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MASTER'S THESIS

Alpha Particles in Effective Field Theory

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Resumo

Instituto de Física da Universidade de São Paulo

Mestre em Ciências

Partículas Alfa em Teorias de Campo Efetivas

por Cristian Caniu Barros

Nesta tese, nós trabalhamos sobre o problema do sistema de duas partículas alfa utilizando uma teoria de campos efetiva. O nosso objetivo é abordar os observáveis e a ressonância do sistema alfa-alfa de baixa energia identificada como o estado fundamental do berílio-8. Neste trabalho nós começamos com uma teoria de campo efetiva em que os graus de liberdade são as partículas alfa interagindo via forças de contato dependentes do momento. Estes, em contraste com as forças que são dependentes da energia, são mais úteis na extensão da teorias para sistemas com mais de duas partículas alfa. Além disso, forças dependentes do momento nos permitem abordar restrições causais nos observáveis, conhecidas como causalidade de Wigner. Nós apresentamos nossos cálculos para o sistema alfa-alfa.

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Abstract

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by Cristian Caniu Barros

In this thesis we work on the problem of the two-alpha-particle system using a halo/-cluster effective field theory (EFT). Our goal is to address the alpha-alpha scattering observables and its low-energy resonance identified as the ground state of Beryllium-8. In this work we start with an EFT in which the degrees of freedom are the alpha particles interacting via momentum-dependent contact forces. These, in contrast to forces that are energy-dependent, are more useful in extending the theory to systems with more than two alpha particles. Additionally, momentum-dependent forces allow us to address causal restrictions on scattering observables, known as the Wigner's causality bound. We present our EFT calculations for the alpha-alpha system.

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Dedicado a mis padres y familia.

Chapter 1

Introduction

Weakly bound nuclear systems has received enormous attention in the last 20 years. Numerous reactions involving such systems play central roles in understanding astronomical phenomena such as the creation of interstellar material, supernova explosions, formation of neutron stars and stellar evolution. In massive stars, it would not be an exaggeration to say that the most important reaction is the triple-alpha (3α) process through which carbon (^{12}C) and the heaviest elements in the universe are formed.

The reaction among three alphas always had a rich history of investigative interest. It is extremely difficult to occur directly due to the Coulomb repulsion. The reaction rate is significantly increased by the existence of a two-alpha resonance identified as the ground state of beryllium (^8Be). However, this increase is still insufficient to explain the abundance of carbon in the universe. Addressing this problem, Hoyle [1] suggested the existence of a genuine three-alpha resonance, through which a beryllium and an alpha particle would transit before decaying into the ground state of carbon-12. The so-called Hoyle state was confirmed experimentally three years later [2]. This state, having an energy very close to the three-alpha threshold, is often cited as strong fact that favors the anthropic principle [3].

From a theoretical perspective, the Hoyle state remains a challenge. *Ab-initio* calculations starting from the interaction between two (NN) and three nucleons (3N) are not yet capable of reproducing this state without considerable level of approximation. The main reason is the fact that this state involves very low energies (~ 0.3 MeV) compared with the binding energy of the stable ground state of ^{12}C (~ 7.5 MeV). Variational calculations [4, 5] or cluster approximation [6] have achieved better agreement with experimental data.

This thesis aims to address the two-alpha-particles ($\alpha\alpha$) system using the formalism known as halo/cluster effective field theory [7], where the degrees of freedom are the cores and/or nucleons forming a weakly bound nuclear system.

The first ideas of effective field theories came from chiral perturbation theory [8] and its extensions to systems with a few nucleons [9, 10]. They incorporate significant advantages such as to establish a model-independent approach, to preserve the symmetries of the fundamental theory and to estimate robust theoretical uncertainties at a given energy scale. It is quite suitable for the description of an interesting phenomenon in few body physics called the Efimov effect. This is a genuine three-body outcome in the limit when, in the two-body subsystem, the corresponding scattering length goes to infinity. In the three-body system, this limit generates a spectrum of geometrically spaced bound states, with the n -th state being determined by the $(n-1)$ -th state through the relation $E^{(n)} = E^{(n-1)}/\lambda_0$, where $\lambda_0 \cong 515$. In the language of effective field theory (EFT), this phenomenon is closely related to the discrete symmetry scale, anomalously broken down from a non-relativistic conformal symmetry in the two-body subsystem [11].

