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Laços de Wilson em teorias super-simétricas de Chern-Simons com matéria

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Wilson Loops in supersymmetric Chern-Simons-matter theories

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Abstract

The goal of this thesis is to study Wilson loops in 3 dimensional Chern-Simons-matter theories aiming at applications in holography. In particular, we study operators preserving some amount of supersymmetry in ABJ(M) theory and in $\mathcal{N} = 4$ Chern-Simons-matter theories. In both cases, we seek for a systematic construction of these operators to understand the moduli space of the theory. In the case of ABJ(M), we also make a perturbative detour aiming at some universal information as, for example, the Bremsstrahlung function.

Key-Words: “supersymmetry”, “Wilson loops”, “Chern-Simons-matter theories”.

Resumo

O objetivo desta tese é estudar laços de Wilson em teoria supersimétrica de Chern-Simons com matéria em 3 dimensões visando aplicações em holografia. Em particular, nós estudamos operadores preservando alguma quantidade de supersimetria em teoria de $ABJ(M)$ e em teorias com $\mathcal{N} = 4$. Em ambos os casos, nossa intenção é obter uma construção sistemática desses operadores para entender o espaço de módulos da teoria. No caso de $ABJ(M)$, nós também fazemos cálculos perturbativos visando informações universais como, por exemplo, a função de Bremsstrahlung.

Palavras-Chave: “supersimetria”, “laços de Wilson”, “teorias supersimétricas de Chern-Simons com matéria”.

Contents

1	Introduction	1
2	Basic toolkit	5
2.1	Supersymmetric quiver theories	5
2.2	Supersymmetric Wilson loops	7
2.3	Quivers and Wilson loops	9
3	$\mathcal{N}=6$ Chern-Simons-matter theory	14
3.1	Supersymmetric Wilson loops	16
3.1.1	Circular loops	16
3.1.2	Latitude loops	21
3.2	From 1-node to 2-node loops	23
3.3	New 1/6 BPS operator	40
3.4	Perturbative detour: deforming the 1/2 BPS loop	48
3.4.1	1-loop contribution	49
3.4.2	2-loops contribution	55
4	$\mathcal{N} = 4$ Chern-Simons-matter theories	60
4.1	1-node Wilson loops	64
4.2	Hyperloops at $\theta = 0$	66
4.2.1	Moduli spaces	70
4.3	Hyperloops at $\theta \neq 0$	73
4.3.1	Construction	73
4.3.2	A closer look at the hyperloops	77
4.3.3	Further examples	79
4.4	A matrix model proposal	81
4.5	Comments	82
4.6	New hyperloops	84
	Appendices	89

Chapter 1

Introduction

Wilson loops (WLs) are fundamental objects in any gauge theory. Mathematically, they represent the holonomy of the gauge connection A around a closed path γ ,

$$W_{\mathcal{R}}[\gamma] = \text{Tr}_{\mathcal{R}} \mathcal{P} \exp \left(i \oint_{\gamma} A \right) . \quad (1.1)$$

The symbol $\mathcal{P}$ stands for the path-ordering of the operator. The gauge field takes value on the Lie algebra of the corresponding gauge group and the trace is taken with respect to the representation $\mathcal{R}$ of the gauge field. From the physical point of view, these loops give information, for example, about the Schwinger pair production probability [1] and about confinement [2].

In the context of supersymmetric theories, Wilson loops may acquire an extra and special feature. When these operators are made supersymmetric, their expectation value may be computed exactly. This is extremely valuable since exact results are rare in quantum field theory (QFT). Such results are obtained using a technique called localization [3–5]. In fact, localization can be used to compute the expectation value of any supersymmetry preserving operator and it relies on the fact that in some cases path integrals are exactly equal to their semiclassical approximations [6, 7]. Roughly speaking, a supersymmetric theory admits a Lie superalgebra of symmetries. This superalgebra contains odd generators and, for a theory with $\mathcal{N}$ supersymmetry, there are $4 \times \mathcal{N}$ of those. An operator $\mathcal{O}$ that is annihilated by at least one odd element $\mathcal{Q}$ of this superalgebra is named after Evgeny Bogomolny, M. K. Prasad, and Charles Sommerfield as BPS and denoted as $\mathcal{O}_{\text{BPS}}$. The spirit of localization is then to deform the action S of the theory by a term generated by the action of $\mathcal{Q}$ on some field configuration A , called $\mathcal{Q}$ -exact term, such that

$$S \rightarrow S + t\mathcal{Q}A . \quad (1.2)$$

For Q^2 -invariant A , this deformation does not change the vacuum expectation value of $\mathcal{O}_{\text{BPS}}$. In terms of the Feynman path integral we have that the vacuum expectation value of $\mathcal{O}_{\text{BPS}}$ is given by

$$\langle \mathcal{O}_{\text{BPS}} \rangle = \int D[a] e^{-S[a] + tQ^A} \mathcal{O}_{\text{BPS}}, \quad (1.3)$$

where a are the fields of the theory. Finally, for sufficiently well-behaved A , the expectation value is independent of t so this path integral can be evaluated in the limit $t \rightarrow \infty$, where the integral is dominated by the saddle points of the deformed term.

These exact results provide a rich arena to test the conjectured duality between string theory in anti-de Sitter (AdS) space and conformal field theories (CFTs), the so called AdS/CFT correspondence [8, 9]. The duality is an explicit realization of the holographic principle [10, 11] and for this reason is sometimes referred to as holography. More precisely, it states a weak-strong coupling duality where, on the one hand, we have a string theory defined in terms of three fundamental quantities: the radius L of a $d + 1$ dimensional AdS space, the string length l_s and the string coupling constant g_s . On the other hand, we have a d dimensional field theory living in the boundary of AdS with coupling constant g_{YM} and $SU(N)$ gauge symmetry, such that the 't Hooft coupling constant is $\lambda \equiv Ng_{\text{YM}}^2$. For this reason subscripts are often included to specify the dimensions as $\text{AdS}_{d+1}/\text{CFT}_d$ and we often refer to the respective theories as bulk and boundary theories. The correspondence then states that quantities from both sides are related through

$$\left(\frac{L}{l_s}\right)^4 = \lambda, \quad g_{\text{YM}}^2 = 2\pi g_s.$$

Holography has been mostly implemented in the limit where the number N of fields involved is very large and the 't Hooft coupling λ is kept fixed and large. In this limit, referred to as large N limit, only planar diagrams in perturbation theory survive and the gauge theory is substantially simplified [12]. From the bulk theory point of view, this implies that the string coupling constant is small, $g_s \ll 1$, such that the string theory is weakly coupled and non-perturbative effects can be safely neglected. Moreover, having $\lambda \gg 1$, implies that the radius L of AdS is large. Thus the AdS curvature becomes small and string theory is effectively described by classical supergravity — so this side of the duality is also often referred to as the gravity side. Therefore one use of the AdS/CFT correspondence is the possibility to use an easier description rather than the other for computing

interesting physical observables, since questions on one side of the duality can be reformulated in the language of the other side.

Being a weak-strong coupling duality makes the AdS/CFT correspondence very powerful, but it also makes it hard to be tested. Testing it requires a computation of the expectation value of some observable at all orders in the coupling constant on one side of the correspondence. This is where the aforementioned exact results become particularly relevant. Holographically, Wilson loop operators are mapped to fundamental strings and their exact results allow us to probe the duality at both weak and strong coupling.

This non-trivial check of the duality was first done in four-dimensional $\mathcal{N} = 4$ super Yang-Mills (SYM), which is dual to type IIB strings on $\text{AdS}_5 \times S^5$ [9]. The circular 1/2 BPS loop was computed exactly through localization [3, 4]. The weak coupling expansion of the exact result matches the perturbative computation whereas the strong coupling limit matches the fundamental string result [13].

In the same spirit, this non-trivial check can be extended to three-dimensional supersymmetric theories. In this work we will focus on the particular cases of $\mathcal{N} = 6$ Chern-Simons-matter (CSm) theory [14, 15], known as ABJ(M) theory, and $\mathcal{N} = 4$ Chern-Simons-matter theories [16–19]. Even though most of our attention will be driven to the construction of supersymmetric operators itself, it is clear how this fits into the holography program.

The thesis is organized as follows. In Chapter 2 we begin by covering some common ground concerning generic CSM theories. Hopefully it provides, in a pedagogical way, the required tools to later discuss the particular cases of $\mathcal{N} = 6$ and $\mathcal{N} = 4$ theories.

The case of $\mathcal{N} = 6$ is addressed in Chapter 3. We begin it with a review of the theory on $\mathbb{R}^3$ and the known BPS WLs. We explain how one can reformulate circular operators and in this way address their moduli space. This is based on results presented in the second chapter of [20], but here we provide much more detail. Then we introduce a new 1/6 BPS operator that to our knowledge was not known in the literature. Finally, we make an attempt of a perturbative computation concerning a deformed version of the 1/2 BPS loop.

Chapter 4 is then dedicated to $\mathcal{N} = 4$ theories on S^3 and it is mostly based in [21]. We begin by introducing the theory and presenting BPS operators that couple to a single node. These are presented as great circles that are continuously connected by marginal deformations to the usual bosonic 1/4 BPS Gaiotto-Yin WLs [22]. Extending the proposal of [20] to $\mathcal{N} = 4$, we use these operators to

derive their multiple nodes counterparts and then we study their moduli spaces. Finally, we include some preliminary results of an ongoing project [23] that started from our studies of the $\mathcal{N} = 4$ analogue of the new 1/6 BPS operator derived in ABJ(M).

All details about supersymmetry transformations, Feynman rules and other technicalities are presented in three appendices.

Chapter 2

Basic toolkit

The main content of this work is divided into two parts, one about ABJ(M) theory and one about $\mathcal{N} = 4$ CSM theories. There is, however, some overlap between these two, since both consist on supersymmetric quiver theories and in both cases we are interested in WLs preserving some amount of supersymmetry. We dedicate this chapter to cover this common ground and equip the reader with a basic toolkit for what follows.

2.1 Supersymmetric quiver theories

Quiver diagrams are pictorial representations that illustrate the field content and symmetries of a theory in a simple way. For supersymmetric theories, the kind of quiver we can draw varies depending on the amount of supersymmetry and, once augmented by the superpotential, mass terms and Fayet-Iliopoulos (FI) terms, the quiver diagram is enough to completely fix the Lagrangian of a particular theory.

The general idea is to associate a symbol to each element of the theory: nodes to gauge group factors and arrows to matter fields. Within each node we have a number N_I specifying the $U(N_I)$ gauge group. An arrow connecting a node I and a neighbor node $I + 1$ represents bifundamental fields in the fundamental of the starting node and anti-fundamental representation of the target node, i.e., in $(\square, \bar{\square})$ of $U(N_I) \times U(N_{I+1})$. It will not be the case for the particular theories considered in the following chapters, but for completeness let us also consider the possibility of having global (flavor) symmetry and, in this case, $U(M_I)$ global factors are associated to square nodes.

Take for instance the simplest case where $\mathcal{N} = 1$. As explained above, vector superfields are represented as nodes and chiral superfields as arrows. To each vector multiplet V transforming in the adjoint representation of the gauge group

we associate a circle node. To each global symmetry we associate a square node. To each chiral superfield Φ we associate an arrow.

Going one step further, we have $\mathcal{N} = 2$ supersymmetry. Its vector multiplet can be decomposed in terms of $\mathcal{N} = 1$ superfields as the direct sum of a vector and a chiral multiplet,

$$\mathcal{N} = 2 \text{ vector multiplet} \longrightarrow V \oplus \Phi ,$$

whereas its hypermultiplet can be split into the direct sum of two $\mathcal{N} = 1$ chiral multiplets,

$$\mathcal{N} = 2 \text{ hypermultiplet} \longrightarrow \Psi \oplus \bar{\Psi} .$$

Thus one way of drawing a quiver for a theory with $\mathcal{N} = 2$ is simply by following the same rules used for the $\mathcal{N} = 1$ case. However, we can also simplify these rules by considering the decomposition above. In particular, every circle node (associated to V) has an arrow (associated to Φ) that starts and ends in the same circle node, so we can remove arrows that start and end on the same circle node. Furthermore, every arrow starting on node I and ending on node $I + 1$ (associated to Ψ) has a counterpart, i.e., an arrow starting on node $I + 1$ and ending on node I (associated to $\bar{\Psi}$). So for $\mathcal{N} = 2$ we can replace two arrows between nodes by an edge with no orientation.

To clarify the ideas explained so far, let us consider as an example 3d $\mathcal{N} = 4$ SQED with N flavors. Since this theory can be obtained from 4d $\mathcal{N} = 2$ SQED through dimensional reduction, we can apply the set of rules explained above and see how one would draw the corresponding quiver. The field content of 4d $\mathcal{N} = 2$ SQED is composed by one vector multiplet and one hypermultiplet. Being it a supersymmetric version of Quantum Electrodynamics, there is an $U(1)$ gauge symmetry and, in addition to it, we also include an $U(N)$ flavor symmetry. The quiver diagrams representing the theory are shown in Figure 1, where in (a) the quiver is drawn in $\mathcal{N} = 1$ language whereas in (b) we use the simplified rules of $\mathcal{N} = 2$.

Let us pause for a moment to comment on the consistency of these diagrams. First, note that Ψ and $\bar{\Psi}$ transform in the same representations of both gauge and flavor groups, as expected from the $\mathcal{N} = 2$ hypermultiplet decomposition. Therefore Ψ and $\bar{\Psi}$ transform in conjugate representations, namely $(\square, \bar{\square})$ and $(\bar{\square}, \square)$ of $U(1) \times U(N)$. This is captured by the orientation of the arrows representing them in Figure 1a. Also, note that Φ belongs to the adjoint representation of $U(1)$, as can be seen from the $\mathcal{N} = 2$ vector multiplet decomposition. This is captured by

Figure 1: Quiver diagram of 4d $\mathcal{N} = 2$ SQED in (a) $\mathcal{N} = 1$ and (b) $\mathcal{N} = 2$ languages. In addition to the quiver, we also need the superpotential to completely fix the Lagrangian of the theory.

the fact that its arrow starts and ends on the circle node in Figure 1a. Of course, once you are familiarized with the theory and its field content, it is easy to see that the $\mathcal{N} = 2$ quiver in Figure (1b) is enough to capture all essential features present in Figure 1a.

Throughout this work, we will make a minor change to the rules presented above. In order to follow the standard notation in the literature of supersymmetric WLs, instead of writing N_I inside each node corresponding to a $U(N_I)$ gauge group factor, we will place the gauge field A_I itself inside the node. Nevertheless, it will soon become clear how quiver diagrams are useful tools to study supersymmetric operators.

2.2 Supersymmetric Wilson loops

In supersymmetric theories, a Wilson loop operator may preserve some amount of supersymmetry once we allow it to couple not only to the gauge field but also to other fields of the theory. For example, in four-dimensional $\mathcal{N} = 4$ SYM in Euclidean space, this is achieved by coupling the loop to scalar fields Φ_I , $I = 1, \dots, 6$, such that

$$W_{\mathcal{R}}[\gamma, \theta] = \text{Tr}_{\mathcal{R}} \mathcal{P} \exp \left(-i \oint_{\gamma} (A_{\mu} \dot{x}^{\mu} - i |\dot{x}| \theta^I \Phi_I) ds \right). \quad (2.1)$$

Note that, in comparison with (1.1), the operator now depends on the coupling θ .¹ Generically, this coupling varies along the contour γ along which the loop is supported. For a circular loop, we find that the particular form $\theta^I = (1, 0, 0, 0, 0, 0)$

¹The notation $W_{\mathcal{R}}[\gamma, \theta]$ is precise because it makes explicit the dependence of the loop on three features: the representation $\mathcal{R}$ of the gauge group, the contour γ along which it is supported and the matter coupling θ . Nevertheless, to avoid cluttering, from now on we will denote the operator simply as W and the trace as Tr . Specifications will be included only when needed.

is such that

$$\delta (A_\mu \dot{x}^\mu - i|\dot{x}| \theta^I \Phi_I) = 0, \quad (2.2)$$

for 8 out of the 16 supercharges of the theory. Therefore, for this choice of coupling, the resulting loop is 1/2 BPS.

For supersymmetric loops that couple to the gauge field and scalars, as in (2.1), the argument of the integral is usually referred to as the “dressed bosonic connection” and denoted as $\mathcal{A}$. In general, the corresponding WLs are called bosonic and written as

$$W = \text{Tr } \mathcal{P} \exp \left(-i \oint \mathcal{A} ds \right). \quad (2.3)$$

As in (2.2), the condition they need to fulfill to be supersymmetric is that, under the action of some supersymmetry transformation, the variation of $\mathcal{A}$ is null.

There is, however, the possibility of allowing the loop to also couple to fermions, giving rise to operators that are called fermionic. These are constructed by promoting the dressed bosonic connection to a “superconnection” $\mathcal{L}$ that includes gauge, scalar and fermi fields of the theory. In this case, the loop is written as

$$W = \text{Tr } \mathcal{P} \exp \left(-i \oint \mathcal{L} ds \right), \quad (2.4)$$

where $\mathcal{L}$ is a supermatrix and the trace is used loosely, in the sense that it could be a supertrace or something more intricate, depending on the particular structure of $\mathcal{L}$. Actually, in the first construction of a fermionic loop [24], the authors obtained that, in order to preserve supersymmetry, the operator should be traced. However, tracing supermatrices does not seem like a reasonable operation. Indeed, later in [25] the authors argued that the operator should be written in terms of a supertrace. We postpone the details of this discussion to the next chapter.

In contrast to the bosonic case, supersymmetry invariance of fermionic loops is more subtle. In particular, the condition $\delta \mathcal{A} = 0$ is replaced by the weaker requirement that $\delta \mathcal{L}$ translates into a supergauge transformation, such that the traced/supertraced loop operator is invariant. To wit, a fermionic loop is supersymmetric if

$$\delta \mathcal{L} = \mathcal{D}_s \mathcal{G} \equiv \partial_s \mathcal{G} + i[\mathcal{L}, \mathcal{G}]. \quad (2.5)$$

Here $\mathcal{D}_s \mathcal{G}$ represents the supercovariant derivative constructed out of the super-

connection $\mathcal{L}$ acting on a supermatrix $\mathcal{G}$.

Historically, the first fermionic loop was constructed in ABJ(M) theory in [24]. The proof of supersymmetry invariance was carried out explicitly, i.e., by expanding the path-ordered exponential in (2.4) and showing that terms from different orders of the expansion non-trivially cancelled each other. Later in [26], it was understood that this was actually a consequence of having equation (2.5) satisfied. This allowed the authors of [25] to construct loops supported along arbitrary curves on S^2 .

2.3 Quivers and Wilson loops

When reviewing the literature of supersymmetric WLs, we usually come across a narrative that, as the one of the previous section, presents bosonic and fermionic loops as distinct objects. Such a distinction is comprehensible if we look at the developments within the field through chronological lenses. However, here we would like to follow a different approach. In this section our goal is to give a tour that starts from $\mathcal{A}$ in (2.3) and goes to $\mathcal{L}$ in (2.4) in a intuitive way, hopefully offering a unified overview of the underlying physics. To make our journey as smooth as possible, let us put aside particularities of a theory and focus on the narrative itself.

Suppose we have a certain three-dimensional Chern-Simons-matter theory with multiple gauge fields and matter fields, scalars and fermions, charged under them. In $\mathcal{N} = 2$ language, such theory would be schematically described by the quiver diagram of Figure 2.

Figure 2: Schematic representation of a quiver gauge theory in $\mathcal{N} = 2$ language.

Now imagine that within this theory, we have supersymmetric WLs with the same structure as the one in (2.3). Since our hypothetical theory has multiple nodes, we already have the generalization where there is an operator like that for each node of index I ,

$$W = \text{Tr } \mathcal{P} \exp \left(-i \oint \mathcal{A}_I ds \right) = \text{Tr } \mathcal{P} \exp \left(-i \oint A_{I\mu} \dot{x}^\mu + \dots ds \right).$$

Let us focus on what could arise in the ellipsis piece. Whatever that is, it should

feature some of the properties of the gauge field: it should have mass dimension equal to 1 and belong to the adjoint representation of $U(N_I)$. Recall that matter fields in this theory belong either to the bifundamental or to the anti-bifundamental representation. So if we want to include them, it should be done as a bilinear combination belonging to the same representation as A_I . Since in 3 dimensions scalars have dimension 1/2 and fermions have dimension 1, we conclude that scalar bilinears should be the perfect fit for the ellipsis above, giving

$$W = \text{Tr } \mathcal{P} \exp \left(-i \oint A_{I\mu} \dot{x}^\mu + \text{scalar bilinears } ds \right).^2$$

In particular, such linear combination can be made for all scalars that are charged under the I -th node, either the ones to the left or to the right of it. At this point it is enlightening to mix languages for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ quivers and open each edge in Figure 2 as the corresponding two arrows of $\mathcal{N} = 1$. This results in the quiver diagram shown in Figure 3 where, in addition to turning edges into arrows, we also highlight the matter fields involved in red. From it we see that we could think of $\mathcal{A}_I$ as starting from the I -th node and traveling through (scalar) red arrows such that by the end of the day the path takes us back to the starting point.

Figure 3: Schematic representation of a quiver gauge theory mixing $\mathcal{N} = 2$ and $\mathcal{N} = 1$ languages where we highlight matter fields involved in the definition of the 1-node WL.

So far we have the possibility of defining an operator that couples to one node of the quiver theory and, in particular, we have an operator like that for each node I . We can then think of each of these as one Lego piece and, in the same spirit as the game, try to explore the possibility of gluing two of these pieces together. Physically, this would correspond to allowing the resulting operator to couple to two nodes of the quiver theory, giving rise to something that could be sketched as

$$W = \text{Tr } \mathcal{P} \exp \left(-i \oint \begin{pmatrix} \mathcal{A}_I & 0 \\ 0 & \mathcal{A}_{I+1} \end{pmatrix} ds \right).$$

Whatever this object is, we already see that it has dimensions $(N_I + N_{I+1}) \times (N_I +$

²The analogous analysis holds for (2.1). The theory is four-dimensional, thus scalars have mass dimension 1. These belong to the adjoint representation, so they appear linearly in $\mathcal{A}$.

N_{I+1}). Its diagonal blocks are $N_I \times N_I$ and $N_{I+1} \times N_{I+1}$ matrices, since these are in the adjoint of $U(N_I)$ and $U(N_{I+1})$. Consequently, its off-diagonal entries are $N_I \times N_{I+1}$ and $N_{I+1} \times N_I$ dimensional. Now recall that matter fields connecting nodes I and $I+1$ have these exact matrix dimensions, since they belong to $(\square, \bar{\square})$ and $(\bar{\square}, \square)$ of $U(N_I) \times U(N_{I+1})$. So we can consider linear appearances of them as suitable candidates to be placed in the off-diagonal entries of this matrix. By dimensional analysis, such candidate should have mass dimension 1 and this is the precise case for fermions in 3d. So these represent a perfect fit, giving rise to an operator that looks like

$$W = \text{Tr } \mathcal{P} \exp \left(-i \oint \begin{pmatrix} \mathcal{A}_I & \text{fermions} \\ \text{fermions} & \mathcal{A}_{I+1} \end{pmatrix} ds \right).$$

This motivates the form of $\mathcal{L}$ in (2.4), showing that its off-diagonal blocks are Grassmann odd and therefore $\mathcal{L}$ is actually a $(N_I + N_{I+1}) \times (N_I + N_{I+1})$ supermatrix. As in the 1-node case, looking at the quiver diagram, we could think of this operator as diagonal entries consisting of paths starting and ending at each node. In particular, we would have the red path of Figure 3 for $\mathcal{A}_I$ and the analogous one for $\mathcal{A}_{I+1}$, as show in Figure 4a. These are augmented by off-diagonal entries consisting of open paths, the (fermionic) arrows, connecting neighbor nodes. These are denoted in blue in Figure 4b. The top-right entry of the operator is represented by the blue arrow that goes from node I to node $I+1$ whereas the bottom-left entry is the conjugate arrow pointing in the opposite direction.

Figure 4: Highlighted paths illustrating the new pieces that are now part of $\mathcal{L}$. In (a) we highlight the structure of $\mathcal{A}_{I+1}$. In (b) we highlight the structure of the off-diagonal entries of the superconnection.

As long as we follow the consistency conditions outlined above, we could keep generalizing the procedure by, for example, gluing more than two pieces together. This would lead to operators that couple to three or more nodes of the quiver

theory and written in terms of a superconnection with the generic form

$$\mathcal{L} = \begin{pmatrix} \ddots & & & \\ & \mathcal{A}_{I-1} & \text{fermions} & \\ & \text{fermions} & \mathcal{A}_I & \text{fermions} \\ & & \text{fermions} & \mathcal{A}_{I+1} \\ & & & \ddots \end{pmatrix}.$$

In the same spirit, another possibility would be to include some of the nodes more than once. Suppose we did so by including the I -th node twice. This would give us an operator with

$$\mathcal{L} = \begin{pmatrix} \ddots & & & \\ & \mathcal{A}_{I-1} & \text{fermions} & \\ & \text{fermions} & \mathcal{A}_I & \\ & & & \mathcal{A}_I & \text{fermions} \\ & & & \text{fermions} & \mathcal{A}_{I+1} \\ & & & & \ddots \end{pmatrix},$$

which seems to be the most general structure we can construct using the building blocks outlined so far.

There is, however, a further possibility that we can consider. That is the possibility of placing scalar bilinears above and below the fermionic entries in $\mathcal{L}$ [27]. As previously argued, such bilinears automatically fit the dimensional analysis. Now in order to fit the matrix representation they should consist of scalars belonging to neighboring edges of the quiver. To illustrate this possibility in more detail, let us consider for instance a 3-node operator with a superconnection of the form

$$\mathcal{L} = \begin{pmatrix} & \mathcal{A}_{I-1} & \text{fermions} & \text{scalar bilinears} \\ & \text{fermions} & \mathcal{A}_I & \text{fermions} \\ \text{scalar bilinears} & & \text{fermions} & \mathcal{A}_{I+1} \end{pmatrix}.$$

Above we colored the new structures appearing in the superconnection. These are denoted with the respective colors in Figure 5. Note that, as in the case of fermions, their respective highlighted paths are open since these scalar bilinears connect consecutive nodes.

Figure 5: Highlighted paths illustrating the possibility of including scalar bilinears outside of $\mathcal{A}$.

Though we have intentionally skipped details and considered a generic theory, this game sketches the idea behind all known examples of supersymmetric WLs in three-dimensional Chern-Simons-matter theories. By now, we hope that the reader is convinced that quiver diagrams can offer an intuitive view on supersymmetric loops. In the following chapters we will focus on the particular case of ABJ(M) and $\mathcal{N} = 4$ theories. In particular, in Chapter 4 we will associate a diagram to the Wilson loop operator itself, allowing us to increment this Lego game with extra features, providing an even richer overview.

Chapter 3

$\mathcal{N}=6$ Chern-Simons-matter theory

$\mathcal{N} = 6$ Chern-Simons-matter theory is a three-dimensional quiver gauge theory constructed from two copies of Chern-Simons actions coupled to matter, one with gauge groups $U(N_1)$ at level k and the other with gauge group $U(N_2)$ at level $-k$. The construction of such theory was first done in 2008 by Aharony, Bergman, Jafferis and Maldacena [14] for the case where $N_1 = N_2$ and it was named ABJM theory. Later in the same year the generalisation for $N_1 \neq N_2$ was done by Aharony, Bergman and Jafferis [15] and it was named ABJ theory. In both cases, the gauge theory is dual to type IIA strings on $\text{AdS}_4 \times \mathbb{CP}^3$. Here, as mostly done in the literature, we consider gauge groups with different ranks and refer to the theory as ABJ(M).

The field content of ABJ(M) is depicted in the quiver diagram of Figure 6. It includes two gauge fields, A_1 and A_2 , in the adjoint representation of the respective gauge groups $U(N_1)$ and $U(N_2)$. The matter sector has $SU(4)$ R-symmetry and is composed by scalars C_I and fermions $\bar{\psi}^I$, $I = \{1, 2, 3, 4\}$, in the $(\square, \bar{\square})$ representation of $U(N_1) \times U(N_2)$. By conjugation there are also the fields $\bar{C}^I$ and ψ_I in $(\bar{\square}, \square)$ of $U(N_1) \times U(N_2)$.

Figure 6: Quiver diagram of ABJ(M) theory. Below each node we include the level of the respective copy of the Chern-Simons action.

In Euclidean space, the theory is described by the action

$$S = S_{\text{CS}} + S_{\text{matter}} + S_{\text{gf}} , \quad (3.1)$$

where¹

$$\begin{aligned}
S_{\text{CS}} &= -\frac{ik}{4\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \text{Tr} \left[A_{1\mu} \partial_\nu A_{1\rho} + \frac{2}{3} i A_{1\mu} A_{1\nu} A_{1\rho} - A_{2\mu} \partial_\nu A_{2\rho} - \frac{2}{3} i A_{2\mu} A_{2\nu} A_{2\rho} \right], \\
S_{\text{matter}} &= \int d^3x \, \text{Tr} \left[D_\mu C_I D^\mu \bar{C}^I + i \bar{\psi}^I \gamma^\mu D_\mu \psi_I \right] + S_{\text{int}}, \\
S_{\text{gf}} &= \frac{k}{4\pi} \int d^3x \, \text{Tr} \left[\frac{1}{\xi} (\partial_\mu A_1^\mu)^2 + \partial_\mu \bar{c} D^\mu c - \frac{1}{\xi} (\partial_\mu A_2^\mu)^2 - \partial_\mu \bar{\hat{c}} D^\mu \hat{c} \right].
\end{aligned}$$

The covariant derivatives are defined as

$$\begin{aligned}
D_\mu C_I &= \partial_\mu C_I + i(A_{1\mu} C_I - C_I A_{2\mu}), & D_\mu \bar{C}^I &= \partial_\mu \bar{C}^I - i(\bar{C}^I A_{1\mu} - A_{2\mu} \bar{C}^I), \\
D_\mu \bar{\psi}^I &= \partial_\mu \bar{\psi}^I + i(A_{1\mu} \bar{\psi}^I - \bar{\psi}^I A_{2\mu}), & D_\mu \psi_I &= \partial_\mu \psi_I - i(\psi_I A_{1\mu} - A_{2\mu} \psi_I).
\end{aligned} \tag{3.2}$$

The theory has $\mathcal{N} = 6$ supersymmetry and is invariant under 24 supersymmetry transformations generated by Q_{IJ} and S_{IJ} . Generators Q come accompanied by their respective parameters $\bar{\theta}$ and generate super-Poincaré transformations. Analogously, generators S come accompanied by parameters $\bar{\epsilon}$ and generate superconformal transformations. The fact that the theory is invariant under these supersymmetry transformations means that

$$\delta S = (\bar{\theta} Q + \bar{\epsilon} S) S = \bar{\theta}^{IJ\alpha} Q_{IJ\alpha} S + \bar{\epsilon}^{IJ\alpha} S_{IJ\alpha} S = 0,$$

for any Q and S or, equivalently, any $\bar{\theta}$ and $\bar{\epsilon}$. Note that indices I and J go from 1 to 4 and are anti-symmetric, giving rise to $\frac{4 \times 3}{2} = 6$ parameters. In addition to that, there are two possibilities for the spinor index α , which we denote as $+$ and $-$. Thus there is a total of $6 \times 2 = 12$ parameters $\bar{\theta}$ and 12 parameters $\bar{\epsilon}$ that leave S invariant, giving the 24 supercharges preserved by the theory.

In what follows we will focus on supersymmetric WLs within this theory. So we should clarify that for any BPS operator $\mathcal{O}_{\text{BPS}}$, its invariance under supersymmetry transformations is shown to hold only for certain linear combinations of generators Q and S . So, in contrast to the notation above, we denote a particular preserved linear combination of generators as $\mathcal{Q}$. In this case we have that, under the action of a preserved supercharge $\mathcal{Q}$, the operator satisfies

$$\delta \mathcal{O}_{\text{BPS}} = \bar{\Theta} \mathcal{Q} \mathcal{O}_{\text{BPS}} = 0.$$

¹The interaction terms in S_{int} will only play a role in the perturbative computation and, in particular, through the interaction vertices in (A.3) and (A.4). Therefore we do not write it explicitly. Also, the piece S_{gf} will play no role since in our perturbative computation we use Feynman rules in Landau gauge and therefore ghost fields $\xi, c, \hat{c}$ are removed.

The parameter $\bar{\Theta}$ is read from $\mathcal{Q}$ through the appropriate index contractions, namely $\bar{\Theta}^{IJ\alpha} \mathcal{Q}_{IJ\alpha}$. Moreover, for any preserved supercharge $\mathcal{Q}$, the corresponding parameter $\bar{\Theta}$ is a conformal Killing spinor. In particular, gauge and matter fields transform under supersymmetry transformations generated by these parameters as (A.2).

3.1 Supersymmetric Wilson loops

As explained in the previous chapter, one can construct supersymmetric Wilson loops within this theory and, in particular, there are two main ways of doing so. One way is to allow the loop to couple only to one node of the quiver in Figure 6, such that it contains vector and scalar fields [28–30]. The other way is to allow the loop to couple to the two nodes of the theory, including not only vector and scalar fields, but also fermions [24, 31].

To avoid cluttering, we intentionally skipped the details in the narrative of Chapter 2, but recall that every loop is defined along some particular contour. In what follows we specialise to either the circle C ,

$$C = (0, \cos \tau, \sin \tau), \quad (3.3)$$

or the latitude Γ ,

$$\Gamma = (\sin \theta_0, \cos \theta_0 \cos \tau, \cos \theta_0 \sin \tau) \quad \text{with} \quad -\frac{\pi}{2} \leq \theta_0 \leq \frac{\pi}{2}, \quad (3.4)$$

as shown in Figure 7. We will mainly state the form of the operators and include the corresponding references for more details.

Before proceeding, let us point that we do not include normalisation factors in the definition of the operators. These are usually included to guarantee that, at zero order in the perturbative computation, the expectation value of W is normalised to 1. Since a perturbative analysis is restricted to Section 3.4, we remove the normalisation everywhere else.

3.1.1 Circular loops

When defined along the circle C , the 1-node loop of ABJ(M) is invariant under 4 out of the 24 supercharges and therefore is 1/6 BPS. This operator was first discovered as the 1/2 BPS loop of the $\mathcal{N} = 2$ version of Chern-Simons-matter

Figure 7: Circle and latitude curves with their respective parameters.

theory [22]. It takes the form

$$W = \text{Tr } \mathcal{P} \exp \left(-i \oint \mathcal{A} d\tau \right), \quad \mathcal{A} \equiv A_\mu \dot{x}^\mu - i\sigma |\dot{x}|, \quad (3.5)$$

where, in accordance with [24], the path ordering is taken to be left-to-right. In $\mathcal{N} = 2$ theory, the σ field is the auxiliary field in the vector multiplet. In $\mathcal{N} = 6$ theory, the loops coupling to each node can be defined independently as

$$W_{N_1} = \text{Tr}_{N_1} \mathcal{P} \exp \left(-i \oint \mathcal{A}_1 d\tau \right) \quad \text{and} \quad W_{N_2} = \text{Tr}_{N_2} \mathcal{P} \exp \left(-i \oint \mathcal{A}_2 d\tau \right), \quad (3.6)$$

where

$$\mathcal{A}_1 = A_{1\mu} \dot{x}^\mu - i\sigma_1 |\dot{x}| \quad \text{and} \quad \mathcal{A}_2 = A_{2\mu} \dot{x}^\mu - i\sigma_2 |\dot{x}|. \quad (3.7)$$

Scalar bilinears are then obtained by integrating out the auxiliary fields σ_1 and σ_2 . In particular, in [28–30] these were fixed to be

$$\sigma_1 = \frac{2\pi}{k} M_J^I C_I \bar{C}^J \quad \text{and} \quad \sigma_2 = \frac{2\pi}{k} M_J^I \bar{C}^J C_I, \quad (3.8)$$

with $M = \text{diag}(-1, -1, +1, +1)$.

The scalar coupling matrix M dictates the form in which scalar bilinears appear in the loop. With the particular form above, the resulting loops preserve four supercharges parametrised by $\bar{\theta}^{12}$ and $\bar{\theta}^{34}$, accompanied by superconformal supersymmetries that are fixed to

$$\bar{\epsilon}^{12} = -i \frac{\sigma_3}{|x|} \bar{\theta}^{12} \quad \text{and} \quad \bar{\epsilon}^{34} = i \frac{\sigma_3}{|x|} \bar{\theta}^{34}. \quad (3.9)$$

We can explicitly check that the loops W_{N_1} and W_{N_2} are invariant under supersymmetry transformations generated by these parameters.

