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A teoria efetiva de campo para estruturas em larga escala e múltiplos traçadores

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The effective field theory of large-scale structures and multiple tracers

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In dedication to my parents, Elianete and Sávio.

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Resumo

A cosmologia moderna possui vários problemas em aberto, sendo muitos deles ligados a questões de física fundamental. Dois exemplos desses problemas são a natureza da matéria escura e qual é o mecanismo por trás da expansão acelerada do Universo. Uma fonte de informação importantíssima para a busca por soluções para tais problemas são as estruturas em larga escala do Universo. Com isso em mente, neste trabalho nós propomos uma nova maneira de modelar essas estruturas. Esse novo modelo foi obtido juntando idéias de teorias efetivas de campo para as estruturas em larga escala com a técnica de múltiplos traçadores. Para testá-lo, realizamos uma análise do espectro de potência utilizando simulações de N-corpos. Nossos resultados, que neste trabalho se limitam ao espaço real, mostram de maneira clara a superioridade da estratégia de dividir os traçadores em subcategorias e fazer uma análise conjunta de todos estes subgrupos ao invés de tratá-los como sendo um único traçador, mesmo considerando o número maior de parâmetros que essa divisão acarreta. Nossas conclusões devem se estender a situações mais realísticas, como em análises feitas em espaço de redshift e com dados reais.

Palavras chave: Cosmologia, Estrutura em Larga Escala, Teoria de Perturbação, Teorias Efetivas de Campo.

Abstract

Modern cosmology has several open problems, many of them stemming from fundamental physics. Two examples of these problems are the lack of information about the nature of dark matter particles, and the mechanism responsible for the accelerated cosmic expansion that we observe. A key source of information that can contribute to solving these issues and many other problems is the distribution of the large-scale structures of the Universe. This motivated us to employ a new model for the description of structure formation, which is based in effective field theories, and to generalize it to treat multiple tracers of the cosmic structures. In order to test our model we performed a series of power-spectrum analysis using results from N-body simulations in real space. Our results show a clear advantage in performing parameter extraction by splitting the tracers into different classes, as compared to the one usually employed in the literature – despite the larger number of free parameters implied by this subdivision. We suppose that our main conclusions will hold in more realistic cases, e.g. in analysis performed in redshift space, as well as with real data.

Keywords: Cosmology, Large-Scale Structure, Perturbation Theory, Effective Field Theories.

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0 Introduction

Cosmology has entered an era of high precision data. With past, current and upcoming cosmological surveys we have been able to better understand the history of the universe and to test many theories of physics. This cosmological chronology is tremendously complex, with even its simplest aspects deserving a whole research area on their own. In this introduction we briefly review the main ideas behind the current standard cosmological model, Λ CDM, and help the reader to situate where the ideas developed in this dissertation find a place in the panorama of modern cosmology.

0.1 The Big-Bang model

Although in our history we always tried to speculate about the nature of the universe, only recently we were capable of developing tools to promote these speculations into real science. With the advent of modern technology we were capable of extending our view of the universe, and nowadays we are fortunate to have data like the one shown in Figure 1. In this figure we see the distribution of galaxies obtained by the surveys 2dF [1], CfA2 [2] and SDSS [3], as well as a map of large-scale structure from the Millennium simulation [4]. These plots represent a 3D map of our universe in recent times. One of the first conclusions one can derive from this figure is the presence of some web pattern, usually referred to as the cosmic web. The presence of such structures requires that some regions of the universe must be more populated than others, and that this distribution should not be completely random. However, it is a fact that when we look at scales much larger than the ones being depicted in Fig. 1, our universe looks more uniform. In the words that cosmologists usually employ, on large-scales the universe is then assumed to be *homogeneous*.

A similar conclusion can be made from the *Cosmic-Microwave Background* (CMB), in Fig. 2. The CMB is a temperature map of the universe, right after the epoch when the photons decoupled from the matter, and could finally free-stream. Different regions in

Figure 1: Galaxy maps obtained by the 2dF, CfA2 and SDSS surveys. In the right there is also an illustration of a synthetic galaxy map obtained from the Millenium simulation. Extracted from [5].

this figure represent different directions in the sky, so we can readily conclude that there is no perfect isotropy. Nevertheless, the disparities between the blue and red regions are about $10^{-5}K$, so the deviations from isotropy are quite small. These two hypothesis, that our universe is homogeneous on large-scales but with deviations from homogeneity as we probe smaller-scales, and that it is almost isotropic, are the basis of modern cosmology. In mathematical language this setup is modeled by assuming an homogenous and isotropic background. On top of that we introduce small perturbations that account for the deviations from homogeneity and isotropy. The study of the dynamics of these perturbations is what defines cosmological perturbation theory and the tools we develop in this work furnishes, in some extent, a new perspective on how to better model them.

In the framework of general relativity (GR), the homogeneous and isotropic back-

Figure 2: The temperature map of the Cosmic Microwave Background (CMB) measured by the Planck satellite. The usual temperature changes between the reddish and the bluish regions are of order μK . This image is from the Planck collaboration.

ground is described by the *Friedmann–Lemaître–Robertson–Walker* (FLRW) metric,

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \quad (0.1)$$

where the parameter k is associated with the curvature of the universe. Although we may show it in some equations, all results of this work were obtained by setting $k = 0$.

The dynamics is present in the *scale factor*, $a(t)$, and as will prove to be convenient in the subsequent discussion, we define the *Hubble rate* as

$$H(t) \equiv \frac{1}{a} \frac{da}{dt}, \quad (0.2)$$

which measures how the scale factor changes as the universe evolve. We will also use the *conformal time*, $\mathcal{H} \equiv aH(t)$. The time evolution of the Hubble rate is obtained by solving the equations of motion for the FLRW metric.

In this work we will assume general relativity (GR) for describing gravitational dynamics. As usual in GR we also need to specify what are the sources for gravity, i.e. the energy momentum tensor. In order to discuss these sources it is convenient to define

Figure 3: The running of the comoving Hubble parameter with the redshift. The grey regions are associated to the 65% and 95% confidence intervals of the Planck data. Extracted from [6].

the normalized densities, bringing all densities to the same units. The usual normalizing factor is the *critical density* today,

$$\rho_{\text{cr}} \equiv \frac{3H_0^2}{8\pi G}, \quad (0.3)$$

where G is the gravitational constant and H_0 is the Hubble rate today. Thus for some component i with density $\rho_i(a)$ the associated *density parameter* is defined as:

$$\Omega_i \equiv \frac{\rho_i(t_0)}{\rho_{\text{cr}}}. \quad (0.4)$$

The density of component i at any other time can be found by solving the conservation equation in an expanding universe:

$$\dot{\rho}_i + \frac{\dot{a}}{a}[3\rho_i + 3P_i] = \dot{\rho}_i + 3\frac{\dot{a}}{a}\rho_i[1 + w_i] = 0. \quad (0.5)$$

It is easy to notice that if the *equation of state*, $w_i = P_i/\rho_i$, is time independent, the

density can then be written as

$$\rho_i(a) = \Omega_i \rho_{\text{cr}} a^{-3(1+w_i)} . \quad (0.6)$$

Thus, any information about ρ_i entails some constraint in the combination $\omega_i \equiv \Omega_i h^2$ [7]. Some of the most relevant sources for cosmological dynamics are matter and radiation, corresponding to $w_m = 0$ and $w_r = 1/3$, respectively.

However, it is well established today that our universe is going through some epoch of accelerated expansion. The first evidences for this acceleration were presented in [8, 9]. Neither matter nor radiation can be responsible for such accelerated expansion, so understanding the mechanism causing such behavior is one of the most intriguing problem in physics nowadays. The current standar model, Λ CDM, assigns this accelerated expansion with a *cosmological constant*, Λ , which is added to the Einstein equation. This contribution can be equally moved into the energy-momentum tensor, so it can also be viewed as a new energy component. Moreover, since the energy density of the cosmological constant occupies space independently of any other matter or radiation components, it can be regarded as the vacuum energy.

In general, any kind of “dark energy” with a negative equation of state ($w < -1/3$) is able to drive an accelerated expansion. These are more general approaches that allow for deviations from the scenario with a cosmological constant. In particular, there are models called $w_0 w_a$ CDM, where dark energy has an equation of state that is parametrized by:

$$w(z) = w_0 + (1 - a(z))w_a . \quad (0.7)$$

In addition to dark energy, modified theories of gravity could also account for such expansion. These theories include corrections of the form $f(\mathcal{R})$ to the usual Einstein-Hilbert action, where $\mathcal{R}$ is the Ricci scalar (e.g. see [10]). Current data imposes strong constraints on modified theories of gravity, which imply that they must be very close to GR with a dark energy component [11, 12]. Therefore, in practice the addition of

Figure 4: Some examples from the gain obtained by performing a joint analysis between CMB, BAO and Full Shape. Extracted from [16].

a new source of energy appears to be inevitable [13, 14]. Moreover, in addition to the robust fact the universe is going through an expanding epoch, we also know that current experiments sets $\Omega_m \approx 0.3$. Since radiation is completely negligible nowadays, the dark-energy must have $\Omega_\Lambda \approx 0.7$, being the dominant component in the present. Both matter and radiation have sub-categories, such that the total contributions to the energy density can be summarized as [15]:

$$\underbrace{\Omega_\gamma + \Omega_{\nu,r}}_{\Omega_r} + \underbrace{\Omega_{\text{cdm}} + \Omega_b + \Omega_{\nu,m}}_{\Omega_m} + \Omega_\Lambda = 1 ,$$

where in the radiation sector we have γ and ν standing for photons and massless neutrinos, whereas in the matter sector we have cdm, b and ν, m standing for cold-dark matter, baryons (ordinary matter) and massive neutrinos, respectively.

With all these facts in hand we can get the evolution of the scale factor by using the Einstein equation. It turns out to be [7]:

$$H(t)^2 = H_0^2 \left(\Omega_r a(t)^{-4} + \Omega_m a(t)^{-3} + \Omega_k a(t)^{-2} + \Omega_\Lambda \right) , \quad (0.8)$$

where Ω_k is the density parameter associated with the spatial curvature of the universe. In Figure 3 we show the comoving Hubble rate against data. The shaded regions are associated with the 1- and 2- σ confidence regions around the Λ CDM model adjusted to

Figure 5: Constraint obtained in the dark-energy equation of state by including a full shape (FS) analysis. Extracted from [16].

the Planck measurements. The error bars represent other cosmological probes besides CMB, such as quasars and Ly- α forest. It is evident the onset of the accelerated epoch at $z \approx 0.6$.

Once the background equations are solved, the next step is to include perturbations. This is a very complicated topic, and basically all modern cosmology relies on solving for these perturbations. The general theory behind it can be found in [7, 17, 18, 19]. For a good review on how these perturbations explain the CMB we refer the reader to [20]. For now on we shall only be concerned with the universe in its most recent phases, i.e., after the radiation-dominated epoch. In other words, we will be mainly concerned with the perturbations of the matter field during the matter-dominated era.

0.2 The Large-Scale Structures: what can we learn from it?

The point of this work is to motivate, in some sense, a new technique that can better explain the spatial distributions of tracers, such as galaxies, quasars and dark-matter halos. As we will show, the model we developed here leads to significant improvements in the description of the non-linear clustering, which in turn leads to a better measurements of some cosmological parameters. Here we briefly motivate why one should be concerned in

seeking new elaborate theories that better explain non-linear clustering in the first place, and what kind of physics we can learn in the process.

Cosmology is a branch of science that can give us physical insights in theories ranging from particle physics to gravitation. However, the most striking questions that remain to be answered are related to the cosmic expansion. As we briefly discussed in the last subsection, the universe is going through an accelerated expansion phase, and we have very few clues about what is causing this. Moreover this energy source, called dark-energy, contributes with approximately 70% to energy budget of the universe nowadays. The next generation of cosmological surveys will aim at shedding light on this problem – see, e.g., the Euclid¹ and DESI² projects. Since structure formation is a dance between accelerated expansion and gravitational collapsing, it is quite natural to seek for constraints on dark energy by studying the structures like the one in Fig. 1.

Although the outstanding measurements of the cosmological parameters performed by the Planck satellite may be superior to the ones obtained by clustering alone, their joint analysis produces stronger constraints. For instance, it is usual to find in the literature joint analysis of the CMB with the imprints of the Baryonic Acoustic Oscillations (BAO) in galaxy clustering. Some examples of such analysis can be seen in Figs. 4 and 5, which were extracted from the SDSS-III [16]. Also in Fig. 6 we show some results from the third year of the Dark Energy Survey (DES) [21]. From all these figures it is evident that the joint analysis of Planck and BAO produces slightly better measurements than Planck alone. Nevertheless, it has been noted that an additional *Full-Shape* (FS) analysis of the distribution of galaxy clustering can produce even better constraints. The main objective of the FS approach is to use as data points the whole domain of the estimated n-point functions of the measured galaxy distribution, both the broadband and BAO wiggles. This can only be done with a robust modeling of the clustering around the non-linear scales. Information about the cosmological parameters is then obtained by adjusting the curves predicted by these models to the measured data points. The BAO analysis, in contrast,

¹<https://www.euclid-ec.org/>

²<https://www.desi.lbl.gov/>

Figure 6: The most recent results of the DES survey. This figures shows how important it is to perform joint analysis of distinct sources of information, including the information about galaxy clustering that enters in the 3x2pt data set, which includes a FS analysis. Extracted from [21].

is limited to narrow band around the BAO signature. See [22] for a joint analysis using only FS and BAO, without CMB, and, for instance, [23, 24, 25, 26, 27, 28] for other FS procedures. Thus, we can affirm that:

- *The information contained in the scales near the non-linear regime is key to deriving strong constraints in the cosmological model as a whole, including the parameters modeling the dark-energy dynamics.*

For a more detailed discussion on how non-linear clustering can be used to study dark-energy we refer the reader to [29, 30]. In fact, in that reference the authors make use of the Effective Field Theory of Large-Scale Structure (EFTofLSS), which will be one of the main topics in this dissertation. It is important to mention there are another ways to model the non-linear regime. The EFTofLSS is description that is built solely with analytical techniques of perturbation theory (they are scrutinized in Section 2). Nevertheless, there are another surrogate modelling for these fields at the same scales. These alternative approaches, such as Halofit [31] and Cosmic Emulators³ [32], relies on interpolation procedures and phenomenological fits. Although these models can come in handy sometimes, they are a *black box* and not so versatile as the perturbative approach – the latter not only has a natural way to include features, such as modified gravity, massive neutrinos, primordial non-Gaussianities and so on, but also furnishes analytical expressions to compute any n-point statistics up to any order in perturbation theory. The alternative approaches, on the other hand, require a whole new set of expensive simulations and interpolations for all distinct scenarios.

In this work we will assume GR, so both the EFTofLSS and biased tracer theory developed here are applicable to cosmological models that also assume GR. However, there are works that extend this formalism to include modified gravity theories. Therefore, it is possible to obtain relevant constraints in gravitational theories when using an analysis very similar to the one performed here [29, 30, 33, 34, 35, 36, 37]. Thus we can also affirm that:

³<https://www.hep.anl.gov/cosmology/CosmicEmu/index.html>

- *The Large-Scale structures dynamics is an indispensable source of information to study modified gravity theories.*

From the perspective of particle physics it is established that:

- *A robust knowledge of structure formation in both the linear and the non-linear regimes can provide valuable information about the nature of dark-matter particles, the mass scale of massive neutrinos and even about theories beyond the Standard Model of Particles and Fields.*

The dominating matter component in the universe today is the dark matter, contributing approximately by 85%. Thus, it is expected that different models of dark matter particles should somehow leave imprints in the clustering of biased tracers clustering. In the standard Λ CDM model, dark matter is assumed to be cold (i.e. having negligible thermal energy, which is necessary for formation of small-scale structures), collisionless (very small cross-section with other particles or with itself), and stable. For a long time this more idealized CDM model has provided an excellent fit to data. Nevertheless, with the advent of precision cosmology we will have the opportunity to check if more elaborate models are required, which in turn may lead to a better understanding about the nature of dark matter. For instance, the Generalised Dark Matter (GDM) model, first introduced by [38], encapsulates deviations from the CDM by introducing a speed of sound and viscosity to the matter fluid. This model was extensively studied in [39], where it was concluded that CMB alone is not enough to produce stringent constraints in the GDM parameters, with the nonlinear regime of structure formation being a promising probe to break some degeneracies [40, 41, 42, 43, 44]. This is possible because the interactions of dark matter with baryons suppress structure formation with respect to the Λ CDM, so a detection of such damping would result in direct measurements of the dark matter cross section [45, 46, 47, 48]. In the same vein, the impact of massive neutrinos can also lead to such suppression on small scales, being a source of information that is complementary to the particle physics experiments [49]. The required approach to probe all these signals are very similar to the one we will perform in this work [50, 51, 52, 53, 54, 55].

1 The Initial Conditions of the Universe

The standard model introduced in the last Section is able to explain basically all current data with only six free parameters: the Hubble parameter, H_0 , the baryonic density, Ω_b , the total matter density, Ω_m , the optical depth to reionization, τ , the primordial scalar perturbation, A_s and the primordial spectral index, n_s . Two of these parameters, A_s and n_s , are related to the very early dynamics of the universe during inflation. This theory was developed around the end of 70's as an attempt to solve problems with the standard hot Big Bang scenario [56]. One of these problems is known as *the horizon problem*, and is related to the fact the Cosmic Microwave Background (CMB), in the hot Big Bang scenario, should not be as uniform as we observe it to be [7, 57, 58, 59]. Inflationary physics not only solve this problem but also furnishes a natural mechanism that explains the primordial fluctuations needed to seed structure formation. Since the whole point of this dissertation is to propose a model that can produce a good description of the dynamics of the evolved form of those seeds, we decided to provide a brief review on this topic.

In the inflationary epochs of the universe the energy scales were so high and the usual spatial scales were so small (near the Planck scale) that the predominant scalar field at that time, the *inflaton field*, was subjected to strong quantum effects that caused it to possess *intrinsic quantum fluctuations*. Moreover, as will see, in the inflationary epoch the universe goes through an accelerated expansion, with the scale factor having at least 60 *e*-folding of expansion. Thus the quantum fluctuations are stretched to scales larger than the Hubble horizon, where they freeze-out⁴, and continue their evolution after reentering the horizon, when they start sourcing structure formation.

The simplest model of inflation is described by a single real-valued scalar field, minimally coupled to gravity. A simple second-order expansion of that field's action appears to be enough to describe the plethora of data available nowadays. Together with the slow-roll assumption, which is a well motivated approximation for the model and will be fairly discussed in this Section, some predictions can be made about the density perturbations.

⁴There is a well understood conservation theorem guaranteeing this [59, 60, 61].

Firstly, they are *almost scale-invariant*. As will be clear in the subsequent discussions, the equations governing the evolution of the perturbations have a mild spatial dependence, so basically all modes experiment the same expansion. The parameter capturing departures from a perfect scale invariant scenario is the spectral index, n_s . Secondly, as mentioned before, *the primordial density perturbations are intrinsically connected to the inflaton perturbations*. Without inflation, we would not have any prior knowledge about how to introduce these quantities. Finally, *the density perturbation (or the inflaton perturbations) are approximately Gaussian*.

Inflation is still a theoretical scenario, since we have no definitive idea what it really is. In principle, the only prerequisite is any particle candidate that can imitate a slowly-moving scalar field during the early universe. For instance, some previous works tried to conciliate the Standard Model (SM) with inflation by assuming the Higgs boson as a candidate for the inflaton. Its Lagrangian in the Standard Model be written as

$$\mathcal{L}_h = |D_\mu H|^2 + \mu^2 H^\dagger H - \lambda (H^\dagger H)^2, \quad (1.1)$$

where H is the Higgs doublet, λ is the Higgs self-coupling, and μ is the Higgs mass parameter. The self-interaction, however, leads to an extremely long inflationary period and do not generate the desired spectrum of primordial perturbations. To circumvent this problem, a new term must be added to the Lagrangian, a non-minimal coupling of the Higgs to gravity, leading to the so-called *Higgs inflation* [62, 63],

$$\mathcal{L}_{HR} = \xi H^\dagger H R, \quad (1.2)$$

where ξ is a constant to be fixed by comparing the model with data. Since we know all the types and strengths of the interactions between the Higgs field and SM particles, we have precise inflationary predictions that are in good agreement with current observations. Nevertheless, this theory as formulated above require $\xi \approx 10^4$ to explain the correct amplitude of density fluctuations. There are more elaborate Higgs inflation models that

circumvent this unitarity problem – e.g. see [64, 65].

Higgs inflation is one drop in an ocean of other possible candidates to explain inflation. We expect to have stronger restrictions in these models with the data of the next generation of surveys, where the information coming from galaxy clustering will be indispensable. For a review on the plethora of inflationary models see, for instance, [66, 67].

1.1 Introducing the Inflaton field

The usual matter content, such as radiation or baryonic matter, satisfies the strong energy condition (SEC). Consider a matter source with P being its pressure and ρ its density. This condition is defined as $1 + 3P/\rho > 0$. As will be discussed in this section, SEC necessarily implies that at the last scattering surface, which is basically what we measure in the CMB, many regions should be causally disconnected. But this cannot be true, since the remarkable measurements of the CMB show that the universe was approximately thermalized in this epoch. We will show how inflation can solve this problem.

1.2 Particle and Event Horizons

The origin of the famous horizon problem is related to the inconsistent particle horizon size at the last scattering surface. The size of causal patches is determined by how far the light can travel in a certain amount of time. Under the assumptions of homogeneity and isotropy, the line element describing the metric of spacetime is given by:

$$ds^2 = a^2(\tau)[d\tau^2 - d\chi^2], \quad (1.3)$$

where $a(\tau)d\tau = dt$, with τ the conformal time. In this way, the path length followed by photons is given by $\Delta\chi(\tau) = \pm\Delta\tau$, and the definitions of horizons refer to this relationship.

The *particle horizon* defines a region in the spacetime diagram of χ versus τ representing all the region in the past from where a particle could have sent a signal in order for

an observer receive it today. It defines the causal past region, so any particle that has never intersected it cannot send information to the observer. More precisely, for two instants in time τ_1 and τ_2 , it is simply $\Delta\tau = \tau_2 - \tau_1$. If the Big Bang occurred at some point $\tau_i \equiv 0$, then the largest comoving distance from which an observer at some previous time t will be able to receive signals is given by:

$$\chi_{\text{ph}}(\tau) = \tau - \tau_i = \int_{\tau_i}^{\tau} \frac{dt}{a(t)}. \quad (1.4)$$

The *event horizon* has a definition similar to the particle horizon. While the latter defines a region where events that happened far away, in the past, will never reach the observer today, the former does the same, but for the future. The largest distance from which an observer at some future time t_f can receive signals that were emitted at some time than t is:

$$\chi_{\text{eh}}(\tau) = \tau_f - \tau = \int_{\tau}^{\tau_f} \frac{dt}{a(t)}. \quad (1.5)$$

The horizon problem refers to the size of particle horizon at the time of the last scattering surface, if one assumes the standard hot Big Bang scenario. To elucidate it, we rewrite Eq. (1.4) in the following way:

$$\chi_{\text{ph}}(\tau) = \int_{\tau_i}^{\tau} \frac{dt}{a} = \int_{a_i}^a \frac{da}{a\dot{a}} = \int_{\ln a_i}^{\ln a} (aH)^{-1} d \ln a, \quad (1.6)$$

where $\mathcal{H} = (aH)^{-1}$ is the Hubble radius. Using the Friedmann equations, it can be written as:

$$\mathcal{H} = H_0^{-1} a^{\frac{1}{2}(1+3w)}. \quad (1.7)$$

From this equation we can see why the matter source for the horizon dynamics is so important: it appears explicitly in the evolution of the Hubble radius. Therefore, different matter contents will produce different horizon evolutions. Going back to the SEC, it is now clear why it leads to the wrong horizon evolution. If we plug the condition $1+3w > 0$ in Eq. (1.6), the integral is dominated by the upper limit, since $\tau_i \rightarrow 0$ when both $a_i \rightarrow 0$

and $w > -1/3$ holds. This is the so-called horizon problem, and constitutes one of the main reasons why one should try to modify the standard Big Bang in light of the CMB results.

1.3 Solving the Horizon Problem

The SEC implies an increasing Hubble sphere, which causes the particle horizon to have a vanishing contribution from early times. Using the results from the previous subsection we have a clue to what must be done to solve the horizon problem. *If we state that in the early universe the matter content does not satisfy the SEC, the Big Bang singularity is pushed to negative values of conformal time, $\tau_i \rightarrow -\infty$ (with $a_i \rightarrow 0$ and $w < -1/3$).* This imposition can create a time window that is large enough for all particles to have had time to exchange information. In other words, we are imposing a shrinking Hubble radius:

$$\frac{d}{dt}(aH)^{-1} < 0. \quad (1.8)$$

The shrinking Hubble sphere necessarily implies many other features, so here we display some of them.

- In the literature, it is quite common to read that inflation is an epoch of accelerated expansion. This can be seen by computing the acceleration:

$$\frac{d}{dt}(aH)^{-1} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2}. \quad (1.9)$$

- Another feature is that, at the inflation time, the Hubble parameter changes slowly with time:

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1 - \varepsilon) < 0, \quad (1.10)$$

where ε is defined as $-\dot{H}/H^2$.

- The last feature we would like to comment about is the fact that inflation requires a nearly constant energy density. From the continuity equation,

$$\left| \frac{d \ln \rho}{d \ln a} \right| = 2\varepsilon < 1, \quad (1.11)$$

where ε is usually assumed to be small. As will be later discussed, spacetimes with nearly constant density are seen as quasi-de Sitter. On top of this assumption we will construct an Effective Field Theory for the inflaton field.

Usual estimations state that for inflation to solve the horizon problem the Hubble horizon must shrink by a factor of 10^{28} during inflation [7, 68]. In other words, if a_E and a_I are, respectively, the scale factor computed at the end and at beginning of inflation, we must have $\ln(a_E/a_I) > 64$. Therefore, in order to solve the horizon problem we require at least ~ 60 e -foldings of expansion.

1.4 Free Scalar Field

Now we present the simplest model that can have the required characteristics presented in the last subsection to solve the horizon problem. We will consider the inflaton field to be a single scalar field, satisfying

$$S = \frac{M_{\text{P}}^2}{2} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \quad (1.12)$$

hence gravity is minimally coupled to the inflaton. The background is usually assumed to be the spatially flat FRW,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\mathbf{x}^2. \quad (1.13)$$

With all these ingredients, the equations of motions turn out to be [7, 68]:

$$H^2 = \frac{1}{3M_{\text{P}}^2} \left(\frac{1}{2} \dot{\phi}^2 + V \right), \quad (1.14)$$

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_{\text{P}}^2}, \quad (1.15)$$

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0. \quad (1.16)$$

We re-impose the condition on ε from Eq. (1.10), since this guarantees the shrinking of the Hubble radius. Moreover, we also require that ε should remain small during inflation in order to obtain a sufficient amount of e -foldings. This last imposition can be written as

$$\eta \equiv \frac{\dot{\varepsilon}}{\varepsilon H} \ll 1. \quad (1.17)$$

Eqs. (1.10) and (1.17) form the so-called *slow-roll* conditions. To understand what is the physics behind these conditions we can do the following. Using the fact a scalar field has a density, ρ_ϕ , given by

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (1.18)$$

the Friedmann equation, $H^2 = \rho_\phi/(3M_{\text{pl}}^2)$, can be rewritten as

$$H^2 = \frac{1}{3M_{\text{pl}}^2} \left[\frac{1}{2} \dot{\phi}^2 + V \right]. \quad (1.19)$$

In this manner, the slow-roll condition $\varepsilon \ll 1$ is translated into $\dot{\phi}^2 \ll V(\phi)$. *So the inflaton field is dominated by its potential energy.* Moreover, for this to be true for a desirable amount of time, the kinetic energy should only vary slowly⁵. This is the motivation for the second imposed condition above. Therefore, the picture one can make about it is to

⁵The shape of the potential should be sufficiently flat. The slow-roll parameters can be translated into the potential parameters,

$$\epsilon_V \equiv \frac{M_{\text{P}}^2}{2} \left(\frac{V'}{V} \right)^2 \ll \mathcal{O}(1), \quad \eta_V \equiv M_{\text{P}}^2 \frac{V''}{V} \ll \mathcal{O}(1). \quad (1.20)$$

imagine a body slowly rolling down into a potential well.

The inflaton perturbations are introduced via

$$\phi(\mathbf{x}, \tau) = \phi_0(\tau) + \frac{\delta\phi(\mathbf{x}, \tau)}{a(\tau)}. \quad (1.21)$$

After plugging these perturbations into the action, neglecting terms suppressed by the slow-roll parameters and going to momentum space, we obtain the famous *Mukhanov-Sasaki* equation [7, 68]

$$\ddot{\delta\phi}(\mathbf{k}, \tau) + \left(k^2 - \frac{\ddot{a}}{a}\right) \delta\phi = 0 \quad (1.22)$$

The quantum nature of the primordial universe requires the quantization of the inflaton field. We refer the reader to Ref. [68] for a detailed discussion of this procedure. The end result of quantizing these fluctuations is that the modes of the scalar field are given by:

$$\delta\phi(\mathbf{k}, \tau) = A \frac{H}{\sqrt{2k^3}} (1 + ik\tau) e^{-ik\tau}, \quad (1.23)$$

where $\langle A \rangle = 0$ and $\langle A^* A \rangle = 1$. On super-horizon modes ($k\tau \ll 1$), the amplitude $\delta\phi$ is approximately equal to its conjugate, and the inflaton fluctuations can be regarded as a classical stochastic field, with quantum expectations identified with classical ensemble averages. Notice that the dynamics of the Eq.(1.23) can be summarize in the following way. Inside the horizon, for modes satisfying $k\tau \gg 1$, we have an oscillatory behaviour (this is obvious, since in these cases the equations are the same as those for a harmonic oscillator). However, as we move to larger scales, its amplitude is frozen. It means that different comoving superhorizon regions will evolve classically, but with different values of $\delta\phi$. This also means they will cross the horizon at different times, since $\delta t \approx \delta\phi/\dot{\phi}_0$.

We can compute the two-point function of the inflaton perturbations,

$$\langle |\delta\hat{\phi}|^2 \rangle = \int d\ln k \frac{k^3}{2\pi^2} \langle |\delta\phi_k(\tau)|^2 \rangle \equiv \int d\ln k \Delta_{\delta\phi}(k, \tau). \quad (1.24)$$

In Eq.(1.24), we defined the *power spectrum* of the inflaton perturbations as

$$\Delta_{\delta\phi}(k, \tau) = \frac{k^3}{2\pi^2} |\delta\phi_k(\tau)|^2. \quad (1.25)$$

At horizon crossing the power spectrum is $\Delta_{\delta\phi}^2 \approx (H/2\pi)^2$. In summary, all modes oscillate when they are inside the Hubble horizon. As time goes by, they are stretched and reach the horizon at different moments. Once outside the horizon, the fields become frozen, classical stochastic fields, with variance given by Eq. (1.24). They will remain in this way until they re-enter the horizon again, when they start sourcing the formation of structures in the universe.

The inflaton so far has been treated as a free field, and for this reason it satisfies Gaussian statistics (at least as long as we truncate the action to second order in the perturbation). Most recent measurements of the Planck survey can be remarkably understood by using this simple model. Nevertheless, as is discussed in [69], it is possible to introduce higher order operators into the action by making use of effective field theory (EFT) arguments. This theory is constructed within two relevant scales, $\dot{\phi}^{\frac{1}{2}}$ and H . The COBE satellite has fixed $H/\dot{\phi}^{\frac{1}{2}} \sim 10^{-2}$, so it is important to know at which values we are allowed to construct an EFT. Moreover, it is important to also consider operators that do not spoil the slow-roll assumptions. They concluded that a good first try is to consider

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_P^2 R - \frac{1}{2} (\nabla\phi)^2 - V(\phi) + \frac{1}{8M^4} (\nabla\phi)^2 (\nabla\phi)^2 + \dots \right], \quad (1.26)$$

where M captures the scale of new physics. An useful redefinition is to work with the curvature perturbations, which in the spatially flat gauge are simply

$$\mathcal{R}_g = -\frac{\mathcal{H}}{\dot{\phi}} \delta\phi. \quad (1.27)$$

It is convenient to work with these perturbations because it is established that $\mathcal{R}_g$ is conserved outside the horizon (the underscript g says that the perturbation is Gaussian). The

deviations from Gaussianity can be introduced by [69]:

$$\mathcal{R} = \mathcal{R}_g - \frac{3}{5} f_{\text{NL}} \mathcal{R}_g^2. \quad (1.28)$$

Also in [69] it was concluded that the single-field non-Gaussian signal introduced by the interaction term described above imposes the upper bound

$$f_{\text{NL}} \lesssim 1. \quad (1.29)$$

The Planck measurements are far from allowing a clear detection of this and other models, since it can only constraint $\Delta f_{\text{NL}} = -1 \pm 5$.