The $\alpha\alpha$ system is an example where the continuous conformal symmetry takes place. In [12], this system was studied under halo/cluster EFT. In their proposal, the authors considered the alpha particle as the relevant degree of freedom and an auxiliary dimeron field representing multiple interactions of two alphas in S -wave. The latter leads to an energy-dependent description of the strong force among alpha particles. It was shown that the nearly conformal symmetry in the $\alpha\alpha$ system rises from a very fine balance between the strong and electromagnetic forces. As in [12], we work on the problem of the $\alpha\alpha$ system using a halo/cluster EFT. Our goal is to readdress the $\alpha\alpha$ scattering observables and its low-energy resonance identified as the ^8Be ground state. However, in this work we start with an EFT in which the degrees of freedom are the alpha particles interacting via momentum-dependent contact forces. There are two main reasons for this work. First, an energy-dependent interaction is extremely difficult, from both technical and conceptual reasons, to implement in systems with more than two particles. The situation is the opposite with a momentum-dependent one. Second, contrary to the energy-dependent case, a momentum-dependent force allows us to address causal restrictions on scattering observables, known as the Wigner's causality bound [13].

This work is organized as follows. In Chapter 2, we review some basic concepts from quantum mechanics with focus on scattering theory at low energies. We review the main elements of the pure Coulomb scattering process such as the Coulomb wave functions and the Coulomb propagator, which are essential tools for the description of the $\alpha\alpha$ system. Furthermore, we review the so-called two-potential formalism that allows a

simple treatment of the scattering amplitude, with its clean separation into a pure Coulomb and a Coulomb-modified strong part.

In Chapter 3, we present the main ideas behind effective field theories. In order to understand the EFT for two alpha particles, we review the EFT for nucleons proposed by Kaplan *et al.* [14] and the application to the proton-proton system studied by Kong and Ravndal [15].

In Chapter 4, we present the strong effective Lagrangian with momentum-dependent contact interactions and discuss how electromagnetic interactions are included. There is a subtle difference from Kong and Ravndal's work due to the presence of a low-energy S -wave resonance. To renormalize the theory, we compute the $\alpha\alpha$ amplitude to match the amplitude under the effective-range parametrization. We address the experimental situation, comparing our predictions to scattering data. Finally, we present our analysis of the Wigner bound for this specific system.

Our conclusions are outlined in Section 5.

Chapter 2

Scattering Theory

2.1 The general theory of elastic scattering

We begin with a brief overview of scattering theory where the scattered particles, together with their internal structure, are left unchanged. Emphasis is given to scattering at relatively small energies which, for nuclear systems, comprises energies of the order of a few MeVs. The material consulted to prepare this review is well known in the literature [16–18].

One knows that a two-body scattering problem is equivalent to the motion of a single particle, with reduced mass, under the influence of a potential $V(\mathbf{r})$, with $\mathbf{r}$ being the distance between the particles. We consider a short-range central potential $V(r)$ with $r = |\mathbf{r}|$, to represent the force between two identical particles of mass M . To make the discussion as simple as possible, we assume the particles spinless, allowing us to focus on the most important features of the scattering without the complications of a more realistic description. We carry the analysis in the center-of-mass (c.m.) system. Throughout this work we adopt a unit system where $\hbar = c = 1$.

We call by ψ the solution of the Schrödinger equation describing the relative motion, with reduced mass $m_r = M/2$ and positive energy $E = p^2/2m_r$, with p the relative momentum. We require that ψ describes fluxes of incident plus scattered particles, the latter moving away from the scattering center.

In the presence of a short-range central potential the most general expression for the wave function reads,

$$\psi(\mathbf{r}) = \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) \varphi_l(r), \quad (2.1)$$

where $\varphi_l(r)$ is the solution of the radial Schrödinger equation, and $P_l(\cos\theta)$ are the Legendre polynomials (one has $m = 0$ due to azimuthal symmetry of the problem). The polar angle θ provides the direction of the scattered particle with respect to the incident beam. Outside of the range R of the potential, $\psi(\mathbf{r})$ has to match the most general free solution in terms of spherical Bessel functions, $j_l(pr)$ and $n_l(pr)$, whose asymptotic form is known in terms of sine and cosine functions [19].

An usual way to write the asymptotic form of $\psi(\mathbf{r})$ is as follows

$$\psi(\mathbf{r}) \approx \exp(ipz) + f(\theta) \frac{\exp(ipr)}{r}, \quad (2.2)$$

where the first term corresponds to the incident beam moving in the z direction with momentum p , and the second term is the scattered spherical wave modulated by the so-called scattering amplitude $f(\theta)$. The scattering amplitude $f(\theta)$ is related to the differential cross section for elastic scattering within a solid angle element $d\Omega$ in the θ direction through

$$d\sigma = |f(\theta)|^2 d\Omega. \quad (2.3)$$

In order to understand the relation between the Eq. (2.2) and the asymptotic form of (2.1) it is convenient to compare them with the asymptotic form of the partial-wave expansion of the free-particle solution,

$$\exp(ipz) \approx \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\theta) \frac{\sin(pr - l\pi/2)}{pr}. \quad (2.4)$$