For instance, let us take W_{N_1} . We begin by introducing projectors

$$\Pi_{\pm} \equiv \frac{1}{2} \left(1 \pm \frac{\dot{x}^\mu \gamma_\mu}{|\dot{x}|} \right) = \frac{1}{2} \begin{pmatrix} 1 & \mp i e^{-i\tau} \\ \pm i e^{i\tau} & 1 \end{pmatrix}. \quad (3.10)$$

Then, (3.9) implies that each Killing spinor $\bar{\Theta}$ can be written as

$$\bar{\Theta}^{12} = 2\Pi_+ \bar{\theta}^{12} = 2\bar{\theta}^{12} \Pi_- \quad \text{and} \quad \bar{\Theta}^{34} = 2\Pi_- \bar{\theta}^{34} = 2\bar{\theta}^{34} \Pi_+, \quad (3.11)$$

since the projectors satisfy $(\Pi_{\pm})_{\alpha}^{\beta} = -(\Pi_{\mp})_{\beta}^{\alpha}$.² Plugging these into the variation of each field, according to the rules of Appendix A, we obtain

$$\begin{aligned} \delta A_{1\mu} &= \frac{4\pi i}{k} \bar{\Theta}^{12\alpha} (\gamma_\mu)_\alpha{}^\beta \left[C_1 \psi_{2\beta} - C_2 \psi_{1\beta} + \bar{\psi}_\beta^3 \bar{C}^4 - \bar{\psi}_\beta^4 \bar{C}^3 \right] \\ &\quad + \frac{4\pi i}{k} \bar{\Theta}^{34\alpha} (\gamma_\mu)_\alpha{}^\beta \left[C_3 \psi_{4\beta} - C_4 \psi_{3\beta} + \bar{\psi}_\beta^1 \bar{C}^2 - \bar{\psi}_\beta^2 \bar{C}^1 \right], \end{aligned} \quad (3.12)$$

$$\delta C_I = 2\bar{\Theta}^{12\alpha} \epsilon_{12IJ} \bar{\psi}_\alpha^J + 2\bar{\Theta}^{34\alpha} \epsilon_{34IJ} \bar{\psi}_\alpha^J, \quad (3.13)$$

$$\delta \bar{C}^J = 2\bar{\Theta}^{JK\alpha} \psi_{K\alpha}. \quad (3.14)$$

Now, the particular combination

$$\mathcal{A}_1 = A_{1\mu} \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| M_J^I C_I \bar{C}^J \quad (3.15)$$

is such that

$$\begin{aligned} \delta \mathcal{A}_1 &= \frac{4\pi i}{k} \bar{\Theta}^{12\alpha} (1 + \dot{x}^\mu \gamma_\mu)_\alpha{}^\beta \left[C_1 \psi_{2\beta} - C_2 \psi_{1\beta} + \bar{\psi}_\beta^3 \bar{C}^4 - \bar{\psi}_\beta^4 \bar{C}^3 \right] \\ &\quad - \frac{4\pi i}{k} \bar{\Theta}^{34\alpha} (1 - \dot{x}^\mu \gamma_\mu)_\alpha{}^\beta \left[C_3 \psi_{4\beta} - C_4 \psi_{3\beta} + \bar{\psi}_\beta^1 \bar{C}^2 - \bar{\psi}_\beta^2 \bar{C}^1 \right] \\ &= \frac{8\pi i}{k} \bar{\Theta}^{12\alpha} (\Pi_+)_\alpha{}^\beta \left[C_1 \psi_{2\beta} - C_2 \psi_{1\beta} + \bar{\psi}_\beta^3 \bar{C}^4 - \bar{\psi}_\beta^4 \bar{C}^3 \right] \\ &\quad - \frac{8\pi i}{k} \bar{\Theta}^{34\alpha} (\Pi_-)_\alpha{}^\beta \left[C_3 \psi_{4\beta} - C_4 \psi_{3\beta} + \bar{\psi}_\beta^1 \bar{C}^2 - \bar{\psi}_\beta^2 \bar{C}^1 \right]. \end{aligned}$$

²As explained in Appendix A, spinor indices are omitted as in (A.1) and are raised and lowered with the appropriate ϵ symbol. Explicitly, $\bar{\Theta}_\alpha^{12} = 2(\Pi_+)_\alpha{}^\beta \bar{\theta}_\beta^{12} = -2(\Pi_-)_\alpha{}^\beta \bar{\theta}_\beta^{12} = -2\bar{\theta}_\beta^{12} (\Pi_-)^\beta{}_\alpha = 2\bar{\theta}_\beta^{12} (\Pi_-)_{\beta\alpha}$.

From (3.11), it follows that

$$\bar{\Theta}^{12\alpha}(\Pi_+)_{\alpha}{}^{\beta} = 0 \quad \text{and} \quad \bar{\Theta}^{34\alpha}(\Pi_-)_{\alpha}{}^{\beta} = 0, \quad (3.16)$$

since projectors satisfy $\Pi_- \Pi_+ = \Pi_+ \Pi_- = 0$. Consequently $\delta \mathcal{A}_1 = 0$ and the loop W_{N_1} indeed preserves 4 supercharges parametrised by $\bar{\theta}^{12}$ and $\bar{\theta}^{34}$ accompanied by special parameters fixed as in (3.9). Thus the operator is indeed 1/6 BPS. An analogous computation can be carried for W_{N_2} , giving $\delta \mathcal{A}_2 = 0$.

It is worth noticing that in this check of supersymmetry invariance, it is crucial to write $\bar{\Theta}$ in terms of the projectors $\Pi_{\pm}$ and to have the specific form of $M = \text{diag}(-1, -1, +1, +1)$. These ingredients appear with mild variations throughout this work. We should also comment that, as can be seen from the structure of M , the loop has a residual $SU(2) \times SU(2)$ R-symmetry. This residual symmetry is relevant because it should be also manifest in the dual setting, i.e., in the string theory side the corresponding semi-classical string surface should have $SU(2) \times SU(2)$ bosonic symmetry.

This gives the simplest example of a circular supersymmetric loop. Its vev was computed exactly through localization in [5]. On the string theory side, this 1/6 BPS solution matched a $\mathbb{CP}^3$ smeared string. This was the first non-trivial check of the AdS/CFT duality in the context of ABJ(M).

The 2-node version of this loop is based on a superconnection $\mathcal{L}$ that is a $(N_1 + N_2) \times (N_1 + N_2)$ -dimensional supermatrix. For a loop supported along a particular curve in Euclidean space, $\mathcal{L}$ has the generic form

$$\mathcal{L} = \begin{pmatrix} \mathcal{A}_1 & -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\eta_I^{\alpha}\bar{\psi}_{\alpha}^I \\ -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\psi_I^{\alpha}\bar{\eta}_{\alpha}^I & \mathcal{A}_2 \end{pmatrix}, \quad \begin{aligned} \mathcal{A}_1 &= A_{1\mu}\dot{x}^{\mu} - \frac{2\pi i}{k}|\dot{x}|M_J^I C_I \bar{C}^J, \\ \mathcal{A}_2 &= A_{2\mu}\dot{x}^{\mu} - \frac{2\pi i}{k}|\dot{x}|M_J^I \bar{C}^J C_I. \end{aligned} \quad (3.17)$$

The fermi fields appear through fermionic couplings η and $\bar{\eta}$ that are Grassman even quantities, such that the off-diagonal blocks of $\mathcal{L}$ are Grassmann odd and $\mathcal{L}$ is indeed a supermatrix.

When the loop is supported along the circle C , the couplings

$$\eta_I^{\alpha}(\tau) = \begin{pmatrix} e^{i\tau/2} & -ie^{-i\tau/2} \end{pmatrix} \delta_I^1, \quad \bar{\eta}_{\alpha}^I(\tau) = \begin{pmatrix} ie^{-i\tau/2} \\ -e^{i\tau/2} \end{pmatrix} \delta_1^I, \quad (3.18)$$

and

$$M = \text{diag}(-1, +1, +1, +1),$$

make the traced loop,

$$W = \text{Tr } \mathcal{P} \exp \left(-i \oint_C \mathcal{L}(\tau) d\tau \right), \quad (3.19)$$

invariant under half of the supersymmetries of the theory [24]. In particular, this operator is invariant under the action of supersymmetry transformations parameterised by $\bar{\theta}^{1I}$ and $\bar{\theta}^{IJ}$, $I, J \neq 1$, with superconformal parameters fixed to

$$\bar{\epsilon}^{1I} = -i \frac{\sigma_3}{|x|} \bar{\theta}^{1I} \quad \text{and} \quad \bar{\epsilon}^{IJ} = i \frac{\sigma_3}{|x|} \bar{\theta}^{IJ}, \quad I, J \neq 1. \quad (3.20)$$

The parametrisation (3.20) is such that $\bar{\Theta}^{1I} = 2\Pi_+ \bar{\theta}^{1I}$ and $\bar{\Theta}^{IJ} = 2\Pi_- \bar{\theta}^{IJ}$, $I, J \neq 1$. In terms of these, the variation of each component of the loop is given by

$$\begin{aligned} \delta \mathcal{A}_1 &= \frac{8\pi i}{k} C_I \psi_1 \bar{\Theta}^{1I} + \frac{4\pi i}{k} \epsilon_{1IJK} \bar{\Theta}^{IJ} \bar{\psi}^1 \bar{C}^K, \\ \delta \mathcal{A}_2 &= \frac{8\pi i}{k} \psi_1 C_I \bar{\Theta}^{1I} + \frac{4\pi i}{k} \epsilon_{1IJK} \bar{\Theta}^{IJ} \bar{C}^K \bar{\psi}^1, \\ \eta_1(\tau) \delta \bar{\psi}^1 &= \mathcal{D}_\tau (4i \eta_1 \bar{\theta}^{1I} C_I), \\ \delta \psi_1 \bar{\eta}^1(\tau) &= \mathcal{D}_\tau (-2i \epsilon_{1IJK} \bar{\theta}^{IJ} \bar{\eta}^1 \bar{C}^K). \end{aligned} \quad (3.21)$$

The covariant derivatives above are written in terms of the dressed bosonic connections, i.e., instead of the usual definition in (3.2) we have

$$\begin{aligned} \mathcal{D}_\tau C_I &\equiv \dot{x}^\mu \partial_\mu C_I + i(\mathcal{A}_1 C_I - C_I \mathcal{A}_2), \\ \mathcal{D}_\tau \bar{C}^I &\equiv \dot{x}^\mu \partial_\mu \bar{C}^I - i(\bar{C}^I \mathcal{A}_1 - \mathcal{A}_2 \bar{C}^I). \end{aligned} \quad (3.22)$$

Plugging (3.21) into the variation of the superconnection,

$$\delta \mathcal{L} = \begin{pmatrix} \delta \mathcal{A}_1 & -i \sqrt{\frac{2\pi}{k}} |\dot{x}| \eta_I \delta \bar{\psi}^I \\ -i \sqrt{\frac{2\pi}{k}} |\dot{x}| \delta \psi_I \bar{\eta}^I & \delta \mathcal{A}_2 \end{pmatrix}, \quad (3.23)$$

we find that (2.5) is satisfied with

$$\mathcal{G} = 2 \sqrt{\frac{2\pi}{k}} \begin{pmatrix} 0 & 2\eta \bar{\theta}^{1I} C_I \\ -\epsilon_{1IJK} \bar{\theta}^{IJ} \bar{\eta} \bar{C}^K & \end{pmatrix}. \quad (3.24)$$

This loop is dual to a fundamental $\mathbb{CP}^1$ localized string solution preserving 1/2 of the supercharges of the vacuum and $U(1) \times SL(2, \mathbb{R}) \times SU(3)$ bosonic symmetry. Accordingly, note that the scalar coupling $M = \text{diag}(-1, +1, +1, +1)$ of this 2-node loop has precisely $SU(3)$ residual symmetry. Chronologically, this

string solution was known before the construction of the corresponding loop on the gauge theory side. Finding the gauge theory dual relied on allowing, for the first time, the loop to couple to fermions.

3.1.2 Latitude loops

When defined along the latitude Γ in (3.4) the 1-node loop of ABJ(M) is invariant under 2 out of the 24 supercharges and is 1/12 BPS. As in the circular case, it can be defined independently for each gauge group and the respective dressed bosonic connections are

$$\begin{aligned}\mathcal{A}_1 &= A_{1\mu}\dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}|M_J^I C_I \bar{C}^J, \\ \mathcal{A}_2 &= A_{2\mu}\dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}|M_J^I \bar{C}^J C_I,\end{aligned}$$

with

$$M = \begin{pmatrix} -\nu & e^{-i\tau}\sqrt{1-\nu^2} & 0 & 0 \\ e^{i\tau}\sqrt{1-\nu^2} & \nu & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here the M matrix is written in terms of what is called the effective “latitude parameter”,

$$\nu \equiv \sin 2\alpha \cos \theta_0, \quad (3.25)$$

where α is an arbitrary parameter that can be freely chosen in $[0, \frac{\pi}{2}]$ and θ_0 is the latitude angle. The operator is invariant under the action of supersymmetry transformations parameterised by two spinorial parameters ω_i , $i = 1, 2$. The non-vanishing components of $\bar{\Theta}^{IJ}$ are

$$\begin{aligned}\bar{\theta}_+^{13} &= e^{\frac{i\theta_0}{2}}\sqrt{1+\nu}\omega_1, & \bar{\epsilon}_+^{13} &= -ie^{-\frac{i\theta_0}{2}}\sqrt{1+\nu}\omega_1, \\ \bar{\theta}_-^{23} &= -ie^{\frac{i\theta_0}{2}}\sqrt{1-\nu}\omega_1, & \bar{\epsilon}_-^{23} &= e^{-\frac{i\theta_0}{2}}\sqrt{1-\nu}\omega_1, \\ \bar{\theta}_+^{14} &= e^{-\frac{i\theta_0}{2}}\sqrt{1-\nu}\omega_2, & \bar{\epsilon}_+^{14} &= ie^{\frac{i\theta_0}{2}}\sqrt{1-\nu}\omega_2, \\ \bar{\theta}_-^{24} &= -ie^{-\frac{i\theta_0}{2}}\sqrt{1+\nu}\omega_2, & \bar{\epsilon}_-^{24} &= -e^{\frac{i\theta_0}{2}}\sqrt{1+\nu}\omega_2.\end{aligned}$$

Going to its 2-node version, we find that for a superconnection $\mathcal{L}$ of the form (3.17) with couplings

$$\begin{aligned}
\eta_I^\alpha(\tau) &= \frac{e^{\frac{i\nu\tau}{2}}}{\sqrt{2}} \begin{pmatrix} \sqrt{1+\nu} \\ -\sqrt{1-\nu}e^{i\tau} \\ 0 \\ 0 \end{pmatrix}_I (1 \quad -ie^{-i\tau})^\alpha, \\
\bar{\eta}_\alpha^I(\tau) &= \frac{e^{-\frac{i\nu\tau}{2}}}{\sqrt{2}} (\sqrt{1+\nu} \quad -\sqrt{1-\nu}e^{-i\tau} \quad 0 \quad 0)^I \begin{pmatrix} 1 \\ ie^{i\tau} \end{pmatrix}_\alpha, \\
M_J^I &= \begin{pmatrix} -\nu & e^{-i\tau}\sqrt{1-\nu^2} & 0 & 0 \\ e^{i\tau}\sqrt{1-\nu^2} & \nu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{3.26}$$

the supertraced³ operator

$$W = \text{sTr} \left(\mathcal{P} \exp \left(-i \oint_{\Gamma} \mathcal{L} d\tau \right) \mathcal{T} \right) \tag{3.27}$$

is 1/6 BPS. Here the matrix $\mathcal{T}$, called “twist matrix”, is explicitly inserted in the supertrace to guarantee invariance of W under supergauge transformations. In the latitude case, it is given by

$$\mathcal{T} = \begin{pmatrix} e^{-\frac{i\pi\nu}{2}} \mathbb{I}_{N_1} & 0 \\ 0 & e^{\frac{i\pi\nu}{2}} \mathbb{I}_{N_2} \end{pmatrix}. \tag{3.28}$$

In particular, the operator in (3.27) is invariant under the action of supersymmetry transformations parameterised by four constants ω_i , $i = 1, \dots, 4$. The non-vanishing components of $\bar{\Theta}^{IJ}$ are

$$\begin{aligned}
\bar{\theta}_+^{13} &= e^{-\frac{i\theta_0}{2}} \sqrt{1-\nu} \omega_1 + e^{\frac{i\theta_0}{2}} \sqrt{1+\nu} \omega_2, & \bar{\epsilon}_+^{13} &= ie^{\frac{i\theta_0}{2}} \sqrt{1-\nu} \omega_1 - ie^{-\frac{i\theta_0}{2}} \sqrt{1+\nu} \omega_2, \\
\bar{\theta}_-^{23} &= -ie^{-\frac{i\theta_0}{2}} \sqrt{1+\nu} \omega_1 - ie^{\frac{i\theta_0}{2}} \sqrt{1-\nu} \omega_2, & \bar{\epsilon}_-^{23} &= e^{-\frac{i\theta_0}{2}} \sqrt{1-\nu} \omega_2 - e^{\frac{i\theta_0}{2}} \sqrt{1+\nu} \omega_1, \\
\bar{\theta}_+^{14} &= e^{-\frac{i\theta_0}{2}} \sqrt{1-\nu} \omega_3 + e^{\frac{i\theta_0}{2}} \sqrt{1+\nu} \omega_4, & \bar{\epsilon}_+^{14} &= ie^{\frac{i\theta_0}{2}} \sqrt{1-\nu} \omega_3 - ie^{-\frac{i\theta_0}{2}} \sqrt{1+\nu} \omega_4, \\
\bar{\theta}_-^{24} &= -ie^{-\frac{i\theta_0}{2}} \sqrt{1+\nu} \omega_3 - ie^{\frac{i\theta_0}{2}} \sqrt{1-\nu} \omega_4, & \bar{\epsilon}_-^{24} &= e^{-\frac{i\theta_0}{2}} \sqrt{1-\nu} \omega_4 - e^{\frac{i\theta_0}{2}} \sqrt{1+\nu} \omega_3.
\end{aligned}$$

This operator is actually a particular example within a larger family of loops defined along arbitrary curves on S^2 [25]. For $\alpha = 0$, the limit $\nu \rightarrow 0$ corresponds to the “Zarembo-like” loop [32]. For $\alpha = \pi/4$, we have the ABJ(M) analogue of the DGRT loops of $\mathcal{N}=4$ SYM [33]. For more details, we refer to [34].

³The supertrace sums the first and subtracts the second diagonal entry: given some diagonal matrix $\text{diag}(a, b)$, its supertrace gives $a - b$.

3.2 From 1-node to 2-node loops

With the 1-node and 2-node circular loops properly introduced, we have the required ingredients to address what was developed in the second chapter of [20]. In this collaboration we show how these two operators can be connected through a continuous deformation. As every chapter of [20] consists on a short review, here we provide more details on the construction and results we found.

The starting point of the project was the formulation of the 1/2 BPS circular Wilson loop according to [25, 34]. In contrast with the original formulation of [24], there the authors argue that a supersymmetric WL operator should be defined in terms of a supertrace instead of a trace, since the operator is constructed with supermatrices and only supertraces are invariant under cyclic permutations of their arguments. So they rewrite (3.19) as

$$W = \text{sTr} \left[\mathcal{P} \exp \left(-i \int_C \mathcal{L}(\tau) d\tau \right) \mathcal{T} \right], \quad (3.29)$$

where $\mathcal{T} = -i \text{diag}(\mathbb{I}_{N_1}, -\mathbb{I}_{N_2})$ is a twist matrix that, for short, we denote as $\mathcal{T} = -i\sigma_3$ with the understanding that each non-zero entry of σ_3 is actually a block diagonal matrix with the required dimensions. Note that this twist matrix is recovered from the latitude one in (3.28) when $\nu \rightarrow 1$.

As mentioned before, the twist matrix is included in the definition of the WL to enforce that the supertraced operator is invariant under supergauge transformations. Let us dedicate a few words to explain what we mean by that. Recall that under supersymmetry transformations preserved by the loop, the superconnection transforms as the supercovariant derivative of some supermatrix $\mathcal{G}$, i.e., $\delta\mathcal{L} = \mathcal{D}_\tau \mathcal{G}$. First, if we look at the structure of $\delta\mathcal{L}$ we see that derivative terms should appear only in its off-diagonal entries, since we only expect derivative terms to arise from the variation of fermions, see (A.2). Therefore $\mathcal{G}$ is always block off-diagonal. Furthermore, since we integrate this supercovariant derivative along the loop, we have to take into account its boundary conditions.

In the circular case, for instance, we have $\mathcal{G}$ given by (3.24) and, due to the anti-periodicity of the fermionic couplings in (3.18), it satisfies $\mathcal{G}(2\pi) = -\mathcal{G}(0)$. Therefore under finite supergauge transformations $U = \exp(-i\mathcal{G})$, we have

$$U(2\pi) = \mathcal{T}U(0)\mathcal{T}^{-1},$$

with $\mathcal{T} = -i\sigma_3$. Under the action of a preserved supercharge, the path ordered

exponential in (3.29) transforms as

$$w \equiv \exp \left(-i \int_C \mathcal{L}(\tau) d\tau \right) \mapsto U(0)wU^{-1}(2\pi),$$

such that the operator itself transforms as

$$\begin{aligned} \text{sTr}(w\mathcal{T}) &\mapsto \text{sTr}(U(0)wU^{-1}(2\pi)\mathcal{T}) = \text{sTr}(U(0)w\mathcal{T}\mathcal{T}^{-1}U^{-1}(2\pi)\mathcal{T}) \\ &= \text{sTr}(U(0)w\mathcal{T}U^{-1}(0)) \\ &= \text{sTr}(w\mathcal{T}). \end{aligned}$$

Thus the supertraced operator $\text{sTr}(w\mathcal{T})$ is indeed invariant supergauge transformations generated by $\mathcal{G}$ and is supersymmetric.

The circular loop has a twist matrix whose effect can be seen as multiplying the upper diagonal block by 1 and the lower by -1, thus taking the supertrace in (3.29) back to original trace of (3.19). So by looking only at this particular case, it seems that the two formulations are straightforwardly equivalent to each other. Nevertheless, the argument above carries over to all loops and, in particular, to the latitude case where $\mathcal{G}$ now satisfies $\mathcal{G}(2\pi) = \mathcal{T}\mathcal{G}(0)\mathcal{T}^{-1}$ with the non-trivial twist matrix (3.28) due to the boundary conditions of the fermionic couplings (3.4).

The punchline is that, due to the fact that η and $\bar{\eta}$ are not periodic around the loop, neither is $\mathcal{G}$ and we need to enforce periodicity by adding a twist matrix. Keeping this in mind, let us go back to the supertraced operator (3.29). Our initial idea when we started developing the second chapter of [20] was to make a gauge transformation parameterised by

$$\Lambda = \frac{(2\tau - \pi)}{8} \cdot \sigma_3,$$

where once more we apply the shorthand notation $\sigma_3 = \text{diag}(\mathbb{I}_{N_1}, \mathbb{I}_{N_2})$. I.e., we made abelian gauge transformations $U_1 = \mathbb{I}_{N_1} e^{-i\frac{(2\tau-\pi)}{8}} \in U(N_1)$ and $U_2 = \mathbb{I}_{N_2} e^{i\frac{(2\tau-\pi)}{8}} \in U(N_2)$, where $\pi/8$ accounts for different i factors we introduced for $\eta, \bar{\eta}$ compared to the original formulation. Under these gauge transformations the fields transform as

$$\begin{aligned} A_1 &\rightarrow U_1 A_1 U_1^{-1} - iU_1 dU_1^{-1} = A_1 + \frac{|\dot{x}|}{4|x|}, \\ A_2 &\rightarrow U_1 A_2 U_2^{-1} - iU_2 dU_2^{-1} = A_2 - \frac{|\dot{x}|}{4|x|}, \\ C_I &\rightarrow U_1 C_I U_2^{-1} = \sqrt{-i}e^{-i\tau/2} C_I \quad \bar{C}^I \rightarrow U_2 \bar{C}^I U_1^{-1} = \sqrt{i}e^{i\tau/2} \bar{C}^I, \end{aligned}$$

$$\bar{\psi}^I \rightarrow U_1 \bar{\psi}^I U_2^{-1} = \sqrt{-i} e^{-i\tau/2} \bar{\psi}^I, \quad \psi_I \rightarrow U_2 \psi_I U_1^{-1} = \sqrt{i} e^{i\tau/2} \psi_I.$$

The discontinuity of Λ at 2π yields a delta function term which can be integrated to cancel the twist matrix $\mathcal{T}$. Indeed, Λ is not a periodic function on $[0, 2\pi]$. It has a discontinuity, which we account for such that

$$\partial_\tau \Lambda = \left(\frac{1}{4} - \frac{\pi}{2} \delta(\tau - 2\pi) \right) \cdot \sigma_3. \quad (3.30)$$

We can account for this delta function contribution via

$$\mathcal{P} \exp \left(-i \int_{2\pi-\epsilon}^{2\pi+\epsilon} \mathcal{L} d\tau \right) = \exp \left(i \frac{\pi}{2} \sigma_3 \right) = i \sigma_3 = \mathcal{T}^{-1}, \quad (3.31)$$

such that in the new gauge the Wilson loop is

$$W = \text{sTr} \mathcal{P} \exp \left(-i \int_C \mathcal{L}'(\tau) d\tau \right), \quad (3.32)$$

with

$$\mathcal{L}' = \begin{pmatrix} A_{1\mu} \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| M_J^I C_I \bar{C}^J + \frac{1}{4} & \sqrt{-\frac{2\pi i}{k}} \eta'_I \bar{\psi}^I \\ \sqrt{-\frac{2\pi i}{k}} \psi_I \bar{\eta}'^I & A_{2\mu} \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| M_J^I \bar{C}^J C_I - \frac{1}{4} \end{pmatrix}. \quad (3.33)$$

The M matrix is unchanged, since scalars C and $\bar{C}$ change oppositely, whereas the fermionic couplings become

$$\eta'^\alpha_I = \begin{pmatrix} 1 & -ie^{-i\tau} \end{pmatrix} \delta^1_I \quad \text{and} \quad \bar{\eta}'^I_\alpha(\tau) = \begin{pmatrix} 1 \\ ie^{i\tau} \end{pmatrix} \delta^I_1. \quad (3.34)$$

The advantage of the formulation (3.32) is that it provides a manifestly (super)gauge invariant operator without the aid of a twist matrix, since the couplings (3.34) are periodic along the circle. This comes, of course, at the expense of introducing the constant $\pm 1/4$ pieces in the dressed bosonic connections $\mathcal{A}_1$ and $\mathcal{A}_2$ that can be viewed as the coupling to a background field. Moreover, it also gives a reparametrisation invariant formulation of the couplings by allowing us to write them in terms of eigenstates of the projector Π_+ . This happens in contrast to all previous formulations, where the couplings contained extra phase factors depending on the parametrisation of the contour along which the loop is defined. To be precise, the new formulation gives fermionic couplings that are expressed in

terms of Π_+ as

$$\eta_I'^\alpha(\tau) = (s \, 2\Pi_+)^{\alpha} \delta_I^1 \quad \text{and} \quad \bar{\eta}'^I_\alpha(\tau) = (2\Pi_+ s)_{\alpha} \delta_1^I,$$

whereas the ones in the original formulation are written as

$$\eta_I^\alpha = e^{i\tau/2} (s \, 2\Pi_+)^{\alpha} \delta_I^1 \quad \text{and} \quad \bar{\eta}_\alpha^I = i e^{-i\tau/2} (2\Pi_+ s)_{\alpha} \delta_1^I,$$

for a constant spinor $s^\alpha = \begin{pmatrix} 1 & 0 \end{pmatrix}$.⁴ Since the observable should depend on its contour, but not on how we choose to parametrise it, reparametrisation invariance of the couplings and, consequently, of the loop is naturally expected.

Furthermore, this new formulation allowed us to go one step further and view the circular 2-node loop as a deformation of the 1-node one. The idea behind this deformation procedure is to start from the 1/6 BPS loops in (3.6) and construct a composite bosonic connection $\mathcal{L}_{\text{bos}}$ containing the constant $\pm 1/4$ shifts,

$$\mathcal{L}_{\text{bos}} = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix} + \frac{|\dot{x}|}{4|x|} \sigma_3.$$

We want to add some deformation $\Delta\mathcal{L}$ to $\mathcal{L}_{\text{bos}}$ such that the resulting superconnection,

$$\mathcal{L} = \mathcal{L}_{\text{bos}} + \Delta\mathcal{L},$$

varies as a supercovariant derivative of some supermatrix. The resulting operator,

$$W = \text{sTr} \, \mathcal{P} \exp \left(-i \int_C \mathcal{L} \, d\tau \right), \quad (3.35)$$

should then be supersymmetric and couple also to fermions. In particular, the deformation $\Delta\mathcal{L}$ that we propose is given by

$$\Delta\mathcal{L} = i\mathcal{Q}\mathcal{G} + \mathcal{G}^2, \quad \text{where} \quad \mathcal{G} = \begin{pmatrix} 0 & \bar{\alpha}^i C_i \\ \alpha_i \bar{C}^i & 0 \end{pmatrix},$$

$\alpha_i, \bar{\alpha}_i, i = 1, 2$, are constant complex parameters that are not necessarily complex conjugates and $\mathcal{Q}$ is some supercharge that we will define in what follows.

Before moving on to the details of the construction, let us discuss the general idea behind the expression above. First, the $\mathcal{G}^2$ piece adds to $\mathcal{L}_{\text{bos}}$ some diagonal contribution of scalar bilinears. To that we also add some off-diagonal contribution composed by fermions coming from $\mathcal{Q}\mathcal{G}$. In the same spirit as the fermionic

⁴This formulation was extended to DGRT-like loops in Chapter 7 of [20].

variations in (3.21), this piece should be such that its variation gives a covariant derivative of scalars written in terms of dressed bosonic connections. Namely we should have

$$\delta(\mathcal{Q}\mathcal{G}) \sim \mathcal{D}_\tau(\theta\mathcal{G}),$$

which in terms of $\mathcal{Q}$ translates to

$$\mathcal{Q}^2\mathcal{G} \sim \mathcal{D}_\tau\mathcal{G}.$$

We choose $\mathcal{Q}$ to be given by a linear combination of the supercharges preserved by the 1-node loops that we started up with, such that the resulting operator should also be invariant under its action. At the end of the day we expect the following to hold,

$$\delta\mathcal{L} = \delta(\mathcal{L}_{\text{bos}} + \Delta\mathcal{L}) = \delta(\Delta\mathcal{L}) \stackrel{!}{\sim} \mathcal{D}_\tau(\theta\mathcal{G}) = \partial_\tau(\theta\mathcal{G}) + i[\mathcal{L}, \theta\mathcal{G}], \quad (3.36)$$

such that the resulting operator is supersymmetric. Note that at this point we are constantly referring to three different types of derivatives, so let us review these before proceeding. First we have the usual covariant derivative of the theory D , defined in (3.2). Then we have its version with dressed bosonic connections denoted as $\mathcal{D}$, see (3.22). Finally, we have the supercovariant derivative $\mathfrak{D}$ in (2.5) which, in contrast to the previous two, acts on supermatrices instead of individual fields.

With this scenario in mind, we are ready to move on to the details of the computation. Our goal is to explicitly check that the resulting superconnection $\mathcal{L}$ is invariant under all supersymmetry transformations preserved by $\mathcal{L}_{\text{bos}}$. We begin by writing the operator $\mathcal{L}$ explicitly.

Recall that the 1-node loops containing $\mathcal{A}_1$ and $\mathcal{A}_2$ are invariant under supersymmetry transformations generated by $\bar{\theta}^{12}$ and $\bar{\theta}^{34}$ with superconformal parameters fixed as (3.9). We choose the supercharge $\mathcal{Q}$, with respect to which we will build our deformation $\Delta\mathcal{L}$, to be parametrised by $\bar{\theta}_+^{12}$ and $\bar{\theta}_-^{34}$, fixing superconformal parameters to

$$\bar{\epsilon}_+^{12} = -i\bar{\theta}_+^{12} \quad \text{and} \quad \bar{\epsilon}_-^{34} = -i\bar{\theta}_-^{34}.$$

We also rescale this supercharge by a factor of $\frac{1}{2\sqrt{2}}$, such that⁵

$$\mathcal{Q} = \frac{1}{2\sqrt{2}} \left((Q_{12}^+ - iS_{12}^+) + (Q_{34}^- - iS_{34}^-) \right).$$

According to (3.11), this choice of parameters fixes conformal Killing spinors $\bar{\Theta}$ to

$$\bar{\Theta}_\alpha^{12} = \begin{pmatrix} 1 \\ ie^{i\tau} \end{pmatrix} \bar{\theta}_+^{12} \quad \text{and} \quad \bar{\Theta}_\alpha^{34} = \begin{pmatrix} ie^{-i\tau} \\ 1 \end{pmatrix} \bar{\theta}_-^{34}.$$

Equivalently,

$$\bar{\Theta}^{12\alpha} = \begin{pmatrix} ie^{i\tau} & -1 \end{pmatrix} \bar{\theta}_+^{12} \quad \text{and} \quad \bar{\Theta}^{34\alpha} = \begin{pmatrix} 1 & -ie^{-i\tau} \end{pmatrix} \bar{\theta}_-^{34}.$$

Also, for later convenience, we write these in terms of $\Pi_\pm$ as

$$\begin{aligned} \bar{\Theta}_\alpha^{12} &= (2\Pi_+)_\alpha{}^\beta \hat{s}_\beta \bar{\theta}_+^{12}, & \bar{\Theta}^{12\alpha} &= \hat{s}^\beta (2\Pi_-)_\beta{}^\alpha \bar{\theta}_+^{12}, \\ \bar{\Theta}_\alpha^{34} &= (2\Pi_-)_\alpha{}^\beta s_\beta \bar{\theta}_-^{34}, & \bar{\Theta}^{34\alpha} &= s^\beta (2\Pi_+)_\beta{}^\alpha \bar{\theta}_-^{34}, \end{aligned}$$

where $\hat{s}$ and s are constant spinors,

$$\hat{s}_\beta \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad s_\beta \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In terms of these Killing spinors, the variation of scalars is given by

$$\begin{aligned} \delta C_1 &= 2\bar{\Theta}^{34}\bar{\psi}^2, & \delta C_2 &= -2\bar{\Theta}^{34}\bar{\psi}^1, \\ \delta \bar{C}^1 &= 2\bar{\Theta}^{12}\psi_2, & \delta \bar{C}^2 &= -2\bar{\Theta}^{12}\psi_1. \end{aligned} \tag{3.37}$$

Thus, recalling the factor of $\frac{1}{2\sqrt{2}}$ in the definition of $\mathcal{Q}$, its action on each scalar is

$$\begin{aligned} \mathcal{Q}C_1 &= \frac{1}{\sqrt{2}}(s \, 2\Pi_+)\bar{\psi}^2, & \mathcal{Q}C_2 &= -\frac{1}{\sqrt{2}}(s \, 2\Pi_+)\bar{\psi}^1, \\ \mathcal{Q}\bar{C}^1 &= \frac{1}{\sqrt{2}}(\hat{s} \, 2\Pi_-)\psi_2, & \mathcal{Q}\bar{C}^2 &= -\frac{1}{\sqrt{2}}(\hat{s} \, 2\Pi_-)\psi_1, \end{aligned}$$

⁵The prefactor in $\mathcal{Q}$ is precisely what is required to have $i\mathcal{Q}^2\mathcal{G} = \mathcal{D}_\tau\mathcal{G}$. This will become evident after we derive (3.45).

giving

$$i\mathcal{Q}\mathcal{G} = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}}(s\,2\Pi_+)(\bar{\alpha}^1\bar{\psi}^2 - \bar{\alpha}^2\bar{\psi}^1) \\ \frac{i}{\sqrt{2}}(\hat{s}\,2\Pi_-)(\alpha_1\psi_2 - \alpha_2\psi_1) & 0 \end{pmatrix}.$$

Therefore we can write $\mathcal{L}$ explicitly as

$$\mathcal{L} = \begin{pmatrix} \mathcal{A}'_1 & \frac{i}{\sqrt{2}}(s\,2\Pi_+)(\bar{\alpha}^1\bar{\psi}^2 - \bar{\alpha}^2\bar{\psi}^1) \\ \frac{i}{\sqrt{2}}(\hat{s}\,2\Pi_-)(\alpha_1\psi_2 - \alpha_2\psi_1) & \mathcal{A}'_2 \end{pmatrix}, \quad (3.38)$$

where

$$\mathcal{A}'_1 \equiv \mathcal{A}_1 + \bar{\alpha}^i\alpha_j C_i \bar{C}^j + \frac{1}{4} \quad \text{and} \quad \mathcal{A}'_2 \equiv \mathcal{A}_2 + \bar{\alpha}^i\alpha_j \bar{C}^j C_i - \frac{1}{4}.$$

To check whether the variation of $\mathcal{L}$ is equal to the aforementioned superco-variant derivative, we have to derive how each piece of $\mathcal{L}$ varies under $\mathcal{Q}$. In what follows we drop its $\frac{1}{2\sqrt{2}}$ factor since it appears equally in every piece of the calculation and therefore its is meaningless. Let us begin by computing the variation of the off-diagonal components of the superconnection. In particular, let us start from the top-right component containing $\bar{\psi}$. Using (A.2), we have

$$\begin{aligned} \delta\bar{\psi}^1 &= 2i\gamma^\mu\bar{\Theta}^{12}D_\mu C_2 - \frac{4\pi i}{k}\bar{\Theta}^{12}(C_2\bar{C}^I C_I - C_I\bar{C}^I C_2) \\ &\quad - \frac{8\pi i}{k}\left(\bar{\Theta}^{12}(C_1\bar{C}^1 C_2 - C_2\bar{C}^1 C_1) + \bar{\Theta}^{34}(C_3\bar{C}^1 C_4 - C_4\bar{C}^1 C_3)\right) - 2i\bar{\epsilon}^{12}C_2, \\ \delta\bar{\psi}^2 &= -2i\gamma^\mu\bar{\Theta}^{12}D_\mu C_1 + \frac{4\pi i}{k}\bar{\Theta}^{12}(C_1\bar{C}^I C_I - C_I\bar{C}^I C_1) \\ &\quad - \frac{8\pi i}{k}\left(\bar{\Theta}^{12}(C_1\bar{C}^2 C_2 - C_2\bar{C}^2 C_1) + \bar{\Theta}^{34}(C_3\bar{C}^2 C_4 - C_4\bar{C}^2 C_3)\right) + 2i\bar{\epsilon}^{12}C_1. \end{aligned} \quad (3.39)$$

In $\mathcal{L}$ these appear multiplied by $(s\,2\Pi_+)$, so it is worth listing the following properties of the projectors,

$$\Pi_\pm\gamma^\mu\Pi_\pm = \pm\Pi_\pm\dot{x}^\mu, \quad (\Pi_\pm)^2 = \Pi_\pm \quad \text{and} \quad \Pi_+\Pi_- = \Pi_-\Pi_+ = 0,$$

to derive the relations

$$\begin{aligned}
(s \, 2\Pi_+) \gamma^\mu \bar{\Theta}^{12} &= (s \, 2\Pi_+) \gamma^\mu (2\Pi_+ \hat{s}) \bar{\theta}_+^{12} = 2(s \, 2\Pi_+ \hat{s}) \dot{x}^\mu \bar{\theta}_+^{12} = 2\dot{x}^\mu \bar{\theta}_+^{12}, \\
(s \, 2\Pi_+) \bar{\Theta}^{12} &= (s \, 2\Pi_+) (2\Pi_+ \hat{s}) \bar{\theta}_+^{12} = 2(s \, 2\Pi_+ \hat{s}) \bar{\theta}_+^{12} = 2\bar{\theta}_+^{12}, \\
(s \, 2\Pi_+) \bar{\Theta}^{34} &= (s \, 2\Pi_+) (2\Pi_- s) \bar{\theta}_-^{34} = 0, \\
(s \, 2\Pi_+) \bar{\epsilon}^{12} &= \bar{\epsilon}_+^{12} = -i\bar{\theta}_+^{12}.
\end{aligned} \tag{3.40}$$

With these in hand, we find

$$\begin{aligned}
(s \, 2\Pi_+) \delta \bar{\psi}^1 &= 4i\bar{\theta}_+^{12} \left\{ \dot{x}^\mu D_\mu C_2 - \frac{2\pi}{k} (C_2 \bar{C}^I C_I - C_I \bar{C}^I C_2) \right. \\
&\quad \left. - \frac{4\pi}{k} (C_1 \bar{C}^1 C_2 - C_2 \bar{C}^1 C_1) + \frac{i}{2} C_2 \right\}, \\
(s \, 2\Pi_+) \delta \bar{\psi}^2 &= -4i\bar{\theta}_+^{12} \left\{ \dot{x}^\mu D_\mu C_1 - \frac{2\pi}{k} (C_1 \bar{C}^I C_I - C_I \bar{C}^I C_1) \right. \\
&\quad \left. + \frac{4\pi}{k} (C_1 \bar{C}^2 C_2 - C_2 \bar{C}^2 C_1) + \frac{i}{2} C_1 \right\},
\end{aligned}$$

such that the variation of the top-right component of $\mathcal{L}$ can be written as

$$\begin{aligned}
\frac{i}{\sqrt{2}} (s \, 2\Pi_+) (\bar{\alpha}^1 \delta \bar{\psi}^2 - \bar{\alpha}^2 \delta \bar{\psi}^1) &= 2\sqrt{2} \bar{\theta}_+^{12} \bar{\alpha}^i \left\{ D_\tau C_i - \frac{2\pi}{k} (C_i \bar{C}^I C_I - C_I \bar{C}^I C_i) \right. \\
&\quad \left. + \frac{4\pi}{k} (C_i \bar{C}^j C_j - C_j \bar{C}^j C_i) + \frac{i}{2} C_i \right\}.
\end{aligned}$$

The expression above can be put in the form:

$$\begin{aligned}
&\bar{\alpha}^i \left\{ D_\tau C_i - \frac{2\pi}{k} (C_i \bar{C}^I C_I - C_I \bar{C}^I C_i) + \frac{4\pi}{k} (C_i \bar{C}^j C_j - C_j \bar{C}^j C_i) + \frac{i}{2} C_i \right\} \\
&= \bar{\alpha}^i \left\{ \partial_\tau C_i + i \left(A_{1\mu} \dot{x}^\mu - \frac{2\pi i}{k} (C_I \bar{C}^I - 2C_j \bar{C}^j) + \frac{1}{4} \right) C_i \right. \\
&\quad \left. - i C_i \left(A_{2\mu} \dot{x}^\mu - \frac{2\pi i}{k} (C_I \bar{C}^I - 2C_j \bar{C}^j) - \frac{1}{4} \right) \right\} \\
&= \bar{\alpha}^i \left\{ \partial_\tau C_i + i \left(A_{1\mu} \dot{x}^\mu - \frac{2\pi i}{k} (C_I \bar{C}^I - 2C_j \bar{C}^j) + \bar{\alpha}^j \alpha_k C_j \bar{C}^k + \frac{1}{4} \right) C_i \right. \\
&\quad \left. - i C_i \left(A_{2\mu} \dot{x}^\mu - \frac{2\pi i}{k} (C_I \bar{C}^I - 2C_j \bar{C}^j) + \bar{\alpha}^j \alpha_k \bar{C}^k C_j - \frac{1}{4} \right) \right\}.
\end{aligned}$$

From the first to the second expression, we simply open up the definition of the supercovariant derivative. Then, from the second to the third, we include the contractions $\bar{\alpha}^j \alpha_k C_j \bar{C}^k$ and $\bar{\alpha}^j \alpha_k \bar{C}^k C_j$ since they cancel each other when multiplying

$\bar{\alpha}^i C_i$ from the left and from the right, respectively. Finally, note that

$$(C_I \bar{C}^I - 2C_J \bar{C}^J) = M_J^I C_I \bar{C}^J \quad \text{and} \quad (\bar{C}^I C_I - 2\bar{C}^J C_J) = M_J^I \bar{C}^J C_I.$$

Thus we find that the top-right component of $\mathcal{L}$ is such that

$$\begin{aligned} \delta \left(i \mathcal{Q}(\bar{\alpha}^i C_i) \right) &= \frac{i}{\sqrt{2}} (s \, 2\Pi_+) (\bar{\alpha}^1 \delta \bar{\psi}^2 - \bar{\alpha}^2 \delta \bar{\psi}^1) \\ &= 2\sqrt{2} \bar{\theta}_+^{12} \bar{\alpha}^i \left\{ \partial_\tau C_i + i \mathcal{A}'_1 C_i - i C_i \mathcal{A}'_2 \right\} \\ &= \mathcal{D}_\tau \left(2\sqrt{2} \bar{\theta}_+^{12} \bar{\alpha}^i C_i \right), \end{aligned} \tag{3.41}$$

where, as anticipated, the derivative is written in terms of the new dressed bosonic connections of $\mathcal{L}$.

