UNIVERSIDADE DE SÃO PAULO
Instituto de Física - Departamento de Física Matemática

Gauge and Matter Fields on a Lattice - Generalizing Kitaev's Toric Code Model

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Campos de Gauge e Matéria na Rede - Generalizando o *Toric Code*

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Abstract

Topological phases of matter are characterized for having a topologically dependent ground state degeneracy, anyonic quasi-particle bulk excitations and gapless edge excitations. Different topologically ordered phases of matter can not be distinguished by te usual *Ginzburg-Landau* scheme of symmetry breaking. Therefore, a new mathematical framework for the study of such phases is needed. In this dissertation we present the simplest example of a topologically ordered system, namely, the **Toric Code** (TC) introduced by A. Kitaev in [1]. Its ground state is 4-fold degenerate when embedded on the surface of a torus and its elementary excited states are interpreted as quasi-particle anyons. The TC is a particular case of a more general class of lattice models known as **Quantum Double Models** (QDMs) which can be interpreted as an implementation of $(2+1)$ Lattice Gauge Theories in the Hamiltonian formulation with discrete gauge group G. We generalize these models by the inclusion of matter fields at the vertices of the lattice. We give a detailed construction of such models, we show they are exactly solvable and explore the case when the gauge group is set to be the abelian $\mathbb{Z}_2$ cyclic group and the matter degrees of freedom to be elements of a 2-dimensional vector space V_2. Furthermore, we show that the ground state degeneracy is not topologically dependent and obtain the most elementary excited states.

Resumo

Fases topológicas da matéria são caracterizadas por terem uma degenerescência do estado fundamental que depende da topologia da variedade em que o sistema físico é definido, além disso apresentam estados excitados no interior do sistema que são interpretados como sendo quase-partículas com estatística de tipo *anyonica*. Estes sistemas apresentam também excitações sem gap de energia em sua borda. Fases topologicamente ordenadas distintas não podem ser distinguidas pelo esquema usual de quebra de simetria de *Ginzburg-Landau*. Nesta dissertação apresentamos como exemplo o modelo mais simples de um sistema com *Ordem Topológica*, a saber, o **Toric Code** (TC), introduzido originalmente por A. Kitaev em [1]. O estado fundamental deste modelo apresenta degenerescência igual a 4 quando incorporado à superfície de um toro. As excitações elementares são interpretadas como sendo quase-partículas com estatística do tipo anyonica. O TC é um caso especial de uma classe mais geral de models chamados de **Quantum Double Models** (QDMs), estes modelos podem ser entendidos como sendo uma implementação de Teorias de gauge na rede em $(2+1)$ dimensões na formulação Hamiltoniana, em que os graus de liberdade vivem nas arestas da rede e são elementos do grupo de gauge G. Nós generalizamos estes modelos com a inclusão de campos de matéria nos vértices da rede. Também apresentamos uma construção detalhada de tais modelos e mostramos que eles são exatamente solúveis. Em particular, exploramos o modelo que corresponde à escolher o grupo de gauge como sendo o grupo cíclico $\mathbb{Z}_2$ e os graus de liberdade de matéria como sendo elementos de um espaço vetorial bidimensional V_2. Além disso, mostramos que a degenerescência do estado fundamental não depende da topologia da variedade e obtemos os estados excitados mais elementares deste modelo.

Contents

List of Figures

Chapter 1

Introduction

Matter as we observe it on our everyday life, can take many different states. The most known of them being gas, liquid and solid; besides these *states of matter* we can distinguish other more intrincate states that can occur in different situations and scales. For instance: plasmas, Bose-Einstein condensates, quantum spin liquids or superfluids to mention a few.

States of matter are, in general, classified into phases that can be connected to each other via phase transitions. This can be seen as follows: different states of matter are distinguished by their internal structure (order), to visualize this we can take the example of a solid at some finite temperature where the atoms are arranged in a regular (or almost regular) pattern depending on its constituents, their interactions and the external conditions such as pressure and temperature. If we choose to vary some of these conditions, say, the temperature, eventually the *crystal order* will be destroyed and the solid will suffer a transition into a *liquid phase* where the motion of the atoms is now less correlated. If the temperature continues to be raised the system will again go through a transition into a very disorderd phase, namely, it will become a gas, where the motion of an atom hardly depends on the motion of the other constituents.

Thus, phases of matter in principle could be classified by means of the phase transitions, under this scheme different states or phases of matter have different internal structure. One key step in order to develop a general theory that could ultimately classify these phases of matter was the realization that the internal orders of a system are related to the symmetries of its elementary constituents. As a material undergoes a phase transition the internal symmetries of the system change. This is the fundamental idea in what is known as the *Ginzburg-Landau* theory of phase transitions [2, 3, 4] that was originally developed to describe the transition to a superconductor phase of matter by means of a local order parameter and its fluctuations. For a long time it was thought that this theory could describe all phases of matter and their phase transitions.

In the 1980's, a major discovery regarding a system of electrons confined on an interface of two different semiconductors was made, creating what is known as *2-dimensional electron gas* (2DEG), which when subject to strong magnetic fields at very low temperatures [5] forms a new kind of order, that although being a strongly correlated state it is a special kind of liquid called *quantum liquid*. These states are characterized by a quantity coined *filling fraction* that measures the ratio of the electron density and the flux quanta of the external magnetic field that is being applied, this is

$$\nu = \frac{nhc}{eB}, \tag{1.1}$$

where n is the electron density. Those states for which this quantity is an integer number are called **Integral Quantum Hall** states (IQH) wheras those states for which ν is a fractional number are, correspondingly, called **Fractional Quantum Hall** states (FQH). While the former can be understood from the Landau level structure[1] the latter requires an entirely new theory. The theoretical study of the FQH states was started by Laughlin [6]. The novelty brought by the FQH states is that they show internal orders or "patterns" that do not have any relation with any kind of symmetries (or the breaking of them) and thus cannot be described by the usual Ginzburg-Landau symmetry-breaking scheme. These patterns consist on a highly correlated motion of the electrons around each other such that they do their own cyclotron motion in the first Landau level, an electron always takes integer steps to go around another neighboring electron and they tend to be apart from each other as much as possible (which makes the fluid an incompressible one). It is this global motion pattern that corresponds to the **topological order** in FQH states [7]. Aditionally FQH states have a very special feature, as their ground state is degenerate and this degeneracy depends on the topology of the space [8, 9, 10] that can not be modified by perturbations or impurities [7], for instance the Laughlin state with $\nu = 1/q$ has a denegeracy of q^g where g is the genus of the manifold the system is embedded into; thus, only a change in the internal pattern or topological order (implying a change on ν for the cited state) can induce a change in the ground state degeneracy, since topological order is a property of the ground state of the system.

The features of topologically ordered systems are not restricted to the topology dependent ground state degeneracy. The excitations of such systems exhibit characteristic properties such as the fractionalization of the charge and anyonic statistics. In particular, for the FQH states, which arise in systems whose constituents are electrons each one with charge e, the excitations carry a charge that is a fraction of e. Although this might seem to be disconnected with the topological order at first glance, it is closely related to the degeneracy as it can be shown that fractionalization implies the degeneracy of the ground state [11, 12]. Another feature involving the quasi particle excitations of such systems is that they obbey exotic statistics. In 3 spatial dimensions it is known that the quantum states of identical particles behave either as bosons or fermions under an exchange of a pair, even though this fact may seem simple it is fundamental for the understanding of nature, and it is at the root of the classification of elements as we know them as the Pauli exclusion principle holds only for fermions. Nevertheless, in two dimensional systems, such as the FQH states, there are new possibilities for quantum statistics that interpolate continuously between those of bosons and fermions. Under an exchange of two quasiparticles the quantum state can acquire and overall phase $e^{i\theta}$, where the special cases $\theta = 0, \pi$ correspond to the bosonic and fermionic statistic respectively. The **statistical angle** θ can take different values, and the particles obeying these generalized statistics are called **anyons**[13, 14, 15, 16, 17, 18].
Another phenomenon related to topological order can be exemplified through the excitations arising at the edges of the FQH liquids. The excitations on the bulk of the system are gapped, meaning there is a finite energy difference between the ground state and the elementary excited states. Nevertheless, in the boundary of the system (the FQH liquid)

[1]The motion of a single electron subject to a magnetic field consists in a circular orbit which is quantized (due to the wave-particle duality) in terms of the wavelenght of the electron, so if an electron takes n steps to go around a circular orbit we say it is in its n^{th} Landau level.

gappless excitations arise [1]. The structure of the edge excitations depend on the bulk topological order, as for different types of order in the bulk there are different edge excitations structures, that can be understood as surface (edge) waves propagating along the border of the FQH liquid [19, 20].

Thus, FQH liquids are very different from any other state of matter because they ordered in a different way, they contain topological order. Hence there is a need for a general theory of topological phases and consequently the need for a mathematical framework that could ultimately characterize and classify these topological phases of matter. In the past few years there has been a major interest on the study of these phases of matter via a detailed analysis of exactly solvable lattice models that exhibit the features of having **topological order**. The simplest example is the so called **Toric Code** model introduced by A. Kitaev in [1] which is constructed as a many body interacting system defined over a 2-dimensional lattice. It exhibits the features of a topologically order system as its ground state is 4-fold degenerate when the lattice is embedded on the surface of a Torus, hence part of its name. The degeneracy is protected from local perturbations that come as the elementary excitations of the model. These elementary excited states can be interpreted as quasi-particle anyonic excitations located at the vertices and faces of the lattice, they display both bosonic and fermionic statistics when braided among themselves. This model can be interpreted as a particular lattice gauge theory [21] where the gauge group is the abelian $\mathbb{Z}_2$ group [22]. Furthermore, for any finite group G, in [1] Kitaev introduces a more general class of models called **Quantum Double Model** (QDM) defined through a Hamiltonian that is written as a sum of mutually commuting projectors [23, 24, 25, 26]. The elementary excitations of this models are anyons whose fusion and braiding properties depend on the specific choice of the group G giving rise to the possibility of having non-abelian anyons that can be used to implement a fault-tolerant quantum computation process [27, 28, 29, 30], where unitary transformations are obtained by the braiding of anyons and the final measurement is performed by the joining of pairs of excitations. Moreover, a large class of topological orders were identified by the systematic construction of the so called **string-net** models [20, 31] where it is shown that each topological phase is associated to a *fusion category*. The QDM being a subclass of these models as shown in [32].

The QDM are a direct implementation of **Lattice Gauge Theories** in $(2+1)$ dimensions with a finite and discrete gauge group G in the Hamiltonian formalism [21, 26, 33], where the gauge fields are degrees of freedom living on the edges of the lattice $\mathcal{L}$ and they are elements of the group algebra $\mathbb{C}(G)$. In general, a $(2+1)$ lattice gauge theory is described by a partition function defined on $\mathcal{L}$ by:

$$Z_G = \sum_{\text{conf.}} e^{-\beta S(\text{conf.})}, \tag{1.2}$$

where β is a constant and $S(\text{conf})$ is the gauge invariant action of the theory defined for a fixed configuration of the gauge degrees of freedom on the lattice [34, 35, 36]. This action is associated to a quantity called *holonomy* U_f of a plaquette (face) which is defined to be the product of the gauge degrees of freedom along the face starting and ending at the same vertex (following a given orientation) although this quantity is not gauge invariant,

[1] A *gapless* excitation is such that its energy goes to zero as its momentum goes to zero, or equivalently, only an infinitesimal amount of energy is required to go from the ground state to an excited state.

the action, which has to be invariant under gauge transformations is defined in terms of a class function $T[U_f]$[1],

$$S_{\text{conf.}} = \sum_f (T[U_f] + T[U_f^{-1}]), \tag{1.3}$$

such that the partition function defined in Eq.(1.2) can be written as:

$$Z_G = \sum_{\text{conf.}} \prod_{f\in\mathcal{L}} M(f), \tag{1.4}$$

where $M(f) = e^{-\beta(T[U_f]+T[U_f^{-1}])}$ can be thought as an statistical weight associated to the faces of the lattice $\mathcal{L}$ where we added the inverse orientation holonomy U_f^{-1} such that the action is invariant under an overall change of plaquette orientation.

Our approach to the phenomena and models encompassed by the term **Topological Order** is based on the fact that a large class of lattice models, lattice gauge theories being a sub-class of these, can be described by an algebraic structure on a vector space $\mathcal{A}$ over a field $\mathbb{K}$. This algebraic structure consists on a multiplication map $m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$, a co-multiplication given by $\Delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ and an endomorphism $S : \mathcal{A} \to \mathcal{A}$ such that $S^2 \equiv \mathbb{1}_{\mathcal{A}}$. Moreover, we assume that the multiplication and co-multiplication map possess a unit η and a co-unit ϵ, respectively. The structure constants of these maps can be regarded as being tensors $m_i^{jk} \in \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}^*$, $\Delta_a^{bc} \in \mathcal{A}\otimes\mathcal{A}^*\otimes\mathcal{A}^*$ and $S_a^b \in \mathcal{A}\otimes\mathcal{A}^*$. By associating contractions of these tensors to the faces and edges of the lattice it is possible to construct a *partition function* that can reproduce those of lattice gauge theories in certain cases, where the statistical weights $M(f)$ are associated to the m tensors, the Δ tensors are associated to the edges of the lattice ensuring that the gauge degrees of freedom on edges that are shared by two plaquettes agree on the same variables and the S tensors play the role of a change of orientation such that it squares to identity. In [37, 38] it is shown that when the set $(\mathcal{A}, m, \Delta, S, \eta, \epsilon)$ defines a **Hopf Algebra** the *partition function* $Z^{\text{top}}(\mathcal{M}, \mathcal{A})$. resulting from the contraction of the aforementioned tensors as a **Tensor Network** [39] is a topological invariant, in the sense that it does not depend on the details of the lattice $\mathcal{L}$ discretizing the manifold $\mathcal{M}$, thus defining the partition function of a Topological Lattice Field Theory, such invariant is known as **Kuperberg Invariant**. The partition functions obtained following this procedure can be generalized by the introduction of parameters in the form of central elements of the algebra $z \in \mathcal{A}$ and the dual algebra $z^* \in \mathcal{A}^*$, respectively. In particular, if $z = \eta$ and $z^* = \epsilon$ are the unit and co-unit of the algebra the partition function $Z(\mathcal{A}, \eta, \epsilon)$ reproduces the Kuperberg invariant. Furthermore, in [40] it is shown that, departing from the same algebraic structure it is possible to describe a Lattice Gauge Theory where the gauge group is $\mathbb{Z}_2$.

In [41] the vector space $\mathcal{A}$ is set to be the group algebra $\mathbb{C}(G)$ and the QDM Hamiltonian is obtained from the partition function $Z(\mathbb{C}(G), z, z^*)$ of a three dimensional Lattice Gauge Theory. The procedure of obtaining such Hamiltionans relies on the fact that we can associate the partition function to a transfer matrix T [21], this is,

$$Z(\mathbb{C}(G), z, z^*) = tr(T^N), \tag{1.5}$$

[1] A class function on a group is such that it is invariant under a conjugation transformation, for instance, if $f(g) = f(h^{-1}gh)$, where $g, h \in G$ then we say f is class function since it is a *constant* function on conjugacy classes of G.

where it is assumed that the manifold on which the lattice gauge theory is defined is of the form $\Sigma \times S^1$, Σ being a $2D$ surface and there is a time direction that is split in N steps. At the same time, the transfer matrix can be regarded as a time evolution operator, thus relating a Hamiltonian H with the partition function,

$$T = e^{-\Delta t H}. \tag{1.6}$$

Thus the Hamiltionans are obtained by taking the logarithm of the transfer matrix which in turn depends on the z and z^* parameters, hence a large class of models can be obtained following this procedure, the QDM Hamiltonians included. Also in [41] the particular case where $G = \mathbb{Z}_2$ is obtained which reproduces the Toric Code, the formalism is also useful to explicitly write the ground states of the model as tensor networks. More general choices of parameters of the theory can be interpreted as adding perturbations that induce quantum phase transitions [42] making unsolvable the otherwise solvable models (as the QDM models); in [43] are considered more general choices of parameters at the level of the transfer matrix, the Hamiltionans obtained from such transfer matrices are explored for the abelian $\mathbb{C}(\mathbb{Z}_n)$ and the non-abelian $\mathbb{C}(S_3)$ where we show that the inclusion of these parameters result on modified vertex operators although the models remain solvable and in the case of $\mathbb{C}(\mathbb{Z}_n)$ the model is still in the QDM phase. When the gauge degrees of freedom are chosen to be elements of the (non-abelian) group algebra $\mathbb{C}(S_3)$ the model is not equivalent to the Quantum Double model of S_3 as the braiding statistics of the anyonic excitations arising in this new model differ from those of the QDM for S_3. Also in [43] a more intricate deformation has been made, by associating weights to the volumes and the vertices of a $3D$ lattice it is possible to get transfer matrices that are not those of lattice gauge theories anymore. Likewise, following the procedure in [41] of splitting the transfer matrix one obtains a Hamiltonian defining quantum models that reproduce more general class of models known as **Double Semion** [44, 45] and **Twisted Quantum Double** models [46]. Summarizing, this formalism has shown to be very versatile and useful for the study of topological order as different topological phases can be realized via the systematical obtainment of exactly solvable models from certain choices of the parameters z and z^*. The QDM models arise from this formalism in a natural manner and the different phases are distinguished by the analysis of the elementary excitation's braiding statistics. Moreover, the formalism can be extended as in [43] in order to reproduce a more general class of models that exhibit topological order.

Up to this point we just considered pure gauge theories as the partition function in Eq.(1.2) describes Lattice Gauge Theories with degrees of freedom associated to the edges of the lattice and interpreted as being gauge fields. However, within the framework of lattice gauge theories [47, 48] **Matter Fields** are naturally implemented as being degrees of freedom sitting at the vertices of $\mathcal{L}$, the gauge fields are then realized as degrees of freedom sitting on the edges of $\mathcal{L}$ that act as parallel transporters between adjacent vertices. This is, say we want to compare the configuration of two adjacent matter degrees of freedom, or equivalentlym two adjancent vertex configurations, then the gauge field living in the edge between the two vertices is the object that makes the parallel transformation that allows us to compare the two. This has to do with the geometrical interpretation of gauge fields as being the **connection** of the theory. To perform a "*long distance*" comparison we define a path that is restricted to be composed of elementary links where the gauge degrees of freedom lie, when the path corresponds

to a closed loop it is said to be a *Wilson Loop*, when going around a closed loop that does not bring the vertex configuration to the initial one the connection is said to have a *curvature*. Therefore, theories for which the vertex configurations are not changed after a transport along Wilson Loops are said to have **Flat Connection**, this is the case of the QDM hamiltonians for which the plaquette operator, which measures the holonomy of the smallest possible Wilson Loop, projects out the states that have holonomy different from the identity element $e \in G$ of the gauge group. A gauge transformation is a change of the configurations on the vertices that induces a compensating change in the gauge degrees of freedom. Therefore, a *gauge invariant* object is such that its properties are not changed after a gauge transformation.

In order to describe matter fields, we need to extend the algebraic structure used in the case of pure gauge. In addition to the set $(\mathcal{A}, m, \Delta, S, \eta, \epsilon)$ new objects have to be included. We extend this algebraic data by the inclusion of the set $(V, \mu, t, \mathcal{G})$ where V is a vector space over a field $\mathbb{K}$, $t : V \to V \otimes V$ is a co-multiplication map that can be regarded as a tensor on $V \otimes V^* \otimes V^*$, where V^* is the dual vector space; the action of the algebra $\mathcal{A}$ on V is given by the map $\mu : \mathcal{A} \otimes V \to V$ and finally $\mathcal{G}$ is a bilinear form $\mathcal{G} : V \otimes V \to \mathbb{K}$. In [49] we show how to obtain Transfer Matrices for Lattice Gauge Theories coupled with matter, where we associate the tensors $t_i^{jk} \in V \otimes V^* \otimes V^*$ to the vertices of $\mathcal{L}$ and include a module structure through the map μ that defines the action of the gauge group on the matter degrees of freedom. In particular, we show that when the gauge degrees of freedom are chosen to be elements of the **Group Algebra** $\mathbb{C}(G)$, the matter degrees of freedom are elements of a n-dimensional vector space, which we label V_n, that carries the action of $\mathbb{C}(G)$ and the set (V, μ) is a left $\mathbb{K}G$-module, the transfer matrix is conveniently written as a product of operators parametrized by z, z^* and the bilinear form $\mathcal{G}$, i.e.

$$T(z, z^*, \mathcal{G}) = \prod_{v \in \mathcal{L}} A_v(z^*) \prod_{l \in \mathcal{L}} C_l(\mathcal{G}) \prod_{p \in \mathcal{L}} B(z). \tag{1.7}$$

By following the same procedure of [41] we obtain Hamiltonians from this transfer matrix that for certain parameters are made of mutually commuting projectors operators similar to the ones of the QDM with the inclusion of a new operator that plays the role of a parallel transporter acting on an edge and its two adjacent vertices. Such models are exactly solvable and they can describe new phases, topological or not. In this dissertation we will not be concerned with the details of how to go from the algebraic data to the Hamiltonians. Instead we will start from specific models and explore them. In particular, the model obtained by choosing the gauge group as being $\mathbb{Z}_2$ and the matter degrees of freedom are elements of a bi-dimensional vector space which we call V_2, in this sense this model is interpreted as a generalization of the **Toric Code** model.

This dissertation is organized as follows, chapter 2 is divided in two parts, in the first half we thoroughly describe the lattice model called Toric Code as it was originally defined by Kitaev in [1]. We explicitly write its ground state and show the topological nature of its degeneracy, furthermore, we construct the elementary excited states and show that they can be seen as quasi-particle localized excitations that are created by string-like operators acting on the global Hilbert space. The anyonic statistical behavior of such excitations is shown by the explicit braiding of pairs of quasi-particles and their fusion rules, defining the $\mathbb{Z}_2$ topological order in two dimensions. In the second half of

Chapter 2 we describe the algebraic structure of the **Quantum Double Models** for any finite and discrete group G, starting from the most elementary operators that act over on-site Hilbert spaces we construct the vertex and plaquette operators that define the Hamiltonian and develop on their algebraic properties, we end the chapter showing how the Toric Code is an special case of these more general models. As mentioned above these Hamiltonians can be obtained from transfer matrices of lattice gauge theories, the details of such procedure can be found in [41] and we do not include them here as it is not the main focus of this work.

On chapter 3 we extend the QDM by including the possibility of having degrees of freedom on the vertices of the lattice which are interpreted to be matter fields, these degrees of freedom are elements of a finite n-dimensional vector space labeled V_n. Moreover, the matter degrees of freedom are coupled via a gauge interaction that is defined through the map $\mu : \mathbb{C}(G) \otimes V_n \to V_n$, this is the role played by the new operator C_l. Consequently, the operator that performs the gauge transformations has its action extended to the matter degrees of freedom via the aforementioned map μ. The plaquette operator is not changed as it still acts on the gauge degrees of freedom that lie at the edges of $\mathcal{L}$. As in the second part of Chapter 2, we thoroughly describe the algebraic construction of these models departing form the most fundamental operators that act on local Hilbert spaces corresponding to the edges and vertices of the lattice; furthermore, we develop in detail the algebra of these operators showing that they are commuting projectors, making the Hamiltonian defined as a sum of such operators a solvable one. In the second half of the chapter we solve the model coined $V_2/\mathbb{C}(\mathbb{Z}_2)$ which we interpret as being a generalized version of the Toric Code model. This is the simplest model coming from the formalism. We start constructing the elementary operators and the three main operators namely, A_v, B_p and C_l, writing them in terms of the well known **Pauli Matrices**. More general operators are also defined that later will be responsible of creating the elementary excited states and the domain walls which are a novelty coming from this model. We end the chapter writing the ground state of the model and showing that, unlike the Toric Code, its degeneracy cannot depend on the topology of the surface the lattice $\mathcal{L}$ is embedded into.

Chapter 4 is a brief description of the elementary excited states of the $V_2/\mathbb{C}(\mathbb{Z}_2)$ model that come as violations of the *ground state conditions*, we show the operators that create excitations and the way they perform this task. Moreover, we show the braiding statistics of such elementary excitations and their fusion rules are extracted from the algebra of the operators that create them, very much in the same fashion as in chapter 2.

We conclude the dissertation with a final remarks chapter where we comment on the physical consequences of extending the Toric Code model by the inclusion of the matter degrees of freedom. We briefly discuss the new phenomena arising in these models and some of the on-going research problems being studied in the group led by Prof. Teotônio-Sobrinho. I would like to end this introductory chapter by mentioning that this work, which is mainly focused on the specific models arising from the formalism developed in [49], is a small part of the result of a joint research work that includes the efforts of the entire research group integrated by the Post-Doctoral fellow Pramod Padmanabhan, the Ph.D. students Miguel Jorge Bernabé Ferreira, Javier Ignacio Lorca Espiro, Maria Fernanda Araujo de Resende and Hudson Kazuo Teramoto Mendonça and myself.

Chapter 2

Toric Code and Quantum Double Models

The toric code is an exactly solvable model realized over a two dimensional square lattice that can be embedded on a non-trivial surface such as a Torus. It exhibits a ground state degeneracy of 4^g, where g is the genus of the surface in question. Many of the topological features of the model were essentially understood by Read and Chakraborty [50] where they considered a two-dimensional square lattice spin 1/2 model originally proposed by Anderson [51] and showed that the states are fourfold degenerate however, no exactly solvable model was proposed until A. Kitaev writes a two-dimensional quantum spin Hamiltonian [1] that is exactly solvable and exhibits the features of a topologically ordered system. The model was initially proposed as a system that can be considered as a quantum computer, where the logic gates are implemented by moving the excitations around each other, measurements can be performed by joining or *fusing* these excitations in pairs and observing the resulting state. In this chapter we will thoroughly describe this model as originally proposed by A. Kitaev [1], its ground state and excitations, the rules for fusing these excitations together. Then we briefly describe a more general class of models from which the Toric Code is an special case. This models are coined **Quantum Double Models** [1, 52, 53] as some releveant operators in the model are isomorphic to the Drinfeld's Quantum Double $\mathcal{D}(G)$ of a finite group G [23, 24, 25, 26]. As an example, we show that the Toric Code model is recovered when the group is $\mathbb{Z}_2$.

2.1 The Toric Code Model

Consider a lattice $\mathcal{L}$ that can possibly be embedded in an arbitrary two-dimensional surface. The simplest example consists on a $k \times k$ lattice with periodic boundary conditions thus forming a torus. On each edge of the lattice we define a bi-dimensional local Hilbert space $\mathcal{H}_i$ that is spanned by the orthonormal vectors $|\phi_1\rangle$ and $|\phi_{-1}\rangle$. We denote each of these degrees of freedom as a hollow circle on the edges of the lattice as seen in Fig.2.1. Thus the Hilbert space of the model is a tensor product of local Hilbert spaces $\mathcal{H} = \bigotimes_{i\in\mathcal{L}} \mathcal{H}_i$. There are then altogether $N_v = k^2$ vertices and $N_p = k^2$ plaquettes. For each vertex v and each plaquette p, consider the following operators:

$$A_v = \bigotimes_{i\in \mathrm{star}(v)} \sigma_i^x = \sigma_{i_1}^x \otimes \sigma_{i_2}^x \otimes \sigma_{i_3}^x \otimes \sigma_{i_4}^x, \qquad B_p = \bigotimes_{j\in\partial p} \sigma_j^z = \sigma_{j_1}^z \otimes \sigma_{j_2}^z \otimes \sigma_{j_3}^z \otimes \sigma_{j_4}^z, \quad (2.1)$$

where the index $i \in \text{star}(v)$ runs over all four edges around an specific vertex v and $j \in \partial p$ stands for all four edges in the boundary of a plaquette p of the lattice, as shown in Fig. 2.1 and the Pauli matrices are given by:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

We note that both the vertex and plaquette operators actually act over the entire Hilbert space $\mathcal{H}$ although they do it trivially on the local Hilbert spaces corresponding to the sites that are not part of v or p respectively, where by trivial action we mean the action of the identity operator $\mathbb{1}$.

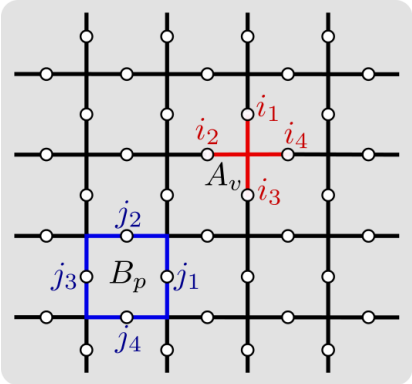


Figure 2.1: A piece of the square lattice; The spin degrees of freedom lie on the edges of the lattice and are represented by hollow circles. The sites being acted by the *Plaquette Operator* B_p and the *Vertex Operator* A_v are explicitly shown.

2.1.1 Algebra of Operators

The algebra of our vertex and plaquette operators is induced by the one of Pauli operators, recalling the commutation (and anticommutation) relations of the latter

$$\left[\sigma^a, \sigma^b\right] = 2i\epsilon_{abc}\sigma^c \tag{2.3}$$
$$\left\{\sigma^a, \sigma^b\right\} = 2\delta_{ab}\mathbb{1}, \tag{2.4}$$

where, a, b and c can be x, y or z; we can immediately see that for σ^x and σ^z the above expressions give us the following relations,

$$[\sigma^x, \sigma^x] = 0 \tag{2.5}$$
$$[\sigma^z, \sigma^z] = 0 \tag{2.6}$$
$$\sigma^x\sigma^z = -\sigma^z\sigma^x. \tag{2.7}$$

These expressions fix the commutation relations of the operators B_p and A_v, It is clear that all B_p commmute with each other because of eq.(2.6), similarly all A_v commute with each other since eq.(2.5) holds; The commutation relation between A_v and B_p is slightly less trivial. Consider an arbitrary plaquette together with some of its vertex, for example

the ones shown in Fig. (2.2). The operators:

$$\begin{aligned} A_v B_p &= \sigma^x_{i_1} \otimes \sigma^x_{i_2} \otimes \sigma^x_{i_3}\sigma^z_{i_3} \otimes \sigma^z_{i_4} \otimes \sigma^z_{i_5} \otimes \sigma^x_{i_6}\sigma^z_{i_6} \bigotimes_{j \notin v,p} \mathbb{1}_j, \\ &= \cdots \otimes -\sigma^z_{i_3}\sigma^x_{i_3} \otimes \cdots \otimes -\sigma^z_{i_6}\sigma^x_{i_6} \otimes \ldots, \\ &= (-1)^2 B_p A_v, \end{aligned} \tag{2.8}$$

where in the second line we used the commutation relation in Eq.(2.7) on edges i_3 and i_6 thus getting the two minus signs in the last line. This will be true for any two adjacent vertex and plaquette for they will always have common support on two of their edges. Therefore,

$$[A_v, B_p] = 0, \tag{2.9}$$

for any vertex v and plaquette p in $\mathcal{L}$.

