \geq 0$$

quantifies the fluctuation, around the expected value $\rho(M)$, of the physical quantity associated with the observable M in the state ρ .

In the next, we define *interactions* and *short-range interactions* in the fermionic context:

2.3.37 DEFINITION (FERMIONIC INTERACTIONS). An “interaction” is a function $\Phi: \mathcal{P}_f(\mathfrak{L}) \rightarrow \mathcal{U}$ such that for $\Lambda \subset \mathcal{P}_f(\mathfrak{L})$: $\Phi_\Lambda = \Phi_\Lambda^* \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$ and $\Phi_\emptyset = 0$. We also define:

1. If $\text{diam } \Lambda \doteq \max\{d(x, y): x, y \in \Lambda\}$ is the “diameter” of Λ , Φ is “finite-range” if for $\Lambda \subset \mathcal{P}_f(\mathfrak{L})$ and some $R > 0$, $\text{diam } \Lambda > R$ implies that $\Phi_\Lambda = 0$. If R is infinite we say that Φ is “long-range”.
2. Φ is “translationally invariant” if for all $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ and $x \in \mathfrak{L}$ we have

$$\chi_x(\Phi_\Lambda) \doteq \Phi_{\Lambda+x},$$

where $\Lambda + x \doteq \{y + x: y \in \Lambda\}$. If, in addition, Φ is finite range, we say that Φ is “translationally invariant finite range”.

3. For two interactions Φ, Ψ , the full interaction is endowed by the vector space structure

$$(\lambda_1\Phi + \lambda_2\Psi)_\Lambda \doteq \lambda_1\Phi_\Lambda + \lambda_2\Psi_\Lambda,$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\Lambda \in \mathcal{P}_f(\mathfrak{L})$. ▀

Definition of interaction in Fermionic Lattice Systems is different to that given for Quantum Spin Systems (QSS) in one crucial fact². In the latter, disjoint local algebras commute, i.e., for $\Lambda_1, \Lambda_2 \in \mathcal{P}_f(\mathfrak{L})$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$, implies $[\mathcal{U}_{\Lambda_1}, \mathcal{U}_{\Lambda_2}] = 0$. However, when dealing with fermions such a relation not necessarily is true. Instead, we impose interactions to belong to even elements of the algebra $\mathcal{U}$. This is justified as follows: let $\Lambda_1, \Lambda_2 \in \mathcal{P}_f(\mathfrak{L})$ be disjoint regions of $\mathfrak{L}$. For $i = \{1, 2\}$, $A_i \in \mathcal{U}_{\Lambda_i}$ are self-adjoint elements, with even and odd elements $A_i^\pm$

²In the context of QSS, for the region $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, we copy a Hilbert space $\mathcal{H}$ over all point $x \in \Lambda$ in such a way that the Hilbert space associated to Λ is $\mathcal{H}_\Lambda \equiv \bigotimes_{x \in \Lambda} \mathcal{H}_x$. Thus, the local C^* -algebra $\mathcal{U}_\Lambda$ related to $\mathcal{H}_\Lambda$ is isomorphic to $\mathcal{B}(\mathcal{H}_\Lambda)$. In this case the interactions belong to $\mathcal{U}^+ = \mathcal{U}$.

and A_i^- respectively. By the identity

$$[AB, CD] = A\{B, C\}D - \{A, C\}BD + CA\{B, D\} - C\{A, D\}B, \quad A, B, C, D \in \mathcal{U}$$

we have $[A_1^+, A_2^+] = [A_1^+, A_2^-] = [A_1^-, A_2^+] = \{A_1^-, A_2^-\} = 0$ while $[A_1^-, A_2^-] \neq 0$. Thus, if an interaction has odd elements, the *infinitesimal generator* of the time evolution τ_t (see (2.3.70) and Definition 2.3.46 below), would not be a physical observable and could not be a limit of commutators. See [AM03, §5.1] and [Rue74, §7.1].

For the box Λ_L , defined by (2.3.21), $\mathcal{U}_L \equiv \mathcal{U}_{\Lambda_L} \subset \mathcal{U}$ will denote the local C^* -subalgebra. Thus, Definition 2.3.37 allows to define space average of interactions $\{\Psi_\Lambda\}_{\Lambda \subset \Lambda_L} \in \mathcal{U}^+ \cap \mathcal{U}_L$ in the box Λ_L :

$$(2.3.38) \quad A_L^\Psi \doteq \frac{1}{|\Lambda_L|} \sum_{\Lambda \subset \Lambda_L} \Psi_\Lambda \in \mathcal{U}^+ \cap \mathcal{U}_L,$$

compare with (2.3.36).

2.3.39 LEMMA (ERGODIC STATES). *A state $\rho \in \mathcal{U}^*$ is ergodic iff, for any translation-invariant and finite-range interaction Ψ , $\rho(A_L^\Psi)$ converges, as $L \rightarrow \infty$, and*

$$\lim_{L \rightarrow \infty} \rho \left(\left(A_L^\Psi - \rho(A_L^\Psi) \mathbb{1} \right)^2 \right) = 0. \quad \blacklozenge$$

PROOF. See [ABPM17, Lemma 8.2].

End

SHORT-RANGE INTERACTIONS

We define Banach spaces of short-range interactions by introducing specific norms for interactions, taking into account space decay.

To this end, following [NOS06, Equations. (1.3)–(1.4)], we consider positive-valued and *non-increasing* decay functions $\mathbf{F}: \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ satisfying the following properties:

- *Summability on $\mathfrak{Q}$.*

$$(2.3.40) \quad \|\mathbf{F}\|_{1, \mathfrak{Q}} \doteq \sup_{y \in \mathfrak{Q}} \sum_{x \in \mathfrak{Q}} \mathbf{F}(|x - y|) = \sum_{x \in \mathfrak{Q}} \mathbf{F}(|x|) < \infty.$$

- *Bounded convolution constant.*

$$(2.3.41) \quad \mathbf{D} \doteq \sup_{x, y \in \mathfrak{Q}} \sum_{z \in \mathfrak{Q}} \frac{\mathbf{F}(|x - z|) \mathbf{F}(|z - y|)}{\mathbf{F}(|x - y|)} < \infty.$$

- *Logarithmic superadditivity.*

$$(2.3.42) \quad \mathbf{F}(r_1 + r_2) \geq \mathbf{F}(r_1) \mathbf{F}(r_2), \quad r_1, r_2 \in \mathbb{R}_0^+.$$

In the case $\mathcal{Q}$ would be a general countable set with infinite cardinality and some metric d , the existence of such a function $\mathbf{F}$ satisfying (2.3.40)–(2.3.41) with $d(\cdot, \cdot)$ instead of $|\cdot - \cdot|$ refers to the so-called *regular* property of $\mathcal{Q}$. For any $d \in \mathbb{N}$, $\mathcal{Q} \doteq \mathbb{Z}^d$ is in this sense regular with the metric $d(\cdot, \cdot) = |\cdot - \cdot|$. Indeed, a typical example of such a $\mathbf{F}$ for $\mathcal{Q} = \mathbb{Z}^d$, $d \in \mathbb{N}$, and the metric induced by $|\cdot|$ is the function

$$(2.3.43) \quad \mathbf{F}(r) \doteq (1 + r)^{-(d+\varsigma)} \quad r \in \mathbb{R}_0^+,$$

which has convolution constant $\mathbf{D} \leq 2^{d+1+\varsigma} \|\mathbf{F}\|_{1,\mathcal{Q}}$ for $\varsigma \in \mathbb{R}^+$. See [NOS06, Eq. (1.6)] or [Sim11, Example 3.1]. Note that the exponential function $\mathbf{F}(r) = e^{-\varsigma r}$, $\varsigma \in \mathbb{R}^+$, satisfies (2.3.40) and (2.3.42) but not (2.3.41). Nevertheless, as observed in [Sim11, §3.1], the multiplication of such a function $\mathbf{F}$ with a non-increasing weight $f: \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ satisfying (2.3.42) does not increase the convolution constant $\mathbf{D}$. An example of a family of logarithmically superadditive functions is given by

$$f(x) = \left(1 + \sum_{n=1}^N \frac{(\varepsilon x)^n}{(n!)^\vartheta} \right)^{-1}, \quad \varepsilon \in \mathbb{R}^+, \vartheta \geq 1, N \in \mathbb{N} \cup \{\infty\}.$$

Observe that Condition (2.3.42) is not necessary for the existence of infinite volume dynamics, but to obtain later estimates on correlation functions.

The function $\mathbf{F}$ encodes the short-range property of interactions. Indeed, an interaction Φ is said to be *short-range* if

$$(2.3.44) \quad \|\Phi\|_{\mathcal{W}} \doteq \sup_{x,y \in \mathcal{Q}} \sum_{\Lambda \in \mathcal{P}_f(\mathcal{Q}), \Lambda \supset \{x,y\}} \frac{\|\Phi_\Lambda\|_{\mathcal{Q}}}{\mathbf{F}(|x-y|)} < \infty.$$

Since the map $\Phi \mapsto \|\Phi\|_{\mathcal{W}}$ defines a norm on interactions, the space of short-range interactions w.r.t. to the decay function $\mathbf{F}$ is the real separable Banach space $\mathcal{W} \equiv (\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ of all interactions Φ with $\|\Phi\|_{\mathcal{W}} < \infty$. Note that a short-range interaction $\Phi \in \mathcal{W}$ is not necessarily weak away from the origin of $\mathcal{Q}$: Generally, the element $\Phi_{x+\Lambda}$, $x \in \mathcal{Q}$, does not vanish when $|x| \rightarrow \infty$.

2.3.4 TIME *-AUTOMORPHISMS AND TEMPORAL EVOLUTION

Dynamics in Quantum Mechanics is understood as the change of observables, states, wave functions, etc., which describe a physical system. In order to study this, different approaches are well-known, namely *Schrödinger*, *Heisenberg* and *Dirac pictures* (or *Interaction picture*). Schrö-

dinger picture studies the evolution of state vectors, called *kets*, by assuming that the observables, e.g., position, momentum or kinetic energy, are fixed at all time. Thus, the evolution of a system is entirely determined by solving the *Schrödinger equation* involving state vectors [SN11]. On the other hand, the Heisenberg picture studies systems when the observables evolve over time while the state vectors remain stationary. Therefore, in this picture the observables are responsible for the dynamics of the system. This approach does that the equations to be used have a similar form that in *Classical Mechanics* [CDL77, Chapter III, Complement G]. Recall that if $\mathcal{U}$ is a C^* -algebra describing a system, both approaches suppose that there is a self-adjoint operator called *Hamiltonian* $H \in \mathcal{U}$, related to the energy of the system. Roughly speaking, the latter provides a family of *temporal evolution* operators $\{O_t^{(H)}\}_{t \in \mathbb{R}} \in \mathcal{U}$, which are employed in such a way that state vectors or operators evolve depending explicitly on these. In particular, if there is not interaction between particles, the dynamics of $\mathcal{U}$ is studied via unitary operators. Finally, the Dirac picture is an intermediary representation between Schrödinger and Heisenberg pictures, and it is generally applied when there are explicitly time-dependent interaction terms in the Hamiltonian.

Based on above comments, it is important to highlight that when working with operators the Heisenberg picture is used, while when working with states the Schrödinger picture is the chosen. As follows, we briefly introduce the mathematical formalism of the temporal evolution of operator algebras [EBN⁺06]:

Firstly, note that if $\mathcal{U}$ is a C^* -algebra, the $*$ -automorphism $\tau: \mathcal{U} \rightarrow \mathcal{U}$, satisfies that $\tau(A^*) = (\tau(A))^*$ and $\tau(AB) = \tau(A)\tau(B)$, for all $A, B \in \mathcal{U}$ as well as $\|\tau(A)\| = \|A\|$. $*$ -automorphisms τ are closely linked to the time evolution of the elements of $\mathcal{U}$. With this in mind we define:

2.3.45 DEFINITION (C_0 -GROUP OF $*$ -AUTOMORPHISMS). Let $\mathcal{U}$ be a C^* -algebra with identity $\mathbb{1}$. We denote to the family $\tau \doteq \{\tau_t\}_{t \in \mathbb{R}}$ of $*$ -automorphisms on $\mathcal{U}$ as the “strongly continuous group” of $*$ -automorphisms—or “ C_0 -group” of $*$ -automorphisms—if $\tau_0 = \mathbb{1}$ and $\tau_s \circ \tau_t = \tau_{s+t}$ for all $s, t \in \mathbb{R}$. Here, we understand that τ is a strongly continuous group $*$ -automorphisms if the function

$$\tau(A): \mathbb{R} \rightarrow \mathcal{U}, \quad t \mapsto \tau_t(A)$$

is continuous in norm, i.e., $\lim_{t \rightarrow 0} \|\tau_t(A) - A\| = 0$, for every $A \in \mathcal{U}$. ▀

2.3.46 DEFINITION (GENERATOR). Let $\mathcal{U}$ be a C^* -algebra with identity $\mathbb{1}$. A C^* -dynamical system is a pair $(\mathcal{U}, \tau)$, where $\tau \doteq \{\tau_t\}_{t \in \mathbb{R}}$ is a strongly continuous group of $*$ -automorphisms. Define the linear subspace

$$\mathcal{D}(\delta) \doteq \{A \in \mathcal{U} : t \mapsto \tau_t(A) \text{ is differentiable at } t = 0\} \subset \mathcal{U}$$

and the linear operator $\delta: \mathcal{D} \rightarrow \mathcal{U}$ by

$$\delta(A) \doteq \left. \frac{d\tau_t(A)}{dt} \right|_{t=0}.$$

The operator δ is called the (unique, generally unbounded) *generator* of τ and $\mathcal{D}(\delta)$ is the (dense) domain of definition of δ . $\blacksquare$

Remarks 2.3.47.

1. In the sequel we will suppose that δ is a *symmetric unbounded derivation* [BR03a, Definition 3.2.21], i.e., the domain $\mathcal{D}(\delta)$ of δ is a dense $*$ -subalgebra of $\mathcal{U}$ and, for all $A, B \in \mathcal{D}(\delta)$,

$$\delta(A)^* = \delta(A^*), \quad \delta(AB) = \delta(A)B + A\delta(B).$$

2. As remarked in [AM03], in order to define dynamics in Lattice Fermion System in general is taken $\mathcal{D}(\delta) = \mathcal{U}_0$ as the domain of δ , where $\mathcal{U}_0$ is given by Expression (2.3.23) or (2.3.24).
3. Every generator of a C_0 -group τ of $*$ -automorphisms is a derivation on some dense $*$ -subalgebra of $\mathcal{U}$, but the converse does not generally hold.
4. Note that the set of all symmetric derivation on $\mathcal{U}_0$ can be endowed with a real vector space structure: For any symmetric derivations δ_1 and δ_2 and all real numbers α_1, α_2 , the definition

$$(\alpha_1\delta_1 + \alpha_2\delta_2)(A) \doteq \alpha_1\delta_1(A) + \alpha_2\delta_2(A), \quad A \in \mathcal{U}_0,$$

gives rise to another symmetric derivation $\alpha_1\delta_1 + \alpha_2\delta_2$ on $\mathcal{U}_0$. $\ast$

Regarding to the fermion case, to analyse its dynamics we recall the increasing sequence of cubic boxes $\Lambda_L, L \in \mathbb{R}_0^+$ given by (2.3.21). Thus, $\mathcal{U}_L \equiv \mathcal{U}_{\Lambda_L}$ is the local C^* -subalgebra, generated by the identity $\mathbb{1}$ and $\{a(e_x)\}_{x \in \Lambda_L}$. In this case, the *finite volume* dynamics corresponds to the continuous group $\{\tau_t^{(L)}\}_{t \in \mathbb{R}}$ of $*$ -automorphisms of $\mathcal{U}$ defined for $A \in \mathcal{U}$ by:

$$(2.3.48) \quad \tau_t^{(L)}(A) \doteq e^{itH_L} A e^{-itH_L},$$

with the generator given via *Duhamel's identity* (see Proposition A.2.26):

$$(2.3.49) \quad \delta^{(L)}(A) = i[H_L, A].$$

Here $H_L = H_L^* \in \mathcal{U}_L$ is the Hamiltonian in Λ_L , which explicitly is

$$(2.3.50) \quad H_L \equiv H_{\Lambda_L}(\Phi) \doteq \sum_{\Lambda \subset \Lambda_L} \Phi_\Lambda,$$

where $\{\Phi_\Lambda\}_{\Lambda \subset \Lambda_L} \in \mathcal{U}^+ \cap \mathcal{U}_L$, is an interaction in the region $\Lambda \subset \Lambda_L$, Definition 2.3.37. Compare last expression with (2.3.38). We are interested in knowing if the *infinite-volume* dynamics exists when $L \rightarrow \infty$, i.e., if there is a $*$ -automorphism $\{\tau_t\}_{t \in \mathbb{R}}$ of $\mathcal{U}$, with generator δ , when $L \rightarrow \infty$.

Note that the choice of Λ_L determines that for $L \rightarrow \infty$, Λ_L will contain all the finite subsets, $\mathcal{P}_f(\mathfrak{L})$ of $\mathfrak{L}$. However, it is not *a priori* clear if the limits of (2.3.48) and (2.3.49) are bounded for all the Hamiltonians H_L in $L \rightarrow \infty$. In the scope of Lattice Spin Systems, Bratteli and Robinson [BR03b, Chapter 6] shown for different interactions that indeed both limits exist. The Lattice Fermion System (LFS) case is slightly more complicated, and the convergence is studied using a broad variety of techniques (see Example 2.3.59 below for a complete description at the free-fermion setting). Moreover, recently Bru and Pedra studied the existence of dynamics in free fermion systems, in presence of an external electrical field and disorder on the crystal, and also study short range interactions in the scope of *linear response theory*.

2.3.5 DISORDERED MEDIA WITHIN ELECTROMAGNETIC FIELDS AND LINEAR RESPONSE CURRENT OBSERVABLE

We use the mathematical framework of [BP15, BP16a].

1. As already mentioned, the crystal lattice is represented by $\mathfrak{L} \doteq \mathbb{Z}^d$ and $\mathcal{P}_f(\mathfrak{L}) \subset 2^{\mathfrak{L}}$ is the set of all finite subsets of $\mathfrak{L}$. Further,

$$\mathbb{D} \doteq \{z \in \mathbb{C} : |z| \leq 1\} \quad \text{and} \quad \mathfrak{b} \doteq \{\{x, x'\} \subset \mathfrak{L} : |x - x'| = 1\}$$

is the set of (non-oriented) edges of the cubic lattice $\mathfrak{L}$.

2. Disorder in the crystal is modeled by a random variable taking values in the measurable space $(\Omega, \mathfrak{A}_\Omega)$, with distribution $\mathfrak{a}_\Omega$:

Ω : Elements of Ω are pairs $\omega = (\omega_1, \omega_2) \in \Omega$, where ω_1 is a function on lattice sites with values in the interval $[-1, 1]$ and ω_2 is a function on edges with values in the complex closed unit disc $\mathbb{D}$. I.e.,

$$\Omega \doteq [-1, 1]^{\mathfrak{L}} \times \mathbb{D}^{\mathfrak{b}}.$$

$\mathfrak{A}_\Omega$: Let $\Omega_x^{(1)}$, $x \in \mathbb{Z}^d$, be an arbitrary element of the Borel σ -algebra $\mathfrak{A}_x^{(1)}$ of the interval $[-1, 1]$ w.r.t. the usual metric topology. Define

$$\mathfrak{A}_{[-1, 1]^{\mathfrak{L}}} \doteq \bigotimes_{x \in \mathfrak{L}} \mathfrak{A}_x^{(1)},$$

i.e., $\mathfrak{A}_{[-1, 1]^{\mathfrak{L}}}$ is the σ -algebra generated by the cylinder sets $\prod_{x \in \mathfrak{L}} \Omega_x^{(1)}$, where $\Omega_x^{(1)} = [-1, 1]$

for all but finitely many $x \in \mathfrak{L}$. In the same way, let

$$\mathfrak{A}_{\mathbb{D}^b} \doteq \bigotimes_{x \in \mathfrak{b}} \mathfrak{A}_x^{(2)},$$

where $\mathfrak{A}_x^{(2)}$, $x \in \mathfrak{b}$, is the Borel σ -algebra of the complex closed unit disc $\mathbb{D}$ w.r.t. the usual metric topology. Then

$$\mathfrak{A}_\Omega \doteq \mathfrak{A}_{[-1,1]^{\mathfrak{L}}} \otimes \mathfrak{A}_{\mathbb{D}^b}.$$

α_Ω : The distribution α_Ω is an arbitrary *ergodic* probability measure on the measurable space $(\Omega, \mathfrak{A}_\Omega)$. I.e., it is invariant under the action

$$(2.3.51) \quad (\omega_1, \omega_2) \mapsto \chi_x^{(\Omega)}(\omega_1, \omega_2) \doteq \left(\chi_x^{(\mathfrak{L})}(\omega_1), \chi_x^{(\mathfrak{b})}(\omega_2) \right), \quad x \in \mathfrak{L},$$

of the group $(\mathfrak{L}, +)$ of translations on Ω and $\alpha_\Omega(\mathcal{X}) \in \{0, 1\}$ whenever $\mathcal{X} \in \mathfrak{A}_\Omega$ satisfies $\chi_x^{(\Omega)}(\mathcal{X}) = \mathcal{X}$ for all $x \in \mathfrak{L}$. Here, for any $\omega = (\omega_1, \omega_2) \in \Omega$, $x \in \mathfrak{L}$ and $y, y' \in \mathfrak{L}$ with $|y - y'| = 1$,

$$(2.3.52) \quad \chi_x^{(\mathfrak{L})}(\omega_1)(y) \doteq \omega_1(y + x), \quad \chi_x^{(\mathfrak{b})}(\omega_2)(\{y, y'\}) \doteq \omega_2(\{y + x, y' + x\}).$$

$\mathbb{E}[\cdot]$ denotes the expectation value associated with α_Ω .

3. As is usual, the one-particle Hilbert space is $\mathcal{H} \doteq \ell^2(\mathfrak{L})$ with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Its canonical orthonormal basis is denoted by $\{e_x\}_{x \in \mathfrak{L}}$, which is defined by $e_x(y) \doteq \delta_{x,y}$ for all $x, y \in \mathfrak{L}$. ($\delta_{x,y}$ is the Kronecker delta.) To any $\omega \in \Omega$ and strength $\vartheta \in \mathbb{R}_0^+$ of hopping disorder, we associate a self-adjoint operator $\Delta_{\omega, \vartheta} \in \mathcal{B}(\ell^2(\mathfrak{L}))$ describing the hoppings of a single particle in the lattice:

$$(2.3.53) \quad \begin{aligned} [\Delta_{\omega, \vartheta}(\psi)](x) &\doteq 2d\psi(x) - \sum_{j=1}^d \left((1 + \vartheta \overline{\omega_2(\{x, x - e_j\})}) \psi(x - e_j) \right. \\ &\quad \left. + \psi(x + e_j)(1 + \vartheta \omega_2(\{x, x + e_j\})) \right) \end{aligned}$$

for any $x \in \mathfrak{L}$ and $\psi \in \ell^2(\mathfrak{L})$, with $\{e_k\}_{k=1}^d$ being the canonical orthonormal basis of the Euclidian space $\mathbb{R}^d$. In the case of vanishing hopping disorder $\vartheta = 0$, (up to a minus sign) $\Delta_{\omega, 0}$ is the usual d -dimensional discrete Laplacian. Since the hopping amplitudes are complex-valued (ω_2 takes values in $\mathbb{D}$), note additionally that random electromagnetic potentials can be implemented in our model. Then, the random tight-binding model is

the one-particle Hamiltonian defined by

$$(2.3.54) \quad h^{(\omega)} \doteq \Delta_{\omega, \vartheta} + \lambda \omega_1, \quad \omega = (\omega_1, \omega_2) \in \Omega, \lambda, \vartheta \in \mathbb{R}_0^+,$$

where the function $\omega_1: \mathfrak{L} \rightarrow [-1, 1]$ is identified with the corresponding (self-adjoint) multiplication operator. We use this operator to define a (infinite volume) dynamics, by the unitary group $\{e^{ith^{(\omega)}}\}_{t \in \mathbb{R}}$, in the one-particle Hilbert space $\mathcal{H}$. Note that the tight-binding Anderson model corresponds to the special case $\vartheta = 0$.

4. Let

$$\begin{aligned} \mathfrak{Z} &\doteq \left\{ \mathcal{Z} \in 2^{\mathfrak{L}} : (\forall Z_1, Z_2 \in \mathcal{Z}) \ Z_1 \neq Z_2 \implies Z_1 \cap Z_2 = \emptyset \right\}, \\ \mathfrak{Z}_f &\doteq \left\{ \mathcal{Z} \in \mathfrak{Z} : |\mathcal{Z}| < \infty \text{ and } (\forall Z \in \mathcal{Z}) \ 0 < |Z| < \infty \right\}. \end{aligned}$$

One can restrict the dynamics to collections $\mathcal{Z} \in \mathfrak{Z}$ of disjoint subsets of the lattice by using the orthogonal projections P_Λ , $\Lambda \subset \mathfrak{L}$, defined on $\mathcal{H}$ by

$$(2.3.55) \quad [P_\Lambda(\varphi)](x) \doteq \begin{cases} \varphi(x) & , \text{ if } x \in \Lambda. \\ 0 & , \text{ else.} \end{cases}$$

Then, the one-particle Hamiltonian within $\mathcal{Z} \in \mathfrak{Z}$ is

$$(2.3.56) \quad h_{\mathcal{Z}}^{(\omega)} \doteq \sum_{Z \in \mathcal{Z}} P_Z h^{(\omega)} P_Z,$$

leading to the unitary group $\{e^{ith_{\mathcal{Z}}^{(\omega)}}\}_{t \in \mathbb{R}}$. This kind of decomposition over collections of disjoint subsets of the lattice is important in the technical proofs.

5. By the Combes–Thomas estimate (Appendix D),

$$(2.3.57) \quad \left| \left\langle e_x, e^{ith_{\mathcal{Z}}^{(\omega)}} e_y \right\rangle_{\mathcal{H}} \right| \leq 36e^{|t\eta| - 2\mu_\eta|x-y|}$$

for any $\eta, \mu \in \mathbb{R}^+$, $x, y \in \mathbb{Z}^d$, $\mathcal{Z} \in \mathfrak{Z}$, $\omega \in \Omega$, and $\lambda, \vartheta \in \mathbb{R}_0^+$, where

$$(2.3.58) \quad \mu_\eta \doteq \mu \min \left\{ \frac{1}{2}, \frac{\eta}{8d(1+\vartheta)e^\mu} \right\}.$$

See Corollary D.0.9, by observing that the parameter $\mathbf{S}$ defined by (D.0.4) is bounded in this case by $\mathbf{S}(h_{\mathcal{Z}}^{(\omega)}, \mu) \leq 2d(1+\vartheta)e^\mu$.

2.3.59 Example (Dynamics of Noninteracting Lattices Fermion Systems). As already mentioned, free fermions are nothing but individual electrons that are not affected by the existence of others: they only interact with each other via the Pauli exclusion principle, forming an ideal lattice fermion

system [BPH14a], and whose Hamiltonians $h^{(\omega)}$ and $h^{(\omega, \Lambda)}$ in the infinite and finite setting are given by Expressions (2.3.54) and (2.3.56). In this case, for $\Lambda \in \mathcal{P}_t(\mathfrak{V})$ and the C^* -algebra $\mathcal{U}$, the finite-dynamics is described by the C_0 -group of $*$ -automorphisms $\{\tau_t^{(\Lambda)}\}_{t \in \mathbb{R}}$ of $\mathcal{U}$, such that the temporal evolution of any operator $A \in \mathcal{U}$ is given by

$$(2.3.60) \quad \tau_t^{(\Lambda)}(a^\#(\psi)) \equiv \tau_t^{(\omega, \Lambda)}(a^\#(\psi)) = a^\#((U_t^{(\Lambda)})^* \psi),$$

for $t \in \mathbb{R}, \psi \in \ell^2(\Lambda)$ and the symbol $\#$ stands for either creation or annihilation element of the C^* -algebra $\mathcal{U}_\Lambda$. Thus, annihilation and creation operators on $\mathcal{U}_\Lambda$, evolve through a *one-parameter group of Bogoliubov automorphism* (see Expression (2.3.17) and Chapter 4–Expression (C.1.6)). In 2.3.60, $\{U_t^{(\Lambda)} \equiv U_t^{(\omega, \Lambda)}\}_{t \in \mathbb{R}} \doteq \{e^{it h^{(\Lambda)}} \equiv e^{it h^{(\omega, \Lambda)}}\}_{t \in \mathbb{R}}$, where $h^{(\omega, \Lambda)} \in \ell^2(\Lambda)$ denotes the (one-particle) Hamiltonian (restricted to Λ). Note that $U_t^{(\Lambda)} \in \mathcal{B}(\Lambda)$ is a strongly continuous group of unitary operators, which for a fix $\Lambda \in \mathcal{P}_t(\mathfrak{V})$, is nothing but a unitary operator $U_\Lambda \in \mathcal{B}(\mathcal{H})$ such that $U_t^{(\Lambda)}(U_t^{(\Lambda)})^* = (U_t^{(\Lambda)})^* U_t^{(\Lambda)} = \mathbb{1}$ and $(U_t^{(\Lambda)})^* = (U_t^{(\Lambda)})^{-1}$. This highlights the importance of one-parameter groups, at least in the non-interacting case and we define these in Appendix A. The second quantization operator $H^{(\Lambda)} \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$ associated to $h^{(\Lambda)}$ is given by (Definition 2.3.15):

$$H^{(\Lambda)} \doteq \sum_{x, y \in \Lambda} \langle e_x, h^{(\Lambda)} e_y \rangle a(e_x)^* a(e_y).$$

In particular, by Expression (2.3.17) note that (2.3.60) also can be written as:

$$(2.3.61) \quad \tau_t^{(\Lambda)}(A) = (O_t^{(\Lambda)})^* A O_t^{(\Lambda)},$$

where $\{O_t^{(\Lambda)}\}_{t \in \mathbb{R}} \doteq \{e^{it H^{(\Lambda)}}\}_{t \in \mathbb{R}} \in \mathcal{U}$.

Next Lemma shows the existence of the infinite-dynamics for free fermions embedded into $\mathfrak{V}$:

2.3.62 LEMMA (EXISTENCE OF DYNAMICS). *Let $\{\tau_t^{(\Lambda)}\}_{t \in \mathbb{R}}$ be the one-parameter group of Bogoliubov automorphism of $\mathcal{U}$ associated with the unitary one-parameter group $\{e^{it h^{(\Lambda)}}\}_{t \in \mathbb{R}}$, and $h^{(\Lambda)}$ given by Expression (2.3.56). Then for any $t \in \mathbb{R}, \Lambda \in \mathcal{P}_t(\mathfrak{V}), A \in \mathcal{U}_\Lambda$ and $L_1, L_2 \in \mathbb{R}_0^+$ with $\Lambda \subset \Lambda_{L_1} \subsetneq \Lambda_{L_2}$,*

$$(2.3.63) \quad \|\tau_t^{(L_2)}(A) - \tau_t^{(L_1)}(A)\|_{\mathcal{U}} \leq 2\|A\|_{\mathcal{U}} |t| \sum_{y \in \Lambda_{L_2} \setminus \Lambda_{L_1}} \sum_{x \in \Lambda} \left| \langle e_x, h^{(L_2)} e_y \rangle_{\mathcal{H}} \right|.$$

With $\tau_t^{(L_i)} \equiv \tau_t^{(\Lambda_{L_i})}$ for $i = \{1, 2\}$. In particular, for $L_1, L_2 \rightarrow \infty$, the right hand side of (2.3.63) goes to zero. ♦

PROOF. Firstly, note that:

$$\begin{aligned} \tau_t^{(L_2)}(A) - \tau_t^{(L_1)}(A) &= \int_0^t \frac{d}{ds} (\tau_s^{(L_2)}(\tau_{t-s}^{(L_1)}(A))) ds \\ &= \int_0^t \tau_s^{(L_2)}([H^{(L_2)} - H^{(L_1)}, \tau_{t-s}^{(L_1)}(A)]) ds \end{aligned}$$

By Expression (2.2.6) and due to $\Lambda \subset \Lambda_1$ one has:

$$\begin{aligned}
 \|\tau_t^{(L_2)}(A) - \tau_t^{(L_1)}(A)\|_{\mathcal{U}} &\leq \sum_{y \in \Lambda_{L_2} \setminus \Lambda_{L_1}} \sum_{x \in \Lambda_{L_1}} \int_0^{|t|} \left\| \left[\langle e_x, h^{(L_2)} e_y \rangle a_x^* a_y, \tau_s^{(L_1)}(A) \right] \right\|_{\mathcal{U}} ds \\
 (2.3.64) \quad &\leq 2\|A\|_{\mathcal{U}} |t| \sum_{y \in \Lambda_{L_2} \setminus \Lambda_{L_1}} \sum_{x \in \Lambda} \left| \langle e_x, h^{(L_2)} e_y \rangle \right|.
 \end{aligned}$$

Thus, for $L_1, L_2 \rightarrow \infty$ the right hand side of (2.3.64) goes to 0 by the *Combes–Thomas estimate* (see Appendix D, Theorem D.0.8) applied to the function $f(h^{(L_2)}) = h^{(L_2)}$ such that $\left| \langle e_x, h^{(L_2)} e_y \rangle \right| \leq D e^{-\mu|x-y|}$. End

2.3.65 LEMMA (QUASI-FREE STATES AND KMS STATES). *Let $\{\tau_t^{(\Lambda)}\}_{t \in \mathbb{R}}$ be the one-parameter group of Bogolioubov automorphism of $\mathcal{U}$ associated with the unitary one-parameter group $\{e^{it h^{(\Lambda)}}\}_{t \in \mathbb{R}}$, and $h^{(\Lambda)}$ given by Expression (2.3.56). Then the quasi free state $\rho_\Lambda^{(\beta)} \in E_\Lambda \subset \mathcal{U}^*$ (cf. (2.3.28)) is a $(\tau_t^{(\Lambda)}, \beta)$ -KMS state, which is unique.* ♦

PROOF. $A, B \in \mathcal{U}$ satisfy the KMS condition if for inverse temperature $\beta \neq 0$ we have (see Appendix B–Expression (B.1.7))

$$\rho_\Lambda^{(\beta)}(A \tau_{i\beta}(B)) = \rho_\Lambda^{(\beta)}(BA).$$

In particular, for $A = a(\varphi_1)^*$ and $B = a(\varphi_2)$

$$\rho_\Lambda^{(\beta)}(a(\varphi_1)^* \tau_{i\beta}(a(\varphi_2))) = \rho_\Lambda^{(\beta)}(a(\varphi_2) a(\varphi_1)^*).$$

It is not hard to show that the dynamic expression (2.3.60) can be extended analitically for $z \in \mathbb{C}$ satisfying

$$\tau_z^{(\Lambda)}(a^\#(\psi)) \doteq \begin{cases} a^*(e^{izh^{(\Lambda)}}(\psi)), & \text{for } a^\# = a^*, \\ a(e^{i\bar{z}h^{(\Lambda)}}(\psi)), & \text{for } a^\# = a. \end{cases}$$

Thus, using CAR (2.2.5) we rewrite

$$\rho_\Lambda^{(\beta)}(a(\varphi_1)^* a(e^{\beta h^{(\Lambda)}} \varphi_2)) = \rho_\Lambda^{(\beta)}(a(\varphi_2) a(\varphi_1)^*) \Rightarrow \rho_\Lambda^{(\beta)}(a(\varphi_1)^* a((e^{\beta h^{(\Lambda)}} + \mathbb{1}) \varphi_2)) = \langle \varphi_2, \varphi_1 \rangle_{\mathcal{H}}.$$

Finally, since $h^{(\Lambda)} \in \mathcal{B}(\mathcal{H})$ is self-adjoint we are able to redefine φ_2 in order to have a *unique* quasi-free state (2.3.28) for $\beta \neq 0$, i.e.,

$$\rho_\Lambda^{(\beta)}(a(\varphi_1)^* a(\varphi_2)) = \left\langle \varphi_2, \frac{\mathbb{1}}{\mathbb{1} + e^{\beta h^{(\Lambda)}}} \varphi_1 \right\rangle_{\mathcal{H}}.$$

End

If the (one-particle) Hamiltonian is time-dependent, denoted by $h_t^{(L)} \in \mathcal{B}(\ell^2(\Lambda_L))$, we use the Schrödinger equation given for *two-parameter group of unitary operators* $\{U_{t,s}^{(L)} \equiv U_{t,s}^{(\Lambda_L)}\}_{t \geq s} \in \mathcal{B}(\ell^2(\Lambda_L))$ [SN11, Chapter 2]:

$$(2.3.66) \quad \forall s, t \in \mathbb{R}, t \geq s: \quad \partial_t U_{t,s}^{(L)} = -i h_t U_{t,s}^{(L)}, \quad U_{s,s}^{(L)} \doteq \mathbb{1},$$

where ∂_t denotes the derivative w.r.t. time and $U_{t,r}^{(L)} U_{r,s}^{(L)} = U_{t,s}^{(L)}$ for all $r, s, t \in \mathbb{R}$. Using the Heisenberg picture, the time evolution of any observable $A_s^{(L)} \in \mathcal{B}(\ell^2(\Lambda_L))$ at initial time $t = s \in \mathbb{R}$ equals $A_t^{(L)} =$

$(U_{t,s}^{(L)})^* A_s^{(L)} U_{t,s}^{(L)}$ for $t \geq s$, which yields to the Schrödinger equation

$$(2.3.67) \quad \forall s, t \in \mathbb{R}, t \geq s: \quad \partial_t A_t^{(L)} = (U_{t,s}^{(L)})^* i[h_t^{(L)}, A_s^{(L)}] U_{t,s}^{(L)}.$$

Similarly to one-parameter groups, two-parameter groups are related to a unique family of strongly continuous *two-parameter automorphisms* of $\mathcal{U}_L$, namely, $\{\tau_{t,s}^{(L)} \equiv \tau_{t,s}^{(\Lambda_L)}\}_{t \geq s}$. Note that in (2.3.67) $\tau_{t,s}^{(L)}$ obeys the *nonautonomous evolution equation* [EBN⁺06, Chapter 6, §9], [BPH14a, Eq. 2.12]

$$\forall s, t \in \mathbb{R}, t \geq s: \quad \partial_t \tau_{t,s}^{(L)} = \tau_{t,s}^{(L)} \circ \delta_t^{(L)}, \quad \tau_{s,s}^{(L)} \doteq \mathbb{1}.$$

In the finite-volume case, as well as in the infinite-volume case, the existence of $\{\tau_{t,s}^{(L)}\}_{t \geq s}$ satisfying the above Expression is proven using *Dyson–Phillips* series. See details in [BPH14a, §5 and Appendix A].*

Remarks 2.3.68.

1. Let $h \in \mathcal{H} \equiv \ell^2(\mathfrak{L})$ be the Hamiltonian of the whole system with second quantization $H \in \mathcal{U}$ and $\{\tau_t\}_{t \in \mathbb{R}}$ be the $*$ -Bogoliubov automorphisms of $\mathcal{U}$. [BR03b, Theorem 5.2.4] states that for $A \in \mathcal{U}_L$, the limit $\lim_{L \rightarrow \infty} \|\tau_t(A) - \tau_t^{(L)}(A)\|$ goes uniformly to zero w.r.t. t in compact intervals. Thus, the existence of dynamics is guaranteed if the Hamiltonian $H^{(L)} \in \mathcal{H}_L$ converges strongly to the Hamiltonian H of the whole system. In Lemma 2.3.62 we use an explicit form for the Hamiltonian to tackle the systems of interest in this Thesis.
2. In regard to Lemma 2.3.65, [BR03b, Theorem 5.2.24] shows the weak*-convergence $\rho_\Lambda \in E_\Lambda$ to $\rho \in E$ as $\Lambda \rightarrow \mathfrak{L}$ (with symbol S) given by

$$\rho(a(\varphi_1)^* a(\varphi_2)) = \langle \varphi_2, S\varphi_1 \rangle_{\mathcal{H}}, \quad \varphi_1, \varphi_2 \in \mathcal{H}. \quad *$$

For the Banach space $\mathcal{W}$ defined in §§2.3.3, short-range interactions $\Phi \in \mathcal{W}$ defines symmetric derivations on $\mathcal{U}_0$ which generates C_0 -group τ of $*$ -automorphisms, and then an infinite-volume dynamics given by the following Theorem proven in [BP16a, Theorem 4.8]:

Theorem 2.3.69 (Infinite-volume dynamics and its generator):

Assume (2.3.40)–(2.3.41) and let $\Phi \in \mathcal{W}$.

1. *Infinitesimal generator.* There is a unique conservative closed symmetric derivation δ of $\mathcal{U} \equiv \mathcal{U}(\ell^2(\mathfrak{L}))$ with core $\mathcal{U}_0$ and

$$(2.3.70) \quad \delta(A) = i \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} [\Phi_\Lambda, A], \quad A \in \mathcal{U}_0.$$

The above sum is absolutely convergent.

2. *Infinite-volume dynamics.* δ is the generator of a C_0 -group $\tau \doteq \{\tau_t\}_{t \in \mathbb{R}}$ of $*$ -automorphisms. ♣

We now apply an external electric field $\mathbf{E}$ into the region Λ_L satisfying the Weyl gauge (also

named temporal gauge). We use the compactly supported potential

$$\mathbf{A} \in \mathbf{C}_0^\infty \doteq \bigcup_{L \in \mathbb{R}^+} C_0^\infty(\mathbb{R} \times [-L, L]^d; (\mathbb{R}^d)^*)$$

in order to define

$$E_{\mathbf{A}}(t, x) \doteq -\partial_t \mathbf{A}(t, x), \quad t \in \mathbb{R}, x \in \mathbb{R}^d.$$

Here, $(\mathbb{R}^d)^*$ is the set of one-forms on $\mathbb{R}^d$ that take values in $\mathbb{R}$ and $\mathbf{A}(t, x) \equiv 0$ whenever $x \notin [-L, L]^d$ and $\mathbf{A} \in C_0^\infty(\mathbb{R} \times [-L, L]^d; (\mathbb{R}^d)^*)$ [BPH15]. Note that $\mathbf{A}(t, x) = 0$ for all $t \leq t_0$, where $t_0 \in \mathbb{R}$ is some initial time. Since, $\mathbf{A}$ is compactly supported, the AC-condition is satisfied

$$\int_{t_0}^t E_{\mathbf{A}}(s, x) ds = 0, \quad x \in \mathbb{R}^d,$$

for sufficiently large times $t \geq t_1 \geq t_0$. Here t_1 is given by

$$t_1 \doteq \min \left\{ t \geq t_0 : \int_{t_0}^{t'} E_{\mathbf{A}}(s, x) ds = 0 \quad \text{for all } x \in \mathbb{R}^d \text{ and } t' \geq t \right\}$$

is the time at which the electric field is turned off [BPH16].

Let the set of *edges* given by:

$$(2.3.71) \quad \mathfrak{R} \doteq \left\{ \mathbf{x} = (x^{(1)}, x^{(2)}) \in \mathfrak{Q}^2 : |x^{(1)} - x^{(2)}| = 1 \right\},$$

so that, for fix $\omega \in \Omega$ and $\vartheta \in \mathbb{R}_0^+$, the “paramagnetic” and “diamagnetic” current observables $I_{\mathbf{x}}^{(\omega)}$ and $I_{\mathbf{x}}^{(\omega, \mathbf{A})}$ for $\mathbf{A} \in C_0^\infty$ at time $t \in \mathbb{R}$ are defined respectively by

$$(2.3.72) \quad I_{\mathbf{x}}^{(\omega)} \doteq -2\Im \left(\langle e_{x^{(1)}}, \Delta_{\omega, \vartheta} e_{x^{(2)}} \rangle_{\mathcal{H}} a(e_{x^{(1)}})^* a(e_{x^{(2)}}) \right) \quad \text{and}$$

$$(2.3.73) \quad I_{\mathbf{x}}^{(\omega, \mathbf{A})} \doteq -2\Im \left(\left(e^{-i \int_0^1 [\mathbf{A}(t, \alpha x^{(2)} + (1-\alpha)x^{(1)})](x^{(2)} - x^{(1)}) d\alpha} - 1 \right) \langle e_{x^{(1)}}, \Delta_{\omega, \vartheta} e_{x^{(2)}} \rangle_{\mathcal{H}} a(e_{x^{(1)}})^* a(e_{x^{(2)}}) \right)$$

where $\Im(A)$ is the imaginary part of A . Observe, that if (2.3.66) and (2.3.60) are satisfied, the above Expressions can be seen as currents, i.e., flow of electrons moving in a region $\Lambda \in \mathcal{P}_f(\mathfrak{Q})$, because one has the *discrete continuity* equation:

$$\partial_t (\tau_{t, t_0}^{(\omega)} (a_x^* a_x)) = - \sum_{z \in \Lambda, |z|=1} \tau_{t, t_0} (I_{(x, x+z)}^{(\omega)} + I_{(x, x+z)}^{(\omega, \mathbf{A})}),$$

for $x \in \Lambda$ and $t \geq t_0$. Note that the potential $\mathbf{A}$ depending on t generates nonautonomous dynamics, described by the family of two-parameter automorphisms $\{\tau_{t,s}\}_{s,t \in \mathbb{R}}$ on $\mathcal{U}$. Negative sign in last expression comes from the fact that the particles are negatively charged. Electric fields accelerate charged particles and induce so-called diamagnetic currents, which correspond

to the ballistic movement of particles. In fact, as explained in [BPH15, Sections III and IV], this component of the total current creates a kind of *wave front* that destabilizes the whole system by changing its physical state. The presence of diamagnetic currents leads then to the progressive appearance of paramagnetic currents which are responsible for heat production and the in-phase AC-conductivity of the system. Hence, the paramagnetic current observable I_x is intrinsic to the system and it is related, in absence of external electromagnetic potentials, to the flow of particles from the lattice site $x^{(1)}$ to the lattice site $x^{(2)}$ or the current from $x^{(2)}$ to $x^{(1)}$ (for negatively charged fermions). On the other hand, the diamagnetic current observable I_x^A is only non-vanishing in presence of electromagnetic potentials.