We can carry out the same computation for the bottom-left entry containing ψ . In this case, from (A.2), we have

$$\begin{aligned} \delta \psi_1 &= 2i\gamma^\mu \bar{\Theta}^{34} D_\mu \bar{C}^2 + \frac{4\pi i}{k} \bar{\Theta}^{34} (\bar{C}^2 C_I \bar{C}^I - \bar{C}^I C_I \bar{C}^2) \\ &\quad + \frac{8\pi i}{k} \left(\bar{\Theta}^{12} (\bar{C}^3 C_1 \bar{C}^4 - \bar{C}^4 C_1 \bar{C}^3) + \bar{\Theta}^{34} (\bar{C}^1 C_1 \bar{C}^2 - \bar{C}^2 C_1 \bar{C}^1) \right) - 2i\bar{\epsilon}^{34} \bar{C}^2, \\ \delta \psi_2 &= -2i\gamma^\mu \bar{\Theta}^{34} D_\mu \bar{C}^1 - \frac{4\pi i}{k} \bar{\Theta}^{34} (\bar{C}^1 C_I \bar{C}^I - \bar{C}^I C_I \bar{C}^1) \\ &\quad + \frac{8\pi i}{k} \left(\bar{\Theta}^{12} (\bar{C}^3 C_2 \bar{C}^4 - \bar{C}^4 C_2 \bar{C}^3) + \bar{\Theta}^{34} (\bar{C}^1 C_2 \bar{C}^2 - \bar{C}^2 C_2 \bar{C}^1) \right) + 2i\bar{\epsilon}^{34} \bar{C}^1. \end{aligned} \tag{3.42}$$

In $\mathcal{L}$ these appear multiplied by $(\hat{s} \, 2\Pi_-)$, so we use the following relations

$$\begin{aligned} (\hat{s} \, 2\Pi_-) \gamma^\mu \bar{\Theta}^{34} &= (\hat{s} \, 2\Pi_-) \gamma^\mu (2\Pi_- s) \bar{\theta}^{34}_- = -2(\hat{s} \, 2\Pi_- s) \dot{x}^\mu \bar{\theta}^{34}_- = 2\dot{x}^\mu \bar{\theta}^{34}_-, \\ (\hat{s} \, 2\Pi_-) \bar{\Theta}^{34} &= (\hat{s} \, 2\Pi_-) (2\Pi_- s) \bar{\theta}^{34}_- = 2(\hat{s} \, 2\Pi_- s) \bar{\theta}^{34}_- = -2\bar{\theta}^{34}_-, \\ (\hat{s} \, 2\Pi_-) \bar{\Theta}^{12} &= (\hat{s} \, 2\Pi_-) (2\Pi_+ \hat{s}) \bar{\theta}^{12}_+ = 0, \\ (\hat{s} \, 2\Pi_-) \bar{\epsilon}^{34} &= -\bar{\epsilon}^{34}_- = i\bar{\theta}^{34}_-, \end{aligned} \tag{3.43}$$

to write

$$\begin{aligned} (\hat{s} \, 2\Pi_-) \delta \psi_1 &= 4i\bar{\theta}^{34}_- \left\{ \dot{x}^\mu D_\mu \bar{C}^2 - \frac{2\pi}{k} (\bar{C}^2 C_I \bar{C}^I - \bar{C}^I C_I \bar{C}^2) \right. \\ &\quad \left. - \frac{4\pi}{k} (\bar{C}^1 C_1 \bar{C}^2 - \bar{C}^2 C_1 \bar{C}^1) - \frac{i}{2} \bar{C}^2 \right\}, \\ (\hat{s} \, 2\Pi_-) \delta \psi_2 &= -4i\bar{\theta}^{34}_- \left\{ \dot{x}^\mu D_\mu \bar{C}^1 - \frac{2\pi}{k} (\bar{C}^1 C_I \bar{C}^I - \bar{C}^I C_I \bar{C}^1) \right. \end{aligned}$$

$$+ \frac{4\pi}{k} (\bar{C}^1 C_2 \bar{C}^2 - \bar{C}^2 C_2 \bar{C}^1) - \frac{i}{2} \bar{C}^1 \Big\}.$$

Then the bottom-left component of $\mathcal{L}$ varies as

$$\begin{aligned} \frac{i}{\sqrt{2}} (\hat{s} 2\Pi_-) (\alpha_1 \delta \psi_2 - \alpha_2 \delta \psi_1) &= 2\sqrt{2} \bar{\theta}_-^{34} \alpha_i \left\{ D_\tau \bar{C}^i - \frac{2\pi}{k} (\bar{C}^i C_I \bar{C}^I - \bar{C}^I C_I \bar{C}^i) \right. \\ &\quad \left. + \frac{4\pi}{k} (\bar{C}^i C_j \bar{C}^j - \bar{C}^j C_j \bar{C}^i) - \frac{i}{2} C_i \right\}. \end{aligned}$$

Doing the analogous algebraic steps we did for $\delta \bar{\psi}$, we find

$$\begin{aligned} \delta (i \mathcal{Q}(\alpha_i \bar{C}^i)) &= \frac{i}{\sqrt{2}} (\hat{s} 2\Pi_-) (\alpha_1 \delta \psi_2 - \alpha_2 \delta \psi_1) \\ &= 2\sqrt{2} \bar{\theta}_-^{34} \alpha_i \left\{ \partial_\tau \bar{C}^i - i \bar{C}^i \mathcal{A}'_1 + i \mathcal{A}'_2 \bar{C}^i \right\} \\ &= \mathcal{D}_\tau (2\sqrt{2} \bar{\theta}_-^{34} (\alpha_i \bar{C}^i)), \end{aligned} \tag{3.44}$$

and again the derivative is written in terms of the new dressed bosonic connections of $\mathcal{L}$.

Putting (3.41) and (3.44) together, we have that the variation of the off-diagonal components of the superconnection are given by

$$\delta(i \mathcal{Q} \mathcal{G}) = \begin{pmatrix} 0 & \mathcal{D}_\tau (2\sqrt{2} \bar{\theta}_+^{12} \bar{\alpha}^i C_i) \\ \mathcal{D}_\tau (2\sqrt{2} \bar{\theta}_-^{34} \alpha_i \bar{C}^i) & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \mathcal{D}_\tau g \\ \mathcal{D}_\tau \bar{g} & 0 \end{pmatrix}. \tag{3.45}$$

Note that if we keep the $\frac{1}{2\sqrt{2}}$ factor in the variation above, we recover precisely $\delta(i \mathcal{Q} \mathcal{G}) = \mathcal{D}_\tau(\theta \mathcal{G})$ and equivalently $i \mathcal{Q}^2 \mathcal{G} = \mathcal{D}_\tau \mathcal{G}$.

Now we can move to the variation of the diagonal entries of $\mathcal{L}$. In this case, we already know by construction that for the supercharges we are considering $\delta \mathcal{A}_1 = \delta \mathcal{A}_2 = 0$. Thus we only need to compute the variation of the piece arising from $\mathcal{G}^2$, namely

$$\begin{aligned} \bar{\alpha}^i \alpha_j \delta(C_i \bar{C}^j) &= \bar{\alpha}^i \alpha_j \left((\delta C_i) \bar{C}^j + C_i (\delta \bar{C}^j) \right), \\ \bar{\alpha}^i \alpha_j \delta(\bar{C}^j C_i) &= \bar{\alpha}^i \alpha_j \left((\delta \bar{C}^j) C_i + \bar{C}^j (\delta C_i) \right). \end{aligned}$$

Using (3.37), we write them as

$$\begin{aligned} \bar{\alpha}^i \alpha_j \delta(C_i \bar{C}^j) &= 2\bar{\alpha}^i \alpha_j \left(\epsilon_{ik} \bar{\Theta}^{34} \bar{\psi}^k \bar{C}^j + C_i \epsilon^{jk} \bar{\Theta}^{12} \psi_k \right), \\ \bar{\alpha}^i \alpha_j \delta(\bar{C}^j C_i) &= 2\bar{\alpha}^i \alpha_j \left(\epsilon^{jk} \bar{\Theta}^{12} \psi_k C_i + \bar{C}^j \epsilon_{ik} \bar{\Theta}^{34} \bar{\psi}^k \right). \end{aligned}$$

Focusing on the first expression, for instance, we can plug in the particular form of the Killing spinors and anti-commute Grassmann-odd quantities to rewrite it as

$$\begin{aligned}\bar{\alpha}^i \alpha_j \delta(C_i \bar{C}^j) &= 2\bar{\alpha}^i \alpha_j \left(\epsilon_{ik} (s \, 2\Pi_+) \bar{\theta}_-^{34} \bar{\psi}^k \bar{C}^j + C_i \epsilon^{jk} (\hat{s} \, 2\Pi_-) \bar{\theta}_+^{12} \psi_k \right) \\ &= i \left[\left(\frac{i}{\sqrt{2}} (s \, 2\Pi_+) \epsilon_{ik} \bar{\alpha}^i \bar{\psi}^k \right) (2\sqrt{2} \bar{\theta}_-^{34} \alpha_j \bar{C}^j) \right. \\ &\quad \left. - (2\sqrt{2} \bar{\theta}_+^{12} \bar{\alpha}^i C_i) \left(\frac{i}{\sqrt{2}} (\hat{s} \, 2\Pi_-) \epsilon^{jk} \alpha_j \psi_k \right) \right].\end{aligned}$$

We organized the expression above to make it easier to identify the quantities

$$\left(\frac{i}{\sqrt{2}} (s \, 2\Pi_+) \epsilon_{ik} \bar{\alpha}^i \bar{\psi}^k \right) = i\mathcal{Q}(\bar{\alpha}^i C_i) \quad \text{and} \quad (2\sqrt{2} \bar{\theta}_-^{34} \alpha_j \bar{C}^j) = \bar{g},$$

as well as

$$(2\sqrt{2} \bar{\theta}_+^{12} \bar{\alpha}^i C_i) = g \quad \text{and} \quad \left(\frac{i}{\sqrt{2}} (\hat{s} \, 2\Pi_-) \epsilon^{jk} \alpha_j \psi_k \right) = i\mathcal{Q}(\alpha_j \bar{C}^j),$$

to write

$$\bar{\alpha}^i \alpha_j \delta(C_i \bar{C}^j) = i \left[i\mathcal{Q}(\bar{\alpha}^i C_i) \bar{g} - g \, i\mathcal{Q}(\alpha_j \bar{C}^j) \right]. \quad (3.46)$$

Analogously, for the second expression, we find

$$\bar{\alpha}^i \alpha_j \delta(\bar{C}^j C_i) = i \left[i\mathcal{Q}(\alpha_j \bar{C}^j) g - \bar{g} \, i\mathcal{Q}(\bar{\alpha}^i C_i) \right]. \quad (3.47)$$

Therefore, for the superconnection $\mathcal{L}$ in (3.38), we find that its variation gives

$$\delta\mathcal{L} = \begin{pmatrix} i \left(i\mathcal{Q}(\bar{\alpha}^i C_i) \bar{g} - g \, i\mathcal{Q}(\alpha_j \bar{C}^j) \right) & \mathcal{D}_\tau g \\ \mathcal{D}_\tau \bar{g} & i \left(i\mathcal{Q}(\alpha_j \bar{C}^j) g - \bar{g} \, i\mathcal{Q}(\bar{\alpha}^i C_i) \right) \end{pmatrix},$$

which exactly matches the supercovariant derivative of

$$\begin{pmatrix} 0 & g \\ \bar{g} & 0 \end{pmatrix}. \quad (3.48)$$

Indeed,

$$\begin{aligned}
\mathfrak{D}_\tau \begin{pmatrix} 0 & g \\ \bar{g} & 0 \end{pmatrix} &= \partial_\tau \begin{pmatrix} 0 & g \\ \bar{g} & 0 \end{pmatrix} + i \left[\mathcal{L}, \begin{pmatrix} 0 & g \\ \bar{g} & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & \partial_\tau g \\ \partial_\tau \bar{g} & 0 \end{pmatrix} + i \left[\begin{pmatrix} \mathcal{A}'_1 & i\mathcal{Q}(\bar{\alpha}^i C_i) \\ i\mathcal{Q}(\alpha_i \bar{C}^i) & \mathcal{A}'_2 \end{pmatrix}, \begin{pmatrix} 0 & g \\ \bar{g} & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} i \left(i\mathcal{Q}(\bar{\alpha}^i C_i) \bar{g} - g i\mathcal{Q}(\alpha_j \bar{C}^j) \right) & \partial_\tau g + i(\mathcal{A}'_1 g - g \mathcal{A}'_2) \\ \partial_\tau \bar{g} + i(\mathcal{A}'_2 \bar{g} - \bar{g} \mathcal{A}'_1) & i \left(i\mathcal{Q}(\alpha_j \bar{C}^j) g - \bar{g} i\mathcal{Q}(\bar{\alpha}^i C_i) \right) \end{pmatrix} \\
&= \begin{pmatrix} i \left(i\mathcal{Q}(\bar{\alpha}^i C_i) \bar{g} - g i\mathcal{Q}(\alpha_j \bar{C}^j) \right) & \mathcal{D}_\tau g \\ \mathcal{D}_\tau \bar{g} & i \left(i\mathcal{Q}(\alpha_j \bar{C}^j) g - \bar{g} i\mathcal{Q}(\bar{\alpha}^i C_i) \right) \end{pmatrix}.
\end{aligned}$$

The computation above was performed for supersymmetry transformations generated by $\bar{\theta}_+^{12}$ and $\bar{\theta}_-^{34}$ with superconformal parameters fixed as in (3.11). It shows us that $\mathcal{L}$ preserves $Q_{12}^+ - iS_{12}^+$ and $Q_{34}^- - iS_{34}^-$ separately, since for the former we get

$$\delta \mathcal{L} = \mathfrak{D}_\tau \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix},$$

whereas for the latter

$$\delta \mathcal{L} = \mathfrak{D}_\tau \begin{pmatrix} 0 & 0 \\ \bar{g} & 0 \end{pmatrix}.$$

Nevertheless, recall that our goal is to show that $\mathcal{L}$ is invariant under all supersymmetry transformations preserved by $\mathcal{L}_{\text{bos}}$. Thus we still need to check whether $\mathcal{L}$ is invariant under supersymmetry transformations generated by $\bar{\theta}_-^{12}$ and $\bar{\theta}_+^{34}$. As we did above, we will do this by carrying out a single computation to show invariance of $\mathcal{L}$ under both $Q_{12}^- + iS_{12}^-$ and $Q_{34}^+ + iS_{34}^+$. Namely, we will use

$$\mathcal{Q}' = (Q_{12}^- + iS_{12}^-) + (Q_{34}^+ + iS_{34}^+). \quad (3.49)$$

For clarity, we will refer to the corresponding variations as δ' .

First, turning on $\bar{\theta}_-^{12}$ and $\bar{\theta}_+^{34}$ implies on superconformal parameters $\bar{\epsilon}_-^{12} = i\bar{\theta}_-^{12}$ and $\bar{\epsilon}_+^{34} = i\bar{\theta}_+^{34}$, such that the Killing spinors are given by

$$\bar{\Theta}_\alpha^{12} = \begin{pmatrix} -ie^{-i\tau} \\ 1 \end{pmatrix} \bar{\theta}_-^{12} \quad \text{and} \quad \bar{\Theta}_\alpha^{34} = \begin{pmatrix} 1 \\ -ie^{i\tau} \end{pmatrix} \bar{\theta}_+^{34}. \quad (3.50)$$

Equivalently,

$$\bar{\Theta}^{12\alpha} = \begin{pmatrix} 1 & ie^{-i\tau} \end{pmatrix} \bar{\theta}_-^{12} \quad \text{and} \quad \bar{\Theta}^{34\alpha} = \begin{pmatrix} -ie^{i\tau} & -1 \end{pmatrix} \bar{\theta}_+^{34}. \quad (3.51)$$

In terms of spinors s and $\hat{s}$, these can be written as

$$\begin{aligned} \bar{\Theta}_\alpha^{12} &= (2\Pi_+)_\alpha{}^\beta s_\beta \bar{\theta}_-^{12}, & \bar{\Theta}^{12\alpha} &= s^\beta (2\Pi_-)_\beta{}^\alpha \bar{\theta}_-^{12}, \\ \bar{\Theta}_\alpha^{34} &= (2\Pi_-)_\alpha{}^\beta \hat{s}_\beta \bar{\theta}_+^{34}, & \bar{\Theta}^{34\alpha} &= \hat{s}^\beta (2\Pi_+)_\beta{}^\alpha \bar{\theta}_+^{34}. \end{aligned}$$

With these in hand, we are ready to reproduce the analogues of (3.40) and (3.43).

For the Killing spinors above, we find

$$\begin{aligned} (s \, 2\Pi_+) \gamma^\mu \bar{\Theta}^{12} &= (s \, 2\Pi_+) \gamma^\mu (2\Pi_+ s) \bar{\theta}_-^{12} = 2(s \, 2\Pi_+ s) \dot{x}^\mu \bar{\theta}_-^{12} = -2ie^{-i\tau} \dot{x}^\mu \bar{\theta}_-^{12}, \\ (s \, 2\Pi_+) \bar{\Theta}^{12} &= (s \, 2\Pi_+) (2\Pi_+ s) \bar{\theta}_-^{12} = 2(s \, 2\Pi_+ s) \bar{\theta}_-^{12} = -2ie^{-i\tau} \bar{\theta}_-^{12}, \\ (s \, 2\Pi_+) \bar{\Theta}^{34} &= (s \, 2\Pi_+) (2\Pi_- \bar{s}) \bar{\theta}_+^{34} = 0, \\ (s \, 2\Pi_+) \bar{\epsilon}^{12} &= -ie^{-i\tau} \bar{\epsilon}_-^{12} = e^{-i\tau} \bar{\theta}_-^{12}. \end{aligned} \quad (3.52)$$

while for the second we have

$$\begin{aligned} (\hat{s} \, 2\Pi_-) \gamma^\mu \bar{\Theta}^{34} &= (\hat{s} \, 2\Pi_-) \gamma^\mu (2\Pi_- \hat{s}) \bar{\theta}_+^{34} = -2(\hat{s} \, 2\Pi_- \hat{s}) \dot{x}^\mu \bar{\theta}_+^{34} = -2ie^{i\tau} \dot{x}^\mu \bar{\theta}_+^{34}, \\ (\hat{s} \, 2\Pi_-) \bar{\Theta}^{34} &= (\hat{s} \, 2\Pi_-) (2\Pi_- \hat{s}) \bar{\theta}_+^{34} = 2(\hat{s} \, 2\Pi_- \hat{s}) \bar{\theta}_+^{34} = -2ie^{i\tau} \bar{\theta}_+^{34}, \\ (\hat{s} \, 2\Pi_-) \bar{\Theta}^{12} &= (\hat{s} \, 2\Pi_-) (2\Pi_+ s) \bar{\theta}_-^{12} = 0, \\ (\hat{s} \, 2\Pi_-) \bar{\epsilon}^{34} &= ie^{i\tau} \bar{\epsilon}_+^{34} = -e^{i\tau} \bar{\theta}_+^{34}. \end{aligned} \quad (3.53)$$

Then, using (3.39), we have

$$\begin{aligned} (s \, 2\Pi_+) \delta' \bar{\psi}^1 &= 4\bar{\theta}_-^{12} e^{-i\tau} \left\{ \dot{x}^\mu D_\mu C_2 - \frac{2\pi}{k} (C_2 \bar{C}^I C_I - C_I \bar{C}^I C_2) \right. \\ &\quad \left. - \frac{4\pi}{k} (C_1 \bar{C}^1 C_2 - C_2 \bar{C}^1 C_1) - \frac{i}{2} C_2 \right\}, \\ &= 4\bar{\theta}_-^{12} \left\{ \dot{x}^\mu D_\mu (e^{-i\tau} C_2) - \frac{2\pi}{k} ((e^{-i\tau} C_2) \bar{C}^I C_I - C_I \bar{C}^I (e^{-i\tau} C_2)) \right. \\ &\quad \left. - \frac{4\pi}{k} (C_1 \bar{C}^1 (e^{-i\tau} C_2) - (e^{-i\tau} C_2) \bar{C}^1 C_1) + \frac{i}{2} (e^{-i\tau} C_2) \right\}, \end{aligned}$$

and

$$\begin{aligned} (s \, 2\Pi_+) \delta' \bar{\psi}^2 &= -4\bar{\theta}_-^{12} e^{-i\tau} \left\{ \dot{x}^\mu D_\mu C_1 - \frac{2\pi}{k} (C_1 \bar{C}^I C_I - C_I \bar{C}^I C_1) \right. \\ &\quad \left. + \frac{4\pi}{k} (C_1 \bar{C}^2 C_2 - C_2 \bar{C}^2 C_1) - \frac{i}{2} C_1 \right\} \end{aligned}$$

$$\begin{aligned}
&= -4\bar{\theta}_-^{12} \left\{ \dot{x}^\mu D_\mu (e^{-i\tau} C_1) - \frac{2\pi}{k} \left((e^{-i\tau} C_1) \bar{C}^I C_I - C_I \bar{C}^I (e^{-i\tau} C_1) \right) \right. \\
&\quad \left. + \frac{4\pi}{k} \left((e^{-i\tau} C_1) \bar{C}^2 C_2 - C_2 \bar{C}^2 (e^{-i\tau} C_1) \right) + \frac{i}{2} (e^{-i\tau} C_1) \right\}.
\end{aligned}$$

Note that, when going from the first to the second equality, in both expressions, we included the phase factor $e^{-i\tau}$ inside the derivatives, leading to the change of sign of the $i/2$ factor. Thus, following the same steps as before, we obtain that the top-right component of $\mathcal{L}$ varies as

$$\delta' \left(i \mathcal{Q}(\bar{\alpha}^i C_i) \right) = \frac{i}{\sqrt{2}} (s 2\Pi_+) (\bar{\alpha}^1 \delta' \bar{\psi}^2 - \bar{\alpha}^2 \delta' \bar{\psi}^1) = \mathcal{D}_\tau \left(-2\sqrt{2} i \bar{\theta}_-^{12} \bar{\alpha}^i (e^{-i\tau} C_i) \right).$$

Furthermore, using (3.42), we have

$$\begin{aligned}
(\hat{s} 2\Pi_-) \delta' \psi_1 &= 4e^{i\tau} \bar{\theta}_+^{34} \left\{ \dot{x}^\mu D_\mu \bar{C}^2 - \frac{2\pi}{k} (\bar{C}^2 C_I \bar{C}^I - \bar{C}^I C_I \bar{C}^2) \right. \\
&\quad \left. - \frac{4\pi}{k} (\bar{C}^1 C_1 \bar{C}^2 - \bar{C}^2 C_1 \bar{C}^1) + \frac{i}{2} \bar{C}^2 \right\}, \\
&= 4\bar{\theta}_+^{34} \left\{ \dot{x}^\mu D_\mu (e^{i\tau} \bar{C}^2) - \frac{2\pi}{k} \left((e^{i\tau} \bar{C}^2) C_I \bar{C}^I - \bar{C}^I C_I (e^{i\tau} \bar{C}^2) \right) \right. \\
&\quad \left. - \frac{4\pi}{k} \left(\bar{C}^1 C_1 (e^{i\tau} \bar{C}^2) - (e^{i\tau} \bar{C}^2) C_1 \bar{C}^1 \right) - \frac{i}{2} (e^{i\tau} \bar{C}^2) \right\},
\end{aligned}$$

and

$$\begin{aligned}
(\hat{s} 2\Pi_-) \delta' \psi_2 &= -4e^{i\tau} \bar{\theta}_+^{34} \left\{ \dot{x}^\mu D_\mu \bar{C}^1 - \frac{2\pi}{k} (\bar{C}^1 C_I \bar{C}^I - \bar{C}^I C_I \bar{C}^1) \right. \\
&\quad \left. + \frac{4\pi}{k} (\bar{C}^1 C_2 \bar{C}^2 - \bar{C}^2 C_2 \bar{C}^1) + \frac{i}{2} \bar{C}^1 \right\} \\
&= -4\bar{\theta}_+^{34} \left\{ \dot{x}^\mu D_\mu (e^{i\tau} \bar{C}^1) - \frac{2\pi}{k} \left((e^{i\tau} \bar{C}^1) C_I \bar{C}^I - \bar{C}^I C_I (e^{i\tau} \bar{C}^1) \right) \right. \\
&\quad \left. + \frac{4\pi}{k} \left((e^{i\tau} \bar{C}^1) C_2 \bar{C}^2 - \bar{C}^2 C_2 (e^{i\tau} \bar{C}^1) \right) - \frac{i}{2} (e^{i\tau} \bar{C}^1) \right\}.
\end{aligned}$$

Then the bottom-left component of $\mathcal{L}$ varies as

$$\delta' \left(i \mathcal{Q}(\alpha_i \bar{C}^i) \right) = \frac{i}{\sqrt{2}} (\hat{s} 2\Pi_-) (\alpha_1 \delta' \psi_2 - \alpha_2 \delta' \psi_1) = \mathcal{D}_\tau \left(-2\sqrt{2} i \bar{\theta}_+^{34} \alpha_i (e^{i\tau} \bar{C}^i) \right).$$

Moving on to the diagonal entries of $\mathcal{L}$, we have that

$$\bar{\alpha}^i \alpha_j \delta' (C_i \bar{C}^j) = 2\bar{\alpha}^i \alpha_j \left(\epsilon_{ik} \bar{\Theta}^{34} \bar{\psi}^k \bar{C}^j + C_i \epsilon^{jk} \bar{\Theta}^{12} \psi_k \right),$$

now gives

$$\begin{aligned}\bar{\alpha}^i \alpha_j \delta'(C_i \bar{C}^j) &= 2\bar{\alpha}^i \alpha_j \left(\epsilon_{ik} (\hat{s} 2\Pi_+) \bar{\theta}_+^{34} \bar{\psi}^k \bar{C}^j + C_i \epsilon^{jk} (s 2\Pi_-) \bar{\theta}_-^{12} \psi_k \right) \\ &= i \left[\left(\frac{i}{\sqrt{2}} (s 2\Pi_+) \epsilon_{ik} \bar{\alpha}^i \bar{\psi}^k \right) \left(-2\sqrt{2} i e^{i\tau} \bar{\theta}_+^{34} \alpha_j \bar{C}^j \right) \right. \\ &\quad \left. - \left(-2\sqrt{2} i e^{-i\tau} \bar{\theta}_-^{12} \bar{\alpha}^i C_i \right) \left(\frac{i}{\sqrt{2}} (\hat{s} 2\Pi_-) \epsilon^{jk} \alpha_j \psi_k \right) \right].\end{aligned}$$

Above we used that $(\hat{s} 2\Pi_+) = -ie^{i\tau}(s 2\Pi_+)$ and $(s 2\Pi_-) = -ie^{-i\tau}(\hat{s} 2\Pi_-)$. Also, we organized the expression to make it easier to identify the quantities

$$\left(\frac{i}{\sqrt{2}} (s 2\Pi_+) \epsilon_{ik} \bar{\alpha}^i \bar{\psi}^k \right) = i\mathcal{Q}(\bar{\alpha}^i C_i) \quad \text{and} \quad \left(-2\sqrt{2} i e^{i\tau} \bar{\theta}_+^{34} \alpha_j \bar{C}^j \right) \equiv \bar{g}',$$

as well as

$$\left(-2\sqrt{2} i e^{-i\tau} \bar{\theta}_-^{12} \bar{\alpha}^i C_i \right) \equiv g' \quad \text{and} \quad \left(\frac{i}{\sqrt{2}} (\hat{s} 2\Pi_-) \epsilon^{jk} \alpha_j \psi_k \right) = i\mathcal{Q}(\alpha_j \bar{C}^j),$$

to write

$$\bar{\alpha}^i \alpha_j \delta'(C_i \bar{C}^j) = i \left[i\mathcal{Q}(\bar{\alpha}^i C_i) \bar{g}' - g' i\mathcal{Q}(\alpha_j \bar{C}^j) \right]. \quad (3.54)$$

Analogously, for $\bar{\alpha}^i \alpha_j \delta'(\bar{C}^j C_i)$, we find

$$\bar{\alpha}^i \alpha_j \delta'(\bar{C}^j C_i) = i \left[i\mathcal{Q}(\alpha_j \bar{C}^j) g' - \bar{g}' i\mathcal{Q}(\bar{\alpha}^i C_i) \right]. \quad (3.55)$$

Thus we have that for this second pair of supercharges the analysis holds in complete analogy to the one derived for the first pair. In this case, instead of (3.48), we have that $\delta'\mathcal{L}$ equates to the supercovariant derivative of

$$\theta \mathcal{G}' \equiv \begin{pmatrix} 0 & g' \\ \bar{g}' & 0 \end{pmatrix}. \quad (3.56)$$

Therefore we find that, as claimed from the beginning, the 2-node loop we constructed is 1/6 BPS and is indeed invariant under the same supercharges as the 1-node loops we started from.

Having carried out the explicit computation of supersymmetry invariance of $\mathcal{L}$, we hope to have convinced the reader about the power of the deformation proposal. It does not only give a well defined procedure to find fermionic operators,

but it also spares the long calculation above. That is, instead of performing the explicit supersymmetry check of the superconnection, we simply need to find $\mathcal{Q}$ and $\mathcal{G}$ satisfying $i\mathcal{Q}^2\mathcal{G} = \mathcal{D}_\tau\mathcal{G}$. Invariance of $\mathcal{L}$ under $\mathcal{Q}$ is then straightforwardly guaranteed. On the one hand we have

$$\delta\mathcal{L} = \delta(\mathcal{L}_{\text{bos}} + i\mathcal{Q}\mathcal{G} + \mathcal{G}^2) = \delta(i\mathcal{Q}\mathcal{G}) + \{\delta\mathcal{G}, \mathcal{G}\},$$

and on the other we have

$$\begin{aligned}\mathfrak{D}_\tau(\theta\mathcal{G}) &= \partial_\tau(\theta\mathcal{G}) + i[\mathcal{L}, \theta\mathcal{G}] = \partial_\tau(\theta\mathcal{G}) + i[\mathcal{L}_{\text{bos}} + i\mathcal{Q}\mathcal{G} + \mathcal{G}^2, \theta\mathcal{G}] \\ &= \partial_\tau(\theta\mathcal{G}) + i[\mathcal{L}_{\text{bos}}, \theta\mathcal{G}] - [\mathcal{Q}\mathcal{G}, \theta\mathcal{G}].\end{aligned}$$

If $i\mathcal{Q}^2\mathcal{G} = \mathcal{D}_\tau\mathcal{G}$, it follows that $\delta(i\mathcal{Q}\mathcal{G}) = \partial_\tau(\theta\mathcal{G}) + i[\mathcal{L}_{\text{bos}}, \theta\mathcal{G}]$. Then the two equations above are indeed equal, since

$$\{\delta\mathcal{G}, \mathcal{G}\} = \delta\mathcal{G}\mathcal{G} + \mathcal{G}\delta\mathcal{G} = \theta\mathcal{Q}\mathcal{G}\mathcal{G} + \mathcal{G}\theta\mathcal{Q}\mathcal{G} = -\mathcal{Q}\mathcal{G}\theta\mathcal{G} + \theta\mathcal{G}\mathcal{Q}\mathcal{G} = -[\mathcal{Q}\mathcal{G}, \theta\mathcal{G}],$$

and $[\mathcal{G}^2, \theta\mathcal{G}] = 0$.

As for $\mathcal{Q}'$, invariance of $\mathcal{L}$ under it follows because we have

$$\delta'\mathcal{L} = \delta'(i\mathcal{Q}\mathcal{G}) + \{\delta'\mathcal{G}, \mathcal{G}\},$$

and

$$\mathfrak{D}_\tau(\theta\mathcal{G}') = \partial_\tau(\theta\mathcal{G}') + i[\mathcal{L}_{\text{bos}}, \theta\mathcal{G}'] - [\mathcal{Q}\mathcal{G}, \theta\mathcal{G}'] + i[\mathcal{G}^2, \theta\mathcal{G}'].$$

What happens in this case is that $\delta'\mathcal{G} = A\delta\mathcal{G}$ and $\mathcal{G}' = B\mathcal{G}$ with

$$A = \begin{pmatrix} -ie^{i\tau} & 0 \\ 0 & -ie^{-i\tau} \end{pmatrix}, \quad \text{and} \quad B = 2\sqrt{2} \begin{pmatrix} -ie^{-i\tau} & 0 \\ 0 & -ie^{i\tau} \end{pmatrix}.$$

The matrices A and B are such that $A\mathcal{G} = \mathcal{G}B$ and $A\delta\mathcal{G} = \delta\mathcal{G}B$. We then have

$$\begin{aligned}\{\delta'\mathcal{G}, \mathcal{G}\} &= A\delta\mathcal{G}\mathcal{G} + \mathcal{G}A\delta\mathcal{G} = 2\sqrt{2}(A\theta\mathcal{Q}\mathcal{G}\mathcal{G} + \mathcal{G}A\theta\mathcal{Q}\mathcal{G}) \\ &= 2\sqrt{2}(-\mathcal{Q}\mathcal{G}\theta B\mathcal{G} + \theta B\mathcal{G}\mathcal{Q}\mathcal{G}) \\ &= -\mathcal{Q}\mathcal{G}\theta\mathcal{G}' + \theta\mathcal{G}'\mathcal{Q}\mathcal{G} \\ &= -[\mathcal{Q}\mathcal{G}, \theta\mathcal{G}'].\end{aligned}$$

Together with the fact that $[\mathcal{G}^2, \theta\mathcal{G}'] = 0$, we see that the second pair of supercharges is also preserved. More directly, one could already conclude this by noting that $\mathcal{G}'$ is related to $\mathcal{G}$ by a gauge transformation.