Since the Pauli matrices are involutory, i.e., each one of them is its own inverse, or

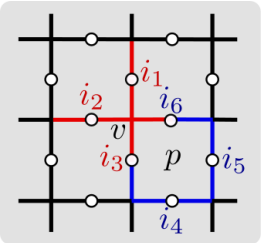


Figure 2.2: The configuration considered for the commutation relation between A_v and B_p in Eq.(2.8), any of the four vertices around the plaquette can be chosen.

equivalently, $(\sigma^x)^2 = (\sigma^y)^2 = (\sigma^z)^2 = \mathbb{1}$, where $\mathbb{1}$ is the 2×2 identity matrix; it is straightforward to see that the operators A_v and B_p also square to identity, namely:

$$A_v^2 = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} = B_p^2, \tag{2.10}$$

this is, if we apply the vertex (plaquette) operator twice on a given state it will remain unchanged. One can wonder what happens if we apply vertex (plaquette) operators on adjacent vertices (plaquettes), to answer this, note that two adjacent vertices[1] share one common edge, if we then apply the corresponding vertex operators we will act on this common edge with σ^x twice, recalling the involutory nature of these operators we see that when acting on two adjacent vertices the state lying on the common edge remains unchanged. Analogously, it is clear that since two adjacent plaquettes share one edge, if we apply the plaquette operators the state living on the shared edge will remain unchanged (it is being acted with the identity operator). If we now think of the lattice as having periodic boundary conditions on both space directions (thus forming a Torus) we can extend this analysis for the case when we apply vertex (plaquette) operators to the entire lattice, since each pair of adjacent vertex (plaquette) operators leave the state on the shared edge invariant, the result of applying vertex (plaquette) operators on the whole

[1]Note that by vertex we actually mean the 4 edges around it, since they are the states the vertex operator is acting upon.

lattice will leave all edges without any change, we can write this fact as the following overall constraints

$$\prod_{\text{all } v} A_v = \bigotimes_{i\in\mathcal{L}} \mathbb{1}_i \qquad \prod_{\text{all } p} B_p = \bigotimes_{j\in\mathcal{L}} \mathbb{1}_j. \tag{2.11}$$

Now, let us consider an eigenstate $|\psi_v\rangle$ for the Vertex Operator A_v and $|\psi_p\rangle$ an eigenstate of the Plaquette Operator B_p, such that:

$$A_v |\psi_v\rangle = a |\psi_v\rangle \qquad B_p |\psi_p\rangle = b |\psi_v\rangle ,$$

because of the involutory nature of both operators we notice that

$$(A_v)^2 |\psi_v\rangle = \mathbb{1} |\psi_v\rangle = a^2 |\psi_v\rangle \tag{2.12}$$
$$(B_p)^2 |\psi_p\rangle = \mathbb{1} |\psi_p\rangle = b^2 |\psi_v\rangle , \tag{2.13}$$

therefore, the only values a and b can take are ± 1. Given the Hilbert space on the edges is spanned by the states $|\phi_1\rangle$ and $|\phi_{-1}\rangle$ that can be chosen to be the eigenstates of the σ^z Pauli operator, this is, we choose a representation for the basis states where

$$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\phi_{-1}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.14}$$

such that the action of the Pauli matrices on them states is the following:

$$\begin{array}{ll} \sigma^x |\phi_1\rangle = |\phi_{-1}\rangle & \sigma^x |\phi_{-1}\rangle = |\phi_1\rangle \\ \sigma^z |\phi_1\rangle = |\phi_1\rangle & \sigma^z |\phi_{-1}\rangle = - |\phi_{-1}\rangle . \end{array}$$

2.1.2 Path and Loop Operators

Apart from the operators defined in Eqs.(2.1) we can define operators acting on bigger pieces of the Hilbert space $\mathcal{H}$ constructed in a similar fashion, i.e., from Pauli operators. Note that the vertex operator can be seen as a tensor product of σ^x along a loop on the dual lattice enclosing the vertex in question; likewise, the plaquette operator can be thought of as being a tensor product of σ^z operators along a closed path on the direct lattice. In this sense we can think on constructing bigger closed paths, both on the direct and the dual lattice, and then apply tensors products of Pauli operators, associating σ^x to paths on the dual lattice and σ^z to paths on the direct lattice, as the ones labeled c and c' in Fig.(2.3), the operators associated to these two paths would then be:

$$Z(c); = \bigotimes_{i\in c} \sigma_i^z, \tag{2.15}$$
$$X(c'); = \bigotimes_{j\in c'} \sigma_j^x, \tag{2.16}$$

as it can be seen in Fig.(2.3) the loop defined by path c is enclosing three plaquettes, namely, p_1, p_2 and p_3. Let us take the product of the plaquette operators corresponding to the above mentioned plaquettes; on doing this we note that on the edges that are being shared by each pair of plaquettes we will have a product of two σ^z operators coming from each B_p's; we recall the involutory nature of the Pauli operators (this is, they square to

the identity operator $\mathbb{1}$) to note that the edges that are shared by two plaquette operators will be acted upon by identity operators. Therefore the operator defined in Eq.(2.15) can be written as a product of the plaquette operators that correspond to the plaquettes being enclosed by the path on the direct lattice c, i.e.

$$Z(c) = B_{p_1} B_{p_2} B_{p_3}. \tag{2.17}$$

Using the same arguments it is straightforward to see that the operator defined in Eq.(2.16) can be written as a product of the vertex operators corresponding to the vertices that are being enclosed by the dual path c', this is,

$$X(c') = A_{v_1} A_{v_2} A_{v_3}. \tag{2.18}$$

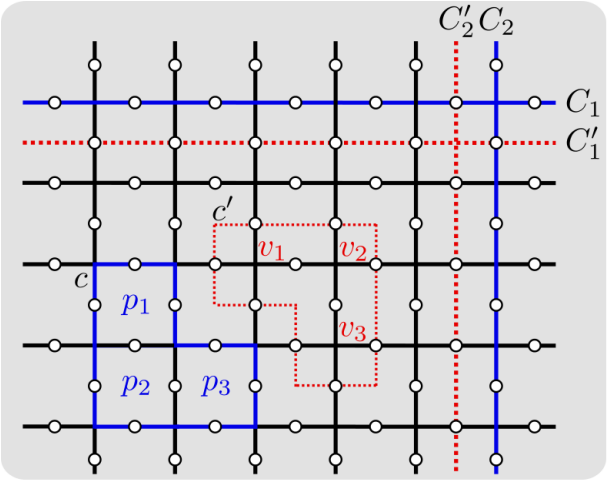


Figure 2.3: Four paths and two contractible loops are shown, the path c (solid blue) along the direct lattice forms a closed loop around the plaquettes p_1, p_2 and p_3; while the dual path c' (dotted line) is enclosing three vertices, namely v_1, v_2 and v_3. The paths C_1 and C_1' wind the Torus along the horizontal direction; conversely the paths C_2 and C_2' wind the torus in the vertical direction.

Therefore, any operator $X(c')$, where c' defines a closed (contractible) loop on the dual lattice can be written as a product of operators A_v corresponding to the vertices that are enclosed by the path in question. Likewise, for a path c that defines a contractible loop on the direct lattice, the operator $Z(c)$ can be written as a product of operators B_p, where p is any plaquette that is being enclosed by c.

Let us now consider the case when the lattice $\mathcal{L}$ is embedded on the surface of a Torus, on doing this we note that on such surface we can define closed loops (both on the dual and direct lattice) that are non contractible, in the sense that they cannot be deformed into a point, on Fig.(2.3) we label these paths as C_1', C_2', C_1 and C_2. The paths C_1' and C_1 wind the Torus along the horizontal direction whereas the paths C_2' and C_2 wind it along the vertical direction. The operators related to these paths, namely, $X(C_1')$, $X(C_2')$, $Z(C_1)$ and $Z(C_2)$ cannot be

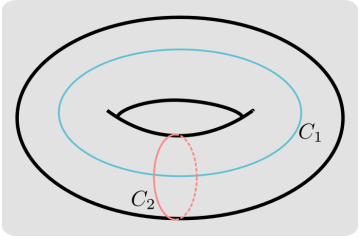


Figure 2.4: The two non-contractible paths on a Torus.

written in terms of products of A_v and B_p, although this fact does not seem relevant at this point it will be very important when looking at the explicit form of the ground state of the model defined below.

The above operators can, as well, be defined as to act on domains corresponding to open paths. If γ defines an open path on the direct lattice and γ^* defines an open path on the dual lattice, the operators are given by:

$$Z(\gamma) := \bigotimes_{i\in\gamma} \sigma_i^z, \tag{2.19}$$

$$X(\gamma^*) := \bigotimes_{j\in\gamma^*} \sigma_j^x, , \tag{2.20}$$

these operators will be responsible of the creation of the elementary excitations as we will see in § 2.3.

2.1.3 Hamiltonian

Let us now define the Hamiltonian of the *Toric Code* as a sum of plaquette and vertex operators over all plaquettes and vertices of the lattice,

$$H_{TC} = -\sum_v A_v - \sum_p B_p, \tag{2.21}$$

since all terms in this Hamiltonian commute with each other, (2.21) can be diagonalized in the basis of the eigenstates of the A_v and B_p operators, therefore it can be exactly solved, this is, given that we know the eigenvalues and eigenstates of both vertex and plaquette operators, the ground state and the excited states of (2.21) can be found, as we will show in the following two sections.

2.2 Ground States

To find the ground state $|\Psi_0\rangle$ of H, we need to find the condition for the energy to be minimum in Eq. (2.21), or equivalently (because of the negative sign in front of each term in Eq. (2.21)) we need to maximize the eigenvalues of each A_v and B_p operators, given that their eigenvalues are ± 1, it is clear that a lower bound on the energy is obtained when the ground state $|\Psi_0\rangle$ satisfies

$$B_p |\Psi_0\rangle = |\Psi_0\rangle \quad \forall p, \quad \text{and} \quad A_v |\Psi_0\rangle = |\Psi_0\rangle \quad \forall v, \tag{2.22}$$

therefore any state that fulfills these two conditions will be a ground state of the system. Leaving for the ground state energy $E_0 = -(N_p + N_v)$, where N_p and N_v stand for the number of plaquettes and vertices present on the lattice.

Consider a plaquette operator acting on the states living on the boundary of an arbitrary plaquette p, let us call this state $|\phi_p\rangle$, such that

$$B_p |\phi_p\rangle = \pm |\phi_p\rangle , \tag{2.23}$$

notice that the eigenvalue of B_p will depend on the configuration of the four spin degrees of freedom living on the edges along the plaquette p. In particular, if the plaquette operator

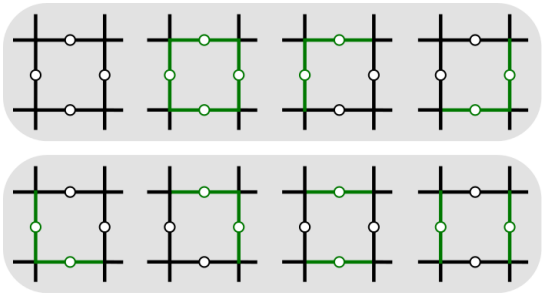


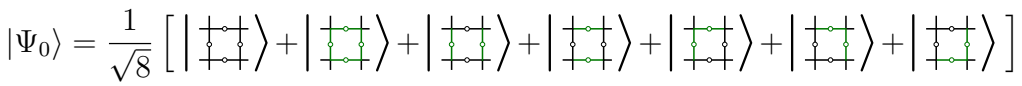
Figure 2.5: All possible vortex-free spin configurations for a single plaquette are shown, where we consider an arbitrary plaquette p and only down spin $|\phi_{-1}\rangle$ states are green colored.

has eigenvalue 1 when applied to a particular state we say the state is *vortex-free*, the reason for this name will become clear in the § 2.3 where we study the excitations of the model. Recalling that the plaquette operator is a tensor product of Pauli matrices σ^z whose action on the basis states $\{|\phi_1\rangle, |\phi_{-1}\rangle\}$ is given by:

$$\sigma^z |\phi_1\rangle = |\phi_1\rangle , \qquad \sigma^z |\phi_{-1}\rangle = -|\phi_{-1}\rangle , \tag{2.24}$$

it is clear that the state will be a *vortex-free* one for configurations with an even number of $\{|\phi_1\rangle, |\phi_{-1}\rangle\}$ states. We show all possible *vortex-free* states for a single plaquette in Fig.(2.5) where we adopt a graphical representation in which the edges containing a $|\phi_{-1}\rangle$ state are green colored.

Thus, if we consider a single plaquette the most general ground state for this system will be a linear combination of the states shown in Fig.(2.5) satisfying $B_p |\Psi_0\rangle = |\Psi_0\rangle$ where

$|\Psi_0\rangle =$  (2.25)

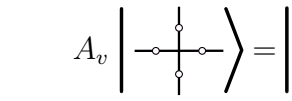
Let us now look at the action of the vertex operator on a given state, from Eq.(2.1) we know the vertex operator acts with σ^x on the 4 edges adjacent to the vertex in question, the action of this Pauli operator on the basis states $\{|\phi_1\rangle, |\phi_{-1}\rangle\}$ is given by

$$\sigma^x |\phi_1\rangle = |\phi_{-1}\rangle , \qquad \sigma^x |\phi_{-1}\rangle = |\phi_1\rangle , \tag{2.26}$$

this is, the vertex operator action consists on flipping the states on each edge from $|\phi_1\rangle$ to $|\phi_{-1}\rangle$ and viceversa, for instance let $|v\rangle = |\phi_1\rangle_{i_1} \otimes |\phi_1\rangle_{i_2} \otimes |\phi_1\rangle_{i_3} \otimes |\phi_1\rangle_{i_4}$, where i_1, i_2, i_3 and i_4 are the four edges adjacent to an arbitrary vertex, the action of the vertex operator on this state is given by:

$$\begin{aligned} A_v |v\rangle =& \sigma^x_{i_1} \otimes \sigma^x_{i_2} \otimes \sigma^x_{i_3} \otimes \sigma^x_{i_4} \left[|\phi_1\rangle_{i_1} \otimes |\phi_1\rangle_{i_2} \otimes |\phi_1\rangle_{i_3} \otimes |\phi_1\rangle_{i_4} \right] \\ =& \sigma^x_{i_1} |\phi_1\rangle_{i_1} \otimes \sigma^x_{i_2} |\phi_1\rangle_{i_2} \otimes \sigma^x_{i_3} |\phi_1\rangle_{i_3} \otimes \sigma^x_{i_4} |\phi_1\rangle_{i_4} \\ =& |\phi_{-1}\rangle_{i_1} \otimes |\phi_{-1}\rangle_{i_2} \otimes |\phi_{-1}\rangle_{i_3} \otimes |\phi_{-1}\rangle_{i_4} , \end{aligned} \tag{2.27}$$

if instead we choose to use the graphical representation of these states, the above action will look like:

A_v  (2.28)

where the edges holding $|\phi_{-1}\rangle$ states are green colored. In this sense, the vertex operator can be understood as a local gauge transformation, equivalently we say that two configurations that are related to each other by the action of a vertex operator are *gauge equivalent*. In Fig.(2.6) we show several actions of the vertex operator on arbitrary states of a given vertex.

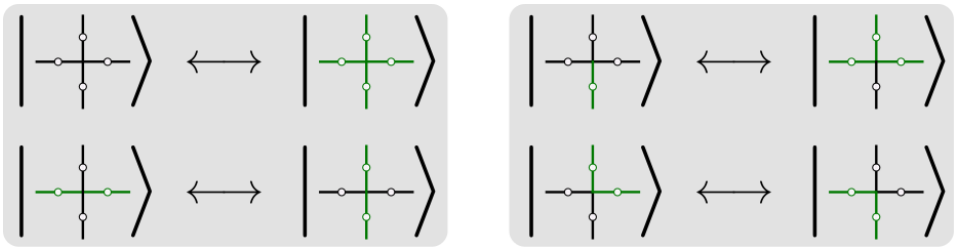


Figure 2.6: Some illustrative actions of the Vertex Operator A_v are shown using the graphical representation for the states, where the colored edges hold down spin $|\phi_{-1}\rangle$ states. We say that two states that are related by the action of the vertex operator have *gauge equivalent* configurations.

Let us go back to the analysis of the ground state $|\Psi_0\rangle$ for the Hamiltonian defined in Eq.(2.21), the first condition on Eq.(2.22) involving the plaquette operator restricts the configurations of the ground states as being *vortex-free*, such as the ones shown in Fig.(2.5), therefore we can write the most general ground state as [22, 42]

$$|\Psi_0\rangle = \sum_{s:B_p|s\rangle=|s\rangle\ \forall p} c_s\,|s\rangle\,, \tag{2.29}$$

where $|s\rangle$ is a *vortex-free* state and the ground state $|\Psi_0\rangle$ is a linear combination of all *vortex-free* configurations. The second condition in Eq.(2.22) involving the vertex operator tells us that any two configurations on a given vertex that are related to each other by the action of a vertex operator will appear in the ground state with the same weight, this is, all coefficients c_s are required to be equal. Hence, a ground state of the toric code is an equal-weight superposition of vortex free configurations.

To find the explicit form of such state, we use the involutory property of A_v and B_p, this is

$$A_v^2 = \bigotimes_{i\in v} \mathbb{1}_i \quad \text{and} \quad B_p^2 = \bigotimes_{j\in\partial p} \mathbb{1}_j, \tag{2.30}$$

or equivalently, $A_v(\mathbb{1}+A_v) = A_v + \mathbb{1}$[1]. Similarly ,for the plaquette operator $B_p(\mathbb{1}+B_p) = \mathbb{1}+B_p$. With this in mind, consider the following state

$$|\Psi_0^v\rangle = \prod_{v\in\mathcal{L}} \frac{1}{\sqrt{2}}\,(\mathbb{1}+A_v) \bigotimes_{l\in\mathcal{L}} |\phi_1\rangle_l\,, \tag{2.31}$$

where the product runs over all vertices in the lattice, the $1/\sqrt{2}$ factor in front is nothing but a normalization factor and, by the tensor product we mean that each edge l of the lattice $\mathcal{L}$ carries a $|\phi_1\rangle$ state. The reason why we want the states on each edge to be $|\phi_1\rangle$

[1] Actually this is an abuse of notation, the $\mathbb{1}$ stands for identity on every site of the lattice

will become clear when we show that $|\Psi_0^v\rangle$ is indeed a ground state of the Hamiltonian in Eq.(2.21). Let us first act on this state with an arbitrary vertex operator $A_{v'}$, this is

$$\begin{aligned} A_{v'}|\Psi_0^v\rangle &= A_{v'}\prod_{v\in\mathcal{L}}\frac{1}{\sqrt{2}}(\mathbb{1}+A_v)\bigotimes_{l\in\mathcal{L}}|\phi_1\rangle_l \\ &= \prod_{v\in\mathcal{L}}\frac{1}{\sqrt{2}}(\mathbb{1}+A_v)\bigotimes_{l\in\mathcal{L}}|\phi_1\rangle_l \\ &= |\Psi_0^v\rangle\,, \end{aligned} \tag{2.32}$$

where in the second line we used $A_{v'}(\mathbb{1}+A_{v'})=(A_{v'}+\mathbb{1})$ as v' is an arbitrary vertex that is already contained in the product. Thus, this state satisfies the ground state condition for the vertex operator. Let us now consider the case when we act on this state with an arbitrary plaquette operator B_p, i.e.

$$\begin{aligned} B_p|\Psi_0^v\rangle &= B_p\prod_{v\in\mathcal{L}}\frac{1}{\sqrt{2}}(\mathbb{1}+A_v)\bigotimes_{l\in\mathcal{L}}|\phi_1\rangle_l \\ &= \prod_{v\in\mathcal{L}}\frac{1}{\sqrt{2}}(\mathbb{1}+A_v)\,B_p\bigotimes_{l\in\mathcal{L}}|\phi_1\rangle_l \\ &= \prod_{v\in\mathcal{L}}\frac{1}{\sqrt{2}}(\mathbb{1}+A_v)\bigotimes_{l\in\mathcal{L}}|\phi_1\rangle_l \\ &= |\Psi_0^v\rangle\,, \end{aligned} \tag{2.33}$$

where in the second line we used $[A_v,B_p]=0$ for any $v,p\in\mathcal{L}$, and in the third line we note that the states on which the plaquette operator is acting are *vortex-free*, this is:

$$B_p\bigotimes_{l\in\partial p}|\phi_1\rangle_l=\bigotimes_{l\in\partial p}|\phi_1\rangle_l\,, \tag{2.34}$$

since $\sigma^z|\phi_1\rangle=|\phi_1\rangle$. In this sense, any product of vertex (plaquette) operators will act trivially on $|\Psi_0^v\rangle$, equivalently, any operator $Z(\gamma)$ or $X(\gamma^*)$ where γ and γ^* define contractible loops on the direct and dual lattice, respectively, will act on $|\Psi_0^v\rangle$ in a trivial way. We can interpret this ground state as a **Loop Gas**, to see this, let us assume the graphical representation where an uncolored edge holds a $|\phi_1\rangle$ state, we recall that the action of a σ^x operator on this state will take it to $|\phi_{-1}\rangle$ as shown in Eq.(2.26), we will represent any edge holding a $|\phi_{-1}\rangle$ state by making it green. Thus, in our graphical representation the action of a σ^x operator on any uncolored edge will consist on changing its color. Let us now look at the product in Eq.(2.31), namely, $\prod_{v\in\mathcal{L}}(\mathbb{1}+A_v)$. The first term coming out of this product is an identity operator acting on every edge degree of freedom, this term leaves the $\bigotimes_{l\in\mathcal{L}}|\phi_1\rangle_l$ invariant. The following cross terms will have products of A_v's, as they consist on σ^x operators acting on the 4 adjacent edges to the vertex v they will flip $|\phi_1\rangle$ to $|\phi_{-1}\rangle$ correspondingly (c.f. Fig.(2.6)). Therefore the ground state as written in Eq.(2.31) is a linear combination of all such states, some of them are shown in Fig.(2.7).

The red closed loops on the graphical representation stand for the action of products of A_v operators on the state with all edges holding $|\phi_1\rangle$, it is in this sense that the ground

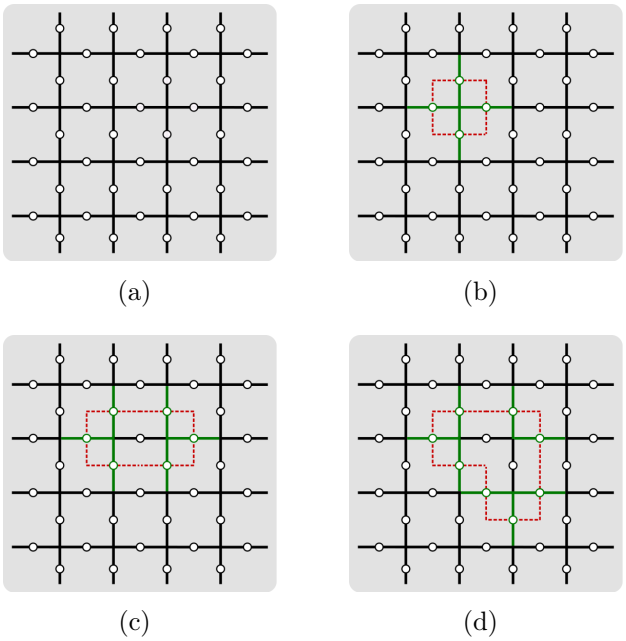


Figure 2.7: Some illustrative constituents of the ground state $|\Psi_0^v\rangle$ are shown, where (a) corresponds to the first term in the expansion of Eq.(2.31), (b) corresponds to a term in this expansion for which there is a single A_v acting on the $|\phi_1\rangle$ states around v, in (c) and (d) we show terms that include the action of several vertex operators.

state of the Toric Code can be interpreted as a **Loop Gas**, containing all possible contractible loops that could be defined on the lattice. From now on we consider the lattice to be embedded on the surface of a Torus, from section 2.1 we know there are two non-contractible loops (c.f. Fig.(2.3)). Thus, the operators $X(C_1')$, $X(C_2')$, $Z(C_1)$ and $Z(C_2)$ can be defined and will not be made of products of vertex or plaquette operators, correspondingly. Let us study the action of these operators on the state defined in Eq.(2.31). First we note that $[A_v, X(C_{1,2}')] = 0$ as both operators have σ^x as their constituents. Now, since operators $Z(C_1)$ and $Z(C_2)$ are constructed in terms of σ^z operators we might in principle be tempted to think they do not commute with the vertex operators that lie along the non contractible loop, we note though that each vertex operator along that loop will share two edges with $Z(C_{1,2})$, this means $[A_v, Z(C_{1,2})] = 0$. Hence, the above path operators do indeed commute with the vertex operator A_v (so they do with B_p) [1, 44, 54]. Consider the operator $X(C_1')$ where the path C_1' winds the Torus along a horizontal non-contractible loop on the dual lattice, the action of this operator on the ground state $|\Psi_0^v\rangle$ is given by:

$$\begin{aligned} X(C_1')\,|\Psi_0^v\rangle &= X(C_1')\prod_{v\in\mathcal{L}}\frac{1}{\sqrt{2}}\,(\mathbb{1}+A_v)\bigotimes_{l\in\mathcal{L}}|\phi_1\rangle_l \\ &= \prod_{v\in\mathcal{L}}\frac{1}{\sqrt{2}}\,(\mathbb{1}+A_v)\,X(C_1')\bigotimes_{l\in\mathcal{L}}|\phi_1\rangle_l \\ &= \prod_{v\in\mathcal{L}}\frac{1}{\sqrt{2}}\,(\mathbb{1}+A_v)\bigotimes_{l\notin C_1'}|\phi_1\rangle_l\bigotimes_{l'\in C_1'}|\phi_{-1}\rangle_{l'} \\ &= |\Psi_1^v\rangle\,, \end{aligned} \tag{2.35}$$

where in the second line we used the commutation relation $[X(C_1'), A_v] = 0$, and the action of the winding operator $X(C_1')$ on the basis states $|\phi_1\rangle$ along the non-contractible loop C_1' consists on changing them into $|\phi_{-1}\rangle$. Clearly this is still a ground state under the action of any vertex operator A_v, let us see if it remains satisfying the ground state condition under the action of an arbitrary plaquette operator, this is:

$$\begin{aligned} B_p \left|\Psi_1^v\right\rangle &= B_p \prod_{v\in\mathcal{L}} \frac{1}{\sqrt{2}} \left(\mathbb{1} + A_v\right) \bigotimes_{l\notin C_1'} |\phi_1\rangle_l \bigotimes_{l'\in C_1'} |\phi_{-1}\rangle_{l'} \\ &= \prod_{v\in\mathcal{L}} \frac{1}{\sqrt{2}} \left(\mathbb{1} + A_v\right) B_p \bigotimes_{l\notin C_1'} |\phi_1\rangle_l \bigotimes_{l'\in C_1'} |\phi_{-1}\rangle_{l'} \,, \end{aligned} \tag{2.36}$$

at this point we note the following: If the plaquette operator in question is acting on any plaquette that does not have an edge crossing the loop C_1' (since it is a loop on the dual lattice) will act trivially on the above state, so we have to look at the plaquettes that have edges being crossed by C_1', here we note that the dual loop C_1' will necessarily cross any plaquette through two of its edges, therefore the configuration of the degrees of freedom on the plaquettes will be those shown in Fig.(2.5), this is, vortex free ones. This ensures that:

$$B_p \left|\Psi_1^v\right\rangle = \left|\Psi_1^v\right\rangle . \tag{2.37}$$

Thus, we have shown that this new state is a ground state of the Toric Code Hamiltonian. Again it can be interpreted as a **Loop Gas**, some of its constituents are depicted in Fig.(2.8). Notice that the particular path defined by C_1' along the dual lattice is not relevant as long as it winds the Torus through a non-contractible one, since all possible deformations of the path are already contained in the Loop Gas. Since this dual path cannot be written as a product of vertex operators, the ground state $|\Psi_1^v\rangle$ cannot be written in terms of $|\Psi_0^v\rangle$ Likewise, acting with $X(C_2')$ on $|\Psi_0^v\rangle$ we can create another ground state, namely:

$$X(C_2') \left|\Psi_0^v\right\rangle = \left|\Psi_2^v\right\rangle . \tag{2.38}$$

Also by acting with $X(C_1')$ and $X(C_2')$ simultaneously on $|\Psi_0^v\rangle$ we create another ground state, i.e.:

$$X(C_1')X(C_2') \left|\Psi_0^v\right\rangle = \left|\Psi_{1,2}^v\right\rangle . \tag{2.39}$$

Thus, $|\Psi_0^v\rangle$, $|\Psi_1^v\rangle$, $|\Psi_2^v\rangle$ and $\left|\Psi_{1,2}^v\right\rangle$ are the four ground states of the Toric Code, and their existence is guaranteed as long as the lattice is embedded on a Torus. Note that if we allow the lattice to be embedded on a more general surface with genus g more ground states can be constructed, depending on the number of homotopically inequivalent non-contractible loops that can be defined on such surface. Hence the dependence of the ground state on the topological properties of the surface the model is defined in.

The toric code Hamiltonian can be seen as a lattice gauge theory [21], in this framework the degrees of freedom living on the edges of the lattice correspond to $\mathbb{Z}_2$ valued gauge degrees of freedom, the vertex operator corresponds to a gauge transformation, and since it commutes with the plaquette operator implies an overall gauge invariance. It is from this equivalence to a gauge theory that the excitations of the toric code that come from violating one of the conditions in Eq.(2.22) are commonly known as *charges* and *fluxes* as we will see in the next section.

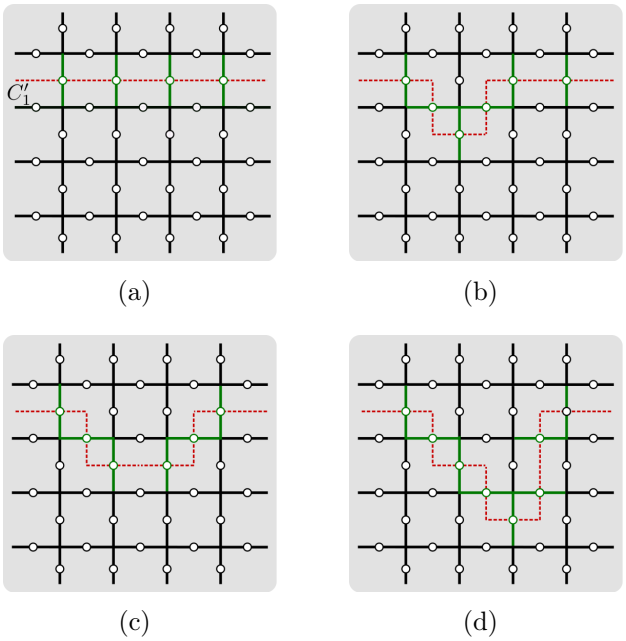


Figure 2.8: The path C_1' winds the torus along a non-contractible loop, the operator $X(C_1')$ defined on this path transforms the degrees of freedom that lie on C_1', the state $|\Psi_1^v\rangle$ is composed of all such transformations that can be gotten by the action of vertex operators in $\mathcal{L}$.