1.5 The Effective Field Theory of Inflation

As was previously discussed, one of the features of inflation is that it has an almost constant energy density. This allows us to model it as a quasi-de Sitter space time, whose metric can be written as

$$ds^2 \simeq -dt^2 + e^{2Ht} d\mathbf{x}^2. \quad (1.30)$$

In a perfect de Sitter spacetime there are the time translation symmetry and the dilation symmetry:

$$t \mapsto t + \lambda, \quad x \mapsto e^{-H\lambda} x \quad (1.31)$$

which implies the existence of time-like Killing vectors. Therefore, since inflation is a synonym of quasi de Sitter space time, we could interpret this as if the inflaton field spontaneously breaks the spacetime-dependent time shift. In other words, if we consider some action, it would mean that a shift

$$t \mapsto t + \pi(t, \mathbf{x}) \quad (1.32)$$

should leave the Lagrangian invariant. As a consequence, we could use all the tools from Effective Field Theories and spontaneous breaking of gauge symmetries to develop this theory ⁶. The most general Lorentz-invariant Lagrangian for the field $U \equiv t + \pi$ would be something like

$$S = \int d^4x \sqrt{-g} \mathcal{L} [U, (\partial_\mu U)^2, \square U, \dots] . \quad (1.33)$$

It turns out that the most effective way to proceed is by making use of geometrical quantities. Since time diffeomorphism symmetry is broken, we now have a preferred slicing of spacetime. This suggests a geometrical way to approach the problem by projecting all relevant quantities into these spacelike hypersurfaces. Hence, after mapping all higher dimension tensors into these surfaces, the relevant quantities are the *extrinsic curvature tensor*, given by $K_{\mu\nu} = h_\mu^\rho \nabla_\rho n_\nu$, and the *induced Riemann curvature*, $\hat{R}_{\alpha\beta\gamma\theta}$. They can be written in terms of four-dimensional quantities through the Gauss-Codazzi relation:

$$\hat{R}_{\alpha\beta\gamma\delta} = h_\alpha^\mu h_\beta^\nu h_\gamma^\rho h_\delta^\sigma R_{\mu\nu\rho\sigma} - K_{\alpha\gamma} K_{\beta\delta} + K_{\alpha\delta} K_{\beta\gamma} , \quad (1.34)$$

where $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ is the induced metric on the hypersurfaces orthogonal to $n_\mu \equiv -\delta_\mu^0 / \sqrt{-g^{00}}$.

We now have all ingredients in our hands to construct an EFT for the inflaton field. Basically, at this point we just need to specify at which order the calculations must be considered. The expansion in terms of the perturbations goes like

$$\begin{aligned} \Delta S = & \int d^4x \sqrt{-g} \left[\frac{M_2^4(t)}{2} (\delta g^{00})^2 + \frac{M_3^4(t)}{3!} (\delta g^{00})^3 + \frac{M_4^4(t)}{4!} (\delta g^{00})^4 + \dots \right. \\ & - \frac{\bar{M}_1^3(t)}{2} \delta g^{00} \delta K - \frac{\bar{M}_2^2(t)}{2} \delta K^2 - \frac{\bar{M}_3^2(t)}{2} (\delta K_\nu^\mu)^2 + \dots \\ & \left. - \frac{\hat{M}_1^2(t)}{2} \delta g^{00} \hat{R} + \dots \right] , \end{aligned} \quad (1.35)$$

where $\delta g^{00} = 1 + g^{00}$, $\delta K_{\mu\nu} = K_{\mu\nu} - a^2 H h_{\mu\nu}$ and $\hat{R}$ is the Ricci scalar of the induced metric, defined in Eq. 1.34. At this point we would like to make some comments. As it

⁶Here, the metric plays the role of the gauge field

should be expected, the leading contributions from the above equations are simply

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{pl}}^2}{2} R - M_{\text{pl}}^2 (3H^2 + \dot{H}) + M_{\text{pl}}^2 \dot{H} g^{00} \right], \quad (1.36)$$

which is basically the Lagrangian of a homogeneous, free scalar field ⁷.

Secondly, we can also recover the results related to primordial non-Gaussianities, which were previously introduced by the operator $\mathcal{L}_\phi = (\nabla^2 \phi)^2$. To see this, we simply organize the model in the following way:

$$\mathcal{L}_\pi = \mathcal{L}_\pi^{(0)} + \mathcal{L}_\pi^{(2)} + \mathcal{L}_\pi^{(3)} + \dots. \quad (1.37)$$

In this manner, the second-order Lagrangian becomes basically :

$$\mathcal{L}_\pi^{(2)} = \frac{M_{\text{pl}}^2 |\dot{H}|}{c_s^2} \left(\dot{\pi}^2 - c_s^2 \frac{(\partial_i \pi)^2}{a^2} \right), \quad (1.38)$$

with the speed of sound defined as

$$c_s^2 \equiv \frac{M_{\text{pl}}^2 \dot{H}}{M_{\text{pl}}^2 \dot{H} - 2M_2^4}. \quad (1.39)$$

If $c_s^2 = 1$ we recover single-field scalar inflation. Therefore, the most basic model we have to describe inflationary physics, the scalar field satisfying the slow-roll conditions, is given by a Goldstone boson generated from the spontaneous symmetry breaking of time diffeomorphism with $c_s^2 = 1$. Using the tools developed in this Section, one can extend the formalism to any desired order. This is a natural way, with respect to perturbation theory, to develop the inflaton models.

⁷If it is not clear that this is the case, remember that from the Friedmann equations we have

$$\begin{aligned} 3M_{\text{pl}}^2 H^2 &= \rho \\ 6M_{\text{pl}}^2 (\dot{H} + H^2) &= -(\rho + 3P) \end{aligned}$$

After using the slow-roll approximation, we recover exactly the Lagrangian of a scalar field coupled to gravity.

2 Modeling the Large-Scale Structures of the Universe

Figure 7: The large-scale structure of the universe has a characteristic pattern, sometimes referred to as *cosmic web*. This image was obtained from a complex N-body simulation and it is believed to be a faithful illustration of how matter is distributed in our universe today. For comparison, the Milky way galactic disk is approximately $18\text{kpc}/h$. Figure extracted from [70].

In the previous Section we showed that the primordial universe, due to its quantum nature, is naturally endowed with fluctuations. These fluctuations, which are remarkably well described by quantum harmonic oscillators, turn out to be the seeds for the large structures in the universe. By “structure” we mean the spatial distribution of observed objects in the sky, such as galaxies, galaxy clusters, quasars, etc. Our task in this Section is to take the first step towards a robust theoretical model to explain the evolution of those structures.

Although what we see in the universe are baryonic structures, we know that matter component is not the dominant one. The idea that we have another underlying matter component, referred to as *dark matter*, due to its weakly interaction nature with ordinary

matter, is one of the cornerstones of the standard Λ CDM model (where, of course, the “CDM” stands for *cold dark matter*). Moreover, since the dark matter dominates, it is suggestive to first try to understand its dynamics and then introduce the baryonic structures *a posteriori*. This is precisely what we will do in the next Sections.

As we will see, to analytically solve the system of equations describing the matter field⁸ is not an easy task. Throughout the past four decades, many candidate models appeared. They all can be meticulously examined by comparing their predictions against the results obtained from N-body simulations, whose results are widely believed to be a faithful representation of the matter field we see in nature. We show an illustration of such numerical results in Figure 7.

Unfortunately, the majority of these analytical approaches presented some stark limitations, such as not presenting an acceptable agreement with data⁹ Even so, we have one perturbative approach that has been developed over the last decade, and which has been proving to be the most consistent way to model matter clustering analytically: the *Effective Field Theory of Large-Scale Structure* (EFTofLSS). We dedicate this Section to succinctly introduce this EFT approach to structure formation. In 2.1 we mathematically formalize the concept of large-scale structure and derive the main equations that need to be solved. In 2.2 we show a simplified way to solve the LSS equations, with a limited range of applicability. In 2.3 we present a perturbative approach, the Standard Perturbation Theory, which, despite some flaws, is the starting point for many improvements that improve our ability to describe structure formation on smaller scales. Finally, in 2.4 we present the EFTofLSS in all its glory, and show its remarkable agreement with data – a few percent agreement up to $k_{\text{max}} \approx 0.25h/\text{Mpc}$ and $z = 0$.

⁸It is customary to refer to the dark matter field simply as “matter field”.

⁹we will present a more detailed discussion about this later.

2.1 The Large-Scale Structure equations

Our discussions start right after the radiation-dominated era, in an expanding universe dominated by dark matter – which, in the current standard, are non-relativistic. Moreover, we also consider that dark matter is collisionless.

As usual in cosmology, we are not interested in describing the dynamics of a few particles or the dynamics of a limited region in space. Instead, we make our conclusions based on the *statistical properties of the universe as a whole*. For this reason, the modeling of dark matter (and also radiation) dynamics is developed on the basis of the usual statistical mechanics approach [7], with equations of motion being described by Einstein and relativistic Boltzmann equation. We have a solid understanding about how to solve these equations in the linear regime – and as we go back in time, the perturbations are small enough for the linear theory to be relevant. There are many public numerical codes that solve these equations in linear order (for instance, CAMB [71] and CLASS [72]). Linear theory can indeed explain extremely well some phenomena that happened in the past, such as the Cosmic Microwave Background (CMB) [6]. However, in order to describe the large-scale structures, including the formation of halos and galaxies, it is not enough to limit ourselves to the linear regime, since we are ultimately interested in understanding the numbers of those objects as well as their distributions in the recent universe, and over the smallest scales possible.

The relevant scales for structure formation are well inside the Hubble horizon, where gravity can act to drive structure formation, so we are motivated to simplify the theory and take the Newtonian limit of both relativistic phase-space and gravity [73, 74]. The Newtonian equations of motion tell us that, in the limit of a smooth potential, the equations of motion are:

$$\ddot{\mathbf{r}} = -\nabla_r \phi(\mathbf{r}) = -\nabla_r \left(G \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} \right), \quad (2.1)$$

where a dot represents a time derivative, $\mathbf{r}$ is the physical position, ϕ is the gravitational potential, and ρ is the matter density. The comoving position, $\mathbf{x}$ is defined through $\mathbf{r} = a\mathbf{x}$.

In our discussion we will be working with the comoving position and the conformal time, τ , defined in terms of cosmic time and the scale factor as $ad\tau = dt$. Using this last equation we can derive some useful relations. For instance, the first and second derivatives of the physical position with respect to the cosmological time are given by:

$$\dot{\mathbf{r}} = \mathcal{H}\mathbf{x} + a\dot{\mathbf{x}} \equiv \mathcal{H}\mathbf{x} + \mathbf{x}', \quad (2.2)$$

$$\ddot{\mathbf{r}} = \frac{1}{a} (\mathcal{H}'\mathbf{x} + \mathcal{H}\mathbf{x}' + \mathbf{x}'') . \quad (2.3)$$

We are mainly interested in considering the motions relative to the Hubble expansion, so it would be great to factor it out from the equations. This can be done by realizing that in Eq.(2.3) there is a term proportional to the particle position that accounts for this expansion. We can think about it as a ‘‘Hubble force’’ associated with some potential, and absorb it. By imposing in Eq.(2.1) that $\phi \rightarrow \Phi = \phi + \phi_{\text{Hubble}}$, where:

$$\phi_{\text{Hubble}} = -\frac{1}{2a^2}\mathcal{H}'r^2, \quad (2.4)$$

we obtain:

$$\ddot{\mathbf{r}} = \frac{1}{a} (\mathcal{H}'\mathbf{x} + \mathcal{H}\mathbf{x}' + \mathbf{x}'') = -\nabla_r(\phi + \phi_{\text{Hubble}}) = \frac{1}{a}\nabla_x\phi + \frac{1}{a}\mathcal{H}'\mathbf{x}, \quad (2.5)$$

which in turn leads to:

$$\mathbf{x}'' + \mathcal{H}\mathbf{x}' = -\nabla_x\phi. \quad (2.6)$$

It is straightforward to check that the Poisson equation for both ϕ and ϕ_{Hubble} are

$$\nabla_x^2\phi_{\text{Hubble}} = 4\pi G\bar{\rho}, \quad \nabla_x^2\phi = 4\pi G\bar{\rho}\delta = \frac{3}{2}\Omega_m(\tau)\mathcal{H}^2(\tau)\delta, \quad (2.7)$$

where, as usual, we consider $\rho(x, \tau) \equiv \bar{\rho}(1 + \delta(x, \tau))$. In other words, the ϕ_{Hubble} sources the background, whereas the ϕ sources the fluctuations. From Eq. (2.6) it is easy to see that the appropriate choice for the particle peculiar momentum is $\mathbf{p} = am\mathbf{u}$, where

the comoving peculiar velocity, $\mathbf{u}$, was defined as $\mathbf{x}' = \mathbf{u}$. For now on we shall only work with comoving quantities, so I will drop the x sub-script in the differential operators.

The phase-space conservation implies that the matter field probability density function (PDF), $f(\mathbf{x}, \mathbf{p}, \tau)$, satisfies the *Vlasov equation*,

$$\begin{aligned} \frac{df}{d\tau} &= \frac{\partial f}{\partial \tau} + \frac{d\mathbf{x}}{d\tau} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{d\mathbf{p}}{d\tau} \frac{\partial f}{\partial \mathbf{p}} \\ &= \frac{\partial f}{\partial \tau} + \frac{\mathbf{p}}{ma} \cdot \frac{\partial f}{\partial \mathbf{x}} - am \nabla \phi \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \end{aligned} \quad (2.8)$$

where in the last line we substituted the equation of motion, Eq. (2.6). This equation is tremendously difficult to solve, as it is a non-linear partial differential equation on seven variables (p_x, p_y, p_z, x, y, z and τ). However, we are not interested in having a complete knowledge of the distribution in phase space, so we proceed by taking the momenta of that distribution with respect to the momentum. The first three momenta of $f(\mathbf{x}, \mathbf{p}, \tau)$ define the density field, $\rho(\mathbf{x}, \tau)$, the peculiar velocity flow, $\mathbf{u}(\mathbf{x}, \tau)$, and the stress tensor, $\sigma_{ij}(\mathbf{x}, \tau)$, respectively, as:

$$\int d^3\mathbf{p} f(\mathbf{x}, \mathbf{p}, \tau) \equiv \rho(\mathbf{x}, \tau), \quad (2.9)$$

$$\int d^3\mathbf{p} \frac{\mathbf{p}}{am} f(\mathbf{x}, \mathbf{p}, \tau) \equiv \rho(\mathbf{x}, \tau) \mathbf{u}(\mathbf{x}, \tau), \quad (2.10)$$

$$\int d^3\mathbf{p} \frac{p_i p_j}{a^2 m^2} f(\mathbf{x}, \mathbf{p}, \tau) \equiv \rho(\mathbf{x}, \tau) u_i(\mathbf{x}, \tau) u_j(\mathbf{x}, \tau) + \sigma_{ij}(\mathbf{x}, \tau), \quad (2.11)$$

where again we used that $\mathbf{p} = am\mathbf{u}$.

To obtain the equations of motion for these fields one just needs to take the momenta of Eq.(2.8). Its zeroth- and first-order momenta lead to the *continuity* and *Euler* equations [7, 75, 76]:

$$\partial_\tau \rho(\mathbf{x}, \tau) + \nabla \cdot \{[1 + \delta(\mathbf{x}, \tau)] \rho \mathbf{u}(\mathbf{x}, \tau)\} = 0, \quad (2.12)$$

$$\partial_\tau \mathbf{u}(\mathbf{x}, \tau) + \mathcal{H}(\tau) \mathbf{u}(\mathbf{x}, \tau) + \mathbf{u}(\mathbf{x}, \tau) \cdot \nabla \mathbf{u}(\mathbf{x}, \tau) = -\nabla \phi(\mathbf{x}, \tau) - \frac{1}{\rho} \nabla_j (\rho \sigma_{ij}), \quad (2.13)$$

where we introduced the overdensity in the usual way, $\rho = \bar{\rho}(\tau)[1 + \delta(\mathbf{x}, \tau)]$.

Eqs. (2.7), (2.12) and (2.13) contain all the information we need to describe the large-scale structure of the universe. Although they still can seem tremendously complex, we will make further well-motivated assumptions that will simplify these equations. Even so, we could have ended with a much more daunting set of equations. Notice that we mysteriously stopped at the third momentum of Eq.(2.8). In principle, this procedure must continue to higher momenta, with a new field introduced at each step. So, why not consider higher momenta? As is discussed in [77], inhomogeneities and anisotropies can be encoded in the PDF as:

$$f(\mathbf{x}, \mathbf{p}, \tau) = f_0(p) [1 + \delta_f(\mathbf{x}, p, \hat{\mathbf{p}}, \tau)] . \quad (2.14)$$

By expanding the Fourier modes of the of the perturbations δ_f in Legendre polynomials P_n ,

$$\delta_f(\mathbf{k}, p, \hat{\mathbf{p}}, \tau) \equiv \sum_{n=0}^{\infty} (-i)^n (2n+1) \delta_f^{[n]}(\mathbf{k}, p, \tau) P_n(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}) , \quad (2.15)$$

the Vlasov equation, Eq.(2.8), implies a hierarchical evolution given by [77]:

$$\dot{\delta}_f^{[n]} = k v_p \left[\frac{n+1}{2n+1} \delta_f^{[n+1]} - \frac{n}{2n+1} \delta_f^{[n-1]} \right] , \quad n \geq 2 , \quad (2.16)$$

where $v_p = \mathbf{p}/am = \dot{\mathbf{x}}$ is the particle peculiar velocity. At long wavelengths the time derivatives are of order of the Hubble time, so we can estimate that

$$\delta_f^{[n]} \sim k v_p \mathcal{H}^{-1} \left[\frac{n+1}{2n+1} \delta_f^{[n+1]} - \frac{n}{2n+1} \delta_f^{[n-1]} \right] \quad n \geq 2 . \quad (2.17)$$

Therefore, if $k v_p \mathcal{H}^{-1} \ll 1$, we have a natural suppression of momenta higher than 2:

$$\delta_f^{[n]} \sim (k v_p \mathcal{H}^{-1})^{n-2} \delta_f^{[2]} . \quad (2.18)$$

Usually, in fluid dynamics, this same procedure leads to the similar hierarchy as in

Eq. (2.18), with the prefactor being $kv_p\tau_c \approx k\tau_d$, where τ_c is the characteristic collision time, and τ_d is the mean free path of the particles¹⁰. In our specific case, $\tau_d = v_p\mathcal{H}^{-1}$ is the average distance traveled by dark matter particles in a Hubble time. What Eq. (2.18) is saying, therefore, is that, as long as $kv_p\mathcal{H}^{-1}$ is small, the truncation at the second moment is a good approximation. Before checking whether it is really the case, it is instructive to define the *non-linear matter scale*, k_{NL} . Since we are expanding with respect to the matter field, one can readily see that as long as δ is small our approach is valid. We can project this reasoning to the two point function of the matter field. For this purpose we now introduce the dimensionless power spectrum, Δ_δ , defined as

$$\Delta_\delta(k) = P(k) \frac{k^3}{2\pi^2}, \quad (2.19)$$

where $P(k)$ is the linear power spectrum (if the reader is not familiarized with it, see Eq. 2.58 for a definition). As will become clear later, the higher-order corrections to the linear theory can be seen as constructed with powers of Δ_δ . In this way, in order to have some notion of convergence, we should have that $\Delta_\delta \leq 1$. Since the Λ CDM dimensionless linear power spectrum is an increasing function, the non-linear matter scale is defined as the scale in which $\Delta_\delta = 1$. With that in mind we can proceed to check whether $kv_p\mathcal{H}^{-1}$ is indeed small. For this purpose, we can estimate that v_p is, at most, as large as the velocity field at non-linear scales [78],

$$v_p \leq \Delta_v(k_{\text{NL}}) \sim \Delta_\delta^2(k_{\text{NL}}) \frac{\mathcal{H}^2}{k_{\text{NL}}^2} \sim \frac{\mathcal{H}^2}{k_{\text{NL}}^2}, \quad (2.20)$$

which implies that:

$$kv_p\mathcal{H}^{-1} \lesssim \frac{k}{k_{\text{NL}}}. \quad (2.21)$$

Hence, as long as we work with scales larger than the non-linear scales, the Boltzmann truncation is well-motivated, and the fluid description is valid. As discussed in [78], this

¹⁰This hierarchy is a vital ingredient for a fluid behavior. It states that, up to some given precision, the number of equations required to describe the dynamics of the particle, in a statistical sense, is limited.

is a manifestation of the fact that dark matter particles do not have high velocities, so they do not move a considerable distance during a Hubble time ¹¹.

Making contact with fluid dynamics, this finite amount of distance traveled by dark matter particles can be seen as an effective mean-free path. Gravity induces an important hierarchy between the Hubble horizon, $\mathcal{H}^{-1}$ and the non-linear scales, k_{NL} . Hence, as long as we limit ourselves to scales $k_{\text{NL}} < k < \mathcal{H}^{-1}$, we expect our modeling to be a good description of the matter field dynamics.

To summarize, the fluid equations describing the LSS dynamics are Eqs. (2.7), (2.12) and (2.13). This fluid description is valid if for modes satisfying $k_{\text{NL}} < k < \mathcal{H}^{-1}$. These equations are widely known in fluid dynamics for being tremendously difficult to solve, and of course, it will not be different here. In the next subsections we will present two distinct perturbative approaches that have been extensively debated in the literature: the Standard Perturbation Theory (SPT) and the EFTofLSS. But first, we consider the linearized equations of motion.

2.2 Linear solution

At large scales we expect the universe to be nearly homogeneous and isotropic, so all perturbations can be considered to be very small [7]. In this way, we linearize Eqs. (2.12), (2.13) with respect to the matter overdensity, δ , and the divergence of the velocity field,

$$\theta = \nabla \cdot \mathbf{u}(\mathbf{x}, \tau)$$

¹¹For conventional fluids, the particles can have large velocities compared with the sound speed. However, they usually present strong interactions, which limits their mean free path.

. We then obtain the following equations for the linear regime [74]:

$$\frac{\partial \mathbf{u}(\mathbf{x}, \tau)}{\partial \tau} + \mathcal{H}(\tau) \mathbf{u}(\mathbf{x}, \tau) = -\nabla \Phi(\mathbf{x}, \tau), \quad (2.22)$$

$$\frac{\partial \delta(\mathbf{x}, \tau)}{\partial \tau} + \theta(\mathbf{x}, \tau) = 0, \quad (2.23)$$

$$(2.24)$$

To completely describe the velocity field we choose to work with its divergence and vorticity, $\mathbf{w} \equiv \nabla \times \mathbf{u}$, whose equations of motions can be obtained from Eq. (2.22):

$$\frac{\partial \theta(\mathbf{x}, \tau)}{\partial \tau} + \mathcal{H}(\tau) \theta(\mathbf{x}, \tau) + \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2(\tau) \delta(\mathbf{x}, \tau) = 0, \quad (2.25)$$

$$\frac{\partial \mathbf{w}(\mathbf{x}, \tau)}{\partial \tau} + \mathcal{H}(\tau) \mathbf{w}(\mathbf{x}, \tau) = 0. \quad (2.26)$$

From Eq.(2.26), we can easily see that if there is some vorticity in the initial conditions, that component will decay due to the expansion of the universe, since $\mathbf{w} \sim a^{-1}$. Thus, we assume the LSS matter fluid to be irrotational. It is then completely described by its overdensity field, δ , and the divergence part of its velocity field, θ . Moreover, we impose the solution to be separable, $\delta(\mathbf{x}, \tau) = \delta(\mathbf{x}, 0) D_1(\tau)$, where $\delta(\mathbf{x}, 0)$ is the initial condition for the matter field, obtained by evolving the primordial seeds from inflation through the radiation epoch. The $D_1(\tau)$ is called *linear growth factor*. It is possible to find a second-order differential equation for $D_1(\tau)$ by first taking the time derivative of Eq. (2.23) and then substituting in Eq. (2.25):

$$\frac{d^2 D_1(\tau)}{d\tau^2} + \mathcal{H}(\tau) \frac{dD_1(\tau)}{d\tau} = \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2(\tau) D_1(\tau). \quad (2.27)$$

As a second-order ordinary differential equation, the solutions to Eq. (2.27) allow for two linear independent modes, defined as $D_+(\tau)$ and $D_-(\tau)$, which are associated with the fast and slow solutions, respectively. Thus, the most general solution can be written

as a linear combination of them,

$$\delta(\mathbf{x}, \tau) = D_1^{(+)}(\tau)A(\mathbf{x}) + D_1^{(-)}(\tau)B(\mathbf{x}), \quad (2.28)$$

where $A(\mathbf{x})$ and $B(\mathbf{x})$ are arbitrary functions to be determined by the initial conditions.

We can substitute Eq. (2.28) in Eq. 2.23 to obtain the solution for the velocity field,

$$\theta(\mathbf{x}, \tau) = -\mathcal{H}(\tau) [f(\Omega_m, \Omega_\Lambda) A(\mathbf{x}) + g(\Omega_m, \Omega_\Lambda) B(\mathbf{x})], \quad (2.29)$$

where we followed [74] and defined

$$f(\Omega_m, \Omega_\Lambda) \equiv \frac{d \ln D_1^{(+)}}{d \ln a} = \frac{1}{\mathcal{H}} \frac{d \ln D_1^{(+)}}{d \tau}, \quad g(\Omega_m, \Omega_\Lambda) = \frac{1}{\mathcal{H}} \frac{d \ln D_1^{(-)}}{d \tau}. \quad (2.30)$$

Eqs.(2.28,2.29) completely describe the matter field at linear order. There are no general analytic solutions for both $D_1^{(+)}$ and $D_1^{(-)}$, but they can be easily computed numerically. Notwithstanding, the matter dominated universe $\Omega_m = 1$ does allow an analytic solution, given by

$$D_1^{(+)} = a, \quad D_1^{(-)} = a^{-3/2}, \quad f(1, 0) = 1 \quad (\text{EdS. solution}). \quad (2.31)$$

This matter-dominated cosmology is usually referred to as the *Einstein-de Sitter universe* (EdS), and is very useful due to the simple functional form of $D_1^{(+)}$ and $D_1^{(-)}$.

Now that we have the linear solution in our hands, we can compare its predictions with simulations. We will make many such comparisons later in this Section (see, e.g., Fig. 16), but it is clear that the linear regime fails at some scales. E.g., comparing the linear matter power spectrum with Halofit [79, 80], a phenomenological model used to describe non-linear matter clustering, we find that the linear theory prediction fails at all scales except the largest, near the Hubble radius. The physical reason for this is clear: we neglected all non-linear terms in Eqs. (2.12, 2.13) by assuming that all deviations from a

homogeneous and isotropic background can be considered to be small, but this is true only on the largest scales. As we start to include smaller modes, both $\delta(x, \tau)$ and $\theta(x, \tau)$ are no longer so small, causing the contributions coming from non-linear terms to become more relevant. In the next sections, we make further steps and propose more elaborate solutions to tackle this issue.

2.3 Standard Perturbation Theory

We will now make a first attempt at solving the complete system of equations, described by:

$$\partial_\tau \mathbf{u}(\mathbf{x}, \tau) + \mathcal{H}(\tau) \mathbf{u}(\mathbf{x}, \tau) + \mathbf{u}(\mathbf{x}, \tau) \cdot \nabla \mathbf{u}(\mathbf{x}, \tau) = -\nabla \phi(\mathbf{x}, \tau) - \frac{1}{\rho} \nabla_j (\rho \sigma_{ij}), \quad (2.32)$$

$$\partial_\tau \delta(\mathbf{x}, \tau) + \nabla \cdot \{[1 + \delta(\mathbf{x}, \tau)] \mathbf{u}(\mathbf{x}, \tau)\} = 0. \quad (2.33)$$

In the linearized system we could see that if there is any sort of vorticity on the initial conditions, the universe's expansion will naturally suppress it, so we discarded it. However, when one consider the whole system, the equation of motion for the vorticity becomes

$$\frac{\partial \mathbf{w}(\mathbf{x}, \tau)}{\partial \tau} + \mathcal{H}(\tau) \mathbf{w}(\mathbf{x}, \tau) - \nabla \times [\mathbf{u}(\mathbf{x}, \tau) \times \mathbf{w}(\mathbf{x}, \tau)] = \nabla \times \left(\frac{1}{\rho} \nabla \cdot \vec{\sigma} \right), \quad (2.34)$$

where we can see that the non-linear dynamics sources $\mathbf{w}(\mathbf{x}, \tau)$. So we now have the opposite - even if there is no vorticity on initial conditions, it will inevitably be generated by non-linear clustering. This is a consequence of the intersection of cold dark matter particles trajectories, a feature known as *shell-crossing*. This is due to the collisionless nature of dark matter and causes the large-scale velocity field to be multi-valued and discontinuous at caustics [81, 82, 83, 84, 85]. Nevertheless, throughout this section we still impose $\sigma_{ij} \approx 0$, which is known as the *single-stream* approximation. This is reasonable on large enough scales, but it is one of the main reasons for the breakdown of perturbation

Figure 8: Evolution of the system from initial condition to evolved time τ , seen by an Eulerian observer. For an observer sitting in the fluid line, the Lagrangian frame, there is no displacement at all, $\Psi = 0$. The fluid line in Lagrangian coordinates is always $\mathbf{q}$.

theory on small-scales [86, 87].

Now, we claim the fields δ and θ admit a perturbative form given by

$$\delta(\mathbf{x}, t) = \sum_{n=1}^{\infty} \delta^{(n)}(\mathbf{x}, t), \quad \theta(\mathbf{x}, t) = \sum_{n=1}^{\infty} \theta^{(n)}(\mathbf{x}, t). \quad (2.35)$$

At this point we could simply substitute these expressions in Eqs. (2.33-2.32), grouping all terms of same order and solving for the coupled partial differential equations. However, it is more enlightening to work with these equations in Fourier space. After having taken the Fourier transform, Eqs. (2.33-2.32) turn out to be [76]:

$$\partial_{\tau} \tilde{\delta}(\mathbf{k}, \tau) + \tilde{\theta}(\mathbf{k}, \tau) = - \int d^3 \mathbf{k}_1 d^3 \mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \alpha(\mathbf{k}_1, \mathbf{k}_2) \tilde{\theta}(\mathbf{k}_1, \tau) \tilde{\delta}(\mathbf{k}_2, \tau), \quad (2.36)$$

$$\begin{aligned} \partial_{\tau} \tilde{\theta}(\mathbf{k}, \tau) + \mathcal{H}(\tau) \tilde{\theta}(\mathbf{k}, \tau) + \frac{3}{2} \Omega_m \mathcal{H}^2(\tau) \tilde{\delta}(\mathbf{k}, \tau) = & - \int d^3 \mathbf{k}_1 d^3 \mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \\ & \times \beta(\mathbf{k}_1, \mathbf{k}_2) \tilde{\theta}(\mathbf{k}_1, \tau) \tilde{\theta}(\mathbf{k}_2, \tau), \end{aligned} \quad (2.37)$$

where $\mathbf{k}_{12} = \mathbf{k}_1 + \mathbf{k}_2$ and the mode-coupling α and β were defined as

$$\alpha(\mathbf{k}_1, \mathbf{k}_2) \equiv \frac{\mathbf{k}_{12} \cdot \mathbf{k}_1}{k_1^2}, \quad \beta(\mathbf{k}_1, \mathbf{k}_2) \equiv \frac{k_{12}^2 (\mathbf{k}_1 \cdot \mathbf{k}_2)}{2k_1^2 k_2^2}. \quad (2.38)$$

Figure 9: Diagrammatic representation of the perturbative solutions in Eqs.(2.40,2.41). The solution at order n is the contraction of n incoming modes (the ones being integrated) that convolve according to F_n , producing the outgoing mode k . Extracted from [75].