In addition to what was outlined above, there are aspects about the construction that are worth mentioning. First, we find that the resulting family of 1/6 BPS loops is parametrised by $\alpha_i, \bar{\alpha}^i$, $i = 1, 2$, and it can be explicitly written as (3.35) with superconnection (3.38). We can identify the moduli space of 1/6 BPS deformations by noting that any rescaling of α and $\bar{\alpha}$ such that their product is invariant can be absorbed by a constant gauge transformation. Thus the resulting manifold is spanned by four complex parameters minus the trivial action of a constant gauge transformation acting on the superconnection as $\mathbb{C}^*$. This is the space of singular complex matrices $\mathbb{C}^4 // \mathbb{C}^*$, which is the conifold.⁶ Also, this construction reproduces previous loops presented in the literature. In particular, it matches Class I of [31], while Class II is obtained by breaking the other $SU(2)$, i.e., by coupling to C_3 , $\bar{C}^3$, C_4 and $\bar{C}^4$ in $\mathcal{G}$. Therefore, the corresponding two branches are spanned by constants $\alpha_i, \bar{\alpha}^i$ with i begin either 1, 2 or 3, 4. These intersect at the origin singularity $\mathcal{L}_{\text{bos}}$, since for $\{\alpha_i, \bar{\alpha}^i\} \rightarrow 0$ they approach the same point of the moduli space.

Through this deformation procedure, the scalar coupling of the bosonic connections, $M_J^I = \delta_J^I - 2\delta_J^i \delta_i^I$, gain an extra contribution of

$$\Delta M_J^i \equiv \left(-\frac{2\pi i}{k} \right)^{-1} \bar{\alpha}^i \alpha_j.$$

In particular, when ΔM has eigenvalues 2 and 0, the full matrix $M + \Delta M$ has enhanced $SU(3)$ symmetry. Commuting the 4 preserved supercharges with this $SU(3)$ symmetry gives rise to 12 supercharges, so these operators are 1/2 BPS. More precisely, they correspond to the loops of [24] apart from a gauge transformation $\Lambda = \frac{(2\tau - \pi)}{8} \cdot \sigma_3$, as explained in the beginning of this section.

When we wrote the second chapter of [20], one question that was raised was whether the deformation $\Delta\mathcal{L} = i\mathcal{Q}\mathcal{G} + \mathcal{G}^2$ covers the full moduli space of 1/6 BPS loops. To that we can already say that the answer is no. The reason is that we found a new example of 1/6 BPS loop that is not connected to $\mathcal{L}_{\text{bos}}$ through $\Delta\mathcal{L}$. We dedicate the next section to present this new operator.

⁶We leave a more detailed discussion about these conical spaces to Section 4.2.1.

3.3 New 1/6 BPS operator

After extensively studying the moduli space of circular loops, which resulted in the content presented in the previous section, we tried constructing a 1/3 BPS WL. The motivation for this comes from the fact that Wilson loop operators are expected to become vortex operators under mirror symmetry [35]. A 1/3 BPS loop in ABJ(M) was never found, even though vortex operators preserving the same amount of supersymmetry have already been constructed [36]. While seeking for such operator, we ended up finding a new 1/6 BPS one instead.

The content of this section follows closely the original construction of the 1/2 BPS loop of [24]. We begin by defining the operator supported along the line in Lorentzian space-time.⁷ Then we move to the one supported along the circle in Euclidean space. These are not built in the language of Section 3.2, i.e., not as a deformation of a composite bosonic connection. Quite oppositely, the construction relies heavily on a particular guess of ansätze for the couplings. In this section we simply derive the operators and leave a proper analysis of them to the next chapter, in particular to Section 4.6 where we study their $\mathcal{N} = 4$ analogues.

Line

Here we take the theory to be defined in Lorentzian space-time. The operator is supported along the time-like line $x^\mu = (\tau, 0, 0)$ and it is written as

$$W = \text{Tr } \mathcal{P} \exp \left(-i \int \mathcal{L} d\tau \right).$$

The superconnection has the generic form

$$\mathcal{L} = \begin{pmatrix} A_1 + \frac{2\pi}{k} M_J^I C_I \bar{C}^J & \sqrt{\frac{2\pi}{k}} \eta_I^\alpha \bar{\psi}_\alpha^I \\ \sqrt{\frac{2\pi}{k}} \psi_I^\alpha \bar{\eta}_\alpha^I & A_2 + \frac{2\pi}{k} M_J^I \bar{C}^J C_I \end{pmatrix}.$$

For the fermionic couplings we take the following ansatz

$$\eta_I^\alpha = \eta \begin{pmatrix} \delta_I^1 & -\delta_I^2 \end{pmatrix} \quad \text{and} \quad \bar{\eta}_\alpha^I = \bar{\eta} \begin{pmatrix} \delta_1^I \\ -\delta_2^I \end{pmatrix},$$

⁷So far we only considered operators defined along closed curves γ , but recall that under a conformal transformation a line is transformed into a circle. It is actually interesting that, even though this is a symmetry of the theory, it changes the topology of the curve and, as it turns out, also the expectation value of the WL. In the case of the 1/2 BPS WLs of $\mathcal{N} = 4$ SYM, for example, one finds that, whereas the straight line has trivial expectation value, the circular loop depends on the coupling constant of the theory.

where η and $\bar{\eta}$ are constants. The matrix M is given by $M = \text{diag}(-1, +1, 0, 0)$.

At the beginning we expected this operator to preserve $\bar{\theta}_+^{13}, \bar{\theta}_+^{14}, \bar{\theta}_-^{23}, \bar{\theta}_-^{24}$ out of the 12 Poincaré supercharges. If this was the case, the set $\bar{\epsilon}_+^{13}, \bar{\epsilon}_+^{14}, \bar{\epsilon}_-^{23}, \bar{\epsilon}_-^{24}$ would also be preserved — since the operator is defined along the line, Poincaré and superconformal parameters are preserved separately. Thus there would also be 4 out of 12 superconformal supercharges and the operator would be 1/3 BPS. However, in the end what we found was that the loop is 1/6 BPS. This fact will become clear after we go through the computation and see that some constraints between these parameters arise.

First, we need to compute the variation of the loop when the parameters $\bar{\theta}_+^{13}, \bar{\theta}_+^{14}, \bar{\theta}_-^{23}$ and $\bar{\theta}_-^{24}$ are turned on. We use the rules of (A.2), with the caveat that Killing spinors $\bar{\Theta}$ are replaced by Poincaré parameters $\bar{\theta}$ and, in the variation of fermions, we lose the linear $\bar{\epsilon}$ term. For superconformal transformations, we have the same rules with $\bar{\theta}$ now replaced by $\bar{\epsilon}$. Thus in the following we perform the computation for $\bar{\theta}$ and the proof for $\bar{\epsilon}$ follows straightforwardly.

We begin by computing the variation of the top-right entry of the superconnection, $\bar{f} \equiv \eta\bar{\psi}$. Using (A.2), it takes the following form

$$\begin{aligned} \delta \bar{f} = \eta \left\{ 2i\bar{\theta}_+^{1I}(\gamma^\mu)_+^\alpha D_\mu C_I - \frac{4\pi i}{k}\bar{\theta}_+^{1I}(C_I\bar{C}^J C_J - C_J\bar{C}^J C_I) - \frac{8\pi i}{k}\bar{\theta}_+^{IJ}C_I\bar{C}^1 C_J \right\} \\ - \eta \left\{ 2i\bar{\theta}_-^{2I}(\gamma^\mu)_-^\alpha D_\mu C_I - \frac{4\pi i}{k}\bar{\theta}_-^{2I}(C_I\bar{C}^J C_J - C_J\bar{C}^J C_I) - \frac{8\pi i}{k}\bar{\theta}_-^{IJ}C_I\bar{C}^2 C_J \right\}. \end{aligned}$$

The equation above holds for any Poincaré parameter. Now looking at the set of parameters we want to turn on, namely $\bar{\theta}_+^{13}, \bar{\theta}_+^{14}, \bar{\theta}_-^{23}$ and $\bar{\theta}_-^{24}$, the equation above becomes

$$\begin{aligned} \delta \bar{f} = \eta\bar{\theta}_+^{1j} \left\{ 2i(\gamma^\mu)_+^\alpha D_\mu C_j - \frac{4\pi i}{k}(C_j\bar{C}^J C_J - C_J\bar{C}^J C_j) - \frac{8\pi i}{k}(C_1\bar{C}^1 C_j - C_j\bar{C}^1 C_1) \right\} \\ - \eta\bar{\theta}_-^{2j} \left\{ 2i(\gamma^\mu)_-^\alpha D_\mu C_j - \frac{4\pi i}{k}(C_j\bar{C}^J C_J - C_J\bar{C}^J C_j) - \frac{8\pi i}{k}(C_2\bar{C}^2 C_j - C_j\bar{C}^2 C_2) \right\}, \end{aligned}$$

where lower case indices i, j belong to the set $\{3, 4\}$. From now on, adopt this notation throughout the section. Note that $(i\gamma^\mu)_+^+ = \delta_0^\mu$ and $(i\gamma^\mu)_-^- = -\delta_0^\mu$, so

$$\begin{aligned} \delta \bar{f} = \eta\bar{\theta}_+^{1j} \left\{ 2D_0 C_j - \frac{4\pi i}{k}(C_j\bar{C}^J C_J - C_J\bar{C}^J C_j) - \frac{8\pi i}{k}(C_1\bar{C}^1 C_j - C_j\bar{C}^1 C_1) \right\} \\ - \eta\bar{\theta}_-^{2j} \left\{ -2D_0 C_j - \frac{4\pi i}{k}(C_j\bar{C}^J C_J - C_J\bar{C}^J C_j) - \frac{8\pi i}{k}(C_2\bar{C}^2 C_j - C_j\bar{C}^2 C_2) \right\} \\ = 2\eta \left\{ \left(\bar{\theta}_+^{1i} + \bar{\theta}_-^{2i} \right) D_0 C_i - \frac{2\pi i}{k} \left[\left(\bar{\theta}_+^{1i} + \bar{\theta}_-^{2i} \right) C_i (-\bar{C}^1 C_1 + \bar{C}^2 C_2) + \left(\bar{\theta}_+^{1i} - \bar{\theta}_-^{2i} \right) \right] \right\} \end{aligned}$$

$$C_i(\bar{C}^j C_j) \Big] + \frac{2\pi i}{k} \left[(-C_1 \bar{C}^1 + C_2 \bar{C}^2) \left(\bar{\theta}_+^{1i} + \bar{\theta}_-^{2i} \right) C_i + (C_j \bar{C}^j) \left(\bar{\theta}_+^{1i} - \bar{\theta}_-^{2i} \right) C_i \right] \Big\}.$$

To combine this as a covariant derivative with a dressed bosonic connection we impose that

$$\bar{\theta}_+^{1i} = \bar{\theta}_-^{2i}. \quad (3.57)$$

This means that the loop is actually 1/6 BPS, since it is supersymmetric if we turn on $\bar{\theta}_+^{13}, \bar{\theta}_+^{14}, \bar{\theta}_-^{23}, \bar{\theta}_-^{24}$ and also impose the constraint above. In particular, in this case it follows that

$$\delta \bar{f} = 4\eta \bar{\theta}_+^{1i} \left\{ D_0 C_i - \frac{2\pi i}{k} C_i \left(-\bar{C}^1 C_1 + \bar{C}^2 C_2 \right) + \frac{2\pi i}{k} \left(-C_1 \bar{C}^1 + C_2 \bar{C}^2 \right) C_i \right\}.$$

Using the definition of the covariant derivative with dressed bosonic connection, $\mathcal{D}_0 C_i = \partial_0 C_i + i(\mathcal{A}_1 C_i - C_i \mathcal{A}_2)$, we have simply

$$\delta \bar{f} = \mathcal{D}_0 \left(4\eta \bar{\theta}_+^{1i} C_i \right). \quad (3.58)$$

For the bottom-left entry of the superconnection, $f \equiv \psi \bar{\eta}$, we go through the same steps as $\delta \bar{f}$ and find that its variation gives

$$\delta f = \mathcal{D}_0 \left(-4\eta \theta_{2i}^- \bar{C}^i \right). \quad (3.59)$$

For the diagonal entries, we start from the top-left one. We have $\mathcal{A}_1 \equiv A_1 + \frac{2\pi}{k} M_J^I C_I \bar{C}^J$ and its variation gives as

$$\begin{aligned} \delta \mathcal{A}_1 = \frac{4\pi i}{k} & \left[\bar{\theta}^{2j+} (\delta_j^i - M_j^i) C_i \psi_{2+} - \bar{\theta}^{1j-} (\delta_j^i + M_j^i) C_i \psi_{1-} - \theta_{1i}^+ (\delta_j^i + M_j^i) \bar{\psi}_+^1 \bar{C}^j \right. \\ & \left. + \theta_{2i}^- (\delta_j^i - M_j^i) \bar{\psi}_-^2 \bar{C}^j \right]. \end{aligned}$$

Using the fact that $M_j^i = 0$ and imposing (3.57), it becomes

$$\delta \mathcal{A}_1 = \frac{4\pi i}{k} \left[\bar{\theta}_+^{1i} C_i (\psi_1^+ - \psi_2^-) - \theta_{2i}^- (\bar{\psi}_+^1 - \bar{\psi}_-^2) \bar{C}^i \right]. \quad (3.60)$$

Analogously, for $\mathcal{A}_2 \equiv A_2 + \frac{2\pi}{k} M_J^I \bar{C}^J C_I$ the variation is given by

$$\delta \mathcal{A}_2 = \frac{4\pi i}{k} \left[\bar{\theta}_+^{1i} (\psi_1^+ - \psi_2^-) C_i - \theta_{2i}^- \bar{C}^i (\bar{\psi}_+^1 - \bar{\psi}_-^2) \right]. \quad (3.61)$$

With (3.58) and (3.59) in hand, we see that $\delta\mathcal{L}$ should be equal to the super-covariant derivative of

$$\begin{pmatrix} 0 & 4\eta\sqrt{\frac{2\pi}{k}}\bar{\theta}_+^{1i}C_i \\ -4\bar{\eta}\sqrt{\frac{2\pi}{k}}\theta_{2i}^-\bar{C}^i & 0 \end{pmatrix}.$$

This means that we expect $\delta\mathcal{L}$ to be equal to $\mathcal{D}_0$ acting on this matrix, i.e.,

$$\begin{pmatrix} \delta\mathcal{A}_1 & \sqrt{\frac{2\pi}{k}}\delta\bar{f} \\ \sqrt{\frac{2\pi}{k}}\delta f & \delta\mathcal{A}_2 \end{pmatrix} = \begin{pmatrix} i\sqrt{\frac{2\pi}{k}}(\bar{f}\bar{g} - gf) & \mathcal{D}_0 g \\ \mathcal{D}_0 \bar{g} & i\sqrt{\frac{2\pi}{k}}(fg - \bar{g}\bar{f}) \end{pmatrix}. \quad (3.62)$$

with $g = 4\eta\sqrt{\frac{2\pi}{k}}\bar{\theta}_+^{1i}C_i$ and $\bar{g} = -4\bar{\eta}\sqrt{\frac{2\pi}{k}}\theta_{2i}^-\bar{C}^i$. Off-diagonal equalities are thus automatically satisfied. As for the diagonal ones, we plug g and $\bar{g}$ into the equations for $\delta\mathcal{A}_1$ and $\delta\mathcal{A}_2$,

$$\delta\mathcal{A}_1 = i\sqrt{\frac{2\pi}{k}}(\bar{f}\bar{g} - gf) \quad \text{and} \quad \delta\mathcal{A}_2 = i\sqrt{\frac{2\pi}{k}}(fg - \bar{g}\bar{f}),$$

and it gives us the condition $\eta\bar{\eta} = -\frac{1}{2}$.

Summarizing, we find that the superconnection for this line operator is given by

$$\mathcal{L} = \begin{pmatrix} A_1 + \frac{2\pi}{k}M_J^I C_I \bar{C}^J & \sqrt{\frac{2\pi}{k}}\eta_I^\alpha \bar{\psi}_\alpha^I \\ \sqrt{\frac{2\pi}{k}}\psi_I^\alpha \bar{\eta}_\alpha^I & A_2 + \frac{2\pi}{k}M_J^I \bar{C}^J C_I \end{pmatrix},$$

with

$$\eta_I^\alpha = \eta \left(\delta_I^1 \delta_+^\alpha - \delta_I^2 \delta_-^\alpha \right), \quad \bar{\eta}_\alpha^I = \bar{\eta} \left(\delta_1^I \delta_+^\alpha - \delta_2^I \delta_-^\alpha \right), \quad M_J^I = \text{diag}(-1, +1, 0, 0),$$

and $\eta\bar{\eta} = -\frac{1}{2}$. The resulting loop is invariant under supersymmetry transformations generated by $\bar{\theta}_+^{13}$, $\bar{\theta}_+^{14}$, $\bar{\theta}_-^{23}$ and $\bar{\theta}_-^{24}$ subject to the constraint $\bar{\theta}_+^{1i} = \bar{\theta}_-^{2i}$. Therefore the operator is 1/6 BPS.

Circle

To construct the loop supported along the circle we go to Euclidean space. The operator is supported along the curve C and it is defined as

$$W = \text{Tr} \mathcal{P} \exp \left(-i \oint \mathcal{L} d\tau \right)$$

where

$$\mathcal{L} = \begin{pmatrix} A_{1\mu}\dot{x}^\mu - \frac{2\pi i}{k} M_J^I C_I \bar{C}^J & -i\sqrt{\frac{2\pi}{k}} \eta_I^\alpha \bar{\psi}_\alpha^I \\ -i\sqrt{\frac{2\pi}{k}} \psi_I^\alpha \bar{\eta}_\alpha^I & A_{2\mu}\dot{x}^\mu - \frac{2\pi i}{k} M_J^I \bar{C}^J C_I \end{pmatrix}.$$

For the couplings we make the following ansatz,

$$\begin{aligned} \eta_I^\alpha &= \eta_1(\tau)(\Pi_+)_+^\alpha \delta_I^1 - \eta_2(\tau)(\Pi_-)_-^\alpha \delta_I^2, \\ \bar{\eta}_\alpha^I &= i\bar{\eta}_1(\tau)(\Pi_+)_\alpha^+ \delta_1^I - i\bar{\eta}_2(\tau)(\Pi_-)_\alpha^- \delta_2^I, \\ M_J^I &= \text{diag}(-1, +1, 0, 0), \end{aligned}$$

where $\eta_1(\tau)$ and $\eta_2(\tau)$ are functions of τ .

This loop is constructed based on the 1/6 BPS line. For the line, we found that the parameters that should be turned on were $\bar{\theta}_+^{13}, \bar{\theta}_+^{14}, \bar{\theta}^{23+}, \bar{\theta}^{24+}$ subject to the constraint $\bar{\theta}_+^{1i} = \theta_-^{2i}$. For the circle, we will turn on the same supercharges but with both chiralities, i.e., we will turn on $\bar{\theta}_\alpha^{13}, \bar{\theta}_\alpha^{14}, \bar{\theta}_\alpha^{23}, \bar{\theta}_\alpha^{24}$. As happened for the line, for the circular case we also expect some constraint between these parameters to arise.

First, let us go through the preliminary choices. We begin by choosing the following form for the superconformal parameters:

$$\bar{\epsilon}^{1i} = -i\sigma_3 \bar{\theta}^{1i} \quad \text{and} \quad \bar{\epsilon}^{2i} = i\sigma_3 \bar{\theta}^{2i}. \quad (3.63)$$

With this choice, it follows that

$$\bar{\Theta}^{1i} = 2(\Pi_+) \bar{\theta}^{1i} \quad \text{and} \quad \bar{\Theta}^{2i} = 2(\Pi_-) \bar{\theta}^{2i}. \quad (3.64)$$

Since we will need to use it later, it is also useful to write down

$$\epsilon_{1i} = -i\theta_{1i}\sigma_3, \quad \epsilon_{2i} = i\theta_{2i}\sigma_3 \quad \text{and} \quad \Theta_{1i} = 2\theta_{1i}(\Pi_+), \quad \Theta_{2i} = 2\theta_{2i}(\Pi_-).$$

Moreover, we will need the following property

$$(\Pi_\pm)\gamma^\mu(\Pi_\pm) = \pm(\Pi_\pm)\dot{x}^\mu. \quad (3.65)$$

Now let us compute the variation of the loop when the parameters $\bar{\theta}^{1i}$ and $\bar{\theta}^{2i}$ are turned on. Following the same steps taken for the line operator, we will sketch each piece of the variation separately. We begin with the variation of $\bar{f} \equiv \eta\bar{\psi}$ and see that it takes the following form

$$\delta \bar{f} = \eta_1(\tau)(\Pi_+)_+^\alpha \delta \bar{\psi}_\alpha^1 - \eta_2(\tau)(\Pi_-)_-^\alpha \delta \bar{\psi}_\alpha^2$$

$$\begin{aligned}
&= \eta_1(\tau)(\Pi_+)_+^\alpha \left\{ -2i\bar{\Theta}_\beta^{1I}(\gamma^\mu)_{\beta\alpha} D_\mu C_I - \frac{4\pi i}{k} \bar{\Theta}_\alpha^{1I}(C_I \bar{C}^J C_J - C_J \bar{C}^J C_I) \right. \\
&\quad \left. - \frac{8\pi i}{k} \bar{\Theta}_\alpha^{IJ} C_I \bar{C}^1 C_J - 2i\bar{\epsilon}_\alpha^{1I} C_I \right\} - \eta_2(\tau)(\Pi_-)_-^\alpha \left\{ -2i\bar{\Theta}_\beta^{2I}(\gamma^\mu)_{\beta\alpha} D_\mu C_I \right. \\
&\quad \left. - \frac{4\pi i}{k} \bar{\Theta}_\alpha^{2I}(C_I \bar{C}^J C_J - C_J \bar{C}^J C_I) - \frac{8\pi i}{k} \bar{\Theta}_\alpha^{IJ} C_I \bar{C}^2 C_J - 2i\bar{\epsilon}_\alpha^{2I} C_I \right\}.
\end{aligned}$$

Using (3.63), (3.64) and (3.65), the expression above becomes

$$\begin{aligned}
\delta \bar{f} &= \eta_1(\tau)(\Pi_+)_+^\alpha \left\{ 4i\bar{\theta}_\alpha^{1i} \dot{x}^\mu D_\mu C_i - \frac{8\pi i}{k} \bar{\theta}_\alpha^{1i}(C_i \bar{C}^J C_J - C_J \bar{C}^J C_i) \right. \\
&\quad \left. - \frac{16\pi i}{k} \bar{\theta}_\alpha^{1i}(C_1 \bar{C}^1 C_i - C_i \bar{C}^1 C_1) - 2i(-i\sigma_3)_\alpha^\beta \bar{\theta}_\beta^{1i} C_i \right\} \\
&\quad - \eta_2(\tau)(\Pi_-)_-^\alpha \left\{ -4i\bar{\theta}_\alpha^{2i} \dot{x}^\mu D_\mu C_i - \frac{8\pi i}{k} \bar{\theta}_\alpha^{2i}(C_i \bar{C}^J C_J - C_J \bar{C}^J C_i) \right. \\
&\quad \left. - \frac{16\pi i}{k} \bar{\theta}_\alpha^{2i}(C_2 \bar{C}^2 C_i - C_i \bar{C}^2 C_2) - 2i(i\sigma_3)_\alpha^\beta \bar{\theta}_\beta^{2i} C_i \right\}.
\end{aligned}$$

To be able to sum everything as a single covariant derivative we impose

$$\eta_1(\tau)(\Pi_+)_+^\alpha \bar{\theta}_\alpha^{1i} = \eta_2(\tau)(\Pi_-)_-^\alpha \bar{\theta}_\alpha^{2i}. \quad (3.66)$$

This is our first constraint on the circle. Once satisfied, we have

$$\begin{aligned}
\delta \bar{f} &= 8i\eta_1(\tau)(\Pi_+)_+^\alpha \bar{\theta}_\alpha^{1i} \left\{ D_\tau C_i - \frac{2\pi}{k} \left(C_i(-\bar{C}^1 C_1 + \bar{C}^2 C_2) - (-C_1 \bar{C}^1 + C_2 \bar{C}^2) C_i \right) \right\} \\
&\quad - 2\eta_1(\tau)(\Pi_+)_+^\alpha (\sigma_3)_\alpha^\beta \bar{\theta}_\beta^{1i} C_i - 2\eta_2(\tau)(\Pi_-)_-^\alpha (\sigma_3)_\alpha^\beta \bar{\theta}_\beta^{2i} C_i \\
&= 8i\mathcal{D}_\tau \left(\eta_1(\tau)(\Pi_+)_+^\alpha \bar{\theta}_\alpha^{1i} C_i \right) - 8i\partial_\tau \left(\eta_1(\tau)(\Pi_+)_+^\alpha \bar{\theta}_\alpha^{1i} \right) C_i - 2\eta_1(\tau)(\Pi_+)_+^\alpha (\sigma_3)_\alpha^\beta \\
&\quad \bar{\theta}_\beta^{1i} C_i - 2\eta_2(\tau)(\Pi_-)_-^\alpha (\sigma_3)_\alpha^\beta \bar{\theta}_\beta^{2i} C_i.
\end{aligned}$$

By requiring that

$$8i\partial_\tau \left(\eta_1(\tau)(\Pi_+ \bar{\theta}^{1i})_+ \right) C_i + 2\eta_1(\tau)(\Pi_+ \sigma_3 \bar{\theta}^{1i})_+ C_i + 2\eta_2(\tau)(\Pi_- \sigma_3 \bar{\theta}^{2i})_- C_i = 0, \quad (3.67)$$

we have

$$\delta \bar{f} = 8i\mathcal{D}_\tau \left(\eta_1(\tau)(\Pi_+)_+^\alpha \bar{\theta}_\alpha^{1i} C_i \right). \quad (3.68)$$

Equation (3.67) is our second constraint on the circle. One possible solution to it is

$$\eta_1(\tau) = e^{i\tau/2}, \quad \eta_2(\tau) = e^{-i\tau/2}, \quad \bar{\theta}_-^{1i} = i\bar{\theta}_-^{2i} \quad \text{and} \quad \bar{\theta}_+^{1i} = -i\bar{\theta}_+^{2i}. \quad (3.69)$$

Note that this solution is consistent in the sense that it does not violate (3.66).

Moving on to the derivation of $\delta f \equiv \delta(\psi\bar{\eta})$, we follow the same steps we did for $\delta\bar{f}$. We come across an analogous version of (3.66),

$$\theta_{1i}^\alpha(\Pi_+)_{\alpha}{}^+ \bar{\eta}_1(\tau) = \theta_{2i}^\alpha(\Pi_-)_{\alpha}{}^- \bar{\eta}_2(\tau). \quad (3.70)$$

We also come across an analogous version of (3.67),

$$8i\partial_\tau \left((\theta_{1i}\Pi_+)^+ \bar{\eta}_1(\tau) \right) \bar{C}^i - 2(\theta_{1i}\sigma_3\Pi_+)^+ \bar{\eta}_1(\tau) \bar{C}^i - 2(\theta_{2i}\sigma_3\Pi_-)^- \bar{\eta}_2(\tau) \bar{C}^i = 0.$$

These conditions are automatically satisfied as a consequence of (3.69), since it implies that

$$\bar{\eta}_1(\tau) = e^{-i\tau/2}, \quad \bar{\eta}_2(\tau) = e^{i\tau/2}, \quad \theta_{1i}^+ = i\theta_{2i}^+ \quad \text{and} \quad \theta_{1i}^- = -i\theta_{2i}^-.$$

Thus we find

$$\delta f = 8\mathcal{D}_\tau \left(\bar{\eta}_1(\tau) \theta_{1i}^\alpha(\Pi_+)_{\alpha}{}^+ \bar{C}^i \right). \quad (3.71)$$

Now the variation of $\mathcal{A}_1 \equiv A_{1\mu}\dot{x}^\mu - \frac{2\pi i}{k} M_J^I C_I \bar{C}^J$ can be written in terms of $\bar{\theta}^{1i}$ and $\bar{\theta}^{2i}$ as

$$\delta\mathcal{A}_1 = \frac{8\pi i}{k} C_i \left[\psi_1(\Pi_+) \bar{\theta}^{1i} - \psi_2(\Pi_-) \bar{\theta}^{2i} \right] + \frac{8\pi i}{k} \left[\theta_{1i}(\Pi_+) \bar{\psi}^1 - \theta_{2i}(\Pi_-) \bar{\psi}^2 \right] \bar{C}^i. \quad (3.72)$$

Analogously, the variation of $\mathcal{A}_2 \equiv A_{2\mu}\dot{x}^\mu - \frac{2\pi i}{k} M_J^I \bar{C}^J C_I$ can be written as

$$\delta\mathcal{A}_2 = \frac{8\pi i}{k} \left[\psi_1(\Pi_+) \bar{\theta}^{1i} - \psi_2(\Pi_-) \bar{\theta}^{2i} \right] C_i + \frac{8\pi i}{k} \bar{C}^i \left[\theta_{1i}(\Pi_+) \bar{\psi}^1 - \theta_{2i}(\Pi_-) \bar{\psi}^2 \right]. \quad (3.73)$$

With all variations in hand, we can check whether $\delta\mathcal{L}$ equates to the supercovariant derivative of the block off-diagonal supermatrix

$$\begin{pmatrix} 0 & g \\ \bar{g} & 0 \end{pmatrix}, \quad \text{with} \quad g = 8\sqrt{\frac{2\pi}{k}} \eta_1(\tau) (\Pi_+)_{\alpha}{}^+ \bar{\theta}_\alpha^{1i} C_i, \quad \bar{g} = -8i\sqrt{\frac{2\pi}{k}} \bar{\eta}_1(\tau) \theta_{1i}^\alpha (\Pi_+)_{\alpha}{}^+ \bar{C}^i. \quad (3.74)$$

This means checking that the diagonal entries of

$$\begin{pmatrix} \delta \mathcal{A}_1 & -i\sqrt{\frac{2\pi}{k}}\delta \bar{f} \\ -i\sqrt{\frac{2\pi}{k}}\delta f & \delta \mathcal{A}_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2\pi}{k}}(\bar{f}\bar{g} - gf) & \mathcal{D}_0 g \\ \mathcal{D}_0 \bar{g} & \sqrt{\frac{2\pi}{k}}(fg - \bar{g}\bar{f}) \end{pmatrix},$$

hold. We find that

$$\delta \mathcal{A}_1 = \sqrt{\frac{2\pi}{k}}(\bar{f}\bar{g} - gf) \quad \text{and} \quad \delta \mathcal{A}_2 = \sqrt{\frac{2\pi}{k}}(fg - \bar{g}\bar{f}),$$

hold automatically if we use (3.66) and (3.70).

Summarizing, the superconnection for the circular Wilson loop operator is

$$\mathcal{L} = \begin{pmatrix} A_{1\mu}\dot{x}^\mu - \frac{2\pi i}{k}M_J^I C_I \bar{C}^J & -i\sqrt{\frac{2\pi}{k}}\eta_I^\alpha \bar{\psi}_\alpha^I \\ -i\sqrt{\frac{2\pi}{k}}\psi_I^\alpha \bar{\eta}_\alpha^I & A_{2\mu}\dot{x}^\mu - \frac{2\pi i}{k}M_J^I \bar{C}^J C_I \end{pmatrix}, \quad (3.75)$$

with

$$\begin{aligned} \eta_I^\alpha &= e^{i\tau/2}(\Pi_+)_+^\alpha \delta_I^1 - e^{-i\tau/2}(\Pi_-)_-^\alpha \delta_I^2, \\ \bar{\eta}_\alpha^I &= ie^{-i\tau/2}(\Pi_+)_\alpha^+ \delta_1^I - ie^{i\tau/2}(\Pi_-)_\alpha^- \delta_2^I, \\ M_J^I &= \text{diag}(-1, +1, 0, 0). \end{aligned}$$

By imposing $\bar{\epsilon}^{1i} = -i\sigma_3 \bar{\theta}^{1i}$, $\bar{\epsilon}^{2i} = i\sigma_3 \bar{\theta}^{2i}$, $\bar{\theta}_-^{1i} = i\bar{\theta}_-^{2i}$ and $\bar{\theta}_+^{1i} = -i\bar{\theta}_+^{2i}$, the superconnection is such that $\delta \mathcal{L} = \mathfrak{D}_\tau(\theta \mathcal{G})$ with

$$\theta \mathcal{G} = 8\sqrt{\frac{2\pi}{k}} \begin{pmatrix} 0 & e^{i\tau/2}(\Pi_+)_+^\alpha \bar{\theta}_\alpha^{1i} C_i \\ -ie^{-i\tau/2}\theta_{1i}^\alpha (\Pi_+)_\alpha^+ \bar{C}^i & 0 \end{pmatrix},$$

where the variation is taken with respect to the parameters $\bar{\theta}^{1i}$. Therefore the loop is invariant under the action of supersymmetry transformations generated by,

$$(Q_{1i}^+ - iS_{1i}^+) + i(Q_{2i}^+ + iS_{2i}^+) \quad \text{and} \quad (Q_{1i}^- + iS_{1i}^-) - i(Q_{2i}^- - iS_{2i}^-), \quad (3.76)$$

with $i = 3, 4$ and it is 1/6 BPS. As in the case of the line, we see from the scalar coupling matrix M that the loop has a residual $U(1) \times U(1)$ R-symmetry. In the context of the AdS/CFT correspondence, this operator should then have a dual description on the string theory side with $U(1) \times U(1)$ bosonic symmetry.

Finally, it is worth mentioning that this new operator does not belong to any previously known examples and, in particular, not to any classification of [49].

3.4 Perturbative detour: deforming the 1/2 BPS loop

The 1/2 BPS loop of ABJ(M) theory was presented in Section 3.1, see (3.19). Here our goal is to compute the expectation value of a deformed version of this operator perturbatively. We should clarify that by deformation we no longer mean the one built out of $\mathcal{G}$, namely $\Delta\mathcal{L} = i\mathcal{Q}\mathcal{G} + \mathcal{G}^2$, but actually the one where we change the contour along which the loop is supported.

The deformation we consider is analogous to the one introduced in [37] in the context of $\mathcal{N} = 4$ SYM. The idea is to deform the circle C by making small perturbations parameterised by a real valued function $g(\tau)$. Then the closed path $\tilde{C}$ is given by

$$\tilde{C} = (0, e^{g(\tau)} \cos(\tau), e^{g(\tau)} \sin(\tau)). \quad (3.77)$$

We carry the perturbative computation by expanding the observable not only in the t'Hooft coupling $\lambda = N/k$, but also in the deformation parameter $g(\tau)$. In particular, we write the deformation parameter in terms of its Fourier series,

$$g(\tau) = \sum_{n=-\infty}^{\infty} b_n e^{in\tau}. \quad (3.78)$$

The Wilson loop operator we are interested in is defined in terms of the superconnection

$$\mathcal{L}(\tau) = \begin{pmatrix} \mathcal{A}_1 & -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\eta_I\bar{\psi}^I \\ -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\psi_I\bar{\eta}^I & \mathcal{A}_2 \end{pmatrix}, \quad (3.79)$$

with dressed bosonic connections

$$\mathcal{A}_1 = A_{1\mu}\dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}|M_J^I C_I \bar{C}^J \quad \text{and} \quad \mathcal{A}_2 = A_{2\mu}\dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}|M_J^I \bar{C}^J C_I. \quad (3.80)$$

In the case of the perfect circle, i.e., when $g(\tau) = 0$, the fermionic couplings are

$$\eta_I^\alpha(\tau) = \begin{pmatrix} e^{i\tau/2} & -ie^{-i\tau/2} \end{pmatrix} \delta_I^1 \quad \text{and} \quad \bar{\eta}_\alpha^I(\tau) = \begin{pmatrix} ie^{-i\tau/2} \\ -e^{i\tau/2} \end{pmatrix} \delta_1^I. \quad (3.81)$$

The scalar coupling is $M = \text{diag}(-1, 1, 1, 1)$. These couplings are such that the traced loop is 1/2 BPS [24]. Here we are interested on a generically deformed circle, where $g(\tau) \neq 0$. We keep the scalar coupling as the original one, but

consider new fermionic couplings given by

$$\eta_I^\alpha(\tau) = e^{g(\tau)/2} \begin{pmatrix} e^{i\tau/2} & -ie^{-i\tau/2} \end{pmatrix} \delta_I^1 \quad \text{and} \quad \bar{\eta}_\alpha^I(\tau) = e^{g(\tau)/2} \begin{pmatrix} ie^{-i\tau/2} \\ -e^{i\tau/2} \end{pmatrix} \delta_1^I. \quad (3.82)$$

Since the couplings depend on the loop's contour, it is natural to expect them to change when the contour is deformed. However, it is not completely clear why they should take the form in (3.82). The motivation to make this particular choice comes from the Feynman diagram computation, which we will present subsequently. We expect $\langle W \rangle$ at second order in the deformation parameter $g(\tau)$ (and consequently at second order in b_n) to behave as

$$\langle W \rangle|_{b^2} \sim \sum_{n=2}^{\infty} n(n^2 - 1) |b_n|^2.$$

This particular form is expected because it gives what is called the Bremsstrahlung function, denoted as $B(\lambda)$. We will explain the meaning of $B(\lambda)$ at the end of this section, but for now let us just blindly assume that the behavior above should hold. We find that this is obtained for couplings in (3.82) and we rely on this result to justify our choice. Nevertheless, a deeper understanding of its structure is still lacking.

With the given definitions the expectation value of the Wilson loop reads

$$\langle W \rangle = \frac{1}{N_1 + N_2} \int D[A_1, A_2, C, \bar{C}, \psi, \bar{\psi}] e^{-S} \text{Tr} \mathcal{P} \exp \left(-i \oint_{\mathcal{C}} \mathcal{L}(\tau) d\tau \right). \quad (3.83)$$

In the following we present the results obtained so far up to two loops, i.e., evaluating $\langle W \rangle$ up to second order in $\lambda_1 = N_1/k$ and $\lambda_2 = N_2/k$. We use the Feynman rules shown in Appendix A.

