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An introduction to the superstring in flat and $AdS_5 \times S^5$ backgrounds

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Resumo

Apresentamos uma revisão dos elementos básicos do estudo da teoria clássica das supercordas em *backgrounds* planos e curvos, dando ênfase ao caso importante em que o *background* é a variedade $AdS_5 \times S^5$. Nós incluímos um estudo da corda bosônica para revisarmos alguns conceitos básicos da teoria de campos conforme em duas dimensões. Em seguida estudamos a teoria das supercordas em um espaço plano onde apresentamos uma introdução pedagógica ao formalismo de espinores puros. A última parte é dedicada à generalização da ação de Green-Schwarz para o caso de $AdS_5 \times S^5$ e uma apresentação do modelo sigma do formalismo de espinores puros no mesmo *background*.

Abstract

We present a review of the basic elements of the study of classical superstring theory in flat and curved backgrounds, giving emphasis to the very important case of the $AdS_5 \times S^5$ background. We include a study of the bosonic string to review some basic concepts of two dimensional conformal field theory. We then move on to the superstring in flat space where we present a pedagogical introduction to the pure spinor formalism of superstrings. The last part is devoted to the generalization of the Green-Schwarz action to $AdS_5 \times S^5$ and a presentation of the pure spinor sigma model in the same background.

Contents

Chapter 1

Introduction

The study of holographic theories has became a very active field of research in high energy physics. It all started with the Maldacena conjecture, the so-called AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence which can be regarded as a duality between certain four dimensional quantum gauge theories and a theory of closed strings moving in a ten dimensional curved space-time.

To be precise, the example proposed by Maldacena suggested the duality between the four dimensional maximally-supersymmetric $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ and Type IIB superstring theory defined in an $AdS_5 \times S^5$ background, which is the product of a five dimensional anti-de Sitter space (the maximally symmetric space of constant negative curvature) and a five-sphere.

The $\mathcal{N} = 4$ super Yang-Mills theory is an exact conformal field theory in four dimensions with conformal algebra given by $\mathfrak{so}(4, 2)$, which includes the Poincaré algebra generators together with the generators of dilations and special conformal transformations. The supersymmetry generators extend the conformal algebra to the superconformal algebra $\mathfrak{psu}(2, 2|4)$ which is the full algebra of global symmetries of the $\mathcal{N} = 4$ theory. $\mathfrak{psu}(2, 2|4)$ also plays the role of symmetry algebra of Type IIB superstrings in the $AdS_5 \times S^5$ background. As you can see, the gauge and string theory share the same symmetry algebra. However this, in principle, does not imply their undoubted equivalence.

In order to achieve an explicit proof of the AdS/CFT conjecture, it is necessary to fully understand superstring theory when formulated in AdS backgrounds. In this work, we will be focusing on studying the very basic ingredients to formulate superstring theory in $AdS_5 \times S^5$. The plan of this dissertation is to review the basics of bosonic string theory and superstring theory in flat space and then give an introduction to curved backgrounds and the very important case of the $AdS_5 \times S^5$ background. Here I will give a short intro-

duction to the topics included in each of the chapters of this work.

Bosonic strings

The dynamics of a bosonic string propagating in a D-dimensional flat space is given by the Polyakov action which can be interpreted as proportional to the area of the two dimensional surface described by the string. The Polyakov action is given by

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{-g} \, g^{ab} \partial_a X^\mu \partial_b X_\mu \, . \tag{1.1}$$

One can show that the number of dimensions in which the bosonic string propagates is equal to 26, we will see this by requiring the vanishing of the so called Weyl anomaly. The bosonic string theory is a very simple theory, but it presents some inconsistencies. For instance, the quantum spectrum of the theory presents tachyonic states which are a signal of inconsistency[1]. However, the lack of fermionic states in the quantum spectrum is the downfall of bosonic string theory. The inclusion of supersymmetry leads us to superstring theory.

Superstrings

The Ramond-Neveu-Schwarz (RNS) formalism of the superstring is a field theory which maps a supersymmetric worldsheet to a bosonic space-time, while the Green-Schwarz formalism maps a bosonic worldsheet to a supersymmetric space-time. On the one hand, the RNS action is rather easy to quantize but the quantum spectrum of the theory requires an ad-hoc prescription in order to make it supersymmetric, namely the Gliozzi-Scherk-Olive (GSO) projection. Apart from that, the procedure to compute amplitudes higher than one loop is very complicated because of the lack of manifest space-time supersymmetry. On the other hand, the Green-Schwarz formalism possesses manifest space-time supersymmetry and can be easily generalized to a generic supergravity background. However, due to the nature of the constraints of the theory, the covariant quantization for this model is not known, and one is forced to use the kappa symmetry of the theory to quantize it in the light cone gauge, thus losing manifest Lorentz covariance. Furthermore, computing amplitudes in light-cone gauge is extremely difficult. We should point out here, that we will not be studying the RNS formalism in this dissertation.

The Berkovits' pure spinor formalism possess manifest space-time supersymmetry and the covariant quantization can be achieved by means of the inclusion of a BRST charge. In

[1] In fact, the open string tachyon has been related to the instability of D-branes.

this sense, the pure spinor formalism takes the advantages of the RNS and Green-Schwarz formalisms and it does not suffer from their difficulties.

Curved backgrounds

In generic supergravity backgrounds, both Green-Schwarz and pure spinor formalisms can be used to describe a string at the classical level. The equations of motion for the background fields are implied by the kappa symmetry in the Green-Schwarz case and by BRST symmetry in the pure spinor one. As in the flat space case, the kappa symmetry is a complicated gauge symmetry and the Green-Schwarz superstring is difficult to discuss quantum-mechanically, whereas in the pure spinor formalism, kappa symmetry is replaced by BRST symmetry.

The Type IIB supergravity background $AdS_5 \times S^5$ is supported by a self-dual Ramond-Ramond five-form flux. The presence of this background flux does not allow us to use the standard RNS formalism for the superstring. Then one needs to use a formalism with manifest target supersymmetry. One may use the Green-Schwarz formalism or the pure spinor formalism. Just as in the flat space case where the Green Schwarz-formalism can be interpreted as a Wess-Zumino-Witten like sigma model on the coset superspace being a quotient of the ten dimensional super-Poincaré group (SUSY($\mathcal{N} = 2$)) over its Lorentz subgroup $SO(9,1)$, one can make use of a similar approach in the $AdS_5 \times S^5$ case but this time with a target space given by the supercoset $\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$. This supercoset has a $\mathbb{Z}_4$ structure which can be used to write the action model as a bilinear form of currents. A classical action for the pure spinor formalism can be explicitly written down by applying the same technique and by introducing pure spinor variables adapted to $AdS_5 \times S^5$.

Chapter 2

CFTs and the bosonic string theory

This chapter is intended to introduce the basic elements of conformal field theories (CFTs). The techniques presented here will be widely used throughout the whole text. We start by looking at the conformal group and then we develop the basics of CFTs. Since our objective is to study the bosonic string, we work with two major examples; the first, which we will call the XX system, is none other than Polyakov's action after conformal gauge fixing, and the bc system, which arises as the Faddeev-Popov ghost system. Some very useful operator product expansions (OPEs) are computed explicitly. At the end of the chapter we give a comprehensive although short presentation of the BRST quantization of the bosonic string.

The main references we used to develop the basics of CFTs were the excellent books by Polchinski [1] and Blumenhagen and Plauschinn [2], some other references we used include [3] and the very useful lectures by Tong [4]. Some clear presentations of the bosonic string theory include [5], [6] and [7].

2.1 The conformal group

We start by defining the so-called conformal transformations and we will see that these transformations form a group which for the special case of two dimensions possess an infinite number of generators. At the end of this section we will define the Virasoro algebra as a central extension of the Witt algebra.

2.1.1 Conformal transformations

We consider a manifold $\mathcal{M}$ provided with a metric g. We define a conformal transformation as the coordinate transformations defined by the mapping $x \to x'$ such that

$$g'_{\rho\sigma}(x') \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\, g_{\mu\nu}(x)\,. \tag{2.1}$$

We will consider flat spaces, so we can write the condition for conformal transformations as follows

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\, \eta_{\mu\nu}\,, \tag{2.2}$$

where $\Lambda(x)$ is a local scale factor. In other words, conformal transformations are coordinate transformations that leave the metric tensor invariant up to a scale factor. The set of conformal transformations form a group which contains the Poincaré group as a subgroup, since the latter correspond to the case when $\Lambda(x) = 1$.

Let us consider now an infinitesimal coordinate transformation, with a small parameter $\epsilon(x)$, up to first order we can write

$$x'^{\rho} \;=\; x^{\rho} + \epsilon^{\rho}(x)\,. \tag{2.3}$$

In order to know the consequences of equation (2.2) on the infinitesimal coordinate transformations we have just defined, we replace the transformation above in (2.2)

$$\begin{aligned} \eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} &= \eta_{\rho\sigma} \left(\delta^{\rho}_{\mu} + \partial_{\mu}\epsilon^{\rho}\right) \left(\delta^{\sigma}_{\nu} + \partial_{\nu}\epsilon^{\sigma}\right) \\ &= \left(\eta_{\mu\sigma} + \eta_{\rho\sigma}\, \partial_{\mu}\epsilon^{\rho}\right) \left(\delta^{\sigma}_{\nu} + \partial_{\nu}\epsilon^{\rho}\right) \\ &= \eta_{\mu\nu} + \left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right)\,. \end{aligned} \tag{2.4}$$

Then, for this transformation to be conformal, we should require that[1]

$$\begin{aligned} \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} &= f(x)\, \eta_{\mu\nu} \\ \Rightarrow\; 2\, \partial_{\alpha}\epsilon^{\alpha} &= f(x)\, D \\ \Rightarrow\; f(x) &= \frac{2}{D}\, \partial_{\alpha}\epsilon^{\alpha}\,, \end{aligned} \tag{2.5}$$

where in the second line we multiplied both sides by $\eta^{\mu\nu}$. Then replacing it back in (2.4) we have:

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \left(1 + \frac{2}{D}\, \partial_{\alpha}\epsilon^{\alpha}\right) \eta_{\mu\nu}\,. \tag{2.6}$$

[1] Let us stress here that $\mu, \nu = 0, 1, \ldots, D-1$.

In this way, we can conclude that for a coordinate transformation with parameter ϵ to be conformal, the following relation must hold

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \;=\; \frac{2}{D}\,\partial_\alpha\,\epsilon^\alpha \eta_{\mu\nu}\,. \tag{2.7}$$

2.1.2 A look at the conformal group in $D \geq 3$

Taking two derivatives $\partial^\mu \partial^\nu$ from both sides of equation (2.7) we get

$$(D-1)\,\Box(\partial\cdot\epsilon) = 0\,, \tag{2.8}$$

where $\Box = \eta_{\mu\nu}\,\partial^\mu\partial^\nu$. It is easy to note from equation (2.8) that ϵ_μ is at most quadratic in the coordinates, for this reason we can write a general expression for it as follows

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}\,x^\nu x^\rho\,, \qquad c_{\mu\nu\rho} = c_{\mu\rho\nu}\,. \tag{2.9}$$

To study the generators of the conformal group, we need to study each of the terms in (2.9) separately. We take the Table 2.1 appearing in the second chapter of reference [2]

Transformations		Generators		
Translation	$x'^\mu = x^\mu + a^\mu$	$\mathcal{P}_\mu$	$=$	$-i\partial_\mu$
Dilation	$x'^\mu = \alpha x^\mu$	$\mathcal{D}$	$=$	$-ix^\mu\partial_\mu$
Rotation	$x'^\mu = M^\mu{}_\nu x^\nu$	$\mathcal{L}_{\mu\nu}$	$=$	$i\,(x_\mu\partial_\nu - x_\nu\partial_\mu)$
SCFT	$x'^\mu = \frac{x^\mu - (x\cdot x)b^\mu}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)}$	$\mathcal{K}_\mu$	$=$	$-i(2x_\mu x^\nu\partial_\nu - (x\cdot x)\partial_\mu)$

In the table above we can see each of the generators of the conformal group, and the conformal transformations they are associated with. [2]

Now that we have the generators, let us determine the conformal group for $D \geq 3$. In fact we will concentrate on the conformal algebra (the Lie algebra corresponding to the conformal group). Defining

$$\begin{aligned} \mathcal{J}_{\mu,\nu} &= \mathcal{L}_{\mu\nu}\,, \qquad & \mathcal{J}_{-1,\mu} &= \tfrac{1}{2}\,(\mathcal{P}_\mu - \mathcal{K}_\mu)\,, \\ \mathcal{J}_{-1,0} &= \mathcal{D}\,, \qquad & \mathcal{J}_{0,\mu} &= \tfrac{1}{2}\,(\mathcal{P}_\mu + \mathcal{K}_\mu)\,. \end{aligned} \tag{2.10}$$

It is possible to verify that $\mathcal{J}_{mn}$ with $m,n = -1,0,1,\ldots,(D-1)$ satisfy the following commutation relations

$$[\mathcal{J}_{mn},\mathcal{J}_{rs}] = i\,(\eta_{ms}\mathcal{J}_{nr} + \eta_{nr}\mathcal{J}_{ms} - \eta_{mr}\mathcal{J}_{ns} - \eta_{ns}\mathcal{J}_{mr})\,. \tag{2.11}$$

Let us finish this part of the discussion about the conformal group with the following statement. For the case of dimensions $D = p+q \geq 3$, the conformal group of $\mathbb{R}^{p,q}$ is $SO(p+1,q+1)$.

[2] SCFT stands for special conformal transformations.

2.1.3 The conformal group in two dimensions

We now look at the very special case of two dimensions. Let us make a crucial observation, the condition (2.7) in two dimensions reads as follows

$$\partial_0\epsilon_0 = +\partial_1\epsilon_1 \,, \qquad \partial_0\epsilon_1 = -\partial_1\epsilon_0 \,. \tag{2.12}$$

Conditions (2.12) are none other than the well known Cauchy-Riemann equations which appears in complex analysis and state that a complex function whose real and imaginary parts satisfy (2.12) is a holomorphic function. Let us introduce complex variables in the following way

$$z = x^0 + ix^1 \,, \qquad \epsilon = \epsilon^0 + i\epsilon^1 \,, \qquad \partial_z = \frac{1}{2}\left(\partial_0 - i\partial_1\right) \,, \tag{2.13}$$

$$\bar{z} = x^0 - ix^1 \,, \qquad \bar{\epsilon} = \epsilon^0 - i\epsilon^1 \,, \qquad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_0 + i\partial_1\right) \,. \tag{2.14}$$

Since $\epsilon(z)$ is holomorphic, so is $f(z) = z + \epsilon(z)$, then we can conclude that any change of coordinates given by a holomorphic function $f(z)$ gives rise to an infinitesimal two-dimensional conformal transformation $z \mapsto f(z)$.

Now we show that the algebra of infinitesimal conformal transformations in two dimensions is infinite dimensional. Let us consider some general conformal transformations

$$z' = z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n \left(-z^{n+1}\right) \,, \tag{2.15}$$

$$\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \left(-\bar{z}^{n+1}\right) \,, \tag{2.16}$$

where ϵ_n and $\bar{\epsilon}_n$ are just constants. The generators corresponding to a transformation for a particular n are

$$l_n = -z^{n+1}\partial_z \,, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \,. \tag{2.17}$$

It is easy to note that since $n \in \mathbb{Z}$, the number of independent infinitesimal conformal transformations is infinite. Let us now compute the commutator of the generators con-

sidered above

$$\begin{aligned}
[l_m, l_n] &= z^{m+1}\partial_z\left(z^{n+1}\partial_z\right) - z^{n+1}\partial_z\left(z^{m+1}\partial_z\right) \\
&= (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z \\
&= -(m-n)z^{m+n+1}\partial_z \\
&= (m-n)l_{m+n}\,, \qquad (2.18)
\end{aligned}$$

$$\left[\bar{l}_m, \bar{l}_n\right] = (m-n)\bar{l}_{m+n}\,, \qquad (2.19)$$

$$\left[l_m, \bar{l}_n\right] = 0\,. \qquad (2.20)$$

The first commutation relations define one copy of the so-called *Witt algebra*, and because of the other two relations, there is a second copy which commute with the first one. From the observation we made earlier this algebra is infinite dimensional.

Let us now define the so-called *Virasoro algebra*. The central extension $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ of a Lie algebra $\mathfrak{g}$ by $\mathbb{C}$ is defined by the following commutation relations

$$\begin{aligned}
[\,\tilde{x}, \tilde{y}\,]_{\tilde{\mathfrak{g}}} &= [x, y]_{\mathfrak{g}} + c\,p(x, y)\,, & \tilde{x}, \tilde{y} \in \tilde{\mathfrak{g}}\,, \\
[\,\tilde{x}, c\,]_{\tilde{\mathfrak{g}}} &= 0\,, & x, y \in \mathfrak{g}\,, \\
[\,c, c\,]_{\tilde{\mathfrak{g}}} &= 0\,, & c \in \mathbb{C}\,,
\end{aligned} \qquad (2.21)$$

where $p : \mathfrak{g} \times \mathfrak{g} \mapsto \mathbb{C}$ is bilinear. Let the elements of the central extension of the Witt algebra be L_n with $n \in \mathbb{Z}$. We define the Virasoro algebra as the central extension of the Witt algebra, where the constant c in the following equation is called the *central charge*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0}\,. \qquad (2.22)$$

2.2 Aspects of two dimensional conformal field theory

In this section we will review some aspects of two dimensional conformal field theories. We start by looking at how to define Ward identities in CFTs, then we move on to the very important concept of Operator Product Expansions (OPEs), finally we work on some important results which will be very useful to us when dealing with bosonic string theory, such as some energy momentum tensor computations.

2.2.1 Massless scalars, the XX CFT

Let us consider D massless scalar fields X^μ in two dimensions, these fields define the so-called XX system. We will get the action for the XX system by fixing the so-called *conformal gauge* ($g_{ab} = \eta_{ab}$), and performing a Wick rotation on the Polyakov action (1.1)

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu\right) . \tag{2.23}$$

The gauge defined above is called conformal gauge because after gauge fixing the world-sheet metric, we still have some residual symmetry called conformal symmetry, that is, our action is invariant under conformal transformations.

Now let us write the action (2.23) in complex coordinates, in order to achieve this let us consider the following definitions

$$z = \sigma^1 + i\,\sigma^2 , \qquad \bar{z} = \sigma^1 - i\,\sigma^2 . \tag{2.24}$$

We also need to define the derivatives in complex coordinates as follows

$$\partial_z = \partial = \frac{1}{2}\left(\partial_1 - i\,\partial_2\right) , \qquad \partial_{\bar{z}} = \bar{\partial} = \frac{1}{2}\left(\partial_1 + i\,\partial_2\right) . \tag{2.25}$$

Let us now write the explicit form of the metric tensor in the new variables we have just defined

$$g_{ab} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} , \qquad g^{ab} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} . \tag{2.26}$$

Making use of all the notation defined above we can rewrite (2.23) in the following way

$$S = \frac{1}{2\pi\alpha'} \int d^2z\, \partial X^\mu \bar{\partial} X_\mu . \tag{2.27}$$

We can now apply the variational principle and we get the following equations of motion

$$\partial(\bar{\partial} X^\mu) = \bar{\partial}(\partial X^\mu) = 0 . \tag{2.28}$$

It is very easy to see from (2.28) that ∂X^μ is holomorphic, and $\bar{\partial} X^\mu$ is anti-holomorphic (holomorphic in $\bar{z}$). The next step we will take is to find the propagator of the XX system, and define the so-called *Conformal Normal Ordering*.

2.2.2 The XX propagator

In order to find the propagator for our theory, we make use of a very simple property of path integrals. As we know, the path integral of a total derivative vanishes, then:

$$\begin{aligned} 0 &= \int \mathcal{D}X \, \frac{\delta}{\delta X_\mu(z,\bar{z})} \left[\exp\left(-S\right) X^\nu(w,\bar{w})\right] \\ &= \int \mathcal{D}X \, \exp\left(-S\right) \left[\eta^{\mu\nu}\delta^2(z-w,\bar{z}-\bar{w}) + \frac{1}{\pi\alpha'}\,\partial_z\partial_{\bar{z}} X^\mu(z,\bar{z}) X^\nu(w,\bar{w})\right] \\ &= \eta^{\mu\nu}\left\langle \delta^2(z-w,\bar{z}-\bar{w})\right\rangle + \frac{1}{\pi\alpha'}\,\partial_z\partial_{\bar{z}}\left\langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w})\right\rangle \,, \end{aligned} \tag{2.29}$$

The equation (2.29) tells us that the classical equations of motion still holds at the quantum level except at coincident points ($z = w$). If we consider additional fields far away from z and w, we can actually write:

$$\frac{1}{\pi\alpha'}\,\partial_z\partial_{\bar{z}}\left\langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) \ldots\right\rangle = -\eta^{\mu\nu}\left\langle \delta^2(z-w,\bar{z}-\bar{w}) \ldots\right\rangle \,, \tag{2.30}$$

where the dots represent additional insertions (fields) far away from z and w. Then, we can write:

$$\frac{1}{\pi\alpha'}\,\partial_z\partial_{\bar{z}} X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) = -\eta^{\mu\nu}\delta^2(z-w,\bar{z}-\bar{w}) \,, \tag{2.31}$$

which of course holds as an operator equation. Solving equation (2.31) we can write

$$\left\langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w})\right\rangle = -\eta^{\mu\nu}\frac{\alpha'}{2}\ln|z-w|^2 \,. \tag{2.32}$$

Equation (2.32) is what we will refer to as the propagator of the XX system. We will use this result when computing OPEs since we just need to replace the propagator of our theory when contracting fields inside OPEs as we will see later.

Now let us define the *conformal normal ordering* of a general operator $\mathcal{O}$ denoted by $:\mathcal{O}:$, as follows

$$:X^\mu(z,\bar{z}): \;=\; X^\mu(z,\bar{z}) \,, \tag{2.33}$$

$$:X^\mu(z,\bar{z})X^\nu(w,\bar{w}): \;=\; X^\mu(z,\bar{z})X^\nu(w,\bar{w}) + \frac{\alpha'}{2}\,\eta^{\mu\nu}\ln|z-w|^2 \,. \tag{2.34}$$

The whole point with this definition is that the expression above satisfies the equations of motion as we see bellow

$$\begin{aligned} \partial_z\partial_{\bar{z}} :X^\mu(z,\bar{z})X^\nu(w,\bar{w}): &= \partial_z\partial_{\bar{z}} X^\mu(z,\bar{z})X^\nu(w,\bar{w}) + \frac{\alpha'}{2}\,\eta^{\mu\nu}\partial_z\partial_{\bar{z}}\ln|z-w|^2 \\ &= -\pi\alpha'\,\eta^{\mu\nu}\delta^2(z-w,\bar{z}-\bar{w}) + \frac{\alpha'}{2}\,\eta^{\mu\nu}\,2\pi\,\delta^2(z-w,\bar{z}-\bar{w}) \\ &= 0 \,, \end{aligned} \tag{2.35}$$

where in the second line we made use of equation (2.31) as well as the identity

$$\partial_z \partial_{\bar{z}} \ln |z|^2 = 2\pi\, \delta^2(z, \bar{z})\,. \tag{2.36}$$

You should note that equations like (2.34) are supposed to hold inside path integrals, so the first term in the right is actually time ordered in the usual sense of ordinary quantum field theory. For simplicity we will write many expressions in the same fashion, so you should remember that all the operator equations we write are supposed to hold inside path integrals.

2.2.3 Operator product expansion

Let us now define a very useful CFT tool here, the so-called *Operator Product Expansion* (OPE). Consider a product of two local operators, if we take them defined at two points sufficiently close together we can actually approximate this product to arbitrary accuracy by a sum of a string of local operators at one of this points, that is

$$\mathcal{O}_i(z, \bar{z})\, \mathcal{O}_j(w, \bar{w}) = \sum_k c_{ij}^k\, (z - w, \bar{z} - \bar{w})\, \mathcal{O}_k(w, \bar{w})\,. \tag{2.37}$$

As we said earlier, expressions like this are supposed to work inside path integrals, so we can write

$$\langle \mathcal{O}_i(z, \bar{z})\, \mathcal{O}_j(w, \bar{w})\, \ldots \rangle = \sum_k c_{ij}^k\, (z - w, \bar{z} - \bar{w})\, \langle \mathcal{O}_k(w, \bar{w})\, \ldots \rangle\ , \tag{2.38}$$

where again, the dots stand for operator insertions at points far away from z and w.

Now let us find some OPEs for the XX system. We can compactly generalize the definition (2.34) for any functional $\mathcal{F}$ of the fields as follows

$$:\mathcal{F}: = \exp\left(\frac{\alpha'}{4} \int d^2 z_1 d^2 z_2 \ln |z_{12}|^2 \frac{\delta}{\delta X^\rho(z_1, \bar{z}_1)} \frac{\delta}{\delta X_\rho(z_2, \bar{z}_2)}\right) \mathcal{F}\,, \tag{2.39}$$

where $z_{12} = z_1 - z_2$. We now give a very simple example of how this expression works,

$$\begin{aligned}
:X^\mu(z, \bar{z})\, X^\nu(w, \bar{w}): \;&=\; \exp\left(\frac{\alpha'}{4} \int d^2 z_1 d^2 z_2 \ln |z_{12}|^2 \frac{\delta}{\delta X^\rho(z_1, \bar{z}_1)} \frac{\delta}{\delta X_\rho(z_2, \bar{z}_2)}\right) X^\mu(z, \bar{z})\, X^\nu(w, \bar{w}) \\
&=\; X^\mu(z, \bar{z})\, X^\nu(w, \bar{w}) + \\
&\quad + \frac{\alpha'}{4} \int d^2 z_1 d^2 z_2 \ln |z_{12}|^2 \left(\frac{\delta}{\delta X^\rho(z_1, \bar{z}_1)} \frac{\delta}{\delta X^\rho(z_2, \bar{z}_2)}\, X^\mu\, X^\nu\right) \\
&=\; X^\mu(z, \bar{z})\, X^\nu(w, \bar{w}) + \\
&\quad + \alpha' \int d^2 z_1 d^2 z_2 \ln |z_{12}|^2 \left(\eta^{\mu\rho} \delta(z - z_2, \bar{z} - \bar{z}_2)\, \eta^\nu{}_\rho \delta(w - z_1, \bar{w} - \bar{z}_1)\right) \\
&=\; X^\mu(z, \bar{z})\, X^\nu(w, \bar{w}) + \frac{\alpha'}{2}\, \ln |z - w|^2 \eta^{\mu\nu}\,.
\end{aligned} \tag{2.40}$$

Which of course agrees with (2.34). If we invert equation (2.39), we get

$$\begin{aligned}\mathcal{F} &= \exp\left(-\frac{\alpha'}{4}\int d^2z_1 d^2z_2 \ln|z_{12}|^2 \frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)}\frac{\delta}{X^\mu(z_2,\bar{z}_2)}\right) :\mathcal{F}: \\ &= :\mathcal{F}: + \sum \text{contractions}\,. \end{aligned} \tag{2.41}$$

These kind of expressions are what we will refer to as OPEs, where the term contractions stand for taking a couple of fields an *contract* them, that is replacing them by the propagator of the theory we are working with. We can generalize the expression above to find the OPE of any pair of operators as:

$$:\mathcal{F}::\mathcal{G}:=:\mathcal{F}\,\mathcal{G}: + \sum \text{cross-contractions}\,. \tag{2.42}$$

We will heavily use equation (2.42) to compute OPEs in a rather simple way in the rest of this text.

2.2.4 Conformal Ward identities

We now derive the so-called Ward identities for a CFT, we first consider a symmetry in general and then we particularize for the case of conformal symmetry. Let us consider some general field theory with action $S[\phi]$ in D space-time dimensions. The action is defined in such a way that it is invariant under the following symmetry transformations

$$\phi'_\alpha(\sigma) = \phi_\alpha(\sigma) + \delta\phi_\alpha(\sigma)\,, \tag{2.43}$$

where the second term on the right hand side is supposed to be proportional to an infinitesimal parameter. Because of this symmetry, the following product is invariant

$$\mathcal{D}\,\phi' \exp(-S[\phi']) = \mathcal{D}\,\phi \exp(-S[\phi])\,. \tag{2.44}$$

As we know from Noether's theorem, a continuous symmetry in field theory automatically implies the existence of a conserved current, but as we will see in the following, it also implies Ward identities. To see this, let us consider the following transformations

$$\phi'_\alpha(\sigma) = \phi_\alpha(\sigma) + \rho(\sigma)\delta\phi_\alpha(\sigma)\,, \tag{2.45}$$

of course these transformations do not represent a symmetry, so the change in the product (2.44) must be proportional to $\partial_a\rho$, that is

$$\mathcal{D}\,\phi' \exp(-S[\phi']) = \mathcal{D}\,\phi \exp(-S[\phi])\left[1 + \frac{i\,\epsilon}{2\,\pi}\int d^d\sigma\, g^{1/2} j^a \partial_a\rho(\sigma) + O(\epsilon^2)\right]\,, \tag{2.46}$$

where the additional factor $\frac{i\,\epsilon}{2\,\pi}$ has been included for simplicity. For the transformation (2.45) to be a symmetry, it must be true that

$$\frac{\epsilon}{2\,\pi\,i}\int d^d\sigma\, g^{1/2}\rho(\sigma)\,\langle\partial_a j^a\ldots\rangle = 0\,, \tag{2.47}$$

this is actually Noether's theorem, where again, the dots represent additional insertions away from σ. The procedure we sketched above is what we will refer to as *Noether's procedure*; we take a global symmetry, promote the transformation parameter to depend on the coordinates, and finally we identify the current associated to our symmetry which should be proportional to the divergence of the parameter associated to it.

Now we let ρ to be 1 in some region $\mathcal{C}(R)$ of radius R, and 0 outside it. Let us also include some local operator $\mathcal{O}(\sigma_0)$ (σ_0 is allowed to be inside $\mathcal{C}(R)$) in the path integral. The operator transforms as follows

$$\mathcal{O}'(\sigma_0) = \mathcal{O}(\sigma_0) + \epsilon\,\delta\mathcal{O}(\sigma_0)\,. \tag{2.48}$$

In a similar way to before, we can write

$$\begin{aligned}\mathcal{D}\,\phi'\exp(-S[\phi'])\,\mathcal{O}'(\sigma_0) \;&=\; \mathcal{D}\,\phi\exp(-S[\phi])\left[1-\frac{i\,\epsilon}{2\,\pi}\int_{\mathcal{C}(R)} d^d\sigma\, g^{1/2}\nabla_a j^a(\sigma)\,\rho(\sigma)\right]\times\\ &\qquad\times\left(\mathcal{O}(\sigma_0)+\epsilon\,\delta\mathcal{O}(\sigma_0)\right).\end{aligned} \tag{2.49}$$

In order to have a symmetry, it is easy to see that the following expression should vanish

$$-\frac{i}{2\,\pi}\int_{\mathcal{C}(R)} d^d\sigma\, g^{1/2}\nabla_a j^a(\sigma)\mathcal{O}(\sigma_0) + \delta\mathcal{O}(\sigma_0)\,, \tag{2.50}$$

lastly we make use of the divergence theorem to get the so-called Ward identities

$$\int_{\partial\mathcal{C}} d\,A\,n_a j^a\mathcal{O}(\sigma_0) = \frac{2\,\pi}{i}\,\delta\mathcal{O}(\sigma_0)\,, \tag{2.51}$$

where $\partial\mathcal{C}$ is the boundary of $\mathcal{C}(R)$, dA is the area element and n^a is the normal vector to it. We now specialize to two dimensions and complex coordinates getting

$$\oint_{\mathcal{C}(R)} (j\,d\,z - \bar{j}\,d\,\bar{z})\,\mathcal{O}(z_0,\bar{z}_0) = 2\,\pi\delta\mathcal{O}(z_0,\bar{z}_0)\,. \tag{2.52}$$

This equation holds as an operator equation, so considering additional insertions we can write

$$\oint_{\mathcal{C}} d\,z\,\langle j_z(z,\bar{z})\,\mathcal{O}(z_0,\bar{z}_0)\ldots\rangle - \oint_{\mathcal{C}} d\,\bar{z}\,\langle\bar{j}_{\bar{z}}(z,\bar{z})\,\mathcal{O}(z_0,\bar{z}_0)\ldots\rangle = 2\,\pi\,\langle\delta\mathcal{O}(z_0,\bar{z}_0)\ldots\rangle\,. \tag{2.53}$$

In CFTs, the currents j and $\bar{j}$ are holomorphic and anti-holomorphic respectively, so the integral above just picks up the residues in the following way

$$\mathrm{Res}\,\{j(z)\,\mathcal{O}(z_0,\bar{z}_0)\} + \mathrm{Res}\,\{\bar{j}(\bar{z})\,\mathcal{O}(z_0,\bar{z}_0)\} = \frac{1}{i}\,\delta\mathcal{O}(z_0,\bar{z}_0)\,. \tag{2.54}$$

In the left hand side of (2.54) we have the rest picked from the OPEs $j\,\mathcal{O}$ and $\bar{j}\,\mathcal{O}$, and in the right hand side we just have the symmetry transformations of $\mathcal{O}$. In other words, the equation above tells us that, in conformal field theories, we have a very close relation between the symmetry properties of an operator and the OPE of it with the conserved current associated to such a symmetry.

We now apply the results obtained here to our example, the XX system, we will derive the energy momentum tensor of the theory by considering Noether's procedure, and then we study the conformal properties of the fields involved.

2.2.5 The XX energy momentum tensor

The energy momentum tensor for the XX system can be found by using the Noether procedure which we described before. We first consider the worldsheet translations $\delta z = \epsilon$, $\delta\bar{z} = \bar{\epsilon}$. Under these translations the fields transform as follows

$$\delta X^\mu(z,\bar{z}) = \epsilon\,\partial X + \bar{\epsilon}\,\bar{\partial} X\,. \tag{2.55}$$

We now promote ϵ and $\bar{\epsilon}$ to depend on the worldsheet variables and vary the action as follows

$$\begin{aligned}\delta S &= \frac{1}{2\pi\alpha'}\int d^2z\,\left[\bar{\partial}X_\mu\partial\left(\partial X^\mu\epsilon + \bar{\partial}X^\mu\bar{\epsilon}\right) + \partial X_\mu\bar{\partial}\left(\partial X^\mu\epsilon + \bar{\partial}X^\mu\bar{\epsilon}\right)\right]\\ &= \frac{1}{2\pi}\int d^2z\left[-\frac{1}{\alpha'}\,:\partial X^\mu\partial X_\mu:\bar{\partial}\epsilon - \frac{1}{\alpha'}\,:\bar{\partial}X^\mu\bar{\partial}X_\mu:\partial\bar{\epsilon} + \right.\\ &\qquad\left. + \partial(\bar{\partial}X_\mu\partial X^\mu\epsilon) + \bar{\partial}(\partial X_\mu\bar{\partial}X^\mu\bar{\epsilon})\right]\,.\end{aligned} \tag{2.56}$$

We can drop the last two terms in the expression above since they are total derivatives. As you can see, there is an holomorphic and anti-holomorphic part of the energy momentum tensor

$$T(z) = -\frac{1}{\alpha'}\,:\partial X^\mu\partial X_\mu:\,, \qquad \bar{T}(\bar{z}) = -\frac{1}{\alpha'}\,:\bar{\partial}X^\mu\bar{\partial}X_\mu:\,. \tag{2.57}$$

The fact that Noether currents split into holomorphic and anti-holomorphic parts is a general property of CFTs an not only of the XX system we are studying.

We will now derive the energy momentum tensor by making use of the usual method in field theory, that is we vary the action with respect to the metric

$$T_{\alpha\beta} = -\frac{4\,\pi}{\sqrt{-g}}\,\frac{\delta\,S}{\delta\,g^{\alpha\beta}}\,. \tag{2.58}$$

Let us apply (2.58) to Polyakov's action (1.1). The part of the variation which involves the metric field is the following

$$\begin{aligned}\delta S &= -\frac{1}{4\,\pi\alpha'}\int d\tau d\sigma\,\left[\delta(\sqrt{-g})\,g^{\alpha\beta}\,\partial_\alpha X^\mu\partial_\beta X_\mu + \sqrt{-g}\,\delta g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + 2\sqrt{-g}\,g^{\alpha\beta}\partial_\alpha\delta X^\mu\partial_\beta X_\mu\right]\\ &= -\frac{1}{4\,\pi\alpha'}\int d\tau d\sigma\,\left[\frac{1}{2}\sqrt{-g}\,g_{ab}\partial^\alpha X^\mu\partial_\alpha X_\mu + \sqrt{-g}\,\partial_\alpha X^\mu\partial_\beta X_\mu\right]\delta g^{\alpha\beta} + O(\delta X^\mu)\,.\end{aligned} \tag{2.59}$$

And then we have:

$$T_{\alpha\beta} = -\frac{1}{\alpha'}\,:\left(\partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{2}\,\delta_{\alpha\beta}\,\partial_\rho X^\mu\partial^\rho X_\mu\right):\,. \tag{2.60}$$

A key property of CFTs is that the energy momentum tensor is traceless, for instance, look at (2.60)

$$\eta^{\alpha\beta}\,T_{\alpha\beta} = -\frac{1}{\alpha'}:\left(\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{2}\,\eta^{\alpha\beta}\delta_{\alpha\beta}\,\partial_\rho X^\mu\partial^\rho X_\mu\right):=0\,. \tag{2.61}$$

Furthermore, we can write the energy momentum tensor (2.60) in complex coordinates by using the following coordinate transformations

$$T_{\alpha'\beta'} = \frac{\partial\sigma^a}{\partial x^{\alpha'}}\frac{\partial\sigma^b}{\partial x^{\beta'}}\,T_{ab}\,. \tag{2.62}$$

For instance, let us compute $T_{z\bar{z}}$

$$\begin{aligned}T_{z\bar{z}} &= \frac{\partial\sigma^a}{\partial z}\frac{\partial\sigma^b}{\partial\bar{z}}T_{ab}\\ &= \frac{\partial\sigma^1}{\partial z}\frac{\partial\sigma^1}{\partial\bar{z}}T_{11} + \frac{\partial\sigma^2}{\partial z}\frac{\partial\sigma^1}{\partial\bar{z}}T_{21} + \frac{\partial\sigma^1}{\partial z}\frac{\partial\sigma^2}{\partial\bar{z}}T_{12} + \frac{\partial\sigma^2}{\partial z}\frac{\partial\sigma^2}{\partial\bar{z}}T_{22}\,.\end{aligned} \tag{2.63}$$

We should note that $\frac{\partial\sigma^1}{\partial z} = \frac{\partial\sigma^1}{\partial\bar{z}} = \frac{1}{2}$, $\frac{\partial\sigma^2}{\partial z} = -\frac{\partial\sigma^2}{\partial\bar{z}} = \frac{1}{2\,i}$, as well as

$$T_{11} = -\frac{1}{\alpha'}\left(\partial_1 X^\mu\partial_1 X_\mu - \frac{1}{2}\partial_1 X^\mu\partial_1 X_\mu - \partial_2 X^\mu\partial_2 X_\mu\right)\,, \tag{2.64}$$

$$T_{22} = -\frac{1}{\alpha'}\left(\partial_2 X^\mu\partial_2 X_\mu - \frac{1}{2}\partial_1 X^\mu\partial_1 X_\mu - \partial_2 X^\mu\partial_2 X_\mu\right)\,, \tag{2.65}$$

$$T_{12} = T_{21} = -\frac{1}{\alpha'}\left(\partial_1 X^\mu\partial_2 X_\mu\right)\,. \tag{2.66}$$

Combining equations (2.63), (2.64), (2.65) and (2.66) we get

$$T_{z\bar{z}} = T_{\bar{z}z} = 0\,. \tag{2.67}$$

In a similar way we can write the other components

$$T_{zz} = T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu :\,, \tag{2.68}$$

$$T_{\bar{z}\bar{z}} = \bar{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu :\,. \tag{2.69}$$

Using the equations of motion it follows that $\bar{\partial}\, T_{zz} = \partial\, \bar{T}_{\bar{z}\bar{z}}$, then T and $\bar{T}$ are holomorphic and anti-holomorphic respectively. As you can see, the equations above agree with the ones we found using Noether's procedure.