As previously discussed, the EdS scenario is very useful due to the simple, closed expressions for matter growth in the linear regime, so we proceed by assuming $\Omega_m = 1$ and $\Omega_\Lambda = 0$, so $a \sim \tau^2$ and $\mathcal{H} = 2/\tau$. As a result, Eqs. (2.36) and (2.37) become homogeneous in τ , admitting a solution given by [88, 89, 90]:

$$\tilde{\delta}(\mathbf{k}, \tau) = \sum_{n=1}^{\infty} a^n(\tau) \delta_n(\mathbf{k}), \quad \tilde{\theta}(\mathbf{k}, \tau) = -\mathcal{H}(\tau) \sum_{n=1}^{\infty} a^n(\tau) \theta_n(\mathbf{k}). \quad (2.39)$$

The specific form of both δ_n and θ_n is obtained by substituting Eq. (2.39) into Eqs. (2.36) and (2.37). Hence, we obtain the well-known SPT iterative result for cosmological perturbations:

$$\delta_n(\mathbf{k}) = \int d^3\mathbf{q}_1 \dots \int d^3\mathbf{q}_n \delta_D(\mathbf{k} - \mathbf{q}_{1\dots n}) F_n(\mathbf{q}_1, \dots, \mathbf{q}_n) \delta_1(\mathbf{q}_1) \dots \delta_1(\mathbf{q}_n), \quad (2.40)$$

$$\theta_n(\mathbf{k}) = \int d^3\mathbf{q}_1 \dots \int d^3\mathbf{q}_n \delta_D(\mathbf{k} - \mathbf{q}_{1\dots n}) G_n(\mathbf{q}_1, \dots, \mathbf{q}_n) \delta_1(\mathbf{q}_1) \dots \delta_1(\mathbf{q}_n), \quad (2.41)$$

whose kernels F_n and G_n are completely determined by [88, 89, 90]

$$\begin{aligned}
F_n(\mathbf{q}_1, \dots, \mathbf{q}_n) &= \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, \dots, \mathbf{q}_m)}{(2n+3)(n-1)} [(2n+1)\alpha(\mathbf{k}_1, \mathbf{k}_2) F_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) \\
&\quad + 2\beta(\mathbf{k}_1, \mathbf{k}_2) G_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n)], \\
G_n(\mathbf{q}_1, \dots, \mathbf{q}_n) &= \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, \dots, \mathbf{q}_m)}{(2n+3)(n-1)} [3\alpha(\mathbf{k}_1, \mathbf{k}_2) F_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) \\
&\quad + 2n\beta(\mathbf{k}_1, \mathbf{k}_2) G_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n)].
\end{aligned} \tag{2.42}$$

Hence, the n -th solution is given by a integral convolution of n linear matter fields. This convolution is specified by the explicit form of the integration kernels F_n and G_n . They contain all symmetries of the large-scale structure dynamics, namely invariance under Galilean transformations. This is the case because, back in the beginning of this Section, we took the Newtonian limit of the relativistic phase-space. With that said, it is enlightening to take a closer look at some expressions and understand the physics behind each contribution. By using Eqs. (2.42) one can easily obtain the second-order integrating Kernels,

$$F_2(\mathbf{q}_1, \mathbf{q}_2) = \frac{5}{7} + \frac{1}{2} \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right) \mu + \frac{2}{7} \mu^2, \tag{2.43}$$

$$G_2(\mathbf{q}_1, \mathbf{q}_2) = \frac{3}{7} + \frac{1}{2} \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1 q_2} \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{4}{7} \frac{(\mathbf{q}_1 \cdot \mathbf{q}_2)^2}{q_1^2 q_2^2}. \tag{2.44}$$

Or, in real space [5], we get:

$$\delta^{(2)}(\mathbf{x}, \tau) = \frac{17}{21} [\delta^{(1)}]^2 + \frac{2}{7} [K_{ij}^{(1)}]^2 - \Psi_{(1)}^i \partial_i \delta^{(1)}, \tag{2.45}$$

$$-\frac{1}{\mathcal{H}(\tau)f(\tau)} \theta^{(2)}(\mathbf{x}, \tau) = \frac{13}{21} [\delta^{(1)}]^2 + \frac{4}{7} [K_{ij}^{(1)}]^2 - \Psi_{(1)}^i \partial_i \delta^{(1)}, \tag{2.46}$$

where $\mu \equiv \mathbf{q}_1 \cdot \mathbf{q}_2 / q_1 q_2$, $K_{ij}^{(1)}$ is the first order *tidal field*, defined as

$$K_{ij}^{(1)} = \left[\frac{\partial_i \partial_j}{\nabla^2} - \frac{1}{3} \delta_{ij} \right] \delta^{(1)}, \quad (2.47)$$

and $\Psi_{(1)}(\mathbf{q}, \tau)$ is the first order displacement field (see discussion around Eqs. (2.55, 2.56)) given by

$$\Psi_{(1)}(\mathbf{q}, \tau) = -\frac{\nabla}{\nabla^2} \delta^{(1)}(\mathbf{q}, \tau). \quad (2.48)$$

Some important comments are in order about Eqs. (2.45-2.46). For starters, let's try to understand the first term. Following [91, 92] we can impose the existence of a spherical symmetry of δ around $\mathbf{x} = 0$. Hence, the overdensity field in Fourier space will only depend on the norm of $\mathbf{k}$. The central density for this initial condition is then given by [91, 92]:

$$\delta_{\text{sc}}(a) = \sum_n a^n \int d^3 \mathbf{q}_1 \dots \int d^3 \mathbf{q}_n F_n(\mathbf{q}_1, \dots, \mathbf{q}_n) \delta_1(|\mathbf{q}_1|) \dots \delta_1(|\mathbf{q}_n|). \quad (2.49)$$

The linear overdensity field, δ_1 , does not depend on any angle, so the only contribution for this part of the volume integration arises from the integrating kernel F_n . For this purpose, it is instructive to work with the spherical averages of the kernels, defined as [76]:

$$\nu_n \equiv n! \int \frac{d\Omega_1}{4\pi} \dots \frac{d\Omega_n}{4\pi} F_n(\mathbf{k}_1, \dots, \mathbf{k}_n), \quad (2.50)$$

$$\mu_n \equiv n! \int \frac{d\Omega_1}{4\pi} \dots \frac{d\Omega_n}{4\pi} G_n(\mathbf{k}_1, \dots, \mathbf{k}_n). \quad (2.51)$$

Both F_n and G_n are constructed in terms of $\alpha(\mathbf{k}_1, \mathbf{k}_2)$ and $\beta(\mathbf{k}_1, \mathbf{k}_2)$, whose spherical average are respectively 1 and $\frac{1}{3}$. Hence, one can use it together with Eq. (2.42) to

conclude that

$$\nu_n = \sum_{m=1}^{n-1} \binom{n}{m} \frac{\mu_m}{(2n+3)(n-1)} \left[(2n+1)\nu_{n-m} + \frac{2}{3}\mu_{n-m} \right], \quad (2.52)$$

$$\mu_n = \sum_{m=1}^{n-1} \binom{n}{m} \frac{\mu_m}{(2n+3)(n-1)} \left[3\nu_{n-m} + \frac{2}{3}n\mu_{n-m} \right]. \quad (2.53)$$

Using Eq. (2.52) into Eq. (2.49) allows us to obtain for the central density the well-known *spherical collapse solution*,

$$\delta_{\text{sc}}(a) = \sum_n \frac{\nu_n}{n!} a^n \left(\int d^3\mathbf{q} \delta(|\mathbf{q}|) \right)^n. \quad (2.54)$$

For $n = 2$ the prefactor is $34/21 \times 1/2$, which is precisely the first contribution in Eq. (2.45). Therefore, the spherical collapse result is built into the SPT solution. This part of the solution at order n will be proportional to $(\delta_1)^n$. A similar argument can be made for the central velocity field, $\theta_{\text{sc}}(a)$.

Finally, we have a contribution coming from the displacement field, $\Psi_{(1)}(\mathbf{q}, \tau)$. This feature of large-scale displacements of small-scales physics due to long-wavelength perturbations is usually referred to as *bulk flows* (see Figure 8). Here, $\mathbf{q}$ is the initial position (i.e. Lagrangian position). The SPT solution turns out to be a perturbative expansion also on the displacement field, which is a quantity that is not small in the standard Λ CDM model. A simple way to see this is to first note that the Lagrangian and Eulerian positions can be related through

$$\mathbf{x}(\tau) = \mathbf{q} + \Psi(\mathbf{q}, \tau). \quad (2.55)$$

Hence, as is discussed in [93], the equation of motion for the $\Psi_{(1)}(\mathbf{q}, \tau)$ is given by

$$\left[\frac{d^2}{d\tau^2} + \mathcal{H} \frac{d}{d\tau} - \frac{3}{2} \Omega_m(a) \mathcal{H}^2 \right] \Psi_{i,i}^{(1)} = 0, \quad (2.56)$$

Figure 10: Diagrammatic representation of the 1-loop SPT power spectrum, as described in Eq. (2.58). It is possible to represent any n -point statistics at any loop order using the Feynman rules, as discussed in the main text. Extracted from [75].

whose solution in real space is precisely Eq. (2.48). With this and Eq. (2.55) we can schematically state that, when one start with some field $\tilde{f}(\mathbf{q}, \tau_*)$ in the initial conditions, its time-evolved form would be something like:

$$\tilde{f}(\mathbf{q}, \tau_*) \xrightarrow{\text{timeevolution}} \left(\frac{D_*}{D} \right) f(\mathbf{q} + \Psi(\mathbf{q}, \tau), \tau) = \left(\frac{D_*}{D} \right) f(\mathbf{q}, \tau) + \left(\frac{D_*}{D} \right) \Psi_{(1)} \cdot \nabla f \dots, \quad (2.57)$$

where we truncated the expansion at leading order. For the SPT case this expansion is built-in, due to the fact that we work in Euler coordinates. In Lagrangian perturbation theory, on the other hand, there is no displacement field due to the fact the observer is co-moving with the fluid, so this problem is not present. This expansion with respect to the displacement field is corrected by performing an *IR-Resummation* [94, 95, 96].

Now that we know the SPT solution for both δ and θ at any order in perturbation theory, we can proceed to see how well it explains the data. As is customarily used in standard cosmology, we will work with n -point functions. During the discussion we will always be assuming Gaussian initial conditions. We start with the matter power spectrum, $P(k)$, which can be written as

$$\begin{aligned} (2\pi^3)\delta^{(D)}(\mathbf{k}, \mathbf{k}')P(k) &\equiv \langle \delta(\mathbf{k}, \tau)\delta(\mathbf{k}', \tau) \rangle = \left\langle \sum_n \delta_n(\mathbf{k}, \tau) \sum_m \delta_m(\mathbf{k}', \tau) \right\rangle \\ &= \underbrace{\langle \delta_0 \delta_0 \rangle}_{\text{linear}} + \underbrace{[\langle \delta_1 \delta_3 \rangle + \langle \delta_2 \delta_2 \rangle]}_{1\text{-loop}} + \underbrace{[2\langle \delta_1 \delta_5 \rangle + 2\langle \delta_2 \delta_4 \rangle + \langle \delta_3 \delta_3 \rangle_I + \langle \delta_3 \delta_3 \rangle_{II}]}_{2\text{-loop}} + \dots \quad (2.58) \end{aligned}$$

Figure 11: All diagrams contributing to the 1-loop SPT power spectrum. The dashed lines represents negative values, so what is being plotted is its absolute value. Extracted from [75].

Notice that all correlators displayed above always contain an even number of linear matter fields (recall that $\delta_n \sim \delta_0^n$). This happens due to the fact that we are assuming Gaussian initial conditions, so $\langle \delta_a \delta_b \rangle = 0$ if $a + b$ is odd. We can now plug Eqs. (2.40 - 2.41) in the above average. In the remaining discussions we will use the definition

$$\langle \delta_n(\mathbf{k}) \delta_m(\mathbf{k}') \rangle \equiv (2\pi)^3 \delta^{(D)}(\mathbf{k}, \mathbf{k}') P_{nm}. \quad (2.59)$$

At one loop, the expressions above turns out to be [75, 76, 97]:

$$P_{22}(k) = 2 \int \frac{d^3 q}{(2\pi)^3} P(q) P(|\mathbf{k} - \mathbf{q}|) |F_2(\mathbf{q}, \mathbf{k} - \mathbf{q})|^2. \quad (2.60)$$

For numerical purposes we follow [75] and perform the integral in the coordinates $\mathbf{k} = (0, 0, k)$ and $\mathbf{q} = rk \left(\sqrt{1 - \mu^2}, 0, \mu \right)$. We then obtain

$$P_{22} = \frac{k^3}{2\pi^2} \int dr r^2 \int d\mu P(rk) P(\psi(r, \mu)k) |F_{2, d}(q, \mu)|^2, \quad (2.61)$$

with $\psi(r, \mu) = \sqrt{1 + r^2 - 2r\mu}$ and

$$F_{2, d}(r, \mu) = \frac{7\mu + (3 - 10\mu^2)r}{14r(r^2 - 2\mu r + 1)}. \quad (2.62)$$

The remaining term contributing to the 1-loop power spectrum is simply

$$P_{13}(k) = 3P_{\text{lin}}(k) \int \frac{d^3 q}{(2\pi)^3} P_{\text{lin}}(q) F_3(\mathbf{k}, \mathbf{q}, -\mathbf{q}). \quad (2.63)$$

In Figure 11 we show each term individually, as well as the resulting 1-loop prediction for the matter power spectrum. The dotted lines represent negative values, so what is being plotted are the absolute values. On large scales it is evident that the leading contribution is the linear theory, as it should. Also in this regime the total 1-loop contribution is negative and quite negligible with two orders of magnitude below the linear theory. As we go to

Figure 12: Contribution of each diagram to the total 1-loop SPT bispectrum, defined in Eq. (2.65). As explained in the main text, there is a cancellation in the IR-regime due to the large-scale structure symmetry group. The squeezed limit is computed with $\Delta k = 0.013 h \text{Mpc}^{-1}$.

Figure 13: Diagrammatic representation of the 2-loop SPT power spectrum, as described in Eq. (2.58). Extracted from [98].

higher momenta the linear power decays, whereas the perturbation theory contributions grows and start to be prominent.

Before moving on it is important to formalize the Feynman rules for the SPT diagrams, since they will be extensively used throughout this work. Let's suppose someone is willing to compute the i -th order to the n -spectrum. The recipe for constructing the relevant diagrams are the following [75, 5, 99]:

1. Draw all distinct connected diagrams with n external lines and with the vertex going up to the desired i -th order in $\delta^{(1)}$. All fields must be connected by lines with a dot representing the linear propagation.
2. For each vertex F one must impose momentum conservation. For instance, suppose a vertex with some in-going momenta $\mathbf{q}_i$ and an out-going momenta $\mathbf{p}$. The momentum conservation would be then $(2\pi)^3 \delta^{(D)}(\mathbf{p} - \sum_i \mathbf{q}_i) F(\mathbf{q}_1, \dots, \mathbf{q}_n)$. The number of in-going momenta gives the order associated to the vertex. In Figure 10, the P_{22} has two second order couplings, whereas P_{13} presents one third order as well as a free propagation.
3. Each dot represents a linear propagator, and they are always associated with the out-going momenta. For each dot with out-going $\mathbf{q}$ and $\mathbf{q}'$ you must include $(2\pi)^3 \delta^{(D)}(\mathbf{q} + \mathbf{q}') P_{\text{lin}}(q)$.
4. Integrate over all internal momenta, $\int \prod_i (d^3 q_i / (2\pi)^3)$, and multiply by the diagram symmetry factor.
5. Sum over all possible distinct labels of external momenta and then sum over all possible diagrams.

With these rules we can systematically transform all diagrams in Figures 10, 13 and 14 into integrals and implement them numerically. There are some public codes to evaluate these diagrams. Some examples of these are COPTER¹² [100], FnFast¹³ [101], FAST-PT¹⁴ [102, 103] and CLASS-PT¹⁵ [104]. The implementation of these diagrams requires complex numerical techniques, so we refer the reader to the references above for more details. In the present work we used the CLASS-PT, since it is an efficient implementation of the 1-loop biased tracers power spectrum. However, we stress that the other

¹²<http://mwhite.berkeley.edu/Copter/>

¹³<https://github.com/jrwalsh1/FnFast>

¹⁴<https://github.com/JoeMcEwen/FAST-PT>

¹⁵<https://github.com/Michalychforever/CLASS-PT>

codes can compute diagrams of higher statistics, but may not have the cheap numerical cost needed for an MCMC analysis.

Figure 14: Diagrammatic representation of the tree level and 1-loop corrections for the SPT bispectrum. Extracted from [105]

Another relevant statistics is the *Bispectrum*. Although in this work we will only use the power spectrum, we decided to briefly show some results associated to the Bispectrum. It is defined as:

$$\langle \delta(\mathbf{k}_1) \delta(\mathbf{k}_2) \delta(\mathbf{k}_3) \rangle \equiv (2\pi)^3 \delta_D^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3). \quad (2.64)$$

In Figure 14 we show the Feynman diagram associated to its tree level and 1-loop contributions, that must be summed as [106, 105]

$$B(k_1, k_2, k_3) = B_{112} + B_{222} + B_{321}^I + B_{321}^{II} + B_{411}, \quad (2.65)$$

and in Figure 12 we show a typical plot of them.

Now that we have defined all useful quantities we can proceed to see how they perform in explaining data. The SPT has been extensively tested in the past few decades, so there

Figure 15: The theoretical power spectrum up to 2-loops in comparison with the power spectrum measured from a N-body simulations (black boxes) at two different redshifts. It is shown the linear (black; dotted), 1-loop (red; solid) and the 2-loop (blue; dashed) SPT predictions. The total power spectrum is divided by its associated non-wiggly contribution (see [100] for more details). The SPT performance is highly dependent on the redshift. At $z = 0$, the 1-loop departs from the reference power spectrum by 1% at $k_{\text{max}} = 0.08 \text{Mpc}^{-1}$, whereas for the 2-loop it happens at $k_{\text{max}} = 0.04 \text{Mpc}^{-1}$. At $z = 1.5$, on the other hand, the k_{max} for 1-loop and 2-loop are, respectively, $0.13 h \text{Mpc}^{-1}$ and 0.23Mpc^{-1} . This figure was extracted from [100]

Models	$k_{\max}(z)h\text{Mpc}^{-1}$				
	$z = 0.0,$	$0.3,$	$0.7,$	$1.0,$	1.5
Linear	0.03	0.08	0.09	0.09	0.09
1-loop	0.08	0.10	0.11	0.13	0.14
2-loop	0.04	0.06	0.09	0.23	0.20

Table 1: Values of $k_{\max}$ in which the SPT power spectrum deviates from the reference one, measured from N-body simulations, by 1%. This value is computed at different redshifts and for different models. Extracted from [100].

are plenty of interesting and important conclusions. Here, we briefly summarize the most striking points and show why this perturbative approach breaks on smaller redshifts and scales.

In Figures 15 and 16 we show how much the 1-loop SPT power spectrum agrees with the one obtained from an N-body simulation and from the Halofit model that is built-in in the CAMB module, respectively. Fig. 15 gives us an indication that the SPT model works better at higher redshifts. This can be confirmed by looking at Table 1, where we show, as a function of the redshift, the scales in which the SPT power spectrum deviate by 1% from the reference one. This feature does not show up as a surprise. Since the gravitational collapse causes all primordial perturbations to grow with time, they all will be smaller at higher redshift. In addition, one of the vital assumptions behind SPT is that the matter perturbations are small. In this way, a higher redshift will cause the SPT to be a better model for large-scale structure. What is cumbersome is the fact that at $z = 0, 0.3$ and 0.7 the 2-loop corrections worsen the agreement between theory and data, causing the 1-loop power spectrum to be more accurate. It is a strong indication of the presence of some flaw in the perturbative series, since we have lost the notion of convergence. This problem is not present for $z \geq 1.0$. The failure of SPT power spectrum is also shown by the blue line in Figure 16. Although it produces a reasonably good description at $z = 0$ up to $k \approx 0.06h\text{Mpc}^{-1}$, it rapidly blows up.

The bispectrum is more tricky, since it is a function of three distinct modes. In Figure 17 we show the *squeezed limit*, which formally means one of the modes going to zero. For practical purposes, this momentum is usually fixed in one of the lowest modes available.

Figure 16: The relative deviation of the theoretical prediction of the SPT 1-loop power spectrum from a non-Linear matter obtained using the built-in Halofit in the CAMB. Clearly the SPT curve fails in recovering the non-linear matter power spectrum, where the SPT overestimate the clustering power. A similar feature occurs in Figure 17. As we will see, this overestimation is due to the strong UV-dependence.

Figure 17: The theoretical Bispectrum up to 1-loop compared with N-body simulations (red points). We show the squeezed limit with $\Delta k = 0.013 h\text{Mpc}^{-1}$. The total SPT prediction discussed in this section is represented by the thin black line, all at $z = 0$. We can notice a partial agreement of the SPT modeling. Nevertheless, the theoretical model gets more accurate when we insert the effective field theory corrections (dubbed as EFT in the legend). The green explicitly shows these corrections as a function of k . On large-scales (small values of k) the SPT prediction is basically unchanged, whereas a power subtractions starts to appear in small-scales (larger values of k). These EFT corrections will be succinctly introduced in the subsequent subsection. This figure was extracted from [105]

In this case, the squeezed limit was computed using $\Delta k = 0.013h\text{Mpc}^{-1}$. The red points correspond to the data obtained from N-body simulations, whereas the thin black line shows the SPT prediction. Clearly, the SPT overestimation also occurs for the SPT 1-loop bispectrum, as it fails to agree with data around $k = 0.13h\text{Mpc}^{-1}$.

Now that we have witnessed the SPT failing in describing realistic data, it is important to take a few steps back and recognize what we did wrong. Right in the beginning of this Section we started with the Boltzmann equation for dark matter particles. The time evolution of its PDF is described by the Vlasov equation, where we started taking moments, producing the Boltzmann hierarchy. We proceeded by truncating this hierarchy, considering only the first two momenta. Hence, our perturbative model is supposed to work in the same regime where this truncation is valid, basically for modes $k < k_{\text{NL}}$. It may have gone unnoticed, but we have included modes in the theory that do not satisfy this condition. To realize it, let's consider again the 1-loop power spectrum, described in Figure 10. Each loop stands for an internal momentum integration in q with $q \rightarrow \infty$ (see Eqs.(2.60, 2.63)). In this way we are obtaining contributions from $q > k_{\text{NL}}$, where our theory is supposed to break. Usually in field theory such integration would lead to divergences. Indeed, the convergence range for the 1-loop diagrams is quite narrow. By assuming $P_{11} \approx k^n$ one can conclude that

	UV-divergent	IR-divergent
P_{13}	$n \geq -1$	$n \leq -1$
P_{22}	$n \geq 1/2$	$n \leq -1$
$P_{1-\text{loop}}$	$n \geq -1$	$n \leq -3$

Notwithstanding, the ΛCDM linear power spectrum “luckily” lies inside that range. So we obtain an UV contribution that, although being finite, is in fact wrong. The theory is then claiming for a *renormalization*, which is a procedure that removes all non-physical contributions that arise in loop integrals. This is what we will construct in the follow-

Figure 18: An illustration of the Wilsonian renormalization scheme for the large-scale overdensity field. The time evolution goes vertically and $\mathbf{q}$ are the particles initial position. The smoothing scale is Λ^{-1} , whereas R_* is the particle (or tracer) characteristic non-local scale. Extracted from [5].

ing section, obtaining a more robust theory that accesses small scales with a remarkable agreement with data.

2.4 The Effective Field Theory of Large-Scale Structure

As we discussed in the last subsection, the SPT results fail in describing realistic data. All loop integrals receive contributions from scales smaller than k_{NL} , where the theory is supposed to break. With that said, one must find a way to remove these undesired contributions and it is exactly what we will do now.

We take the Wilsonian approach to renormalization, so our task is to split the relevant fields as

$$X = X_l(k) + X_s(k), \quad (2.66)$$

where the sub-scripts s and l stands for small (or high-energy; UV) and large (or small-energy; IR) scales contributions to X . The next step then is simply to integrate out all

small-scales dynamics, producing an effective field theory description of the large-scale dynamics. The large-scale contribution for some field X is defined as

$$X_\ell \equiv [X]_\Lambda(\mathbf{x}) = \int d^3\mathbf{x}' W_\Lambda(|\mathbf{x} - \mathbf{x}'|) X(\mathbf{x}'), \quad (2.67)$$

where Λ is the scale that split the fields in its large- and small-scales contributions. The small-scale contribution will be $X_s = X - X_\ell$. Now we proceed by applying this splitting in the field in the Euler and continuity equations. It can be achieved by smoothing both continuity and Euler equation and using Eq. (2.67). For the Euler equation the smoothing is simply

$$[\rho_m \partial_t v^i(\mathbf{x}') + \rho_m v^j(\mathbf{x}') \partial_{j'} v^i(\mathbf{x}')]_\Lambda = 0. \quad (2.68)$$

Notice that this smoothing is far from being trivial and the result will not be an equation similar to the above with the fields having the sub-script l . A good exercise is to see what the smoothing of bilinear and trilinear functionals produces. It is useful to pick a Gaussian window, in such a way that

$$\int_{\mathbf{x}'} W_\Lambda(\mathbf{x} - \mathbf{x}')^i (\mathbf{x} - \mathbf{x}')^j = \frac{1}{\Lambda^2} \delta^{ij}, \quad (2.69)$$

$$\partial_{j'} W_\Lambda = -\partial_j W_\Lambda = \Lambda^2 (\mathbf{x} - \mathbf{x}')^j W_\Lambda, \quad (2.70)$$

$$\partial_{i'} \partial_{j'} W_\Lambda = \partial_i \partial_j W_\Lambda = -\Lambda^2 \delta_{ij} W_\Lambda + \Lambda^4 (\mathbf{x} - \mathbf{x}')^i (\mathbf{x} - \mathbf{x}')^j W_\Lambda. \quad (2.71)$$

For two functions f and g we have

$$[fg]_\Lambda = [f_l g_l]_\Lambda + [f_l g_s]_\Lambda + [f_s g_l]_\Lambda + [f_s g_s]_\Lambda. \quad (2.72)$$

Notice that f_l is already smoothed, so it is associated to long-wavelength contributions (in our case, both v^i and δ_l are to be seen as small perturbations). Thus, f_l in the first term

above can be Taylor expanded around $\mathbf{x}$ as

$$f_l(\mathbf{x}') = f_l(\mathbf{x}) + \partial_i f_l(\mathbf{x}) (\mathbf{x}' - \mathbf{x})^i + \frac{1}{2} \partial_i \partial_j f_l(\mathbf{x}) (\mathbf{x}' - \mathbf{x})^i (\mathbf{x}' - \mathbf{x})^j + \dots \quad (2.73)$$

with a similar expression being valid for g_l . After integrating by parts and discarding the boundary terms we have then

$$[f_l g_l]_\Lambda = f_\ell g_\ell + \frac{1}{\Lambda^2} \left(\nabla f_\ell \cdot \nabla g_\ell + \frac{1}{2} f_\ell \nabla^2 g_\ell + \frac{1}{2} g_\ell \nabla^2 f_\ell \right) + \dots \quad (2.74)$$

with the dots representing higher-derivative corrections.

For the mixed long-short terms, it is useful to rewrite them as

$$[f_\ell g_s]_\Lambda = [f_\ell g]_\Lambda - [f_\ell g_\ell]_\Lambda, \quad (2.75)$$

where the last term above was already computed. For the first we just need to do the same procedure and Taylor expand f_l , leading to

$$\begin{aligned} [f_\ell g]_\Lambda &= f_\ell g_\ell - \nabla f_\ell \cdot [(\mathbf{x} - \mathbf{x}') g(\mathbf{x}')]_\Lambda + \frac{1}{2} \nabla_i \nabla_j f_\ell \cdot \left[(\mathbf{x} - \mathbf{x}')^i (\mathbf{x} - \mathbf{x}')^j g(\mathbf{x}') \right]_\Lambda + \dots \\ &= f_\ell g_\ell + \frac{1}{\Lambda^2} \left(\nabla f_\ell \cdot \nabla g_\ell + \frac{1}{2} g_\ell \nabla^2 f_\ell \right) + \dots \end{aligned} \quad (2.76)$$

Therefore, we can conclude that

$$[f g]_\Lambda = f_\ell g_\ell + [f_s g_s]_\Lambda + \frac{1}{\Lambda^2} \nabla f_\ell \cdot \nabla g_\ell + \dots \quad (2.77)$$

In a similar way, the smoothing of a trilinear functional is given by [107]

$$\begin{aligned}
[fgh]_\Lambda &= f_l g_l h_l + [f_s g_s h_s]_\Lambda + f_l [g_s h_s]_\Lambda + g_l [f_s h_s]_\Lambda + h_l [f_s g_s]_\Lambda \\
&+ \frac{1}{\Lambda^2} (f_l \nabla g_l \cdot \nabla h_l + g_l \nabla f_l \cdot \nabla h_l + h_l \nabla f_l \cdot \nabla g_l) \\
&+ \frac{1}{\Lambda^2} (\nabla f_l \cdot \nabla [g_s h_s]_\Lambda + \nabla g_l \cdot \nabla [f_s h_s]_\Lambda + \nabla h_l \cdot \nabla [f_s g_s]_\Lambda) \\
&+ \frac{1}{2\Lambda^2} (\nabla^2 f_l [g_s h_s]_\Lambda + \nabla^2 g_l [f_s h_s]_\Lambda + \nabla^2 h_l [f_s g_s]_\Lambda) + \dots
\end{aligned} \tag{2.78}$$

Eqs. (2.77 - 2.78) are basically all information needed to obtain the smoothed Euler equation. After some lengthy algebra one obtain that Eq. (2.68) can be written as [73, 78, 107, 108]

$$\rho_\ell [\dot{v}_\ell^i + v_\ell^j \nabla_j v_\ell^i] + \rho_\ell \nabla_i \phi_\ell = -\nabla_j [\tau_i^j]_\Lambda - \nabla_j [\tau_i^j]^{\partial^2}, \tag{2.79}$$

where we defined

$$\tau_j^i = \rho_m v_i^s v_j^s - \frac{\phi_{,k}^s \phi_{,k}^s \delta_j^i - 2\phi_{,i}^s \phi_{,j}^s}{8\pi G}, \tag{2.80}$$

$$[\tau_j^i]^{\partial^2} = \rho_\ell \frac{\nabla v_\ell^i \cdot \nabla v_\ell^j}{\Lambda^2} + \frac{\nabla \phi_{,k}^\ell \cdot \nabla \phi_{,k}^\ell \delta_j^i - 2\nabla \phi_{,i}^\ell \cdot \nabla \phi_{,j}^\ell}{8\pi G \cdot \Lambda^2}. \tag{2.81}$$

This is a strong result. Even though we have discarded the velocity dispersion, *the smoothing procedure naturally introduces an effective stress tensor*, causing the large-scale matter field to be seen as an imperfect fluid. Nevertheless, the equations still have an explicit dependence on the UV physics, which is non-perturbative. In QFT this problem is solved by integrating out the UV degrees of freedom, and here we proceed in the same way. After having neglected the higher-derivative terms, the UV-averaged Euler equation is given by

$$\dot{v}_\ell^i + v_\ell^j \nabla_j v_\ell^i + \nabla_i \phi_\ell = -\frac{1}{\rho_\ell} \nabla_j \langle [\tau_i^j]_\Lambda \rangle_s. \tag{2.82}$$

The UV averaging does not act on the large-scale fields, since they have negligible variations on small scales. However, the $\langle [\tau_i^j]_\Lambda \rangle_s$ on the RHS is quite subtle, so we must carefully understand what it really means. Firstly, notice that it contains a large-

scale smoothing of the effective stress tensor, denoted by $[\tau_i^j]_\Lambda$, followed by the UV average. The first operation can be seen as capturing some sort of coupling between the small-scales dynamics with long-wavelength perturbations. The subsequent UV averaging should somehow eliminate the explicit short-wavelength modes from the equations, producing corrections to the IR dynamics. Our task then is to find a way to write down the final result in terms only of large-scale fields. The starting point is to realize that the small-scales dynamics is not disconnected from the long-wavelength fluctuations, since the presence of the latter can cause changes in the particles geodesics, which in turn induces changes in the short-scale stress-tensor. This UV-IR coupling is a vital feature for the effective description of structure formation. At this point, an experienced EFT theorist could simply make use of the usual EFT approach and write down the most general combinations of differential operators applied to allowed combinations of large-scale fields in such a way the result is a scalar, which is exactly what we have in Eq. (2.92). However, to have some intuition about what are the building blocks of $\langle [\tau_j^i] \rangle$, we follow an enlightening example from [78]. The reader who is comfortable with EFT results is suggested to go directly to Eq. (2.92).