3.4.1 1-loop contribution

A 1-loop computation of the Wilson loop requires expanding the path-ordered exponential in (3.83) up to second order:

$$\text{Tr} \mathcal{P} \exp \left(-i \int \mathcal{L}(\tau) d\tau \right) \cong \text{Tr} \left(1 - i \int d\tau_1 \mathcal{L}(\tau_1) + (-i)^2 \int d\tau_{1>2} \mathcal{L}(\tau_1) \mathcal{L}(\tau_2) \right). \quad (3.84)$$

Here the trace is taken over the corresponding representation. In particular, the superconnection $\mathcal{L}$ is a $(N_1 + N_2) \times (N_1 + N_2)$ dimensional supermatrix. The upper left entry has dimensions $N_1 \times N_1$ and the bottom right $N_2 \times N_2$. These diago-

nal blocks belong to the adjoint representation of $U(N_1)$ and $U(N_2)$ respectively. Moreover, the term quadratic in $\mathcal{L}$ reads

$$\begin{pmatrix} \mathcal{A}_1(\tau_1)\mathcal{A}_1(\tau_2) - \frac{2\pi}{k}|\dot{x}_1||\dot{x}_2|(\eta\bar{\psi})_1(\psi\bar{\eta})_2 & -i\sqrt{\frac{2\pi}{k}}\left(|\dot{x}_2|\mathcal{A}_1(\tau_1)(\eta\bar{\psi})_2 + |\dot{x}_1|(\eta\bar{\psi})_1\mathcal{A}_2(\tau_2)\right) \\ -i\sqrt{\frac{2\pi}{k}}\left(|\dot{x}_1|(\psi\bar{\eta})_1\mathcal{A}_1(\tau_2) + |\dot{x}_2|\mathcal{A}_2(\tau_1)(\psi\bar{\eta})_2\right) & \mathcal{A}_2(\tau_1)\mathcal{A}_2(\tau_2) - \frac{2\pi}{k}|\dot{x}_1||\dot{x}_2|(\psi\bar{\eta})_1(\eta\bar{\psi})_2 \end{pmatrix},$$

where we have introduced the shorthand notation $(\eta\bar{\psi})_i \equiv \eta_I^\alpha(\tau_i)\bar{\psi}_\alpha^I(\tau_i)$ and $(\psi\bar{\eta})_i \equiv \psi_I^\alpha(\tau_i)\bar{\eta}_\alpha^I(\tau_i)$.

Plugging (3.84) in (3.83), we get three non-vanishing correlators. In what follows, we focus on the upper $N_1 \times N_1$ block contribution of the supermatrix, which is the $U(N_1)$ sector. The result for the $N_2 \times N_2$ block can be obtained from that simply by exchanging $N_1 \leftrightarrow N_2$ in the final result.

For the $N_1 \times N_1$ block, we have three different diagrams coming from Wick contracting the fields. These are shown in Figure 8.

Figure 8: One loop diagrams: (a) the scalar tadpole, (b) the single gluon exchange and (c) the single fermion exchange. The blue wavy line represents the contour, the black solid line is the scalar propagator, the black wavy line is the gauge propagator and the black dashed line is the fermion propagator.

The first diagram represents the exchange of a single scalar coming from $\mathcal{A}$. This term contains $\sim C(x)\bar{C}(x)$, which generates a scalar tadpole graph. The second one is the exchange of a gauge boson coming from the term quadratic on $\mathcal{A}$. Schematically we have

$$\mathcal{A}(\tau_1)\mathcal{A}(\tau_2) \sim A_\mu A_\nu \dot{x}^\mu(\tau_1)\dot{x}^\nu(\tau_2) + \text{scalars}, \quad (3.85)$$

and the gauge field Wick contraction comes from the first term on the right-hand side, generating the single gauge field exchange. Finally, the third diagram is the single fermion exchange that comes from Wick contracting $(\eta\bar{\psi})_1(\psi\bar{\eta})_2$. This term arises from the piece that is quadratic in $\mathcal{L}$.

For a circular contour, the contribution of each diagram was computed in [38].

There the integrals were computed using dimensional regularisation with a dimensional reduction scheme (DRED). This scheme preserves gauge invariance and supersymmetry of Chern-Simons theories up to two loops [39] and we will also use it as one of the techniques to compute our integrals.

In the DRED scheme, we perform tensor manipulations and apply the respective Feynman rules in three dimensions. Then, once the loop integrals are obtained, we promote them to $(3 - 2\epsilon)$ dimensions. There is a prescription to contract three dimensional object with $(3 - 2\epsilon)$ dimensional ones. For example, whenever we have a three dimensional metric $\eta^{\mu\nu}$ contracted with a $(3 - 2\epsilon)$ dimensional one $\hat{\eta}^{\mu\nu}$, we assign the following rules:

$$\eta^{\mu\nu}\eta_{\mu\nu} = 3, \quad \hat{\eta}^{\mu\nu}\hat{\eta}_{\mu\nu} = 3 - 2\epsilon, \quad \eta^{\mu\nu}\hat{\eta}_{\nu\rho} = \hat{\eta}^\mu{}_\rho. \quad (3.86)$$

Let us briefly resume the results obtained for the circular case, i.e., for the non-deformed contour. The scalar tadpole vanishes in dimensional reduction. The single gluon exchange also vanishes, not due to DRED, but due to the antisymmetry of the ϵ tensor contracted with three vectors lying on the plane. The single fermion exchange is a bit more intricate, but in the end of the day it gives no finite contribution.

For the deformed circle, both scalar tadpole and single gauge exchange diagrams must still evaluate to zero. In the first case because the contour does not change the tadpole nature of the scalar contraction and in the second because the deformation keeps the contour lying on the equatorial plane. Therefore, the only possibly non-vanishing contribution must come from the single fermion exchange.

The single fermion exchange contribution is given by

$$\text{Tr} \left\langle \frac{2\pi}{k} \int d\tau_{1>2} |\dot{x}_1| |\dot{x}_2| (\eta\bar{\psi})_1 (\psi\bar{\eta})_2 \right\rangle, \quad (3.87)$$

where

$$(\psi\bar{\eta})_2 = (\psi_I^\alpha)_{\hat{j}}^i(\tau_2) \bar{\eta}_\alpha^I(\tau_2), \quad (3.88)$$

$$(\eta\bar{\psi})_1 = \eta_J^\alpha(\tau_1) (\bar{\psi}_\alpha^J)_{\hat{k}}^j(\tau_1), \quad (3.89)$$

and we introduced the short-hand notation $x_i \equiv x(\tau_i)$. Using the Feynman rules of Appendix A, we have

$$\frac{2\pi}{k} \text{Tr} \left\langle \int d\tau_{1>2} |\dot{x}_1| |\dot{x}_2| \eta_J^\alpha(\tau_1) (\bar{\psi}_\alpha^J)_{\hat{k}}^j(\tau_1) (\psi_I^\beta)_{\hat{j}}^i(\tau_2) \bar{\eta}_\beta^I(\tau_2) \right\rangle$$

$$= -\frac{2\pi}{k} i N_1 N_2 \frac{\Gamma(\frac{3}{2} - \epsilon)}{2\pi^{\frac{3}{2} - \epsilon}} \int d\tau_{1>2} |\dot{x}(\tau_1)| |\dot{x}(\tau_2)| (\eta_1 \gamma^\mu \bar{\eta}_2) \frac{(x(\tau_1) - x(\tau_2))_\mu}{[(x(\tau_1) - x(\tau_2))^2]^{3/2}},$$

where $\Gamma(x)$ is the Euler Gamma function. The $U(N_2)$ sector contributes with exactly same integral, so the single fermion exchange diagram is twice the result above:

$$\text{Diagram} = -\frac{4\pi}{k} i N_1 N_2 \frac{\Gamma(\frac{3}{2} - \epsilon)}{2\pi^{\frac{3}{2} - \epsilon}} \int d\tau_{1>2} |\dot{x}(\tau_1)| |\dot{x}(\tau_2)| (\eta_1 \gamma^\mu \bar{\eta}_2) \frac{(x(\tau_1) - x(\tau_2))_\mu}{[(x(\tau_1) - x(\tau_2))^2]^{3/2}}.$$

We want to evaluate this integral for small perturbations $g(\tau)$ that deform the circle. We consider the contour $\tilde{C}$, assume the couplings (3.82) and expand the integrand in powers of $g(\tau)$. The 0-th order term gives us an integral of the form

$$\sim \int d\tau_{1>2} \frac{1}{\sin^2\left(\frac{\tau_1 - \tau_2}{2}\right)},$$

that matches the one obtained for the circular contour, see equation (3.2) of [38]. As mentioned before, when DRED is used, it gives a contribution proportional to the regularisation parameter ϵ and it goes to zero for $\epsilon \rightarrow 0$. The 2-nd order expansion in $g(\tau)$ gives a sum of terms of the form

$$\int d\tau_{1>2} \frac{e^{in_1 \tau_1 + in_2 \tau_2}}{(e^{i\tau_1} - e^{i\tau_2})^4}, \quad (3.90)$$

for integer valued n_1 and n_2 . This structure appears also in [37] and there the authors propose a recursive method to solve this kind of integral. We refer to this method as the A^p method and a review of it can be found in Appendix B.

Thus we can compute the contribution of the single fermion exchange diagram either through DRED, as in [38], or using the A^p method, as in [37]. Using DRED, we find that (c) gives a finite result proportional to

$$\sum_{n=2}^{\infty} n(n^2 - 1) |b_n|^2, \quad (3.91)$$

where n are the Fourier modes and b_n the Fourier coefficients of the deformation parameter $g(\tau)$. To double check this result, we also use the A^p method, namely rule (B.4), to compute (c). In this case, we find a result proportional to

$$\sum_{n=2}^{\infty} n(n^2 - 1) |b_n|^2 + 16 J_4^{(2)} \sum_{n=2}^{\infty} (n^2 - 1) |b_n|^2, \quad (3.92)$$

where $J_4^{(2)}$ is the arbitrary and possibly infinite constant $J_{n_1+n_2}^{(2)}$ for $n_1 + n_2 = 4$ that

arises through the A^p method. Note that, compared to the result found through DRED, there is an extra piece proportional to $J_4^{(2)}$ that was not expected.

What is expected from the A^p method is a final result that is independent of constants. These J 's account for infinite contributions and can appear individually for each diagram but they are expected to cancel after summing all diagrams arising in the perturbative computation. So, to investigate the appearance of $J_4^{(2)}$, we went one step back in our analysis and reconsidered the 1-loop diagrams (a) and (b). The idea was to check whether they could contribute with a constant piece that would cancel the one present at the single fermion exchange result. For the scalar tadpole (a), we have a contraction of the form $\langle C(x)\bar{C}(y) \rangle$, where

$$\langle C(x)\bar{C}(y) \rangle \sim \frac{1}{[(x-y)^2]^{\frac{1}{2}}}.$$

The integral for the tadpole is given by this propagator at $y = x$ times $|\dot{x}|$, i.e.,

$$\text{tadpole diagram} \sim \int_0^{2\pi} \frac{|\dot{x}|}{[(x-x)^2]^{\frac{1}{2}}} d\tau.$$

We choose to rewrite it as

$$\int_0^{2\pi} \frac{|\dot{x}|}{[(x(0)-x)^2]^{\frac{1}{2}}} d\tau - \frac{1}{2} \int_0^{2\pi} \int_0^{\tau_1} \frac{(x_2-x_1)}{[(x_2-x_1)^2]^{\frac{3}{2}}} \cdot \left(\frac{\dot{x}_1}{|\dot{x}_1|} + \frac{\dot{x}_2}{|\dot{x}_2|} \right) |\dot{x}_1||\dot{x}_2| d\tau_2 d\tau_1,$$

giving two integrals that we need to solve:

$$I_0 \equiv \int_0^{2\pi} \frac{|\dot{x}|}{[(x(0)-x)^2]^{\frac{1}{2}}} d\tau,$$

$$I_1 \equiv -\frac{1}{2} \int_0^{2\pi} \int_0^{\tau_1} \frac{(x_2-x_1)}{[(x_2-x_1)^2]^{\frac{3}{2}}} \cdot \left(\frac{\dot{x}_1}{|\dot{x}_1|} + \frac{\dot{x}_2}{|\dot{x}_2|} \right) |\dot{x}_1||\dot{x}_2| d\tau_2 d\tau_1.$$

For the single gluon exchange (b) we still get a null result due to a propagator containing the contraction $\epsilon_{\mu\nu\rho}(x-y)^\rho$ that vanishes for the planar contour $\tilde{C}$ we are considering so we do not worry about it.

Note that I_0 is a single integral, whereas the A^p method solves double integrals. To solve it we developed another recursive method, named B^p , which is the single integral analogue of the A^p method. The I_1 term, on the other hand, is a double integral, so in principle it could be solved using A^p symbols. However, at second order in $g(\tau)$, we find terms of the form (3.90) with half-integer valued n_1 and n_2 . So we generalised the A^p method for half-integers to solve I_1 . These developments are all presented in Appendix B. In particular, using (B.10) and (B.6) we find

(quite non-trivial) finite results for both I_0 and I_1 that precisely cancel each other,

$$I_0 = -\frac{(1+4n^2)}{4} \left[\text{HarmonicNumber}\left(-\frac{1}{2}-n\right) + \text{HarmonicNumber}\left(-\frac{1}{2}+n\right) \right] \\ - \log(2) - n^2(-3 + \log(16)) = -I_1.$$

Therefore we find no infinite constants that could possibly cancel $J_4^{(2)}$ appearing in the single fermion exchange diagram when computed through the A^p method, see (3.92).

Since reconsidering the scalar tadpole diagram was not enough to clarify the presence of this constant, we took yet another step back and applied these recursive methods to compute the integrals for the non-deformed loop. In this case, we find

$$I_0 = 2 \text{ PolyGamma}\left(0, \frac{1}{2}\right), \\ I_1 = 2\pi^2 K_2^{(1)} + \left[2 - \text{PolyGamma}\left(0, \frac{1}{2}\right) - \text{PolyGamma}\left(0, \frac{3}{2}\right) \right],$$

giving

$$\bigcirc = 2\pi^2 K_2^{(1)},$$

whereas for the single fermion exchange we find

$$\bigcirc = 2\pi^2 J_2^{(1)}.$$

Constants $K_2^{(1)}$ and $J_2^{(1)}$ are the ones arising from the respective substitution rule, (B.5) and (B.3). Therefore, not even for the circular loop we recover a finite result that is free from constants. Since J 's and K 's are not related to each other, we cannot affirm that the contributions from (a) and (c) cancel each other and give a null result. Thus we argue that setting the constants to zero should be scheme equivalent to DRED. This way we recover $\langle W \rangle^{1\text{-loop}} = 0$ for the circular loop and for the deformed loop we find

$$\langle W \rangle^{1\text{-loop}} \sim \sum_{n=2}^{\infty} n(n^2 - 1) |b_n|^2.$$

3.4.2 2-loops contribution

A computation of the expectation value of the Wilson loop at 2-loops requires expanding the path-ordered exponential up to fourth order:

$$\begin{aligned} \text{Tr } \mathcal{P} \exp \left(-i \int \mathcal{L}(\tau) d\tau \right) &\cong \text{Tr} \left[1 + (-i) \int d\tau \mathcal{L}(\tau) + (-i)^2 \int d\tau_{1>2} \mathcal{L}(\tau_1) \mathcal{L}(\tau_2) \right. \\ &\quad \left. + (-i)^3 \int d\tau_{1>2>3} \mathcal{L}(\tau_1) \mathcal{L}(\tau_2) \mathcal{L}(\tau_3) + (-i)^4 \int d\tau_{1>2>3>4} \mathcal{L}(\tau_1) \mathcal{L}(\tau_2) \mathcal{L}(\tau_3) \mathcal{L}(\tau_4) \right]. \end{aligned}$$

In particular, the $N_1 \times N_1$ block has the following form

$$\begin{aligned} \text{Tr} \left[1 - i \int d\tau \mathcal{A}_1 - \int d\tau_{1>2} \left(\mathcal{A}_1(\tau_1) \mathcal{A}_2(\tau_2) - \frac{2\pi}{k} |\dot{x}_1| (\eta \bar{\psi})_1 |\dot{x}_2| (\psi \bar{\eta})_2 \right) \right. \\ + i \int d\tau_{1>2>3} \left(\mathcal{A}_1(\tau_1) \mathcal{A}_1(\tau_2) \mathcal{A}_1(\tau_3) - \frac{2\pi}{k} |\dot{x}_1| (\eta \bar{\psi})_1 |\dot{x}_2| (\psi \bar{\eta})_2 \mathcal{A}_1(\tau_3) \right. \\ \left. - \frac{2\pi}{k} \mathcal{A}_1(\tau_1) |\dot{x}_2| (\eta \bar{\psi})_2 |\dot{x}_3| (\psi \bar{\eta})_3 - \frac{2\pi}{k} |\dot{x}_1| (\eta \bar{\psi})_1 \mathcal{A}_2(\tau_2) |\dot{x}_3| (\psi \bar{\eta})_3 \right) \\ + \int d\tau_{1>2>3>4} \left(\mathcal{A}_1(\tau_1) \mathcal{A}_1(\tau_2) \mathcal{A}_1(\tau_3) \mathcal{A}_1(\tau_4) + \left(\frac{2\pi}{k} \right)^2 (\eta \bar{\psi})_1 (\psi \bar{\eta})_2 (\eta \bar{\psi})_3 (\psi \bar{\eta})_4 \right. \\ \left. - \frac{2\pi}{k} \mathcal{A}_1(\tau_1) \mathcal{A}_1(\tau_2) |\dot{x}_3| (\eta \bar{\psi})_3 |\dot{x}_4| (\psi \bar{\eta})_4 - \frac{2\pi}{k} \mathcal{A}_1(\tau_1) |\dot{x}_2| (\eta \bar{\psi})_2 \mathcal{A}_2(\tau_3) |\dot{x}_4| (\psi \bar{\eta})_4 \right. \\ \left. - \frac{2\pi}{k} |\dot{x}_1| (\eta \bar{\psi})_1 \mathcal{A}_2(\tau_2) \mathcal{A}_2(\tau_3) |\dot{x}_4| (\psi \bar{\eta})_4 - \frac{2\pi}{k} \mathcal{A}_1(\tau_1) |\dot{x}_2| (\eta \bar{\psi})_2 |\dot{x}_3| (\psi \bar{\eta})_3 \mathcal{A}_1(\tau_4) \right. \\ \left. - \frac{2\pi}{k} |\dot{x}_1| (\eta \bar{\psi})_1 \mathcal{A}_2(\tau_2) |\dot{x}_3| (\psi \bar{\eta})_3 \mathcal{A}_1(\tau_4) - \frac{2\pi}{k} |\dot{x}_1| (\eta \bar{\psi})_1 |\dot{x}_2| (\psi \bar{\eta})_2 \mathcal{A}_1(\tau_3) \mathcal{A}_1(\tau_4) \right) \Big]. \end{aligned}$$

In this case, there are six different diagrams to consider and these are represented in Figure 9. Three of them are purely bosonic and we will refer to those as the bosonic sector. The other three contain fermions and will be referred to as the fermionic sector.

Bosonic sector

Diagram (a) in Figure 9 comes from the third order expansion of the WL contracted with the gauge cubic vertex⁸. Its value depends only on the topology of the loop. Since our deformation preserves the unknot structure of the contour, we can borrow the result from the literature [40] and, for the $N_1 \times N_1$ block, we have

$$-\frac{N_1(N_1^2 - 1)}{N_1 + N_2} \frac{\pi^2}{6k^2}.$$

⁸The interaction vertices appearing in the action are shown in Appendix A

Figure 9: Bosonic (a-c) and fermionic diagrams (d-f) from the 2-loop expansion.

Considering also the $N_2 \times N_2$ block, we get that the first diagram gives

$$\text{(a)} = -\frac{N_2(N_2^2 - 1) + N_1(N_1^2 - 1)}{N_1 + N_2} \frac{\pi^2}{6k^2}.$$

Diagram (b) comes from the bosonic part of the quadratic term in the WL expansion with a one-loop gauge propagator inserted. More specifically, it comes from the one-loop insertion in

$$-\text{Tr} \left[\int d\tau_{1>2} A_{1\mu}(x_1) \dot{x}_1^\mu A_{1\nu}(x_2) \dot{x}_2^\nu \right],$$

which gives

$$-\frac{\Gamma^2(\frac{1}{2} - \epsilon)}{k^2 \pi^{1-2\epsilon}} N_1^2 N_2 \int d\tau_{1>2} \frac{\dot{x}_1 \cdot \dot{x}_2}{[(x_1 - x_2)^2]^{1-2\epsilon}}.$$

Summing the $N_2 \times N_2$ contribution to it, we get

$$\text{(b)} = -\frac{N_1^2 N_2 + N_2^2 N_1}{N_1 + N_2} \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{k^2 \pi^{1-2\epsilon}} \int d\tau_{1>2} \frac{\dot{x}_1 \cdot \dot{x}_2}{[(x_1 - x_2)^2]^{1-2\epsilon}}. \quad (3.93)$$

As in the single fermion exchange diagram, the second order expansion of the integrand in the deformation parameter $g(\tau)$ gives

$$\sim \sum_{n=2}^{\infty} n(n^2 - 1) |b_n|^2. \quad (3.94)$$

We were able to obtain this result through the A^p method and also through the DRED regularisation scheme after setting the constants to zero.

Diagram (c) comes from the bosonic part of the quadratic term in the WL expansion with contractions between scalars:

$$-\left(\frac{2\pi i}{k}\right)^2 \text{Tr} \left[\int d\tau_{1>2} |\dot{x}_1| M_J^I C_I \bar{C}^J |\dot{x}_2| M_K^L C_L \bar{C}^K \right].$$

It gives

$$\left(\frac{2\pi}{k} \frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 N_1^2 N_2 \int d\tau_{1>2} \left[M_J^I M_I^J \frac{|\dot{x}_1| |\dot{x}_2|}{[(x_1 - x_2)^2]^{1-2\epsilon}} \right].$$

Using that $M_J^I M_I^J = 4$ and also summing the $N_2 \times N_2$ contribution, we find

$$\text{Diagram (c)} = \left(\frac{2\pi}{k} \frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 \frac{N_1^2 N_2 + N_2^2 N_1}{N_1 + N_2} \int d\tau_{1>2} \left[4 \frac{|\dot{x}_1| |\dot{x}_2|}{[(x_1 - x_2)^2]^{1-2\epsilon}} \right]. \quad (3.95)$$

Up to second order in the deformation parameter $g(\tau)$, the integral above gives zero. This result was obtained both through the A^p method and DRED.

Fermionic sector

Diagram (d) comes from the fermionic part of the second order expansion of the WL with a one-loop contraction between two fermions, i.e., it comes from the one-loop insertion in

$$\left(\frac{2\pi}{k}\right) \text{Tr}_{N_1} \left[\int d\tau_{1>2} |\dot{x}_1| |\dot{x}_2| (\eta \bar{\psi})_1 (\psi \bar{\eta})_2 \right].$$

It gives

$$i \left(\frac{2\pi}{k}\right)^2 \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{16\pi^{3-2\epsilon}} (N_2 - N_1) N_1 N_2 \int d\tau_{1>2} \frac{|\dot{x}_1| |\dot{x}_2| \eta_I^\alpha(x_1) \bar{\eta}_\alpha^I(x_2)}{[(x_1 - x_2)^2]^{1-2\epsilon}}.$$

Taking into account the contribution from both blocks, we get

$$\text{Diagram (d)} \sim \int d\tau_{1>2} \frac{|\dot{x}_1| |\dot{x}_2|}{[(x_1 - x_2)^2]^{1-2\epsilon}} \left(\eta_I^\alpha(x_1) \bar{\eta}_\alpha^I(x_2) - \eta_I^\alpha(x_2) \bar{\eta}_\alpha^I(x_1) \right).$$

Note that $\eta_I^\alpha(x_1) \bar{\eta}_\alpha^I(x_2) = \eta_I^\alpha(x_2) \bar{\eta}_\alpha^I(x_1)$, so diagram (d) gives zero.

Diagram (e) comes from the double contraction of fermions in the fourth order

expansion of the WL,

$$\begin{aligned} \text{Diagram (e)} &= -\frac{1}{N_1 + N_2} \left(\frac{2\pi}{k} \right)^2 \frac{\Gamma^2\left(\frac{3}{2} - \epsilon\right)}{4\pi^{3-2\epsilon}} \int d\tau_{1>2>3>4} |\dot{x}_1| |\dot{x}_2| |\dot{x}_3| |\dot{x}_4| \times \\ &\left\{ \left[N_2 N_1^2 (\eta_1 \gamma^\mu \bar{\eta}_2) (\eta_3 \gamma^\nu \bar{\eta}_4) + N_2^2 N_1 (\eta_2 \gamma^\mu \bar{\eta}_1) (\eta_4 \gamma^\nu \bar{\eta}_3) \right] \frac{(x_1 - x_2)_\mu (x_3 - x_4)_\nu}{[(x_1 - x_2)^2 (x_3 - x_4)^2]^{\frac{3}{2}-\epsilon}} \right. \\ &\left. + \left[N_2^2 N_1 (\eta_1 \gamma^\mu \bar{\eta}_4) (\eta_3 \gamma^\nu \bar{\eta}_2) + N_2 N_1^2 (\eta_4 \gamma^\mu \bar{\eta}_1) (\eta_2 \gamma^\nu \bar{\eta}_3) \right] \frac{(x_1 - x_4)_\mu (x_2 - x_3)_\nu}{[(x_1 - x_4)^2 (x_2 - x_3)^2]^{\frac{3}{2}-\epsilon}} \right\}, \end{aligned}$$

and diagram (f) comes from the six gauge-fermion mixed terms in the third order expansion of the WL,

$$\begin{aligned} \text{Diagram (f)} &= -\frac{1}{N_1 + N_2} \frac{2\pi}{k} \int d\tau_{1>2>3} \text{Tr} \left\{ \eta_{2I} \bar{\eta}_3^J \langle A_{1\mu}(\tau_1) \bar{\psi}_2^I \psi_{3J} \rangle \dot{x}_1^\mu |\dot{x}_2| |\dot{x}_3| + \bar{\eta}_2^I \eta_{3J} \times \right. \\ &\langle A_{2\mu}(\tau_1) \psi_{2I} \bar{\psi}_3^J \rangle \dot{x}_1^\mu |\dot{x}_2| |\dot{x}_3| + \eta_{3I} \bar{\eta}_1^J \langle \psi_{1I} A_{1\mu}(\tau_2) \bar{\psi}_3^J \rangle |\dot{x}_1| \dot{x}_2^\mu |\dot{x}_3| + \bar{\eta}_3^I \eta_{1J} \times \\ &\langle \bar{\psi}_1^J A_{2\mu}(\tau_2) \psi_{3I} \rangle |\dot{x}_1| \dot{x}_2^\mu |\dot{x}_3| + \eta_{1I} \bar{\eta}_2^J \langle \bar{\psi}_1^I \psi_{2J} A_{1\mu}(\tau_3) \rangle |\dot{x}_1| |\dot{x}_2| \dot{x}_3^\mu + \bar{\eta}_1^I \eta_{2J} \times \\ &\left. \langle \psi_{1I} \bar{\psi}_2^J A_{2\mu}(\tau_3) \rangle |\dot{x}_1| |\dot{x}_2| \dot{x}_3^\mu \right\}. \end{aligned}$$

We still have to compute these last two diagrams to finish the 2-loop computation of the Wilson loop. This is work in progress.

Comments

The $\mathcal{O}(b_n^2)$ terms appearing in the computation of the expectation value of W give what is known as the wavy-line approximation. This approximation was considered in the context of $\mathcal{N}=4$ SYM theory in [41]. There the authors work with the WL supported along a wavy line, which deviates by a small amount from an infinite straight line. They parameterise it as

$$x^\mu(s) = (s, \xi(s)), \quad s \in (-\infty, +\infty),$$

where the three-dimensional vector $\xi(s)$ is a smooth function of the curve parameter s with small magnitude. They found that, relying on symmetry properties of the theory, a universal term of the form

$$\int ds \int ds' \frac{(\dot{\xi}(s) - \dot{\xi}(s'))^2}{(s - s')^2} \quad (3.96)$$

appears in the computation of the expectation value of the Wilson loop up to second order in $\xi(s)$. They also found that this universal term comes accompanied by

a non-trivial function of the coupling constant, which should receive contributions from all orders in perturbation theory. Later in [42] this non-trivial function of the coupling constant was named Bremsstrahlung function $B(\lambda)$. It was shown that the B function appears not only in the two point function of the displacement operator,

$$\langle\langle \mathbb{D}_i(t)\mathbb{D}_j(0) \rangle\rangle = \frac{12B\delta_{ij}}{t^4},$$

where the double brackets denote the expectation value evaluated on the Wilson line, but also in the the energy emitted by a moving quark in the small velocity limit (which justifies the name Bremsstrahlung),

$$\Delta E = 2\pi B \int dt (\dot{v})^2,$$

and at the derivative of the cusp anomalous dimension $\Gamma_{\text{cusp}}(\phi)$ at zero angle⁹

$$B = \frac{1}{2} \partial_\phi^2 \Gamma_{\text{cusp}}(\phi) \Big|_{\phi=0}.$$

In fact, the appearance of B in these three observables holds for any conformal gauge theory with a Wilson loop operator. Here, since we have a deformed Wilson loop in $\mathcal{N} = 6$ Chern-Simons-matter theory, we expect the $\mathcal{O}(b_n^2)$ contribution to be proportional to the Bremsstrahlung function. Moreover, since we are computing $\langle W \rangle$ up to two loops, we expect to find $B(\lambda)$ up to second order in the coupling constant λ . In particular, recall that the t'Hooft couplings are $\lambda_1 = N_1/k$ and $\lambda_2 = N_2/k$. In the large N limit, we have $\lambda_1 \sim \lambda_2$ and

$$\langle W \rangle^{(2)} \sim B(\lambda) \sum_{n=2}^{\infty} n(n^2 - 1) |b_n|^2. \quad (3.97)$$

After computing the remaining diagrams listed above, we expect to recover a closed formula at two loops for the function $B(\lambda)$. This will allow us to compare our result to the ones of [34]. If these results match, not only our claim that the fermionic couplings should take the form (3.18) would be strongly supported, but also the use of the substitution methods of Appendix B with constants set to zero. With this settled, we could for instance use the results to do the perturbative computation at fourth order in the deformation parameter g , which has not been considered before in the literature.

⁹For an operator supported along a straight line, a cusp is a region where the line makes a sudden turn by an angle ϕ . Then the cusp anomalous dimension is the logarithmic divergence that arises in the expectation value of the Wilson line due to the sudden turn [43].

Chapter 4

$\mathcal{N} = 4$ Chern-Simons-matter theories

This chapter is mostly based on [21]. At the end we also include some preliminary results of an ongoing project that should be presented in the near future [23].

An $\mathcal{N} = 4$ Chern-Simons-matter theory is a quiver theory that can be either circular or linear. In this chapter most of our discussion is carried for a generic theory and we do not worry about the endpoints of the quiver. We consider nodes labeled by an index I and these represent the respective gauge field A_I . An edge connecting nodes I and $I + 1$ represents a hypermultiplet $(q_I^a, \psi_{I\dot{a}})$, whereas an edge connecting I and $I - 1$ represents a twisted hypermultiplet $(\tilde{q}_{I-1\dot{a}}, \tilde{\psi}_{I-1}^a)$. This structure is carried over along the quiver in an alternated fashion. The field content is summarized in the quiver diagram of Figure 10, where the thick solid lines represent the matter fields.

Figure 10: The quiver and field content of the $\mathcal{N} = 4$ theory.

The (twisted) hyper multiplets can be decomposed into pairs of chiral multiplets. Figure 11 shows the chiral scalar in this decomposition explicitly. Here and throughout chiral fields are denoted as solid arrows and when needed, the anti-chiral fields are represented by dashed arrows. The orientation of the arrows stands for the field's representation. For example, the fields q_I^2 is in the $(\square, \bar{\square})$ of $U(N_I) \times U(N_{I+1})$ and $\bar{q}_{I1}$ is in the conjugate representation.

Figure 11: The decomposition of the $\mathcal{N} = 4$ matter multiplets into pairs of chiral multiplets. Only the chiral scalar of each multiplet is indicated explicitly.

The $SU(2)_L \times SU(2)_R$ R-symmetry indices $a, b = 1, 2$ and $\dot{a}, \dot{b} = \dot{1}, \dot{2}$ are raised and lowered using the appropriate epsilon symbols: $v^a = \epsilon^{ab}v_b$ and $v_a = \epsilon_{ab}v^b$ with $\epsilon^{12} = \epsilon_{21} = 1$, and similarly for the dotted indices.

The covariant derivatives acting on scalars, for example, are defined as

$$\begin{aligned} D_\varphi q_I^a &= \partial_\varphi q_I^a - iA_{\varphi,I}q_I^a + iq_I^a A_{\varphi,I+1}, \\ D_\varphi \bar{q}_{Ia} &= \partial_\varphi \bar{q}_{Ia} - iA_{\varphi,I+1}\bar{q}_{Ia} + i\bar{q}_{Ia} A_{\varphi,I}. \end{aligned} \quad (4.1)$$

Naturally, for twisted scalars we have

$$\begin{aligned} D_\varphi \tilde{q}_{I-1\dot{a}} &= \partial_\varphi \tilde{q}_{I-1\dot{a}} - iA_{\varphi,I-1}\tilde{q}_{I-1\dot{a}} + i\tilde{q}_{I-1\dot{a}} A_{\varphi,I}, \\ D_\varphi \tilde{\bar{q}}_{I-1}^{\dot{a}} &= \partial_\varphi \tilde{\bar{q}}_{I-1}^{\dot{a}} - iA_{\varphi,I}\tilde{\bar{q}}_{I-1}^{\dot{a}} + i\tilde{\bar{q}}_{I-1}^{\dot{a}} A_{\varphi,I-1}. \end{aligned}$$

For later convenience we define moment maps and currents, which are bilinears of the hypermultiplets and twisted hypermultiplets in the adjoint representation of $U(N_I)$, as follows

$$\begin{aligned} \mu_I^a{}_b &= q_I^a \bar{q}_{Ib} - \frac{1}{2} \delta_b^a q_I^c \bar{q}_{Ic}, & j_I^{a\dot{b}} &= q_I^a \bar{\psi}_I^{\dot{b}} - \epsilon^{ac} \epsilon^{\dot{b}\dot{c}} \psi_{I\dot{c}} \bar{q}_{Ic}, \\ \tilde{\mu}_I^{\dot{a}}{}_{\dot{b}} &= \tilde{\bar{q}}_{I-1}^{\dot{a}} \tilde{q}_{I-1\dot{b}} - \frac{1}{2} \delta_{\dot{b}}^{\dot{a}} \tilde{\bar{q}}_{I-1}^{\dot{c}} \tilde{q}_{I-1\dot{c}}, & \tilde{j}_I^{\dot{b}a} &= \tilde{\bar{q}}_{I-1}^{\dot{b}} \tilde{\psi}_{I-1}^a - \epsilon^{\dot{b}\dot{c}} \epsilon^{ac} \tilde{\psi}_{I-1\dot{c}} \tilde{q}_{I-1\dot{c}}, \\ \nu_I &= q_I^a \bar{q}_{Ia}, & \tilde{\nu}_I &= \tilde{\bar{q}}_{I-1}^{\dot{a}} \tilde{q}_{I-1\dot{a}}. \end{aligned}$$

Similar bilinears (with the appropriate replacement of hypermultiplets and twisted hypermultiplets) exist also for the other nodes. For example, for the $I+1$ node one can define $\nu_{I+1} = \bar{q}_{I+1} q_{I+1}^a$. Note that in this notation the index of the moment maps represents the node under which they are charged, rather than the fields they are made of, as is the case in the notation of [17, 44]. In particular, $\mu_{I+1}^b{}_a$ is made of the same fields $\bar{q}_{I+1}$ and q_{I+1}^b as $\mu_I^b{}_a$, but it is charged under a different group. The moment maps are triplets of the respective $SU(2)$ R-symmetry group and are used below to construct the basic 1-node Wilson loops. The moment maps can be thought of as the generalization to $\mathcal{N} = 4$ of the $\mathcal{N} = 2$ scalar σ , though the latter is an auxiliary field in an off-shell formulation while we work in an on-shell formulation. We provide some details on this correspondence in Appendix C.

We define the theory on S^3 and the Wilson loops we construct are supported along the equator of this sphere. The corresponding $\mathcal{N} = 4$ supersymmetry transformations are given in the appendix in (C.1), along with details about how they relate to the ones for the $\mathcal{N} = 2$ theory [45, 46]. The theory is invariant under 16 supersymmetry transformations generated by parameters ξ_{ab} . Each parameter is a linear combination of four Killing-spinors on S^3 such that, together with the two possible values for a and b , we indeed have a total of $2 \times 2 \times 4 = 16$ parameters. We label each Killing-spinor as ξ^l , $\xi^{\bar{l}}$, ξ^r , $\xi^{\bar{r}}$, according to their chiralities. They obey

$$\nabla_\mu \xi^{l,\bar{l}} = \frac{i}{2} \gamma_\mu \xi^{l,\bar{l}}, \quad \nabla_\mu \xi^{r,\bar{r}} = -\frac{i}{2} \gamma_\mu \xi^{r,\bar{r}}. \quad (4.2)$$

Along the circle we take $\gamma_\varphi = \sigma_3$ and these reduce to [35]

$$\xi_\alpha^l = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_\alpha^{\bar{l}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi_\alpha^r = \begin{pmatrix} e^{-i\varphi} \\ 0 \end{pmatrix}, \quad \xi_\alpha^{\bar{r}} = \begin{pmatrix} 0 \\ e^{i\varphi} \end{pmatrix}. \quad (4.3)$$

Compared to ABJ(M) theory, in $\mathcal{N} = 4$ the allowed set of theories is wider and different loops that are equivalent under the $SO(6)$ R-symmetry of ABJ(M) may be on disconnected branches of the moduli space in the presence of only $SO(4)$ R-symmetry. Operators preserving 4 and 8 out of the 16 supercharges were considered in [47]. The 1/2 BPS ones were found to present a finite degeneracy that was recognized in [44] and was soon realized holographically in [48]. As for less supersymmetric operators [31, 49, 27, 50], the study of their degeneracy has blossomed into an independent research program whose full scope is still unclear.

In CSM theories in general, the past several years were marked by more and more examples of BPS loops being found. In the last couple of years, in particular, some initial steps to reorganize the subject were taken. First, the roadmap paper [20] reviewed what was known at the time about BPS Wilson loops in three dimensions, introduced some new formalism and, for the first time, properly addressed the moduli spaces of the 1/6 BPS Wilson loops in ABJ(M) theory. Second, the moduli spaces of BPS Wilson loops in $\mathcal{N} = 2$ theories were studied in [51] and identified with quiver varieties. Most recently, the symmetries of BPS line operators in diverse dimensions were analyzed in [52], where the naturalness of marginal defect couplings in three dimensions was stressed. Our work in [21] aims to continue on that path, focusing on $\mathcal{N} = 4$ Chern-Simons-matter theories [16–19].

We study circular Wilson loops in three-dimensional $\mathcal{N} = 4$ theories that are

continuously connected by marginal deformations to the usual 1/4 BPS Gaiotto-Yin (“bosonic”) Wilson loop [22]. One type of deformation arising in quiver gauge theories involves couplings to the matter fields in bifundamental representations of the gauge groups. This follows closely previous studies of the moduli spaces of loops in $\mathcal{N} = 2$ theories and in ABJ(M) [20, 51]. The other deformation is often called the “latitude” deformation and is similar to the four-dimensional Wilson loops in [53] and to the ABJM latitudes [27]. This construction uses that $\mathcal{N} = 4$ theories have triplets of bifundamental bilinears, the moment-maps that generalize the scalar coupling of the Gaiotto-Yin Wilson loop.