In the presence of a short-range central potential, the asymptotic expression for ψ can be expressed in a similar way as Eq. (2.4), the only difference being a phase in the argument of the sine function in each partial wave,

$$\psi \approx \sum_{l=0}^{\infty} A_l i^l (2l+1) P_l(\cos\theta) \frac{\sin(pr - l\pi/2 + \delta_l)}{pr}. \quad (2.5)$$

The coefficient A_l must be chosen so that this expression has the form (2.2). Doing so, one obtains $A_l = \exp(i\delta_l)$ and the following partial-wave expansion for the scattering amplitude

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \left[\frac{e^{2i\delta_l} - 1}{2ip} \right] = \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \left[\frac{1}{p \cot \delta_l - ip} \right], \quad (2.6)$$

where δ_l is known as the partial-wave phase shift. It is a phase in the scattered wave relative to the free solution, and its dependence with the energy provides useful information about the interacting potential.

At very low energies, in the case where the velocities of the particles under scattering are so small that their wavelengths are large compared with the typical range R of the potential (*i.e.* $pR \ll 1$), the phase shift $\delta_l(p)$ approaches zero like p^{2l+1} [16]. Since p is small, all the phases δ_l are small. According to (2.6) in the limit of low energies,

$$f_l \equiv \frac{1}{2ip} [\exp(2i\delta_l) - 1] \sim \frac{1}{2ip} [\exp(2ip^{2l+1}) - 1] \sim p^{2l}. \quad (2.7)$$

Therefore, the amplitude is dominated by S-wave and can be written as

$$f(\theta) \approx f_0 = \frac{e^{2i\delta_0} - 1}{2ip} = \frac{1}{p \cot \delta_0(p) - ip}. \quad (2.8)$$

It is worth pointing out that there are some examples where such behavior is not fulfilled. For instance, $f(\theta)$ is dominated by an $l \neq 0$ component if this channel contains a low-energy resonance. Another example is two identical fermions in a symmetric spin state, where the Pauli exclusion principle prevents them to be in S -wave.

At sufficiently low energies the effective-range function $K_l(p^2)$ [20, 21] is defined by

$$K_l(p^2) \equiv p^{2l+1} \cot \delta_l(p). \quad (2.9)$$

It is known to be an analytic function of p^2 in a large domain of the complex p -plane, for a large class of potentials. Its expansion in p^2 is called the effective-range expansion (ERE), and for $l = 0$, it is conventionally expressed in the form

$$K_0(p^2) = -\frac{1}{a} + \frac{1}{2}r_0p^2 - \frac{1}{4}\mathcal{P}_0p^4 + \dots, \quad (2.10)$$

where the coefficient a is known as the scattering length, r_0 as the effective range, and $\mathcal{P}_0$ as the shape parameter. The scattering length governs the zero-energy limit $p \rightarrow 0$ and from Eqs. (2.8) to (2.10) we see that the scattering amplitude is determined uniquely by a ,

$$f(\theta) = -a. \quad (2.11)$$

By inserting Eq. (2.11) into (2.3) and integrating it, we obtain the total scattering cross section at zero energy:

$$\sigma_0 = 4\pi a^2. \quad (2.12)$$

This is identical to the scattering cross section of an impenetrable sphere of radius a .

It is possible to establish a connection between a and the zero-energy limit of the asymptotic form of the wave function, Eq. (2.5). In this limit, the angular dependence of the wave function ψ is isotropic, since only the $l = 0$ contribution remains and $\delta_0 \sim p$. Let

FIGURE 2.1: Schematic figure of a ground and unbound state.

us first define the radial function $u^{(p)}(r) \equiv r\varphi_0(r)$ which satisfies

$$\frac{d^2 u^{(p)}}{dr^2} - [2m_r V(r) - p^2] u^{(p)} = 0 \quad (2.13)$$

One can easily find a solution to this equation, $w(r)$, valid in the limit $p \rightarrow 0$ and outside the range R of the potential, $w(r) = \lim_{p \rightarrow 0} u^{(p)}(r \geq R)$. It satisfies $w''(r) \approx 0$ whose solution is just a straight line, $w(r) = C(r + \alpha)$ with C and α constants.