Consider an expanding universe filled with dark matter particles. We then slice this universe in boxes of side Λ^{-1} . The relevant field for the particle's motion is the gravitational potential, whose short-modes are well inside each box. By imposing $\phi = \phi_s + \phi_l$ the equation of motion of some particle α inside a box is

$$\frac{d^2 \mathbf{x}_\alpha}{d\tau^2} + \mathcal{H} \frac{d\mathbf{x}_\alpha}{d\tau} = -\nabla \phi_s(\mathbf{x}_\alpha) - \nabla \phi_l(\mathbf{x}_\alpha). \quad (2.83)$$

The real position of a particle α can be decomposed as the position of this particle's box center of mass, $\mathbf{x}_{CM}$, plus its distance to that center, $\mathbf{r}_\alpha$. Thus, summing Eq. (2.83)

over all particles leads to the equation of motion of the center of mass,

$$\frac{d^2 \mathbf{x}_{CM}}{d\tau^2} + \mathcal{H} \frac{d\mathbf{x}_{CM}}{d\tau} = - \sum_{\alpha} \nabla \phi_l(\mathbf{x}_{\alpha}) - \sum_{\alpha} \nabla \phi_s(\mathbf{x}_{\alpha}) \approx - \sum_{\alpha} \nabla \phi_l(\mathbf{x}_{\alpha}) = - \nabla \phi_l(\mathbf{x}_{CM}). \quad (2.84)$$

Our first approximation above was to set $\sum_{\alpha} \phi_s(\mathbf{x}_{\alpha}) \approx 0$. It is motivated by the fact that the internal forces of a system cannot generate a non-vanishing net force to its own center of mass. If we take the difference between Eq. (2.83) and Eq. (2.84), this gives us the dynamics of the particle α relative to the center of mass,

$$\frac{d^2 \mathbf{r}_{\alpha}}{d\tau^2} + \mathcal{H} \frac{d\mathbf{r}_{\alpha}}{d\tau} = - \nabla \phi_s(\mathbf{x}_{\alpha}) - \nabla \phi_{\ell}(\mathbf{x}_{\alpha}) + \nabla \phi_l(\mathbf{x}_{CM}). \quad (2.85)$$

Now, a simple Taylor expansion of $\phi_l(\mathbf{x}_{\alpha})$ around $\mathbf{x}_{CM}$ allows us to conclude that:

$$\frac{d^2 r_{\alpha}^i}{d\tau^2} + \mathcal{H} \frac{dr_{\alpha}^i}{d\tau} = - \nabla \phi_s - \frac{\partial^2 \phi_l}{\partial x_i \partial x_j} r_{\alpha}^j + \dots \quad (2.86)$$

Therefore, the effect of a long-wavelength mode in the short-scale dynamics is captured by $\kappa_{ij} = \partial_i \partial_j \phi_l$, which is the tidal field. We can take an extra step and use the Zel'dovich approximation. According to it the displacement field relative to the center of mass can be written as:

$$r_{\alpha}^i = -l(\tau) [\partial_i \phi_s] |_{\mathbf{r}_{\alpha}(\tau=0)}, \quad \text{with } l(\tau) \sim \mathcal{H}^{-2} \quad (2.87)$$

where the last term is the short-modes potential computed at the initial position of particle α . Now, the effect of the tidal field on short-scale dynamics is transparent – it is responsible for the UV-IR connection. We are now in a position to make our last approximation and finally obtain the desired answer. Although we understand all relevant ingredients describing the UV dynamics in the presence of long-wavelength perturbations, we still need to solve the differential equation:

$$\frac{d^2 r_{\alpha}^i}{d\tau^2} + \mathcal{H} \frac{dr_{\alpha}^i}{d\tau} = - \partial_i \phi_s - l(\tau) \frac{\partial^2 \phi_l}{\partial x_i \partial x_j} \partial_j \phi_s. \quad (2.88)$$

During the matter era ϕ is constant, so the only function depending on time in the RHS is $l(\tau)$. Moreover, if we assume $\mathcal{H}$ to be approximately constant, the solution of the above equation can be written as

$$r_\alpha^i \approx -l(\tau)\partial_i\phi_s - g(\tau)\frac{\partial^2\phi_\ell}{\partial x_i\partial x_j}\partial_j\phi_s. \quad (2.89)$$

Using this last equation we compute the particle velocities, $v_\alpha^i = \frac{dr_\alpha^i}{d\tau} \approx \frac{r_\alpha^i}{\mathcal{H}^{-1}}$, as being

$$\boxed{v_\alpha^i = n_1\mathcal{H}^{-1}\partial_i\phi_s + n_2\frac{\partial^2\phi_\ell}{\partial x_i\partial x_j}\partial_j\phi_s}, \quad (2.90)$$

where n_1 and n_2 are $\sim \mathcal{O}(1)$.

The Eq. (2.90) is the desired result, since it explicitly shows how the long- and short-modes influence the particle dynamics. Recall from our expression for the effective stress tensor, Eq. (2.80), that the first contributing term is $\sim \rho_m v^i v^j$. Using our new expression and taking the UV averaging we obtain

$$\langle \rho v_i v_j \rangle \sim \delta_{ij} \langle \rho v_s^2 \rangle + \frac{\partial_i \partial_j \phi_\ell}{\mathcal{H}^2} \langle \rho v_s^2 \rangle \sim \delta_{ij} \langle \rho v_s^2 \rangle + \frac{\partial_i v_j^\ell + \partial_j v_i^\ell}{\mathcal{H}} \langle \rho v_s^2 \rangle \quad (2.91)$$

where we used that $\langle \partial_i \phi_s \partial_j \phi_s \rangle \sim \mathcal{H}^2 \delta_{ij} \langle v_s^2 \rangle$ and $\partial_i \partial_j \phi_\ell \sim \mathcal{H} (\partial_i v_j^\ell + \partial_j v_i^\ell)$. The first term accounts for an isotropic pressure, whereas the second one is the anisotropic stress and it is associated to the breaking of large-scale isotropy generated by large-scale perturbations. Now we are in a position to understand that the general form of $\langle \tau_{ij} \rangle$ must be

$$\langle \tau_{ij} \rangle = \rho \left[c_1 \left(\frac{\langle v_s^2 \rangle \partial^2}{\mathcal{H}^2} \right)_{ij} + c_2 \left(\frac{\langle v_s^2 \rangle \partial^2}{\mathcal{H}^2} \right)_{ij}^2 + \dots \right] \phi_\ell + \dots, \quad (2.92)$$

where c_i are time-dependent coefficients of order 1. The index ij stands for the specific form of each differential operator. Notice that due to the equivalence principle the contributions for the effective stress tensor can only depend on terms composed of second order derivatives of the gravitational potential¹⁶. Also notice that in the expression above for

¹⁶Linear and first derivatives of the potential can be eliminated by Galilean transformations.

each ∂ in the numerator we have the combination $\langle v_s \rangle \mathcal{H}^{-1} \approx k_{NL}^{-1}$ in the denominator. It explicitly exhibit we are performing an expansion with respect to k/k_{NL} as well. One can extend these results up to second order in large-scale fields as

$$\langle \tau_{ij} \rangle = \delta_{ij}^{(K)} \langle p_b \rangle + \underbrace{\tilde{c}_1 \partial_i \partial_j \phi_\ell + \tilde{c}_2 \delta_{ij}^{(K)} \partial^2 \phi_\ell}_{\text{leading order}} + \quad (2.93)$$

$$+ \underbrace{\tilde{c}_3 \partial_i \partial_m \phi_\ell \partial_m \partial_j \phi_\ell + \tilde{c}_4 \delta_{ij}^{(K)} \partial^2 \phi_\ell \partial^2 \phi_\ell + \tilde{c}_5 \partial_i \partial_j \phi_\ell \partial^2 \phi_\ell + \dots}_{\text{next to leading order}} \quad (2.94)$$

where we kept the pressure generated by the short-scales [see Eq. (2.91)], absorbed the multiplicative factors by defining the tilde coefficients and represented some of the missing terms we will introduce later by the dots. It just happens that we can make use of the equations of motion to substitute the derivatives acting on the large-scale gravitational potential in terms of δ_l and v_ℓ^i . For instance, we can straightaway identify $\partial^2 \phi_\ell \sim \delta_\ell$ and $\partial_i \phi_\ell \sim v_\ell^i$. In the end we end up having many terms which are common in fluid dynamics: a pressure perturbation $\sim \delta_{ij}^{(K)} \delta_\ell$, a shear-viscosity contribution $\sim \partial_i v_\ell^j + \partial_j v_\ell^i - \frac{2}{3} \delta_{ij}^{(K)} \partial_k v_\ell^k$ and a bulk-viscosity $\sim \delta_{ij}^{(K)} \partial_k v_\ell^k$, leading to [73, 75, 108, 109]:

$$\langle \tau_{ij} \rangle = p_b \delta_{ij}^{(K)} + p_b \left[c_s^2 \delta_l \delta^{ij} - \frac{c_{bv}^2}{\mathcal{H}} \delta^{ij} \partial_k v_\ell^k - \frac{3}{4} \frac{c_{sv}^2}{\mathcal{H}} \left(\partial^j v_\ell^i + \partial^i v_\ell^j - \frac{2}{3} \delta^{ij} \partial_k v_\ell^k \right) \right] + \dots \quad (2.95)$$

Before proceeding to solve the equations for the large-scale fields with the effective stress-tensor just derived, we have two comments about the above expression. Firstly, notice that there was an $[\dots]_\Lambda$ in the effective stress-tensor. These terms would have appeared in Eq. (2.92) as

$$\rho \left[\left(d_1^{(n)} \left(\frac{\partial^2}{\Lambda^2} \right) + d_2^{(n)} \left(\frac{\partial^2}{\Lambda^2} \right)^2 + \dots \right) \{ v_\ell^2, \delta_\ell \phi_\ell, \dots \} \right]_{ij}. \quad (2.96)$$

They arises solely due to the smoothing procedure over scales Λ^{-1} , being suppressed by powers of k/Λ^{-1} . Moreover, they need to start at second-order in the fields, causing them to be quite negligible. The second comment is about stochastic contributions to

Eq. (2.95). These terms arise from deviations of the short-scale dynamics, represented by $\tilde{\delta}$ from the large-scale mean, δ . They can be understood as a local reshuffling of the linear matter field into a non-linear configuration, $\delta_J = \delta - \tilde{\delta}$. By imposing momentum and mass conservation one can conclude that the contributions of this stochastic field, δ_J , can be approximated as:

$$\begin{aligned}\delta_J(\mathbf{k}) &= \int_{\Lambda^{-1}} d^3x \exp[-i\mathbf{k} \cdot (\mathbf{x}_0 + \mathbf{x})] \left[\delta(\mathbf{x}_0 + \mathbf{x}) - \tilde{\delta}(\mathbf{x}_0 + \mathbf{x}) \right] \\ &= \exp[-i\mathbf{k} \cdot \mathbf{x}_0] \int_{\Lambda^{-1}} d^3x \left[1 + i\mathbf{k} \cdot \mathbf{x} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{x})^2 \right] \left[\delta(\mathbf{x}_0 + \mathbf{x}) - \tilde{\delta}(\mathbf{x}_0 + \mathbf{x}) \right] \\ &= \mathcal{O}(k^2).\end{aligned}\tag{2.97}$$

Later we will see that this contribution is sub-leading for the 1-loop matter power spectrum ($P_{JJ} \sim k^4/k_{\text{NL}}^4$).

Now we move on, and try to understand the structure of our EFT solutions. For convenience we display below the set of coupled equations we need to solve:

$$\nabla^2 \phi_\ell = \frac{a}{2} \bar{\rho} \delta_\ell = \frac{3}{2} \mathcal{H}^2 \Omega_m \delta_\ell = \frac{3}{2} \frac{\mathcal{H}_0^2 \Omega_{m,0}}{a} \delta_\ell,\tag{2.98}$$

$$\partial_\tau \delta_\ell(\mathbf{x}, \tau) + \nabla \cdot \{[1 + \delta_\ell(\mathbf{x}, \tau)] \mathbf{u}_\ell(\mathbf{x}, \tau)\} = 0,\tag{2.99}$$

$$\begin{aligned}\partial_\tau \mathbf{u}_\ell(\mathbf{x}, \tau) + \mathcal{H}(\tau) \mathbf{u}_\ell(\mathbf{x}, \tau) + \mathbf{u}_\ell(\mathbf{x}, \tau) \cdot \nabla \mathbf{u}_\ell(\mathbf{x}, \tau) &= -c_s^2 \nabla \delta_\ell + \frac{3}{4} \frac{c_{sv}^2}{\mathcal{H}} \partial^2 \mathbf{u}_\ell + \\ &+ \frac{4c_{bv}^2 + c_{sv}^2}{4\mathcal{H}} \nabla (\nabla \cdot \mathbf{u}_\ell) - \Delta J^i \dots\end{aligned}\tag{2.100}$$

where ΔJ^i represents stochastic contributions inherent to auto-correlation of the short-modes contributions.

Before continuing, we take a step further and only consider the leading contributions arising from the effective stress-tensor in Eq. (2.100). These leading contribution can be obtained by substituting the fields in the RHS by their linear expressions. Recall from the SPT solutions we derived in the last subsection that we usually work with the divergence of the velocity field, θ . When one takes the divergence of Eq. (2.100), the terms in the

Figure 19: In this Figure we show how the EFTofLSS organizes itself in perturbation theory when we assume a power law universe. The Λ CDM power spectrum has $n = -1.5$ around the non-linear regime. Thus we can see that for a consistent modeling at 1-loop we only need to include the SPT 1-loop results and the $c_s^2 k^2$ counterterm. Moreover, the stochastic field of the matter field is completely negligible, being less important than the 2-loop contributions in the relevant scales. Extracted from [75].

RHS will be $\sim \nabla^2 \delta_\ell, \nabla^2 \theta, \dots$. Furthermore, since $\delta^{(1)} \sim \theta^{(1)}$, one can write the leading order contributions of the effective-stress tensor in terms only of δ . For the 1-loop power spectrum this is enough.

The LHS of all equations above have the same form as the SPT expressions. The only difference now is the fact that we are working with smoothed fields, denoted by the ℓ sub-script, and the corrections appearing in the smoothed Euler equation. It turns out that these corrections to the Euler equation can be seen as *counter-terms* arising from the small-scales dynamics. To have a glimpse on how relevant these corrections are, the authors in [109] made use of the scaling symmetry present in the EdS universe to estimate the relevance of the counter-terms. The Ward identity associated to this scaling symmetry is that the dimensionless power spectrum, defined as

$$\Delta^2(k, \tau) \equiv \frac{k^3 P(k, \tau)}{2\pi^2}, \quad (2.101)$$

should only be a function of k/k_{NL} [110, 109]. This quantity is precisely one of the expan-

sion parameters we found in Eq. (2.92). Moreover, by imposing the EFT counter-terms to be invariant under this symmetry and considering again a scaling universe, $P(k) \sim k^n$, one obtains [109]:

$$\Delta_{\text{lin}}^2 = \left(\frac{k}{k_{\text{NL}}} \right)^{3+n}, \quad \Delta_{l-\text{loop}}^2 = \left(\frac{k}{k_{\text{NL}}} \right)^{(3+n)(1+l)}, \quad (2.102)$$

$$\Delta_{c_s^2}^2 = \left(\frac{k}{k_{\text{NL}}} \right)^{(5+n)}, \quad \Delta_{\text{stoch}}^2 = \left(\frac{k}{k_{\text{NL}}} \right)^7. \quad (2.103)$$

The contributions with the smallest exponents are the most relevant ones for the total power spectrum, which can then be written as:

$$\Delta_{\text{tot}}^2 = \Delta_{\text{lin}}^2 + \sum \Delta_{N-\text{loop}}^2 + \Delta_{c_s^2}^2 + \Delta_{\text{noise}}^2. \quad (2.104)$$

In Figure 19 we show the running of all contributions exponents as a function of the power spectrum slope. For the usual Λ CDM power spectrum around the non-linear scales we have $n \sim -1.5$. Therefore, we can see that the leading contribution to the power spectrum arising from the effective stress-tensor is the term $\sim c_s^2 k^2$, which is associated to the term $-c_s^2 \nabla \delta_\ell$ in Eq. (2.92). The next-to-leading order contribution would be $\sim c_s^2 \nabla^2 \delta_\ell$, but from Figure 19 we can see this term is negligible. Therefore, the EFTofLSS matter field is simply

$$\delta_\ell(\mathbf{k}, \eta) = \delta_\ell^{(1)}(\mathbf{k}, \eta) + \delta_\ell^{(2)}(\mathbf{k}, \eta) + \delta_\ell^{(3)}(\mathbf{k}, \eta) - k^2 c_s^2(\eta) \delta_\ell^{(1)}(\mathbf{k}, \eta) + \delta_J(\mathbf{k}, \eta), \quad (2.105)$$

which in turn allows us to obtain the desired EFTofLSS 1-loop power spectrum,

$$P(k) = P_{\text{lin}}(k) + P_{22,\Lambda}(k) + 2P_{13,\Lambda}(k) - 2c_{s,\Lambda}^2 k^2 P_{\text{lin}}(k) + P_{JJ,\Lambda}(k). \quad (2.106)$$

Recall that all large-scale fields, denoted by the sub-script ℓ , have a smoothing over scales Λ^{-1} , which is why we are now explicitly showing the cutoff dependence in the above equation. The cut-off dependence is indeed a problem in any theory, since the

scale Λ^{-1} is completely arbitrary – our conclusions about nature can not depend on such arbitrary choices. However, as we will show now, this dependence is only apparent. First, it is important to know that all SPT diagrams admit analytical asymptotic forms. For instance, in the low- k limit for P_{13} (external momentum k much smaller than the internal momentum q) we have that:

$$P_{13}(k) = -\frac{1}{3}k^2 P(k) \int \frac{d^3q}{(2\pi)^3} \frac{P(q)}{q^2} \left(\frac{61}{210} - \frac{2}{35} \frac{k^2}{q^2} + \dots \right) = \quad (2.107)$$

$$= -\frac{61}{210}k^2 P(k) \int \frac{dq}{(2\pi)^3} P(q) \equiv -\frac{61}{210}k^2 P(k) \sigma_d^2, \quad (2.108)$$

where σ_d is usually referred to as the *displacement dispersion*, and the integral above runs in the interval $q \in [0, \infty]$. Thus, we can write the following for P_{13} :

$$\begin{aligned} P_{13,\infty}(k) &= 3P_{\text{lin}}(k) \int_0^\Lambda \frac{d^3q}{(2\pi)^3} F_{3,s}(\mathbf{k}, \mathbf{q}, -\mathbf{q}) P_{\text{lin}}(q) + \\ &+ 3P_{\text{lin}}(k) \int_\Lambda^\infty \frac{d^3q}{(2\pi)^3} F_{3,s}(\mathbf{k}, \mathbf{q}, -\mathbf{q}) P_{\text{lin}}(q) \quad (2.109) \\ &= P_{13,\Lambda}(k) - k^2 P_{\text{lin}}(k) \frac{61}{210} \frac{1}{6\pi^2} \int_\Lambda^\infty dq P_{\text{lin}}(q), \end{aligned}$$

where we assumed the smoothing is roughly a sharp cutoff at Λ and used the fact that our modes of interest, $\mathbf{k}$, are much smaller than $q \in [\Lambda, \infty]$, allowing us to use the approximation presented in Eq. (2.107). Notice the k -dependence of the second term above, which is the term capturing contributions from the UV sector, where our usual perturbative series is not valid, is *exactly* the same as the counter-term, both $\sim k^2 P_{\text{lin}}(k)$.

To be even more explicit, we can restate the above conclusion as

$$\begin{aligned} P_{13,\Lambda} - c_{s,\Lambda}^2 k^2 P_{\text{lin}} &= \left(P_{13,\infty} + k^2 P_{\text{lin}}(k) \frac{61}{210} \frac{1}{6\pi^2} \int_\Lambda^\infty dq P_{\text{lin}}(q) \right) - c_{s,\Lambda}^2 k^2 P_{\text{lin}} = \\ P_{13,\infty} - k^2 P_{\text{lin}} \left(c_{s,\Lambda}^2 - k^2 P_{\text{lin}}(k) \frac{61}{210} \frac{1}{6\pi^2} \int_\Lambda^\infty dq P_{\text{lin}}(q) \right) &\equiv P_{13,\infty} - k^2 P_{\text{lin}} c_{s,\infty}^2, \end{aligned} \quad (2.110)$$

Figure 20: The 1-loop power spectrum in the Effective Field Theory of Large-Scale Structure approach. Notice the diagrams in the first line are exactly the same as the SPT case, the only difference is the introduction of a counter-term with vertex $\tilde{F}_1 = -c_s^2 k^2$. The same will be true for all other n-point functions, so the EFTofLSS is a consistent way to include the needed counter-terms to fix the SPT loop integrals.

where we defined

$$c_{s,\infty}^2 = c_{s,\Lambda}^2 - \frac{61}{210} \frac{1}{6\pi^2} \int_{\Lambda}^{\infty} dq P_{\text{lin}}(q). \quad (2.111)$$

Now, even if we take the limit $\Lambda \rightarrow \infty$ we have a residual correction to the usual SPT result. A similar argument holds for P_{22} and P_{JJ} , since both are $\sim k^4$ [75, 98]. Therefore, our final result for the 1-loop power spectrum in the EFTofLSS is

$$P(k) = P_{\text{lin}}(k) + P_{22,\infty}(k) + 2P_{13,\infty}(k) - 2c_{s,\infty}^2 k^2 P_{\text{lin}}(k) + P_{JJ,\infty}(k). \quad (2.112)$$

Let's briefly summarize what we did in order to arrive at the result shown above. The usual SPT approach fails in describing the matter field at small scales because the internal loop integrals capture contributions from all scales, including the ones in which the theory is supposed to break. To mend this problem, we adopted the Wilsonian renormalization approach, allowing us to obtain the equations of motion for the long-modes of the fields, Eqs. (2.98, 2.99, 2.100). Although these equations are the desired ones for the long-modes, they still have an (apparent) dependence on the cut-off scale Λ . However, as we just argued for the 1-loop power spectrum, once one include the relevant counter-

terms from the effective stress-tensor, the limit $\Lambda \rightarrow \infty$ naturally recovers the usual SPT diagrams, plus additional counter-terms that do the job we need, by capturing all UV contributions in the loop integrals. At 1-loop the price we paid for this was the insertion of our first free parameter in the large-scale structure modeling, c_s^2 . As usual for EFT's, these parameters are fitted when testing the theoretical model against data. In Figure 16 we showed it is worth including this parameter, since the deviations from the target non-linear power spectrum decrease significantly. These residual counter-terms are natural, so the EFTofLSS approach end up being a consistent way to include the counter-terms needed to *renormalize* the SPT solutions [109, 111].

Therefore, in order to extend the above procedure to higher loops or higher n-point statistics, one should first find the scaling of all UV contributions in each SPT diagram, and then check if the counter-term producing this specific scaling is allowed in the effective stress-tensor. This procedure has been successfully done for both the 2-loop [112, 108] and 3-loop [113] power spectrum, the 1-loop bispectrum [105, 114] and the 1-loop trispectrum [115, 116]. The main results of this work will be obtained using only the 1-loop power spectrum.

3 Biased Tracers

In the last section, we followed the dynamics during the matter-dominated epoch and developed the EFTofLSS, a robust theory we have up to date with a satisfactory agreement with simulations. However, notice that this matter is not observed directly. What is seen in the cosmological surveys are light-emitting objects, such as galaxies, galaxy clusters, quasars, the Lyman- α forest, etc. Then, the following question naturally arises: once we have a matter theory in hands, how can we link it with the observed distribution of galaxies in the universe? It is what we pursue in this section.

3.1 The Large-Scale galaxy bias expansion

The biased tracers are formed as a result of a highly non-linear process that takes place over a good fraction of the Hubble time. To locally understand this formation process is still an open problem that is far from being satisfactorily understood. It is then not amenable to construct a theory that naturally accounts for the tracer's formation. Thus, we need to make further assumptions based on what we see in nature in order to include these objects in the theory. Already in the first accurate measurements of the 2-point function from cosmological data, it was clear that different tracers do not have the same power spectrum. On large-scales, a possible way to describe such discrepancy is to assume that the tracer field is simply a linear response to the local matter field [117],

$$\delta_g(\mathbf{x}, \tau) \equiv \left(\frac{n_g(\mathbf{x}, \tau)}{\bar{n}_g} - 1 \right) = b_1(z) \delta(\mathbf{x}, \tau). \quad (3.1)$$

This simple linear relation between the matter field and matter was enough to remarkably explain data [118, 16, 119]. However, the price paid for using such a simplistic model is that its range of applicability is very limited. As one includes smaller scales into the analysis the linear bias is not sufficient. Moreover, since the upcoming (and even for some actual) surveys will furnish us with a wealth of data reaching considerably small scales, it is of utter importance to find a way to improve the model.

The modern description of galaxy biasing is constructed on a very similar approach we did in the last section. As we showed, linear biasing appeared as a suggestive way to understand the clustering on large scales. This model consisted of a direct connection between matter and tracers, and such a connection is intuitive. For instance, consider dark-matter halos. One naturally expects that overdense locations will attract more and more matter, producing a favorable region for halo formation. On the other hand, for regions scarce of matter we expect a smaller number of halos. A similar argument can also be made for voids, which have the opposite behavior. From this perspective, it is understandable that such a simple equation as Eq. (3.1) works so well. This connection between tracer formation and the matter field is the vital idea behind the concept of galaxy bias¹⁷. Given the knowledge of this relatedness between galaxy and matter, *the whole purpose of galaxy bias is then write down the most general expression for the tracer field in terms of the relevant degrees in its environment* [5]. The EFT mindset built-in in this interpretation is evident. With that said, we hope to have motivated the reader that the starting point for describing the tracer field using this EFT perspective is then

$$\delta_g(\mathbf{x}, \tau) = \sum_{\mathcal{O}} b_{\mathcal{O}}(\tau) \mathcal{O}(\mathbf{x}, \tau), \quad (3.2)$$

where $b_{\mathcal{O}}$ are called as *bias parameters*¹⁸ associated to the operator $\mathcal{O}(\mathbf{x}, \tau)$. These bias parameters in some data analysis are considered as nuisance parameters. Once we find what these operators are, we will have a model that connects all the theories previously constructed with the observed distribution of objects.

From Eq. (3.1), we can already see that $\delta \in \{\mathcal{O}\}$, but at this point, the reason for that may not be evident. To find the other contributions and the mechanism behind them, we must elaborate a little more on the picture of structure formation. In Figure 21 we show the usual picture used to represent the tracer formation history. Again, $\mathbf{q}$ represents

¹⁷Recall this model is valid for all large-scale structure tracers. Thus, although we may refer to it as *galaxy bias*, bear in mind we are referring to the whole bias model.

¹⁸We emphasize all bias parameters are time dependent, but sometimes we omit its explicit dependence to avoid clutter.

Figure 21: An illustration of the tracer formation. The R_* represent the non-locality radius. The relevant fields must, in principle, be integrated over this region.

the initial position of some agglomerate of cosmic dust, gas, dark matter, or any ingredient needed for some specific tracer to form. This agglomerate then evolves in time, following some path described by $\mathbf{x}_{\text{fl}}$. Whatever happens in the formation history of this tracer, the relevant operators for it, due to the equivalence principle, must be constructed as combinations of $\partial_i \partial_j \phi$, where ϕ is treated perturbatively. In General Relativity it is possible to also have contributions like $\phi \partial_i \partial_j \phi$. However, since tracer formation is non-relativistic, the contributions of all these terms are sub-leading. We have two important comments now. The first one is about the non-locality in time. It is well established that tracer formation takes a considerable fraction of the Hubble time to occur. In this way, all operators are supposed to be integrated over the whole tracer history [120]. However, the corrections arising from non-locality in time are only relevant for two-loop calculations¹⁹ [120, 112], so here we neglect these complications. The second one is about spatial non-locality. Imagine again the agglomerate of ingredients traveling in the universe. It happens that treating the spatial distribution of these constituents as being punctual is not correct. The relevant information for tracer formation is encapsulated in a non-negligible

¹⁹This is why we did not comment on this when discussing the 1-loop matter power spectrum.

region around it, referred to as *non-locality radius*. It is denoted by R_* in Figure 21 and obviously will be tracer-dependent. This effect was consistently detected for the first time in a bias analysis during the first steps of this work [121], and we shall include it later ²⁰. For the moment, we assume that tracer formation is effectively local in space.

Relying on all features discussed above, it was proved that it is possible to iteratively construct a complete set of independent operators up to any order in cosmological perturbation theory [122]. More specifically, they proved that starting from

$$\Pi_{ij}^{[1]}(\mathbf{x}, \tau) = \frac{2}{3\Omega_m \mathcal{H}^2} \partial_i \partial_j \Phi(\mathbf{x}, \tau) = K_{ij}(\mathbf{x}, \tau) + \frac{1}{3} \delta_{ij} \delta(\mathbf{x}, \tau), \quad (3.3)$$

where K_{ij} is the tidal field as defined in Eq. (2.47) (recall it is traceless), it is possible to obtain the next *generation* of tensors as

$$\Pi_{ij}^{[n]} = \frac{1}{(n-1)!} \left[(\mathcal{H}f)^{-1} \frac{D}{D\tau} \Pi_{ij}^{[n-1]} - (n-1) \Pi_{ij}^{[n-1]} \right]. \quad (3.4)$$

Let us clarify the notation above. The superscript $[n]$ indicates that this tensor belongs to the n -th generation, which in turn means that it starts its contribution in cosmological perturbation theory at order (n) . For instance, the quantity δ^2 belongs to the second generation since its leading order is second order in perturbation theory. The convective derivative, $D/D\tau$, is the time derivative along the particles' trajectories,

$$\frac{D}{D\tau} \equiv \frac{\partial}{\partial \tau} + v^i \frac{\partial}{\partial x^i} \quad (3.5)$$

In this way, the relevant degrees of freedom for tracer formation is, effectively, all scalars formed from combinations of these tensors. Up to third order in perturbation theory we

²⁰In this reference, it is motivated that the halos non-locality scale coincides with the Lagrangian halo bias.

have [5, 122]

$$\begin{aligned}
& 1^{\text{st}} \text{ tr } [\Pi^{[1]}] \\
& 2^{\text{nd}} \text{ tr } \left[(\Pi^{[1]})^2 \right], (\text{tr } [\Pi^{[1]}])^2 \\
& 3^{\text{rd}} \text{ tr } \left[(\Pi^{[1]})^3 \right], \text{tr } \left[(\Pi^{[1]})^2 \right] \text{tr } [\Pi^{[1]}], (\text{tr } [\Pi^{[1]}])^3, \text{tr } [\Pi^{[1]} \Pi^{[2]}]
\end{aligned} \tag{3.6}$$

It is a sturdy theoretical result since we managed to encapsulate all complexity present in tracer formation in a finite number of terms.

The first order contribution in Eq. (3.6) is simply $\text{tr } [\Pi_{ij}^{[1]}] = \delta$, the linear in matter contribution we started discussing about in this section. Moving to second order, we find the square of the first order, $(\text{tr } [\Pi_{ij}^{[1]}])^2 = \delta^2$ and $\text{tr } [(\Pi_{ij}^{[1]})^2] = K^2 + \frac{1}{3}\delta^2$. The form of this second order can be easily read from Eq. (3.3). The majority of the third-order contributions can also be easily translated in cosmological fields. The first three terms can be directly written as $[(\Pi^{[1]})^3] = K^3 + \frac{1}{3}\delta^3$, $\text{tr } [(\Pi^{[1]})^2] \text{tr } [\Pi^{[1]}] = \delta K^2 + \frac{1}{3}\delta^3$ and $(\text{tr } [\Pi_{ij}^{[1]}])^3 = \delta^3$. The last term requires the explicit form of $\Pi_{ij}^{[2]}$ and the computation of the contribution $\text{tr } [\Pi^{[1]} \Pi^{[2]}]$ requires a little of algebra, so we refer the reader to Appendix C of [5] for an explicit derivation. The final answer is

$$\text{tr } [\Pi^{[1]} \Pi^{[2]}] \Big|^{(3)} = \frac{5}{4}\Gamma_3 + \frac{37}{63}\delta^3 + \frac{6}{21}\delta K^2 + (K_{ij})^3, \tag{3.7}$$

where $K^3 \equiv K_{ij}K_{jl}K_{li}$ and

$$\Gamma_3 \equiv \left(\frac{2}{3\Omega_m \mathcal{H}^2} \right)^2 \left[(\partial_i \partial_j \Phi)^2 - (\nabla^2 \Phi)^2 \right] - \left[(\partial_i \partial_j \Phi_v)^2 - (\nabla^2 \Phi_v)^2 \right], \tag{3.8}$$

where $\Phi_v \equiv -(f\mathcal{H})^{-1}\nabla^{-2}\theta$ is the velocity potential.