We will now define the so-called *Primary Fields*, we then realize that there is a close relation between them and the energy momentum tensor of the theory that is been studied.

2.2.6 Primary fields

As we will see later, the energy momentum tensor of CFTs defines the conformal weight of some especial fields called *Primary fields*. Let us define these fields as follows: If a field $\phi(z,\bar{z})$ transforms under conformal transformations $z \to f(z)$ according to

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z}))\,, \tag{2.70}$$

then it is called a *Primary Field* of conformal dimension $(h,\bar{h})$. Now let us investigate how a primary field $\phi(z,\bar{z})$ behaves under infinitesimal conformal transformations. Consider a map $f(z) = z + \epsilon(z)$ with $\epsilon(z)$ very small, up to first order in $\epsilon(z)$ we have

$$\left(\frac{\partial f}{\partial z}\right)^h = (1+\partial\epsilon(z))^h = 1 + h\,\partial\epsilon(z), \tag{2.71}$$

$$\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} = (1+\bar{\partial}\bar{\epsilon}(\bar{z}))^{\bar{h}} = 1 + \bar{h}\,\bar{\partial}\bar{\epsilon}(\bar{z}), \tag{2.72}$$

$$\phi(z+\epsilon(z),\bar{z}+\bar{\epsilon}(\bar{z})) = \phi(z,\bar{z}) + \epsilon(z)\,\partial\phi(z,\bar{z}) + \bar{\epsilon}(\bar{z})\,\bar{\partial}\phi(z,\bar{z})\,, \tag{2.73}$$

and then making use of definition (2.70) for a primary field we get

$$\begin{aligned} \phi'(z,\bar{z}) &= (1+h\,\partial\epsilon)(1+\bar{h}\,\bar{\partial}\bar{\epsilon})(\phi + \epsilon\,\partial\phi + \bar{\epsilon}\,\bar{\partial}\phi) \\ &= (1+\bar{h}\,\bar{\partial}\bar{\epsilon} + h\,\partial\epsilon)(\phi + \epsilon\partial\phi + \bar{\epsilon}\bar{\partial}\phi) \\ &= \phi + (h\,\partial\epsilon + \bar{h}\,\bar{\partial}\bar{\epsilon} + \epsilon\partial + \bar{\epsilon}\bar{\partial})\,\phi\,. \end{aligned} \tag{2.74}$$

Finally, from the equation above we can write how a primary field of conformal weight $(h, \bar{h})$ transforms under infinitesimal conformal transformations

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = \left(h\partial\epsilon + \epsilon\partial + \bar{h}\,\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\right)\phi(z,\bar{z})\,. \tag{2.75}$$

We will now make use of Ward identities to derive some properties of primary fields as well as the energy momentum tensor

2.2.7 Conserved charges and radial ordering

As we saw earlier, when deriving Ward identities, in ordinary field theory, conserved charges are generators of symmetry transformations

$$\delta\phi = [\mathcal{Q}, \phi]\ . \tag{2.76}$$

In the case of conformal transformations, the conserved currents can be written in terms of the energy momentum tensor of the theory, then using equation (2.52) we can write the conserved charges as follows

$$\mathcal{Q} = \frac{1}{2\,\pi\,i}\oint_{\mathcal{C}}\left[dz\,T(z)\epsilon(z) + d\bar{z}\,\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\right]\ . \tag{2.77}$$

This expression will allow us to determine the infinitesimal transformations of a field $\phi(z,\bar{z})$ generated by a conserved charge $\mathcal{Q}$. To do so, we compute $\delta\phi$ according to (2.76)

$$\delta\phi(w,\bar{w}) = \frac{1}{2\,\pi\,i}\oint_{\mathcal{C}} dz\ \left[T(z)\epsilon(z), \phi(w,\bar{w})\right] + \frac{1}{2\,\pi\,i}\oint_{\mathcal{C}} d\bar{z}\ \left[\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \phi(w,\bar{w})\right]\ . \tag{2.78}$$

The expression we wrote above is none other than the Ward identities we derived before, we just need to be careful at defining the commutators inside the integrals since expression like this are supposed to work inside path integrals, so they are time ordered expressions, or as can be equivalently considered, radial ordered.

In CFTs, time ordering can be regarded as radial ordering and thus products like $\mathcal{A}(z)\mathcal{B}(w)$ inside path integrals only make sense for $|z| > |w|$. For this reason, we define the *radial ordering* of two operators as follows

$$\mathcal{R}\left(\mathcal{A}(z)\mathcal{B}(w)\right) := \begin{cases} \mathcal{A}(z)\mathcal{B}(w) & \text{for} \quad |z| > |w| \\ \mathcal{B}(w)\mathcal{A}(z) & \text{for} \quad |w| > |z| \end{cases}\ . \tag{2.79}$$

Taking this into account, expressions like the ones appearing in (2.78) can be written as follows

$$\begin{aligned} \oint dz\ [\mathcal{A}(z), \mathcal{B}(w)] &= \oint_{|z|>|w|} dz\,\mathcal{A}(z)\mathcal{B}(w) - \oint_{|z|<|w|} dz\,\mathcal{B}(w)\mathcal{A}(z) \\ &= \oint_{\mathcal{C}(w)} dz\,\mathcal{R}\left(\mathcal{A}(z)\mathcal{B}(w)\right)\ , \end{aligned} \tag{2.80}$$

where $\mathcal{C}(2)$ is some contour of integration cetered at w. We can make use of this observation to write (2.78) in the following way

$$\delta\phi(w,\bar{w}) = \frac{1}{2\,\pi\,i}\oint_{\mathcal{C}(w)} dz\,\epsilon(z)\mathcal{R}\left(T(z)\phi(w,\bar{w})\right) + \frac{1}{2\,\pi\,i}\oint_{\mathcal{C}(\bar{w})} d\bar{z}\,\bar{\epsilon}(\bar{z})\mathcal{R}\left(\bar{T}(\bar{z})\phi(w,\bar{w})\right)\,. \tag{2.81}$$

If the field $\phi(z,\bar{z})$ we are considering above is primary, we can make use of (2.75), and then compare it to the Ward identities we wrote above

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}}\,\phi(w,\bar{w}) &= h(\partial_w\epsilon(w))\phi(w,\bar{w}) + \epsilon(w)(\partial_w\phi(w,\bar{w})) + \text{anti-holomorphic}\\ &= \frac{1}{2\,\pi\,i}\oint_{\mathcal{C}(w)} dz\,\epsilon(z)\mathcal{R}\left(T(z)\phi(w,\bar{w})\right) + \text{anti-holomorphic}\,. \end{aligned}\tag{2.82}$$

We can compare this two expressions by making use of the well known Cauchy's differentiation formula of complex analysis, the left hand side of the first line of 2.82 can be written as follows

$$h(\partial_w\epsilon(w))\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} dz\,h\,\frac{\epsilon(z)}{(z-w)^2}\,\phi(w,\bar{w})\,, \tag{2.83}$$

$$\epsilon(w)(\partial_w\phi(w,\bar{w})) = \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} dz\,\frac{\epsilon(z)}{(z-w)}\,\partial_w\phi(w,\bar{w})\,. \tag{2.84}$$

Comparing (2.82), (2.83) and (2.84), we can write for a primary field of weight $(h,\bar{h})$

$$\mathcal{R}\left(T(z)\phi(w,\bar{w})\right) = \frac{h}{(z-w)^2}\,\phi(w,\bar{w}) + \frac{1}{z-w}\,\partial_w\phi(w,\bar{w}) + \ldots\,. \tag{2.85}$$

From now on, we will drop the $\mathcal{R}$ form this kind of expressions, since for computation purposes, they can be regarded as the OPEs we have been working with since the beginning.

Based on our last results, let us present an alternative definition of a primary field as follows: A field $\phi(z,\bar{z})$ is a primary field with conformal dimensions $(h,\bar{h})$, if the OPE between the energy momentum tensor and $\phi(z,\bar{z})$ takes the following form

$$T(z)\phi(w,\bar{w}) = \frac{h}{(z-w)^2}\,\phi(w,\bar{w}) + \frac{1}{z-w}\,\partial_w\phi(w,\bar{w}) + \ldots\,, \tag{2.86}$$

$$\bar{T}(\bar{z})\phi(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\,\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\,\partial_{\bar{w}}\phi(w,\bar{w}) + \ldots \tag{2.87}$$

We now go back to our main example, the XX system. By making use of this definition and the propagator we found in (2.32), we compute the conformal weight (dimension) of

some characteristic fields of this system

$$\begin{aligned} T(z)X^\rho(w,\bar{w}) &= -\frac{1}{\alpha'} : \partial X^\nu(z,\bar{z})\partial X_\nu(z,\bar{z}) : \; X^\rho(w,\bar{w}) \\ &= \partial_z \ln|z-w|^2\, \partial X^\rho(z,\bar{z}) + \ldots \\ &= \partial_z(\ln(z-w) + \ln(\bar{z}-\bar{w}))\, \partial X^\rho(z,\bar{z}) + \ldots \\ &= \frac{\partial X^\rho(z,\bar{z})}{z-w} + \ldots \\ &\sim \frac{\partial X^\rho(w,\bar{w})}{z-w}\,. \end{aligned} \tag{2.88}$$

Similarly, we can compute

$$\begin{aligned} T(z)\partial X^\rho(w,\bar{w}) &= -\frac{1}{\alpha'} : \partial X^\nu(z,\bar{z})\partial X_\nu(z,\bar{z}) : \partial X^\rho(w,\bar{w}) \\ &= \partial_w\partial_z \ln|z-w|^2 \partial_z X^\rho(z,\bar{z}) + \ldots \\ &= \frac{\partial_z X^\rho(z,\bar{z})}{(z-w)^2} + \ldots \\ &\sim \frac{\partial X^\rho(w,\bar{w})}{(z-w)^2} + \frac{\partial^2 X^\rho(w,\bar{w})}{(z-w)}\,. \end{aligned} \tag{2.89}$$

It is very easy to see from (2.88) and (2.89), that the fields X and ∂X have conformal weights $(0,0)$ and $(1,0)$ respectively.

2.2.8 Conformal properties of the energy momentum tensor

Now we want to investigate how the energy momentum tensor transforms under conformal transformations. Is it a primary field? The answer is no. For a general CFT, the energy momentum tensor transforms as follows

$$\delta T(z) = \frac{c}{12}\,\partial^3 v(z) + 2\,\partial_z v(z)\,T(z) + v(z)\partial_z T(z)\,, \tag{2.90}$$

where c is a constant known as the *central charge* of the system. We will prove this result for our case of interest, the XX system. First of all let us compute the TT OPE

$$\begin{aligned}
T(z)T(w) &= \frac{1}{\alpha'^2} : \partial_z X^\mu(z,\bar{z})\partial_z X_\mu(z,\bar{z}) :: \partial_w X^\mu(w,\bar{w})\partial_w X_\mu(w,\bar{w}) : \\
&= \frac{4}{\alpha'}\,\partial_w X^\mu(w,\bar{w})\partial_z X_\mu(z,\bar{z})\partial_z\partial_w \ln|z-w|^2 + \frac{1}{2}\,\frac{1}{(z-w)^4}\eta^\mu{}_\mu + \ldots \\
&= \frac{D}{2}\frac{1}{(z-w)^4} - \frac{2}{\alpha'}\frac{1}{(z-w)^2} : \partial_w X^\mu(w,\bar{w})\partial_z X_\mu(z,\bar{z}) : + \ldots \\
&\sim \frac{D}{2}\frac{1}{(z-w)^4} - \frac{2}{\alpha'}\frac{1}{(z-w)^2} : \partial_w X^\mu(w,\bar{w})\partial_w X_\mu(w,\bar{w}) : + \\
&\quad - \frac{2}{\alpha'}\frac{1}{(z-w)} : \partial_w X^\mu(w,\bar{w})\partial_w^2 X_\mu(w,\bar{w}) : \\
&\sim \frac{D/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,. \qquad (2.91)
\end{aligned}$$

As we already know, there is a close relation between this kind of OPE and how the field transforms under conformal transformations, so by making use of conformal Ward identities we can write

$$\begin{aligned}
\delta T(z) &= \frac{1}{2\,\pi\,i}\oint_{\mathcal{C}(z)} dw\, v(w)T(w)T(z) \\
&= \frac{1}{2\,\pi\,i}\oint_{\mathbb{C}(z)} dw\, v(w)\left(\frac{D/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\right) \\
&= \frac{D}{12}\,\partial^3 v(z) + 2\,T(z)\partial v(z) + \partial T(z) v(z)\,. \qquad (2.92)
\end{aligned}$$

This is a very important result, as you can see for the XX theory, the central charge of the theory coincides with the number of fields we are considering, namely the number of degrees of freedom of the system.

2.2.9 Another example, the bc CFT

Until now we have just considered the XX system as our main example. However, we will encounter more systems of interest to us, for instance the ghost system which arises when gauge fixing the bosonic string theory is a bc CFT. Siegel's modification of the Green-Schwarz action that we will review in Chapter 3 is also a bc CFT. We will also see a little of the so-called $\beta\gamma$ system, since the two are pretty similar. Examples of this system in string theory include the ghost system which arises when gauge fixing the RNS

superstring and the Berkovits pure spinor ghost system, which is a curved $\beta\gamma$ system as we will see later.

Let us start with the bc CFT, we consider two holomorphic anti-commutative fields b and c with conformal weights $(\lambda, 0)$ and $(1-\lambda, 0)$ with the following action

$$S_g = \frac{1}{2\pi}\int d^2z\, b\,\bar{\partial}\, c\,. \tag{2.93}$$

As you can see the action is, by construction, invariant under conformal transformations. It is very easy to deduce the equations of motion for the fields of our system which are given by

$$\bar{\partial} b = \bar{\partial} c = 0\,. \tag{2.94}$$

These equations tell us that the fields we are considering are indeed holomorphic, so we can now write down how they transform under infinitesimal conformal transformations. Using (2.75) and the conformal weights of the fields of our system we have

$$\delta b(z) = \lambda\,\partial\epsilon\, b(z) + \epsilon\,\partial b(z)\,, \tag{2.95}$$

$$\delta c(z) = (1-\lambda)\,\epsilon\, c(z) + \epsilon\,\partial c(z)\,. \tag{2.96}$$

We can use these transformations to apply Noether's procedure in order to find the energy-momentum tensor of our system

$$\begin{aligned}
\delta S_g &= \frac{1}{2\pi}\int d^2z\,\left[\delta b\,\bar{\partial} c + b\,\bar{\partial}\delta c\right] \\
&= \frac{1}{2\pi}\int d^2z\,\left[(\lambda\,\partial\epsilon\, b + \epsilon\,\partial b)\,\bar{\partial} c + b\,\bar{\partial}((1-\lambda)\epsilon c + \epsilon\,\partial c)\right] \\
&= \frac{1}{2\pi}\int d^2z\,\left[b(1-\lambda)\,\partial\bar{\partial}\epsilon\, c + \bar{\epsilon}\, b\partial c\right] \\
&= \frac{-1}{2\pi}\int d^2z\,[(1-\lambda)\,\partial b\, c - \lambda\, b\,\partial c]\,\bar{\partial}\epsilon\,.
\end{aligned} \tag{2.97}$$

As you can see, the anti-holomorphic part of the energy momentum tensor of the bc CFT vanishes, then from (2.97) we can write

$$T^g(z) = (1-\lambda) : \partial b\, c : -\lambda : b\,\partial c : \qquad , \qquad \bar{T}^g(\bar{z}) = 0\,. \tag{2.98}$$

Before computing any interesting OPEs in this system, let us find the propagator for the bc CFT, the procedure we follow is identical to the one employed in the XX system before

$$\begin{aligned}
0 &= \int \mathcal{D}\, b\,\mathcal{D}\, c\,\frac{\delta}{\delta c(z)}\,[\exp(-S_g)\, c(w)] \\
&= \int \mathcal{D}\, b\,\mathcal{D}\, c\,\exp(-S_g)\left[-\frac{1}{2\pi}\,\bar{\partial} b(z)\, c(w) + \delta^2(z-w, \bar{z}-\bar{w})\right]\,.
\end{aligned} \tag{2.99}$$

Since the equations above hold inside path integrals, we can write them as operator equations, from (2.99) we deduce

$$\left\langle \bar{\partial} b(z)\, c(w) \ldots \right\rangle = 2\pi\delta^2(z-w, \bar{z}-\bar{w})\,. \tag{2.100}$$

Where the dots represent additional insertions away from z and w. Solving the equation above, we find the propagator for the bc CFT which is given by

$$\langle b(z)\, c(w) \rangle = \frac{1}{z-w}\,. \tag{2.101}$$

As we did for the XX system, we can now define conformal normal ordering for the bc system as follows

$$: b(z)c(w) := b(z)c(w) - \frac{1}{z-w}\,, \tag{2.102}$$

of course this is defined in such a way that it satisfies the equations of motion

$$\begin{aligned} \bar{\partial}_{\bar{z}} : b(z)c(w) : &= \bar{\partial}_{\bar{z}} b(z)c(w) - 2\pi\delta^2(z-w, \bar{z}-\bar{w}) \\ &= 0\,. \end{aligned} \tag{2.103}$$

We saw before that the XX OPE is non-singular (up to infrared divergences), in contrast, the basic fields in the bc system have singular OPEs

$$b(z)c(w) \sim \frac{1}{z-w}\,, \qquad c(z)b(w) \sim \frac{1}{z-w}\,, \tag{2.104}$$

the other OPEs are non-singular.

Now, as an exercise, let us verify the conformal weights of the basic fields of the bc system by computing their corresponding OPEs with the energy-momentum tensor T^g

$$\begin{aligned} T^g(z)c(w) &= (1-\lambda) : (\partial b(z))c(z) : \, c(w) - \lambda : b(z)\partial c(z) : c(w) \\ &\sim (1-\lambda)\frac{c(z)}{(z-w)^2} + \lambda\frac{\partial c(z)}{(z-w)} \\ &\sim (1-\lambda)\frac{c(w)}{(z-w)^2} + (1-\lambda)\frac{\partial c(w)}{(z-w)} + \lambda\frac{\partial c(w)}{(z-w)} \\ &\sim (1-\lambda)\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{(z-w)}\,, \end{aligned} \tag{2.105}$$

then c is indeed a primary field of conformal dimensions $(1-\lambda, 0)$. On the other hand

$$\begin{aligned} T^g(z)b(w) &= (1-\lambda) : (\partial b(z))c(z) : \, b(w) - \lambda : b(z)\partial c(z) : b(w) \\ &\sim (1-\lambda)\frac{\partial b(z)}{(z-w)^2} - \lambda b(z)\frac{-1}{(z-w)^2} \\ &\sim (1-\lambda)\frac{\partial b(w)}{(z-w)} + \lambda\frac{b(w)}{(z-w)^2} + \lambda\frac{\partial b(w)}{(z-w)} \\ &\sim \lambda\frac{b(w)}{(z-w)^2} + \frac{\partial b(w)}{(z-w)}\,, \end{aligned} \tag{2.106}$$

and b is a primary field of dimensions $(\lambda, 0)$.

Now we establish a very important result, we compute the central charge for the bc CFT, we need to be very careful with signs, because of the anti-commutative nature of the fields b and c

$$\begin{aligned} T^g(z)T^g(w) &= [(1-\lambda) : \partial b(z)c(z) : -\lambda : b(z)\partial c(z) :] \, [(1-\lambda) : \partial b(w)c(w) : -\lambda : b(w)\partial c(w) :] \\ &= (1-\lambda)^2 \, (: \partial b(z)c(z) : : \partial b(w)c(w) :) - \lambda(1-\lambda) \, (: \partial b(z)c(z) : : b(w)\partial c(w) :) + \\ &\quad - \lambda(1-\lambda) \, (: b(z)\partial c(z) : : \partial b(w)c(w) :) + \lambda^2 \, (: b(z)\partial c(z) : : b(w)\partial c(w) :) \\ &\sim (1-\lambda)^2 \left[\frac{1}{(z-w)^4} - \frac{: \partial b(z)c(w) :}{(z-w)^2} - \frac{: c(z)\partial b(w) :}{(z-w)^2}\right] + \\ &\quad - \lambda(1-\lambda) \left[\frac{2}{(z-w)^3}\frac{1}{(z-w)} + \frac{: \partial b(z)\partial c(w) :}{(z-w)} + 2\,\frac{: c(z)b(w) :}{(z-w)^3}\right] + \\ &\quad - \lambda(1-\lambda) \left[\frac{1}{(z-w)}\frac{2}{(z-w)^3} + 2\,\frac{: b(z)c(w) :}{(z-w)^3} + \frac{: \partial c(z)\partial b(w) :}{(z-w)}\right] + \\ &\quad + \lambda^2 \left[\frac{-1}{(z-w)^2}\frac{-1}{(z-w)^2} - \frac{: b(z)\partial c(w) :}{(z-w)^2} - \frac{: \partial c(z)b(w) :}{(z-w)^2}\right] \\ &\sim \frac{-c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,, \end{aligned} \tag{2.107}$$

where the central charge of the bc system is given by

$$c^g = -3\,(2\lambda - 1)^2 + 1\,. \tag{2.108}$$

Of course, since $\bar{T}$=0, the anti-holomorphic central charge of the system vanishes, $\bar{c} = 0$.

By looking at the action of the bc CFT we can see that it possesses the so-called *ghost number* symmetry, that is it is invariant under the following global transformations

$$\delta b = -i\epsilon b\,, \qquad \delta c = i\epsilon c\,. \tag{2.109}$$

In fact

$$\begin{aligned}\delta S_g &= \frac{1}{2\pi}\int d^2z\,\big(\delta b\,\bar\partial c + b\,\partial\bar\delta c\big)\\ &= \frac{1}{2\pi}\int d^2z\,\big(-i\epsilon b\,\bar\partial c + ib\,\epsilon\,\bar\partial c\big) = 0\,. \end{aligned}\tag{2.110}$$

We can use Noether's procedure to find the current associated to the ghost number symmetry

$$\begin{aligned}\delta S_g &= \frac{1}{2\pi}\int d^2z\,\big(\delta b\,\bar\partial c + b\,\bar\partial\delta c\big)\\ &= \frac{1}{2\pi}\int d^2z\,\big(-i\epsilon\,b\,\bar\partial c + ib\,\epsilon\,\bar\partial c + ib\,\bar\partial\epsilon\,c\big)\\ &= -\frac{i}{2\pi}\int d^2z\,(-:bc:)\,\bar\partial\epsilon\,. \end{aligned}\tag{2.111}$$

We define the *ghost number current* as follows

$$j = -:bc:\ .\tag{2.112}$$

Finally, let us compute some easy OPEs involving the ghost current we have just defined above

$$j(z)b(w) = -:b(z)c(z):b(w) = -\frac{b(w)}{z-w}\,,\tag{2.113}$$

$$j(z)c(w) = -:b(z)c(z):c(w) = -\frac{c(w)}{z-w}\,.\tag{2.114}$$

We now give a very short look at the so-called $\beta\gamma$ system. Basically, all the results found for the bc system apply here (up to some sign changes). We consider commuting fields this time, β is a $(\lambda,0)$ primary field and γ a $(1-\lambda,0)$ primary field. The action is

$$S_{\beta\gamma} = \frac{1}{2\pi}\int d^2z\,\beta\,\bar\partial\gamma\,.\tag{2.115}$$

The equations of motion are similar to the bc system, so these fields are both holomorphic. We can define the OPEs for the fields, which will be similar to the bc system, up to some sign changes, due to the different statistics of the fields

$$\beta(z)\gamma(w) \sim -\frac{1}{z-w}\,,\qquad \gamma(z)\beta(w)\sim\frac{1}{z-w}\,.\tag{2.116}$$

The energy momentum tensor is given by

$$T(z) = (1-\lambda):\partial\beta\,\gamma: -\lambda:\beta\,\partial\gamma:\qquad,\qquad \bar T(\bar z) = 0\,.\tag{2.117}$$

And the central charge is simply reversed in sign

$$c = 3\,(2\lambda-1)^2 - 1\,,\qquad \bar c = 0\,.\tag{2.118}$$

2.2.10 The Virasoro algebra

Thus far we have been studying the classical aspects of a two dimensional conformal field theory in flat space. However we know that the worldsheet of a string is in general a cylinder. For this reason it is useful to show the map that relates this two spaces. To do so, we perform a change of variables from the cylinder to the complex plane as follows

$$z = e^{w} = e^{\sigma^1} \cdot e^{i\sigma^2} \,, \tag{2.119}$$

which is a map from the two dimensional world sheet described by σ^1 and σ^2 to the complex plane described by z. As a consequence of this map, we should note that time translations on the world-sheet $\sigma'^1 = \sigma^1 + a$ are then mapped to complex dilations $z' = e^a z$, and space translations $\sigma'^2 = \sigma^2 + b$ are mapped to rotations in the complex plane $z' = e^{ib} z$.

Since, in usual quantum field theory we are interested in time-ordered correlation functions, with the help of the observation above we can realise that time ordering on the cylinder becomes radial ordering on the plane. For this reason, operators in correlation functions are ordered so that those inserted at larger radial distance are moved to the left, this is why we were referring to radial ordering as time ordering since the beginning.

Considering this, the energy momentum tensor of our theory, as we saw earlier, is an holomorphic function on the complex plane. In general we can perform Laurent expansions when dealing with holomorphic or anti-holomorphic (holomorphic in $\bar{z}$) functions, so we can write for the energy-momentum tensor as

$$T_{zz}(z) \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \,, \qquad \bar{T}_{\bar{z}\bar{z}}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\bar{L}_m}{\bar{z}^{\,m+2}} \,. \tag{2.120}$$

The coefficients L_m, known as *Laurent modes* can be found by inverting the expansions above, they are given by

$$L_m = \oint_{\mathcal{C}} \frac{dz}{2\pi i} \, z^{m+1} \, T_{zz}(z) \,, \qquad \bar{L}_m = \oint_{\mathcal{C}} \frac{d\bar{z}}{2\pi i} \, \bar{z}^{m+1} \, \bar{T}_{\bar{z}\bar{z}}(\bar{z}) \,. \tag{2.121}$$

Let us now use equation (2.77) and the expansion for the energy momentum tensor we wrote above. If we choose a particular conformal transformation $\epsilon(z) = -\epsilon_n z^{n+1}$, we find that

$$\begin{aligned} \mathcal{Q}_n = \oint \frac{dz}{2\pi i} T(z) \left(-\epsilon_n z^{n+1}\right) &= -\epsilon_n \sum_{m=-\infty}^{\infty} \oint \frac{dz}{2\pi i} \, L_m z^{n-m-1} \\ &= -\epsilon_n \sum_m \delta_{n,m} L_m \\ &= -\epsilon_n L_n \,. \end{aligned} \tag{2.122}$$

We have found that the Laurent modes are actually the generators of infinitesimal conformal transformations. For this reason they should satisfy the Virasoro algebra, as we show next

$$\begin{aligned}
[L_m, L_n] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}\, z^{m+1} w^{n+1}\, [T(z), T(w)] \\
&= \oint \frac{dw}{2\pi i}\, w^{n+1} \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i}\, z^{n+1}\, \mathcal{R}\left(T(z)T(w)\right) \\
&= \oint \frac{dw}{2\pi i}\, w^{n+1} \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i}\, z^{m+1} \left[\frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}\right] \\
&= \oint \frac{dw}{2\pi i} \left[\frac{c}{12}\,(m^3 - m) w^{m+n-1} + 2(m+1) w^{m+n+1} T(w) + w^{m+n+2} \partial_w T(w)\right] \\
&= \frac{c}{12}\,(m^3 - m)\delta_{m,-m} + 2(m+1) L_{m+n} - \oint \frac{dw}{2\pi i}\,(m+n+2) w^{m+n+1} T(w) \\
&= (m-n) L_{m+n} + \frac{c}{12}\,(m^3 - m)\,\delta_{m,-n}\,, \qquad (2.123)
\end{aligned}$$

and they indeed satisfy the Virasoro algebra (2.22). In a similar computation, we can show that $\bar{L}_m$ satisfy the same algebra with central charge $\bar{c}$.

Let us make some observations when looking at the Virasoro algebra. Generally we work with eigenstates of L_0 and $\bar{L}_0$. The generator L_0 satisfies

$$[L_0, L_n] = -n\, L_n\,. \qquad (2.124)$$

Then if $|\psi\rangle$ is an eigenstate of L_0 with eigenvalue h, we have

$$\begin{aligned}
L_0 L_n |\psi\rangle &= \left([L_0, L_n] + L_n L_0\right)|\psi\rangle \\
&= \left(-n\, L_n + L_n L_0\right)|\psi\rangle \\
&= L_n (L_0 - n)|\psi\rangle \\
&= (h-n) L_n |\psi\rangle\,, \qquad (2.125)
\end{aligned}$$

thus, $L_n|\psi\rangle$ is an eigenstate of L_0 with eigenvalue $(h-n)$. As you can see, the generators with $n < 0$ raise the L_0 eigenvalue and those with $n > 0$ lower it. We can also note that the three generators L_0, and $L_{\pm 1}$ form a closed algebra without central charge

$$[L_0, L_1] = -L_1\,; \qquad [L_0, L_{-1}] = L_{-1}\,; \qquad [L_1, L_{-1}] = 2\, L_0\,. \qquad (2.126)$$

This is the algebra of $SL(2, \mathbb{R})$.

Let us now consider the Laurent expansion for a general holomorphic tensor field (primary field) $\mathcal{O}$ of dimensions $(h, 0)$

$$\mathcal{O}(z) = \sum_{m=-\infty}^{\infty} \frac{\mathcal{O}_m}{z^{m+h}}\,. \qquad (2.127)$$

Inverting this expansion we get

$$\begin{aligned}\oint \frac{z^n}{2\pi i}\,\mathcal{O}(z)dz &= \sum_{m=-\infty}^{\infty}\oint \frac{dz}{2\pi i}\,\mathcal{O}_m z^{m-n-h}\\ &= \sum_{m=-\infty}^{\infty}\mathcal{O}_m\,\delta_{m,n-h+1}\\ &= \mathcal{O}_{n-h+1}\,,\end{aligned} \tag{2.128}$$

performing a change of variables in the indices, we finally get for the Laurent modes of the primary field $\mathcal{O}$

$$\mathcal{O}_m = \frac{1}{2\pi i}\oint dz\, z^{m+h-1}\mathcal{O}(z)\,. \tag{2.129}$$

We can make use of this modes and the general OPE for a primary field we found earlier to compute the following commutator

$$\begin{aligned}[L_m,\mathcal{O}_n] &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\, z^{m+1}w^{n+h-1}\,[T(z),\mathcal{O}(w)]\\ &= \oint \frac{dz}{2\pi i}\, w^{n+h-1}\oint \frac{dw}{2\pi i}\, z^{m+1}\left[\frac{h}{(z-w)^2}\,\mathcal{O}_w + \frac{\partial\mathcal{O}(w)}{(z-w)}\right]\\ &= \oint \frac{dz}{2\pi i}\, w^{n+h-1}\left[(m+1)\,hw^n\mathcal{O}(w) + w^{n+1}\partial\mathcal{O}(w)\right]\\ &= \oint \frac{dz}{2\pi i}\,\left[(m+1)\,hw^{m+n+h-1}\mathcal{O}(w) + w^{m+n+h}\partial\mathcal{O}(w)\right]\\ &= (m+1)h\mathcal{O}_{m+n} - (m+n+h)\mathcal{O}_{m+n}\\ &= [(h-1)m-n]\,\mathcal{O}_{m+n}\,.\end{aligned} \tag{2.130}$$

Again, modes with $n>0$ reduce L_0, while modes with $n<0$ increase it.

2.3 BRST quantization of the bosonic string theory

We now work on the BRST procedure to quantize the bosonic string. We will not show the details of the Faddeev-Popov method when gauge fixing Polyakov's action, we rather choose to show the partition function after gauge fixing the worldsheet metric ($g_{ab}=\delta_{ab}$), which is given by the sum of D free massless scalars and the ghost sector which is given by the sum of an holomorphic and anti-holomorphic bc systems with $\lambda=2$

$$Z = \int \mathcal{D}X\mathcal{D}\,b\mathcal{D}\,c\,\exp\left(-S_m - S_g\right), \tag{2.131}$$

where the term $S = S_m + S_g$ is given by

$$S = \frac{1}{2\pi\alpha'} \int d^2z\, \partial X^\mu \bar{\partial} X_\mu + \frac{1}{2\pi} \int d^2z \left(b\, \bar{\partial} c + \bar{b}\, \partial \bar{c} \right) . \tag{2.132}$$

This is the quantum worldsheet action of the gauge-fixed theory. The associated energy-momentum tensor is given by

$$T(z) = T^X(z) + T^g(z) , \tag{2.133}$$

where the energy momentum tensor for the matter sector and the ghost sector are given by equations (2.68) and (2.98) respectively (with $\lambda = 2$). Then we can compute the total central charge of the theory, since we are considering D scalar fields, the central charge of the matter sector is just D, and the central charge of the ghost sector is given by (2.108) with $\lambda = 2$, then the total central charge is given by

$$c = c^X + c^g = D - 26 . \tag{2.134}$$

2.3.1 The Weyl anomaly

At the classical level, one of the defining features of a CFT is the vanishing of the trace of the energy-momentum tensor. This is a general property of Weyl invariant Lagrangians. We will see that for this symmetry to hold at the quantum level, the total central charge of the theory must vanish.

By a suitable choice of coordinates, we can always put any two dimensional metric in the form $g_{ab} = e^{2w}\delta_{ab}$. So the only non-vanishing Christoffel symbols are the following

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = \partial_1 w , \tag{2.135}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\Gamma^2_{11} = \partial_2 w . \tag{2.136}$$

By looking at the Christoffel symbols, there are just two non-vanishing components of the Ricci tensor

$$R_{11} = R_{22} = -\partial^2 w , \tag{2.137}$$

where $\partial^2 = \partial_1\partial_1 w + \partial_2\partial_2 w$. We can now easily compute the Ricci scalar as follows

$$\begin{aligned} R &= g^{11}R_{11} + g^{22}R_{22} \\ &= -2e^{-2w}\partial^2 w . \end{aligned} \tag{2.138}$$

We claim that, at the quantum level

$$\langle T^\alpha{}_\alpha \rangle = -\frac{c}{12} R . \tag{2.139}$$

Let us prove this result. Our first step will be to find an expression for the $T_{z\bar{z}}T_{w\bar{w}}$ OPE. To do so, we start with the energy conservation equation $\partial^a T_{ab} = 0$, which when written in complex coordinates takes the following form

$$\begin{aligned} \partial^z T_{z\bar{z}} + \partial^{\bar{z}} T_{\bar{z}z} &= 0 \\ \Rightarrow -\partial_{\bar{z}} T_{zz} &= \partial_z T_{z\bar{z}} \,. \end{aligned} \tag{2.140}$$

With the help of the result above and the TT OPE encountered in (2.91), we can find our desired OPE as follows

$$\begin{aligned} \partial_z T_{z\bar{z}}(z,\bar{z})\partial_w T_{w,\bar{w}}(w,\bar{w}) &= \partial_{\bar{z}} T_{zz}(z,\bar{z})\partial_{\bar{w}} T_{ww}(w,\bar{w}) \\ &= \partial_{\bar{z}}\partial_{\bar{w}}\left[\frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots\right] . \end{aligned} \tag{2.141}$$

The only non-vanishing term in the expression above is the first one, as it is easy to see that

$$\begin{aligned} \bar{\partial}_{\bar{z}}\bar{\partial}_{\bar{w}}\frac{1}{(z-w)^4} &= \frac{1}{6}\,\bar{\partial}_{\bar{z}}\bar{\partial}_{\bar{w}}\left[\partial_z^2\partial_w\frac{1}{z-w}\right] \\ &= \frac{\pi}{3}\,\partial_z^2\partial_w\bar{\partial}_{\bar{w}}\delta(z-w,\bar{z}-\bar{w}) \,. \end{aligned} \tag{2.142}$$

Inserting this result into the OPE (2.141) and integrating the $\partial_z\partial_w$ derivatives on both sides we end up with

$$T_{z\bar{z}}(z,\bar{z})T_{w,\bar{w}}(w,\bar{w}) = \frac{c\pi}{6}\partial_z\bar{\partial}_{\bar{w}}\delta(z-w,\bar{z}-\bar{w}) \,. \tag{2.143}$$

So the OPE above almost vanishes, but there is a singular behaviour when z is close enough to w. We assume that $\langle T^\alpha{}_\alpha\rangle = 0$ in flat space. We will also consider spaces which are close to flat space. Let us now compute the following correlation function

$$\begin{aligned} \delta\langle T^\alpha{}_\alpha(\sigma)\rangle &= \delta\int \mathcal{D}\phi\, e^{-S}\, T^\alpha{}_\alpha(\sigma) \\ &= \frac{1}{4\pi}\int \mathcal{D}\phi\, e^{-S}\left[T^\alpha{}_\alpha(\sigma)\int d^2\sigma'\,\sqrt{g}\,\delta g^{ab}T_{ab}(\sigma')\right] , \end{aligned} \tag{2.144}$$

where we made use of the definition of the energy-momentum tensor we gave at (2.58). We will now restrict ourselves to Weyl transformations, so the change in a flat metric is $\delta g_{ab} = 2w\delta_{ab}$. Similarly the change in the inverse metric is $\delta g^{ab} = -2w\delta^{ab}$, so we can write

$$\delta\langle T^\alpha{}_\alpha(\sigma)\rangle = -\frac{1}{2\pi}\int \mathcal{D}\phi\, e^{-S}\left[T^\alpha{}_\alpha(\sigma)\int d^2\sigma\, w(\sigma')T^b{}_b(\sigma')\right] . \tag{2.145}$$

We will need to change the OPE (2.143) to Cartesian coordinates. In order to do this, we should remember that we need to make a change of coordinates similar to the one performed at (2.62). It is not hard to show that

$$T^a{}_a(\sigma)T^b{}_b(\sigma') = 16\, T_{z\bar{z}}(z,\bar{z})T_{w,\bar{w}}(w,\bar{w}) \quad \text{and} \quad \delta^2(\sigma-\sigma') = -8\,\partial_z\bar{\partial}_{\bar{w}}\delta(z-w,\bar{z}-\bar{w})\,. \tag{2.146}$$

With the help of the results above, we can finally write the OPE (2.143) in Cartesian coordinates as follows

$$T^a{}_a(\sigma)T^b{}_b(\sigma') = -\frac{c\,\pi}{3}\,\delta^2(\sigma-\sigma')\,. \tag{2.147}$$

To finish the computation, we use the result above to calculate the integral (2.145), integrating by parts and using the Dirac delta to compute the σ' integral we get

$$\delta\langle T^\alpha{}_\alpha\rangle = \frac{c}{6}\,\partial^2 w\,. \tag{2.148}$$

We finally compare this result with equation (2.138), and our proof is finished

$$\langle T^\alpha{}_\alpha\rangle = -\frac{c}{12}\,R\,, \tag{2.149}$$

where we considered $e^{-2w} \sim 1$. Thus, only if $R = 0$ or if $c = 0$ we have that the expectation value of the trace vanishes. However, if we set $R = 0$ then we are only allowed to work in flat space , so in order to include curved space-times, the central charge of the theory must vanish, $c = 0$. Since the trace of the energy-momentum tensor vanishes classically, due to the CFT being Weyl invariant, and since when quantizing the CFT this vanishing is no longer guaranteed, we refer to (2.149) as the *Weyl anomaly.*

Let us look at equation (2.134), as you ca see, the easiest way to get a vanishing central charge is to consider 26 bosonic free scalar fields as the matter sector, since each one will contribute with a value of 1 to c^X. However, this is not the only way to get this value because we only need to consider a theory which has central charge 26 so it cancels the ghost contribution. Then the space of CFTs with central charge equal to 26 can be thought as the space of classical solutions to the bosonic string.