The usual 1/4 BPS loop involves couplings to scalar bilinears of both the hyper and twisted hyper fields of the three-dimensional Chern-Simons-matter theories. These couplings break the $SO(4) \simeq SU(2)_L \times SU(2)_R$ to $U(1)_L \times U(1)_R$, while the latitude deformation further breaks one of the $U(1)$ ’s (we choose it to be $U(1)_R$). This is a continuous deformation with a parameter θ and for generic values of this parameter the loop preserves 1/8 of the supercharges. Given this as a starting point, we can deform the bosonic loops by introducing couplings to more bifundamental fields through superconnections and get “fermionic” operators, in the spirit of [24, 20]. This produces richer moduli spaces of BPS loops, which also include 1/16 BPS operators. As this construction involves in an intimate way the hypermultiplet fields and their $SU(2)$ R-symmetries, we name these operators *hyperloops*.

As mentioned above, we define our theories on S^3 and our loops are supported along great circles of this space. What we call “latitude” deformation is then a slight misnomer, as for us this deformation only affects the internal space of moment map couplings and not the geometric contour on which these operators are defined. As a consequence, the two supercharges preserved by the latitude loops are not a subset of the four preserved by the Gaiotto-Yin loop, but this can be resolved by a conformal transformation mapping the original circle at the equator to an actual latitude of the S^3 . The alternative formulation would follow the four-dimensional construction in [33] or the ABJ(M) analog in [25] with the operators defined from the start along latitudes and with the matter field couplings dictated by this choice of geometric contour. In that case, the latitude loops would be a subset of loops with arbitrary shapes on $S^2 \subset S^3$, all preserving a fixed subset of the supercharges. So, barring the fact that we have made the choice to keep the circle fixed in space, we can view all the loops presented here as continuous deformations of the Gaiotto-Yin loop preserving a subset of the supercharges.

The classification of the hyperloops depends on (i) some discrete data and (ii) some continuous parameters. The discrete data is a choice of vector fields that appear in the (super)connection (and their multiplicities) and a subset of the matter fields that we allow to couple to the loop. This information can be conveniently conveyed by certain quiver diagrams, as we explain below. Restricting to only half of the matter fields enhances supersymmetry and gives hyperloops that are (at least) 1/8 BPS, so there are two choices of which half of these fields to include, with each option spanning a separate branch of the moduli space. One can also allow a coupling to all the fields at the price of preserving less supersymmetry and obtaining 1/16 BPS operators. The continuous parameters are the latitude angle θ mentioned above, an azimuthal angle φ_0 (which we mostly ignore) and the continuous couplings to the matter fields. In the simplest case, there are two or four complex parameters per edge in the quiver, depending on how many supercharges one wants to preserve. These parameters are subject to a global gauge symmetry, reducing the moduli space to a cone, similar to the conifold in the case of ABJ(M) theory [20].

4.1 1-node Wilson loops

We start by constructing Wilson loops coupling to a single gauge field and hence suppress the I index on the fields. The most symmetric such loop is a circle coupling to both the untwisted and twisted moment maps through the connection

$$\mathcal{A} = A_\varphi - \frac{i}{k}(\mu^1_1 - \mu^2_2 + \tilde{\mu}^1_1 - \tilde{\mu}^2_2). \quad (4.4)$$

This choice of scalar coupling can be motivated by considering what is the natural generalization to the $\mathcal{N} = 4$ case of the scalar σ appearing in the Wilson loop in $\mathcal{N} = 2$ Chern-Simons-matter theory [22]. Since the moment maps are triplets of the R-symmetry group for $\mathcal{N} = 4$, the coupling above corresponds to picking the moment maps along the ‘third’ direction of the R-symmetry triplet, so can be denoted alternatively as μ^3 and $\tilde{\mu}^3$.

From (C.1) one finds the supersymmetry transformations of the moment maps

$$\delta\mu_I^a{}_b = \xi_{bc}\tilde{j}_I^{ac} - \frac{1}{2}\delta_b^a\xi_{cc}\tilde{j}_I^{cc}, \quad \delta\tilde{\mu}_I^{\dot{a}}{}_b = -\xi_{cb}\tilde{j}_I^{\dot{a}c} + \frac{1}{2}\delta_b^{\dot{a}}\xi_{cc}\tilde{j}_I^{\dot{c}c}, \quad (4.5)$$

so the variation of the connection in (4.4) is

$$\delta\mathcal{A} = \frac{i}{k}\xi_{ab}\gamma_\varphi(j^{ab} - \tilde{j}^{ba}) - \frac{i}{k}(\xi_{1b}j^{1b} - \xi_{2b}j^{2b} - \xi_{a1}\tilde{j}^{1a} + \xi_{a2}\tilde{j}^{2a}), \quad (4.6)$$

with γ_φ along the equatorial circle on which the loop is supported. There are no solutions to $\delta\mathcal{A} = 0$ for $\xi_{1\dot{2}}$ and $\xi_{2\dot{1}}$, while for the other components one finds the conditions

$$\xi_{1\dot{1}}(1 - \gamma_\varphi) = \xi_{2\dot{2}}(1 + \gamma_\varphi) = 0. \quad (4.7)$$

The resulting Wilson loop is hence 1/4 BPS, and it is in fact the same as the 1/2 BPS loop in $\mathcal{N} = 2$ theories [22] or the 1/6 BPS loop in ABJ(M) theory, as in [28–30].

Equation (4.7) restricts $\xi_{1\dot{1}}$ to the two chiralities $\xi^{\bar{l}}$ and $\xi^{\bar{r}}$, while $\xi_{2\dot{2}}$ is a linear combination of ξ^l and ξ^r . We can write the corresponding four supersymmetries as

$$Q_{\bar{l}}^{\dot{1}1}, \quad Q_{\bar{r}}^{\dot{1}1}, \quad Q_l^{\dot{2}2}, \quad Q_r^{\dot{2}2}. \quad (4.8)$$

In the subsequent sections we construct families of Wilson loop operators that preserve all or particular linear combinations of those supercharges. These loops couple to two or more nodes of the quiver and involve combining several gauge connections and couplings to more bifundamental fields. However, there is still a family of Wilson loops involving just a single node that preserve only two supercharges, being therefore 1/8 BPS. Following the logic of the 1/4 BPS Wilson loops in $\mathcal{N} = 4$ SYM in 4d [53], the connection (4.4) is naturally generalized introducing a “latitude”¹ angle θ and an azimuthal angle φ_0 , as follows

$$\mathcal{A}^\theta = A_\varphi - \frac{i}{k} \left(\mu^1_1 - \mu^2_2 + \cos\theta (\tilde{\mu}^{\dot{1}}_1 - \tilde{\mu}^{\dot{2}}_2) + \sin\theta (e^{-i(\varphi-\varphi_0)} \tilde{\mu}^{\dot{1}}_2 + e^{i(\varphi-\varphi_0)} \tilde{\mu}^{\dot{2}}_1) \right). \quad (4.9)$$

Notice that only the couplings to the twisted hypermultiplets are modified, via their moment maps. This operator is now coupled to three different moment maps, which can alternatively be written as $\tilde{\mu}^3$ and $\tilde{\mu}^\pm$. This deformation is not possible in theories with only $\mathcal{N} = 2$ symmetry, with only one σ field in the vector multiplet. An analogous construction where the untwisted moment maps are modified and the twisted ones remain as in (4.4) also works, but deforming both at the same time does not give BPS operators. This deformation involves two parameters, θ and φ_0 , which define an S^2 . For simplicity we set $\varphi_0 = 0$ in the following.

¹Note that this is only a latitude in the internal space of scalar couplings, while the loop is still an equator of the S^3 .

The supersymmetry variation of (4.9) gives

$$\begin{aligned} \delta \mathcal{A}^\theta = & \frac{i}{k} \left(\xi_{ab} \gamma_\varphi (j^{ab} - \tilde{j}^{ba}) - (\xi_{1b} j^{1b} - \xi_{2b} j^{2b}) \right. \\ & \left. + \cos \theta (\xi_{a1} \tilde{j}^{1a} - \xi_{a2} \tilde{j}^{2a}) + \sin \theta (e^{-i\varphi} \xi_{a2} \tilde{j}^{1a} + e^{i\varphi} \xi_{a1} \tilde{j}^{2a}) \right). \end{aligned}$$

Requiring this to vanish and collecting terms according to the components of the currents, one finds conditions on the supersymmetry parameters which are all solved imposing

$$\begin{aligned} \xi_{1\dot{a}}(1 - \gamma_\varphi) &= \xi_{2\dot{a}}(1 + \gamma_\varphi) = 0, \\ \xi_{a1}(\gamma_\varphi - \cos \theta) &= \sin \theta e^{-i\varphi} \xi_{a2}. \end{aligned} \quad (4.10)$$

The first line of (4.10) sets $\xi_{1\dot{a}}^l = \xi_{1\dot{a}}^r = \xi_{2\dot{a}}^{\bar{l}} = \xi_{2\dot{a}}^{\bar{r}} = 0$. The second line also eliminates $\xi_{a2}^r = \xi_{a2}^{\bar{l}} = \xi_{a1}^l = \xi_{a1}^{\bar{r}} = 0$. The remaining $\xi_{11}^{\bar{l}}, \xi_{12}^{\bar{r}}, \xi_{21}^r, \xi_{22}^l$ are related by

$$\xi_{11}^{\bar{l}} = \frac{\sin \theta}{1 - \cos \theta} \xi_{12}^{\bar{r}} = \cot \frac{\theta}{2} \xi_{12}^{\bar{r}}, \quad \xi_{21}^r = -\frac{\sin \theta}{1 + \cos \theta} \xi_{22}^l = -\tan \frac{\theta}{2} \xi_{22}^l. \quad (4.11)$$

We find finally the two independent supercharges preserved by the loop

$$\mathcal{Q}_1 = \cos \frac{\theta}{2} Q_l^{i1} + \sin \frac{\theta}{2} Q_{\bar{r}}^{\dot{2}1}, \quad \mathcal{Q}_2 = \cos \frac{\theta}{2} Q_l^{\dot{2}2} - \sin \frac{\theta}{2} Q_r^{i2}, \quad (4.12)$$

which is then 1/8 BPS, as advertised. Notice that the supercharges in (4.12) are not a subset of those in (4.8). The reason for this is that we kept the circle on the equator of S^3 . Were we to follow the logic of [25] and place the loop at a latitude angle of $\pi/2 - \theta$, which can be done via a conformal transformation on the S^3 , the resulting loops would preserve Q_l^{i1} and $Q_l^{\dot{2}2}$, which are indeed a subset of (4.8).

4.2 Hyperloops at $\theta = 0$

In this and the next section we construct what we dub *hyperloops*: BPS Wilson loops involving multiple gauge fields and couplings to the hypermultiplets beyond their bilinears. We apply the deformation procedure that was introduced in [20], and explained in detail in Section 3.2, based on the 1-node loop (4.4). We also explain what the moduli space of the resulting operator is and its relation to quiver varieties according to [51].

We begin by defining two linear combinations of the four supercharges pre-

served by the $1/4$ BPS 1-node loop,

$$\mathcal{Q}_+^L = Q_I^{11} + Q_I^{22}, \quad \mathcal{Q}_-^L = Q_I^{11} - Q_I^{22}.$$

These are such that, when acting twice on scalars, give back their covariant derivatives. Therefore, these are appropriate candidates to construct a deformation $\Delta\mathcal{L} = i\mathcal{Q}_\pm\mathcal{G} + \mathcal{G}^2$ for some matrix $\mathcal{G}$.

In the context of ABJM, the matrix $\mathcal{G}$ is block 2×2 because the theory is a 2-node quiver theory and the resulting loops always couple to these 2-node. Here, as anticipated in Chapter 2, our loop is allowed to couple to multiple nodes and consequently the matrix $\mathcal{G}$ may have a richer structure.

Suppose that our loop couples to 3 nodes, say $I - 1$, I and $I + 1$. Then $\mathcal{G}$ should contain scalars q_I , $\bar{q}_I$, as well as twisted ones $\tilde{q}_{I-1}$, $\bar{\tilde{q}}_{I-1}$. Moreover, suppose we want to include some of the nodes more than once and choose, for example, node I to appear twice. Then our composite bosonic connection is

$$\mathcal{L}_{\text{bos}} = \begin{pmatrix} \mathcal{A}_{I-1} + \frac{1}{2} & 0 & 0 & 0 \\ 0 & \mathcal{A}_I & 0 & 0 \\ 0 & 0 & \mathcal{A}_I & 0 \\ 0 & 0 & 0 & \mathcal{A}_{I+1} + \frac{1}{2} \end{pmatrix}, \quad (4.13)$$

where we already include the constant shifts. Note that instead of adding $\pm 1/4$ to every node, as in Section 3.2, we chose to move these around in such a way that they appear in an alternated fashion. This is harmless to the derivation since what matters is the difference between two neighboring nodes, but it will be useful in the analysis that follows. Next we define the matrix

$$\mathcal{G} = \begin{pmatrix} 0 & \bar{\alpha}_{I-1}^1 \tilde{q}_{I-1\dot{2}} & \bar{\alpha}_{I-1}^2 \tilde{q}_{I-1\dot{2}} & 0 \\ \alpha_{I-1}^1 \bar{\tilde{q}}_{I-1}^{\dot{2}} & 0 & 0 & \alpha_I^1 q_I^2 \\ \alpha_{I-1}^2 \bar{\tilde{q}}_{I-1}^{\dot{2}} & 0 & 0 & \alpha_I^2 q_I^2 \\ 0 & \bar{\alpha}_I^1 \bar{q}_{I2} & \bar{\alpha}_I^2 \bar{q}_{I2} & 0 \end{pmatrix}, \quad (4.14)$$

which is parameterised by constants α and $\bar{\alpha}$ that are complex, but not necessarily complex conjugates of each other. These carry the capital node index I and a lower-case multiplicity index i . From this matrix we construct the superconnection

$$\mathcal{L}_{\alpha, \bar{\alpha}} = \mathcal{L}_{\text{bos}} + i\mathcal{Q}_+^L \mathcal{G} + \mathcal{G}^2. \quad (4.15)$$

The hyperloop is then²

$$W_{\alpha, \bar{\alpha}} = \text{sTr } \mathcal{P} \exp \left[i \oint \mathcal{L}_{\alpha, \bar{\alpha}} |\dot{x}| ds \right],$$

and it preserves all four supersymmetry transformations generated by (4.8).

Another possibility is to consider the composite bosonic connection with swapped constant shifts,³

$$\mathcal{L}_{\text{bos}} = \begin{pmatrix} \mathcal{A}_{I-1} & 0 & 0 & 0 \\ 0 & \mathcal{A}_I + \frac{1}{2} & 0 & 0 \\ 0 & 0 & \mathcal{A}_I + \frac{1}{2} & 0 \\ 0 & 0 & 0 & \mathcal{A}_{I+1} \end{pmatrix}. \quad (4.16)$$

In this case we find that for

$$\mathcal{G} = \begin{pmatrix} 0 & \beta_{I-1}^1 \tilde{q}_{I-1\text{i}} & \beta_{I-1}^2 \tilde{q}_{I-1\text{i}} & 0 \\ \bar{\beta}_{I-1}^1 \tilde{q}_{I-1}^{\text{i}} & 0 & 0 & \bar{\beta}_I^1 q_I^1 \\ \bar{\beta}_{I-1}^2 \tilde{q}_{I-1}^{\text{i}} & 0 & 0 & \bar{\beta}_I^2 q_I^1 \\ 0 & \beta_I^1 \tilde{q}_{I1} & \beta_I^2 \tilde{q}_{I1} & 0 \end{pmatrix}, \quad (4.17)$$

we can follow the construction exactly as above with a superconnection and hyperloop that could be denoted as $\mathcal{L}_{\beta, \bar{\beta}}$ and $W_{\beta, \bar{\beta}}$ (with the parameters β and $\bar{\beta}$ being complex, but, again, not complex conjugates of each other).

Applying the analysis of [21], these two hyperloops can be visualized by quiver diagrams, which may include some or all of the nodes and edges of the original quiver defining the gauge theory. Let us pause for a moment to explain the machinery to associate a quiver diagram to a hyperloop. First, we include a node for each vector multiplet the loop couples to, solid arrows for the chiral fields and dashed arrows for the anti-chirals. Some of the nodes are denoted by squiggly circles and some by unsquiggly ones. This represents that the connection of the gauge field in the squiggly nodes has an extra shift by $1/2$,⁴ and supersymmetry requires alternating squiggly and unsquiggly nodes. Each node is decorated by

²Note the $+i$ factor in the path-ordered exponential, in contrast with $-i$ from (2.4). The difference is due to the conventions we follow in this chapter. See for instance the sign difference in the covariant derivatives (3.2) and (4.1). This sign difference is also manifest in the supercovariant derivative definition, which here becomes $\mathfrak{D}_\varphi(\cdot) \equiv \partial_\varphi(\cdot) - i[\mathcal{L}, (\cdot)]$.

³To avoid cluttering the notation too much, we denote the bosonic connections and the matrices $\mathcal{G}$ of all these examples with the same symbols: $\mathcal{L}_{\text{bos}}$ and $\mathcal{G}$. We always refer to explicit equations, so this should hopefully not lead to confusion.

⁴Here and throughout we set the radius of the sphere to $R = 1$. For a generic radius the shifts would be $\pm \frac{1}{2R}$, and similarly for the shifts in the next section.

integers p_I , indicating the multiplicity of the gauge field in the hyperloop. We indicate the chiral fields coupling to the hyperloop (according to the decomposition in Figure 11) by solid arrows and the anti-chirals by dashed arrows. Arrows between the I and $I + 1$ nodes are decorated by $p_I \times p_{I+1}$ complex parameters α_I, β_I for chirals and $\bar{\alpha}_I, \bar{\beta}$ for anti-chirals, though they are not complex conjugates. Applied to $W_{\alpha, \bar{\alpha}}$ and $W_{\beta, \bar{\beta}}$, these rules give the quiver diagrams drawn in Figure 12 and we see that the supertrace in the definition of the operators treats the unsquiggly nodes (I) as even and the squiggly ones ($I \pm 1$) as odd.

Figure 12: Quiver diagrams representing 1/4 BPS Wilson loops (a) $W_{\alpha, \bar{\alpha}}$ and (b) $W_{\beta, \bar{\beta}}$.

Other hyperloops are also allowed and at this point the power of these quiver diagrams starts manifesting. Say that we want to use a $\mathcal{G}$ that couples to all scalars, instead of only those with indices $1, \dot{1}$ or $2, \dot{2}$ as above. In this case, what we would be seeking is a diagram containing edges from Figure 12a and Figure 12b at the same time. However, by looking at these diagrams, we see that we need to be careful due to their different “squigglyness”. This means that we need to combine scalars to match the constant shifts of a single bosonic connection. To be more explicit, suppose that we want to use $\mathcal{L}_{\text{bos}}$ in (4.13). In this case, we can use $\mathcal{G}$ in (4.14) but for the one in (4.17) we would have to include some extra phases

$$\mathcal{G} \rightarrow \begin{pmatrix} 0 & e^{i\varphi} \beta_{I-1}^1 \tilde{q}_{I-1 \dot{1}} & e^{i\varphi} \beta_{I-1}^2 \tilde{q}_{I-1 \dot{1}} & 0 \\ e^{-i\varphi} \bar{\beta}_{I-1}^1 \tilde{q}_{I-1}^{\dot{1}} & 0 & 0 & e^{-i\varphi} \bar{\beta}_I^1 q_I^1 \\ e^{-i\varphi} \bar{\beta}_{I-1}^2 \tilde{q}_{I-1}^{\dot{1}} & 0 & 0 & e^{-i\varphi} \bar{\beta}_I^2 q_I^1 \\ 0 & e^{i\varphi} \beta_I^1 \bar{q}_{I1} & e^{i\varphi} \beta_I^2 \bar{q}_{I1} & 0 \end{pmatrix}, \quad (4.18)$$

in order to preserve supersymmetry⁵. This would lead to a new $\mathcal{G}$ given by

$$\begin{pmatrix} 0 & \bar{\alpha}_{I-1}^1 \tilde{q}_{I-1\dot{2}} + e^{i\varphi} \beta_{I-1}^1 \tilde{q}_{I-1\dot{1}} & \bar{\alpha}_{I-1}^2 \tilde{q}_{I-1\dot{2}} + e^{i\varphi} \beta_{I-1}^2 \tilde{q}_{I-1\dot{1}} & 0 \\ \alpha_{I-1}^1 \tilde{q}_{I-1}^{\dot{2}} + e^{-i\varphi} \bar{\beta}_{I-1}^1 \tilde{q}_{I-1}^{\dot{1}} & 0 & 0 & \alpha_I^1 q_I^2 + e^{-i\varphi} \bar{\beta}_I^1 q_I^1 \\ \alpha_{I-1}^2 \tilde{q}_{I-1}^{\dot{2}} + e^{-i\varphi} \bar{\beta}_{I-1}^2 \tilde{q}_{I-1}^{\dot{1}} & 0 & 0 & \alpha_I^2 q_I^2 + e^{-i\varphi} \bar{\beta}_I^2 q_I^1 \\ 0 & \bar{\alpha}_I^1 \bar{q}_{I2} + e^{i\varphi} \beta_I^1 \bar{q}_{I1} & \bar{\alpha}_I^2 \bar{q}_{I2} + e^{i\varphi} \beta_I^2 \bar{q}_{I1} & 0 \end{pmatrix}. \quad (4.19)$$

The rest of the construction follows as before where the superconnection and the hyperloop would now carry indices $\alpha, \beta, \bar{\alpha}, \bar{\beta}$. However, the resulting operator will only be invariant under $\mathcal{Q}_+^L$ so it is 1/16 BPS. We can also use $\mathcal{Q}_-^L$ to build our hyperloop and in this case only this supercharge would be preserved. The resulting operator can be represented graphically as in Figure 13.

Figure 13: A quiver diagram for a 1/16 BPS hyperloop.

For any quiver diagram we can always exchange squiggled and unsquiggled nodes (*i.e.* grading) at the price of adding phases to the matter fields, as in (4.18). This leads to the same loop operator, but in a different gauge. Note that in the case of 1/16 BPS loops shown in Figure 13, the two gradings of the quiver are gauge equivalent. This is not the case for families of 1/4 BPS loops, where the two gradings also represent which of the chiral fields are included, as illustrated in Figure 12.

These diagrams represent 1/4 BPS hyperloops, preserving all four supercharges in (4.8) if all solid arrows point into squiggly nodes.⁶ If the decorated quiver contains solid arrows pointing both in and out of the squiggly circles, the loop preserves only one linear combination of the four supercharges in (4.8), either $\mathcal{Q}_+^L$ or $\mathcal{Q}_-^L$ according to how we choose to build the hyperloop, and is 1/16 BPS.

4.2.1 Moduli spaces

The hyperloops we construct are built upon a connection $\mathcal{L}_{\text{bos}}$ to which we add bifundamental couplings encoded in the matrix $\mathcal{G}$. The deformation is thus de-

⁵From the point of view of the chiral decomposition of $\mathcal{N} = 4$ to $\mathcal{N} = 2$ theories, the need to include the extra phases can be seen as matching the behavior of the chiral scalar q_1 and the anti-chiral scalar q_2 .

⁶In a different gauge, they all point out of squiggly nodes.

scribed by the parameters in this matrix, which are a set of complex numbers α , $\bar{\alpha}$, etc. This description is however redundant, so the moduli space of hyperloops is a quotient of the space of these parameters, as described below.

Wilson loops are intimately related to gauge invariance, and in particular are gauge invariant observables. The invariance under $\prod_I U(N_I)$ gauge transformations are built into the definition of the hyperloops, but they possess a larger (global) gauge symmetry. $\mathcal{L}$ can be thought of as a superconnection, which in the example above is in the superalgebra $GL(N_{I-1} + N_{I+1} | 2N_I)$. We do not expect local-gauge symmetry under this group, as it is not a symmetry of the theory, but constant gauge transformations do not require extra gauge fields. Of those, the transformations that preserve our formulation and are not the gauge symmetries of the quiver theory are the centralizer of $\mathcal{L}_{\text{bos}}$, which in our examples is

$$(\mathbb{C}^*)^2 \times GL(2, \mathbb{C}).$$

Concretely this symmetry acts by conjugation

$$\mathcal{L}_{\text{bos}} \rightarrow \begin{pmatrix} x^{-1} & 0 & 0 & 0 \\ 0 & & S^{-1} & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & y^{-1} \end{pmatrix} \mathcal{L}_{\text{bos}} \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & & S & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & y \end{pmatrix} = \mathcal{L}_{\text{bos}}, \quad S \in GL(2, \mathbb{C}).$$

The action on $\mathcal{G}$ and consequently on $\mathcal{L}_{\alpha, \bar{\alpha}}$ is generally non-trivial, mapping superconnections with different parameters to each other. Note though that matrices in the center of $GL(4, \mathbb{C})$, i.e. where $y = x$ and $S = \text{diag}(x, x)$ do commute with all matrices, so should be excluded, thus the gauge symmetry is really

$$S((\mathbb{C}^*)^2 \times GL(2, \mathbb{C})). \quad (4.20)$$

Remaining with our examples above, we describe the 1/4 BPS loops with eight complex parameters, see (4.14) and (4.17), and the 1/16 BPS loops with sixteen, see (4.19). As argued, this means that the moduli space of 1/4 BPS loops correspond to two copies of

$$\mathbb{C}^8 // S((\mathbb{C}^*)^2 \times GL(2, \mathbb{C})),$$

while for 1/16 BPS operators we have a single copy of

$$\mathbb{C}^{16} // S((\mathbb{C}^*)^2 \times GL(2, \mathbb{C})).$$

These are 3- and 11-(complex) dimensional conical spaces, respectively.

These spaces are the usual quiver varieties associated to the quiver representations in Figures 12 and 13, see [54–58]. We use the double slash notation mirroring the concept of geometric invariant theory, in order to point out that we need to be careful when identifying the singular orbits of the resulting manifolds. In particular, hyperloops with off-diagonal components that are exclusively upper or lower triangular are identical as quantum operators for all values of α and therefore should be identified.

The analysis can be carried out for more general quivers as follows. In the case of 1/4 BPS loops, each edge between nodes of multiplicities p_I and p_{I+1} has $2p_I p_{I+1}$ complex parameters. For 1/16 BPS operators, this becomes $4p_I p_{I+1}$. From these we need to remove the symmetries, which amount to a factor of $GL(p_I, \mathbb{C})$ for each node with multiplicity p_I , apart for the trivial action of the center, as explained above (4.20).

Therefore, for a linear quiver of length L we find the moduli space of 1/4 BPS loops to be two copies of

$$\mathbb{C}^{2p_1 p_2} \times \mathbb{C}^{2p_2 p_3} \times \dots \times \mathbb{C}^{2p_{L-1} p_L} // S(GL(p_1) \times GL(p_2) \times \dots \times GL(p_L)). \quad (4.21)$$

For the 1/16 BPS loops, we loose the second copy and we have the same as above where the 2s in the exponents become 4s.

For a circular quiver, the moduli space of 1/4 BPS loops is two copies of

$$\mathbb{C}^{2p_1 p_2} \times \mathbb{C}^{2p_2 p_3} \times \dots \times \mathbb{C}^{2p_{L-1} p_L} \times \mathbb{C}^{2p_L p_1} // S(GL(p_1) \times GL(p_2) \times \dots \times GL(p_L)). \quad (4.22)$$

For 1/16 BPS loops, we again loose the second copy and the manifold is the one above with 2s replaced by 4s.

All these moduli spaces are the quiver varieties associated to the quiver representations that the hyperloops furnish. Before moving on to the next section, where we study the $\theta \neq 0$ versions of these hyperloops, let us specialize to the moduli space of 1/6 BPS loops in ABJ(M) theory with $p_1 = p_2 = 1$. As it is a circular two-node quiver, we set $L = 2$ in (4.22), giving two copies

$$\mathbb{C}^2 \times \mathbb{C}^2 // S((\mathbb{C}^*) \times (\mathbb{C}^*)) = \mathbb{C}^4 // \mathbb{C}^*.$$

This quotient is the conifold, as already found in [20].

4.3 Hyperloops at $\theta \neq 0$

Having reviewed the formulation of the loops in terms of quiver representations and implemented it for theories with $\mathcal{N} = 4$ supersymmetry, we generalize it to loops based on the bosonic connections deformed as in (4.9). We provide here much more detail, as most of these loops are novel.

4.3.1 Construction

To construct hyperloops with $\theta \neq 0$ we apply the same logic of [20], involving a deformation built from the supercharges in (4.12). More specifically, we consider the combinations $\mathcal{Q}_\pm = \mathcal{Q}_1 \pm \mathcal{Q}_2$. Note that when $\theta = 0$ it follows that $\mathcal{Q}_\pm = \mathcal{Q}_\pm^L$. The whole procedure relies on being able to write $(\mathcal{Q}_\pm)^2 \mathcal{G} = \mathcal{D}_\varphi \mathcal{G}$, for some matrix of matter fields $\mathcal{G}$ and an appropriate covariant derivative $\mathcal{D}_\varphi$. Since $(\mathcal{Q}_1)^2 = (\mathcal{Q}_2)^2 = 0$, this is the same as acting with $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ on $\mathcal{G}$. In Appendix C we compute the explicit form of this double transformation acting on the untilded scalar fields q_I^a and on a specific spatially-dependent rotation of the tilded ones, defined by

$$\tilde{r}_{I\pm 1\dot{1}} \equiv \cos \frac{\theta}{2} \tilde{q}_{I\pm 1\dot{1}} + e^{-i\varphi} \sin \frac{\theta}{2} \tilde{q}_{I\pm 1\dot{2}}, \quad \tilde{r}_{I\pm 1\dot{2}} \equiv \cos \frac{\theta}{2} \tilde{q}_{I\pm 1\dot{2}} - e^{i\varphi} \sin \frac{\theta}{2} \tilde{q}_{I\pm 1\dot{1}}. \quad (4.23)$$

These combinations are nice because the connection in (4.9) can be written compactly in their terms as

$$\mathcal{A}_I^\theta = A_{\varphi, I} - \frac{i}{k} (\mu_I^1{}_{\dot{1}} - \mu_I^2{}_{\dot{2}} + \bar{\tilde{r}}_{I-1}^{\dot{1}} \tilde{r}_{I-1\dot{1}} - \bar{\tilde{r}}_{I-1}^{\dot{2}} \tilde{r}_{I-1\dot{2}}). \quad (4.24)$$

Moreover, one can show that the double transformation acting on q_I^a and $\tilde{r}_{I\pm 1\dot{a}}$ can be recast as a covariant derivative, see (C.6) and (C.7). These fields are then the natural ingredients to write down $\mathcal{G}$.

The connections appearing in the double deformations are as in (4.24), with extra shifts of $\pm \frac{1}{2}$ and $\pm \frac{1}{2} \cos \theta$ that can be viewed as the coupling to a background field on the sphere. This is implemented by shifting the original connections, as in the previous section. Let us note that the effect of these shifts is to introduce phases like $e^{\pm i\pi \cos \theta}$ in the definition of the Wilson loop, which are compensated for in the definition of the trace. Recall that in the original formulation of the $1/2$ BPS Wilson loop of [24] a trace was required to make them gauge invariant, but in the gauge introduced in [20] and in Section 4.2, this is replaced with a more

natural supertrace. For gauge invariance of hyperloops at $\theta \neq 0$, the definition of supertrace should include ± 1 and $e^{\pm i\pi \cos \theta}$ gradings. The generic Wilson loop with both α and β couplings is then defined as

$$W_{\alpha, \bar{\alpha}, \beta, \bar{\beta}}^\theta = \text{sTr}_\theta \mathcal{P} \exp \left[i \oint \mathcal{L}_{\alpha, \bar{\alpha}, \beta, \bar{\beta}}^\theta |\dot{x}| ds \right],$$

where sTr_θ includes the generalized gradings mentioned above to compensate for the effect of the shifts.

To be concrete, we illustrate this explicitly in an example of a hyperloop involving the three nodes $I-1$, I and $I+1$, as before. For $p_{I-1} = p_I = p_{I+1} = 1$, we now have

$$\mathcal{L}_0^\theta = \begin{pmatrix} \mathcal{A}_{I-1}^\theta + \frac{1}{2} & 0 & 0 \\ 0 & \mathcal{A}_I^\theta & 0 \\ 0 & 0 & \mathcal{A}_{I+1}^\theta + \frac{1}{2} \cos \theta \end{pmatrix}. \quad (4.25)$$

The explicit expression for $\mathcal{A}_{I\pm 1}^\theta$ is written in (C.5). We associate to this choice of connection the quiver diagram in Figure 12a, though there are now some differences: the loop preserves only two supercharges, so it is 1/8 BPS; the right node has a shift of $\frac{1}{2} \cos \theta$ instead of $\frac{1}{2}$ and the left arrows are couplings to the rotated fields $\tilde{r}_{I-1\dot{2}}$ and their conjugates, rather than to the $\tilde{q}$'s. The corresponding hyperloop is

$$\begin{aligned} W_0^\theta &= \text{sTr}_\theta \mathcal{P} \exp \left[i \oint \mathcal{L}_0^\theta |\dot{x}| ds \right] \equiv -\text{Tr} \mathcal{P} \exp \left[i \oint \left(\mathcal{A}_{I-1}^\theta + \frac{1}{2} \right) |\dot{x}| ds \right] \\ &\quad + \text{Tr} \mathcal{P} \exp \left[i \oint \mathcal{A}_I^\theta |\dot{x}| ds \right] \\ &\quad + e^{-i\pi \cos \theta} \text{Tr} \mathcal{P} \exp \left[i \oint \left(\mathcal{A}_{I+1}^\theta + \frac{1}{2} \cos \theta \right) |\dot{x}| ds \right]. \end{aligned}$$

To proceed with the deformation, we define as before the matrix

$$\mathcal{G} = \begin{pmatrix} 0 & \bar{\alpha}_{I-1} \tilde{r}_{I-1\dot{2}} & 0 \\ \alpha_{I-1} \tilde{\bar{r}}_{I-1}^{\dot{2}} & 0 & \alpha_I q_I^2 \\ 0 & \bar{\alpha}_I \bar{q}_{I2} & 0 \end{pmatrix}. \quad (4.26)$$

Crucially, this matrix is such that $i(Q_+)^2 \mathcal{G} = \partial_\varphi \mathcal{G} - i[\mathcal{L}_{\text{bos}}^\theta, \mathcal{G}]$. The expression in (4.26) can then be used to construct a deformed superconnection

$$\mathcal{L}_{\alpha, \bar{\alpha}}^\theta = \mathcal{L}_{\text{bos}}^\theta + iQ_+ \mathcal{G} + \mathcal{G}^2. \quad (4.27)$$

Under the action of $\mathcal{Q}_+$

$$\mathcal{Q}_+ \mathcal{L}_{\alpha, \bar{\alpha}}^\theta = i(\mathcal{Q}_+)^2 \mathcal{G} + \{\mathcal{Q}_+ \mathcal{G}, \mathcal{G}\} = \partial_\varphi \mathcal{G} - i[\mathcal{L}_{\text{bos}}^\theta, \mathcal{G}] + \{\mathcal{Q}_+ \mathcal{G}, \mathcal{G}\}. \quad (4.28)$$

This is easily seen to correspond to the supercovariant derivative

$$\mathcal{D}_\varphi \mathcal{G} \equiv \partial_\varphi \mathcal{G} - i\{\mathcal{L}_{\alpha, \bar{\alpha}}^\theta, \mathcal{G}\},$$

so that the hyperloop with superconnection $\mathcal{L}_{\alpha, \bar{\alpha}}^\theta$ transforms as a total derivative under $\mathcal{Q}_+$.

To analyse the behavior under the action of $\mathcal{Q}_-$ it is useful to split $\mathcal{G} = G + \bar{G}$ with

$$G = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_{I-1} \tilde{r}_{I-1}^2 & 0 & \alpha_I q_I^2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \bar{G} = \begin{pmatrix} 0 & \bar{\alpha}_{I-1} \tilde{r}_{I-1}^2 & 0 \\ 0 & 0 & 0 \\ 0 & \bar{\alpha}_I \bar{q}_{I2} & 0 \end{pmatrix}, \quad (4.29)$$

which satisfy $\mathcal{Q}_- G = \mathcal{Q}_+ G$ and $\mathcal{Q}_- \bar{G} = -\mathcal{Q}_+ \bar{G}$. One has then

$$\begin{aligned} \mathcal{Q}_- \mathcal{L}_{\alpha, \bar{\alpha}}^\theta &= i(\mathcal{Q}_+)^2 (-G + \bar{G}) + \{\mathcal{Q}_+ (G - \bar{G}), \mathcal{G}\} \\ &= \partial_\varphi (-G + \bar{G}) - i[\mathcal{L}_0^\theta, -G + \bar{G}] + \{\mathcal{Q}_+ (G - \bar{G}), \mathcal{G}\} \\ &= \mathcal{D}_\varphi (-G + \bar{G}) + \{\mathcal{Q}_+ (G - \bar{G}), \mathcal{G}\} - \{\mathcal{Q}_+ \mathcal{G}, -G + \bar{G}\} + i[\mathcal{G}^2, -G + \bar{G}], \end{aligned} \quad (4.30)$$

where we attempted to write the last line as a covariant derivative with a $\mathcal{L}_{\alpha, \bar{\alpha}}^\theta$ superconnection. For the loop to be invariant under $\mathcal{Q}_-$, the three extra (anti)commutators should vanish and while there are some cancellations among them, the remaining terms are eliminated when G and $\bar{G}$ are nilpotent of degree 2 as in (4.29). This implies that for generic α and $\bar{\alpha}$ these Wilson loops are 1/8 BPS, preserving exactly the same supercharges as $\mathcal{L}_{\text{bos}}^\theta$.