On the other hand, the $l = 0$ term from Eq. (2.5) becomes

$$\lim_{p \rightarrow 0} \frac{\sin(pr + \delta_0)}{p} = \lim_{p \rightarrow 0} \frac{\cos \delta_0 (\sin pr + \cos pr \tan \delta_0)}{p} = r - a \quad (2.14)$$

The asymptotic form of $u^{(0)}(r) \approx w(r) = C(r - a)$ vanishes at $r = a$. Fig. 2.1 shows the zero-energy wave function $u^{(0)}(r)$ and $w(r)$ for bound and unbound states. Thus, the scattering length a can be generally interpreted as the value of r where the function $w(r)$ becomes zero.

The measurement of the cross section at zero energy determines the absolute value of the scattering length, but not its sign. Conventionally, the sign of the scattering length is set by the existence or non-existence of a bound state. When a system has a bound state at some near zero-energy the radial wave function $u^{(0)}(r)$ decays exponentially with increasing r in the forbidden zone $r > R$ (the zone at which classically the particles do not have access) with a decay length b , *i.e.*, $u^{(0)}(r) \approx \exp(-r/b)$. We could say that b is the size of the ground state and relates to the binding energy B as $b = 1/\sqrt{2m_r B}$. For a weakly-bound system $b \sim a$. Fig. 2.1 (A) shows the ground state of a bound system. The existence of a bound state implies a positive value of a . Conversely, when there is no bound state, $w(r)$ grows linearly with r , leading to a negative a as shown in Fig. 2.1 (B).

In the general theory of scattering, a quantity of interest is the probability amplitude for a transition from the initial to the final state, the so-called S-matrix:

$$S(\mathbf{p}', \mathbf{p}) = \delta(\mathbf{p}' - \mathbf{p}) - 2\pi i \delta(E_{p'} - E_p) T \equiv \langle \mathbf{p}' | V | \psi_{\mathbf{p}}^{(+)} \rangle, \quad (2.15)$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final c.m. momenta, respectively. When $\mathbf{p}^2 = \mathbf{p}'^2$, with $\mathbf{p} \cdot \mathbf{p}' = p^2 \cos \theta$ the scattering amplitude $f(\theta)$ is related to the T -matrix by

$$f(\theta) = -\frac{M}{4\pi} T. \quad (2.16)$$

2.2 Coulomb scattering

Charged particles, such as alpha particles, interact via electromagnetic forces. In addition, alpha particles are subjected to the short-range nuclear forces. Throughout this thesis we take advantage of a formalism that allows a clear separation of pure-Coulomb and Coulomb-modified nuclear terms. This Section is dedicated to the first part, reviewing the main elements of scattering of charged particles interacting only through the Coulomb potential.

For pure Coulomb scattering it is possible to calculate the differential scattering cross-section exactly, without the need of relying on the Born approximation or even on the partial wave expansion. The Schrödinger equation for the Coulomb potential $V(r) = Z_1 Z_2 e^2 / r$ and a positive energy $E = \mathbf{p}^2 / 2m_r$ takes the form

$$-\frac{1}{2m_r} \nabla^2 \psi + \frac{Z_1 Z_2 e^2}{r} \psi = \frac{\mathbf{p}^2}{2m_r} \psi. \quad (2.17)$$

The solutions can be expressed in terms of an in-state (one that develops out from a free state in the infinite past) with outgoing spherical waves $\chi_{\mathbf{p}}^{(+)}$ and an auxiliary mathematical one (an out-state which develops out backwards in time from a specific free state in the infinite future) with incoming spherical waves $\chi_{\mathbf{p}}^{(-)}$ [22]:

$$\chi_{\mathbf{p}}^{(+)}(\mathbf{r}) = e^{-\frac{1}{2}\pi\eta} \Gamma(1 + i\eta) M(-i\eta, 1, ipr - i\mathbf{p} \cdot \mathbf{r}) e^{i\mathbf{p} \cdot \mathbf{r}}, \quad (2.18)$$

$$\chi_{\mathbf{p}}^{(-)}(\mathbf{r}) = e^{-\frac{1}{2}\pi\eta} \Gamma(1 - i\eta) M(i\eta, 1, -ipr - i\mathbf{p} \cdot \mathbf{r}) e^{i\mathbf{p} \cdot \mathbf{r}}, \quad (2.19)$$

where η is the dimensionless quantity

$$\eta = \frac{Z_1 Z_2 e^2 \mu}{p}, \quad (2.20)$$

and $M(a, b, z)$ is the Kummer function (or well-known confluent hypergeometric function of first kind ${}_1F_1(a, b, z)$).