For $\mathbf{A} \in \mathbf{C}_0^\infty$ and $\Delta_{\omega, \vartheta}$ (see (2.3.53)), the discrete magnetic Laplacian is (up to a minus sign) the self-adjoint operator

$$\Delta_{\omega, \vartheta}^{(\mathbf{A})} \equiv \Delta_{\omega, \vartheta}^{(\mathbf{A}(t, \cdot))} \in \mathcal{B}(\ell^2(\mathcal{Q})), \quad t \in \mathbb{R}$$

defined by

$$\langle e_x, \Delta_{\omega, \vartheta}^{(\mathbf{A})} e_y \rangle \doteq e^{i \int_0^1 [\mathbf{A}(t, \alpha y + (1-\alpha)x)](y-x) d\alpha} \langle e_x, \Delta_{x,y} e_y \rangle$$

for all $t \in \mathbb{R}$, $\omega \in \Omega$, $\vartheta \in \mathbb{R}_0^+$ and $x, y \in \mathcal{Q}$. The one-particle Hamiltonian $(\Delta_{\omega, \vartheta} + \lambda \omega_1)$ at fixed $\omega \in \Omega$ and $\vartheta, \lambda \in \mathbb{R}_0^+$ is now for $t \in \mathbb{R}$

$$\Delta_{\omega, \vartheta}^{(\mathbf{A})} + \lambda \omega_1 \equiv \Delta_{\omega, \vartheta}^{(\mathbf{A}(t, \cdot))} + \lambda \omega_1.$$

For more details, see [BPH15, BP14a, BP15].

For any finite subset $\Lambda \in \mathcal{P}_f(\mathcal{Q})$, we define the space-averaged transport coefficient observable $\mathcal{C}_\Lambda^{(\omega)} \in C^1(\mathbb{R}; \mathcal{B}(\mathbb{R}^d; \mathcal{U}))$, w.r.t. the canonical orthonormal basis $\{e_q\}_{q=1}^d$ of the Euclidian space $\mathbb{R}^d$, by the corresponding matrix entries

$$(2.3.74) \quad \begin{aligned} \{\mathcal{C}_\Lambda^{(\omega)}(t)\}_{k,q} &\doteq \frac{1}{|\Lambda|} \sum_{\{x,y,x+e_k,y+e_q\} \in \Lambda} \int_0^t i \left[\tau_{-\alpha}^{(\omega)}(I_{(y+e_q,y)}^{(\omega)}), I_{(x+e_k,x)}^{(\omega)} \right] d\alpha \\ &\quad + \frac{2\delta_{k,q}}{|\Lambda|} \sum_{x \in \Lambda} \Re \left(\langle e_{x+e_k}, \Delta_{\omega, \vartheta} e_x \rangle_{\mathcal{H}} a(e_{x+e_k})^* a(e_x) \right) \end{aligned}$$

for any $\omega \in \Omega$, $t \in \mathbb{R}$, $\lambda, \vartheta \in \mathbb{R}_0^+$ and $k, q \in \{1, \dots, d\}$. This object is the conductivity observable matrix associated with the lattice region Λ and time t . In fact, the first term in the right-hand side of (2.3.74) corresponds to the paramagnetic coefficient, whereas the second one is the diamagnetic component. For more details, see [BP14a, Theorem 3.7].

We now fix a direction $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$ and a (time-dependent) continuous, compactly supported, electric field $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$. Then, [BP14a, BP15]³ shows that the space-averaged linear response current observable in the lattice region Λ and at time $t = 0$ in the direction $\vec{w}$ is

³Strictly speaking, these papers use smooth electric fields, but the extension to the continuous case is straightforward.

equal to

$$(2.3.75) \quad \mathbb{I}_{\Lambda}^{(\omega, \mathcal{E})} \doteq \sum_{k,q=1}^d w_k \int_{-\infty}^0 \{\mathcal{E}(\alpha)\}_q \{\mathcal{C}_{\Lambda}^{(\omega)}(-\alpha)\}_{k,q} d\alpha.$$

To obtain the current density at any time $t \in \mathbb{R}$, it suffices to replace $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ in this equation with

$$(2.3.76) \quad \mathcal{E}_t(\alpha) \doteq \mathcal{E}(\alpha + t), \quad \alpha \in \mathbb{R}.$$

In order to have an useful estimate to handle Expression (2.3.74) we define for $x \in \Lambda \subset \mathcal{P}_f(\mathfrak{Q})$ the *shift* bounded linear operator S_x on $\ell^2(\Lambda)$ such that for any $y \in \Lambda$ we have:

$$s_x(e_y) \doteq e_{x+y},$$

satisfying $s_x^* = s_{-x} = s_x^{-1}$. Then, for any $\omega \in \Omega$ and $\vartheta \in \mathbb{R}_0^+$, the single-hopping operators are

$$S_{x,y}^{(\omega)} \doteq \langle e_y, \Delta_{\omega, \vartheta} e_x \rangle_{\mathcal{H}} P_{\{y\}} s_{y-x} P_{\{x\}}, \quad x, y \in \mathfrak{Q},$$

where $P_{\{x\}}$ is the orthogonal projection defined by (2.3.55) for $\Lambda = \{x\}$. Observe that

$$d \langle Q, S_{x,y}^{(\omega)} Q \rangle = \langle e_y, \Delta_{\omega, \vartheta} e_x \rangle_{\mathcal{H}} a(e_y)^* a(e_x), \quad x, y \in \mathfrak{Q}.$$

Similarly, the paramagnetic current observables defined by (2.3.72) equal

$$I_{(x,y)}^{(\omega)} = -2d \langle Q, \Im \{S_{x,y}^{(\omega)}\} Q \rangle,$$

for any $\omega \in \Omega$ and $\vartheta \in \mathbb{R}_0^+$. Compare with (2.3.20). From straightforward, albeit cumbersome, computations we get that

$$\begin{aligned} \{\mathcal{C}_{\Lambda}^{(\omega)}(t)\}_{k,q} &= -\frac{8}{|\Lambda|} d \left\langle Q, \int_0^t d\alpha \Im \left\{ \langle e_y, s_{\mp e_q} (U_{-\alpha}^{(\Lambda)})^* s_{\mp e_k} U_{-\alpha}^{(\Lambda)} e_x \rangle_{\mathcal{H}} \right\} Q \right\rangle \\ &\quad + \frac{2\delta_{k,q}}{|\Lambda|} d \langle Q, \Re \{S_{x+e_k, x}^{(\omega)}\} Q \rangle. \end{aligned}$$

Thus, we define local current observables, for any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, any collection $\mathcal{X}^{(\tau)} \in \mathfrak{Z}$, $\mathcal{Z} \in \mathfrak{Z}_f$, and $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$ (see Definition 2.3.15)

$$(2.3.77) \quad \mathfrak{K}_{\mathcal{Z}, \mathcal{X}^{(\tau)}}^{(\omega, \mathcal{E})} = d \langle Q, K_{\mathcal{Z}, \mathcal{X}^{(\tau)}}^{(\omega, \mathcal{E})} Q \rangle,$$

where

$$\begin{aligned}
 K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} &\doteq -8 \sum_{k,q=1}^d w_k \sum_{e_k, e_q \in \mathcal{Z}} \int_{-\infty}^0 \{\mathcal{E}(\alpha)\}_q d\alpha \\
 &\quad \int_0^{-\alpha} ds \Im \left\{ \left\langle e_y, s_{\mp e_q} \left(U_{-\alpha}^{(\mathcal{Z}^{(\tau)})} \right)^* s_{\mp e_k} U_{-\alpha}^{(\mathcal{Z}^{(\tau)})} e_x \right\rangle_{\mathcal{H}} \right\} \\
 (2.3.78) \quad &+ 2 \sum_{k=1}^d w_k \sum_{e_k \in \mathcal{Z}} \left(\int_{-\infty}^0 \{\mathcal{E}(\alpha)\}_q d\alpha \right) \Re \{ S_{x+e_k, x}^{(\omega)} \}.
 \end{aligned}$$

is an operator acting on $\mathcal{H}$ whose range is finite dimensional, so that

$$\mathfrak{R}_{\{\Lambda\}, \{\mathfrak{L}\}}^{(\omega, \mathcal{E})} = |\Lambda| \mathbb{I}_{\Lambda}^{(\omega, \mathcal{E})}, \quad \Lambda \in \mathcal{P}_f(\mathfrak{L}).$$

2.4 LARGE DEVIATION THEORY AND LATTICE FERMION SYSTEMS

As remarked in §2.1, in the scope of classical statistical mechanics at thermodynamic equilibrium we are often interested in finding rate functions. In such a context, the sequence $\{X_n\}_{n \in \mathbb{N}}$ of $\mathbb{R}^d$ -valued random variables commute and its density measure $\{\mathfrak{m}_n\}_{n \in \mathbb{N}}$ is related with the *partition function* of the system. Such a commutative property is due to the commutativity of the quantities involved in the *Hamiltonian* and other observables, e.g., position, momentum, spin, etc. In contrast, in quantum statistical mechanics, in general, the observables used to describe systems does not commute between them and then Definitions and Theorems of §2.1 must be modified to tackle quantum systems.

Let ρ be a state, i.e., a normalized positive linear functional with $\rho(\mathbb{1}) = 1$, on a C^* -algebra $\mathcal{U}$ and $A \in \mathcal{U}$ a selfadjoint operator. We use the Measurable Functional Calculus and the Riesz–Markov Theorem to associate a probability measure to ρ and A , see Appendix A. Indeed, exists a unique probability measure $\mathfrak{m}_{\rho, A}$ on $\mathbb{R}$ such that $\mathfrak{m}_{\rho, A}(\text{spec}(A)) = 1$ and, for all $f \in C(\mathbb{R}; \mathbb{C})$,

$$(2.4.1) \quad \rho(f(A)) = \int_{\mathbb{R}} f(x) \mathfrak{m}_{\rho, A}(dx).$$

In the context of our work, we will call $\mathfrak{m}_{\rho, A}$ the “fluctuation measure” associated to ρ and A . Therefore, given a sequence of self-adjoint operators $\{A_n\}_{n \in \mathbb{N}}$ of $\mathcal{U}$, and a state $\rho \in \mathcal{U}^*$, we say that these satisfy an LDP, see Definition 2.1.4, if for all Borel measurable $\Gamma \subset \mathbb{R}$,

$$(2.4.2) \quad -\inf_{x \in \bar{\Gamma}} I(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{a_n} \ln \mathfrak{m}_{\rho, A_n}(\Gamma) \leq \limsup_{n \rightarrow \infty} \frac{1}{a_n} \ln \mathfrak{m}_{\rho, A_n}(\Gamma) \leq -\inf_{x \in \bar{\Gamma}} I(x),$$

where $\mathring{\Gamma}$ and $\bar{\Gamma}$ are the interior and closure of Γ , respectively and a_n is as defined in Lemma

2.1.11. Quoting [OR11], the probability measure $m_{\rho, A_n}(\Gamma)$ is interpreted as:

... the probability that the observables A_n takes value in Γ if the system is in the state ρ ...

To fix ideas, we will see in the rest of this Thesis, that A_n can be taken as a sort of well-defined space average of observables in the subalgebra $\mathcal{U}_n \subset \mathcal{U}$. In fact, in order to use LDT to study fermionic systems, one usually take space average on an interaction Ψ given by (2.3.38), i.e.,

$$A_L^\Psi \doteq \frac{1}{|\Lambda_L|} \sum_{\Lambda \subset \Lambda_L} \Psi_\Lambda \in \mathcal{U}^+ \cap \mathcal{U}_L,$$

where Λ_L refers to the sequence of boxes (2.3.21) and $\mathcal{U}_L \equiv \mathcal{U}_{\Lambda_L} \in \mathcal{U}$. Thus, the LDP (2.4.2) now is written as:

$$(2.4.3) \quad -\inf_{x \in \bar{\Gamma}} I(x) \leq \liminf_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln m_{\rho, A_L^\Psi}(\Gamma) \leq \limsup_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln m_{\rho, A_L^\Psi}(\Gamma) \leq -\inf_{x \in \bar{\Gamma}} I(x),$$

Note that in this case, Expression (2.1.12) reduce to a sequence depending of the sequence of probability measures $\{m_{\rho, A_L^\Psi}\}_{L \in \mathbb{R}_0^+}$. Hence,

$$(2.4.4) \quad \begin{aligned} J(s) &\doteq \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln \left(\int_{\mathbb{R}} e^{s|\Lambda_L|x} m_{\rho, A_L^\Psi}(dx) \right) \\ &= \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln \rho(e^{s|\Lambda_L|A_L^\Psi}), \end{aligned}$$

where $s \in \mathbb{R}$ is a parameter. Note that if ρ is a KMS state, taking into account Expressions (B.1.3)–(B.1.4) it follows that

$$(2.4.5) \quad J(s) = \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln \frac{\text{tr}(e^{-\beta H_L} e^{s|\Lambda_L|A_L^\Psi})}{\text{tr}(e^{-\beta H_L})}$$

where $H_L \equiv H_{\Lambda_L}(\Phi)$ is the Hamiltonian given by (2.3.50). Here, $\text{tr} \in \mathcal{U}^*$ is the tracial state satisfying (2.3.28) at $\beta = 0$. We will prove in Chapter 5 that KMS states of weakly interacting fermions on the lattice are exponentially ergodic states (cf. Definition 2.3.35) defined by:

2.4.6 DEFINITION (EXPONENTIALLY ERGODIC STATES). Exponentially ergodic states are ergodic states ρ with the property that, for any translation-invariant and finite-range interaction Ψ , the sequence $\{A_L^\Psi\}_{L \in \mathbb{R}^+}$ (2.3.38) satisfies an LDP in some closed interval $\mathcal{I} \ni \lim_{L \rightarrow \infty} \rho(A_L^\Psi)$ with a good rate function I . $\blacksquare$

Additionally, by Definition 2.1.17, we say in this context that the family $\{A_L^\Psi\}_{L \in \mathbb{R}_0^+} \in \mathcal{U}^+ \cap \mathcal{U}_L$ is exponentially tight in the state $\rho \in \mathcal{U}^*$ if, for every $\alpha \in \mathbb{R}$, there exists a compact set $K_\alpha \subset \mathbb{R}$

such that

$$(2.4.7) \quad \limsup_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln m_{\rho, A_L^\Psi}(\mathbb{C}K_\alpha) < -\alpha,$$

where $\mathbb{C}K_\alpha$ stands for the complement of K_α . Recall that a sufficient condition to ensure that a sequence of observables satisfies an LDP is given by the Gärtner–Ellis Theorem. Here, we can formulate Gärtner–Ellis Theorem 2.1.14 for lattice fermion systems:

Theorem 2.4.8 (Gärtner–Ellis):

Let Λ_L be the sequence of boxes (2.3.21), $\rho \in \mathcal{U}^*$ be a state and $C \subset \mathbb{R}$ and $O \subset \mathbb{R}$ be, respectively, closed and open sets. We suppose that for the exponentially tight family $\{A_L^\Psi\}_{L \in \mathbb{R}_0^+} \in \mathcal{U}^+ \cap \mathcal{U}_L$ in the state ρ the free energy J (2.4.4) exists. Then,

1. $\{A_L^\Psi\}_{L \in \mathbb{R}_0^+}$ satisfies a large deviation upper bound for C

$$\limsup_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln m_{\rho, A_L^\Psi}(C) \leq -\inf_{x \in C} I(x)$$

Here, I is the Legendre–Fenchel transform of $J(s)$, i.e.,

$$I(x) = \sup_{s \in \mathbb{R}} \{sx - J(s)\}.$$

2. If J is differentiable for all $s \in \mathcal{D}_J$ and $\mathcal{D}_J = \mathbb{R}$, then $\{A_L^\Psi\}_{L \in \mathbb{R}_0^+}$ satisfies a large deviation lower bound

$$\limsup_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln m_{\rho, A_L^\Psi}(O) \geq -\inf_{x \in O} I(x). \quad \clubsuit$$

In a similar way we can rewrite Bryc’s Theorem 2.1.19 as follows

Theorem 2.4.9 (Bryc):

Let Λ_L be the sequence of boxes (2.3.21), $\rho \in \mathcal{U}^*$ be a state, ϵ a positive number and z_ϵ be a complex number with $|z_\epsilon| < \epsilon$. We consider the space average family $\{A_L^\Psi\}_{L \in \mathbb{R}_0^+} \in \mathcal{U}^+ \cap \mathcal{U}_L$ in the state ρ such that for z_ϵ the free energy J (2.4.4) exists and has analytic continuation. Then the family of measures $\{m_{\rho, \sqrt{|\Lambda_L|}(A_L^\Psi - \rho(A_L^\Psi)\mathbb{1})}\}_{L \in \mathbb{R}_0^+}$ converges in distribution to the normal distribution $\mathcal{N}_{0, \sigma^2}$ with variance $\sigma^2 = \frac{d^2}{ds^2} J(s) \Big|_{s=0}$. $\clubsuit$

Remark 2.4.10 (LDP and uniqueness of KMS states). [NR04] is a study of LD in quantum spin systems at inverse temperatures $\beta < \beta_0$ done by “polymer–cluster expansions”, without explicitly using the uniqueness of KMS states. For sufficiently small $\beta > 0$, however, the KMS state is unique. See discussions of [NR04, Remark 2.14]. In [HMO07] LD associated with particular ergodic states (finitely correlated ergodic states) are studied in dimension 1. See also [Oga10]. In [OR11] the authors impose strong conditions on the state, which are known to be satisfied in concrete models only when the KMS state is unique. In [ABPM16], similar

to [GLM02], we consider weakly interacting fermions on the lattice at any inverse temperature $\beta \in \mathbb{R}^+$. In this situation, the uniqueness of KMS is generally unknown. $\ast$

LARGE DEVIATIONS PRINCIPLE FOR CURRENT DISTRIBUTIONS OF FREE FERMIONS

THIS chapter is a revised version of the work [ABPR17]. Here, we prove Conjecture 2 for free-fermion systems at equilibrium. In fact, we show that quantum uncertainty of microscopic electric current densities (around their classical, macroscopic values) is suppressed, *exponentially fast* w.r.t. the volume $|\Lambda_L| = \mathcal{O}(L^d)$ (in lattice units (l.u.), $d \in \mathbb{N}$ being the space dimension) of the region of the lattice where an external electric field is applied. In order to achieve this, we use the large deviation formalism [DS89, DZ98], which has been adopted in quantum statistical mechanics since the eighties [ABPM17, Section 7]. Other mathematical results which are pivotal in our analysis are the Combes–Thomas estimates [CT73, AW15], the Akcoglu-Krengel ergodic theorem [CL12] and the (Arzelà–) Ascoli theorem [Rud91, Theorem A5]. Indeed, combined with the celebrated Gärtner–Ellis theorem (see, e.g., [DZ98, Corollary 4.5.27]), they allow us to prove a large deviation principle (LDP) for the current density distributions, which quantify the probability of deviations, due to quantum uncertainty, from the expected value. Our analysis use the mathematical structure introduced in Chapter 2. For more details and proofs see [ABPR17].

3.1 MAIN RESULTS

We study large deviations (LD) for the microscopic current density produced by any fixed, time-dependent electric field $\mathcal{E}$. Thus, we use same notation that in Chapter 2–§2.3.5. In particular, recall that $\vec{w} \in \mathbb{R}^d$ is a vector such that $\|\vec{w}\|_{\mathbb{R}^d} = 1$ and $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ is a (time-dependent) continuous, compactly supported, electric field. Then using the measurable space $(\Omega, \mathfrak{A}_\Omega)$ defined in Chapter 2–§2.3.5, we show an LDP for the KMS-state $\rho^{(\omega)}$ and the linear response current $\mathbb{I}_\Lambda^{(\omega, \mathcal{E})}$ defined by Expression (2.3.75) with $\omega \in \Omega$ and $\Lambda \in \mathbb{Z}^d$ (see discussions around it). Via the Gärtner–Ellis theorem (see, e.g., [DZ98, Corollary 4.5.27]), this is a consequence of the following result:

Theorem 3.1.1 (Generating functions for currents):

There is a measurable subset $\tilde{\Omega} \subset \Omega$ of full measure such that, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, the limit

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln \varrho^{(\omega)} \left(e^{|\Lambda_L| \mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}} \right)$$

exist and equals

$$J^{(\mathcal{E})} \doteq \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \mathbb{E} \left[\ln \varrho^{(\cdot)} \left(e^{|\Lambda_L| \mathbb{I}_{\Lambda_L}^{(\cdot, \mathcal{E})}} \right) \right].$$

Moreover, for any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, the map $s \mapsto J^{(s\mathcal{E})}$ from $\mathbb{R}$ to itself is continuously differentiable and convex. ♣

PROOF. The assertions directly follow from Corollaries 3.2.24 and 3.2.25. Note that the map $s \mapsto J^{(s\mathcal{E})}$ is a limit of convex functions, and hence, it is also convex. End

Note that a combination of Theorem 3.1.1 and Theorem 2.4.8 yields to:

3.1.2 COROLLARY (LARGE DEVIATION PRINCIPLE FOR CURRENTS). Let $\tilde{\Omega} \subset \Omega$ be the measurable subset of full measure of Theorem 3.1.1. Then, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, the sequence $\{\mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}\}_{L \in \mathbb{R}^+}$ of microscopic current densities satisfies an LDP, in the KMS state $\varrho^{(\omega)}$, with speed $|\Lambda_L|$ and good rate function $I^{(\mathcal{E})}$ defined on $\mathbb{R}$ by

$$I^{(\mathcal{E})}(x) \doteq \sup_{s \in \mathbb{R}} \{sx - J^{(s\mathcal{E})}\} \geq 0. \quad \blacksquare$$

Remark 3.1.3. By direct estimates, one verifies that, for any fixed state ρ , $\{\mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}\}_{L \in \mathbb{R}^+}$ yields an exponentially tight family of probability measures, defined by (2.4.7) for $A = \mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}$. Therefore, by [DZ98, Lemma 4.1.23], $\{\mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}\}_{L \in \mathbb{R}^+}$ satisfies, along some subsequence, an LDP, in any state ρ , with speed $|\Lambda_L|$ and a good rate function. However, it is not clear whether this rate function depends on the choice of subsequences and $\omega \in \Omega$. Moreover, no information on minimizers of the rate function, like in [ABPR17, Theorem 3.4], can be deduced from [DZ98, Lemma 4.1.23]. *

As mentioned in Chapter 2–§2.1 if an LDP holds true, then the law of large numbers follows. Therefore, by [BPH16, BP15] and Corollary 3.1.2, the distributions of the microscopic current density observables, in the state $\varrho^{(\omega)}$, weak* converges, for $\omega \in \Omega$ almost surely, to the delta distribution at the (classical value of the) macroscopic current density. Using Theorem 3.1.1, we sharpen this result by proving that the microscopic current density converges *exponentially fast* to the macroscopic one, w.r.t. the volume $|\Lambda_L|$ of the region of the lattice where an external electric

field is applied. To this end, in [ABPR17, Corollary 3.5] it is shown that the microscopic current density converges *exponentially fast* to the macroscopic one, w.r.t. the volume $|\Lambda_L|$ (in lattice units (l.u.)) of the region of the lattice where the electric field is applied. This is in accordance with the low temperature (4.2K) experiment [WMR⁺12] on the resistance of nanowires with lengths down to approximately 20 l.u. ($L \simeq 10$).

3.2 TECHNICAL PROOFS

3.2.1 PRELIMINARY ESTIMATES

As in Chapter 2–§2.3, we denote by $\mathcal{H}$ to the (one–particle) Hilbert space, by $\mathcal{U} \equiv \mathcal{U}(\mathcal{H})$ the CAR C^* –algebra associated to $\mathcal{H}$ and by $\mathfrak{Q} \doteq \mathbb{Z}^d$ the d –dimensional crystal lattice where the conducting fermions are embedded.

We first show an elementary observation:

3.2.1 LEMMA (OPERATOR NORM ESTIMATE). *For any operator $C \in \mathcal{B}(\mathcal{H})$,*

$$\|C\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{x \in \mathfrak{Q}} \sum_{y \in \mathfrak{Q}} \left| \langle e_x, C e_y \rangle_{\mathcal{H}} \right|. \quad \blacklozenge$$

PROOF. By the Cauchy–Schwarz inequality, for all $\varphi, \psi \in \mathcal{H}$,

$$\begin{aligned} \left| \langle \varphi, C \psi \rangle_{\mathcal{H}} \right| &\leq \sum_{x, y \in \mathfrak{Q}} \left| \varphi(x) \psi(y) \langle e_x, C e_y \rangle_{\mathcal{H}} \right| \\ &= \sum_{x, y \in \mathfrak{Q}} \left(|\varphi(x)| \left| \langle e_x, C e_y \rangle_{\mathcal{H}} \right|^{1/2} \right) \left(|\psi(y)| \left| \langle e_x, C e_y \rangle_{\mathcal{H}} \right|^{1/2} \right) \\ &\leq \sqrt{\sum_{x, y \in \mathfrak{Q}} \left(|\varphi(x)|^2 \left| \langle e_x, C e_y \rangle_{\mathcal{H}} \right| \right)} \sqrt{\sum_{x, y \in \mathfrak{Q}} \left(|\psi(y)|^2 \left| \langle e_x, C e_y \rangle_{\mathcal{H}} \right| \right)} \\ &\leq \|\varphi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} \sup_{x \in \mathfrak{Q}} \sum_{y \in \mathfrak{Q}} \left| \langle e_x, C e_y \rangle_{\mathcal{H}} \right|. \end{aligned} \quad \boxed{\text{End}}$$

The second one is a version of the Bogoliubov inequality. Recall that the tracial state $\text{tr} \in \mathcal{U}^*$ is the quasi–free state satisfying (2.3.28) at $\beta = 0$.

3.2.2 LEMMA (BOGOLIUBOV–TYPE INEQUALITIES). *Let $C \in \mathcal{U}$ be any strictly positive element.*

1. *For any continuously differentiable family $\{H_\alpha\}_{\alpha \in \mathbb{R}} \subset \mathcal{U}$ of self–adjoint elements,*

$$\left| \partial_\alpha \ln \text{tr} (C e^{H_\alpha}) \right| \leq \sup_{u \in [-1/2, 1/2]} \left\| e^{u H_\alpha} \{ \partial_\alpha H_\alpha \} e^{-u H_\alpha} \right\|_{\mathcal{U}}.$$

2. Similarly, for any self-adjoint $H_0, H_1 \in \mathcal{U}$,

$$\left| \ln \operatorname{tr} (C e^{H_1}) - \ln \operatorname{tr} (C e^{H_0}) \right| \leq \sup_{\alpha \in [0,1]} \sup_{u \in [-1/2, 1/2]} \left\| e^{u(\alpha H_1 + (1-\alpha)H_0)} (H_1 - H_0) e^{-u(\alpha H_1 + (1-\alpha)H_0)} \right\|_{\mathcal{U}}.$$

3. Moreover, suppose that the family $\{H_\alpha\}_{\alpha \in \mathbb{R}}$ is twice continuous differentiable and $\{H_\alpha\}_{\alpha \in \mathbb{R}}$ and that $\{\partial_\alpha H_\alpha\}_{\alpha \in \mathbb{R}}$ commute. Then

$$\left| \partial_\alpha^{(2)} \ln \operatorname{tr} (C e^{H_\alpha}) \right| \leq \left\| \partial_\alpha^{(2)} H_\alpha \right\|_{\mathcal{U}} + \left| \left[\partial_\mu^{(2)} \left(\ln \left(\operatorname{tr} (C e^{H_\alpha + \mu \partial_\alpha(H_\alpha)}) \right) \right) \right]_{\mu=0} \right|. \quad \blacklozenge$$

PROOF.

1. By Duhamel's formula, for any continuously differentiable family $\{H_\alpha\}_{\alpha \in \mathbb{R}} \subset \mathcal{U}$ of self-adjoint elements,

$$\partial_\alpha \{e^{H_\alpha}\} = \int_0^1 e^{uH_\alpha} \{\partial_\alpha H_\alpha\} e^{(1-u)H_\alpha} du,$$

which implies that

$$\partial_\alpha \ln \operatorname{tr} (C e^{H_\alpha}) = \int_0^1 \frac{\operatorname{tr} (C e^{uH_\alpha} \{\partial_\alpha H_\alpha\} e^{(1-u)H_\alpha})}{\operatorname{tr} (C e^{H_\alpha})} du.$$

Using the cyclicity of the trace, we then get

$$\begin{aligned} \partial_\alpha \ln \operatorname{tr} (C e^{H_\alpha}) &= \int_0^1 \frac{\operatorname{tr} \left(e^{\frac{H_\alpha}{2}} C e^{\frac{H_\alpha}{2}} e^{(u-\frac{1}{2})H_\alpha} \{\partial_\alpha H_\alpha\} e^{(\frac{1}{2}-u)H_\alpha} \right)}{\operatorname{tr} \left(e^{\frac{H_\alpha}{2}} C e^{\frac{H_\alpha}{2}} \right)} du \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\operatorname{tr} \left(e^{\frac{H_\alpha}{2}} C e^{\frac{H_\alpha}{2}} e^{uH_\alpha} \{\partial_\alpha H_\alpha\} e^{-uH_\alpha} \right)}{\operatorname{tr} \left(e^{\frac{H_\alpha}{2}} C e^{\frac{H_\alpha}{2}} \right)} du, \end{aligned}$$

which yields 1.

2. To prove the second assertion, it suffices to apply Assertion (i) to the family defined by

$$H_\alpha = H_0 + \alpha (H_1 - H_0), \quad \alpha \in [0, 1].$$

3. For the final assertion note that

$$\partial_\alpha^{(2)} \ln \operatorname{tr} (C e^{H_\alpha}) = \frac{\operatorname{tr} (C \partial_\alpha^{(2)} \{e^{H_\alpha}\}) \operatorname{tr} (C e^{H_\alpha}) - \left(\operatorname{tr} (C \partial_\alpha \{e^{H_\alpha}\}) \right)^2}{(\operatorname{tr} (C e^{H_\alpha}))^2}.$$

By using again the Duhamel's formula for $\{H_\alpha\}_{\alpha \in \mathbb{R}} \subset \mathcal{U}$ we find

$$\begin{aligned} \partial_\alpha^{(2)} \{e^{H_\alpha}\} &= \int_0^1 \int_0^1 \left[e^{uH_\alpha} \{\partial_\alpha^{(2)} H_\alpha\} e^{(1-u)H_\alpha} + ue^{uvH_\alpha} \{\partial_\alpha H_\alpha\} e^{(1-v)uH_\alpha} \{\partial_\alpha H_\alpha\} e^{(1-u)H_\alpha} \right. \\ &\quad \left. + ue^{(1-u)H_\alpha} \{\partial_\alpha H_\alpha\} e^{(1-v)uH_\alpha} \{\partial_\alpha H_\alpha\} e^{uvH_\alpha} \right] dudv. \end{aligned}$$

In particular, since we assume that H_α and $\partial_\alpha H_\alpha$ commute, H_α also commutes with its second derivative. Hence,

$$\partial_\alpha \{e^{H_\alpha}\} = \{\partial_\alpha H_\alpha\} e^{H_\alpha} \quad \text{and} \quad \partial_\alpha^{(2)} \{e^{H_\alpha}\} = \{\partial_\alpha^{(2)} H_\alpha\} e^{H_\alpha} + \{\partial_\alpha H_\alpha\}^2 e^{H_\alpha},$$

such that

$$(3.2.3) \quad \mathfrak{p}_\alpha^{(2)} \ln \operatorname{tr}(C e^{H_\alpha}) = \frac{\operatorname{tr}(C \{\partial_\alpha^{(2)} H_\alpha\} e^{H_\alpha})}{\operatorname{tr}(C e^{H_\alpha})} + \frac{\operatorname{tr}(C \{\partial_\alpha H_\alpha\}^2 e^{H_\alpha})}{\operatorname{tr}(C e^{H_\alpha})} - \frac{(\operatorname{tr}(C \{\partial_\alpha H_\alpha\} e^{H_\alpha}))^2}{(\operatorname{tr}(C e^{H_\alpha}))^2}$$

With assertion 1 and some manipulations we end the proof. End

Observe that Lemma 3.2.2–2 is proven in [LR05, Lemma 3.6]. Here, we give a proof of this estimate for completeness. These Bogoliubov–type inequalities are useful because we deal with quasi-free dynamics. In this case, we have a very good control on the norm of

$$e^{uH_\alpha} \{\partial_\alpha H_\alpha\} e^{-uH_\alpha},$$

because H_α can be view as the second quantization of some operator $h_\alpha \in \mathcal{H}$, see Definition 2.3.15.

We also observe that the second term on the right-hand side of Expression (3.2.3) allows to study the *fluctuations* of the family of self-adjoint elements $\{\partial_\alpha H_\alpha\}_{\alpha \in \mathbb{R}}$. In order to estimate such a behaviour, note that for $\mu \in \mathbb{R}$ the family of functions $\{\mathfrak{f}_\alpha(\mu)\}_{\alpha \in \mathbb{R}}$ defined by

$$\mathfrak{f}_\alpha(\mu) = \ln \left(\frac{\operatorname{tr}(C e^{H_\alpha + \mu \partial_\alpha H_\alpha})}{\operatorname{tr}(C e^{H_\alpha})} \right),$$

in particular is convex [OP04]. In contrast to Proposition 3.2.18, in a similar way to [LR05, Corollary 3.5] we also can use assertion 3.2.2–3 combined with convexity arguments to prove that the family of functions $\left\{ J_{\mathcal{L}, \mathcal{L}^{(2)}, \mathcal{L}^{(3)}}^{(\omega, s, \mathcal{E})} \right\}_{s \in \mathbb{R}}$, defined by (3.2.12), is continuously differentiable w.r.t. s . Note that Proposition 3.2.18 will permit to claim that the generating function $J^{(\omega, s, \mathcal{E})}$ is differentiable and then there will be an LDP for the sequence $\{\Pi_{\Lambda_L}^{(\omega, \mathcal{E})}\}_{L \in \mathbb{R}^+}$ of microscopic conductivities following Theorem 2.4.8.

3.2.2 QUASI-FREE DYNAMICS AND SECOND QUANTIZATION PROPERTIES

Because of the identities (2.3.16), namely

$$e^{d\langle Q, CQ \rangle} a(\varphi)^* e^{-d\langle Q, CQ \rangle} = a(e^C \varphi)^* \quad \text{and} \quad e^{d\langle Q, CQ \rangle} a(\varphi) e^{-d\langle Q, CQ \rangle} = a(e^{-C^*} \varphi),$$

second quantization operators can be used to represent the dynamics $\{\tau_t^{(\omega, \mathcal{Z})}\}_{t \in \mathbb{R}}$ for any $\omega \in \Omega$ and $\mathcal{Z} \in \mathfrak{Z}_f$. See Example 2.3.59 (Expressions (2.3.61)–(2.3.60)) replacing $h^{(\Lambda)}$ with $h_{\mathcal{Z}}^{(\omega)}$ (cf. (2.3.56)), and observe that the range of $h_{\mathcal{Z}}^{(\omega)} \in \mathcal{B}(\mathcal{H})$ is finite dimensional whenever $\mathcal{Z} \in \mathfrak{Z}_f$. Additionally, by using the tracial state $\text{tr} \in \mathcal{U}^*$, i.e., the quasi-free state satisfying (2.3.28) for $\beta = 0$, the corresponding KMS state defined by (2.3.28) by replacing $h^{(\omega)}$ in this equation with $h_{\mathcal{Z}}^{(\omega)}$ (see (2.3.56) and Lemma 2.3.65) is explicitly given by

$$(3.2.4) \quad \varrho_{\mathcal{Z}}^{(\omega)}(B) = \frac{\text{tr}\left(Be^{-\beta d\langle Q, h_{\mathcal{Z}}^{(\omega)} Q \rangle}\right)}{\text{tr}\left(e^{-\beta d\langle Q, h_{\mathcal{Z}}^{(\omega)} Q \rangle}\right)}, \quad B \in \mathcal{U},$$

for any $\omega \in \Omega$, $\lambda, \vartheta \in \mathbb{R}_0^+$, $\beta \in \mathbb{R}^+$ and $\mathcal{Z} \in \mathfrak{Z}_f$. We conclude now by an additional observation used later to control quantum fluctuations:

3.2.5 LEMMA. *For any self-adjoint operators $C_1, C_2 \in \mathcal{B}(\mathcal{H})$ whose ranges are finite dimensional, let $C \doteq \ln(e^{C_2} e^{C_1} e^{C_2})$. Then,*

$$\text{ran}(C) \subset \text{lin}\{\text{ran}(C_1) \cup \text{ran}(C_2)\}$$

and there is a constant $D \in \mathbb{R}$ such that

$$e^{d\langle Q, C_2 Q \rangle} e^{d\langle Q, C_1 Q \rangle} e^{d\langle Q, C_2 Q \rangle} = e^{d\langle Q, CQ \rangle + D\mathbb{1}}. \quad \blacklozenge$$

PROOF. Fix all parameters of the lemma. We give the proof in two steps:

Step 1: Let

$$\mathcal{H}_0 \doteq \text{lin}\{\text{ran}(C_1) \cup \text{ran}(C_2)\}$$

and $\mathcal{U}_{\mathcal{H}_0} \equiv \mathcal{U}(\mathcal{H}_0) \subset \mathcal{U}$ be the (finite dimensional) CAR C^* -subalgebra generated by the identity $\mathbb{1}$ and $\{a(\varphi)\}_{\varphi \in \mathcal{H}_0}$, for $\mathcal{H}_0 \subset \mathcal{H}$. Take two strictly positive elements M_1, M_2 of $A_{\mathcal{H}_0}$ satisfying the conditions

$$M_1 a(\varphi) M_1^{-1} = M_2 a(\varphi) M_2^{-1} \quad \text{and} \quad M_1 a(\varphi)^* M_1^{-1} = M_2 a(\varphi)^* M_2^{-1}$$

for any $\varphi \in \mathcal{H}_0$. From this we conclude that

$$M_1 A M_1^{-1} = M_2 A M_2^{-1}, \quad A \in \mathcal{U}_{\mathcal{H}_0},$$

because all elements of $\mathcal{U}_{\mathcal{H}_0}$ are polynomials in $\{a(\varphi), a(\varphi)^*\}_{\varphi \in \mathcal{H}_0}$, by definition of $\mathcal{U}_{\mathcal{H}_0}$ and finite dimensionality of $\mathcal{H}_0$. In particular, by choosing, respectively, $A = M_2^{-1}$ and $A = M_2^{-1} B M_2$ for $B \in \mathcal{U}_{\mathcal{H}_0}$, it follows that

$$M_1 M_2^{-1} = M_2^{-1} M_1 \quad \text{and} \quad M_1 M_2^{-1} B = B M_1 M_2^{-1}.$$

Hence, since any element of $\mathcal{U}_{\mathcal{H}_0}$ commuting with all elements of $\mathcal{U}_{\mathcal{H}_0}$ is a multiple of the identity, there is $D \in \mathbb{C}$ such that

$$M_1 M_2^{-1} = M_2^{-1} M_1 = D \mathbb{1}.$$

The constant D is non-zero because M_1, M_2 are assumed to be invertible. In fact, $M_1 = D M_2$ with $D > 0$ because $M_1, M_2 > 0$.

Step 2: Observe that $e^{C_2} e^{C_1} e^{C_2} > 0$ because C_1, C_2 are both self-adjoint operators. In particular, $C \doteq \ln(e^{C_2} e^{C_1} e^{C_2})$ is well-defined as a bounded self-adjoint operator acting on $\mathcal{H}$ with $\text{ran}(C) \subset \mathcal{H}_0$. Using (2.3.16), we obtain that

$$e^{d\langle Q, CQ \rangle} a(\varphi) e^{-d\langle Q, CQ \rangle} = e^{d\langle Q, C_2 Q \rangle} e^{d\langle Q, C_1 Q \rangle} e^{d\langle Q, C_2 Q \rangle} a(\varphi) e^{-d\langle Q, C_2 Q \rangle} e^{-d\langle Q, C_1 Q \rangle} e^{-d\langle Q, C_2 Q \rangle}$$

and

$$e^{d\langle Q, CQ \rangle} a(\varphi)^* e^{-d\langle Q, CQ \rangle} = e^{d\langle Q, C_2 Q \rangle} e^{d\langle Q, C_1 Q \rangle} e^{d\langle Q, C_2 Q \rangle} a(\varphi)^* e^{-d\langle Q, C_2 Q \rangle} e^{-d\langle Q, C_1 Q \rangle} e^{-d\langle Q, C_2 Q \rangle}.$$

By Step 1, the assertion follows. End

3.2.3 DECAY BOUNDS FOR CURRENT OPERATORS

For simplicity, below we fix $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, and $\eta, \mu \in \mathbb{R}^+$ once for all. For any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, any collection $\mathcal{Z}^{(\tau)} \in \mathfrak{Z}$, $\mathcal{Z} \in \mathfrak{Z}_f$, and $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$, we recall Expression

(2.3.78)

$$\begin{aligned}
K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} &\doteq -8 \sum_{k, q=1}^d w_k \sum_{e_k, e_q \in \mathcal{Z}} \int_{-\infty}^0 \{\mathcal{E}(\alpha)\}_q d\alpha \\
&\quad \int_0^{-\alpha} ds \Im \left\{ \left\langle e_y, s_{\mp e_q} \left(U_{-\alpha}^{(\mathcal{Z}^{(\tau)})} \right)^* s_{\mp e_k} U_{-\alpha}^{(\mathcal{Z}^{(\tau)})} e_x \right\rangle_{\mathcal{H}} \right\} \\
&\quad + 2 \sum_{k=1}^d w_k \sum_{e_k \in \mathcal{Z}} \left(\int_{-\infty}^0 \{\mathcal{E}(\alpha)\}_q d\alpha \right) \Re \{ S_{x+e_k, x}^{(\omega)} \}.
\end{aligned}$$

This one-particle operator satisfies the following decay bounds:

3.2.6 LEMMA (DECAY OF LOCAL CURRENTS). *For any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$, $x, y \in \mathfrak{L}$, and two collections $\mathcal{Z} \in \mathfrak{Z}_f$ and $\mathcal{Z}^{(\tau)} \in \mathfrak{Z}_f$*

$$\begin{aligned}
\left| \left\langle e_x, K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} e_y \right\rangle_{\mathcal{H}} \right| &\leq D_{3.2.6} \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d} e^{2|\alpha|\eta} d\alpha \right) (e^{-\mu_\eta |x-y|} + \eta \delta_{1, |x-y|}), \\
\frac{1}{|\cup \mathcal{Z}|} \sum_{x, y \in \mathfrak{L}} \left| \left\langle e_x, K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} e_y \right\rangle_{\mathcal{H}} \right| &\leq D_{3.2.6} \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d} e^{2|\alpha|\eta} d\alpha \right) \sum_{z \in \mathfrak{L}} e^{-\mu_\eta |z|} (1 + \eta),
\end{aligned}$$

where

$$D_{3.2.6} \doteq 4d\eta^{-1} \times 36^2 (1 + \vartheta)^2 \sum_{z \in \mathfrak{L}} e^{2\mu_\eta (1-|z|)} < \infty.$$

Recall that μ_η is defined by (2.3.58). ♦

PROOF. Fix the parameters of the lemma. By (2.3.57), note that for any $z_1, z_2, x, y \in \mathfrak{L}$, $\omega \in \Omega$, $\vartheta \in \mathbb{R}_0^+$ and $s \in \mathbb{R}$,

$$(3.2.7) \quad \left| \left\langle e_x, s_{\mp e_q} \left(U_{-\alpha}^{(\mathcal{Z}^{(\tau)})} \right)^* s_{\mp e_k} U_{-\alpha}^{(\mathcal{Z}^{(\tau)})} e_y \right\rangle_{\mathcal{H}} \right| \leq 36^2 (1 + \vartheta)^2 e^{2|s|\eta} e^{-2\mu_\eta (|x-e_q|+|y+e_k|)} \delta_{y,0}$$

By the triangle inequality, observe also that

$$(3.2.8) \quad \sum_{z \in \mathfrak{L}} e^{-2\mu_\eta (|x-z|+|y-z|)} \leq e^{-\mu_\eta |x-y|} \sum_{z \in \mathfrak{L}} e^{-\mu_\eta (|x-z|+|y-z|)} \leq e^{-\mu_\eta |x-y|} \left(\sum_{z \in \mathfrak{L}} e^{-2\mu_\eta |z|} \right).$$

From (3.2.7)–(3.2.8), we obtain the bound

$$\begin{aligned}
&\sum_{e_q, e_k \in \mathcal{Z}} \left| \left\langle e_x, s_{\mp e_q} \left(U_{-\alpha}^{(\mathcal{Z}^{(\tau)})} \right)^* s_{\mp e_k} U_{-\alpha}^{(\mathcal{Z}^{(\tau)})} e_y \right\rangle_{\mathcal{H}} \right| \\
(3.2.9) \quad &\leq 36^2 (1 + \vartheta)^2 e^{2|s|\eta} e^{-\mu_\eta |x-y|} \left(\sum_{z \in \mathfrak{L}} e^{2\mu_\eta (1-|z|)} \right),
\end{aligned}$$

using that $|z - e_k| \geq |z| - 1$ for any $z \in \mathfrak{L}$ and $k \in \{1, \dots, d\}$. The other terms computed from (2.3.78) are estimated in the same way. We omit the details. This yields the first bound of the lemma. The second estimate is also proven in the same way.