Next we turn to the hyperloops with the quiver diagram in Figure 12b, but again we take multiplicity equal to one in the central node. In this case⁷

$$\mathcal{L}_0^\theta = \begin{pmatrix} \mathcal{A}_{I-1}^\theta & 0 & 0 \\ 0 & \mathcal{A}_I^\theta + \frac{1}{2} & 0 \\ 0 & 0 & \mathcal{A}_{I+1}^\theta + \frac{1-\cos\theta}{2} \end{pmatrix}. \quad (4.31)$$

Note that now the connection in the bottom right corner is shifted by $(1 - \cos \theta)/2$,

⁷Again, we always use the same symbols $\mathcal{L}_0^\theta$ and $\mathcal{G}$ and refer to explicit equations to avoid confusion.

though the corresponding node in the diagram is not squiggly.⁸ This does not exactly match the shifts, which are no longer all equal to $1/2$. However, we retain the same notation in the figure to indicate the allowed couplings to matter fields that preserve the supercharges. These are now organized in the matrix

$$\mathcal{G} = \begin{pmatrix} 0 & \beta_{I-1} \tilde{r}_{I-1\dot{1}} & 0 \\ \bar{\beta}_{I-1} \tilde{r}_{I-1}^{\dot{1}} & 0 & \bar{\beta}_I q_I^1 \\ 0 & \beta_I \bar{q}_{I1} & 0 \end{pmatrix}, \quad (4.32)$$

and the construction proceeds exactly as before to produce another family of $1/8$ BPS loops.

To construct $1/16$ BPS loops we again gauge transform the connection in (4.31) to bring it to the form in (4.25). This results in the extra phases in the corresponding $\mathcal{G}$,

$$\mathcal{G} \rightarrow \begin{pmatrix} 0 & \beta_{I-1} e^{i\varphi} \tilde{r}_{I-1\dot{1}} & 0 \\ \bar{\beta}_{I-1} e^{-i\varphi} \tilde{r}_{I-1}^{\dot{1}} & 0 & \bar{\beta}_I e^{-i\varphi \cos \theta} q_I^1 \\ 0 & \beta_I e^{i\varphi \cos \theta} \bar{q}_{I1} & 0 \end{pmatrix}. \quad (4.33)$$

We can now add this expression to the $\mathcal{G}$ in (4.26) to get

$$\mathcal{G} = \begin{pmatrix} 0 & \bar{\alpha}_{I-1} \tilde{r}_{I-1\dot{2}} + \beta_{I-1} e^{i\varphi} \tilde{r}_{I-1\dot{1}} & 0 \\ \alpha_{I-1} \tilde{r}_{I-1}^{\dot{2}} + \bar{\beta}_{I-1} e^{-i\varphi} \tilde{r}_{I-1}^{\dot{1}} & 0 & \alpha_I q_I^2 + \bar{\beta}_I e^{-i\varphi \cos \theta} q_I^1 \\ 0 & \bar{\alpha}_I \bar{q}_{I2} + \beta_I e^{i\varphi \cos \theta} \bar{q}_{I1} & 0 \end{pmatrix}. \quad (4.34)$$

This is similar to the sum of the two matrices in (4.19), but there is an obstruction to using this $\mathcal{G}$ to construct a deformation, as the combination $\alpha_I q_I^2 + \bar{\beta}_I e^{-i\varphi \cos \theta} q_I^1$ is not periodic. This is worse than the non-periodicity in (4.33), as it cannot be fixed by a gauge transformation. The only way to overcome this is to set either $\alpha_I = \bar{\alpha}_I = 0$ or $\beta_I = \bar{\beta}_I = 0$, giving two branches of $1/16$ BPS Wilson loops.

For $\beta_I = \bar{\beta}_I = 0$ we can represent it by the quiver in Figure 14.

Figure 14: A quiver diagram for a $1/16$ BPS Wilson loop.

⁸The particular shift of $(1 - \cos \theta)/2$ appears because supersymmetry requires a relative shift between nodes I and $I + 1$ of $(-\cos \theta)/2$.

In the case of $\alpha_I = \bar{\alpha}_I = 0$, it is a bit nicer to employ the gauge in (4.31) to avoid the awkward phase multiplying q_I^1 . The diagram is in Figure 15.

Figure 15: The other class of 1/16 BPS loops.

4.3.2 A closer look at the hyperloops

Let us examine in detail the hyperloops with superconnection $\mathcal{L}_{\alpha, \bar{\alpha}}$ in (4.27). To do so we need the explicit expressions of $\mathcal{Q}_+$ acting on the scalar fields q_I^a and $\tilde{r}_{I-1 \dot{a}}$. For the latter, we get back components of $\tilde{\psi}_{I-1}^a$ as (the $\pm$ on the $\tilde{\psi}$ represent spinor indices)

$$\begin{aligned} \mathcal{Q}_+ \tilde{r}_{I-1 \dot{1}} &= -\tilde{\psi}_{I-1, +}^1, & \mathcal{Q}_+ \tilde{r}_{I-1 \dot{2}} &= \tilde{\psi}_{I-1, -}^2, \\ \mathcal{Q}_+ \tilde{r}_{I-1}^{\dot{1}} &= \tilde{\bar{\psi}}_{I-1, 1, -}, & \mathcal{Q}_+ \tilde{r}_{I-1}^{\dot{2}} &= -\tilde{\bar{\psi}}_{I-1, 2, +}. \end{aligned}$$

For the action on the q_I^a fields it is convenient to define rotated fermions $\rho_{I\dot{a}}$ via

$$\rho_{I\dot{1}} \equiv \cos \frac{\theta}{2} \psi_{I\dot{1}} + e^{-i\varphi} \sin \frac{\theta}{2} \psi_{I\dot{2}}, \quad \rho_{I\dot{2}} \equiv \cos \frac{\theta}{2} \psi_{I\dot{2}} - e^{i\varphi} \sin \frac{\theta}{2} \psi_{I\dot{1}}, \quad (4.35)$$

such that

$$\mathcal{Q}_+ q_I^1 = -\rho_{I\dot{1}, -}, \quad \mathcal{Q}_+ q_I^2 = \rho_{I\dot{2}, +}, \quad \mathcal{Q}_+ \bar{q}_{I1} = \bar{\rho}_{I, +}^{\dot{1}}, \quad \mathcal{Q}_+ \bar{q}_{I2} = -\bar{\rho}_{I, -}^{\dot{2}}.$$

Hence we find

$$\mathcal{L}_{\alpha, \bar{\alpha}}^\theta = \mathcal{L}_{\text{bos}}^\theta + \begin{pmatrix} \bar{\alpha}_{I-1} \alpha_{I-1} \tilde{r}_{I-1 \dot{2}} \tilde{r}_{I-1}^{\dot{2}} & i \bar{\alpha}_{I-1} \tilde{\psi}_{I-1, -}^2 & \bar{\alpha}_{I-1} \alpha_I \tilde{r}_{I-1 \dot{2}} q_I^2 \\ -i \alpha_{I-1} \tilde{\bar{\psi}}_{I-1, 2, +} & \alpha_{I-1} \bar{\alpha}_{I-1} \tilde{r}_{I-1}^{\dot{2}} \tilde{r}_{I-1 \dot{2}} + \alpha_I \bar{\alpha}_I q_I^2 \bar{q}_{I2} & i \alpha_I \rho_{I\dot{2}, +} \\ \bar{\alpha}_I \alpha_{I-1} \bar{q}_{I2} \tilde{r}_{I-1}^{\dot{2}} & -i \bar{\alpha}_I \bar{\rho}_{I, -}^{\dot{2}} & \bar{\alpha}_I \alpha_I \bar{q}_{I2} q_I^2 \end{pmatrix}. \quad (4.36)$$

An interesting question is whether there are any points of enhanced supersymmetry along our moduli space. A simple guide is to look for points of enhanced bosonic symmetry that does not commute with the preserved supercharges, so either $SU(2)_L$ or $SU(2)_R$.

Looking to impose the $SU(2)_R$ symmetry, recall that it acts on the dotted indices of $\tilde{r}$ and $\tilde{\rho}$. Examining (4.36), we immediately see that we should impose $\alpha_I = \bar{\alpha}_I = 0$ to eliminate the off-diagonal entries where dotted indices appear. This

has the effect of eliminating the entire third row and column of the supermatrix. The corresponding diagonal entry in $\mathcal{L}_0^\theta$, which is $\mathcal{A}_{I+1}^\theta$, is also clearly not $SU(2)_R$ symmetric, as can be seen from (C.5). So we should remove it from $\mathcal{L}_0^\theta$ as well and focus on a loop coupling to only two nodes. Of course, it is possible to generalize this to any even number of nodes, but with only couplings between pairs.

Focusing then on the upper-left 2×2 block of (4.36), there are still explicit dotted indices in the diagonal parts, but those appear also in $\mathcal{L}_0^\theta$. Let us write them together, say for the I -th node:

$$\mathcal{A}_I^\theta + \alpha_{I-1} \bar{\alpha}_{I-1} \tilde{r}_{I-1}^{\dot{2}} \tilde{r}_{I-1\dot{2}} + \alpha_I \bar{\alpha}_I q_I^2 \bar{q}_{I2} = A_{\varphi,I} - \frac{i}{k} \tilde{r}_{I-1}^{\dot{a}} \tilde{M}_{\dot{a}}^{\dot{b}} \tilde{r}_{I-1\dot{b}} - \frac{i}{k} q_I^a M_a^{\dot{b}} \bar{q}_{I\dot{b}}, \quad (4.37)$$

with

$$\tilde{M}_{\dot{a}}^{\dot{b}} = \begin{pmatrix} 1 & 0 \\ 0 & ik\alpha_{I-1}\bar{\alpha}_{I-1} - 1 \end{pmatrix} \quad \text{and} \quad M_a^{\dot{b}} = \begin{pmatrix} 1 & 0 \\ 0 & ik\bar{\alpha}_I\alpha_I - 1 \end{pmatrix}. \quad (4.38)$$

We chose to write the matrix $\tilde{M}$ in the basis of the twisted $\tilde{r}$, but regardless of the basis, in order to preserve $SU(2)_R$, it has to be proportional to the identity, so $ik\alpha_{I-1}\bar{\alpha}_{I-1} = 2$.

The exact same structure follows for the top left entry, which is now also symmetric. Recalling the $\mathbb{C}^*$ symmetry, we can further fix $\alpha_{I-1} = \bar{\alpha}_{I-1} = \sqrt{2i/k}$ and the full form of the connection becomes

$$\mathcal{L}^\theta = \begin{pmatrix} A_{\varphi,I-1} + \tilde{q}_{I-1\dot{a}} \tilde{q}_{I-1}^{\dot{a}} + \bar{q}_{I1} q^1 - \bar{q}_{I2} q^2 + \frac{1}{2} & -\frac{1-i}{\sqrt{k}} \tilde{\psi}_{I-1,-}^2 \\ \frac{1-i}{\sqrt{k}} \tilde{\psi}_{I-1,2,+} & A_{I,\varphi} + \tilde{q}_{I-1}^{\dot{a}} \tilde{q}_{I-1\dot{a}} + q^1 \bar{q}_{I1} - q^2 \bar{q}_{I2} \end{pmatrix}.$$

Note that the θ -dependence completely dropped out of this expression, so it is within the class of 1/4 BPS operators presented in Section 4.2, but with extra $SU(2)_R$ symmetry, so this is in fact the 1/2 BPS loop of [44], now adapted to the 3-sphere.

Imposing $SU(2)_L$ symmetry is similar, but the results are different. Looking for undotted indices in (4.36), we see that now we should take $\alpha_{I-1} = \bar{\alpha}_{I-1} = 0$. This eliminates the top line and left column from the matrix and, as before, we should remove the top left entry in $\mathcal{L}_{\text{bos}}^\theta$ as well.

We should again also examine the diagonal blocks, as we again have the expression in (4.37). Now $\tilde{M} = \text{diag}(1, -1)$ and we want M to be proportional to the identity, so we set $ik\bar{\alpha}_I\alpha_I = 2$, or $\alpha_I = \bar{\alpha}_I = \sqrt{2i/k}$. The resulting 2×2

connection is

$$\mathcal{L}^\theta = \begin{pmatrix} A_{I,\varphi} + \tilde{r}_{I-1}^{\dot{1}} \tilde{r}_{I-1\dot{1}} - \tilde{r}_{I-1}^{\dot{2}} \tilde{r}_{I-1\dot{2}} + q^a \bar{q}_{Ia} & -\frac{1-i}{\sqrt{k}} \rho_{I\dot{2},+} \\ \frac{1-i}{\sqrt{k}} \bar{\rho}_{I,-}^{\dot{2}} & A_{I+1,\varphi} + \tilde{r}_{I-1\dot{1}} \tilde{r}_{I-1}^{\dot{1}} - \tilde{r}_{I-1\dot{2}} \tilde{r}_{I-1}^{\dot{2}} + \bar{q}_{Ia} q^a + \frac{1}{2} \end{pmatrix}. \quad (4.39)$$

The fermions in the off-diagonal entries are defined in terms of the original fields in (4.35) and the scalar bilinears are

$$\tilde{r}_{I-1}^{\dot{1}} \tilde{r}_{I-1\dot{1}} - \tilde{r}_{I-1}^{\dot{2}} \tilde{r}_{I-1\dot{2}} = \begin{pmatrix} \tilde{q}_{I-1}^{\dot{1}} & \tilde{q}_{I-1}^{\dot{2}} \end{pmatrix} \begin{pmatrix} \cos \theta & e^{-i\varphi} \sin \theta \\ e^{i\varphi} \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \tilde{q}_{I-1\dot{1}} \\ \tilde{q}_{I-1\dot{2}} \end{pmatrix},$$

We recognize the same structure as the “fermionic latitude” of [25], so this is its generalization to arbitrary $\mathcal{N} = 4$ Chern-Simons-matter theories. For $\theta = 0$ it becomes another class of 1/2 BPS loop of [44].

In addition to the supercharges $\mathcal{Q}_\pm$, this loop is also invariant under the ones obtained by swapping undotted indices,

$$\cos \frac{\theta}{2} Q_l^{\dot{1}2} + \sin \frac{\theta}{2} Q_{\tilde{r}}^{\dot{2}2}, \quad \cos \frac{\theta}{2} Q_l^{\dot{2}1} - \sin \frac{\theta}{2} Q_{\tilde{r}}^{\dot{1}1}.$$

The examples thus far were for the grading in Figure 12a. The story for the second grading in Figure 12b is analogous, with the β parameters coupling to the remaining fields. In particular, the matrices $\tilde{M}$ and M in (4.38) would have the upper left corner shifted by bilinears of β . To preserve $SU(2)_R$ we now set $\beta_I = \bar{\beta}_I = 0$, and again have to focus on a superconnection involving only the $I-1$ and I nodes with M as before and $\tilde{M} = \text{diag}(-1, -1)$. This is the second $SU(2)_R$ invariant 1/2 BPS Wilson loop described in [44]. Likewise, the $SU(2)_L$ invariant hyperloop on this branch is another version of the “fermionic latitude” loops described in [25] (in the notation of that paper it has $l = -1$).

The analysis here relies on a bosonic symmetry to indicate enhanced supersymmetry. In principle there could be further points with accidental or more subtle supersymmetry enhancement. We leave the study of that to the future.

4.3.3 Further examples

So far we focused on hyperloops involving three nodes of a long quiver. The simplest generalization arising already in that case is taking multiple copies of the connections, i.e. $p_J > 1$ in Figure 12, forming larger $\mathcal{L}_{\text{bos}}^\theta$ connections and involving more α and/or β parameters in the appropriate matrices $\mathcal{G}$. This mirrors

the examples in Section 4.2, where the central node had multiplicity 2.

In the case of a linear quiver one may worry about the final nodes which couple either to hypermultiplets or to twisted hypermultiplets, although both kinds of fields appear in (4.9) and (C.5). This turns out not to be a problem, and the construction proceeds as before with the missing moment maps removed. For example, suppose that there is no hypermultiplet to the left of the $I - 1$ node in the quiver of the underlying theory. When constructing the hyperloop as in (4.25), this would mean that $\mathcal{A}_{I-1}^\theta$ would lose the μ contribution and become

$$A_{\varphi, I-1} = \frac{i}{k} (\tilde{r}_{I-1,1} \bar{\tilde{r}}_{I-1}^1 - \tilde{r}_{I-1,2} \bar{\tilde{r}}_{I-1}^2),$$

but the construction of the hyperloop would follow exactly as before. Likewise if the underlying theory had no twisted hypermultiplet to the right of the node $I + 1$, then $\mathcal{A}_{I+1}^\theta$ would become

$$A_{\varphi, I+1} = \frac{i}{k} (\mu_{I+1,1}^1 - \mu_{I+1,2}^2),$$

and again the construction would follow as before.

Of course, if we have only a 2-node quiver and no twisted hypermultiplets, there would not be a way to θ -deform their couplings. In that case there would only be the analog deformation of μ , which is completely parallel to the constructions based on $\tilde{\mu}$.

Next we examine what happens for longer quivers and for circular quivers like ABJ(M) theory. Consider first the hyperloop coupling to 4 nodes as in Figure 16.

Figure 16: A quiver diagram for a 4-node 1/8 BPS hyperloop.

The story proceeds exactly as before, except that we should remember that the shifts are relative to neighboring nodes, so in this case (for all $p_J = 1$) the starting point is

$$\mathcal{L}_{\text{bos}}^\theta = \begin{pmatrix} \mathcal{A}_{I-1} + \frac{1}{2} & 0 & 0 & 0 \\ 0 & \mathcal{A}_I & 0 & 0 \\ 0 & 0 & \mathcal{A}_{I+1} + \frac{1}{2} \cos \theta & 0 \\ 0 & 0 & 0 & \mathcal{A}_{I+2} + \frac{\cos \theta - 1}{2} \end{pmatrix}.$$

It should be clear how to deform this loop by adding couplings to the fermions and also construct the loop corresponding to the quiver with the second possible grading.

In the case of circular quivers, turning on $\theta \neq 0$ poses a challenge. In the simplest case of ABJ(M) we have two nodes, so $L = 2$. Then in our figures and expressions for $\mathcal{L}_{\text{bos}}$, etc. we can take $I = 2$ and identify the nodes $I - 1 = 1$ and $I + 1 = 3$. If we consider a hyperloop coupling only to one edge in the quiver—so the quiver for the hyperloop is a linear 2-node quiver—there is no problem. If we want to couple to matter from both nodes, we face the fact that while we identify nodes 1 and 3, the shift of $\mathcal{A}_1$ is $\frac{1}{2}$ and that of $\mathcal{A}_3$ it should be $\frac{1}{2} \cos \theta$, see (4.25).

The solution to this problem was already anticipated in [51] (in other contexts where the shifts were not $1/2$) and it amounts to taking a cover of the original gauge theory quiver. So we can couple the Wilson loop to both edges of the original quiver as long as we consider a 3-node hyperloop with $\mathcal{L}_{\text{bos}}^\theta$ given by

$$\mathcal{L}_0^\theta = \begin{pmatrix} \mathcal{A}_1 + \frac{1}{2} & 0 & 0 \\ 0 & \mathcal{A}_2 & 0 \\ 0 & 0 & \mathcal{A}_1 + \frac{1}{2} \cos \theta \end{pmatrix}. \quad (4.40)$$

Here $\mathcal{A}_1$ appears twice with different shifts. One can continue further with another copy of $\mathcal{A}_2$ with a shift of $(\cos \theta - 1)/2$, and so on.

For $\theta \neq 0$ the connection in (4.40) has $(\mathbb{C}^*)^3$ symmetry (with one copy acting trivially), so the moduli space of $1/4$ BPS loops according to (4.21) is

$$\mathbb{C}^2 \times \mathbb{C}^2 // \mathbb{C}^* \times \mathbb{C}^* \simeq \mathbb{C}^2.$$

The $\theta = 0$ case has enhanced symmetry $(\mathbb{C}^*)^3 \rightarrow GL(2, \mathbb{C}) \times \mathbb{C}^*$. The moduli space according to (4.22) is now four-dimensional

$$\mathbb{C}^6 // GL(2, \mathbb{C}).$$

In addition to the very special case of $\theta = 0$, for any rational $\cos \theta$, a hyperloop based on a long enough quiver will have also some enhanced symmetry.

4.4 A matrix model proposal

The construction of the hyperloops is based on a deformation of a bosonic loop and all the loops of fixed θ are cohomologically equivalent under the supercharge $\mathcal{Q}_+$ used to define them. This means that any localization computation for any loop

on the moduli space is immediately applicable to any one. The proof following [24, 44] requires expanding the exponentials and checking order-by-order that the difference between the different operators is $\mathcal{Q}_+$ -exact. We do not reproduce this computation here since it is essentially identical.

We propose now a matrix model that we hope captures the expectation value of our operators. The matrix model partition function can be motivated by considering the usual ingredients due to the vector multiplets (hyperbolic sines) and hypermultiplets (hyperbolic cosines) at each node [5] and the proposal in [59] on how to introduce the θ -deformation:

$$Z = \prod_I \frac{1}{N_I!} \int \prod_{i=1}^{N_I} \frac{d\lambda_{Ii}}{2\pi} e^{i \frac{k_I}{4\pi} \lambda_{Ii}^2} \prod_{i < j}^{N_I} 4 \sinh \frac{\nu(\lambda_{Ii} - \lambda_{Ij})}{2} \sinh \frac{\lambda_{Ii} - \lambda_{Ij}}{2\nu} \\ \times \left(\prod_{i=1}^{N_I} \prod_{j=1}^{N_{I+1}} 2 \cosh \frac{\nu^{(-1)^I} (\lambda_{Ii} - \lambda_{I+1,j})}{2} \right)^{-1}.$$

The explicit value of the parameter ν can be fixed by the comparison with a perturbative computation and turns out to be given, in our notation, by $\nu = \sqrt{\cos \theta}$ [59]. Note that twisted and untwisted hypermultiplets contribute differently, with ν either in the numerator or in the denominator of the argument of the hyperbolic cosines. The expectation value of the θ -deformed hyperloops is given by inserting in the partition function above $\sum_I \sum_{i=1}^{N_I} e^{\nu \lambda_{Ii}}$.

4.5 Comments

So far we reorganized the space of known Wilson loop operators in $\mathcal{N} = 4$ Chern-Simons-matter theories in three dimensions, which we now call hyperloops, and generalized it considerably to include loops preserving 1, 2, 4 and 8 supercharges. Our findings clarify and elaborate the intricate structure of the supersymmetric line operators and their moduli spaces. The strategy, adapted from [20, 51], is to start with a choice of diagonal (bosonic) superconnection, which is at the apex of a conical moduli space, and describes all other superconnections as deformations of it. The operators we have thus obtained are classified in terms of quiver diagrams encoding which gauge fields are involved and the couplings to the matter fields. This, together with a “latitude” parameter θ , completes the set of data necessary for the classification.

More concretely, we need to choose some or all of the vector fields and the number of times they are represented in the Wilson loop. The next step is to

choose a grading, which plays two roles: it indicates the constant shifts in some of the diagonal connections and it implies which half of the chiral fields we couple to, in order to get hyperloops with 4 preserved supercharges. The two possible gradings then also give the two branches of the moduli space. Finally, for each of the included chiral (antichiral) fields we have couplings α_I, β_I ($\bar{\alpha}_I, \bar{\beta}_I$) subject to a global gauge invariance, reducing the moduli space to a quotient of $\mathbb{C}^p$ for some p . These moduli spaces are known as quiver varieties and generically are conical, as first observed in [20] for the conifold in the case of the 1/6 BPS loops of the ABJ(M) theory.

The hyperloops with $\theta = 0$ were previously found in [51], as well as in [27, 44, 47–49], but our description is much more algorithmic and their moduli space was never studied in such detail. Loops with $\theta \neq 0$ were only studied in ABJ(M) theory and not in theories with $\mathcal{N} = 4$ supersymmetry. Furthermore, the constructions focused on the bosonic loops and on the analogues of the 1/4 BPS loops (which are 1/6 BPS in ABJ(M)) presented in Section 4.3.2. The continuous family of hyperloops interpolating between those two cases and all the other directions in the moduli space have not been previously described.

As mentioned in Section 4.1, the Gaiotto-Yin loop breaks $SU(2)_L \times SU(2)_R \rightarrow U(1)_L \times U(1)_R$. The parameter θ further breaks $U(1)_R$ by including extra couplings to the moment maps arising from the twisted hypermultiplets. All the examples that are presented in the preceding sections have analogues with the roles of hypermultiplets and twisted hypermultiplets reversed.

There are many possible directions that can be pursued from here. Among the most obvious ones is to attempt the complete exploration of the full moduli space of line operators in three-dimensional Chern-Simons-matter theories. In addition to the tried and tested approach of making ansätze and restricting them to be BPS, we can try to extend the point of view introduced in [20] and employed here, of constructing the loops as deformations of previously identified ones. One should first verify whether there are further line operators involving only single nodes. Then whether there are any further deformations of them with more complicated forms than considered here. Finally, one should examine other points along the moduli space to see whether there are other branches that may intersect those points, but not pass through the origin we employ here.

Still, these explorations cannot answer the question of what is the full space of BPS line operators, which would require new tools to address. Moreover, there are other types of line operators, known as vortex loops [36, 60, 61]. In some cases they

are known to be dual under mirror symmetry to Wilson loops [35], so it should be exciting to understand their moduli spaces as well.

In Section 4.3.2 we looked at some special examples of these loops which have enhanced supersymmetry. It is not clear that the ones identified there, which were all previously known, are the only points of enhanced supersymmetry on the moduli space.

Also in this spirit of discovering new hyperloops would be the construction of operators supported along generic curves on a $S^2 \subset S^3$, following the four-dimensional example of [62, 63, 33] or the ABJ(M) analog of [25].

Another question worth asking is what happens to these operators at the quantum level and whether the classical moduli spaces described here receive corrections. The analysis of [52] suggests that moduli spaces are natural for line operators in three dimensions, but it does not predict their dimensions, as we found here, nor that their classical structure is not subject to quantum corrections. The heroic 3-loop calculation of [64]⁹ suggests that the degeneracy among pairs of 1/2 BPS Wilson loops of [44] may sometimes get lifted. Such a perturbative analysis would also be useful to test the matrix model proposal we put forward in Section 4.4.

Yet another angle is to study these moduli spaces as defect conformal manifolds in the context of defect CFT. Explicit analysis of this type for line operators in four-dimensional $\mathcal{N} = 4$ theory include [66–68] and in three dimensions [69].

Wilson loops are also interesting in the holographic context, for they provide a rich dictionary between gauge theory and string theory objects. It would then be interesting to understand the holographic realization of the operators constructed here. Very little has been done in this direction since the original proposal for the holographic dual in [28–30]. Proposals for the holographic duals of 1/2 BPS loops in some $\mathcal{N} = 4$ theories were put forward in [48] and a first examination of a possible moduli space of 1/6 BPS loops in ABJ(M) theory was done in [70].

4.6 New hyperloops

To conclude this chapter, we present some preliminary results concerning more general hyperloops in $\mathcal{N} = 4$ CSM theories. This is part of an ongoing project [23] that started from the $\mathcal{N} = 4$ analogue of the loops presented in Section 3.3. So let us begin by presenting such operators and, in particular, by translating the supercharges in (3.76) to our setting. To do so we have to map not only R-

⁹See also [65] for a previous attempt limited to a two-loop computation.

symmetry indices from $SU(4)$ to $SU(2)_L \times SU(2)_R$ but also supercharges in $\mathbb{R}^3$ to S^3 . We map R-symmetry indices as

$$\{1, 2, 3, 4\} \mapsto \{\dot{1}, \dot{2}, 1, 2\},$$

and supercharges as

$$Q^+ - iS^+ \sim Q_{\bar{l}}, \quad Q^- - iS^- \sim Q_l, \quad Q^- + iS^- \sim Q_{\bar{r}}, \quad Q^+ + iS^+ \sim Q_r,$$

to find that the $\mathcal{N} = 4$ analogue should be invariant under 4 supercharges that are linear combinations of $Q_{\bar{l}}^{\dot{1}1}$ and $Q_r^{\dot{2}1}$, $Q_{\bar{r}}^{\dot{1}1}$ and $Q_l^{\dot{2}1}$, $Q_{\bar{l}}^{\dot{1}2}$ and $Q_r^{\dot{2}2}$, $Q_{\bar{r}}^{\dot{1}2}$ and $Q_l^{\dot{2}2}$.

The null entries of the scalar coupling matrix appearing in (3.75) translate to $\mathcal{N} = 4$ as the absence of scalars from one of the matter multiplets. Through the map of R-symmetry indices above, this should be the untwisted multiplet, such that the dressed bosonic connection appears as

$$\mathcal{A}_I = A_{I,\varphi} + \frac{i}{k}(\tilde{\mu}_I^{\dot{1}1} - \tilde{\mu}_I^{\dot{2}2}). \quad (4.41)$$

As for the off-diagonal entries we find that the two fermi fields appearing through η and $\bar{\eta}$ should then translate to fermi fields of $\mathcal{N} = 4$ carrying dotted indices, so the operator should couple to fermions in the hypermultiplet and not to ones in the twisted hypermultiplet. At the end we find a superconnection given by

$$\mathcal{L} = \begin{pmatrix} A_{\varphi,I} - \frac{i}{k}(\tilde{\mu}_I^{\dot{1}1} - \tilde{\mu}_I^{\dot{2}2}) & -\frac{i}{2}(\bar{\beta}_I \psi_{I\dot{1}-} - \alpha_I e^{i\varphi} \psi_{I\dot{2}+}) \\ \frac{i}{2}(\beta_I \bar{\psi}_{I+}^{\dot{1}} - \bar{\alpha}_I e^{-i\varphi} \bar{\psi}_{I-}^{\dot{2}}) & A_{\varphi,I+1} - \frac{i}{k}(\tilde{\mu}_{I+1}^{\dot{1}1} - \tilde{\mu}_{I+1}^{\dot{2}2}) - \frac{1}{2} \end{pmatrix}, \quad (4.42)$$

and, for constants $\{\bar{\beta}_I, \beta_I\}$ and $\{\bar{\alpha}_I, \alpha_I\}$ satisfying $\bar{\beta}_I \beta_I = -\bar{\alpha}_I \alpha_I = 2i/k$, the resulting loop is invariant under the action of the following supercharges

$$Q_{\bar{l}}^{\dot{1}1} - \frac{\alpha_I}{\beta_I} Q_r^{\dot{2}1}, \quad Q_{\bar{l}}^{\dot{1}2} - \frac{\alpha_I}{\beta_I} Q_r^{\dot{2}2}, \quad Q_{\bar{r}}^{\dot{1}1} - \frac{\alpha_I}{\beta_I} Q_l^{\dot{2}1}, \quad Q_{\bar{r}}^{\dot{1}2} - \frac{\alpha_I}{\beta_I} Q_l^{\dot{2}2}. \quad (4.43)$$

One novelty here is that the set of preserved supercharges carry this unusual α, β dependence. To try to understand the structure of this operator and its preserved supercharges, we attempted to employ the logic proposed in [20]. However, note that supercharges in (4.43), are not linear combinations of the ones preserved by the 1-node loop, see (4.8). So we should not expect to connect $\mathcal{L}$ above with $\mathcal{L}_{\text{bos}}$ through $\Delta\mathcal{L} = i\mathcal{Q}\mathcal{G} + \mathcal{G}^2$.

What we find is that this operators can actually be seen as a composition of

two hyperloops that are 1/2 BPS. These two are defined in terms of

$$\mathcal{L}_{\beta,\bar{\beta}} = \begin{pmatrix} A_{\varphi,I} - \frac{i}{k}(-\nu_I + \tilde{\mu}_I^{\dot{1}\dot{1}} - \tilde{\mu}_I^{\dot{2}\dot{2}}) & -i\bar{\beta}_I\psi_{I\dot{1}-} \\ i\beta_I\bar{\psi}_{I+}^{\dot{1}} & A_{\varphi,I+1} - \frac{i}{k}(-\nu_{I+1} + \tilde{\mu}_{I+1}^{\dot{1}\dot{1}} - \tilde{\mu}_{I+1}^{\dot{2}\dot{2}}) - \frac{1}{2} \end{pmatrix},$$

and

$$\mathcal{L}_{\alpha,\bar{\alpha}} = \begin{pmatrix} A_{\varphi,I} - \frac{i}{k}(\nu_I + \tilde{\mu}_I^{\dot{1}\dot{1}} - \tilde{\mu}_I^{\dot{2}\dot{2}}) - \frac{1}{2} & i\alpha_I\psi_{I\dot{2}+} \\ -i\bar{\alpha}_I\bar{\psi}_{I-}^{\dot{2}} & A_{\varphi,I+1} - \frac{i}{k}(\nu_{I+1} + \tilde{\mu}_{I+1}^{\dot{1}\dot{1}} - \tilde{\mu}_{I+1}^{\dot{2}\dot{2}}) \end{pmatrix}.$$

The corresponding operators, to which we refer below respectively as $W_{\beta,\bar{\beta}}$ and $W_{\alpha,\bar{\alpha}}$, are $SU(2)_L$ symmetric. They preserve all supersymmetry transformations preserved by $\mathcal{L}_{\text{bos}}$ plus the ones obtained by exchanging undotted indices, namely,

$$Q_l^{\dot{1}\dot{1}}, \quad Q_{\bar{r}}^{\dot{1}\dot{1}}, \quad Q_l^{\dot{2}\dot{2}}, \quad Q_{\bar{r}}^{\dot{2}\dot{2}} \quad \text{and} \quad Q_l^{\dot{1}\dot{2}}, \quad Q_{\bar{r}}^{\dot{1}\dot{2}}, \quad Q_l^{\dot{2}\dot{1}}, \quad Q_{\bar{r}}^{\dot{2}\dot{1}}. \quad (4.44)$$

In particular, they match what in [44] was named ψ_1 -loop and ψ_2 -loop. Employing the analysis of [20] and [51], they can be obtained from $\mathcal{L}_{\text{bos}}$ using Q_L^+ and

$$\mathcal{G}_{\beta,\bar{\beta}} = \begin{pmatrix} 0 & \bar{\beta}_I q_I^1 \\ \beta_I \bar{q}_{I1} & 0 \end{pmatrix} \quad \text{or} \quad \mathcal{G}_{\alpha,\bar{\alpha}} = \begin{pmatrix} 0 & \alpha_I q_I^2 \\ \bar{\alpha}_I \bar{q}_{I2} & 0 \end{pmatrix},$$

together with the respective constraints $\bar{\beta}_I\beta_I = 2i/k$ or $\bar{\alpha}_I\alpha_I = -2i/k$, responsible for rendering the loop $SU(2)_L$ symmetric. To them we can associate the quiver diagrams in Figure 17.

Figure 17: Quiver diagrams representing 1/2 BPS Wilson loops (a) $W_{\beta,\bar{\beta}}$ and (b) $W_{\alpha,\bar{\alpha}}$.

Now the superconnection in (4.42) can be seen as a composition of $\mathcal{L}_{\beta,\bar{\beta}}$ with

the following gauge transformed version of $\mathcal{L}_{\alpha, \bar{\alpha}}$,

$$\mathcal{L}'_{\alpha, \bar{\alpha}} = \begin{pmatrix} A_{\varphi, I} - \frac{i}{k}(\nu_I + \tilde{\mu}_I^{\dot{1}} - \tilde{\mu}_I^{\dot{2}}) & i\alpha_I e^{i\varphi} \psi_{I\dot{2}+} \\ -i\bar{\alpha}_I e^{-i\varphi} \bar{\psi}_{I-}^{\dot{2}} & A_{\varphi, I+1} - \frac{i}{k}(\nu_{I+1} + \tilde{\mu}_{I+1}^{\dot{1}} - \tilde{\mu}_{I+1}^{\dot{2}}) - \frac{1}{2} \end{pmatrix}.$$

Indeed, we precisely have that it corresponds to $\frac{\mathcal{L}_{\beta, \bar{\beta}} + \mathcal{L}'_{\alpha, \bar{\alpha}}}{2}$.

The reasoning behind the particular choice of $\mathcal{L}_{\beta, \bar{\beta}}$ and $\mathcal{L}'_{\alpha, \bar{\alpha}}$ and supercharges in (4.44) is that, under the variation generated by these, the superconnections vary as the supercovariant derivative of precisely the same supermatrix. More precisely, under $Q_l^{\dot{1}1} - \frac{\alpha_I}{\beta_I} Q_r^{\dot{2}1}$, for example, we have

$$\begin{aligned} \delta \mathcal{L}_{\beta, \bar{\beta}} &= \mathfrak{D}_{\varphi} H_1 = \partial_{\varphi} H_1 - i[\mathcal{L}_{\beta, \bar{\beta}}, H_1], \\ \delta \mathcal{L}'_{\alpha, \bar{\alpha}} &= \mathfrak{D}_{\varphi} H_1 = \partial_{\varphi} H_1 - i[\mathcal{L}'_{\alpha, \bar{\alpha}}, H_1], \end{aligned} \tag{4.45}$$

with

$$H_1 = \begin{pmatrix} 0 & -\bar{\beta}_I(\xi^{\bar{l}})_{-} q_I^1 \\ -\bar{\alpha}_I(\xi^r)_{+} \bar{q}_{I2} & 0 \end{pmatrix}.$$

For the other three supercharges we have analogous versions of (4.45). In particular we have that under the variation generated by

$$Q_l^{\dot{1}2} - \frac{\alpha_I}{\beta_I} Q_r^{\dot{2}2}, \quad Q_{\bar{r}}^{\dot{1}1} - \frac{\alpha_I}{\beta_I} Q_l^{\dot{2}1}, \quad Q_{\bar{r}}^{\dot{1}2} - \frac{\alpha_I}{\beta_I} Q_l^{\dot{2}2},$$

we obtain the corresponding supercovariant derivatives of

$$H_2 = \begin{pmatrix} 0 & -\bar{\beta}_I(\xi^{\bar{l}})_{-} q_I^2 \\ -\bar{\alpha}_I(\xi^r)_{+} \bar{q}_{I1} & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 0 & -\bar{\beta}_I(\xi^{\bar{r}})_{-} q_I^1 \\ -\bar{\alpha}_I(\xi^l)_{+} \bar{q}_{I2} & 0 \end{pmatrix},$$

and

$$H_4 = \begin{pmatrix} 0 & -\bar{\beta}_I(\xi^{\bar{r}})_{-} q_I^2 \\ -\bar{\alpha}_I(\xi^l)_{+} \bar{q}_{I1} & 0 \end{pmatrix},$$

respectively.