2.3 Elementary Excitations

The elementary excitations of the Toric Code are the result of violating the constraints on Eq.(2.22). From Eqs.(2.12) and (2.13) we know the eigenvalues of the vertex and plaquette operators when acting on an eigenstate can only be ± 1. Thus, an elementary excitation corresponds to an eigenstate $|\Psi\rangle \in \mathcal{H}$ such that for some vertex and/or plaquette operator it has eigenvalue -1.
This can be seen as follows, an operator that *creates* an excitation at some vertex and/or plaquette of the lattice, is such that it anticommutes with the vertex (plaquette) operator in question.

2.3.1 Vertex Excitations: Charges

Consider the following state

$$|\Psi_i^z\rangle = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \sigma_i^z \otimes \cdots \otimes \mathbb{1} \, |\Psi_0\rangle \,, \tag{2.40}$$

where $|\Psi_0\rangle$ is some of the four ground states of the Toric Code Hamiltonian. This new state is no longer a ground state of the model for the condition in Eq.(2.22) involving the vertex operator A_v does not hold anymore since the two vertex operators A_{v_1} and A_{v_2} sharing the i-th (see Fig.(2.9)) edge do not commute with the σ^z that is acting on it , this is:

$$\begin{aligned} A_{v_1,v_2}\left|\psi_i^z\right\rangle &= A_{v_1,v_2}\sigma_i^z\left|\Psi_0\right\rangle \\ &= -\sigma_i^z A_{v_1,v_2}\left|\Psi_0\right\rangle \\ &= -\sigma_i^z\left|\Psi_0\right\rangle \\ &= -\left|\Psi_i^z\right\rangle \end{aligned}$$

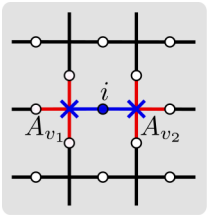


Figure 2.9: We denote the action of the σ^z operator on i by coloring it blue and the presence of an excitation of this type is represented by a blue "x".

In this sense we say that the Pauli operator σ_i^z **creates** two excitations located at vertices v_1 and v_2. The energy of such excited state is then

$$E_v = -(N_p + N_v - 2). \tag{2.41}$$

Let us now consider an operator of the form:

$$Z(\gamma) = \bigotimes_{j\in\gamma} \sigma_j^z, \tag{2.42}$$

where γ stands for am open path on the direct lattice, although we do not write it explicitly it is understood that this operator is acting as identity on the lattice sites that are not part of the path γ. This **string operator** as defined in Eq.(2.42) will anticommute with the vertex operators acting on the endpoints of γ thus creating a pair of localized excitations which henceforth will be called **charges**, as a consequence the excitations can be "moved" by simply extending the path γ on the lattice, the energy of the excited state does not change since the number of vertex operators being affected by the path operator is the same, eventually we can think on closing the path thus creating a loop, in this case no pair of excitations is created since such operator can be written as the product of B_p's where the plaquettes are those enclosed by the path γ, in §2.2 we have already shown that this kind of operators act trivially on a ground state.

2.3.2 Plaquette Excitations: Fluxes

Likewise, the excitations coming from the violation of the first constraint in Eq.(2.22) involving the plaquette operator B_p are created by a **string operator** given by

$$X(\gamma^*) = \bigotimes_{i\in\gamma^*} \sigma_i^x, \tag{2.43}$$

where γ^* defines a path in the dual lattice such as the one shown in Fig.(2.10). The state given by the action of this operator on a ground state $\left|\Psi_0\right\rangle$ defined as:

$$\left|\Psi^x(\gamma^*)\right\rangle = X(\gamma^*)\left|\Psi_0\right\rangle, \tag{2.44}$$

is an excited state of the Hamiltonian carrying two localized excitations on the plaquettes where the string operator $X(\gamma^*)$ has its endpoints. As in the case of charges the energy of this excited state depends only on the number of plaquettes that are being afected by the path operator $X(\gamma^*)$ and since they always come in pairs the energy of the state is given by:

$$E_p = -(N_p - 2 + N_v). \tag{2.45}$$

Again, if the γ^* forms a closed loop no excitation is created and the operator maps between ground state constituents.
Note that the states $|\Psi^z(\gamma)\rangle$ and $|\Psi^x(\gamma^*)\rangle$ and their corresponding energy do not depend on the path itself, to illustrate this consider the state $|\Psi^x(\gamma^*)\rangle$ that has a pair of fluxes living on two plaquettes p_1 and p_2, there are many paths on the dual lattice that join the aforesaid plaquettes, analogously there is an equal number of path operators $X(\gamma^*)$ that create this pair of fluxes. We say that the state $|\Psi^x(\gamma^*)\rangle$ (and its energy) rather depend on the homotopy class[1] of the path γ^* whereas the operators that map a given ground state into the excited states depend on the path itself.

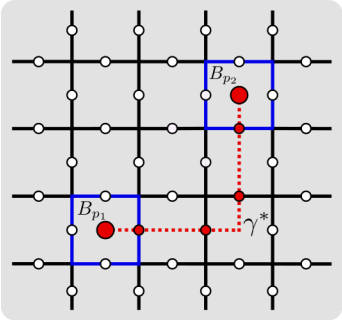


Figure 2.10: The operator $X(\gamma^*)$ creates two localized excitations (red circles) at the plaquettes p_1 and p_2, we denote the action of the σ^x operators on the edges of the lattice by coloring them red.

2.3.3 Fusion and the Dyonic Excitation

Let us now consider the case when we have pair of disconnected charges, by this we mean they were created by means of two path operators $Z(\gamma_1)$ and $Z(\gamma_2)$ as depicted in Fig.(2.11), this state is the result of the action of two string operators $Z(\gamma_1)$ and $Z(\gamma_2)$ on a ground state $|\Phi_0\rangle$, i.e.

$$|\Psi^z(\gamma_{1,2})\rangle = Z(\gamma_1)Z(\gamma_2)\,|\Psi_0\rangle\,, \tag{2.46}$$

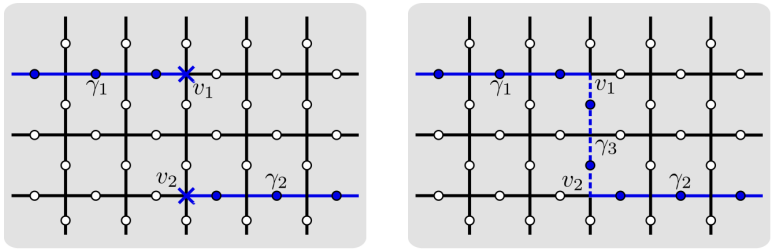
consider this as being our initial state, now we decide to connect these two paths by extending either γ_2 or γ_1 through an additional path γ_3, on doing so note that the excitations that in principle were sitting at the endpoints of both γ_1 and γ_2 annihilate with themselves as the resulting operator $Z(\gamma_1+\gamma_2+\gamma_3) = Z(\gamma_1)Z(\gamma_2)Z(\gamma_3)$ now commutes with the vertex operators acting at v_1 and v_2. Thus, we say that the charge excitations are their own antiparticles, as when we **fuse** them together they cancel each other:

$$e \times e = 1, \tag{2.47}$$

here e stands for the presence of a charge excitation at some vertex and 1 for the absence of excitations at the given vertex.

In a similar manner, we can imagine the process of creating two disconnected flux excitations each one sitting at arbitrary plaquettes p_1 and p_2 as shown in Fig.(2.12). This state, which we call $\left|\Psi^x(\gamma^*_{1,2})\right\rangle$, is the result of applying the string operators $X(\gamma^*_1)$ and

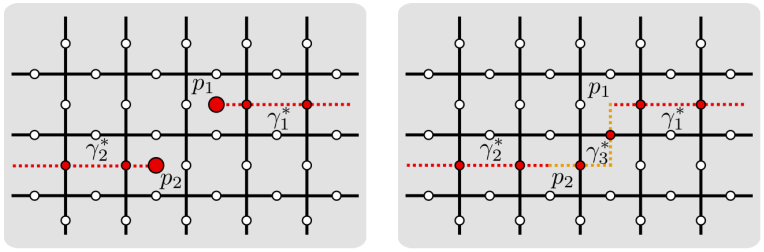
[1] Two maps are said to be homotopically equivalent or to belong to the same homotopy class if one can be deformed into the another.



(a) (b)

Figure 2.11: In (a) we show the initial configuration of the excited state $|\Psi^z(\gamma_{1,2})\rangle$. Then in (b) the path γ_3 is an extension of either γ_1 or γ_2.

$X(\gamma_2^*)$ on a ground state $|\Psi_0\rangle$ of H_{TC}. By extending one of the dual paths we move the corresponding flux, such that if we join the paths γ_1^* and γ_2^* the two fluxes that in principle were sitting at plaquettes p_1 and p_2 annihilated themselves as they are their own antiparticle.



(a) (b)

Figure 2.12: Two fluxes sitting at p_1 and p_2 are brought together by extending one of the paths γ_1^* or γ_2^* via the additional path γ_3^* (in orange) as a result of the process the two fluxes are annihilated.

This induces the following **fusion rule**:

$$m \times m = 1, \tag{2.48}$$

where m stands for the presence of a flux on a particular plaquette. Moreover, consider a state that is the outcome of a simultaneous application of string operators $Z(\gamma)$ and $X(\gamma^*)$ such that they share one (or both) of its endpoints (see Fig.(2.13)), this is, there are operators σ^x and σ^z being applied at the edge where the paths end. As discussed in §2.3.1 and §2.3.2 this process creates a charge and a flux which in this case are regarded as a single excitation called **Dyon**.

This can be interpreted as the **fusion rule**:

$$e \times m = m \times e = \epsilon. \tag{2.49}$$

From these three fusion rules we can deduce the following ones:

$$\epsilon \times e = m, \quad \epsilon \times m = e, \tag{2.50}$$

and for a matter of completeness we add the trivial fusion rules

$$1 \times 1 = 1, \quad 1 \times e = e \times 1 = e, \quad 1 \times m = m \times 1 = m, \quad 1 \times \epsilon = \epsilon \times 1 = \epsilon. \tag{2.51}$$

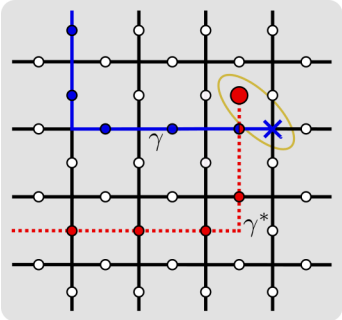


Figure 2.13: The dyonic excitation is the result of fusing a charge together with a flux.

2.3.4 Braiding Statistics

We now study the statistical behavior of the elementary excitations of the model, namely the charge e, the flux m and the dyon ϵ. This is, what happens to the overall state when we move one of the excitations around another. To begin with, consider an state that has a pair of localized charge excitations such as the initial configuration in Fig.(2.11); instead of fusing these two charges together consider a process by which we **exchange** their position by applying two path extensions $Z(\gamma_1')$ and $Z(\gamma_2')$ (see Fig.(2.14)).

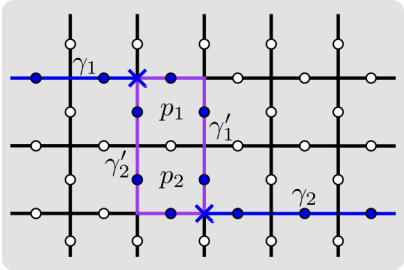


Figure 2.14: The braiding of two charges by transporting them along the paths γ_1' and γ_2' (in purple). Note that both these extension paths can be written as products of plaquette operators acting at p_1 and p_2.

Thus, the initial state $|\Psi^z(\gamma_{1,2})\rangle$ is the result of applying string operators $Z(\gamma_1)$ and $Z(\gamma_2)$ on a ground state, as in Eq.(2.46), now, the exchange procedure is given by the action of operators $Z(\gamma_1')$ and $Z(\gamma_2')$, such that the final state would be:

$$|\Psi^z(\gamma_{1,2,1',2'})\rangle = Z(\gamma_1')Z(\gamma_2')\,|\Psi^z(\gamma_{1,2})\rangle\,, \tag{2.52}$$

the two additional paths form a loop enclosing two plaquettes, namely p_1 and p_2, by the considerations done in §2.1.2 we can write them as the product of plaquette operators, this is:

$$Z(\gamma_1')Z(\gamma_2') = B_{p_1}B_{p_2}, \tag{2.53}$$

thus the state defined in Eq.(2.54) can be written as:

$$|\Psi^z(\gamma_{1,2,1',2'})\rangle = B_{p_1}B_{p_2}\,|\Psi^z(\gamma_{1,2})\rangle = |\Psi^z(\gamma_{1,2})\rangle\,, \tag{2.54}$$

in the last equality we note that the action of B_{p_1} and B_{p_2} on the initial state is trivial since there are no flux excitations sitting at the mentioned plaquettes. Hence, the final

state of the system is the same as the initial one. The exchange of two charges leaves the state invariant, thus signaling the bosonic nature of charges relative to themselves. A similar argument holds for the exchange of two plaquette excitations or fluxes.

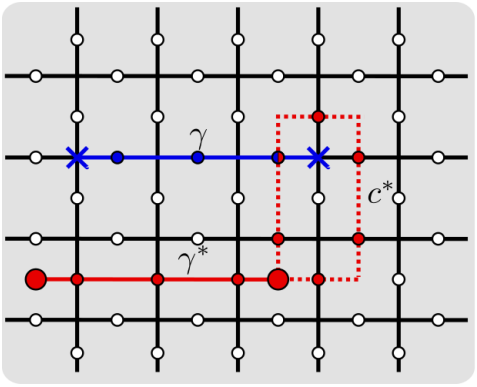


Figure 2.15: The braiding of a flux around a charge is described, the initial state contains a pair of charges at the endpoints of the path γ (in blue) and a pair of fluxes at the endpoints of the dual path γ^* (in red). Then one of the fluxes is taken around one charge by means of the operator $X(c^*)$.

On the other hand, consider the exchange of a charge and a flux. The initial state, containing a pair of charges and a pair of fluxes, is given by

$$|\Psi_i\rangle = Z(\gamma)\,|\Psi^x(\gamma^*)\rangle = Z(\gamma)X(\gamma^*)\,|\Psi_0\rangle\,, \tag{2.55}$$

where the paths γ and γ^* are those depicted in Fig.(Fig.(2.15)), to move one of the fluxes around one of the charges we extend the dual path γ^* by means of another path on the dual lattice which we call c^*. Thereby the final state will be given by

$$\begin{aligned} |\Psi_f\rangle &= X(c^*)\,|\Psi_i\rangle \\ &= X(c^*)Z(\gamma)\,|\Psi^x(\gamma^*)\rangle \\ &= -Z(\gamma)X(c^*)\,|\Psi^x(\gamma^*)\rangle \\ &= -\,|\Psi_i\rangle\,, \end{aligned} \tag{2.56}$$

where the minus sign in the third equality comes from the fact that the operators $Z(\gamma)$ and $X(c^*)$ anti commute at the edge where they cross each other and, in the last equality we used $X(c^*)\,|\Psi^x(\gamma^*)\rangle = |\Psi^x(\gamma^*)\rangle$ since the paths γ^* and c^* belong to the same homotopy class. Comparing Eq.(2.55) with Eq.(2.56) we notice that the process of moving one of the fluxes around one charge induces an overall -1 phase to the state. It is because of this property that these particles are called *abelian anyons*. The overall phase gained can be interpreted as a Aharanov-Bohm effect. The existence of anyons is at the root of the topologically dependent ground state degeneracy as argued in [16] and both are manifestations of the underlying **Topological Order** of the model.

2.4 Algebraic Structure: The Quantum Double Model

In this section we briefly describe a more general class of models, the Toric Code being an special case of these so called **Quantum Double Models**, these models were originally

analyzed in [55, 56] from a field theoretical point of view. We construct the models as in [1], where Kitaev writes them in a many body interaction Hamiltonian formulation. The model is defined on a two dimensional oriented lattice that is embedded on an arbitrary orientable surface. Every edge of the lattice is oriented and it joins two adjacent vertices. Every vertex can be connected to at least three edges. So the lattice consists on vertices, edges and plaquettes. Although both the lattice and the orientation of the edges can be arbitrary, for simplicity reasons we consider a square lattice and the orientation shown in Fig.(2.16) where each vertex is connected to four edges, two of them are *incoming* whereas the other two are *outgoing*. To each edge i we associate a local Hilbert space $\mathcal{H}_i = \mathbb{C}(G)$ that is the Group Algebra of the finite group G (cf. Appendix C), i.e., the vector space over the complex numbers spanned by the basis elements $\{|\phi_g\rangle\}$ labeled by the elements $g \in G$. The dimension of this local vector space is $D = |G|$, the order of the group. The full Hilbert space is a tensor product of the local Hilbert spaces $\mathcal{H}_i$, namely,

$$\mathcal{H} := \bigotimes_{i \in \mathcal{L}} \mathcal{H}_i. \tag{2.57}$$

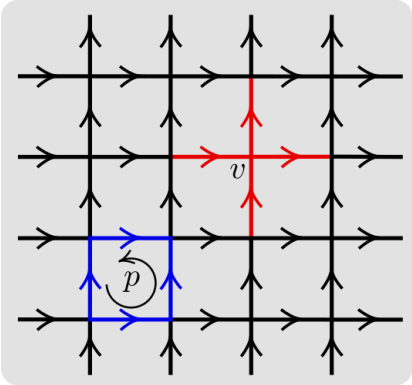


Figure 2.16: The square lattice with oriented edges is shown, where we highlight arbitrary vertex and plaquette, note that we fix he orientation of each plaquette of the lattice as being **counterclockwise**.

2.4.1 Elementary Operators

Let us now define operators acting on the local Hilbert spaces $\mathcal{H}_i$, these operators are labeled by elements ϕ_g of the group algebra $\mathbb{C}(G)$ and elements ξ^g of the dual algebra $\mathbb{C}(G)^*$ (cf. Appendix C). They come in two species, one that performs a shift induced by a group algebra element and given by the group multiplication itself and the other that performs a projective measure of the degrees of freedom configuration on an arbitrary edge of $\mathcal{L}$.

- **Shift Operators:**

$$L_l(\phi_g)\,|\phi_k\rangle = |\phi_g\phi_k\rangle = |\phi_{gk}\rangle\,, \qquad R_l(\phi_g)\,|\phi_k\rangle = |\phi_k\phi_g\rangle = |\phi_{kg}\rangle\,. \tag{2.58}$$

  where the product $\phi_g\phi_k = \phi_{gk}$ in $\mathbb{C}(G)$ is naturally induced by the group multiplication on G, i.e., they form a representation of G.

- **Projection Operator:**

$$\begin{aligned} T(\psi^g)\,|\phi_k\rangle &= \psi^g(\phi_k)\,|\phi_k\rangle\,, \\ &= \delta(g,k)\,|\phi_k\rangle\,. \end{aligned} \tag{2.59}$$

2.4.2 Algebra of Operators

The elementary operators defined in Eqs.(2.58) and (2.59) satisfy the following relations when acting on the same local Hilbert space:

$$L(\phi_g)L(\phi_h) = L(\phi_{gh}), \tag{2.60}$$
$$R(\phi_g)R(\phi_h) = R(\phi_{hg}), \tag{2.61}$$
$$T(\xi^g)T(\xi^h) = \delta(g,h)T(\xi^g), \tag{2.62}$$
$$L(\phi_g)T(\xi^h) = T(\xi^{gh})L(\phi_g), \tag{2.63}$$
$$R(\phi_g)T(\xi^h) = T(\xi^{hg})R(\phi_g), \tag{2.64}$$
$$L(\phi_g)R(\phi_h) = R(\phi_h)L(\phi_g), \tag{2.65}$$

for all $\phi^g \in \mathbb{C}(G)$ and all $\xi^h \in \mathbb{C}(G)^*$. The action of the operators $L(\phi_g)$, $R(\phi_g)$ on the local Hilbert spaces $\mathcal{H}_i$ depends on the orientation given to each edge of the lattice, from now on we consider a square lattice where the edges are oriented as shown in Fig.(2.16). If j is an arbitrary oriented edge of $\mathcal{L}$ and v one of the two vertices it joins, we associate to it an operator $L_j(\phi_g)$ or $R_j(\phi_{g^{-1}})$ depending on where the endpoint v is at. If v is the origin of the arrow (corresponding to the orientation of j) then we act on the edge with $L_j(\phi_g)$, and we act with $R_j(\phi_{g^{-1}})$ otherwise, as depicted in Fig.(2.17). From these elementary operators we can construct the operators that make part of the Hamiltonian of the model.

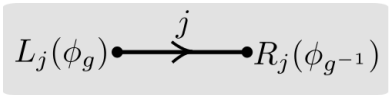


Figure 2.17: The attribution of the shift operators $L_j(\phi_g)$ and $R_j(\phi_{g^{-1}})$ depends on the orientation of the edge j.

2.4.3 Vertex and Plaquette Operators

For any vertex v and plaquette p of the lattice $\mathcal{L}$ we define a set of local operators A_v^g and $B_{p,v}^h$ that act trivially on all edges in $\mathcal{L}$ except for those contained in v and p respectively. Consider an arbitrary vertex $v \in \mathcal{L}$, there are four local Hilbert spaces H_i, $i \in v$ corresponding to the four edges adjacent edges to the vertex. According to the convention given on §2.4.2 we can write the following operator:

$$A_v^g := R_{i_1}(\phi_{g^{-1}}) \otimes R_{i_2}(\phi_{g^{-1}}) \otimes L_{i_3}(\phi_g) \otimes L_{i_4}(\phi_g) \bigotimes_{i \notin v} \mathbb{1}_i, \tag{2.66}$$

that acts non trivially on the four edges around the vertex v as shown in Fig.(2.18). Equivalently we say this operator acts on the subspace $\mathcal{H}_v \subset \mathcal{H}$ consisting on the four

edges around the vertex in question. This operator can be interpreted as a local gauge transformation by the group element $g \in G$ as it shifts the degrees of freedom labeled by the group elements living in edges around the vertex in question. To see this schematically, let us use a graphical representation similar to the one used when describing the **Toric Code** in §2.1. Although the degrees of freedom lying on the edges of the lattice are elements of the group algebra $\mathbb{C}(G)$ whose basis elements are $\{\phi_g\}$, let us label the edges of the lattice by just the group element $|g\rangle$ for no other reason than simplicity. Thus, the action of A_v^g on an arbitrary state with basis elements on its edges is:

$$A_v^g \left| \; |a\rangle, |d\rangle, |b\rangle, |c\rangle \; \right\rangle = \left| \; |ag^{-1}\rangle, |gd\rangle, |bg^{-1}\rangle, |gc\rangle \; \right\rangle, \tag{2.67}$$

where we only represent the relevant part of the lattice, or equivalently, of the Hilbert space that is being acted upon with the vertex operator A_v^g. In an algebraic form this action would look like:

$$\begin{aligned} A_v^g \, |\phi_a\rangle_{i_1} \otimes |\phi_b\rangle_{i_2} \otimes |\phi_c\rangle_{i_3} \otimes |\phi_d\rangle_{i_4} &= |\phi_a \phi_{g^{-1}}\rangle_{i_1} \otimes |\phi_b \phi_{g^{-1}}\rangle_{i_2} \otimes |\phi_g \phi_c\rangle_{i_3} \otimes |\phi_g \phi_d\rangle_{i_4} \\ &= |\phi_{ag^{-1}}\rangle_{i_1} \otimes |\phi_{bg^{-1}}\rangle_{i_2} \otimes |\phi_{gc}\rangle_{i_3} \otimes |\phi_{gd}\rangle_{i_4} \, , \end{aligned} \tag{2.68}$$

The operator A_v that goes to the Hamiltonian of the model is a sum of A_v^g over all $g \in G$, this is

$$A_v := \frac{1}{|G|} \sum_{g \in G} A_v^g. \tag{2.69}$$

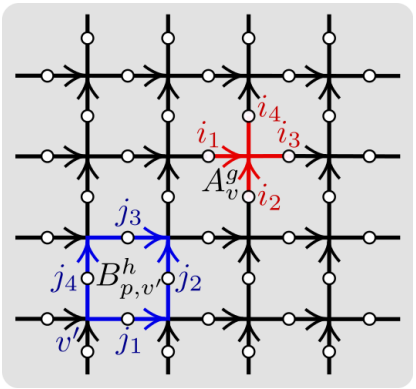
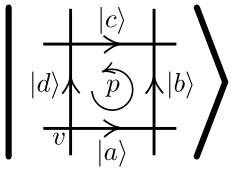
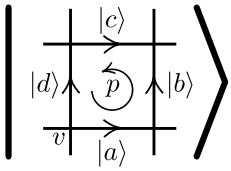


Figure 2.18: The local Hilbert spaces $\mathcal{H}_v$ and $\mathcal{H}_p$ on the lattice over which the A_v^g and $B_{p,v'}^h$ operators act are shown in red and blue respectively.

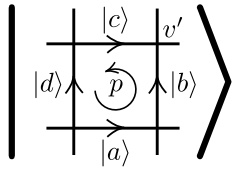
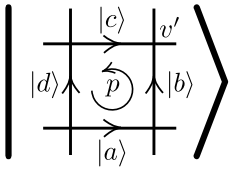
Likewise, let us consider an arbitrary plaquette $p \in \mathcal{L}$ and choose one of the vertices around the plaquette as being v', define an operator that acts non trivially on the Hilbert space $\mathcal{H}_p$ consisting on the four edges at the boundary of the plaquette p

$$B_{p,v'}^h := \sum_{\{g_i\}_{j=1}^{j=4}} \delta(g_1 g_2 g_3^{-1} g_4^{-1}, h) T_{j_1}(\xi^{g_1}) \otimes T_{j_2}(\xi^{g_2}) \otimes T_{j_3}(\xi^{g_3}) \otimes T_{j_4}(\xi^{g_4}) \bigotimes_{j \notin \partial p} \mathbb{1}_j, \tag{2.70}$$

where the sum is taken over all combinations of elements $g_1, g_2, g_3, g_4 \in G$ such that $g_1 g_2 g_3^{-1} g_4^{-1} = h$, and $j_1, \ldots, j_4$ are the edges at the boundary of p listed following a counterclockwise order, starting and ending at the same vertex v. In a schematic manner, the action of $B^h_{p,v}$ on a basis state would be:

$B^h_{p,v}$  $= \delta(abc^{-1}d^{-1}, h)$  , (2.71)

from the above expression is clear that the $B^h_{p,v}$ will project out the states for which the product of group elements along the plaquette's edges is not equal to h. Notice that the product of group elements taken along the plaquette's edges depends on the orientation of each edge, in the above example we start at vertex v following a counterclockwise order, since edges at which $|\phi_a\rangle$ and $|\phi_b\rangle$ are sitting agree with the plaquette orientation they go as a and b to the product, conversely the edges at which the basis elements are $|\phi_c\rangle$ and $|\phi_d\rangle$ do not agree with the plaquette orientation they enter the product as inverse elements $c^{-1}d^{-1}$ Thus, the holonomy of the plaquette ends up being $abc^{-1}d^{-1}$. Consider now, a plaquette operator $B^h_{p,v'}$ where $v \neq v'$ acting on an arbitrary state,

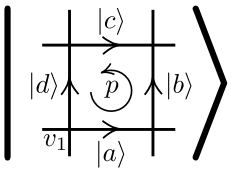
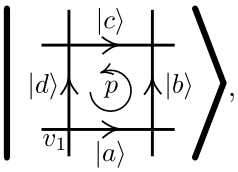
$B^h_{p,v'}$  $= \delta(c^{-1}d^{-1}ab, h)$  . (2.72)

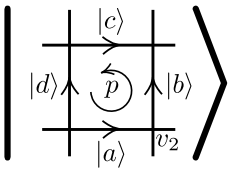
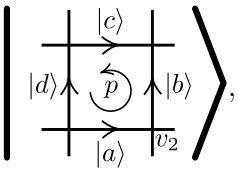
if we compare the expressions within the deltas in Eq.(2.71) and (2.72), in general we have $abc^{-1}d^{-1} \neq c^{-1}d^{-1}ab$ for any group G, the converse holds in specific cases as we will see later. Thus, the choice of the vertex from which we start taking the product along the plaquette p is important, as different results can be gotten for different choices of the vertex v. In particular, the projector to the "identity flux" $B^e_{p,v}$ would be

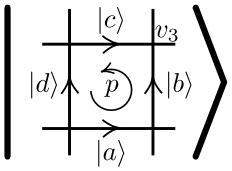
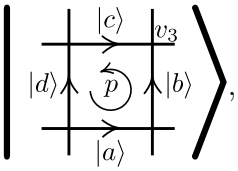
$$\begin{aligned} B^e_{p,v} &= \sum_{\{g_i\}_{j=1}^{j=4}} \delta(g_1 g_2 g_3^{-1} g_4^{-1}, e) T_{j_1}(\xi^{g_1}) \otimes T_{j_2}(\xi^{g_2}) \otimes T_{j_3}(\xi^{g_3}) \otimes T_{j_4}(\xi^{g_4}) \\ &= \sum_{\{g_i\}_{j=1}^{j=3}} T_{j_1}(\xi^{g_1}) \otimes T_{j_2}(\xi^{g_2}) \otimes T_{j_3}(\xi^{g_3}) \otimes T_{j_4}(\xi^{g_1 g_2 g_3^{-1}}), \end{aligned} \tag{2.73}$$

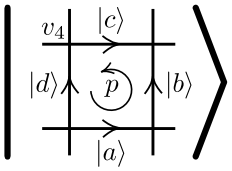
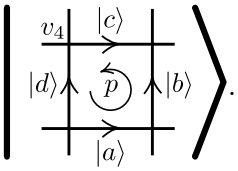
where e is the identity element of the group G. Note that since the element to which this operator is projecting is the identity element e the choice of the vertex v from which start taking the product along the edges of the plaquette is arbitrary, thus we can write $B^e_{p,v} = B^e_p := B_p$. To ilustrate this fact let us consider the state of an arbitrary plaquette p such as the one in Eq.(2.71). To each choice of the vertex v, from which we start measuring the holonomy of the plaquette, we can associate a different plaquette operator, namely B^e_{p,v_1}, B^e_{p,v_2}, B^e_{p,v_3} and B^e_{p,v_4}, where v_i $(i = 1, 2, 3, 4)$ are the four vertices around the plaquette p. The action of these four plaquette operators on the corresponding state

would then be:

B^e_{p,v_1}  $= \delta(abc^{-1}d^{-1}, e)$  , (2.74)

B^e_{p,v_2}  $= \delta(bc^{-1}d^{-1}a, e)$  , (2.75)

B^e_{p,v_3}  $= \delta(c^{-1}d^{-1}ab, e)$  , (2.76)

B^e_{p,v_4}  $= \delta(d^{-1}abc^{-1}, e)$  . (2.77)

Eq.(2.74) tells us that in fact states for which $abc^{-1}d^{-1} = e$ will be eigenstates of B^e_{p,v_1}. Notice that:

$$\begin{aligned} abc^{-1}d^{-1} &= e, \\ a^{-1}abc^{-1}d^{-1}a &= a^{-1}ea, \\ bc^{-1}d^{-1}a &= ea^{-1}a, \\ bc^{-1}d^{-1}a &= e, \end{aligned} \tag{2.78}$$

which is the condition for the states in Eq.(2.75) to be eigenstates of B^e_{p,v_2}. Hence the eigenstates of B^e_{p,v_2} are those of B^e_{p,v_1}. Likewise, take the expression in Eq.(2.78) and multiply both sides by b^{-1} from the left and by b from the right, by performing this operation we get the condition for a state to be eigenstate of B^e_{p,v_3}, this is,

$$\begin{aligned} bc^{-1}d^{-1}a &= e, \\ b^{-1}abc^{-1}d^{-1}ab &= b^{-1}eb, \\ c^{-1}d^{-1}ab &= eb^{-1}b, \\ c^{-1}d^{-1}ab &= e. \end{aligned} \tag{2.79}$$

In a similar fashion, we can show that the condition in Eq.(2.79) is equivalent to $d^{-1}abc^{-1} = e$. Therefore, all conditions are equivalent, and the action of each of the plaquette operators above shown is the same on the arbitrary state of Eqs.(2.74), (2.75), (2.76) and (2.77). This shows the independence of the plaquette operator $B^e_{p,v}$ on the choice of the vertex v from which one starts taking the product of basis elements along the edges of p, due to the fact that $ge = eg$ for any $g \in G$. As a matter of fact, the plaquette operator $B^z_{p,v}$ is independent of the choice of the vertex v for any central element $z \in Z(G)$[1].