We said that bias formation can be written as a general expansion in terms of the tensors $\Pi_{ij}^{[n]}$. However, up to some order in some cosmological perturbative scheme, one can find an explicit dependence of the scalars formed from these tensors on the usual cosmological fields, such as δ and K_{ij} . Therefore, by making use of SPT, we found that a

bias expansion using the terms in Eq. (3.6) can be translated into an expansion concerning the fields

$$\begin{aligned}
& 1^{\text{st}} \delta \\
& 2^{\text{nd}} \delta^2, K^2 \\
& 3^{\text{rd}} \delta^3, \delta K^2, K^3, \Gamma_3
\end{aligned} \tag{3.9}$$

Thus, from Eq. (3.2), we can finally write down our first bias expansion as

$$\delta_g(\mathbf{x}, \tau) = \underbrace{\tilde{b}_1 \delta}_{\text{linear}} + \underbrace{\frac{\tilde{b}_2}{2!} \delta^2 + \tilde{b}_{K^2} (K^{(1)})^2}_{\text{second-order}} + \underbrace{\frac{\tilde{b}_3}{3!} \delta^3 + \tilde{b}_{\delta K^2} \delta (K^{(1)})^2 + \tilde{b}_{K^3} K^3 + \tilde{b}_{\Gamma_3} \Gamma_3}_{\text{third-order}}. \tag{3.10}$$

In the equation above the linear matter field is the same we discussed in 2.4, so it was already renormalized for the 2-point function in the EFTofLSS scheme. However, all new *composite* operators (i.e. operators formed by the convolution of others) we introduced to model the tracer field also requires a renormalization [123, 124, 125]. Recall that the main conclusion of the EFTofLSS approach was to motivate the insertion of counter-terms to the SPT results to remove undesired contributions arising in the loop integrals. Since these composite operators introduce new diagrams having new loops, they must be renormalized as well. The field δ^n , for example, should be formally renormalized as

$$[\delta^n](\mathbf{x}, \tau) \equiv \delta^n(\mathbf{x}, \tau) + \sum_{\tilde{\mathcal{O}}} Z_{\tilde{\mathcal{O}}}^{(\delta^n)} \tilde{\mathcal{O}}(\mathbf{x}, \tau), \tag{3.11}$$

where the second contribution accounts for the needed operators for the renormalization. To single-out the UV-dependence that arises from these new contractions, the renormalization condition is defined as [124]

$$\left\langle [\delta^2]_{\mathbf{q}} \delta_{\mathbf{q}_1}^{(1)} \cdots \delta_{\mathbf{q}_m}^{(1)} \right\rangle = \left\langle (\delta^2)_{\mathbf{q}} \delta_{\mathbf{q}_1}^{(1)} \cdots \delta_{\mathbf{q}_m}^{(1)} \right\rangle_{\text{tree}} \quad \text{at} \quad q_i = 0, \forall i. \tag{3.12}$$

The reason for this is clear after substituting Eq. (3.11) in its RHS

$$\begin{aligned}
\left\langle [\delta^2]_{\mathbf{q}} \delta_{\mathbf{q}_1}^{(1)} \dots \delta_{\mathbf{q}_m}^{(1)} \right\rangle &= \left\langle (\delta^2)_{\mathbf{q}} \delta_{\mathbf{q}_1}^{(1)} \dots \delta_{\mathbf{q}_m}^{(1)} \right\rangle + \sum_{\tilde{\mathcal{O}}} Z_{\tilde{\mathcal{O}}}^{(\delta^n)} \left\langle \tilde{\mathcal{O}}_{\mathbf{q}} \delta_{\mathbf{q}_1}^{(1)} \dots \delta_{\mathbf{q}_m}^{(1)} \right\rangle = \\
&\left\langle (\delta^2)_{\mathbf{q}} \delta_{\mathbf{q}_1}^{(1)} \dots \delta_{\mathbf{q}_m}^{(1)} \right\rangle_{\text{tree}} + \left\langle (\delta^2)_{\mathbf{q}} \delta_{\mathbf{q}_1}^{(1)} \dots \delta_{\mathbf{q}_m}^{(1)} \right\rangle_{\text{loop}} + \sum_{\tilde{\mathcal{O}}} Z_{\tilde{\mathcal{O}}}^{(\delta^n)} \left\langle \tilde{\mathcal{O}}_{\mathbf{q}} \delta_{\mathbf{q}_1}^{(1)} \dots \delta_{\mathbf{q}_m}^{(1)} \right\rangle.
\end{aligned} \tag{3.13}$$

The first term in the last equation precisely cancels the tree contributions in Eq. (3.12), leading to

$$\sum_{\tilde{\mathcal{O}}} Z_{\tilde{\mathcal{O}}}^{(\delta^n)} \left\langle \tilde{\mathcal{O}}_{\mathbf{q}} \delta_{\mathbf{q}_1}^{(1)} \dots \delta_{\mathbf{q}_m}^{(1)} \right\rangle_{\text{tree}} = - \left\langle (\delta^n)_{\mathbf{q}} \delta_{\mathbf{q}_1}^{(1)} \dots \delta_{\mathbf{q}_m}^{(1)} \right\rangle_{\text{loop}}, \tag{3.14}$$

The role of the counter-terms is now evident, to absorb only the new divergences arising from the contractions. The worked example in [124] for the operator δ^2 is clarifying and an undeniable corroboration that *bias renormalization is required* to obtain physical results, that is having no UV-dependence (or not being cut-off dependent). Once this renormalized bias is considered the tracer power-spectrum from these fields is [124]

$$\begin{aligned}
P^{AA}(z, k) &= (b_1^A)^2 [P_0(z, k) + P_1(z, k)] + b_1^A b_2^A \mathcal{I}_{\delta^2}(z, k) + 2b_1^A b_{\mathcal{G}_2}^A \mathcal{I}_{\mathcal{G}_2}(z, k) \\
&+ \left(2b_1^A b_{\mathcal{G}_2}^A + \frac{4}{5} b_1^A b_{\Gamma_3}^A \right) \mathcal{F}_{\mathcal{G}_2}(z, k) + \frac{1}{4} (b_2^A)^2 \mathcal{I}_{\delta^2 \delta^2}(z, k) \\
&+ (b_{\mathcal{G}_2}^A)^2 \mathcal{I}_{\mathcal{G}_2 \mathcal{G}_2}(z, k) + b_2^A b_{\mathcal{G}_2}^A \mathcal{I}_{\delta^2 \mathcal{G}_2}(z, k),
\end{aligned} \tag{3.15}$$

where the diagrams $\mathcal{I}$ and $\mathcal{F}$ are defined as

$$\mathcal{I}_{\delta^2}(z, k) \equiv 2 \int_{\mathbf{q}} F_2(\mathbf{q}, \mathbf{k} - \mathbf{q}) P_{\text{lin}}(z, |\mathbf{k} - \mathbf{q}|) P_{\text{lin}}(z, q), \quad (3.16)$$

$$\mathcal{I}_{\mathcal{G}_2}(z, k) \equiv 2 \int_{\mathbf{q}} \sigma^2(\mathbf{q}, \mathbf{k} - \mathbf{q}) F_2(\mathbf{q}, \mathbf{k} - \mathbf{q}) P_{\text{lin}}(z, |\mathbf{k} - \mathbf{q}|) P_{\text{lin}}(z, q), \quad (3.17)$$

$$\mathcal{F}_{\mathcal{G}_2}(z, k) \equiv 4 P_{\text{lin}}(z, k) \int_{\mathbf{q}} \sigma^2(\mathbf{q}, \mathbf{k} - \mathbf{q}) F_2(\mathbf{k}, -\mathbf{q}) P_{\text{lin}}(z, q), \quad (3.18)$$

$$\mathcal{I}_{\delta^2 \delta^2}(z, k) \equiv 2 \int_{\mathbf{q}} P_{\text{lin}}(z, |\mathbf{k} - \mathbf{q}|) P_{\text{lin}}(z, q) - 2 \int_{\mathbf{q}} P_{\text{lin}}^2(z, q), \quad (3.19)$$

$$\mathcal{I}_{\mathcal{G}_2 \mathcal{G}_2}(z, k) \equiv 2 \int_{\mathbf{q}} \sigma^4(\mathbf{q}, \mathbf{k} - \mathbf{q}) P_{\text{lin}}(z, |\mathbf{k} - \mathbf{q}|) P_{\text{lin}}(z, q), \quad (3.20)$$

$$\mathcal{I}_{\delta^2 \mathcal{G}_2}(z, k) \equiv 2 \int_{\alpha} \sigma^2(\mathbf{q}, \mathbf{k} - \mathbf{q}) P_{\text{lin}}(z, |\mathbf{k} - \mathbf{q}|) P_{\text{lin}}(z, q), \quad (3.21)$$

with

$$\sigma^2(\mathbf{k}_1, \mathbf{k}_2) = \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} - 1. \quad (3.22)$$

We emphasize that in this work all diagrams in Eqs. (3.16 - 3.21) were performed using the CLASS-PT code.

3.2 The stochastic field

From the sampling perspective, it is straightforward to perceive that there must be some intrinsic noise field associated with the discrete sampling of the matter field. Consider a collection of N tracers in a finite volume V , whose overdensity field can be written as

$$\delta_{\text{g}}(\mathbf{x}) = \frac{1}{\bar{n}} \sum_i \delta^{(\text{D})}(\mathbf{x} - \mathbf{x}_i). \quad (3.23)$$

Its Fourier transform is

$$\delta_{\text{g}}(\mathbf{k}) = \frac{1}{\bar{n}} \sum_i \exp[i\mathbf{k}\mathbf{x}_i]. \quad (3.24)$$

Thus we can write the following for the power spectrum

$$\begin{aligned}
P_g(k) &= \frac{1}{V} \langle \delta(\mathbf{k}) \delta(-\mathbf{k}) \rangle \\
&= \frac{V}{N^2} \sum_{i=j} \langle \exp[i\mathbf{k}(\mathbf{x}_i - \mathbf{x}_j)] \rangle + \frac{V}{N^2} \sum_{i \neq j} \langle \exp[i\mathbf{k}(\mathbf{x}_i - \mathbf{x}_j)] \rangle \\
&= \frac{1}{\bar{n}} + P_{g,det},
\end{aligned} \tag{3.25}$$

where we can see an inherent white-noise contribution, usually referred to as *shot-noise*. The $P_{g,det}$ denotes the other contributions, where we loosely call the *deterministic* part. Although this uncertainty can be subtracted by hand from the measured power spectrum, we will see that it is inherent to the covariance matrix.

The stochasticity also arises from phenomena of another nature. For instance, we know for a fact that the distance between two distinct dark-matter halos must be at the very least the sum of their radius. It can be mathematically described by writing the correlation function of the discrete tracer as [126]

$$\xi_{hh}^{(d)}(r) = \begin{cases} -1 & \text{for } r < R \\ \xi_{hh}^{(c)}(r) & \text{for } r \geq R \end{cases}, \tag{3.26}$$

where the continuous correlation function is to be seen somehow connected to the matter one, for instance through $\xi_{hh}^{(c)}(r) \approx b_1^2 \xi_m^{(c)}(r)$. In this way, the power spectrum can be computed to be

$$\begin{aligned}
\int_0^\infty d^3r \xi_{hh}^{(d)}(r) j_0(kr) &= - \int_0^R d^3r j_0(kr) + \overbrace{\int_R^\infty d^3r \xi_{hh}^{(c)}(r) j_0(kr)}^{f_R^\infty = f_0^\infty - f_0^R} = \\
&= -V_{\text{excl}} W_R(k) - \int_0^R d^3r \xi_{hh}^{(c)}(r) j_0(kr) + \int_0^\infty d^3r \xi_{hh}^{(c)}(r) j_0(kr) \\
&= -V_{\text{excl}} W_R(k) - V_{\text{excl}} \left[W_R * P_{hh}^{(c)} \right](k) + P_{hh}^{(c)}(k),
\end{aligned} \tag{3.27}$$

where the exclusion volume, V_{excl} , is the halo volume, which can be approximately spherical, j_0 is the zeroth-order spherical Bessel function, and W_R is the Fourier transform

of the top-hat window function. The notation $[A * B]$ represent the usual convolution between the functions A and B . With this new source of noise, our prediction for the discrete power spectrum is

$$P_{\text{hh}}^{(d)}(k) = \frac{1}{\bar{n}} + P_{\text{hh}}^{(c)}(k) - V_{\text{excl}} W_R(k) - V_{\text{excl}} [W_R * P_{\text{hh}}^{(c)}](k), \quad (3.28)$$

where the last two terms account for the exclusion effect. By taking the asymptotic $k \rightarrow 0$ behavior, the authors in [126] concluded that

$$\begin{aligned} [W_R * P_{\text{hh}}^{(c)}](k) &\xrightarrow{k \rightarrow 0} \int \frac{d^3 q}{(2\pi)^3} P_{\text{hh}}^{(c)}(q) W_R(q) + \\ &+ k^2 R^2 \int \frac{d^3 q}{(2\pi)^3} P_{\text{hh}}^{(c)}(q) \left[W_R(q) \left(\frac{1}{(qR)^2} - \frac{1}{6} \right) - \frac{\sin(qR)}{(qR)^3} \right]. \end{aligned} \quad (3.29)$$

It shows that exclusion alone also produces a white-noise contribution that is completely degenerated with the shot-noise. In addition, it also introduces a noise contribution $\sim k^2$.

For realistic surveys, the effect of *Satellite Galaxies* is also a source of the noise. Suppose a population of N dark-matter halos. Inside these halos, we can associate a single galaxy that coincides with the halo center. This central galaxy sample will have the same population as the halo, so $N_{\text{central}} = N$. Moreover, we can populate each halo with more $N_{\text{satellite}}$ galaxies. In this way, the total number of galaxies is given by $N_{\text{total}} = N_{\text{central}} N_{\text{satellite}}$. If the satellites galaxies are arranged following some profile $\rho_s(r)$, with normalized Fourier transform being $u_s(k)$, the satellite galaxies power spectrum is [126]

$$P_{ss}^{(d)} = \overbrace{\frac{1}{\bar{n}_s} [1 + (N_{\text{satellite}} - 1) u_s^2(k)]}^{\text{1-halo term}} - \overbrace{\frac{4\pi R^3}{3} u_s^2(k) W_R(k)}^{\text{2-halo term}}, \quad (3.30)$$

where the deviations from the usual Poisson shot-noise are evident. For the reader acquainted with the Halo model, the first term above is associated with the 1-halo term – it captures the pairings of galaxies inside the same halo. The second term, on the other hand, accounts for the 2-halo term – it is associated with the pairing between galaxies liv-

ing in different halos. In the limit of very large-scales, the distribution of galaxies inside each halo can be neglected, so they give contributions as if they were a single particle. Thus, in this limit, the stochastic contribution coincides with the usual halo shot-noise corrected by the exclusion effect. On the other hand, for sub-halo scales, we expect the usual shot-noise for the satellite number density, $\frac{1}{\bar{n}_s}$.

These two effects proved to be complex and are difficult to be analytically modeled. However, as we just showed, the corrections of the noise field to the 2-point functions have the form $c_0 + c_2 k^2$. The constant noise, c_0 , can receive contributions from the usual shot-noise, exclusion effect, and satellite fraction. The k -dependent noise received contributions from the exclusion effect. All these effects are more on the analysis side like if they were inherent to the statistical sampling one performs to compare the model with our theory. Suppose, for instance, that only the Poissonian shot-noise is relevant. We could subtract the usual $\frac{1}{\bar{n}}$ from the data at the measurement level and then forget this boring talk about stochastic field. It turns out that it is not how we should proceed. We now argue that even in this unrealistic case, the stochastic field is still present and is built-in in our theoretical censorship of what happens on small scales.

The large-scale bias expansion is an effective description of the response of the tracer field to the long-wavelength perturbations. Although it may sound complete, notice that *structure formation depends on the initial conditions on small-scales*. Moreover, by assuming Gaussian initial conditions, these small-scale initial conditions are not correlated over long distances. Therefore, even though the gravitational time evolution is deterministic, the initial conditions are not. On top of statistical uncertainties we previously discussed, such as the Poisson shot-noise and exclusion effect, we also need to include a noise field to account for this intrinsic uncertainty we just commented [5, 127, 128, 129]. We are now in a position to propose that the large-scale galaxy bias should be

$$\delta_g(\mathbf{x}, \tau) = \sum_{\mathcal{O}} b_{\mathcal{O}}(\tau) \mathcal{O}(\mathbf{x}, \tau) + \varepsilon(\mathbf{x}, \tau), \quad (3.31)$$

where we introduced the *stochastic bias* field, ε . This stochastic field is not correlated with large-scale fields. Thus, by making use of non-locality arguments and the EFT spirit the allowed contribution of the stochastic power spectrum for some tracer A is [5]

$$\langle \varepsilon_A \varepsilon_A \rangle = \left(\frac{1}{\bar{n}_A} \right) (c_0^A + c_2^A k^2 + c_4 k^4 + \dots) . \quad (3.32)$$

In this equation, we factored out the usual Poissonian shot-noise, so the terms $c_0^A, c_2^A, \dots$ are measured in units of shot-noise. They are seen as nuisance parameters as well, *so they encompass all sources of stochasticity*. In practice, it is enough to truncate the above expansion at second-order, adding then two new free parameters to the tracer auto-spectrum.

3.3 Higher-derivative bias

Recall that in Subsection 3.1 we considered tracer formation as being effectively local. However, it has been noted that the leading contribution arising from non-locality may be relevant so we decided to introduce it in the model. Here we will briefly reproduce the arguments present in [5, 121] to motivate the structure of the leading contribution. For a more detailed discussion on this topic, we refer the reader to [129, 130].

The role of the higher-derivative bias, i.e. the manifestation of tracer formation not being precisely local, can be seen in Figure 21. Instead of assuming that the formation occurs precisely along the $\mathbf{x}_{\text{fl}}$, we impose that what matters is some region of radius R_* around the trajectory of the tracer ingredients. Therefore, all large-scale operators $\mathcal{O}$ are supposed to be integrated around this region. Since we are only considering the leading contribution we shall only study this averaging for the leading field, δ ,

$$b_1(\tau) \delta_m(\mathbf{x}, \tau) \rightarrow \int d^3 \mathbf{y} F(\mathbf{y}, \tau) \delta_m(\mathbf{x} + \mathbf{y}, \tau). \quad (3.33)$$

Due to the homogeneity of the universe, the kernel F is approximated as being indepen-

dent of $\mathbf{x}$. A subsequent Taylor expansion of δ around $\mathbf{x}$ leads to

$$b_1(\tau)\delta_m(\mathbf{x}, \tau) \rightarrow b_1(\tau)\delta_m(\mathbf{x}, \tau) + \hat{b}_{\nabla^2\delta}(\tau)\nabla^2\delta_m(\mathbf{x}, \tau) + \dots \quad (3.34)$$

where $\hat{b}_{\nabla^2\delta}(\tau)$ is the higher-derivative bias. Notice it has dimensions of $[length]^2$ and is usually approximated as being the tracer non-local radius, R_* . For halos, it is then expected to be the halo Lagrangian radius, as was verified to be true in [121]. In Fourier space the higher-derivative bias takes the form

$$-\frac{k^2}{k_M^2}\delta_m, \quad (3.35)$$

where k_M is the Fourier mode associated to the non-locality scale. This contribution to the tracer field was detected for the first time in [121]. We refer the reader to this work and references therein for a more detailed discussion.

Before proceeding we have a comment about this higher-derivative bias. For some tracer A we would have that

$$\delta_A \supset b_1^A \overbrace{(\delta_m - c_s^2 k^2 \delta_m)}^{\text{EFTofLSS}} - \hat{b}_{\nabla^2\delta}^A k^2 \delta_m, \quad (3.36)$$

where the matter entering in the bias expansion is the one renormalized by the EFTofLSS.

The contribution of these terms to the power spectrum is

$$P_{AA} \supset (b_1^A)^2 P_0 - 2 \left[(b_1^A)^2 c_s^2 + b_1^A \left(\hat{b}_{\nabla^2\delta}^A \right)^2 \right] k^2 P_0 \equiv (b_1^A)^2 P_0 + P_{\nabla^2\delta}^{AA}. \quad (3.37)$$

This specific functional form motivated us to use the following definition

$$P_{\nabla^2\delta}^{AA} = -2b_1^A \left[b_1^A c_s^2 + \left(\hat{b}_{\nabla^2\delta}^A \right)^2 \right] k^2 P_0 \equiv -2b_1^A \hat{b}_{\nabla^2\delta}^A k^2 P_0. \quad (3.38)$$

3.4 The multi-tracer bias expansion

In all the current and next-generation surveys the measured data encompasses a considerable class of objects, each one having its history and large-scale dynamics. For instance, future surveys such as DESI [131] and J-PAS [132] will map large swaths of the sky using galaxies of different types (Luminous Red Galaxies, Emission Line Galaxies) as well as quasars. In the "usual" analysis, all these objects are treated as being the same. That is, the same set of bias parameters are used to describe both Luminous Red Galaxies (LRG) and Emission Line Galaxies (ELG). The disadvantage of doing this is that although they are tracers of the same underlying matter field, their response to the perturbations is quite disparate. Therefore, as we will motivate in this subsection and prove in the results section, it is better to discern the tracers than join them together as being the same.

One of the advantages of discerning the tracers according to some physical property, i.e. applying the multi-tracer technique, is the suppression of the cosmic variance when we measure observables that depend on the relative clustering between the tracers [133, 134]. The idea is that, in any volume containing a realization of the Gaussian processes that give rise to the fluctuations in the density field, the statistical noise from this random process is highly correlated for all tracers of the density field, which means that we may be able to cancel cosmic variance by comparing directly the clustering of the tracers – e.g., by taking ratios of the power spectra or correlation functions. Indeed, it was shown in [135] that ratios of power spectra of different tracers can be measured to an accuracy that is not constrained by cosmic variance – only by the numbers of tracers and, of course, by the accuracy of the fiducial model. Moreover, in [135] it was also shown that the degrees of freedom corresponding to those ratios are completely independent (in the linear and Gaussian approximations) from the degrees of freedom corresponding to the bandpowers of the matter power spectrum – which means that they can be measured independently.

That idea has been explored as a means to enhance the constraints on parameters from non-Gaussianities to redshift-space distortions, both in simulations and in real data [136, 137, 138, 139, 140, 141, 142, 143, 144]. It has also provided a blueprint for combining

the clustering information from different types of galaxies in the GAMA [145] and SDSS [146, 147, 148] surveys, as well as surveys on different wavelengths [149, 150, 151, 152, 153, 154, 155].

Although there are many advantages associated with the multi-tracer well established in the literature, there is still a lot to be explored. In this work, we take a step further and implement this technique into the EFT description of cosmological probes. Suppose we have two distinct tracers population, tracer A and tracer B . We can use the whole theory we developed in this section and apply it to each tracer separately. The bias expansion for each tracer is

$$\delta_A(\mathbf{x}, \tau) = \tilde{b}_1^A \delta + \frac{\tilde{b}_2^A}{2!} \delta^2 + \tilde{b}_{K^2}^A (K^{(1)})^2 + \frac{\tilde{b}_3^A}{3!} \delta^3 + \tilde{b}_{\delta K^2}^A \delta (K^{(1)})^2 + \tilde{b}_{K^3}^A K^3 + \tilde{b}_{\Gamma_3}^A \Gamma_3 + \varepsilon_A, \quad (3.39)$$

$$\delta_B(\mathbf{x}, \tau) = \tilde{b}_1^B \delta + \frac{\tilde{b}_2^B}{2!} \delta^2 + \tilde{b}_{K^2}^B (K^{(1)})^2 + \frac{\tilde{b}_3^B}{3!} \delta^3 + \tilde{b}_{\delta K^2}^B \delta (K^{(1)})^2 + \tilde{b}_{K^3}^B K^3 + \tilde{b}_{\Gamma_3}^B \Gamma_3 + \varepsilon_B. \quad (3.40)$$

The tilde represents an expansion before renormalization. We associated each tracer population with its own set of bias parameters. From a theoretical point of view, it makes total sense. For this splitting of the tracer population into sub-groups to be possible, the tracers must have some characteristics that single them out. The fact that tracers present such dissimilarity is a strong indication that they may dwell in a specific physical environment different from the others and have a different history. Putting in another way, their response to the cosmological dynamics are different. As we will see, including this step in the analysis leads to an ameliorated measurement of cosmological parameters, despite having to introduce more nuisance parameters.

With two distinct tracers populations we have three data vectors, P_{AA} , P_{BB} and P_{AB} . The first two are simply the auto-power spectrum for each population, whose generic form

for the tracer A after renormalization is

$$\begin{aligned}
P^{AA}(z, k) &= (b_1^A)^2 [P_0(z, k) + P_1(z, k)] + b_1^A b_2^A \mathcal{I}_{\delta^2}(z, k) + 2b_1^A b_{\mathcal{G}_2}^A \mathcal{I}_{\mathcal{G}_2}(z, k) \\
&+ \left(2b_1^A b_{\mathcal{G}_2}^A + \frac{4}{5} b_1^A b_{\Gamma_3}^A \right) \mathcal{F}_{\mathcal{G}_2}(z, k) + \frac{1}{4} (b_2^A)^2 \mathcal{I}_{\delta^2 \delta^2}(z, k) \\
&+ (b_{\mathcal{G}_2}^A)^2 \mathcal{I}_{\mathcal{G}_2 \mathcal{G}_2}(z, k) + b_2^A b_{\mathcal{G}_2}^A \mathcal{I}_{\delta^2 \mathcal{G}_2}(z, k) + P_{\nabla^2 \delta}^{AA}(z, k) + P_{\varepsilon^A \varepsilon^A}(z, k).
\end{aligned} \tag{3.41}$$

The expression for P_{BB} is obtained by simply exchanging the tracer label in the bias parameters. The higher-derivative and stochastic power spectrum are defined in the same way as in Eqs. (3.38) and (3.32), respectively. The crossed power spectrum can be easily obtained from the auto-power spectrum by performing a simple symmetrization of the tracer index. For tracers A and B its form is

$$\begin{aligned}
P^{AB}(z, k) &= b_1^A b_1^B [P_0(z, k) + P_1(z, k)] + \frac{1}{2} (b_1^A b_2^B + b_1^B b_2^A) \mathcal{I}_{\delta^2}(z, k) \\
&+ (b_1^A b_{\mathcal{G}_2}^B + b_1^B b_{\mathcal{G}_2}^A) \mathcal{I}_{\mathcal{G}_2}(z, k) + \left[(b_1^A b_{\mathcal{G}_2}^B + b_1^B b_{\mathcal{G}_2}^A) \frac{2}{5} (b_1^A b_{\Gamma_3}^B + b_1^B b_{\Gamma_3}^A) \right] \mathcal{F}_{\mathcal{G}_2}(z, k) \\
&+ \frac{1}{4} b_2^A b_2^B \mathcal{I}_{\delta^2 \delta^2}(z, k) + b_{\mathcal{G}_2}^B b_{\mathcal{G}_2}^A \mathcal{I}_{\mathcal{G}_2 \mathcal{G}_2}(z, k) + \frac{1}{2} (b_2^A b_{\mathcal{G}_2}^B + b_2^B b_{\mathcal{G}_2}^A) \mathcal{I}_{\delta^2 \mathcal{G}_2}(z, k) \\
&+ P_{\nabla^2 \delta}^{AB}(z, k) + P_{\varepsilon^A \varepsilon^B}(z, k).
\end{aligned} \tag{3.42}$$

The higher-order contributions to the crossed-power spectrum is

$$P_{\nabla^2 \delta}^{AB} \equiv - (b_1^A b_{\nabla^2 \delta}^B + b_1^B b_{\nabla^2 \delta}^A) k^2 P_0. \tag{3.43}$$

In previous works, it has been verified that the contributions from the crossed noise field, $P_{\varepsilon^A \varepsilon^B}$ are quite negligible. However, we decided to take the most conservative approach and modeled it as

$$\langle \varepsilon_A \varepsilon_B \rangle = \frac{1}{2} \left(\frac{1}{\bar{n}_A} + \frac{1}{\bar{n}_B} \right) [c_0^{AB} + c_2^{AB} k^2], \tag{3.44}$$

where we factored out the average of the Poissonian shot-noise of the tracers.

Thus, the most general multi-tracer bias analysis for these two tracers, A and B , re-

quires a set of sixteen parameters,

$$\{b_1^A, b_2^A, b_{\mathcal{G}_2}^A, b_{\Gamma_3}^A, b_{\nabla^2\delta}^A, b_1^B, b_2^B, b_{\mathcal{G}_2}^B, b_{\Gamma_3}^B, b_{\nabla^2\delta}^B, c_0^{AA}, c_2^{AA}, c_0^{BB}, c_2^{BB}, c_0^{AB}, c_2^{AB}\}. \quad (3.45)$$

In comparison, the single-tracer bias expansion have only 7 parameters,

$$\{b_1^A, b_2^A, b_{\mathcal{G}_2}^A, b_{\Gamma_3}^A, b_{\nabla^2\delta}^A, c_0^{AA}, c_2^{AA}\}. \quad (3.46)$$

As we will show in our results, even though the multi-tracer model has sixteen parameters, it remarkably produces smaller error bars for some cosmological parameters than the single-tracer setup, which has only 7.

Notice that multi- and single-tracer have a different set of bias parameters. Since the whole purpose of this work is to compare the constraining power of both approaches, we need to find a way to put them on the same ground. Thus, to be able to have a fair comparison between their results, we will now define the *effective tracer* and its associated *effective bias*. The effective tracer is defined as

$$\delta^{\text{eff}} = \frac{1}{\bar{n}} \sum_i \bar{n}^i \delta^i, \quad (3.47)$$

where $\bar{n} = \sum_i \bar{n}^i$ is the mean number density of the total tracer population and $\bar{n}^i$ is the number density for a specific sub-population. In that way, for two tracers, the effective tracer parameters are given by

$$b_{[\mathcal{O}]}^{\text{eff}} = \frac{1}{\bar{n}} (\bar{n}^A b_{[\mathcal{O}]}^A + \bar{n}^B b_{[\mathcal{O}]}^B), \quad (3.48)$$

$$c^{\text{eff}} = \frac{1}{\bar{n}^2} \left[(\bar{n}^A)^2 c^{AA} + (\bar{n}^B)^2 c^{BB} + 2\bar{n}^A \bar{n}^B c^{AB} \right]. \quad (3.49)$$

4 Bias Co-evolution relations and Multi-tracer

Although we started to model large-scale structure using the dark matter field, we do not have direct access to it. Since what we observe in cosmological surveys are tracers of the underlying field, such as galaxies, galaxy clusters and Ly- α systems, it is necessary to somehow connect them with the matter field.

In the previous Section we argued that the the bias expansion turns out to be a good way to achieve this. The flaw of doing so is the fact that each tracer requires its own set of bias parameters. Up to third order in perturbation theory the relevant bias parameter to describe the power spectrum in 1-loop are $b_1, b_2, b_{\mathcal{G}_2}, b_{\Gamma_3}$ and $b_{\nabla^2\delta}$. Therefore, for N-tracers one would need $N \times 5$ bias parameters plus the coefficients arising from combinations of the noise fields. Needless to say, a theoretical prior on some of these parameters would be extremely useful. Thus, in this Section we discuss how one can derive deterministic priors linking many high-order parameters to the lower-order ones. As we will show later, these relations not only speed up the MCMC analysis but also result in a considerable decrease of cosmological parameters error bars.

We start by studying the co-evolution model, which consists in the evolution of an N-component fluid evolving gravitationally in an expanding background. By identifying tracers as components of this multi-component fluid, we show how one can easily obtain the desired bias relations. Although our calculations here are up to second order in perturbation theory, their extension to higher order are trivial.

4.1 N-Component Fluid Dynamics

The dynamics of a multiple component fluid in an uniformly expanding spacetime was studied in an enlightening way in [156]. We will re-state many of their results here and refer the reader to their paper for more details.

The relevant equations that describe the system are Poisson's equation, the Euler equation, associated with the conservation of momentum, and the continuity equation, associ-

ated with the conservation of mass or number density. For an N-component fluid the set of equations that need to be solved are:

$$\partial_\tau \delta_i(\mathbf{x}, \tau) + \nabla \cdot [(1 + \delta_i(\mathbf{x}, \tau)) \mathbf{v}_i(\mathbf{x}, \tau)] = 0, \quad (4.1)$$

$$\partial_\tau \mathbf{v}_i(\mathbf{x}, \tau) + \mathcal{H}(\tau) \mathbf{v}_i(\mathbf{x}, \tau) + [\mathbf{v}_i(\mathbf{x}, \tau) \cdot \nabla] \mathbf{v}_i(\mathbf{x}, \tau) = -\nabla \Phi(\mathbf{x}, \tau), \quad (4.2)$$

$$\nabla^2 \Phi(\mathbf{x}, \tau) = 4\pi G a^2 \sum_{i=1}^N \bar{\rho}_i(\tau) \delta_i(\mathbf{x}, \eta) = \frac{3}{2} \Omega_m(\eta) \mathcal{H}^2(\eta) \sum_{i=1}^N \omega_i \delta_i(\mathbf{x}, \eta), \quad (4.3)$$

where the index i refers to some fluid component, τ is the conformal time and a is the scale factor. The density perturbation is defined in the usual way for each component, $\delta_i = \rho_i/\bar{\rho} - 1$. As will turn out to be useful, we also introduce the fractional contribution of each fluid species to the total matter density²¹, $\omega_i = \Omega_i/\Omega_m$, so the overdensities can be written as $\delta_i = \rho_i/\bar{\rho}_m \omega_i - 1$.

Following the same steps as in Section 2, it can be assumed that the velocity field is irrotational. Therefore, after having taken the Fourier transform we obtain a set of equations similar to Eqs. (2.40) and (2.41):

$$\frac{\partial \delta_i(\mathbf{k}, \eta)}{\partial \eta} - \theta_i(\mathbf{k}, \eta) + \delta_{12}^D \alpha(\mathbf{k}_1, \mathbf{k}_2) \delta_i(\mathbf{k}_1, \eta) \theta_i(\mathbf{k}_2, \eta) = 0, \quad (4.4)$$

$$\frac{\partial \theta_i(\mathbf{k}, \eta)}{\partial \eta} + \theta_i(\mathbf{k}, \eta) \left[1 - \frac{\Omega_m}{2} + \Omega_\Lambda \right] - \frac{3}{2} \Omega_m \sum_{j=1}^N \omega_j \delta_j = \delta_{12}^D \beta(\mathbf{k}_1, \mathbf{k}_2) \theta_i(\mathbf{k}_1, \eta) \theta_i(\mathbf{k}_2, \eta), \quad (4.5)$$

where we used that $\delta_{12}^D = \delta^D(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)$, and the definition

$$\theta_i(\mathbf{x}, \tau) = -\frac{\nabla \cdot \mathbf{v}_i(\mathbf{x}, \tau)}{\mathcal{H}}. \quad (4.6)$$

In this expression we also defined the time variable, $\eta = \ln a$, and we assumed that

²¹The energy density in each species is in units of critical density,

$$\Omega_i \equiv \frac{\bar{\rho}_i}{\rho_{\text{crit}}}$$

repeated $\mathbf{k}$ -vectors are implicitly integrated over, e.g.:

$$\int d\mathbf{k}_1 d\mathbf{k}_2 \alpha(\mathbf{k}_1, \mathbf{k}_2) f(\mathbf{k}_1, \mathbf{k}_2) = \alpha(\mathbf{k}_1, \mathbf{k}_2) f(\mathbf{k}_1, \mathbf{k}_2). \quad (4.7)$$

The interaction between the different fluid components is represented by the third term in the LHS of Eq.(4.5). Previously, in Section 2, we studied a simplified version of this set of equations which had only one fluid component, and the couplings α and β were defined in Eq. (2.38).