Since the Coulomb wave functions $\langle \mathbf{r} | \chi_{\mathbf{p}}^{(\pm)} \rangle = \chi_{\mathbf{p}}^{(\pm)}(\mathbf{r})$ form a complete set in the repulsive case, we can write an useful expression of the Coulomb propagator $\hat{G}_C^{(\pm)}$,

$$\hat{G}_C^{(\pm)}(E) \equiv \frac{1}{(E_p - \hat{H}_0 - \hat{V}_C \pm i\epsilon)} = 2m_r \int \frac{d^3q}{(2\pi)^3} \frac{|\chi_{\mathbf{q}}^{(\pm)}\rangle \langle \chi_{\mathbf{q}}^{(\pm)}|}{2m_r E - \mathbf{q}^2 \pm i\epsilon}. \quad (2.21)$$

The Sommerfield factor [16, 23]

$$C_\eta^{(0)2} \equiv \left| \chi_{\mathbf{p}}^{(\pm)}(0) \right|^2 = e^{-\pi\eta} \Gamma(1 + i\eta) \Gamma(1 - i\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}, \quad (2.22)$$

becomes an important parameter in theories containing Coulomb interactions and represents the probability density to find the two particles at zero separation.

The asymptotic behavior of the wave function for large r is

$$\chi_{\mathbf{p}}^{(+)}(\mathbf{r}) \approx e^{i\mathbf{p} \cdot \mathbf{r} + i\eta \ln[pr(1 - \cos\theta)]} + f(\theta) \frac{e^{ipr - i\eta \ln[pr(1 - \cos\theta)]}}{r}, \quad (2.23)$$

where

$$f(\theta) = -\frac{\Gamma(1 + i\eta)}{\Gamma(1 - i\eta)} \frac{\eta}{p(1 - \cos\theta)}. \quad (2.24)$$

The contribution of the logarithmic terms to the phases in Eq. (2.23) makes this wave function very different from that of Eq. (2.2). The reason arises from the $1/r$ dependence of the Coulomb potential. The very slow decrease of this long-range potential influences the particles even at infinity, leading to a divergent phase proportional to $\eta \ln(pr)$. Realistically one knows that the Coulomb potential is shielded at infinity by the existence of other particles. An appropriate shielding that makes the Coulomb potential vanish beyond a very large radius R_C is enough to eliminate this divergent phase and restore the form (2.2) [24]. This is straightforward for two-body systems, but becomes technically very challenging for systems with three and more particles [25].

Due to the symmetry around the $\mathbf{p}$ -direction, the partial-wave expansion of $\chi_{\mathbf{p}}^{(\pm)}$ is independent on the azimuthal angle ϕ , (*i.e.* $m = 0$):

$$\chi_{\mathbf{p}}^{(\pm)}(\mathbf{r}) = \sum_{l=0}^{\infty} i^l (2l+1) e^{i\sigma_l} R_l^\pm(pr) P_l(\cos\theta), \quad (2.25)$$

where σ_l is the partial-wave Coulomb phaseshift

$$\sigma_l = \frac{1}{2i} \ln \frac{\Gamma(1 + l + i\eta)}{\Gamma(1 + l - i\eta)} \equiv \arg \Gamma(1 + l + i\eta) \quad (2.26)$$

and $R_l^\pm(pr)$ is the solution of the radial Schrödinger equation in the repulsive channel [16],

$$R_l^\pm(pr) = e^{-\pi\eta/2} \frac{|\Gamma(1+l+i\eta)|}{(2l+1)!} (2pr)^l e^{ipr} {}_1F_1(1+l+i\eta, 2l+2, -2ipr). \quad (2.27)$$

The partial-wave form of the amplitude (2.24) can be written in terms of the Coulomb phase shifts,

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) \left[\frac{e^{2i\sigma_l} - 1}{2ip} \right] \quad (2.28)$$

with σ_l given by (2.26). However, one should keep in mind that this is just a formal expression, useful to analyze processes within a conserved angular momentum channel. For observable like cross-sections, that requires the sum over all angular momenta, there is no guarantee that the sum (2.28) converges. A feel for this may be grasped by looking at Eq. (2.24) at forward angles, $\theta \approx 0$. This has a close connection to the fact that the $1/r$ Coulomb potential is felt at infinity and therefore, scattering theoretically never ceases to happen [24].

2.3 Two-potential formalism

The scattering of alpha particles, which are under the combined influence of Coulomb and nuclear forces, may be simplified by the two-potential formalism [24]. Including the strong and Coulomb potential via the local operators $\hat{V}_S$ and $\hat{V}_C$ respectively, the system is described by the wave function $|\Psi_p\rangle$ satisfying

$$(\hat{H}_0 + \hat{V}_C + \hat{V}_S)|\Psi_p\rangle = E_p|\Psi_p\rangle. \quad (2.29)$$

We now define [26]

$$|\Psi_p^{(\pm)}\rangle \equiv \lim_{\epsilon \rightarrow 0} \frac{i\epsilon}{(E_p - \hat{H} \pm i\epsilon)} |\mathbf{p}\rangle, \quad (2.30)$$

where $\hat{H} = \hat{H}_0 + \hat{V}_C + \hat{V}_S$. The full Green's function $\hat{G}_{SC}^{(\pm)} \equiv 1/(E_p - \hat{H} \pm i\epsilon)$ is the resolvent operator to the Hamiltonian $\hat{H}$. Eq. (2.29) is the operation by which stationary eigenstates of $\hat{H}_0$, denoted by $|\mathbf{p}\rangle$, are mapped into specific eigenstates of $\hat{H}$.