[1]The center of a group G, denoted $Z(G)$, is defined by $Z(G) = \{z \in G : zg = gz,\ \forall g \in G\}$.

Define the Hamiltonian of the **Quantum Double model** as an operator that acts on the entire lattice $\mathcal{L}$, or equivalently $H_{QD} : \mathcal{H} \to \mathcal{H}$, where:

$$H_{QD} := -\sum_{v\in\mathcal{L}} A_v - \sum_{p\in\mathcal{L}} B_p, \tag{2.80}$$

as in the case of the **Toric Code** model this Hamiltonian is a sum of commuting projectors, where the A_v operators project out states that are not invariant under the action of A_v^g for all elements $g \in G$ and for each vertex $v \in \mathcal{L}$. Likewise the operator B_p projects out the states that have non-vanishing flux passing through the plaquette $p \in \mathcal{L}$. In the following section we show the algebra of these operators and that they indeed commute with each other, thus they can be simultaneously diagonalized [57].

2.4.4 Algebra of vertex and plaquette operators

The A_v^g and $B_{p,v}^h$ operators act (non-trivially) on subspaces that correspond to the edges adjacent to a vertex v and to the boundary of a plaquette p respectively, thus it is natural to wonder about the algebra of such operators when acting on subspaces with some common support, this is:

- Vertex operators A_v^g and A_v^h, both acting on the same subspace $\mathcal{H}_v$.
- Vertex operators A_v^g and $A_{v'}^h$, where the subspaces $\mathcal{H}_v$ and $\mathcal{H}_{v'}$ have common support, this is the case when v and v' are neighboring vertices.
- Plaquette operators $B_{p,v}^g$ and $B_{p,v}^h$ acting on the same subspace $\mathcal{H}_p$.
- Plaquette operators $B_{p,v}^g$ and $B_{p',v'}^h$ acting on the subspaces $\mathcal{H}_p$ and $\mathcal{H}_{p'}$ that have one edge as a common support, this is the case of two adjacent plaquettes.
- Vertex operator A_v^g and plaquette operator $B_{p,v}^h$ acting on subspaces $\mathcal{H}_v$ and $\mathcal{H}_p$ that have as a common support two edges belonging to both the vertex v and the plaquette p.

We point out that when the subspaces corresponding to vertices and/or plaquettes that do not share any common support the algebra of the corresponding operators is trivial, namely:

$$[A_v^g\ , A_{v'}^h] = 0 \qquad [B_{p,v}^g\ , B_{p',v'}^h] = 0, \tag{2.81}$$

where v, v', p and p' neither are equal nor they share any common support (for any $g, h \in G$). Let us consider now the cases listed above.

Vertex operators A_v^g and A_v^h acting on the same subspace $\mathcal{H}_v$.- From the definition of the vertex operator in Eq.(2.67), we have:

$$\begin{aligned} A_v^g A_v^h &= [R_{i_1}(\phi_{g^{-1}}) \otimes R_{i_2}(\phi_{g^{-1}}) \otimes L_{i_3}(\phi_g) \otimes L_{i_4}(\phi_g)] \times \\ &\quad [R_{i_1}(\phi_{h^{-1}}) \otimes R_{i_2}(\phi_{h^{-1}}) \otimes L_{i_3}(\phi_h) \otimes L_{i_4}(\phi_h)]\,, \\ &= R_{i_1}(\phi_{(gh)^{-1}}) \otimes R_{i_2}(\phi_{(gh)^{-1}}) \otimes L_{i_3}(\phi_{gh}) \otimes L_{i_4}(\phi_{gh}), \\ &= A_v^{gh}, \end{aligned} \tag{2.82}$$

where in the third line we used Eq.(2.60) from §2.4.2. Therefore, they form a representation of G on the Hilbert space $\mathcal{H}$. Because of this fact the vertex operator A_v is a projector, i,e.

$$\begin{aligned} A_v A_v &= \left(\frac{1}{|G|}\sum_{g\in G} A_v^g\right)\left(\frac{1}{|G|}\sum_{h\in G} A_v^h\right), \\ &= \frac{1}{|G|^2}\sum_{g,h\in G} A_v^{gh}, \\ &= \frac{1}{|G|}\sum_{g'\in G} A_v^{g'} \\ &= A_v, \end{aligned} \tag{2.83}$$

where $g' = gh$ and the double sum $\sum_{g,h} A_v^{gh} = |G|\sum_{g'} A_v^{g'}$.

Vertex operators $A_{v_1}^g$ and $A_{v_2}^h$, where v_1 and v_2 are neighboring vertices.- Let us take two vertex operators $A_{v_1}^g$ and $A_{v_2}^h$ acting on the subspaces $\mathcal{H}_{v_1}$ and $\mathcal{H}_{v_2}$ that correspond to the neighboring vertices v_1 and v_2 as depicted in Fig.(2.19). Both operators will act on the common edge i_3, this is

$$\begin{aligned} A_{v_1}^g A_{v_2}^h &= R_{i_1}(\phi_{g^{-1}}) \otimes R_{i_2}(\phi_{g^{-1}}) \otimes L_{i_3}(\phi_g) R_{i_3}(\phi_{h^{-1}}) \otimes L_{i_4}(\phi_g) \otimes R_{i_5}(\phi_{h^{-1}}) \otimes L_{i_6}(\phi_h) \otimes L_{i_7}(\phi_h) \\ &= \cdots \otimes R_{i_3}(\phi_{h^{-1}}) L_{i_3}(\phi_g) \otimes \ldots \\ &= A_{v_2}^h A_{v_1}^g, \end{aligned} \tag{2.84}$$

where in the second equality we used Eq.(2.65). Therefore

$$[A_{v_1}^g, A_{v_2}^h] = 0, \tag{2.85}$$

for any two adjacents vertices v_1 and v_2 in the lattice $\mathcal{L}$. Now, the A_v operator is a sum of A_v^g for all $g \in G$. Let us consider the commutation between two vertex operators A_{v_1} and A_{v_2}, where v_1 and v_2 are two neighboring vertices,

$$\begin{aligned} [A_{v_1}, A_{v_2}] &= [\sum_{g\in G} A_{v_1}^g, \sum_{h\in G} A_{v_2}^h\ ], \\ &= \sum_{g\in G} [A_{v_1}^g, \sum_{h\in G} A_{v_2}^h\ ], \\ &= \sum_{g\in G}\sum_{h\in G} [A_{v_1}^g, A_{v_2}^h], \end{aligned} \tag{2.86}$$

because of Eq.(2.85) we note that the last term of the above expression vanishes, leaving us with

$$[A_{v_1}, A_{v_2}] = 0, \tag{2.87}$$

for any two adjacent vertices $v_1, v_2 \in \mathcal{L}$.

Plaquette operators $B_{p,v}^g$ and $B_{p,v}^h$ acting on the same subspace $\mathcal{H}_p$. The plaquette operator $B_{p,v}^h$ defined in Eq.(2.70) is constructed from the projection operator $T(\xi^g)$, because of Eq.(2.62) these projection operators commute, this is:

$$[T(\xi^g), T(\xi^h)] = 0, \tag{2.88}$$

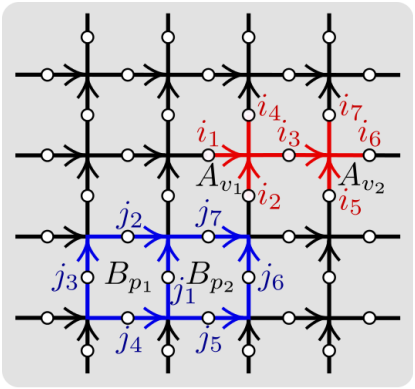


Figure 2.19: Two neighboring vertices v_1, v_2 and two adjacent plaquettes p_1, p_2 are shown on the oriented lattice.

for any $g, h \in G$. Let us consider now two plaquette operators $B^g_{p,v}$ and $B^h_{p,v}$ that act on the local Hilbert space $\mathcal{H}_p$ which corresponds to the four edges around a plaquette p, the action of these two plaquette operators on a given arbitrary basis state being:

$$B^g_{p,v}B^h_{p,v}\Big|\ |c\rangle,\ |d\rangle,\ p,\ |b\rangle,\ v,\ |a\rangle\ \Big\rangle = \delta(abc^{-1}d^{-1},h)B^g_{p,v}\Big|\ |c\rangle,\ |d\rangle,\ p,\ |b\rangle,\ v,\ |a\rangle\ \Big\rangle,$$

$$= \delta(abc^{-1}d^{-1},h)\delta(abc^{-1}d^{-1},g)\Big|\ |c\rangle,\ |d\rangle,\ p,\ |b\rangle,\ v,\ |a\rangle\ \Big\rangle,$$

$$= \delta(abc^{-1}d^{-1},h)\delta(g,h)\Big|\ |c\rangle,\ |d\rangle,\ p,\ |b\rangle,\ v,\ |a\rangle\ \Big\rangle. \tag{2.89}$$

From the last expression we notice that the plaquette operator itself is a projector, i.e.

$$B^g_{p,v}B^h_{p,v} = \delta(g,h)B^h_{p,v}, \tag{2.90}$$

for any plaquette $p \in \mathcal{L}$ and any two group elements $g, h \in G$. Consequently, it is clear that the following commutation relation holds

$$[B^g_{p,v}, B^h_{p,v}] = 0. \tag{2.91}$$

Plaquette operators $B^g_{p,v}$ and $B^h_{p',v'}$, where p and p' are neighboring plaquettes.
For two adjacent plaquettes sharing an edge j_1 as shown in Fig.(2.19), the product of the corresponding plaquette operators would be:

$$B^g_{p_1,v}B^h_{p_2.v'} = \sum_{\{g_i\}} \delta(g_4g_1g_2^{-1}g_3^{-1},g)T_{j_1}(\xi^{g_1})\otimes T_{j_2}(\xi^{g_2})\otimes T_{j_3}(\xi^{g_3})\otimes T_{j_4}(\xi^{g_4})\times$$
$$\sum_{\{g_j\}} \delta(g_5g_6g_7^{-1}g_1^{-1},h)T_{j_5}(\xi^{g_5})\otimes T_{j_6}(\xi^{g_6})\otimes T_{j_7}(\xi^{g_7})\otimes T_{j_1}(\xi^{g_1}), \tag{2.92}$$

Since Eq.(2.62) implies $[T(\xi^g), T(\xi^h)] = 0$ for any $g, h \in G$, the terms that act on the common link j_1 will commute, such that the above expression acts on the corresponding states the same way the following one does

$$B^h_{p_2,v'} B^g_{p_1,v} = \sum_{\{g_j\}} \delta(g_5 g_6 g_7^{-1} g_1^{-1}, h) T_{j_5}(\xi^{g_5}) \otimes T_{j_6}(\xi^{g_6}) \otimes T_{j_7}(\xi^{g_7}) \otimes T_{j_1}(\xi^{g_1}) \times$$
$$\sum_{\{g_i\}} \delta(g_1 g_2^{-1} g_3^{-1} g_4, g) T_{j_1}(\xi^{g_1}) \otimes T_{j_2}(\xi^{g_2}) \otimes T_{j_3}(\xi^{g_3}) \otimes T_{j_4}(\xi^{g_4}), \tag{2.93}$$

this fact is better seen when considering the action of these operators on a basis state, i.e.

$$B^g_{p_1,v} B^h_{p_2,v'} \Big| \; = \delta(g_5 g_6 g_7^{-1} g_1^{-1}, h) B^g_{p_1,v} \Big| \;$$
$$= \delta(g_5 g_6 g_7^{-1} g_1^{-1}, h) \delta(g_4 g_1 g_2^{-1} g_3^{-1}, g) \Big| \;$$
$$= \delta(g_4 g_1 g_2^{-1} g_3^{-1}, g) \delta(g_5 g_6 g_7^{-1} g_1^{-1}, h) \Big| \;$$
$$= B^h_{p_2,v'} B^g_{p_1,v} \Big| \; . \tag{2.94}$$

Hence

$$[B^g_{p_1,v}\ , B^h_{p_2,v'}] = 0, \tag{2.95}$$

for any two adjacent plaquettes $p_1, p_2 \in \mathcal{L}$ and any $g, h \in G$. In particular the commutation relation holds for the *indentity flux* plaquete operator, the one that goes into the Hamiltonian of the model, this is.

$$[B^e_p\ , B^e_{p'}] = [B_p, B_{p'}] = 0, \tag{2.96}$$

for any two plaquettes p and p' in $\mathcal{L}$.

Vertex operator A^g_v and plaquette operator $B^h_{p,v}$ acting on subspaces $\mathcal{H}_v$ and $\mathcal{H}_p$.

The local Hilbert spaces $\mathcal{H}_v$ and $\mathcal{H}_p$, where v is one of the four vertices around the plaquette p, share two edges as common support as depicted in Fig.(2.20). In this case, the joint action of the vertex operator A^g_v and plaquette operator $B^h_{p,v}$ depends on the order they act on the state, this is because the shift operators $L(\phi_g)$, $R(\phi_g)$ in general do not commmute with the projection operator $T(\xi^h)$ as exhibited in Eqs.(2.63) and (2.64). This is:

$$A^g_v B^h_{p,v} = \sum_{\{g_i\}} \delta(g_1^{-1} g_2 g_3 g_4^{-1}, h) R_{j_1}(\phi_{g^{-1}}) T_{j_1}(\xi^{g_1}) \otimes T_{j_2}(\xi^{g_2}) \otimes T_{j_3}(\xi^{g_3}) \otimes$$
$$L_{j_4}(\phi_g) T_{j_4}(\xi^{g_4}) \otimes L_{j_5}(\phi_g) \otimes R_{j_6}(\phi_{g^{-1}}), \tag{2.97}$$

from Eqs.(2.63) and (2.64) the operators acting on edges j_1 and j_4 can be written as $T_{j_1}(\xi^{g_1 g^{-1}})R_{j_1}(\phi_{g^{-1}})$ and $T_{j_4}(\xi^{gg_4})L_{j_4}(\phi_g)$ respectively, so the above expression would now be:

$$A_v^g B_{p,v}^h = \sum_{\{g_i\}} \delta(g_1^{-1}g_2g_3g_4^{-1}, h) T_{j_1}(\xi^{g_1 g^{-1}})R_{j_1}(\phi_{g^{-1}}) \otimes T_{j_2}(\xi^{g_2}) \otimes T_{j_3}(\xi^{g_3}) \otimes \\ \otimes T_{j_4}(\xi^{gg_4})L_{j_4}(\phi_g) \otimes L_{j_5}(\phi_g) \otimes R_{j_6}(\phi_{g^{-1}}),$$

by performing the following change of variables: $g_1' = g_1 g^{-1}$ and $g_4' = gg_4$, the above summation changes into:

$$A_v^g B_{p,v}^h = \sum_{\{g_i\}} \delta(gg_1'^{-1}g_2g_3g_4'^{-1}g^{-1}, h) T_{j_1}(\xi^{g_1'})R_{j_1}(\phi_{g^{-1}}) \otimes T_{j_2}(\xi^{g_2}) \otimes T_{j_3}(\xi^{g_3}) \otimes \\ \otimes T_{j_4}(\xi^{g_4'})L_{j_4}(\phi_g) \otimes L_{j_5}(\phi_g) \otimes R_{j_6}(\phi_{g^{-1}}),$$

or equivalently

$$A_v^g B_{p,v}^h = \sum_{\{g_i\}} \delta(g_1^{-1}g_2g_3g_4^{-1}, g^{-1}hg) T_{j_1}(\xi^{g_1})R_{j_1}(\phi_{g^{-1}}) \otimes T_{j_2}(\xi^{g_2}) \otimes T_{j_3}(\xi^{g_3}) \otimes \\ \otimes T_{j_4}(\xi^{g_4})L_{j_4}(\phi_g) \otimes L_{j_5}(\phi_g) \otimes R_{j_6}(\phi_{g^{-1}}), \tag{2.98}$$

where we dropped the $'$ notation since g_1' and g_4' are now dummy variables. Notice that the right hand side of the above equation is actually the action of the vertex operator A_v^g followed by the action of a plaquette operator $B_{p,v}^{g^{-1}hg}$, hence the following relation

$$A_v^g B_{p,v}^h = B_{p,v}^{g^{-1}hg} A_v^g. \tag{2.99}$$

In particular, if we choose $h = e$ in the above relation, it is straightforward to check that the operators commute, i.e.

$$A_v^g B_{p,v}^e = B_{p,v}^{g^{-1}eg} A_v^g \tag{2.100}$$
$$= B_{p,v}^e A_v^g, \tag{2.101}$$

for any $g \in G$, moreover, the $B_{p,v}^e$ operator commute with the A_v one since it does it with every one of A_v's components as we just have shown. Thus the Hamiltonian defined in Eq.(2.80) is a sum of commuting projectors as claimed in §2.4.3.

2.5 An Example: $\mathbb{Z}_2$ Quantum Double Model

In this section we intend to describe the way in which the Toric Code model is an special case of the class of models described in §2.4. Let us choose the group G as being the cyclic group of order 2 usually denoted as $\mathbb{Z}_2 = \{e, a : a^2 = e\}$, e being the identity element of the group. Canonically, the group algebra $\mathbb{C}(\mathbb{Z}_2)$ is a 2-dimensional vector space over the complex numbers spanned by the basis vectors $\{\phi_e, \phi_a\}$. In the following lines we systematically construct the operators composing the model, departing from the elementary operators of §2.4.1 and arriving to the vertex and plaquette operators that make up the Hamiltonian of this model.

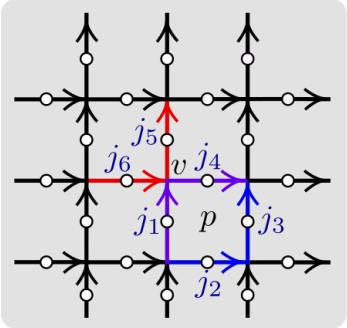


Figure 2.20: The local Hilbert spaces $\mathcal{H}_v$ and $\mathcal{H}_p$ are shown, they correspond to the four edges around vertex v and the four that lie along the plaquette p, we highlight the two common edges by coloring them purple.

2.5.1 Elementary Operators

As stated in §2.4.1 there are three elementary operators, namely the two shift operators labeled by the basis elements of the group algebra $\mathbb{C}(\mathbb{Z}_2)$ and the projection operator labeled by elements of the dual algebra $\mathbb{C}^*(\mathbb{Z}_2)$.

- **Shift Operators:** The action of these operators is given by:

$$L(\phi_e)\,|\phi_k\rangle = |\phi_k\rangle = R(\phi_e)\,|\phi_k\rangle\,, \tag{2.102}$$
$$L(\phi_a)\,|\phi_e\rangle = |\phi_a\rangle = R(\phi_a)\,|\phi_e\rangle\,, \tag{2.103}$$
$$L(\phi_a)\,|\phi_a\rangle = |\phi_e\rangle = R(\phi_a)\,|\phi_a\rangle\,. \tag{2.104}$$

  Where $k = e, a \in G$ in the first expression.

- **Projection Operator:**

$$T(\xi^e)\,|\phi_e\rangle = |\phi_e\rangle\,, \qquad T(\xi^e)\,|\phi_a\rangle = 0, \tag{2.105}$$
$$T(\xi^a)\,|\phi_e\rangle = 0, \qquad T(\xi^a)\,|\xi_a\rangle = |\phi_a\rangle\,, \tag{2.106}$$

Let us choose a 2-dimensional representation in which the basis elements are given by:

$$|\phi_e\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}, \qquad |\phi_a\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}, \tag{2.107}$$

the action of the elementary operators in Eqs.(2.102), (2.103) and (2.104) induce the representations of such operators as being:

$$L(\phi_e) \equiv \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbb{1}_{2\times 2}, \qquad L(\phi_a) \equiv \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \sigma^x, \tag{2.108}$$

for the right shift operator

$$R(\phi_e) \equiv \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbb{1}_{2\times 2}, \qquad R(\phi_a) \equiv \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \sigma^x, \tag{2.109}$$

and the projection operators of Eqs.(3.57) and (3.58) would then be:

$$T(\xi^e) \equiv \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \qquad T(\xi^a) \equiv \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}. \tag{2.110}$$

Notice that the representations of the projection operators can be expressed using the σ^z Pauli operator, this is:

$$T(\xi^e) = \frac{\mathbb{1} + \sigma^z}{2}, \qquad T(\xi^a) = \frac{\mathbb{1} - \sigma^z}{2}. \tag{2.111}$$

Now that we have written the elementary operators of the model we can consruct the vertex and plaquette operators.

2.5.2 Vertex and Plaquette Operators

Because of the model being based on the $\mathbb{Z}_2$ group, from Eq.(2.66) we get two vertex operators, the first being:

$$A_v^e := R_{i_1}(\phi_e) \otimes R_{i_2}(\phi_e) \otimes L_{i_3}(\phi_e) \otimes L_{i_4}(\phi_e) = \mathbb{1}_{i_1} \otimes \mathbb{1}_{i_2} \otimes \mathbb{1}_{i_3} \otimes \mathbb{1}_{i_4}, \tag{2.112}$$

where i_j for $j = 1, 2, 3, 4$ are the four vertices around an arbitrary vertex v counted in the order shown in Fig.(2.18). The second vertex operator of the model is the one labeled by the ϕ_a element of the group algebra $\mathbb{C}(\mathbb{Z}_2)$, this is

$$A_v^a := R_{i_1}(\phi_a) \otimes R_{i_2}(\phi_a) \otimes L_{i_3}(\phi_a) \otimes L_{i_4}(\phi_a) = \sigma_{i_1}^x \otimes \sigma_{i_2}^x \otimes \sigma_{i_3}^x \otimes \sigma_{i_4}^x. \tag{2.113}$$

Therefore the vertex operator A_v that is part of the Hamiltonian H_{QD} is given by:

$$A_v := \frac{1}{2}\left(A_v^e + A_v^a\right) = \frac{1}{2}\left(\mathbb{1}_{i_1} \otimes \mathbb{1}_{i_2} \otimes \mathbb{1}_{i_3} \otimes \mathbb{1}_{i_4} + \sigma_{i_1}^x \otimes \sigma_{i_2}^x \otimes \sigma_{i_3}^x \otimes \sigma_{i_4}^x\right). \tag{2.114}$$

Let us construct the plaquette operator that projects to the identity flux, although in general this operator might be difficult to write in a compact form, the simplicity of the group in which this model is based upon together with the relations in Eqs.(2.111) allows us to write $B_p = B_p^e$ in terms of Pauli operators. From eq.(2.70) we have

$$B_p^e := \sum_{g_i} \delta(g_1 g_2 g_3^{-1} g_4^{-1}, e) T_{j_1}(\xi^{g_1}) \otimes T_{j_2}(\xi^{g_2}) \otimes T_{j_3}(\xi^{g_3}) \otimes T_{j_4}(\xi^{g_4}), \tag{2.115}$$

to expand the summation we first need to know the configurations of the g_i elements such that the delta gives in the above expression is satisfied. The following table summarizes all possible combinations that give the identity element after taking their product.

From this table, the projectors that make the plaquette operator B_p^e:

$$\begin{aligned} B_p^e = {} & T_{i_1}(\xi^e) \otimes T_{i_2}(\xi^e) \otimes T_{i_3}(\xi^e) \otimes T_{i_4}(\xi^e) + T_{i_1}(\xi^e) \otimes T_{i_2}(\xi^e) \otimes T_{i_3}(\xi^a) \otimes T_{i_4}(\xi^a) + \\ & T_{i_1}(\xi^e) \otimes T_{i_2}(\xi^a) \otimes T_{i_3}(\xi^e) \otimes T_{i_4}(\xi^a) + T_{i_1}(\xi^a) \otimes T_{i_2}(\xi^e) \otimes T_{i_3}(\xi^e) \otimes T_{i_4}(\xi^a) + \\ & T_{i_1}(\xi^e) \otimes T_{i_2}(\xi^a) \otimes T_{i_3}(\xi^a) \otimes T_{i_4}(\xi^e) + T_{i_1}(\xi^a) \otimes T_{i_2}(\xi^a) \otimes T_{i_3}(\xi^e) \otimes T_{i_4}(\xi^e) + \\ & T_{i_1}(\xi^a) \otimes T_{i_2}(\xi^a) \otimes T_{i_3}(\xi^e) \otimes T_{i_4}(\xi^e) + T_{i_1}(\xi^a) \otimes T_{i_2}(\xi^a) \otimes T_{i_3}(\xi^a) \otimes T_{i_4}(\xi^a), \end{aligned} \tag{2.116}$$

by replacing the expressions of Eq.(2.111) into the above equation and expanding the tensor products the plaquette operator reduces to:

$$B_p = B_p^e = \frac{1}{2}\left(\mathbb{1}_{j_1} \otimes \mathbb{1}_{j_2} \otimes \mathbb{1}_{j_3} \otimes \mathbb{1}_{j_4} + \sigma_{j_1}^z \otimes \sigma_{j_2}^z \otimes \sigma_{j_3}^z \otimes \sigma_{j_4}^z\right). \tag{2.117}$$

g_1	g_2	g_3	g_4
e	e	e	e
e	e	a	a
e	a	e	a
a	e	e	a
e	a	a	e
a	e	a	e
a	a	e	e
a	a	a	a

Table 2.1: Identity flux configurations.

2.5.3 Ground States

The Hamiltonian of the $\mathbb{Z}_2$ Quantum Double model is defined as:

$$H_{QD} := -\sum_{v\in\mathcal{L}} A_v - \sum_{p\in\mathcal{L}} B_p, \tag{2.118}$$

where the plaquette and vertex operators are those obtained in Eqs.(2.117) and (2.114). The states of this model are elements of the Hilbert space $\mathcal{H}$ which in turn is a tensor product of local Hilbert spaces $\mathcal{H}_i = \mathbb{C}(\mathbb{Z}_2)$ corresponding to each edge of the lattice $\mathcal{L}$. Clearly any state $|\Psi_0\rangle \in \mathcal{H}$ such that

$$A_v |\Psi_0\rangle = |\Psi_0\rangle = B_p |\Psi_0\rangle , \tag{2.119}$$

for any vertex v and plaquette p on the lattice $\mathcal{L}$, will be a ground state of the model. Consider the following state:

$$|\Psi_0\rangle = \prod_{v\in\mathcal{L}} A_v \bigotimes_{i\in\mathcal{L}} |\phi_e\rangle_i , \tag{2.120}$$

this state is exactly the same to the one defined on Eq.(2.31) (with the identification $\phi_e \mapsto \phi_1$), and it corresponds to a linear combination of **Loop Gas** states. If the lattice is embedded on a surface with non-zero genus, say the surface of a torus, it is possible to construct more ground states departing from this one by using the loop operators in §2.1.2 defined along non-contractible paths on the surface.

Chapter 3

The Quantum Double with Matter Fields

In this chapter we introduce a new class of models that can be thought of as a generalization of the models described in chapter 2. These models are, as well, defined on a 2-dimensional lattice, that can be embedded on a non-trivial surface such as a torus, and are also based on a group algebra $\mathbb{C}(G)$ of a group G. The generalization procedure includes associating degrees of freedom to the vertices of the lattice in addition to those associated to the edges of the lattice as in the case of the **Quantum Double Models** (QDM). The vertex degrees of freedom will later be interpreted as Matter fields whereas the edge degrees of freedom as being Gauge fields. As a next step we define an **action** of the Gauge degrees of freedom on the Matter ones, which is naturally interpreted as an on-site **Gauge Transformation**, it is through this action that we define an operator that resembles the vertex operator in the **QDM** case, together with a new kind of operator that plays the role of a parallel transport. We show the way the operators are constructed, their properties and then define one of the many possible Hamiltonians that can be constructed out of these operators. In particular, we solve the model for a special choice of the group G and the vector space on vertices V_n aiming towards a generalization of the **Toric Code**, we construct the ground state of this model and show that the ground state degeneracy becomes trivial and not dependent on the topology of the surface the lattice is embedded in.

3.1 General Description of the Model

The model is defined on a 2-dimensional oriented lattice that can be embedded on any closed orientable surface as in the case of the **QDM** model. The lattice consists on vertices, edges and plaquettes. For simplicity we consider the lattice $\mathcal{L}$ shown in Fig.(3.1), so each vertex is connected to four edges and every edge connects two vertices. To each vertex $v \in \mathcal{L}$ we associate a finite n-dimensional vector space $\mathcal{H}_v = V_n$ whose orthonormal basis elements are $\{\chi_\alpha\}_{\alpha=1}^n$. Likewise, to each edge $l \in \mathcal{L}$ associate a vector space $\mathcal{H}_l = \mathbb{C}(G)$, the group algebra of a group G (cf. Appendix C) spanned by the orthonormal basis $\{\phi_g\}_{g\in G}$, indexed by the group elements $g \in G$. Thus, the full Hilbert space is a tensor

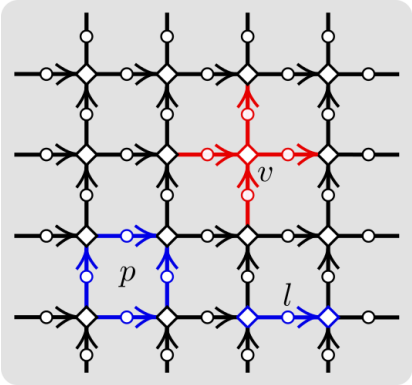


Figure 3.1: We consider an oriented square lattice, it is made of vertices which hold degrees of freedom ($\Diamond$) belonging to the vector space V_n and joined together by edges that hold degrees of freedom ($\circ$) belonging to the group algebra $\mathbb{C}(G)$. The subspaces corresponding to a plaquette p and a link l (together with its two endpoint vertices) are highlighted in blue. The subspace corresponding to a vertex v is highlighted in red.

product of vertex and link Hilbert spaces, namely:

$$\mathcal{H} := \left[\bigotimes_{v\in\mathcal{L}} \mathcal{H}_v\right] \otimes \left[\bigotimes_{l\in\mathcal{L}} \mathcal{H}_l\right] \tag{3.1}$$

From now on we will refer to the vertex Hilbert spaces as *Matter sector* and the edges Hilbert spaces as *Gauge sector* to differentiate one from another, in the following sections we describe the elementary operators that make the model, very much in the same fashion of the models described in chapter 2.