2.3.2 XX Laurent mode expansions

Let us now look at the Laurent mode expansions for the fields in free bosonic closed string theory, the XX system. As we have seen, ∂X and $\bar{\partial} X$ are holomorphic and anti-holomorphic respectively, and so we can expand them in Laurent modes just as follows

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m=-\infty}^{\infty}\frac{\alpha^\mu_m}{z^{m+1}}\,, \qquad \bar{\partial} X^\mu(\bar{z}) = -i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m=-\infty}^{\infty}\frac{\bar{\alpha}^\mu_m}{\bar{z}^{m+1}}\,. \tag{2.150}$$

Inverting these expansions the modes can be written as follows

$$\begin{aligned}\alpha_\mu^m &= \left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{dz}{2\pi}\, z^m \partial X_\mu(z)\,, \\ \bar{\alpha}_\mu^m &= -\left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{d\bar{z}}{2\pi}\, \bar{z}^m \bar{\partial} X_\mu(\bar{z})\,. \end{aligned} \tag{2.151}$$

The single-valuedness of X^μ implies that $\alpha_0^\mu = \bar{\alpha}_0^\mu$. Moreover, under space-time translations $\delta X^\mu = a^\mu$, we have for the space-time momentum

$$p^\mu = \frac{1}{2\pi i} \oint_{\mathcal{C}} (dz\, j^\mu - d\bar{z}\, \bar{j}^\mu) = \left(\frac{2}{\alpha'}\right)^{1/2} \alpha_0^\mu = \left(\frac{2}{\alpha'}\right)^{1/2} \bar{\alpha}_0^\mu\,. \tag{2.152}$$

Integrating the mode expansions (2.150) we get

$$\begin{aligned} X^\mu(z,\bar{z}) &= \int dz d\bar{z}\, \left(\partial X^\mu(z) + \bar{\partial} X^\mu(\bar{z})\right) \\ &= x^\mu - i\frac{\alpha'}{2}\, p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m\neq 0}^{\infty} \frac{1}{m}\left(\frac{\alpha_\mu^m}{z^m} + \frac{\bar{\alpha}_m^\mu}{\bar{z}^m}\right)\,. \end{aligned} \tag{2.153}$$

Now let us compute the following commutators

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu] &= \left(\frac{2}{\alpha'}\right) \oint \frac{dz}{2\pi} \oint \frac{dw}{2\pi}\, z^m w^n\, [\partial X^\mu(z), \partial X^\nu(w)] \\ &= -\oint \frac{dw}{2\pi}\, w^n \oint \frac{dz}{2\pi}\, z^m \frac{1}{(z-w)^2} \eta^{\mu\nu} \\ &= m\, \delta_{m,-n}\, \eta^{\mu\nu}\,, \end{aligned} \tag{2.154}$$

where in the second line we made use of the XX propagator derived earlier. In a similar way it can be shown that

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}\,. \tag{2.155}$$

These results could have also been obtained from standard canonical quantization. The spectrum can be obtained by starting with a state $|0,k\rangle$ that has momentum k^μ and which is annihilated by all of the lowering modes, α_n^μ for $n > 0$ and acting in all possible ways with the raising ($n < 0$) modes.

We can now replace the mode expansions (2.150) in the energy momentum tensor, as

follows

$$\begin{aligned} T(z) &= -\frac{1}{\alpha'} : \partial X^\mu(z) \partial X_\mu(z) : \\ &= \frac{1}{\alpha'} \left(\frac{\alpha'}{2}\right)^{1/2} \sum_m \sum_n \frac{\alpha^\mu_m}{z^{m+1}} \frac{\alpha_{n\ \mu}}{z^{n+1}} \\ &= \frac{1}{2} \sum_m \sum_n \alpha^\mu_m \alpha_{n\ \mu} \frac{1}{z^{m+n+2}} \\ &= \sum_m \frac{\left(\frac{1}{2} \sum_n \alpha^\mu_{m-n} \alpha_{n\ \mu}\right)}{z^{m+2}} , \end{aligned} \tag{2.156}$$

comparing this result with the Laurent expansion of the energy momentum tensor we wrote in (2.120), we can identify the Laurent modes as follows

$$L_m \sim \frac{1}{2} \sum_n \alpha^\mu_{m-n} \alpha_{n\ \mu} . \tag{2.157}$$

We put the symbol $\sim$ here because of the ordering of the modes.

- For $m \neq 0$;

$$[\alpha^\mu_{m-n}, \alpha^\nu_n] = (m-n)\delta_{m-n,-n}\eta^{\mu\nu} = 0 , \tag{2.158}$$

  then the mode operators in each term commute and thus the ordering is irrelevant. So the expansion is well defined and correct as it stands.

- For $m = 0$, we put the lowering operators on the right, and introduce the normal ordering constant a^X

$$\begin{aligned} L_0 &= \frac{1}{2} \alpha^\mu_0 \alpha_{0\ \mu} + \sum_{n=1}^{\infty} \alpha^\mu_{-n} \alpha_{n\ \mu} + a^X \\ \Rightarrow L_0 &= \frac{\alpha' p^2}{4} + \sum_{n=1}^{\infty} \alpha^\mu_{-n} \alpha_{n\ \mu} + a^X . \end{aligned} \tag{2.159}$$

  Since we know $2\, L_0 |0;0\rangle = (L_1 L_{-1} - L_{-1} L_1)|0;0\rangle = 0$, then it follows that the normal ordering constant a^X vanishes

$$a^X = 0 . \tag{2.160}$$

2.3.3 bc Laurent mode expansions

Le us now work with the bc system. We start by writing the Laurent expansions of the fields b and c

$$b(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+\lambda}}\,, \qquad c(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+1-\lambda}}\,. \tag{2.161}$$

Using the OPEs of the fields we can easily find the commutator

$$\{b_m, c_n\} = \delta_{m,-n}\,. \tag{2.162}$$

As we will see later, the BRST method splits the theory into a part containing matter fields and a part containing ghost fields, thus the vacuum of our theory $|0\rangle$ will be given by the tensor product of the matter vacuum and the ghost vacuum $|0\rangle = |0,k\rangle \otimes |0\rangle_g$.

We have already defined the matter vacuum $|0,k\rangle$ in the preceding discussion, so we now need to define the ghost vacuum. So we consider the ground state as the state annihilated by all of the $n > 0$ modes. The b_0, c_0 oscillator algebra generates two such ground states $|\downarrow\rangle$ and $|\uparrow\rangle$, with the following properties

$$b_0\,|\downarrow\rangle = 0\,, \qquad b_0\,|\uparrow\rangle = |\downarrow\rangle\,, \tag{2.163}$$

$$c_0\,|\downarrow\rangle = |\uparrow\rangle\,, \qquad c_0\,|\uparrow\rangle = 0\,, \tag{2.164}$$

$$b_n\,|\downarrow\rangle = b_n\,|\uparrow\rangle = c_n\,|\downarrow\rangle = c_n\,|\uparrow\rangle = 0\,, \qquad n > 0\,. \tag{2.165}$$

The general state is obtained by acting on this states with the rising modes ($n < 0$). It is convenient to group b_0 with the lowering operators and c_0 with the raising operators, so we will single out $|\downarrow\rangle$ as the ghost vacuum $|0\rangle_g$. Thus the BRST vacuum is given by

$$|0\rangle = |0,k\rangle \otimes |\downarrow\rangle\,. \tag{2.166}$$

The physical states are constructed by acting with the BRST operator $\mathcal{Q}_B$, but let us postpone this discussion for the next section.

We can find the Virasoro generators for the ghost sector as we did for the matter sector, they are given by

$$L_m^g \sim \sum_{n=-\infty}^{\infty} (m\lambda - n) : b_n c_{m-n} : + \delta_{m,0}\, a^g\,, \tag{2.167}$$

where the expression inside the dots stand for the ordinary normal ordering of field theory,[3] and the constant a^g is a normal ordering constant which can be determined in a similar

[3]Note that we are treating the ordinary normal ordering and conformal normal ordering as if they were equivalent. We should emphasise that this is not always the case.

way than the normal ordering constant in the matter sector. In fact

$$\begin{aligned} 2L_0 \left|\downarrow\right\rangle &= (L_1 L_{-1} - L_{-1} L_1) \left|\downarrow\right\rangle \\ &= (\lambda\, b_0\, c_1)\, [(1-\lambda) b_{-1} c_0] \left|\downarrow\right\rangle = \lambda(1-\lambda) \left|\downarrow\right\rangle . \end{aligned} \tag{2.168}$$

Thus $a^g = \frac{\lambda(1-\lambda)}{2}$ and

$$L^g_m = \sum_{n=-\infty}^{\infty} (m\lambda - n) : b_n c_{m-n} : + \frac{\lambda(1-\lambda)}{2}\, \delta_{m,0} . \tag{2.169}$$

Let us now define the ghost number charge.

$$\mathcal{N}^g = \frac{1}{2\pi i} \oint dz\, j(z) , \tag{2.170}$$

where j is the ghost number current defined in (2.112). With the help of the OPEs (2.113) and (2.114) we can compute the following commutators

$$[\mathcal{N}^g, b(w)] = -b(w) , \tag{2.171}$$

$$[\mathcal{N}^g, c(w)] = c(w) , \tag{2.172}$$

$$[\mathcal{N}^g, X^\mu(w)] = 0 . \tag{2.173}$$

Which of course are none other than the ghost number transformations we defined in (2.109). By looking at the equations above, we can see that the ghost number charge counts the number of c minus the number of b excitations of the state considered. Now we show that the ground state $\left|\downarrow\right\rangle$ has ghost number $-\frac{1}{2}$. First of all we expand the ghost number charge in ghost modes to obtain

$$\mathcal{N}^g = \frac{1}{2} (c_0\, b_0 - b_0\, c_0) + \sum_{n=1}^{\infty} (c_{-n} b_n - b_{-n} c_n) . \tag{2.174}$$

Now let us use this operator to compute the ghost number of the ground state of our theory

$$\begin{aligned} \mathbb{I} \otimes \mathcal{N}^g \left|0\right\rangle &= \mathbb{I} \otimes \left(\frac{1}{2} (c_0\, b_0 - b_0\, c_0) + \sum_{n=1}^{\infty} (c_{-n} b_n - b_{-n} c_n) \right) \left|0,k\right\rangle \otimes \left|\downarrow\right\rangle \\ &= \frac{1}{2} c_0\, b_0 \left(\left|0,k\right\rangle \otimes \left|\downarrow\right\rangle\right) - \frac{1}{2} b_0\, c_0 \left(\left|0,k\right\rangle \otimes \left|\downarrow\right\rangle\right) + \sum_{n=1}^{\infty} c_{-n} b_n \left(\left|0,k\right\rangle \otimes \left|\downarrow\right\rangle\right) + \\ &\quad - \sum_{n=1}^{\infty} b_{-n} c_n \left(\left|0,k\right\rangle \otimes \left|\downarrow\right\rangle\right) \\ &= -\frac{1}{2} \left(\left|0,k\right\rangle \otimes \left|\downarrow\right\rangle\right) , \end{aligned} \tag{2.175}$$

where in the computation above $\mathbb{I}$ is the identity operator which acts on the matter sector, and we made use of equations (2.163), (2.164) and (2.165).

We now have enough material to explain the BRST method for quantizing the bosonic string which we do next.

2.3.4 BRST symmetry in general

We consider a path integral in general provided with some local symmetry and let us denote the path integral fields as $\phi_i(X^\mu(\sigma), g_{ab}(\sigma))$. By assumption, the gauge parameters ϵ^α are real. The gauge transformations satisfy the following algebra

$$[\delta_\alpha, \delta_\beta] = f^\gamma{}_{\alpha\beta}\, \delta_\gamma \,, \tag{2.176}$$

and the gauge fixing condition is given by

$$F^A(\phi) = 0 \,. \tag{2.177}$$

Following the Faddeev Poppov procedure, the path integral becomes

$$\int \frac{\mathcal{D}\phi_i}{V_{gauge}} \exp(-S_1) \to \int \mathcal{D}\phi_i \mathcal{D}\, B_A \mathcal{D}\, b_A \mathcal{D}\, c^\alpha \, \exp(-S_1 - S_2 - S_3) \,, \tag{2.178}$$

where S_1 is the original gauge invariant action, S_2 is the gauge fixing action

$$S_2 = -iB_A F^A(\phi) \,, \tag{2.179}$$

and S_3 is the Faddeev Poppov action[4]

$$S_3 = b_A c^\alpha \delta_\alpha F^A(\phi) \,. \tag{2.180}$$

The action written above is invariant under the Becchi-Rouet-Stora-Tyutin (BRST) transformations

$$\begin{aligned}
\delta_B \phi_i &= -i\epsilon\, c^\alpha \delta_\alpha \phi_i \,, && (2.181)\\
\delta_B B_A &= 0 \,, && (2.182)\\
\delta_B b_A &= \epsilon\, B_A \,, && (2.183)\\
\delta_B c^\alpha &= \frac{i}{2}\, \epsilon\, f^\alpha{}_{\beta\gamma} c^\beta c^\gamma \,. && (2.184)
\end{aligned}$$

[4]Note that we are using a very condensed notation here, for instance we can write

$$S_2 = -iB_A F^A(\phi) = -i \int d\sigma \sqrt{-g}\, B_A(\sigma) F^A(\phi;\sigma) \,.$$

In fact, to see this we just need to note that the following result holds

$$\begin{aligned}
\delta_B\left(b_A F^A(\phi)\right) &= \delta_B b_A F^A(\phi) + b_A \delta_B F^A(\phi) \\
&= \epsilon\, B_A F^A(\phi) + b_A\left(-i\epsilon\, c^\alpha \delta_\alpha F^A(\phi)\right) \\
&= i\epsilon\left(-iB_A F^A(\phi) + b_A c^\alpha \delta_\alpha F^A(\phi)\right) \\
&= i\epsilon\left(S_2 + S_3\right) . \qquad (2.185)
\end{aligned}$$

It is very easy to compute the variation under BRST symmetry transformations of the full action by using the relations above

$$\delta_B\left(S_1 + S_2 + S_3\right) = \delta_B\left(S_1 + \frac{1}{i\epsilon}\,\delta_B\left(b_A F^A(\phi)\right)\right) = 0\,, \qquad (2.186)$$

where we used the fact that the original gauge action S_1 is clearly BRST invariant, as well as the nilpotency of the BRST symmetry ($\delta_B^2 = 0$), which can easily be checked by looking at the transformations (2.181), (2.182), (2.183) and (2.184). And as you can see the total action is constructed in such a way that it is indeed BRST invariant.

2.3.5 BRST cohomology and physical states

In the procedure of BRST quantization, the BRST symmetry is used to derive the physical spectrum of quantum states of the gauge theory. A quantum amplitude is of the form

$$\langle f|i\rangle = \int \mathcal{D}\phi_i \mathcal{D}\, B_A \mathcal{D}\, b_A \mathcal{D}\, c^\alpha\, \exp(-S_1 - S_2 - S_3)\,. \qquad (2.187)$$

For this amplitude to be physical, it should be independent of the gauge which one chooses to calculate the path integral. Now consider a small change δF in the gauge-fixing condition. The change in the ghost and gauge-fixing action gives

$$\begin{aligned}
\epsilon\,\delta\langle f|i\rangle &= -\epsilon \int \mathcal{D}\phi_i \mathcal{D}\, B_A \mathcal{D}\, b_A \mathcal{D}\, c^\alpha\, \exp(-S_1 - S_2 - S_3)\delta\left(S_1 + S_2 + S_3\right) \\
&= \epsilon \int \mathcal{D}\phi_i \mathcal{D}\, B_A \mathcal{D}\, b_A \mathcal{D}\, c^\alpha\, \exp(-S_1 - S_2 - S_3)\left(-iB_A \delta F^A(\phi) + b_A c^\alpha \delta_\alpha \delta F(\phi)\right) \\
&= \int \mathcal{D}\phi_i \mathcal{D}\, B_A \mathcal{D}\, b_A \mathcal{D}\, c^\alpha\, \exp(-S_1 - S_2 - S_3)\left[i\epsilon B_A \delta F^A(\phi) + b_A \epsilon c^\alpha \delta_\alpha \delta F^A(\phi)\right] \\
&= \int \mathcal{D}\phi_i \mathcal{D}\, B_A \mathcal{D}\, b_A \mathcal{D}\, c^\alpha\, \exp(-S_1 - S_2 - S_3)\left[i\delta_B b_A \delta F^A(\phi) + i b_A \delta_B\left(\delta F^A(\phi)\right)\right] \\
&= i\langle f|\delta_B\left(b_A \delta F^A\right)|i\rangle \\
&= -\epsilon\langle f|\left\{\mathcal{Q}_B, b_A \delta F^A\right\}|i\rangle\,, \qquad (2.188)
\end{aligned}$$

where in the last line we have written the BRST variation as an anti-commutator with the corresponding conserved charge $\mathcal{Q}_B$.

Since this equality must hold for any arbitrary change $\delta F^A(\phi)$ in the gauge choice, it is required that all physical states $|\psi\rangle$ are closed with respect to the BRST charge

$$\langle\psi| \left\{\mathcal{Q}_B, b_A \delta F^A\right\} |\psi'\rangle = 0\,, \tag{2.189}$$

then for the states to be physical they must be BRST invariant

$$\mathcal{Q}_B|\psi\rangle = \mathcal{Q}_B|\psi'\rangle = 0\,. \tag{2.190}$$

Let us now point out some important properties of the BRST charge without proving them for now.

- The BRST charge is nilpotent
$$\mathcal{Q}_B^2 = 0\,. \tag{2.191}$$

- The nilpotence of $\mathcal{Q}_B$ has an important consequence. A state of the form $\mathcal{Q}_B|\chi\rangle$ will be annihilated by $\mathcal{Q}_B$ for any χ and so it is a physical state. However it is orthogonal to all physical states including itself
$$\langle\psi|(\mathcal{Q}_B|\chi\rangle) = (\langle\psi|\mathcal{Q}_B)|\chi\rangle = 0\,. \tag{2.192}$$

- If $\mathcal{Q}_B|\psi\rangle = 0$, then all physical amplitudes involving such a null state vanish. Two physical states that differ by a null state
$$|\psi'\rangle = |\psi\rangle + \mathcal{Q}_B|\chi\rangle\,, \tag{2.193}$$
will have the same inner products with all physical states and are therefore physically equivalent. In this way, we identify the true physical space with a set of equivalence classes, where states which differ by a null state belong to the same equivalence class. In other words, physical states belong to the cohomology of $\mathcal{Q}_B$.

We will use the term *BRST closed* for states that are annihilated by Q_B, and the term *BRST exact* for the states of the form $\mathcal{Q}_B|\chi\rangle$, then the prescription for the Hilbert space of our theory will be as follows: the BRST Hilbert space $\mathcal{H}_{BRST}$ is given by taking the quotient of the Hilbert space formed from BRST closed states, $\mathcal{H}_{closed}$, with the Hilbert space formed from BRST exact states, $\mathcal{H}_{exact}$, so our physical Hilbert space is given by

$$\mathcal{H}_{BRST} = \frac{\mathcal{H}_{closed}}{\mathcal{H}_{exact}}\,. \tag{2.194}$$

2.3.6 BRST quantization of the bosonic string

After conformal gauge fixing, the total BRST invariant action is given by

$$S_T = S_X + S_{GF} + S_g \,, \tag{2.195}$$

where

$$S_X = \frac{1}{2\pi\alpha'} \int d^2z \, \partial X^\mu \bar{\partial} X_\mu \,, \tag{2.196}$$

$$S_g = \frac{1}{2\pi} \int d^2z \, \left(b\bar{\partial}c + \bar{b}\partial\bar{c} \right) \,. \tag{2.197}$$

So the full path integral becomes:

$$\int \mathcal{D}X \mathcal{D}g \mathcal{D}b \mathcal{D}\bar{b} \mathcal{D}c \mathcal{D}\bar{c} \, e^{(-S_X - S_g)} \,. \tag{2.198}$$

You should note that in this path integral we have already integrated the auxiliary field B_{ab} as a result of gauge fixing $g_{ab} = \delta_{ab}$. For the sake of completeness, let us write the gauge fixing term for the total action

$$S_{GF} = \frac{i}{4\pi} \int d^2\sigma \, g^{1/2} B^{ab} \left(\delta_{ab} - g_{ab} \right) \,. \tag{2.199}$$

The total action is invariant under the following BRST symmetry transformations

$$\begin{aligned}
\delta_B X^\mu &= i\epsilon \left(c\partial - \bar{c}\bar{\partial} \right) X^\mu \,, \\
\delta_B b &= i\epsilon \left(T^X + T^g \right) \,, \\
\delta_B \bar{b} &= i\epsilon \left(\bar{T}^X + \bar{T}^g \right) \,, \\
\delta_B c &= i\epsilon \left(c\partial - \bar{c}\bar{\partial} \right) c \,, \\
\delta_B \bar{c} &= i\epsilon \left(c\partial - \bar{c}\bar{\partial} \right) \bar{c} \,,
\end{aligned} \tag{2.200}$$

where we have used the equations of motion of B_{ab} when writing the BRST transformations of the ghosts b and $\bar{b}$.

Let us now employ Noether's procedure to find the BRST current associated to the BRST symmetry transformations I just wrote above

$$\delta_B S_T = \delta_B S_X + \delta_B S_g \,. \tag{2.201}$$

As usual, we promote the symmetry parameter ϵ to depend on the worldsheet variables z and $\bar{z}$. For the matter part we have

$$\begin{aligned}
\delta_B S_X &= \frac{1}{2\pi\alpha'}\int d^2z\,\left[\partial\delta_B X^\mu\bar{\partial}X_\mu + \partial X^\mu\bar{\partial}\delta_B X_\mu\right] \\
&= \frac{1}{2\pi\alpha'}\int d^2z\,\Big[\left(\partial\epsilon c\partial X^\mu + \epsilon\partial(c\partial X^\mu) + \partial\epsilon\bar{c}\bar{\partial}X^\mu\right)\bar{\partial}X_\mu + \\
&\qquad + \partial X_\mu\left(\bar{\partial}\epsilon c\partial X^\mu + \bar{\partial}\epsilon\bar{c}\bar{\partial}X^\mu + \epsilon\bar{\partial}(\bar{c}\bar{\partial}X^\mu)\right)\Big] \\
&= \frac{1}{2\pi\alpha'}\int d^2z\,\Big\{\partial\epsilon\left[\bar{c}\bar{\partial}X^\mu\bar{\partial}X_\mu\right] + \bar{\partial}\epsilon\left[c\partial X^\mu\partial X_\mu\right] + \left(\partial\epsilon c\partial X^\mu\bar{\partial}X_\mu + \epsilon\partial(c\partial X^\mu)\bar{\partial}X_\mu\right) \\
&\qquad + \left(\bar{\partial}\epsilon\bar{c}\bar{\partial}X^\mu\partial X_\mu + \partial X_\mu\epsilon\bar{\partial}(\bar{c}\bar{\partial}X^\mu)\right)\Big\}. \qquad (2.202)
\end{aligned}$$

Similarly, for the ghost part of the action we can write

$$\begin{aligned}
\delta_B S_g &= \frac{1}{2\pi}\int d^2z\,\left[\delta_B b\bar{\partial}c + b\bar{\partial}\delta_B c + \delta_B\bar{b}\partial\bar{c} + \bar{b}\partial\delta_B\bar{c}\right] \\
&= \frac{i}{2\pi}\int d^2z\,\left[\epsilon(T^X + T^g)\bar{\partial}c + b\bar{\partial}(\epsilon c\partial c) + \epsilon(\bar{T}^X + \bar{T}^g)\partial\bar{c} + \bar{b}\partial(\epsilon\bar{c}\bar{\partial}\bar{c})\right] \\
&= \frac{i}{2\pi}\int d^2z\,\left[(bc\partial c)\bar{\partial}\epsilon + (\bar{b}\bar{c}\bar{\partial}\bar{c})\partial\epsilon\right]. \qquad (2.203)
\end{aligned}$$

The two terms at the end of (2.202) are just total derivatives, so we can write the total variation of the action as follows

$$\begin{aligned}
\delta S_T &= \frac{i}{2\pi}\int d^2z\,\Bigg[\left(-\bar{c}\frac{1}{\alpha'} : \bar{\partial}X^\mu\bar{\partial}X_\mu : + : \bar{b}\bar{c}\bar{\partial}\bar{c} :\right)\partial\epsilon + \\
&\qquad + \left(-c\frac{1}{\alpha'} : \partial X^\mu\partial X_\mu : + : bc\partial c :\right)\bar{\partial}\epsilon\Bigg]. \qquad (2.204)
\end{aligned}$$

From the variation computed above, it is pretty easy to identify the holomorphic and anti-holomorphic parts of the BRST current

$$j_B = c\,T^X + : b\,c\,\partial c : + \frac{3}{2}\,\partial^2 c\,, \qquad (2.205)$$

$$\bar{j}_B = \bar{c}\,\bar{T}^X + : \bar{b}\,\bar{c}\,\bar{\partial}\bar{c} : + \frac{3}{2}\,\bar{\partial}^2\bar{c}\,. \qquad (2.206)$$

Where the final terms in the currents are just total derivatives, so they do not contribute to the BRST charge, they have been added by hand in order to make the BRST current transform as a primary field.

Let us now compute some useful OPEs of the BRST current with the ghost fields and a general primary field

$$\begin{aligned}
j_B(z)b(w) &= :c(z)T^X(z):b(w)+:b(z)c(z)\partial c(z):b(w)+\frac{3}{2}\partial^2 c(z)b(w) \\
&= \frac{T^X(z)}{(z-w)}-\frac{:b(z)c(z):}{(z-w)^2}-\frac{:b(z)\partial c(z):}{(z-w)}+\frac{3}{(z-w)^3}+\dots \\
&= \frac{T^X(w)}{(z-w)}-\frac{:b(w)c(w):}{(z-w)^2}-\frac{:\partial(b(w)c(w)):}{(z-w)}-\frac{:b(w)\partial c(w):}{(z-w)}+\frac{3}{(z-w)^3}+\dots \\
&= \frac{T^X(w)}{(z-w)}+\frac{T^g(w)}{(z-w)}+\frac{j^g(w)}{(z-w)^2}+\frac{3}{(z-w)^3}+\dots \\
&\sim \frac{3}{(z-w)^3}+\frac{j^g(w)}{(z-w)^2}+\frac{T^{X+g}}{(z-w)}\,, \qquad (2.207)
\end{aligned}$$

where of course, thanks to Ward identities, it is easy to see from the single pole of this OPE the way the ghost field b transforms under BRST transformations. Similarly we can work out the OPE with the c ghost field as follows

$$\begin{aligned}
j_B(z)c(w) &= :c(z)T^X(z):c(w)+:b(z)c(z)\partial c(z):c(w)+\frac{3}{2}\partial^2 c(z)c(w) \\
&= \frac{:c(z)\partial c(z):}{z-w}+\dots \\
&\sim \frac{c(w)\partial c(w)}{(z-w)}\,. \qquad (2.208)
\end{aligned}$$

And finally, computing the OPE with a general primary field operator $\mathcal{O}$, we can investigate how it transforms under BRST transformations

$$\begin{aligned}
j_B(z)\,\mathcal{O}(w) &= :c(z)T^X(z):\mathcal{O}(w)+:b(z)c(z)\partial c(z):\mathcal{O}(w) \\
&= c(z)\left[\frac{h}{(z-w)^2}\mathcal{O}(w)+\frac{1}{(z-w)}\partial\mathcal{O}(w)+\dots\right] \\
&\sim \frac{h}{(z-w)^2}c(w)\mathcal{O}(w)+\frac{1}{(z-w)}\left[h(\partial c(w))\mathcal{O}(w)+c(w)\partial\mathcal{O}(w)\right] \qquad (2.209)
\end{aligned}$$

Let us now define the BRST charge in the following way

$$\mathcal{Q}_B=\frac{1}{2\pi i}\oint\left(dz j_b-d\bar{z}\bar{j}_B\right)\,. \qquad (2.210)$$

With the help of the $j_B b$ OPE showed in (2.207) and the fact that the $\bar{j}_B b$ OPE is non-

singular, we can compute the following anti-commutator

$$\begin{aligned}
\{\mathcal{Q}_B, b_m\} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \{j_B(z), b(w)\} w^{m+1} \\
&= \oint \frac{dw}{2\pi i} w^{m+1} \oint \frac{dz}{2\pi i} \left[\frac{T^{X+g}}{z-w} + \frac{3}{(z-w)^3} + \frac{j^g}{(z-w)^2} \right] \\
&= \oint \frac{dw}{2\pi i} w^{m+1} T^{X+g}(w) \\
&= L_m^X + L_m^g \,. \qquad (2.211)
\end{aligned}$$

We now find the explicit form of the BRST charge in terms of mode expansions and Virasoro generators

$$\begin{aligned}
\mathcal{Q}_B &= \frac{1}{2\pi i} \oint (dz\, j_B(z) - d\bar{z}\, \bar{j}_B(\bar{z})) \\
&= \oint \frac{dz}{2\pi i} \left[\sum_{m,n} \frac{c_m L_n^X}{z^{m+n+1}} - \sum_{m,n,l} (l-1) \frac{b_m c_n c_L}{z^{m+n+l+1}} \right] + \text{anti-holomorphic part} \\
&= \sum_{n=-\infty}^{\infty} c_n L_{-n}^X - \sum_{n,l} (l-1)\, b_{-n-l} c_n c_l + \text{anti-holomorphic part} \\
&= \sum_{n=-\infty}^{\infty} c_n L_{-n}^X + \sum_{n,m} (n-m) c_n c_m b_{-n-m} + \text{anti-holomorphic part}\,. \qquad (2.212)
\end{aligned}$$

With a similar contribution from the anti-holomorphic part, we can finally write

$$\begin{aligned}
\mathcal{Q}_B &= \sum_{n=-\infty}^{\infty} \left(c_n L_{-n}^X + \bar{c}_n \bar{L}_{-n}^X \right) + \\
&\quad + \sum_{m,n} \frac{(m-n)}{2} : \left(c_m c_n b_{-m-n} + \bar{c}_m \bar{c}_n \bar{b}_{-m-n} \right) : + a^B (c_0 + \bar{c}_0)\,, \qquad (2.213)
\end{aligned}$$

where the terms are normal ordered and the normal ordering constant a^B is just $a^B = a^g = -1$. As we saw earlier, there is an anomaly in the gauge symmetry when $c^X \neq 26$, so we should expect a breakdown in the BRST formalism. It turns out that the BRST charge $\mathcal{Q}_B$ is not nilpotent unless $c^X = 26$

$$\{\mathcal{Q}_B, \mathcal{Q}_B\} = 0 \quad \text{only if} \quad c^X = 26\,. \qquad (2.214)$$

We can prove this very easily by using the definition (2.210). We just need to compute the $j_B j_B$ OPE, since the BRST charge anti-commutator just depends on the simple poles

of this OPE. The computation goes as follows

$$\begin{aligned} j_B(z)j_B(w) &= \left(c(z)T^X(z) + :b(z)c(z)\partial c(z): + \frac{3}{2}\partial^2 c(z)\right) \times \\ &\quad \times \left(c(w)T^X(w) + :b(w)c(w)\partial c(w): + \frac{3}{2}\partial^2 c(w)\right) \\ &= :c(z)c(w): T^X(z)T^X(w) + c(z)T^X(z) :b(w)c(w)\partial c(w): + \\ &\quad + :b(z)c(z)\partial c(z): c(w)T^X(w) + (:b(z)c(z)\partial c(z):)(:b(w)c(w)\partial c(w):) + \\ &\quad + \frac{3}{2} :b(z)c(z)\partial c(z): \partial^2 c(w) + \frac{3}{2}\partial^2 c(z) :b(w)c(w)\partial c(w): + \dots . \end{aligned} \tag{2.215}$$

As you can see, the computation is quite lengthy, and we should be very careful with signs, so we will perform this computation term by term. The first out of 6 terms in (2.215) goes as follows

$$\begin{aligned} :c(z)c(w): T^X(z)T^X(w) &= :c(z)c(w): \left[\frac{c^X/2}{(z-w)^4} + \frac{2\,T^X(w)}{(z-w)^2} + \frac{\partial T^X(w)}{(z-w)}\right] \\ &\sim \partial c(w)c(w)\left[\frac{c^X/2}{(z-w)^3} + \frac{2\,T^X(w)}{(z-w)}\right] + \frac{\partial^2 c(w)c(w)}{2}\left[\frac{c^X/2}{(z-w)^2}\right] + \\ &\quad + \frac{\partial^3 c(w)c(w)}{6}\left[\frac{c^X/2}{(z-w)}\right], \end{aligned} \tag{2.216}$$

where in the first line we made use of the $T^X T^X$ OPE of the matter part of the bosonic string and in the second line a Taylor expansion was performed. We now work out the second and third terms

$$\begin{aligned} c(z)T^X(z) :b(w)c(w)\partial c(w): &= T^X(z)\frac{c(w)\partial c(w)}{(z-w)} \\ &\sim \frac{T^X(w)c(w)\partial c(w)}{(z-w)}. \end{aligned} \tag{2.217}$$

$$\begin{aligned} :b(z)c(z)\partial c(z): c(w)T^X(w) &= \frac{:c(z)\partial c(z): T^X(w)}{(z-w)} \\ &\sim \frac{T^X(w)c(w)\partial c(w)}{(z-w)}, \end{aligned} \tag{2.218}$$

where we have been using the bc propagator which we derived earlier in this chapter. The computation for the fourth term is a bit lengthier since we find triple, double and single

poles

$$\begin{aligned}
\frac{3}{2} : b(z)c(z)\partial c(z) : \partial^2 c(w) &= 3\,\frac{: c(z)\partial c(z) :}{(z-w)^3} \\
&\sim 3\,\frac{c(w)\partial c(w)}{(z-w)^3} + 3\left[\frac{\partial c(w)\partial c(w)}{(z-w)^2} + \frac{c(w)\partial^2 c(w)}{(z-w)^2}\right] + \\
&\quad + \frac{3}{2}\left[\frac{\partial c(w)\partial^2 c(w)}{(z-w)} + \frac{c(w)\partial^3 c(w)}{(z-w)}\right] \\
&\sim 3\,\frac{c(w)\partial c(w)}{(z-w)^3} + 3\,\frac{c(w)\partial^2 c(w)}{(z-w)^2} + \frac{3}{2}\frac{\partial c(w)\partial^2 c(w)}{(z-w)} + \\
&\quad + \frac{3}{2}\frac{c(w)\partial^3 c(w)}{(z-w)}\,. \qquad (2.219)
\end{aligned}$$

The fifth term is an easy one, we just need to use the bc propagator to get

$$\frac{3}{2}\partial^2 c(z) : b(w)c(w)\partial c(w) := 3\,\frac{c(w)\partial c(w)}{(z-w)^3}\,. \qquad (2.220)$$

The last term requires a bit more of careful work, since there are triple, double and single poles involved

$$\begin{aligned}
: b(z)c(z)\partial c(z) :: b(w)c(w)\partial c(w) : &= -\frac{: c(z)\partial c(z)b(w)\partial c(w) :}{(z-w)} + \frac{c(z)\partial : c(z)b(w)c(w) :}{(z-w)^2} + \\
&\quad - \frac{: b(z)c(z)c(w)\partial c(w) :}{(z-w)^2} - \frac{: b(z)\partial c(z)c(w)\partial c(w) :}{(z-w)} + \\
&\quad + \frac{: c(z)\partial c(w) :}{(z-w)^3} + \frac{: \partial c(z)\partial c(w) :}{(z-w)^2} - \frac{: c(z)c(w) :}{(z-w)^4} + \\
&\quad - \frac{: \partial c(z)c(w) :}{(z-w)^3} \\
&\sim 3\,\frac{c(w)\partial c(w)}{(z-w)^3} + \frac{3}{2}\frac{\partial^2 c(w)\partial c(w)}{(z-w)} - \frac{3}{2}\frac{\partial^2 c(w)c(w)}{(z-w)^2} \\
&\quad - \frac{2}{3}\frac{\partial^3 c(w)c(w)}{(z-w)}\,. \qquad (2.221)
\end{aligned}$$

Putting pieces together, we can finally write

$$\begin{aligned}
j_B(z)j_B(w) &\sim \partial c(w)c(w)\left[\frac{c^X/2}{(z-w)^3} - \frac{9}{(z-w)^3}\right] + \partial^2 c(w)c(w)\left[\frac{c^X/4}{(z-w)^2} - \frac{9}{2}\frac{1}{(z-w)^2}\right] \\
&\quad + \partial^3 c(w)c(w)\left[\frac{c^X/12}{(z-w)} - \frac{13}{6}\frac{1}{(z-w)}\right]\,. \qquad (2.222)
\end{aligned}$$

Reducing this expression a bit more we are left with

$$j_B(z)j_B(w) \sim -\frac{(c^X-18)}{2\,(z-w)^3}\,c\,\partial c(w) - \frac{(c^X-18)}{4\,(z-w)^2}\,c\,\partial^2 c(w) - \frac{(c^X-26)}{12\,(z-w)}\,c\,\partial^3 c(w)\,. \quad (2.223)$$

As we said earlier, when computing the BRST charge anti-commutator with itself, we will only be interested in the single pole of the OPE above, which as you can see only vanishes when $c^X = 26$, then the BRST charge is nilpotent only if the central charge of the matter theory takes the critical value needed to cancel the Weyl anomaly.