There are some useful relations between the full, the Coulomb and the free Green's functions, where the free operator is $\hat{G}_0^{(\pm)} \equiv 1/(E_p - \hat{H}_0 \pm i\epsilon)$,

$$\hat{G}_{SC}^{(\pm)} = \hat{G}_0^{(\pm)} + \hat{G}_0^{(\pm)}(\hat{V}_C + \hat{V}_S)\hat{G}_{SC}^{(\pm)} \quad (2.31)$$

and

$$\hat{G}_{SC}^{(\pm)} = \hat{G}_C^{(\pm)} + \hat{G}_C^{(\pm)} \hat{V}_S \hat{G}_{SC}^{(\pm)}. \quad (2.32)$$

We use (2.31) in (2.30) to separate the full state $|\Psi_p^{(\pm)}\rangle$ into a free and scattered part, and (2.32) in (2.30) to separate it into a Coulomb and the scattered part,

$$|\Psi_p^{(\pm)}\rangle = \begin{cases} |\mathbf{p}\rangle + \hat{G}_0^{(\pm)}(\hat{V}_C + \hat{V}_S)|\Psi_p^{(\pm)}\rangle \\ |\chi_p^{(\pm)}\rangle + \hat{G}_C^{(\pm)}\hat{V}_S|\Psi_p^{(\pm)}\rangle. \end{cases} \quad (2.33)$$

Solving in terms of the corresponding asymptotic states we have

$$|\Psi_p^{(\pm)}\rangle = \begin{cases} [1 + \hat{G}_{SC}^{(\pm)}(\hat{V}_C + \hat{V}_S)]|\mathbf{p}\rangle, \\ [1 + \hat{G}_{SC}^{(\pm)}\hat{V}_S]|\chi_p^{(\pm)}\rangle. \end{cases} \quad (2.34)$$

From the first line of both Eqs. (2.33) and (2.34), the S-matrix element takes the standard form

$$S(\mathbf{p}, \mathbf{p}') = \langle \Psi_p^{(-)} | \Psi_{p'}^{(+)} \rangle = \delta(\mathbf{p} - \mathbf{p}') - 2\pi i \delta(E - E') \langle \mathbf{p} | (\hat{V}_C + \hat{V}_S) | \Psi_{p'}^{(+)} \rangle. \quad (2.35)$$

Where $T(\mathbf{p}, \mathbf{p}') \equiv \langle \mathbf{p} | (\hat{V}_C + \hat{V}_S) | \Psi_{p'}^{(+)} \rangle$ is the total transition amplitude.

Using the following equation

$$\langle \mathbf{p} | = \langle \chi_p^{(-)} | - \langle \chi_p^{(-)} | \hat{V}_C \hat{G}_0^{(+)} \quad (2.36)$$

and the first line of Eq. (2.33), $T(\mathbf{p}, \mathbf{p}')$ becomes

$$\langle \mathbf{p} | (\hat{V}_C + \hat{V}_S) | \Psi_{p'}^{(+)} \rangle = \langle \chi_p^{(-)} | \hat{V}_C | \mathbf{p}' \rangle + \langle \chi_p^{(-)} | \hat{V}_S | \Psi_{p'}^{(+)} \rangle. \quad (2.37)$$

At this point we can recognize

$$T_C(\mathbf{p}, \mathbf{p}') \equiv \langle \chi_p^{(-)} | \hat{V}_C | \mathbf{p}' \rangle \quad (2.38)$$

as the pure Coulomb scattering amplitude and

$$T_{SC}(\mathbf{p}, \mathbf{p}') \equiv \langle \chi_p^{(-)} | \hat{V}_S | \Psi_{p'}^{(+)} \rangle \quad (2.39)$$

as the strong scattering amplitude modified by Coulomb corrections.

It will be useful to express the full state in terms of Coulomb states alone,

$$|\Psi_p^{(\pm)}\rangle = \sum_{n=0}^{\infty} (\hat{G}_C^{(\pm)} \hat{V}_S)^n |\chi_p^{(\pm)}\rangle, \quad (2.40)$$

where we used the identity (2.32).