It is convenient to introduce at this point the notation

$$(3.2.10) \quad \partial_\Lambda(\tilde{\Lambda}) \doteq \left\{ \{x, y\} \subset \Lambda : |x - y| = 1, \{x, y\} \cap \tilde{\Lambda} \neq \emptyset \text{ and } \{x, y\} \cap \tilde{\Lambda}^c \neq \emptyset \right\}$$

for any set $\tilde{\Lambda} \subset \Lambda \subset \mathfrak{L}$ with complement $\tilde{\Lambda}^c \doteq \mathfrak{L} \setminus \tilde{\Lambda}$, while, for any $\mathcal{Z} \in \mathfrak{Z}$,

$$\partial_\Lambda(\mathcal{Z}) \doteq \{\partial_\Lambda(Z) : Z \in \mathcal{Z}\}.$$

Then, the one-particle operators (2.3.78) also satisfy the following bounds:

3.2.11 LEMMA (BOX DECOMPOSITION OF LOCAL CURRENTS). *For any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\Lambda, \tilde{\Lambda} \in \mathcal{P}_f(\mathfrak{L})$, $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$, and $\mathcal{Z} \in \mathfrak{Z}_f$ with $\cup \mathcal{Z} \subset \tilde{\Lambda}$,*

1.

$$\begin{aligned} & \sum_{x, y \in \mathfrak{L}} \left| \left\langle \mathbf{e}_x, \left(K_{\{\Lambda\}, \{\tilde{\Lambda}\}}^{(\omega, \mathcal{E})} - K_{\{\Lambda\}, \mathcal{Z}}^{(\omega, \mathcal{E})} \right) \mathbf{e}_y \right\rangle_{\mathcal{H}} \right| \\ & \leq D_1 \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d} \alpha^2 e^{2|\alpha\eta|} d\alpha \right) \left(\sum_{x \in \Lambda} \sum_{z \in \tilde{\Lambda} \setminus \cup \mathcal{Z}} e^{-\mu_\eta |x-z|} + \sum_{z \in \mathfrak{L}} e^{-\mu_\eta |z|} \sum_{x \in \cup \partial_\Lambda(\mathcal{Z})} 1 \right), \end{aligned}$$

where

$$D_1 \doteq 8 \times 36^4 (1 + \vartheta)^3 (4d + \lambda) e^{3\mu_\eta} \left(\sum_{z \in \mathfrak{L}} e^{-\mu_\eta |z|} \right)^3 < \infty.$$

2.

$$\sum_{x, y \in \mathfrak{L}} \left| \left\langle \mathbf{e}_x, \left(K_{\{\Lambda\}, \mathcal{Z}_\tau}^{(\omega, \mathcal{E})} - K_{\mathcal{Z}, \mathcal{Z}_\tau}^{(\omega, \mathcal{E})} \right) \mathbf{e}_y \right\rangle_{\mathcal{H}} \right| \leq D_2 \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d}^2 |\alpha| e^{2|\alpha\eta|} d\alpha \right) \sum_{z \in (\Lambda \setminus \cup \mathcal{Z}) \cup (\cup \partial_\Lambda(\mathcal{Z}))} 1,$$

where

$$D_2 \doteq 16 \times 36^2 (1 + \vartheta)^2 d e^{4\mu_\eta} \left(\sum_{z \in \mathfrak{L}} e^{-2\mu_\eta |z|} \right)^2 + d (1 + \vartheta) < \infty. \quad \blacklozenge$$

PROOF. See proof of Lemmata 4.6 and 4.7 in [ABPR17].

3.2.4 FINITE-VOLUME GENERATING FUNCTIONS

Fix $\beta \in \mathbb{R}^+$ and $\lambda, \vartheta \in \mathbb{R}_0^+$. Given $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\omega \in \Omega$ and three finite collections $\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)} \in \mathfrak{Z}_f$, we define the finite-volume generating function

$$(3.2.12) \quad J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} \doteq g_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} - g_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, 0)},$$

where

$$(3.2.13) \quad g_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} \doteq \frac{1}{|\cup \mathcal{Z}|} \ln \operatorname{tr} \left(e^{-\beta d \langle Q, h_{\mathcal{Z}^{(\vartheta)}}^{(\omega)} Q \rangle} e^{\mathfrak{R}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}} \right).$$

Recall that the tracial state $\operatorname{tr} \in \mathcal{U}^*$ is the quasi-free state satisfying (2.3.28) at $\beta = 0$, and $h_{\mathcal{Z}^{(\vartheta)}}^{(\omega)}$ is the one-particle Hamiltonian defined by (2.3.56). See also Definition 2.3.15 and (2.3.77). By construction, note that

$$(3.2.14) \quad \frac{1}{|\Lambda_L|} \ln \varrho^{(\omega)} \left(e^{|\Lambda_L| \mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}} \right) = \lim_{L_\varrho \rightarrow \infty} \lim_{L_\tau \rightarrow \infty} J_{\{\Lambda_L\}, \{\Lambda_{L_\varrho}\}, \{\Lambda_{L_\tau}\}}^{(\omega, \mathcal{E})}.$$

The family of functions $\mathcal{E} \mapsto J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}$ is equicontinuous with uniformly bounded second derivative:

3.2.15 PROPOSITION (EQUICONTINUITY OF GENERATING FUNCTIONS). *Fix $n \in \mathbb{N}$. The family of maps $\mathcal{E} \mapsto J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}$ from $C_0^0([-n, n]; \mathbb{R}^d) \subset C_0^0(\mathbb{R}; \mathbb{R}^d)$ to $\mathbb{R}$, for $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)} \in \mathfrak{Z}_f$, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, and ϑ in a compact set of $\mathbb{R}_0^+$, is equicontinuous w.r.t. the sup norm for $\mathcal{E}$ in any bounded set of $C_0^0([-n, n]; \mathbb{R}^d)$.* $\blacksquare$

PROOF. Fix $n \in \mathbb{N}$, $\beta \in \mathbb{R}^+$, $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)} \in \mathfrak{Z}_f$. By using Lemma 3.2.2 (ii), for any $\mathcal{E}_0, \mathcal{E}_1 \in C_0^0([-n, n]; \mathbb{R}^d)$,

$$(3.2.16) \quad \left| g_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E}_1)} - g_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E}_0)} \right| \leq \frac{1}{|\cup \mathcal{Z}|} \sup_{\alpha \in [0, 1]} \sup_{u \in [-1/2, 1/2]} \left\| e^{u \mathfrak{R}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \alpha \mathcal{E}_1 + (1-\alpha) \mathcal{E}_0)}} \mathfrak{R}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E}_1 - \mathcal{E}_0)} e^{-u \mathfrak{R}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \alpha \mathcal{E}_1 + (1-\alpha) \mathcal{E}_0)}} \right\|_{\mathcal{U}}.$$

Recall that, for any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\mathfrak{R}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}$ is the second quantization associated with the operator $K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}$. See (2.3.77) and (2.3.78). In particular, from (2.3.16), we deduce the inequality

$$(3.2.17) \quad \sup_{u \in [-1/2, 1/2]} \sup_{x, y \in \mathcal{U}} \left\| e^{u \mathfrak{R}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}} a(e_x)^* a(e_y) e^{-u \mathfrak{R}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}} \right\|_{\mathcal{U}} \leq e^{\|K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}\|_{\mathcal{B}(\mathcal{H})}}.$$

The assertion then follows by combining (2.3.77), (3.2.16) and Definition 2.3.15 with (3.2.17) and Lemmata 3.2.1, 3.2.6. End

3.2.18 PROPOSITION (UNIFORM BOUNDEDNESS OF SECOND DERIVATIVES). Fix $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\beta_1, s_1, \vartheta_1, \lambda_1 \in \mathbb{R}^+$. Then,

$$\sup_{\substack{\beta \in (0, \beta_1], \vartheta \in [0, \vartheta_1], \lambda \in [0, \lambda_1] \\ \omega \in \Omega, s \in [-s_1, s_1], \mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)} \in \mathcal{Z}_f}} \left\{ \left| \partial_s J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, s\mathcal{E})} \right| + \left| \partial_s^2 J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, s\mathcal{E})} \right| \right\} < \infty. \quad \blacksquare$$

PROOF. Fix the parameters of the proposition. Then, by cyclicity of the tracial state,

$$\partial_s J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, s\mathcal{E})} = \frac{1}{|\cup \mathcal{Z}|} \omega_s \left(\mathfrak{K}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} \right)$$

and

$$\partial_s^2 J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, s\mathcal{E})} = \frac{1}{|\cup \mathcal{Z}|} \left(\omega_s \left(\left(\mathfrak{K}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} \right)^2 \right) - \omega_s \left(\mathfrak{K}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} \right)^2 \right),$$

where ω_s is the state defined, for any $B \in \mathcal{U}$, by

$$\omega_s(B) = \frac{\text{tr} \left(B e^{\frac{s}{2} \mathfrak{K}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}} e^{-\beta d(Q, h_{\mathcal{Z}^{(\vartheta)}}^{(\omega)})} e^{\frac{s}{2} \mathfrak{K}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}} \right)}{\text{tr} \left(e^{\frac{s}{2} \mathfrak{K}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}} e^{-\beta d(Q, h_{\mathcal{Z}^{(\vartheta)}}^{(\omega)})} e^{\frac{s}{2} \mathfrak{K}_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}} \right)}.$$

By Lemma 3.2.5 and (2.3.77), observe that ω_s is the quasi-free state satisfying

$$(3.2.19) \quad \omega_s(a^*(\varphi) a(\psi)) = \left\langle \psi, \frac{1}{1 + e^{-\frac{s}{2} K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}} e^{\beta h_{\mathcal{Z}^{(\vartheta)}}^{(\omega)}} e^{-\frac{s}{2} K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}}} \varphi \right\rangle_{\mathcal{H}}, \quad \varphi, \psi \in \mathcal{H}.$$

Therefore, by (2.3.77) and Definition 2.3.15, we directly compute that

$$\partial_s J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, s\mathcal{E})} = \frac{1}{|\cup \mathcal{Z}|} \sum_{x, y \in \mathcal{U}} \left\langle e_x, K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} e_y \right\rangle_{\mathcal{H}} \omega_s(a(e_x)^* a(e_y))$$

and

$$\begin{aligned} \partial_s^2 J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, s\mathcal{E})} &= \frac{1}{|\cup \mathcal{Z}|} \sum_{x, y, u, v \in \mathcal{U}} \left\langle e_x, K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} e_y \right\rangle_{\mathcal{H}} \left\langle e_u, K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} e_v \right\rangle_{\mathcal{H}} \\ &\quad \times \omega_s(a(e_y) a(e_u)^*) \omega_s(a(e_x)^* a(e_v)), \end{aligned}$$

because of the identity

$$\omega_s(a(e_x)^* a(e_y) a(e_u)^* a(e_v)) = \omega_s(a(e_x)^* a(e_y)) \omega_s(a(e_u)^* a(e_v)) + \omega_s(a(e_y) a(e_u)^*) \omega_s(a(e_x)^* a(e_v)),$$

for $x, y, u, v \in \mathfrak{L}$. As a consequence,

$$\left| \partial_s J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, s\mathcal{E})} \right| \leq \frac{1}{|\cup \mathcal{Z}|} \sum_{x, y \in \mathfrak{L}} \left| \left\langle e_x, K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} e_y \right\rangle_{\mathcal{H}} \right|$$

and

$$\begin{aligned} \left| \partial_s^2 J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, s\mathcal{E})} \right| &\leq \sup_{u, v \in \mathfrak{L}} \left| \left\langle e_u, K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} e_v \right\rangle_{\mathcal{H}} \right| \left(\frac{1}{|\cup \mathcal{Z}|} \sum_{x, y \in \mathfrak{L}} \left| \left\langle e_x, K_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})} e_y \right\rangle_{\mathcal{H}} \right| \right) \\ &\quad \times \sup_{y \in \mathfrak{L}} \sum_{u \in \mathfrak{L}} \left| \omega_s(a(e_y) a(e_u)^*) \right| \sup_{x \in \mathfrak{L}} \sum_{v \in \mathfrak{L}} \left| \omega_s(a(e_x)^* a(e_v)) \right|, \end{aligned}$$

which, by Lemma 3.2.6, implies that

$$(3.2.20) \quad \left| \partial_s J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, s\mathcal{E})} \right| \leq D_{3.2.6} \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d} e^{2|\alpha\eta|} d\alpha \right) \sum_{z \in \mathfrak{L}} e^{-\mu_\eta |z|} (1 + \eta)$$

as well as

$$(3.2.21) \quad \begin{aligned} \left| \partial_s^2 J_{\mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)}}^{(\omega, s\mathcal{E})} \right| &\leq D_{3.2.6}^2 \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d} e^{2|\alpha\eta|} d\alpha \right)^2 (1 + \eta)^2 \sum_{z \in \mathfrak{L}} e^{-\mu_\eta |z|} \\ &\quad \times \sup_{y \in \mathfrak{L}} \sum_{u \in \mathfrak{L}} \left| \omega_s(a(e_y) a(e_u)^*) \right| \sup_{x \in \mathfrak{L}} \sum_{v \in \mathfrak{L}} \left| \omega_s(a(e_x)^* a(e_v)) \right|. \end{aligned}$$

Again by Lemma 3.2.6 together with (2.3.57)–(2.3.58), for any $\mu > \mu_\eta$,

$$\sup_{\substack{\beta \in (0, \beta_1], \vartheta \in [0, \vartheta_1], \lambda \in [0, \lambda_1] \\ \omega \in \Omega, s \in [-s_1, s_1], \mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)} \in \mathfrak{Z}_f}} \left\{ \mathbf{S}_0(sK_{\mathcal{Z}, \mathcal{Z}^{(\tau)}}^{(\omega, \mathcal{E})}, \mu) + \mathbf{S}_0(\beta h_{\mathcal{Z}^{(\vartheta)}}^{(\omega)}, \mu) \right\} < \infty.$$

See (D.0.1). We thus infer from (3.2.19) and Corollary D.0.12 that there is a constant $\mu_1 \in \mathbb{R}^+$ such that, for any $x, y \in \mathfrak{L}$,

$$\sup_{\substack{\beta \in (0, \beta_1], \vartheta \in [0, \vartheta_1], \lambda \in [0, \lambda_1] \\ \omega \in \Omega, s \in [-s_1, s_1], \mathcal{Z}, \mathcal{Z}^{(\vartheta)}, \mathcal{Z}^{(\tau)} \in \mathfrak{Z}_f}} \left| \omega_s(a(e_x)^* a(e_y)) \right| \leq 2e^{-\mu_1 |x-y|}.$$

Combining this estimate with (3.2.20)–(3.2.21), one gets the assertion. End

The local generating functionals (3.2.12) can be approximately decomposed into boxes of fixed volume: By using the boxes (2.3.21) for any subset $\Lambda \subset \mathfrak{L}$ and $l \in \mathbb{N}$, we define the l -th box decomposition $\mathcal{Z}^{(\Lambda, l)}$ of Λ by

$$\mathcal{Z}^{(\Lambda, l)} \doteq \{\Lambda_l + (2l+1)x : x \in \mathfrak{L} \text{ with } (\Lambda_l + (2l+1)x) \subset \Lambda\} \in \mathfrak{Z}.$$

Then, we get the following assertion:

3.2.22 PROPOSITION (BOX DECOMPOSITION OF GENERATING FUNCTIONS). *Fix $n \in \mathbb{N}$ and $\beta_1, \lambda_1, \vartheta_1 \in \mathbb{R}^+$. Then,*

$$\lim_{l \rightarrow \infty} \limsup_{L_\tau \geq L_\vartheta \geq L \rightarrow \infty} \left| J_{\{\Lambda_L\}, \{\Lambda_{L_\vartheta}\}, \{\Lambda_{L_\tau}\}}^{(\omega, \mathcal{E})} - \frac{1}{|\mathcal{Z}^{(\Lambda_L, l)}|} \sum_{Z \in \mathcal{Z}^{(\Lambda_L, l)}} J_{\{Z\}, \{Z\}, \{Z\}}^{(\omega, \mathcal{E})} \right| = 0,$$

uniformly w.r.t. $\beta \in [0, \beta_1]$, $\vartheta \in [0, \vartheta_1]$, $\lambda \in [0, \lambda_1]$, $\omega \in \Omega$ and $\mathcal{E}$ in any bounded set of $C_0^0([-n, n]; \mathbb{R}^d)$. $\blacksquare$

PROOF. The proof of this statement is divided in a series of Lemmata, see [ABPR17] for details. End

3.2.5 AKCOGLU–KRENGEL ERGODIC THEOREM AND EXISTENCE OF GENERATING FUNCTIONS

The Ackoglu–Krengel (superadditive) ergodic theorem, cornerstone of ergodic theory, generalizes the celebrated Birkhoff additive ergodic theorem (see Appendix E, Theorem E.2.4). It is used to deduce, via Proposition 3.2.15, the following Corollary:

3.2.23 COROLLARY (AKCOGLU–KRENGEL ERGODIC THEOREM FOR GENERATING FUNCTIONS). *There is a measurable subset $\tilde{\Omega} \subset \Omega$ of full measure such that, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$,*

$$\lim_{L \rightarrow \infty} \frac{1}{|\mathcal{Z}^{(\Lambda_L, l)}|} \sum_{Z \in \mathcal{Z}^{(\Lambda_L, l)}} J_{\{Z\}, \{Z\}, \{Z\}}^{(\omega, \mathcal{E})} = \mathbb{E} \left[J_{\{\Lambda_l\}, \{\Lambda_l\}, \{\Lambda_l\}}^{(\cdot, \mathcal{E})} \right]. \quad \blacksquare$$

PROOF. Fix $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \Omega$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$. For any $\Gamma \in \mathcal{P}_f(\mathfrak{Q})$, let

$$\mathfrak{F}_l^{(\omega, \mathcal{E})}(\Gamma) \doteq \sum_{x \in \Gamma} J_{\{\Lambda_l + (2l+1)x\}, \{\Lambda_l + (2l+1)x\}, \{\Lambda_l + (2l+1)x\}}^{(\omega, \mathcal{E})}.$$

Then, if

$$\Lambda^{(L)} \equiv \Lambda^{(L, l)} \doteq \{x \in \mathfrak{Q} : (\Lambda_l + (2l+1)x) \subset \Lambda_L\} \subset \Lambda_L,$$

observe that

$$|\Lambda^{(L)}|^{-1} \mathfrak{F}_l^{(\omega, \mathcal{E})}(\Lambda^{(L)}) = \frac{1}{|\mathcal{Z}^{(\Lambda_L, l)}|} \sum_{Z \in \mathcal{Z}^{(\Lambda_L, l)}} J_{\{Z\}, \{Z\}, \{Z\}}^{(\omega, \mathcal{E})}.$$

Therefore, since $\{\Lambda^{(L)}\}_{L \in \mathbb{R}^+}$ is clearly a regular sequence, by Theorem E.2.4, for any $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, there is a measurable subset $\hat{\Omega} \equiv \hat{\Omega}^{(\beta, \vartheta, \lambda, l, \mathcal{E}, \vec{w})} \subset \Omega$ of full measure such that, for all $\omega \in \hat{\Omega}$,

$$\lim_{L \rightarrow \infty} \frac{1}{|\mathcal{Z}^{(\Lambda_L, l)}|} \sum_{Z \in \mathcal{Z}^{(\Lambda_L, l)}} J_{\{Z\}, \{Z\}, \{Z\}}^{(\omega, \mathcal{E})} = \mathbb{E} \left[J_{\{\Lambda_l\}, \{\Lambda_l\}, \{\Lambda_l\}}^{(\cdot, \mathcal{E})} \right].$$

Observe that, for any $n \in \mathbb{N}$, there is a countable dense set $\mathcal{D}_n \subset C_0^0(\mathbb{R}; \mathbb{R}^d)$. Let $\mathbb{S}^{d-1}$ be a dense countable subset of the $(d-1)$ -dimensional sphere. Hence, by Proposition 3.2.15, we arrive at the assertion for any realization $\omega \in \tilde{\Omega} \subset \Omega$, where

$$\tilde{\Omega} \doteq \bigcap_{\vartheta, \lambda \in \mathbb{Q} \cap \mathbb{R}_0^+} \bigcap_{\beta \in \mathbb{Q} \cap \mathbb{R}^+} \bigcap_{\vec{w} \in \mathbb{S}^{d-1}} \bigcap_{n \in \mathbb{N}} \bigcap_{\mathcal{E} \in \mathcal{D}_n} \bigcap_{l \in \mathbb{N}} \hat{\Omega}^{(\beta, \vartheta, \lambda, l, \mathcal{E}, \vec{w})}.$$

[Recall that any countable intersection of measurable sets of full measure has full measure]. End

3.2.24 COROLLARY (ALMOST SURELY EXISTENCE OF GENERATING FUNCTIONS). *Let $\tilde{\Omega} \subset \Omega$ be the measurable subset of Corollary 3.2.23. Then, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$,*

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \mathbb{E} \left[\ln \varrho^{(\cdot)} \left(e^{|\Lambda_L| \mathbb{I}_{\Lambda_L}^{(\cdot, \mathcal{E})}} \right) \right] = \lim_{L_\tau \geq L_\varrho \geq L \rightarrow \infty} J_{\{\Lambda_L\}, \{\Lambda_{L_\varrho}\}, \{\Lambda_{L_\tau}\}}^{(\omega, \mathcal{E})} \doteq J^{(\mathcal{E})}.$$

For all $n \in \mathbb{N}$, the convergence is uniform w.r.t. $\beta, \vartheta, \lambda$ in compact sets, $\omega \in \tilde{\Omega}$, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$ and $\mathcal{E}$ in any bounded set of $C_0^0([-n, n]; \mathbb{R}^d)$. ▀

PROOF. By translation invariance of the distribution α_Ω ,

$$\mathbb{E} \left[J_{\{\Lambda_l\}, \{\Lambda_l\}, \{\Lambda_l\}}^{(\cdot, \mathcal{E})} \right] = \mathbb{E} \left[\frac{1}{|\mathcal{Z}^{(\Lambda_L, l)}|} \sum_{Z \in \mathcal{Z}^{(\Lambda_L, l)}} J_{\{Z\}, \{Z\}, \{Z\}}^{(\cdot, \mathcal{E})} \right].$$

Hence,

$$\left\{ \mathbb{E} \left[J_{\{\Lambda_l\}, \{\Lambda_l\}, \{\Lambda_l\}}^{(\cdot, \mathcal{E})} \right] \right\}_{l \in \mathbb{N}}$$

is a Cauchy sequence, by [ABPR17, Expressions (47)–(48)]. By Proposition 3.2.22 and Corollary 3.2.23, there is a measurable subset $\tilde{\Omega} \subset \Omega$ of full measure such that, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$,

$$\lim_{L_\tau \geq L_\varrho \geq L \rightarrow \infty} J_{\{\Lambda_L\}, \{\Lambda_{L_\varrho}\}, \{\Lambda_{L_\tau}\}}^{(\omega, \mathcal{E})} = \lim_{l \rightarrow \infty} \mathbb{E} \left[J_{\{\Lambda_l\}, \{\Lambda_l\}, \{\Lambda_l\}}^{(\cdot, \mathcal{E})} \right].$$

For all $n \in \mathbb{N}$, the convergence is uniform w.r.t. $\beta, \vartheta, \lambda$ in compact sets, $\omega \in \tilde{\Omega}$, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$ and $\mathcal{E}$ in any bounded set of $C_0^0([-n, n]; \mathbb{R}^d)$. By (3.2.14), the assertion then follows. End

3.2.25 COROLLARY (DIFFERENTIABILITY OF GENERATING FUNCTIONS). *Fix $\beta, \lambda, \vartheta \in \mathbb{R}^+$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$. For any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, the map $s \mapsto J^{(s, \mathcal{E})}$ from $\mathbb{R}$ to itself is continuously differentiable.* ■

PROOF. Take any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\omega \in \tilde{\Omega}$. See Corollary 3.2.24. Then, for any $s \in \mathbb{R}$,

$$J^{(s, \mathcal{E})} = \lim_{L_\tau \geq L_\vartheta \geq L \rightarrow \infty} J_{\{\Lambda_L\}, \{\Lambda_{L_\vartheta}\}, \{\Lambda_{L_\tau}\}}^{(\omega, s, \mathcal{E})}.$$

By Proposition 3.2.18 combined with the mean value theorem and the (Arzelà–) Ascoli theorem [Rud91, Theorem A5], there are three sequences $\{L_\tau^{(n)}\}_{n \in \mathbb{N}}, \{L_\vartheta^{(n)}\}_{n \in \mathbb{N}}, \{L^{(n)}\}_{n \in \mathbb{N}} \subset \mathbb{R}_0^+$, with $L_\tau^{(n)} \geq L_\vartheta^{(n)} \geq L^{(n)}$, such that the maps

$$s \mapsto J_{\{\Lambda_{L^{(n)}}\}, \{\Lambda_{L_\vartheta^{(n)}}\}, \{\Lambda_{L_\tau^{(n)}}\}}^{(\omega, s, \mathcal{E})} \quad \text{and} \quad s \mapsto \partial_s J_{\{\Lambda_{L^{(n)}}\}, \{\Lambda_{L_\vartheta^{(n)}}\}, \{\Lambda_{L_\tau^{(n)}}\}}^{(\omega, s, \mathcal{E})}$$

converge uniformly for s in any compact set of $\mathbb{R}$. In particular, the map $s \mapsto J^{(s, \mathcal{E})}$ from $\mathbb{R}$ to itself is continuously differentiable with

$$(3.2.26) \quad \partial_s J^{(s, \mathcal{E})} = \lim_{L_\tau \geq L_\vartheta \geq L \rightarrow \infty} \partial_s J_{\{\Lambda_L\}, \{\Lambda_{L_\vartheta}\}, \{\Lambda_{L_\tau}\}}^{(\omega, s, \mathcal{E})} = \lim_{L \rightarrow \infty} \varrho^{(\omega)} \left(\frac{\mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}}{|\Lambda_L|} e^{s|\Lambda_L| \mathbb{I}_{\Lambda_L}^{(\omega, \mathcal{E})}} \right). \quad \text{End}$$

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GENERATING FUNCTIONS AS GAUSSIAN BEREZIN INTEGRALS AND DETERMINANT BOUNDS

HERE we show a revised version of the article [ABPM17]. Then, we prove that for *any* finite range interaction fermion system, the generating function (2.4.4) related to *any* self-adjoint operator $K \in \mathcal{U} \cap \mathcal{U}^+$ can be written as Berezin (Grassmann) integral with covariance C_h for $h \in \mathcal{B}(\mathcal{H})$ a self-adjoint operator. For technical reasons it is useful to bound the determinant of C_h and we do it efficiently for *any* inverse temperature $\beta \in \mathbb{R}^+$. As in Chapter 3, our analysis use the mathematical structure introduced in Chapter 2. For more details and proofs see [ABPM17].

The main results here we show are Theorems 4.1.13–4.2.7–4.3.7 and Corollaries 4.1.17–4.2.13. In the next Chapter we will see that these are related to the existence and analiticity of the logarithmic moment generating function (2.4.4), namely,

$$J(s) = \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln \rho(e^{s|\Lambda_L|A_L^\Psi}),$$

for $A_L^\Psi = (A_L^\Psi)^* \in \mathcal{U}^+ \cap \mathcal{U}$ a space average of interactions $\{\Psi_\Lambda\}_{\Lambda \subset \Lambda_L} \in \mathcal{U}^+ \cap \mathcal{U}_L$ on the box Λ_L given by the sequence of boxes (2.3.21) and ρ is a KMS-state. By following Expression (2.4.5) we rewrite

$$J(s) = \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \ln \frac{\text{tr}(e^{-\beta H_L} e^{s|\Lambda_L|A_L^\Psi})}{\text{tr}(e^{-\beta H_L})},$$

where H_L is the Hamiltonian of the systems restricted to the region Λ_L , see (2.4.5).

4.1 GENERATING FUNCTIONS AS BEREZIN INTEGRALS

In order to state the main results, we show for $\mathcal{H}$, a finite Hilbert space, the Chernoff product approximation for exponentials in the CAR C^* -algebra $\mathcal{U} \equiv \mathcal{U}(\mathcal{H})$:

4.1.1 DEFINITION (A CHERNOFF PRODUCT APPROXIMATION). Let $\mathcal{H}$ be a Hilbert space. For every

element $A \in \mathcal{U}$ and all $n \in \mathbb{N}$,

$$X_A^{(n)} \doteq \left[\kappa^{-1} \left(e^{\frac{1}{n} \kappa(A)} \right) \right]^n \in \mathcal{U},$$

where we recall that κ is the canonical isomorphism of $*$ -algebras from Definition 2.2.41, while e^ξ is defined by (2.2.16)¹ for all $\xi \in \wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$. $\blacksquare$

In the limit $n \rightarrow \infty$, $X_A^{(n)}$ gives the usual exponential of A :

Theorem 4.1.2 (Chernoff product formula):

Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{U}$,

$$(X_A^{(n)})^* = X_{A^*}^{(n)} \quad \text{and} \quad \lim_{n \rightarrow \infty} X_A^{(n)} = e^A.$$

This convergence is uniform on bounded subsets of the C^* -algebra $\mathcal{U}$. $\clubsuit$

PROOF. Since

$$\kappa^{-1} \left(e^{\kappa(0)} \right) = \mathbb{1} \in \mathcal{U} \quad \text{and} \quad \partial_s \left\{ \kappa^{-1} \left(e^{s \kappa(A)} \right) \right\}_{s=0} = A,$$

the assertion is a usual Chernoff product formula. See, e.g., [EBN⁺06, 5.2 Theorem]. Although general results proving the Chernoff product formula are available, we provide here an explicit proof to extract properties that are particular to our choice of the Chernoff product approximation.

By finite dimensionality of $\mathcal{H}$, the involution of $\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ is continuous. By Lemma 2.2.38 and Equation (2.2.16),

$$[e^\xi]^* = e^{\xi^*}, \quad \xi \in \wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}}).$$

Since the map κ is a $*$ -isomorphism (Definition 2.2.41), for every $n \in \mathbb{N}$ it follows that $X_A^{(n)*} = X_{A^*}^{(n)}$ for any $A \in \mathcal{U}$.

For every $n \in \mathbb{N}$ and $A \in \mathcal{U}$ observe next that

$$(4.1.3) \quad X_A^{(n)} = \left(1 + \frac{A}{n} \right)^n + \sum_{k=1}^n \binom{n}{k} \tilde{X}_A^k \left(1 + \frac{1}{n} A \right)^{n-k},$$

¹In other words, we use the exterior product $\wedge$ (and *not* the circle product from Definition 2.2.34) of the Grassmann algebra $\wedge^* \mathcal{H}$ (see (2.2.12)) to define the exponential.

where

$$\tilde{X}_A \doteq \sum_{m=2}^{\infty} \frac{1}{m!} \left(\frac{1}{n}\right)^m \kappa^{-1}(\kappa(A)^m).$$

So, it suffices to show that the last term of (4.1.3) converges to zero when $n \rightarrow \infty$ to prove the Chernoff product formula. By finite dimensionality of $\mathcal{H}$, note that the linear map κ from $\mathcal{U}$ to $\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ is continuous. Moreover, its inverse κ^{-1} is also continuous, by Definition (2.2.42). From (2.2.17), we obtain that

$$\begin{aligned} \|\tilde{X}_A\|_{\mathcal{U}} &\leq \sum_{m=2}^{\infty} \frac{1}{m!} \left(\frac{1}{n}\right)^m \|\kappa(A)^m\|_{\wedge^* \mathcal{H}} \\ &\leq \frac{1}{n^2} D_{\kappa(A)}^2 \sum_{m=2}^{\infty} \frac{1}{m!} \left(\frac{1}{n}\right)^{m-2} D_{\kappa(A)}^{m-2} \\ (4.1.4) \quad &\leq \frac{1}{n^2} D_{\kappa(A)}^2 e^{\frac{1}{n} D_{\kappa(A)}}. \end{aligned}$$

Now, by using the definition

$$D_A \doteq \max\{D_{\kappa(A)} e^{D_{\kappa(A)}}, \|A\|_{\mathcal{U}}\},$$

we infer from (4.1.4) that

$$\begin{aligned} \left\| \sum_{k=1}^n \binom{n}{k} \tilde{X}_A^k \left(1 + \frac{1}{n} A\right)^{n-k} \right\|_{\mathcal{U}} &\leq \sum_{k=1}^n \binom{n}{k} \left(\frac{D_A^2}{n^2}\right)^k \left(1 + \frac{D_A}{n}\right)^{n-k} \\ &= \left(1 + \frac{D_A}{n} + \frac{D_A^2}{n^2}\right)^n - \left(1 + \frac{D_A}{n}\right)^n \\ &\leq \left(1 + \frac{D_A}{n}\right)^n \left(\left(1 + \frac{D_A^2}{n^2}\right)^n - 1 \right) \\ (4.1.5) \quad &\leq e^{D_A} \left(e^{\frac{D_A^2}{n}} - 1 \right). \end{aligned}$$

By finite dimensionality of $\mathcal{H}$, for any bounded subset $\mathfrak{B}$ of $\mathcal{U}$,

$$\sup_{A \in \mathfrak{B}} D_A < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{A \in \mathfrak{B}} \left\| \left(1 + \frac{A}{n}\right)^n - e^A \right\|_{\mathcal{U}} = 0.$$

As a consequence, by (4.1.3) and (4.1.5), the convergence, as $n \rightarrow \infty$, of $X_A^{(n)}$ to e^A is uniform on bounded subsets of $\mathcal{U}$. End

Remark 4.1.6 (Self-adjoint case). If $\mathfrak{B} \subset \mathcal{U}$ is a bounded subset of self-adjoint elements, then there is $N_{\mathfrak{B}} \in \mathbb{N}$ such that, for all $n \geq N_{\mathfrak{B}}$, $s \in \mathbb{R}$ and $A \in \mathfrak{B}$, the element $(X_A^{(n)})^s$ is well-defined and self-adjoint,

by Theorem 4.1.2 and the spectral theorem. Like in Theorem 4.1.2, it tends to e^{sA} in the limit $n \rightarrow \infty$. Moreover, in this case, again by the spectral theorem, the family of maps $s \mapsto \left(X_A^{(n)}\right)^s$, $n \geq N_{\mathfrak{B}}$, $A \in \mathfrak{B}$, is equicontinuous. $\star$

We now recall that if $\mathcal{H}$ is a finite Hilbert space with orthonormal basis $\{\psi_j\}_{j \in J}$, then for $h \in \mathcal{B}(\mathcal{H})$, its (free) second-quantization is given by (see Definition 2.3.15)

$$d\langle Q, hQ \rangle = \sum_{i,j \in J} \langle \psi_j, h\psi_i \rangle_{\mathcal{H}} a(\psi_j)^* a(\psi_i).$$

Note by the isomorphism κ defined in 2.2.41, we have the following identity:

$$\kappa(d\langle Q, hQ \rangle) = -\frac{1}{2} \langle \mathcal{H}, h\mathcal{H} \rangle + \text{Tr}_{\mathcal{H}}(h)\mathbb{1}.$$

We also note that if one applies the above Chernoff product formula to the second quantization (Definition 2.3.15) then one gets an expression similar to a *Bernoulli or Euler limit*: Since, by Definition 2.2.29, for any $h \in \mathcal{B}(\mathcal{H})$,

$$(4.1.7) \quad e^{\frac{1}{2}\langle \mathcal{H}, h\mathcal{H} \rangle} = \prod_{i,j \in J} \left(\mathbb{1} + \langle \psi_j, h\psi_i \rangle_{\mathcal{H}} \psi_j^* \wedge \psi_i \right),$$

for $\mathcal{H}$ and operator $h \in \mathcal{B}(\mathcal{H})$, by Definition 2.2.41,

$$(4.1.8) \quad X_{-d\langle Q, hQ \rangle}^{(n)} = e^{-\text{Tr}_{\mathcal{H}}(h)} \left[\kappa^{-1} \left(\prod_{i,j \in J} \left(\mathbb{1} + \frac{1}{n} \langle \psi_j, h\psi_i \rangle_{\mathcal{H}} \psi_j^* \wedge \psi_i \right) \right) \right]^n, \quad n \in \mathbb{N}.$$

Because of Theorem 4.1.2, $X_{-d\langle Q, hQ \rangle}^{(n)}$ converges to $e^{-d\langle Q, hQ \rangle}$ in the limit $n \rightarrow \infty$.

Note that for a self-adjoint operator $h \in \mathcal{B}(\mathcal{H})$ the Chernoff product approximation (4.1.8) has an explicit formulation in terms of the generators $\{a(\varphi)\}_{\varphi \in \mathcal{H}}$ of $\mathcal{U}$:

4.1.9 LEMMA (CHERNOFF PRODUCT APPROXIMATION AND SECOND QUANTIZATION). *Let $\mathcal{H}$ be a finite dimensional Hilbert space, and $h \in \mathcal{B}(\mathcal{H})$ a self-adjoint operator. Let $\{\psi_j\}_{j \in J}$ be any orthonormal basis of $\mathcal{H}$. Then, for all $n \in \mathbb{N}$ with $n > 2\|h\|_{\mathcal{B}(\mathcal{H})}$ and any $s \in \mathbb{R}$, $X_{-d\langle Q, hQ \rangle}^{(n)} > 0$ and*

$$\left(X_{-d\langle Q, hQ \rangle}^{(n)}\right)^s = e^{-s\text{Tr}_{\mathcal{H}}(h)} \prod_{j \in J} \left(\mathbb{1} + \frac{1}{n} \langle \psi_j, h\psi_j \rangle_{\mathcal{H}} a(\psi_j)^* a(\psi_j) \right)^{sn}. \quad \blacklozenge$$

PROOF. See [ABPM17, Lemma 4.5]. End

Secondly, through the isomorphism κ , defined for any $\mathcal{H}$ by Definition 2.2.41, the tracial state $\text{tr} \in \mathcal{U}^*$, Expression (2.3.29), can be represented in terms of Berezin integrals. To explain this, we introduce the following definitions:

1. For a finite Hilbert space $\mathcal{H}$ and integers $n \in \mathbb{N}_0$ and $k \in \{0, \dots, n\}$, we define the map

$$(4.1.10) \quad \kappa^{(k)} \doteq \kappa_{(0,0)}^{(k,k)} \circ \kappa$$

from $\mathcal{U}$ to the k -copy $\wedge^* (\mathcal{H}^{(k)} \oplus \bar{\mathcal{H}}^{(k)})$ of $\wedge^* (\mathcal{H} \oplus \bar{\mathcal{H}})$. Note that $\kappa^{(0)} = \kappa$, see (2.2.25) and Definition 2.2.41.

2. Recall Definition given by Expression (2.2.24), namely,

$$\mathbf{H}^{(n)} \oplus \bar{\mathbf{H}}^{(n)} \doteq \bigoplus_{k=0}^{n-1} (\mathcal{H}^{(k)} \oplus \bar{\mathcal{H}}^{(k)}).$$

3. For any $n \in \mathbb{N}$, a map A from $(\mathcal{H} \oplus \bar{\mathcal{H}})$ to itself is canonically extended to a map $\hat{A}$ from $\mathbf{H}^{(n)} \oplus \bar{\mathbf{H}}^{(n)}$ to itself by the conditions

$$(4.1.11) \quad \hat{A}\varphi^{(k)} \doteq (A\varphi)^{(k)}, \quad k \in \{0, \dots, n-1\}.$$

Then, we get the well-known *tracial state formula* in the context of Grassmann algebra:

Theorem 4.1.12 (Tracial state formula):

Let $\mathcal{H}$ be a finite Hilbert space. Then, for all $n \in \mathbb{N}$ and $A_0, \dots, A_{n-1} \in \mathcal{U}$,

$$\mathrm{tr}(A_0 \cdots A_{n-1}) \mathbb{1} = 2^{-\dim(\mathcal{H})} \int d(\mathbf{H}^{(n)}) e^{\frac{1}{2} \langle \mathbf{H}^{(n)}, \partial \mathbf{H}^{(n)} \rangle} \left(\prod_{k=0}^{n-1} \kappa^{(k)}(A_k) \right).$$

where,

$$\frac{1}{2} \langle \mathbf{H}^{(n)}, \partial \mathbf{H}^{(n)} \rangle = \langle \mathcal{H}^{(0)}, \mathcal{H}^{(0)} \rangle + \langle \mathcal{H}^{(0)}, \mathcal{H}^{(n-1)} \rangle + \sum_{k=1}^{n-1} (\langle \mathcal{H}^{(k)}, \mathcal{H}^{(k)} \rangle - \langle \mathcal{H}^{(k)}, \mathcal{H}^{(k-1)} \rangle). \quad \spadesuit$$

PROOF. Observe that Expression (2.3.14) holds true. Then, by using linearity properties and explicit computations on arbitrary, normally-ordered monomials, we prove that

$$\mathrm{tr}(A) \mathbb{1} = 2^{-\dim \mathcal{H}} \int d(\mathcal{H}) \kappa(A) e^{2 \langle \mathcal{H}, \mathcal{H} \rangle}, \quad A \in \mathcal{U}.$$

Next, we use the equality

$$\kappa(A_0 \cdots A_{n-1}) = \kappa(A_0) \circ \cdots \circ \kappa(A_{n-1}),$$

together with Lemma 2.2.35–1 and a renumbering, to get the assertion. Note that

$$\int d(\mathbb{H}^{(n)}) = \prod_{k=0}^{n-1} \int d(\mathcal{H}^{(k)}), \quad n \in \mathbb{N}.$$

For more details we recommend [BP] and [ABPM17].

End

We are now in position to state the main results. We restrict our analysis to even elements, that is, elements $A \in \mathcal{U}^+ \cap \mathcal{U}$, see Chapter 2–§2.3.2.

Theorem 4.1.13 (Feynman–Kac–like formula for the tracial state):

Let $H, K \in \mathcal{U}^+ \cap \mathcal{U}$ be two even elements, K being also self-adjoint. Then, for all $\beta \in \mathbb{R}^+$, $s \in \mathbb{R}$ and any basis projection P ,

$$\mathrm{tr}(e^{-\beta H} e^{sK}) \mathbb{1} = 2^{-\dim \mathcal{H}} \lim_{n \rightarrow \infty} \int d(\mathbb{H}^{(n_\beta)}) e^{\frac{1}{2} \langle \mathbb{H}^{(n_\beta)}, \partial \mathbb{H}^{(n_\beta)} \rangle} \wedge \exp \left(\frac{\beta s}{n} \sum_{k=n}^{n_\beta-1} \chi^{(k)}(K) - \frac{\beta}{n} \sum_{k=0}^{n-1} \chi^{(k)}(H) \right)$$

with $\chi^{(k)}$ and $\frac{1}{2} \langle \mathbb{H}^{(n_\beta)}, \partial \mathbb{H}^{(n_\beta)} \rangle$ respectively defined by (4.1.10) and Theorem 4.1.12. Here, $n_\beta \doteq n + \lfloor n/\beta \rfloor$, with $\lfloor x \rfloor$ being the largest natural number smaller than $x \in \mathbb{R}^+$. $\clubsuit$

PROOF. Fix $\beta \in \mathbb{R}^+$, $s \in \mathbb{R}$, two elements $H, K = K^* \in \mathcal{U}^+ \cap \mathcal{U}$. We then infer from Theorem 4.1.2 and Remark 4.1.6 that

$$\mathrm{tr}(e^{-\beta H} e^{sK}) = \lim_{n \rightarrow \infty} \mathrm{tr} \left(X_{-\beta H}^{(n)} \left(X_{s\beta K}^{(n)} \right)^{n^{-1} \lfloor n/\beta \rfloor} \right),$$

which, combined with (4.1.10), Theorem 4.1.12 and Definition 4.1.1, in turn implies that

$$(4.1.14) \quad \mathrm{tr}(e^{-\beta H} e^{sK}) = 2^{-\dim \mathcal{H}} \lim_{n \rightarrow \infty} \int d(\mathbb{H}^{(n_\beta)}) e^{\frac{1}{2} \langle \mathbb{H}^{(n_\beta)}, \partial \mathbb{H}^{(n_\beta)} \rangle} \left[\prod_{k=0}^{n-1} \exp \left(-\frac{\beta}{n} \chi^{(k)}(H) \right) \right] \\ \wedge \prod_{k=n}^{n_\beta-1} \exp \left(\frac{\beta s}{n} \chi^{(k)}(K) \right).$$

Now, if H and K are even, then this last equation implies the assertion. Observe that the products involved in the right-hand side of Equation (4.1.14) all refer to the exterior product $\wedge$ of $\wedge^* \mathcal{H}$. Thus, this space is viewed as Grassmann algebra (Definition 2.2.13). End

Remark 4.1.15 (Non-even and non-self-adjoint case). Equation (4.1.14) gives a Feynman–Kac–like formula for traces of the form $\mathrm{tr}(e^{-\beta H} e^{sK})$ for all $H, K = K^* \in \mathcal{U}$. We focus our study on the cases for which

H and K are even, which is the typical case for fermionic systems, see Definition 2.3.37 and the subsequent discussions. Note also that the self-adjointness of K could be relaxed. We omit the details. $\star$

We focus on second quantization operators (Definition 2.3.15) perturbed by an even, self-adjoint element $W = W^* \in \mathcal{U}$, that is, fermionic Hamiltonians of the form

$$(4.1.16) \quad H = d\langle Q, hQ \rangle + W, \quad h = h^* \in \mathcal{B}(\mathcal{H}).$$

Define now the linear operator $A_h^{(n)} \in \mathcal{B}(\mathbf{H}^{(n_\beta)})$, with $n \in \mathbb{N}$, $\beta \in \mathbb{R}^+$ and $n_\beta \doteq n + \lfloor n/\beta \rfloor$, that satisfies the conditions:

$$A_h^{(n)}(\psi_i^{(0)}) = \psi_i^{(0)} + \psi_i^{(2n-1)} - \frac{\beta}{n} \sum_{j \in I} \psi_i^*(h(\psi_j)) \psi_j^{(0)}$$

for all $i \in I$,

$$A_h^{(n)}(\psi_i^{(k)}) = \psi_i^{(k)} - \psi_i^{(k-1)} - \frac{\beta}{n} \sum_{j \in I} \psi_i^*(h(\psi_j)) \psi_j^{(k)}$$

for all $i \in I$ and $k \in \{1, \dots, n-1\}$, as well as

$$A_h^{(n)}(\psi_i^{(k)}) = \psi_i^{(k)} - \psi_i^{(k-1)},$$

for all $i \in I$ and $k \in \{n, \dots, n_\beta - 1\}$. The Feynman–Kac–like formula for traces can be written in terms of a limit of a Gaussian Berezin integral (Definition 2.2.32):

4.1.17 COROLLARY (FEYNMAN–KAC–A FORMULA FOR GENERATING FUNCTIONS). *Let $h \in \mathcal{B}(\mathcal{H})$ be any self-adjoint operator, and let $W, K \in \mathcal{U}$ be even elements, with K being self-adjoint. Then, for all $\beta \in \mathbb{R}^+$, $s \in \mathbb{R}$,*

$$\frac{\text{tr}\left(e^{-\beta(d\langle Q, hQ \rangle + W)} e^{sK}\right)}{\text{tr}\left(e^{-\beta d\langle Q, hQ \rangle}\right)} \mathbb{1} = \lim_{n \rightarrow \infty} \int d\mu_{C_h^{(n)}}\left(\mathbf{H}^{(n_\beta)}\right) \wedge \exp\left(\frac{\beta s}{n} \sum_{k=n}^{n_\beta-1} \chi^{(k)}(K) - \frac{\beta}{n} \sum_{k=0}^{n-1} \chi^{(k)}(W)\right).$$

Here, for any integer $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$,

$$(4.1.18) \quad C_h^{(n)} \doteq \left(A_h^{(n)}\right)^{-1} \in \mathcal{B}\left(\mathbf{H}^{(n_\beta)}\right).$$

Recall that $n_\beta \doteq n + \lfloor n/\beta \rfloor$. $\blacksquare$

PROOF. Fix all the parameters of the corollary. On the one hand, we deduce from Lemmata 2.2.35 (4–5) and 4.1.9, together with Definition 2.2.41 and elementary computations like (2.2.30)

and (4.1.7), that, for all $n \in \mathbb{N}$ such that $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$,

$$(4.1.19) \quad e^{\frac{\beta}{2n} \langle H^{(n)}, \hat{h} H^{(n)} \rangle} = e^{\frac{\beta}{2} \text{Tr}_{\mathcal{H}}(h)} \prod_{k=0}^{n-1} \chi^{(k)} \left(\left[X_{-\beta d \langle Q, hQ \rangle}^{(n)} \right]^{\frac{1}{n}} \right).$$

The product here refers to the exterior one, $\wedge$ (Notation 2.2.14). On the other hand, we deduce from (2.2.31) that

$$(4.1.20) \quad \det [A_h^{(n)}] \mathbb{1} = 2^{-\dim \mathcal{H}} \int d(H^{(n_\beta)}) e^{\frac{1}{2} \langle H^{(n_\beta)}, \partial H^{(n_\beta)} \rangle} e^{\frac{\beta}{2n} \langle H^{(n)}, \hat{h} H^{(n)} \rangle}, \quad n \in \mathbb{N}.$$

Therefore, we infer from (4.1.19) and (4.1.20), together with Theorem 4.1.12, that

$$(4.1.21) \quad \det [A_h^{(n)}] = e^{\frac{\beta}{2} \text{Tr}_{\mathcal{H}}(h)} \text{tr} \left(X_{-\beta d \langle Q, hQ \rangle}^{(n)} \right) > 0$$

for any integer $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$, since the tracial state tr is faithful and $X_{-\beta d \langle Q, hQ \rangle}^{(n)} > 0$ (Lemma 4.1.9). In particular, for $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$, the operator $A_h^{(n)}$ is invertible in $H^{(n)}$. Hence, $A_h^{(n)}$ is invertible for $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$ and the covariance $C_h^{(n)}$ is, in this case, well-defined. Now, we infer from Theorem 4.1.13 that

$$(4.1.22) \quad e^{\frac{\beta}{2} \text{Tr}_{\mathcal{H}}(h)} \text{tr} \left(e^{-\beta(d \langle Q, hQ \rangle + W)} e^{sK} \right) \mathbb{1} \\ = \lim_{n \rightarrow \infty} \int d(H^{(n_\beta)}) e^{\frac{1}{2} \langle H^{(n_\beta)}, \partial H^{(n_\beta)} \rangle + \frac{\beta}{2n} \langle H^{(n)}, \hat{h} H^{(n)} \rangle} e^{\frac{\beta s}{n} \sum_{k=n}^{n_\beta-1} \chi^{(k)}(K) - \frac{\beta}{n} \sum_{k=0}^{n-1} \chi^{(k)}(W)}.$$

Therefore, by Definition 2.2.32, it suffices to combine (4.1.20) with (4.1.22) to get the assertion. End

Corollary 4.1.17 refers to a Feynman–Kac–like formula by viewing the variable $k \in \{0, \dots, n-1\}$ in the definition (2.2.24) of the finite-dimensional Hilbert space $H^{(n)}$, $n \in \mathbb{N}$, as a “time coordinate”. Each element of a (copied) orthonormal basis $\{\psi_i^{(k)}\}_{i \in I}$ of $\mathcal{H}^{(k)}$ is associated with a “space coordinate” $i \in I$. For more details see [ABPM17].