In a very similar way, it can be shown that the BRST current is primary only for $c^X = 26$ by looking at the following OPE

$$T^{X+g}(z)j_B(w) \sim \frac{c^X-26}{2\,(z-w)^4}\,c(w) + \frac{1}{(z-w)^2}\,j_B(w) + \frac{1}{z-w}\,\partial j_B(w)\,. \quad (2.224)$$

We will not compute this OPE explicitly, but after all the OPEs we have computed so far, it should not be hard to convince you that the cubic term we added by hand in (2.205) was crucial to cancel the total contribution to the quadruple pole showed in (2.224) when $c^X = 26$.

To finish this chapter let us spend a few words to talk about the spectrum of the bosonic string. The main purpose was to develop some basic elements of conformal field theory and the BRST method, so we will not compute the spectrum of the theory. However, we have already defined the vacuum to be the direct product of the matter part and the ghost part as follows

$$|0\rangle = |0,k\rangle \otimes |\downarrow\rangle\,. \quad (2.225)$$

The physical states of the theory are constructed by acting on the vacuum with the BRST operator $\mathcal{Q}_B$. For instance we have

$$\begin{aligned} \mathcal{Q}_B|0\rangle &= \mathcal{Q}_B\left(|0,k\rangle \otimes |\downarrow\rangle\right) \\ &= \left((L_0^X-1)|0,k\rangle\right)\left(c_0\,|\downarrow\rangle\right) + \sum_{m>0}(L_m^X|0\rangle)(c_{-m}\,|\downarrow\rangle)\,, \end{aligned} \quad (2.226)$$

where in the last line, we only kept the terms for $m > 0$ since the $m < 0$ terms for c_{-m} annihilate $|\downarrow\rangle$. As you can see, when the BRST charge annihilate the ground state we have that for all $m > 0$

$$(L_0^X-1)|0,k\rangle = 0\,, \qquad L_m^X|0\rangle = 0\,. \quad (2.227)$$

These are none other than the known physical state conditions which lead to the tachyon of the bosonic string. Furthermore, imposing the extra condition $b_n|\psi\rangle = 0$ on physical states implies that a physical state can not contain c oscillator modes. On the other hand, fixing the ghost number prohibits the physical states from having any b oscillator modes.

Chapter 3

Superstrings in flat space

The main purpose of this chapter is to formulate the superstring action in a ten-dimensional flat-space. We start by studying the Green-Schwarz superstring as a generalization of the Brink-Schwarz superparticle. We will explain the problems with the covariant quantization of this particular superstring model. After that we give a pedagogical introduction to the pure spinor superstring by constructing the pure spinor ghost action from some consistency conditions which we will establish later.

The best reference to study the Green-Schwarz superstring is [5]. On the other hand, there are some excellent lectures on the pure spinor superstring which include [8] and [9], apart from that, [10] presents a very nice introduction, in fact, some of the computation details that we omit can be found there.

3.1 The Brink-Schwarz superparticle

We will start by studying the superparticle action. Even though it is much simpler than the superstring, it is extremely useful to review it since there are some important similarities between them. Let us write the Brink-Schwarz action for the superparticle [11] [1]

$$S = \int d\tau \left(\Pi^m P_m + e P_m P^m \right) , \tag{3.1}$$

[1] The standard description of the massless superparticle can be obtained from (3.1) by using the equation of motion for P_m (3.3) to get

$$S = -\frac{1}{4} \int d\tau \, e^{-1} \Pi^m \Pi_m \, .$$

where

$$\Pi^m = \dot{X}^m - \frac{i}{2}\dot{\theta}^\alpha \gamma^m_{\alpha\beta}\,\theta^\beta\,, \qquad m = 0, 1, \ldots, 9\,, \tag{3.2}$$

here the fermionic variables θ^α are Majorana-Weyl spinors, $\gamma^m_{\alpha\beta}$ are sixteen by sixteen matrices, P^m are the canonical momenta associated to X^m and e is a Lagrange multiplier. Furthermore, the equations of motion can be easily found by varying (3.1)

$$P^2 = 0\,, \qquad \dot{P}^m = 0\,, \qquad (\gamma^m\dot{\theta})_\alpha P_m = 0\,, \qquad \Pi^m + 2eP^m = 0\,. \tag{3.3}$$

The variables Π^m are constructed in such a way that the action (3.1) possess manifest supersymmetry, that is it is invariant under the following global transformations

$$\delta_\epsilon\theta^\alpha = \epsilon^\alpha\,, \tag{3.4}$$

$$\delta_\epsilon X^m = \frac{i}{2}\,\theta^\alpha\,\gamma^m_{\alpha\beta}\,\epsilon^\beta\,, \tag{3.5}$$

$$\delta_\epsilon P^m = 0\,, \tag{3.6}$$

$$\delta_\epsilon e = 0\,, \tag{3.7}$$

where ϵ^α is a constant fermionic parameter. We can check that (3.1) is indeed invariant under the transformations above, it is straightforward to write

$$\begin{aligned}
\delta_\epsilon S &= \int d\tau\,(\delta_\epsilon\,\Pi^m P_m + \Pi^m\delta_\epsilon P_m + 2e\,\delta_\epsilon P^m P_m) \\
&= \int d\tau\left(\delta_\epsilon\dot{X}^m - \frac{i}{2}\delta_\epsilon\dot{\theta}\,\gamma^m\theta - \frac{i}{2}\dot{\theta}\,\gamma^m\delta_\epsilon\theta\right) \\
&= \int d\tau\left(\frac{i}{2}\,\theta\,\gamma^m\epsilon - \frac{i}{2}\,\dot{\theta}\,\gamma^m\epsilon\right) \\
&= 0\,.
\end{aligned} \tag{3.8}$$

3.1.1 The problem with covariant quantization

The Brink-Schwarz superparticle represents an example of a constrained system, let us find the constraints of this theory. In order to do this we first compute the canonical momenta for the variables X^m and θ^α

$$\frac{\delta L}{\delta\dot{X}^m} = P_m\,, \tag{3.9}$$

$$\frac{\delta L}{\delta\dot{\theta}^\alpha} = -\frac{i}{2}P_m\,(\gamma^m\theta)_\alpha \equiv p_\alpha\,. \tag{3.10}$$

As you can see, the canonical momenta associated to θ_α depends explicitly on the variables, this represent a constraint on the theory and for this reason, it should be treated using Dirac's prescription to quantize constrained systems [12].

In order to identify the nature of the constraints we are dealing with, we need to impose canonical Poisson brackets as follows

$$\left\{p_\alpha, \theta^\beta\right\}_P = -i\delta_\alpha^\beta\,, \qquad \{X^m, P^n\}_P = \eta^{mn}\,. \tag{3.11}$$

We will denote the constraints of the system as d_α defined by

$$d_\alpha = p_\alpha + \frac{i}{2}\,P_m\,(\gamma^m\theta)_\alpha = 0\,. \tag{3.12}$$

The Poisson bracket of the constraints is given by the following matrix

$$\begin{aligned} c_{\alpha\beta} &= \{d_\alpha, d_\beta\}_P \\ &= \frac{i}{2}\,P_m\big\{p_\alpha, (\gamma^m\theta)_\beta\,\big\}_P + \frac{i}{2}\,P_n\big\{\,(\gamma^n\theta)_\alpha\,, p_\beta\big\}_P \\ &= P_m\,\gamma^m_{\alpha\beta}\,. \end{aligned} \tag{3.13}$$

We should remember that in general, if we have a set of constrains ϕ_A with $c_{AB} = \{\phi_A, \phi_B\}_P$, ϕ_m are *first class* constraints if c_{AB} is zero or a linear combination of the constraints (weakly zero), otherwise ϕ_α are *second class* constraints. From relation (3.13), we can note that half of the constraints are first class and the other half are second class. We can see this as follows: because of the equation of motion $P^2 = 0$, one can chose a reference frame in which $P^m = (P, 0, \ldots, 0, P)$, so that $c_{\alpha,\beta} \sim (\gamma^0 - \gamma^9)_{\alpha\beta} \sim (\gamma^-)_{\alpha\beta}$. The rank of this matrix is 8, and for this reason, $c_{\alpha\beta}$ only has 8 eigenvalues different from zero.

In order to covariantly quantize the superparticle one should covariantly separate the first and second class constraints, something that until now has not been achieved. However, in order to deal with the second class constraints, one can use the light cone gauge, but the cost is the breaking of manifest Lorentz covariance.

3.1.2 Kappa symmetry

The eight first class constraints generate gauge symmetries, thus we can fix eight θ^α components reducing the fermionic degrees of freedom. In order to deal with the first class constraints we define

$$D^\alpha = iP^m\,\gamma_m^{\alpha\beta}\,d_\beta\,. \tag{3.14}$$

It is easy to show that

$$\left\{D^\alpha, D^\beta\right\} = 0\,. \tag{3.15}$$

These first class constraints D^α generate the so-called *kappa symmetry* discovered by Warren Siegel [13], defined by the following transformations

$$\begin{aligned}
\delta_\kappa \theta^\alpha &= P^m \left(\gamma_m \, \kappa\right)^\alpha && (3.16)\\
\delta_\kappa X^m &= -\frac{i}{2} \left(\theta \, \gamma^m \, \delta_\kappa \theta\right) && (3.17)\\
\delta_\kappa P^m &= 0 && (3.18)\\
\delta_\kappa e &= i\dot{\theta}^\alpha \kappa_\alpha \, . && (3.19)
\end{aligned}$$

Here κ is a local fermionic parameter. Let us show that the action (3.1) is indeed invariant under these transformations. First let me find how the supersymmetric object Π^m transforms

$$\begin{aligned}
\delta_\kappa \Pi^m &= \delta_\kappa \dot{X}^m - \frac{i}{2} \, \delta_\kappa \dot{\theta}^\alpha \gamma^m_{\alpha\beta} \theta^\beta - \frac{i}{2} \, \dot{\theta}^\alpha \gamma^m_{\alpha\beta} \delta_\kappa \theta^\beta \\
&= -i \left(\dot{\theta} \gamma^m \delta_\kappa \theta\right) \\
&= -i \, \dot{\theta}^\alpha \gamma^m_{\alpha\beta} \, \gamma^{\beta\delta}_n P^n \kappa_\delta \\
&= -i\dot{\theta}^\alpha \kappa_\alpha \, P^m \, . && (3.20)
\end{aligned}$$

Then the effect of the kappa transformations on the action can be computed easily as follows

$$\begin{aligned}
\delta_\kappa S &= \int d\tau \, \left(\delta_\kappa \Pi^m P_m + \delta_k e \, P_m P^m\right) \\
&= \int d\tau \, \left(-i\dot{\theta}^\alpha \kappa_\alpha \, P^m P_m + i\dot{\theta}^\alpha \kappa_\alpha \, P_m P^m\right) \, = \, 0 \, . && (3.21)
\end{aligned}$$

3.2 The Green-Schwarz superstring

We will describe the motion of a string in a ten dimensional Minkowski superspace with two supersymmetries described by ten bosonic coordinates $X^m \qquad (m = 0, \ldots 9)$ and two fermionic ones $\theta^{1\,\alpha}$ and $\theta^{2\,\alpha}$, each of them with 32 components $(\alpha = 1, \ldots 32)$.[2]

There are two string theories with $N = 2$ supersymmetry in ten dimensions, Type IIA and Type IIB superstring theories. The difference between the two of them is the relative chirality of the two spinorial coordinates in each of them

$$\begin{aligned}
&\text{Type IIA :} \qquad \Gamma\theta^1 = \theta^1 \qquad ; \qquad \Gamma\theta^2 = -\theta^2 \, , && (3.22)\\
&\text{Type IIB :} \qquad \Gamma\theta^1 = \theta^1 \qquad ; \qquad \Gamma\theta^2 = \theta^2 \, , && (3.23)
\end{aligned}$$

[2]In fact, we will be working with Majorana-Weyl spinors. As we know a Dirac spinor has $2^{\frac{D}{2}}$ components in D dimensions, but because of the Majorana-Weyl conditions, the number of degrees of freedom is reduced to 16. That is why we have been considering sixteen by sixteen gamma matrices γ^m.

where $\Gamma = \Gamma^0 \dots \Gamma^9$ is the chirality matrix, you can see Appendix A for our conventions.

3.2.1 Generalizing the superparticle action

In order to define the Green-Schwarz action, we consider a bosonic action and then we make it supersymmetric by defining manifest supersymmetric variables. Let us start with Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int_{\mathcal{M}} d^2\sigma \, \sqrt{-g} \, g^{ab} \, \partial_a X^\mu \partial_b X_\mu \, . \tag{3.24}$$

If we want to make (3.24) supersymmetric, we write the action above in the following way

$$S_1 = -\frac{1}{4\pi\alpha'} \int_{\mathcal{M}} d^2\sigma \, \sqrt{-g} \, g^{ab} \, \Pi_a^m \Pi_b^n \, \eta_{mn} \, , \tag{3.25}$$

where g^{ab} is the world-sheet metric, η_{mn} is the flat space metric and

$$\Pi_a^m = \partial_a X^m - i\delta_{\mathcal{AB}} \, \bar{\theta}^{\mathcal{A}} \gamma^m \partial_a \theta^{\mathcal{B}} \, ; \qquad \mathcal{A}, \mathcal{B} = 1, 2 \, , \tag{3.26}$$

here $\delta_{\mathcal{AB}}$ is the usual Kronecker delta. The action (3.25) is, of course, supersymmetric by construction, that is, it is invariant under the following global transformations

$$\delta_\epsilon X^m = i \, \delta_{\mathcal{AB}} \, \bar{\theta}^{\mathcal{A}} \gamma^m \epsilon^{\mathcal{B}} \, , \tag{3.27}$$

$$\delta_\epsilon \theta^{\mathcal{A}} = \epsilon^{\mathcal{A}} \, . \tag{3.28}$$

Of course Π_a^m is supersymmetric by construction

$$\begin{aligned} \delta_\epsilon \Pi_a^m &= \partial_a \delta_\epsilon X^m - i\delta_{\mathcal{AB}} \, \delta_\epsilon \bar{\theta}^{\mathcal{A}} \gamma^m \partial_a \theta^{\mathcal{B}} \\ &= i\partial_a \bar{\theta}^1 \gamma^m \epsilon^1 + i\partial_a \bar{\theta}^2 \gamma^m \epsilon^2 - i\bar{\epsilon}^1 \gamma^m \partial_a \theta^1 - i\bar{\epsilon}^2 \gamma^m \partial_a \theta^2 \\ &= 0 \, . \end{aligned} \tag{3.29}$$

Then the action (3.25) we considered above has manifest supersymmetry

$$\delta_\epsilon S_1 = 0 \, . \tag{3.30}$$

The action (3.25) is just a generalization of the superparticle action we considered before, but it is not the action we are interested in since the so-called kappa symmetry has been lost in this procedure. As a result, the fermionic variables θ describe twice as many degrees of freedom as they should. For this reason, we need to recover the local kappa symmetry of the theory. Apart from that, this symmetry is also of crucial importance for the quantum spectrum of the theory to be supersymmetric.

In the next section, we show how to add a second term to the action (3.25) to make it invariant under kappa transformations similar to the ones considered for the Brink-Schwarz superparticle.

3.2.2 Recovering kappa symmetry

Motivated by the Brink-Schwarz superparticle, let us establish the kappa transformations for the bosonic coordinates X^m as follows

$$\delta_\kappa X^m = i\delta_{\mathcal{AB}}\bar{\theta}^{\mathcal{A}}\gamma^m\delta_\kappa\theta^B\,, \tag{3.31}$$

so, it is not difficult to find the transformation rule for the superfields Π_a^m which I show next

$$\begin{aligned}\delta_\kappa\Pi_a^m &= \delta_\kappa\left(\partial_a X^m - i\delta_{\mathcal{AB}}\,\bar{\theta}^{\mathcal{A}}\gamma^m\partial_a\theta^{\mathcal{B}}\right)\\ &= i\delta_{\mathcal{AB}}\,\partial_a\bar{\theta}^{\mathcal{A}}\gamma^m\delta_\kappa\theta^{\mathcal{B}} - i\delta_{\mathcal{AB}}\,\delta_\kappa\bar{\theta}^{\mathcal{A}}\gamma^m\partial_a\theta^{\mathcal{B}}\\ &= 2i\,\delta_{\mathcal{AB}}\,\partial_a\bar{\theta}^{\mathcal{A}}\gamma^m\delta_\kappa\theta^{\mathcal{B}}\,.\end{aligned} \tag{3.32}$$

Now, with the help of the expression above, we can write the variation of S_1 under kappa transformations as follows

$$\begin{aligned}\delta_\kappa S_1 = -\frac{i}{2\pi\alpha'}\int d^2\sigma\Big\{&\sqrt{-g}\,g^{ab}\,\Pi_a^m\,\delta_{\mathcal{AB}}\,\bar{\theta}^{\mathcal{A}}\gamma^n\partial_b\theta^{\mathcal{B}}\,\eta_{mn} +\\ &+\delta_\kappa\left(\sqrt{-g}\,g^{ab}\right)\Pi_a^m\Pi_b^n\,\eta_{mn}\Big\}\,,\end{aligned} \tag{3.33}$$

where the transformation rule for the world sheet metric under kappa transformations is still to be determined.

The procedure to recover kappa symmetry will be the following: we will add a term S_2 to the action in such a way that the total action $S_{GS} = S_1 + S_2$ is invariant under kappa transformations without breaking any existent symmetry of the theory, namely super-Poincaré invariance and diffeomorphism invariance.

There is a general method to achieve this (see Chapter 4 for details). We will write S_2 as the integral of a two-form Ω_2 which does not depend on the worldsheet metric, that is

$$S_2 = \int_{\mathcal{M}}\Omega_2 = \frac{1}{2}\int_{\mathcal{M}} d^2\sigma\,\epsilon^{\alpha\beta}\Omega_{\alpha\beta}\,. \tag{3.34}$$

Now let us define the three-form Ω_3 as follows

$$\Omega_3 = d\,\Omega_2\,, \tag{3.35}$$

using Stoke's theorem we can write

$$S_2 = \int_{\mathcal{D}}\Omega_3 = \int_{\mathcal{M}}\Omega_2\,, \tag{3.36}$$

where $\mathcal{D}$ is a three dimensional manifold and $\mathcal{M}$ is his border which can be interpreted as the string worldsheet, that is, $\partial\mathcal{D} = \mathcal{M}$. The main advantage of writing the action in this fashion is that the symmetries of the theory appear manifest.

In order to have manifestly supersymmetry, we need to construct our three form using manifest supersymmetric objects, the ones we count with are the following

$$\Pi^m = d\,X^m + i\delta_{\mathcal{AB}}\,\bar{\theta}^{\mathcal{A}}\gamma^m\,d\,\theta^{\mathcal{B}} \qquad \text{and} \qquad d\,\theta^A\,. \tag{3.37}$$

The correct form we need to consider turns out to be

$$\Omega_3 = \mathfrak{s}_{\mathcal{AB}}\,d\,\bar{\theta}^{\mathcal{A}}\,\gamma^m d\,\theta^{\mathcal{B}}\,\Pi_m\,, \tag{3.38}$$

where $\mathfrak{s}_{\mathcal{AB}}$ is a 2×2 matrix which will be determined by the requirement of Ω_3 to be exact, this is

$$\begin{aligned} d\,\Omega_3 &= \mathfrak{s}_{\mathcal{AB}}\,d\,\bar{\theta}^{\mathcal{A}}\,\gamma^m d\,\theta^{\mathcal{B}}\,d\,\Pi_m \\ &= \mathfrak{s}_{11}\,d\,\bar{\theta}^1\,\gamma^m d\,\theta^1\,d\,\Pi_m + \mathfrak{s}_{22}\,d\,\bar{\theta}^2\,\gamma^m d\,\theta^2\,d\,\Pi_m + \\ &\quad + \mathfrak{s}_{12}\,d\,\bar{\theta}^1\,\gamma^m d\,\theta^2\,d\,\Pi_m + \mathfrak{s}_{21}\,d\,\bar{\theta}^2\,\gamma^m d\,\theta^1\,d\,\Pi_m \\ &= \mathfrak{s}_{11}\,(d\,\bar{\theta}^1\,\gamma^m d\,\theta^1)(i d\,\bar{\theta}^1\,\gamma^m d\,\theta^2) + \mathfrak{s}_{22}\,(d\,\bar{\theta}^2\,\gamma^m d\,\theta^2)(i d\,\bar{\theta}^1\,\gamma^m d\,\theta^1)\,. \end{aligned} \tag{3.39}$$

It is very easy to see from the equation above, that for Ω_3 to be exact, we need to have

$$\mathfrak{s}_{11} = -\mathfrak{s}_{22} = c\,, \tag{3.40}$$

then

$$\Omega_3 = c\left(d\,\bar{\theta}^1\,\gamma^m d\,\theta^1 - d\,\bar{\theta}^2\,\gamma^m d\,\theta^2\right)\Pi_m\,. \tag{3.41}$$

Let us now compute how Ω_3 transforms under kappa transformations

$$\begin{aligned} \delta_\kappa\Omega_3 &= d\left[2c\left(\delta_\kappa\bar{\theta}^1\,\gamma^m d\,\theta^1 - \delta_\kappa\bar{\theta}^2\,\gamma^m d\,\theta^2\right)\Pi_m\right] \\ \Rightarrow \quad \delta_\kappa\Omega_2 &= 2c\left(\delta_\kappa\bar{\theta}^1\,\gamma^m d\,\theta^1 - \delta_\kappa\bar{\theta}^2\,\gamma^m d\,\theta^2\right)\Pi_m\,. \end{aligned} \tag{3.42}$$

We can now add the variation of the two pieces of our total action $S_{GS} = S_1 + S_2$, this is

$$\begin{aligned} \delta_\kappa S_{GS} &= \delta_\kappa S_1 + \delta_\kappa S_2 \\ &= -\frac{i}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\left\{\sqrt{-g}g^{ab}\Pi^m_a\delta_{\mathcal{AB}}\,\delta_\kappa\bar{\theta}^{\mathcal{A}}\gamma^n\partial_b\theta^{\mathcal{B}}\eta_{mn} + \delta_\kappa(\sqrt{-g}g^{ab})\Pi^m_a\Pi^n_b\eta_{mn}\right\} + \\ &\quad +2c\int_{\mathcal{M}} d^2\sigma\,\epsilon^{ab}\left(\delta_\kappa\bar{\theta}^1\gamma_m\partial_a\theta^1 - \delta_\kappa\bar{\theta}^2\gamma_m\partial_a\theta^2\right)\,. \end{aligned} \tag{3.43}$$

The constant c is defined in such a way that the action is invariant under kappa transformations. Let us define $c = -\frac{i}{2\pi\alpha}$, replacing in the expression above and rearranging some of the terms we get

$$\begin{aligned}\delta_\kappa S_{GS} &= -\frac{i}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\,\sqrt{-g}\left[\Pi_a^m\left(g^{ab}+\frac{\epsilon^{ab}}{\sqrt{-g}}\right)\delta_\kappa\bar{\theta}^1\gamma_m\partial_b\theta^1\right]+\\ &\quad -\frac{i}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\,\sqrt{-g}\left[\Pi_a^m\left(g^{ab}-\frac{\epsilon^{ab}}{\sqrt{-g}}\right)\delta_\kappa\bar{\theta}^2\gamma_m\partial_b\theta^2\right]+\\ &\quad +\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\,\delta_\kappa\left(\sqrt{-g}g^{ab}\right)\Pi_a^m\Pi_b^n\,\eta_{mn}\,. \end{aligned} \tag{3.44}$$

Let us now define the following tensor operators

$$P_\pm^{ab} = \frac{1}{2}\left(g^{ab}\pm\frac{\epsilon^{ab}}{\sqrt{-g}}\right)\,, \tag{3.45}$$

these objects are called duality projectors in the worldsheet. Considering these definitions we are allowed to write the following expression

$$\begin{aligned}\delta_\kappa S_{GS} &= -\frac{i}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\,\sqrt{-g}\left\{\Pi_a^m\,P_-^{ab}\,\delta_\kappa\bar{\theta}^1\gamma_m\partial_b\theta^1+\Pi_a^m\,P_+^{ab}\,\delta_\kappa\bar{\theta}^2\gamma_m\partial_b\theta^2\right\}+\\ &\quad +\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\,\delta_\kappa\left(\sqrt{-g}g^{ab}\right)\Pi_a^m\Pi_b^n\,\eta_{mn}\,. \end{aligned} \tag{3.46}$$

Using the superparticle as a guide we write the kappa transformations for the fermionic variables as $\delta_\kappa\theta^{\mathcal{A}} = \Gamma_m\Pi_a^m\kappa^{\mathcal{A}\,a}$, using these transformations an some properties of gamma matrices we end up with the following result

$$\begin{aligned}\delta_\kappa S_{GS} &= -\frac{i}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\,\sqrt{-g}\,\Pi_a^m\Pi_b^n\left\{\partial_c\bar{\theta}^1\gamma_m\gamma_n\left(P_-^{ac}\kappa^{1\,b}-P_-^{bc}\kappa^{1\,a}\right)\right\}+\\ &\quad -\frac{i}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\,\sqrt{-g}\,\Pi_a^m\Pi_b^n\left\{\partial_c\bar{\theta}^2\gamma_m\gamma_n\left(P_+^{ac}\kappa^{2\,b}-P_+^{bc}\kappa^{2\,a}\right)\right\}+\\ &\quad +\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\,\delta_\kappa\left(\sqrt{-g}g^{ab}\right)\Pi_a^m\Pi_b^n\,\eta_{mn}\,. \end{aligned} \tag{3.47}$$

We can now define some constraints on the kappa parameters and the transformation rule for the world sheet metric tensor under kappa transformations so that the total Green-Schwarz action is invariant under kappa transformations

$$\kappa^{1\,a} = P_-^{ab}\kappa_b^1\,, \tag{3.48}$$

$$\kappa^{2\,a} = P_+^{ab}\kappa_b^2\,, \tag{3.49}$$

$$\delta_\kappa\left(\sqrt{-g}g^{ab}\right) = 8i\sqrt{-g}\left(P_-^{ac}\partial_c\bar{\theta}^1\kappa^{1\,b}+P_+^{ac}\partial_c\bar{\theta}^2\kappa^{2\,b}\right)\,. \tag{3.50}$$

With the conditions established above the expression (3.47) vanishes and thus we can define an action invariant under kappa transformations. To conclude the discussion let us summarize the procedure by writing the total Green-Schwarz action for the superstring

$$\begin{aligned} S_{GS} &= S_1 + S_2 \\ &= -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma \left\{ \sqrt{-g} g^{ab}\, \Pi_a \cdot \Pi_b - 2\epsilon^{ab}\left(\bar{\theta}^1\gamma^m\partial_a\theta^1\right)\left(\bar{\theta}^2\gamma_m\partial_b\theta^2\right) + \right. \\ &\qquad \left. +2i\epsilon^{ab}\partial_a X_m \left(\bar{\theta}^1\gamma^m\partial_b\theta^1 - \bar{\theta}^2\gamma^m\partial_b\theta^2\right) \right\} . \end{aligned} \tag{3.51}$$

3.2.3 The problem with covariant quantization

The equations of motion which follow from the Green-Schwarz action are heavily non linear, for this reason, it is very difficult to solve them. The local kappa symmetry we talked about last section allows us to define a special gauge in which these equations simplify considerably, but before going into details, let us show that the Green-Schwarz superstring suffers a similar illness as the Brink-Schwarz superparticle, namely, the constraints of the theory can not be disentangled in a manifestly covariant way.

Let us first go to conformal gauge in equation (3.51), that is, we first Wick rotate the signature and then fix the gauge according to $g^{ab} = \delta^{ab}$. Using the same definitions for the worldsheet coordinates and the derivatives that we used in (2.24) and (2.25), the Green-Schwarz action in conformal gauge can be written as follows

$$\begin{aligned} S_{GS} &= \frac{1}{2\pi\alpha'}\int d^2z \left[\partial X^m\bar{\partial}X_m - 2i\partial X_m\left(\bar{\theta}^1\gamma^m\bar{\partial}\theta^1\right) - 2i\bar{\partial}X_m\left(\bar{\theta}^2\gamma^m\partial\theta^2\right) + \right. \\ &\qquad - \left(\theta^1\gamma_m\bar{\partial}\theta^1\right)\left(\bar{\theta}^1\gamma^m\partial\theta^1 + \bar{\theta}^2\gamma^m\partial\theta^2\right) + \\ &\qquad \left. - \left(\theta^2\gamma_m\bar{\partial}\theta^2\right)\left(\bar{\theta}^1\gamma^m\bar{\partial}\theta^1 + \bar{\theta}^2\gamma^m\bar{\partial}\theta^2\right)\right] . \end{aligned} \tag{3.52}$$

In order to quantize the Green-Schwarz action, we need to compute the conjugate momenta $p^{\mathcal{A}}$ for the fermionic variables $\theta^{\mathcal{A}}$ and then impose canonical anti commutators between them. To give an example of how this computation works, let us find the conjugate momentum for θ^1

$$p^1_\alpha \equiv \pi\frac{\delta S_{GS}}{\delta\,\partial_1\theta^{1\,\alpha}} = \frac{i}{\alpha'}(\bar{\theta}^1\gamma_m)_\alpha\left[\Pi^m + \frac{i}{2}(\bar{\theta}^1\gamma^m\partial_1\theta^2)\right] , \tag{3.53}$$

where we can define the so-called Green-Schwarz constraints for the fermionic variables θ^1 as follows

$$d^1_\alpha \equiv p^1_\alpha - \frac{i}{\alpha'}\left(\bar{\theta}^1\,\gamma^m\right)_\alpha \Pi_m + \frac{i}{2\alpha'}\left(\bar{\theta}^1\,\gamma_m\right)_\alpha\left(\bar{\theta}^1\,\gamma^m\partial_1\theta^1\right) , \tag{3.54}$$

and the matrix for the fermionic constraints d^1_α follows to be

$$d^1_\alpha(z)d^1_\beta(w) \quad \sim \quad -\frac{i\,\gamma^m_{\alpha\beta}\Pi_m}{z-w}\,, \tag{3.55}$$

where we made use of the following OPE

$$p^1_\alpha\,\theta^{1\,\beta}(w) \quad \sim \quad \frac{\delta^\beta_\alpha}{z-w}\,. \tag{3.56}$$

By following a similar reasoning than the one we considered for the Brink-Schwarz superparticle, considering the so-called Virasoro constraints, $\Pi^m\Pi_m = 0$,[3] we conclude that half of the constraints are first class and half of them are second class. There is no prescription to disentangle the first and second class constraints in a covariant way. In the following, let us look at a convenient gauge to work with, the so-called light-cone gauge.

3.2.4 Light cone gauge

As we said earlier, the equations of motion which follow from (3.51) are very hard to solve, and as we have just seen, the covariant quantization is not known. However, even though we will not do it here, it is possible to find the spectrum of the theory if we use a convenient gauge. Let us introduce the so-called light cone coordinates in the following way

$$X^\pm = \frac{1}{2}\left(X^0 \pm X^9\right)\,, \qquad X^I \qquad \text{with} \qquad I = 1,\ldots 8\,, \tag{3.57}$$

we also need to make the following definitions

$$\eta^{+-} \;=\; -\eta^{II} \;=\; -1\,, \tag{3.58}$$

$$\gamma^\pm \;=\; \frac{(\gamma^0 \pm \gamma^9)}{\sqrt{2}}\,. \tag{3.59}$$

The light cone gauge consists on taking the plus coordinate in the following way

$$X^+ = x^+ + p^+\tau\,, \tag{3.60}$$

where x^+ and p^+ are just constants. As a result of this choice for the bosonic coordinates, the theory contains eight bosonic degrees of freedom.

[3] Where Π^m are the momentum (3.26) written in complex coordinates. Furthermore, the equations of motion for the metric are the Virasoro constraints

$$\Pi^m\Pi_m = \bar{\Pi}^m\bar{\Pi}_m = 0\,.$$

Let us now analyse the fermionic degrees of freedom. As we know, a spinor in ten dimensions has 32 complex components, since we are dealing with Majorana-Weyl spinors, we in fact only have 16 real components. Since we have been studying Type IIB superstrings, we need to consider two Majorana-Weyl spinors (θ^1 and θ^2), which makes a total of 32 real components. As we said earlier, the local kappa symmetry transformations can be used to gauge away half of the 32 fermionic degrees of freedom, so the light cone gauge condition enables us to reduce the number of degrees of freedom to eight for each spinor $\theta^{\mathcal{A}}$. So let us show how to use the local kappa symmetry to make our gauge choice. Let us consider the following expression

$$\begin{aligned} \gamma^+ \bar{\theta}'^{\mathcal{A}} &= \gamma^+ \left(\bar{\theta}^{\mathcal{A}} + \delta_\kappa \bar{\theta}^{\mathcal{A}}\right) \\ &= \gamma^+ \bar{\theta}^{\mathcal{A}} + \gamma^+ \Gamma_m \Pi_i^m \kappa^{\mathcal{A}\, i} \,, \end{aligned} \tag{3.61}$$

where in the first line we performed a kappa transformation. Then, it is always possible to use kappa transformations to make the following gauge choice

$$\gamma^+ \bar{\theta}'^{\mathcal{A}} = 0 \,, \tag{3.62}$$

which is the so-called light cone gauge for the fermionic variables.

With the use of the light cone gauge conditions (3.60) and (3.62), the action takes a very simple form. In order to notice this, we first need to realise that the only non zero component of expressions like $\bar{\theta}^{\mathcal{A}} \gamma^m \partial \theta^{\mathcal{A}}$ (here the repeated indexes do not imply any kind of sum) is when $m = -$ since the transverse and plus components vanish, that is,

$$\bar{\theta}^{\mathcal{A}} \gamma^m \partial \theta^{\mathcal{A}} = 0 \,, \qquad \text{except for } m = - \,. \tag{3.63}$$

In fact, the plus component vanishes because of (3.62). On the other hand, for the transverse components we can write

$$\bar{\theta}^{\mathcal{A}} = \theta^{\dagger\, \mathcal{A}} \gamma^0 = \theta^\dagger \left(\gamma^+ + \gamma^-\right) \,, \tag{3.64}$$

and since $\left\{\gamma^I, \gamma^+\right\} = 0$ and $\gamma^- = -(\gamma^+)^\dagger$, we can write

$$\bar{\theta}^{\mathcal{A}} \gamma^I \partial \theta^{\mathcal{A}} = \theta^{\dagger\, \mathcal{A}} \gamma^I \partial \left(\gamma^+ \theta^{\mathcal{A}}\right) - \left(\gamma^+ \theta^{\mathcal{A}}\right)^\dagger \gamma^I \partial \theta^{\mathcal{A}} = 0 \,. \tag{3.65}$$

It follows from the light cone gauge condition for the bosonic coordinates that

$$\bar{\partial} X^+ = -\partial X^+ = \frac{1}{2} p^+ \,. \tag{3.66}$$

Using the two identities (3.63) and (3.66), the Green-Schwarz action (3.52) takes the form

$$S_{GS} = \frac{1}{2\pi\alpha'} \int d^2z \left[-\partial X^I \bar{\partial} X^I - \frac{1}{2} p^+ \bar{\partial} X^- + \frac{1}{2} p^+ \partial X^- + ip^+ (\bar{\theta}^1 \gamma^- \bar{\partial} \theta^1) - ip^+ (\bar{\theta}^2 \gamma^- \partial \theta^2) \right] , \tag{3.67}$$

where the second and third terms can be dropped since they are total derivatives. Using the representation for γ^- that appears in Appendix A, (3.67) becomes

$$S_{GS} = \frac{1}{2\pi\alpha'} \int d^2z \left(-\partial X^I \bar{\partial} X^I + \bar{S}^{1\,\alpha} \bar{\partial} S^{1\,\alpha} + \bar{S}^{2\,\alpha} \bar{\partial} S^{2\,\alpha} \right) , \tag{3.68}$$

here $I = 1, \ldots, 8\,;\ \alpha = 1, \ldots, 8$ and

$$S^{1\,\alpha} = \frac{1}{2^{1/4}} \sqrt{ip^+} \theta^{1\,\alpha} , \qquad S^{2\,\alpha} = \frac{1}{2^{1/4}} \sqrt{ip^+} \theta^{2\,\alpha} . \tag{3.69}$$

3.3 The pure spinor formulation of superstrings

We now review the construction of the so-called pure spinor formalism of superstrings. We start by presenting Siegel's approach to the Green-Schwarz superstring and then we move on to construct the pure spinor formalism by requiring some consistency conditions.

3.3.1 Siegel's approach to the Green-Schwarz superstring

As we saw in the last section when studying the Green-Schwarz superstring, the covariant quantization of (3.51) can not be achieved due to the nature of the set of constraints of the theory as it was explained. When trying to quantize the theory, it is necessary to use kappa symmetry in order to fix the gauge (the so-called light cone gauge) thus breaking the manifest Lorentz covariance.