3.1.1 Elementary Operators

The operators that act on the Gauge sector are those of the Quantum Double models, we just mention them as they were discussed in detail in §2.4.1.

3.1.1.1 Gauge Operators

We distinguish two types of operators, one that acts as a gauge transformation and the other as a projection operator, defined as:

- **Shift Operators:**

$$L_l(\phi_g)\,|\phi_h\rangle = |\phi_{gh}\rangle \quad \text{and} \quad R_l(\phi_g)\,|\phi_h\rangle = |\phi_{hg}\rangle\,. \tag{3.2}$$

  They induce the left and right multiplication by the group element $g \in G$.

- **Projection Operator:**

$$\begin{aligned} T(\xi^g)\,|\phi_h\rangle &= \xi^g(\phi_h)\,|\phi_h\rangle\,. \\ &= \delta(g,h)\,|\phi_h\rangle\,, \end{aligned} \tag{3.3}$$

  This operator projects to the element that labels it.

The algebra of these operators is described in detail in §2.4.2.

3.1.1.2 Matter Operators

Likewise, we would like to define operators that have the same action but on the matter degrees of freedom, this is, operators that perform gauge transformations and that measure a given configuration by means of projections. To construct the operator that makes a shift on the matter degrees of freedom $\mathcal{H}_v$ induced by an element of the Gauge sector $\phi_g \in \mathbb{C}(G)$, we need to define the action of $\mathbb{C}(G)$ on the matter sector V_n. Let $\mu : \mathbb{C}(G) \otimes V_n \to V_n$ be a map that takes the pair $(\phi_g, \chi_\alpha) \mapsto \chi_\beta$, we denote this map as the left action $\mu(\phi_g)\chi_\alpha = \chi_\beta$, where χ_β is another basis element resulting from a permutation induced by the group algebra element $\mu(\phi_g)$. Thus we restrict the action to be a permutation between basis elements in V_n. This map is chosen to be a group homomorphism, it is unitary and the action induced by the identity element ϕ_e of $\mathbb{C}(G)$ is the trivial action, i.e.

$$\textbf{P1} \qquad \mu(\phi_a)\mu(\phi_b) = \mu(\phi_a\phi_b) = \mu(\phi_{ab}), \tag{3.4}$$

$$\textbf{P2} \qquad \mu(\phi_a)^\dagger = \mu(\phi_{a^{-1}}), \tag{3.5}$$

$$\textbf{P3} \qquad \mu(\phi_e) = \mathbb{1}. \tag{3.6}$$

Provided with this map we are able to define the shift operator on the matter sector.

Shift Operator:

$$M_v^g \left|\chi_\alpha\right\rangle = \left|\mu(\phi_g)\chi_\alpha\right\rangle. \tag{3.7}$$

The projection operator is very similar to the one of the gauge sector and it measures the configuration of a matter degree of freedom on a particular vertex $v \in \mathcal{L}$, it is parametrized by a basis element of the dual vector space V_n^* (cf. Appendix B).

Projection Operator:

$$\begin{aligned} T_v(\chi^\alpha)\left|\chi_\beta\right\rangle &= \chi^\alpha(\chi_\beta)\left|\chi_\beta\right\rangle, \\ &= \delta(\alpha,\beta)\left|\chi_\beta\right\rangle. \end{aligned} \tag{3.8}$$

Where $\chi_\beta \in V_n$ is a basis element.

3.1.1.3 Algebra of Matter Operators

From the first property of the μ map, Eq.(3.4), follows that the shift operator is, as well, a group homomorphism, i.e.

$$M_v^g M_v^h = M_v^{gh}, \qquad \forall g, h \in G. \tag{3.9}$$

The projection operator in the matter sector is a projector since:

$$T_v(\chi^\alpha)T_v(\chi^\beta) = \delta(\alpha,\beta)T_v(\chi^\beta), \tag{3.10}$$

from this relation follows that the projection operators commute when acting on the same vertex Hilbert space $\mathcal{H}_v$, i.e.

$$[T_v(\chi^\alpha), T_v(\chi^\beta)] = 0. \tag{3.11}$$

Let us consider now the action of a shift operator M_v^g and a projection operator $T_v(\chi^\alpha)$ acting on the same vertex Hilber space $\mathcal{H}_v$, this action is order dependent as:

$$M_v^g T_v(\chi^\alpha) = T_v(\chi^{\alpha'})M_v^g, \tag{3.12}$$

where $\chi^{\alpha'}$ is such that $\chi^{\alpha'}(\mu(\phi_g)\chi_\alpha) = 1$.

Since these operators act on the vertex Hilbert spaces $\mathcal{H}_v$ it is clear that two operators acting on spaces with no common support will trivially commute, this is

$$[M_v^g, M_{v'}^h] = 0, \tag{3.13}$$

$$[T_v(\chi^\alpha), M_{v'}^h] = 0, \tag{3.14}$$

$$[T_v(\chi^\alpha), T_{v'}(\chi^\beta)] = 0, \tag{3.15}$$

where $v \neq v'$ and for all $g, h \in G$, $\chi^\alpha, \chi^\beta \in V_n^*$.

3.1.2 Plaquette, Vertex and Link Operators

In order to build the model we define operators very similar to the ones that make the **Quantum Double Model** in addition to a new operator that plays the role of a parallel transport. In the following lines we thoroughly describe them together with their properties and the algebra between them. The first operator we consider is one that acts only on the gauge degrees of freedom, and it measures their configuration along a plaquette, it can be interpreted as the connection of the theory since it measures the holonomy of the minimum closed loop that can be defined on the lattice, this is, the holonomy of a plaquette. This operator is exactly the same of the pure gauge model or **Quantum Double Model**, for this reason we will just mention it here, for further detail we refer the reader to §2.4.3.

Consider the subspace $\mathcal{H}_p \subset \mathcal{H}$ consisting on the four links along an arbitrary plaquette p, label these links by j_i where $i = 1, 2, 3, 4$ and choose one of the four vertices v as shown in Fig.(3.2) to define the operator

$$B_{p,v}^h := \sum_{\{g_i\}_{i=1}^4} \delta(g_1 g_2 g_3^{-1} g_4^{-1}, h)\; T_{j_1}(\xi^{g_1}) \otimes T_{j_2}(\xi^{g_2}) \otimes T_{j_3}(\xi^{g_3}) \otimes T_{j_4}(\xi^{g_4}), \tag{3.16}$$

as in the **QDM** case this operator projects out the gauge degrees of freedom on links along the plaquette for which their product is different from the element that labels the operator, in this case $h \in G$. Although we do not write it explicitly this operator acts as identity on the rest of the Hilbert space. In particular, as shown in §2.4.3 the plaquette operator labeled by the identity element $e \in G$ does not depend on the choice of the vertex v and we set it to be the one that goes into the Hamiltonian of the model,

$$B_{p,v}^e = B_p^e := B_p, \tag{3.17}$$

for any of the four vertices v around the plaquette p, the action of this operator on a basis state is the one given by Eq.(2.71).

Now, we would like to define an operator that performs a gauge transformation, in the **QDM** case this was done by the vertex operator A_v. As now we are considering a more general model for which there are degrees of freedom living on the vertices of the lattice $\mathcal{L}$, the vertex operator that performs the gauge transformation has its action extended to the vertex degree of freedom by means of the shift operator in the matter sector M_v^g, therefore it is defined as:

$$A_v^g := M_v^g \otimes R_{i_1}(\phi_{g^{-1}}) \otimes R_{i_2}(\phi_{g^{-1}}) \otimes L_{i_3}(\phi_g) \otimes L_{i_4}(\phi_g), \tag{3.18}$$

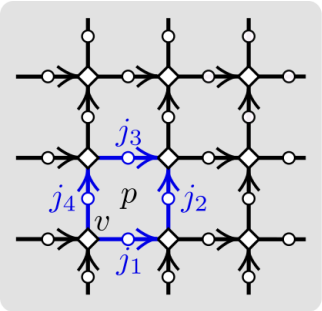
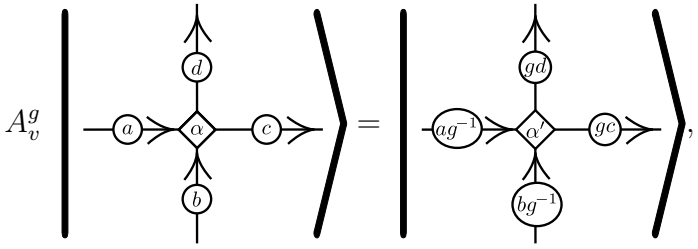


Figure 3.2: The subspace $\mathcal{H}_p$ on which the plaquette operator $B^h_{p,v}$ acts is shown in blue.

where $\phi_g \in \mathbb{C}(G)$. This operator acts non trivially on the Hilbert space corresponding to the four edges adjacent to the vertex v and the vertex degree of freedom itself, as shown in Fig.(3.3).

In order to illustrate the action of this operator on a basis state we use a graphical representation where the labels on the edges are the group indices $g \in G$. The vertex labels are greek letters representing the matter degrees of freedom $\chi_\alpha \in V_2$, such that the action of the elementary vertex operator A^g_v on a basis state is given by:



(3.19)

where α' on the vertex stands for $\chi_{\alpha'} = \mu(\phi_g)\chi_\alpha$, and $a\overline{g}$ stands for $\phi_{ag^{-1}}$.

The vertex operator that makes part of the model is a sum of the elementary vertex operators A^g_v over all $g \in G$, this is:

$$A_v := \frac{1}{|G|} \sum_{g \in G} A^g_v. \tag{3.20}$$

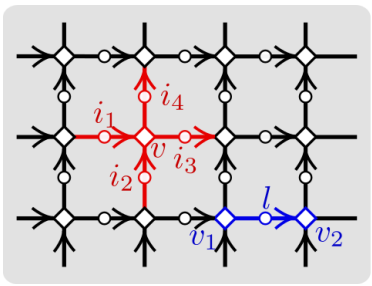


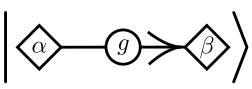
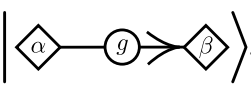
Figure 3.3: The vertex operator A^g_v acts on a vertex v and its four adjacents edges.

Let us now define an operator that measures the configuration of two adjacent vertex degrees of freedom, and the link degree of freedom in between. We call it **Link Operator** as it acts on the Hilbert spaces corresponding to an edge and the two vertices it joins.

Its action is not restricted to the measurement of these configurations as it also compares the degrees of freedom on vertices v_1 and v_2 after gauge transforming one of them by the action induced by the gauge degree of freedom sitting at the edge in between; such that it will project out configurations for which the two vertex degrees of freedom are not equal after gauge transforming one of them, the choice of which vertex degree of freedom is going to be transformed depends on the orientation. The operator is defined as:

$$C_l := \sum_{\chi_\alpha,\chi_\beta \in V_n} \sum_{g\in G} \delta(\chi_\alpha, \mu(\phi_g)\chi_\beta) T_{v_1}(\chi^\alpha) \otimes T_l(\xi^g) \otimes T_{v_2}(\chi^\beta), \tag{3.21}$$

where the two vertices v_1, v_2 and the link l are depicted in Fig.(3.3). Using our graphical representation the action of the link operator C_l on a basis state is:

C_l  $= \delta(\chi_\alpha, \mu(\phi_g)\chi_\beta)$  .

(3.22)

3.1.3 Algebra of Operators

In this section we discuss the algebra of operators A_v^g, $B_{p,v}^h$ and C_l, in particular, we show that the operators A_v, B_p and C_l are projectors and commute with each other as well as with themselves. As a consequence, a model that has its Hamiltonian written as a sum of such operators can be exactly solved, which is our ultimate goal. Notice that the algebra of these operators is trivial when they act on subspaces that do not share any common support. This is, $[A_v^g, A_{v'}^h] = 0$ where the vertices v and v' are not adjacent to each other, such that the corresponding vertex operators do not share any common support. Thus it follows that

$$[A_v, A_{v'}] = 0. \tag{3.23}$$

For the plaquette operators,

$$[B_{p,v}^g, B_{p',v'}^h] = 0, \tag{3.24}$$

where p and p' are two non adjacent plaquettes and the vertices v, v' are any of the four vertices on each plaquette respectively. In particular, for the *identity flux* plaquette operator $[B_p, B_{p'}] = 0$. Likewise, when considering two link operators such that the edges l and l' do not share any common vertex, it is clear that:

$$[C_l, C_{l'}] = 0. \tag{3.25}$$

Moreover, the following relations hold:

$$[A_v^g, B_{p',v'}^h] = [A_v^g, C_l] = [B_{p,v}^h, C_l] = 0, \tag{3.26}$$

where the operators do not share any common support whatsoever. So far the commutation relations between our operators is trivial since we just considered operators acting on different subspaces, in the following lines we show how these commutation relations are modified when considering operators acting in subspaces that share common support. For this matter we distinguish two cases, namely, the algebra of alike operators and the one of different operators. **Vertex operators A_v^g and A_v^h acting in the same subspace**

$\mathcal{H}_v$.-

As in the **QDM** case, in general, these operators do not commute as they form a representation of the group, this is:

$$A_v^g A_v^h = A_v^{gh}, \tag{3.27}$$

and in general $gh \neq hg$. However, the vertex operator A_v that is a sum of the elementary A_v^g's is a projector since

$$\begin{aligned} A_v A_v =& \left(\frac{1}{|G|}\sum_{g\in G} A_v^g\right)\left(\frac{1}{|G|}\sum_{h\in G} A_v^h\right), \\ =& \frac{1}{|G|^2}\sum_{g\in G}\sum_{h\in G} A_v^{gh} \\ =& \frac{1}{|G|}\sum_{g'\in G} A_v^{g'}, \\ =& A_v. \end{aligned} \tag{3.28}$$

for all $g, h \in G$.

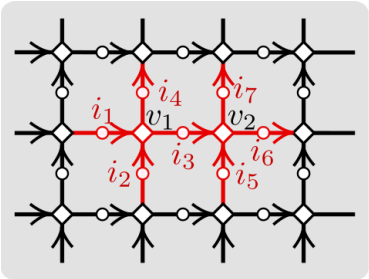


Figure 3.4: Two adjacent vertices v_1 and v_2 in $\mathcal{L}$ are shown, the vertex operators A_{v_1} and A_{v_2} act on the i_3 edge simultaneously.

Vertex operators $A_{v_1}^g$ and $A_{v_2}^h$, where v_1 and v_2 are neighbouring vertices.- Consider the vertex operators $A_{v_1}^g$ and $A_{v_2}^h$ acting on adjacent vertices v_1 and v_2 as shown in Fig.(3.4), equivalently they act on the subspaces corresponding to the two vertices and their edges. Note that both operators have the i_3 edge as common support, this is, they both act on the Hilbert space $\mathcal{H}_{i_3}$ as in the **QDM** case, so $A_{v_1}^g A_{v_2}^h = A_{v_2}^h A_{v_1}^g$. Therefore

$$[A_{v_1}^g, A_{v_2}^h] = 0 \tag{3.29}$$

for any two adjacents vertices $v_1, v_2 \in \mathcal{L}$. As a consequence:

$$[A_{v_1}, A_{v_2}] = 0. \tag{3.30}$$

Plaquette operator $B_{p,v}^g$ all cases.-

As this operator is essentially not modified from the **QDM** case so we just mention its algebra (cf. §2.4.4 on chapter 2 for details). Consider the case of two plaquette operators $B_{p,v}^g$ and $B_{p,v}^h$ acting on the same subspace $\mathcal{H}_p$, from Eq.(2.90) we know that these operators are projectors as $B_{p,v}^g B_{p,v}^h = \delta(g,h) B_{p,v}^h$, therefore

$$[B_{p,v}^g, B_{p,v}^h] = 0. \tag{3.31}$$

Similarly, when the two plaquette operators act on neighbouring plaquettes p_1 and p_2, as shown in Fig.(3.5) we have:

$$[B^g_{p_1,v}, B^h_{p_2,v'}] = 0, \tag{3.32}$$

for any p_1, p_2 adjacent plaquettes in $\mathcal{L}$, any $g, h \in G$ and any vertex v, v' of the two plaquettes respectively. In particular for the *identity flux* plaquette operator:

$$[B^e_{p_1}, B^e_{p_2}] = [B_{p_1}, B_{p_2}] = 0, \tag{3.33}$$

Thus, we can conclude that the plaquette operator is a projector and it commutes with each other for any label $g \in G$ and any plaquette $p \in \mathcal{L}$.

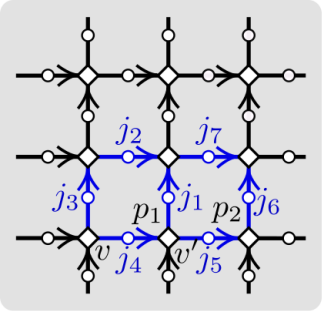


Figure 3.5: Two adjacent plaquettes p_1 and p_2 are shown, the corresponding plaquette operators act simultaneously on the edge j_1.

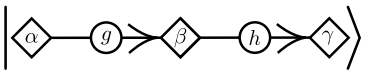
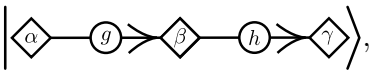
Link Operator C_l acting on the same subspace.-

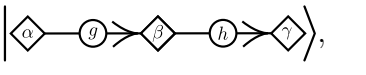
In this case, we consider two link operators acting on the same Hilbert subspace corresponding to an edge l and the two vertices v_1, v_2 it joins as shown in Fig.(3.6), from Eq.(3.21) we have:

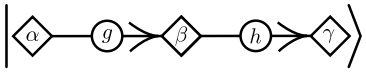
$$\begin{aligned}
C_l C_l = &\left(\sum_{\chi_\alpha, \chi_\beta \in V_n} \sum_{g \in G} \delta(\chi_\alpha, \mu(\phi_g)\chi_\beta) T_{v_1}(\chi^\alpha) \otimes T_l(\xi^g) \otimes T_{v_2}(\chi^\beta) \right) \times \\
&\left(\sum_{\chi_{\alpha'}, \chi_{\beta'} \in V_n} \sum_{h \in G} \delta(\chi_{\alpha'}, \mu(\phi_h)\chi_{\beta'}) T_{v_1}(\chi^{\alpha'}) \otimes T_l(\xi^h) \otimes T_{v_2}(\chi^{\beta'}) \right), \\
= &\sum_{\substack{\chi_\alpha, \chi_\beta \in V_n \\ \chi_{\alpha'}, \chi_{\beta'} \in V_n}} \sum_{\substack{g \in G \\ h \in G}} \delta(\chi_\alpha, \mu(\phi_g)\chi_\beta) \delta(\chi_{\alpha'}, \mu(\phi_h)\chi_{\beta'}) T_{v_1}(\chi^\alpha) T_{v_1}(\chi^{\alpha'}) \otimes T_l(\xi^g) T_l(\xi^h) \otimes T_{v_2}(\chi^\beta) T_{v_2}(\chi^{\beta'}), \\
= &\sum_{\substack{\chi_\alpha, \chi_\beta \in V_n \\ \chi_{\alpha'}, \chi_{\beta'} \in V_n}} \sum_{\substack{g \in G \\ h \in G}} \delta(\chi_\alpha, \mu(\phi_g)\chi_\beta) \delta(\chi_{\alpha'}, \mu(\phi_h)\chi_{\beta'}) \delta(\alpha, \alpha') \delta(g, h) \delta(\beta, \beta') T_{v_1}(\chi^\alpha) \otimes T_l(\xi^g) \otimes T_{v_2}(\chi^\beta), \\
= &\sum_{\chi_\alpha, \chi_\beta \in V_n} \sum_{g \in G} \delta(\chi_\alpha, \mu(\phi_g)\chi_\beta) T_{v_1}(\chi^\alpha) \otimes T_l(\xi^g) \otimes T_{v_2}(\chi^\beta), \\
= &C_l,
\end{aligned} \tag{3.34}$$

for any edge $l \in \mathcal{L}$.

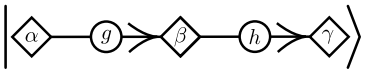
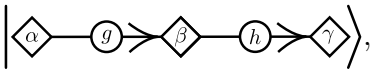
Link Operators C_{l_1} and C_{l_2} acting on adjacent edges l_1 and l_2.-
Consider two link operators that act nontrivially on adjacent edge subspaces as shown in Fig.(3.6). They both act on the Hilbert subspace that corresponds to the vertex v_2. To see this let us adopt a graphical representation of the states where the vertices are labeled by the basis states χ_α, χ_β and χ_γ respectively and the edges are labelled by the gauge basis states ϕ_g and ϕ_h, so the joint action of these two operators on a basis state is given by:

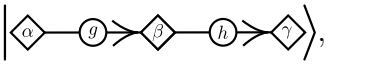
$C_{l_1}C_{l_2}$  $=\delta(\chi_\beta, \mu(\phi_h)\chi_\gamma)$ C_{l_1}  ,

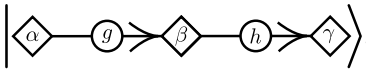
$=\delta(\chi_\beta, \mu(\phi_h)\chi_\gamma)\delta(\chi_\alpha, \mu(\phi_g)\chi_\beta)$  ,

$=\delta(\chi_\alpha, \mu(\phi_g)\mu(\phi_h)\chi_\gamma)$  . (3.35)

On the other hand,

$C_{l_2}C_{l_1}$  $=\delta(\chi_\alpha, \mu(\phi_g)\chi_\beta)C_{l_2}$  ,

$=\delta(\chi_\alpha, \mu(\phi_g)\chi_\beta)\delta(\chi_\beta, \mu(\phi_h)\chi_\gamma)$  ,

$=\delta(\chi_\alpha, \mu(\phi_g)\mu(\phi_h)\chi_\gamma)$  , (3.36)

comparing the last expression with Eq.(3.35) we note that the action of operators $C_{l_1}C_{l_2}$ is exactly the same as the one of $C_{l_1}C_{l_2}$, equivalently, subtracting these two equations we note that:

$$[C_{l_1}, C_{l_2}] = 0, \tag{3.37}$$

for any two adjacent links $l_1, l_2 \in \mathcal{L}$.

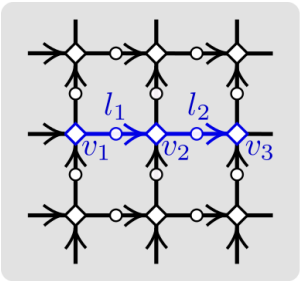


Figure 3.6: Two neighbouring edges l_1 and l_2 are shown together with the three vertices v_1, v_2 and v_3. The link operators C_{l_1} and C_{l_2} act simultaneously on the vertex degree of freedom sitting at v_2.

Link operator C_l and vertex operator A_v^g acting on subspaces with common support.-
Consider now the case when a vertex operator A_v^g and a link operator C_l act on subspaces with common support. We distinguish two cases as the vertex operator action

depends on the orientation of the link. Let us call v_1 the vertex at the left of edge l and v_2 the one at the right. We will treat the two cases separately.

$$A^g_{v_1} C_l \left| \right\rangle = \delta(\chi_\alpha, \mu(\phi_c)\chi_\beta) A^g_{v_1} \left| \right\rangle,$$

$$= \delta(\chi_\alpha, \mu(\phi_c)\chi_\beta) \left| \right\rangle, \tag{3.38}$$

on the other hand

$$C_l A^g_{v_1} \left| \right\rangle = C_l \left| \right\rangle, \tag{3.39}$$

the action of the link operator would now be:

$$C_l \left| \right\rangle = \delta(\mu(\phi_g)\chi_\alpha, \mu(\phi_{gc})\chi_\beta) \left| \right\rangle,$$

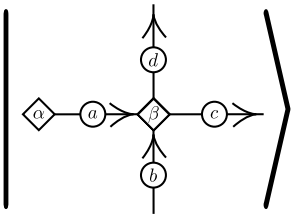
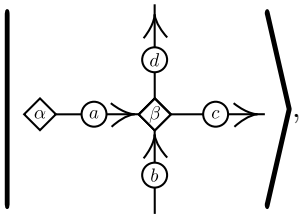
$$= \delta(\chi_\alpha, \mu(\phi_{g^{-1}})\mu(\phi_{gc})\chi_\beta) \left| \right\rangle,$$

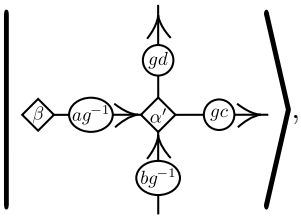
$$= \delta(\chi_\alpha, \mu(\phi_c)\chi_\beta) \left| \right\rangle, \tag{3.40}$$

where in the first and the second equality we used properties of Eqs.(3.4) and (3.5), comparing Eqs.(3.38) and (3.40) it is clear the the action of both operators commute, this is

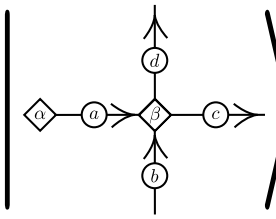
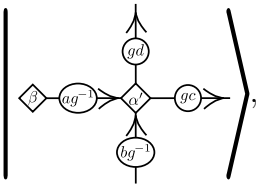
$$[A_{v_1}, C_l] = 0. \tag{3.41}$$

Now, let us look at the case when the vertex operator is acting at the right of the link l, namely

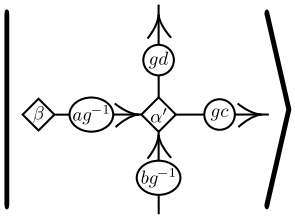
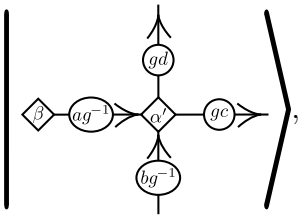
$A^g_{v_2} C_l$  $=\delta(\chi_\alpha, \mu(\phi_a)\chi_\beta) A^g_{v_2}$  ,

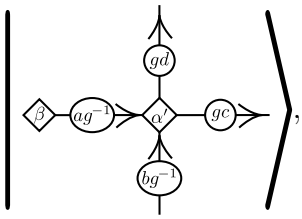
$=\delta(\chi_\alpha, \mu(\phi_a)\chi_\beta)$  , (3.42)

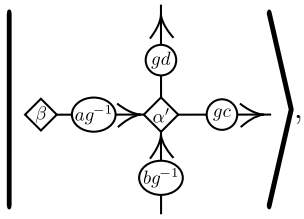
conversely,

$C_l A^g_{v_2}$  $= C_l$  , (3.43)

now the action of the link operator on the gauge transformed state would be:

C_l  $=\delta(\chi_\alpha, \mu(\phi_{ag^{-1}})\mu(\phi_g)\chi_\beta)$  ,

$=\delta(\chi_\alpha, \mu(\phi_{ag^{-1}g})\chi_\beta)$  ,

$=\delta(\chi_\alpha, \mu(\phi_a)\chi_\beta)$  , (3.44)

from the last expression and Eq.(3.42) it is clear that these operators commute,

$$[A_{v_2}, C_l] = 0 \tag{3.45}$$

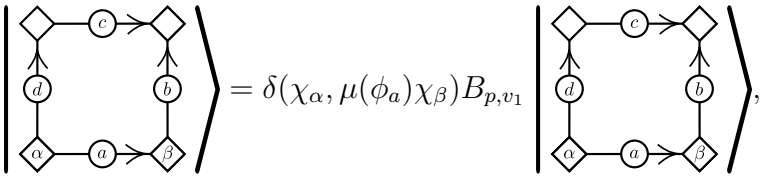
Therefore, without lost of generality we can write:

$$[A_v, C_l] = 0, \quad \forall\ v, l \in \mathcal{L}. \tag{3.46}$$

Link operator C_l and plaquette operator $B^h_{p,v}$ acting on subspaces with common support.-

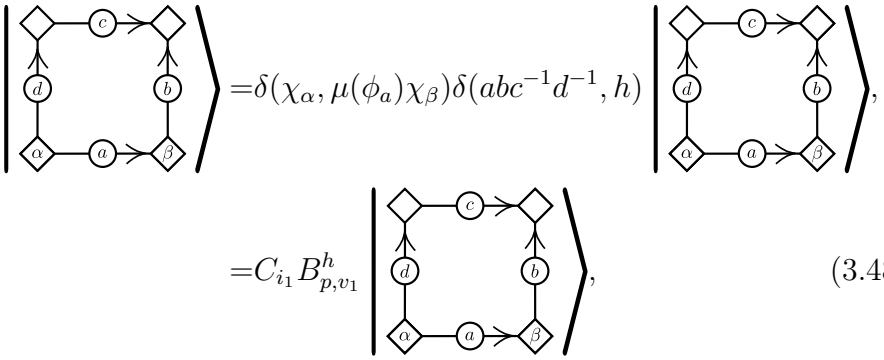
Both operators are projectors since they are constructed from the T operators whose algebra both on the gauge and the matter sector is given in Eqs.(2.62) and (3.10). Consider an arbitrary plaquette p, correspondingly a plaquette operator $B^h_{p,v}$ acting on the gauge degrees of freedom. Moreover, consider one of its links i_1 and a link operator C_{i_1}, the joint action of these operators is given by:

$$B^h_{p,v_1} C_{i_1} \left| \; \right\rangle = \delta(\chi_\alpha, \mu(\phi_a)\chi_\beta) B_{p,v_1} \left| \; \right\rangle, \tag{3.47}$$



now, the action of the plaquette operator on this state consists on measuring the holonomy of the gauge degrees of freedom along the plaquette, leaving us with:

$$\begin{aligned} \delta(\chi_\alpha, \mu(\phi_a)\chi_\beta) B_{p,v_1} \left| \; \right\rangle &= \delta(\chi_\alpha, \mu(\phi_a)\chi_\beta)\delta(abc^{-1}d^{-1}, h) \left| \; \right\rangle, \\ &= C_{i_1} B^h_{p,v_1} \left| \; \right\rangle, \end{aligned} \tag{3.48}$$



this is, the action of these operators on the common Hilbert space does not depend on the order they are being applied, in other words:

$$[B^h_{p,v}, C_l] = 0, \tag{3.49}$$

for any $p, v, l \in \mathcal{L}$ and any $h \in G$, in particular this is true for the *identity flux* plaquette operator B_p. To summarize, let us write all commutation relations together

$$\begin{array}{lll} [A_v, B_p] = 0, & [A_v, C_l] = 0, & [C_l, B_p] = 0, \\ A_v A_v = A_v, & B_p B_p = B_p, & C_l C_l = C_l, \\ A^g_v A^h_v = A^{gh}_v, & B^g_p B^h_p = \delta(g,h) B^g_p, & \end{array} \tag{3.50}$$

3.1.4 Hamiltonian of the Model

Consider the Hamiltonian $H : \mathcal{H} \to \mathcal{H}$, given by:

$$H := -\sum_v A_v - \sum_p B_p - \sum_l C_l, \tag{3.51}$$

because all its constituent operators commute with each other this Hamiltonian is exactly solvable. An arbitrary state $|\Psi\rangle \in \mathcal{H}$ is a simultaneous eigenstate of A_v, B_p and C_l. The vertex operator A_v performs gauge transformations at each vertex (and its four adjacent

edges), the plaquette operator B_p projects out the states for which the holonomy along a plaquette p is different from $\phi_e \in \mathbb{C}(G)$ and the link operator projects out the states for which the two vertex degrees of freedom joined by a link l are not the same after gauge transforming one of them.