The system of $2N$ equations in Eqs.(4.4,4.5) can be rewritten in a more elegant form using matrix notation. This can be done by defining the vector $\Psi^T(\mathbf{k}, \eta) = [\delta_1, \theta_1, \dots, \delta_N, \theta_N]$. Using this notation the system above can be cast into the form:

$$\partial_\eta \Psi^a(\mathbf{k}, \eta) + \Omega_{ab} \Psi_b(\mathbf{k}, \eta) = \gamma_{abc}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \Psi^b(\mathbf{k}_1, \eta) \Psi^c(\mathbf{k}_2, \eta), \quad (4.8)$$

with γ_{abc} defined as:

$$\gamma_{(2j-1)(2j-1)(2j)}^{(s)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = \frac{\alpha(\mathbf{k}_2, \mathbf{k}_1)}{2} \delta^D(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2), \quad (4.9)$$

$$\gamma_{(2j-1)(2j)(2j-1)}^{(s)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = \frac{\alpha(\mathbf{k}_1, \mathbf{k}_2)}{2} \delta^D(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2), \quad (4.10)$$

$$\gamma_{(2j)(2j)(2j)}^{(s)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = \beta(\mathbf{k}_1, \mathbf{k}_2) \delta^D(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2). \quad (4.11)$$

for $j = 1, 2, \dots, N$.

The $2N \times 2N$ matrix Ω_{ab} is given by:

$$\Omega_{ab} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -\frac{3}{2}\Omega_m w_1 & [1 - \frac{\Omega_m}{2} + \Omega_\Lambda] & -\frac{3}{2}\Omega_m w_2 & 0 & -\frac{3}{2}\Omega_m w_3 & 0 & \dots & -\frac{3}{2}\Omega_m w_N & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ -\frac{3}{2}\Omega_m w_1 & 0 & -\frac{3}{2}\Omega_m w_2 & [1 - \frac{\Omega_m}{2} + \Omega_\Lambda] & -\frac{3}{2}\Omega_m w_3 & 0 & \dots & -\frac{3}{2}\Omega_m w_N & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ -\frac{3}{2}\Omega_m w_1 & 0 & -\frac{3}{2}\Omega_m w_2 & 0 & -\frac{3}{2}\Omega_m w_3 & 0 & \dots & -\frac{3}{2}\Omega_m w_N & [1 - \frac{\Omega_m}{2} + \Omega_\Lambda] \end{bmatrix}.$$

The system of equations above is quite general, and may be used to study different

situations. The authors in [156] used it to study the non-linear evolution of the matter-baryon perturbations, so one could already fix $N = 2$, standing for δ_m and δ_b . In this work, we will proceed in a slightly different manner, as will become clear in the next pages.

Although we constructed the concise set of equations (4.8), we still need to solve them. The most useful assumption one could make is to consider an Einstein-de Sitter (EdS) universe. We did this in Section 2 when solving only for the matter dynamics. In our case here we can see that assuming the EdS universe makes the Ω_{ab} time independent. Hence, the Eq.(4.8) can be algebraically solved by taking a Laplace transform [156, 157]:

$$(s\tilde{\Psi}^a + \psi^a) + \Omega_{ab}\tilde{\Psi}^b = \gamma_{abc}\mathcal{L}[\Psi^b \cdot \Psi^c](s) \quad (4.12)$$

$$(s\delta_{ab} + \Omega_{ab})\tilde{\Psi}^a \equiv M_{ab}\tilde{\Psi}^b = \psi^a + \gamma_{abc}\mathcal{L}[\Psi^b \cdot \Psi^c](s) \quad (4.13)$$

$$\rightarrow \boxed{\tilde{\Psi}^a = \psi^b(M^{-1})_{ab} + (M^{-1})_{ab}\gamma_{bcd}\mathcal{L}[\Psi^c \cdot \Psi^d](s)} \quad (4.14)$$

The ψ above is the initial condition Ψ . Finally, it is possible to go back to the η -space by applying the inverse Laplace transform, $\mathcal{L}^{-1}$:

$$\boxed{\Psi(\eta) = G_{ab}\psi^b + \int_0^\eta d\eta' G_{ab}(\eta - \eta')\gamma_{bcd}\Psi^c(\eta')\Psi(\eta')^d} \quad (4.15)$$

The G_{ab} above, called *linear propagator*, is defined as:

$$G_{ab}(\eta) = \mathcal{L}^{-1}(M_{ab}) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{x-iT}^{x+iT} ds (M^{-1})_{ab} e^{\eta s}. \quad (4.16)$$

Therefore, by assuming an EdS universe we were able to formally solve Eq.(4.8). This assumption was routinely used in the previous decade, mainly by authors working on perturbative techniques [74, 158, 88]. In fact, we encountered this approximation previously when working with the matter-only dynamics where we witnessed that it caused the

solutions to be separable:

$$\delta(\mathbf{x}, \tau) \xrightarrow{(\text{EdS})} a(\tau)\delta(\mathbf{x}, 0) = D_+^{\text{EdS}}\delta(\mathbf{x}, 0). \quad (4.17)$$

As was recently noted by [159], the relative errors associated with this approximation in redshift-space are $P_0^{\text{Exact}}/P_0^{\text{EdS-approx}} \sim 1.003$ and $P_2^{\text{Exact}}/P_2^{\text{EdS-approx}} \sim 1.007$ for the monopole and quadrupole at $z = 0.57$, respectively. Moreover, we can make another approximation. One can solve the equations assuming an EdS universe and, in the end, exchange the $D_{(+)}^{(\text{EdS})}$ by the linear growth factor of the correct cosmology, $D_+(\Omega_m, \Omega_\Lambda|\tau)$ [156, 160, 161]. Recall that $\eta = \ln a(\tau) = \ln D_+^{(\text{EdS})}$, so the new time variable reads $\eta = \ln D_+(\Omega_m, \Omega_\Lambda|\tau)$, satisfying

$$\frac{d^2 D}{d\tau^2} + \mathcal{H} \frac{dD}{d\tau} - \frac{3}{2} \mathcal{H}^2 \Omega_m D = 0. \quad (4.18)$$

After solving the time dependent part of the problem, we still have the implicit integrations over the internal momentum in Eq.(4.15). We proceed here in the same fashion as before by assuming that the fields can be expanded in powers of the the linear overdensity²², δ_0 ,

$$\psi^a(\eta, \mathbf{q}) = \sum_n \delta_{1,\dots,n}^{(D)} \mathcal{I}_a^{(n)}(\mathbf{q}_1, \dots, \mathbf{q}_n) \prod_{i=1}^n \delta_0^i(\eta, \mathbf{q}_i), \quad (4.19)$$

$$\Psi^a(\eta, \mathbf{q}) = \sum_n \delta_{1,\dots,n}^{(D)} \mathcal{F}_a^{(n)}(\mathbf{q}_1, \dots, \mathbf{q}_n) \prod_{i=1}^n \delta_0^i(\eta, \mathbf{q}_i), \quad (4.20)$$

in such a way that after substitution in Eq. (4.15) the solutions for the final kernels are

$$\begin{aligned} \mathcal{F}_a^{(n)}(\eta, \mathbf{q}_1, \dots, \mathbf{q}_n) &= G_{ab}(\eta) \mathcal{I}_b^{(n)}(\mathbf{q}_1, \dots, \mathbf{q}_n) + \sum_{l=1}^{n-1} \int_0^\eta d\eta' G_{ab}(\eta - \eta') \times \\ &\times \gamma_{bcd}(\mathbf{q}_{1,\dots,j}, \mathbf{q}_{j+1,\dots,n}) \mathcal{F}_c^{(j)}(\mathbf{q}_1, \dots, \mathbf{q}_j, \eta') \mathcal{F}_d^{(n-j)}(\mathbf{q}_{j+1}, \dots, \mathbf{q}_n, \eta'). \end{aligned} \quad (4.21)$$

²²In what follows, I will be using many results from Standard Perturbation Theory. The reader not acquainted with such results is referred to Section 2.

Eqs. (4.19), (4.20) and (4.21) completely solve the problem. In the subsequent subsections we work with these equations in an increasing level of complexity. We go from the solutions for only matter, having $N = 1$, to matter and two interacting tracers, $N = 3$. As far as we know, there is no other mention in the literature for the solutions we find for this last case.

All computations done in this sections were performed using a code implemented in the software `Mathematica`²³.

4.2 Solution for the matter field

For $N = 1$ the matrix Ω_{ab} becomes simply:

$$\Omega_{ab} = \begin{bmatrix} 0 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}, \quad (4.22)$$

whereas the propagator is given by performing the integral in Eq.(4.16):

$$\begin{aligned} G_{ab}(\eta) &= \mathcal{L}^{-1} \left[\begin{pmatrix} \frac{2s+1}{2s^2+s-3} & \frac{2}{2s^2+s-3} \\ \frac{3}{2s^2+s-3} & \frac{2s}{2s^2+s-3} \end{pmatrix} \right] (\eta) = \begin{pmatrix} \frac{2}{5}e^{-\frac{3\eta}{2}} + \frac{3e^\eta}{5} & \frac{1}{5}(-2)e^{-\frac{3\eta}{2}} + \frac{2e^\eta}{5} \\ \frac{-3}{5}e^{-\frac{3\eta}{2}} + \frac{3e^\eta}{5} & \frac{3}{5}e^{-\frac{3\eta}{2}} + \frac{2e^\eta}{5} \end{pmatrix} \\ &= \frac{e^\eta}{5} \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} - \frac{e^{-3\eta/2}}{5} \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}. \end{aligned} \quad (4.23)$$

Eq.(4.23) shows that the linear propagation of the system is composed of a growing mode, $e^{(1)} = e^\eta$, and a decaying mode, associated with the eigenvalue $e^{(2)} = e^{-3\eta/2}$. The

²³<https://www.wolfram.com/mathematica/>

eigenmodes are given by

$$\zeta^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow G \cdot \zeta_{(1)} = e^\eta \zeta_{(1)}, \quad (4.24)$$

$$\zeta^{(2)} = \begin{bmatrix} \frac{-2}{3} \\ 1 \end{bmatrix} \rightarrow G \cdot \zeta_{(2)} = e^{-3\eta/2} \zeta_{(2)}. \quad (4.25)$$

Let's suppose that our initial conditions can be written as some linear combination of the growing and decaying modes [see Eq. (4.19)]:

$$\mathcal{I}^{(1)} = c_1 \zeta_{(1)} + c_2 \zeta_{(2)}, \quad (4.26)$$

$$\mathcal{I}^{(n)} = 0 \quad \forall \quad n \geq 2. \quad (4.27)$$

The linear solution is straightforward, since it is only the application of the linear propagator to the initial field,

$$\mathcal{F}^{(1)}(\eta, \mathbf{k}) = c_1 e^\eta \zeta_{(1)} + c_2 e^{-3\eta/2} \zeta_{(2)}. \quad (4.28)$$

The second-order solution is more lengthy, so instead of looking at the complete expression we examine only the solution projected into Legendre polynomials $\mathcal{P}_l(\mu)$,

$$\mathcal{F}_a^{(2)}(\eta, \mathbf{k}_1, \mathbf{k}_2) = \sum_{l=0} \mathcal{P}_l(\mu) \mathcal{F}_{a,l}^{(2)}(k_1, k_2), \quad (4.29)$$

where $\mu \equiv \hat{k}_1 \cdot \hat{k}_2$.

Starting with the matter component, we obtain the following:

$l = 0$:

$$\begin{aligned}\mathcal{F}_{a,0}^{(2)} = & c_2^2 \left(e^{-3\eta} \frac{1}{3} - \frac{4}{15} e^{-3\eta/2} - e^\eta \frac{1}{15} \right) + c_1 c_2 \left(\frac{2}{15} e^{-3\eta/2} - e^{-\eta/2} \frac{4}{9} + e^\eta \frac{14}{45} \right) \\ & + c_1^2 \left(\frac{-8}{105} e^{-3\eta/2} - e^\eta \frac{11}{15} + e^{2\eta} \frac{17}{21} \right)\end{aligned}\quad (4.30)$$

$l = 1$:

$$\begin{aligned}\mathcal{F}_{a,1}^{(2)} = & c_1^2 \left(-\frac{k_1 e^\eta}{2k_2} + \frac{k_1 e^{2\eta}}{2k_2} + \frac{k_2 e^{2\eta}}{2k_1} - \frac{k_2 e^\eta}{2k_1} \right) + \\ & c_2 c_1 \left(\frac{k_1 e^{-\frac{3\eta}{2}}}{3k_2} - \frac{2k_1 e^{-\frac{\eta}{2}}}{3k_2} + \frac{k_1 e^\eta}{3k_2} + \frac{k_2 e^{-\frac{3\eta}{2}}}{3k_1} + \frac{k_2 e^\eta}{3k_1} - \frac{2k_2 e^{-\frac{\eta}{2}}}{3k_1} \right) + \\ & c_2^2 \left(-\frac{2k_1 e^{-\frac{3\eta}{2}}}{9k_2} + \frac{2k_1 e^{-3\eta}}{9k_2} + \frac{2k_2 e^{-3\eta}}{9k_1} - \frac{2k_2 e^{-\frac{3\eta}{2}}}{9k_1} \right)\end{aligned}\quad (4.31)$$

$l = 2$:

$$\begin{aligned}c_1^2 \left(\frac{8}{105} e^{-\frac{3\eta}{2}} - \frac{4e^\eta}{15} + \frac{4e^{2\eta}}{21} \right) + c_2 c_1 \left(\frac{8}{15} e^{-\frac{3\eta}{2}} - \frac{8e^{-\frac{\eta}{2}}}{9} + \frac{16e^\eta}{45} \right) + \\ c_2^2 \left(\frac{e^{-3\eta}}{9} - \frac{8}{45} e^{-\frac{3\eta}{2}} + \frac{e^\eta}{15} \right)\end{aligned}\quad (4.32)$$

The first feature worth noticing from the above solutions is the fact that we now have transient effects. By “transient” we mean terms in the solutions that have sub-leading contributions with respect to the dominant term in the limit $\eta \rightarrow \infty$. Hence, the solution above is different than the second order solution we computed before when using SPT. For instance, let us compare the two monopoles of the second order solutions. The SPT solution is simply

$$\mathcal{F}_{a,0}^{(2),SPT} = \frac{17}{21} e^{2\eta}. \quad (4.33)$$

When comparing Eq.(4.30) with Eq.(4.33), we see that the results agree only if one considers the growing mode in the initial conditions, $c_2 = 0$, and assume $\eta \gg 1$, in such a way that $e^{2\eta}$ dominates over the other terms. This is very instructive, since it shows that,

back in Section 2, we made some implicit assumptions without even realizing it. These transients effects are very important, and were studied by [162]. As a matter of fact, these new *temporal structures* in the solutions will prove to be an important piece in the bias co-evolution relations. Now that we understood how to work with this simple model, let us move on and work with two fluid components.

4.3 Co-evolving matter and one (massless) tracer

In this part we adapt the formalism described above to include biased tracers. Its most simplistic construction for a matter field and one tracer species is the following. We assume that from some time $\tau = 0$ to $\tau = \tau_*$ the universe is filled only by matter. Then, at time $\tau = \tau_*$, a tracer field instantaneously appears in the universe. Therefore, the solution for any time $\tau > \tau_*$ is obtained by *co-evolving* this 2-component fluid.

The first step is always to compute the linear propagator. It is usually assumed that $\omega_{\text{tracer}} = 0$, so $\omega_{\text{matter}} = 1$ (we will comment about this choice in the next subsection). Thus, from Eq. 4.16 we obtain

$$G_{ab} = e^{-3\eta/2} \left[\mathcal{N}_{(-\frac{3}{2})} \right]_{ab} + e^{-\eta/2} \left[\mathcal{N}_{(-\frac{1}{2})} \right]_{ab} + [\mathcal{N}_{(0)}]_{ab} + e^{\eta} [\mathcal{N}_{(1)}]_{ab}, \quad (4.34)$$

with the matrices above being

$$\mathcal{N}_{(-\frac{3}{2})} = \begin{pmatrix} \frac{2}{5} & -\frac{2}{5} & 0 & 0 \\ -\frac{3}{5} & \frac{3}{5} & 0 & 0 \\ \frac{2}{5} & -\frac{2}{5} & 0 & 0 \\ -\frac{3}{5} & \frac{3}{5} & 0 & 0 \end{pmatrix}, \quad \mathcal{N}_{(-\frac{1}{2})} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad (4.35)$$

$$\mathcal{N}_{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathcal{N}_{(1)} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} & 0 & 0 \\ \frac{3}{5} & \frac{2}{5} & 0 & 0 \\ \frac{3}{5} & \frac{2}{5} & 0 & 0 \\ \frac{3}{5} & \frac{2}{5} & 0 & 0 \end{pmatrix} \quad (4.36)$$

whose eigenvalues and eigenmodes are

$$\zeta^{(1)} = \begin{bmatrix} -\frac{2}{3} \\ 1 \\ -\frac{2}{3} \\ 1 \end{bmatrix} \rightarrow G \cdot \zeta_{(1)} = e^{-3\eta/2} \zeta_{(1)}, \quad \zeta^{(2)} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \rightarrow G \cdot \zeta_{(2)} = e^{-\eta/2} \zeta_{(2)}, \quad (4.37)$$

$$\zeta^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow G \cdot \zeta_{(3)} = 1 \zeta_{(3)}, \quad \zeta^{(4)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow G \cdot \zeta_{(4)} = e^{\eta} \zeta_{(4)}. \quad (4.38)$$

To simplify the problem we assume that matter is only on its growing mode. Furthermore, since $\omega_{\text{tracer}} = 0$, the matter evolves as if the tracer does not exist²⁴. Hence, we expect the usual SPT solutions for the matter not only for $\tau < \tau_*$ but also for $\tau > \tau^*$. We can then forget it and pay careful attention to what's happening in the tracer sector.

Just as it does for the matter, the tracer is completely described by its density and velocity field. The only difference is that, as we extensively discussed in Section 3, the tracers and the underlying matter field are intrinsically connected. Hence, we do not need to associate a completely new field to the tracer, but instead to link it with the matter by constructing an effective theory. This connection is formally described, in a mathematical sense, by the bias expansion we discussed before. We showed that at second order in perturbation theory the relevant operators for the tracer were $\{\delta, \delta^2, (K_{ij})^2\}$, so the most general expression to describe the tracer density field at formation time is:

$$\delta_g^{(1+2)}(\mathbf{x}_*, \tau_*) = b_1^* [\delta^{(1)} + \delta^{(2)}](\mathbf{x}_*, \tau_*) + \varepsilon^*(\mathbf{x}_*, \tau_*) + \frac{1}{2} b_2^* [(\delta^{(1)})_*]^2 + b_{K^2}^* \left[(K_{ij}^{(1)})_* \right]^2. \quad (4.39)$$

²⁴This can be explicitly seen in the Euler equation for the matter field.

Our first task is to reach some simplified conclusions, so we proceed by setting $b_{K^2}^* = 0$. In the literature this approximation is called a local-in-matter-density (LIMD) expansion. We also drop the stochastic field, since its coupling with other fields is sub-leading. In this way, the initial condition for the tracer field becomes simply

$$\delta_g^{(1+2)}(\mathbf{x}_*, \tau_*) = b_1^* [\delta^{(1)} + \delta^{(2)}](\mathbf{x}_*, \tau_*) + \frac{1}{2} b_2^* [(\delta^{(1)})_*]^2. \quad (4.40)$$

For the tracer velocity field we can make use of the equivalence principle: since both tracer and matter feels the same gravitational potential, they must “fall” in the same way. However, it is possible to have momentum exchange between the components due to baryonic feature. This leads to a slight departure between them, which can be captured by simply assuming that $\theta_g = b_v \theta$. We stick with this for our computations at linear order, but at second order we assume $b_v = 1$. The reason for this is to make contact with the usual bias co-evolution relations found in the literature.

Now that we know the initial conditions for the tracer field we must write it according to Eq. (4.19), which yields:

$$\mathcal{I}_a^{(1)} = \begin{pmatrix} 1 \\ 1 \\ b_1^* \\ b_v^* \end{pmatrix}, \quad \mathcal{I}_a^{(2)} = \begin{pmatrix} F_2(\mathbf{k}_1, \mathbf{k}_2) \\ G_2(\mathbf{k}_1, \mathbf{k}_2) \\ b_1^* F_2(\mathbf{k}_1, \mathbf{k}_2) + b_2^*/2 \\ b_v^* G_2(\mathbf{k}_1, \mathbf{k}_2) \end{pmatrix}. \quad (4.41)$$

The linear solution is simply the application of the linear propagator to the linear initial conditions. The tracer density and velocity field are respectively given by

$$\mathcal{F}_3^{(01)} = \left(2 - 2e^{-\frac{\eta}{2}}\right) b_v^* + b_1^* + 2e^{-\frac{\eta}{2}} + e^\eta - 3, \quad (4.42)$$

$$\mathcal{F}_4^{(01)} = e^{-\frac{\eta}{2}} \left(b_v^* + e^{\frac{3\eta}{2}} - 1\right). \quad (4.43)$$

The first feature we notice from the pair of solutions above is the emergence of different

Figure 22: Time evolution of the linear bias. The presence of velocity bias has a considerable impact on b_1 . Extracted from [163].

time scaling. Remember that all these fields are linked to the matter, which is only in the growing mode e^η . So, if we factor out this term, the remaining time dependences are identified as *the time evolution of the bias coefficients*. According to Eqs. (4.42) and (4.43) these evolved bias factors are:

$$b_1 \equiv e^{-\eta} (2b_v^* + b_1^* - 3) + e^{-\frac{3\eta}{2}} (2 - 2b_v^*) + 1, \quad (4.44)$$

$$b_v \equiv e^{-\frac{3\eta}{2}} (b_v^* - 1) + 1. \quad (4.45)$$

Both Eqs. (4.44) and (4.45) can be inverted, so in the following expressions we write b_1^* in terms of b_1 . In Figure 22 we show how these coefficients evolve for different values of b_v^* and b_1^* . The presence of velocity bias implies that the tracer will be biased with respect to the matter at all times.

Let's now proceed to the second order solution. At this stage the computations become more lengthy, since the couplings γ_{abcd} mix everything. The solution for the matter density is simply:

$$\mathcal{F}_1^{(02)} = e^{2\eta} \left[\frac{\mu}{2} \left(\frac{k_2}{k_1} + \frac{k_1}{k_2} \right) + \frac{2\mu^2}{7} + \frac{5}{7} \right] = e^\eta F_2, \quad (4.46)$$

which is exactly the SPT prediction, as we expected (recall that, although we are evolving

Figure 23: The time evolution of the linear and velocity bias parameters for different initial conditions. Notice that the presence of velocity bias causes the tracer to be biased with respect to the matter for all time. This figure was extracted from [163].

a 2-component field, the matter does not feel the tracer). However, for the tracer overdensity field we find an interesting result. Before showing the solution, let's make a naive assumption. This solution is simply the gravitational evolution of Eq. (4.40), so we could be expecting to see something like

$$\delta_g^{(2)*} \rightarrow \delta_g^{(2)} = b_1 \delta^{(2)} + \frac{1}{2} b_2 (\delta^{(1)})^2. \quad (4.47)$$

That is, if the tracer only depends on δ and δ^2 at the initial conditions, this would still be valid at some time later. If that were true, then the *relative field* at second order,

$$\delta_{g,\text{rel}}^{(2)} = \delta_g^{(2)} - b_1 \delta^{(2)}, \quad (4.48)$$

should only be proportional to $\delta^{(1)}$ according to Eq. (4.47). However, when we expand

the relative field for the exact answer in Legendre polynomials we find that:

$$[\delta_{g,\text{rel}}^{(2)}]_{l=0} = \frac{b_2^*}{2} + \frac{4}{21}\epsilon_\delta + \frac{2}{21}\epsilon_v \left[3 + 14e^{\eta/2} - 14\epsilon_v + \frac{21\epsilon_\delta}{e^\eta - 1} \right], \quad (4.49)$$

$$[\delta_{g,\text{rel}}^{(2)}]_{l=1} = \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \epsilon_v \left[-1 + 2e^{\eta/2} + \frac{\epsilon_\delta}{e^\eta - 1} \right], \quad (4.50)$$

$$[\delta_{g,\text{rel}}^{(2)}]_{l=2} = -\frac{4}{21}\epsilon_\delta + \frac{4}{21}\epsilon_v \left[-12 + 14e^{\eta/2} + 7\epsilon_v \right], \quad (4.51)$$

where, following [163], we defined:

$$\begin{aligned} \epsilon_\delta &\equiv (b_1 - 1) e^\eta (e^\eta - 1), \\ \epsilon_v &\equiv (b_v - 1) e^\eta (e^{\eta/2} - 1). \end{aligned} \quad (4.52)$$

This result is extremely important for a better understanding of galaxy clustering. It simply says that, even if tracer formation depends only on $\{\delta^i\}$ during formation time, *the gravitational evolution naturally induces the dependence on non-local in matter operators at any time later*. This effect was first noted by [164, 165]. The monopole, dipole and quadrupole stand for spherical growth, bulk displacements and tidal field effects, respectively. These contributions to the tracer field normalized by the multipoles of the matter field are shown in Figure 23. The first thing we can notice from this figure is the fact that in the absence of velocity bias at the formation time, $b_v^* = 1$, the dipole vanishes for all later times. This is simply the equivalence principle in action, since in that case both tracer and matter feel the same potential, so there must be no relative displacement. The second interesting feature is that, as we increase b_v^* , the normalized monopole increases, whereas the quadrupole decreases. All multipoles present their maximum for lower values of a and asymptotically go to some constants as time goes by.

For a long time only simplistic models were used to describe galaxy statistics, such as the LIMD approximation [166]. In Section 2 we developed the theory by making use of effective theory arguments, so the relevant operators were identified by symmetry arguments. In this way, we concluded that besides the powers of the density field, new operators must be considered. Here we had the opportunity to witness the emergence

of a tidal field influence in a natural way. At the time when Ref. [166] appeared, such perturbation theory tools were not developed yet, so their solutions only encapsulate part of the total answer, namely the monopole (i.e. the spherical collapse solution). But we already knew that the tidal field should be included in the bias expansion, so even if it is a beautiful application of perturbation theory, we have found nothing new. The whole point of developing these equations is that they give us an important tool to relate many of the bias parameters. When performing data analysis, these bias parameters should be considered as completely free – at least in principle. Thus, having some theoretical priors would be of great use.

Figure 24: The points are measured values of the tidal field bias and the continuum red line is the co-evolution prediction, given by Eq. 4.54. Although the points are close to the co-evolution prediction, there is a consistent shift to more negative values. Extracted from [167].

To clearly see that this formalism generates such relations we get the second order solution for the tracer overdensity field and set $b_v^* = 1$. Now, the expression is much more simple,

$$\mathcal{F}_3^{(2)} = e^{2\eta} \left(b_1 \delta^2 - \frac{2}{7} (b_1 - 1) (\mu^2 - 1) e^{-\eta} (e^\eta - 1) \right). \quad (4.53)$$

As we showed in previous sections, the second term in the equation above is simply the tidal field in momentum space. Thus, all other multiplicative factors are effectively the tidal field bias at evolved time,

$$b_{K^2} = -\frac{2}{7} (b_1 - 1) (1 - e^{-\eta}) . \quad (4.54)$$

According to this result, all tracers with $b_1 \neq 1$ should have a bias tidal field at any time after their formation. This emergence of non-local in matter operators was first pointed out by [163], and the corresponding co-evolution was been extensively studied in [167, 168]. We refer the reader to the latter for a complete set of co-evolution relations up to third-order. They also included a tidal-field dependence in formation time. We could have done the same here by simply introducing it in Eq. (4.41), but we decided to stick with the simplistic approach. However, we point out that there are no strong physical reasons to neglect this tidal field at formation time [164, 165].

Before continuing, an important comment is in order. Eq. (4.54) has a clear time dependence. When changing back to the correct time dependence, we have $e^\eta \rightarrow D_+(\tau)/D_+(\tau_*)$, where $D_+(\tau_*)$ is the growth factor at formation time. Usually in the literature it is assumed that tracer formation occurs in the far past, so $D_+(\tau_*) \ll D_+(\tau)$. In this approximation, Eq. (4.54) becomes:

$$b_{K^2} = -\frac{2}{7} (b_1 - 1) . \quad (4.55)$$

In Fig. 24 we show the measured values of the tidal field bias as a function of the same tracer linear bias. The continuum red line is Eq. (4.55). During the last phases of this work, a new accurate measurement of the second-order biases was published [169], where the recently developed EFT-based forward modeling formalism was used [170]. Their results are shown in Figure 25, where we see that there is a partial agreement between Eq. (4.55) and the data. Notwithstanding, a shift to more negative values in some data sets is clear. Still, the most striking feature is the fact that any relaxation of $D_+(\tau_*) \ll D_+(\tau)$ shifts the red line to higher values. The more realistic situation thus produces a

worse agreement with data. In the next subsections we discuss how this problem can be circumvented by using the multi-tracer bias co-evolution relations.

Figure 25: A test of the bias co-evolution relation using the EFTofLSS Forward Modelling technique. A concise explanation of this tool can be found in [169], which is the same reference we extracted this figure from.

4.4 Massive tracers and bias co-evolution relations

The purpose of this work is to see how the multi-tracer technique can improve probes of the non-linear tracer dynamics. As we discussed in the previous Subsection, it is possible to derive theoretical links between higher-order bias parameters with the linear one, which can help us to improve the analysis we will perform later. Nevertheless, these relations were obtained by assuming that a *single-tracer population* suddenly appears in the universe. This assumption is tremendously simplistic for many reasons, and one of them is related to the fact that only a single tracer is considered. It is evident our universe is not composed of a single tracer population, but a plethora of them that interact gravitationally. This puts us in a position to propose the addition of more tracers in the co-evolution model and to jointly evolve them, considering their mutual interactions. However, notice

that if we extend the theory from the previous subsection to this more elaborate model we would simply end up with the same result as before but applied independently to each tracer. For instance, if we do the necessary changes and include the tracer B in the game we would have obtained that:

$$b_{K^2}^A = -\frac{2}{7}(b_1^A - 1), \quad (4.56)$$

$$b_{K^2}^B = -\frac{2}{7}(b_1^B - 1). \quad (4.57)$$

These equations may look fine, but consider the following. Take a box filled with matter and dark matter halos, and suppose that both tracers A and B are halo populations. What happens if the mass of, say, halo B starts to increase, whereas the mass of A is fixed? Since the total mass in the box is constant, this can only be the case if the continuous matter field loses mass in order to feed the halo B . As the halo mass increases, the linear biases increase as well, which in turn would cause $b_{K^2}^B$ to have a higher absolute value. However, according to Eq. (4.56), the tracer A appears not to feel any of that, since the bias relation remains the same. This insensitivity of the tracer A only occurs because in the last subsection, in order to derive the equations above, *both tracers A and B were treated as massless*. Since there are no gravitational interactions between them, they only respond to the matter field, which explains why the tracers are blind to each other.

What we propose in the following discussion is to allow the tracer field to contribute to the large-scale Poisson equation, and see how that impacts the co-evolution relation. In this way, the gravitational interactions between the tracers will be integrated since their time of formation, and can cause a cross-dependence in the bias relations – e.g. $b_{K^2}^A = -2/7(b_1^A - 1) + \alpha f(b_1^B, \tau)$, where α is some small parameter and f is some function of the bias parameters of the other tracer.

Treating the tracers as being massless is a customary assumption in literature – see, e.g., Refs. [122, 163]. In this last reference there is a comment on alleviating this assumption, stating that corrections due to the tracer mass are small for halos, and may be

completely negligible for galaxies. Since, as far as we know, there are no references in the literature explicitly showing these results, we decided to compute them. As we will show, there may be a misconception in treating the tracers as being massless, and it is associated with the fact that although the tracer mass corrections are indeed small at first, they can excite a growing mode, so even if the interaction is weak it can be relevant when integrated over sufficiently long times. The idea to work on this topic came to us during the final steps of the dissertation and it is not directly relevant for our main results, which we show in Section 5. Therefore, although the idea is still in its first steps, we decided to show some preliminary results. We start by revisiting the single-tracer case but including the tracer mass instead. Finally, we include two massive tracers and show how one can obtain a multi-tracer bias co-evolution relation.

4.4.1 Matter and one massive tracer

In Subsection 4.3 we explicitly derived all relevant results up to second order in perturbation theory. The only difference at this time will be the fact we also include the tracer mass into the Poisson equation, so its mass becomes relevant. The formalism is very much like the one we presented before, and for this reason we shall be more straightforward.