In 1986, Siegel proposed a modification to the Green-Schwarz action in such a way that the fermionic canonical momenta (3.53) were independent variables [14]. In order to relate the Green-Schwarz superstring to Siegel's proposal, we rewrite the action (3.52) by considering only one spinor, that is we make $\theta^1 = \theta$ and $\theta^2 = 0$, in this way we get [4]

$$S_{GS} = \frac{1}{4\pi} \int d^2z \left[\partial X^m \bar{\partial} X_m - 2i \partial X_m (\theta \gamma^m \bar{\partial} \theta) - (\theta \gamma_m \bar{\partial} \theta)(\theta \gamma^m \partial \theta) \right] . \tag{3.70}$$

We can define the conjugate momenta p_α as follows

$$p_\alpha \equiv 2\pi \frac{\delta S_{GS}}{\delta\, \bar{\partial} \theta^\alpha} = \frac{1}{2} \left(-2i \partial X_m - \theta \gamma_m \partial \theta \right) (\theta \gamma^m)_\alpha , \tag{3.71}$$

[4] From now on, in order to simplify the calculations, we make $\alpha' = 2$.

and for this case, the Green-Schwarz constraints would be

$$d_\alpha \equiv p_\alpha - \frac{1}{2}\left(-2i\partial X_m - \theta\gamma_m\partial\theta\right)\left(\theta\gamma^m\right)_\alpha\,, \tag{3.72}$$

and the Green-Schwarz action becomes

$$\begin{aligned} S_{GS} &= \frac{1}{4\pi}\int d^2z\,\left[\partial X^m\bar{\partial}X_m - 2(d_\alpha - p_\alpha)\bar{\partial}\theta^\alpha\right] \\ &= \frac{1}{2\pi}\int d^2z\,\left(\frac{1}{2}\,\partial X^m\bar{\partial}X_m + p_\alpha\bar{\partial}\theta^\alpha\right) - \frac{1}{2\pi}\int d^2z\,d_\alpha\bar{\partial}\theta^\alpha\,. \end{aligned} \tag{3.73}$$

Let us define *Siegel action* as follows

$$S_1 \equiv \frac{1}{2\pi}\int d^2z\,\left(\frac{1}{2}\,\partial X^m\bar{\partial}X_m + p_\alpha\bar{\partial}\theta_\alpha\right)\,, \tag{3.74}$$

which is of course related to the Green-Schwarz action according to

$$S_1 = S_{GS} + \frac{1}{2\pi}\int d^2z\,d_\alpha\bar{\partial}\theta^\alpha\,. \tag{3.75}$$

From the relation above, we can note that if p_α is constrained by $d_\alpha = 0$, then the actions (3.70) and (3.74) are completely equivalent. Otherwise, if we relax the constraint and consider p_α as an independent variable, we obtain an alternative action, Siegel's action.

The main problem with Siegel's attempt is that the theory has too many degrees of freedom, thus, when trying to quantize (3.74) we need to write a set of constraints which we should impose in order to truncate the spectrum. The Virasoro constraints $T = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha$ and the kappa symmetry generators of the Green-Schwarz formalism $G^\alpha = \Pi^m\left(\gamma_m d\right)^\alpha$ should certainly belong to such a set of constraints. However, a complete set of constraints that reproduces the superstring spectrum has not been found until now.

A way to solve this issue was proposed by Berkovits in 2000 [15]. Basically, we will add new ghost fields to the theory in such a way that the physical spectrum of the theory can be defined by a BRST charge Q to be defined properly. However, the procedure is not that straightforward, we need to add new terms to Siegel's action but they must obey some consistency conditions.

It is very useful to note that Siegel's action S_1 can be thought as the sum of ten free scalar fields which conform a XX system and sixteen bc-like systems $(p_\alpha, \theta^\alpha)$ where we can make the identification $b \to p$ and $c \to \theta$ with conformal weights $(1,0)$ and $(0,0)$ respectively. We have studied these systems extensively in Chapter 2 and we are very

familiar with some results like OPEs and other computations associated to them. For this reason, we are allowed to write the following OPEs

$$X^m(z,\bar{z})\,X^n(w,\bar{w}) \sim -\frac{1}{2}\,\eta^{mn}\ln|z-w|^2\,, \tag{3.76}$$

$$p_\alpha(z)\,\theta^\beta(w) \sim \frac{\delta_\alpha^\beta}{(z-w)}\,, \tag{3.77}$$

$$d_\alpha(z)\,d_\beta(w) \sim -\frac{1}{2}\frac{\gamma^m_{\alpha\beta}\Pi_m}{(z-w)}\,, \tag{3.78}$$

$$d_\alpha(z)\,\Pi^m(w) \sim -\frac{1}{2}\frac{(\gamma^m\partial\theta)_\alpha}{(z-w)}\,. \tag{3.79}$$

We want our model to be free of anomalies, as a consequence, the central charge c_1 associated to Siegel's action should vanish. However, as we will see next, this is not the case. This represent a big problem with Siegel's attempt to modify the Green-Schwarz formalism.

Let us compute explicitly the central charge to see that it is indeed non-vanishing. First of all let me write the energy momentum tensor T^1 associated to Siegel's model, from the identification we did before it is straightforward to write the following relation

$$T^1 = -\frac{1}{2}\partial X^m\partial X_m - p_\alpha\partial\theta^\alpha\,. \tag{3.80}$$

The central charge c^X associated to the XX system is just equal to the number of degrees of freedom of it, thus we can write $c^X = 10$. On the other hand, the central charge $c^{p\theta}$ associated to the sixteen bc-like systems (p_α,θ^α) is $c^{p\theta} = 16\times(-2) = -32$, where $\lambda = 1$. In this way, the total central charge of Siegel's model is

$$c^1 = c^X + c^{p\theta} = -22\,. \tag{3.81}$$

In this way, the central charge of the set of ghosts that we will add, must be such that it cancels the total central charge of the theory.

Let us now find the spin contribution to the Lorentz currents in Siegel's model. As we will only consider the spin contributions, let us write how the fermionic coordinates transform under Lorentz transformations

$$\delta p_\alpha = \frac{1}{4}\epsilon_{mn}(\gamma^{mn})_\alpha{}^\beta\,p_\beta\,, \qquad \delta\theta^\alpha = \frac{1}{4}\epsilon_{mn}(\gamma^{mn})^\alpha{}_\beta\,\theta^\beta\,. \tag{3.82}$$

Now with the help of Noether's procedure we can find the Lorentz currents associated to

the fermionic variables

$$\begin{aligned}
\delta S_1 &= \frac{1}{2\pi}\int d^2z\,\left(\delta p_\alpha\,\bar\partial\theta^\alpha + p_\alpha\,\bar\partial\delta\theta^\alpha\right)\\
&= \frac{1}{2\pi}\int d^2z\,\left[\left(\frac{1}{4}\epsilon_{mn}(\gamma^{mn})_\alpha{}^\beta p_\beta\right)\bar\theta^\alpha + p_\alpha\bar\partial\left(\frac{1}{4}\epsilon_{mn}(\gamma^{mn})^\alpha{}_\beta\theta^\beta\right)\right]\\
&= \frac{1}{2\pi}\int d^2z\,\left[\frac{1}{4}\epsilon_{mn}(\gamma^{mn})_\alpha{}^\beta p_\beta\bar\partial\theta^\alpha + \frac{1}{4}p_\alpha\bar\partial\epsilon_{mn}(\gamma^{mn})^\alpha{}_\beta\theta^\beta + \frac{1}{4}p_\alpha\epsilon_{mn}(\gamma^{mn})^\alpha{}_\beta\bar\partial\theta^\beta\right]\\
&= \frac{1}{2\pi}\int d^2z\,\bar\partial\epsilon_{mn}\left[\frac{1}{4}(p\,\gamma^{mn}\theta)\right]\,,
\end{aligned}\tag{3.83}$$

where we have used the identity $(\gamma^{mn})_\alpha{}^\beta = -(\gamma^{mn})^\beta{}_\alpha$ to reduce terms in the last line. Then we can write the Lorentz currents as follows

$$\Sigma^{mn} = \frac{1}{2}\,(p\,\gamma^{mn}\theta)\,. \tag{3.84}$$

Le us compute the $\Sigma\Sigma$ OPE, the computation is quite tedious so we will not pay that much attention to the details involved

$$\begin{aligned}
\Sigma^{mn}(z)\Sigma^{pq}(w) &= \frac{1}{4}\left[p_\alpha(\gamma^{mn})^\alpha{}_\beta\theta^\beta\right]\left[p_\mu(\gamma^{pq})^\mu{}_\nu\theta^\nu\right]\\
&\sim \frac{1}{4}\Bigg[p_\alpha(\gamma^{mn})^\alpha{}_\beta(\gamma^{pq})^\mu{}_\nu\theta^\nu\frac{\delta^\beta_\mu}{z-w} + (\gamma^{mn})^\alpha{}_\beta\theta^\beta p_\mu(\gamma^{pq})^\mu{}_\nu\frac{\delta^\nu_\alpha}{z-w} +\\
&\qquad + \frac{(\gamma^{mn})^\alpha{}_\beta(\gamma^{pq})^\mu{}_\nu\delta^\beta_\mu\delta^\alpha_\nu}{(z-w)^2}\Bigg]\\
&\sim \frac{1}{4}\,\frac{p\,(\gamma^{mn}\gamma^{pq} - \gamma^{pq}\gamma^{mn})\,\theta}{z-w} + \frac{1}{4}\,\frac{\mathrm{tr}(\gamma^{mn}\gamma^{pq})}{(z-w)^2}\,.
\end{aligned}\tag{3.85}$$

The numerator in the simple pole can be reduced by using the definition $\gamma^{mn} = \frac{1}{2}\left[\gamma^m,\gamma^n\right]$, and then just making use of the properties of gamma matrices. To reduce the double pole numerator, we just need to use the following identity

$$\mathrm{tr}(\gamma^m\gamma^n\gamma^p\gamma^q) = 16\,(\eta^{mn}\eta^{pq} - \eta^{mp}\eta^{nq} + \eta^{mq}\eta^{np})\,. \tag{3.86}$$

The $\Sigma\Sigma$ OPE finally takes the following form

$$\Sigma^{mn}(z)\Sigma^{pq}(w) \sim \frac{\eta^{p[n}\Sigma^{m]q} - \eta^{q[n}\Sigma^{m]p}}{z-w} + 4\frac{\eta^{m[q}\eta^{p]n}}{(z-w)^2}\,. \tag{3.87}$$

Wee need to make a key observation here. In the Ramond-Neveu-Schwarz (RNS) formalism, the spin contribution to the Lorentz generators is given by $\Sigma^{mn}_{RNS} = \psi^m\psi^n$, which

satisfies the following OPE [7]

$$\Sigma^{mn}_{RNS}(z)\Sigma^{pq}_{RNS}(w) \sim \frac{\eta^{p[n}\Sigma^{m]q}_{RNS} - \eta^{q[n}\Sigma^{m]p}_{RNS}}{z-w} + \frac{\eta^{m[q}\eta^{p]n}}{(z-w)^2}\,. \tag{3.88}$$

Comparing equations (3.87) and (3.88), we realize that there is a discrepancy concerning the coefficient of the double pole of both OPEs. Berkovits considered this observation and suggested that we should modify Siegel's model in such a way that the Lorentz generators Σ^{mn} obey the same OPE than those of the RNS formalism. In the procedure to construct the pure spinor formalism we will consider these condition together with the vanishing of the central charge as consistency conditions for the model.

3.3.2 A note on Lorentz ghost currents

The pure spinor formalism solves the two difficulties we discussed earlier, namely, it adds a set of ghosts to Siegel's model (the pure spinor ghost action) which contribute with the right number to cancel the total central charge of the theory and its contribution to the Lorentz generators is such that the OPE of the Lorentz generators is compatible with equation (3.88).

It is easy to guess the OPE that the Lorentz generators associated to the pure spinor ghost system should obey. Let us define the pure spinor Lorentz generators M^{mn} as follows

$$M^{mn} = \Sigma^{mn} + N^{mn}\,, \tag{3.89}$$

where Σ^{mn} is the contribution from Siegel's model that we saw before and N^{mn} are the Lorentz currents due to the pure spinor ghost action. Let us compute the Lorentz currents OPE as follows

$$\begin{aligned} M^{mn}(z)M^{pq}(w) &= (\Sigma^{mn}(z) + N^{mn}(z))\,(\Sigma^{pq}(w) + N^{pq}(w)) \\ &= \Sigma^{mn}(z)\Sigma^{pq}(w) + N^{mn}(z)N^{pq}(w) + \Sigma^{mn}(z)N^{pq}(w) + \\ &\quad + N^{mn}(z)\Sigma^{pq}(w)\,. \end{aligned} \tag{3.90}$$

For this expression to take the desired form of the OPE (3.88), the OPEs of the Lorentz currents of the ghost system must be the following

$$N^{mn}(z)N^{pq}(w) \sim \frac{\eta^{p[n}N^{m]q} - \eta^{q[n}N^{m]p}}{z-w} - 3\frac{\eta^{m[q}\eta^{p]n}}{(z-w)^2}\,, \tag{3.91}$$

$$\Sigma^{mn}(z)N^{pq}(w) \sim \text{regular}\,. \tag{3.92}$$

Replacing equations (3.87), (3.91) and (3.92) in (3.90) we finally get our desired OPE

$$M^{mn}(z)M^{pq}(w) \sim \frac{\eta^{p[n}M^{m]q} - \eta^{q[n}M^{m]p}}{z-w} + \frac{\eta^{m[q}\eta^{p]n}}{(z-w)^2}\,. \tag{3.93}$$

As we said earlier, the pure spinor formalism adds a set of ghost fields *à la* BRST to Siegel's model, from the relations above we know some consistency conditions that these additional terms must satisfy, in particular, the Lorentz currents associated to them must obey the OPEs (3.91) and (3.92).

3.3.3 Pure spinor ghosts and the BRST charge

Let us now proceed to define the BRST operator of the pure spinor formalism. Berkovits proposed the BRST operator as a linear combination of the constraints d_α and some bosonic variables[5] λ^α

$$Q_{BRST} = \oint \frac{dz}{2\pi i}\,\lambda^\alpha(z)\,d_\alpha(z)\,, \tag{3.94}$$

where the supersymmetric Green-Schwarz constraints are given by

$$d_\alpha = p_\alpha - \frac{1}{2}\left(\gamma^m\theta\right)_\alpha \partial X_m - \frac{1}{8}\left(\gamma^m\theta\right)_\alpha\left(\theta\gamma_m\partial\theta\right)\,. \tag{3.95}$$

As we saw when studying the BRST quantization of the bosonic string in Chapter 2, for the spectrum of the theory to be defined as the cohomology of the BRST operator, the operator must be nilpotent $Q^2_{BRST} = 0$. Applying this condition to the pure spinor BRST operator we have just defined we get

$$\begin{aligned} Q^2_{BRST} &= \frac{1}{2}\left\{Q_{BRST}, Q_{BRST}\right\} \\ &= \oint \frac{dz}{2\pi i}\frac{dw}{2\pi i}\,\lambda^\alpha(z)\,d_\alpha(z)\,\lambda^\beta(w)\,d_\beta(w) \\ &= \oint \frac{dz}{2\pi i}\frac{dw}{2\pi i}\,\lambda^\alpha(z)\lambda^\beta(w)\left(-\frac{1}{2}\frac{\gamma^m_{\alpha\beta}\,\Pi_m}{z-w}\right) \\ &= -\frac{1}{2}\oint \frac{dz}{2\pi i}\,\left(\lambda^\alpha\,\gamma^m_{\alpha\beta}\,\lambda^\beta\right)\Pi_m\,, \end{aligned} \tag{3.96}$$

where in the second line, we made use of the OPE (3.78). Looking at the expression above, we can note that for the BRST operator to be nilpotent, the bosonic fields λ^α must satisfy the so-called *pure spinor constraints*, that is

$$\left(\lambda^\alpha\,\gamma^m_{\alpha\beta}\,\lambda^\beta\right) = \left(\lambda\,\gamma^m\,\lambda\right) = 0\,. \tag{3.97}$$

[5] As we know, the BRST operator rises the ghost number, hence it has to contain λ^α, which has ghost number +1, so it is natural to define the BRST operator as we did in (3.94).

We will call a ten dimensional Weyl spinor λ^α a *pure spinor* if the relation above holds for $m = 0, \ldots, 9$.

The ten constraints contained in (3.97) are not independent. The pure spinor formalism relies on the properties of the pure spinor λ^α. In order to solve the constraints (3.97), it turns out to be necessary to perform a Wick rotation from $SO(9,1)$ to $SO(10)$ and break $SO(10)$ to $U(5)$. For this reason, we will spend a few words on how to do this.

3.3.4 Breaking $SO(10)$ to $U(5)$

In order to solve the pure spinor constraints, we need a convenient group representation. What we want is to write vectors, tensors and spinors of the Euclidean Lorentz group $SO(10)$ in terms of $U(5)$ variables.

We can decompose any $SO(10)$ vector V^m in his fundamental ($v^a \equiv \mathbf{5}$) and anti-fundamental ($v_a \equiv \bar{\mathbf{5}}$) representations of $U(5)$ in the following way

$$V^m \to (v^a \oplus v_a)\,, \tag{3.98}$$

with

$$v^a = \frac{1}{\sqrt{2}}\left(V^a + iV^{a+5}\right)\,, \tag{3.99}$$

$$v_a = \frac{1}{\sqrt{2}}\left(V^a - iV^{a+5}\right)\,, \tag{3.100}$$

where in the expressions above, $a = 1, \ldots, 5$. We denote this decomposition as $\mathbf{10} = \mathbf{5} \oplus \bar{\mathbf{5}}$. Furthermore, the scalar product of two vectors V^m and W^m can be written in the following way

$$V^m W_m = v^a w_a + v_a w^a\,. \tag{3.101}$$

Following the same reasoning as above, we can deduce the $U(5)$ decomposition of any antisymmetric rank-2 $SO(10)$ tensor N^{mn} as follows

$$\begin{aligned} N^{mn} &\to (v^a \oplus v_a) \otimes \left(w^b \oplus w_b\right) \\ &\to \left(n^{ab} \oplus n^a{}_b \oplus n_{ab} \oplus n\right)\,, \end{aligned} \tag{3.102}$$

with

$$
\begin{aligned}
n^{ab} &= \frac{1}{2}\left(N^{ab} + iN^{a(b+5)} + iN^{(a+5)b} - N^{(a+5)(b+5)}\right) , && (3.103)\\
n_{ab} &= \frac{1}{2}\left(N^{ab} - iN^{a(b+5)} - iN^{(a+5)b} - N^{(a+5)(b+5)}\right) , && (3.104)\\
n^{a}{}_{b} &= \frac{1}{2}\left(N^{ab} - iN^{a(b+5)} + iN^{(a+5)b} + N^{(a+5)(b+5)}\right) - i\frac{\delta^a_b}{5}\sum_a N^{(a+5)a} , && (3.105)\\
n &= \frac{i}{\sqrt{5}}\sum N^{(a+5)a} . && (3.106)
\end{aligned}
$$

We can prove the results above pretty easily by using definitions (3.99) and (3.100). For instance

$$
\begin{aligned}
n^{ab} &\equiv v^a \otimes w^b \\
&= \frac{1}{2}\left(V^a + iV^{a+5}\right) \otimes \left(W^b + iW^{b+5}\right) \\
&= \frac{1}{2}\left(N^{ab} + iN^{a(b+5)} + iN^{(a+5)b} - N^{(a+5)(b+5)}\right) . && (3.107)
\end{aligned}
$$

In this way, the 45 components of N^{mn} are decomposed according to $\mathbf{45} \to \mathbf{10}\oplus\mathbf{24}\oplus\bar{\mathbf{10}}\oplus\mathbf{1}$.

In order to decompose a spinor, it is necessary to decompose first the 10-dimensional gamma matrices as follows

$$
\begin{aligned}
a^i &= \frac{1}{2}\left(\Gamma^i + i\Gamma^{i+5}\right) ; \qquad i = 1, \ldots, 5 , && (3.108)\\
a_i &= \frac{1}{2}\left(\Gamma^i - i\Gamma^{i+5}\right) . && (3.109)
\end{aligned}
$$

We can use the Clifford algebra $\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$ to write the following relations

$$
\left\{a_i, a^j\right\} = \delta^i_j , \qquad \{a_i, a_j\} = \left\{a^i, a^j\right\} = 0 , \tag{3.110}
$$

We recognize this algebra as the harmonic oscillator algebra, then we can identify a^i and a_i as creation and annihilation operators respectively. In this way, we can define a vacuum state $|0\rangle$ by $a_i|0\rangle = 0$. A generic state $|A\rangle$ can be constructed by applying a^i repeatedly as follows

$$
\begin{aligned}
|A\rangle &= \Big[A_0 + \sum_i A_i a^i + \sum_{i<j} A_{ij} a^i a^j + \sum_{i<j<k} A_{ijk} a^i a^j a^k + \\
&\qquad + \sum_{i<j<k<l} A_{ijkl} a^i a^j a^k a^l + A_5 a^1 a^2 a^3 a^4 a^5\Big]|0\rangle . && (3.111)
\end{aligned}
$$

We should note that the number of components of this state is actually $1+5+10+10+5+1=32$, as we could have suspected for a generic spinor in ten dimensions.

We are interested in how to decompose Weyl and anti-Weyl spinors, for this reason, let us define the chirality matrix as

$$\Gamma = i\prod_{m=1}^{10}\Gamma^m\,, \tag{3.112}$$

which can also be written in the following fashion

$$\Gamma = -\prod_{i=1}^{5}(a_i+a^i)(a_i-a^i) = -\prod_{i=1}^{5}(2a^ia_i-1)\,, \tag{3.113}$$

where the following relations hold

$$\{\Gamma, a_i\} = \left\{\Gamma, a^i\right\} = 0\,. \tag{3.114}$$

Since, as it is easy to note, $\Gamma|0\rangle = |0\rangle$, a positive chirality state ($\Gamma|\lambda\rangle_+ = |\lambda\rangle_+$) contains only terms with 0, 2 or 4 creation operators a^i, whereas a negative chirality state ($\Gamma|\omega\rangle_- = -|\omega\rangle_-$) contains only terms with 1, 3 or 5 a^i. Therefore, for a Weyl spinor we have

$$|\lambda\rangle_+ = \lambda_+|0\rangle + \frac{1}{2}\lambda_{ij}a^ja^i|0\rangle + \frac{1}{4!}\,\lambda^i\epsilon_{ijklm}a^ma^la^ka^j|0\rangle\,, \tag{3.115}$$

where the components are given by

$$\lambda_+ = \langle 0|\lambda\rangle\,, \qquad \lambda_{ij} = \langle 0|a_ia_j|\lambda\rangle\,, \qquad \lambda^i = \frac{1}{4!}\,\epsilon^{ijklm}\langle 0|a_ja_ka_la_m|\lambda\rangle\,. \tag{3.116}$$

On the other hand, for an anti-Weyl spinor we can write

$$|\omega\rangle_- = \omega_ia^i|0\rangle + \frac{1}{2\cdot 3!}\,\omega^{ij}\epsilon_{ijklm}a^ka^la^m|0\rangle + \omega_+a^1a^2a^3a^4a^5|0\rangle\,, \tag{3.117}$$

where this time, the components are given by

$$\omega_+ = \langle 0|a_5a_4a_3a_2a_1|\omega\rangle\,, \qquad \omega^{ij} = \frac{1}{3!}\,\epsilon^{ijklm}\langle 0|a_ka_la_m|\omega\rangle\,, \qquad \omega_i = \langle 0|a_i|\omega\rangle\,. \tag{3.118}$$

As you have seen, we have obtained the $U(5)$ decomposition of a Weyl spinor as

$$\lambda^\alpha \to \left(\lambda^+ \oplus \lambda_{ij} \oplus \lambda^i\right)\,; \qquad (\mathbf{16} = \mathbf{1} \oplus \bar{\mathbf{10}} \oplus \mathbf{5})\,, \tag{3.119}$$

whereas, for an anti-Weyl spinor we can write

$$\omega_\alpha \to \left(\omega_i \oplus \omega^{ij} \oplus \omega_+\right)\,; \qquad (\bar{\mathbf{16}} = \bar{\mathbf{5}} \oplus \mathbf{10} \oplus \bar{\mathbf{1}})\,. \tag{3.120}$$

It is relatively easy to solve the pure spinor constraints in these variables, we will do this in the next sub-section, where we will be following [16].

3.3.5 Solving the pure spinor constraints

To begin with, let us define the charge conjugation matrix C as follows

$$C\Gamma^m C^{-1} = -(\Gamma^m)^T \,, \tag{3.121}$$

which in ten dimensions can be written in the following way

$$C = -i\,\Gamma^6\Gamma^7\Gamma^8\Gamma^9\Gamma^{10} = \prod_{i=1}^{5}(a_i - a^i)\,, \tag{3.122}$$

where the matrices a^i and a_i are defined in (3.108) and (3.109) respectively. What is more, the following relations are valid

$$a_i C = -C a^i \,, \qquad a^i C = -C a_i \,. \tag{3.123}$$

Furthermore, we should note that since $(\gamma^m)_{\alpha\beta} = (\gamma^m)_\alpha{}^\gamma C_{\gamma\beta}$, the constraints (3.97) are obtained from the 32-dimensional expression $\lambda\,(\Gamma^m C\lambda) = 0$ which under the decomposition $SO(10) \to U(5)$ goes to two independent equations

$$\langle\lambda|\gamma^i C|\lambda\rangle \;=\; \langle\lambda|(a_i + a^i)C|\lambda\rangle \;=\; 0\,, \tag{3.124}$$

$$\langle\lambda|\gamma^{i+5} C|\lambda\rangle \;=\; i\langle\lambda|(a_i - a^i)C|\lambda\rangle \;=\; 0\,, \tag{3.125}$$

which can be reduced to the following expressions

$$\langle\lambda|C a^i|\lambda\rangle \;=\; 0\,, \tag{3.126}$$

$$\langle\lambda|C a_i|\lambda\rangle \;=\; 0\,. \tag{3.127}$$

In order to solve the two set of constraints above, we just need to consider the $U(5)$ expansion for a Weyl spinor given in (3.115), but before doing this, it is very useful to note that the only non-vanishing terms in the expressions (3.126) and (3.127) are the ones proportional to $\langle 0|C\,a^i a^j a^k a^l a^m|0\rangle$. In fact, from the definition of the charge conjugation matrix, we can write $C|0\rangle = -a^1a^2a^3a^4a^5|0\rangle$ and $\langle 0|C = \langle 0|a^5a^4a^3a^2a^1$. Furthermore, we have

$$\langle 0|C a^1a^2a^3a^4a^5|0\rangle = \langle 0|a^5a^4a^3a^2a^1a^1a^2a^3a^4a^5|0\rangle = 1\,. \tag{3.128}$$

The expression above is totally antisymmetric in the exchange of its indices, so we are allowed to write

$$\langle 0|C a^i a^j a^k a^l a^m|0\rangle = \epsilon^{ijklm}\,. \tag{3.129}$$

We can now proceed to solve the two sets of constraints (3.126) and (3.127). Let us start with the first one

$$\begin{aligned}\langle\lambda|Ca^{i_0}|\lambda\rangle &= \langle 0|C\left(\lambda_+ + \frac{1}{2}\,\lambda_{ij}a^ia^j + \frac{1}{4!}\,\lambda^i a j a^k a^l a^m \epsilon^{ijklm}\right)a^{i_0}|\lambda\rangle \\ &= \lambda_+\langle 0|Ca^{i_0}|\lambda\rangle + \frac{1}{2}\,\lambda_{ij}\langle 0|Ca^ia^ja^{i_0}|\lambda\rangle + \\ &\quad + \frac{1}{24}\,\lambda^i\epsilon_{ijklm}\langle 0|Ca^ja^ka^la^ma^{i_0}|\lambda\rangle\,, \end{aligned} \tag{3.130}$$

where we made use of relations (3.123) which can also be used to compute the brackets involved in the expression above in the following way

$$\begin{aligned}\langle 0|Ca^{i_0}|\lambda\rangle &= \frac{1}{24}\,\lambda^i\epsilon_{ijklm}\langle 0|Ca^{i_0}a^ma^la^ka^j|0\rangle \\ &= \frac{1}{24}\,\lambda^i\epsilon_{ijklm}\epsilon^{i_0mlkj} \\ &= \lambda^{i_0}\,, \end{aligned} \tag{3.131}$$

where in the first line we made use of the statement which carried us to equation (3.129). Analogously we can compute

$$\begin{aligned}\langle 0|Ca^ia^ja^{i_0}|\lambda\rangle &= \frac{1}{2}\,\lambda_{kl}\langle 0|Ca^ia^ja^{i_0}a^ka^l|0\rangle \\ &= \frac{1}{2}\,\lambda_{kl}\epsilon^{iji_0kl} = \frac{1}{2}\,\lambda_{kl}\epsilon^{i_0ijkl}\,. \end{aligned} \tag{3.132}$$

$$\begin{aligned}\langle 0|Ca^ja^ka^la^ma^{i_0}|\lambda\rangle &= \lambda_+\langle 0|Ca^ja^ka^la^ma^{i_0}|0\rangle \\ &= \lambda_+\epsilon^{jklmi_0} = \lambda_+\epsilon^{i_0jklm}\,. \end{aligned} \tag{3.133}$$

Finally, the first set of constraints takes the following form

$$\begin{aligned}\langle\lambda|Ca^{i_0}|\lambda\rangle &= \lambda_+\lambda^{i_0} + \frac{1}{4}\,\lambda_{ij}\lambda_{kl}\epsilon^{i_0ijkl} + \frac{1}{24}\,\lambda^i\lambda_+\epsilon_{ijklm}\epsilon^{i_0jklm} \\ &= 2\lambda_+\lambda^{i_0} + \frac{1}{4}\,\epsilon^{i_0ijkl}\lambda_{ij}\lambda_{kl}\,. \end{aligned} \tag{3.134}$$

from equation (3.126), the solution to the first set of constraints $2\lambda_+\lambda^i + \frac{1}{4}\,\epsilon^{ijklm}\lambda_{jk}\lambda_{lm} = 0$ is given by

$$\lambda^i = -\frac{1}{8\lambda_+}\,\epsilon^{ijklm}\lambda_{jk}\lambda_{lm}\,. \tag{3.135}$$

It can be shown that (3.135) solves also the constraints (3.127) automatically. Therefore a pure spinor in 10 dimensions is given by the decomposition (3.119) in which (3.135)

holds for $\lambda^+ \neq 0$. As a final remark, we should note from (3.135) that a pure spinor has only eleven independent complex components (ten from λ_{ij} and one from λ_+) which, as we will see later, leads to a vanishing central charge in the pure spinor formalism.

3.3.6 Consistency conditions

It turns out that we need to add some extra terms to Sielgel's action (3.74). However, we need these extra terms to be compatible with some consistency conditions which we will summarize next.

- The first consistency condition should be clear at this point, the pure spinor ghost action must be added to Siegel's action in such a way that the resulting action possesses a vanishing central charge and, as a consequence, the theory is free of anomalies. For this reason we claim that the central charge of the pure spinor ghost action must be $c^\lambda = 22$.

- We have talked about the second consistency condition too. The Lorentz currents associated to the pure spinor ghost action must obey the OPEs (3.91) and (3.92). Let us write these OPE in $U(5)$ language. We know how to perform a $U(5)$ decomposition of the Lorentz ghost currents (antisymmetric tensor of rank 2), therefore, we claim that the U(5) Lorentz ghost currents $n^{ab}, n^a{}_b, n_{ab}, n$ must satisfy the following OPEs[6]

$$n_{ab}(z)n^{cd}(w) \sim \frac{-\delta_a^{[c}n_b^{d]}(w) - \frac{2}{\sqrt{5}}\delta_a^{[c}\delta_b^{d]}n(w)}{z-w} + 3\,\frac{\delta_a^{[c}\delta_b^{d]}}{(z-w)^2}\,, \tag{3.136}$$

$$n_b^a(z)n_d^c(w) \sim \frac{\delta_c^b n_a^d(w) - \delta_a^b\delta_c^d(w)}{z-w} - 3\,\frac{\delta_a^d\delta_c^b - \frac{1}{5}\delta_a^b\delta_c^d}{(z-w)^2}\,, \tag{3.137}$$

$$n(z)n(w) \sim -\frac{3}{(z-w)^2}\,, \tag{3.138}$$

$$n(z)n_{ab}(w) \sim -\frac{2}{\sqrt{5}}\frac{n_{ab}(w)}{z-w}\,, \tag{3.139}$$

$$n(z)n^{ab}(w) \sim \frac{2}{\sqrt{5}}\frac{n^{ab}(w)}{z-w}\,, \tag{3.140}$$

$$n(z)n_b^a(w) \sim \text{regular}\,. \tag{3.141}$$

[6] We used [10] to write these OPEs. Some of the computations details we are omitting can be found there.

Proving relations (3.136)-(3.141) is rather simple, we just need to use (3.103)-(3.106) together with (3.91). For instance, the proof of (3.138) goes as follows

$$\begin{aligned} n(z)n(w) &= -\frac{1}{5}\sum_{a,b} N^{(a+5)a}N^{(b+5)b} \\ &\sim -\frac{1}{5}\Bigg[\sum_{ab}\frac{\delta^{(b+5)[a}N^{(a+5)]b}(w)-\delta^{b[a}N^{(a+5)](b+5)}(w)}{z-w}+ \\ &\qquad -3\,\frac{\delta^{(a+5)b}\delta^{a(b+5)}-\delta^{(a+5)(b+5)}\delta^{ab}}{(z-w)^2}\Bigg] \\ &\sim -\frac{3}{5}\frac{\sum_a \delta^{(a+5)(a+5)}\delta^{aa}}{(z-w)^2} \\ &\sim -\frac{3}{(z-w)^2}\,, \end{aligned} \tag{3.142}$$

as you can see, we have used δ instead of η in the OPE (3.91) since we are working in $SO(10)$. The proof of the rest of the OPEs is pretty similar but the computation details are really tedious and we will not perform them here.

- We need to require one *extra* consistency condition. Since the pure spinor variables λ^α must transform as spinors under the action of the Lorentz currents $M^{mn} = \Sigma^{mn} + N^{mn}$, that is

$$\delta\lambda^\alpha = \frac{1}{2}\oint\frac{dz}{2\pi i}\,\epsilon_{mn}\,[M^{mn},\lambda^\alpha] = \frac{1}{4}\,\epsilon_{mn}(\gamma^{mn})^\alpha{}_\beta\,\lambda^\beta\,. \tag{3.143}$$

With the help of our experience when dealing with this kind of integrals, we know that we need to replace the commutator inside the integral with its respective OPE. Since the OPE of λ^α and Σ^{mn} has no poles, we can deduce that the pure spinor variables must satisfy

$$N^{mn}(z)\lambda^\alpha(w) \sim \frac{1}{2}\frac{(\gamma^{mn})^\alpha{}_\beta\,\lambda^\beta(z)}{z-w}\,, \tag{3.144}$$

we can use the $U(5)$ decomposition of the pure spinor variables $(\lambda^+,\lambda_{ij},\lambda^i)$ and the Lorentz ghost currents $\left(n^{ab}\oplus n^a{}_b\oplus n_{ab}\oplus n\right)$ to find the $U(5)$ version of the OPE

written above. Let me summarize the results as follows

$$
\begin{aligned}
n(z)\lambda_+(w) &\sim -\frac{\sqrt{5}}{2}\frac{\lambda_+(w)}{z-w}\,, && (3.145)\\
n(z)\lambda_{cd}(w) &\sim -\frac{1}{2\sqrt{5}}\frac{\lambda_{cd}(w)}{z-w}\,, && (3.146)\\
n(z)\lambda^c(w) &\sim \frac{3}{2\sqrt{5}}\frac{\lambda^c(w)}{z-w}\,, && (3.147)\\
n^a_b(z)\lambda_+(w) &\sim \text{regular}\,, && (3.148)\\
n^a_b(z)\lambda_{cd}(w) &\sim \frac{\delta^a_d\lambda_{cb}-\delta^a_c\lambda_{db}}{z-w}-\frac{2}{5}\frac{\delta^a_b\lambda_{cd}}{z-w}\,, && (3.149)\\
n^a_b(z)\lambda^c(w) &\sim \frac{1}{5}\,\delta^a_b\lambda^c-\delta^c_b\lambda^a\,, && (3.150)\\
n_{ab}(z)\lambda_+(w) &\sim \frac{\lambda_{ab}(w)}{z-w}\,, && (3.151)\\
n_{ab}(z)\lambda_{cd}(w) &\sim \epsilon_{abcde}\lambda^e\,, && (3.152)\\
n_{ab}(z)\lambda^c(w) &\sim \text{regular}\,, && (3.153)\\
n^{ab}(z)\lambda_+(w) &\sim \text{regular}\,, && (3.154)\\
n^{ab}(z)\lambda_{cd}(w) &\sim -\frac{\delta^{[a}_c\delta^{b]}_d\lambda_+(w)}{z-w}\,, && (3.155)\\
n^{ab}(z)\lambda^c(w) &\sim -\frac{1}{2}\,\epsilon^{abcde}\lambda_{de}\,. && (3.156)
\end{aligned}
$$

It is not that hard to prove all of these results, we just need to use the OPE (3.144) together with some gamma matrices properties. For instance, the proof of (3.151) goes as follows

$$
\begin{aligned}
n_{ab}(z)\lambda^+(w) &= \frac{1}{2}\left(N^{ab}-iN^{a(b+5)}-iN^{(a+5)b}-N^{(a+5)(b+5)}\right)\langle 0|\lambda\rangle\\
&\sim \frac{1}{4}\frac{\langle 0|\left(\gamma^{ab}-i\gamma^{a(b+5)}-i\gamma^{(a+5)b}-\gamma^{(a+5)(b+5)}\right)|\lambda\rangle}{z-w}\\
&\sim \frac{\langle 0|a_a a_b|\lambda\rangle}{z-w}\\
&\sim \frac{\lambda_{ab}}{z-w}\,, \qquad (3.157)
\end{aligned}
$$

where in the second line we made use of the OPE (3.144), and in the third line we made use of the following identity

$$
4\,a_i a_j = \Gamma^{ij}-i\Gamma^{i(j+5)}-i\Gamma^{(i+5)j}-\Gamma^{(i+5)(j+5)}\,, \qquad (3.158)
$$

which is valid for $i \neq j$. The rest of the OPEs can be computed in a similar way and will not be performed here.