Thus we find that supersymmetry of $\mathcal{L}$ in (4.42) automatically holds. Indeed, for the four supercharges in (4.8), we find

$$\begin{aligned} \delta \mathcal{L} &= \frac{1}{2}(\delta \mathcal{L}_{\beta, \bar{\beta}} + \delta \mathcal{L}'_{\alpha, \bar{\alpha}}) = \frac{1}{2} \left((\partial_{\varphi} H_i - i[\mathcal{L}_{\beta, \bar{\beta}}, H_i]) + (\partial_{\varphi} H_i - i[\mathcal{L}'_{\alpha, \bar{\alpha}}, H_i]) \right) \\ &= \partial_{\varphi} H_i - i \left[\frac{\mathcal{L}_{\beta, \bar{\beta}} + \mathcal{L}'_{\alpha, \bar{\alpha}}}{2}, H_i \right] \\ &= \mathfrak{D}_{\varphi} H_i, \end{aligned}$$

with $i = 1, 2, 3, 4$. From this derivation we were able to see that we can actually relax the condition on $\mathcal{L}_{\bar{\beta},\beta}$ and $\mathcal{L}'_{\bar{\alpha},\alpha}$ appearing with the same weights in $\mathcal{L}$. We find that, as long as their weights sum to one, the derivation above holds. I.e., for some constant parameter λ , the operator built out of

$$\mathcal{L} = (1 - \lambda) \mathcal{L}_{\bar{\beta},\beta} + \lambda \mathcal{L}'_{\bar{\alpha},\alpha}, \quad (4.46)$$

is invariant under all four supercharges since, following the same steps as above, we obtain

$$\delta \mathcal{L} = \partial_\varphi H_i - i \left[(1 - \lambda) \mathcal{L}_{\bar{\beta},\beta} + \lambda \mathcal{L}'_{\bar{\alpha},\alpha}, H_i \right] = \mathfrak{D}_\varphi H_i,$$

with $i = 1, 2, 3, 4$. Thus we find a one parameter family of operators parameterised by λ that generically is 1/4 BPS. At the particular points $\lambda = 0$ and $\lambda = 1$ the corresponding operators are 1/2 BPS.

Of course the presentation within this section is mostly based on the construction introduced in Section 3.3 in the context of ABJ(M) theory. Consequently it also relies heavily on a particular guess of ansätze for the couplings. However, the discussion here is neater specially because we are on S^3 . This spares us from working with projectors and all the spinorology we painfully faced in Section 3.2. Fortunately, it allowed us to understand further aspects of the operators we found and in particular find the one parameter family in (4.46). Furthermore, it opened our eyes to the possibility of generalizing the deformation procedure introduced in [20]. As exemplified by this new 1/4 BPS operator, not all supersymmetric loops are continuously connected to $\mathcal{L}_{\text{bos}}$ through $\Delta \mathcal{L} = i \mathcal{Q} \mathcal{G} + \mathcal{G}^2$. Indeed, the example above points towards the direction of using the superconnection of a 1/2 BPS operator as the starting point upon which we should build our deformation. Since a 1/2 BPS operator furnishes a wider set of preserved supercharges than $\mathcal{L}_{\text{bos}}$, we should be able to recover from it a wider spectrum of supersymmetric loops. This is currently under investigation and should be presented soon in [23].

Appendix A

ABJM(M) theory

We denote the coordinates as $x^\mu = \{x^0, x^1, x^2\}$. Spinor indices $\alpha = +, -$ are lowered and raised as $(\gamma^\mu)^\alpha{}_\beta = \epsilon^{\alpha\gamma}(\gamma^\mu)_\gamma{}^\delta \epsilon_{\beta\delta}$ where

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

When spinor indices are omitted, the contraction is understood as

$$\phi\psi = \phi^\alpha\psi_\alpha, \quad (\text{A.1})$$

In Lorentzian space $\mathbb{R}^{1,2}$ the metric is given by $g_{\mu\nu} = \text{diag}(-1, 1, 1)$ and the gamma-matrices are chosen as $(\gamma^\mu)_\alpha{}^\beta = \{-i\sigma^3, \sigma^1, \sigma^2\}$.

In Euclidean space, the metric is $g_{\mu\nu} = \text{diag}(1, 1, 1)$ and the gamma-matrices are chosen as $(\gamma^\mu)_\alpha{}^\beta = \{-\sigma^3, \sigma^1, \sigma^2\}$.

The supersymmetry transformations are as written in [25]:

$$\begin{aligned} \delta A_\mu &= \frac{4\pi i}{k} \bar{\Theta}^{IJ\alpha} (\gamma_\mu)_\alpha{}^\beta \left(C_I \psi_{J\beta} + \frac{1}{2} \epsilon_{IJKL} \bar{\psi}_\beta^K \bar{C}^L \right), \\ \delta \hat{A}_\mu &= \frac{4\pi i}{k} \bar{\Theta}^{IJ\alpha} (\gamma_\mu)_\alpha{}^\beta \left(\psi_{J\beta} C_I + \frac{1}{2} \epsilon_{IJKL} \bar{C}^L \bar{\psi}_\beta^K \right), \\ \delta \bar{\psi}_\beta^I &= -2i \bar{\Theta}^{IJ\alpha} (\gamma^\mu)_{\alpha\beta} D_\mu C_J - \frac{4\pi i}{k} \bar{\Theta}_\beta^{IJ} (C_J \bar{C}^K C_K - C_K \bar{C}^K C_J) \\ &\quad - \frac{8\pi i}{k} \bar{\Theta}_\beta^{JK} C_J \bar{C}^I C_K - 2i \bar{\epsilon}^{IJ} C_J, \\ \delta \psi_I^\beta &= -i \bar{\Theta}^{KL\alpha} \epsilon_{IJKL} (\gamma^\mu)_\alpha{}^\beta D_\mu \bar{C}^J + \frac{2\pi i}{k} \bar{\Theta}^{KL\beta} \epsilon_{IJKL} (\bar{C}^J C_M \bar{C}^M \\ &\quad - \bar{C}^M C_M \bar{C}^J) + \frac{4\pi i}{k} \bar{\Theta}^{KL\beta} \epsilon_{KLMN} \bar{C}^M C_I \bar{C}^N - i \bar{\epsilon}^{KL\beta} \epsilon_{IJKL} \bar{C}^J, \\ \delta C_I &= \bar{\Theta}^{KL\alpha} \epsilon_{IJKL} \bar{\psi}_\alpha^J, \\ \delta \bar{C}^I &= 2 \bar{\Theta}^{IJ\alpha} \psi_{J\alpha}, \end{aligned} \quad (\text{A.2})$$

where $\Theta^{IJ} = \bar{\theta}^{IJ} - (x^\mu \gamma_\mu) \bar{\epsilon}^{IJ}$ is a conformal Killing spinor. The Poincaré ($\bar{\theta}^{IJ}$) and superconformal parameters ($\bar{\epsilon}^{IJ}$) satisfy the relations

$$\begin{aligned}\bar{\theta}^{IJ} &= -\bar{\theta}^{JI}, & \theta_{IJ} &= \frac{1}{2} \epsilon_{IJKL} \bar{\theta}^{KL}, \\ \bar{\epsilon}^{IJ} &= -\bar{\epsilon}^{JI}, & \epsilon_{IJ} &= \frac{1}{2} \epsilon_{IJKL} \bar{\epsilon}^{KL}.\end{aligned}$$

In Lorentzian space, they obey the reality condition $\bar{\theta}^{IJ} = (\theta_{IJ})^*$ and $\bar{\epsilon}^{IJ} = (\epsilon_{IJ})^*$. In Euclidean space, there is no reality condition, i.e., $\bar{\theta} \neq \theta^\dagger$ and $\bar{\epsilon} \neq \epsilon^\dagger$.

From the Euclidean action (3.1) one has the following Feynman rules in Landau gauge:

- Tree-level vector propagators:

$$\begin{aligned}\langle A_\mu^a(x) A_\nu^b(y) \rangle^{(0)} &= \delta^{ab} \left(\frac{2\pi i}{k} \right) \frac{\Gamma\left(\frac{3}{2} - \epsilon\right)}{2\pi^{\frac{3}{2} - \epsilon}} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{[(x-y)^2]^{\frac{3}{2} - \epsilon}}, \\ \langle \hat{A}_\mu^a(x) \hat{A}_\nu^b(y) \rangle^{(0)} &= -\delta^{ab} \left(\frac{2\pi i}{k} \right) \frac{\Gamma\left(\frac{3}{2} - \epsilon\right)}{2\pi^{\frac{3}{2} - \epsilon}} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{[(x-y)^2]^{\frac{3}{2} - \epsilon}};\end{aligned}$$

- One-loop vector propagators:

$$\begin{aligned}\langle A_\mu^a(x) A_\nu^b(y) \rangle^{(1)} &= \delta^{ab} \left(\frac{2\pi}{k} \right)^2 N_2 \frac{\Gamma^2\left(\frac{3}{2} - \epsilon\right)}{4\pi^{3-2\epsilon}} \left[\frac{\delta_{\mu\nu}}{[(x-y)^2]^{1-2\epsilon}} - \partial_\mu \partial_\nu \frac{(x-y)^{2\epsilon}}{4\epsilon(1+2\epsilon)} \right], \\ \langle \hat{A}_\mu^a(x) \hat{A}_\nu^b(y) \rangle^{(1)} &= \delta^{ab} \left(\frac{2\pi}{k} \right)^2 N_1 \frac{\Gamma^2\left(\frac{3}{2} - \epsilon\right)}{4\pi^{3-2\epsilon}} \left[\frac{\delta_{\mu\nu}}{[(x-y)^2]^{1-2\epsilon}} - \partial_\mu \partial_\nu \frac{(x-y)^{2\epsilon}}{4\epsilon(1+2\epsilon)} \right];\end{aligned}$$

- Scalar propagator:

$$\langle (C_I)_i^{\hat{j}}(x) (\bar{C}^J)_k^{\hat{l}}(y) \rangle^{(0)} = \delta_I^J \delta_i^{\hat{l}} \delta_k^{\hat{j}} \frac{\Gamma\left(\frac{1}{2} - \epsilon\right)}{4\pi^{\frac{3}{2} - \epsilon}} \frac{1}{[(x-y)^2]^{\frac{1}{2} - \epsilon}};$$

- Tree-level fermion propagator:

$$\langle (\psi_I^\alpha)_i^{\hat{j}}(x) (\bar{\psi}_\beta^J)_k^{\hat{l}}(y) \rangle^{(0)} = -i \delta_I^J \delta_i^{\hat{l}} \delta_k^{\hat{j}} \frac{\Gamma\left(\frac{3}{2} - \epsilon\right)}{2\pi^{\frac{3}{2} - \epsilon}} \frac{(\gamma^\mu)^\alpha{}_\beta (x-y)_\mu}{[(x-y)^2]^{\frac{3}{2} - \epsilon}};$$

- One-loop fermion propagator:

$$\langle (\psi_I^\alpha)_i^{\hat{j}}(x) (\bar{\psi}_\beta^J)_k^{\hat{l}}(y) \rangle^{(1)} = -i \delta_I^J \delta_i^{\hat{l}} \delta_k^{\hat{j}} \left(\frac{2\pi}{k} \right) (N_2 - N_1) \frac{\Gamma^2\left(\frac{1}{2} - \epsilon\right)}{16\pi^{3-2\epsilon}} \frac{1}{[(x-y)^2]^{1-2\epsilon}}.$$

The interaction piece of the action, S_{int} , contains the following vertices:

- Gauge cubic vertex

$$-\frac{ik}{12\pi}\epsilon^{\mu\nu\rho}\int d^3x\ f^{abc}A_\mu^aA_\nu^bA_\rho^c, \quad (\text{A.3})$$

where f^{abc} is the structure constant of $U(N_1)$;

- Gauge-fermion cubic vertex

$$-\int d^3x\ \text{Tr}\left(\bar{\psi}^I\gamma^\mu\psi_iA_\mu-\bar{\psi}^I\gamma^\mu\hat{A}_\mu\psi_I\right). \quad (\text{A.4})$$

Appendix B

A^p and B^p methods

The A^p method was proposed in [37] as a recursive algorithm to compute integrals of the form

$$A_{n_1, n_2}^p = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\tau_1} d\tau_1 d\tau_2 \frac{e^{in_1\tau_1 + in_2\tau_2}}{(e^{i\tau_1} - e^{i\tau_2})^p}, \quad (\text{B.1})$$

where $p \in \mathbb{Z}_+$ and $n_1, n_2 \in \mathbb{Z}$. For $p = 0$ this is simply $A_{n_1, n_2}^0 = \delta_{0, n_1} \delta_{0, n_2}$. For $p > 0$, instead of performing the integrals, we view them as formal objects satisfying the parity condition and recurrence relations, arising from combinations of integrands with factorisable numerators,

$$A_{n_1, n_2}^p = (-1)^p A_{n_2, n_1}^p, \quad \sum_{k=0}^p \binom{p}{k} (-1)^k A_{n_1+p-k, n_2+k}^p = A_{n_1, n_2}^0. \quad (\text{B.2})$$

For $p = 2$ this is solved by

$$A_{n_1, n_2}^2 = \frac{1}{4} |n_1 - n_2| \delta_{2, n_1+n_2} + J_{n_1+n_2}^{(1)}, \quad (\text{B.3})$$

with arbitrary $J_n^{(1)}$. For $p = 4$ we have instead

$$A_{n_1, n_2}^4 = \frac{1}{96} |n_1 - n_2| ((n_1 - n_2)^2 - 4) \delta_{4, n_1+n_2} + J_{n_1+n_2}^{(2)} (n_1 - n_2)^2 + J_{n_1+n_2}^{(3)}. \quad (\text{B.4})$$

The arbitrary J 's ($J_n^{(1)}$, $J_n^{(2)}$, $J_n^{(3)}$ and so on) should contain the divergences of the integrals.

For half-integer valued n_1 and n_2 , we have that A_{n_1, n_2}^p should still satisfy parity condition and recurrence relations outlined in (B.2), but now with boundary condition

$$A_{n_1, n_2}^0 = -\frac{1}{\pi^2} \frac{1}{n_1 n_2}.$$

In this case, for $p = 2$ we find

$$A_{n_1, n_2}^2 = \frac{(1 - n_2)\text{PolyGamma}[0, n_2] + (1 - n_1)\text{PolyGamma}[0, 2 - n_1]}{(-2 + n_1 + n_2)\pi^2} + K_{n_1 + n_2}^{(1)}, \quad (\text{B.5})$$

and for $p = 4$ we have instead

$$\begin{aligned} A_{n_1, n_2}^4 = & -\frac{1}{12(4 - n_1 - n_2)\pi^2} \left[(-3 + n_2)(-2 + n_2)(-1 + n_2) \left(\text{PolyGamma}[0, 4 - n_2] \right. \right. \\ & + \left. \text{PolyGamma}[0, n_2] \right) + (-3 + n_1)(-2 + n_1)(-1 + n_1) \left(\text{PolyGamma}[0, 4 - n_1] \right. \\ & + \left. \text{PolyGamma}[0, n_1] \right) \left. \right] + \frac{3n_2^2 + 3n_1^2 - 3(n_2 + n_1) - 4n_2n_1}{24\pi^2} \\ & + K_{n_1 + n_2}^{(2)}(n_1 - n_2)^2 + K_{n_1 + n_2}^{(3)}, \end{aligned} \quad (\text{B.6})$$

with arbitrary $K_n^{(i)}$.

In analogy to the derivation above, the B^p method we propose is a recursive algorithm to compute integrals of the form

$$B_n^p = \frac{1}{2\pi} \int_0^{2\pi} d\tau \frac{e^{in\tau}}{(e^{i\tau} - 1)^p}, \quad (\text{B.7})$$

for $p \in \mathbb{Z}_+$. Here we view B_n^p symbols as formal objects satisfying the recurrence relations

$$\sum_{k=0}^p \binom{p}{k} (-1)^{k+p} B_{n+k}^p = B_n^0, \quad (\text{B.8})$$

where $B_n^0 = \frac{i}{\pi n}$. For $p = 1$ we find

$$B_n^1 = \begin{cases} \frac{(\text{Sign}[n] + (n - 1)\delta_{n,0})}{2} & , n \in \mathbb{Z} \\ \frac{i\text{PolyGamma}[0, n]}{\pi} & , n + \frac{1}{2} \in \mathbb{Z} \end{cases}, \quad (\text{B.9})$$

while for $p = 3$ we have

$$B_n^3 = \begin{cases} \frac{|n-1|(n-2)}{4} + nT^{(0)} + n^2T^{(1)} & , n \in \mathbb{Z} \\ \frac{(6n - 2n^2 - 4)\text{PolyGamma}[0, n]}{4\pi i} + nR^{(0)} + n^2R^{(1)} & , n + \frac{1}{2} \in \mathbb{Z} \end{cases}, \quad (\text{B.10})$$

with arbitrary $T^{(i)}$ and $R^{(i)}$.

Constants J , K , T and R that arise in the recursive methods outlined above are not related to each other.

Appendix C

$\mathcal{N} = 4$ theories

The supersymmetry transformations of the $\mathcal{N} = 4$ Chern-Simons-matter theory on S^3 can be seen to be given by

$$\begin{aligned}
\delta A_{\mu I} &= \frac{i}{k} \xi_{ab} \gamma_\mu (j_I^{ab} - \tilde{j}_I^{ba}), \\
\delta q_I^a &= \xi^{ab} \psi_{I\dot{b}}, \\
\delta \bar{q}_{I\dot{a}} &= \xi_{ab} \bar{\psi}_I^{\dot{b}}, \\
\delta \tilde{q}_{I-1\dot{b}} &= -\xi_{ab} \tilde{\psi}_{I-1}^a, \\
\delta \bar{\tilde{q}}_{I-1}^{\dot{b}} &= -\xi^{ab} \bar{\tilde{\psi}}_{I-1\dot{a}}, \\
\delta \psi_{I\dot{a}} &= i\gamma^\mu \xi_{ba} D_\mu q_I^b + i\zeta_{ba} q_I^b - \frac{i}{k} \xi_{ba} (\nu_I q_I^b - q_I^b \nu_{I+1}) + \frac{2i}{k} \xi_{bc} (\tilde{\mu}_I^{\dot{c}} q_I^b - q_I^b \tilde{\mu}_{I+1\dot{a}}^{\dot{c}}), \\
\delta \bar{\psi}_I^{\dot{a}} &= i\gamma^\mu \xi^{ba} D_\mu \bar{q}_{I\dot{b}} + i\zeta^{ba} \bar{q}_{I\dot{b}} - \frac{i}{k} \xi^{ba} (\bar{q}_{I\dot{b}} \nu_I - \nu_{I+1} \bar{q}_{I\dot{b}}) + \frac{2i}{k} \xi^{bc} (\bar{q}_{I\dot{b}} \tilde{\mu}_I^{\dot{a}} - \tilde{\mu}_{I+1\dot{c}}^{\dot{a}} \bar{q}_{I\dot{b}}), \\
\delta \tilde{\psi}_{I-1}^a &= -i\gamma^\mu \xi^{ab} D_\mu \tilde{q}_{I-1\dot{b}} - i\zeta^{ab} \tilde{q}_{I-1\dot{b}} + \frac{i}{k} \xi^{ab} (\tilde{q}_{I-1\dot{b}} \tilde{\nu}_I - \tilde{\nu}_{I-1} \tilde{q}_{I-1\dot{b}}) \\
&\quad - \frac{2i}{k} \xi^{bc} (\tilde{q}_{I-1\dot{c}} \mu_{I\dot{b}}^a - \mu_{I-1\dot{b}}^a \tilde{q}_{I-1\dot{c}}), \\
\delta \bar{\tilde{\psi}}_{I-1\dot{a}} &= -i\gamma^\mu \xi_{ab} D_\mu \bar{\tilde{q}}_{I-1}^{\dot{b}} - i\zeta_{ab} \bar{\tilde{q}}_{I-1}^{\dot{b}} + \frac{i}{k} \xi_{ab} (\tilde{\nu}_I \bar{\tilde{q}}_{I-1}^{\dot{b}} - \bar{\tilde{q}}_{I-1}^{\dot{b}} \tilde{\nu}_{I-1}) \\
&\quad - \frac{2i}{k} \xi_{bc} (\mu_{I\dot{a}}^b \bar{\tilde{q}}_{I-1}^{\dot{c}} - \bar{\tilde{q}}_{I-1}^{\dot{c}} \mu_{I-1\dot{a}}^b).
\end{aligned} \tag{C.1}$$

where $\zeta_{ab} = \frac{1}{3} \gamma^\mu \nabla_\mu \xi_{ab}$. More specifically, from (4.2) one finds $\zeta_{ab}^{l,\bar{l}} = \frac{i}{2} \xi_{ab}^{l,\bar{l}}$ and $\zeta_{ab}^{r,\bar{r}} = -\frac{i}{2} \xi_{ab}^{r,\bar{r}}$. We work in Euclidean signature and take the gamma-matrices, $(\gamma^\mu)_\alpha^\beta$, to be given by the Pauli matrices. As usual, the spinor contractions are such that

$$\xi_1 \xi_2 \equiv \xi_1^\alpha \xi_{2,\alpha} = +\xi_2 \xi_1, \quad \xi_1 \gamma^\mu \xi_2 \equiv \xi_1^\alpha (\gamma^\mu)_\alpha^\beta \xi_{2,\beta} = -\xi_2 \gamma^\mu \xi_1, \quad \alpha, \beta = \pm.$$

Then it follows that the Killing spinors on S^3 of (4.3) satisfy $\xi^{\bar{l}}\xi^l = \xi^l\xi^{\bar{l}} = 1$ and $\xi^{\bar{l}}\gamma^\mu\xi^l = -\xi^l\gamma^\mu\xi^{\bar{l}} = \delta_\varphi^\mu$, and similarly for the contractions involving ξ^r and $\xi^{\bar{r}}$.

The expressions in (C.1) can be motivated by relating them to the transformations of the $\mathcal{N} = 2$ Chern-Simons-matter theory on S^3 written down in [45, 46]. The off-shell supersymmetry transformation of the physical fields from the vector multiplets $(A_\mu, \sigma, D, \lambda, \bar{\lambda})$ and chiral multiplets (ϕ, ψ, F) in the $(\square, \bar{\square})$ bifundamental and their conjugates are

$$\begin{aligned}\delta A_{\mu I} &= -\frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda_I - \bar{\lambda}_I\gamma_\mu\epsilon), & \delta\phi_I &= \bar{\epsilon}\psi_I, & \delta\bar{\phi}_I &= \epsilon\bar{\psi}_I, \\ \delta\psi_I &= i\gamma^\mu\epsilon D_\mu\phi_I + i\epsilon(\sigma_I\phi_I - \phi_I\sigma_{I+1}) + \frac{i}{3}\gamma^\mu D_\mu\epsilon\phi_I + \bar{\epsilon}F_I, \\ \delta\bar{\psi}_I &= i\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi}_I - i(\sigma_{I+1}\bar{\phi}_I - \bar{\phi}_I\sigma_I)\bar{\epsilon} + \frac{i}{3}\bar{\phi}_I\gamma^\mu D_\mu\bar{\epsilon} + \bar{F}_I\epsilon.\end{aligned}\tag{C.2}$$

To match with $\mathcal{N} = 4$ theories we go on-shell, so use the actions¹

$$\begin{aligned}S_{CS}^{(I)} &= -\frac{k_I}{4\pi} \int d^3x \sqrt{g} \text{Tr} \left[\frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} (A_{\mu I} \partial_\nu A_{\rho I} - \frac{2i}{3} A_{\mu I} A_{\nu I} A_{\rho I}) - \bar{\lambda}_I \lambda_I + 2D_I \sigma_I \right], \\ iS_{\text{matter}}^{(I)} &= \int d^3x \sqrt{g} \text{Tr} \left[iD_\mu \bar{\phi}_I D^\mu \phi_I + \bar{\psi}_I \gamma^\mu D_\mu \psi_I + \frac{3i}{4l^2} \bar{\phi}_I \phi_I - \bar{\psi}_I (\sigma_I \psi_I - \psi_I \sigma_{I+1}) \right. \\ &\quad \left. - \bar{\psi}_I (\lambda_I \phi_I - \phi_I \lambda_{I+1}) + \bar{\phi}_I (\bar{\lambda}_I \psi_I - \psi_I \bar{\lambda}_{I+1}) - \bar{\phi}_I (D_I \phi_I - \phi_I D_{I+1}) \right. \\ &\quad \left. + i\bar{\phi}_I (\sigma_I \sigma_I \phi_I - 2\sigma_I \phi_I \sigma_{I+1} + \phi_I \sigma_{I+1} \sigma_{I+1}) + i\bar{F}_I F_I \right],\end{aligned}$$

to integrate out the auxiliary fields $\lambda, \bar{\lambda}, \sigma, D, F$ and $\bar{F}$.

This yields the on-shell transformations

$$\begin{aligned}\delta A_{\mu I} &= -\frac{2\pi i}{k_I} \left[\bar{\epsilon}\gamma_\mu (\bar{\phi}_{I-1} \psi_{I-1} - \psi_I \bar{\phi}_I) - (\phi_I \bar{\psi}_I - \bar{\psi}_{I-1} \phi_{I-1}) \gamma_\mu \epsilon \right], \\ \delta\phi_I &= \bar{\epsilon}\psi_I, \\ \delta\bar{\phi}_I &= \epsilon\bar{\psi}_I, \\ \delta\psi_I &= i\gamma^\mu\epsilon D_\mu\phi_I + \frac{2\pi i}{k_I} \epsilon \left[(\bar{\phi}_{I-1} \phi_{I-1} - \phi_I \bar{\phi}_I) \phi_I - \phi_I (\phi_{I+1} \bar{\phi}_{I+1} - \bar{\phi}_I \phi_I) \right] + \frac{i}{3} \gamma^\mu D_\mu \epsilon \phi_I, \\ \delta\bar{\psi}_I &= i\gamma^\mu \bar{\epsilon} D_\mu \bar{\phi}_I - \frac{2\pi i}{k_I} \left[(\phi_{I+1} \bar{\phi}_{I+1} - \bar{\phi}_I \phi_I) \bar{\phi}_I - \bar{\phi}_I (\bar{\phi}_{I-1} \phi_{I-1} - \phi_I \bar{\phi}_I) \right] \bar{\epsilon} + \frac{i}{3} \bar{\phi}_I \gamma^\mu D_\mu \bar{\epsilon}.\end{aligned}\tag{C.3}$$

Now we use the chiral decomposition of $\mathcal{N} = 4$ hypermultiplets and twisted hypermultiplets in Figure 11. The chiral fields in this representation are q^2 and

¹Note that for the $\mathcal{N} = 4$ theory the CS-levels are alternating, $k_I = (-1)^I k$.

$\tilde{q}_i$. Extending to the other fields in the multiplets we match (C.1) to (C.3) with the replacements

$$\tilde{q}_i, q^2 \rightarrow \phi, \quad \psi_2, -\tilde{\psi}^1 \rightarrow \psi, \quad \bar{\tilde{q}}^i, \bar{q}_2 \rightarrow \bar{\phi}, \quad \bar{\psi}^2, -\bar{\tilde{\psi}}_1 \rightarrow \bar{\psi}.$$

where we also identified the supersymmetry parameters as $\xi^{1i} \rightarrow \epsilon$ and $\xi^{2\dot{2}} \rightarrow \bar{\epsilon}$. A mismatch by 2π in the non-linear terms can be fixed by rescaling the fields.

The other fields transform in the conjugate $(\bar{\square}, \square)$ representation, and with the same choice of supersymmetry parameter identification they would be matched to $\mathcal{N} = 2$ fields in this representation according to

$$\bar{\tilde{q}}^2, \bar{q}_1 \rightarrow \phi, \quad \bar{\psi}^1, -\bar{\tilde{\psi}}_2 \rightarrow \psi, \quad \tilde{q}_2, q^1 \rightarrow \bar{\phi}, \quad \psi_1, -\tilde{\psi}^2 \rightarrow \bar{\psi}.$$

C.1 Double transformations of the fields

The main ingredient in the construction of fermionic Wilson loops are the double transformations of scalar fields. Using (C.1), we write these as

$$\begin{aligned} [\delta_1, \delta_2] q_I^a &= -i(\xi_1^{ab} \gamma^\mu \xi_{2bb} - \xi_2^{ab} \gamma^\mu \xi_{1bb}) D_\mu q_I^b - i(\xi_1^{ab} \zeta_{2bb} - \xi_2^{ab} \zeta_{1bb}) q_I^b \\ &\quad + \frac{i}{k} (\xi_1^{ab} \xi_{2bb} - \xi_2^{ab} \xi_{1bb}) (\nu_I q_I^b - q_I^b \nu_{I+1}) - \frac{2i}{k} (\xi_1^{a\dot{a}} \xi_{2bb} - \xi_2^{a\dot{a}} \xi_{1bb}) (\tilde{\mu}_I^{\dot{b}} q_I^b - q_I^b \tilde{\mu}_{I+1}^{\dot{a}}) \\ [\delta_1, \delta_2] \tilde{q}_{I-1\dot{a}} &= -i(\xi_{1a\dot{a}} \gamma^\mu \xi_2^{ab} - \xi_{2a\dot{a}} \gamma^\mu \xi_1^{ab}) D_\mu \tilde{q}_{I-1\dot{b}} - i(\xi_{1a\dot{a}} \zeta_2^{ab} - \xi_{2a\dot{a}} \zeta_1^{ab}) \tilde{q}_{I-1\dot{b}} \\ &\quad + \frac{i}{k} (\xi_{1a\dot{a}} \xi_2^{ab} - \xi_{2a\dot{a}} \xi_1^{ab}) (\tilde{q}_{I-1\dot{b}} \tilde{\nu}_I - \tilde{\nu}_{I-1} \tilde{q}_{I-1\dot{b}}) \\ &\quad - \frac{2i}{k} (\xi_{1a\dot{a}} \xi_2^{bb} - \xi_{2a\dot{a}} \xi_1^{bb}) (\tilde{q}_{I-1\dot{b}} \mu_I^a{}_b - \mu_{I-1}^a{}_b \tilde{q}_{I-1\dot{b}}). \end{aligned} \quad (C.4)$$

We specialize the double transformations to the supercharges (4.12), whose corresponding parameters are given by

$$\xi_{1,a\dot{b}} = \delta_a^1 \left(\delta_b^{\dot{1}} \cos \frac{\theta}{2} \xi^{\bar{l}} + \delta_b^{\dot{2}} \sin \frac{\theta}{2} \xi^{\bar{r}} \right), \quad \xi_{2,a\dot{b}} = \delta_a^2 \left(\delta_b^{\dot{2}} \cos \frac{\theta}{2} \xi^l - \delta_b^{\dot{1}} \sin \frac{\theta}{2} \xi^r \right).$$

The Killing spinors in (4.3) obey (4.2), from which one sees that the second term in (C.4) becomes

$$-i(\xi_1^{ab} \zeta_{2bb} - \xi_2^{ab} \zeta_{1bb}) q_I^b = -\frac{1}{2} \cos \theta (\delta_1^a q_I^1 - \delta_2^a q_I^2).$$

Combining with the rest of (C.4) one finds

$$i\{\mathcal{Q}_1, \mathcal{Q}_2\} q_I^1 = D_\varphi q_I^1 - \frac{i}{2} \cos \theta q_I^1 + \frac{1}{k} (\nu_I q_I^1 - q_I^1 \nu_{I+1})$$

$$\begin{aligned}
& -\frac{1}{k} \left(\bar{\tilde{r}}_{I-1}^1 \tilde{r}_{I-1\dot{1}} - \bar{\tilde{r}}_{I-1}^2 \tilde{r}_{I-1\dot{2}} \right) q_I^1 + \frac{1}{k} q_I^1 \left(\tilde{r}_{I+1\dot{1}} \bar{\tilde{r}}_{I+1}^1 - \tilde{r}_{I+1\dot{2}} \bar{\tilde{r}}_{I+1}^2 \right), \\
i\{\mathcal{Q}_1, \mathcal{Q}_2\} q_I^2 &= D_\varphi q_I^2 + \frac{i}{2} \cos \theta q_I^2 - \frac{1}{k} (\nu_I q_I^2 - q_I^2 \nu_{I+1}) \\
& - \frac{1}{k} \left(\bar{\tilde{r}}_{I-1}^1 \tilde{r}_{I-1\dot{1}} - \bar{\tilde{r}}_{I-1}^2 \tilde{r}_{I-1\dot{2}} \right) q_I^2 + \frac{1}{k} q_I^2 \left(\tilde{r}_{I+1\dot{1}} \bar{\tilde{r}}_{I+1}^1 - \tilde{r}_{I+1\dot{2}} \bar{\tilde{r}}_{I+1}^2 \right),
\end{aligned}$$

with fields $\tilde{r}$ defined in (4.23). Noting that

$$\begin{aligned}
\nu_I q_I^1 - q_I^1 \nu_{I+1} &= -(\mu_{I1}^1 - \mu_{I2}^2) q_I^1 + q_I^1 (\mu_{I+11}^1 - \mu_{I+12}^2), \\
\nu_I q_I^2 - q_I^2 \nu_{I+1} &= (\mu_{I1}^1 - \mu_{I2}^2) q_I^2 - q_I^2 (\mu_{I+11}^1 - \mu_{I+12}^2)
\end{aligned}$$

we can write

$$i\{\mathcal{Q}_1, \mathcal{Q}_2\} q_I^a = D_\varphi q_I^a - \frac{i}{2} \cos \theta \left(\delta_1^a q_I^1 - \delta_2^a q_I^2 \right) - i(\mathcal{A}_I^\theta - A_{\varphi,I}) q_I^a + i q_I^a (\mathcal{A}_{I+1}^\theta - A_{\varphi,I+1}).$$

where $\mathcal{A}_{I+1}^\theta$ is given by the natural generalization of (4.24) to the $I+1$ -th node

$$\mathcal{A}_{I\pm 1}^\theta = A_{\varphi,I\pm 1} - \frac{i}{k} (\mu_{I\pm 11}^1 - \mu_{I\pm 12}^2 + \tilde{r}_{I\pm 1\dot{1}} \bar{\tilde{r}}_{I\pm 1}^1 - \tilde{r}_{I\pm 1\dot{2}} \bar{\tilde{r}}_{I\pm 1}^2). \quad (\text{C.5})$$

The covariant derivative is $D_\varphi q_I^a = \partial_\varphi q_I^a - i A_{\varphi,I} q_I^a + i q_I^a A_{\varphi,I+1}$, so that the double transformation can be recast as a total covariant derivative with respect to the $\mathcal{A}^\theta$ -connection, including the coupling to a background field

$$\begin{aligned}
i\{\mathcal{Q}_1, \mathcal{Q}_2\} q_I^1 &= \mathcal{D}_\varphi q_I^1 \equiv \partial_\varphi q_I^1 - \frac{i}{2} \cos \theta q_I^1 - i \mathcal{A}_I^\theta q_I^1 + i q_I^1 \mathcal{A}_{I+1}^\theta, \\
i\{\mathcal{Q}_1, \mathcal{Q}_2\} q_I^2 &= \mathcal{D}_\varphi q_I^2 \equiv \partial_\varphi q_I^2 + \frac{i}{2} \cos \theta q_I^2 - i \mathcal{A}_I^\theta q_I^2 + i q_I^2 \mathcal{A}_{I+1}^\theta.
\end{aligned} \quad (\text{C.6})$$

Note that q_I^1 and q_I^2 are charged oppositely with respect to the background field.

We repeat the computation for the $\tilde{r}_a$'s arriving at

$$\begin{aligned}
i\{\mathcal{Q}_1, \mathcal{Q}_2\} \tilde{r}_{I-1\dot{1}} &= D_\varphi \tilde{r}_{I-1\dot{1}} + \frac{i}{2} \tilde{r}_{I-1\dot{1}} - \frac{1}{k} (\tilde{r}_{I-1\dot{1}} \tilde{\nu}_I - \tilde{\nu}_{I-1} \tilde{r}_{I-1\dot{1}}) \\
& + \frac{1}{k} \left(\tilde{r}_{I-1\dot{1}} (\mu_{I1}^1 - \mu_{I2}^2) - (\mu_{I-11}^1 - \mu_{I-12}^2) \tilde{r}_{I-1\dot{1}} \right), \\
i\{\mathcal{Q}_1, \mathcal{Q}_2\} \tilde{r}_{I-1\dot{2}} &= D_\varphi \tilde{r}_{I-1\dot{2}} - \frac{i}{2} \tilde{r}_{I-1\dot{2}} + \frac{1}{k} (\tilde{r}_{I-1\dot{2}} \tilde{\nu}_I - \tilde{\nu}_{I-1} \tilde{r}_{I-1\dot{2}}) \\
& + \frac{1}{k} \left(\tilde{r}_{I-1\dot{2}} (\mu_{I1}^1 - \mu_{I2}^2) - (\mu_{I-11}^1 - \mu_{I-12}^2) \tilde{r}_{I-1\dot{2}} \right).
\end{aligned}$$

It is easy to see that these are also given by total derivatives

$$\begin{aligned} i\{\mathcal{Q}_1, \mathcal{Q}_2\}\tilde{r}_{I-1\dot{1}} &= \mathcal{D}_\varphi \tilde{r}_{I-1\dot{1}} \equiv \partial_\varphi \tilde{r}_{I-1\dot{1}} + \frac{i}{2}\tilde{r}_{I-1\dot{1}} - i\mathcal{A}_{I-1}^\theta \tilde{r}_{I-1\dot{1}} + i\tilde{r}_{I-1\dot{1}}\mathcal{A}_I^\theta, \\ i\{\mathcal{Q}_1, \mathcal{Q}_2\}\tilde{r}_{I-1\dot{2}} &= \mathcal{D}_\varphi \tilde{r}_{I-1\dot{2}} \equiv \partial_\varphi \tilde{r}_{I-1\dot{2}} - \frac{i}{2}\tilde{r}_{I-1\dot{2}} - i\mathcal{A}_{I-1}^\theta \tilde{r}_{I-1\dot{2}} + i\tilde{r}_{I-1\dot{2}}\mathcal{A}_I^\theta, \end{aligned} \quad (\text{C.7})$$

with, again, the two components oppositely charged with respect to the background field. We also need the corresponding expressions for the conjugate fields, which are given by

$$\begin{aligned} i\{\mathcal{Q}_1, \mathcal{Q}_2\}\bar{q}_{Ia} &= \mathcal{D}_\varphi \bar{q}_{Ia} \equiv \partial_\varphi \bar{q}_{Ia} - (-1)^a \frac{i}{2} \cos \theta \bar{q}_{Ia} - i\mathcal{A}_{I+1}^\theta \bar{q}_{Ia} + i\bar{q}_{Ia}\mathcal{A}_I^\theta, \\ i\{\mathcal{Q}_1, \mathcal{Q}_2\}\bar{\tilde{r}}_{I-1}^{\dot{a}} &= \mathcal{D}_\varphi \bar{\tilde{r}}_{I-1}^{\dot{a}} \equiv \partial_\varphi \bar{\tilde{r}}_{I-1}^{\dot{a}} + (-1)^{\dot{a}} \frac{i}{2} \bar{\tilde{r}}_{I-1}^{\dot{a}} - i\mathcal{A}_I^\theta \bar{\tilde{r}}_{I-1}^{\dot{a}} + i\bar{\tilde{r}}_{I-1}^{\dot{a}}\mathcal{A}_{I-1}^\theta. \end{aligned} \quad (\text{C.8})$$

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