4.2 DETERMINANT BOUNDS

Having in mind further applications of the Feynman–Kac–like formula of Corollary 4.1.17, we aim to bound determinants of the form

$$(4.2.1) \quad \det \left[\mathfrak{M}_{j_q, j_{N+1}} \bar{\varphi}_q^{(k_q)} \left(C_h^{(n)} \varphi_{N+1}^{(k_{N+1})} \right) \right]$$

for $\beta \in \mathbb{R}^+$, $n, m, N \in \mathbb{N}$ with n being sufficiently large, $\mathfrak{M} \in \text{Mat}(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$, and all $k_1, \dots, k_{2N} \in \{0, \dots, n_\beta - 1\}$, $j_1 \dots j_{2N} \in \{1, \dots, m\}$, $\bar{\varphi}_1, \dots, \bar{\varphi}_N \in \bar{\mathcal{H}}$, $\varphi_{N+1}, \dots, \varphi_{2N} \in \mathcal{H}$. Recall

that $n_\beta \doteq n + \lfloor n/\beta \rfloor$ and $C_h^{(n)}$ is the covariance defined by (4.1.18) for integers $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$. Observe also that the matrix $\mathfrak{M}$ is symmetric, for it is positive.

In next Chapter we use $\mathfrak{M}$ to prove the analyticity of logarithmic moment generating functions. In fact, the positive, real matrix $\mathfrak{M}$ appears in the so-called Brydges–Kennedy tree expansions. See, e.g., [BP16b, Section 1.3] and [BP, ABPM16] for more details. For previous applications of this expansion, see also [BGPS94, Section 3], [GM10, Section 3.2], and more recently [GMP16, Section 5.A.]. In *all* previous results, the covariance refers to the special case $k < n$ in Corollary 4.1.17.

We start by bounding determinants of the form

$$(4.2.2) \quad \det \left[\bar{\varphi}_q^{(k_q)} \left(C_h^{(n)} \varphi_{N+l}^{(k_{N+l})} \right) \right]_{q,l=1}^N$$

for $\beta \in \mathbb{R}^+$, $n, N \in \mathbb{N}$ with n being sufficiently large, and all $k_1, \dots, k_{N+M} \in \{0, \dots, n_\beta - 1\}$, $\bar{\varphi}_1, \dots, \bar{\varphi}_N \in \mathcal{H}$, $\varphi_{N+1}, \dots, \varphi_{2N} \in \mathcal{H}$. Then, we show below that the general case $m \in \mathbb{N}$ can always be reduced to the situation $m = 1$ and $\mathfrak{M} = 1$, by redefining the Hilbert space $\mathcal{H}$.

The ingredients to bound (4.2.2) are: (I) Corollary 2.2.33, which states that

$$(4.2.3) \quad \det \left[\bar{\varphi}_q^{(k_q)} \left(C_h^{(n)} \varphi_{N+l}^{(k_{N+l})} \right) \right]_{q,l=1}^N \mathbb{1} = \int d\mu_{C_h^{(n)}}(\mathbf{H}^{(n_\beta)}) \bar{\varphi}_1^{(k_1)} \dots \bar{\varphi}_N^{(k_N)} \varphi_{2N}^{(k_{2N})} \dots \varphi_{N+1}^{(k_{N+1})};$$

(II) the relationship between the exterior product $\wedge$ and the circle product $\circ$ as stated in Lemma 2.2.35 (4–5); (III) the Chernoff product approximation of Lemma 4.1.9; (IV) the tracial state formula (Theorem 4.1.12), which allows us to represent determinants (4.2.3) as traces; and (V) Hölder inequalities for Schatten norms. This approach based on Grassmann–algebra methods turns out to be more efficient, in the finite-dimensional case, than the construction done in [BP16b], which involves quasi-free states in suitably-chosen CAR algebras.

We define Schatten norms in the context of CAR algebras by

$$(4.2.4) \quad \|A\|_s \doteq (\operatorname{tr}(|A|^s))^{\frac{1}{s}}, \quad A \in \mathcal{U}, \quad s \geq 1,$$

and

$$(4.2.5) \quad \|A\|_\infty \doteq \lim_{s \rightarrow \infty} (\operatorname{tr}(|A|^s))^{\frac{1}{s}} = \|A\|_{\mathcal{U}}, \quad A \in \mathcal{U},$$

where, as is usual for $\mathcal{H}$, a Hilbert space, $\mathcal{U} \equiv \mathcal{U}(\mathcal{H})$ is its CAR C^* -algebra associated, $|A| \doteq (A^*A)^{1/2}$ and $\operatorname{tr} \in \mathcal{U}^*$ is the tracial state of Definition 2.3.29. See also Remark 2.3.14. Hölder inequalities for Schatten norms then refer to the following bounds: For any $m \in \mathbb{N}$,

$r, s_1, \dots, s_m \in [1, \infty]$ such that $\sum_{j=1}^m \frac{1}{s_j} = \frac{1}{r}$, and all elements $A_1, \dots, A_m \in \mathcal{U}$,

$$(4.2.6) \quad \|A_1 \cdots A_m\|_r \leq \prod_{j=1}^m \|A_j\|_{s_j}.$$

This type of inequality, combined with (4.2.3) and Lemmata 2.2.35 and 4.1.9, yields a *sharp* bound on the determinant (4.2.2):

Theorem 4.2.7 (Determinant bounds – I):

Let h be a self-adjoint operator on $\mathcal{H}$. Then, for $\beta \in \mathbb{R}^+$, $n, M, N \in \mathbb{N}$ with $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$, all $k_1, \dots, k_{N+M} \in \{0, \dots, n_\beta - 1\}$ ($n_\beta \doteq n + \lfloor n/\beta \rfloor$) and $\bar{\varphi}_1, \dots, \bar{\varphi}_N \in \mathcal{H}$, $\varphi_{N+1}, \dots, \varphi_{N+M} \in \mathcal{H}$, the following bound holds true:

$$\left| \det \left[\bar{\varphi}_q^{(k_q)} \left(C_h^{(n)} \varphi_{N+l}^{(k_{N+l})} \right) \right]_{q,l=1}^N \right| \leq \delta_{N,M} \prod_{q=1}^N \|\bar{\varphi}_q\|_{\mathcal{H}} \prod_{l=1}^N \|\varphi_{N+l}\|_{\mathcal{H}}. \quad \clubsuit$$

Remark 4.2.8. In this theorem the Kronecker delta means that if $N \neq M$ the left-hand side of the inequality is zero, see Theorem 2.2.33. ✱

PROOF. Fix all parameters of the theorem. Pick any orthonormal basis $\{\psi_j\}_{j \in J}$ of $\mathcal{H}$ of eigenvectors for the self-adjoint operator $h \in \mathcal{B}(\mathcal{H})$. Similar to the derivation of (4.1.19), by Lemmata 2.2.35–4 and 4.1.9, together with CAR Definition (2.2.5), Definition 2.2.41, and Equation (2.2.9), for any $n_1, n_2 \in \mathbb{N}$, and all indices $j_1, \dots, j_{n_1+n_2} \in J$,

$$\begin{aligned} & \psi_{j_1}^* \cdots \psi_{j_{n_1}}^* \kappa \left(\left[X_{-\beta d\langle Q, hQ \rangle}^{(n)} \right]^{\frac{1}{n}} \right) \psi_{j_{n_1+1}} \cdots \psi_{j_{n_1+n_2}} \\ &= \kappa \left(a(\psi_{j_1}) \cdots a(\psi_{j_{n_1}}) \left[X_{-\beta d\langle Q, hQ \rangle}^{(n)} \right]^{\frac{1}{n}} a(\psi_{j_{n_1+1}})^* \cdots a(\psi_{j_{n_1+n_2}})^* \right), \end{aligned}$$

where $\psi_{j_i}^* \doteq \bar{\psi}_{j_i}$ for all $i \in \{1, \dots, n_1 + n_2\}$ (Definition 2.2.37). By taking linear combinations of the above equalities, we thus deduce that, for any $n_1, n_2 \in \mathbb{N}$ and all $\varphi_1, \dots, \varphi_{n_1+n_2} \in \mathcal{H}$,

$$(4.2.9) \quad \begin{aligned} & \varphi_1^* \cdots \varphi_{n_1}^* \kappa \left(\left[X_{-\beta d\langle Q, hQ \rangle}^{(n)} \right]^{\frac{1}{n}} \right) \varphi_{n_1+1} \cdots \varphi_{n_1+n_2} \\ &= \kappa \left(a(\varphi_1) \cdots a(\varphi_{n_1}) \left[X_{-\beta d\langle Q, hQ \rangle}^{(n)} \right]^{\frac{1}{n}} a(\varphi_{n_1+1})^* \cdots a(\varphi_{n_1+n_2})^* \right). \end{aligned}$$

We are now in a position to write the determinant as a trace in order to next use Hölder

inequalities: By Definition 2.2.32 and Equations (4.1.18)–(4.1.21) and (4.2.3),

$$(4.2.10) \quad \det \left[\bar{\varphi}_q^{(k_q)} \left(C_h^{(n)} \varphi_{N+l}^{(k_{N+l})} \right) \right]_{q,l=1}^N \mathbb{1} = \frac{1}{\operatorname{tr} \left(X_{-\beta d\langle Q, hQ \rangle}^{(n)} \right)} \int d \left(\mathbb{H}^{(n_\beta)} \right) e^{\frac{1}{2} \langle \mathbb{H}^{(n_\beta)}, \partial \mathbb{H}^{(n_\beta)} \rangle} \mathfrak{Q}$$

with

$$\mathfrak{Q} \doteq \left(\prod_{k=0}^{n-1} \varkappa^{(k)} \left(\left[X_{-\beta d\langle Q, hQ \rangle}^{(n)} \right]^{\frac{1}{n}} \right) \right) (\varphi_1^*)^{(k_1)} \cdots (\varphi_N^*)^{(k_N)} \varphi_{2N}^{(k_{2N})} \cdots \varphi_{N+1}^{(k_{N+1})}.$$

To get the assertion, it suffices now to reorganize the exterior products $\wedge$ in $\mathfrak{Q}$ by regrouping all terms associated with the same index $k \in \{0, \dots, n_\beta - 1\}$. Note that an additional minus sign can appear, because of the antisymmetry of $\wedge$. Doing this, we rewrite the element $\mathfrak{Q}$ in the form $\pm \varkappa^{(0)}(A_0) \wedge \cdots \wedge \varkappa^{(n_\beta-1)}(A_{n_\beta-1})$ and, by Lemma 2.2.35–4 and Theorem 4.1.12, together with (4.2.10),

$$(4.2.11) \quad \det \left[\bar{\varphi}_q^{(k_q)} \left(C_h^{(n)} \varphi_{N+l}^{(k_{N+l})} \right) \right]_{q,l=1}^N = \pm \frac{1}{\operatorname{tr} \left(X_{-\beta d\langle Q, hQ \rangle}^{(n)} \right)} \operatorname{tr}(A_0 \cdots A_{n_\beta-1}).$$

More precisely, using Definition 2.2.41 and 4.1.10, for any $k \notin \{k_1, \dots, k_{2N}\}$,

$$A_k = \left[X_{-\beta d\langle Q, hQ \rangle}^{(n)} \right]^{\frac{1}{n}}$$

unless $k \in \{n, \dots, n_\beta - 1\}$ in which case $A_k = \mathbb{1}$. If $k \in \{k_1, \dots, k_{2N}\}$ then $\varkappa(A_k)$ is of the form (4.2.9), with h being zero whenever $k \in \{n, \dots, n_\beta - 1\}$. Therefore, one can apply the inequality

$$|\operatorname{tr}(B)| \leq \operatorname{tr}(|B|), \quad B \in \mathcal{U},$$

and Hölder inequalities (4.2.6) to the right-hand side of (4.2.11), with $m = n + 2N$, $r = 1$, $s_j = n$ for each term $[X_{-\beta d\langle Q, hQ \rangle}^{(n)}]^{\frac{1}{n}}$ and $s_j = \infty$ for every generator $a(\varphi)$. Since, by inequality (2.2.6), Lemma 4.1.9 and Equation (4.2.5),

$$\left\| \left[X_{-\beta d\langle Q, hQ \rangle}^{(n)} \right]^{\frac{1}{n}} \right\|_n = \left(\operatorname{tr} \left(X_{-\beta d\langle Q, hQ \rangle}^{(n)} \right) \right)^{\frac{1}{n}},$$

the assertion then follows.

Bounds for general determinants of the form (4.2.1) can be deduced from Theorem 4.2.7, which corresponds to the special case $m = 1$ and $\mathfrak{M} = 1$. This is done by extending the Hilbert space $\mathcal{H}$, similar to what is done in [BP16b, Section 1.3]: For any fixed $m \in \mathbb{N}$, a (generic) non-vanishing

real positive matrix $\mathfrak{M} \in \text{Mat}(m, \mathbb{R})$ gives rise to a positive sesquilinear form on $\mathbb{C}^m$ defined by

$$(4.2.12) \quad \langle (x_1, \dots, x_m), (y_1, \dots, y_m) \rangle_{\mathbb{C}^m}^{\mathfrak{M}} \doteq \sum_{j,l=1}^m \bar{x}_j y_l \mathfrak{M}_{j,l}.$$

In general, this sesquilinear form may be degenerated. Then, one can define a Hilbert space

$$\mathbb{M} \doteq \mathbb{C}^m / \{x \in \mathbb{C}^m : \langle x, x \rangle_{\mathbb{C}^m}^{\mathfrak{M}} = 0\}$$

whose scalar product is defined by

$$\langle [x], [y] \rangle_{\mathbb{M}} \doteq \langle x, y \rangle_{\mathbb{C}^m}^{\mathfrak{M}}, \quad x, y \in \mathbb{C}^m.$$

Then, it suffices to replace $\mathcal{H}$ with $\mathcal{H} \otimes \mathbb{M}$ and $\tilde{\mathcal{H}}$ with $\tilde{\mathcal{H}} \otimes \bar{\mathbb{M}}$ to get, from Theorem 4.2.7, bounds for determinants of the form (4.2.1):

4.2.13 COROLLARY (DETERMINANT BOUNDS – II). *Let h be a self-adjoint operator on $\mathcal{H}$. Then, for $\beta \in \mathbb{R}^+$, $n, m, M, N \in \mathbb{N}$ with $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$, all $k_1, \dots, k_{N+M} \in \{0, \dots, n_\beta - 1\}$ ($n_\beta \doteq n + \lfloor n/\beta \rfloor$), $j_1 \dots j_{N+M} \in \{1, \dots, m\}$, $\mathfrak{M} \in \text{Mat}(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$, and $\bar{\varphi}_1, \dots, \bar{\varphi}_N \in \tilde{\mathcal{H}}$, $\varphi_{N+1}, \dots, \varphi_{N+M} \in \mathcal{H}$, the following bound holds true:*

$$\left| \det \left[\mathfrak{M}_{j_q, j_{N+l}} \bar{\varphi}_q^{(k_q)} \left(C_h^{(n)} \varphi_{N+l}^{(k_{N+l})} \right) \right]_{q,l=1}^N \right| \leq \delta_{N,M} \prod_{q=1}^N \|\bar{\varphi}_q\|_{\tilde{\mathcal{H}}} \mathfrak{M}_{j_q, j_q}^{1/2} \prod_{l=1}^N \|\varphi_{l+N}\|_{\mathcal{H}} \mathfrak{M}_{j_{l+N}, j_{l+N}}^{1/2}. \quad \blacksquare$$

PROOF. Fix $m \in \mathbb{N}$ and $\mathfrak{M} \in \text{Mat}(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$. Let, for instance, $\tilde{\mathcal{H}} \doteq \mathcal{H} \otimes \mathbb{M}$. Any operator $A \in \mathcal{B}(\mathcal{H})$ can be extended to an operator acting on $\tilde{\mathcal{H}}$ by the definition $\tilde{A} \doteq A \otimes \mathbf{1}_{\mathbb{M}} \in \mathcal{B}(\tilde{\mathcal{H}})$. Meanwhile, we use the canonical identification

$$\tilde{\mathbf{H}}^{(n)} \doteq \bigoplus_{k=0}^{n-1} \tilde{\mathcal{H}}^{(k)} \equiv \mathbf{H}^{(n)} \otimes \mathbb{M}, \quad n \in \mathbb{N},$$

via the unitary map uniquely defined by the conditions

$$(\varphi \otimes [x])^{(k)} \mapsto \varphi^{(k)} \otimes [x], \quad \varphi \in \mathcal{H}, \quad x \in \mathbb{C}^m,$$

for any $k \in \{0, \dots, n-1\}$. It is then straightforward to verify that

$$C_{\tilde{h}}^{(n)} = C_h^{(n)} \otimes \mathbf{1}_{\mathbb{M}}.$$

See Equations (2.2.24) and Definition of $C_h^{(n)}$ in Corollary 4.1.17.

Now, using the notation $e_j \doteq [e_j] \in \mathbb{M}$, where $\{e_j\}_{j=1}^m$ is the canonical basis of $\mathbb{C}^m$, note that

$$\mathfrak{M}_{j,l} = \langle e_j, e_l \rangle_{\mathbb{M}}, \quad j, l \in \{1, \dots, m\}.$$

Then, the result follows as a direct application of Theorem 4.2.7. End

4.3 SUMMABILITY OF THE COVARIANCE

In this section we derive bounds on the decay of the covariances (4.1.18) for general fermion systems on the crystal lattice $\mathfrak{Q} \doteq \mathbb{Z}^d, d \in \mathbb{N}$, from the celebrated Combes–Thomas estimates.

4.3.1 LEMMA (SUMMABILITY OF THE FERMI DISTRIBUTION). *Fix $\beta \in \mathbb{R}^+$ and $h = h^* \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathbf{D}_{h,\beta} \doteq \sup_{\alpha \in [0,\beta]} \sup_{x \in \mathfrak{Q}} \left\{ \sum_{y \in \mathfrak{Q}} \left| \left\langle e_x, \frac{e^{\alpha h}}{1 + e^{\beta h}} e_y \right\rangle_{\mathcal{H}} \right| \right\} \leq 96 \inf_{\mu \in \mathbb{R}_0^+} \sum_{x \in \mathfrak{Q}} e^{-\mu \min\{1, \frac{\pi}{4\beta \mathbf{S}(h,\mu)}\} |x|} = \mathcal{O}((\beta + 1)^d). \quad \blacklozenge$$

PROOF. The proof is a simple adaptation of the one from [AG98, Theorem 3]: Fix all parameters of the lemma and observe that Corollary D.0.7 combined with Inequality (D.0.5) yields

$$(4.3.2) \quad \begin{aligned} & \left| \left\langle e_x, ((h - u)^2 + \eta^2)^{-1} e_y \right\rangle_{\mathcal{H}} \right| \\ & \leq 12e^{-\frac{\mu\eta}{2\mathbf{S}(h,\mu)} |x-y|} \left\langle e_x, ((h - u)^2 + \eta^2)^{-1} e_x \right\rangle_{\mathcal{H}}^{1/2} \left\langle e_y, ((h - u)^2 + \eta^2)^{-1} e_y \right\rangle_{\mathcal{H}}^{1/2} \end{aligned}$$

for $x, y \in \mathfrak{Q}$, $u \in \mathbb{R}$ and $\eta \in (0, 2\mathbf{S}(h, \mu)]$. On the other hand, at fixed $\alpha \in [0, \beta]$ and $\beta \in \mathbb{R}^+$ the function on the stripe

$$\mathbb{R} + \frac{\pi i}{2\beta} [-1, 1] \subset \mathbb{C}$$

defined by

$$G(z) \doteq \frac{e^{\alpha z}}{1 + e^{\beta z}}$$

is analytic and uniformly bounded by $\sqrt{2}$. Using Cauchy's integral formula and some translation by $\pm i\eta$, we write this function as

$$(4.3.3) \quad \begin{aligned} G(E) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{G(u - i\eta)}{u - i\eta - E} - \frac{G(u + i\eta)}{u + i\eta - E} \right) du \\ &= \frac{\eta}{\pi} \int_{\mathbb{R}} \frac{G(u - i\eta) + G(u + i\eta)}{(E - u)^2 + \eta^2} du - \frac{2\eta}{\pi} \int_{\mathbb{R}} \frac{G(u)}{(E - u)^2 + 4\eta^2} du \end{aligned}$$

for all $E \in \mathbb{R}$ and $\eta \in (0, \pi/(2\beta)]$. By spectral calculus, together with (4.3.2)–(4.3.3) and the Cauchy–Schwarz inequality, it follows that

$$\sup_{\alpha \in [0, \beta]} \left| \left\langle e_x, \frac{e^{\alpha h}}{1 + e^{\beta h}} e_y \right\rangle_{\mathcal{H}} \right| \leq 48 \exp \left(-\mu \min \left\{ 1, \frac{\pi}{4\beta S(h, \mu)} \right\} |x - y| \right)$$

for all $x, y \in \mathfrak{Q}$, $\mu \in \mathbb{R}_0^+$, and $\epsilon \in (0, 1]$. This in turn implies the assertion. End

SUMMABILITY OF COVARIANCES – GENERAL BOUND

In order to show the analiticity of the logarithmic moment generating function $J(s)$ we take two boxes Λ_{L_i} and Λ_{L_f} such that $\Lambda_{L_i} \subset \Lambda_{L_f} \in \mathcal{P}_f(\mathfrak{Q})$, where the Hamiltonian $h \in \mathcal{B}(\mathcal{H})$ is projected in $h_{L_i} \in \mathcal{B}(\mathcal{H}_{L_i})$ and $h_{L_f} \in \mathcal{B}(\mathcal{H}_{L_f})$ respectively, with $\mathcal{H}_{L_i} \equiv \mathcal{H}_{\Lambda_{L_i}}$ and $\mathcal{H}_{L_f} \equiv \mathcal{H}_{\Lambda_{L_f}}$. Hence, we define the decay parameter

$$(4.3.4) \quad \omega_h \doteq \limsup_{L_i \rightarrow \infty} \limsup_{L_f \rightarrow \infty} \lim_{n \rightarrow \infty} \sup_{k_1 \in \{0, \dots, n\beta-1\}} \sup_{x_1 \in \Lambda_{L_i}} \left\{ \beta n^{-1} \sum_{k_2=0}^{n\beta-1} \sum_{x_2 \in \Lambda_{L_i}} \left| \left\langle e_{x_1}^{(k_1)}, C_{h_{L_f}} e_{x_2}^{(k_2)} \right\rangle_{\mathcal{H}^{(n\beta)}} \right| \right\}.$$

Recall that $C_{h_{L_f}}^{(n)}$ is the covariance (4.1.18) for $\mathcal{H}_{L_f}$ and $n_\beta \doteq n + \lfloor n/\beta \rfloor$. Similarly to [ABPM17, Equations (92) and (93)], define the function

$$(4.3.5) \quad F_{u_1, u_2}(h) \doteq \begin{cases} \frac{e^{-\alpha(u_1, u_2)h}}{1 + e^{-\beta t}} & \text{for } u_1 < u_2, \\ \frac{e^{\alpha(u_2, u_1)h}}{1 + e^{\beta t}} & \text{for } u_1 \geq u_2, \end{cases}$$

for any $\beta \in \mathbb{R}^+$, $u_1, u_2 \in [0, \beta + 1]$, $h \in \mathcal{B}(\mathcal{H})$ where

$$\alpha(u_1, u_2) \doteq [\min \{\beta - u_1, u_2 - u_1\}]_+ \in [0, \beta], \quad u_1, u_2 \in [0, \beta + 1].$$

Then, one gets the following expression for the decay parameter ω_h :

4.3.6 LEMMA (EXPLICIT EXPRESSION OF THE DECAY PARAMETER). *Fix $d \in \mathbb{N}$ and $\beta \in \mathbb{R}^+$. Let $h \in \mathcal{B}(\mathcal{H})$ be any self-adjoint operator on $\mathcal{H}$. Then*

$$\omega_h = \sup_{u_1 \in [0, \beta+1]} \sup_{x_1 \in \mathfrak{Q}} \sum_{x_2 \in \mathfrak{Q}} \int_0^{\beta+1} |\langle e_{x_1}, F_{u_1, u_2}(h) e_{x_2} \rangle_{\mathcal{H}}| du_2. \quad \blacklozenge$$

PROOF. Fix all parameters of the Lemma and define:

$$h^{(n)} \doteq \frac{\beta^{-1}n}{2} \ln \left(\frac{1 + n^{-1}\beta h}{1 - n^{-1}\beta h} \right).$$

Explicit computations using the spectral theorem and [ABPM17, Inequality (91)] show that, for any $\beta \in \mathbb{R}^+$, $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$, $L_f \in \mathbb{R}_0^+ \cup \{\infty\}$ and $\alpha \in [0, \beta]$,

$$\begin{aligned} \left\| \frac{e^{\pm \alpha h_{L_f}}}{1 + e^{\pm \beta h_{L_f}}} - \frac{e^{\pm \alpha h_{L_f}^{(n)}}}{1 + e^{\pm \beta h_{L_f}^{(n)}}} \right\|_{\mathcal{B}(\mathcal{H})} &\leq \sup_{|\lambda| \leq \|h_{L_f}\|_{\mathcal{B}(\mathcal{H})}} \left| \frac{e^{\pm \alpha \lambda}}{1 + e^{\pm \beta \lambda}} - \frac{e^{\pm \frac{\alpha \beta^{-1} n}{2} \left(\ln \left(\frac{1+n^{-1} \beta \lambda}{1-n^{-1} \beta \lambda} \right) \right)}}{1 + e^{\pm \frac{n}{2} \left(\ln \left(\frac{1+n^{-1} \beta \lambda}{1-n^{-1} \beta \lambda} \right) \right)}} \right| \\ &\leq 2\beta \|h_{L_f} - h_{L_f}^{(n)}\|_{\mathcal{B}(\mathcal{H})} \leq D n^{-2} \beta^3 \|h\|_{\mathcal{B}(\mathcal{H})}^3, \end{aligned}$$

for some finite constant D . It follows that

$$\omega_h = \limsup_{L_i \rightarrow \infty} \limsup_{L_f \rightarrow \infty} \sup_{u_1 \in [0, \beta+1)} \sup_{x_1 \in \Lambda_{L_i}} \left\{ \sum_{x_2 \in \Lambda_{L_i}} \int_0^{\beta+1} \left| \langle e_{x_1}, F_{u_1, u_2}(h_{L_f}) e_{x_2} \rangle_{\mathcal{H}} \right| du_2 \right\}.$$

Therefore, because the set Λ_{L_i} is finite and the family

$$E \mapsto \int_0^{\beta+1} \frac{e^{-\alpha(u_1, u_2)E}}{1 + e^{-\beta E}} du_2, \quad u_1 \in [0, \beta+1),$$

of functions is uniformly equicontinuous on compact subsets of $\mathbb{R}$, we deduce from the last equality that

$$\omega_h = \limsup_{L_i \rightarrow \infty} \sup_{u_1 \in [0, \beta+1)} \sup_{x_1 \in \Lambda_{L_i}} \left\{ \sum_{x_2 \in \Lambda_{L_i}} \int_0^{\beta+1} \left| \langle e_{x_1}, F_{u_1, u_2}(h) e_{x_2} \rangle_{\mathcal{H}} \right| du_2 \right\}.$$

Finally, by using approximate maximizers of the suprema over $u_1 \in [0, \beta+1)$ and $x_1 \in \mathfrak{L}$, and the monotone convergence, the lemma follows. End

Now we are in a position to prove the summability of the covariance:

Theorem 4.3.7 (Summability of the covariance):

Fix $d \in \mathbb{N}$ and let $h \in \mathcal{B}(\mathcal{H})$ be any self-adjoint operator on $\mathcal{H}$. Then, for any $\beta \in \mathbb{R}^+$,

$$\omega_h \leq 8\mathbf{D}_{h, \beta}(\beta+1).$$

PROOF. The theorem is a consequence of Lemmata 4.3.1 and 4.3.6, and Equation (4.3.5) together with straightforward computations. End

By Lemma 4.3.1, we observe from this theorem that

$$(4.3.8) \quad \omega_h = \mathcal{O}\left((\beta+1)^{d+1}\right),$$

for *all* self-adjoint operators $h \in \mathcal{B}(\mathcal{H})$ as soon as $\mathbf{S}(h, \mu) < \infty$ for some $\mu \in \mathbb{R}^+$. The scaling (4.3.8) with respect to the inverse temperature $\beta \in \mathbb{R}^+$ is exactly the same one obtained from the Fourier analysis of the two-point Green function of translation-invariant, free fermi systems at positive density. Our result shows that this estimate does *not* depend on translation invariance.

SUMMABILITY OF COVARIANCES – GAPPED CASE

Estimate (4.3.8) does not allow for the control of the decay parameter ω_h (4.3.4) in the zero-temperature limit, i.e., when $\beta \rightarrow \infty$. However, in the special case of *gapped* self-adjoint operators, that is, self-adjoint operators h satisfying

$$(4.3.9) \quad g_h \doteq \inf \{ \varepsilon > 0 : [-\varepsilon, \varepsilon] \cap \text{spec}(h) \neq \emptyset \} > 0,$$

we can uniformly bound ω_h at arbitrarily large $\beta \gg 1$. To demonstrate this, we need to adapt Lemma 4.3.1 to this gapped situation:

4.3.10 LEMMA (SUMMABILITY OF THE FERMI DISTRIBUTION – GAPPED CASE). *Fix $d \in \mathbb{N}$, $\beta \in \mathbb{R}^+$ and $h = h^* \in \mathcal{B}(\mathcal{H})$ such that $g_h > 0$. Then,*

$$\begin{aligned} & \sup_{u_1 \in [0, \beta+1]} \sup_{x_1 \in \mathfrak{U}} \sum_{x_2 \in \mathfrak{U}} \int_0^{\beta+1} \left| \langle e_{x_1}, F_{u_1, u_2}(h, \mathbf{1}[h > 0]) e_{x_2} \rangle_{\mathcal{H}} \right| du_2 \\ & \leq 152 \left\| \sup_{u_1 \in [0, \beta+1]} \inf_{\mu \in \mathbb{R}_0^+} \left\{ \int_0^{\beta+1} \frac{e^{\alpha(u_1, u_2) \frac{g_h}{2}}}{e^{\frac{\beta g_h}{2}} - 1} du_2 \sum_{x \in \mathfrak{U}} e^{-\mu \min \left\{ 1, \frac{g_h}{4\mathbf{S}(h, \mu)} \right\} |x|} \right\} \right\|. \quad \blacklozenge \end{aligned}$$

PROOF. Fix all parameters of the lemma. By Inequality (D.0.5), Theorem D.0.6 implies the bound

$$(4.3.11) \quad \left| \langle e_x, (z - h)^{-1} e_y \rangle \right| \leq 4g_h^{-1} \exp \left(-\mu \min \left\{ 1, \frac{g_h}{4\mathbf{S}(h, \mu)} \right\} |x - y| \right)$$

for any $x, y \in \mathfrak{U}$ and $z \in \mathbb{C}$ such that $\Delta(h, z) \geq g_h/2 > 0$. On the other hand, for every $\eta \in (0, g_h/2]$, the function defined by

$$G(z) \doteq \frac{e^{\alpha z}}{1 + e^{\beta z}}, \quad z \in \mathbb{R}_0^+ + \eta + i\eta[-1, 1],$$

is analytic and uniformly bounded by $e^{\alpha\eta}(e^{\beta\eta} - 1)^{-1}$. Similar to (4.3.3), we again use Cauchy's integral formula to write, for all real $E \in \mathbb{R} \setminus \{\eta\}$,

$$\mathbf{1}[E > \eta] G(E) = \frac{1}{2\pi i} \int_{\eta}^{\infty} \left(\frac{G(u - i\eta)}{u - E - i\eta} - \frac{G(u + i\eta)}{u - E + i\eta} \right) du - \frac{1}{2\pi} \int_{-\eta}^{\eta} \frac{G(\eta + iu)}{\eta - E + iu} du,$$

which yields

$$\begin{aligned} \mathbf{1}[E > \eta] G(E) &= \frac{\eta}{\pi} \int_{\eta}^{\infty} \frac{G(u - i\eta) + G(u + i\eta)}{(u - E)^2 + \eta^2} du - \frac{2\eta}{\pi} \int_{\eta}^{\infty} \frac{G(u)}{(u - E)^2 + 4\eta^2} du \\ &+ \frac{1}{2\pi} \int_0^{\eta} \frac{G(\eta - iu)}{\eta - iu - E + 2i\eta} du + \frac{1}{2\pi} \int_0^{\eta} \frac{G(\eta + iu)}{\eta + iu - E - 2i\eta} du \\ &- \frac{1}{2\pi} \int_{-\eta}^{\eta} \frac{G(\eta + iu)}{\eta - E + iu} du. \end{aligned}$$

$E \mapsto \mathbf{1}[E > \eta]$ is the characteristic function of the set (η, ∞) . By spectral calculus, together with the last equality, Inequalities (4.3.2) and (4.3.11) and the Cauchy–Schwarz inequality, it follows that

$$\left| \left\langle \mathbf{e}_x, \mathbf{1}[h > 0] \frac{e^{\pm \alpha h}}{1 + e^{\pm \beta h}} \mathbf{1}[h > 0] \mathbf{e}_y \right\rangle_{\mathcal{H}} \right| \leq \frac{38e^{\frac{\alpha g_h}{2}}}{e^{\frac{\beta g_h}{2}} - 1} e^{-\mu \min\left\{1, \frac{g_h}{4S(h, \mu)}\right\} |x-y|}$$

for all $x, y \in \mathfrak{L}$, $\mu \in \mathbb{R}_0^+$ and $\alpha \in [0, \beta)$. By (4.3.5), this in turn implies the assertion. End

We are now in a position to prove the uniform summability of the covariance with respect to the inverse temperature $\beta \in \mathbb{R}^+$, in the special case of gapped Hamiltonians:

Theorem 4.3.12 (Summability of the covariance – gapped case):

Fix $d \in \mathbb{N}$ and let $h \in \mathcal{B}(\mathcal{H})$ be any self-adjoint operator on $\mathcal{H}$ such that $g_h > 0$. Then

$$\omega_h \leq 152 \sup_{u_1 \in [0, \beta+1)} \inf_{\mu \in \mathbb{R}_0^+} \left\{ \int_0^{\beta+1} \frac{e^{\alpha(u_1, u_2) \frac{g_h}{2}}}{e^{\frac{\beta g_h}{2}} - 1} du_2 \times \sum_{x \in \mathfrak{L}} e^{-\mu \min\left\{1, \frac{g_h}{4S(h, \mu)}\right\} |x|} \right\} = \mathcal{O}\left(\left(g_h^{-1} + 1\right)^{d+1}\right). \quad \spadesuit$$

PROOF. See proof in [ABPM17, Theorem 5.13]. End

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ANALITICITY OF GENERATING FUNCTIONS FROM BRYDGES–KENNEDY TREE EXPANSIONS

BRYDGES–KENNEDY Tree Expansions are pivotal in the study of analyticity properties in constructive quantum field theory. Here we present these tools as presented in [ABPM16], which take as main bibliography the works [Ped05, BP]. Thus, here we derive the so-called tree expansion for the logarithm

$$(5.0.1) \quad \log \left(\int d\mu_C(\mathcal{H}) e^{\mathcal{W}} \right),$$

where $\mathcal{W}$ is an even element of some finite dimensional Grassmann algebra and $\int d\mu_C$ is a Grassmann (Gaussian) Berezin integral. We call such an object a “Grassmann generating function”. Here, for $z > 0$,

$$\log(z\mathbb{1}) \doteq \log(z)\mathbb{1}.$$

5.1 FINITE MOMENT GENERATING FUNCTIONS AND ANALITICITY

Before presenting the formalism of Brydges–Kennedy as provided in [ABPM16, BP], we first discuss its relation with the logarithm moment generating functions in the scope of Fermion Lattice Systems:

In the previous chapter we show in Corollary 4.1.17 that for a Hilbert space $\mathcal{H}$ with CAR C^* -algebra associated $\mathcal{U} \equiv \mathcal{U}(\mathcal{H})$ and elements $h = h^* \in \mathcal{B}(\mathcal{H})$, $K = K^*$, $W \in \mathcal{U}^+ \cap \mathcal{U}$, the expression

$$(5.1.1) \quad \frac{\mathrm{tr} \left(e^{-\beta(d\langle Q, hQ \rangle + W)} e^{sK} \right)}{\mathrm{tr} \left(e^{-\beta d\langle Q, hQ \rangle} \right)} \mathbb{1},$$

can be written as a Gaussian Berezin integral in the form of the term inside of $\log(\cdot)$ in (5.0.1), for some covariance C_h , which we show in Theorem 4.2.7 has a sharp determinant bound. Recall

that $\text{tr} \in \mathcal{U}^*$ is the tracial state of the C^* -algebra $\mathcal{U}$, satisfying (2.3.29) for $\beta = 0$. These results are relevant for us because allow to tackle the finite moment generating function or KMS state of lattice fermions in equilibrium, i.e.,

$$(5.1.2) \quad \rho_{L_f, L_i} \left(e^{s|\Lambda_{L_0}|A_{L_0}} \right) = \frac{\text{tr} \left(e^{s|\Lambda_{L_0}|A_{L_0}} e^{-\beta H_{L_f, L_i}} \right)}{\text{tr} \left(e^{-\beta H_{L_f, L_i}} \right)},$$

where

$$(5.1.3) \quad H_{L_f, L_i} \doteq \sum_{\Lambda_1 \subseteq \Lambda_{L_f}} \Phi_{f, \Lambda_1} + \sum_{\Lambda_2 \subseteq \Lambda_{L_i}} \Phi_{i, \Lambda_2} = |\Lambda_{L_f}| E_{L_f}^{\Phi_f} + |\Lambda_{L_i}| E_{L_i}^{\Phi_i} \quad L_f, L_i \in \mathbb{R}_0^+.$$

is the *internal* energy observable with Φ_f and Φ_i two interactions respectively representing the free (or unperturbed) and interparticle components of the total interaction $\Phi = \Phi_f + \Phi_i$. Here, for $\mathcal{U}_{L_f} \equiv \mathcal{U}_{\Lambda_{L_f}}$ and $\mathcal{U}_{L_i} \equiv \mathcal{U}_{\Lambda_{L_i}}$, $E_{L_f}^{\Phi_f} \in \mathcal{U}^+ \cap \mathcal{U}_{L_f}$ and $E_{L_i}^{\Phi_i} \in \mathcal{U}^+ \cap \mathcal{U}_{L_i}$ denote two sequence of (internal) energy observables, see Expression (2.3.38). In (5.1.2), $A_{L_0} \equiv A_{\Lambda_{L_0}}$ refers to the space average of interactions $\{\Psi_{\Lambda_3}\}_{\Lambda_3 \subseteq \Lambda_{L_0}} \in \mathcal{U}^+ \cap \mathcal{U}_{L_0}$ in the box Λ_{L_0} :

$$(5.1.4) \quad A_{L_0} \doteq \frac{1}{|\Lambda_{L_0}|} \sum_{\Lambda_3 \subseteq \Lambda_{L_0}} \Psi_{\Lambda_3} \in \mathcal{U}^+ \cap \mathcal{U}_{L_0}.$$

From (5.1.1), we notice now that for $s \in \mathbb{R}$ it is useful to define the map $s \mapsto J_{L_0}(s)$, as the finite logarithmic moment generating function $J_{L_0, L_i, L_f}(s) \equiv J_{\Lambda_{L_0}, \Lambda_{L_i}, \Lambda_{L_f}}(s)$ given by

$$(5.1.5) \quad J_{L_0, L_i, L_f}(s) \doteq g_{L_0, L_i, L_f}(s) - g_{L_0, L_i, L_f}(0),$$

where

$$(5.1.6) \quad g_{L_0, L_i, L_f}(s) \doteq \frac{1}{|\Lambda_{L_0}|} \ln \frac{\text{tr} \left(e^{-\beta(d\langle Q, h_{L_f} Q \rangle + W_{L_i})} e^{sK_{L_0}} \right)}{\text{tr} \left(e^{-\beta d\langle Q, h_{L_f} Q \rangle} \right)}.$$

Here, for $\mathcal{H}_{L_f} \equiv \mathcal{H}_{\Lambda_{L_f}}$: (i) $h_{L_f} \in \mathcal{B}(\mathcal{H}_{L_f})$ is a self-adjoint operator on $\mathcal{H}_{L_f}$, which has second quantization $d\langle Q, h_{L_f} Q \rangle \in \mathcal{U}^+ \cap \mathcal{U}_{L_f}$ describing the non-interacting particle interaction of Expression (5.1.3), (ii) $W_{L_i} \in \mathcal{U}^+ \cap \mathcal{U}_{L_i}$ describes the interparticle interaction in (5.1.3) and (iii) $\mathcal{U}^+ \cap \mathcal{U}_{L_0} \ni K_{L_0} = |\Lambda_{L_0}| A_{L_0}$, where A_{L_0} is defined by (5.1.4). Note that we can do the inclusion of CAR C^* -algebras $\mathcal{U}_{L_0} \subset \mathcal{U}_{L_i} \subset \mathcal{U}_{L_f}$ because the comments around Expression (2.3.23) in Chapter 2. Compare Expressions (5.1.5)–(5.1.6) with (3.2.12)–(3.2.13) in Chapter 3.

In regards to (5.1.2) (see [ABPM17, §8]), the limit $(L_f, L_i) \rightarrow (\infty, \infty)$ refers to the thermodynamic limit. Similar to the set of probability measures in the commutative setting, the set of all states on any C^* -algebra is a weak*-compact (convex) set and so, for any interaction Φ_f ,

Φ_i , the family $\{\rho_{L_f, L_i}\}_{L_f, L_i \in \mathbb{R}_0^+}$ has at least one weak*-accumulation point, which is a state. If Φ_f, Φ_i are finite-range interactions (Definition 2.3.37), the groups $\{\tau_t^{(L_f, L_i)}\}_{t \in \mathbb{R}}, L_f, L_i \in \mathbb{R}_0^+$, converge strongly, as $(L_f, L_i) \rightarrow (\infty, \infty)$, to a strongly continuous group $\{\tau_t\}_{t \in \mathbb{R}}$ of *-automorphisms on $\mathcal{U}$. See for instance [BP16a, Theorem 4.8]. By [BR03b, Proposition 5.3.25], in this case, any weak*-accumulation point ρ of $\{\rho_{L_f, L_i}\}_{L_f, L_i \in \mathbb{R}_0^+}$ is a KMS state associated with $\{\tau_t\}_{t \in \mathbb{R}}$ and the inverse temperature $\beta \in \mathbb{R}^+$. Note that, in our study, we take the limit $L_i \rightarrow \infty$ after $L_f \rightarrow \infty$. See, for instance, Equation (4.3.4).