3.3.7 The pure spinor ghost action

It turns out that the action which is compatible with the three consistency conditions we have stated, is a curved $\beta\gamma$-like system called *Pure Spinor Ghost Action*, which we define next

$$S_\lambda \equiv \frac{1}{2\pi}\int d^2z\, \omega_\alpha \bar{\partial}\lambda^\alpha\,, \tag{3.159}$$

where λ^α are the pure spinor variables defined at (3.97) and ω_α are their conjugate momenta. We will spend the rest of this chapter trying to convince you that the pure spinor ghost action indeed satisfies the consistency conditions we have talked before. As we said earlier, the pure spinor variables are defined to guarantee the nilpotency of the BRST operator defined in (3.94), however, because of the constraints (3.97), we can not use the naive OPE (that for the $\beta\gamma$-like systems) between ω_α and λ^α as

$$\omega_\alpha(z)\lambda^\beta(w) \sim \frac{\delta^\alpha_\beta}{z-w}\,, \tag{3.160}$$

because we get a contradiction with the pure spinor constraints, as you can see in the following equations

$$\omega_\alpha(z)\left(\lambda^\beta\gamma^m_{\beta\gamma}\lambda^\gamma\right)(w) \sim 2\,\frac{\gamma^m_{\alpha\gamma}\lambda^\gamma}{z-w} \neq 0\,. \tag{3.161}$$

For this reason, we need to write the pure spinor ghost action S_λ in terms of unconstrained fields, that is, we need to use our solution to the pure spinor constraints and write the action using $U(5)$ variables.

To begin with, we need to realize that the action S_λ is invariant under the following local transformations

$$\delta_{\mathcal{Z}}\omega_\alpha = \mathcal{Z}_m(\gamma^m)_{\alpha\beta}\lambda^\beta\,, \qquad \delta\lambda^\alpha = 0\,, \tag{3.162}$$

where $\mathcal{Z}_m$ is a local vector parameter. We just need to vary the action S_λ as follows

$$\begin{aligned}
\delta_{\mathcal{Z}}S_\lambda &= \frac{1}{2\pi}\int d^2z\, \mathcal{Z}_m(\gamma^m)_{\alpha\beta}\lambda^\beta\bar{\partial}\lambda^\alpha \\
&= -\frac{1}{2\pi}\int d^2z\, \left\{\bar{\partial}\left[\mathcal{Z}_m\lambda^\alpha(\gamma^m)_{\alpha\beta}\lambda^\beta\right] + \mathcal{Z}_m(\gamma^m)_{\beta\alpha}\lambda^\alpha\bar{\partial}\lambda^\beta\right\} \\
&= -\delta_{\mathcal{Z}}S_\lambda \\
\Rightarrow \delta_{\mathcal{Z}}S_\lambda &= 0\,,
\end{aligned} \tag{3.163}$$

where in the second line we used integration by parts and the pure spinor constraints. If we perform a $U(5)$ decomposition of the vector parameters as $\mathcal{Z}^m \to (\zeta^i, \zeta_i)$, then the gauge transformations we defined in (3.162), can be written as follows

$$\delta_{\mathcal{Z}}\omega_\alpha = \zeta^i (a_i\lambda)_\alpha + \zeta_i (a^i\lambda)_\alpha \,, \tag{3.164}$$

where we have written the scalar product according to (3.101). Let us combine the expression above together with (3.118) to get

$$\begin{aligned} \delta_{\mathcal{Z}}\omega_i &= \langle 0|a_i|\delta_{\mathcal{Z}}\omega\rangle \\ &= \zeta^j \langle 0|a_i a_j|\lambda\rangle + \zeta_j \langle 0|a_i a^j|\lambda\rangle \\ &= \zeta^j \, \lambda_{ij} + \zeta_i \, \lambda^+ \,, \end{aligned} \tag{3.165}$$

where we made use of (3.116) to write the $U(5)$ components of λ^α. We can realize from the expression above, that it is always possible to make a gauge choice such that ω_i vanishes. For instance, if we choose

$$\mathcal{Z}^i = -\frac{\omega_i}{\sqrt{2}\lambda^+} \,, \qquad \mathcal{Z}^{i+5} = -i\,\frac{\omega_i}{\sqrt{2}\lambda^+} \,, \tag{3.166}$$

the $U(5)$ components follow immediately

$$\zeta^i = 0 \,, \qquad \zeta_i = -\frac{\omega_i}{\lambda^+} \,. \tag{3.167}$$

Using this gauge choice, it is straightforward to write

$$\delta_{\mathcal{Z}}\omega_i = -\omega_i \qquad \Rightarrow \qquad \omega_i \to \omega_i + \delta_{\mathcal{Z}}\omega_i = \omega_i - \omega_i = 0 \,, \tag{3.168}$$

then, it is always possible to make a gauge choice, such that the $U(5)$ components ω_i vanish. We will consider this observation when writing the action S_λ in $U(5)$ variables, but before going on, let us investigate how does the scalar product between an anti-chiral and a chiral spinor looks like. The charge conjugation matrix in ten dimensions admits the following anti-diagonal form

$$C = \begin{pmatrix} 0 & 1_{16} \\ -1_{16} & 0 \end{pmatrix} \,, \tag{3.169}$$

then we can define the scalar product as

$${}_-\langle\omega|C|\lambda\rangle_+ = \begin{pmatrix} 0 & \omega_\beta \end{pmatrix} \begin{pmatrix} 0 & \delta_\alpha^\beta \\ -\delta_\alpha^\beta & 0 \end{pmatrix} \begin{pmatrix} \lambda^\alpha \\ 0 \end{pmatrix} = -\omega_\alpha \lambda^\alpha \,. \tag{3.170}$$

On the other hand, the same computation can be done by considering the expansions for a Weyl and anti-Weyl spinors (3.115) and (3.117) respectively

$$
\begin{aligned}
{}_-\langle\omega|C|\lambda\rangle_+ &= \left(\omega_i^*\langle 0|a_i + \frac{1}{2\cdot 3!}\omega^{ij\,*}\epsilon_{ijklm}\langle 0|a_m a_l a_k + \omega_+^*\langle 0|a_5 a_4 a_3 a_2 a_1\right)\times \\
&\quad \times C\left(\lambda_+|0\rangle + \frac{1}{2}\lambda_{pq}a^q a^p|0\rangle + \frac{1}{4!}\,\lambda^p\epsilon_{pqrst}a^t a^s a^r a^q|0\rangle\right) \\
&= -\frac{1}{4!}\,\omega_i^*\lambda^p\epsilon_{pqrst}\langle 0|Ca^i a^t a^s a^r a^q a^p|0\rangle - \frac{1}{4!}\,\omega^{ij\,*}\lambda_{pq}\epsilon_{ijklm}\langle 0|Ca^m a^l a^k a^q a^p|0\rangle + \\
&\quad -\omega_+^*\lambda^+\langle 0|Ca^5 a^4 a^3 a^2 a^1|0\rangle \\
&= -\omega_+^*\lambda^+ - \omega_i^*\lambda^i - \frac{1}{2}\,\omega^{ij\,*}\lambda_{ij}\,. \qquad (3.171)
\end{aligned}
$$

Thus, comparing (3.170) and (3.171), we can write

$$\omega_\alpha\lambda^\alpha = \omega_+^*\lambda^+ + \omega_a^*\lambda^a + \frac{1}{2}\,\omega^{ab\,*}\lambda_{ab}\,, \qquad (3.172)$$

and as a consequence

$$\omega_\alpha\bar\partial\lambda^\alpha = \omega_+^*\bar\partial\lambda^+ + \omega_a^*\bar\partial\lambda^a + \frac{1}{2}\,\omega^{ab\,*}\bar\partial\lambda_{ab}\,. \qquad (3.173)$$

Let us now define

$$\lambda^+ = e^s\,, \qquad \lambda_{ab} = u_{ab}\,, \qquad \lambda^a = -\frac{1}{8}\,e^{-s}\epsilon^{abcde}u_{bc}u_{de}\,, \qquad (3.174)$$

and

$$\omega_+^* = \partial t\, e^{-s}\,, \qquad \omega^{ab\,*} = v^{ab}\,, \qquad \omega_a^* = 0\,. \qquad (3.175)$$

for any s, t and antisymmetric u, v. You should note that we chose $\omega_a^* = 0$ which is justified from the discussion we had earlier in (3.168). Taking into consideration all of these definitions, it should not be very hard to convince you that the pure spinor ghost action can be written as follows

$$S_\lambda = \frac{1}{2\pi}\int d^2z\left(-\partial t\,\bar\partial s + \frac{1}{2}\,v^{ab}\bar\partial u_{ab}\right)\,, \qquad (3.176)$$

where the following free field OPEs hold

$$
\begin{aligned}
t(z)s(w) &\sim \ln(z-w)\,, \qquad (3.177)\\
v^{ab}(z)u_{cd}(w) &\sim \frac{\delta_c^{[a}\delta_d^{b]}}{z-w}\,. \qquad (3.178)
\end{aligned}
$$

The fields v, u, ∂t, and ∂s have conformal weights (1,0), (0,0), (1,1) and (1,1) respectively, so we can write how these fields transforms under conformal transformations as follows

$$\begin{aligned}
\delta v^{ab} &= \partial\epsilon\, v^{ab} + \epsilon\, \partial v^{ab} + \bar{\epsilon}\, \bar{\partial} v^{ab}\,, && (3.179)\\
\delta u_{ab} &= \epsilon\, \partial u_{ab} + \bar{\epsilon}\, \bar{\partial} u_{ab}\,, && (3.180)\\
\delta\partial t &= \partial\epsilon\partial t + \epsilon\, \partial^2 t + \bar{\partial}\bar{\epsilon}\partial t + \bar{\epsilon}\, \bar{\partial}\partial t\,, && (3.181)\\
\delta\bar{\partial} s &= \partial\epsilon\bar{\partial} s + \epsilon\, \partial\bar{\partial} s + \bar{\partial}\bar{\epsilon}\bar{\partial} s + \bar{\epsilon}\, \bar{\partial}^2 s\,. && (3.182)
\end{aligned}$$

We now make use of these transformations to employ Noether's procedure in order to compute the energy momentum tensor T_λ of the pure spinor ghost action. It is rather easy to note from the transformations above, that u and v form a $\beta\gamma$-like system under the identification $\beta \to -1/2\, v^{ab}$ and $\gamma \to u_{ab}$, with conformal weights (1,0) and (0,0) respectively. Therefore we can write its contribution to the energy-momentum tensor directly. On the other hand, we vary the action in order to find the contribution from the fields ∂t and ∂s, that is

$$\begin{aligned}
\delta S_\lambda &= -\frac{1}{2\pi}\int d^2z\, (\delta\partial t\, \bar{\partial} s + \partial t\, \delta\bar{\partial} s) + \text{other contributions}\\
&= -\frac{1}{2\pi}\int d^2z\, \big[(\partial\epsilon\partial t + \epsilon\, \partial^2 t + \bar{\partial}\bar{\epsilon}\partial t + \bar{\epsilon}\, \bar{\partial}\partial t)\bar{\partial} s\\
&\qquad\qquad + \partial t(\partial\epsilon\bar{\partial} s + \epsilon\, \partial\bar{\partial} s + \bar{\partial}\bar{\epsilon}\bar{\partial} s + \bar{\epsilon}\, \bar{\partial}^2 s)\big] + \text{other contributions}\\
&= -\frac{1}{2\pi}\int d^2z\, \big[(\partial t\partial s)\bar{\partial}\epsilon + (\bar{\partial} s\bar{\partial} t)\partial\bar{\epsilon}\big] + \text{other contributions}\,, && (3.183)
\end{aligned}$$

Since we already know the contribution of the $\beta\gamma$-like system (see Chapter 2), it is straightforward to write the total energy-momentum tensor for the pure spinor ghost action

$$T_\lambda(z) = \frac{1}{2}\, v^{ab}\partial u_{ab} + \partial t\partial s + \partial^2 s\,, \qquad (3.184)$$

where the last term is a total derivative that has been added by hand in order to make the Lorentz currents transform as primary fields. We are interested in the central charge, for this reason we need to compute the $T_\lambda T_\lambda$ OPE which we do next

$$\begin{aligned}
T_\lambda(z)T_\lambda(w) &= \left[\frac{1}{2}\, v^{ab}\partial u_{ab}(z) + \partial t\partial s(z)\right]\left[\frac{1}{2}\, v^{cd}\partial u_{cd}(w) + \partial t\partial s(w)\right]\\
&\sim \frac{1}{4}\,\frac{\delta_c^{[a}\delta_d^{b]}\delta_a^{[c}\delta_b^{d]}}{(z-w)^4} + \frac{1}{4}\,\frac{v^{ab}(z)\partial u_{cd}(w)\delta_a^{[c}\delta_b^{d]}}{(z-w)^2} + \frac{1}{4}\,\frac{\partial u_{ab}(z)v^{cd}(w)\delta_c^{[a}\delta_d^{b]}}{(z-w)^2} +\\
&\quad + \frac{1}{(z-w)^4} + \frac{\partial t(z)\partial s(w)}{(z-w)^2} + \frac{\partial s(z)\partial t(w)}{(z-w)^2}\\
&\sim \frac{11}{(z-w)^4} + \frac{2T_\lambda(w)}{(z-w)^2} + \frac{\partial T_\lambda(w)}{z-w}\,. && (3.185)
\end{aligned}$$

And as you can see from the quadruple pole, the central charge of the pure spinor ghost system is just

$$c^\lambda = 22\,, \tag{3.186}$$

which is the value we established earlier as a consistency condition. We now move on to the consistency conditions involving the Lorentz ghost currents. We will make a strong claim here and then we will prove it by computing the OPEs explicitly.

Claim: If we use the $U(5)$ ghost variables s, t, u and v to construct the Lorentz ghost currents n, $n^a{}_b$, n^{ab} and n_{ab} as follows

$$\begin{aligned}
n &= -\frac{1}{\sqrt{5}}\left(\frac{1}{4}\,u_{ab}v^{ab} + \frac{5}{2}\,\partial t - \frac{5}{2}\,\partial s\right)\,, && (3.187)\\
n^a_b &= u_{bc}v^{ac} - \frac{1}{5}\,\delta^a_b u_{cd}v^{cd}\,, && (3.188)\\
n^{ab} &= -e^s v^{ab}\,, && (3.189)\\
n_{ab} &= e^{-s}\left(2\partial u_{ab} - u_{ab}\partial t - 2u_{ab}\partial s + u_{ac}u_{bd}v^{cd} - \frac{1}{2}u_{ab}u_{cd}v^{cd}\right)\,, && (3.190)
\end{aligned}$$

then, their OPEs among themselves and with λ^+, λ_{ab} and λ^a correctly reproduce the relations (3.136)-(3.141) and (3.145)-(3.156). In other words, all the consistency conditions are satisfied.

Let us prove our claim. We need to compute explicitly the OPEs involved, of course we will not compute all of them, but we rather compute some of them to show the calculation procedure, since they are very similar. Let us start with (3.140) which is pretty simple

$$\begin{aligned}
n(z)n^{ab}(w) &= -\frac{1}{\sqrt{5}}\left(\frac{1}{4}u_{ab}v^{ab}(z) + \frac{5}{2}\partial t(z) - \frac{5}{2}\partial s(z)\right)\left(e^{-s(w)}v^{cd}(w)\right)\\
&\sim -\frac{1}{\sqrt{5}}\left(\frac{e^{s(w)}v^{ab}(z)\delta^{[a}_c\delta^{b]}_d}{z-w} - \frac{5}{2}\,\frac{e^{s(w)}v^{cd}(w)}{z-w}\right)\\
&\sim \frac{2}{\sqrt{5}}\,\frac{n^{ab}}{z-w}\,. && (3.191)
\end{aligned}$$

The next will be (3.138), we have plenty of experience at computing OPEs, for this reason

some of the computation details will be omitted

$$
\begin{aligned}
n(z)n(w) &= \frac{1}{5}\left(\frac{1}{4}u_{ab}v^{ab}(z)+\frac{5}{2}\partial t(z)-\frac{5}{2}\partial s(z)\right)\left(\frac{1}{4}u_{acd}v^{cd}(w)+\frac{5}{2}\partial t(w)-\frac{5}{2}\partial s(w)\right)\\
&\sim \frac{1}{5}\Bigg[\frac{1}{16}u_{ab}(z)v^{cd}(w)\frac{\delta_c^{[a}\delta_d^{b]}}{z-w}-\frac{1}{16}v^{ab}(z)u_{cd}(w)\frac{\delta_c^{[a}\delta_d^{b]}}{z-w}+\\
&\qquad -\frac{1}{16}\,\frac{\delta_c^{[a}\delta_d^{b]}\delta_a^{[c}\delta_b^{d]}}{(z-w)^2}-\frac{25}{2}\,\frac{1}{(z-w)^2}\Bigg]\\
&\sim -\frac{3}{(z-w)^2}\,.
\end{aligned}
\tag{3.192}
$$

The last OPE from the Lorentz current OPEs we compute will be (3.136), this computation is a bit longer and requires some tricks to arrive to the desired form of it

$$
\begin{aligned}
n_{ab}(z)n^{cd}(w) &= e^{-s(z)}\Bigg(2\partial u_{ab}(z)-u_{ab}\partial t(z)-2u_{ab}\partial s(z)+u_{ae}u_{bf}v^{ef}(z)+\\
&\qquad -\frac{1}{2}u_{ab}u_{ef}v^{ef}(z)\Bigg)\times\left(-e^{s(w)}v^{cd}(w)\right)\\
&\sim 3\,\frac{\delta_a^{[c}\delta_b^{d]}}{(z-w)^2}+\frac{(\frac{1}{2}u_{ef}v^{ef}+\partial t-\partial s)\delta_a^{[c}\delta_b^{d]}}{z-w}-\frac{u_{bf}v^{ef}\delta_a^{[c}\delta_e^{d]}}{z-w}+\\
&\quad \frac{-u_{ae}v^{ef}\delta_b^{[c}\delta_f^{d]}}{z-w}\\
&\sim \frac{-\delta_a^{[c}n_b^{d]}-\frac{2}{\sqrt{5}}\delta_a^{[c}\delta_b^{d]}n}{z-w}+3\,\frac{\delta_a^{[c}\delta_b^{d]}}{(z-w)^2}\,,
\end{aligned}
\tag{3.193}
$$

We now move on to the second set of OPEs which are a bit simpler, we start with (3.145), the computation goes as follows

$$
\begin{aligned}
n(z)\lambda^+(w) &= -\frac{1}{\sqrt{5}}\left(\frac{1}{4}u_{(}ab)v^{ab}(z)+\frac{5}{2}\partial t(z)-\frac{5}{2}\partial s(z)\right)e^{s(w)}\\
&\sim -\frac{5}{2\sqrt{5}}\,\frac{e^s}{z-w}\\
&\sim -\frac{\sqrt{5}}{2}\,\frac{\lambda^+}{z-w}\,.
\end{aligned}
\tag{3.194}
$$

In a similar way, we can compute (3.148) and (3.156)

$$\begin{aligned} n^a_b(z)\lambda^+(w) &= (u_{be}v^{ae}(z) - \frac{1}{5}\delta^a_b u_{ef}v^{ef}(z))u_{cd}(w) \\ &\sim u_{be}(z)\,\frac{\delta^{[a}_c\delta^{e]}_d}{z-w} - \frac{1}{5}\delta^a_b u_{ef}(z)\,\frac{\delta^{[e}_c\delta^{f]}_d}{z-w} \\ &\sim \frac{\delta^a_c u_{bd} - \delta^a_d u_{bc}}{z-w} - \frac{1}{5}\,\frac{(\delta^a_b u_{cd} - \delta^a_b u_{dc})}{z-w} \\ &\sim \frac{\delta^a_c \lambda_{bd} - \delta^a_d \lambda_{bc}}{z-w} - \frac{2}{5}\,\frac{\delta^a_b \lambda_{cd}}{z-w}\,. \end{aligned} \tag{3.195}$$

$$\begin{aligned} n^{ab}(z)\lambda^c(w) &= \left(e^{-s(z)}v^{ab}(z)\right)\left(\frac{-1}{8}e^{-s(w)}\epsilon^{cmnpq}u_{mn}u_{pq}(w)\right) \\ &\sim \frac{1}{8}\epsilon^{cmnpq}\left(\delta^{[a}_m\delta^{b]}_n u_{pq} + \delta^{[a}_p\delta^{b]}_q u_{mn}\right) \\ &\sim \frac{1}{8}\left(4\epsilon^{cabpq}u_{pq}\right) \\ &\sim -\frac{1}{2}\epsilon^{abcpq}\lambda_{pq}\,. \end{aligned} \tag{3.196}$$

To finish our prove let us compute one more OPE, since (3.147) is a bit tricky we compute it here

$$\begin{aligned} n(z)\lambda^c(w) &= -\frac{1}{\sqrt{5}}\left(\frac{1}{4}u_{ab}v^{ab}(z) + \frac{5}{2}\partial t(z) - \frac{5}{2}\partial s(z)\right)\left(\frac{-1}{8}e^{-s(w)}\epsilon^{cdefg}u_{de}u_{fg}(w)\right) \\ &\sim \frac{1}{32\sqrt{5}}u_{ab}(z)\left[\frac{\delta^{[a}_d\delta^{b]}_e}{z-w}\epsilon^{cdefg}u_{fg} + \frac{\delta^{[a}_f\delta^{b]}_g}{z-w}\epsilon^{cdefg}u_{de}(w)\right]e^{-s(w)} + \frac{5}{2\sqrt{5}}\,\frac{\lambda^c(w)}{z-w} \\ &\sim \frac{1}{8\sqrt{5}}\,\frac{1}{z-w}e^{-s(z)}\epsilon^{cabfg}u_{ab}u_{fg}(z) + \frac{5}{2\sqrt{5}}\,\frac{\lambda^c(w)}{z-w} \\ &\sim \frac{3}{2\sqrt{5}}\,\frac{\lambda^c}{z-w}\,. \end{aligned} \tag{3.197}$$

To finish this chapter, let me write the full action for the Pure Spinor formalism of superstrings[7]

$$S_{PS} = \frac{1}{2\pi}\int d^2z\left[\frac{1}{2}\,\partial X^m\bar{\partial}X_m + p_\alpha\bar{\partial}\theta^\alpha - \partial t\partial s + \frac{1}{2}v^{ab}\bar{\partial}u_{ab}\right]\,. \tag{3.198}$$

[7]In fact, in this chapter we were working only with the holomorphic part of the action, if one for instance wants to write the action for Type II superstrings, we would need to include an anti-holomorphic part with superspace variables $(\hat{p}_{\hat{\alpha}}, \hat{\theta}^{\hat{\alpha}})$ and a pure spinor system $(\hat{\omega}_{\hat{\alpha}}, \hat{\lambda}^{\hat{\alpha}})$. The relative chirality of the hatted and unhatted spinors define either Type IIA or Type IIB superstrings.

where the total energy momentum tensor is given by

$$T^{PS}(z) = -\frac{1}{2}\partial X^m \partial X_m - p_\alpha \partial\theta^\alpha + \frac{1}{2}v^{ab}\partial u_{ab} + \partial t \partial s + \partial^2 s\,, \tag{3.199}$$

where the central charge vanishes and the spinorial contribution to the Lorentz current agrees with that of the RNS formalism. In this way we have constructed the pure spinor ghost action which saved us from the inconsistencies of Siegel's model, and allowed us to formulate the *Pure Spinor Superstring*. Let us finish this section by remarking that this chapter was intended to be a short introduction to the pure spinor superstring, we are interested in formulating superstrings in $AdS_5 \times S^5$ so this should be enough for our purpose. However, the interested reader should review the references listed at the end of this work in order to study the computation of amplitudes which is probably the most interesting application of the formalism.

Chapter 4

Superstrings in $AdS_5 \times S^5$

In order to have a better understanding of the AdS/CFT duality, it is important to study how to formulate superstrings in AdS backgrounds. For this reason we devote our final chapter to give a rather short but pedagogical introduction to the required tools to formulate the superstring in curved backgrounds, in particular the Type IIB supergravity background $AdS_5 \times S^5$. We start by generalizing the Green-Schwartz superstring to curved backgrounds and then we review the supercoset sigma-model formulation of the Green-Schwarz superstring. At the end of the chapter we give a short presentation of the pure spinor superstring in a generic supergravity background.

The best reference to study superstrings in AdS is the review by Mazzucato [17] which we will follow closely. The supercoset sigma model formulation of the Green-Schwarz superstring is explained in detail in [18]. On the other hand, the reader can consult [19] and [20] for a basic approach to the theory of manifolds and superalgebras respectively.

4.1 Superstrings in general backgrounds

Until now, we have been studying superstring theory formulated in a flat target space. However, our goal is to review the study of Type II superstrings in $AdS_5 \times S^5$ so we will start by generalizing the Green-Schwarz action to a generic curved background.

4.1.1 Green-Schwarz superstring in general backgrounds

Let us start by considering a target superspace provided with curved supercoordinates $Z^M = \{X^m, \theta^\mu, \hat{\theta}^{\hat{\mu}}\}$,[1] where just as we considered in Chapter 3, the Grassmann fermionic coordinates are Majorana-Weyl spinors[2] and $M = (m, \mu, \hat{\mu})$; $m = 0, \ldots, 9$. The Green-Schwarz Type II superstring can be naturally extended to curved backgrounds [21]

$$S_{GS} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \left(\sqrt{-g}\, g^{ij} G_{MN}(Z) + \epsilon^{ij} B_{NM}(Z)\right) \partial_i Z^M \partial_j Z^N \,. \tag{4.1}$$

The action presented above represents a generalization of the action (3.51) we constructed in Chapter 3. The first term corresponds to the kinetic term of (3.51), while the second one corresponds to the Wess-Zumino term. G_{MN} and B_{NM} are background superfields.

The superspace can be regarded as a supermanifold, so at every point Z we can define a tangent superspace with flat metric η_{ab} and a cotangent superspace. The latter admits coordinate dual basis $\{dZ^M\}$ and orthonormal basis $\{J^{\mathcal{A}}\}$, where $\mathcal{A} = (a, \alpha, \hat{\alpha})$ with $a = 0, \ldots, 9\,; \alpha, \hat{\alpha} = 1, \ldots, 16$ are indices on the tangent superspace. The change of basis define the *supervielbein* $E_M{}^{\mathcal{A}}$ as follows

$$J^{\mathcal{A}} = E_M{}^{\mathcal{A}} dZ^M \,. \tag{4.2}$$

The superfield G_{MN} represents a generalization of the metric to the superspace, then we can write

$$G_{MN}(Z) = E_M{}^a(Z) E_N{}^b(Z) \eta_{ab} \,. \tag{4.3}$$

On the other hand, in a general supergravity background the Wess-Zumino term is given by the 2-superform

$$B = \frac{1}{2} B_{MN} dZ^M \wedge dZ^N = \frac{1}{2} B_{\mathcal{AB}} J^{\mathcal{A}} \wedge J^{\mathcal{B}} \,, \tag{4.4}$$

where

$$B_{MN}(Z) = E_M{}^{\mathcal{A}}(Z) E_N{}^{\mathcal{B}}(Z) B_{\mathcal{AB}}(Z) \,. \tag{4.5}$$

[1] We will make a little change in the notation with respect to Chapter 3, we use hats instead of bars to discriminate the chirality of the spinorial coordinates.

[2] As we said earlier in this work, Type IIB corresponds to spinors of the same chirality, while Type IIA for those of opposite chiralities.

The Wess-Zumino term can be written as an integral of the 2-superform B as follows

$$\begin{aligned} S_{WZ} &= \frac{1}{2\pi\alpha'}\int B \;=\; \frac{1}{4\pi\alpha'}\int B_{MN}dZ^M\wedge dZ^N \\ &= +\frac{1}{4\pi\alpha'}\int B_{MN}\partial_i Z^M\partial_j Z^N d\sigma^i\wedge d\sigma^j \\ &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\epsilon^{ij}B_{MN}\partial_i Z^M\partial_j Z^N\,. \end{aligned} \tag{4.6}$$

The expression (4.2) defining the supervielbein can be written in components in the following way

$$J_i{}^{\mathcal{A}} = E_M{}^{\mathcal{A}}\partial_i Z^M\,, \tag{4.7}$$

so using equations (4.3), (4.6) and (4.7), the Green-Schwarz sigma model action in curved backgrounds can be written as follows

$$S_{GS} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\left(\sqrt{-g}\,g^{ij}\eta_{ab}\,J_i{}^a J_j{}^b + \epsilon^{ij}B_{\mathcal{AB}}J_i{}^{\mathcal{A}}J_j{}^{\mathcal{B}}\right)\,. \tag{4.8}$$

4.1.2 Coset formulation of superspace

As we have seen in the previous sub-section, in order to formulate the Green-Schwarz action in curved backgrounds, we need to know the supervielbein $E_M{}^{\mathcal{A}}$ and the 2-superform $B_{\mathcal{AB}}$. The most important case occurs when the superspace $\mathcal{M}$ can be regarded as a coset manifold, so we need to first understand the theory of cosets.

If we consider a group G with a subgroup H,[3] the coset group denoted by G/H is defined as the group G modulo the elements related by equivalence relations under H, that is elements $g\in G$, such that $g\sim gh$ for $h\in H$. If G and H are continuous groups, then $\mathcal{M}\cong G/H$ is a manifold (supermanifold) called *coset manifold*. The best known example is the n-sphere which can be defined as

$$S^n \cong \frac{SO(n+1)}{SO(n)}\,, \tag{4.9}$$

$SO(n+1)$ plays the role of the group of isometries on the sphere, while $SO(n)$ is the group of local relations on the sphere which can be regarded as "the local Lorentz group". Let us now consider the Lie algebra $\mathcal{G}$ of the group G, with generators T_A satisfying

$$[T_A, T_B] = C_{AB}{}^C\,T_C\,. \tag{4.10}$$

[3]In fact, in order to construct a coset supermanifold $\mathcal{M}$, we consider G as a supergroup and H as a bosonic subgroup.

Here ${C_{AB}}^C$ are structure constants. Consider now that we can split the generators as $T_A = \big(T_{(ab)}, T_{\mathcal{A}}\big)$, here $T_{(ab)}$ are the generators of H and $T_{\mathcal{A}}$ remain in the quotient G/H. As the generators span the tangent space of a group manifold, T_A describes the tangent superspace of $\mathcal{M}$. By definition a coset element is

$$e^{Z^{\mathcal{A}} T_{\mathcal{A}}} h\,, \quad \forall h\,. \tag{4.11}$$

We have a coset representative for $h = 1$, and Z^M being the coordinates on the coset, defining it as a manifold (supermanifold). In order to construct the supervielbein, we first define the so-called *canonical form* or *Maurer-Cartan form*

$$J \equiv g^{-1} dg\,, \qquad g \in G\,, \tag{4.12}$$

which takes values in the Lie algebra of G and, as a consequence, can be decomposed as

$$J = J^A T_A = J^{\mathcal{A}} T_{\mathcal{A}} + J^{(ab)} T_{(ab)}, \tag{4.13}$$

from the previous sub-section we can write the expression above as follows

$$J = {J_M}^A dZ^M T_A = {J_M}^{\mathcal{A}} dZ^M T_{\mathcal{A}} + {J_M}^{(ab)} dZ^M T_{(ab)}\,. \tag{4.14}$$

We can see equation (4.14) as a definition for the supervielbein ${J_M}^{\mathcal{A}}$ and the spin connections ${J_M}^{(ab)}$. By taking $dZ^M = \partial_i Z^M d\sigma^i$, we find

$${J_i}^A = {J_M}^A \partial_i Z^M\,, \tag{4.15}$$

which are the same as in (4.7). In this way, the kinetic term of the action (4.8) can be constructed with no major difficulties when we consider target spaces which can be regarded as a coset manifold.

The Wess-Zumino term for a supergroup manifold can be obtained by generalizing the bosonic analogue on a group manifold [22]. For the case of flat space, we have already constructed the Wess-Zumino term for the Green-Schwarz superstring (see section 2.2 of Chapter 3). Let us consider the 3-form

$$\Omega_3 = \mathrm{str}\,(J \wedge [J \wedge J]) = C_{ABC}\, J^A \wedge J^B \wedge J^C\,, \tag{4.16}$$

with

$$C_{ABC} = {C_{AB}}^D\, \eta_{DC}\,, \tag{4.17}$$

where $\eta_{AB} = \mathrm{str}(T_A T_B)$. Then the Wess-Zumino contribution is given by

$$S_{WZ} = \int_{\mathcal{D}} \Omega_3\,, \tag{4.18}$$

where $\mathcal{D}$ is a three dimensional manifold whose boundary can be interpreted as the string worldsheet. For the case of a supergroup one can make use of the Jacobi identity and the Maurer-Cartan equation

$$dJ + \frac{1}{2}\,[J \wedge J] = 0\,, \tag{4.19}$$

to show that Ω_3 is closed, that is $d\Omega_3 = 0$. In this way, Ω_3 can be regarded as locally exact, which means that it is possible to find a 2-form B depending on the coordinates of the supergroup such that $\Omega_3 = d\,B$. On the other hand, when dealing with supercoset manifolds, in general the Wess-Zumino term can not be written as in (4.16), with J^m restricted to $\mathcal{G}/\mathcal{H}$, because $\mathcal{G}/\mathcal{H}$ is not a superalgebra and, as a consequence one can not make use of Jacobi and Maurer-Cartan equations and Ω_3 is in general not closed. This problem can be solved for some particular cosets, we are particularly interested in the coset model for $AdS_5 \times S^5$, but we will first use the flat ten dimensional superspace ($\mathcal{N} = 2$) as an example.

4.1.3 Flat Green-Schwarz superstrings

Before trying to formulate the superstring action in the $AdS_5 \times S^5$ background, we will use the flat superspace background as a warm up. Let us construct the ten dimensional space with two supersymmetries that we used in Chapter 3 to formulate the Green-Schwarz superstring. Denoting by SUSY($\mathcal{N} = 2$) the super-Poincaré group with two supersymmetries in ten dimensions and with $SO(9,1)$ its Lorentz subgroup, the flat ten dimensional superspace with $\mathcal{N} = 2$ is given by the following coset

$$\frac{\mathrm{SUSY}(\mathcal{N} = 2)}{SO(9,1)}\,. \tag{4.20}$$

The superstring can be defined as a mapping from a two dimensional space-time (the worldsheet) into the coset manifold SUSY($\mathcal{N} = 2$)/$SO(9,1)$ [23]. The bosonic generators in the coset are given by P_m and the fermionic ones are $Q_{\alpha\,I}$ where $m = 0, \ldots, 9$ are space-time indices and $\alpha = 1, \ldots, 16$ are spinorial indices and $I = 1, 2$ correspond to the two supersymmetries.

The first step to formulate the superstring sigma model on (4.20) is to build the SUSY-invariant Maurer-Cartan form. In order to do this we need to note that a coset element is given by

$$g = e^{X^m P_m + \theta^{\alpha\,I} Q_{\alpha\,I}}\,, \tag{4.21}$$

where the flat $\mathcal{N} = 2$ superalgebra is given by

$$\{Q_{\alpha\, I}, Q_{\beta\, J}\} = -2i\delta_{IJ}\,(\gamma^m)_{\alpha\beta}\,P_m\,, \tag{4.22}$$

$$[P_m, P_n] = 0\,, \tag{4.23}$$

$$[Q_{\alpha\, I}, P_m] = 0\,. \tag{4.24}$$

The Maurer-Cartan form can be constructed by making use of the Baker-Campbell-Hausdorff formula

$$e^{-A}de^{A} = dA + \frac{1}{2}\,[dA, A] + \frac{1}{3!}\,[[dA, A]\,, A] + \dots\,. \tag{4.25}$$

Since we have

$$\begin{aligned}\left[dX^m P_m + d\theta^{\alpha\, I}Q_{\alpha\, I}, X^n P_n + \theta^{\beta\, J}Q_{\beta\, J}\right] &= -d\theta^{\alpha\, I}\theta^{\beta\, J}\,\{Q_{\alpha\, I}, Q_{\beta\, J}\} \\ &= -2i\,\theta^{\alpha\, I}\,(\gamma^m)_{\alpha\beta}\,d\theta^{\beta\, I}P_m\,.\end{aligned} \tag{4.26}$$

Thus, using the equation above, we can realize that only the first two terms of the expansion (4.25) survive and then we find that the Maurer-Cartan form can be written as follows

$$J = g^{-1}dg = \left(dX^m - i\,\theta^I\gamma^m d\theta^I\right)P_m + d\theta^{\alpha\, I}Q_{\alpha\, I}\,. \tag{4.27}$$

Comparing equations (4.13) and (4.27) and taking into consideration that for the case of flat space, the indices on the supermanifold and on the tangent superspace are the same, we can write

$$J^m = dX^m - i\,\theta^I\gamma^m d\theta^I\,; \qquad J^{\alpha\, I} = d\theta^{\alpha\, I}\,, \tag{4.28}$$

which can also be written in components as follows

$$J_i{}^m = \partial_i X^m - i\,\theta^I\gamma^m\partial_i\theta^I\,; \qquad J_i{}^{\alpha\, I} = \partial_i\theta^{\alpha\, I}\,. \tag{4.29}$$

Equation (4.29) is none other than the supersymmetric objects Π_a^m which we considered in Chapter 3 when constructing the Green-Schwarz action. Thus we can use (4.29) and the superstring sigma model action (4.8) to reproduce the kinetic part of Green-Schwarz action (3.51) up to a normalization constant.

To construct the Wess-Zumino part of the action (4.8), we need to find the closed three form Ω_3 which is invariant under $\mathcal{N} = 2$ supersymmetry transformations,[4] such a form is given by

$$\Omega_3 = f_{\mathcal{ABC}}\,J^{\mathcal{A}} \wedge J^{\mathcal{B}} \wedge J^{\mathcal{C}}\,, \tag{4.30}$$

[4]We have already done this in Chapter 2 when trying to recover the kappa symmetry of the Green-Schwarz action.

where $f_{\mathcal{ABC}}$ are constants. J^m and $J^{\alpha\, I}$ transform respectively as a vector and a spinor under Lorentz transformations $SO(9,1)$, so to make Ω_3 Lorentz-invariant we need to consider the constants $f_{\mathcal{ABC}}$ as the components of an appropriate Lorentz-covariant quantity which provides Ω_3 with the following Lorentz-covariant structure

$$\Omega_3 = i\, \mathfrak{s}_{IJ} J^m \wedge J^{\alpha\, I} \left(\gamma_m\right)_{\alpha\beta} \wedge J^{\beta\, J}\,, \tag{4.31}$$

where $\mathfrak{s}_{IJ}$ is symmetric, traceless, and in order to make Ω_3 not only closed but exact, we should take $\mathfrak{s}_{11} = -\mathfrak{s}_{22} = 1$. Then we can write

$$\Omega_3 = dB\,, \tag{4.32}$$

where B is a 2-form, given by (See section 3.2 of Chapter 3)

$$B = \epsilon^{IJ} \left[-i\partial_i X^m \left(\theta^1 \gamma_m \partial_j \theta^1 - \theta^2 \gamma_m \partial_j \theta^2\right) + \left(\theta^1 \gamma^m \partial_i \theta^1\right)\left(\theta^2 \gamma_m \partial_j \theta^2\right)\right] d^2\sigma\,. \tag{4.33}$$

The Wess-Zumino part of the Lagrangian is given by -up to a constant- $\int B$. This shows that the procedure we made in Section 3.2 of Chapter 3 to recover the kappa symmetry of the action, was not trivial since we can interpret such a term as a Wess-Zumino term.