By inserting Eq. (2.40) into (2.39) we have

$$T_{SC}(\mathbf{p}, \mathbf{p}') = \sum_{n=0}^{\infty} \langle \chi_{\mathbf{p}}^{(-)} | \hat{V}_S (\hat{G}_C^{(+)} \hat{V}_S)^n | \chi_{\mathbf{p}'}^{(+)} \rangle. \quad (2.41)$$

The partial wave expansion for T_{SC} is found from Eq. (2.28) and imposing that $T(\mathbf{p}, \mathbf{p}')$ acquires the phase $\sigma_l + \delta_l$ relative to the free solution,

$$T_{SC}(\mathbf{p}, \mathbf{p}') = -\frac{4\pi}{m} \sum_{l=0}^{\infty} (2l+1) e^{2i\sigma_l} P_l(\cos \theta) \left[\frac{e^{2i\delta_l} - 1}{2ip} \right]. \quad (2.42)$$

In the above expression δ_l is the phase shift generated by the strong interaction in the presence of the Coulomb potential.

We end this section with the effective range expansion of T_{SC} . In the presence of the Coulomb potential, the K -function defined in (2.9) is no longer the analytic function to be expanded. Instead, the function $K_{SC,l}(p^2)$ suitable for the effective-range expansion is defined as [21, 27, 28]

$$K_{SC,l}(p^2) \equiv p^{2l+1} \frac{C_{\eta}^{(l)2}}{C_{\eta}^{(0)2}} \left[2\eta H(\eta) + C_{\eta}^{(0)2} (\cot \delta_l - i) \right], \quad (2.43)$$

where

$$C_{\eta}^{(l)2} = \binom{l+i\eta}{l} \binom{l-i\eta}{l} C_{\eta}^{(0)2} = e^{-\eta\pi} \frac{(1+l+i\eta)(1+l-i\eta)}{\Gamma(1+l)^2}, \quad (2.44)$$

and the H -function is related to the digamma function ψ by

$$H(\eta) \equiv \psi(i\eta) + \frac{1}{2i\eta} - \ln(i\eta). \quad (2.45)$$

The expansion of this Coulomb-modified effective-range function becomes

$$K_{SC,l}(p^2) = -\frac{1}{a_l} + \frac{1}{2} r_l p^2 \dots \quad (2.46)$$

With this, the S -wave part of the amplitude (2.42) is written as

$$T_{SC}(\mathbf{p}, \mathbf{p}') = -\frac{4\pi}{m} \left[\frac{C_{\eta}^2 e^{2i\sigma_0}}{K_{SC}(p^2) - 2k_C H(\eta)} \right]. \quad (2.47)$$

2.4 Natural length scale and systems with large scattering length

As shown in Chapter 3, the low and high energy scales of a system are the first ingredients in the construction of an effective field theory. They provide an expansion parameter used to identify the operators that are dominant at low energies and to estimate the size of loop contributions in the calculation of physical observables.

Before going to the main aspects in the construction of an EFT, let us review the concept of natural length scale ℓ associated with an interaction potential. This is set by the typical range R of the potential, *i.e.* $\ell \sim R$. For the interacting system, ℓ sets the high-momentum scale Q_H by the de Broglie relation, $Q_H = 1/\ell$. The low-momentum scale Q_L is established by the energy of the process.

The effective range expansion can be expressed in terms of the small parameter given by the ratio between the low- and high-momentum scales, $Q_L/Q_H \ll 1$,

$$K(p^2) = -\frac{1}{a} + \frac{1}{2}Q_H^2 \sum_{n=1}^{\infty} r_n \left(\frac{Q_L^2}{Q_H^2} \right)^n. \quad (2.48)$$

For each term to be smaller than the preceding one, the size of the coefficients must be of the order of the natural length scale, $r_n \sim \ell$. If the magnitude $|a|$ of the scattering length is comparable to ℓ , we say that a has a *natural size*. In the case $|a| \gg \ell$, we say that a is *unnaturally large*.

Systems having a bound state close to zero energy have a positive scattering length, but what defines a system with a large scattering length? It is a system where the size b of the bound state is considerably larger than the range R of the potential, and there is a considerable probability of finding the two particles in the bound state at a distance larger than the range of the forces which hold them together. Therefore such bound state is a rather weakly bound state.

For example, in the spin-singlet channel of the proton-neutron system the scattering length and the effective range are $a_s \approx -23.7$ fm and $r_s \approx 2.7$ fm [29]. Moreover, for nucleon energies much less than the pion mass the effective Lagrangian contains only contact interactions and the natural length scale of such interactions is established by the range of the one-pion exchange potential $\ell_\pi = 1/m_\pi \approx 1.4$ fm (see Chapter 3). Hence, we see that the effective range is comparable to the natural length scale. However, the scattering length a_s is much larger than ℓ_π . If a_s was positive, the bound singlet state of the system would have a correspondingly large value of R and hence a very low binding energy, $B_s \approx 100$ keV. However, there is no bound singlet state, thus, a_s is negative.