Any state $|\Psi_0\rangle$ that fulfills the following conditions:

$$A_v |\Psi_0\rangle = B_p |\Psi_0\rangle = C_l |\Psi_0\rangle = |\Psi_0\rangle , \tag{3.52}$$

minimizes the energy of the Hamiltonian in question and therefore it is a **Ground State** of the model, with ground state energy $E_0 = -N_v - N_p - N_l$, where N_v, N_p and N_l are the number of vertices, plaquettes and links on the lattice $\mathcal{L}$ respectively. Thus the following operator:

$$P_0 := \prod_v A_v \prod_p B_p \prod_l C_l, \tag{3.53}$$

will project any state $|\Psi\rangle$ in the Hilbert space $\mathcal{H}$ to the ground state subspace $\mathcal{H}_0$.

3.2 The $V_2/\mathbb{Z}_2$ Model

Let us look at an specific model, we choose the matter degrees of freedom to belong to the 2-dimensional vector space V_2 spanned by the basis elements $\{\chi_1, \chi_{-1}\}$. Similarly the gauge degrees of freedom belong to the group algebra $\mathbb{C}(\mathbb{Z}_2)$ which is the complex vector space spanned by the basis elements $\{\phi_e, \phi_a\}$. In the following lines we construct the elementary operators and consequently the operators that make up the model.

3.2.1 Elementary Operators

3.2.1.1 Gauge Sector

Acting on the gauge degrees of freedom in $\mathbb{C}(\mathbb{Z}_2)$, as defined in § 3.1.1.1 there are two kind of operators namely, the **Shift Operators** and the **Projection Operators**. If we choose a representation in which the basis elements of the vector space $\mathbb{C}(\mathbb{Z}_2)$ are given by:

$$|\phi_e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\phi_a\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.54}$$

then the shift operators acting on edge degrees of freedom l would be given by:

$$L_l(\phi_e) \equiv R_l(\phi_e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}, \tag{3.55}$$

$$L_l(\phi_a) \equiv R_l(\phi_a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^x, \tag{3.56}$$

Given the abelian nature of the $\mathbb{Z}_2$ group there is no difference between the left and right action, thus the two shift operators $L(\phi_g)$ and $R(\phi_g)$ are equivalent. The projection operators are parametrized by elements of the dual vector space $\mathbb{C}^*(\mathbb{Z}_2)$ such that their action on the basis states is the following:

$$T_l(\xi^e) |\phi_e\rangle = |\phi_e\rangle , \quad T(\xi^e) |\phi_a\rangle = 0, \tag{3.57}$$

$$T_l(\xi^a) |\phi_e\rangle = 0, \quad T(\xi^a) |\phi_a\rangle = |\phi_a\rangle , \tag{3.58}$$

and their matrix representation is given by:

$$T_l(\xi^e) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad T_l(\xi^a) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.59}$$

let us write this operators as a linear combinations of Pauli matrices $\mathbb{1}$ and σ^z, this will be useful when writing the plaquette and link operators of the actual model.

$$T_l(\xi^e) = \frac{\mathbb{1} + \sigma^z}{2}, \qquad T_l(\xi^a) = \frac{\mathbb{1} - \sigma^z}{2}. \tag{3.60}$$

3.2.1.2 Matter Sector

On each vertex v of the lattice $\mathcal{L}$ we associated a 2-dimensional vector space V_2 whose basis elements are $\{\chi_1, \chi_{-1}\}$. The action of the gauge degrees of freedom, elements of $\mathbb{C}(\mathbb{Z}_2)$, on this vector space is defined through the μ map, and it is given by:

$$\begin{aligned} \mu(\phi_1)\chi_1 = \chi_1, \quad \mu(\phi_1)\chi_{-1} = \chi_{-1} \\ \mu(\phi_{-1})\chi_1 = \chi_{-1}, \quad \mu(\phi_{-1})\chi_{-1} = \chi_1. \end{aligned} \tag{3.61}$$

Let us choose a representation where the basis elements of V_2 are given by:

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.62}$$

such that:

$$\mu(\phi_a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}, \qquad \mu(\phi_a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^x. \tag{3.63}$$

The **Shift Operators** on the matter degrees of freedom are induced by the actions shown above, this is:

$$M_v^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}, \qquad M_v^a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^x. \tag{3.64}$$

The only ingredient that is left in order to construct the operators of the model are the **Projection Operators**, they are very similar in form to the ones of the gauge sector as their action on the basis states of the matter vector space V_2 resembles the action in Eqs.(3.57) and (3.58) and they can, as well, be written in terms of Pauli matrices, this is

$$T_v(\chi^1) = \frac{\mathbb{1} + \sigma^z}{2}, \qquad T_v(\chi^{-1}) = \frac{\mathbb{1} - \sigma^z}{2}. \tag{3.65}$$

3.2.2 Plaquette, Vertex and Link Operators

In general the Plaquette operator is the one defined in Eq.(3.16), though the one that goes into the Hamiltonian is the *identity flux* one, this is, it is parametrized by the identity element of the group $e \in \mathbb{Z}_2$, the configurations that make the operator are those shown in table 2.1 as this operator is essentially the same of the **QDM** in § 2.4.3, thus it is given by:

$$B_p = \frac{\mathbb{1}_{j_1} \otimes \mathbb{1}_{j_2} \otimes \mathbb{1}_{j_3} \otimes \mathbb{1}_{j_4} + \sigma^z_{j_1} \otimes \sigma^z_{j_2} \otimes \sigma^z_{j_3} \otimes \sigma^z_{j_4}}{2}, \tag{3.66}$$

where the edges j_1, j_2, j_3 and j_4 are those shown in Fig.(3.7). Canonically associated to this operator there is an operator that is **orthogonal** to B_p in the following sense:

$$B_p + B_p^{\perp} = \mathbb{1}, \qquad B_p B_p^{\perp} = \mathbb{0}, \tag{3.67}$$

where $\mathbb{0}$ is the 2×2 null matrix whose entries are all zero and the orthogonal operator $B_p^{\perp}$ is given by:

$$B_p^{\perp} = \frac{\mathbb{1}_{j_1} \otimes \mathbb{1}_{j_2} \otimes \mathbb{1}_{j_3} \otimes \mathbb{1}_{j_4} - \sigma_{j_1}^z \otimes \sigma_{j_2}^z \otimes \sigma_{j_3}^z \otimes \sigma_{j_4}^z}{2}. \tag{3.68}$$

It is straightforward to see that this operator fulfills the conditions in Eq.(3.67) and that $[B_p, B_p^{\perp}] = 0$.

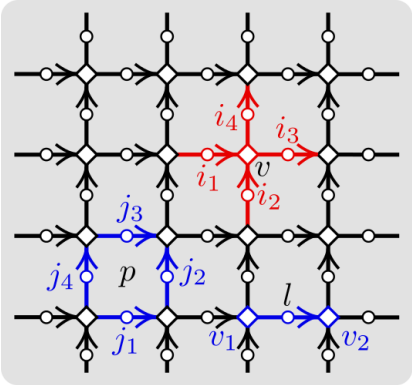


Figure 3.7: The local Hilbert spaces upon the operators A_v, B_p and C_l act are shown.

As the $\mathbb{Z}_2$ group has two elements namely $\{e, a\}$, correspondingly there are two elementary vertex operators A_v^e and A_v^a, the vertex operator as defined in Eq.(3.20) is the sum of these two:

$$A_v = \frac{\mathbb{1}_v \otimes \mathbb{1}_{i_1} \otimes \mathbb{1}_{i_2} \otimes \mathbb{1}_{i_3} \otimes \mathbb{1}_{i_4} + \sigma_v^x \otimes \sigma_{i_1}^x \otimes \sigma_{i_2}^x \otimes \sigma_{i_3}^x \otimes \sigma_{i_4}^x}{2}, \tag{3.69}$$

where v is an arbitrary vertex upon which the A_v is acting, and i_1, i_2, i_3 and i_4 are the four links adjacent to the vertex v as shown in Fig.(3.7). Likewise, the operator orthogonal to this one is given by:

$$A_v^{\perp} = \frac{\mathbb{1}_v \otimes \mathbb{1}_{i_1} \otimes \mathbb{1}_{i_2} \otimes \mathbb{1}_{i_3} \otimes \mathbb{1}_{i_4} - \sigma_v^x \otimes \sigma_{i_1}^x \otimes \sigma_{i_2}^x \otimes \sigma_{i_3}^x \otimes \sigma_{i_4}^x}{2}, \tag{3.70}$$

such that $A_v + A_v^{\perp} = \mathbb{1}$ and $A_v A_v^{\perp} = \mathbb{0}$.

The link operator is defined in Eq.(3.21), it acts on a link l and its two adjacent vertices v_1 and v_2 as shown in Fig.(3.7). Its action consists in projecting the states that have the following configurations on the aforementioned vertices and links.

Therefore, the link operator is given by:

$$\begin{aligned} C_l =& T_{v_1}(\chi^1) \otimes T_l(\xi^e) \otimes T_{v_2}(\chi^1) + T_{v_1}(\chi^1) \otimes T_l(\xi^a) \otimes T_{v_2}(\chi^{-1}) + \\ & T_{v_1}(\chi^{-1}) \otimes T_l(\xi^e) \otimes T_{v_2}(\chi^{-1}) + T_{v_1}(\chi^{-1}) \otimes T_l(\xi^a) \otimes T_{v_2}(\chi^1), \end{aligned} \tag{3.71}$$

$\mathbf{v_1}$	l	$\mathbf{v_2}$
1	e	1
1	a	-1
-1	e	-1
-1	a	1

Table 3.1: Allowed configurations for the vertex and link degrees of freedom.

from Eqs.(3.60) and (3.65) we know that the projectors can be written in terms of Pauli operators, replacing these expression into the above equation and expanding the tensor products the link operator reduces to:

$$C_l = \frac{\mathbb{1}_{v_1} \otimes \mathbb{1}_l \otimes \mathbb{1}_{v_2} + \sigma^z_{v_1} \otimes \sigma^z_l \otimes \sigma^z_{v_2}}{2}. \tag{3.72}$$

As in the case of the plaquette and vertex operator, the link operator has an orthogonal partner given by:

$$C_l^{\perp} = \frac{\mathbb{1}_{v_1} \otimes \mathbb{1}_l \otimes \mathbb{1}_{v_2} - \sigma^z_{v_1} \otimes \sigma^z_l \otimes \sigma^z_{v_2}}{2}. \tag{3.73}$$

All the above operators commute with each other and they are projectors as it has been shown in § 3.1.3. The Hamiltonian of the model is a sum of these operators acting on each vertex v, plaquette p and link l of the lattice $\mathcal{L}$, i.e.,

$$H_{V_2/\mathbb{Z}_2} := -\sum_v A_v - \sum_p B_p - \sum_l C_l. \tag{3.74}$$

3.2.3 Path and Surface Operators

In § 2.1.2 we introduced the path operators $Z(\gamma)$ and $X(\gamma^*)$ that were responsible for creating excitations (open paths) of the **Toric Code** model and mapping between the different ground states of the model (when defined along the non-contractible paths on the Torus). In the present model we can still define such operators on the direct an dual lattice, moreover, as we now include degrees of freedom on the vertices of the lattice we can define more general operators that ultimately will be the ones creating the elementary excitations of the $V_2/\mathbb{Z}_2$ model. Acting exclusively on the gauge degrees of freedom there are two kinds of operators, namely

$$Z(\gamma_l) := \bigotimes_{i \in \gamma_l} \sigma^z_i, \tag{3.75}$$

where the path γ_l can define either an open or a closed one in $\mathcal{L}$. In the case the path c forms a closed loop as shown in Fig. 3.8 the loop operator $Z(c)$ can be written as a product of the plaquette operator and its orthogonal partner, as follows:

$$Z(c) = \bigotimes_{i \in c} \sigma^z_i = \prod_{p \in c} (B_p - B_p^{\perp}), \tag{3.76}$$

where the plaquettes p are those being enclosed by the closed path c.

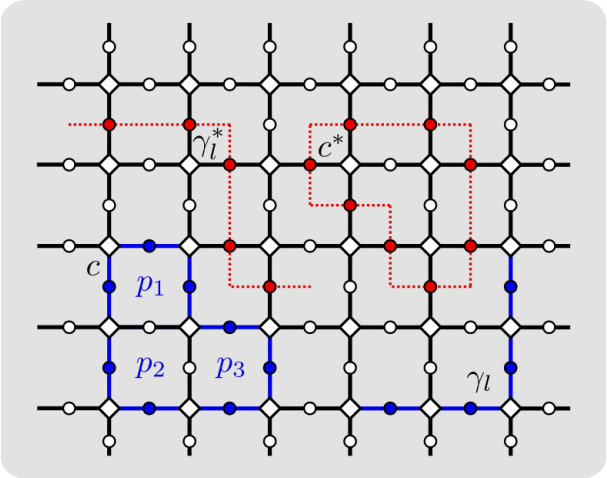


Figure 3.8: A path γ_l and a loop c on the direct lattice are shown in blue. Similarly, a dual path γ_l^* and a dual loop c^* are shown in red.

In the same fashion, we can define operators that act on tensor products of vertex Hilbert spaces $\mathcal{H}_v$, they can be constructed by means of paths on the direct lattice γ_v, i.e.

$$Z(\gamma_v) := \bigotimes_{v \in \gamma_v} \sigma_v^z, \tag{3.77}$$

they cannot be written as products of neither link nor plaquette operators, as for the latter the σ^z operators on the vertices cancel each other and the for the former they do not act on the vertex Hilbert spaces. Naturally we can construct an operator that acts on both vertex and link degrees of freedom along a path $\gamma_{l,v}$ on the direct lattice as:

$$Z(\gamma_{v,l}) := Z(\gamma_v) \otimes Z(\gamma_l). \tag{3.78}$$

Let us define a more general kind of operator acting on surface-like domains, the reason for the existence of this operator will be clear when studying the excitations of the model. Let D be a surface covering some connected region of the lattice $\mathcal{L}$ such as the one depicted in Fig.(3.9), let $v \in D_z$ denote the set of vertices enclosed by the region D, then define the operator:

$$Z(D_z) := \bigotimes_{v \in D_z} \sigma_v^z. \tag{3.79}$$

Analogously, let us define path and surface operators using the σ^x Pauli operator. Recall that in the **Toric Code** case the $X(\gamma^*)$ operator was the responsible for the creation of **Fluxes** (plaquette-like excitations). At this point it is important to mention that, while in the Toric Code and QDM models the lattice is self dual (the dual lattice is equivalent to the direct lattice) it is not the case of the present model as the inclusion of degrees of freedom on the vertices eliminates this symmetry. This fact has important consequences in the model as it will be seen in the following section when studying the ground state of the model. Thus, we will define X type path operators along dual paths, intending to create plaquette excitations in the model, this is, if γ_l^* is a dual path on $\mathcal{L}$ then define the operator:

$$X(\gamma_l^*) := \bigotimes_{i \in \gamma_l^*} \sigma_i^x. \tag{3.80}$$

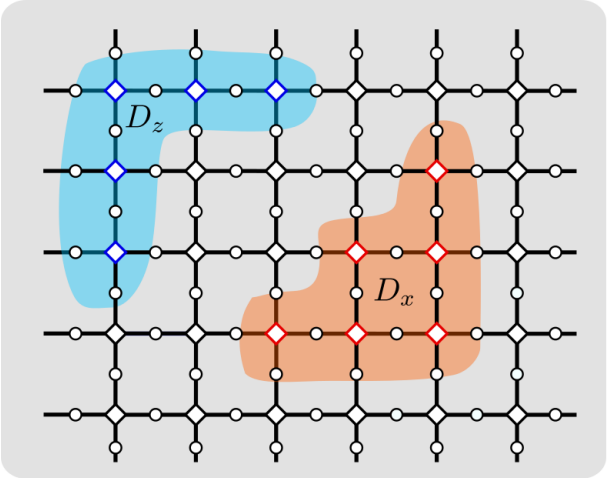


Figure 3.9: Two illustrative surfaces are shown, the first labeled D_z in pale blue, is used to define the operator $Z(D_z)$, the vertices being enclosed by D_z are also highlighted in blue. Similarly, the surface D_x used to define $X(D_x)$ is shown in pale orange and the vertices therein are highlighted in red.

Likewise, let us define an operator similar to the one in Eq.(3.79) but made of σ^x Pauli operators. If D_x determines a surface on the lattice as depicted in Fig. 3.9 and $v \in D_x$ denotes the vertices being enclosed by this surface, then:

$$X(D_x) := \bigotimes_{v \in D_x} \sigma_v^x, \tag{3.81}$$

note that when the lattice is embedded on a closed surface, as a sphere or a torus, and D_x encloses the complete lattice the above operator can be written in terms of the vertex operator A_v and its orthogonal projector, this is:

$$X(D_x) \equiv \prod_{v \in D_x} (A_v - A_v^{\perp}). \tag{3.82}$$

3.2.4 Ground state

To find the ground state of the model we first need to find the eigenvalues of each of the constituent operators, this is, A_v, B_p and C_l. From Eq.(3.50) we know these operators commute with each other, so they form a complete set of commuting operators, that is, a state $|\psi\rangle \in \mathcal{H}$ is a simultaneous eigenstate of the three aforementioned operators[57]. Consider an arbitrary simultaneous state $|\psi\rangle$ such that:

$$B_p |\psi_p\rangle = \omega_p |\psi_p\rangle , \qquad A_v |\psi_v\rangle = \omega_v |\psi_v\rangle , \qquad C_l |\psi_l\rangle = \omega_l |\psi_l\rangle , \tag{3.83}$$

where ω_p, ω_v and ω_l stand for the eigenvalues of B_p, A_v and C_l, respectively. In § 3.1.3 we have shown that these operators are projectors, this is, they square to themselves, this means their eigenvalues are either 1 or 0, for instance:

$$(B_p)^2 |\psi_p\rangle = \omega_p^2 |\psi_p\rangle ,$$

$$\text{but since:} \qquad B_p^2 = B_p, \tag{3.84}$$

$$\omega_p^2 - \omega_p = \omega_p(\omega_p - 1) = 0, \tag{3.85}$$

therefore $\omega_p = 0$ or $\omega_p = 1$. The same holds for ω_l and ω_v as this only depends on the fact the operators are projectors and not on their specific form. Now that we know the spectrum of these operators it is easy to see that the minimum eigenvalue of the Hamiltonian in Eq.(3.74) is $E_0 = -N_v - N_p - N_l$ and corresponds to an state $|\Psi_0\rangle \in \mathcal{H}$ for which:

$$A_v |\Psi_0\rangle = B_p |\Psi_0\rangle = C_l |\Psi_0\rangle = |\Psi_0\rangle \,, \ \forall p, v, l \in \mathcal{L}, \tag{3.86}$$

an state that has eigenvalue 1 for each vertex, plaquette and link operator.

As in the Toric Code case we can explicitly write an state that satisfies the conditions in Eq.(3.52), namely:

$$|\Psi_0\rangle = \prod_v A_v \bigotimes_v |\chi_1\rangle \bigotimes_l |\phi_e\rangle \,, \tag{3.87}$$

where the product runs over the entire lattice, meaning all vertices $v \in \mathcal{L}$. In the tensor products, at each vertex v of the lattice there is a $\chi_1 \in V_2$. Likewise, sitting at each edge l of the lattice there is an identity element of the group algebra $\phi_e \in \mathbb{C}(\mathbb{Z}_2)$. This state is indeed a ground state of the model, this is, it fulfills the conditions on Eq.(3.86). To see this, let us first consider the action of an arbitrary vertex operator on this state, as the vertex operator is a projector ($A_v^2 = A_v$), the action of the arbitrary vertex operator on this state is "absorbed" by the product. Therefore,

$$A_v |\Psi_0\rangle = |\Psi_0\rangle \,, \tag{3.88}$$

for any vertex $v \in \mathcal{L}$. Moreover, consider a plaquette operator B_p acting on an arbitrary plaquette $p \in \mathcal{L}$, since $[A_v, B_p] = 0$ as shown in § 3.1.3,

$$\begin{aligned} B_p |\Psi_0\rangle =& B_p \prod_v A_v \bigotimes_v |\chi_1\rangle \bigotimes_l |\phi_e\rangle \,, \\ =& \prod_v A_v B_p \bigotimes_v |\chi_1\rangle \bigotimes_l |\phi_e\rangle \,, \end{aligned} \tag{3.89}$$

now we note that the plaquette operator act only on the gauge degrees of freedom and since the initial configuration holds $|\phi_e\rangle$ states on every link, it is clear that

$$\begin{aligned} B_p |\Psi_0\rangle =& \prod_v A_v B_p \bigotimes_v |\chi_1\rangle \bigotimes_l |\phi_e\rangle \,, \\ =& \prod_v A_v \bigotimes_v |\chi_1\rangle \bigotimes_l |\phi_e\rangle \,, \end{aligned} \tag{3.90}$$

for any plaquette $p \in \mathcal{L}$. This is so because we intendedly constructed the state as being *vortex-free* (a flat connection) for every plaquette. Now let us look at the action of an arbitrary link operator C_l on the ground state:

$$\begin{aligned} C_l |\Psi_0\rangle =& C_l \prod_v A_v \bigotimes_v |\chi_1\rangle \bigotimes_l |\phi_e\rangle \,, \\ =& \prod_v A_v C_l \bigotimes_v |\chi_1\rangle \bigotimes_l |\phi_e\rangle \,, \\ =& \prod_v A_v \bigotimes_v |\chi_1\rangle \bigotimes_l |\phi_e\rangle \,, \\ =& |\Psi_0\rangle \,, \end{aligned} \tag{3.91}$$

where we used the fact that the link and vertex operator commute, this is, $[A_v, C_l] = 0$. Likewise, the initial configuration is an allowed one for the action of the link operator as it is one of the configurations shown in table 3.1. Thus, we have shown that this state is indeed a ground state of the model defined by the Hamiltonian in Eq.(3.51).

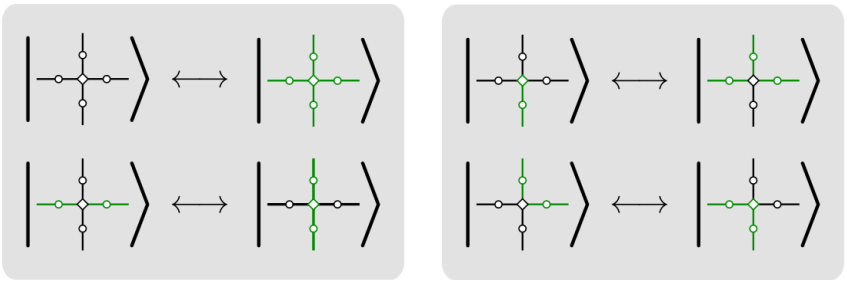


Figure 3.10: Illustration of the action of elementary vertex operator A_v^a made of σ^x Pauli operators on basis states that are represented graphically, where uncolored edges/vertices hold $|\phi_e\rangle$ and $|\chi_1\rangle$ respectively, while green colored edges/vertices hold $|\phi_a\rangle$ and $|\chi_{-1}\rangle$ respectively.

In § 2.2 we found a similar state for the Toric Code model, it was interpreted as being a gas of loops, where the loops were merely a graphical representation of the action of vertex operators as a result of the product $\prod_v(\mathbb{1} + A_v)$. For the actual model, as a consequence of the inclusion of degrees of freedom on the vertices of the lattice, the interpretation of the ground state is consequently extended to being a gas of **Domain Walls**. To develop on this let us assume a graphical representation analogous to the one given in § 2.2. In this graphical representation any uncolored (black) edge hold a $|\phi_e\rangle$ element of the group algebra $\mathbb{C}(\mathbb{Z}_2)$, conversely if the edge is green colored then it means it holds the $|\phi_a\rangle$ element. Likewise for the vertex degrees of freedom, an uncolored vertex holds a $|\chi_1\rangle$ element whereas a green colored vertex holds a $|\chi_{-1}\rangle$ element. Thus, the action of the vertex operator A_v^a consists on interchanging between these two as shown in Fig.(3.10), also the allowed configurations for the link operator C_l of table 3.1 are shown in Fig.(3.11).

Figure 3.11: The allowed configurations of the link operator in table 3.1 are graphically represented. For instance, in (a) the state being represent is such that holds $|\chi_1\rangle$ at the vertices and $|\phi_e\rangle$ at the edge.

The first term coming out from the product in Eq.(3.87) corresponds to identity operators acting on every vertex and edge of the lattice, this means that the first configuration of the ground state is just the one we called *initial configuration*, this is, $|\phi_e\rangle$ elements on every edge of the lattice and $|\chi_1\rangle$ sitting at every vertex of the lattice, as in Fig. 3.12(a). The following terms include products of σ^x acting the *initial configuration*, as their action consists on flipping the configurations they will flip both matter and gauge degrees of freedom on vertices and the four adjacent edges as in Fig. 3.12(b). Moreover, consider a

more intricate term being the result of a product of many A_v^a acting on different vertices, the Hilbert subspaces these operators a ct upon define domain walls (dotted red lines in our graphical representation) that separate regions where the matter degrees of freedom holding $|\chi_1\rangle$ from regions where the matter degrees of freedom are in the $|\chi_{-1}\rangle$ state. Finally we will have a term that is the result of the application of A_v^a on every vertex (and its four adjacent edges) resulting on a state that has all vertex degrees of freedom set at $|\chi_{-1}\rangle$ whereas all the edges hold $|\phi_e\rangle$. In this sense, as the ground state of the model $|\Psi_0\rangle$ is a linear combination of such states it can be thought as being a gas of domain walls, where all configurations are both *vortex-free* and the allowed configurations depicted in Fig. 3.11 for the link operator, thus minimizing the energy of the system.

The ground state of the Toric Code defined in § 2.2 has a topology dependent degeneracy since the state defined in Eq.(2.31) is the unique ground state of the model when the lattice is embedded on the surface of a sphere while it is 4-fold degenerate when defined over the surface of a torus as argued also in § 2.2. In the case of the torus the other degenerate ground states are obtained from the state written in Eq.(2.31) by means of the action of path operator $X(\gamma^*)$ on the *original* ground state and where the dual path is defined along the non-contractible loops shown in Fig. 2.4.

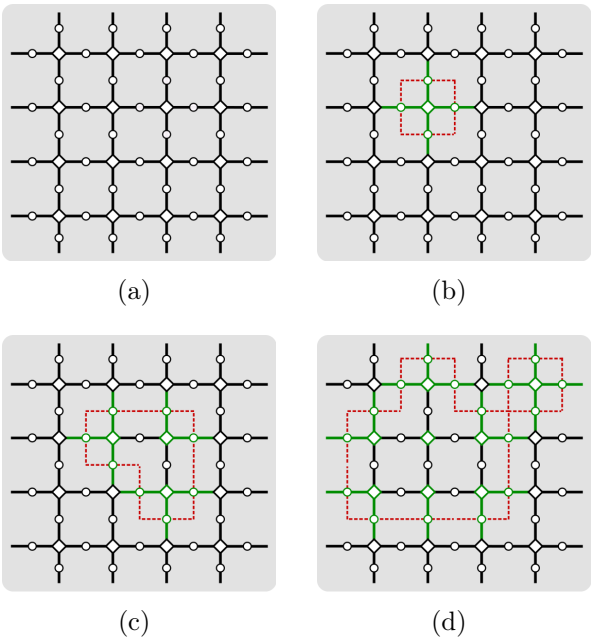


Figure 3.12: We show some illustrative constituents of the ground state defined in Eq.(3.87). In (a) we represent the action of the first term coming out of the product $\prod_v A_v$ on the initial configuration, in (b) we represent a term in which an A_v^a is acting on a single vertex, while in (c) we include the action of three adjacent A_v^a. The last figure (d) shows a bigger **domain wall**, the vertices that are contained by the wall hold $|\chi_{-1}\rangle$ states. Also, note that all the gauge and matter configurations at the boundary of the domain wall are the *allowed* configurations for the link operator shown in Fig.(3.11)

Let us try to extend this argument to our actual model, in principle we can be tempted to apply the path operators $X(\gamma_l^*)$ on $|\Psi_0\rangle$ as they were responsible of mapping between different ground states in the Toric Code; note, however, that in the $V_2/\mathbb{C}(\mathbb{Z}_2)$ case the application of the path operator $X(\gamma_l^*)$ does not map between ground states as the configurations along the path γ_l^* are not the *allowed* configurations of the link operator shown

in Fig. 3.11 both when the dual path defines a contractible and non-contractible loop. In Fig. 3.13(a) we show the action of $X(\gamma^*)$ on a representative configuration of the ground state $|\Psi_0\rangle$, namely the one where there are $|\chi_1\rangle$ states at every vertex and $|\phi_e\rangle$ at every edge. Note that the action of the path operator consist on flipping every $|\phi_e\rangle$ state into $|\phi_a\rangle$ that sits along the loop γ^* making the resulting configurations not being the ones shown in Fig. 3.11. Likewise, when the path defines a contractible loop as shown in Fig. 3.13(b) the resulting path operator $X(\gamma_c^*)$ make the gauge degrees of freedom sitting at the edges being crossed by the path to go from $|\phi_e\rangle$ to $|\phi_a\rangle$, in Fig. 3.13(b) we purposely made the path γ_c^* to resemble the domain wall in Fig. 3.12(d) so we can compare these two configurations. While the domain wall in Fig. 3.12(d) satisfies the ground state conditions in Eq.(3.86) at every local Hilbert space corresponding to the domains of operators A_v, B_p and C_l, the configuration in Fig. 3.13(b) does not satisfies these conditions as the configurations along the path are not the allowed for the link operators that act along the loop. This comes from the fact that the loop operators $X(\gamma^*)$ and $X(\gamma_c^*)$ do not commute with the link operators C_l where $l \in \gamma^*$. This is,

$$[X(\gamma^*), C_l] \neq 0, \tag{3.92}$$

where l is any link that is being crossed by the dual path γ^* open or closed, contractible or non-contractible. Actually, for any of such link operators the following holds:

$$X(\gamma^*)C_l = C_l^{\perp} X(\gamma^*), \tag{3.93}$$

this relation is at the root of the process of creating the elementary excitations, for the plaquette B_p and link C_l operators, when the operator $X(\gamma_l^*)$ is defined along an open path as we will see in detail in the next chapter.