We now move on to construct the superstring sigma model in $AdS_5 \times S^5$. As we have seen in the example that we sketched above, the construction relies heavily in the properties of the coset manifold, so the first step will be to study the properties of the superalgebra $\mathfrak{psu}(2,2|4)$ which is what we will do next.

4.2 The $\mathfrak{psu}(2,2|4)$ superalgebra

A Lie Superalgebra $\mathcal{G}$ is a $\mathbb{Z}_2$-graded Lie algebra which admits a unique decomposition $\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}$, where the subspaces $\mathcal{G}_{\bar{0}}$ and $\mathcal{G}_{\bar{1}}$ are regarded as even (bosonic) and odd (fermionic) respectively, and is equipped with a graded commutator defined as follows

$$[A, B] = AB - (-1)^{\alpha\beta} BA\,, \tag{4.34}$$

which satisfies the following generalized Jacobi identity

$$(-1)^{\alpha\gamma}\,[A,[B,C]] + (-1)^{\alpha\beta}\,[B,[C,A]] + (-1)^{\beta\gamma}\,[C,[A,B]] = 0\,, \tag{4.35}$$

where the indices α, β, γ correspond to the $\mathbb{Z}_2$-gradings of A, B, $C \in \mathcal{G}$ respectively. The bosonic subspace $\mathcal{G}_{\bar{0}}$ forms an ordinary Lie algebra, and is usually referred to as the *maximal bosonic subalgebra*. On the other hand, the fermionic subspace $\mathcal{G}_{\bar{1}}$ transforms

under a representation of $\mathcal{G}_{\bar{0}}$ induced by the commutator, if this representation is reducible, then $\mathcal{G}$ is referred to as a *classical Lie superalgebra*. If $\mathcal{G}$ is classical and also has a non-degenerate invariant bilinear form, then $\mathcal{G}$ is referred to as a *basic Lie superalgebra*. Basic Lie superalgebras have very similar properties to ordinary Lie algebras and are of greatest interest in physics.

A real form of a Lie superalgebra $\mathcal{G}$ over $\mathbb{C}$ is defined as follows. Let ϕ be an involution from $\mathcal{G}$ to $\mathcal{G}$, such that: it preserves the $\mathbb{Z}_2$-grading of $\mathcal{G}$, the graded commutator and is a semi-linear map

$$\begin{aligned} \phi(\mathcal{G}_\alpha) &= \mathcal{G}_\alpha & , \quad & \alpha = \bar{0}, \bar{1}\,, \\ \phi([X,Y]) &= [\phi(X), \phi(Y)] & , \quad & X, Y \in \mathcal{G}\,, \\ \phi(aX + bY) &= a^* \phi(X) + b^* \phi(Y) & , \quad & a, b \in \mathbb{C}\,, \end{aligned} \tag{4.36}$$

where a^* and b^* stand for the usual complex conjugates of the complex numbers a and b. We can regard ϕ as a generalized complex conjugate operation. We now construct a matrix realization for the superalgebra $\mathfrak{su}(2,2|4)$ which can be defined as the real form of $\mathfrak{sl}(4|4)$.

4.2.1 Matrix realization of $\mathfrak{su}(2,2|4)$

Some Lie superalgebras can be defined in terms of matrix realizations, which are very useful when defining the real forms of them. Let $\mathcal{V} = \mathcal{V}_{\bar{0}} \oplus \mathcal{V}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space, where $\dim(\mathcal{V}_{\bar{0}}) = m$ and $\dim(\mathcal{V}_{\bar{1}}) = n$. We define a supermatrix M as a linear map of $\mathcal{V}$ into $\mathcal{V}$ which may be decomposed as follows

$$M \begin{pmatrix} \mathcal{V}_{\bar{0}} \\ \mathcal{V}_{\bar{1}} \end{pmatrix} \to \begin{pmatrix} \mathcal{V}_{\bar{0}} \\ \mathcal{V}_{\bar{1}} \end{pmatrix}, \qquad M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}. \tag{4.37}$$

where m and n are of even grading and of dimension $m \times m$ and $n \times n$ respectively, in contrast, θ and η are of odd grading and of dimension $m \times n$ and $n \times m$ respectively.

We define the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ as the set of all supermatrices (4.37) over the field $\mathbb{C}$. The supertrace and supertranspose are defined as follows

$$\mathrm{str} M \equiv \mathrm{tr}\, m - \mathrm{tr}\, n\,, \qquad M^{st} = \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix} \qquad . \tag{4.38}$$

The matrix representation of $\mathfrak{sl}(m|n)$ over $\mathbb{C}$ is given by

$$\mathfrak{sl}(m|n) = \{M \in \mathfrak{gl}(m|n)\,; \quad \mathrm{str} M = 0\}\ . \tag{4.39}$$

We now specialize to the case of the superalgebra $\mathfrak{sl}(4,4)$ over the field $\mathbb{C}$. The representation of this superalgebra is in terms of 8×8 supermatrices (4.37), which are constructed in terms of 4×4 blocks, where m and n are regarded as even (bosonic) and θ and η are regarded as odd (fermionic).[5] As we established in (4.39), we define the superalgebra $\mathfrak{sl}(4,4)$ as spanned by the matrices M with vanishing supertrace $\mathrm{str}M \equiv \mathrm{tr}\, m - \mathrm{tr}\, n = 0$.

We define the superalgebra $\mathfrak{su}(2,2|4)$ as a non-compact real form of $\mathfrak{sl}(4,4)$. We consider the so-called *Cartan involution*

$$\phi(M) \equiv M^\star = M\,, \tag{4.40}$$

where $M^\star$ is defined as follows

$$M^\star = -HM^\dagger H^{-1}\,, \tag{4.41}$$

here $M^\dagger$ stands for the adjoint of the supermatrix M defined by $M^\dagger = (M^t)^*$. Combining equations (4.40) and (4.41) we find that a supermatrix M from $\mathfrak{su}(2,2|4)$ satisfies the following reality condition

$$MH + HM^\dagger = 0\,. \tag{4.42}$$

The Hermitian matrix H is defined as follows[6]

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & 1_4 \end{pmatrix}\,, \tag{4.43}$$

and Σ is a 4×4 matrix defined as

$$\Sigma = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}\,. \tag{4.44}$$

We can find how the reality condition (4.42) acts on the block entries of M by expanding it as follows

$$\begin{pmatrix} m\Sigma & \theta \\ \eta\,\Sigma & n \end{pmatrix} = \begin{pmatrix} -\Sigma m^\dagger & -\Sigma\eta^\dagger \\ -\theta^\dagger & -n^\dagger \end{pmatrix}\,, \tag{4.45}$$

then the condition (4.42) implies

$$m^\dagger = -\Sigma m \Sigma\,, \qquad n^\dagger = -n\,, \qquad \eta^\dagger = -\Sigma\,\theta\,, \tag{4.46}$$

[5]The entries of θ and η can be thought as Grassmann anti-commuting variables

[6]Here 1_n stands for the $n\times n$ identity matrix.

from these conditions we can easily see that the matrix blocks m and n span the unitary algebras $\mathfrak{u}(2,2)$ and $\mathfrak{u}(4)$ respectively. From this simple analysis we can deduce that the bosonic subalgebra of $\mathfrak{su}(2,2|4)$ is given by

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)\,, \tag{4.47}$$

where we have added the final factor since the $\mathfrak{u}(1)$ generator $i1_4$ is also part of $\mathfrak{su}(2,2|4)$ because it satisfies (4.42) and possess vanishing supertrace. We define the $\mathfrak{psu}(2,2|4)$ superalgebra as a quotient algebra of $\mathfrak{su}(2,2|4)$ over this $\mathfrak{u}(1)$ factor.

We will now try to find an explicit basis for the bosonic part of the superalgebra $\mathfrak{su}(2,2|4)$. Let us start by defining the 8×8 Dirac matrices as follows

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & i & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \Sigma\,, \tag{4.48}$$

which satisfy the $SO(5)$ Clifford algebra relations

$$\{\gamma^i, \gamma^j\} = \delta^{ij}\,, \qquad i,j = 1,\ldots,5. \tag{4.49}$$

It can be easily proven that $\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4$. Since these matrices are Hermitian $(\gamma^i)^* = (\gamma^i)^t$, the anti-Hermitian matrices constructed as $i\gamma^i$ belong to $\mathfrak{su}(4)$. As we know, we can construct the spinor representation of $\mathfrak{so}(5)$ by constructing the generators $n^{ij} = \frac{1}{4}\,[\gamma^i, \gamma^j]$ satisfying the relations

$$\left[n^{ij}, n^{kl}\right] = \delta^{jk}n^{il} - \delta^{ik}n^{jl} - \delta^{jl}n^{ik} + \delta^{il}n^{jk} \qquad n^{ij} = -n^{ji}\,. \tag{4.50}$$

The Weyl spinor representation $\mathfrak{so}(6) \sim \mathfrak{su}(4)$ can be found by adding $n^{i6} = \frac{i}{2}\gamma^i$ to the set of generators, satisfying the same relations (4.50), but now $i,j = 1,\ldots 6$.

On the other hand, the set $\{i\gamma^5, \gamma^i\}$ with $i = 1,\ldots,4$ generates the Clifford algebra for $SO(4,1)$. Analogously, if we introduce $\gamma^0 = i\gamma^5$, then the generators $m^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$, with $i,j = 0,\ldots,4$ satisfy the $\mathfrak{so}(4,1)$ relations

$$\left[m^{ij}, m^{kl}\right] = \eta^{jk}m^{il} - \eta^{ik}m^{jl} - \eta^{jl}m^{ik} + \eta^{il}m^{jk} \qquad m^{ij} = -m^{ji}\,. \tag{4.51}$$

which compared to (4.50) involves $\eta = \mathrm{diag}(-1,1,1,1)$ instead of the Kronecker delta. We can enlarge the set of generators by defining $m^{i5} = \frac{1}{2}\gamma^i$, where $i = 0,\ldots,4$, thus obtaining a realization of $\mathfrak{so}(4,2) \sim \mathfrak{su}(2,2)$, with the same algebra as in (4.51), but now with the metric $\eta = \mathrm{diag}(-1,1,1,1,1,-1)$ and $i,j = 0,\ldots,5$.

In summary, the bosonic algebras $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ can be thought as spanned by the generators

$$\begin{aligned} \mathfrak{su}(4) \quad &\text{spanned by} \quad \left\{\tfrac{i}{2}\gamma^i, \tfrac{1}{4}[\gamma^i,\gamma^j]\right\}, && i,j = 1,\ldots,5\,, \\ \mathfrak{su}(2,2) \quad &\text{spanned by} \quad \left\{\tfrac{1}{2}\gamma^i, \tfrac{i}{2}\gamma^5, \tfrac{1}{4}[\gamma^i,\gamma^j], \tfrac{i}{4}[\gamma^5,\gamma^j]\right\}, && i,j = 1,\ldots,4\,. \end{aligned} \tag{4.52}$$

If we add the $\mathfrak{u}(1)$ generator $i1$, the full set of generators provides an explicit basis for the bosonic subalgebra of $\mathfrak{su}(2,2|4)$.

4.2.2 $\mathbb{Z}_4$-Grading

Let us consider a basic Lie superalgebra $\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}$. An automorphism $\Omega(\mathcal{G}) \to \mathcal{G}$ is a bijective homomorphism from $\mathcal{G}$ into itself which respects the $\mathbb{Z}_2$-gradation, that is $\Omega(\mathcal{G}_{\bar{0}}) \subset \mathcal{G}_{\bar{0}}$ and $\Omega(\mathcal{G}_{\bar{1}}) \subset \mathcal{G}_{\bar{1}}$. The set of all automorphisms of $\mathcal{G}$ form a group denoted by $\mathrm{Aut}(\mathcal{G})$. The group of inner automorphisms, denoted as $\mathrm{Inn}(\mathcal{G})$, is the group generated by the automorphisms of the form $X \to gXg^{-1}$ with $g = \exp Y$, where $X \in \mathcal{G}$ and $Y \in \mathcal{G}_{\bar{0}}$. The automorphisms of $\mathcal{G}$ which are not inner, are called outer automorphisms. The principal characteristic of $\mathfrak{psu}(2,2|4)$ is that it admits a fourth order automorphism as we will see next.

The automorphism group of $\mathfrak{sl}(4|4)$ is generated by $\Omega(M) = e^{\frac{1}{2}\Upsilon\log\rho} M e^{-\frac{1}{2}\Upsilon\log\rho}$. Where the matrix Υ is the so-called hypercharge defined by

$$\Upsilon = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}, \tag{4.53}$$

working out the exponentiation, it is easy to note that

$$e^{\frac{1}{2}\Upsilon\log\rho} = \begin{pmatrix} \rho^{\frac{1}{2}} 1_4 & 0 \\ 0 & \rho^{-\frac{1}{2}} 1_4 \end{pmatrix}, \tag{4.54}$$

and then we can write

$$\Omega(M) = e^{\frac{1}{2}\Upsilon\log\rho} M e^{-\frac{1}{2}\Upsilon\log\rho} = \begin{pmatrix} m & \rho\,\theta \\ \frac{1}{\rho}\,\eta & n \end{pmatrix}. \tag{4.55}$$

The minus supertransposition $M \to -M^{st}$, where the supertranspose is defined in (4.38), belongs to the group of automorphisms generated by (4.55), and is regarded as a fourth order automorphism. This automorphism allows one to endow $\mathfrak{sl}(4|4)$ with the structure of a $\mathbb{Z}_4$-graded Lie superalgebra.

The superalgebra $\mathfrak{su}(2,2|4)$ admits an equivalent fourth order automorphism defined as

$$M \to \Omega(M) \equiv -\mathcal{K} M^{st} \mathcal{K}^{-1} \,. \tag{4.56}$$

Here $\mathcal{K}$ is the 8×8 matrix $\mathcal{K} = \mathrm{diag}(K, K)$, where K is a 4×4 matrix defined as follows

$$K = -\gamma^2 \gamma^4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \,. \tag{4.57}$$

The definition of the supertransposition and the fact that $\mathcal{K}^2 = 1_8$ automatically implies the fourth order nature of the automorphism Ω defined in (4.56). Such an automorphism decomposes the Lie superalgebra into four subspaces $(\mathcal{H}^{(k)})$ in such a way that each subspace is an eigenspace of the map Ω, that is if we denote $\mathcal{H} = \mathfrak{su}(2,2|4)$ we can write

$$\mathcal{H}^{(k)} = \left\{ M \in \mathcal{H} \,, \quad \Omega(M) = i^k M \right\} \,; \qquad k = 0, 1, 2, 3 \,. \tag{4.58}$$

In this way, as a vector space, $\mathcal{H}$ can be decomposed into a direct sum of graded subspaces as follows

$$\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(3)} \,. \tag{4.59}$$

where the subspaces satisfy

$$\left[\mathcal{H}^{(k)}, \mathcal{H}^{(m)} \right] \subseteq \mathcal{H}^{(k+m)} \quad \text{modulo} \qquad \mathbb{Z}_4 \,. \tag{4.60}$$

If we take a supermatrix $M \in \mathcal{H}$, then its projection $M^{(k)} \in \mathcal{H}^{(k)}$ is given by

$$M^{(k)} = \frac{1}{4} \left(M + i^{3k} \Omega(M) + i^{2k} \Omega^2(M) + i^k \Omega^3(M) \right) \,. \tag{4.61}$$

The projection $M^{(0)}$ and $M^{(2)}$ are even while $M^{(1)}$ and $M^{(3)}$ are odd. In fact, let us work out these results explicitly. For $M^{(0)}$, according to (4.61) we can write

$$M^{(k)} = \frac{1}{4} \left(M + \Omega(M) + \Omega^2(M) + \Omega^3(M) \right) \,, \tag{4.62}$$

where we can write the explicit form of every term as follows

$$\Omega(M) = -\Omega^3(M) = -\begin{pmatrix} Km^tK^{-1} & -K\eta^tK^{-1} \\ K\theta^tK^{-1} & Kn^tK^{-1} \end{pmatrix}, \qquad \Omega^2(M) = \begin{pmatrix} m & -\theta \\ -\eta & n \end{pmatrix}. \tag{4.63}$$

In this way, the projection $M^{(0)}$ takes the following form

$$M^{(0)} = \frac{1}{2}\begin{pmatrix} m - Km^tK^{-1} & 0 \\ 0 & n - Kn^tK^{-1} \end{pmatrix}, \tag{4.64}$$

which is indeed even. In a similar way we can obtain the explicit form of the projection $M^{(2)}$ as follows

$$M^{(2)} = \frac{1}{2}\begin{pmatrix} m + Km^tK^{-1} & 0 \\ 0 & n + Kn^tK^{-1} \end{pmatrix}. \tag{4.65}$$

Analogously, we can write the explicit form of the odd projections $M^{(1)}$ and $M^{(3)}$ as follows

$$M^{(0)} = \frac{1}{2}\begin{pmatrix} 0 & \theta - iK\eta^tK^{-1} \\ \eta + iK\theta^tK^{-1} & 0 \end{pmatrix}, \tag{4.66}$$

$$M^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & \theta + iK\eta^tK^{-1} \\ \eta - iK\theta^tK^{-1} & 0 \end{pmatrix}. \tag{4.67}$$

On the other hand, by using the following property

$$\Omega(M)^\dagger = \Upsilon\Omega(M)\Upsilon^{-1} = -\left(\Upsilon H\right)\Omega(M)\left(\Upsilon H\right)^{-1}, \tag{4.68}$$

one can show that the Hermitian-conjugate of $M^{(k)}$ can be written in the following way

$$M^{(k)\,\dagger} = -\frac{1}{4}H\left[M + i^k\Upsilon\Omega(M)\Upsilon^{-1} + i^{2k}\Omega^2(M) + i^{3k}\Upsilon\Omega^3(M)\Upsilon^{-1}\right]H^{-1}. \tag{4.69}$$

To finish this sub-section we realize that every matrix $M \in \mathfrak{su}(2,2|4)$ can be uniquely decomposed into the sum (4.59), where each component $M^{(k)}$ takes values in $\mathfrak{su}(2,2|4)$.

4.3 Green-Schwarz superstring in $AdS_5 \times S^5$

As we saw in section 1, the flat space Green-Schwarz superstring can be interpreted as a sigma model whose target space is given by the coset (4.20). Remarkably, one can use a

similar reasoning in the $AdS^5 \times S_5$ case, where we can define the Type IIB Green-Schwarz superstring as a non-linear sigma model whose target space is given by

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)} \,. \tag{4.70}$$

The supergroup $PSU(2,2|4)$ contains the bosonic subgroup $SU(2,2) \times SU(4)$ which, as you remember from classical group theory, is locally isomorphic to $SO(4,2) \times SO(6)$, where $SO(4,2)$ and $SO(6)$ play the role of the isometry groups of AdS_5 and S^5 respectively. The quotient $\frac{SO(4,2)\times SO(6)}{SO(4,1)\times SO(5)}$ provides a model of the $AdS_5 \times S^5$ manifold, with $SO(4,1) \times SO(5)$ being the group of local Lorentz transformations. In this way, we can regard the coset (4.70) as a model of the $AdS_5 \times S^5$ superspace with $PSU(2,2|4)$ playing the role of the isometry group and $SO(4,1) \times SO(5)$ representing the generalization of the Lorentz group in the space $AdS_5 \times S^5$ on analogy to the flat case.

4.3.1 The Metsaev-Tseytlin superstring

Just as we saw in the flat space action, the action for the Type IIB superstring propagating in $AdS_5 \times S^5$ will be given by a Wess-Zumino-Witten like sigma-model Lagrangian, where the first term corresponds to a kinetic term an the second one is obtained as the integral of a closed 3-form Ω_3 over a three dimensional manifold $\mathcal{D}$ which has the string worldsheet as its boundary.

The first step is to define the Maurer-Cartan form. Let $\mathfrak{g}$ be an element of the supergroup $SU(2,2|4)$, we define the Maurer-Cartan form as $J = -\mathfrak{g}^{-1} d\mathfrak{g}$, since it takes values in $\mathfrak{su}(2,2|4)$ we can write it as follows

$$J = -\mathfrak{g}^{-1} d\mathfrak{g} = J^{(0)} + J^{(2)} + J^{(1)} + J^{(3)} \,, \tag{4.71}$$

where we have exhibited the $\mathbb{Z}_4$-decomposition of J. We should observe that, by construction J satisfy the Maurer-Cartan equation $dJ - J \wedge J = 0$, which in components can be written as follows[7]

$$\partial_\alpha J_\beta - \partial_\beta J_\alpha - [J_\alpha, J_\beta] = 0 \,. \tag{4.72}$$

Let us now consider local right $SO(4,1) \times SO(5)$ multiplication, that is we consider the following transformation

$$\mathfrak{g} \to \mathfrak{g}\mathfrak{h} \,, \tag{4.73}$$

[7]Here α and β denote worldsheet coordinates, so you should not confuse them with spinorial indices.

where $\mathfrak{h}$ belongs to $SO(4,1) \times SO(5)$. It is easy to see that under this transformation the Maurer-Cartan form transforms as follows

$$\begin{aligned} J &\to -(\mathfrak{g}\mathfrak{h})^{-1}\,(d\mathfrak{g}\,\mathfrak{h} + \mathfrak{g}\,d\mathfrak{h}) \\ &\to \mathfrak{h}^{-1} J\,\mathfrak{h} - \mathfrak{h}^{-1} d\mathfrak{h}\,. \end{aligned} \tag{4.74}$$

which in $\mathbb{Z}_4$ components reads

$$J^{(1,2,3)} \to \mathfrak{h}^{-1} J^{(1,2,3)}\,\mathfrak{h}\,, \qquad J^{(0)} \to \mathfrak{h}^{-1} J^{(0)}\,\mathfrak{h} - \mathfrak{h}^{-1} d\mathfrak{h}\,. \tag{4.75}$$

In order to formulate the Lagrangian, we need to make a key observation here. The transformation on $J^{(0)}$ is typical of a gauge field, so we can understand $J^{(0)}$ as the $SO(4,1) \times SO(5)$ gauge field, while $J^{(1,2,3)}$ transform according to the adjoint representation of $SO(4,1) \times SO(5)$. Furthermore, since only the components $J^{(1)}$, $J^{(2)}$ and $J^{(3)}$ undergo a similarity transformation, then any gauge invariant Lagrangian in the supercoset is given by a bilinear in the J's which can not contain $J^{(0)}$,[8] thus the Lagrangian should only depend on a coset element and not on the group element $\mathfrak{g}$.

We now proceed to construct the Green-Schwarz action on the supercoset (4.70). We consider the kinetic term for the bosonic components $J^{(2)}$, but we can not allow a kinetic term for the fermionic components because it would break kappa symmetry. In this way, the fermionic components of the Maurer-Cartan form enter through the Wess-Zumino term. The sigma model Type II superstring action on $AdS_5 \times S^5$ is given by [24]

$$S_{GS} = \int d\tau d\sigma\, \mathcal{L}\,, \tag{4.76}$$

where $\mathcal{L}$ is given by

$$\mathcal{L} = -\frac{1}{4\pi\alpha'}\left[\gamma^{\alpha\beta}\,\mathrm{str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) + \kappa\,\epsilon^{\alpha\beta}\,\mathrm{str}\left(J^{(1)}_\alpha J^{(3)}_\beta\right)\right]\,, \tag{4.77}$$

where $\gamma^{\alpha\beta} = g^{\alpha\beta}\sqrt{-g}$ is the Weyl invariant combination of the worldsheet metric $g_{\alpha\beta}$ such that $\det\gamma = 1$. The parameter κ is a constant number which must be real in order to guarantee the reality of the Lagrangian. Indeed, if we assume $\kappa = \kappa^*$ and taking into consideration the conjugation rule for the fermionic entries, as well as the cyclic properties of the supertrace we see that

$$\begin{aligned} \mathcal{L}^* &= -\frac{1}{4\pi\alpha'}\left[\gamma^{\alpha\beta}\mathrm{str}\left(J^{(2)\,\dagger}_\beta J^{(2)\,\dagger}_\alpha\right) + \kappa\epsilon^{\alpha\beta}\mathrm{str}\left(J^{(3)\,\dagger}_\beta J^{(1)\,\dagger}_\alpha\right)\right] \\ &= -\frac{1}{4\pi\alpha'}\left[\gamma^{\alpha\beta}\mathrm{str}\left(H J^{(2)}_\beta H^{-1} H J^{(2)}_\alpha H^{-1}\right) + \kappa\epsilon^{\alpha\beta}\mathrm{str}\left(H J^{(3)}_\beta H^{-1} H J^{(1)}_\alpha H^{-1}\right)\right] \\ &= \mathcal{L}\,, \end{aligned} \tag{4.78}$$

[8] The components $J^{(2)}$ define the supervielbeins, so we could have formulated the kinetic part of the Lagrangian by making use of (4.8).

where in the second line we made use of (4.69) and in the third line we used the cyclic property of the supertrace. Furthermore, as we will see later, the requirement of kappa symmetry leaves only two possibilities $\kappa = \pm 1$.

It should be clear that we have been using the realization of the $\mathfrak{su}(2,2|4)$ superalgebra rather than $\mathfrak{psu}(2,2|4)$, as we saw earlier, the difference between the two is due to the appearance of the central element $i1$ in the projection $J^{(2)}$ of the $\mathfrak{su}(2,2|4)$ superalgebra. Thus under a right multiplication of the coset element $\mathfrak{g}$ with a group element from $U(1)$ corresponding to $i1$, the component $J^{(2)}$ is shifted

$$J^{(2)} \to J^{(2)} + c \cdot i1 \,. \tag{4.79}$$

The supertrace of both $J^{(2)}$ and the identity matrix vanishes, then the transformation (4.79) leaves invariant the Lagrangian (4.77). In this way, we recognize an extra local $\mathfrak{u}(1)$ symmetry induced by the central element $i1$. We can use this extra symmetry to gauge away the trace of $J^{(2)}$, and then choosing $J^{(2)}$ to be traceless can be viewed as the gauge fixing condition for these $\mathfrak{u}(1)$ transformations.

We will now find the equations of motion which follow from the Lagrangian (4.77). In order to this, we will make use of the following property

$$\mathrm{str}\left(\Omega^k(M_1)M_2\right) = \mathrm{str}\left(M_1\Omega^{4-k}(M_2)\right) \,. \tag{4.80}$$

So, taking the variation of the Lagrangian (4.77) we get

$$\delta\mathcal{L} = -\frac{1}{4\pi\alpha'}\left[2\,\gamma^{\alpha\beta}\mathrm{str}\left(\delta J^{(2)}_\alpha J^{(2)}_\beta\right) + \kappa\,\epsilon^{\alpha\beta}\mathrm{str}\left(\delta J^{(1)}_\alpha J^{(3)}_\beta + J^{(1)}_\alpha \delta J^{(3)}_\beta\right)\right] \,, \tag{4.81}$$

Let us compute each of the terms of (4.81), for instance the term inside the first supertrace on the right hand side can be written as follows

$$\begin{aligned}\mathrm{str}\left(\delta J^{(2)}_\alpha J^{(2)}_\beta\right)1 &= \frac{1}{4}\mathrm{str}\left(\delta J_\alpha J^{(2)}_\beta - \Omega(\delta J_\alpha)J^{(2)}_\beta + \Omega^2(\delta J_\alpha)J^{(2)}_\beta - \Omega^3(\delta J_\alpha)J^{(2)}_\beta\right)\\ &= \mathrm{str}\left(\delta J_\alpha J^{(2)}_\beta\right) \,,\end{aligned} \tag{4.82}$$

where in the second line we have used

$$\begin{aligned}\mathrm{str}\left(\Omega(\delta J_\alpha)J^{(2)}_\beta\right) &= \mathrm{str}\left(\delta J_\alpha\Omega(J_\beta) - \delta J_\alpha J_\beta + \delta J_\alpha\Omega^3(J_\beta) - \delta J_\alpha\Omega^2(J_\beta)\right)\\ &= -\mathrm{str}\left(\delta J_\alpha J^{(2)}_\beta\right)\end{aligned} \tag{4.83}$$

In a similar way, each of the terms in the first line of (4.82) contributes with one factor of $\mathrm{str}\left(\delta J_\alpha J^{(2)}_\beta\right)$, proving (4.82). The second term of the right hand side of (4.81) can be

found in a similar way as we show next

$$\begin{aligned}
\epsilon^{\alpha\beta}\mathrm{str}\left(\delta J_\alpha^{(1)} J_\beta^{(3)} + J_\alpha^{(1)}\delta J_\beta^{(3)}\right) &= \epsilon^{\alpha\beta}\mathrm{str}\left(\delta J_\alpha^{(3)} J_\beta^{(1)} - \delta J_\alpha^{(1)} J_\beta^{(3)}\right) \\
&= \frac{\epsilon^{\alpha\beta}}{4}\,\mathrm{str}\left(\delta J_\alpha J_\beta^{(1)} + i\Omega(\delta J_\alpha) J_\beta^{(1)} - \Omega^2(\delta J_\alpha) J_\beta^{(1)} - i\Omega^3(J_\alpha) J_\beta^{(1)}\right) + \\
&- \frac{\epsilon^{\alpha\beta}}{4}\,\mathrm{str}\left(\delta J_\alpha J_\beta^{(1)} - i\Omega(\delta J_\alpha) J_\beta^{(1)} - \Omega^2(\delta J_\alpha) J_\beta^{(1)} + i\Omega^3(J_\alpha) J_\beta^{(3)}\right) \\
&= \frac{\epsilon^{\alpha\beta}}{4}\,\mathrm{str}\left(\delta J_\alpha \left(J_\beta^{(1)} - J_\beta^{(3)}\right)\right) , \qquad (4.84)
\end{aligned}$$

where the only computation we did is to carefully apply identity (4.80). With the help of the results (4.82) and (4.84) we can write for the variation of the Lagrangian

$$\delta\mathcal{L} = -\mathrm{str}\left(\delta J_\alpha \Lambda^\alpha\right) , \qquad (4.85)$$

where

$$\Lambda^\alpha = \frac{1}{2\pi\alpha'}\left[\gamma^{\alpha\beta} J_\beta^{(2)} - \frac{1}{2}\kappa\,\epsilon^{\alpha\beta}\left(J_\beta^{(1)} - J_\beta^{(3)}\right)\right] . \qquad (4.86)$$

What is more, the variation of J_α is given by

$$\delta J_\alpha = -\delta\left(\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\right) = -\mathfrak{g}^{-1}\delta\mathfrak{g}\, J_\alpha - \mathfrak{g}^{-1}\partial_\alpha(\delta\mathfrak{g}) , \qquad (4.87)$$

plugging the result above into the variation for the Lagrangian we wrote before, we get

$$\begin{aligned}
\delta\mathcal{L} &= +\mathrm{str}\left[\mathfrak{g}^{-1}\delta\mathfrak{g}\, J_\alpha\Lambda^\alpha + \mathfrak{g}^{-1}\partial_\alpha(\delta\mathfrak{g})\Lambda^\alpha\right] \\
&= -\mathrm{str}\left[\mathfrak{g}^{-1}\delta\mathfrak{g}\left(\partial_\alpha\Lambda^\alpha - [J_\alpha, \Lambda^\alpha]\right)\right] , \qquad (4.88)
\end{aligned}$$

where we made use of integration by parts, and we arrived to the second line by neglecting the total derivatives. Since equation (4.88) must vanish for an arbitrary group element $\mathfrak{g}$ and if we consider $\partial_\alpha\Lambda^\alpha - [J_\alpha, \Lambda^\alpha]$ as an element of $\mathfrak{su}(2,2|4)$ which contains the central element $i1$, we must require that

$$\partial_\alpha\Lambda^\alpha - [J_\alpha, \Lambda^\alpha] = c \cdot i1 \qquad (4.89)$$

where the parameter c can be found by taking the trace of both sides in the expression above. Since we defined $\mathfrak{psu}(2,2|4)$ as the quotient algebra of $\mathfrak{su}(2,2|4)$ over its central element, we may identify any elements proportional to the identity as zero, in this way we can write the equations of motion as follows

$$\partial_\alpha\Lambda^\alpha - [J_\alpha, \Lambda^\alpha] = 0 . \qquad (4.90)$$

We can use $\mathbb{Z}_4$ decomposition and rearrange all the terms of the equation above according to its $\mathbb{Z}_4$ grading. For instance, the 0-grading term is given by

$$\gamma^{\alpha\beta}\left[J^{(2)}_\alpha, J^{(2)}_\beta\right] - \frac{1}{2}\,\kappa\,\epsilon^{\alpha\beta}\left(\left[J^{(3)}_\alpha, J^{(1)}_\beta\right] - \left[J^{(1)}_\alpha, J^{(3)}_\beta\right]\right) = 0\,. \tag{4.91}$$

This equation is trivial, since each of the parts of the left hand side vanishes when using the symmetry and anti-symmetry properties of $\gamma^{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ respectively leading to $0 = 0$. The 2-grading component is given by

$$\partial_\alpha\left(\gamma^{\alpha\beta}J^{(2)}_\beta\right) - \gamma^{\alpha\beta}\left[J^{(0)}_\alpha, J^{(2)}_\beta\right] + \frac{1}{2}\,\kappa\,\epsilon^{\alpha\beta}\left(\left[J^{(1)}_\alpha, J^{(1)}_\beta\right] - \left[J^{(3)}_\alpha, J^{(3)}_\beta\right]\right)\,. \tag{4.92}$$

The 1-grading and 3-grading components of the equations of motion are given respectively by

$$\gamma^{\alpha\beta}\left[J^{(3)}_\alpha, J^{(2)}_\beta\right] + \kappa\,\epsilon^{\alpha\beta}\left[J^{(2)}_\alpha, J^{(3)}_\beta\right] = 0\,, \tag{4.93}$$

$$\gamma^{\alpha\beta}\left[J^{(1)}_\alpha, J^{(2)}_\beta\right] + \kappa\,\epsilon^{\alpha\beta}\left[J^{(2)}_\alpha, J^{(1)}_\beta\right] = 0\,, \tag{4.94}$$

where we made use of the Maurer-Cartan equation (4.72). Let us define the tensors[9]

$$P^{\alpha\beta}_\pm = \frac{1}{2}\left(\gamma^{\alpha\beta} \pm \kappa\,\epsilon^{\alpha\beta}\right)\,. \tag{4.95}$$

Considering this definition, the equations of motion (4.93) and (4.94) can be written in the following way

$$P^{\alpha\beta}_-\left[J^{(2)}_\alpha, J^{(3)}_\beta\right] = 0\,, \tag{4.96}$$

$$P^{\alpha\beta}_+\left[J^{(2)}_\alpha, J^{(1)}_\beta\right] = 0\,. \tag{4.97}$$

Furthermore, for $\kappa = \pm 1$, the tensors $P^{\alpha\beta}_\pm$ are orthogonal projectors, that is they satisfy the relations

$$P^{\alpha\beta}_+ + P^{\alpha\beta}_- = \gamma^{\alpha\beta}\,, \qquad P^{\alpha\delta}_\pm P_{\pm\,\delta}^{\ \ \beta} = P^{\alpha\beta}_\pm\,, \qquad P^{\alpha\delta}_\pm P_{\mp\,\delta}^{\ \ \beta} = 0\,. \tag{4.98}$$

When varying the Lagrangian, we have not considered the variations of the worldsheet metric, so we can easily derive the equations of motion for the worldsheet metric by varying the Lagrangian with respect to $\gamma^{\alpha\beta}$, obtaining the so-called Virasoro constraints

$$\mathrm{str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\,\mathrm{str}\left(J^{(2)}_\rho J^{(2)}_\delta\right) = 0\,. \tag{4.99}$$

Just as we mentioned earlier in Chapter 3, these constraints represent the parametrization invariance of the string action with respect to the worldsheet diffeomorphisms.

[9] These are equivalent to the projector tensors we consider in Chapter 3.

4.3.2 Kappa symmetry revisited

In Chapter 3 we saw that there was a local hidden fermionic symmetry in the Brink-Schwarz superparticle. We saw there that this symmetry was generated by the first class constraints of the theory. We also required kappa invariance when studying the Green-Schwarz superstring where we mentioned that its presence is crucial for the space-time supersymmetry of the physical spectrum. Here we establish the kappa symmetry transformations associated to the coset sigma model formulation of the Green-Schwarz superstring.

Unlike the Brink-Schwarz superparticle and the standard formulation of the Green-Schwarz superstring where, as we saw in Chapter 3, the kappa symmetry arises due to the constraints of the theory, in the coset sigma-model formulation, the additional degrees of freedom arise from the representation of the coset space algebra. Let us start by noting that the canonical form J is trivially invariant under global left $PSU(2,2|4)$ multiplication

$$\mathfrak{g} \to \mathfrak{g}'\mathfrak{g}\,, \qquad \mathfrak{g}' \in PSU(2,2|4)\,. \tag{4.100}$$

The $PSU(2,2|4)$ supergroup plays the role of the global symmetry group which is realized on a coset element by multiplication from the left. Similarly, we can view the kappa symmetry transformations as the right local action of $G = \exp(\epsilon)$ on the coset representative $\mathfrak{g}$ as follows

$$\mathfrak{g} \cdot G = \mathfrak{g}'\,\mathfrak{h}\,, \tag{4.101}$$

where $\epsilon = \epsilon(\tau,\sigma)$ is a local fermionic parameter taking values in $\mathfrak{psu}(2,2|4)$ and $\mathfrak{h}$ is a compensating element from the coset denominator $SO(4,1) \times SO(5)$. Unlike the global symmetry group case, the string action is not invariant for an arbitrary form of the ϵ parameter. However, it is possible to find a set of such parameters that allows one to remove the unphysical degrees of freedom. We now find the conditions on ϵ which guarantee the invariance of the action.