Thus, in a suitable sense we will take in (5.1.5) the limit $L_o \rightarrow \infty$ to get the existence of the logarithm moment generating function $J(s)$, see Expression (2.4.5). Finally, in the context of finite range translation invariant interactions, using Brydges–Kennedy Tree Expansions we proof for weakly interacting fermions in the lattice the analyticity of $J(s)$ in a neighborhood of $s = 0$ ¹. In particular, as explained in §2.4, Theorem 2.4.9, the fluctuation measures $\{m_{\rho, \sqrt{|\Lambda_{L_o}|}(A_{L_o} - \rho(A_{L_o})\mathbb{1})}\}_{L_o \in \mathbb{R}_0^+}$ converge in the weak* sense to the normal distribution $\mathcal{N}_{0, \sigma^2}$, where

$$\sigma^2 = \frac{d^2}{ds^2} J(s) \Big|_{s=0}$$

Last but not least, is that an important consequence of this fact is the existence of the so-called “fluctuation algebra”, for weakly interacting fermions on the lattice. As far as we now, this fact was not proven yet, and we will hence discuss it in detail in [ABPM16].

5.2 BRYDGES–KENNEDY TREE EXPANSIONS

5.2.1 GAUSSIAN CONVOLUTIONS

In order to derive tree expansions for logarithms moment generating functions, it is convenient to rewrite them in term of so-called “Gaussian convolutions”, which are defined as follows:

5.2.1 DEFINITION (GAUSSIAN CONVOLUTIONS). Let $\mathcal{H}$ be a finite-dimensional Hilbert space.

1. Fix an orthonormal basis $\{\psi_i\}_{i \in I}$ of $\mathcal{H}$ and let $C \in \mathcal{B}(\mathcal{H})$ be any (possibly not invertible) operator acting on $\mathcal{H}$. The “Grassmann–Laplace operator with covariance C ” $\Delta_C \in \mathcal{B}(\wedge^*(\mathcal{H} \oplus \mathcal{H}))$ is defined by

$$(5.2.2) \quad \Delta_C \doteq \sum_{i, j \in I} \psi_i^* (C \psi_j) \frac{\delta}{\delta \psi_j} \frac{\delta}{\delta \psi_i^*},$$

where $\delta/\delta(\psi_j)$ is the Berezin derivative acting on $\wedge^* \mathcal{H}$.

¹What is guaranteed by the Vitali convergence Theorem, [Sim15, Part 2A, Theorem 6.2.8], which we recall by the sake of completeness: Let $\Omega \in \mathbb{C}$ and $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of analytic functions on Ω such that $\sup_{n \in \mathbb{N}} \|f_n\|_K < \infty$, for all compact $K \subseteq \Omega$. If $\lim_{n \rightarrow \infty} z_n = z \in \Omega$ and $\lim_{n \rightarrow \infty} f_n(z_m) = w_m$, for $m \in \mathbb{N}$, exist, it follows that there exists an analytic function f on Ω such that $f_n \rightarrow f$.

2. The “Gaussian convolution with covariance C ” $\mu_C^* \in \mathcal{B}(\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}}))$ is the operator

$$\mu_C^* \equiv e^{\Delta_C} \doteq \mathbf{1}_{\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})} + \sum_{m=1}^{\dim \mathcal{H}} \frac{\Delta_C^m}{m!}. \quad \square$$

Remark 5.2.3. Note that Grassmann–Laplace operators do not depend on the particular choice of the orthonormal basis $\{\psi_i\}_{i \in I}$. *

We now introduce the projector P_C on $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ with range $\mathbb{C}\mathbf{1}$ which is uniquely defined by the conditions $P_C(\mathbf{1}) = \mathbf{1}$, and

$$(5.2.4) \quad P_C(\bar{\varphi}_1 \cdots \bar{\varphi}_m \varphi_1 \cdots \varphi_n) = 0$$

for all $m, n \in \mathbb{N}_0$ with $m + n \geq 1$ and all $\bar{\varphi}_1, \dots, \bar{\varphi}_m \in \bar{\mathcal{H}}, \varphi_1, \dots, \varphi_n \in \mathcal{H}$. If $m = 0$ then there is no $\bar{\varphi}$ in the above equation. Mutatis mutandis for $n = 0$. Then, we observe that Gaussian convolutions applied to monomials give determinants of covariances:

5.2.5 LEMMA (GAUSSIAN CONVOLUTIONS AND DETERMINANTS). *For any operator $C \in \mathcal{B}(\mathcal{H})$, $\mu_C^* \mathbf{1} = \mathbf{1}$ and, for any $m, n \in \mathbb{N}_0$ so that $m + n \geq 1$ and all $\bar{\varphi}_1, \dots, \bar{\varphi}_m \in \bar{\mathcal{H}}, \varphi_1, \dots, \varphi_n \in \mathcal{H}$,*

$$P_C \circ \mu_C^* \bar{\varphi}_1 \cdots \bar{\varphi}_m \varphi_1 \cdots \varphi_n = \delta_{m,n} \det \left[\bar{\varphi}_p(C\varphi_q) \right]_{p,q=1}^n \mathbf{1}. \quad \blacklozenge$$

PROOF. Let $\{\psi_i\}_{i \in I}$ be any orthonormal basis of $\mathcal{H}$. By linearity, it suffices to prove the assertion for the special case

$$\bar{\varphi}_1 = \psi_{i_1}^*, \dots, \bar{\varphi}_m = \psi_{i_m}^* \text{ and } \varphi_1 = \psi_{j_1}, \dots, \varphi_n = \psi_{j_n}.$$

The equation $\mu_A^* \mathbf{1} = \mathbf{1}$ and the second assertion for $n \neq m$ are obvious consequences of Definition 5.2.1 and we fix from now on $n = m$. Straightforward computations using again 5.2.1 together with (2.2.20) yield

$$\begin{aligned} & P_C \circ \mu_A^* \psi_{i_1}^* \cdots \psi_{i_n}^* \psi_{j_n} \cdots \psi_{j_1} \\ &= P_C \circ \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \sum_{\pi \in \mathcal{S}_n} \psi_{i_{\pi(1)}}^* (A\psi_{j_{\sigma(1)}}) \cdots \psi_{i_{\pi(n)}}^* (A\psi_{j_{\sigma(n)}}) (-1)^\pi (-1)^\sigma \\ & \quad \frac{\delta}{\delta \psi_{j_1}} \cdots \frac{\delta}{\delta \psi_{j_n}} \frac{\delta}{\delta \psi_{i_n}^*} \cdots \frac{\delta}{\delta \psi_{i_1}^*} \psi_{i_1}^* \cdots \psi_{i_n}^* \psi_{j_1} \cdots \psi_{j_n}, \end{aligned}$$

which, combined with (2.2.21) and Leibniz formula for determinants, implies the assertion when $n = m$. End

Gaussian convolutions are related to Gaussian Grassmann integrals as follows:

Theorem 5.2.6 (Relation between Gaussian integrals and convolutions):

Let $C \in \mathcal{B}(\mathcal{H})$ be any invertible operator. The Gaussian convolution of Definition 5.2.1 equals

$$P_C \circ \mu_C * (\cdot) = \int d\mu_C(\mathcal{H}) \wedge (\cdot).$$

PROOF. The assertion follows from straightforward computations, by using the identities

$$\begin{aligned} \det[C] \int d(\mathcal{H}^{(0)}) e^{\langle \mathcal{H}^{(0)}, C^{-1} \mathcal{H}^{(0)} \rangle} e^{\langle \mathcal{H}^{(0)}, \mathcal{H}^{(1)} \rangle} e^{\langle \mathcal{H}^{(1)}, \mathcal{H}^{(0)} \rangle} &= e^{-\langle \mathcal{H}^{(1)}, C \mathcal{H}^{(1)} \rangle}, \\ \int d(\mathcal{H}^{(0)}) e^{\langle \mathcal{H}, C^{-1} \mathcal{H} \rangle} \mathbb{1} &= \det[C]^{-1} \mathbb{1}. \end{aligned}$$

See Expression (2.2.31) and Definition 2.2.32. For details see [BP, ABPM16].

5.2.2 THE POLCHINSKI INITIAL VALUE PROBLEM

We consider $\mathcal{H}$, as before, any finite dimensional Hilbert space and we denote by $\wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ the even elements of the Grassmann algebra $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ (see Expression (2.2.15)). Let $C \in \mathcal{B}(\mathcal{H})$ be an operator and $W_0 \in \wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$. Our purpose is to find an element $W_1 \in \wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ such that

$$\mu_C * e^{-W_0} \doteq e^{\Delta_C} e^{-W_0} = e^{-W_1},$$

where, for all $W \in \wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ the exponential function in Grassmann algebras is given by (2.2.16), i.e.,

$$(5.2.7) \quad e^W \doteq \mathbb{1} + \sum_{k=1}^{\infty} \frac{W^k}{k!}, \quad W \in \wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}}),$$

where W^n is the n -fold product of W with itself in the algebra $\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})$. The above series absolutely converges if the algebra is finite dimensional (see corresponding discussions around (2.2.16)). Note further that, by Theorem 5.2.6 above,

$$P_C(e^{-W_1}) = P_C(\mu_C * e^{-W_0}) = \int d\mu_C(\mathcal{H}) e^{-W_0}.$$

Note that the equality $P_C(e^{-W_1}) = e^{-P_C(W_1)}$ is true and then we have:

$$W_1 = -\log\left(\int d\mu_C(\mathcal{H}) e^{-W_0}\right).$$

We discuss in the present section the so-called “Brydges–Kennedy tree expansions” [AR98, BK87, SW00, BP, ABPM16], which gives — under certain technical conditions — an explicit expression of W_1 in terms of a series in W_0 and C . To arrive at this expansion, we use the Polchinski (differential) equation, the solution $\{W_t\}_{t \in [0,1]}$ of which equals W_0 and W_1 at parameters $t = 0$ and $t = 1$, respectively. Indeed, the existence of a differentiable map $t \mapsto W_t$ from $[0, 1]$ to $\wedge_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ such that

$$(5.2.8) \quad \mu_{tC} * e^{-W_0} = e^{-W_t}, \quad t \in [0, 1],$$

is equivalent to the existence of a solution of the so-called Polchinski initial value problem in the set of differentiable maps from $[0, 1]$ to $\wedge_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$:

Theorem 5.2.9 (Polchinski initial value problem):

A differentiable map $t \mapsto W_t$ from $[0, 1]$ to $\wedge_+^(\mathcal{H} \oplus \tilde{\mathcal{H}})$ is solution of (5.2.8) if and only if it is the solution of the Polchinski initial value problem:*

$$(5.2.10) \quad \forall t \in [0, 1] : \partial_t W_t = \Delta_C W_t + (\nabla W_t, \nabla W_t)_C,$$

where $W_0 \in \wedge_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ is a fixed initial value and

$$(5.2.11) \quad (\nabla A, \nabla B)_C \doteq \sum_{i,j \in I} \psi_i^*(C\psi_j) \left(\frac{\delta}{\delta \psi_i^*} A \right) \left(\frac{\delta}{\delta \psi_j} B \right)$$

for any orthonormal basis $\{\psi_i\}_{i \in I}$ of $\mathcal{H}$ and $A, B \in \wedge_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$. (The last sum does not depend on the particular choice of the orthonormal basis $\{\psi_i\}_{i \in I}$). If the solution $\{W_t\}_{t \in [0,1]}$ exists it is unique. $\clubsuit$

PROOF.

1. If a differentiable map $t \mapsto W_t$ from $[0, 1]$ to $\wedge_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ is solution of (5.2.8), then, by differentiating both sides of (5.2.8) w.r.t. t , one arrives at (5.2.10). Indeed, by Definition 5.2.1, for any $t \in \mathbb{R}$,

$$(5.2.12) \quad \mu_{tC} * e^{-W_0} = e^{t\Delta_C} e^{-W_0}.$$

By finite dimensionality of $\mathcal{H}$, $e^{t\Delta_C}$ is a bounded linear operator which is analytic w.r.t. $t \in \mathbb{R}$, and one explicitly checks that, for any $t \in \mathbb{R}$,

$$(5.2.13) \quad \partial_t (e^{t\Delta_C} e^{-W_0}) = \Delta_C e^{t\Delta_C} e^{-W_0}.$$

Hence, for any $t \in [0, 1]$,

$$(5.2.14) \quad \partial_t (e^{t\Delta_C} e^{-W_0}) = \Delta_C e^{-W_t},$$

Now,

$$(5.2.15) \quad \Delta_C e^{-W_t} = \sum_{i,j \in I} \psi_i^* (C\psi_j) \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\delta}{\delta\psi_j} \frac{\delta}{\delta\psi_i^*} W_t^k \right).$$

On the other hand, by (2.2.21) and because W_t is even, for any $i, j \in I$ and $k \in \mathbb{N}$, $k \geq 2$,

$$\begin{aligned} \frac{\delta}{\delta\psi_j} \frac{\delta}{\delta\psi_i^*} W_t^k &= k \frac{\delta}{\delta\psi_j} \left(\left(\frac{\delta}{\delta\psi_i^*} W_t \right) W_t^{k-1} \right) \\ &= k \left(\frac{\delta}{\delta\psi_j} \frac{\delta}{\delta\psi_i^*} W_t \right) W_t^{k-1} + k(k-1) \left(\frac{\delta}{\delta\psi_j} W_t \right) \left(\frac{\delta}{\delta\psi_i^*} W_t \right) W_t^{k-2}. \end{aligned}$$

By combining this last computation for any $i, j \in I$ and all integers $k \geq 2$ with (5.2.11), (5.2.2) and (5.2.15), we arrive at the equality

$$(5.2.16) \quad \Delta_C e^{-W_t} = -\{\Delta_C W_t + (\nabla W_t, \nabla W_t)_C\} e^{-W_t}$$

for any $t \in [0, 1]$. By (5.2.12) and (5.2.14), it follows that

$$(5.2.17) \quad \partial_t \left\{ \mu_{tC} * e^{-W_0} \right\} = -\{\Delta_C W_t + (\nabla W_t, \nabla W_t)_C\} e^{-W_t}$$

for any $t \in [0, 1]$.

Now, observe from (5.2.7) that

$$(5.2.18) \quad \partial_t \left\{ e^{-W_t} \right\} = -\{\partial_t W_t\} e^{-W_t}, \quad t \in [0, 1],$$

for any differentiable map $t \mapsto W_t$ from $[0, 1]$ to $\wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$. Therefore, we infer from (5.2.17)–(5.2.18) that any differentiable map $t \mapsto W_t$ from $[0, 1]$ to $\wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ which satisfies (5.2.8) is automatically a solution of the Polchinski initial value problem (5.2.10).

2. Suppose now that the (differentiable) map $t \mapsto W_t$ from $[0, 1]$ to $\wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ is solution of the Polchinski initial value problem (5.2.10). Then, by Equations (5.2.12), (5.2.14), (5.2.16), (5.2.18), and $\mu_{tC} * \mu_{-tC}^* = \mathbf{1}_{\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})}, t \in \mathbb{R}$,

$$\partial_t \left\{ \mu_{-tC} * e^{-W_t} \right\} = 0, \quad t \in [0, 1],$$

which implies that

$$\mu_{-tC} * e^{-W_t} = e^{-W_0}, \quad t \in [0, 1].$$

Now, we use again $\mu_{tC} * \mu_{-tC}^* = \mathbf{1}_{\wedge^*(\mathcal{H} \oplus \bar{\mathcal{H}})}, t \in \mathbb{R}$, to arrive at (5.2.8) for any $t \in [0, 1]$.

3. Uniqueness of the solution of the Polchinski initial value problem (5.2.10) is a consequence of the local Lipschitz continuity of the map

$$(5.2.19) \quad \xi \mapsto \Delta_C \xi + (\nabla \xi, \nabla \xi)_C$$

on $\Lambda^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$.

Using the local Lipschitz continuity of the map (5.2.19) and the contraction mapping principle, one can show the existence of a unique differentiable map $t \mapsto W_t$ solving the Polchinski initial value problem (5.2.10) for small times see [BP]. The Brydges–Kennedy construction gives the solution of (5.2.10) via a tree expansion. Under certain conditions to be discussed below, the construction shows the existence of the solution for t on the whole interval $[0, 1]$. This is explained in the next subsections.

5.2.3 BRYDGES–KENNEDY TREE EXPANSIONS

Take $C \in \mathcal{B}(\mathcal{H})$ and $W_0 \in \Lambda_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$, set

$$(5.2.20) \quad W_t^{(1)} \doteq \mu_{tC} * W_0, \quad t \in [0, 1],$$

and consider the following infinite hierarchy of coupled ODEs in $\Lambda_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ with initial values: For all integers $k \geq 2$,

$$(5.2.21) \quad \forall t \in [0, 1]: \quad \partial_t W_t^{(k)} = \Delta_C W_t^{(k)} + \sum_{l, q \in \mathbb{N}: l+q=k} (\nabla W_t^{(l)}, \nabla W_t^{(q)})_C, \quad W_0^{(k)} = 0.$$

This infinite hierarchy of coupled ODEs is indeed directly related to the Polchinski initial value problem (5.2.10):

5.2.22 LEMMA (THE COUPLED ODES AND THE POLCHINSKI INITIAL VALUE PROBLEM). *Let $C \in \mathcal{B}(\mathcal{H})$ be an operator and $W_0 \in \Lambda_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$. Assume the existence of a solution $\{W_t^{(k)}\}_{t \in [0, 1], k \geq 2}$ of (5.2.21) such that the series*

$$(5.2.23) \quad \tilde{W}_t \equiv \sum_{k=1}^{\infty} W_t^{(k)} \doteq \mu_{tC} * W_0 + \sum_{k=2}^{\infty} W_t^{(k)}$$

of continuous functions from $[0, 1]$ to $\Lambda_+^(\mathcal{H} \oplus \tilde{\mathcal{H}})$ is absolutely convergent, uniformly for $t \in [0, 1]$. Then, $\{\tilde{W}_t\}_{t \in [0, 1]}$ is the unique solution of the Polchinski initial value problem (5.2.10).♦*

PROOF. Fix all parameters of the lemma with the corresponding assumptions. Then, by (5.2.21), we obviously have the equality $\tilde{W}_0 = W_0$ at the initial time $t = 0$. Note meanwhile that

$$\forall \xi, \zeta \in \Lambda^*(\mathcal{H} \oplus \tilde{\mathcal{H}}): \quad \xi \mapsto \Delta_C \xi \quad \text{and} \quad (\xi, \zeta) \mapsto (\nabla \xi, \nabla \zeta)_C$$

respectively define linear and bilinear maps and so, by finite dimensionality of $\mathcal{H}$, there is a constant $D \in \mathbb{R}^+$ such that, for all $\xi, \zeta \in \wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$,

$$\|\Delta_C \xi\|_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})} \leq D \|\xi\|_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})}$$

and

$$\|(\nabla \xi, \nabla \zeta)_C\|_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})} \leq D \|\xi\|_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})} \|\zeta\|_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})}.$$

Therefore,

$$\begin{aligned} & \sum_{k=1}^{\infty} \left\| \Delta_C W_t^{(k)} + \sum_{l,q \in \mathbb{N}: l+q=k} (\nabla W_t^{(l)}, \nabla W_t^{(q)})_C \right\|_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})} \\ & \leq D \sum_{k=1}^{\infty} \|W_t^{(k)}\|_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})} \left(1 + \sum_{k=1}^{\infty} \|W_t^{(k)}\|_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})} \right) \end{aligned}$$

and by the hierarchy (5.2.21) of coupled ODEs, the corresponding series of derivatives converges uniformly. Hence, from (5.2.21), one has

$$(5.2.24) \quad \partial_t \tilde{W}_t = \sum_{k=1}^{\infty} \partial_t W_t^{(k)} = \Delta_C \tilde{W}_t + (\nabla \tilde{W}_t, \nabla \tilde{W}_t)_C, \quad t \in [0, 1].$$

In other words, $\{\tilde{W}_t\}_{t \in [0,1]}$ is a solution of the Polchinski initial value problem (5.2.10). The uniqueness of this solution was already stated in the previous theorem. End

The infinite hierarchy (5.2.21) of coupled ODEs with initial values has an explicit solution which is the so-called *Brydges–Kennedy tree expansion* [AR98, BK87, SW00]. To explain the construction, we need first several definitions 1–6:

1. Fix $k \geq 2$ and $t \in [0, 1]$. Again, $C \in \mathcal{B}(\mathcal{H})$ is an operator and as in Chapter 2–Expression (2.2.25) we identify $\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ with $\wedge^*(\mathcal{H}^{(0)} \oplus \tilde{\mathcal{H}}^{(0)})$. Let

$$(5.2.25) \quad \mathcal{G}_k \doteq 2^{\{\{p,q\} : p,q \in \{1, \dots, k\}, p \neq q\}}$$

be the set of all (non-oriented) graphs with k vertices, while $\mathcal{T}_k \subset \mathcal{G}_k$ is the set of trees with $k \geq 2$ vertices, that is, the subset of all minimal (w.r.t. inclusion) connected graphs, see [ABPM16]. Only minimally connected graphs (trees) are relevant for the Brydges–Kennedy tree expansions.

2. Recall that for $k \in \mathbb{N}$, $\mathbf{H}^{(0)} \oplus \bar{\mathbf{H}}^{(0)}$ is defined by (2.2.24), and let

$$(5.2.26) \quad \Theta_k \in \mathcal{B} \left(\wedge^* \left(\mathbf{H}^{(k)} \oplus \bar{\mathbf{H}}^{(k)} \right), \wedge^* \left(\mathcal{H}^{(0)} \oplus \bar{\mathcal{H}}^{(0)} \right) \right)$$

be the unique algebra homomorphism satisfying, for all $p \in \{1, \dots, k\}$ and $\varphi \in \mathcal{H}, \bar{\varphi} \in \bar{\mathcal{H}}$,

$$(5.2.27) \quad \Theta_k \left(\varphi^{(p)} \right) = \varphi \quad \text{and} \quad \Theta_k \left(\bar{\varphi}^{(p)} \right) = \bar{\varphi}.$$

3. Any element $\ell = \{p, q\} \in \mathfrak{g}$ of a graph $\mathfrak{g} \in \mathcal{G}_k$ is named a “line” of $\mathfrak{g}$. For each non-oriented graph $\mathfrak{g}$ with k vertices, all functions $\alpha \in [0, 1]^{\mathfrak{g}}$ and any parameter $v \in [0, 1]$, we define the subgraph

$$(5.2.28) \quad \mathfrak{g}(\alpha, v) \doteq \mathfrak{g} \setminus \{\ell \in \mathfrak{g} : \alpha(\ell) \geq v\} \subset \mathfrak{g}.$$

Let $\mathcal{R}_{\mathfrak{g}(\alpha, v)} \subset \{1, \dots, k\}^2$ denote the smallest equivalence relation for which one has $(p, q) \in \mathcal{R}_{\mathfrak{g}(\alpha, v)}$ for all $p, q \in \{1, \dots, k\}$ such that the line $\{p, q\}$ belongs to the graph $\mathfrak{g}(\alpha, v)$. Then, for any $t \in [0, 1]$, $\mathfrak{M}(\mathfrak{g}, \alpha, t) \in \text{Mat}(k, \mathbb{R})$ is the symmetric positive $k \times k$ real matrix defined by

$$(5.2.29) \quad [\mathfrak{M}(\mathfrak{g}, \alpha, t)]_{p, q} \doteq \int_0^t \mathbf{1}[(p, q) \in \mathcal{R}_{\mathfrak{g}(\alpha, v)}] dv, \quad p, q \in \{1, \dots, k\}.$$

in which $\mathbf{1}[P]$ yields 1 if proposition P is true, and 0 if it is false.

4. For any $\mathfrak{g} \in \mathcal{G}_k$, $\alpha \in [0, 1]^{\mathfrak{g}}$ and $t \in [0, 1]$, we denote by

$$(5.2.30) \quad C(\mathfrak{g}, \alpha, t) \in \mathcal{B}(\mathbf{H}^{(k)})$$

the operator defined by the conditions

$$(5.2.31) \quad C(\mathfrak{g}, \alpha, t) \psi_j^{(q)} \doteq \sum_{p=1}^k \sum_{i \in I} \langle \psi_i, C \psi_j \rangle [\mathfrak{M}(\mathfrak{g}, \alpha, t)]_{p, q} \psi_i^{(p)}$$

for any $q \in \{1, \dots, k\}$ and $j \in I$, where $\{\psi_i\}_{i \in I}$ is any orthonormal basis of $\mathcal{H}$.

5. For any $\mathfrak{g} \in \mathcal{G}_k$, we denote by $\mathfrak{m}_{\mathfrak{g}}$ the uniform probability measure on the set $[0, 1]^{\mathfrak{g}}$ of functions from $\mathfrak{g}$ to $[0, 1]$.
6. For any graph line $\ell = \{p, q\}$ with $p, q \in \{1, \dots, k\}$ and $p \neq q$, the line Laplace operator with covariance C on $\wedge^* (\mathbf{H}^{(k)} \oplus \bar{\mathbf{H}}^{(k)})$ is defined by

$$(5.2.32) \quad \Delta_C^{(\ell)} \equiv \Delta_C^{(\{p, q\})} \doteq \frac{1}{2} \sum_{i, j \in I} \psi_i^* (C \psi_j) \left(\frac{\delta}{\delta \psi_j^{(q)}} \frac{\delta}{\delta \psi_i^{*(p)}} + \frac{\delta}{\delta \psi_j^{(p)}} \frac{\delta}{\delta \psi_i^{*(q)}} \right).$$

Here, $\{\psi_i\}_{i \in I}$ is again any orthonormal basis of $\mathcal{H}$.

We are now in position to define the Brydges–Kennedy tree expansion:

5.2.33 DEFINITION (BRYDGES–KENNEDY TREE EXPANSION). The Brydges–Kennedy tree expansion is defined, for any operator $C \in \mathcal{B}(\mathcal{H})$, initial value $W_0 \in \Lambda_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$, time $t \in [0, 1]$ and every integer $k \geq 2$, by

$$W_t^{(k)} \doteq \frac{1}{k!} \sum_{T \in \mathcal{T}_k} \int_{[0,t]^T} m_T(d\alpha) \Theta_k \left(\mu_{C(T, \alpha, t)} * \left(\prod_{\ell \in T} \Delta_A^{(\ell)} \right) \left(\prod_{p=1}^k \chi_{(0,0)}^{(p,p)}(W_0) \right) \right)$$

with the isomorphisms $\chi_{(0,0)}^{(p,p)}$ defined by (2.2.27) for all $p \in \{1, \dots, k\}$. $\blacksquare$

This expansion gives rise to a solution of the infinite hierarchy (5.2.21) of coupled ODEs in $\Lambda_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ with initial values:

5.2.34 LEMMA (EXPLICIT SOLUTION OF THE INFINITE HIERARCHY OF COUPLED ODEs). For any operator $C \in \mathcal{B}(\mathcal{H})$, initial value $W_0 \in \Lambda_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$, and time $t \in [0, 1]$, the Brydges–Kennedy tree expansion of Definition 5.2.33 is a solution of the infinite hierarchy (5.2.21) of coupled ODEs in $\Lambda_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ with initial values. $\blacklozenge$

PROOF. Fix all the parameters of the lemma. Obviously,

$$(5.2.35) \quad W_0^{(k)} = 0, \quad k \in \{2, 3, \dots, \infty\}.$$

Moreover, how it is defined the homomorphism Θ_k , this is bounded. Then, we will show that the derivative of the Brydges–Kennedy coefficient $W_t^{(k)}$ is the sum of two terms

$$(5.2.36) \quad \partial_t W_t^{(k)} = \mathbf{X}_1 + \mathbf{X}_2, \quad k \in \{2, 3, \dots, \infty\}, \quad t \in [0, 1],$$

that are respectively equal to

$$(5.2.37) \quad \mathbf{X}_1 \doteq \frac{1}{k!} \sum_{T \in \mathcal{T}_k} \int_{[0,t]^T} m_T(d\alpha) \Theta_k \left(\partial_t \mu_{C(T, \alpha, t)} * \left(\prod_{\ell \in T} \Delta_C^{(\ell)} \right) \left(\prod_{p=1}^k \chi_{(0,0)}^{(p,p)}(W_0) \right) \right)$$

and

$$(5.2.38) \quad \mathbf{X}_2 \doteq \frac{1}{k!} \sum_{T \in \mathcal{T}_k} \sum_{\ell \in T} \int_{[0,t]^{T \setminus \{\ell\}}} m_{T \setminus \{\ell\}}(d\alpha) \Theta_k \left(\mu_{C(T, \alpha_{\ell, t}, t)} * \left(\prod_{\tilde{\ell} \in T} \Delta_C^{(\tilde{\ell})} \right) \left(\prod_{p=1}^k \chi_{(0,0)}^{(p,p)}(W_0) \right) \right).$$

Here, for any $t \in [0, 1]$, graph $g \in \mathcal{G}_k$, line $\ell \in g$, and function $\alpha \in [0, 1]^{g \setminus \{\ell\}}$, $\alpha_{\ell, t}$ is the map defined

on $\mathfrak{g}$ by

$$\alpha_\ell(\ell) \doteq t \quad \text{and} \quad \alpha_\ell|_{\mathfrak{g} \setminus \{\ell\}} \doteq \alpha.$$

The first term $\mathbf{X}_1$ in Equation (5.2.36) corresponds to the term $\Delta_C W_t^{(k)}$ in the coupled ODEs (5.2.21), whereas $\mathbf{X}_2$ leads in (5.2.21) to the sum of $(\nabla W_t^{(l)}, \nabla W_t^{(q)})_C$ for $l, q \in \mathbb{N}$ such that $l + q = k$. Before proving these two assertions, we give a trivial observation which is useful in both cases: For any $i, j \in I, l \in \mathbb{N}$, every finite subset $\mathcal{N} \subset \{1, \dots, l\}$ and all elements

$$\xi \in \Lambda^* \left(\mathbf{H}^{(\mathcal{N})} \oplus \bar{\mathbf{H}}^{(\mathcal{N})} \right),$$

note that the following equalities obviously hold true:

$$(5.2.39) \quad \Theta_l \left(\left(\sum_{q \in \mathcal{N}} \frac{\delta}{\delta \psi_j^{(q)}} \right) \xi \right) = \frac{\delta}{\delta \psi_j} \Theta_l(\xi) \quad \text{and} \quad \Theta_l \left(\left(\sum_{p \in \mathcal{N}} \frac{\delta}{\delta \psi_i^{*(p)}} \right) \xi \right) = \frac{\delta}{\delta \psi_i^*} \Theta_l(\xi).$$

We are now in position to study both terms, $\mathbf{X}_1$ and $\mathbf{X}_2$, starting with the first one.

By Definition 5.2.1, for any integer $k \geq 2$, $C \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{T}_k$, $t \in [0, 1]$ and $\alpha \in [0, t]^T$,

$$(5.2.40) \quad \mu_{C(T, \alpha, t)}^* = e^{\Delta_C(T, \alpha, t)},$$

where, by (5.2.30)–(5.2.31),

$$(5.2.41) \quad \Delta_C(T, \alpha, t) = \sum_{i, j \in I} \psi_i^*(C \psi_j) \sum_{p=1}^k \sum_{q=1}^k [\mathfrak{M}(T, \alpha, t)]_{p, q} \frac{\delta}{\delta \psi_j^{(q)}} \frac{\delta}{\delta \psi_i^{*(p)}}.$$

In particular, because trees $T \in \mathcal{T}_k$ are (minimally) connected graphs, we infer from (5.2.29) and (5.2.40)–(5.2.41) that, for any $t \in [0, 1]$ and $\alpha \in [0, t]^T$,

$$(5.2.42) \quad \partial_t \mu_{C(T, \alpha, t)}^* = \sum_{i, j \in I} \psi_i^*(C \psi_j) \left(\sum_{q=1}^k \frac{\delta}{\delta \psi_j^{(q)}} \right) \left(\sum_{p=1}^k \frac{\delta}{\delta \psi_i^{*(p)}} \right) \mu_{C(T, \alpha, t)}^*.$$

Compare (5.2.40)–(5.2.42) with Equations (5.2.12)–(5.2.13). By Equations (5.2.39) and (5.2.42) together with the fact that

$$\mathfrak{m}_T([0, t]^T \setminus [0, t]^T) = 0, \quad t \in [0, 1], T \in \mathcal{T}_k,$$

the first term of the r.h.s. of (5.2.36) is therefore equal to $\Delta_C W_t^{(k)}$, that is,

$$(5.2.43) \quad \mathbf{X}_1 = \Delta_C W_t^{(k)}, \quad k \in \{2, 3, \dots, \infty\}, \quad t \in [0, 1].$$

See (5.2.37) and again Definition 5.2.1–1.

Concerning the second term $\mathbf{X}_2$ of the r.h.s. of (5.2.36), observe that, for any $k \in \{2, 3, \dots, \infty\}$, tree $T \in \mathcal{T}_k$ and line $\ell \in T$, the new graph

$$(5.2.44) \quad T \setminus \{\ell\} = T_1 \cup T_2, \quad T_1 \cap T_2 = \emptyset,$$

is the union of two disconnected subtrees and that $C(T, \alpha_{\ell, t}, t)$ preserves the subspaces related to the two corresponding clusters of $\mathcal{C}_{T \setminus \{\ell\}}$, because $\alpha_{\ell}(\ell) = t$. See (5.2.28). In particular,

$$(5.2.45) \quad \mu_{C(T, \alpha_{\ell, t}, t)}^* = \mu_{C(T_1, \alpha_{\ell, t}, t)}^* * \mu_{C(T_2, \alpha_{\ell, t}, t)}^*, \quad T \in \mathcal{T}_k, \ell \in T.$$

Now, let $\mathcal{T}_{\mathcal{N}}$ denote the set of all trees constructed from the finite set $\mathcal{N} \subset \mathbb{N}$ of vertices and $\mathcal{V}(T_1), \mathcal{V}(T_2) \subset \{1, \dots, k\}$ be the sets of vertices of T_1 and T_2 , respectively. We recall here that T_1, T_2 are the subtrees defined for any $T \in \mathcal{T}_k$ and $\ell \in T$ via Condition (5.2.44). Then, using (2.2.20) and rewriting in (5.2.38) the sum

$$\sum_{T \in \mathcal{T}_k} \sum_{\ell \in T} \quad \text{as} \quad \sum_{l, q \in \{1, \dots, k\}} \sum_{T \in \mathcal{T}_k : T \ni (l, q)},$$

we arrive at the equality

$$(5.2.46) \quad \begin{aligned} \mathbf{X}_2 &= \frac{1}{k!} \sum_{l, q \in \{1, \dots, k\}} \sum_{\mathcal{N}_1, \mathcal{N}_2 \subset \mathbb{N}} \mathbf{1}[\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset] \mathbf{1}[\mathcal{N}_1 \cup \mathcal{N}_2 = \{1, \dots, k\}] \\ &\quad \mathbf{1}[l \in \mathcal{N}_1] \mathbf{1}[q \in \mathcal{N}_2] \sum_{T_1 \in \mathcal{T}_{\mathcal{N}_1}} \sum_{T_2 \in \mathcal{T}_{\mathcal{N}_2}} \Theta_k(\Delta_C^{(l, q)}) \\ &\quad \int_{[0, t]^{T_1}} m_{T_1}(\mathrm{d}\alpha) \mu_{C(T_1, \alpha_{(l, q), t}, t)}^* \left(\prod_{\ell \in T_1} \Delta_C^{(\ell)} \right) \left(\prod_{p \in \mathcal{V}(T_1)} \kappa_{(0,0)}^{(p, p)}(W_0) \right) \\ &\quad \int_{[0, t]^{T_2}} m_{T_2}(\mathrm{d}\alpha) \mu_{C(T_2, \alpha_{(l, q), t}, t)}^* \left(\prod_{\ell \in T_2} \Delta_C^{(\ell)} \right) \left(\prod_{p \in \mathcal{V}(T_2)} \kappa_{(0,0)}^{(p, p)}(W_0) \right). \end{aligned}$$

Recall that Θ_k is an algebra homomorphism defined via (5.2.26)–(5.2.27). Then, by (5.2.11), (5.2.32), (5.2.39) and Definition 5.2.33, it follows that

$$\mathbf{X}_2 = \sum_{\mathcal{N}_1, \mathcal{N}_2 \subset \mathbb{N}} \frac{|\mathcal{N}_1|! |\mathcal{N}_2|!}{k!} \mathbf{1}[\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset] \mathbf{1}[\mathcal{N}_1 \cup \mathcal{N}_2 = \{1, \dots, k\}] (\nabla W_t^{(l, \mathcal{N}_1)}, \nabla W_t^{(l, \mathcal{N}_2)})_C,$$

which, from some elementary combinatorics, is equal to

$$(5.2.47) \quad \mathbf{X}_2 = \sum_{l, q \in \mathbb{N} : l+q=k} (\nabla W_t^{(l)}, \nabla W_t^{(q)})_C, \quad k \in \{2, 3, \dots, \infty\}, \quad t \in [0, 1].$$

By (5.2.35), (5.2.36), (5.2.43) and (5.2.47), the Brydges–Kennedy tree expansion is a solution of the infinite hierarchy (5.2.21) of coupled ODEs in $\wedge_+^*(\mathcal{H} \oplus \bar{\mathcal{H}})$ with initial values. End

By Lemma 5.2.34, the Brydges–Kennedy tree expansion of Definition 5.2.33 is a solution of the infinite hierarchy (5.2.21), but it leads to a series (5.2.23) of continuous functions on $[0, 1]$ which *may not converge uniformly*. The uniform convergence of the series is important to ensure the existence of the unique solution of the Polchinski initial value problem (5.2.10) for t on the whole interval $[0, 1]$, as explained in the beginning of this section. This is proven in the next subsection under additional assumptions.

5.3 ABSOLUTE CONVERGENCE OF THE BRYDGES–KENNEDY SERIES

5.3.1 INTERACTION KERNELS

In the sequel, it will be convenient to describe interactions in terms of interaction kernels. This requires some preliminary definitions.

For convenience, in contrast to Chapter 2–§§2.3, we consider the set of Spins S , such that we define $\mathfrak{Q} \doteq S \times \mathbb{Z}^d$. Let $\mathbb{X}_{\mathfrak{Q}} \doteq \{+, -\} \times \mathfrak{Q}$, with elements denoted by $X = (v, s, x)$ while $\bar{X} \doteq (\bar{v}, s, x)$ with the convention $\bar{+} \doteq -$ and $\bar{-} \doteq +$. Then “interaction kernels” are defined as follows:

5.3.1 DEFINITION (INTERACTION KERNELS). An interaction kernel is a family $\varphi \doteq \{\varphi_n\}_{n \in 2\mathbb{N}_0} \in \mathcal{V}(\mathbb{X}_{\mathfrak{Q}})$ of antisymmetric functions $\varphi_n: \mathbb{X}_{\mathfrak{Q}}^n \rightarrow \mathbb{C}$ satisfying the selfadjointness property: $\varphi_0 \in \mathbb{R}$ and for any $n \in 2\mathbb{N}$,

$$\varphi_n(X_1, \dots, X_n) = \overline{\varphi_n(\bar{X}_n, \dots, \bar{X}_1)}, \quad X_1, \dots, X_n \in \mathbb{X}_{\mathfrak{Q}}.$$

We say that the interaction kernel φ is translation invariant if, for all $n \in 2\mathbb{N}$, all $x, x_1, \dots, x_n \in \mathbb{Z}^d$, all $s_1, \dots, s_n \in S$, and all $v_1, \dots, v_n \in \{+, -\}$,

$$\varphi_n((v_1, s_1, x + x_1), \dots, (v_n, s_n, x + x_n)) = \varphi_n((v_1, s_1, x_1), \dots, (v_n, s_n, x_n)). \quad \blacksquare$$

The set of all interaction kernels is a real vector space w.r.t. the pointwise sum and multiplication with scalars.

Any interaction kernel φ can be associated with an interaction $\Phi(\varphi) = \{\Phi(\varphi)_{\Lambda}\}_{\Lambda \in \mathcal{P}_f(\mathfrak{Q})}$ defined by

$$\Phi(\varphi)_{\Lambda} \doteq \sum_{n \in 2\mathbb{N}} \sum_{\{X_i = (v_i, s_i, x_i) \in \mathbb{X}_{\mathfrak{Q}}\}_{i=1}^n, \{x_1, \dots, x_n\} = \Lambda} \varphi_n(X_1, \dots, X_n) : a_{(s_1, x_1)}^{v_1} \dots a_{(s_n, x_n)}^{v_n} : \quad (5.3.2)$$

Here, $: a_{(\mathfrak{s}_1, x_1)}^{v_1} \cdots a_{(\mathfrak{s}_n, x_n)}^{v_n} :$ stands for the normal ordered product

$$(5.3.3) \quad (-1)^\pi a_{(\mathfrak{s}_{\pi(1)}, x_{\pi(1)})}^{v_{\pi(1)}} \cdots a_{(\mathfrak{s}_{\pi(n)}, x_{\pi(n)})}^{v_{\pi(n)}}$$

defined via any permutation $\pi \in \mathcal{S}_n$ of the set $\{1, \dots, n\}$ moving all creation operators in the product to the left of all annihilation operators. This permutation is of course not unique. The operator defined by the normal ordering is nevertheless uniquely defined because of the factor $(-1)^\pi$ in (5.3.3) and because of the CAR. The map $\varphi \mapsto \Phi(\varphi)$ is not injective and hence, the choice of kernels $\{\varphi_n\}$ for a given interaction Φ is not unique.

If an interaction is “short range” then it can be associated to the generator of a strongly continuous group of automorphisms of the CAR C^* -algebra $\mathcal{U}$. The short range character of an interaction Φ corresponds to its finiteness w.r.t. a suitable norm for interactions. We hence define short range interaction kernels via conveniently chosen norms in the space $\mathcal{V}(\mathbb{X}_{\mathfrak{G}})$. To this end, we introduce a decay kernel $\mathfrak{d}$ on the disjoint union

$$(5.3.4) \quad \bigcup_{n \in 2\mathbb{N}} \mathbb{X}_{\mathfrak{G}}^n.$$

The following properties are assumed for $\mathfrak{d}$: for all $n \in \mathbb{N}$ and $X_1, \dots, X_n \in \mathbb{X}_{\mathfrak{G}}$,

- *Strict positivity*:

$$(5.3.5) \quad \mathfrak{d}(X_1, \dots, X_n) > 0.$$

- *Boundedness*:

$$(5.3.6) \quad \mathfrak{d}_{\max} \doteq \sup_{n \in \mathbb{N}} \sup_{X_1, \dots, X_n \in \mathbb{X}_{\mathfrak{G}} \in \mathcal{J}} |\mathfrak{d}(X_1, \dots, X_n)| < \infty.$$

- *Symmetry*: For any permutation $\pi \in \mathcal{S}_n$ of $n \in \mathbb{N}$ elements,

$$(5.3.7) \quad \mathfrak{d}(X_{\pi(1)}, \dots, X_{\pi(n)}) = \mathfrak{d}(X_1, \dots, X_n).$$

- *Supermultiplicativity*: For any $m \in \{1, \dots, n\}$,

$$(5.3.8) \quad \mathfrak{d}(X_1, \dots, X_n) \geq \mathfrak{d}(X_1, \dots, X_m) \mathfrak{d}(X_{m+1}, \dots, X_n).$$

This condition implies in particular that $\mathfrak{d}(X) = \mathfrak{d}(X) = 1$ for all $X \in \mathbb{X}_{\mathfrak{G}}$.

- *Two-point monotonicity*: For any $m_1, m_2 \in \{1, \dots, n\}$,

$$(5.3.9) \quad \mathfrak{d}(X_{m_1}, X_{m_2}) \geq \mathfrak{d}(X_1, \dots, X_n).$$

The simplest example of such $\mathfrak{d}$ is the constant function $\mathfrak{d} = 1$. A general procedure giving non-trivial examples of decay kernels which fulfill (5.3.5)–(5.3.8) is explained in the following and uses so-called “tree distances”.

Let $\mathbf{d}: \mathbb{X}_{\mathfrak{Q}}^2 \rightarrow \mathbb{R}^+$ be a pseudometric (i.e., $\mathbf{d}$ satisfies all axioms of a metric except that $\mathbf{d}(X_1, X_2) = 0$ may not yield $X_1 = X_2$) and $\mathbf{F}: \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ a non-increasing function satisfying

$$\mathbf{F}(r_1 + r_2) \geq \mathbf{F}(r_1) \mathbf{F}(r_2), \quad r_1, r_2 \in \mathbb{R}_0^+,$$

like the example given below Equation (2.3.43).