We are still considering two fluid components, so the matrix structures will be the same. The only difference appears in the propagator, since it contains information about the masses of the fluid components. If we set $\omega_{\text{tracer}} + \omega_{\text{matter}} = 1$, the propagator becomes:

$$G_{ab} = e^{-3\eta/2} \left[\mathcal{M}_{(-\frac{3}{2})} \right]_{ab} + e^{-\eta/2} \left[\mathcal{M}_{(-\frac{1}{2})} \right]_{ab} + [\mathcal{M}_{(0)}]_{ab} + e^{\eta} [\mathcal{M}_{(1)}]_{ab}, \quad (4.58)$$

with the matrices above being

$$[\mathcal{M}_{(-\frac{3}{2})}] = \begin{pmatrix} \frac{2}{5}(\Delta_m + 1) & \frac{1}{5}(-2)(\Delta_m + 1) & -\frac{1}{5}(2\Delta_m) & \frac{2\Delta_m}{5} \\ \frac{1}{5}(-3)(\Delta_m + 1) & \frac{3}{5}(\Delta_m + 1) & \frac{3\Delta_m}{5} & -\frac{1}{5}(3\Delta_m) \\ \frac{2}{5}(\Delta_m + 1) & \frac{1}{5}(-2)(\Delta_m + 1) & -\frac{1}{5}(2\Delta_m) & \frac{2\Delta_m}{5} \\ \frac{1}{5}(-3)(\Delta_m + 1) & \frac{3}{5}(\Delta_m + 1) & \frac{3\Delta_m}{5} & -\frac{1}{5}(3\Delta_m) \end{pmatrix}, \quad (4.59)$$

$$[\mathcal{M}_{(-\frac{1}{2})}] = \begin{pmatrix} 0 & 2\Delta_m & 0 & -2\Delta_m \\ 0 & -\Delta_m & 0 & \Delta_m \\ 0 & 2(\Delta_m + 1) & 0 & -2(\Delta_m + 1) \\ 0 & -\Delta_m - 1 & 0 & \Delta_m + 1 \end{pmatrix}, \quad (4.60)$$

$$[\mathcal{M}_{(0)}] = \begin{pmatrix} -\Delta_m & -2\Delta_m & \Delta_m & 2\Delta_m \\ 0 & 0 & 0 & 0 \\ -\Delta_m - 1 & -2(\Delta_m + 1) & \Delta_m + 1 & 2(\Delta_m + 1) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.61)$$

$$[\mathcal{M}_1] = \begin{pmatrix} \frac{3}{5}(\Delta_m + 1) & \frac{2}{5}(\Delta_m + 1) & -\frac{1}{5}(3\Delta_m) & -\frac{1}{5}(2\Delta_m) \\ \frac{3}{5}(\Delta_m + 1) & \frac{2}{5}(\Delta_m + 1) & -\frac{1}{5}(3\Delta_m) & -\frac{1}{5}(2\Delta_m) \\ \frac{3}{5}(\Delta_m + 1) & \frac{2}{5}(\Delta_m + 1) & -\frac{1}{5}(3\Delta_m) & -\frac{1}{5}(2\Delta_m) \\ \frac{3}{5}(\Delta_m + 1) & \frac{2}{5}(\Delta_m + 1) & -\frac{1}{5}(3\Delta_m) & -\frac{1}{5}(2\Delta_m) \end{pmatrix}. \quad (4.62)$$

In the matrices above we used $\Delta_m \equiv -\omega_{\text{tracer}}$. Notice that this linear propagator is very different from the case of a massless tracer, Eq. (4.34). However, if we set $\Delta_m = 0$ the $\mathcal{M}$ matrices above become identical to the $\mathcal{N}$ ones. Although the eigenvalues are the same in both the massless and massive cases, we obtain a different set of eigenmodes. For the

massive case they are:

$$\zeta^{(1)} = \begin{bmatrix} -\frac{2}{3} \\ 1 \\ -\frac{2}{3} \\ 1 \end{bmatrix} \rightarrow G \cdot \zeta_{(1)} = e^{-3\eta/2} \zeta_{(1)}, \quad \zeta^{(2)} = \begin{bmatrix} -\frac{2\Delta_m}{\Delta_m+1} \\ \frac{\Delta_m}{\Delta_m+1} \\ -2 \\ 1 \end{bmatrix} \rightarrow G \cdot \zeta_{(2)} = e^{-\eta/2} \zeta_{(2)}, \quad (4.63)$$

$$\zeta^{(3)} = \begin{bmatrix} \frac{\Delta_m}{\Delta_m+1} \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow G \cdot \zeta_{(3)} = 1 \zeta_{(3)}, \quad \zeta^{(4)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow G \cdot \zeta_{(4)} = e^\eta \zeta_{(4)}. \quad (4.64)$$

Recall that, at the time of tracer formation, a fraction of the matter must be removed from the matter field, and directed towards tracer formation. Hence, we expect a small change in the matter solution, and this is exactly what we find. The solution for the matter field at linear order is:

$$\delta^{(1)}(\tau, k) = \begin{cases} \delta^{(1)}(0, k) D(\tau), & \text{for } \tau < \tau_*, \\ \delta^{(1)}(0, k) \left[e^\eta - \Delta_m (b_1^* - 1) \left(1 - \frac{2}{5} e^{-3\eta/2} - \frac{3}{5} e^\eta \right) \right], & \text{for } \tau \geq \tau_* \end{cases} \quad (4.65)$$

This solution is quite interesting, and it is also a good example to help us understand what is happening in the co-evolution model, so let's make a brief review. The tracer field is assumed to instantaneously appear at $\tau = \tau_*$, and at that time it starts to contribute to the Poisson equation. Hence, we expect that before the tracer appears the matter solution at linear order will be the same as the one we found previously in Eq. (2.28), but with no decaying contribution. The decaying mode for this part of the solution is not present because we are using the same initial condition as we used in Subsection 4.3, which has only linear growth. This is exactly what we have in the solution above.

If there was no tracer, or if the tracer was considered to be massless, this would remain

Figure 26: The relative difference between the linear bias between the massive, in Eq. (4.67), and massless case, in Eq. (4.44). Here, b_1^* corresponds to the linear bias at formation time. We used $a_* = 0.2$ and $\omega_{\text{tracer}} = 0.12$, which is a reasonable estimation based on the halos obtained from the MultiDark 4Gpc box we use in Section 5.

a solution for all times. However, since we allowed a new tracer contribution to the Poisson equation, it also sources gravity, which in turn should somehow modify the matter dynamics. This correction is evident in the part of the solution for $\tau \geq \tau_*$. Also notice this piece-wise solution is continuous. Since $e^\eta = D/D_*$, at formation time we have $e^\eta = D_*/D_* = 1$, so the corrections proportional to Δ_m vanish. Notice that we used matter conservation, $\omega_{\text{matter}} + \omega_{\text{tracer}} = \omega_{\text{matter}} - \Delta_m = 1$, to eliminate one of the ω 's, which is why we only have Δ_m in the solution.

In a similar way, the linear solution for the overdensity of the tracer field is

$$\delta_g = \begin{cases} 0, & \text{for } \tau < \tau_*, \\ \delta^{(1)}(0, k) \left[\left(1 - \frac{2}{5}e^{-3\eta/2} - \frac{2}{5}e^\eta\right) (b_1^* - 1) \Delta_m + b_1^* + e^\eta - 1 \right], & \text{for } \tau \geq \tau_*. \end{cases} \quad (4.66)$$

In linear theory we expect the tracer field to simply be a linear response with respect to the matter, whose linear response coefficient is the linear bias. Thus, this value can be

obtained simply by taking the ratio of the evolved fields,

$$b_1(\tau) = \frac{\delta_g^{(1)}(\tau, k)}{\delta^{(1)}(\tau, k)} = 1 + \frac{b_1^* - 1}{e^\eta + \left(1 - \frac{2}{5}e^{-3\eta/2} - \frac{3}{5}e^\eta\right)(b_1^* - 1)\Delta_m}. \quad (4.67)$$

It is evident that if we set $\Delta_m = 0$ in Eq. (4.67) it becomes exactly Eq. (4.44), i.e., the massless case.

In Figure 26 we show the relative difference between the linear biases for the massive and massless scenarios. In that figure we used $\Delta_m = -\omega_{\text{tracer}} = -0.10$, which is an estimation for halos obtained from the the MultiDark N-body simulation²⁵. We vary the value of b_1^* , and since the mass contribution scales with $b_1^* - 1$, we obtain a stronger effect as the linear bias at tracer formation deviates from unity. However, the mass corrections at linear order are really negligible, giving at most a $\sim 2.5\%$ deviation from the massless case. These contributions increase as we decrease a_* , which makes sense – older tracers will have more time for these integrated effects to grow. It is also clear that all mass corrections lead to a linear bias in the massive case which is smaller than the massless case.

The exact solutions at second order are intricate, so there is no point in completely displaying them here. Thus we propose the convenient approximation to treat Δ_m perturbatively. This is not only to make the equations more friendly, but this is also well motivated, since the relative contribution of the tracer mass to the whole mass budget is indeed small (as we said before, it is about 10% for all the halos we have considered in the MultiDark N-body simulation).

The introduction of massive tracers impact the time evolution of the matter tidal field. The time evolution changes from $D_{K^2} = e^{2\eta} = D^2(\tau)/D_*^2$, the usual SPT second order

²⁵We considered halos with mass higher than $13.2 \log(M/M_\odot)$. See the halo $A + B$ in Table 3 for more details.

Figure 27: Time scaling of the matter tidal field in the massless and massive tracers situations. We assumed tracer formation occurred at $a_* = 0.2$.

time evolution, to $\tilde{D}_{K^2}$, given by:

$$\begin{aligned} \tilde{D}_{K^2} = & e^{2\eta} - \frac{1}{5}((b_1 - 1)e^{-\eta/2}(e^{\eta/2} - 1)^2(-8e^{\eta/2} + 2e^\eta + 7e^{3\eta/2} + \\ & + 12e^{2\eta} + 6e^{5\eta/2} - 4))\Delta_m + O(\Delta_m^2). \end{aligned} \quad (4.68)$$

In Figure 27 we show how the massive tracers impact the time evolution of the matter tidal field. The bias relations are also intricate, so we treat Δ_m perturbatively. Up to first order in Δ_m we obtain:

$$\begin{aligned} b_{K^2} = & -\frac{2}{7}(b_1 - 1)(1 - e^{-\eta}) + \overbrace{\frac{2}{35}(b_1 - 1)^2 e^{-\frac{7\eta}{2}}(e^{\eta/2} - 1)^3}^{\text{mass correction}} \\ & \times \underbrace{\left(12e^{\eta/2} + 12e^\eta + 9e^{\frac{3\eta}{2}} + 3e^{2\eta} + 4\right)}_{\text{mass correction}} \Delta_m. \end{aligned} \quad (4.69)$$

The expressions for the higher order solutions with the correct time dependence are quite cumbersome, so we avoid displaying them explicitly. We compare the predictions of

Figure 28: The bias co-evolution relations for the tidal field bias, b_{K^2} . The data points were extracted from Figure 24 and are at $z = 0$. The usual relation used in the literature is the solid red line, with the formation in the far past. When tracer formation is assumed to occur at $D_* = 0.1$, the agreement with data points is worse. In blue, we show the mass-corrected relations we derived. The solid line assumes time formation at $D_* = 0.1$, whereas the dotted line assume that the time of formation is in the far past. In both cases, the agreement with data is better than the usual massless LIMD relation.

Figure 29: The bias co-evolution for the tracer formation in the far past case. The data points were extracted from the upper left panel of Figure 25. In red we show the usual expression used in the literature, whereas in blue we include the mass corrections we derived. These corrections scales with $(b_1 - 1)^{n+1} \Delta_m^n$, as is shown in Eq. (4.70). The $n = 3$ correction produces excellent agreement with the data points. Here we used $\Delta_m = -0.15$.

n	0	1	2	3
a_n	$-\frac{2}{7}$	$\frac{6}{35}$	$-\frac{278}{175}$	$\frac{579}{350}$

Table 2: The matter correction coefficients for the b_{K^2} co-evolution relation up to third order in Δ_m , to be used in Eq. (4.70).

Eq. (4.69) with data in Figure 28. The data points in this figure were extracted from [167]. It is evident that the mass corrections lead to better agreement with data, regardless of whether we assume that tracer formation occurred in far past or not.

When we assume tracer formation in the far past, the equations are more elegant, and a pattern appears. The mass-corrected bias co-evolution for b_{K^2} can be written as:

$$b_{K^2}(b_1) = \sum_{n=0} a_n (b_1 - 1)^{n+1} (\Delta_m)^n, \quad (4.70)$$

where a_n are coefficients to be determined by solving the equations. We computed these coefficients up to third order in Δ_m , and they are shown in Table 2. We show in Figure 28 how these corrections up to $n = 3$ perform against data extracted from [169]. Again, the mass corrections appears to fix the massless LIMD result.

4.4.2 Matter and two massive tracers

Now we include a second tracer in our model. If we neglect the masses of the tracers in the co-evolution model, there will be no interaction between the tracers. In this manner, the derived bias co-evolution relations for tracers A and B would be simply the usual massless result applied to each tracer individually,

$$b_{K^2, \text{LIMD}}^A = - (b_1^A - 1) \frac{2}{7} e^{-\eta} (e^\eta - 1) \equiv - (b_1^A - 1) f_0(\eta), \quad (4.71)$$

$$b_{K^2, \text{LIMD}}^B = - (b_1^B - 1) \frac{2}{7} e^{-\eta} (e^\eta - 1) \equiv - (b_1^B - 1) f_0(\eta). \quad (4.72)$$

The reason for the definition above will become clear in a moment.

We start by solving all equations analytically and later, after deriving the full bias co-evolution relation, we treat ω_A and ω_B perturbatively.

The multi-tracer co-evolution propagator can be written as:

$$G_{ab} = e^{-3\eta/2} [\mathcal{T}_{(-\frac{3}{2})}]_{ab} + e^{-\eta/2} [\mathcal{T}_{(-\frac{1}{2})}]_{ab} + [\mathcal{T}_{(0)}]_{ab} + e^{\eta} [\mathcal{T}_{(1)}]_{ab}. \quad (4.73)$$

The situation is very much like the ones we showed previously, the only difference being the specific form of the matrices. To avoid clutter we decided to show these matrices in Appendix B.

The initial conditions for the multi-tracer scenario can be obtained following the same reasoning as in the previous case. It is easy to realize that, at second order, it must be that:

$$\delta_A^{(1+2)}(\mathbf{x}_*, \tau_*) = b_1^{A*} [\delta^{(1)} + \delta^{(2)}](\mathbf{x}_*, \tau_*) + \frac{1}{2} b_2^{A*} [(\delta^{(1)})_*]^2, \quad (4.74)$$

$$\delta_B^{(1+2)}(\mathbf{x}_*, \tau_*) = b_1^{B*} [\delta^{(1)} + \delta^{(2)}](\mathbf{x}_*, \tau_*) + \frac{1}{2} b_2^{B*} [(\delta^{(1)})_*]^2, \quad (4.75)$$

so in matrix notation they become

$$\mathcal{I}_a^{(1)} = \begin{pmatrix} 1 \\ 1 \\ b_1^{A*} \\ 1 \\ b_1^{B*} \\ 1 \end{pmatrix}, \quad \mathcal{I}_a^{(2)} = \begin{pmatrix} F_2(\mathbf{k}_1, \mathbf{k}_2) \\ G_2(\mathbf{k}_1, \mathbf{k}_2) \\ b_1^{A*} F_2(\mathbf{k}_1, \mathbf{k}_2) + b_2^{A*}/2 \\ G_2(\mathbf{k}_1, \mathbf{k}_2) \\ b_1^{B*} F_2(\mathbf{k}_1, \mathbf{k}_2) + b_2^{B*}/2 \\ G_2(\mathbf{k}_1, \mathbf{k}_2) \end{pmatrix}. \quad (4.76)$$

The first non-trivial result shows up in linear theory, where the evolved linear biases of tracers A and B are:

$$b_1^A = 1 + \frac{5e^{\frac{3\eta}{2}} (b_1^{A*} - 1)}{2\alpha_{AB} - 5e^{\frac{3\eta}{2}} \alpha_{AB} + e^{\frac{5\eta}{2}} (3\alpha_{AB} + 5)}, \quad (4.77)$$

$$b_1^B = 1 + \frac{5e^{\frac{3\eta}{2}} (b_1^{B*} - 1)}{2\alpha_{AB} - 5e^{\frac{3\eta}{2}} \alpha_{AB} + e^{\frac{5\eta}{2}} (3\alpha_{AB} + 5)}, \quad (4.78)$$

with

$$\alpha_{AB} \equiv \omega_A (b_1^{A*} - 1) + \omega_B (b_1^{B*} - 1). \quad (4.79)$$

Hereafter we refer to α_{AB} as the *multi-tracer effective coupling*, a quantity that is determined at formation time. Notice that we recover the usual Eq. (4.44) with $b_v^A = b_v^B = 1$ as $\alpha_{AB} \rightarrow 0$. We can simultaneously invert Eqs. (4.77, 4.78) and substitute all b_1^{A*} and b_1^{B*} in terms b_1^A and b_1^B , so the evolved linear bias of tracer A depends on the tracer B and vice-versa,

$$b_1^{A*} = \frac{1}{5}(b_1^A - 1) \left((3e^\eta - 5) + 2e^{-\frac{3\eta}{2}} \right) \alpha_{AB} + (b_1^A - 1)e^\eta + 1, \quad (4.80)$$

$$b_1^{B*} = \frac{1}{5}(b_1^B - 1) \left((3e^\eta - 5) + 2e^{-\frac{3\eta}{2}} \right) \alpha_{AB} + (b_1^B - 1)e^\eta + 1. \quad (4.81)$$

With that information in hand we can proceed to take a look at the second order solutions. From now on, ω_A and ω_B will be treated perturbatively, and both will be considered to be of the same order. This is well motivated, since in realistic scenarios the tracer contribution to the total matter is really small. Proceeding in the same fashion as before, the bias co-evolution relation for b_{K^2} can be obtained from the quadrupole term of the relative tracer field. At first order we obtain

$$b_{K^2}^A = - (b_1^A - 1) [f_0(\eta) + f_1(\eta)\alpha_{AB}], \quad (4.82)$$

$$b_{K^2}^B = - (b_1^B - 1) [f_0(\eta) + f_1(\eta)\alpha_{AB}], \quad (4.83)$$

where the first term, f_0 , accounts for the zeroth order expressions Eqs. (4.71, 4.72). The first order time component was computed to be

$$f_1(\eta) = \frac{4}{35}e^{-5\eta/2} \left(9e^{\eta/2} + 11e^\eta + 9e^{3\eta/2} + 3e^{2\eta} + 3 \right) \left(e^{\eta/2} - 1 \right)^3. \quad (4.84)$$

This pattern appears to be valid up to any order in α_{AB} . Hence, we propose the following ansatz:

$$b_{K^2}^A = - (b_1^A - 1) \sum_n f_n(\eta) \alpha_{AB}^n, \quad (4.85)$$

$$b_{K^2}^B = - (b_1^B - 1) \sum_n f_n(\eta) \alpha_{AB}^n. \quad (4.86)$$

Up to third order in the multi-tracer effective coupling coupling we obtain:

$$f_2(\eta) = \frac{1}{350} e^{-4\eta} \left(e^{\eta/2} - 1 \right)^4 \left(-192e^{\eta/2} - 347e^\eta - 308e^{3\eta/2} + 130e^{2\eta} + 588e^{5\eta/2} + \right. \\ \left. 687e^{3\eta} + 432e^{7\eta/2} + 108e^{4\eta} - 48 \right) \quad (4.87)$$

$$f_3(\eta) = \frac{1}{875} e^{-11\eta/2} \left(e^{\eta/2} - 1 \right)^6 \left(4e^{\eta/2} + 6e^\eta + 3e^{3\eta/2} + 2 \right) \left(-96e^{\eta/2} - 163e^\eta - 112e^{3\eta/2} \right. \\ \left. + 170 + e^{2\eta} + 444e^{5\eta/2} + 471e^{3\eta} + 288e^{7\eta/2} + 72e^{4\eta} - 24 \right) . \quad (4.88)$$

As we stated previously, these equations were derived in the last phase of this dissertation, so we did not have the opportunity to further develop and test these ideas. Nevertheless, we consider these results to be very interesting since, as we showed in Figures 28 and 29, they lead to the desired corrections to the usual bias co-evolution relations.

4.5 Other bias relations

Although we only derived the bias co-evolution relation for the b_{K^2} bias parameter, we will consider other relations in our analysis. The previous calculations were performed only up to second order in perturbation theory, but its extension to third order is trivial. The bias co-evolution for the only third order operator contributing to the 1-loop tracer power spectrum, b_{Γ_3} , is:

$$b_{\Gamma_3} = \frac{23}{42} (b_1 - 1) . \quad (4.89)$$

In addition, we will also consider a numerical fit for b_2 obtained using the separate Universes approach [171]:

$$b_2 = 0.412 - 2.143 b_1 + 0.929 (b_1)^2 + 0.008 (b_1)^3 . \quad (4.90)$$

5 Results

In this last section we present all results of the theoretical model previously introduced. We begin with a simple Fisher matrix application that can be used to derive the multi-tracer power spectrum covariance matrix we will use in the analysis. We then proceed to explaining the specifications of the data we use and right after that we discuss the MCMC setup. Finally, we end the section displaying our main results. The discussions are made in the next section.

5.1 The Multi-tracer covariance matrix

We now present a simplified derivation of the multi-tracer Fisher matrix, under the assumption of Gaussianity – i.e., that the power spectrum covariance, which is a 4-point function, can be reduced to combinations of two-point functions using Wick’s theorem. For simplicity, we write the fundamental degrees of freedom as the Fourier modes of the density contrast of the tracers, together with their complex conjugates:

$$d_i^a(\vec{k}) = \{\tilde{\delta}_i(\vec{k}), \tilde{\delta}_i^*(\vec{k})\}, \quad (5.1)$$

where the index a denotes the Fourier mode or its complex conjugate, and i denotes the tracer species. In this form, the data covariance takes the simple form:

$$\langle d_i^a(\vec{k}) d_j^b(\vec{k}') \rangle = D^{ab} C_{ij}(\vec{k}, \vec{k}') = C_{ij}^{ab}(\vec{k}, \vec{k}'), \quad (5.2)$$

where $D^{ab} = 1 - \delta^{ab}$ and the correlation function in Fourier space is:

$$C_{ij}(\vec{k}, \vec{k}') = \delta_{\vec{k}\vec{k}'} (P_{ij} + \delta_{ij} s_i). \quad (5.3)$$

Here $P_{ij}(\vec{k})$ is the power spectrum for tracers i, j , and s_i is the shot noise term, which for Poisson statistics is simply $s_i = 1/\bar{n}_i$, with $\bar{n}_i$ being the number density of the species i . In this context, it is worth stressing that the core of the multi-tracer technique is the fact that cosmic variance is a correlated noise for all tracers, which is expressed by Eqns. (5.2)-(5.3).

The Fisher matrix for some set of observables θ^μ , measured in a volume V and over a band-

Figure 30: The measurements of all the bias parameters for both the multi-tracer setup, with tracers A and B , and the single-tracer setup, dubbed $A + B$.

power (Fourier bin) $\tilde{V}_k$, is given by the trace:

$$F_{\mu\nu} = \frac{1}{4} V \tilde{V}_k \sum_{ijkl} \sum_{abcd} [C_{ij}^{ab}]^{-1} \frac{\partial C_{jk}^{bc}}{\partial \theta^\mu} [C_{kl}^{cd}]^{-1} \frac{\partial C_{li}^{da}}{\partial \theta^\nu}, \quad (5.4)$$

where the extra factor of $1/2$ of the Fisher matrix is due to the fact that the Fourier modes were counted twice when we constructed our degrees of freedom with the Fourier modes and their complex conjugates. Now, using $[C_{ij}^{ab}]^{-1} = [D^{ab}]^{-1} [C_{ij}]^{-1}$, and taking into account that $[D^{ab}]^{-1} = D^{ab}$, the second trace in the equation above gives us back the extra factor of two, and we arrive at:

$$F_{\mu\nu} = \frac{M_k}{2} \sum_{ijkl} [C_{ij}]^{-1} \frac{\partial C_{jk}}{\partial \theta^\mu} [C_{kl}]^{-1} \frac{\partial C_{li}}{\partial \theta^\nu}, \quad (5.5)$$

where $M_k = V \tilde{V}_k$ is the number of modes of the bandpower k in the volume, and

$$C_{ij}^{-1} = \frac{\delta_{ij}}{s_i} - \frac{1}{s_i} \frac{P_{ij}}{1 + \mathcal{P}} \frac{1}{s_j},$$

with $\mathcal{P} = \sum_i P_{ii}/s_i$.

We project this Fisher matrix into the set of bandpowers of all the auto- and cross-spectra, $\{P_{ij}\}$. The partial derivatives in Eq. (5.5) can be written as:

$$\frac{\partial C_{ij}}{\partial P_{kl}} = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \delta_{ij} \delta_{jk} \delta_{kl} \delta_{li},$$

which results in the following expression for the Fisher matrix:

$$F[P_{ij}, P_{kl}] = F_{ij,kl} = \frac{M_k}{4} (2 - \delta_{ij}) (2 - \delta_{kl}) \left(C_{ik}^{-1} C_{jl}^{-1} + C_{il}^{-1} C_{jk}^{-1} \right). \quad (5.6)$$

Since the spectra are symmetric, $P_{ij} = P_{ji}$, this Fisher matrix has redundancies which must be eliminated, so we restrict the degrees of freedom for tracers A, B, C , etc., to

$$\theta^\mu \rightarrow \{P_{AA}, P_{AB}, P_{AC}, \dots, P_{BB}, P_{BC}, \dots\}.$$

Restricting the Fisher matrix to these degrees of freedom and inverting that expression we obtain

the completely general covariance matrix:

$$C_{ij,kl} = \frac{1}{M_k} (C_{ik}C_{jl} + C_{il}C_{jk}) , \quad (5.7)$$

which results in the following terms for two tracers A and B :

$$C_{AA,AA} = \frac{1}{M_k} 2P_{AA}^2 N_A^2, \quad (5.8)$$

$$C_{AB,AB} = \frac{1}{M_k} P_{AB}^2 + \frac{1}{N_k} P_{AA} P_{BB} N_A N_B, \quad (5.9)$$

$$C_{AA,AB} = \frac{1}{M_k} 2P_{AB} P_{AA} N_A, \quad (5.10)$$

$$C_{AA,BB} = \frac{1}{M_k} 2P_{AB}^2, \quad (5.11)$$

where P_{AA} , P_{BB} and P_{AB} are the fiducial spectra and $N_i = 1 + s_i/P_{ii} = 1 + 1/(\bar{n}_i, P_{ii})$ with the assumption of Poissonian shot noise.

5.2 Simulation and data specifications

In order to test and compare the validity of the theoretical models constructed in Section 3 we perform an analysis in real space (i.e., neglecting redshift-space distortions) using dark matter halos extracted from the MultiDark N-body simulation [172]. We use the 4 Gpc/ h box at $z = 0$, with fiducial cosmological parameters $\Omega_m = 0.307$, $\Omega_b = 0.048$, $\Omega_\Lambda = 0.693$, $h = 0.678$, $n_s = 0.96$ and $\sigma_8 = 0.829$.

The most natural choice to construct distinct populations of tracers is divide the halos by mass ranges. In this work we split the halo catalogue in two populations, A and B , with masses just below and just above $13.5 \log M_\odot/h$, respectively. We have also limited the mass range for the two populations, so as to end up with a combined tracer (the union of A and B) which has similar properties compared with its sub-populations. More details on the mass ranges and number densities of those halos can be found in Table 3. For simplicity, in this work we have limited our analysis to these two tracer species, since a sub-division into more tracers would lead to an even larger number of bias parameters.

Figure 31: The measurements of the stochastic and higher derivative parameters for both the multi-tracer setup, having tracers A and B , and the single-tracer setup, dubbed $A + B$.

Halo Data set	Mass range [$\log M_\odot/h$]	$\bar{n}[(Mpc/h)^{-3}]$	number count
Halo A	[13.2, 13.5]	1.44×10^{-4}	9.21×10^6
Halo B	[13.5, 15.7]	1.23×10^{-4}	7.90×10^6
Halo A + B	[13.2, 15.7]	2.67×10^{-4}	1.71×10^7

Table 3: Halo populations used in the analysis. The two sub-catalogues A and B have similar number counts.

The halo cross- and auto-power spectra were measured using a simple quadratic estimator²⁶,

$$P_{XY}(k_i) = \frac{1}{M_{k_i}} \sum_{k \in k_i} \tilde{\delta}_X(k) \tilde{\delta}_Y^*(k), \quad (5.12)$$

where the bandpower k_i is defined by $|k - k_i| < \Delta k/2$, and M_{k_i} is the number of modes in that bandpower. We employed a grid with 512 cells per dimension (i.e., cells of $\ell = 7.8125 h^{-1}$ Mpc), and computed 64 bins with $\Delta k = 4\pi/\ell$, ranging from the fundamental mode to the Nyquist frequency ($k_0 = 2\pi/\ell$).

As for the fiducial model, although the cosmological parameters were fixed in the N-body simulation, this is not the case for the linear biases or the stochastic parameters. For the purposes of determining a reference set of parameters, we performed an MCMC analysis considering the following model:

$$P_{XX} \stackrel{k \rightarrow 0}{=} (b_1^X)^2 P_0 + \alpha_{XX} k^2 P_0 + \left(\frac{1}{\bar{n}_X} \right) c_0^X, \quad (5.13)$$

$$P_{XY} \stackrel{k \rightarrow 0}{=} (b_1^X b_1^Y) P_0 + \alpha_{XY} k^2 P_0 + \frac{1}{2} \left(\frac{1}{\bar{n}_X} + \frac{1}{\bar{n}_Y} \right) c_0^{XY}, \quad (5.14)$$

where the α parameters were introduced in order to absorb the non-linear contributions. We used in that case $k_{\max} = 0.15 h/\text{Mpc}$ and the results are displayed in Table 4.

Although the predicted value for this term is quite small when compared to other stochastic parameters, we decided to include it in our more conservative theoretical setup.

In the left panel of Fig. 32 we show the power spectra for both tracers, normalized by the linear spectra. It is evident that for intermediate values of k the two tracers already start to deviate. These different shapes are, in fact, the source for the additional information that can be obtained

²⁶The credit for this measurement goes to Rodrigo Voivodic and his code, <https://github.com/Voivodic>.

	b_1	$\tilde{C}_0^X$	$\tilde{C}_{0,\text{cross}}^{XY}$
HALO A	1.3307 ± 0.0091	0.142 ± 0.066	-0.044 ± 0.066
HALO B	1.8740 ± 0.011	-0.257 ± 0.093	Same as above
HALO A + B	1.5806 ± 0.0088	-0.1 ± 0.13	-

Table 4: Fiducial linear biases obtained by fitting Eqs. (5.13, 5.14) to the low- k range of all tracer power spectrum.

Figure 32: On the left, the power spectrum (normalized by the linear spectrum) for the tracers A and B, the cross spectrum (AB) and the auto-spectrum of the full set of halos (A+B). On the right, the dimensionless power spectrum for each tracer, defined in Eq. (5.15)

by splitting the tracers into multiple distinct populations²⁷. We also display in the same figure the cross-power spectrum (AB), as well as the auto-power spectrum of the combined halos ($A + B$).

On the right panel of Fig. 32 we show the dimensionless spectra defined by

$$\Delta_{XX} = \frac{k^3 P_{XX}}{2\pi^2}. \quad (5.15)$$

This plot evidences a further advantage of splitting tracers into different populations: the tracers A and B reach the non-linear regime ($\Delta > 1$) at different scales, with the more massive and more highly biased tracer (B) becoming non-linear at larger scales compared with either the less massive and less biased tracer (A), or with the combination of the two. In terms of the EFT expansion, this means that the counter-terms for tracer B should become important for smaller values of k , compared with those for tracer A. Therefore, splitting tracers into sub-populations can in principle

²⁷Note that the efficiency of multi-tracer analysis in extracting information from, e.g., redshift-space distortions or non-gaussianities, relies in the same argument: it is only when the shapes of the spectra are different that we can beat down cosmic variance [135].

lead to greater control of the parametric importance of the UV sector.

5.3 Parameter extraction and data covariance matrix

Parameter extraction is performed by standard Bayesian inference. The likelihood is assumed to be Gaussian,

$$\ln \mathcal{L} = -\frac{1}{2}\chi^2, = -\frac{1}{2}(\mathcal{P} - \mathcal{P}_{\text{data}})^t \cdot [\mathcal{C}]^{-1} \cdot (\mathcal{P} - \mathcal{P}_{\text{data}}), \quad (5.16)$$

where $\mathcal{P}_{\text{data}}$ and $\mathcal{P}$ are the multi-dimensional data and theory vectors, respectively, and $\mathcal{C}$ is the data covariance matrix. For the multi-tracer setup we have the data vector $\mathcal{P} = (\vec{P}_{AA}, \vec{P}_{AB}, \vec{P}_{BB})$, where arrows represent vectors in the sense of the list of N_k bandpowers,

$$\vec{P}_{XX} = \{P_{XX}(k_1), \dots, P_{XX}(k_{N_k})\}.$$

In this way, the covariance matrix will be a $3N_k \times 3N_k$ square matrix constructed as:

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}_{AA,AA} & \mathcal{C}_{AA,AB} & \mathcal{C}_{AA,BB} \\ \mathcal{C}_{AB,AA} & \mathcal{C}_{AB,AB} & \mathcal{C}_{AB,BB} \\ \mathcal{C}_{BB,AA} & \mathcal{C}_{BB,AB} & \mathcal{C}_{BB,BB} \end{bmatrix}, \quad (5.17)$$

where $\mathcal{C}_{ij,kl}$ are computed using Eqs. (5.8 - 5.11).