Let us now investigate how the Maurer-Cartan form transforms under this right multiplication, using (4.101) we can write

$$\begin{aligned} J = -\mathfrak{g}^{-1}d\mathfrak{g} \quad &\to \quad -\left(\mathfrak{g}e^{\epsilon}\right)^{-1} d(\mathfrak{g}e^{\epsilon}) \\ &= \quad -e^{-\epsilon}\mathfrak{g}^{-1}\left(d\mathfrak{g}e^{\epsilon} + \mathfrak{g}e^{\epsilon}d\epsilon\right) \\ &= \quad e^{-\epsilon}Je^{\epsilon} - d\epsilon \end{aligned} \tag{4.102}$$

Using the Baker-Hausdorff formula we can write, up to an infinitesimal parameter ϵ, the following transformations

$$\delta_\epsilon J = -d\epsilon + [J,\epsilon]\ . \tag{4.103}$$

If we consider $\epsilon = \epsilon^{(1)} + \epsilon^{(3)}$, the transformations showed above can be written, upon $\mathbb{Z}_4$ decomposition, as follows

$$\delta_\epsilon \left(J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)}\right) = d\left(\epsilon^{(1)} + \epsilon^{(3)}\right) + \left[J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)}, \epsilon^{(1)} + \epsilon^{(3)}\right] . \tag{4.104}$$

We can rearrange the terms according to its respective gradation, if we do this we end up with the following $\mathbb{Z}_4$ relations

$$\delta_\epsilon J^{(0)} = \left[J^{(3)}, \epsilon^{(1)}\right] + \left[J^{(1)}, \epsilon^{(3)}\right] , \tag{4.105}$$

$$\delta_\epsilon J^{(2)} = \left[J^{(1)}, \epsilon^{(1)}\right] + \left[J^{(3)}, \epsilon^{(3)}\right] , \tag{4.106}$$

$$\delta_\epsilon J^{(1)} = -d\epsilon^{(1)} + \left[J^{(0)}, \epsilon^{(1)}\right] + \left[J^{(2)}, \epsilon^{(3)}\right] , \tag{4.107}$$

$$\delta_\epsilon J^{(3)} = -d\epsilon^{(3)} + \left[J^{(2)}, \epsilon^{(1)}\right] + \left[J^{(0)}, \epsilon^{(3)}\right] . \tag{4.108}$$

We can make use of these formulae to find the variation of the Lagrangian as we do next

$$\begin{aligned}\delta_\epsilon \mathcal{L} = -\frac{1}{4\pi\alpha'}\Bigg[&\delta_\epsilon \gamma^{\alpha\beta} \mathrm{str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) + 2\gamma^{\alpha\beta}\mathrm{str}\left(\delta_\epsilon J^{(2)}_\alpha J^{(2)}_\beta\right) + \\ &+\kappa\, \epsilon^{\alpha\beta}\mathrm{str}\left(\delta_\epsilon J^{(1)}_\alpha J^{(3)}_\beta - \delta_\epsilon J^{(3)}_\alpha J^{(1)}_\beta\right)\Bigg] .\end{aligned} \tag{4.109}$$

The variation of the terms inside the supertraces can be found easily as we show next

$$\begin{aligned}\mathrm{str}\left(\delta_\epsilon J^{(2)}_\alpha J^{(2)}_\beta\right) &= \mathrm{str}\left(\left[J^{(1)}_\alpha, \epsilon^{(1)}\right] J^{(2)}_\beta + \left[J^{(3)}_\alpha, \epsilon^{(3)}\right] J^{(2)}_\beta\right) \\ &= \mathrm{str}\left(\left[J^{(2)}_\beta, J^{(1)}_\alpha\right]\epsilon^{(1)} + \left[J^{(2)}_\beta, J^{(3)}_\alpha\right]\epsilon^{(3)}\right) ,\end{aligned} \tag{4.110}$$

where we made use of the $\mathbb{Z}_4$ kappa transformations together with the cyclic property of the supertrace. Similarly we can write

$$\mathrm{str}\left(\delta_\epsilon J^{(1)}_\alpha J^{(3)}_\beta\right) = \mathrm{str}\left(-\partial_\alpha\epsilon^{(1)} J^{(3)}_\beta + \left[J^{(3)}_\beta, J^{(0)}_\alpha\right]\epsilon^{(1)} + \left[J^{(3)}_\beta, J^{(2)}_\alpha\right]\epsilon^{(3)}\right) , \tag{4.111}$$

$$\mathrm{str}\left(\delta_\epsilon J^{(3)}_\alpha J^{(1)}_\beta\right) = \mathrm{str}\left(-\partial_\alpha\epsilon^{(3)} J^{(1)}_\beta + \left[J^{(1)}_\beta, J^{(2)}_\alpha\right]\epsilon^{(1)} + \left[J^{(1)}_\beta, J^{(0)}_\alpha\right]\epsilon^{(3)}\right) . \tag{4.112}$$

Using all of these results, we can finally write for the variation of the Lagrangian

$$\begin{aligned}\delta_\epsilon \mathcal{L} = -\frac{1}{4\pi\alpha'}\Bigg[&\delta_\epsilon\gamma^{\alpha\beta}\mathrm{str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) - 2\gamma^{\alpha\beta}\mathrm{str}\left(\left[J^{(1)}_\alpha, J^{(2)}_\beta\right]\epsilon^{(1)} + \left[J^{(3)}_\alpha, J^{(2)}_\beta\right]\epsilon^{(3)}\right) + \\ &+\kappa\,\epsilon^{\alpha\beta}\mathrm{str}\Bigg(\partial_\alpha J^{(3)}_\beta\epsilon^{(1)} - \partial_\alpha J^{(1)}_\beta\epsilon^{(3)} + \left[J^{(3)}_\beta, J^{(0)}_\alpha\right]\epsilon^{(1)} + \left[J^{(3)}_\beta, J^{(2)}_\alpha\right]\epsilon^{(3)} + \\ &+ \left[J^{(1)}_\beta, J^{(2)}_\alpha\right]\epsilon^{(1)} + \left[J^{(1)}_\beta, J^{(0)}_\alpha\right]\epsilon^{(3)}\Bigg) ,\end{aligned} \tag{4.113}$$

where we performed integration by parts and we neglected the total derivatives. In order to reduce this expression, let us give a look at the Maurer-Cartan equations (4.72)

$$2\epsilon^{\alpha\beta}\,\partial_\alpha J_\beta = \epsilon^{\alpha\beta}\left[J_\alpha, J_\beta\right]\,, \tag{4.114}$$

which using $\mathbb{Z}_4$ decomposition can be written as follows

$$2\epsilon^{\alpha\beta}\,\partial_\alpha\left(J^{(0)}_\beta + J^{(1)}_\beta + J^{(2)}_\beta + J^{(3)}_\beta\right) = \epsilon^{\alpha\beta}\left[J^{(0)}_\alpha + J^{(1)}_\alpha + J^{(2)}_\alpha + J^{(3)}_\alpha\right]\,. \tag{4.115}$$

Comparing the terms of grading 1 and 3 of both sides of the equation above, we end up with

$$\epsilon^{\alpha\beta}\partial_\alpha J^{(1)}_\beta = \epsilon^{\alpha\beta}\left[J^{(0)}_\alpha, J^{(1)}_\beta\right] + \epsilon^{\alpha\beta}\left[J^{(2)}_\alpha, J^{(3)}_\beta\right]\,, \tag{4.116}$$

$$\epsilon^{\alpha\beta}\partial_\alpha J^{(3)}_\beta = \epsilon^{\alpha\beta}\left[J^{(0)}_\alpha, J^{(3)}_\beta\right] + \epsilon^{\alpha\beta}\left[J^{(1)}_\alpha, J^{(2)}_\beta\right]\,. \tag{4.117}$$

Using these identities we can write the variation of the Lagrangian as follows

$$\delta_\epsilon\mathcal{L} = -\frac{1}{4\pi\alpha'}\,\delta_\epsilon\gamma^{\alpha\beta}\mathrm{str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) + \frac{1}{\pi\alpha'}\,\mathrm{str}\left(P^{\alpha\beta}_+\left[J^{(1)}_\beta, J^{(2)}_\alpha\right]\epsilon^{(1)} + P^{\alpha\beta}_-\left[J^{(3)}_\beta, J^{(2)}_\alpha\right]\epsilon^{(3)}\right)\,. \tag{4.118}$$

where $P^{\alpha\beta}_\pm$ are defined in (4.95). It is easy to see that the last two terms must vanish because they are simply the projections of the equations of motion we wrote in (4.96) and (4.97), whereas the first term vanishes due to the Virasoro constraints (4.99). However, for the kappa transformations to define a true symmetry, the action should be invariant without the use of the equations of motion, therefore we need to find the appropriate transformation rule for the worldsheet metric $\delta_\epsilon\gamma^{\alpha\beta}$.

We now consider the projection $V^\alpha_\pm$ of any vector V^α as follows

$$V^\alpha_\pm = P^{\alpha\beta}_\pm V_\beta\,. \tag{4.119}$$

Taking into consideration the definition we wrote above, the variation of the Lagrangian takes the following form

$$\delta_\epsilon\mathcal{L} = -\frac{1}{4\pi\alpha'}\,\delta_\epsilon\gamma^{\alpha\beta}\mathrm{str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) + \frac{1}{\pi\alpha'}\,\mathrm{str}\left(\left[J^{(1),\,\alpha}_+, J^{(2)}_{\alpha,\,-}\right]\epsilon^{(1)} + \left[J^{(3),\,\alpha}_-, J^{(2)}_{\alpha,\,+}\right]\epsilon^{(3)}\right)\,. \tag{4.120}$$

We will consider the following ansatz for the components of the fermionic parameter ϵ

$$\epsilon^{(1)} = J^{(2)}_{\alpha,\,-}\kappa^{(1),\,\alpha}_+ + \kappa^{(1),\,\alpha}_+ J^{(2)}_{\alpha,\,-}\,, \tag{4.121}$$

$$\epsilon^{(3)} = J^{(2)}_{\alpha,\,+}\kappa^{(3),\,\alpha}_- + \kappa^{(3),\,\alpha}_- J^{(2)}_{\alpha,\,+}\,, \tag{4.122}$$

where $\kappa_{\pm}^{(k),\alpha}$ are new independent parameters of kappa symmetry transformations which are homogeneous elements of gradation $k = 1, 3$ with respect to Ω. It is possible to represent the traceless even component $J^{(2)}$ as follows

$$J^{(2)} = \begin{pmatrix} m^i\gamma^i & 0 \\ 0 & n^i\gamma^i \end{pmatrix} . \tag{4.123}$$

Here γ^i are the $SO(5)$ Dirac matrices. The coefficients n^i are considered as purely imaginary while m^i are real for $i = 1, \ldots, 4$ and imaginary for $i = 5$. We can further write

$$J^{(2)}_{\alpha,\pm} J^{(2)}_{\beta,\pm} = \begin{pmatrix} m^i_{\alpha,\pm} m^j_{\beta,\pm} \gamma^i\gamma^j & 0 \\ 0 & n^i_{\alpha,\pm} n^j_{\beta,\pm} \gamma^i\gamma^j \end{pmatrix} . \tag{4.124}$$

As a consequence of the property $P_{\pm}^{\alpha\beta} J_{\beta,\mp} = 0$ it is possible to show that the components $J_{\tau,\pm}$ and $J_{\sigma,\pm}$ must be proportional to each other, allowing us to rewrite the expression above as follows

$$\begin{aligned} J^{(2)}_{\alpha,\pm} J^{(2)}_{\beta,\pm} &= \begin{pmatrix} m^i_{\alpha,\pm} m^j_{\beta,\pm} \frac{1}{2}\gamma^i\gamma^j & 0 \\ 0 & n^i_{\alpha,\pm} n^j_{\beta,\pm} \frac{1}{2}\gamma^i\gamma^j \end{pmatrix} \\ &= \begin{pmatrix} m^i_{\alpha,\pm} m^i_{\beta,\pm} & 0 \\ 0 & n^i_{\alpha,\pm} n^i_{\beta,\pm} \end{pmatrix} \qquad (4.125) \\ &= \frac{1}{8}\Upsilon \operatorname{str}\left(J^{(2)}_{\alpha,\pm} J^{(2)}_{\beta,\pm}\right) + \frac{1}{2}\left(m^i_{\alpha,\pm} m^i_{\beta,\pm} + n^i_{\alpha,\pm} n^i_{\beta,\pm}\right) 1_8 . \qquad (4.126) \end{aligned}$$

Here 1_8 is the identity matrix and Υ is the hypercharge matrix defined in (4.53). As you can see in the variation (4.120), upon substitution of our ansatz (4.121) and (4.122) we will have products of $J^{(2)}$'s which according to the expression above can be expressed as a linear combination of two matrices, one of them being the identity matrix and the other being Υ. When substituting all of these results in the variation of the Lagrangian, we find that the term proportional to the identity matrix will drop out, leaving

$$\begin{aligned} \delta_\epsilon \mathcal{L} &= -\frac{1}{4\pi\alpha'} \delta_\epsilon \gamma^{\alpha\beta} \operatorname{str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) + \frac{1}{8\pi\alpha'} \operatorname{str}\left(J^{(2)}_{\alpha,-} J^{(2)}_{\beta,-}\right) \operatorname{str}\left(\Upsilon\left[\kappa_+^{(1),\beta}, J_+^{(1),\alpha}\right]\right) + \\ &\quad + \frac{1}{8\pi\alpha'} \operatorname{str}\left(J^{(2)}_{\alpha,+} J^{(2)}_{\beta,+}\right) \operatorname{str}\left(\Upsilon\left[\kappa_-^{(3),\beta}, J_-^{(3),\alpha}\right]\right) . \end{aligned} \tag{4.127}$$

Thus, it is easy to see that for the Lagrangian to be invariant under kappa transformations, the worldsheet metric must transform as follows

$$\delta_\epsilon \gamma^{\alpha\beta} = \frac{1}{4} \operatorname{str}\left(\Upsilon\left(\left[\kappa_+^{(1),\alpha}, J_+^{(1),\beta}\right] + \left[\kappa_+^{(1),\beta}, J_+^{(1),\alpha}\right] + \left[\kappa_-^{(3),\alpha}, J_-^{(3),\beta}\right] + \left[\kappa_-^{(3),\beta}, J_-^{(3),\alpha}\right]\right)\right) . \tag{4.128}$$

Furthermore, using the fact that the insertion of the hypercharge Υ turns the supertrace into a regular trace of the matrix, and making use of the identity $P_{\pm}^{\alpha\gamma}P_{\pm}^{\beta\delta} = P_{\pm}^{\beta\gamma}P_{\pm}^{\alpha\delta}$, we have

$$\delta_\epsilon \gamma^{\alpha\beta} = \frac{1}{2}\,\mathrm{tr}\left(\left[\kappa_+^{(1),\,\alpha}, J_+^{(1),\,\beta}\right] + \left[\kappa_-^{(3),\,\alpha}, J_-^{(3),\,\beta}\right]\right) . \tag{4.129}$$

We need to make a very important observation here; we have considered that $P_{\pm}^{\alpha\beta}$ are orthogonal projectors throughout the derivation we presented here, that is, the projectors satisfy

$$P_{\pm}^{\alpha\gamma} g_{\beta\gamma} P_{\mp}^{\gamma\delta} = 0 . \tag{4.130}$$

Looking at the definition of the projector operators (4.95), it is easy to note that the orthogonality condition is equivalent to require $\kappa^2 = 1$, and then the parameter κ is allowed to take values of one or minus one only.

Just as we did in Chapter 3, one can use the so-called light cone gauge in order to use the kappa symmetry of the action to gauge away the extra fermionic degrees of freedom. Apart from that, one could find an explicit form of the action by giving an appropriate parametrization. We will not do it here but we refer the interested reader to the original work [24] and the review [18] for more details about these issues.

4.4 Pure spinor superstrings in general backgrounds

We finish this chapter by giving a very short presentation of the so-called Berkovits-Howe action, which is none other than the pure spinor sigma model in curved backgrounds. We will present the action and explain the terms involved in it.

4.4.1 The Berkovits-Howe superstring

The sigma model action for Type II pure spinor superstrings in a generic supergravity background is given by [25]

$$\begin{aligned} S \;=\; & \frac{1}{2\pi\alpha'}\int d^2z \left[\frac{1}{2}\left(G_{MN}(Z) + B_{MN}(Z)\right)\partial Z^M \bar{\partial} Z^N + \right. \\ & + \big(E_M{}^{\alpha}(Z)\, d_\alpha + \Omega_{M\,\alpha}{}^{\beta}(Z)\,\lambda^\alpha\omega_\beta\big)\bar{\partial}Z^M + \big(E_M{}^{\hat{\alpha}}(Z)\,\hat{d}_{\hat{\alpha}} + \hat{\Omega}_{M\,\hat{\alpha}}{}^{\hat{\beta}}(Z)\,\hat{\lambda}^{\hat{\alpha}}\hat{\omega}_{\hat{\beta}}\big)\partial Z^M + \\ & + C_\alpha^{\beta\hat{\gamma}}(Z)\lambda^\alpha\omega_\beta\,\hat{d}_{\hat{\gamma}} + \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}(Z)\hat{\lambda}^{\hat{\alpha}}\hat{\omega}_{\hat{\beta}}\,d_\gamma + \\ & \left. + P^{\alpha\hat{\beta}}(Z) d_\alpha \hat{d}_{\hat{\beta}} + S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}(Z)\lambda^\alpha\omega_\beta\hat{\lambda}^{\hat{\gamma}}\hat{\omega}_{\hat{\delta}} + \frac{1}{2}\alpha'\Phi(Z)R^{(2)}\right] + S_\lambda + S_{\hat{\lambda}} . \end{aligned} \tag{4.131}$$

Here the action is written in conformal gauge, so z and $\bar{z}$ are the worldsheet coordinates. On the other hand $M = (m, \mu, \hat{\mu})$ are curved superspace indices and $\mathcal{A} = (a, \alpha, \hat{\alpha})$ are tangent superspace indices. The independent worldsheet fields in this action are $Z^M = (X^m, \theta^\mu, \hat{\theta}^{\hat{\mu}})$ and $(d_\alpha, \hat{d}_{\hat{\alpha}})$ in the matter sector, and $(\omega_\alpha, \lambda^\alpha, \hat{\omega}_{\hat{\alpha}}, \hat{\lambda}^{\hat{\alpha}})$ in the ghost sector. If we vary the action with respect to these fields one could find their respective equations of motion.

The ghost content S_λ and $S_{\hat{\lambda}}$ are given by the same expressions we considered in the flat space case in Chapter 3, that is they are given by

$$S_\lambda + S_{\hat{\lambda}} = \frac{1}{2\pi\alpha'} \int d^2z \left(\omega_\alpha \bar{\partial} \lambda^\alpha + \hat{\omega}_{\hat{\alpha}} \partial \hat{\lambda}^{\hat{\alpha}} \right) . \tag{4.132}$$

If we substitute the factor $S_\lambda + S_{\hat{\lambda}}$ that we wrote above into the action (4.131) then we could define the covariant derivative on λ ($\hat{\lambda}$) as the pullback of the left moving (right moving) spin connection ${\Omega_\alpha}^\beta = dZ^M {\Omega_{M\alpha}}^\beta$ (${\hat{\Omega}_{\hat{\alpha}}}^{\hat{\beta}} = dZ^M {\hat{\Omega}_{M\hat{\alpha}}}^{\hat{\beta}}$) as follows

$$(\nabla\lambda)^\alpha = \partial\lambda^\alpha + {\Omega_\beta}^\alpha \lambda^\beta , \qquad (\nabla\hat{\lambda})^{\hat{\alpha}} = \partial\hat{\lambda}^{\hat{\alpha}} + {\hat{\Omega}_{\hat{\beta}}}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} . \tag{4.133}$$

The first line in the action (4.131) is just the standard Type II Green-Schwarz action we saw in (4.1), whereas the other lines are needed for BRST invariance. $R^{(2)}$ is the worldsheet curvature and G, B, P, C, $\hat{C}$, S and Φ are background superfields. G_{MN} is the background metric superfield which can be written as $G_{MN} = {E_M}^a {E_N}^b \eta_{ab}$; the background superfield B_{AB} is the superspace 2-form potential; the lowest components of $C_\alpha^{\beta\hat{\gamma}}$ and $\hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}$ are related to the gravitini and dilatini; the lowest component of $P^{\alpha\hat{\beta}}$ is the Ramond-Ramond bispinor field strength; $S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}$ is related to the curvature associated to the connections Ω and $\hat{\Omega}$; ${E_M}^{\mathcal{A}}$ are the supervielbeins and the Fradkin-Tseytlin term $\frac{1}{2}\alpha'\Phi(Z)R^{(2)}$ describes the coupling of the dilaton to the worldsheet curvature $R^{(2)}$. If the Fradkin-Tseytlin term is omitted, (4.131) is the most general action with classical worldsheet conformal invariance and zero (left-right)-moving ghost number which can be constructed from the Type II worldsheet variables.

Berkovits and Howe showed that, up to the lowest order in α', the conservation of the BRST currents $\bar{\partial}(\lambda^\alpha d_\alpha) = \partial(\hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}) = 0$ and nilpotency of the BRST charges, which are postulated to have the same structure as in the flat space case, that is $Q \sim \oint \lambda^\alpha d_\alpha$ and $\hat{Q} \sim \oint \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}$, actually imply Type II supergravity equations of motion for the background superfields.

4.4.2 The pure spinor superstring in $AdS_5 \times S^5$

We can construct the Pure Spinor superstring action in $AdS_5 \times S^5$ in a very similar way to the flat space case that we studied in Chapter 3. Before doing this, let us write the

Metsaev-Tseytlin action we presented in (4.76) in conformal gauge. We consider the following definitions

$$J_z^a \equiv J^{(k)}\,; \qquad J_{\bar{z}}^{(k)} \equiv \bar{J}^{(k)}\,. \tag{4.134}$$

We also need to remember that the only non-vanishing components of the worldsheet metric in $z, \bar{z}$ coordinates are $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$, whereas for the Levi-Civita tensor we have $\epsilon^{z\bar{z}} = -\epsilon^{\bar{z}z} = 2i$. Using this information we can easily write

$$S_{GS} = \frac{1}{\pi\alpha'}\int d^2z\ \mathrm{str}\left[\frac{1}{2}J^{(2)}\bar{J}^{(2)} - \frac{1}{4}\left(J^{(3)}\bar{J}^{(1)} - J^{(1)}\bar{J}^{(3)}\right)\right]\,, \tag{4.135}$$

where we used $\kappa = -1$. We recall the relation between the Siegel and Green-Schwarz action we established in (3.75), but now such a relation reads

$$S_{matter} = S_{GS} + \frac{1}{\pi\alpha'}\int d^2z\ \left(d_\alpha \bar{J}^{\alpha\,(1)} + d_{\hat{\alpha}} J^{\hat{\alpha}\,(3)} + P^{\alpha\hat{\alpha}} d_\alpha d_{\hat{\alpha}}\right)\,. \tag{4.136}$$

It is not hard to realize that the first two terms in the integral above break the kappa symmetry of the Green-Schwarz action. The third term in the integral is a coupling term to the Ramond-Ramond superfield $P^{\alpha\hat{\alpha}}$. When the Ramond-Ramond superfield is invertible, d_α and $\hat{d}_{\hat{\beta}}$ become auxiliary fields and their equations of motion can be written as follows

$$d_\alpha = J^{\hat{\alpha}\,(3)}\eta_{\alpha\hat{\alpha}}\,, \qquad \hat{d}_{\hat{\alpha}} = -\bar{J}^{\alpha\,(1)}\eta_{\alpha\hat{\alpha}}\,. \tag{4.137}$$

Furthermore, $P^{\alpha\hat{\beta}}$ can be taken as

$$P^{\alpha\hat{\beta}} = \eta^{\alpha\hat{\beta}}\,, \tag{4.138}$$

where the matrix $\eta^{\alpha\hat{\beta}}$ has rank sixteen and is numerically equal to the identity matrix. We can use the equations of motion (4.137) to integrate out the auxiliary fields and we obtain

$$\begin{aligned} S_{matter} &= S_{GS} + \frac{1}{\pi\alpha'}\int d^2z\ \mathrm{str}\left(J^{(3)}\bar{J}^{(1)}\right) \\ &= \frac{1}{\pi\alpha'}\int d^2z\ \mathrm{str}\left(\frac{1}{2}J^{(2)}\bar{J}^{(2)} + \frac{3}{4}J^{(3)}\bar{J}^{(1)} + \frac{1}{4}J^{(1)}\bar{J}^{(3)}\right)\,. \end{aligned} \tag{4.139}$$

To construct the pure spinor action, we need to add a ghost term to the action (4.139). The ghost fields must take values in the fermionic eigenspaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(3)}$, hence we can define them as $\lambda^{(1)}$ and $\hat{\lambda}^{(3)}$. As we saw in Chapter 3, in flat space the momentum has opposite chirality with respect to its conjugate field, so that the coupling ghost-momentum is Lorentz invariant. Analogously in curved space we take each momentum in a different

eigenspace with respect to its ghost, that is, $\omega^{(3)} \in \mathcal{H}^{(3)}$ is the conjugate momentum of $\lambda^{(1)}$ and $\hat{\omega}^{(1)} \in \mathcal{H}^{(1)}$ is the conjugate momentum of $\hat{\lambda}^{(3)}$. To construct the pure spinor ghost action in $AdS_5 \times S^5$, we have to substitute the canonical derivative with the covariant one. Since we interpreted $J^{(0)}$ as the $SO(4,1) \times SO(5)$ gauge field, it is natural to define

$$\nabla \equiv \partial + \left[J^{(0)}, \dots\right] , \qquad \bar{\nabla} \equiv \bar{\partial} + \left[\bar{J}^{(0)}, \dots\right] . \tag{4.140}$$

Then the pure spinor ghost action can be written in the following way

$$S_{ghost} = \frac{1}{\pi\alpha'} \int d^2z \ \mathrm{str} \left(\omega^{(3)}\bar{\nabla}\lambda^{(1)} + \hat{\omega}^{(1)}\nabla\hat{\lambda}^{(3)}\right) . \tag{4.141}$$

Finally the pure spinor action $S_{PS} = S_{matter} + S_{ghost}$ in $AdS_5 \times S^5$ reads

$$\begin{aligned} S_{PS} \ = \ & \frac{1}{\pi\alpha'} \int d^2z \ \mathrm{str} \Bigg[\frac{1}{2}J^{(2)}\bar{J}^{(2)} + \frac{3}{4}J^{(3)}\bar{J}^{(1)} + \frac{1}{4}J^{(1)}\bar{J}^{(3)} + \\ & + \omega^{(3)}\bar{\nabla}\lambda^{(1)} + \hat{\omega}^{(1)}\nabla\hat{\lambda}^{(3)} - N\hat{N}\Bigg] . \end{aligned} \tag{4.142}$$

Here we added a current-current coupling term in the action where N and $\hat{N}$ are the Lorentz generators in the ghost sector and are given by

$$N = -\left\{\omega^{(3)}, \lambda^{(1)}\right\} , \qquad \hat{N} = -\left\{\hat{\omega}^{(1)}, \hat{\lambda}^{(3)}\right\} , \tag{4.143}$$

where $\lambda^{(1)}$ and $\hat{\lambda}^{(3)}$ satisfy the following $SO(4,1) \times SO(5)$ pure spinor constraints

$$\left\{\lambda^{(1)}, \lambda^{(1)}\right\} = \lambda^{\alpha}\gamma^{m}_{\alpha\beta}\lambda^{\beta} = 0 , \qquad \left\{\hat{\lambda}^{(3)}, \hat{\lambda}^{(3)}\right\} = \hat{\lambda}^{\hat{\alpha}}\gamma^{m}_{\hat{\alpha}\hat{\beta}}\hat{\lambda}^{\hat{\beta}} = 0 . \tag{4.144}$$

It is important to note that the matter sector of the pure spinor superstring action (4.142) is not kappa symmetric since the Green-Schwarz action (4.76) is the unique such action. Another important difference is that the pure spinor action does not produce Virasoro constraints even though it is written in conformal gauge. In the pure spinor formalism, both the kappa symmetry and the Virasoro constraints are replaced by BRST symmetry.

Chapter 5

Final remarks

The main purpose of this dissertation work was to review the very basic tools to start studying superstrings in curved backgrounds. We made a kind of deep review of the classical bosonic string treating it as a two dimensional conformal field theory. We put a lot of emphasis in some computation details that are hard to find in the literature. Even though we have not discussed the quantum spectrum of the theory, a pedagogical introduction to the BRST quantization method was presented.

Our study of superstrings in flat space was focused on the Green-Schwarz and the pure spinor formalisms. The former was constructed by generalizing the superparticle action and we explained the difficulties involved with the covariant quantization of this model. The latter was introduced by requiring some consistency conditions for the pure spinor ghost action. As you may have noticed, we have not talked that much about amplitudes, even though this is one the most important aspects of the pure spinor formalism, that is because our main objective was to formulate a superstring action in $AdS_5 \times S^5$ rather than the calculation of amplitudes.

We devoted the last chapter to the study of superstrings in curved backgrounds giving especial attention to the very important case of $AdS_5 \times S^5$. The study we did was a review based in the works by Frolov [18] and Mazzucato [17]. We offered some computations details, but the interested reader should definitely consult those reviews for a deeper study of the superstring in AdS backgrounds.

I want to finish this work by stressing that the importance of this dissertation relies on the amount of calculation details explained in it, for this reason I hope this work can can provide the basic elements for a comprehensive introduction to the pure spinor formalism in flat space and the superstring in curved backgrounds.

Appendix A

Notations and conventions

A.1 Indices and coordinates

A.1.1 Bosonic strings

$$\mu, \nu, \rho, \sigma = 0, 1, 2, 3, \ldots, D-1 \qquad \text{Cartesian space-time indices}\,, \tag{A.1}$$

$$\sigma^1, \sigma^2 \qquad \text{real worldsheet coordinates}\,, \tag{A.2}$$

$$z, \bar{z} \qquad \text{complex worldsheet coordinates}\,, \tag{A.3}$$

$$m, n = -\infty, \infty \qquad \text{Virasoro modes indices}\,. \tag{A.4}$$

A.1.2 Superstrings

In flat space

$$m, n = 0, 1, 2, 3, \ldots, 9 \qquad \text{flat space-time indices}\,, \tag{A.5}$$

$$a, b = 1, 2 \qquad \text{worldsheet coordinates (Green-Schwarz)}\,, \tag{A.6}$$

$$\alpha, \beta = 1, \ldots, 32 \qquad \text{fermionic coordinates}\,, \tag{A.7}$$

$$I, J = 1, \ldots, 8 \qquad \text{transverse light cone coordinates}\,, \tag{A.8}$$

$$a, b = 1, \ldots, 5 \qquad U(5) \text{ variables (pure spinor)}\,, \tag{A.9}$$

In curved backgrounds

$$M = (m, \mu, \hat{\mu}) \qquad \text{superspace coordinates}\,, \tag{A.10}$$
$$m, n = 0, \ldots, 9 \qquad \text{superspace bosonic coordinates}\,, \tag{A.11}$$
$$\mu, \hat{\mu} = 1, \ldots, 16 \qquad \text{superspace fermionic coordinates}\,, \tag{A.12}$$
$$i, j = 1, 2 \qquad \text{worldsheet coordinates}\,, \tag{A.13}$$
$$\mathcal{A} = (a, \alpha, \hat{\alpha}) \qquad \text{tangent superspace coordinates}\,, \tag{A.14}$$
$$a, b = 0, \ldots, 9 \qquad \text{tangent superspace bosonic coordinates}\,, \tag{A.15}$$
$$\alpha, \beta, \gamma, \hat{\alpha}, \hat{\beta}, \hat{\gamma} = 1, \ldots, 16 \qquad \text{tangent superspace fermionic coordinates}\,, \tag{A.16}$$
$$(0), (1), (2), (3) \qquad \text{grading with respect to } \Omega\,. \tag{A.17}$$

A.2 Gamma matrices in 10 dimensions

The convention for the flat metric that we use in this dissertation is given by $\eta_{mn} = \mathrm{diag}(-1, +1, +1, \ldots, +1)$. In light cone coordinates, the only non-vanishing components of the metric are $\eta_{II} = +1$ and $\eta_{+-} = \eta_{-+} = -1$.

The 32×32 Dirac matrices in ten dimensions Γ^m, where $m = 0, \ldots, 9$ in Minkowski space and $m = 1, \ldots, 10$ in Euclidean space, satisfy the $SO(10)$ Clifford algebra

$$\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}\,. \tag{A.18}$$

In the Weyl representation, the gamma matrices are off-diagonal in 16×16 blocks, then

$$\Gamma^m = \begin{pmatrix} 0 & (\gamma^m)^{\alpha\beta} \\ (\gamma^m)_{\alpha\beta} & 0 \end{pmatrix}\,. \tag{A.19}$$

We will consider the following explicit form for the γ matrices

$$(\gamma^0)^{\alpha\beta} = -(\gamma^0)_{\alpha\beta} = \begin{pmatrix} 1_8 & 0 \\ 0 & 1_8 \end{pmatrix}, \tag{A.20}$$

$$(\gamma^9)^{\alpha\beta} = +(\gamma^9)_{\alpha\beta} = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix}, \tag{A.21}$$

$$(\gamma^i)^{\alpha\beta} = +(\gamma^i)_{\alpha\beta} = \begin{pmatrix} 0 & \sigma^i_{a\dot{a}} \\ \sigma^i_{b\dot{b}} & 0 \end{pmatrix}, \tag{A.22}$$

$$(\gamma^+)^{\alpha\beta} = -(\gamma^-)_{\alpha\beta} = \sqrt{2}\begin{pmatrix} 1_8 & 0 \\ 0 & 0 \end{pmatrix}, \tag{A.23}$$

$$(\gamma^-)^{\alpha\beta} = -(\gamma^+)_{\alpha\beta} = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 0 & 1_8 \end{pmatrix}. \tag{A.24}$$

Where we have defined for both chiralities

$$\gamma^{\pm} = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^9)\,. \tag{A.25}$$

The chirality matrix can be expressed as $\Gamma = i\,\Gamma^1\Gamma^2\ldots\Gamma^{10}$. However, if we consider $SO(9,1)$, it can be expressed as $\Gamma = \Gamma^0\Gamma^1\ldots\Gamma^9$.

Appendix B

Properties of manifolds

If we consider the differentiable manifold $\mathcal{M}$ and the Lie group G with identity e, we can define the *action* of G on $\mathcal{M}$ by the following map $(g,p) \in G \times \mathcal{M} \to gp \in \mathcal{M}$ such that

$$ep = pe = p\,, \tag{B.1}$$

$$g_1(g_2 p) = (g_1 g_2)p\,, \tag{B.2}$$

where we have defined $\forall p \in \mathcal{M}$ and $\forall g_1, g_2 \in G$. The action is *transitive* if $\forall p_1, p_2 \in \mathcal{M}$ there is $g \in G$ so that $gp_1 = p_2$. The *orbit* of $p \in \mathcal{M}$ under the action of G is the subset Gp of $\mathcal{M}$ given by

$$Gp = \{gp : \, g \in G\}\,. \tag{B.3}$$

If G acts transitively on $\mathcal{M}$, then $Gp = \mathcal{M}$. The *little group* (or *isotropy group*) of $p \in \mathcal{M}$ is the subgroup H_p of G so that

$$H_p = \{g \in G : \, gp = p\}\,. \tag{B.4}$$

If $H \subset G$ is a subgroup and $g \in G$, the subset $gH = \{gh \in G : \, h \in H\}$ is the *left coset* of H. Analogously one can define the *right coset* Hg. The set of all gH in G is called the *quotient space*

$$\frac{G}{H} = \{gH \subset G : \, g \in G\}\,, \tag{B.5}$$

and it admits the structure of a group if and only if H is a *normal* subgroup, that is if $gH = Hg$; $\forall g$. However if G is a Lie group, G/H admits a differentiable manifold structure, and we call it *coset manifold*.

If a Lie group G acts on $\mathcal{M}$ transitively, and we choose as subgroup of G the little group H_p of some $p \in \mathcal{M}$, the coset manifold G/H_p is *homeomorphic* to $\mathcal{M}$, that is there is a continuous map one to one between G/H_p and $\mathcal{M}$.

B.1 The n-sphere

If we consider a $(n+1)$-dimensional flat bulk of coordinates (y^μ, y^n), with $\mu = 0, \ldots, n-1$, provided with the following metric

$$ds^2_{bulk} = \eta_{\mu\nu} dy^\mu dy^\nu + (dy^n)^2 \,, \qquad \eta_{\mu\nu} = \mathrm{diag}(\overbrace{+1, \ldots, +1}^{n}) \,, \tag{B.6}$$

the n-dimensional sphere is defined by

$$\eta_{\mu\nu} y^\mu y^\nu + (y^n)^2 = R^2 \,, \tag{B.7}$$

where $R \in \mathbb{R}$ is the curvature radius. The group $SO(n+1)$ is transitive on the sphere, and the group of rotations $SO(n)$ around a point does not shift it, then one can write

$$S^n \cong \frac{SO(n+1)}{SO(n)} \,, \tag{B.8}$$

where we can note that

$$\dim\left(\frac{SO(n+1)}{SO(n)}\right) = \dim SO(n+1) - \dim SO(n) \tag{B.9}$$

$$= \frac{1}{2}(n+1)n - \frac{1}{2}n(n-1) = n \,. \tag{B.10}$$

B.2 The anti-de Sitter space

If we now consider a $(n+1)$-dimensional flat bulk provided with the following metric

$$ds^2_{bulk} = \eta_{\mu\nu} dy^\mu dy^\nu + (dy^n)^2 \,, \qquad \eta_{\mu\nu} = \mathrm{diag}(+1, \overbrace{-1, \ldots, -1}^{n-1}) \,, \tag{B.11}$$

then the n-dimensional Anti-de Sitter space is defined as the hyperboloid

$$\eta_{\mu\nu} y^\mu y^\nu + (y^n)^2 = R^2 \,. \tag{B.12}$$

It corresponds in Lorentzian signature to the Lobachevsky space in Euclidean signature. AdS_n is the orbit of the group $SO(n-1,2)$, that is this group acts transitively on AdS. On the other hand, $SO(n-1,1)$ is the little group with respect to any point of AdS, so

$$AdS_n \cong \frac{SO(n-1,2)}{SO(n-1,1)} \,, \tag{B.13}$$

and we can write

$$\dim\left(\frac{SO(n-1,2)}{SO(n-1,1)}\right) = \dim SO(n-1,2) - \dim SO(n-1,1)$$

$$= \frac{1}{2}(n+1)n - \frac{1}{2}n(n-1) = n \,. \tag{B.14}$$

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