Then, extend $\mathbf{d}$ to the disjoint union (5.3.4) by defining

$$(5.3.10) \quad \mathbf{d}(X) \doteq 0 \quad \text{and} \quad \mathbf{d}(X_1, \dots, X_n) \doteq \min_{T \in \mathcal{T}_n} \sum_{\{q_1, q_2\} \in T} \mathbf{d}(X_{q_1}, X_{q_2})$$

for any $X \in \mathbb{X}_{\mathfrak{Q}}$, $n \in 2\mathbb{N}$ and $(X_1, \dots, X_n) \in \mathbb{X}_{\mathfrak{Q}}^n$. The quantity $\mathbf{d}(X_1, \dots, X_n)$ is named here the *tree distance* of the sequence $X_1, \dots, X_n$ associated to the pseudometric. A decay kernel $\mathfrak{d}$ can be then defined by

$$(5.3.11) \quad \mathfrak{d}(X_1, \dots, X_n) = \mathbf{F}(\mathbf{d}(X_1, \dots, X_n)).$$

for any $n \in 2\mathbb{N}$ and $X_1, \dots, X_n \in \mathbb{X}_{\mathfrak{Q}}$. $\mathfrak{d}$ defined this way satisfies the assumptions (5.3.5)–(5.3.8). To see this, use the following straightforward properties of tree distances:

5.3.12 LEMMA (PROPERTIES OF TREE DISTANCES). *Let $n \in \mathbb{N}$ and $X_1, \dots, X_n \in \mathbb{X}_{\mathfrak{Q}}$.*

1. *Positivity:*

$$\mathbf{d}(X_1, \dots, X_n) \geq 0.$$

2. *Symmetry:* For any permutation $\pi \in \mathcal{S}_n$ of n elements,

$$\mathbf{d}(X_{\pi(1)}, \dots, X_{\pi(n)}) = \mathbf{d}(X_1, \dots, X_n).$$

3. *Triangle inequality:* For any $m \in \{1, \dots, n\}$,

$$\mathbf{d}(X_1, \dots, X_n) \leq \mathbf{d}(X_1, \dots, X_m) + \mathbf{d}(X_m, \dots, X_n).$$

4. *Two-point monotonicity:* For any $m_1, m_2 \in \{1, \dots, n\}$,

$$\mathbf{d}(X_{m_1}, X_{m_2}) \leq \mathbf{d}(X_1, \dots, X_n).$$

♦

In the sequel, we will use the following pseudo metric on $\mathbb{X}_{\mathfrak{Q}}$:

5.3.13 DEFINITION (PSEUDO METRIC ON $\mathbb{X}_{\mathfrak{Q}}$). For all $(v_1, s_1, x_1), (v_2, s_2, x_2) \in \mathbb{X}_{\mathfrak{Q}} = \{+, -\} \times S \times \mathbb{Z}^d$,

$$\mathbf{d}((v_1, s_1, x_1), (v_2, s_2, x_2)) \doteq |x_1 - x_2|_2,$$

where $|x_1 - x_2|_2$ is the usual Euclidean distance from x_1 to x_2 in $\mathbb{Z}^d$. ▀

We introduce new norms, depending now on general decay kernels $\mathfrak{d}$, on $\mathcal{V}(\mathbb{X}_{\mathfrak{Q}})$:

5.3.14 DEFINITION (DECAY NORM ON $\mathbf{V}_n(\mathbb{X}_{\mathfrak{Q}})$). Let w be the discrete measure on $\mathbb{X}_{\mathfrak{Q}}$ given by

$$w(\{X\}) = 1, \quad X \in \mathbb{X}_{\mathfrak{Q}}.$$

Then define

$$\|\varphi_0\|_{\mathfrak{d}, \mathbf{V}_0} \doteq |\varphi_0|, \quad \varphi_0 \in \mathbf{V}_0 = \mathbb{C},$$

whereas, for any integer $n \in 2\mathbb{N}$ and all functions $\varphi_n \in \mathbf{V}_n(\mathbb{X}_{\mathfrak{Q}})$,

$$\|\varphi_n\|_{\mathfrak{d}, \mathbf{V}_n} \doteq \max_{m \in \{1, \dots, n\}} \sup_{X_m \in \mathbb{X}_{\mathfrak{Q}}} \sum_{X_1, \dots, \check{X}_m, \dots, X_n \in \mathcal{J}} \left(\prod_{q=1, q \neq m}^n w(X_q) \right) \frac{|\varphi_n(X_1, \dots, X_n)|}{\mathfrak{d}(X_1, \dots, X_n)}. \quad \blacksquare$$

For fixed $\epsilon \in \mathbb{R}^+$ and any $\varphi \doteq (\varphi_n)_{n \in 2\mathbb{N}_0} \in \mathbf{V}(\mathbb{X}_{\mathfrak{Q}})$,

$$\|\varphi\|_{\mathfrak{d}}^{(\epsilon)} \doteq \sum_{n \in 2\mathbb{N}_0} \epsilon^n \|\varphi_n\|_{\mathfrak{d}, \mathbf{V}_n}, \quad \varphi = (\varphi_n)_{n \in 2\mathbb{N}_0} \in \mathbf{V}(\mathbb{X}_{\mathfrak{Q}}).$$

In the special case $\mathfrak{d} = 1$ we use the $\|\varphi\|_{\mathfrak{d}}^{(\epsilon)}$ for the norm $\|\varphi\|_{\mathfrak{d}}^{(\epsilon)}$.

W.r.t. the norm $\|\varphi\|_{\mathfrak{d}}^{(\epsilon)}$ introduced above, the set of all interaction kernels which are finite w.r.t. the norm $\|\cdot\|_{\mathfrak{d}}^{(\epsilon)}$ forms a real separable Banach space denoted by $\mathcal{K}_{\mathfrak{d}} \equiv (\mathcal{K}_{\mathfrak{d}}, \|\cdot\|_{\mathcal{K}_{\mathfrak{d}}})$, where $\|\cdot\|_{\mathcal{K}_{\mathfrak{d}}}$ denotes the restriction of the norm $\|\cdot\|_{\mathfrak{d}}^{(\epsilon)}$ to the subspace $\mathcal{K}_{\mathfrak{d}} \subset \mathcal{V}(\mathbb{X}_{\mathfrak{Q}}) \subset \mathbf{V}(\mathbb{X}_{\mathfrak{Q}})$. Similar to the notation for the norms, $\mathcal{K}$ stands for $\mathcal{K}_{\mathfrak{d}}$ in the special case $\mathfrak{d} = 1$.

We have the following relation between the norms of the space $\mathcal{K}_{\mathfrak{d}}$ of interaction kernels and the norms of the space $\mathcal{W}$ of interactions:

5.3.15 LEMMA (RELATIONSHIP BETWEEN NORMS IN $\mathcal{K}$ AND $\mathcal{W}$). For all $\epsilon > 1$ and any interaction kernel $\varphi \in \mathcal{K}$,

$$\|\Phi(\varphi)\|_{\mathcal{W}} \leq \frac{2\epsilon^{-\frac{1}{\ln \epsilon}}}{\ln \epsilon} |\mathcal{S}| \|\varphi\|_{\mathcal{K}} < \infty. \quad \blacklozenge$$

PROOF. Let $\varphi \in \mathcal{H}$. Then, by (2.3.44), (5.3.9) and (5.3.2),

$$\begin{aligned}
& \sum_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d), \Lambda \supset \{x, y\}} \frac{\|\Phi_\Lambda(\varphi)\|_{\mathcal{H}}}{F(|x-y|)} \\
& \leq \sum_{n \in 2\mathbb{N}} \sum_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d), \Lambda \supset \{x\}} \sum_{\{X_i=(v_i, s_i, x_i) \in \mathbb{X}_{\mathbb{Q}}\}_{i=1}^n, \{x_1, \dots, x_n\}=\Lambda} \frac{|\varphi_n(X_1, \dots, X_n)|}{\mathfrak{d}(X_1, \dots, X_n)} \\
& \leq \sum_{n \in 2\mathbb{N}} 2|S| n \max_{m \in \{1, \dots, n\}} \sup_{X_m \in \mathbb{X}_{\mathbb{Q}}} \sum_{X_1, \dots, X_m, \dots, X_n \in \mathbb{X}_{\mathbb{Q}}} \frac{|\varphi_n(X_1, \dots, X_n)|}{\mathfrak{d}(X_1, \dots, X_n)} \\
& \leq 2|S| \max_{n \in 2\mathbb{N}_0} \{\epsilon^{-n} n\} \sum_{n \in 2\mathbb{N}} \epsilon^n \|\varphi_n\|_{\mathfrak{d}, \mathbf{v}_n},
\end{aligned}$$

from which we deduce the assertion. End

Because the above estimate, by Theorem 2.3.69, we can associate with any interaction kernel $\varphi \in \mathcal{H}$ an infinite volume dynamics, provided $\epsilon > 1$.

5.3.2 ASSUMPTIONS AND BRYDGES–KENNEDY THEOREM

In Section 5.2.3 we proved that the Brydges–Kennedy tree expansion of Definition 5.2.33 is a solution of the infinite hierarchy (5.2.21). In order to obtain the unique solution of the Polchinski initial value problem (5.2.10), by Lemma 5.2.22, we have now to show that it leads to a (Brydges–Kennedy) series (5.2.23) of continuous functions on $[0, 1]$ which converges uniformly. To prove the uniform convergence, we need the following assumption on the operator $C \in \mathcal{B}(\mathcal{H})$:

Assumption 5.3.16 (Determinant bound).

There exists a constant $\gamma_C \in \mathbb{R}^+$ such that, for any integer $k \geq 2$, $N, M \in \mathbb{N}_0$, $\bar{\varphi}_1, \dots, \bar{\varphi}_N \in \mathcal{H}$, $\varphi_{N+1}, \dots, \varphi_{N+M} \in \mathcal{H}$, such that $\|\bar{\varphi}_1\|_{\mathcal{H}} = \dots = \|\bar{\varphi}_N\|_{\mathcal{H}} = \|\varphi_{N+1}\|_{\mathcal{H}} = \dots = \|\varphi_{N+M}\|_{\mathcal{H}} = 1$, time $t \in [0, 1]$, graph $\mathfrak{g} \in \mathcal{G}_k$, and function $\alpha \in [0, 1]^{\mathfrak{g}}$,

$$\left| \int d\mu_{C(\mathfrak{g}, \alpha, t)} (\mathbf{H}^{(k)}) (\bar{\varphi}_1)^{(k_1)} \dots (\bar{\varphi}_N)^{(k_N)} \varphi_{N+M}^{(k_{N+M})} \dots \varphi_{N+1}^{(k_{N+1})} \right| \leq \delta_{N,M} \gamma_C^N$$

with $C(\mathfrak{g}, \alpha, t)$ defined from $C \in \mathcal{B}(\mathcal{H})$ by (5.2.31), see remark 4.2.8. ♣

This assumption is directly related to determinant bound of Corollary 4.2.13. Indeed, by Lemma 5.2.5, Theorem 5.2.6, (5.2.29) and (5.2.31), we arrive at:

$$\begin{aligned}
(5.3.17) \quad & \int d\mu_{C(\mathfrak{g}, \alpha, t)} (\mathbf{H}^{(k)}) (\bar{\psi}_1)^{(k_1)} \dots (\bar{\psi}_N)^{(k_N)} \psi_{N+M}^{(k_{N+M})} \dots \psi_{N+1}^{(k_{N+1})} = \\
& \delta_{N,M} \det \left[[\mathfrak{M}(\mathfrak{g}, \alpha, t)]_{j_q, j_{N+l}} \psi_{j_q}^* (C \psi_{j_{N+l}}) \right]_{q,l=1}^N.
\end{aligned}$$

In particular, in the case $\mathcal{H}$ and $C \in \mathcal{B}(\mathcal{H})$ are respectively replaced by the finite dimensional space $\mathbf{H}^{(n)}$ and $C_h^{(n)} \in \mathcal{B}(\mathbf{H}^{(n)})$, we observe from (5.2.29) and Corollary 4.2.13 that the fermionic

covariance $C_h^{(n)}$ satisfies Assumption 5.3.16 with $\gamma_{C_h^{(n)}} = 1$ for any one-particle Hamiltonian $h = h^* \in \mathcal{B}(\mathcal{H})$. Note that this makes this hypothesis relevant for our purposes.

We are now in position to bound the Brydges–Kennedy terms of Definition 5.2.33 so as to give afterwards, in Corollary 5.3.20, the conditions ensuring the uniform convergence of the Brydges–Kennedy series. Let $\{\psi_i\}_{i \in \mathcal{J}}$ be a fixed orthonormal basis of the Hilbert space $\mathbb{H}$. For any $C \in \mathcal{B}(\mathcal{H})$, define the quantity

$$(5.3.18) \quad \|C\|_{1,\infty} \doteq \max \left\{ \max_{i \in \mathcal{J}} \sum_{j \in \mathcal{J}} w(\{j\}) |\psi_i^*(C\psi_j)|, \max_{j \in \mathcal{J}} \sum_{i \in \mathcal{J}} w(\{i\}) |\psi_i^*(C\psi_j)| \right\}.$$

Theorem 5.3.19:

Fix $W_0 \in \Lambda_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ and $C \in \mathcal{B}(\mathcal{H})$ satisfying Assumption 5.3.16. Then, for any integer $k \geq 2$, $t \in [0, 1]$ and $\epsilon > \gamma_C$, the k -th Brydges–Kennedy term satisfies

$$|P_C W_t^{(k)}| \leq \frac{2w(\mathcal{J})}{k-1} (4\epsilon^{-2} \|C\|_{1,\infty})^{k-1} (\|W_0\|^{(4\epsilon)})^k$$

while

$$\left\| (\mathbf{1}_{\Lambda^*(\mathbb{H} \oplus \tilde{\mathbb{H}})} - P_C) W_t^{(k)} \right\|^{(\epsilon - \gamma_C)} \leq \frac{1}{k(k-1)} (4\epsilon^{-2} \|C\|_{1,\infty})^{k-1} (\|W_0\|^{(4\epsilon)})^k. \quad \spadesuit$$

5.3.20 COROLLARY. Fix $W_0 \in \Lambda_+^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ and $C \in \mathcal{B}(\mathcal{H})$ satisfying Assumption 5.3.16. Assume that, for some $\epsilon > \gamma_C$,

$$(5.3.21) \quad 4\epsilon^{-2} \|C\|_{1,\infty} \|W_0\|^{(4\epsilon)} < 1.$$

1. The Brydges–Kennedy tree expansion

$$(5.3.22) \quad W_t \doteq \sum_{k=1}^{\infty} W_t^{(k)}, \quad t \in [0, 1],$$

is absolutely convergent, uniformly for $t \in [0, 1]$, and it is the unique solution of the Polchinski initial value problem (5.2.10).

2. For any $t \in [0, 1]$, $e^{-t\Delta_C} e^{-W_0} = e^{-W_t}$.

3. For any $t \in [0, 1]$,

$$|P_{\mathbb{C}} W_t| \leq 2w(\mathcal{J}) \frac{\|W_0\|^{(4\epsilon)}}{1 - (4\epsilon^{-2} \|C\|_{1,\infty}) \|W_0\|^{(4\epsilon)}}$$

while

$$\left\| \left(\mathbf{1}_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})} - P_{\mathbb{C}} \right) W_t \right\|^{(\epsilon - \gamma_C)} \leq \frac{\|W_0\|^{(4\epsilon)}}{1 - (4\epsilon^{-2} \|C\|_{1,\infty}) \|W_0\|^{(4\epsilon)}}. \quad \blacksquare$$

PROOF. Fix all the parameters of the corollary with the corresponding assumptions. Then, because of (5.3.21), the absolute convergence of the Brydges–Kennedy tree expansion, which turns out to be uniform for times $t \in [0, 1]$, is directly deduced from Theorem 5.3.19 by using the norm $\|\cdot\|^{(\epsilon-1)}$ on $\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$. Since $\mathcal{H}$ has finite dimension, the same holds true in the (canonical) norm $\|\cdot\|_{\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})}$ of the fermionic Fock space $\wedge^*(\mathcal{H} \oplus \tilde{\mathcal{H}})$ and, by Lemmata 5.2.22 and 5.2.34, the Brydges–Kennedy tree expansion is the unique solution of the Polchinski initial value problem (5.2.10).

As a consequence, Assertion 1. holds true. Then, 2. and 3. straightforwardly results from Theorems 5.3.19 and 5.2.9, respectively. End

The proof of Theorem 5.3.19 is given in [ABPM16].

5.4 APPLICATION TO GENERATING FUNCTIONS AT WEAK COUPLING

In this section we finally prove existence and analiticity of generating functions associated to fermions which interact weakly and moreover such an interaction is finite range and translationally invariant. As discussed at the beginning of this Chapter we will assume that for the sequence of boxes (2.3.21) $\Lambda_{L_0} \subset \Lambda_{L_i} \subset \Lambda_{L_f}$ where for $p = \{f, i, o\}$, $L_p \in \mathbb{R}^+$. Thus, from now on we will consider for $p = \{f, i, o\}$ the Hilbert space $\mathcal{H}_p$ with CAR C^* -algebra associated $\mathcal{U}_p \equiv \mathcal{U}(\mathcal{H}_{\Lambda_p})$. From (5.1.5)–(5.1.6), we notice now that in order to prove the existence of generating functions, it is enough to show that for $s \in \mathbb{R}$ and $(L_f, L_i, L_o) \rightarrow (\infty, \infty, \infty)$ the function defined by

$$g_{L_o, L_i, L_f}(s) \doteq \frac{1}{|\Lambda_{L_o}|} \ln \frac{\text{tr} \left(e^{-\beta(d\langle Q, h_{L_f} Q \rangle + W_{L_i})} e^{sK_{L_o}} \right)}{\text{tr} \left(e^{-\beta d\langle Q, h_{L_f} Q \rangle} \right)}$$

exists, see discussions and definitions of the symbols around (5.1.6). Before presenting our first result we provide the following definition:

5.4.1 DEFINITION. Fix $\beta \in \mathbb{R}^+$. For all $L, n \in \mathbb{N}$ define the finite set

$$\mathbf{Y}_{\mathcal{Q}}(L, n) \doteq \mathbb{T}_n \times \{+, -\} \times \mathcal{S} \times \Lambda_L,$$

where

$$\mathbb{T}_n \doteq \left\{ (k - n_{\beta} + 1) n^{-1} \beta : k \in \{0, \dots, 2n_{\beta} - 1\} \right\} \subset (-\beta + 1, \beta + 1]$$

with $n_{\beta} \doteq n + \lfloor n/\beta \rfloor$. The set $\mathbf{Y}_{\mathcal{Q}}(L, n)$, $L, n \in \mathbb{N}$, is endowed with the discrete measure w defined by

$$w(\Omega) \doteq \frac{\beta}{n} |\Omega|, \quad \Omega \subset \mathbf{Y}_{\mathcal{Q}}(L, n),$$

where $|\Omega|$ is the cardinality of Ω . I.e., single points $X \in \mathbf{Y}_{\mathcal{Q}}(L, n)$ have measure

$$w(\{X\}) = \frac{\beta}{n} > 0. \quad \blacksquare$$

We introduce the Hilbert space $\mathcal{H}_{\mathbf{f}} \equiv \mathcal{H}_{L_{\mathbf{f}}}$ for the box $\Lambda_{\mathbf{f}} \equiv \Lambda_{L_{\mathbf{f}}}$ as

$$\mathcal{H}_{\mathbf{f}} \doteq \bigoplus_{x \in \Lambda_{\mathbf{f}}} \mathcal{H}_{\mathcal{S}, x},$$

where $\mathcal{H}_{\mathcal{S}, x}$ is a copy of the Hilbert space $\mathcal{H}_{\mathcal{S}}$ in the point x . An orthonormal basis (ONB) of $\mathcal{H}_{\mathbf{f}}$ is given by $\{\psi_{\mathcal{S}, x}\}_{(\mathcal{S}, x) \in \mathcal{S} \times \Lambda_{\mathbf{f}}}$, such that an ONB of $\mathcal{H}_{\mathcal{S}, x}$ is given by $\{\psi_{\mathcal{S}, x}\}_{\mathcal{S} \in \mathcal{S}}$ for all $x \in \Lambda_{\mathbf{f}}$. Because of Definition given by Expression (2.2.24) with $n \rightarrow n_{\beta}, \beta \in \mathbb{R}^+$ and $\mathcal{H} \rightarrow \mathcal{H}_{\mathbf{f}}$ we also define

$$\mathbf{H}_{\mathbf{f}}^{(n_{\beta})} \oplus \bar{\mathbf{H}}_{\mathbf{f}}^{(n_{\beta})} \doteq \bigoplus_{k=0}^{n_{\beta}-1} \left(\mathcal{H}_{\mathbf{f}}^{(k)} \oplus \bar{\mathcal{H}}_{\mathbf{f}}^{(k)} \right).$$

Similar to introduction of §5.3, let $\mathbb{X}_{\Lambda_{\mathbf{f}}} \doteq \{+, -\} \times \mathcal{S} \times \Lambda_{\mathbf{f}}$, with elements denoted by $X = (v, \mathcal{S}, x)$ while $\bar{X} \doteq (\bar{v}, \mathcal{S}, x)$ with the convention $\bar{+} \doteq -$ and $\bar{-} \doteq +$. Hence, the element $\psi(X) \in \wedge^* (\mathcal{H}_{\mathbf{f}} \oplus \bar{\mathcal{H}}_{\mathbf{f}})$ is provided via

$$\psi(+, \mathcal{S}, x) \doteq \bar{\psi}_{\mathcal{S}, x} \quad \text{and} \quad \psi(-, \mathcal{S}, x) \doteq \psi_{\mathcal{S}, x}.$$

Similarly, for any $\psi(Y) \in \wedge^* \left(\mathbf{H}_{\mathbf{f}}^{(n_{\beta})} \oplus \bar{\mathbf{H}}_{\mathbf{f}}^{(n_{\beta})} \right)$, the elements are provided via $Y \in \mathbf{Y}_{\mathcal{Q}}(L_{\mathbf{f}}, n)$ such that for $k \in \{0, \dots, n_{\beta} - 1\}$

$$\psi(k, +, \mathcal{S}, x) \doteq \bar{\psi}_{\mathcal{S}, x}^{(k)} \quad \text{and} \quad \psi(k, -, \mathcal{S}, x) \doteq \psi_{\mathcal{S}, x}^{(k)}.$$

For $p, n \in \mathbb{N}, L_{\mathbf{f}} \in \mathbb{R}^+$ we define the antisymmetric function

$$\varphi_p^{(n, L_{\mathbf{f}})} : \mathbf{Y}_{\mathcal{Q}}(L_{\mathbf{f}}, n)^p \rightarrow \mathbb{C}$$

by

(5.4.2)

$$\varphi_p^{(n, L_f)}((k_1, b_1, s_1, x_1), \dots, (k_p, b_p, s_p, x_p)) = \left(\frac{n}{\beta}\right)^{(p-1)} \delta_{k_1, k_2} \dots \delta_{k_{p-1}, k_p} \tilde{\varphi}_p^{(L_f)}((b_1, s_1, x_1), \dots, (b_p, s_p, x_p)).$$

Note that $\varphi_p^{(n, L_f)}$ provides an element

$$\Phi_p^{(n, L_f)} \in \bigoplus_{k=0}^{n_\beta-1} (\mathcal{H}_f^{(k)} \oplus \mathcal{H}_f^{\bar{(k)}})$$

such that

$$\begin{aligned} \Phi_p^{(n, L_f)} &\doteq \int_{\mathbb{T}_n} d\mu_1 \dots \int_{\mathbb{T}_n} d\mu_p \sum_{(b_1, s_1, x_1), \dots, (b_p, s_p, x_p) \in \mathbb{X}_{\Lambda_f}} \\ &\varphi_p^{(n, L_f)}((k_1, b_1, s_1, x_1), \dots, (k_p, b_p, s_p, x_p)) \psi(k_1, b_1, s_1, x_1) \dots \psi(k_p, b_p, s_p, x_p). \end{aligned}$$

Explicitly we get

(5.4.3)

$$\Phi_p^{(n, L_f)} \doteq \left(\frac{n}{\beta}\right)^{(p-1)} \sum_{(b_1, s_1, x_1), \dots, (b_p, s_p, x_p) \in \mathbb{X}_{\Lambda_f}} \tilde{\varphi}_p^{(L_f)}((b_1, s_1, x_1), \dots, (b_p, s_p, x_p)) \psi(b_1, s_1, x_1) \dots \psi(b_p, s_p, x_p).$$

We now are able to state:

5.4.4 LEMMA. *Let $h_{L_f} \in \mathcal{B}(\mathcal{H}_f)$ be any self-adjoint operator, and let $W_{L_i} = W_{L_i}^* \in \mathcal{U}^+ \cap \mathcal{U}_{L_i}$, $K_{L_o} = K_{L_o}^* \in \mathcal{U}^+ \cap \mathcal{U}_{L_o}$. Then, for all $\beta \in \mathbb{R}^+$, $s \in \mathbb{R}$,*

$$\ln \frac{\text{tr} \left(e^{-\beta(d\langle Q, h_{L_f} Q \rangle + W_{L_i})} e^{sK_{L_o}} \right)}{\text{tr} \left(e^{-\beta d\langle Q, h_{L_f} Q \rangle} \right)} \mathbb{1} = \lim_{n \rightarrow \infty} \ln \int d\mu_{C_{h_f}^{(n)}} \left(H_f^{(n_\beta)} \right) \exp \left(\frac{\beta s}{n} \sum_{k=n}^{n_\beta-1} \chi^{(k)}(K_{L_o}) - \frac{\beta}{n} \sum_{k=0}^{n-1} \chi^{(k)}(W_{L_i}) \right),$$

with $\ln(c\mathbb{1}) \doteq \ln(c)\mathbb{1}$ for any $c \in \mathbb{C}$. Here, for any integer $n > \beta \|h_{L_f}\|_{\mathcal{B}(\mathcal{H})}$, $C_{h_f}^{(n)} \equiv C_{h_{L_f}}^{(n)} \in \mathcal{B}(H_f^{(n_\beta)})$ is given by (4.1.18) and $n_\beta \doteq n + \lfloor n/\beta \rfloor$. $\blacklozenge$

PROOF. Because of the continuity of the logarithm function together with the spectral theorem and the strict positivity of the terms inside $\ln$, the result follows from Corollary 4.1.17. End

Theorem 5.3.19 and Corollary 5.3.20 provide us sufficient conditions to solve the Equation (5.2.8). Thus, with $\gamma_{C_{h_f}^{(n)}} \equiv \gamma_{C_{h_{L_f}}^{(n)}}$, we only need to verify inequality (5.3.21) for some $\epsilon > \gamma_{C_{h_f}^{(n)}}$, the

covariance matrix $C_{h_f}^{(n)} \in \mathcal{B}(\mathbf{H}^{(n_\beta)})$ and the element

$$W_0(s) \doteq \frac{\beta s}{n} \sum_{k=n}^{n_\beta-1} \chi^{(k)}(K_{L_0}) - \frac{\beta}{n} \sum_{k=0}^{n-1} \chi^{(k)}(W_{L_i}) \in \wedge^* \left(\mathbf{H}_f^{(n_\beta)} \oplus \bar{\mathbf{H}}_f^{(n_\beta)} \right),$$

for $s \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. As remarked in Chapter 4–§4.3, Theorem 4.3.7 is even valid for random systems. Note that this summability result is useful for us because coincides with the norm $\|\cdot\|_{1,\infty}$ in Expression 5.3.18 taking $w(\{Y\}) = \frac{\beta}{n} > 0$ for $Y \in \mathbf{Y}_\Psi(L_f, n)$. However, note that uniform boundedness of the norm $\|W_0\|^{(4e)}$ in inequality (5.3.21) can be ensured only for translationally invariant interactions, that is, for $\Psi_\Lambda = \Psi_\Lambda^* \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$, $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$ and all $x \in \mathbb{Z}^d$

$$\Psi_{\Lambda+x} = \chi_x(\Psi_\Lambda),$$

see Definition 2.3.37, and $\{\chi_x\}_{x \in \mathbb{Z}^d}$ is the spatial $*$ -automorphism on $\mathcal{U}$ satisfying (2.3.34).

We now define for $\lambda \in \mathbb{R}$ and the translation invariant observable $W_{L_i} = W_{L_i}^* \in \mathcal{U}^+ \cap \mathcal{U}_{L_i}$, a self-adjoint operator $U_{L_i} \in \mathcal{U}^+ \cap \mathcal{U}_{L_i}$ such that $W_{L_i} = \lambda U_{L_i}$. Then λ provides the strenght of the interparticle interaction. By defining for $s, \lambda \in \mathbb{R}$ and fixed $\beta \in \mathbb{R}^+$

$$(5.4.5) \quad \mathcal{W}_\lambda(s) \doteq \frac{\beta s}{n} \sum_{k=n}^{n_\beta-1} \chi^{(k)}(K_{L_0}) - \lambda \frac{\beta}{n} \sum_{k=0}^{n-1} \chi^{(k)}(U_{L_i}) \in \wedge^* \left(\mathbf{H}^{(n_\beta)} \oplus \bar{\mathbf{H}}^{(n_\beta)} \right),$$

with $\mathbf{H}^{(n_\beta)}$ the Hilbert space as $L_f \rightarrow \infty$ of $\mathbf{H}_f^{(n_\beta)}$, the following Corollary follows directly from Lemma 5.4.4:

5.4.6 COROLLARY. *Let $h \in \mathcal{B}(\mathcal{H})$ be any self-adjoint operator such that h and $\mathcal{H}$ are the limits of h_f and $\mathcal{H}_f$ as $L_f \rightarrow \infty$. Then, for all $\beta \in \mathbb{R}^+$, $s, \lambda \in \mathbb{R}$ and $\mathcal{W}_\lambda(s) \in \wedge^* \left(\mathbf{H}^{(n_\beta)} \oplus \bar{\mathbf{H}}^{(n_\beta)} \right)$ given in (5.4.5), we are able to write*

$$g_{L_0, L_i, \infty}(s) - g_{L_0, L_i, \infty}(0) = \frac{1}{|\Lambda_{L_0}|} \lim_{n \rightarrow \infty} \left(\ln \int d\mu_{C_h^{(n)}} \left(\mathbf{H}^{(n_\beta)} \right) e^{\mathcal{W}_\lambda(s)} - \ln \int d\mu_{C_h^{(n)}} \left(\mathbf{H}^{(n_\beta)} \right) e^{\mathcal{W}_\lambda(0)} \right),$$

where $g_{L_0, L_i, \infty}(s)$ is given by (5.1.6). Last expression holds uniformly w.r.t. the size of the boxes $\Lambda_{L_0}, \Lambda_{L_i}$ and $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$, $C_h^{(n)} \in \mathcal{B}(\mathbf{H}^{(n_\beta)})$ is given by (4.1.18) with $n_\beta \doteq n + \lfloor n/\beta \rfloor$. $\blacksquare$

We finally assert:

Theorem 5.4.7:

Let $h \in \mathcal{B}(\mathcal{H})$ be any self-adjoint operator such that h and $\mathcal{H}$ are the limits of h_f and $\mathcal{H}_f$ as $L_f \rightarrow \infty$. Consider $\mathcal{W}_\lambda(s) \in \wedge^ \left(\mathbf{H}^{(n_\beta)} \oplus \bar{\mathbf{H}}^{(n_\beta)} \right)$ given in (5.4.5) and $J_{L_0, L_i, L_f}(s)$ in (5.1.5). Then for $s \in \mathbb{R}$, $\beta \in \mathbb{R}^+$*

and $\lambda \in \mathbb{R}$ small enough, the generating function

$$\begin{aligned} J(s) &= \lim_{L_0 \rightarrow \infty} \lim_{L_i \rightarrow \infty} \lim_{L_f \rightarrow \infty} J_{L_0, L_i, L_f}(s) \\ &= \lim_{L_0 \rightarrow \infty} \frac{1}{|\Lambda_{L_0}|} \lim_{L_i \rightarrow \infty} \lim_{n \rightarrow \infty} \left(\ln \int d\mu_{C_h^{(n)}}(\mathbf{H}^{(n_\beta)}) e^{\mathcal{W}_\lambda(s)} - \ln \int d\mu_{C_h^{(n)}}(\mathbf{H}^{(n_\beta)}) e^{\mathcal{W}_\lambda(0)} \right), \end{aligned}$$

exists. Moreover, for $s \in \mathbb{C}$ in a neighborhood of zero, $J(s)$ is an analytic function such that for all $m \in \mathbb{N}_0$ we have

$$\left| \frac{d^m}{ds^m} J(s) \right| \leq m!. \quad \blacktriangle$$

PROOF. The proof of the theorem follows as a consequence of Theorems 4.2.13, 4.3.7, 5.3.20, Lemma 5.4.4, Corollary 5.4.6 and the Vitali convergence Theorem. End

As we have seen, throughout this Thesis Large Deviation techniques are useful to understand mathematical properties of physical systems. In particular, as stressing in Chapter 1, experimental measures [WMR⁺12] of conductivity shown that in the thermodynamic limit, quantum coherence can very rapidly disappear w.r.t. growing space-scales. Thus, quantum uncertainty of microscopic electric currents is suppressed, exponentially fast w.r.t. the volume of the region of the lattice where the electric field is applied. In this way, we were able to state Conjecture 2 for the family of current distributions $\{\mathbb{I}_{\Lambda_L}^{(\mathcal{E})}\}_{L \in \mathbb{R}^+} \in \mathcal{U}^+ \cap \mathcal{U}_L$, with Λ_L given by the sequence of boxes (1.2.4), namely,

$$\Lambda_L \doteq \{(x_1, \dots, x_d) \in \mathfrak{L} : |x_1|, \dots, |x_d| \leq L\} \in \mathcal{P}_f(\mathfrak{L}),$$

$\mathcal{U}_L \equiv \mathcal{U}_\Lambda$ is the local CARC*-algebra and $\mathfrak{L} \doteq \mathbb{Z}^d$. Chapter 3 was devoted to analyse the situation for free fermions embedded in disordered media. On the other hand, Chapter 4 deals with weakly interacting fermions, however, it was not until Chapter 5 that we use this hypothesis to prove the analyticity of the generating function. At this point, we can summarize some technical advances we achieve in this work comparing with others (for a complete bibliographic references see [ABPM17]):

1. In [GLM02], Gallavotti, Lebowitz and Mastropietro proof that the particle density of (rarefied) quantum gases fulfills an LDP at thermal equilibrium. For fermions, the proof is done for *weakly interacting* fermions via *Berezin integrals* and tree expansions for *correlation functions* of the equilibrium state, from which the (limiting) logarithmic moment generating function is obtained. This approach is strongly connected to the one we develop presently. They perform a multiscale decomposition of the covariance to bound the resulting series of determinants afterwards. Avoiding the multiscale analysis is one of our main motivations of [ABPM17]. We also show that it is not necessary to use *correlation functions* to construct the (limiting) logarithmic moment generating function J via functional integrals. In contrast to [GLM02], the method presented in Chapters 4 and 5 is *not* restricted to particle density observables $\{D_L\}_{L \in \mathbb{R}^+}$. It can be applied to any thermodynamic sequences $\{A_L^\Psi\}_{L \in \mathbb{R}^+}$ associated with densities of any other physical quantity that can be encoded in

a translation-invariant and finite-range interaction Ψ .

2. In 2004, Netočný and Redig [NR04] studied quantum spin systems at thermal equilibrium. They demonstrated the existence of an LDP and a central limit theorem at sufficiently high temperatures for thermodynamic sequences $\{A_L^\Psi\}_{L \in \mathbb{R}^+}$ of density observables encoded in *zero-range*¹ interactions Ψ . This result thus holds true for more general thermodynamic sequences than just the particle density case used in previous studies. As we stressed here, the central limit theorem in [NR04] is a consequence of the Bryc theorem, as it was used in Chapter 5.
3. The following year, Lenci and Rey-Bellet [LR05] took into consideration space averages $\{A_L^\Psi\}_{L \in \mathbb{R}^+}$ of a finite-volume element $|\Lambda_L|$ in either spin or CAR C^* -algebra. This quantity refers to thermodynamic sequences $\{A_L^\Psi\}_{L \in \mathbb{R}^+}$ of density observables for more general translation-invariant, finite-range interactions Ψ . In contrast to [NR04], they did not exploit cluster expansions, but the inequality

$$\left| \ln \operatorname{tr} (Ae^{H+K}) - \ln \operatorname{tr} (Ae^H) \right| \leq \sup_{0 \leq t \leq 1} \sup_{-\frac{1}{2} \leq w \leq \frac{1}{2}} \left\| e^{-w(H+tK)} K e^{w(H+tK)} \right\|_{\mathcal{B}(\mathbb{C}^n)},$$

for any matrices A, H, K so that $A > 0$, $H = H^*$ and $K = K^*$, where tr denotes the normalized trace. See [LR05, Lemma 3.6] and Chapter 3. The uniqueness of the thermal equilibrium state, i.e., the KMS state, is used as a key argument, in contrast to [NR04] (cf. Remark 2.4.10 and [NR04, Remark 2.14]). They obtain an LDP for the empirical average of finite-volume observables either in dimension one at any temperature, or for any finite dimension at sufficiently high temperatures. Their proof works for both spin and fermionic lattice systems. See [LR05, Theorem 3.2].

4. In 2011, Ogata and Rey-Bellet [OR11] made use of the Ruelle-Lanford function [Rue65, Lan73] to technically simplify and unify the proof of previous LD results for quantum spin systems with a slight extension: Their proofs are based on the notion of “asymptotic decoupled states” as defined in the classical case in [Pfi02, Definitions 3.2 and 3.3], i.e., states ρ on spin C^* -algebra for which there is a function $c: \mathbb{N} \rightarrow [0, \infty)$ such that, for any $L \in \mathbb{N}$, and any non-negative observables A, B with corresponding supports $\Lambda^{(A)} \subset \Lambda_L$ and $\Lambda^{(B)} \subset \mathbb{Z}^d \setminus \Lambda_L$,

$$\lim_{L \rightarrow \infty} \frac{c(L)}{|\Lambda_L|} = 0 \quad \text{and} \quad e^{-c(L)} \rho(A) \rho(B) \leq \rho(AB) \leq e^{c(L)} \rho(A) \rho(B).$$

There are few quantum systems for which KMS states are known to satisfy this condition, for instance at dimension one [Ara69] and at high temperatures [Ara74], like the cases

¹A zero-range interaction Ψ yields a sequence of the form $A_L^\Psi = \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \Psi_{\{x\}}$.

considered in [NR04, LR05].

5. More recently, De Roeck, Maes, Netočný and Schütz studied in [DMNS15] (identically distributed (i.d.)) empirical means of operators associated with zero-range interactions for quantum spin systems at thermal equilibrium, via the corresponding distributions. They focus on Gibbsianness or quasi-locality of the (limiting) distribution, which is a stronger feature than the property of asymptotic decoupling used in [OR11], which in turn is stronger than an LD property. See, e.g., [DMNS15, Theorem 4.4]. Like in [NR04, LR05, OR11] they study the high temperature regime, but also the *low* temperature situation for a class of *gapped* quantum spin systems. In all studied regimes, there is a *unique* KMS state. They use a polymer model together with either a high-temperature cluster expansion or an expansion around the ground state that is reminiscent of the quantum Pirogov–Sinai theory.

As can be seen, most of the results depend on the uniqueness of the KMS states, that is assured for inverse temperature $\beta > 0$ small enough. However, note that for non-interacting fermion systems *always* exists a unique KMS because the Bogoliubov automorphism (see Lemma 2.3.65). That is the reason we did not take any restriction on the temperature. Moreover, from Expression (5.4.5) note that with the parameter λ we can *deal* with β in order to avoid KMS uniqueness questions for the weakly interacting setting. Note that our results regarding free fermions deals essentially with sum of commutators and as far as we are aware it is the first formal result involving these. A possible complication for the interacting case is that the sum of commutators leads with a *non*-Bogoliubov automorphism, see Expression (6.1.8). Thus, as we explain below other kind of technologies are necessary, namely, a combination of *Lieb–Robinson bounds* and the results of the common work [ABPM17].

We now intend to show how to tackle the interacting case combining results and the mathematical framework of Chapters 3, 4 and 5, see [Aza17]. Hence, we will only assume that the total interaction depends on the discrete Laplacian $\Delta_{\omega, \vartheta}$ (where the tight-binding Anderson model is a particular case), as described in Chapter 2–§§2.3.5 and 3 whereas the interparticle interaction is short range $\Psi \in \mathcal{W}$, as defined in Chapter 2–§§2.3.3. If in addition we apply an external electromagnetic potential $\mathbf{A} \in \mathbf{C}_0^\infty$, then, we would deal with a non-autonomous dynamics. However, because such a treatment goes beyond our main issue we omit this, for details we encouragement to the reader see [BP15, BP14a, BP16a].

6.1 DYNAMICS OF INTERACTING FERMIONS IN DISORDER MEDIA

Let $\Delta_{\omega,\vartheta}(\psi) \in \mathcal{B}(\ell^2(\mathfrak{L}))$ be the self-adjoint operator given in §§2.3.5, namely,

$$[\Delta_{\omega,\vartheta}(\psi)](x) \doteq 2d\psi(x) - \sum_{j=1}^d \left((1 + \overline{\vartheta\omega_2(\{x, x - e_j\})})\psi(x - e_j) + \psi(x + e_j)(1 + \vartheta\omega_2(\{x, x + e_j\})) \right),$$

for any $x \in \mathfrak{L}$ and $\psi \in \ell^2(\mathfrak{L})$, with $\{e_k\}_{k=1}^d$ being the canonical orthonormal basis of the Euclidian space $\mathbb{R}^d$. In above Expression $\omega = (\omega_1, \omega_2) \in \Omega$ and $\vartheta \in \mathbb{R}_0^+$ (refers to the strength of hopping disorder). Here, Ω is a set containing information on random external potential in the lattice sites, with values on $[-1, 1]$, and the random hopping in lattice edges, with values on the complex unit disc. We now consider the interparticle interaction such that $\Psi^{\text{IP}} \in \mathcal{W}$, see Expression (2.3.44). Then,

$$(6.1.1) \quad \Psi_{\Lambda}^{(\omega,\vartheta)} \doteq \langle e_x, \Delta_{\omega,\vartheta} e_y \rangle a(e_x)^* a(e_y) + (1 - \delta_{x,y}) \langle e_y, \Delta_{\omega,\vartheta} e_x \rangle a(e_y)^* a(e_x) + \Psi_{x,y}^{\text{IP}} \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda},$$

whenever $\Lambda = \{x, y\}$ for $x, y \in \mathfrak{L}$, and $\Psi_{\Lambda}^{(\omega,\vartheta)} \doteq \Psi_{\Lambda}^{\text{IP}}$ otherwise. By convenience we assume that Ψ^{IP} is translationally invariant, i.e., for all $x \in \mathfrak{L}$ and $\Lambda \in \mathcal{P}_{\text{f}}(\mathfrak{L})$

$$\Psi_{\Lambda+x}^{\text{IP}} = \chi_x \left(\Psi_{\Lambda}^{\text{IP}} \right),$$

see Definition 2.3.37, and $\{\chi_x\}_{x \in \mathfrak{L}}$ is the spatial $*$ -automorphism on $\mathcal{W}$ satisfying (2.3.34). Hence, we define in presence of bounded static potentials, the *internal* energy observable $H_L^{(\omega,\vartheta,\lambda)} \in \mathcal{U}^+ \cap \mathcal{U}_L$ in the box Λ_L as

$$(6.1.2) \quad H_L^{(\omega,\vartheta,\lambda)} \doteq \sum_{\Lambda \subset \Lambda_L} \Psi_{\Lambda}^{(\omega,\vartheta)} + \lambda \sum_{x \in \Lambda_L} \omega_1(x) a(e_x)^* a(e_x)$$

$$(6.1.3) \quad = \sum_{x,y \in \Lambda_L} \langle e_x, (\Delta_{\omega,\vartheta} + \lambda\omega_1) \rangle a(e_x)^* a(e_y) + \sum_{\Lambda \subset \Lambda_L} \Psi_{\Lambda}^{\text{IP}},$$

for $\omega = (\omega_1, \omega_2) \in \Omega$, $\vartheta, \lambda \in \mathbb{R}_0^+$ and $L \in \mathbb{R}^+$. As remarked in [BP14a], the first sum in the right hand side (r.h.s.) of above equality is nothing but the second quantization of the one-particle operator $h^{(\omega,\vartheta,L)} \doteq \Delta_{\omega,\vartheta} + \lambda\omega_1$ restricted to the subspace $\ell^2(\Lambda_L) \subset \ell^2(\mathfrak{L})$. The second sum in that equality encodes all interaction mechanisms involving more than one particle, in the box Λ_L .

For $\Lambda_L \in \mathcal{P}_{\text{f}}(\mathfrak{L})$ and the C^* -algebra $\mathcal{W}$, the finite-dynamics is described by the C_0 -group of $*$ -automorphisms $\{\tau_t^{(\omega,\vartheta,\lambda,L)}\}_{t \in \mathbb{R}}$ of $\mathcal{W}$, such that the temporal evolution of any operator $A \in \mathcal{W}$ is given by

$$(6.1.4) \quad \tau_t^{(\omega,\vartheta,\lambda,L)}(A) = e^{itH_L^{(\omega,\vartheta,\lambda)}} A e^{-itH_L^{(\omega,\vartheta,\lambda)}},$$

for $\omega = (\omega_1, \omega_2) \in \Omega$, $\vartheta, \lambda \in \mathbb{R}_0^+$ and $L \in \mathbb{R}^+$.

The following result was proven in [BP16a] and it is fundamental to study the behavior of fermionic systems as $L \rightarrow \infty$ (see Theorem 2.3.69):

Theorem 6.1.5 (Infinite-volume dynamics and its generator):

Let $\omega \in \Omega$ and $\Theta, \vartheta, \lambda \in \mathbb{R}_0^+$.

1. *Infinite volume dynamics.* The continuous groups $\{\tau_t^{(\omega, \vartheta, \lambda, L)}\}_{t \in \mathbb{R}}$, $L \in \mathbb{R}^+$, converge strongly to a C_0 -group $\{\tau_t^{(\omega, \vartheta, \lambda)}\}_{t \in \mathbb{R}}$ of f^* -automorphisms with generator $\delta^{(\omega, \vartheta, \lambda)}$, as $L \rightarrow \infty$.
2. *Infinitesimal generator.* $\delta^{(\omega, \vartheta, \lambda)}$ is conservative closed symmetric derivation which is equal on its core of $\mathcal{U}_0^2$ to

$$\delta^{(\omega, \vartheta, \lambda)}(A) = i \sum_{x, y \in \mathfrak{L}} \langle \epsilon_x, (\Delta_{\omega, \vartheta} + \lambda \omega_1) \epsilon_y \rangle [a(\epsilon_x)^* a(\epsilon_y), A] + i \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} [\Psi_{\Lambda}^{IP}, A], \quad A \in \mathcal{U}_0.$$

Both infinite sums are absolutely convergent.