Parameter	Prior
ω_{cdm}	Flat [0.095, 0.14]
h	Flat [0.6, 0.75]
A	Flat [1.49, 2.8]

Table 5: Priors in the cosmological parameters and linear bias. They are wide enough with respect to the typical errors we find. All upcoming analysis will be performed using these priors for the cosmological parameters.

The posterior was sampled using the `emcee`[173] Python package with the standard move proposal²⁸. The convergence was monitored using the Gelman-Rubin criteria with three parallel

²⁸This move proposal has a free parameter, the *stretch scale*, denoted by a in the `emcee` documentation. The default value is $a = 2.0$. However, during our analysis, we concluded it must be changed to smaller values in order to obtain a satisfactory acceptance ratio. For the single-tracer case the typical values were $a \approx 1.5 - 1.7$, whereas for multi-tracer we used $a \approx 1.3 - 1.5$.

Figure 33: The residuals of the multi-tracer model with the maximum-likelihood set of parameters. The model proved to consistently recover the data. A similar conclusion remains valid when we consider higher values of k_{max} .

chains. For the scale reduction factor we choose $\epsilon = 0.03$. After having converged, the chain analysis was performed using the `getdist` library [174].

An essential part of Bayesian parameter inference is the choice of priors. For the cosmological parameters that we try to measure, $\{\omega_{cdm}, h, A\}$, we employ the same set of priors in all scenarios, which are detailed in Table 5. For each set of cosmological parameters we need to compute P_0 , P_1 and all $\mathcal{I}$ and $\mathcal{F}$ terms contributing to the 1-loop power spectrum. Since the numerical integrations are quite slow, we boosted the speed of the parameter space exploration by Taylor expanding these spectral corrections, as described in Appendix A

For the bias and stochastic parameters we considered three prior setups. The first one is a conservative flat prior over all parameters (3.45), denoted by *Flat* in Table 6. In this work we also study how the single- and multi-tracer setup responds to the bias co-evolution, which motivate us to introduce two additional priors formed by a mixture of flat and Gaussian priors, denoted by $G_0(\sigma)$ and $G_{\text{co-ev}}(\sigma)$ in Table 6. They use the same flat prior as the *Flat* setup for most parameters, with the exception of b_2 , $b_{\mathcal{G}_2}$ and b_{Γ_3} , for which we use Gaussian priors instead. In the $G_0(\sigma)$ priors the Gaussian PDFs have a variance σ , and are centered around zero, whereas in the

Figure 34: *top panel*: The running of the 68% confidence intervals of the sampled posterior with and without the multi-tracer technique. The values were normalized by the fiducial parameters used in the N-body simulations, so a measurement consistent in 1- σ must have the dashed black line inside it. The prior for both models is the *Flat*, as described in Table 6. *bottom panel*: We explicitly show the size of the error bars as a function of k_{max} .

$G_{\text{co-ev}}(\sigma)$ priors the Gaussians are centered around the co-evolution bias relations introduced in Eqs. (4.54). Notice that the variance of the Gaussian prior on b_{Γ_3} has an additional factor of 2 with respect to the Gaussian prior on b_2 and $b_{\mathcal{G}_2}$. This is because the constraints on b_{Γ_3} are typically weaker compared with b_2 and $b_{\mathcal{G}_2}$. We stress that this factor is purely phenomenological, and was chosen after an analysis of our Markov chains. Therefore, the sets of priors G_0 and $G_{\text{co-ev}}$ have a parameter σ which we can vary, obtaining more stringent constraints as we increase our prior

confidence on the range over which the EFT parameters b_2 , $b_{\mathcal{G}_2}$ and b_{Γ_3} are allowed to vary.

Parameter	Prior <i>Flat</i>	Parameter	Prior $G_0(\sigma)$
b_1	Flat [1.0, 2.2]	b_1	Flat [1.0, 2.2]
b_2	Flat [-5.0, 5.0]	b_2	Gauss.(0, σ)
$b_{\mathcal{G}_2}$	Flat [-5.0, 5.0]	$b_{\mathcal{G}_2}$	Gauss(0, σ)
b_{Γ_3}	Flat [-10.0, 10.0]	b_{Γ_3}	Gauss(0,2 σ)
$b_{\nabla^2\delta}$	Flat [-50, 50]	$b_{\nabla^2\delta}$	Flat [-50, 50]
c_0	Flat [-5.0, 5.0]	c_0	Flat [-5.0, 5.0]
c_2	Flat [-50, 50]	c_2	Flat [-50, 50]

Parameter	Prior $G_{\text{co-ev}}(\sigma)$
b_1	Flat [1.0, 2.2]
b_2	Gauss.(Eq. 4.90, σ)
$b_{\mathcal{G}_2}$	Gauss.(Eq. 4.54, σ)
b_{Γ_3}	Gauss.(Eq. 4.89,2 σ)
$b_{\nabla^2\delta}$	Flat [-50, 50]
c_0	Flat [-5.0, 5.0]
c_2	Flat [-50, 50]

Table 6: Prior over the bias and stochastic parameters considered in our analysis for the tracers A and B .

Before proceeding to our main results we have one last comment about the priors just presented. As we will show in a moment, the multi-tracer setup has a better performance than the single-tracer in the full-shape power spectrum analysis we performed. One could conclude that this improvement comes from the fact that in the multi-tracer case we use the priors twice (once for each tracer), whereas in the single-tracer it is used only once. In fact, from Eq. (3.48) we see that a prior the individual tracers implies a prior on the combined tracer that is given by:

$$(\bar{n}_A + \bar{n}_B)^2 \sigma_{b,A+B}^2 = \bar{n}_A^2 \sigma_{b,A}^2 + \bar{n}_B^2 \sigma_{b,B}^2. \quad (5.18)$$

Therefore, if the individual tracers have approximately equal densities, a prior $\sigma_{b,A} = \sigma_{b,B}$ would be equivalent to a prior $\sigma_{b,A+B} \simeq 0.71 \sigma_{b,A}$. We took this factor into consideration in all our results. For the *Flat* prior setup we checked that the ranges of the parameters are wide enough for this effect to be negligible. For the Gaussian prior setups we adopted a rigorous approach, and compensated the single-tracer σ by the factor computed in the equation above. Hereafter, we will refer to the subscript $\sigma = \sigma_{A+B}$, but bear in mind that we will be using the procedure when comparing the two-tracer and single-tracer results.

Figure 35: *Top panel:* The running of the 68% confidence intervals of the sampled posterior with and without the multi-tracer technique. We kept $k_{\text{max}} = 0.22h\text{Mpc}^{-1}$ and varied the σ . The values were normalized by the fiducial parameters used in the N-body simulations, so a measurement consistent in 1- σ must have the dashed black line inside it. The prior for both models are described in Table 6. *Bottom panel:* We explicitly show the size of the error bars as a function of σ .

5.4 Comparing the constraint power of single- and multi-tracer bias expansion

We started with a sanity check. The large-scale values for b_1 and c_0 , in Table 4 should at least partially agree with the one obtained when performing an analysis including information from the smallest scales. In Figures 30 and 31 we show our results of this exercise when using the *Flat*

prior setup. The colored red regions represent the large-scale values, whereas the other regions are the results obtained when including the information from the smallest scales. The agreement between these two approaches is clear, which indicates that our analysis seems to be consistent.

For the sake of completeness we also show in Figure 33 the residuals of the fits performed when we used the multi-tracer (MT) setup to recover the data presented in Figure 32. We can see that the MT presents a good description to data, but this is not a big surprise since we have a lot of free parameters.

As previously discussed, this analysis will have two distinct parameters that must be studied: σ , which quantifies how much we trust in the bias co-evolution relations and the fitting formulas for b_2 , and k_{max} . It is established in the literature that although higher values of k_{max} lead to smaller error bars for basically all parameters, including information from smaller scales can also bias the parameters, so one must carefully analyse how this choice impacts the results.

We started our computations with the *Flat* prior, which is the most simplistic and perhaps the most conservative one. In Figure 34 we show the 68% confidence intervals of the sampled posterior for both single-tracer (ST) and multi-tracer setup. Somewhat surprisingly, the MT already produced better measurements of ω_{cdm} and h in this simplistic prior. In this figure we did not show the measurements of A_s , since the constraining power on this parameter is considerably poor when performing a real space analysis²⁹. From this analysis we found evidence that the MT approach does not appear to be significantly biased as we move to higher values of k_{max} . Recall the regions shown in these plots are the 1- σ intervals, and in the literature it is not uncommon to find measurements of cosmological parameters that are slightly off from their 1- σ regions.

For the priors $G_0(\sigma)$ and $G_{\text{co-ev}}(\sigma)$ we need to fix a value of σ . After a careful analysis we decided that $\sigma = 2.0$ produced strong, yet unbiased constraints for both MT and ST. With σ fixed we can finally vary k_{max} , which is what we show in Figure 36. Starting with the $G_0(\sigma = 2.0)$, MT (red line; bottom panel) consistently produced better measurements than ST (gray line; bottom panel). To check if these values are consistent with the correct (fiducial) answer, we show in the upper panel of Figure 36 the 68% confidence intervals and its relative difference to the correct values (black dotted line). Both MT and ST produced slightly off measurements of the cosmological parameters in the low- k regime. This is due to the fact a prior $\sigma = 2.0$ is, relatively

²⁹The only situation we detected a better measurement of the MT over ST is presented in Figure 38

Figure 36: *Top panel:* the running of the 68% confidence intervals of the sampled posterior with and without the multi-tracer technique. The values were normalized by the fiducial parameters used in the N-body simulations, so a measurement consistent in 1- σ must have the dashed black line inside it. The prior for both models is described in Table 6. *Bottom panel:* we explicitly show the size of the error bars as a function of k_{max} .

speaking, a much more stringent constraint when we consider the small amount of information and the large error bars for the spectra that we get from those large scales, resulting in poor constraints for the bias parameters in this region. The choice of this specific σ was made aiming at a good performance for higher values of k , and it is clear that the measurements gets better for the smaller scales. For some interval the ST produced consistent measurements of the cosmological parameters (gray regions), but it suddenly gets more biased in ω_{cdm} around $k_{\text{max}} \approx 0.2$. For the

Figure 37: The 68% confidence curves for three distinct models at $k_{\text{max}} = 0.13h/\text{Mpc}$. The parameters not appearing in the legend were fixed at zero.

Figure 38: A corner plot comparing the Multi- and Single-tracer sampled posterior using the *Flat* prior and $k_{\text{max}} = 0.14 h \text{Mpc}^{-1}$.

MT setup, on the other hand, once the constraints enter inside the correct region they remain there for all values of k_{max} that we analyzed. This is another indication that the MT may accommodate in a better way all the unlocked information from the non-linear clustering.

To explain why the MT setup outperformed the ST one in parameter extraction we propose the following reasoning. The whole sample is formed by tracers of the large-scale structure that possess quite different physical properties, and inhabit disparate environments. In this way, when we consider them to be the same tracer and only allow a single set of bias parameters to describe them, we end up performing an average over different responses to the tracer formation process.

This averaging in turn may lead to a poor description of those tracers, since it will try to model distinct behaviours in the non-linear regime as if they were the same. The fact the tracers present different linear and non-linear responses can in fact be seen in the data – see Figure 32. In fact, the dimensionless power spectrum indicates that one of the tracers becomes non-linear first ³⁰, which may bring information from the non-linear regime already in mildly non-linear scales, where the ST still “blind” to them. From this perspective, we should expect both ST and MT to present the same performance if we consider more simplistic bias model. In order to check this we started with a model having only the linear bias, b_1 , and shot-noise, c_0 , and gradually included the other parameters until we recovered the complete model. The result from this test is shown in Figure 37. It shows exactly what we were expecting: the ST model has a bigger degradation compared with the MT one, which leads us to conclude that the improvements arising from the MT setup are due to the fact that it can produce better measurements of the bias parameters. This can also be confirmed by looking at Figure 38, where the MT consistently produces more stringent results for all parameters.

³⁰We believe this is the case for another reason as well. All bias co-evolution relations predict higher values for all non-linear bias parameters as b_1 increases. So the highly biased tracers requires, at least theoretically, stronger contributions from the non-linear operators.

6 Final remarks and next steps

In this work, we studied a description of the large-scale structure of the universe in terms of an Effective Field Theory approach and how we can obtain deeper insights into this description by considering different tracers of those structures. In Section 2 we presented a pedagogical and concise description of the main ideas behind the most robust model we have nowadays that can remarkably describe the matter field dynamics in the universe, the Effective-Field Theory of Large-Scale Structure (EFTofLSS). Using this formalism we provided motivation for why the 1-loop matter power-spectrum should be given in terms of Eq.(2.112). In order to link this matter field with the biased tracers, which is what we really see in cosmological surveys, we presented the large-scale galaxy bias formalism in Section 3. In this framework, we showed that the multi-tracer technique can be naturally implemented. This can be achieved by simply performing a bias expansion for each tracer individually, as in Eqs. (3.39) and (3.40). We then derived the expressions for both auto-power and cross-power spectra for two tracers, yielding three data vectors, namely P_{AA} , P_{BB} , Eq.(3.41) and P_{AB} , Eq.(3.42). The only drawback of doing this multi-tracer bias expansion is a large number of nuisance parameters. To circumvent this problem we considered the bias co-evolution relations as a theoretical prior, Eqs. (4.55) and (4.89). We also used a phenomenological fit for the parameter b_2 , Eq. (4.90).

To test if the multi-tracer technique in perturbation theory outperforms the usual single-tracer modeling we did the following exercise. Using a halo catalog extracted from the $4 h^{-1}\text{Gpc}$ MultiDark simulation we considered two distinct setups. The first one treats all halos as being the same tracer, dubbed as $A + B$. The second setup includes a cut at the mass scale $\log M/(M_\odot h^{-1}) = 13.5$, thereby splitting the halos into two distinct tracer populations. The halos with mass below the cut are considered as being the tracer A , whereas the ones with masses above the cut are the tracer population B – see Table 3 for a summary of these two setups, and Figure 32 for a plot of the data. We then proposed the following question:

Which setup will provide more stringent constraints for the cosmological parameters in a full-shape power spectrum analysis? The first one (ST), which treats all halos as being the same tracer (seven nuisance parameters), or the second one (MT), which splits the tracers into two sub-categories and considers a joint analysis between them (with sixteen nuisance parameters)?

We used three different priors, defined in Table 6, and tried to measure ³¹ ω_{cdm} , h and $A \equiv A_s \times 10^9$. Our first result, for the *Flat* prior, is shown in Figure 34. It is clear from this figure that the multi-tracer outperforms the single-tracer case. This improvement persisted when we employed the two other priors, $G_0(\sigma)$ and $G_{\text{co-ev}}(\sigma)$, and was the case not only for $\sigma = 2.0$ and varying k_{max} , in Figure 36, but also for the fixed k_{max} and varying σ , as shown in Figure 35.

To understand from where the improvements of the multi-tracer are arising, we defined an effective tracer in Eq. (3.47), which allows us to combine the results of the multi-tracer model to compare them with those obtained in the single-tracer model. In this way, the effective tracer has only the same six parameters as the single-tracer, which in the case of the effective tracer result from combining the original sixteen bias parameters of the multi-tracer model. We concluded that this is a fair way to directly compare the single-tracer performance with the multi-tracer one. In Figure 38 we show the sampled posterior for the single-tracer case, halo $A + B$, and for the multi-tracer case, obtained by performing a joint analysis between halo A and halo B . That figure shows that *the multi-tracer approach produces better measurements of the higher-order bias parameters, allowing a finer use of the non-linear clustering information.*

This enhancement of the constraining power of the data can be seen from different perspectives. Already in Figure 32 it is clear that tracers A and B have quite distinct behaviors in the non-linear scales. Thus, when one treats them as being the same tracer, we end up averaging over different responses to the large-scale dynamics. This averaging seems to be the cause of the degradation of the bias measurements in the single-tracer setup. We checked this in Figure 37, where the disparities between single- and multi-tracer increases as we include more parameters.

All these results allow us to conclude that the multi-tracer technique is superior to the single-tracer one when it comes to performing full-shape power spectrum analysis, even if it carries a cost in terms of a much larger number of parameters. Since this type of analysis is quite usual when treating rich data sets, such as the new generation of the upcoming survey, our result seems promising and certainly deserves further investigation as to its applicability in more realistic data.

This work opens up a whole new line of research that may be valuable to a better understanding of tracer clustering. Our next steps in this direction will be

- Derive multi-tracer bias co-evolution relations, as we proposed in Section 4. We had this

³¹Since the constraining power over A is quite poor, we decided not to show it explicitly in many cases.

idea during the final steps of this work, so we will further develop these equations. It is a work in progress, and we expect to accomplish this step in the next few months.

- The analysis performed here was limited to real space. It will be interesting to see what happens when we include redshift space distortions (RSDs).
- After implementing the multi-tracer bias expansion in RSDs, we will have all requirements to finally test our model with more realistic data. We have strong indications that the improvements arising from the multi-tracer approach should persist as we consider more realistic data sets and scenarios.

A Taylor expanding the EFT diagrams

The 1-loop power spectrum analysis requires the computation of many costly integrals, slowing the performance of MCMC analysis. In this Appendix we briefly introduce the finite difference algorithm that can be used to evaluate the Taylor expansion of the EFT operators, namely the $\mathcal{I}$ and $\mathcal{F}$ terms. In the end we perform two tests to check the accuracy of the Taylor expanded operators. The first test directly compare the realative errors of the biased tracer power spectrum using the Taylor expanded set of operators against the one using the correct ones. Finally, we perform a whole MCMC analysis to check if the sampled posterior agrees. Our tests were convincing enough for us to be sure the Taylor expansion is accurate, so we use it in all varying cosmology analysis performed in this whole work.

A.1 Finite Difference

The algorithm we use to compute the derivatives is called *Finite Differences*. In short, it works in the following way. Suppose one need to compute $f'(x)$ using $r_x = 3$ grid points centered in x . We define this *stencil* as being $s = [-1, 0, 1]$. Then, we claim that the derivative can be written as a linear combination of the original function evaluated at the grid points,

$$\frac{df}{dx} = c_0 f(x+h) + c_1 f(x) + c_2 f(x-h). \quad (\text{A.1})$$

After Taylor expanding the RHS of Eq.(A.1) up to second order and rearranging the terms,

$$\begin{aligned} \frac{df}{dx} &= c_0 \left(f(x) + h \frac{df}{dx} + \frac{h^2}{2!} \frac{d^2 f}{dx^2} \right) + c_1 f(x) + c_2 \left(f(x) - h \frac{df}{dx} + \frac{h^2}{2!} \frac{d^2 f}{dx^2} \right) \\ &= (c_0 + c_1 + c_2) f(x) + (c_0 - c_2) h \frac{df}{dx} + (c_0 + c_2) \frac{h^2}{2!} \frac{d^2 f}{dx^2}, \quad (\text{A.2}) \end{aligned}$$

we are motivated to impose that:

$$c_0 + c_1 + c_2 = 0 \quad (\text{A.3})$$

$$(c_0 - c_2)h = 1 \quad (\text{A.4})$$

$$(c_0 + c_2)\frac{h^2}{2!} = 0. \quad (\text{A.5})$$

The solution of the linear system above approximates the derivative as

$$\frac{df}{dx} = \frac{f_{i+1} - f_{i-1}}{2h}, \quad (\text{A.6})$$

where $f(x_i) \equiv f_i$.

The same procedure can be done to compute higher order derivatives or higher or higher number of grid points. In the general, for a given stencil $\mathbf{s} = [s_1, \dots, s_N]$, the associated linear system for a derivative of order $d < N$ will take the form

$$\begin{pmatrix} s_1^0 & \cdots & s_N^0 \\ \vdots & \ddots & \vdots \\ s_1^{N-1} & \cdots & s_N^{N-1} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = d! \begin{pmatrix} \delta_{0,d} \\ \vdots \\ \delta_{i,d} \\ \vdots \\ \delta_{N-1,d} \end{pmatrix}. \quad (\text{A.7})$$

The information about the specific derivative only appears on the RHS of Eq.(A.7). Therefore, the stencils matrix can always be computed beforehand. Moreover, a stencil of size N can only be used to compute derivatives of order $n \leq N - 1$.

This formalism can be easily verified for the worked example above. For $\mathbf{s} = [-1, 0, 1]$, the system becomes

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = d! \begin{pmatrix} \delta_{0,d} \\ \delta_{1,d} \\ \delta_{2,d} \end{pmatrix}, \quad (\text{A.8})$$

so picking $d=1$ leads to the same system.

A.2 Mixed partial derivatives

The scheme above can be extended to higher derivative involving multiple variables. For example, let us compute $f(x, y)_{x,y}$:

$$f_{xy} = \partial_y \left(\frac{f_{1,0} - f_{-1,0}}{h_x} \right) = \frac{f_{1,1} - f_{1,-1} - f_{-1,1} + f_{-1,-1}}{4h_x h_y}. \quad (\text{A.9})$$

The conclusion is that we can build the higher mixed derivatives on the top of single derivatives. With that in mind, I propose the notation

$$\partial_x^{n_x} f(x, y) = \frac{1}{h_x^{n_x}} c_x^i f_{i,j}. \quad (\text{A.10})$$

Using this, the mixed partial derivative can be written as

$$\partial_y^{n_y} \partial_x^{n_x} f(x, y) = \partial_y^{n_y} \left[\frac{1}{h_x^{n_x}} c_x^i f_{i,j} \right] = \frac{1}{h_x^{n_x} h_y^{n_y}} c_y^j c_x^i f_{i,j}, \quad (\text{A.11})$$

where the summation on the indexes runs over the pre-defined grid. It is fixed by choosing the stencil size associated to each variable.

As an example, let's compute $\partial_x^2 \partial_y$. The stencil for the x variable will be $\mathbf{s}_x = [-2, -1, 0, 1, 2]$. The associated coefficients are:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \\ 16 & 1 & 0 & 1 & 16 \end{pmatrix} \begin{pmatrix} c_x^1 \\ c_x^2 \\ c_x^3 \\ c_x^4 \\ c_x^5 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad (\text{A.12})$$

The solution for this case is

$$\mathbf{c}_x = \left(\frac{-1}{12}, \frac{4}{3}, \frac{5}{2}, \frac{4}{3}, \frac{-1}{12} \right), \quad (\text{A.13})$$

or, in components notation,

$$c_x^i = \left(-\frac{1}{12}\right) \delta_{-2}^i + \left(\frac{4}{3}\right) \delta_{-1}^i + \left(\frac{5}{2}\right) \delta_0^i + \left(\frac{4}{3}\right) \delta_1^i + \left(-\frac{1}{12}\right) \delta_2^i. \quad (\text{A.14})$$

For the y variable, lets use $s_y = [-1, 0, 1]$. We computed this case above and the coefficients are

$$c_y^i = \left(\frac{-1}{2}\right) \delta_{-1}^i + 0\delta_0^i + \left(\frac{1}{2}\right) \delta_1^i. \quad (\text{A.15})$$

Finally, the derivative will be

$$\begin{aligned} \partial_x^2 \partial_y f(x, y) &= \frac{1}{h_x^2 h_y} c_x^i c_y^j f_{i,j} = \frac{1}{h_x^2 h_y} \left\{ \left(-\frac{1}{12}\right) \delta_{-2}^i + \left(\frac{4}{3}\right) \delta_{-1}^i + \left(\frac{5}{2}\right) \delta_0^i + \left(\frac{4}{3}\right) \delta_1^i + \left(-\frac{1}{12}\right) \delta_2^i \right\} \\ &+ \left\{ \left(-\frac{1}{2}\right) \delta_{-1}^j + 0\delta_0^j + \left(\frac{1}{2}\right) \delta_1^j \right\} f_{i,j}. \end{aligned} \quad (\text{A.16})$$

The derivative operator evaluated at the reference point $(x_{i=0}, y_{j=0})$ is

$$\begin{aligned} [\partial_x^2 \partial_y]^{i,j} &= \frac{1}{h_x^2 h_y} \left\{ \left(-\frac{1}{12}\right) \delta_{-2}^i + \left(\frac{4}{3}\right) \delta_{-1}^i + \left(\frac{5}{2}\right) \delta_0^i + \left(\frac{4}{3}\right) \delta_1^i + \left(-\frac{1}{12}\right) \delta_2^i \right\} \times \\ &\times \left\{ \left(-\frac{1}{2}\right) \delta_{-1}^j + 0\delta_0^j + \left(\frac{1}{2}\right) \delta_1^j \right\}. \end{aligned} \quad (\text{A.18})$$

We implemented a code in `Python` that can find the multiplicative factors used to evaluate the derivatives at any order, stencil size and number of mixed derivatives.

A.3 Testing the Taylor expanded operator basis

To test our Taylor expanded operator basis we perform some tests to compare its output with the case in which we numerically solve the more costly integrals. The stencil used to compute the derivatives is $s = [-2, -1, 0, 1, 2]$, so we can compute the Taylor expansion up to the fourth order. The reference point is $\Omega_{\text{ref}} = [\omega_{\text{cdm,ref}}, h_{\text{ref}}] = [0.119, 0.678]$ and the parameter setup used in the analysis is described in Table 7. The timing result for a MacBook Pro Retina, Mid 2012 is $t_{\text{CLASS-PT}} \approx 1.5s$, whereas we have $t_{\text{Taylor}} \approx 0.006s$, which is a improvement by a factor of

Parameter	Prior 1T	Prior 2T
ω_{cdm}	Flat(0.08,0.13)	Same
h	Flat(0.6,0.8)	Same
b_1^A	Flat(1.0,1.6)	Flat(1.0,1.6)
b_2^A	Gauss(0,0.2)	Same
$b_{\mathcal{G}_2}^A$	0	0
$b_{\Gamma_3}^A$	0	0
c_0^A	Flat(-2,2)	Same
c_2^A	0	0
$b_{\nabla^2\delta}^A$	0	0
b_1^B	-	Flat(1.6,2.2)
b_2^B	-	Gauss.(0,0.2)
$b_{\mathcal{G}_2}^B$	-	0
$b_{\Gamma_3}^B$	-	0
c_0^B	-	Flat(-2,2)
c_2^B	-	0
$b_{\nabla^2\delta}^B$	0	0

Table 7: Parameter setup used to test the Taylor expansion of the EFT diagrams.

250.

Figure 39: Relative error of the Taylor expanded tracer power spectrum with respect to the output obtained by performing all costly calculations. The error bar is obtained by assuming a Gaussian covariance matrix.

Figure 40: Comparing the MCMC analysis using the priors described in Table 7. The Taylor expanded operator basis can correctly recover the posterior.

First, we computed the relative error for the total tracer power spectrum with fixed bias parameters and varying cosmological parameters. The cosmological parameters were allowed to vary in a predefined grid, in such a way that each parameter X is allowed to vary in a range defined by $X \in [X_{\text{ref}} \pm \sigma_X]$. The values we used were $\sigma_{\omega_{\text{cdm}}} = 0.2\omega_{\text{cdm,ref}}$ and $\sigma_h = 0.15\omega_{h,\text{ref}}$. As is clear in Figure 39, the relative errors are all inside a typical error bar for the cases we considered in this work.

To double-check whether the Taylor expansion is correct, we perform a MCMC analysis and compare the posterior. We did this for both one tracer and two tracers case, whose prior details can be seen in Table 7 and the posterior is in Figure 40.

B The multi-tracer co-evolution matrices

The multi-tracer co-evolution propagator, introduced in Section 4, require cumbersome matrices. In the main text we omitted them, but here we show it explicitly. The multi-tracer co-evolution propagator is,

$$G_{ab} = e^{-3\eta/2} [\mathcal{T}_{(-\frac{3}{2})}]_{ab} + e^{-\eta/2} [\mathcal{T}_{(-\frac{1}{2})}]_{ab} + [\mathcal{T}_{(0)}]_{ab} + e^{\eta} [\mathcal{T}_{(1)}]_{ab} . \quad (\text{B.1})$$

The $\mathcal{T}$ matrices are

$$\mathcal{T}_{-(\frac{3}{2})} = \begin{pmatrix} \frac{2\omega_m}{5} & \frac{1}{5}(-2)\omega_m & \frac{2\omega_A}{5} & -\frac{1}{5}(2\omega_A) & \frac{2\omega_B}{5} & -\frac{1}{5}(2\omega_B) \\ \frac{1}{5}(-3)\omega_m & \frac{3\omega_m}{5} & -\frac{1}{5}(3\omega_A) & \frac{3\omega_A}{5} & -\frac{1}{5}(3\omega_B) & \frac{3\omega_B}{5} \\ \frac{2\omega_m}{5} & \frac{1}{5}(-2)\omega_m & \frac{2\omega_A}{5} & -\frac{1}{5}(2\omega_A) & \frac{2\omega_B}{5} & -\frac{1}{5}(2\omega_B) \\ \frac{1}{5}(-3)\omega_m & \frac{3\omega_m}{5} & -\frac{1}{5}(3\omega_A) & \frac{3\omega_A}{5} & -\frac{1}{5}(3\omega_B) & \frac{3\omega_B}{5} \\ \frac{2\omega_m}{5} & \frac{1}{5}(-2)\omega_m & \frac{2\omega_A}{5} & -\frac{1}{5}(2\omega_A) & \frac{2\omega_B}{5} & -\frac{1}{5}(2\omega_B) \\ \frac{1}{5}(-3)\omega_m & +\frac{3}{5}\omega_m & -\frac{1}{5}(3\omega_A) & \frac{3\omega_A}{5} & -\frac{1}{5}(3\omega_B) & \frac{3\omega_B}{5} \end{pmatrix} , \quad (\text{B.2})$$

$$\mathcal{T}_{-(\frac{1}{2})} = \begin{pmatrix} 0 & -2(\omega_A + \omega_B) & 0 & 2\omega_A & 0 & 2\omega_B \\ 0 & \omega_A + \omega_B & 0 & -\omega_A & 0 & -\omega_B \\ 0 & -2(\omega_A + \omega_B - 1) & 0 & 2(\omega_A - 1) & 0 & 2\omega_B \\ 0 & \omega_A + \omega_B - 1 & 0 & 1 - \omega_A & 0 & -\omega_B \\ 0 & -2(\omega_A + \omega_B - 1) & 0 & 2\omega_A & 0 & 2(\omega_B - 1) \\ 0 & \omega_A + \omega_B - 1 & 0 & -\omega_A & 0 & 1 - \omega_B \end{pmatrix} , \quad (\text{B.3})$$

$$(\text{B.4})$$

$$\mathcal{T}_{(0)} = \begin{pmatrix} \omega_A + \omega_B & 2(\omega_A + \omega_B) & -\omega_A & -2\omega_A & -\omega_B & -2\omega_B \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_A + \omega_B - 1 & 2(\omega_A + \omega_B - 1) & 1 - \omega_A & 2 - 2\omega_A & -\omega_B & -2\omega_B \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_A + \omega_B - 1 & 2(\omega_A + \omega_B - 1) & -\omega_A & -2\omega_A & 1 - \omega_B & 2 - 2\omega_B \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{B.5})$$

$$\mathcal{T}_{(1)} = \begin{pmatrix} \frac{1}{5}(-3)(\omega_A + \omega_B - 1) & \frac{1}{5}(-2)(\omega_A + \omega_B - 1) & \frac{3\omega_A}{5} & \frac{2\omega_A}{5} & \frac{3\omega_B}{5} & \frac{2\omega_B}{5} \\ \frac{1}{5}(-3)(\omega_A + \omega_B - 1) & \frac{1}{5}(-2)(\omega_A + \omega_B - 1) & \frac{3\omega_A}{5} & \frac{2\omega_A}{5} & \frac{3\omega_B}{5} & \frac{2\omega_B}{5} \\ \frac{1}{5}(-3)(\omega_A + \omega_B - 1) & \frac{1}{5}(-2)(\omega_A + \omega_B - 1) & \frac{3\omega_A}{5} & \frac{2\omega_A}{5} & \frac{3\omega_B}{5} & \frac{2\omega_B}{5} \\ \frac{1}{5}(-3)(\omega_A + \omega_B - 1) & \frac{1}{5}(-2)(\omega_A + \omega_B - 1) & \frac{3\omega_A}{5} & \frac{2\omega_A}{5} & \frac{3\omega_B}{5} & \frac{2\omega_B}{5} \\ \frac{1}{5}(-3)(\omega_A + \omega_B - 1) & \frac{1}{5}(-2)(\omega_A + \omega_B - 1) & \frac{3\omega_A}{5} & \frac{2\omega_A}{5} & \frac{3\omega_B}{5} & \frac{2\omega_B}{5} \\ \frac{1}{5}(-3)(\omega_A + \omega_B - 1) & \frac{1}{5}(-2)(\omega_A + \omega_B - 1) & \frac{3\omega_A}{5} & \frac{2\omega_A}{5} & \frac{3\omega_B}{5} & \frac{2\omega_B}{5} \end{pmatrix}. \quad (\text{B.6})$$

We used the relation $\omega_m + \omega_A + \omega_B = 1$ in some cases to simplify the notation.

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