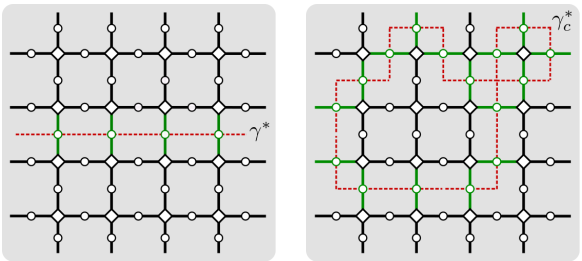


Figure 3.13: We show illustrative configurations after the operator $X(\gamma^*)$ and $X(\gamma_c^*)$ acted on the constituents of the ground state $|\Psi_0\rangle$ we take the configuration of Fig. 3.12(a) as the representative one. The path γ^* defines a non-contractible loop on the torus whereas the path γ^* defines a contractible loop.

The path operators $X(\gamma^*)$ defined over non-contractible loops are not a symmetry of the Hamiltonian anymore and therefore they do not create more ground states as their action on the ground state $|\Psi_0\rangle$ consists on a trail of localized excitations that correspond to the violation of the Eq.(3.86) for the link operators. Therefore, the inclusion of the link operator C_l as a part of the Hamiltonian of the model, which in turn is a consequence of including matter degrees of freedom to the vertices of $\mathcal{L}$, eliminates the mapping between ground states typical of the Toric Code. This means the ground state is unique as the inclusion of matter degrees of freedom on the vertices of the lattice makes the ground state degeneracy to depend on the topology of the manifold in a trivial way.

Chapter 4

Elementary Excitations of the $V_2/\mathbb{Z}_2$ model

In this chapter we briefly expose the process of creating the most elementary excitations of the model developed in § 3.2 and defined by the Hamiltonian of Eq.(3.74) which is a sum of three mutually commuting projectors. The ground state of the model is given in Eq.(3.87) with ground state energy $E_0 = -N_v - N_p - N_l$, where N_v, N_p, N_l stand for the number of vertices, plaquettes and links of the lattice $\mathcal{L}$, respectively. The elementary excitations of the model arise from the violation of the ground state conditions defined in Eq.(3.86), therefore they can come in three types, namely, from the violation of such conditions regarding an specific vertex operator A_v, plaquette operator B_p or link operator C_l. Each elementary excitation is different in nature and so is their energy. These elementary excitations will be created by local operators acting on the ground state $|\Psi_0\rangle$ over single Hilbert spaces $\mathcal{H}_l$ and $\mathcal{H}_v$ corresponding to arbitrary edges and vertices on $\mathcal{L}$.

4.1 Creating Excitations

The Hamiltonian of the model coined $V_2/\mathbb{Z}_2$ is given by:

$$H_{V_2/\mathbb{Z}_2} := -\sum_v A_v - \sum_p B_p - \sum_l C_l. \tag{4.1}$$

Consider the ground state given in Eq.(3.87), in § 3.2.4 we have shown that this state is indeed a ground state of the model as for any vertex, plaquette and link operator the following holds:

$$A_v |\Psi_0\rangle = |\Psi_0\rangle, \quad \forall v \in \mathcal{L}, \tag{4.2}$$
$$B_p |\Psi_0\rangle = |\Psi_0\rangle, \quad \forall p \in \mathcal{L}, \tag{4.3}$$
$$C_l |\Psi_0\rangle = |\Psi_0\rangle, \quad \forall l \in \mathcal{L}, \tag{4.4}$$

Furthermore, in § 3.2.4 we have also shown that, from the projective nature of the A_v, B_p and C_l the spectrum of these operators corresponds to eigenvalues 0 or 1. From the above expressions we know that a ground state is such that the eigenvalue of every vertex, plaquette and link operator is 1. This means that, elementary excited states will be those

for which their eigenvalue, for an arbitrary vertex, plaquette or link operator is zero, this is:

$$A_v \left|\psi_v\right\rangle = 0, \quad \text{for some } v \in \mathcal{L}, \tag{4.5}$$
$$B_p \left|\psi_p\right\rangle = 0, \quad \text{for some } p \in \mathcal{L}, \tag{4.6}$$
$$C_l \left|\psi_l\right\rangle = 0, \quad \text{for some } l \in \mathcal{L}, \tag{4.7}$$

we say the states $|\psi_v\rangle$, $|\psi_p\rangle$ and $|\psi_l\rangle$ are excited states that hold excitations at the particular v, p and l on the lattice. In § 3.2.2 the vertex, plaquette and link operators were written in terms of the well known **Pauli Matrices** in addition we also wrote the operators being orthogonal to them, for instance the plaquette operator and its orthogonal partner:

$$B_p = \frac{\mathbb{1}_{j_1} \otimes \mathbb{1}_{j_2} \otimes \mathbb{1}_{j_3} \otimes \mathbb{1}_{j_4} + \sigma^z_{j_1} \otimes \sigma^z_{j_2} \otimes \sigma^z_{j_3} \otimes \sigma^z_{j_4}}{2}, \tag{4.8}$$
$$B_p^{\perp} = \frac{\mathbb{1}_{j_1} \otimes \mathbb{1}_{j_2} \otimes \mathbb{1}_{j_3} \otimes \mathbb{1}_{j_4} - \sigma^z_{j_1} \otimes \sigma^z_{j_2} \otimes \sigma^z_{j_3} \otimes \sigma^z_{j_4}}{2}, \tag{4.9}$$

where the labels j_1, j_2, j_3 and j_4 stand for the four links around an arbitrary plaquette p and such that the following relations hold

$$B_p + B_p^{\perp} = \mathbb{1}, \qquad B_p B_p^{\perp} = \mathbb{0}, \tag{4.10}$$

Likewise, the vertex operator and its orthogonal partner are given by;

$$A_v = \frac{\mathbb{1}_v \otimes \mathbb{1}_{i_1} \otimes \mathbb{1}_{i_2} \otimes \mathbb{1}_{i_3} \otimes \mathbb{1}_{i_4} + \sigma^x_v \otimes \sigma^x_{i_1} \otimes \sigma^x_{i_2} \otimes \sigma^x_{i_3} \otimes \sigma^x_{i_4}}{2}, \tag{4.11}$$
$$A_v^{\perp} = \frac{\mathbb{1}_v \otimes \mathbb{1}_{i_1} \otimes \mathbb{1}_{i_2} \otimes \mathbb{1}_{i_3} \otimes \mathbb{1}_{i_4} - \sigma^x_v \otimes \sigma^x_{i_1} \otimes \sigma^x_{i_2} \otimes \sigma^x_{i_3} \otimes \sigma^x_{i_4}}{2}, \tag{4.12}$$

where v is an arbitrary vertex on the lattice and i_1, i_2, i_3 and i_4 label the four adjacent links to the vertex in question. Analogously, the link operator and its orthogonal projector are given by:

$$C_l = \frac{\mathbb{1}_{v_1} \otimes \mathbb{1}_l \otimes \mathbb{1}_{v_2} + \sigma^z_{v_1} \otimes \sigma^z_l \otimes \sigma^z_{v_2}}{2}, \tag{4.13}$$
$$C_l^{\perp} = \frac{\mathbb{1}_{v_1} \otimes \mathbb{1}_l \otimes \mathbb{1}_{v_2} - \sigma^z_{v_1} \otimes \sigma^z_l \otimes \sigma^z_{v_2}}{2}. \tag{4.14}$$

An operator $\mathcal{O}$ that creates excited states (of the vertex, plaquette or link type) is such that interchanges between an operator and its orthogonal associate when commuted, this is:

$$A_v \mathcal{O}_v = \mathcal{O}_v A_v^{\perp}, \tag{4.15}$$
$$B_p \mathcal{O}_p = \mathcal{O}_p B_p^{\perp}, \tag{4.16}$$
$$C_l \mathcal{O}_l = \mathcal{O}_l C_l^{\perp}, , \tag{4.17}$$

where $\mathcal{O}_v$, $\mathcal{O}_p$ and $\mathcal{O}_l$ stand for the operators hat create excitation of the vertex, plaquette and link types, respectively. To see this, consider the ground state $|\Psi_0\rangle$ in Eq.(3.87), the action of any arbitrary vertex operator A_v on this state is trivial, since:

$$A_v \left|\Psi_0\right\rangle = \left|\Psi_0\right\rangle,$$

as shown in § 3.2.4. Let us now consider an operator $\mathcal{O}_v$ of the kind shown in Eq.(4.15) that acts on a local Hilbert space corresponding to a vertex v of the lattice (the details of this procees will be exposed in the following sections), moreover consider an state $|E_v\rangle$ that is the result of the action of $\mathcal{O}_v$ on the ground state $|\Psi_0\rangle$, namely

$$|E_v\rangle = \mathcal{O}_v |\Psi_0\rangle , \tag{4.18}$$

this new state is an excitated state of the model that holds an excitation at vertex v for the action of the vertex A_v on this state is given by:

$$\begin{aligned} A_v |E_v\rangle =& A_v \mathcal{O}_v |\Psi_0\rangle , \\ =& \mathcal{O}_v A_v^{\perp} |\Psi_0\rangle , \\ =& 0, \end{aligned} \tag{4.19}$$

in the last line we used the fact that $A_v A_v^{\perp} = \mathbb{0}$ since the ground state is made of product of vertex operators over all vertices in the lattice, this shows the state $|E_v\rangle$ is indeed an excited state for the vertex operator A_v. In the following sections we will describe this procees in detail, showing the several ways the elementary excitations of each type can be constructed.

4.2 Vertex Excitations

As briefly exposed in the last section the excitations arising as violations of the ground state conditions for the vertex operators come by the action of an operator $\mathcal{O}_v$ acting on the Hilbert space corresponding to the vertex v and its four adjacent edges. In the Toric Code, the excitations of vertex type (charges) were created by *string* like operators $Z(\gamma_l)$ composed of Pauli matrices σ^z acting on the edges of the lattice that are part of the path γ (cf. § 2.3). This comes from the fact that at the endpoints of the path γ the operator $Z(\gamma_l)$ plays the role of the operator $\mathcal{O}_v$ in Eq. (4.15) thus creating two localied excitations. This operator still creates such excited states in our current model as the Pauli matrix σ^z interchanges between the A_v and $A_v^{\perp}$ operators defined in Eq. (4.11) and (4.12). Therefore, the string operator $Z(\gamma_l)$ that acts on the edges lying on the direct path γ creates two excitations localized at its endpoints. When the path defines a closed loop no excitation is being created and thus it maps between the ground state constituents. The resulting state

$$|\Psi_v(\gamma_l)\rangle = Z(\gamma_l) |\Psi_0\rangle \tag{4.20}$$

where $Z(\gamma_l)$ is the operator defined in Eq.(3.75) by means of a open path on the direct lattice. The energy of this excited states is the eigenvalue of the Hamiltonian operator, namely

$$H_{V_2/\mathbb{Z}_2} |\Psi_v(\gamma_l)\rangle = -((N_v - 2) - N_p - N_l) |\Psi_v(\gamma_l)\rangle . \tag{4.21}$$

Note that the excitations created at the endpoints of the path γ_l can be *moved* by simply extending the path in question and this process has no energy cost as the energy depends only on the number of vertex operators that are being violated. Besides this kind of excitations, the inclusion of matter degrees of freedom on the vertices allows us to create

excitations at every single vertex v through the action of an operator that acts on the vertex degrees of freedom. Consider the following operator:

$$Z_v := \sigma_v^z, \tag{4.22}$$

that acts non trivially on the Hilbert space corresponding to some arbitrary vertex $v \in \mathcal{L}$ and acts as identity on the rest of the lattice $\mathcal{L}$, we note that this operator act as the $\mathcal{O}_v$ in Eq.(4.15) thus it creates a localized excitation sitting at the vertex $V \in \mathcal{L}$, consequently the state defined by the action of this operator on the ground state

$$|\Psi_v\rangle = Z_v |\Psi_0\rangle , \tag{4.23}$$

is an excited state of the model for the ground state condition of Eq.(4.2) is being violated for the vertex operator A_v. The energy of this state is given by the eigenvalue of the Hamiltonian operator,

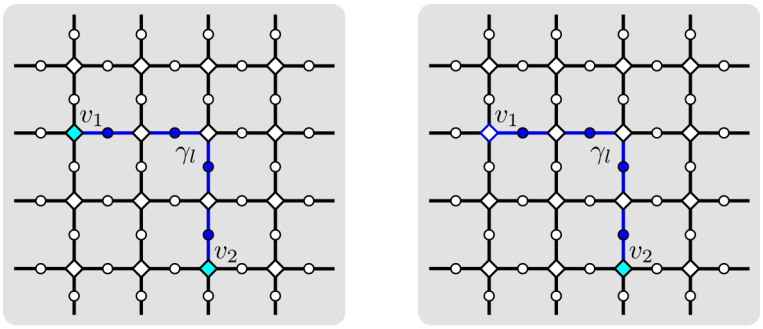
$$H_{V_2/\mathbb{C}(\mathbb{Z}_2)} |\Psi_v\rangle = -((N_v - 1) - N_p - N_l) |\Psi_v)\rangle . \tag{4.24}$$

This excited state holds a single excitation at vertex v. Moreover an excited state $|\Psi_v(\gamma_l)\rangle$ can be turned into an state that holds a single excitation, to see this consider the path in Fig. 4.1(a), the string operator defined over this path $Z(\gamma_l)$ that acts exclusively on the edge degrees of freedom creates two excitations at vertices v_1 and v_2, the endpoints of γ. Thus the state

$$|\Psi_v(\gamma_l)\rangle = Z(\gamma_l) |\Psi_0\rangle ,$$

is an excited state of the model. Now, consider the action of the operator Z_{v_1} on the above state, as a result of the action the configuration on the vertex v_1 becomes an allowed configuration for the vertex operator A_{v_1}, this means he excitation sitting at that vertex is being annihilated leaving a single excitation at vertex v_2,

$$|\Psi_{v_2}\rangle = Z_{v_1} |\Psi_v(\gamma_l)\rangle .$$



(a) (b)

Figure 4.1: In (a) two charges are created at vertices v_1 and v_2 by the action of the path operator $Z(\gamma_l)$, then the charge at vertex v_1 is annihilated by the action of Z_{v_1} leaving a single charge at v_2.

This shows that the action of $Z_{v_1} Z(\gamma_l)$ is equivalent to the action of Z_{v_2} since both create a single excitation sitting at vertex v_2, this symmetry is a new feature of the model

as it was not present in the Toric Code and its origin relies exclusively on the inclusion of matter degrees of freedom. The above argument shows how we can create and annihilate the single vertex excitations at will, this hands us the possibility of creating *hopping* operators that, as their name suggest, will hop the excitations between the vertices of the lattice. Consider an excited state holding an excitation at an arbitrary vertex $v_{i,j}$ of the lattice, where (i, j) are two integer indices that stand for the vertical and horizontal direction on the direct lattice, such that the vertex at the right of $v_{i,j}$ is labeled $v_{i+1,j}$, and the vertex above $v_{i,j}$ is labeled $v_{i,j+1}$. Such state is given by:

$$|\Psi_{v_i}\rangle = Z_{v_{i,j}} |\Psi_0\rangle \,, \tag{4.25}$$

the action of the operator $H_{v_{i,j},v_{i+1,j}} = Z_{v_{i,j}} Z_{v_{i+1,j}}$ on this state:

$$\begin{aligned} H_{v_{i,j},v_{i+1,j}} \left|\Psi_{v_{i.j}}\right\rangle &= Z_{v_{i+1,j}} |\Psi_0\rangle \,, \\ &= \left|\Psi_{v_{i+1,j}}\right\rangle \end{aligned} \tag{4.26}$$

is to annihilate the excitation that as sitting at vertex $v_{i,j}$ and create another at $v_{i+1,j}$ this process in understood as a hopping of excitations and can be carried over any two vertices in the lattice.

4.3 Plaquette and Link Excitations

Besides the excited states that violate the ground state condition for the vertex operator, there are states that do so for the plaquette and link operators, namely the conditions in Eqs.(4.3) and (4.4), respectively. As in the vertex type excitations we can think on operators that create these excited states when acting on the ground state $|\Psi_0\rangle$. We note that both the link and plaquette operator are written in terms of the Pauli matrices σ^z. Therefore, the operators that create excitations of the link and plaquette type are written in terms of the Pauli matrix σ^x. In the Toric Code case, the plaquette excitations (fluxes) were created by string like operators defined via paths on the dual lattice (cf. § 2.3.2) and that therefore were acting on the edges being crossed by such path. In § 3.2.3 we defined the following operator:

$$X(\gamma_l^*) := \bigotimes_{i \in \gamma_l^*} \sigma_i^x, \tag{4.27}$$

where γ_l^* defines a dual path on the lattice $\mathcal{L}$. This operator is responsible for creating excitations both of the plaquette and link type. Let us develop further on this. Consider the simplest operator of thistype that could be constructed, this is, a $X(\gamma_l^*) = \sigma_i^x$ acting on an arbitrary edge $i \in \mathcal{L}$ as shown in Fig. 4.2, this operator plays the role of $\mathcal{O}_p$ and $\mathcal{O}_l$ in Eqs.(4.16) and (4.17), respectively. If p_1 and p_2 are the plaquettes that share the i edge.

The state defined as:

$$|\Psi_{pl}(\gamma_l^*)\rangle = X(\gamma_l^*) |\Psi_0\rangle \,, \tag{4.28}$$

is an excited state for the two plaquette operators B_{p_1}, B_{p_2} and for the link operator C_i. The action of the path operator consists on flipping the gauge configuration at the edge i on the ground state $|\Psi_0\rangle$ such that they are not the *allowed* ones anymore for both the

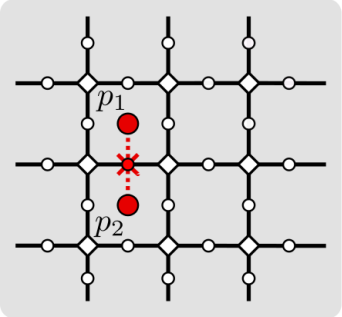


Figure 4.2: Two plaquette excitations (fluxes) are created at plaquettes p_1 and p_2 by the action of a σ^x operator at the link in between, at the same time, a link type excitation is created at that link denoted as a red "$\times$".

plaquette and link operators. Therefore, the operators B_{p_1}, B_{p_2} and C_i will project out the new state, this is:

$$\begin{aligned} B_{p_1,p_2} \left|\Psi_{pl}(\gamma_l^*)\right\rangle &= 0, \\ C_i \left|\Psi_{pl}(\gamma_l^*)\right\rangle &= 0, \end{aligned}$$

equivalently this can be gotten from the algebra of Eqs.(4.16) and (4.17):

$$\begin{aligned} B_{p_1,p_2} \left|\Psi_{pl}(\gamma_l^*)\right\rangle &= B_{p_1,p_2} X(\gamma_l^*) \left|\Psi_0\right\rangle, \\ &= X(\gamma_l^*) B^{\perp}_{p_1,p_2} \left|\Psi_0\right\rangle, \\ &= 0, \end{aligned}$$

since the orthogonal plaquette operators $B^{\perp}_{p_1,p_2}$ has 0 eigenvalue when applied to the ground state $|\Psi_0\rangle$. Likewise, consider the link operator that acts on the Hilbert space corresponding to the i edge, the $X(\gamma^*)$ operator acts as the $\mathcal{O}_l$ in Eq.(4.17) since:

$$\begin{aligned} C_i \left|\Psi_{pl}(\gamma_l^*)\right\rangle &= C_i X(\gamma_l^*) \left|\Psi_0\right\rangle, \\ &= X(\gamma_l^*) C_i^{\perp} \left|\left|\Psi_0\right\rangle\right\rangle, \\ &= 0 \end{aligned} \tag{4.29}$$

where the last equality holds because the ground state is projected out by the orthogonal link operator $C_i^{\perp}$. Therefore the state $|\Psi_{pl}(\gamma_l^*)\rangle$ is an excited state of the model that holds two *plaquette type* excitations (fluxes) at the endpoints of the dual path γ_l^* and a *link type* excitation at the link being crossed by γ_l^*.

Note, however that this excited state does not behave as the old *fluxes* of the Toric Code used to do, we can not freely move the fluxes by means of extending the dual path γ_l^*, as an extension of the path results on more edges being crossed (cf. Fig. 4.3(b)). Thus, if we extend the dual path more excitations of the *link type* will be created leaving a trail of such excitations. Hence, moving fluxes is allowed but it is a process that has an energy cost. The fluxes, that result from the violation of plaquette condition are their own antiparticle, as in the Toric Code case. This means that if two fluxes are joined together they annihilate each other. In particular, we can think on a process of creating two fluxes at the endpoints of a dual path γ_l^* and then, by means of extending such path in order

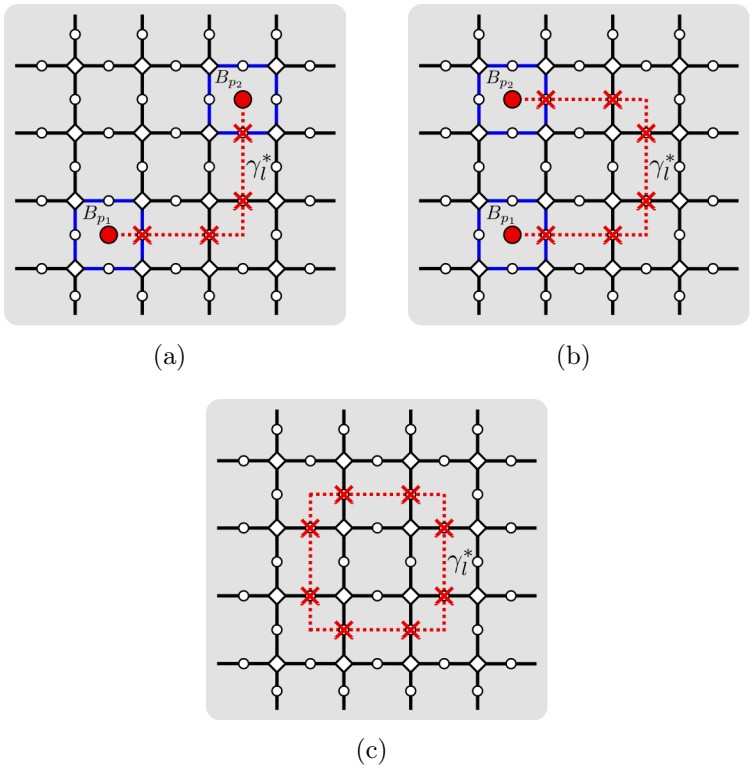


Figure 4.3: In (a), two fluxes are created at plaquettes p_1 and p_2 by the action of $X(\gamma_l^*)$, where the dual path γ_l^* has endpoints in the aforementioned plaquettes. Then, in (b) the fluxes are being *moved* by extending the dual path, as a result more *link type* excitations are created (depicted as red $\times$'s). Finally, in (c) the fluxes are brought together as the dual path becomes a loop, hence the fluxes are annihilated but the trail of *link type* excitations is left.

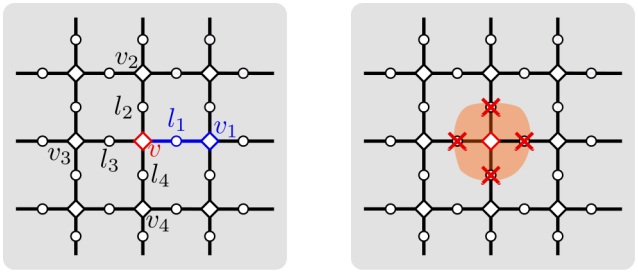
to make it a closed loop, the fluxes are annihilated, however, in spite of not having fluxes anymore the trail of link excitations is left (cf. Fig. 4.3(c)). Therefore, this process does not map between ground state constituents as it was the case in the Toric Code model.

So far we have just explored the creation of such excitations through the action of operators on the gauge degrees of freedom, namely the $X(\gamma_l^*)$ defined through a path on the dual lattice. However, we can still create excitations of the *link type* by acting with σ^x operators on the vertices of the lattice, let us begin by considering the action of a single Pauli σ^x operator on an arbitrary vertex $v \in \mathcal{L}$, we label this operator by σ_v^x. The action of such operator on the ground state $|\Psi_0\rangle$ is to flip the matter configurations on the given vertex, namely, if the matter degree of freedom sitting at v was $|\chi_1\rangle$ after the action of σ_v it will be $|\chi_{-1}\rangle$. As a result the configuration is not an allowed one (cf. Fig. 3.11) for the four links adjcent to the vertex in question. They are interpreted as four excitations of the *link type* as the corresponding four link operators will have eigenvalue 0 when acting on the resulting state. Likewise, we can see this from the operator algebra point of view, for instance, let us pick one the four link operators that act on the four adjacent edges to v, say C_{l_1}, as depicted in Fig. 4.4(a) this link operator act on the local Hilbert spaces $\mathcal{H}_v$, $\mathcal{H}_l$ and $\mathcal{H}_{v_1}$. The state $|\Psi_l\rangle = \sigma_v^x |\Psi_0\rangle$ is an excited state that holds an excitation at

the link l_1 since:

$$\begin{aligned} C_{l_1} |\Psi_l\rangle &= C_{l_1}\sigma_v^x |\Psi_0\rangle , \\ &= \sigma_v^x C_{l_1}^{\perp} |\Psi_0\rangle , \\ &= 0, \end{aligned} \tag{4.30}$$

since the ground state $|\Psi_0\rangle$ is an eigenstate with 0 eigenvalue for the orthogonal link operator $C_{l_1}^{\perp}$. Similarly, the same holds for the other three link operators C_{l_2}, C_{l_3}, and C_{l_4} as they all share te vertex v as common support, andthe operator σ_v^x plays the role of $\mathcal{O}_v$ in Eq. (4.17). Therefore, it is responsible for the creation of four *link type* excitations.



(a) (b)

Figure 4.4: In (a) we show the relevant vertices and edges for the process described in Eq.(4.30), the operator σ_v^x acts on the vertex v while the link operator C_{l_1} acts simultaneously on vertices v, v_1 and the edge l_1. In (b) we illustrate the four *link type* excitations being created by the action of σ_v^x on the ground state $|\Psi_0\rangle$. Furthermore, we draw a surface whose intersections with the edges of the lattice are the places where the localized *link type* excitations arise.

Let us consider a bigger portion of the lattice now delimited by a surface D as shown in Fig.(4.5), note that there are three vertices enclosed by this surface, we call them v_1, v_2 and v_3. Now, consider the action of the operator defined in Eq.(3.81):

$$X(D) = \bigotimes_{v\in D} \sigma_v^x, \tag{4.31}$$

on the ground state $|\Psi_0\rangle$, the resulting state

$$|\Psi_l(D)\rangle = X(D) |\Psi_0\rangle , \tag{4.32}$$

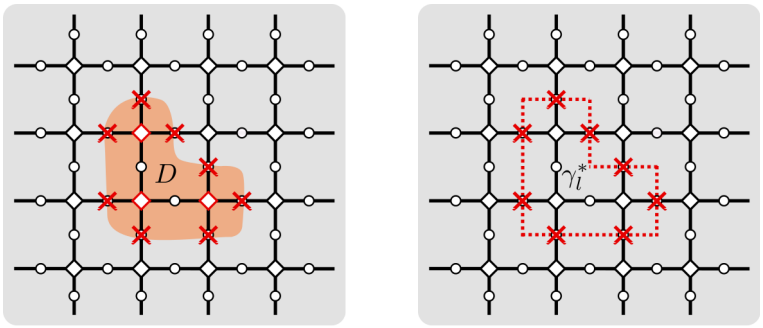
is an excited state that holds localized excitations of the *link type* at the boundary of D, this is, the edges being cut across by the boundary of D, as depicted in Fig.(4.5), since:

$$C_{l\perp D} |\Psi_l(D)\rangle = 0, \tag{4.33}$$

where $l\perp D$ stands for the edges being crossed by the boundary of D.

Notice that a similar excited state can be got from the action of a closed loop operator $X(\gamma_l^*)$ such that the loop γ_l^* coincides with the boundary of the surface D, such operator will create and excited state $|\Psi_l(\gamma_l^*)\rangle = X(\gamma_l^*) |\Psi_0\rangle$ that holds *link type* excitations at the edges it crosses, exactly as the $X(D)$ operator of Eq.(4.31) does. Both states have the same energy, so the following equivalence can be set:

$$X(D) \equiv X(\gamma_l^*), \tag{4.34}$$



(a) (b)

Figure 4.5: In (a) we show the *link type* excitations that are created at the boundary of the highlighted region D, this excited state is the result of the action of $X(D)$ on the ground state $|\Psi_0\rangle$. In (b) we show that a similar state can be created by means of the path operator $X(\gamma_l^*)$ that encloses the region D.

where γ_l^* is a dual closed path that encircles the surface D. This is not true for higher order groups and matter vector spaces with higher dimension, and it is for this reason that we differentiate the two excitation cretion processes by defining both the surface $X(D)$ operator and the closed loop $X(\gamma_l^*)$.

Chapter 5

Concluding Remarks and Outlook

In this final chapter we present a summary of all the content exposed in the preceding chapters. We highlight the important aspects of what was discussed and some that were not included here. This dissertation can be roughly divided in two parts. The first half is devoted to to review well knwon material in the literature. We give a detailed description of the lattice models known as **Quantum Double Models** (QDM). In particular we focus on an special case of such models, namely the **Toric Code**. In the second half of this work, which encompasses chapters 3 and 4, we actually expose our own contribution which consists on an extension of the QDM by the inclusion of vertex degrees of freedom on the lattice that are interpreted as being matter fields. To end this work we mention some directions for future research, as the models here exhibited represent a very small part of the work being developed by the research group led by Prof. Paulo Teotônio-Sobrinho.