3. *Lieb–Robinson bounds.* For any $\vartheta \in [0, \Theta]$, $t \in \mathbb{R}$, $A_1 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(1)}}$ and $A_2 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(2)}}$ with disjoint sets $\Lambda^{(1)}, \Lambda^{(2)} \in \mathcal{P}_f(\mathfrak{L})^3$,

$$\left\| [\tau_t^{(\omega, \vartheta, \lambda)}(A_1), A_2] \right\|_{\mathcal{U}} \leq 2D^{-1} \|A_1\|_{\mathcal{U}} \|A_2\|_{\mathcal{U}} (e^{2D|t|D_{\Theta}} - 1) \sum_{x \in \Lambda^{(1)}} \sum_{y \in \Lambda^{(2)}} F(|x - y|),$$

where

$$D_{\Theta} \doteq \sup \left\{ \|\Psi^{(\omega, \vartheta)}\|_{\mathcal{W}} : \omega \in \Omega \text{ and } \vartheta \in [0, \Theta] \right\} < \infty. \quad \clubsuit$$

As mentioned, the above Theorem is *springboard* when we desire to study the behavior of interacting fermions. In particular, note that the *finite* version of Theorem 6.1.5–3 yields to a similar bound that found in Lemma 2.3.62:

$$\left\| \tau_t^{(\omega, \vartheta, \lambda, L_2)}(A) - \tau_t^{(\omega, \vartheta, \lambda, L_1)}(A) \right\|_{\mathcal{U}} \leq 2 \|A\|_{\mathcal{U}} \|\Psi\|_{\mathcal{U}} |t| e^{4D|t|D_{\Theta}} \sum_{y \in \Lambda_{L_2} \setminus \Lambda_{L_1}} \sum_{x \in \Lambda} F(|x - y|),$$

where $t \in \mathbb{R}$, $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, $A \in \mathcal{U}_{\Lambda}$, $L_1, L_2 \in \mathbb{R}_0^+$ with $\Lambda \subset \Lambda_{L_1} \subsetneq \Lambda_{L_2}$, $\Psi \in \mathcal{W}$ and $\mathbf{V}$ is any independent-time potential [BP16a, Lemma 4.4]. On the other hand, given a C_0 -group of $*$ -automorphisms of $\mathcal{U}$, it is not a priori clear that KMS states exists for this dynamics. However, for all interactions $\Phi \in \mathcal{W}$, we can ensure the existence of $(\tau^{(\omega, \vartheta, \lambda)}, \beta)$ -KMS states ϱ for every $\beta \in \mathbb{R}^+$. Indeed, such KMS states can be constructed as weak*-limits of Gibbs states obtained

²The definition of a conservative operator $S \in \mathcal{U}_0$ can be found in [BR03a, Definition 3.1.13]. Recall that $\mathcal{U}_0$ is given by (2.3.24).

³Recall that D is the bounded convolution constant given by Expression (2.3.41) and $F: \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ is a positive-valued and non-increasing decay function satisfying the properties (2.3.40)–(2.3.42).

by the restriction of the interaction to finite volume regions of the lattice $\mathfrak{L}$. Because any CAR C^* -algebra $\mathscr{U}$ is simple (see [BR03b, Theorem 5.2.5]), (τ, β) -KMS states ϱ are faithful. Recall that the extreme cases $\beta = 0$ and $\beta = \infty$ correspond to the trace (or chaotic) state and ground state respectively.

Indeed, for the continuous group $\{\tau_t^{(\omega, \vartheta, \lambda, L)}\}_{t \in \mathbb{R}}$ of $*$ -automorphism on $\mathscr{U}$ converging strongly to the C_0 -group $\tau_t^{(\omega, \vartheta, \lambda)}$, we can associate some unique KMS state for $\beta \in \mathbb{R}^+$. By [BR03b, Proposition 5.3.25], there is a $(\tau^{(\omega, \vartheta, \lambda)}, \beta)$ -KMS state $\rho^{(\beta, \omega, \vartheta, \lambda)}$ for all $\beta \in \mathbb{R}^+$. As remarked in [BP14a], KMS state does not uniquely define thermal equilibrium. Note that such a uniqueness can be assured if the temperature is high enough for any dimension d or if we restrict to one-dimensional lattice at any temperature.

Just as in [BP15], for any inverse temperature $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$, we impose the following two conditions on the map

$$(6.1.6) \quad \omega \mapsto \rho^{(\beta, \omega, \vartheta, \lambda)}$$

from the set Ω to the dual space $\mathscr{U}^*$:

1. For the family of $*$ -automorphisms $\{\chi_x\}_{x \in \mathfrak{L}}$ we assume for $x \in \mathfrak{L}$

$$\rho^{(\beta, \chi_x^{(\Omega)}(\omega), \vartheta, \lambda)} = \rho^{(\beta, \omega, \vartheta, \lambda)} \circ \chi_x,$$

where $\{\chi_x^{(\Omega)}\}_{x \in \mathfrak{L}}$ denotes the action on Ω , see (2.3.51)–(2.3.52).

2. Thermal equilibrium are random variables. Thus, for any inverse temperature $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$, the state $\rho^{(\beta, \omega, \vartheta, \lambda)}$ given by (6.1.6) is measurable w.r.t. the σ -algebra $\mathfrak{A}_\Omega$ on Ω and the Borel σ -algebra $\mathfrak{A}_{\mathscr{U}^*}$ of $\mathscr{U}^*$ generated by the weak*-topology.

Hence, we are able to define:

6.1.7 DEFINITION (RANDOM INVARIANT STATES). Consider the map given by (6.1.6). We say that this map is measurable w.r.t. $\mathfrak{A}_\Omega$ and $\mathfrak{A}_{\mathscr{U}^*}$ and satisfies $\rho^{(\beta, \chi_x^{(\Omega)}(\omega), \vartheta, \lambda)} = \rho^{(\beta, \omega, \vartheta, \lambda)} \circ \chi_x$. $\blacksquare$

Note that for any inverse temperature $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$, the average state $\bar{\rho}^{(\beta, \vartheta, \lambda)} \in \mathscr{U}^*$ defined for $A \in \mathscr{U}$ by

$$\bar{\rho}^{(\beta, \vartheta, \lambda)}(A) \doteq \mathbb{E} \left[\bar{\rho}^{(\beta, \omega, \vartheta, \lambda)}(A) \right],$$

is translationally invariant, i.e., for any $x \in \mathfrak{L}$

$$\bar{\rho}^{(\beta, \vartheta, \lambda)} = \bar{\rho}^{(\beta, \vartheta, \lambda)} \circ \chi_x.$$

We finally mention that using this algebraic setting we intend in [Aza17] to show the existence of a generating function $J(s)$ w.r.t. the Hamiltonian (6.1.2) of a space average operator $K_L(t) \in$

$\mathcal{U}^+ \cap \mathcal{U}_L$ of the form:

$$(6.1.8) \quad K_L(t) \doteq \frac{1}{|\Lambda_L|} \sum_{\Lambda_1, \Lambda_2 \subset \Lambda_L} i \left[\tau_t^{(\omega, \vartheta, \Lambda, \Lambda_1)}(\Phi_{\Lambda_1}), \Phi_{\Lambda_2} \right],$$

where $\Phi_{\Lambda_1} \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda_1}$ and $\Phi_{\Lambda_2} \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda_2}$ are finite range interactions of the form of Expression (6.1.1).

6.2 OUTLOOK

Possible applications of the methods presented here are motivation for further research works in the scope of fermion lattice systems. In fact, there are several works closer to our approach in which the technologies used here can be useful. W.r.t. this a few remarks are in order:

For example, in the joint work [ABP17] a solution to an open problem stated few years ago it was proposed [SW16], which is related to the *Many-Body Localization* scope. More precisely, in [SW16] it was asserted an exponential decay of many-particle correlations for quasi-free fermions in *one-dimensional* lattices with disorder. As pointed out in [SW16], it is an interesting open question (a) whether [SW16, Theorems 1.1 and 1.2] can be generalized to higher dimensions. Another open question (b) is their generalization for *complex-time* correlation functions. This last point is relevant because such correlation functions (of quasi-free fermions) can be useful to study localization of *weakly interacting* fermion systems on lattices. In fact, (quasi-free) complex-time correlation functions appear in the perturbative expansion of (full) correlations for weakly interacting systems. See, for instance, [BR03b, Section 5.4.1]. Then, in [ABP17], by considering the many-body localization in the sense of the Hausdorff distance and using the *Hadamard three-line Theorem* an answer to both questions (a) and (b) it was proposed, see [ABP17, Corollary 2.3 and Theorem 3.1].

Secondly, in regards to Large Deviation Theory used to understanding quantum systems, in [Rey11] was stated the following related problem: Let $\rho \in \mathcal{U}^*$ be a Gibbs state over a C^* -algebra $\mathcal{U}$ and let $\omega \in \mathcal{U}^*$ be a translation invariant state. Prove the existence of the limit

$$s(\omega, \rho) = \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} S(\omega_L, \rho_L),$$

where $\Lambda_L \in \mathcal{P}_f(\mathcal{Q})$ belongs to the sequence of boxes (1.2.4), $\omega_L, \rho_L \in \mathcal{U}_L^*$ denotes the restriction of the states ω, ρ to the local sub-algebra $\mathcal{U}_L \equiv \mathcal{U}_{\Lambda_L}$, and $S(\omega_L, \rho_L)$ is the relative entropy of the states, see Expression (2.3.33).

In [BP16b], constructions involving quasi-free states in suitably chosen CAR algebras are used, whereas in [ABPM17, ABPM16] Grassmann-algebra computations turn out to be more efficient by allowing us to represent determinants as *traces*. Like in [BP16b, Section 1.6], Hölder inequalities for non-commutative L^p -spaces, in [ABPM17, ABPM16] for Schatten norms, are

pivotal tools and replace the celebrated Gram bounds on which *all* previous results, like [BGPS94, PS08, GLM02, GM10, GMP16, Mas16], are explicitly based. We hope to tackle problems similar to those stressed in the literature, e.g., the works [GMP16] and [Mas16]. In the former the universality of the *Hall conductivity* for two-dimensional interacting fermions systems was proven while in the latter *localization* for *quasi-periodic* fermionic chain was stated. Thus, is an objective to avoid *usual* renormalization expansions while are adapted the tools of this Thesis and similar works.

Last but not least, it is interesting to know if some properties valid in the bosonic setting also hold in the fermionic one. In particular, a few years ago was established [HKV17]:

... a quantum version of the classical isoperimetric inequality relating the Fisher information and the entropy power of a quantum state... in particular, it implies an entropy power inequality for the mentioned convolution operation as well as the isoperimetric inequality,...

Note that the geometric inequalities are related to the quantum Fisher information $J(\rho)$, with ρ a state of a bosonic system. An open problem is knowing if at least for KMS states similar inequalities in the fermionic setting are satisfied.

book [BPT12].

This Appendix is devoted to present some definitions and Theorems relevant in the present work. Several textbooks were used to write these, in particular, we use classic books as [Rud91] and [RS81], and the recent and modern introductory

Notation A.0.1.

1. The symbol $\mathbb{K}$ will represent the field of real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. The elements of $\mathbb{K}$ are called *scalars*.
2. We denote by D any positive and finite generic constant. These constants do not need to be the same from one statement to another.
3. A norm on the generic vector space $\mathcal{X}$ is denoted by $\|\cdot\|_{\mathcal{X}}$ and the identity map of $\mathcal{X}$ by $\mathbb{1}_{\mathcal{X}}$. The space of all bounded linear operators on $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is denoted by $\mathcal{B}(\mathcal{X})$. ♣

A.1 ANALYSIS

A.1.1 DEFINITION (SUPPORT). Let $f: \mathcal{X} \rightarrow \mathbb{R}$ be a continuous function. The support of f , denoted by $\text{supp}(f)$, is the closure of the set of points $x \in \mathcal{X}$ for which $f(x)$ is different from zero, i.e.,

$$\text{supp}(f) = \overline{\{x | f(x) \neq 0\}}.$$

■

A.1.2 DEFINITION. Suppose $\mathcal{X}$ is a set and $f_n: \mathcal{X} \rightarrow \mathbb{R}$ is a real-valued function for every natural number n . We say that the sequence $(f_n)_{n \in \mathbb{N}}$ is uniformly convergent with limit $f: \mathcal{X} \rightarrow \mathbb{R}$, if for every $\epsilon > 0$, there exists a natural number N such that for all $x \in \mathcal{X}$ and all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$. ■

Theorem A.1.3 (Lebesgue's Dominated Convergence Theorem):

Suppose that f_n is a sequence of measurable functions, that $f_n \rightarrow f$ pointwise almost everywhere as

$n \rightarrow \infty$, and that $|f_n| \leq g$ for all n , where g is integrable. Then f is integrable, and

$$\int f(x)\mu(dx) = \lim_{n \rightarrow \infty} \int f_n(x)\mu(dx). \quad \blacktriangle$$

A.1.4 DEFINITION. Let $\mathcal{U}$ be a vector space in $\mathbb{K}$. We say that the map $s: \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{C}$ is a “sesquilinear form” on $\mathcal{U}$ if for all $f, g, h \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{K}$

$$\begin{aligned} s(f, \alpha g + \beta h) &= \alpha s(f, g) + \beta s(f, h), \\ s(\alpha f + \beta g, h) &= \bar{\alpha} s(f, h) + \bar{\beta} s(g, h). \end{aligned}$$

Here, for $z \in \mathbb{K}$, $\bar{z} \in \mathbb{K}$ is its complex conjugate. ▀

A.1.5 DEFINITION (UNIFORM CONTINUITY). A map f from a metric space $M \equiv (M, d)$ to a metric space $N \equiv (N, \rho)$ is said to be uniformly continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ whenever x, y in M satisfy $d(x, y) < \delta$. ▀

A.1.6 DEFINITION. Let $\Omega \in \mathbb{C}$ be an open and connected subset of $\mathbb{C}$. With respect to $f: \Omega \rightarrow \mathbb{C}$ we define:

1. f is “holomorphic” if for each $z \in \Omega$

$$f'(z) \doteq \lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z}$$

exists and is finite.

2. f is “analytic” at each $z_0 \in \Omega$, if and only if can be represented by a power series, i.e., $f(z) = \sum_{n \in \mathbb{N}} a_n (z - z_0)^n$, where $a_n = \frac{1}{n!} f^{(n)}(z_0)$, and as is usual, $f^{(n)}(z_0)$ is the n -derivative of f in z_0 . ▀

Remark A.1.7. Holomorphic functions are known to have derivatives of all orders. For this reason, usually holomorphic and analytic functions are considered synonyms [Sim15, Part 2]. ✱

A.1.8 DEFINITION (REGULAR FUNCTION). A function $f: \Omega \rightarrow \mathbb{C}$ is called “regular” on $\Omega \subset \mathbb{C}$, if f is analytic in Ω . ▀

A.2 FUNCTIONAL ANALYSIS

A.2.1 DEFINITION (SEPARABLE METRIC SPACE). A metric space E is separable if there exists a subset $D \subset E$ that is countable and dense. ▀

Theorem A.2.2 (Fréchet–Riesz Theorem, the Riesz Lemma):

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}^*$ be its dual, i.e., the space of continuous linear functions over $\mathcal{H}$. For any linear functional $\phi: \mathcal{H} \rightarrow \mathbb{C}$ there exists a unique $y \in \mathcal{H}$ such that $\phi(x) = \langle y, x \rangle$, for all $x \in \mathcal{H}$ and $\|y\|_{\mathcal{H}} = \|\phi\|_{\mathcal{H}^*}$. $\clubsuit$

A.2.3 DEFINITION (TOPOLOGICAL VECTOR SPACES). A *topological vector space (TVS)* $\mathcal{X}$ is a vector space equipped with a topology τ for which the vector space operations of $\mathcal{X}$ are continuous and every point of $\mathcal{X}$ defines a closed set. $\blacksquare$

The fact that every point of $\mathcal{X}$ is a closed set is usually not part of the definition of a TVS in many textbooks. It is used here because it is satisfied in most applications and, in this case, the space $\mathcal{X}$ is automatically Hausdorff by [Rud91, Theorem 1.12]. Examples of TVS used in this work are the *dual spaces*. Before to present dual spaces, we invoke the following definitions taken of [Rud91, RS81, BPT12]:

A.2.4 DEFINITION. A TVS $\mathcal{X}$ is called *locally convex* if there is a local base $\mathfrak{B}$ whose members are *convex*. Moreover, a set $C \subset \mathcal{X}$ is said to be *convex* if

$$tC + (1 - t)C \subset C \text{ for } 0 < t < 1. \quad \blacksquare$$

A.2.5 DEFINITION (WEAK AND WEAK*–TOPOLOGIES). Let $\mathcal{X}$ be a TVS whose *dual space* is $\mathcal{X}^*$, i.e., the space of continuous linear functions over $\mathcal{X}$. The *weak topology* of $\mathcal{X}$, denoted by $\sigma(\mathcal{X}, \mathcal{X}^*)$, is the topology generated by the continuous linear functions over $\phi \in \mathcal{X}$. The *weak* topology* of $\mathcal{X}^*$, denoted by $\sigma(\mathcal{X}^*, \mathcal{X})$, is the weakest topology on $\mathcal{X}^*$ in which all the functions $\phi \in \mathcal{X}^* \mapsto \phi(x) \in \mathbb{K}, x \in \mathcal{X}$ are continuous. $\blacksquare$

Remark A.2.6. Note that the weak*–topology is weaker than the weak–topology. $\ast$

The weak and the weak* topologies have several well known basic properties, one of these is that both, $\sigma(\mathcal{X}, \mathcal{X}^*)$ and $\sigma(\mathcal{X}^*, \mathcal{X})$ –topologies, are Hausdorff. For more details and proofs see the references, in particular [Rud91, Theorem 3.10 and page 68] and [BPT12, Propositions 6.2.2 and 6.3.2].

Theorem A.2.7 (Dual space of a TVS):

The dual space $\mathcal{X}^*$ of a TVS $\mathcal{X}$ is a locally convex space (LCS) in the $\sigma(\mathcal{X}^*, \mathcal{X})$ –topology and its dual is $\mathcal{X}$. $\clubsuit$

Since any Banach space is a TVS in the sense of Definition A.2.3, the dual space of a Banach space is a LCS:

A.2.8 COROLLARY. The dual space $\mathcal{X}^*$ of a Banach space $\mathcal{X}$ is a LCS in the $\sigma(\mathcal{X}^*, \mathcal{X})$ –topology and its dual is $\mathcal{X}$. $\blacksquare$

Note that if $\mathcal{X}$ is separable, this yields to the metrizability of any weak*-compact subset K of their dual spaces (cf. [Rud91, Theorem 3.16]):

Theorem A.2.9 (Metrizability of weak*-compact sets):

Let $K \subset \mathcal{X}^$ be any weak*-compact subset of the dual $\mathcal{X}^*$ of a separable TVS $\mathcal{X}$. Then K is metrizable in the weak*-topology.* ❖

A.2.1 OPERATOR ALGEBRAS

Consider the Definition of algebra $\mathcal{U}$ as given in Chapter 2–§2.2, and recall that if $\mathcal{U}$ has identity, we denote this element by $\mathbb{1} \in \mathcal{U}$. We say that an element $A \in \mathcal{U}$ is invertible, if there exists $B \in \mathcal{U}$ such that

$$AB = \mathbb{1} = BA.$$

In this case, as is usual, we write $B = A^{-1}$.

A.2.10 DEFINITION (RESOLVENT AND SPECTRUM). If $\mathcal{U}$ is an algebra with identity, the resolvent set $r_{\mathcal{U}}(A)$ of an element $A \in \mathcal{U}$ is the set of $\lambda \in \mathbb{C}$ such that $\lambda\mathbb{1} - A$ is invertible. The spectrum $\sigma_{\mathcal{U}}(A)$ of A is the complement of $r_{\mathcal{U}}(A)$ in $\mathbb{C}$. The inverse element $(\lambda\mathbb{1} - A)^{-1}$, with $\lambda \in r_{\mathcal{U}}(A)$, is called resolvent of A at λ . ■

A.2.11 DEFINITION (POSITIVITY). Let $\mathcal{U}$ be a $*$ -algebra as given in Chapter 2–§2.2. $A \in \mathcal{U}$ is called positive if it is self-adjoint and its spectrum $\sigma(A)$ is positive. We denote by $\mathcal{U}^+$ the set of positive elements of $\mathcal{U}$. ■

Note that $A \in \mathcal{U}$ is positive if and only if it can be written as $A = BB^*$ for some $B \in \mathcal{U}$.

A.2.12 DEFINITION. A $*$ -morphism between two $*$ -algebras $\mathcal{U}$ and $\mathcal{U}'$ is a homomorphism $\pi: \mathcal{U} \rightarrow \mathcal{U}'$, i.e., for all $A \in \mathcal{U}$ we have $\pi(A) \in \mathcal{U}'$, satisfying

$$\begin{aligned} \pi(\alpha A + \beta B) &= \alpha\pi(A) + \beta\pi(B), \\ \pi(AB) &= \pi(A)\pi(B), \\ \pi(A^*) &= \pi(A)^*, \end{aligned}$$

where $A, B \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{C}$. ■

The $*$ -morphism π in the above definition is positivity preserving because if $A \geq 0$, implies that $A = BB^*$ for some $B \in \mathcal{U}$ and then $\pi(A) = \pi(BB^*) = \pi(B)\pi(B)^* \geq 0$. The kernel $\ker \pi$ of π is defined as

$$\ker \pi = \{A \in \mathcal{U}; \pi(A) = 0\}.$$

We now define a representation of a C^* -algebra, see Definition 2.2.2:

A.2.13 DEFINITION (REPRESENTATION). The pair $(\mathcal{H}, \pi)$ is the representation of a C^* -algebra $\mathcal{U}$. Here, $\mathcal{H}$ is a Hilbert space and π is a $*$ -morphism of $\mathcal{U}$ into $\mathcal{B}(\mathcal{H})$. $(\mathcal{H}, \pi)$ is *faithful* if and only if π is a $*$ -isomorphism between $\mathcal{U}$ and $\pi(\mathcal{U})$, i.e., if and only if $\ker \pi = \{0\}$. $\blacksquare$

The trivial representation, $\pi = 0$, of a C^* -algebra is given by $\pi(A) = 0$ for all $A \in \mathcal{U}$. Note that a representation might be nontrivial but nevertheless have a trivial part. If we define $\mathcal{H}_0$ by

$$\mathcal{H}_0 = \{\psi \in \mathcal{H}; \pi(A)\psi = 0 \text{ for all } A \in \mathcal{U}\},$$

then $\mathcal{H}_0$ is invariant under π . Thus it is said that a representation $(\mathcal{H}, \pi)$ is nondegenerate if $\mathcal{H}_0 = \{0\}$. Alternatively, one says that a set $\mathfrak{M}$ of bounded operators acts nondegenerately on $\mathcal{H}$ if

$$\{\psi; \psi A = 0 \text{ for all } A \in \mathfrak{M}\} = \{0\}.$$

As is usual we denote by $\mathcal{U}^*$ the dual space of $\mathcal{U}$, this means the space of continuous linear functionals over $\mathcal{U}$. The norm for the functional $f \in \mathcal{U}^*$ is

$$\|f\| \doteq \sup_{A \in \mathcal{U}, \|A\|=1} |f(A)|.$$

A.2.14 DEFINITION (STATE). Let $\omega \in \mathcal{U}^*$ be a linear functional on the $*$ -algebra $\mathcal{U}$. ω is positive if

$$\omega(A^*A) \geq 0,$$

for all $A \in \mathcal{U}$. If ω is a positive linear functional over a C^* -algebra $\mathcal{U}$ with $\|\omega\| = 1$, ω is called a state. $\blacksquare$

The fact of the state ω be positive implies its continuity and the reciprocal fact is also true [BR03a], see [Bar, Theorem 38.25]:

Theorem A.2.15:

Let $\mathcal{U}$ a C^ -algebra with identity. The linear functional $\phi: \mathcal{U} \rightarrow \mathbb{C}$ is positive if and only if it is continuous and satisfy $\|\phi\| = \phi(1)$.* $\clubsuit$

Physically the state ω of a physical system S provides the statistical distribution of the mean values of all the physical quantities (observables) $A \in \mathcal{O}_S$. The real number $\omega \in \mathbb{R}$ is the expectation value of the “physical quantity” $A \in \mathcal{O}_S$ when the system S is in the state ω .

Let $\Omega \in \mathcal{H}$ be any nonzero vector. We define ω_Ω by

$$\omega_\Omega = (\Omega, \pi(A)\Omega),$$

for all $A \in \mathcal{U}$. Note that ω_Ω is a linear function over $\mathcal{U}$, which is also positive:

$$\omega_\Omega(A^*A) = (\Omega, \pi(A^*A)\Omega) = (\Omega, \pi(A)^*\pi(A)\Omega) = (\pi(A)\Omega, \pi(A)\Omega) = |\pi(A)\Omega|^2 \geq 0.$$

For $\omega_1, \omega_2 \in \mathcal{U}^*$, we say that $\omega_1 \geq \omega_2$, when $\omega_1 - \omega_2 \geq 0$ is a positive linear functional. In this case we said that ω_2 is majorized by ω_1 . Note that if $\omega_1, \omega_2 \in \mathcal{U}^*$, and $\lambda \in [0, 1]$, then $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ is a state, where $\omega \geq \lambda\omega_1$ and $\omega \geq (1 - \lambda)\omega_2$.

A.2.16 DEFINITION (PURE STATE). Let $\mathcal{U}$ be a C^* -algebra. $\omega \in \mathcal{U}^*$ is a pure state if the only positive linear functionals majorized by ω have the form $\lambda\omega$ for $\lambda \in [0, 1]$. $E_\mathcal{U}$ and $P_\mathcal{U}$ are the set of states and the set of pure states respectively. $\blacksquare$

Theorem A.2.17 (Gelfand–Naimark–Segal (GNS) construction):

Let ω be a state over a C^* -algebra $\mathcal{U}$. It follows that there exists a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ such that

$$\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)$$

for all $A \in \mathcal{U}$ and, consequently, $\|\Omega_\omega\|^2 = \|\omega\| = 1$. Moreover, the representation is unique up to unitary equivalence. $\mathcal{H}_\omega$ is given by $\mathcal{H}_\omega = \overline{\pi_\omega(\mathcal{U})\Omega_\omega}$. $\blacktriangle$

We now introduce “One-parameter groups”, which are used to study non-interacting systems, as remarked in Chapter 2

A.2.18 DEFINITION (ONE-PARAMETER GROUPS). Let $\mathcal{U}$ be a C^* -algebra with identity $\mathbb{1}$. $U \doteq \{U_t\}_{t \in \mathbb{R}_0^+} \in \mathcal{B}(\mathcal{U})$ is a “one-parameter semigroup” —or *linear dynamical system*— if

$$(A.2.19) \quad U_0 = \mathbb{1} \text{ and } U_s U_t = U_{s+t},$$

for all $s, t \in \mathbb{R}_0^+$. If (A.2.19) holds for all $s, t \in \mathbb{R}$, $U \doteq \{U_t\}_{t \in \mathbb{R}}$ is a “one-parameter group” on $\mathcal{U}$. The one-parameter group $\{U_t\}_{t \in \mathbb{R}}$ is “strongly continuous” if the function

$$U_{(\cdot)}A: \mathbb{R} \rightarrow \mathcal{U}, \quad t \mapsto U_t A$$

is continuous for every $A \in \mathcal{U}$.

The “infinitesimal generator” of $\{U_t\}_{t \in \mathbb{R}}$ denoted by S , is defined by using the linear subspace

$$\mathcal{D}(S) \doteq \left\{ A \in \mathcal{U} : t \mapsto U_t A \text{ is differentiable at } t = 0, \text{ i.e., } \lim_{t \rightarrow 0} \frac{U_t A - A}{t} \text{ exists} \right\} \subset \mathcal{U}$$

such that the linear operator $S: \mathcal{D} \rightarrow \mathcal{U}$ is given by

$$iSA \doteq \left. \frac{dU_t}{dt} A \right|_{t=0}.$$

$\blacksquare$

Remark A.2.20. It is relatively straightforward to show that the group $\{U_t\}_{t \in \mathbb{R}}$ of bounded operators on $\mathcal{U}$ is uniformly continuous if, and only if, its generator S is bounded, see [BR03a, Proposition 3.1.1]. $\star$

A.2.2 USEFUL THEOREMS

As follows, we recall two important theorems used in the context of operator algebras, namely, the “Measurable Functional Calculus”, related with the *Spectral Theorem*, and the “Riesz–Markov Theorem”, which associates a measure to every functional [RS81], [Sim15, Parts 1 and 4]:

Theorem A.2.21 (Measurable Functional Calculus):

Let $\mathcal{H}$ be a Hilbert space and let $A = A^* \in \mathcal{B}(\mathcal{H})$ be a bounded operator with spectrum $\text{spec}(A) \subset [-\|A\|, \|A\|]$. If $\sigma(\text{spec}(A))$ denotes the bounded Borel functions on $\text{spec}(A)$, then there is a unique $*$ -homomorphism $\Phi_A: \sigma(\text{spec}(A)) \rightarrow \mathcal{B}(\mathcal{H})$ so that for all $f \in \sigma(\text{spec}(A))$

$$\begin{aligned} \mathbb{1}(x) \equiv 1 &\rightarrow \Phi_A(\mathbb{1}) = \mathbb{1}, \quad \mathbb{1}(x) = x \rightarrow \Phi_A(\mathbb{1}) = A \\ \|\Phi_A(f)\|_{\mathcal{B}(\mathcal{H})} &\leq \|f\|_{\infty}, \end{aligned}$$

where $\Phi_A(f) \doteq f(A)$. Moreover, for $x \in \text{spec}(A)$ and $f, \{f_n\}_{n \in \mathbb{N}} \in \sigma(\text{spec}(A))$, such that $f_n(x) \rightarrow f(x)$ and $\sup_{n \in \mathbb{N}} \|f_n\| < \infty$, we have $\Phi_A(f_n)\psi \rightarrow \Phi_A(f)\psi$ for all $\psi \in \mathcal{H}$. $\clubsuit$

Remark A.2.22. Here Φ_A is called the Measurable Functional Calculus w.r.t. the self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$. $\star$

We now show the following relation between $\sigma(\text{spec}(A))$ and states, Definition A.2.14.

A.2.23 LEMMA. Let $\mathcal{H}$ be a Hilbert space. Consider $\psi \in \mathcal{H}$ such that $\|\psi\| = 1$ and $f \in \sigma(\text{spec}(A))$. If $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ is a C^* -algebra with identity, then

$$\omega_\psi(f(A)) \doteq \langle \psi, f(A)\psi \rangle_{\mathcal{H}} = \langle \psi, \Phi_A(f)\psi \rangle_{\mathcal{H}}$$

is a state. $\blacklozenge$

PROOF. Indeed, by Cauchy–Schwarz inequality note that

$$\left| \langle \psi, \Phi_A(f)\psi \rangle_{\mathcal{H}} \right| \leq \|\Phi_A(f)\| \|\psi\|^2 = \sup_{A \in \mathcal{U}, \|A\|=1} |f(A)|.$$

Therefore, the linear functional $\omega_\psi(f(A))$ is bounded and continuous and since continuity implies positivity we have $\omega_\psi(f(A)) \geq 0$. Finally, take $f = 1$, hence $\Phi_A(f) = \mathbb{1}$ and $\omega_\psi(\mathbb{1}) = 1$. End

We now introduce some notation:

Notation A.2.24.

Let $\mathcal{X}$ be a compact Hausdorff space, we denote by $C_{\mathbb{R}}(\mathcal{X})$ the set of bounded continuous functions from $\mathcal{X}$ to $\mathbb{R}$ and by $C(\mathcal{X})_{+,1}^*$ the normalized positive linear functionals on $C_{\mathbb{R}}(\mathcal{X})$. The Borel probability

measures, $M_{+,1}(\mathcal{X})$, are the probability measures on the Borel sets, see [RS81, Chapter 4], [Sim15, Part 1, Chapter 4]. ♣

Theorem A.2.25 (Riesz–Markov Theorem):

Let $\mathcal{X}$ be a compact Hausdorff space and let $\omega \in C(\mathcal{X})_{+,1}^*$ be a normalized positive linear functional. Then, there is a unique Borel measure μ on $\mathcal{X}$ such that

$$\omega(f) = \int_{\mathcal{X}} f \mu(dx). \quad \spadesuit$$

Throughout this Thesis we use the following identity:

A.2.26 PROPOSITION (DUHAMEL’S IDENTITY). Let $\mathcal{U}$ be a C^* -algebra. If for $t \in [0, 1]$, the map $F: t \mapsto \mathcal{U}$ defined by $F(t) \in \mathcal{U}$ is a bounded operator, then

$$\frac{d}{dt} e^{F(t)} = \int_0^1 e^{uF(t)} F'(t) e^{(1-u)F(t)} du. \quad \blacksquare$$

PROOF. Consider the function $G(u) = e^{uF(t+h)} e^{(1-u)F(t)}$, such that $G(1) - G(0) = \int_0^1 G'(u) du$. Note that $\frac{d}{du} e^{uF(t)} = F(t) e^{uF(t)} = e^{uF(t)} F(t)$. Hence,

$$\begin{aligned} e^{F(t+h)} - e^{F(t)} &= \int_0^1 \left(e^{uF(t+h)} F(t+h) e^{(1-u)F(t)} - e^{uF(t+h)} F(t) e^{(1-u)F(t)} \right) du \\ &= \int_0^1 e^{uF(t+h)} (F(t+h) - F(t)) e^{(1-u)F(t)} du. \end{aligned}$$

By dividing by $h \rightarrow 0$, the proposition follows. End

IN the context we work, a *phase transition* exists if there is more than one *equilibrium state*, i.e., the states of a system which minimize the *free energy* associated to the *variational principle*. Equilibrium states also are known as *Gibbs states*. In classical statistical mechanics, *DLR equations* are a good tool to study the existence of equilibrium states studying the interaction of a region $\Lambda \in \mathcal{P}_f(\mathcal{Q})$ with its exterior $\mathcal{Q} \setminus \Lambda$ [FV, Chapter 6]. Here, $\mathcal{Q} \doteq \mathbb{Z}^d$ denotes the crystal lattice and $\mathcal{P}_f(\mathcal{Q})$ the set of finite subsets of $\mathcal{Q}$. The *boundary conditions* outside of Λ are taken to be fixed and *conditional probabilities* between Λ and its exterior permits to tackle infinite-volume systems at equilibrium. Thus the behaviour of systems is studied using *DLR measures* on infinite probability product spaces [Le 08]. As a consequence the existence of a phase transition is characterized by the number of DLR measures at inverse temperature β . The latter is equivalent to say that if there is just one DLR measure then the system is at equilibrium state or at *DLR state*. W.r.t. classical systems in presence of disorder, there are a few information about “real” problems, in fact, we quote [Bov06]

... From the physical point of view, the former should be more realistic and hence more relevant, so it is natural that we present the general mathematical framework in this context. However, the number of concrete problems one can to this day solve rigorously is quite limited, so that the examples we will treat can mostly be considered as random perturbations of the Ising model. . .

In contrast, the *KMS condition* is a mathematical characterization of thermal equilibrium states for Quantum Many-Body Systems such that a system at equilibrium should satisfy this. These states are called *KMS states*. However, there is a considerable difference in approach between the classic and the quantum cases. For purposes of interest, we consider for the latter, the interaction between Λ and the exterior $\mathcal{Q} \setminus \Lambda$: the system Λ is taken as a subsystem of a larger system [BR03b, Section 6.2.2], [AJP06]. In [Ara74, AI74] the equivalence between the variational principle and the KMS condition for translation invariant states was shown for: (1) the one-dimensional case at all temperatures and (2) all dimensions at high temperature. Note that, as used in this work, KMS states are very useful to study quantum disorder systems, at least random, but ergodic, as described in Chapter 3.

Nevertheless while the KMS states can not be always defined the DLR can, e.g., in systems of

interacting quantum anharmonic oscillators. See details in [AKKR09, Chapter 3], where the authors adapted the DLR approach for quantum spin systems with long-range interactions such that it is possible to construct Gibbs states by means of probability measures as well as in the classic case. In [AM03] and [BP13], equilibrium states using *variational methods* were studied in the context of Fermion Lattice Systems, in particular the long-range interaction case.

B.1 KMS-STATES

We use same notation introduced in Chapter 2-§§2.3.3: Let $\Lambda \in \mathcal{P}_f(\mathfrak{Q})$ be a finite subset of $\mathfrak{Q} \doteq \mathbb{Z}^d$ and let $\mathcal{H}_\Lambda$ and $\mathcal{H}$ the (one-particle) Hilbert space related respectively to Λ and $\mathfrak{Q}$. We denote by $\mathcal{U}_\Lambda \equiv \mathcal{U}(\mathcal{H}_\Lambda)$ and $\mathcal{U} \equiv \mathcal{U}(\mathcal{H})$ the CAR C^* -algebras associated to $\mathcal{H}_\Lambda$ and $\mathcal{H}$ respectively.

We know from physics [CDL77] and mathematics [BR03b], that for the state $\rho_\Lambda \in E_\Lambda$ (see §§2.3.3) there exists a unique *density matrix*, that is a positive operator $d_{\rho_\Lambda} \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$, with $\text{Tr}(d_{\rho_\Lambda}) = 1$, such that the expectation value of $A \in \mathcal{U}_\Lambda$ w.r.t. d_{ρ_Λ} is

$$\langle A \rangle_{d_\Lambda} \doteq \rho_\Lambda(A) = \text{Tr}(d_{\rho_\Lambda} A).$$

Then the von Neumann entropy (2.3.33) is written for any local state ρ_Λ with density matrix d_Λ as

$$S(\rho_\Lambda) = -\text{Tr}(d_\Lambda \ln(d_\Lambda))$$

The finite volume free energy at inverse temperature $\beta \neq 0$ in Λ , and local Hamiltonian $H_\Lambda \in \mathcal{U}_\Lambda$ is

$$(B.1.1) \quad \mathfrak{f}_{\Lambda, H_\Lambda}(\rho_\Lambda) \doteq |\Lambda|^{-1} \rho_\Lambda(H_\Lambda) - (\beta|\Lambda|)^{-1} S(\rho_\Lambda).$$

Thus, the equilibrium state is a minimizer of (B.1.1). Thus, we would like to solve the variational problem $\inf_{\rho \in E} \mathfrak{f}_{\Lambda, H_\Lambda}(\rho) = \inf_{\rho_\Lambda \in E_\Lambda} \mathfrak{f}_{\Lambda, H_\Lambda}(\rho_\Lambda)$, which is equivalent to find the state ρ_Λ such that $\max\{S(\rho_\Lambda) | \text{Tr}(d_{\rho_\Lambda} H_\Lambda) = \mathcal{E}_\Lambda\}$, for some given energy $\mathcal{E}_\Lambda$.

In order to find the minimum free energy we consider the density matrix given by

$$d^{(\beta)} \doteq \frac{e^{-\beta H_\Lambda}}{\text{Tr}(e^{-\beta H_\Lambda})}.$$

Straightforward calculations show that

$$\mathfrak{f}_{\Lambda, H_\Lambda}(\rho_\Lambda) = (\beta|\Lambda|)^{-1} \text{Tr}(d_{\rho_\Lambda} \ln d_{\rho_\Lambda} - d_{\rho_\Lambda} \ln d^{(\beta)}) - (\beta|\Lambda|)^{-1} \ln \text{Tr}(e^{-\beta H_\Lambda}).$$

By the *Klein inequality*, $\text{Tr}(A \ln A - A \ln B) \geq \text{Tr}(A - B)$, for $A, B \geq \mathcal{U}^+$, with the equality iff $A = B$. Hence,

$$\mathfrak{f}_{\Lambda, H_\Lambda}(\rho_\Lambda) \geq -(\beta|\Lambda|)^{-1} \ln \text{Tr}(e^{-\beta H_\Lambda}),$$

because $\text{Tr}(\mathfrak{d}_{\rho_\Lambda}) = \text{Tr}(\mathfrak{d}^{(\beta)}) = 1$. Hence, the matrix density $\mathfrak{d}^{(\beta)}$ minimizes the free energy, which is equal to $-(\beta|\Lambda|)^{-1} \ln \text{Tr}(e^{-\beta H_\Lambda})$. Moreover, since $\mathcal{E}_\Lambda$ is constant we get

$$\beta^{-1} S(\rho_\Lambda) \leq \mathcal{E}_\Lambda + \beta^{-1} \ln \text{Tr}(e^{-\beta H_\Lambda}).$$

Thus, we desire to know when $\beta^{-1} S$ achieves that maximum value for some $\beta \neq 0$. It is easy to prove that the *critical points* β' of the derivative of $g(\beta) = \beta \mathcal{E}_\Lambda + |\Lambda|^{-1} \ln \text{Tr}(e^{-\beta H_\Lambda})$ satisfy

$$\frac{\mathcal{E}_\Lambda}{|\Lambda|} = \frac{\text{Tr}(H_\Lambda e^{-\beta' H_\Lambda})}{\text{Tr}(e^{-\beta' H_\Lambda})} = \text{Tr}(H_\Lambda \mathfrak{d}^{(\beta')}).$$

In particular, note that $g''(\beta) = |\Lambda|^{-1} \left(\langle H_\Lambda^2 \rangle_{\mathfrak{d}^{(\beta)}} - \langle H_\Lambda \rangle_{\mathfrak{d}^{(\beta)}}^2 \right) > 0$, and hence $g(\beta)$ is a *strict convex function*, having g just one critical point.

We denote by $\{E_i\}_{i=1}^n$ the eigenvalues of H_Λ with $E_{\min}$ and $E_{\max}$ their minimum and maximum values, respectively. It follows that

$$\begin{aligned} -E_{\min} &= \lim_{\beta \rightarrow \infty} \beta^{-1} \ln \text{Tr}(e^{-\beta H_\Lambda}), \\ -E_{\max} &= \lim_{\beta \rightarrow -\infty} \beta^{-1} \ln \text{Tr}(e^{-\beta H_\Lambda}), \end{aligned}$$

and due to the positivity of the entropy we have: $E_{\min} < \mathcal{E}_\Lambda < E_{\max}$.

The special case $\beta = 0$, can be seen as $\langle H_\Lambda \rangle_0 = \frac{\text{Tr}(H_\Lambda)}{\text{Tr}(\mathbb{1})} = (\dim \mathcal{H}_\Lambda)^{-1} \sum_{i=1}^{\dim \mathcal{H}_\Lambda} E_i$, where $\dim \mathcal{H}_\Lambda$ is the dimension of the Hilbert space $\mathcal{H}_\Lambda$. Physically $\langle H_\Lambda \rangle_0$ corresponds to the state of maximum entropy and also means that all the energies have the same probability to be achieved. The state in $\beta = 0$ is named “trace state” or “chaotic state”.

For $\beta \neq 0$ the maximum value of $-\mathfrak{f}_{\Lambda, H_\Lambda}$ is called “pressure”, namely

$$(B.1.2) \quad p_{\Lambda, H_\Lambda} \doteq \frac{1}{\beta|\Lambda|} \ln \text{Tr}(e^{-\beta H_\Lambda}).$$

W.r.t. the sequence of boxes (2.3.21), namely,

$$\Lambda_L \doteq \{(x_1, \dots, x_d) \in \mathfrak{Q} : |x_1|, \dots, |x_d| \leq L\} \in \mathcal{P}_f(\mathfrak{Q}),$$

local Gibbs states are defined as

$$(B.1.3) \quad \mathfrak{g}^{(\beta, L)}(B) = \frac{\text{Tr}(B e^{-\beta H_L})}{\text{Tr}(e^{-\beta H_L})}, \quad B \in \mathcal{U}_L$$

where $\beta \in \mathbb{R}$ and $\mathcal{U}_L \equiv \mathcal{U}_{\Lambda_L} \subset \mathcal{U}$. Note, however, that it is not enough to take $L \rightarrow \infty$, in the local Gibbs states to achieve the equilibrium state. As we previously mentioned, this is due to the fact that we must introduce external interactions. Indeed, Araki and Moriya in [AM03, §4 and 7] —we recommend see the details, in particular §4.2 and the Proposition 7.7— shown that the equilibrium state is given by

$$(B.1.4) \quad \rho^{(\beta, L)}(B_1 B_2) = g^{(\beta, L)}(B_1) \text{tr}(B_2), \quad B_1 \in \mathcal{U}_L, B_2 \in \tilde{\mathcal{U}}_L.$$

Here, $\tilde{\mathcal{U}}_L \equiv \mathcal{U}_{\mathcal{Q} \setminus \Lambda_L} \subset \mathcal{U}$ is the C^* -algebra generated by $\{a_x\}_{x \in \mathcal{Q} \setminus \Lambda_L}$ and the identity (here $\text{tr}(\cdot)$ is a *tracial state*, see [AM03, §4.2]). In particular for $B \in \mathcal{U}_L$,

$$(B.1.5) \quad \rho^{(\beta, L)}(B) = g^{(\beta, L)}(B).$$

We now define the KMS states and the KMS condition:

B.1.6 DEFINITION (KMS STATES-CONDITION). Let $z \in \mathbb{C}$ be a complex number, $\tau \doteq \{\tau_t\}_{t \in \mathbb{R}}$ be a family of strongly continuous one-parameter $*$ -automorphisms and ρ be a state on the C^* -algebra $\mathcal{U}$. Then ρ is called a (τ_t, β) -Kubo-Martin-Schwinger $((\tau_t, \beta)$ -KMS) state relative to the group τ if for any $A, B \in \mathcal{U}$, there exists a function $F_{A,B}(z)$ analytic in the open strip (Definition A.1.6 of Appendix A)

$$\mathfrak{D}_\beta = \begin{cases} 0 < \Im z < \beta, & \text{if } \beta \geq 0, \\ \beta < \Im z < 0, & \text{if } \beta \leq 0, \end{cases}$$

and continuous on its closure, which satisfies for all $t \in \mathbb{R}$ the *Kubo-Martin-Schwinger (KMS) condition*

$$F_{A,B}(t) = \rho^{(\beta)}(A \tau_t(B)), \quad F_{A,B}(t + i\beta) = \rho^{(\beta)}(\tau_t(B)A). \quad \blacksquare$$

Several results can be derived from the above definition. Firstly, because of the properties of $F_{A,B}(z)$ in $\mathfrak{D}_\beta$ and its closure, it follows that

$$(B.1.7) \quad \rho^{(\beta)}(A \tau_{t+i\beta}(B)) = \rho^{(\beta)}(\tau_t(B)A).$$

Note that $\rho^{(\beta)}$ is τ invariant, i.e., $\rho^{(\beta)} \circ \tau = \rho^{(\beta)}$. In order to show this we proceed as follows: take $\beta > 0$ and rescale with $\beta = 1$. Define $\rho^{(1)} \doteq \rho$ and suppose that $\mathcal{U}$ has identity $\mathbb{1} \equiv \mathbb{1}_{\mathcal{U}}$. If we take $A = \mathbb{1}$ in (B.1.7) and we define $F_{\mathbb{1},B} \doteq F_B$, we get for any $B \in \mathcal{U}$ that

$$F_B(z + i) = \rho(\tau_{z+i}(B)) = \rho(\tau_z(B)) = F_B(z).$$

Therefore, the analytic function $F_B(z)$ is “entire”, i.e., analytic on the entire complex plane, $\mathbb{C}$. On the other hand, because ρ is state and τ an automorphism we have for $M = \sup\{\|\tau_{i\Im z}(B)\|, 0 <$

$$\Im z < \beta\}$$

$$|F_B(z)| \leq \|\tau_z(B)\| = \|\tau_{i\Im z}(B)\| \leq M.$$

Hence by using the *Liouville's Theorem*, it follows that $F_B(z)$ is constant, and then ρ is τ invariant. If $\mathcal{U}$ has not identity, we use an approximate identity, see [BR03b, Proposition 5.3.3]. Then we can relax Expression (B.1.7)

$$(B.1.8) \quad \rho^{(\beta)}(A\tau_{i\beta}(B)) = \rho^{(\beta)}(BA).$$

The case $\beta = 0$ states that the (τ_t, β) -KMS state is a “trace state” or “chaotic state”, i.e., $\rho^{(0)}(A\tau_t(B)) = \rho^{(0)}(\tau_t(B)A)$, and hence $\rho^{(0)}(AB) = \rho^{(0)}(BA)$ for all $A, B \in \mathcal{U}$. This particular name comes from the fact that physically it corresponds to the state of maximal entropy which is at infinite temperature [AJPP06]. Note that a similar argument done for the $\beta \neq 0$ case can be employed for this state, and therefore also $\rho^0 \circ \tau = \rho^0$. Trace states are studied in the context of *von Neumann algebras* and it is a factor of type II_1 . See details at [Tak02, Chapter XIV.1].

Zero temperature, or physically $\beta \rightarrow \infty$, corresponds to the called *ground states*. These are studied in von Neumann algebras with a factor type I, [Tak02].

By rescaling $\beta \neq 0$, note that mapping the time $t \mapsto \beta t$ and taking $\beta \mapsto 1$, ρ is (τ_t, β) -KMS state iff ρ is $(\tau_{\beta t}, 1)$ -KMS state. Thus $\rho^{(1)}(A\tau_{\beta t+i}(B)) = \rho^{(1)}(\tau_{\beta t}(B)A)$, implies for $\beta \rightarrow -\beta$,

$$\rho^{(1)}(A\tau_{(-\beta)t+i}(B)) = \rho^{(1)}(\tau_{(-\beta)t}(B)A) \rightarrow \rho^{(1)}(A\tau_{\beta(-t)+i}(B)) = \rho^{(1)}(\tau_{\beta(-t)}(B)A),$$

and for this reason $(\tau, -\beta)$ -KMS states are known to correspond with (τ, β) -KMS states with a reversal of time. Finally, the case $\beta = 1$ is related with the *Tomita–Takesaki modular theory* and it is studied in the context of von Neumann algebras. For details, see [BR03b, Sim93].

A number of remarks are in order:

Remarks B.1.9.

1. The set $K_\beta \subset E \subset \mathcal{U}^*$ of (τ, β) -KMS states, form a *convex subset* of the state space E . Moreover, K_β is closed in the weak*-topology. Thus if we call as “extremal points” those points that cannot be written as linear combination of states, these will describe physically the *pure thermodynamic phases* of the model.
2. By [BR03b, Theorem 6.2.42 and discussions on page 294] (quantum spin systems) or [AM03, Theorems 6.4, 11.7, 12.11] (fermion systems), for any translation-invariant and finite-range interaction Φ_f, Φ_i , there is at least one translation-invariant KMS state $\rho \in E_1$. In this case, it is well-known that the set of all translation-invariant KMS states associated with $\{\tau_t\}_{t \in \mathbb{R}}$ and $\beta \in \mathbb{R}^+$ is a weak*-closed face of E_1 and thus, there is at least one such KMS state which is ergodic, see for a complete discussion [ABPM17].

✱

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SELF-DUAL CAR FORMALISM GENERATING FUNCTIONS

he work [ABPM17] provides a pedagogical construction of self-dual CAR algebra starting from a CAR algebra $\mathcal{U} \equiv \mathcal{U}(\mathcal{H})$ associated with some one-particle Hilbert space $\mathcal{H}$. CAR algebra is more widely known than its self-dual analogue. However, the notion of self-dual CAR algebra is more flexible because the one-particle Hilbert space is not fixed anymore.

C.1 SELF-DUAL CAR ALGEBRAS

Let $N \in \mathbb{N}$ be an even number and let $\mathfrak{H}$ be a finite dimensional Hilbert space with dimension $\dim \mathfrak{H} = N$. We will consider an antiunitary antilinear map $\mathfrak{A}: \mathfrak{H} \rightarrow \mathfrak{H}$ with $\mathfrak{A}^2 = \mathbf{1}_{\mathfrak{H}}$ such that

$$\langle \mathfrak{A}\varphi_1, \mathfrak{A}\varphi_2 \rangle_{\mathfrak{H}} \doteq \langle \varphi_2, \varphi_1 \rangle_{\mathfrak{H}} , \quad \varphi_1, \varphi_2 \in \mathfrak{H}.$$

In the sequel, the pairing $(\mathfrak{H}, \mathfrak{A})$ is named a *self-dual Hilbert space*. The latter defines a *self-dual CAR* as follows:

C.1.1 DEFINITION (SELF-DUAL CAR ALGEBRA). Let $(\mathfrak{H}, \mathfrak{A})$ be a self-dual Hilbert space. A self-dual CAR algebra $\mathbb{U} \equiv \text{sCAR}(\mathfrak{H}, \mathfrak{A})$ is a C^* -algebra generated by the identity $\mathbf{1}_{\mathbb{U}}$ and $\{B(\varphi)\}_{\varphi \in \mathfrak{H}}$ satisfying for all $\psi, \varphi \in \mathfrak{H}$:

$$\{B(\psi), B(\varphi)^*\} = \langle \psi, \varphi \rangle_{\mathfrak{H}} \mathbf{1}_{\mathbb{U}}.$$

In this case, the involution element $B(\varphi)^*$ is the linear map $\varphi \mapsto B(\varphi)^*$ defined by $B(\varphi)^* \doteq B(\mathfrak{A}(\varphi))$ for any $\varphi \in \mathfrak{H}$. ▀

Remark C.1.2. Self-dual CAR algebras are also well-defined even if the Hilbert space $\mathfrak{H}$ has infinite or odd dimension. In the odd case, they are $*$ -isomorphic to the direct product of a CAR algebra and a two-dimensional abelian algebra, see [Ara68, Lemmata 3.3, 3.7]. This situation is, however, not important in our study. In [ABPM17] we use the infinite-dimension in order to define the thermodynamic limit on fermion lattice systems. See again [Ara68, Lemma 3.3]. ✱

Note that we can take the Hilbert space $\mathfrak{H}$ in above definition as being

$$(C.1.3) \quad \mathfrak{H} \doteq \mathcal{H} \oplus \mathcal{H}^*,$$

where $\mathcal{H}$ is a Hilbert space (see Definition 2.2.4). Then, as is usual, the scalar product on $\mathfrak{H}$ is given by

$$\langle \varphi, \tilde{\varphi} \rangle_{\mathfrak{H}} \doteq \langle \varphi_1, \tilde{\varphi}_1 \rangle_{\mathcal{H}} + \langle \tilde{\varphi}_2, \varphi_2 \rangle_{\mathcal{H}} , \quad \varphi = (\varphi_1, \varphi_2^*), \tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2^*) \in \mathfrak{H}.$$

Here, φ^* denotes the element of the dual $\mathcal{H}^*$ of the Hilbert space $\mathcal{H}$ which is related to φ via the Riesz representation. In this case, the canonical antiunitary involution $\mathfrak{A}$ of $\mathfrak{H}$ acts as

$$\mathfrak{A}(\varphi_1, \varphi_2^*) \doteq (\varphi_2, \varphi_1^*) , \quad \varphi = (\varphi_1, \varphi_2^*) \in \mathfrak{H}.$$

Note that $\varphi^* = \mathfrak{A}\varphi$ for any $\varphi \in \mathcal{H} \subset \mathfrak{H}$. We now define:

C.1.4 DEFINITION (BASIS PROJECTIONS). A basis projection associated with $(\mathfrak{H}, \mathfrak{A})$ is an orthogonal projection $P \in \mathcal{B}(\mathfrak{H})$, i.e., a self-adjoint projection ($P^2 = P$ and $P^* = P$) satisfying $\mathfrak{A}P\mathfrak{A} = P^\perp \doteq 1_{\mathfrak{H}} - P$. We denote by $\mathfrak{h}_P$ the range $\text{ran}P$ of the basis projection P . $\blacksquare$

Note that $\mathfrak{h}_P$ must satisfy the conditions

$$\mathfrak{A}(\mathfrak{h}_P) = \mathfrak{h}_P^\perp \quad \text{and} \quad \mathfrak{A}(\mathfrak{h}_P^\perp) = \mathfrak{h}_P$$

for any basis projection P . By [Ara68, Lemma 3.3], an explicit basis projection $P \in \mathcal{B}(\mathfrak{H})$ associated with $(\mathfrak{H}, \mathfrak{A})$ can always be constructed because $\dim \mathfrak{H} \in 2\mathbb{N}$. Moreover, $\varphi \mapsto (\mathfrak{A}\varphi)^*$ is a unitary map from $\mathfrak{h}_P^\perp$ to the dual space $\mathfrak{h}_P^*$. In this case we can identify $\mathfrak{H}$ with

$$(C.1.5) \quad \mathfrak{H} \equiv \mathfrak{h}_P \oplus \mathfrak{h}_P^*$$

With this in mind, for the basis projection $P \in \mathcal{B}(\mathfrak{H})$ with range $\text{ran}P = \mathcal{H}$ we can associate the CAR algebra $\mathcal{U}$ and the self-dual CAR algebra $\mathbb{U}$ via

$$B(\varphi) \equiv B_P(\varphi) \doteq a(\varphi_1) + a(\varphi_2)^* , \quad \varphi = (\varphi_1, \varphi_2^*) \in \mathfrak{H}.$$

The elements $B(\varphi + \mathfrak{A}\varphi)$, $\varphi \in \mathcal{H}$, can thus be seen as *field operators* in the context of CAR algebra.

An important set of $*$ -automorphisms are the named *Bogoliubov transformations*, which are defined as follows: For any unitary operator $U \in \mathcal{B}(\mathfrak{H})$ such that $U\mathfrak{A} = \mathfrak{A}U$, the family of elements $B(U\varphi)_{\varphi \in \mathfrak{H}}$ satisfying C.1.1 and, together with the unit 1 , generates $\mathbb{U} \equiv \text{sCAR}(\mathfrak{H}, \mathfrak{A})$. Like in [Ara71, Section 2], such a unitary operator $U \in \mathcal{B}(\mathfrak{H})$ commuting with the antiunitary

map $\mathfrak{U}$ is named a Bogoliubov transformation, and the unique $*$ -automorphism χ_U such that

$$(C.1.6) \quad \chi_U(B(\varphi)) = B(U\varphi), \quad \varphi \in \mathfrak{H},$$

is called in this case a *Bogoliubov $*$ -automorphism*.

C.2 EXISTENCE OF GENERATING FUNCTIONS IN THE SELF-DUAL CAR FORMALISM

By using self-dual CAR formalism, in the sequel we describe the work [ABPM17], which use this approach in contrast to Chapter (4):

Firstly we introduce some notation:

Notation C.2.1.

1. If $h \in \mathcal{B}(\mathcal{H})$ satisfies $h^* = -\mathfrak{U}h\mathfrak{U}$ we say that it is a self-dual operator on $(\mathcal{H}, \mathfrak{U})$.
2. The basis projection P diagonalizes the self-dual operator $h \in \mathcal{B}(\mathcal{H})$ if

$$h \doteq \frac{1}{2} (Ph_P P - P^\perp \mathfrak{U}h_P^* \mathfrak{U}P^\perp), \quad \text{with} \quad h_P \doteq 2PhP.$$

3. For $k \in \mathbb{N}_0$, $\mathcal{H}^{(k)}$ denotes the copy of $\mathcal{H}$ such that $\varphi^{(k)}$ is the copy of $\varphi \in \mathcal{H}$.
4. For any $n \in \mathbb{N}$

$$(C.2.2) \quad \mathfrak{H}^{(n)} \doteq \bigoplus_{k=0}^{n-1} \mathcal{H}^{(k)}.$$

♣

Let $H \in \mathbb{U}$ be the Hamiltonian given by

$$H = d\Gamma(h) + dY(g) + W, \quad W = W^* \in \mathbb{U},$$

where, $h = h^* \in \mathcal{B}(\mathcal{H})$ and $g = -g^*$ is an *antilinear* operator on $\mathcal{H}$. $h = h^* \in \mathcal{B}(\mathcal{H})$ is the free part of the Hamiltonian and $d\Gamma(h) \in \mathbb{U}$ its second quantization. $dY(g) \in \mathbb{U}$ is the non-gauge invariant quadratic part of H , where to simplify the discussion, we consider this term equal to *zero*, and hence $g \equiv 0$. W is the interparticle interaction, which we only require to be finite. Thus, in a first instance, disorder systems are included in our results at least for random, but ergodic, interactions, as described in §§1.3.1. As is usual in the fermionic context, interactions are written as even elements of $\mathbb{U}$. Hence, if K is a self-adjoint element of $\mathbb{U}$ one can deal with *Berezin (Grassmann) integrals* in order to write the quantity

$$\text{tr}(e^{-\beta H} e^{sK}), \quad \beta \in \mathbb{R}^+, s \in \mathbb{R},$$

as a limit of these, where tr is the tracial state on $\mathbb{U}$. This in turns, suitably allows to handle

with *Gaussian* Berezin integrals

$$(C.2.3) \quad \frac{\mathrm{tr}(e^{-\beta(d\Gamma(h)+W)}e^{sK})}{\mathrm{tr}(e^{-\beta d\Gamma(h)})} = \lim_{n \rightarrow \infty} \int d\mu_{C_{h,P}^{(n)}}(\mathfrak{H}^{(n_\beta)}) e^{\mathcal{W}_{W,sK}^{(n_\beta)}}.$$

Here: (i) $\mathfrak{H}^{(n_\beta)}$ is the Hilbert space given by (C.2.2) with $n_\beta \doteq n + \lfloor n/\beta \rfloor$, (ii) P is a basis projection diagonalizing h , (iii) $\int d\mu_{C_{h,P}^{(n_\beta)}}(\mathfrak{H}^{(n_\beta)})$ is the Gaussian Berezin integral on some Grassmann algebra over $\mathfrak{H}^{(n_\beta)}$ associated with an explicit covariance $C_{h,P}^{(n_\beta)}$ only dependent on $h, P, \beta \in \mathbb{R}^+$ and $n \in \mathbb{N}$ and (iv) $\mathcal{W}_{W,sK}^{(n_\beta)}$ is the result of a canonical mapping from $\mathbb{U}$ to the Grassmann algebra over $\mathfrak{H}^{(n_\beta)}$ and only depends on $W, sK, \beta \in \mathbb{R}^+$ and $n_\beta \in \mathbb{N}$. Note the Expression (C.2.3) has a similar form than the factor inside the “ln” in the finite logarithmic moment generating function

$$(C.2.4) \quad J_{\Lambda_L}(s) \doteq \frac{1}{|\Lambda_L|} \ln \frac{\mathrm{tr}(e^{-\beta(d\Gamma(h)+W_L)}e^{sK_L})}{\mathrm{tr}(e^{-\beta d\Gamma(h)+W_L})},$$

where $K_L \equiv K_{\Lambda_L} \in \mathbb{U}$ and $W_L \equiv W_{\Lambda_L} \in \mathbb{U}$ are the operators $K \in \mathbb{U}$ and $W \in \mathbb{U}$ restricted to the box Λ_L given by (1.2.4). In fact, Expressions similar to (C.2.3) will permit to show the analyticity around $s = 0$ of $J_{\Lambda_L}(s)$ for weakly interacting fermion systems, when the interactions involving $H, K \in \mathbb{U}$ are *finite range translation invariant*.

Now, with the intent of showing analyticity we introduce our main results regarding the covariance $C_{h,P}^{(n)}$ (see notations C.2.1):

Theorem C.2.5 (Pfaffian bounds):

Let h be a self-dual Hamiltonian on $(\mathcal{H}, \mathfrak{A})$ and take any basis projection P diagonalizing h . Then, for $\beta \in \mathbb{R}^+$, $n, m, N \in \mathbb{N}$ with $n > \beta \|h\|_{\mathcal{B}(\mathcal{H})}$, $\mathfrak{M} \in \mathrm{Mat}(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$, and all

$$\{(k_q, \varphi_q, j_q)\}_{q=1}^{2N} \subset \{0, \dots, n_\beta - 1\} \times \mathcal{H} \times \{1, \dots, m\},$$

with $n_\beta \doteq n + \lfloor n/\beta \rfloor$, the following sharp bound for Pfaffians holds true:

$$\left| \mathrm{Pf} \left[\mathfrak{M}_{j_q, j_l} \left\langle \mathfrak{A} \varphi_q^{(k_q)}, C_{H,P}^{(n)} \varphi_l^{(k_l)} \right\rangle_{\mathfrak{H}^{(n_\beta)}} \right]_{q,l=1}^{2N} \right| \leq 2^N \prod_{q=1}^{2N} \|\varphi_q\|_{\mathcal{H}} \mathfrak{M}_{j_q, j_q}^{1/2}. \quad \spadesuit$$

Remark C.2.6. As it is well-known, the Pfaffian is defined for a $2N \times 2N$ skew-symmetric complex matrix M with $N \in \mathbb{N}$ by

$$\mathrm{Pf}[M_{k,l}]_{k,l=1}^{2N} \doteq \frac{1}{2^N N!} \sum_{\pi \in \mathcal{S}_{2N}} (-1)^\pi \prod_{j=1}^N M_{\pi(2j-1), \pi(2j)},$$

where $\mathcal{S}_{2N}$ is the set of all permutations of $2N$ elements. It is easy to prove that for $C \in \mathcal{B}(\mathcal{H})$ and $\varphi_1, \dots, \varphi_{2N}$, matrices defined by the elements $\langle \mathfrak{A}\varphi_k, C\varphi_l \rangle_{k,l}^{2N}$ are skew-symmetric. In particular, note that the matrix $\left[\mathfrak{M}_{j,k} \langle \mathfrak{A}\varphi_k, C\varphi_l \rangle \right]_{j,l}^{2N}$ is also skew-symmetric, where for $m \in \mathbb{N}$, $\mathfrak{M} \in \text{Mat}(m, \mathbb{R})$ is positive. *

Theorem C.2.5 does not require of translation invariance, a stronger result than those found in the literature. We now state the second main result w.r.t. covariances $C_{h,P}^{(n)}$. But we must first introduce some notation:

Notation C.2.7.

1. $\{(\mathcal{H}_L, \mathfrak{A}_L)\}_{L \in \mathbb{R}_0^+}$ is a sequence of (one-particle) self-dual Hilbert spaces of *finite* systems restricted to the boxes Λ_L given by (1.2.4). $h_{\Lambda_L} \equiv h_L = h_L^* \in \mathcal{B}(\mathcal{H}_L)$ is the self-adjoint Hamiltonian, i.e., $h_L = -\mathfrak{A}_L h_L \mathfrak{A}_L$.
2. $(\mathcal{H}_\infty, \mathfrak{A}_\infty)$ is the (one-particle) self-dual Hilbert space of the *infinite* fermion system on the lattice $\mathfrak{Q} \doteq \mathbb{Z}^d$, $d \in \mathbb{N}$. Then, $h_\infty = h_\infty^* \in \mathcal{B}(\mathcal{H}_\infty)$ is the self-adjoint Hamiltonian, i.e., $h_\infty = -\mathfrak{A}_\infty h_\infty \mathfrak{A}_\infty$.
3. $\{e_x\}_{x \in \Lambda_L}$ is a canonical orthonormal basis of h_L while $\{e_x\}_{x \in \mathfrak{Q}}$ is of h_∞ , see Expression (1.3.6).
4. $\mathcal{P} \doteq \{P_L\}_{L \in \mathbb{R}_0^+}$ denotes the sequence of basis projections associated to $\{(\mathcal{H}_L, \mathfrak{A}_L)\}_{L \in \mathbb{R}_0^+}$, diagonalizing $\{h_L\}_{L \in \mathbb{R}_0^+}$. Such a sequence strongly converges, as $L \rightarrow \infty$, to the orthogonal projection $P_\infty \in \mathcal{B}(\mathcal{H}_\infty)$ (diagonalizing h_∞), which satisfies

$$\mathbf{D}_{\mathcal{P}} \doteq \sup_{x \in \mathfrak{Q}} \sum_{y \in \mathfrak{Q}} \left| \langle e_x, P_\infty e_y \rangle_{\mathcal{H}_\infty} \right| < \infty.$$

5. Gapped self-dual Hamiltonians h_∞ with spectrum $\text{spec}(h_\infty)$, satisfy

$$\mathfrak{g}_{h_\infty} \doteq \inf \{ \varepsilon > 0 : [-\varepsilon, \varepsilon] \cap \text{spec}(H_\infty) \neq \emptyset \} > 0.$$

6. For $\Lambda_{L_1}, \Lambda_{L_2} \in \mathcal{P}_f(\mathfrak{Q})$ with $\Lambda_{L_1} \subset \Lambda_{L_2}$, $\beta > 0$, $n \in \mathbb{N}$, $n_\beta \doteq n + \lfloor n/\beta \rfloor$, $\mathfrak{H}^{(n_\beta)}$ defined by (C.2.2) and the covariance C as given in Theorem C.2.5, for $h = h_{L_2}$ and $P = P_{L_2}$, the decay parameter is

$$(C.2.8) \quad \omega_{h_\infty, \mathcal{P}} \doteq \limsup_{L_1 \rightarrow \infty} \limsup_{L_2 \rightarrow \infty} \lim_{n \rightarrow \infty} \sup_{k_1 \in \{0, \dots, n_\beta - 1\}} \sup_{x_1 \in \Lambda_{L_1}} \left\{ \beta n^{-1} \sum_{k_2=0}^{n_\beta-1} \sum_{x_2 \in \Lambda_{L_1}} \left| \left\langle e_{x_1}^{(k_1)}, C_{h_{L_2}, P_{L_2}}^{(n)} e_{x_2}^{(k_2)} \right\rangle_{\mathfrak{H}^{(n_\beta)}} \right| \right\}. \quad \clubsuit$$

We finally state:

Theorem C.2.9 (Summability of the covariance):

For any lattice $\mathfrak{Q} \doteq \mathbb{Z}^d$, $d \in \mathbb{N}$, let $h_\infty \in \mathcal{B}(\mathcal{H}_\infty)$ be any self-dual Hamiltonian on $(\mathcal{H}_\infty, \mathfrak{A}_\infty)$ and any $\beta \in \mathbb{R}^+$. Then:

1. For all self-dual Hamiltonians $h_\infty \in \mathcal{B}(\mathcal{H}_\infty)$:

$$\omega_{h_\infty, \mathcal{P}} \leq \mathcal{O}((\beta + 1)^{d+1}).$$

2. For gapped self-dual Hamiltonians:

$$\omega_{h_\infty, \mathcal{P}} \leq \mathcal{O} \left((g_{h_\infty}^{-1} + 1)^{d+1} \right).$$

Note that by the form of the decay parameter, Expression (C.2.8), to prove Theorem (C.2.9) we use Combes–Thomas estimates described in §§1.3.1. In fact, using such estimates we show that for inverse temperature $\beta > 0$ and $h_\infty \in \mathcal{B}(\mathcal{H}_\infty)$ the summability of the Fermi distribution

$$\mathbf{D}_{h_\infty, \beta} \doteq \sup_{\alpha \in [0, \beta]} \sup_{x \in \mathcal{V}} \left\{ \sum_{y \in \mathcal{V}} \left| \left\langle e_x, \frac{e^{\alpha h_\infty}}{1 + e^{\beta h_\infty}} e_y \right\rangle_{\mathcal{H}_\infty} \right| \right\}$$

is bounded by: $\mathcal{O}((\beta + 1)^{d+1})$ for item 1 in Theorem C.2.9 and by $\mathcal{O}((g_{h_\infty}^{-1} + 1)^{d+1})$ for item 2. As a consequence, item 1 is only useful for non-zero temperatures. Nevertheless, in the gapped case 2, we can study systems at zero temperature. In our discussion we notice that such bounds also corresponds to translation-invariant free fermion systems. Indeed, in the same way that in Theorem C.2.5, the above theorem applies even for disordered systems. However, as mentioned, when studying analyticity around zero of $\lim_{L \rightarrow \infty} J_{\Lambda_L}(s)$ for weakly interacting fermions, we will assume that the systems are finite range translation invariants. This in turn, by Bryc's Theorem, implies a central limit theorem for the family of distribution measures $\{\mathfrak{m}_{\rho, K_L}\}_{L \in \mathbb{R}_0^+}$ associated to the observables $\{K_L\}_{L \in \mathbb{R}_0^+} \in \mathbb{U}$ and the KMS-state $\rho \in \mathbb{U}^*$, see subsequent discussion to Expression (1.3.3).

IN the sequel we introduce Combes–Thomas estimates which are useful to analyse the systems we are dealing with. Note that besides Combes–Thomas estimates have not sharp bounds, are taking into account to obtain *good* bounds of two–point correlation *Green functions* while analyzing disorder systems. See [CT73, AG98, AW15] for more details.

For any operator $h \in \mathcal{B}(\mathcal{H})$ and $\mu \in \mathbb{R}_0^+$, let

$$(D.0.1) \quad S_0(h, \mu) \doteq \sup_{x \in \mathfrak{L}} \sum_{y \in \mathfrak{L}} e^{\mu|x-y|} \left| \langle e_x, h e_y \rangle_{\mathcal{H}} \right| \in \mathbb{R}_0^+ \cup \{\infty\}.$$

with $\mathfrak{L} \doteq \mathbb{Z}^d$. Note that

$$(D.0.2) \quad S_0(h_1 h_2, \mu) \leq S_0(h_1, \mu) S_0(h_2, \mu),$$

for any $h_1, h_2 \in \mathcal{B}(\mathcal{H})$ and $\mu \in \mathbb{R}_0^+$. In particular, for any $z \in \mathbb{C}$, $h \in \mathcal{B}(\mathcal{H})$ and $\mu \in \mathbb{R}_0^+$,

$$(D.0.3) \quad S_0(e^{zh}, \mu) \leq e^{S_0(zh, \mu)} = e^{|z| S_0(h, \mu)}$$

and hence,

$$\left| \langle e_x, e^{zh} e_y \rangle_{\mathcal{H}} \right| \leq e^{|z| S_0(h, \mu)} e^{-\mu|x-y|}.$$

The bound obtained here can be sharpened if $z = it$ is imaginary by using Combes–Thomas estimates, first proven in [CT73]. To this end, we present a version of this estimate that is adapted to the present setting: Given a self-adjoint operator $h = h^* \in \mathcal{B}(\mathcal{H})$ whose spectrum is denoted by $\text{spec}(h)$, we define the constants

$$(D.0.4) \quad S(h, \mu) \doteq \sup_{x \in \mathfrak{L}} \sum_{y \in \mathfrak{L}} (e^{\mu|x-y|} - 1) \left| \langle e_x, h e_y \rangle_{\mathcal{H}} \right| \in \mathbb{R}_0^+ \cup \{\infty\},$$

for $\mu \in \mathbb{R}_0^+$, and

$$\Delta(h, z) \doteq \inf \{ |z - \lambda| : \lambda \in \text{spec}(h) \}, \quad z \in \mathbb{C},$$

as the distance from the point z to the spectrum of h . Since the function $x \mapsto (e^{xr} - 1)/x$ is increasing on $\mathbb{R}^+$ for any fixed $r \geq 0$, it follows that

$$(D.0.5) \quad \mathbf{S}(h, \mu_1) \leq \frac{\mu_1}{\mu_2} \mathbf{S}(h, \mu_2), \quad \mu_2 \geq \mu_1 \geq 0.$$

The Combes–Thomas estimate we use is the following:

Theorem D.0.6 (Combes–Thomas):

Let $h = h^* \in \mathcal{B}(\mathcal{H})$, $\mu \in \mathbb{R}_0^+$ and $z \in \mathbb{C}$. If $\Delta(h, z) > \mathbf{S}(h, \mu)$ then, for all $x, y \in \mathfrak{L}$,

$$\left| \left\langle e_x, (z - h)^{-1} e_y \right\rangle \right| \leq \frac{e^{-\mu|x-y|}}{\Delta(h, z) - \mathbf{S}(h, \mu)}. \quad \clubsuit$$

PROOF. This proposition is a version of the first part of [AW15, Theorem 10.5] and is proven in the same way. End

The Combes–Thomas estimate implies the following bound [AG98, Lemma 3]:

D.0.7 PROPOSITION (BOUND ON DIFFERENCES OF RESOLVENTS). Let $h = h^* \in \mathcal{B}(\mathcal{H})$, $\mu \in \mathbb{R}_0^+$ and $\eta \in \mathbb{R}^+$ such that $\mathbf{S}(h, \mu) \leq \eta/2$. Then, for all $x, y \in \mathfrak{L}$ and $u \in \mathbb{R}$,

$$\left| \left\langle e_x, ((h - u)^2 + \eta^2)^{-1} e_y \right\rangle_{\mathcal{H}} \right| \leq 12e^{-\mu|x-y|} \left\langle e_x, ((h - u)^2 + \eta^2)^{-1} e_x \right\rangle_{\mathcal{H}}^{1/2} \left\langle e_y, ((h - u)^2 + \eta^2)^{-1} e_y \right\rangle_{\mathcal{H}}^{1/2}. \quad \blacksquare$$

We are now in a position to introduce the space decay of propagators whose proof can be found in [AG98, Theorem 3]:

Theorem D.0.8 (General space decay of propagators):

Let f be any function bounded by $\|f\|_\infty$ which is analytic in the strip $|\Im m z| \leq \eta$. Then, for any self-adjoint operator $h = h^* \in \mathcal{B}(\mathcal{H})$, $\eta, \mu \in \mathbb{R}^+$, all $x, y \in \mathfrak{L}$ and $t \in \mathbb{R}$,

$$\left| \left\langle e_x, f(h) e_y \right\rangle_{\mathcal{H}} \right| \leq 36\|f\|_\infty e^{-\mu \min\left\{1, \frac{\eta}{2\mathbf{S}(h, \mu)}\right\}|x-y|}. \quad \clubsuit$$

In particular, we are able to prove the following space decay of propagators relevant for the purposes of this work:

D.0.9 COROLLARY (SPACE DECAY OF PROPAGATORS – I). *For any self-adjoint operator $h \in \mathcal{B}(\mathcal{H})$, $\eta, \mu \in \mathbb{R}^+$, all $x, y \in \mathfrak{Q}$ and $t \in \mathbb{R}$,*

$$\left| \left\langle e_x, e^{ith} e_y \right\rangle_{\mathcal{H}} \right| \leq 36e^{|t\eta| - \mu \min\left\{1, \frac{\eta}{2S(h, \mu)}\right\}|x-y|}.$$

□

PROOF. The proof is a simple adaptation of the one from [AG98, Theorem 3]: Fix all parameters of the lemma and observe that Proposition D.0.7 combined with Inequality (D.0.5) yields

$$\begin{aligned} (D.0.10) \quad & \left| \left\langle e_x, ((h-u)^2 + \eta^2)^{-1} e_y \right\rangle_{\mathcal{H}} \right| \\ & \leq 12e^{-\frac{\mu\eta}{2S(h, \mu)}|x-y|} \left\langle e_x, ((h-u)^2 + \eta^2)^{-1} e_x \right\rangle_{\mathcal{H}}^{1/2} \left\langle e_y, ((h-u)^2 + \eta^2)^{-1} e_y \right\rangle_{\mathcal{H}}^{1/2} \end{aligned}$$

for $x, y \in \mathfrak{Q}$, $u \in \mathbb{R}$ and $\eta \in \mathbb{R}^+$. On the other hand, at fixed $\eta \in \mathbb{R}^+$, the function defined by $G(z) \doteq e^{itz}$ on the stripe

$$\mathbb{R} + i\eta[-1, 1] \subset \mathbb{C}$$

is analytic and uniformly bounded by $e^{|t\eta|}$. Using Cauchy's integral formula and some translation by $\pm i\eta$, we write the function G as

$$\begin{aligned} (D.0.11) \quad G(E) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{G(u - i\eta)}{u - i\eta - E} - \frac{G(u + i\eta)}{u + i\eta - E} \right) du \\ &= \frac{\eta}{\pi} \int_{\mathbb{R}} \frac{G(u - i\eta) + G(u + i\eta)}{(E - u)^2 + \eta^2} du - \frac{2\eta}{\pi} \int_{\mathbb{R}} \frac{G(u)}{(E - u)^2 + 4\eta^2} du \end{aligned}$$

for all $E \in \mathbb{R}$ and $\eta \in \mathbb{R}^+$. By spectral calculus, together with (D.0.10)–(D.0.11) and the Cauchy–Schwarz inequality, the assertion follows for e^{ith} . End

D.0.12 COROLLARY (SPACE DECAY OF PROPAGATORS – II). *For any self-adjoint operators $h_1, h_2 \in \mathcal{B}(\mathcal{H})$ and all $x, y \in \mathfrak{Q}$,*

$$\left| \left\langle e_x, \frac{1}{1 + e^{h_2} e^{h_1} e^{h_2}} e_y \right\rangle_{\mathcal{H}} \right| \leq 2 \inf_{\mu \in \mathbb{R}_0^+} \exp\left(-\frac{\mu}{2} e^{-S_0(h_1, \mu) - 2S_0(h_2, \mu)} |x - y|\right).$$

□

PROOF. By (D.0.1)–(D.0.4), note that, for any $\mu \in \mathbb{R}_0^+$,

$$S(e^{h_2} e^{h_1} e^{h_2}, \mu) \leq S_0(e^{h_2} e^{h_1} e^{h_2}, \mu) \leq e^{S_0(h_1, \mu) + 2S_0(h_2, \mu)}.$$

Fix $\mu \in \mathbb{R}_0^+$ and define

$$\mu_1 \doteq \frac{\mu}{2} e^{-S_0(h_1, \mu) - 2S_0(h_2, \mu)}.$$

By (D.0.5), $\mathbf{S}(e^{h_2}e^{h_1}e^{h_2}, \mu_1) < 1/2$. Meanwhile, by using Theorem D.0.6 with $h = e^{h_2}e^{h_1}e^{h_2} \geq 0$,

$$\left| \left\langle e_x, \frac{1}{1 + e^{h_2}e^{h_1}e^{h_2}} e_y \right\rangle_{\mathcal{H}} \right| \leq 2e^{-\mu_1|x-y|}. \quad \boxed{\text{End}}$$

IMPORTANT results in areas of mathematics, e.g., in number theory, or physics, e.g., in statistical mechanics, are studied using ergodic theory. In this appendix, we focus in von Neumann's, Birkhoff's and Kingman's theorems. Birkhoff's theorem is related to additive processes meanwhile Kingman's theorem applies for subadditive processes. It is worth mentioning that the Birkhoff's theorem is a natural consequence of Kingman's theorem as shown, for example, by Avila and Bochi in [AB09]. Furthermore, the result given by von Neumann is a consequence of the Birkhoff's theorem [KB85, Kle13].

E.1 ERGODIC SYSTEMS

Let $(\mathcal{X}, \Sigma, \mu)$ be a probability space and $f: \mathcal{X} \rightarrow \mathcal{X}$ be a measurable map that preserves the measure μ , i.e., $\mu(B) = \mu(f^{-1}(B))$ for all $B \in \Sigma$. We call to $(\mathcal{X}, \Sigma, \mu, f)$ a "dynamical system". This could be thought of as the association of a point $x \in \mathcal{X}$ to the new point $f(x) \in \mathcal{X}$ an instant of time after (or before). By doing this for later times (or earlier), we say that for $n \in \mathbb{Z}$, $f^n(x)$ will be at $\mathcal{X}$; in this case we are making reference to a discrete dynamical system. In the continuous case, i.e., for real time, we say that $f^t: \mathcal{X} \rightarrow \mathcal{X}, t \in \mathbb{R}$, is a "flux", if it satisfies [VO14]

$$f^0 = \text{identity and } f^t \circ f^s = f^{t+s} \text{ for all } t, s \in \mathbb{R}.$$

Dynamical systems with fluxes are associated with differential equations, in the following sense: let $x_0 \in \mathcal{X}$ and consider the differential equation

$$(E.1.1) \quad \frac{dx}{dt} = F(x),$$

where $F(x)$ is a *locally continuous Lipschitz function*. If we denote by $\gamma(t): \mathbb{R} \rightarrow \mathcal{X}$ the solution of (E.1.1) and $f^t: x_0 \mapsto \gamma(t)$, then $(f^t)_{t \in \mathbb{R}}$ is the flux of (E.1.1). However, in the present work we are restricted to the discrete formulation of dynamical systems.

Given the above comments, we will understand by "Ergodic System" $(\mathcal{X}, \Sigma, \mu, f)$ a Dynamical System which satisfies that for every $B \in \Sigma$ such that $f^{-1}(B) = B$, either $\mu(B) = 0$ or $\mu(B) = 1$.

Examples of ergodic systems are easily found in the literature (cf. [KB85, Wal00]). As follows, we enunciate the three theorems as formulated by Viana and Oliveira [VO14]. In order to fix ideas, we recall some basic definitions: Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{h}$ a closed subspace of $\mathcal{H}$. We can write $\mathcal{H} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp$ is the *orthogonal complement* of $\mathfrak{h}$, i.e., the set of all vectors of $\mathcal{H}$ that are orthogonal to $\mathfrak{h}$. Recall that the orthogonal projection to $\mathfrak{h}$, $P_{\mathfrak{h}}: \mathcal{H} \rightarrow \mathfrak{h}$, is nothing but a self-adjoint projection ($P_{\mathfrak{h}}^2 = P_{\mathfrak{h}}$ and $P_{\mathfrak{h}}^* = P_{\mathfrak{h}}$). On the other hand, if μ is a finite Borel measure on a compact Hausdorff space $\mathcal{X}$, the Hilbert space $L^2(\mathcal{X}, \mu)$, denotes the set the complex valued-Borel measurable functions f , such that $\int |f(x)|^2 \mu(dx) < \infty$, with the inner product between two functions $f, g \in L^2(\mathcal{X}, \mu)$ given by

$$\langle f, g \rangle = \int_{\mathcal{X}} \overline{f(x)} g(x) \mu(dx).$$

In the context, if $(\mathcal{X}, \Sigma, \mu)$ is a measurable space, it is said that a property holds at “ μ -almost every point” (a.e.p.) if the set of points in $\mathcal{X}$ where this property fails is contained in a set that has measure zero. Finally, if $(\mathcal{X}, \Sigma, \mu, f)$ is a dynamical system, we say that a measurable function $\varphi: \mathcal{X} \rightarrow \mathbb{R}$ is “invariant” if $\varphi \circ f = \varphi$ for a.e.p. $x \in \mathcal{X}$.

Theorem E.1.2 (von Neumann):

Let $(\mathcal{X}, \Sigma, \mu, f)$ be a dynamical system and take $\varphi \in L^2(\mathcal{X}, \mu)$. If $\widetilde{\varphi}$ denotes the orthogonal projection of φ to the subspace of invariant functions f , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j$$

exists in $L^2(\mathcal{X}, \mu)$ and coincides with $\widetilde{\varphi}$. If f is invertible, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^{-j}$$

exists in $L^2(\mathcal{X}, \mu)$ and coincides with $\widetilde{\varphi}$. ♣

As stated before, Birkhoff’s theorem is a stronger result, because it states that for all $\varphi \in L^1(\mu)$ the convergence holds at μ -a.e.p.

Theorem E.1.3 (Birkhoff):

Let $(\mathcal{X}, \Sigma, \mu, f)$ be a dynamical system and suppose that μ is invariant under f , i.e., $\mu(B) = \mu(f^{-1}(B))$ for all $B \in \Sigma$. For any integrable function $\varphi: \mathcal{X} \rightarrow \mathbb{R}$, the limit

$$\widetilde{\varphi}(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

exists at μ -a.e.p. $x \in \mathcal{X}$. Also, the function $\widetilde{\varphi}$ is integrable and $\int_{\mathcal{X}} \widetilde{\varphi}(x) \mu(dx) = \int_{\mathcal{X}} \varphi(x) \mu(dx)$. ♣

Remark E.1.4. $\tilde{\varphi}$ is known as the “time average” of φ . *

As previously mentioned, Birkhoff’s theorem can be seen as a result of the Kingman’s subadditive theorem. We say that a sequence of functions $\varphi_n: \mathcal{X} \rightarrow \mathbb{R}$ is subadditive for a transformation $f: \mathcal{X} \rightarrow \mathcal{X}$ if for $m, n \geq 1$, $\varphi_{m+n} \leq \varphi_m + \varphi_n \circ f^m$.

Theorem E.1.5 (Kingman):

Let $(\mathcal{X}, \Sigma, \mu, f)$ be a dynamical system and suppose that μ is invariant under f , i.e., $\mu(B) = \mu(f^{-1}(B))$ for all $B \in \Sigma$. Let $\varphi_n: \mathcal{X} \rightarrow \mathbb{R}, n \geq 1$, be a subadditive sequence of measurable functions such that $\varphi_1^+ \in L^1(\mathcal{X}, \mu)$ (where $\varphi^+(x) = \max\{\varphi(x), 0\}$). Then $\left(\frac{\varphi_n}{n}\right)_n$ converges at μ -a.e.p. to some function $\varphi: \mathcal{X} \rightarrow [-\infty, \infty)$ that is invariant under f . Moreover, $\varphi^+ \in L^1(\mathcal{X}, \mu)$ and

$$\int_{\mathcal{X}} \varphi(x) \mu(dx) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\mathcal{X}} \varphi_n(x) \mu(dx) = \inf_{n \rightarrow \infty} \frac{1}{n} \int_{\mathcal{X}} \varphi_n(x) \mu(dx) \in [-\infty, \infty). \quad \clubsuit$$

E.2 ACKOGLU–KRENGEL THEOREM

Statements of previous section are a cornerstone in ergodic theory. However, in the context of our work, we want to use a particular ergodic theorem given by Akcoglu–Krengel, which can be thought like a generalization of Kingman’s theorem, since introduces dimensionality to the system. Additionally, as used in [CL12], Akcoglu–Krengel’s ergodic theorem (AKET) is useful to understanding random operators in the context of quantum systems, which are the main motivation of this work. Note that Kingman’s ergodic theorem can be associated with a sequence of functions that are completing $\mathbb{R}$, while AKET will be associated with a sequence of well-defined functions (depending of some *local* operators) that would complete the cubic lattice $\mathfrak{Q} \doteq \mathbb{Z}^d$. We restrict ourselves to *additive* processes associated with the probability space $(\Omega, \mathfrak{A}_\Omega, \alpha_\Omega)$ defined in Section 2.3.5, even if the AKET holds for superadditive or subadditive ones (cf. [CL12, Definition VI.1.6]).

E.2.1 DEFINITION (ADDITIVE PROCESSES ASSOCIATED WITH RANDOM VARIABLES). $\{\mathfrak{F}^{(\omega)}(\Lambda)\}_{\Lambda \in \mathcal{P}_f(\mathfrak{Q})}$ is an additive process associated with the probability space $(\Omega, \mathfrak{A}_\Omega, \alpha_\Omega)$ if:

1. the map $\omega \mapsto \mathfrak{F}^{(\omega)}(\Lambda)$ is bounded and measurable w.r.t. the σ -algebra $\mathfrak{A}_\Omega$ for any $\Lambda \in \mathcal{P}_f(\mathfrak{Q})$.
2. For all disjoint $\Lambda_1, \Lambda_2 \in \mathcal{P}_f(\mathfrak{Q})$,

$$\mathfrak{F}^{(\omega)}(\Lambda_1 \cup \Lambda_2) = \mathfrak{F}^{(\omega)}(\Lambda_1) + \mathfrak{F}^{(\omega)}(\Lambda_2), \quad \omega \in \Omega.$$

3. For all $\Lambda \in \mathcal{P}_f(\mathfrak{Q})$ and any space shift $x \in \mathfrak{Q}$,

$$(E.2.2) \quad \mathbb{E}[\mathfrak{F}^{(\cdot)}(\Lambda)] = \mathbb{E}[\mathfrak{F}^{(\cdot)}(x + \Lambda)].$$

Recall that $\mathbb{E}[\cdot]$ is the expectation value associated with the distribution α_Ω . ▀

We now define *regular* sequences (cf. [CL12, Remark VI.1.8]) as follows:

E.2.3 DEFINITION (REGULAR SEQUENCES). The family $\{\Lambda^{(L)}\}_{L \in \mathbb{R}^+} \subset \mathcal{P}_f(\mathfrak{Q})$ of non-decreasing (possibly non-cubic) boxes of $\mathfrak{Q}$ is a regular sequence if there is a finite constant $D \in (0, 1]$ and a diverging sequence $\{\ell_L\}_{L \in \mathbb{R}^+}$ such that $\Lambda^{(L)} \subset \Lambda_{\ell_L}$ and $0 < |\Lambda_{\ell_L}| \leq D|\Lambda^{(L)}|$ for all $L \geq 1$. Here, $\{\Lambda_\ell\}_{\ell \in \mathbb{R}^+}$ is the sequence of boxes defined by (2.3.21). ▀

Then, the form of AKET we use in the sequel is the lattice version of [CL12, Theorem VI.1.7, Remark VI.1.8] for additive processes associated with the probability space $(\Omega, \mathfrak{A}_\Omega, \alpha_\Omega)$:

Theorem E.2.4 (Akcoglu–Krengel ergodic theorem):

Let $\{\mathfrak{F}^{(\omega)}(\Lambda)\}_{\Lambda \in \mathcal{P}_f(\mathfrak{Q})}$ be an additive process. Then, for any regular sequence $\{\Lambda^{(L)}\}_{L \in \mathbb{R}^+} \subset \mathcal{P}_f(\mathfrak{Q})$, there is a measurable subset $\tilde{\Omega} \subset \Omega$ of full measure such that, for all $\omega \in \tilde{\Omega}$,

$$\lim_{L \rightarrow \infty} \left\{ |\Lambda^{(L)}|^{-1} \mathfrak{F}^{(\omega)}(\Lambda^{(L)}) \right\} = \mathbb{E} \left[\mathfrak{F}^{(\cdot)}(\{0\}) \right]. \quad \clubsuit$$

See also [KB85].

Remark E.2.5. Recall that a set $\tilde{\Omega} \subset \Omega$ has full measure iff its complement has measure zero. ✱

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