In chapter 2 we begin our study of the topological phases of matter by a detailed description of the simplest exactly solvable model that has **Topological Order** at zero temperature, namely, the **Toric Code** that was introduced by A. Kitaev in [1], which is a many body interaction model defined on a bi-dimensional lattice and described by a Hamiltonian that consists on a sum of two kinds of mutually commuting operators and are written in terms of the well known **Pauli Matrices**. The Hamiltonian acts on the full Hilbert space of the model which is the tensor product of local spaces corresponding to the edges of the lattice, where the degrees of freedom are set to lie. The model is exactly solvable and we explicitly construct the ground state of the model which is a long range entangled gas of loops, where the loops represent the action of the vertex operator on the initial configuration of degrees of freedom, such that it is an equal weight superposition of *vortex-free* states; this is, eigenstates of the plaquette operator with eigenvalue 1. Moreover, we show the topological origin of the ground state degeneracy by means of operators whose domains are determined by paths on the (dual) lattice, such that, when the lattice is embedded on the 2-torus, these operators can wind the torus along two homotopically inequivalent *non-contractible* loops, such operators are responsible of the creation of more linearly independent ground states. The elementary excitations of the model arise as violations of the ground state conditions of Eq. (2.22) for the vertex and plaquette operators, thus they come in two species, labeled **charges** and **fluxes**, the simultaneous violation of both vertex and plaquette operators sharing common support is interpreted as a **dyonic** excitation. Such excitations are localized quasi-particles as they are created at specific vertices and/or plaquettes, they can be

moved through the lattice sites at will without an energy cost. More importantly, they can be braided around each other exhibiting their statistical features. We show that the charges and fluxes are bosons when exchanged between themselves, while the braiding of a charge around a flux (or a flux around a charge) is a fermionic process as the state gains an overall -1 sign after an exchange, the same holds for the braiding of two dyons. In [1] a more general class of models is also introduced, coined **Quantum Double Models**. These are interpreted as the Hamiltonian formulation of a lattice gauge theory for a finite and discrete symmetry group G. The model is defined through a Hamiltonian H_{QD} (cf. § 2.4) that is a sum of quasi-local commuting projectors, thus making the model a exactly solvable one. In § 2.4 we introduce such models starting from the explicit construction of the elementary operators that play the role of a gauge transformation and a projective measure of the configuration for the gauge degrees of freedom. The vertex and plaquette operators that make the model are constructed from these elementary ones. Although we only consider the case when the gauge group is the abelian $\mathbb{Z}_2$ the construction here presented is valid for any abelian or non-abelian finite and discrete group G. We do so since at the level of constructing the model considering non-abelian groups presents no further difficulty. The elementary excitations of the QDM are obtained through the action of **Ribbon Operators** that are a generalization of the **String Operators** defined in § 2.1.2. We do not include them in this manuscript as they are not within the scope of the present work and we refer the reader to Kitaev's original work [1], and [52, 53]. We finish the first half of this dissertation with the construction of the $\mathbb{Z}_2$ Quantum Double model, which is equivalent to the Toric Code defined in the beginning of chapter 2. The QDM's had been widely studied during the recent years from various points of view, they exhibit topological order. Their ground state is protected from local perturbations which makes them suitable for implementing quantum error correcting codes; furthermore, the quasi particle excitations have anyonic statistics whose braiding is used to implement the quantum gates that ultimately lead to a quantum computation process [1, 18, 27, 29, 30, 58].

In chapter 3 we extended the QDMs in the sense that now we allow the vertices of the lattice to hold degrees of freedom that are interpreted to be **Matter fields**. This is naturally done in the framework of Lattice Gauge theories. Although the construction of the model as presented in this work might seem a bit artificial, it comes from a convenient splitting of the Transfer Matrix of Lattice Gauge Theories with matter. The details of such procedure can be found in [49] and we do not include them here since this work mainly focus on presenting the models rather than illustrating the obtaining of the operators that make the model. Such procedure started in [34, 41] and is extended in [49]. The operators obtained are thereby presented in § 3.2.2. They include a vertex operator A_v that is the one in charge of performing gauge transformations. Its nontrivial support consists on the matter field sitting at vertex v and the four gauge edges connected to it (since we are considering square lattices). The plaquette operator B_p is exactly the same as the pure gauge models of § 2.4 and accounts for measuring the holonomy of a single plaquette. The fact that it projects to the identity element $e \in G$ is interpreted as having a flat connection. The inclusion of this operator into the Hamiltonian as defined in Eqs.(2.73) and (3.66) accounts for the dynamics of the gauge field. The novelty brought by the inclusion of matter degrees of freedom comes in the shape of a new operator that was not part of the pure gauge QDMs, namely, the link operator C_l. Its common support involve two

matter sites and the gauge link connecting them. It is a coupling term in the Hamiltonian and it accounts for the parallel transport of the theory. Even though this operator was not part of the pure gauge QDMs, it is usual in the Hamiltonian formulation of lattice gauge theories, also known as Kogut-Susskind formulation [59]. Similar formulations were proposed in the literature, from various points of view, including the increasingly active field of Tensor Networks [39, 60]. A wide class of lattice models can thus be obtained from the Hamiltonian defined in Eq.(3.51), namely,

$$H := -\sum_{v} A_v - \sum_{p} B_p - \sum_{l} C_l, \tag{5.1}$$

where v, p and l stand for the vertices, plaquettes and links of the lattice $\mathcal{L}$. The fact that the vertex operator commutes with every other component of the Hamiltonian implies on an overall gauge invariance. In the second part of chapter 3 we explored an specific model that corresponds to choosing the gauge degrees of freedom to be elements of the group algebra $\mathbb{C}(\mathbb{Z}_2)$ and the matter degrees of freedom are set to be elements of a two dimensional vector space V_2. The model is exactly solvable and hence the ground state is written in the form of a gas of **Domain Walls** that separate regions with distinct configurations of matter and gauge degrees of freedom, always satisfying the ground state conditions, which are translated in a graphical representation in terms of **allowed configurations**. This already is a novelty of the new class of models. We then look for the topological degeneracy of such ground state. Recall that the Toric Code which is the pure gauge version of this model, exhibits a topologically dependent degeneracy which is equal to 4 when the model is defined on the surface of a torus. It seems that the inclusion of the matter degrees of freedom to the models lifts the degeneracy of the ground state. As the operators that mapped between the different ground states in the Toric Code, now map the ground state to an excited state, for the $V_2/\mathbb{C}(\mathbb{Z}_2)$ model. This fact is not true for other choices of the gauge group and the dimension of the matter vector space on the vertices of $\mathcal{L}$. Calculating the exact value of the ground state degeneracy is, in general, not an easy task and efforts are being made towards finding a more systematic and efficient way of finding such values. In [61] a great advance has been achieved in this matter, the ground state degeneracy was obtained by the numerical calculation of the trace of the *ground state projector*, i.e.

$$GSD = Tr\left(\prod_{v} A_v \prod_{p} B_p \prod_{l} C_l\right),$$

for all the vertices, plaquettes and links of $\mathcal{L}$.

In chapter 4 we give a brief glance on the elementary excited states that arise in the gauge+matter models, in particular, the $V_2/\mathbb{C}(\mathbb{Z}_2)$ model. The charges that were present on the Toric Code and, in general, the Quantum Double models are still part of the spectrum of the new model. They come as local violations of the ground state conditions for vertex operators and can be created by means of path operators just as in the QDMs case. The novelty comes in the possibility of creating them on individual

vertices because of the presence of the matter fields, as shown in § 4.2. In § 2.3.2 we illustrated the creation of plaquette type excitations procedure through the action of the dual path operators $X(\gamma^*)$ on a given ground state; furthermore, these dual path operators were responsible for mapping between the different ground states of the Toric Code when defined along the non-contractible loops winding the torus. These fluxes were also created in pairs and localized at the endpoints of the dual path. When we consider the $V_2/\mathbb{C}(\mathbb{Z}_2)$ model, in § 4.3 the *plaquette type* elementary excitations are also created by operators defined on paths over the dual lattice, but in this case, besides the plaquette fluxes *link type* excitations are created at every edge being crossed by the dual path as depicted in Fig. 4.3, because of the presence of link operators C_l. This means the dual path operators $X(\gamma^*)$ will not map between different ground states anymore, since the process of creating and annihilating a pair of fluxes is process that maps a ground state into an excited state that actually depends on the path taken by the fluxes. An argument due to Einarsson [16] and applied to Toric Code in [22] states that the topologically dependent ground state degeneracy has its origins on the existence of at least two particle types (e and m in the TC) with anyonic mutual statistics, the process of creating a pair of particles of one type (say e), winding them around the torus and annihilating the pair; together with a process that does so for the another type of particles (for instance m) but winds around the other nontrivial cycle of the torus, since there are two particle types in the TC and they can be moved around each of the two non-contractible loops on the torus hence the 4-fold ground state degeneracy. However, in our new model coined $V_2/\mathbb{C}(\mathbb{Z}_2)$ there is just one type of quasi-particle excitation that can be created in pairs and freely moved along the non-trivial cycles of the torus meaning that the process mentioned in [16, 17] can not be implemented.

Besides the $V_2/\mathbb{Z}_2$ more general models can be defined and solved exactly. The models for which the order of the group and the dimension of the matter vector space are equal, seem to have trivial ground state degeneracy. The ground state of such models can be interpreted as a more complicated gas of **domain walls** and the elementary excitations come in the same types discussed in chapter 4, this is, from the violation of vertex, plaquettes and link operators, only that for a $V_n/\mathbb{Z}_n$ model there are n types of charges, n fluxes, n *link type* excitations and n ways of constructing the surface-like excitations of Eq. (4.32). The charges have bosonic mutual statistics when braided between themselves, and exhibit abelian anyonic statistics when braided around an existing flux. As in the $V_2/\mathbb{Z}_2$ the fluxes have *link type* excitations associated to the dual path γ^* and thus moving the fluxes is a process that creates more excitations. The fusion rules can be casted from the algebra of the operators that create excitations which are generalizations of the Pauli matrices. More interesting phenomena arises when considering models for which the order of the gauge group and the dimension of the matter vector space are not equal, coined $V_n/\mathbb{C}(\mathbb{Z}_p)$, where $n > p$, in [62] some specific models of this type are analyzed, although the Hamiltonian is not the one defined in Eq.(5.1), the models considered in such work exhibit excitations with non-abelian fusion rules, this is, the fusion of two excitations results in a combination of various excitations and not just a single one. While the models can be treated analytically and can be exactly solved, a systematic classification of the elementary excitations for the models including matter fields is lacking, as in the case of the pure gauge models where the elementary excitations are classified by the irreducible representations of the the quantum double $\mathcal{D}(G)$ which is a Hopf algebra, in principle we expect that,

as in the QDMs, the algebra of local operators identifies the particle sectors, we strongly believe that this algebra is isomorphic to the *Transformation Group Algebra* of G. This is still subject of study. Wether the excitations arising in these models can be used to construct quantum gates and ultimately perform quantum computation process is still an open and very important question and it is an actual object of study within the research group of Prof. Dr. Paulo Teotônio-Sobrinho. Another question regarding the models with matter can be stated in the lines of the works of Bombin et al.[53] and Shor et al.[52], in the latter a generalization of the QDM Hamiltonian is made by means of including terms acting on single edges and they are able to characterize the condensation and confinement of quasi particle excitations. A generalization of this type for the pure gauge models can be naturally achieved in our framework as the Transfer Matrices from which we get the models can include single edge operators by a convinient choice of parameters. The same can be done for the models including matter. In [52] the condensation of quasi-particle excitations is achieved via a definition of the QDM in surfaces with boundaries, moreover a domain wall between two Quantum Doubles can be defined to study the tunneling of excitations. This can, as well, be done for the new models including matter fields, it would be interesting to see how the edge excitations behave aiming towards fiding exactly solvable models that can describe **Topological Insulators** as in [63, 64, 65, 66]. The aforementioned are some of the topics that are currently being considered and developed in our research group and they definitely deserve further study.

Appendix:

Groups, Vector Spaces and the Group Algebra

In this appendix we intend to cover some basic algebraic notions specially regarding groups and vector spaces and more importantly the notion of the **Group Algebra** $\mathbb{C}(G)$ of a group G, which is the central algebraic structure in this work. This appendix is not self contained and it is mainly based on [67, 68, 69, 70, 71] where the proofs of some statements can be found.

A Groups

Definition A.1. *A **group** is a nonempty set G together with a binary operation $G \times G \to G$ that assigns to any two elements of G, say, g and h another element (gh) of G. Such that the following properties are satisfied.*

1. **Associativity.** For all $g, h, k \in G$:

$$(gh)k = g(hk). \tag{A.2}$$

2. **Identity.** There is an special element denoted $e \in G$ and called *identity* such that:

$$eg = ge = g, \tag{A.3}$$

   for all $g \in G$.

3. **Inverses.** For every element $g \in G$, there is an element denoted g^{-1} or $\overline{g}$ in G such that:

$$gg^{-1} = g^{-1}g = e. \tag{A.4}$$

So, a group is a set together with an associative operation, sometimes referred as *product operation*, such that any pair of elements can be combined and the result will always be another element of the set G and every element has an inverse. The number of elements on a group is called the *order* of the group and denoted by $|G|$.

If a group is such that $gh = hg$ for all pairs of elements $g, h \in G$, then the group is called **Abelian**, conversely a group that does not fulfill this property for at least some pair of elements is called **Non-Abelian**.

Definition A.2. *A **function** from one set G to another set H is a rule by which one assigns a **unique** element of H to each element of G. Let $f : G \to H$ be a function such that it maps $g \mapsto h$, and we denote the* image *as:*

$$h = f(g) \tag{A.5}$$

The function is said to be invertible if it exists $f^{-1} : H \to G$ such that:

$$f^{-1}(h) = g, \quad \text{and} \quad f(f^{-1}(h)) = h, \tag{A.6}$$

we say that f^{-1} is the inverse function of f. Also f is said to be a **bijection**.

Definition A.3. *Let G and H be groups. A function $f : G \to H$ is said to be an **homomorphism** if:*

$$f(g_1 g_2) = f(g_1) f(g_2), \qquad \forall g_1, g_2 \in G. \tag{A.7}$$

If the function is a bijection then f is called an **isomorphim**. Therefore, if an isomorphism between two groups G and H exists, they are said to be isomorphic. If e_G and e_H are the identity elements of G and H respectively and $f : G \to H$ is an homomorphism, then $f(e_G) = e_H$. Furthermore, $f(g^{-1}) = f(g)^{-1}$ for all $g \in G$.

B Vector Spaces

Definition B.1. *A **vector space** V over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a nonempty set in which two operations are defined, namely, a rule for adding two elements $+ : V \times V \to V$ such that $u + v \in V$ and a multiplication by an element of $\mathbb{K}$, $\cdot : \mathbb{K} \times V \to V$ (called a **scalar**). The elements (called vectors) of V also satisfy the following properties:*

1. V is an abelian group under the addition operation,

2. $\lambda \cdot (u + v) = \lambda u + \lambda v$,
3. $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$,
4. $(\lambda\mu) \cdot v = \lambda \cdot (\mu \cdot v)$,
5. $1v = v$,

for all $\mu, \lambda \in \mathbb{K}$ and $u, v \in V$ and where 1 is the unit element of $\mathbb{K}$.

Definition B.2. *A vector $v \in V$ is said to be a **linear combination** of the vectors $v_1, \ldots, v_n$ in V provided there exist scalars $\lambda_1, \ldots, \lambda_m$ in $\mathbb{K}$ such that*

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = \sum_{i=1}^{n} \lambda_i v_i. \tag{B.8}$$

In particular, consider a set $\{v_i\}$ a set of n vectors. If the equation

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_k = 0, \tag{B.9}$$

has a non-trivial solution, $\lambda_i \neq 0$ for some index i, the set of vector $\{v_i\}$ is called **linearly dependent**, as this implies that at least one of the vectors can be written as a linear combination of the rest. This motivates the following definition.

Definition B.3. *A set of vectors $\{v_i\}_{i=1}^{k}$ is said to be **linearly independent** over the field $\mathbb{K}$ if the solution to Eq.(B.9) is the trivial one $\lambda_i = 0$ for any i.*

The following are easy consequences of the above definition

1. Any set of vectors that contains a linearly dependent set is linearly dependent.
2. Any subset of linearly independent set of vectors is linearly independent.
3. Any set that contains the 0 vector is linearly independent.
4. A set S of vectors is linearly independent, i.e., if and only if for any distinct vectors $v_1, \ldots, v_n$ of S, $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ implies that each $\lambda_i = 0$.

Definition B.4. *Let V a vector space. A subset B of V is called a **basis** for V if B is linearly independent over $\mathbb{K}$ and spans the vector space V. The vector space V is called **finite dimensional** if B is a finite set.*

Denote the the basis as the set $B = \{e_i\}$, any element $v \in V$ can be written as a linear combination of the elements of B,

$$v = \lambda_1 e_i + \lambda_2 e_2 + \cdots + \lambda_i v_i + \ldots, \tag{B.10}$$

the scalars λ_i are called the *components* of v with respect to the basis $\{e_i\}$. If there are n elements in the basis, the **dimension** of the vector space V is n, denoted $\dim V = n$, furthermore, any two basis B_1 and B_2 have the same number of elements.

Given two vector spaces V and W, a map $f : V \to W$ is called a **linear map** if it satisfies

$$f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2), \tag{B.11}$$

for any $\lambda_1, \lambda_2 \in \mathbb{K}$ and $v_1, v_2 \in V$. In this sense, a linear map is actually an homomorphism that preserves the addition in V and the multiplication by an scalar. If W is the field $\mathbb{K}$ itself then f is called **linear function**. Let $f : V \to \mathbb{K}$ be a linear function on an $n-$dimensional vector space V over a field $\mathbb{K}$. Moreover, let $\{e_i\}$ be a basis and take the arbitrary vector $v = \lambda_1 e_1 + \cdots + \lambda_n e_n$, from the linearity of f we know that $f(v) = \lambda_1 f(e_1) + \ldots \lambda_n f(e_n)$. Therefore, it is sufficient to know $f(e_i)$ for all i, to know the result of the operation $f(v)$ on any vector. Remarkably, the set of linear functions is made into a vector space itself, since a linear combination of two linear functions is also a linear function,

$$(\lambda_1 f_1 + \lambda_2 f_2)(v) = \lambda_1 f_1(v) + \lambda_2 f_2(v). \tag{B.12}$$

Definition B.5. *The **dual vector space** V^* of a vector space V is the space of linear functions $f : V \to \mathbb{K}$. It has the same dimension as V.*

Given a basis $\{e_i\}$ of V the **dual basis** of V^* is the set $\{\epsilon^j\}$ such that:

$$\epsilon^j(e_j) = \delta_{i,j} \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}, \tag{B.13}$$

This completely determines ϵ^j as a linear function, let $v = \lambda_i e_i$, the action of ϵ^j on v would be given by:

$$\epsilon^j(\lambda_i e_i) = \lambda_i \epsilon^j(e_i) = \lambda_i \delta_i^j = \lambda_j,$$

and the set $B^* = \{\epsilon^j\}_{j=1}^n$ is a basis of V^*.

The dual vector space V^* has the same dimension as V, i.e. $\dim V = \dim V^*$. To show this, consider a linear function f called *dual vector*, it can be expanded in terms of the dual basis $\{\epsilon^i\}_{i=1}^n$,

$$f = f_i \epsilon^i,$$

the action of f on an element v of V can be interpreted as a bilinear form [1], which we denote using the notation, $\langle\ ,\ \rangle : V^* \times V \mapsto \mathbb{K}$, defined by the action on the basis elements

$$\langle \epsilon^j, e_i \rangle \doteq \delta_i^j.$$

Now, consider another linear functional $\alpha : V^* \mapsto \mathbb{K}$, this is, $\alpha \in (V^*)^*$, we can write this new functional in terms of the pairing relation we defined above, such that the action on an element of the dual vector space, $f \in V^*$, is given by

$$\alpha(f) = \langle f, v \rangle,$$

where $v \in V$, this means we can relate an element in the vector space V with the linear function α, which basically is telling us that there is an isomorphism relating the double dual vector space $(V^*)^*$ and the vector space V, so $V \simeq (V^*)^*$ and their dimensions are the same. In fact all n-dimensional vector spaces are isomorphic to $\mathbb{K}^n$, and they can be regarded as identical vector spaces [67].

Definition B.6. *A **bilinear form** $\mathcal{B}$ is a map $\mathcal{B} : V \times V \to \mathbb{K}$ such that it is linear in both arguments, this is, it satisfies the following properties:*

- $\mathcal{B}(\lambda v, w) = \mathcal{B}(v, \lambda w) = \lambda \mathcal{B}(v, w)$.
- $\mathcal{B}(v_1 + v_2, w) = \mathcal{B}(v_1, w) + \mathcal{B}(v_2, w)$.
- $\mathcal{B}(v, w_1 + w_2) = \mathcal{B}(v, w_1) + \mathcal{B}(v, w_2)$.

Definition B.7. *Let V and W be two vector spaces over $\mathbb{K}$ with basis $\{v_i\}_{i=1}^n$ and $\{w_j\}_{j=1}^m$ respectively. The map $\otimes : V, W \to V \otimes W$ called **Tensor Product** acts on two (or more) vector spaces and creates another vector space.*

Note that in general both vector spaces have different dimension, as in this case, $\dim V = n$ and $\dim W = m$, the dimension of the new vector space thus being $\dim(V \otimes W) = n \times m$, and its basis elements can be denoted by:

$$v_i \otimes w_j,$$

where the indices i, j run as $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$. Thus an element $\Gamma \in V \otimes W$ is an expression of the form

$$\Gamma = \sum_{i,j=1}^{n,m} \Gamma^{ij} v_i \otimes w_j = \Gamma^{ij} v_i \otimes w_j,$$

[1] The pairing is called "inner product" when both target vector spaces are the same, $V = W$ so it maps a pair of vectors $a, b \in V$ to an element of $\mathbb{K}$.

where the coefficients $\Gamma^{ij} \in \mathbb{K}$ are called *components* of Γ. Since the vector spaces V and W are arbitrary we can choose W as being the dual vector space V^*, basis elements on $V \otimes V^*$ are denoted as $\{v_i \otimes v^j\}$, any vector in this space then, say E, can be written as a linear combination of the basis, i.e.,

$$E = E^i_j v_i \otimes v^j,$$

where the components of this vector are E^i_j, note that this tensor has both subscript and superscript indices which is a consequence of the fact that the tensor product vector space is made out of V and its dual V^*. Following this notation, the tensor $A_{a_1 a_2 \ldots a_n}$ represents an element of the vector space $\underbrace{V \otimes V \otimes \cdots \otimes V}_{\text{n times}}$; likewise the tensor with all indices up $B^{b_1 b_2 \ldots b_m}$ represents an element of the vector space $\underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{\text{m times}}$, and the mixed tensor $C^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_m}$ represents an element of $V^{\otimes n} \otimes V^{* \otimes m}$.

C The Group Algebra $\mathbb{C}(G)$

Given a finite group G, canonically associated to it, there is a vector space of dimension $|G|$ with coefficients on a field $\mathbb{K}$ called the group algebra $\mathbb{K}(G)$, the advantage brought by this structure is that besides the product induced by the group multiplication it is possible to define the addition operation. In order to properly define this algebraic structure let us begin by mentioning some basic definitions, from now on we choose the number field $\mathbb{K}$ to be the complex numbers $\mathbb{C}$.

Definition C.1. *An (associative and unitary)* ***algebra*** *is the triple $(\mathcal{A}, m, \eta)$ where $\mathcal{A}$ is a vector space, a product map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, represented by $m(v \otimes w) = vw$, where $v, w \in \mathcal{A}$, and $\eta : \mathbb{K} \to \mathcal{A}$. Such that the following properties are satisfied:*

1. $u(vw) = (uv)w$;
2. $u(v + w) = uv + uw$;
3. $(u + v)w = uw + vw$,

for all $u, v, w \in \mathcal{A}$. The ***identity*** *element $e \in \mathcal{A}$ such that $eu = ue = u$, for all $v \in \mathcal{A}$.*

The multiplication map is completely defined by its action on the basis elements $\{\phi_i\}$, i.e.

$$m(\phi_a \otimes \phi_b) = \phi_a \phi_b = m^c_{ab} \phi_c, \tag{C.14}$$

where the sum over repeated indices is implied, the coefficients m^c_{ab} are called *structure constants* and can be thought as tensors representing elements of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}^*$.

Analogously, we can consider the dual vector space $\mathcal{A}^*$ and associate an algebra structure over it, by defining an associative multiplication map $\Delta : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$ which is completely defined by the action on two elements of the dual basis $\{\xi^i\}$,

$$\Delta(\xi^i \otimes \xi^j) = \xi^i \xi^j = \Delta^{ij}_k \xi^k, \tag{C.15}$$

where the sum over repeated indices is implicit and the coefficients Δ^k_{ij} are the *structure constants*. Additionally we define a unity element that can be thought of as a map $\epsilon : \mathbb{K} \to \mathcal{A}^*$ such that $\epsilon(1)$ is the unity for the dual multiplicative map Δ. So the triple $\langle \mathcal{A}^*, \Delta, \epsilon \rangle$ defines an algebra structure in $\mathcal{A}^*$; Equivalently we can regard the maps Δ and ϵ as being a co-multiplication and co-unity in $\mathcal{A}$ respectively. In this sense, the map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is defined as

$$\Delta(\phi_i) = \Delta^{jk}_i (\phi_j \otimes \phi_k), \tag{C.16}$$

and the unity map $\epsilon : \mathcal{A} \to \mathbb{K}$ is defined as

$$\epsilon(\phi_i) = \epsilon^i. \tag{C.17}$$

Definition C.2. *A (co-associative and co-unitary)* ***Co-Algebra*** *is a triple $\langle \mathcal{A}, \Delta, \epsilon \rangle$, where $\mathcal{A}$ is a vector space over a field $\mathbb{K}$, $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is an associative co-multiplication map and $\epsilon : \mathcal{A} \to \mathbb{K}$ is a co-unit map.*

Therefore, the algebraic data $\langle \mathcal{A}, m, \eta, \Delta, \epsilon \rangle$ defines an algebra and a co-algebra. If, additionally the **compatibility** conditions between the multiplication, co-multiplication, unit and co-unit are satisfied the quintet is said to define **bi-algebra**. The so called compatibility conditions imply that Δ and ϵ are homomorphism of the algebra [71], this is,

$$\Delta(\phi_g \phi_h) = \Delta(\phi_g)\Delta(\phi_h), \tag{C.18}$$
$$\epsilon(\phi_g \phi_h) = \epsilon(\phi_g)\epsilon(\phi_h). \tag{C.19}$$

Definition C.3. *A* ***bi-algebra*** *is a vector space $\mathcal{A}$ over a flied $\mathbb{K}$, endowed with an algebra structure $\langle \mathcal{A}, m, \eta \rangle$, a co-algebra structure $\langle \mathcal{A}, \Delta, \epsilon \rangle$ such that the co-multiplication map Δ and the co-unit map ϵ are homomorphisms of the algebra structure.*

Definition C.4. *Let $\langle \mathcal{A}, m, \eta, \Delta, \epsilon \rangle$ be a bi-algebra. The linear map $S : \mathcal{A} \to \mathcal{A}$ is called an* ***antipode*** *of the bi-algebra $\mathcal{A}$ if S satisfies $m * (\mathbb{1} \otimes S) * \Delta = \eta * \epsilon = m * (S \otimes \mathbb{1}) * \Delta$,*

where $*$ is a convolution product.

Definition C.5. *The sextet $\langle \mathcal{A}, m, \eta, \Delta, \epsilon, S\rangle$ is a **Hopf algebra**.*

Let G be a finite group whose elements are $g_1, \ldots, g_n$, such that $|G| = n$. The vector space $\mathbb{C}(G)$, whose basis elements are $\{\phi_g : g \in G\}$, is composed of all linear combinations

$$u = \sum_{g\in G} \lambda_g \phi_g, \tag{C.20}$$

where all $\lambda_g \in \mathbb{C}$. The addition of two elements $u = \sum_g \lambda_g\phi_g$ and $v = \sum_g \eta_g\phi_g \in \mathbb{C}(G)$ is given by:

$$u + v = \sum_{g\in G} (\lambda_g + \eta_g)\phi_g. \tag{C.21}$$

The multiplication by an arbitrary element $\lambda \in \mathbb{C}$ is naturally given by:

$$\lambda u = \sum_{g\in G} (\lambda\lambda_g)\phi_g. \tag{C.22}$$

where the multiplication $\lambda\lambda_g$ is that of $\mathbb{C}$. These two facts ensure that $\mathbb{C}(G)$ is a vector space over the field $\mathbb{C}$ with basis elements $\{\phi_g\}$. Besides the vector space structure $\mathbb{C}(G)$ carries the product operation of G and allows us to define a multiplication map $m : \mathbb{C}(G) \otimes \mathbb{C}(G) \to \mathbb{C}(G)$ defined by the action on the basis states:

$$m(\phi_g \otimes \phi_h) := \phi_{gh}, \tag{C.23}$$

for all $g, h \in G$. This is, we choose the *structure constants* of Eq.(C.14) to be $m^c_{ab} = \delta(ab, c)$. This multiplication is indeed associative as:

$$\phi_a(\phi_b\phi_c) = \phi_a(\phi_{bc}) = \phi_{abc}, \tag{C.24}$$

and from Eq.(A.2) we now the group product (abc) is associative. There is an special element ϕ_e labeled by the identity element of the group $e \in G$. From Eq.(A.3) it is clear that the following holds

$$\phi_e\phi_g = \phi_g\phi_e = \phi_g, \tag{C.25}$$

for any $\phi_g \in \mathbb{C}(G)$ or equivalently, any $g \in G$. Therefore the triad $\langle \mathbb{C}(G), m, \eta\rangle$, where $\eta(1) = \phi_e$ is an (associative and unitary) **algebra**, called the **Group Algebra** $\mathbb{C}(G)$ of a group G.

Furthermore, we can define a co-algebra structure on the dual vector space $\mathbb{C}(G)^*$. Consider the dual basis $\{\xi^g : g \in G\}$, such that $\xi^g(\phi_h) = \delta(g, h)$. The co-multiplication map $\Delta : \mathbb{C}(G)^* \otimes \mathbb{C}(G)^* \to \mathbb{C}(G)^*$ is defined through the action on the basis elements, namely,

$$\Delta(\xi^g \otimes \xi^h) = \delta(g, h)\xi^g, \tag{C.26}$$

for any $g, h \in G$. This is, we have chosen the structure constants of Eq.(C.15) to be $\Delta_k^{ij} = \delta(ij, k)$. The co-unity map is then given by

$$\epsilon(1) = \sum_g \xi^g, \qquad \forall\ g \in G, \tag{C.27}$$

where 1 is the identity of the field $\mathbb{C}$. The antipode endomorphism $S : \mathbb{C}(G) \to \mathbb{C}(G)$ is defined to be:

$$S(\phi_g) = \phi_{g^{-1}}, \qquad \forall\ g \in G. \tag{C.28}$$

Note that this map squares to identity, namely, $S^2 = \mathbb{1}$ as the result of applying twice on a basis element gives the same element. Therefore, the sextet $\langle \mathbb{C}(G), m, \eta, \Delta, \epsilon, S \rangle$ forms a **Hopf Algebra**.

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