A bound state example is the deuteron, the spin-triplet channel of the proton neutron system, with $a_t \approx 5.4$ fm and $r_t \approx 1.7$ fm. The deuteron is seen as the simplest halo nuclei with binding energy $B_t \approx 2.2$ MeV, though the numbers are not dramatic as in the singlet channel.

Systems with large scattering length typically require a *fine-tuning* of some parameter in the potential. Even if a is large, we should expect r_0 to have a natural magnitude of order ℓ . However, there are situations with dynamical fine-tuning [30] that requires promotion of formally higher-order terms to the leading one. Analogous fine-tuning takes place in the $\alpha\alpha$ system.

Chapter 3

Nuclear Effective Theories

3.1 Introduction

In chapter 4, we deal with a system of two alpha particles at three-momentum less than an amount Q of about 20 MeV, the momentum at which the ^8Be ground state is reached. More precisely, we calculate the scattering amplitude of such process using an effective Lagrangian in which degrees of freedom at energies above those established by pion exchanges ($m_\pi \approx 140$ MeV) are supposedly integrated out. Such effective Lagrangian depends only on the relevant alpha field and derivatives thereof.

Because of the low-energy nature of the scattering process considered, effective field theory is the appropriate theoretical tool to treat it. It provides a framework to calculate physical observables exploiting the widely separated energy scales of physical systems.

The idea in constructing an effective field theory is not an attempt to reach a theory of everything, but to construct a theory that is appropriate to the energy scales of the experiments which we are interested in and take into account only the relevant degrees of freedom to describe physical phenomena occurring at such energy scales, while ignoring the substructure and degrees of freedom at higher energies.

The starting point is to identify those parameters which are very large compared with the energy E of the process of interest. If there is a single mass scale M in a hypothetical underlying theory, the interactions among the light states can be organized as an expansion in powers of E/M . The underlying idea behind such expansion comes from a local approximation of non-local operators, the latter being a remnant of integrations of the high-energy degrees of freedom at the Lagrangian level. The information about the high-energy dynamics is encoded in the couplings of the resulting low-energy Lagrangian.

Although such expansion contains an infinite number of terms, renormalizability is not a problem because the low-energy theory is specified by a finite number of couplings at a well-established power in p/Λ . This allows for an order-by-order renormalization.

Based on several reviews and lecture notes existing in the literature [31], in Section 3.2, we summarize the general ideas in the construction of an effective theory. In Section 3.3, we describe the application to nuclear physics.

3.2 General ideas

Depending on whether or not an underlying theory is known there are two ways to construct an EFT. When an underlying high-energy theory is known, an effective theory may be obtained in a *top-down* approach by a process in which high-energy effects are systematically eliminated. When an underlying high-energy theory is not known, it may still be possible to obtain an EFT by a *bottom-up* approach where relevant symmetries and naturalness constraints are imposed on candidate Lagrangians.

The *top-down* approach starts with a known theory and then systematically eliminates degrees of freedom associated with energies above some characteristic high-energy scale Q_H . One method to do that was proposed by Wilson and others in the 1970s [32]. It involves roughly two steps: First, the high-energy degrees of freedom are identified and integrated out in the action. These high-energy degrees of freedom are referred to as the high momenta, or *heavy*, fields. The result of this integration is an effective action that describes non-local interactions among the low-energy degrees of freedom (the low momenta, or *light*, fields). Ultimately, the resulting non-local effective action is addressed by expanding the effective action in a set of local operators:

$$S[\varphi_L] = S_0[\varphi_L] + \sum_i \int d^D x g_i \mathcal{O}^i(x), \quad (3.1)$$

where S_0 is the free action and the sum runs over all local operators $\mathcal{O}^i(x)$ allowed by relevant symmetries at low energies. The information on any heavy degrees of freedom is hidden in the couplings g_i .

The above expression involves an infinite number of operators and an infinite number of unknown coefficients. Nevertheless, in order to make any physical predictions, dimensional analysis allows us to determine the level of significance of each local operator, to keep some and reject others.

In units in which the action is dimensionless $\hbar = c = 1$, we start with the dimension of the light field φ_L , which can be obtained from the free action S_0 . After that, if an

operator $\mathcal{O}^i(x)$ has been determined to have units E^{N_i} , the coupling g_i has dimension $D - N_i$ because $d^D x$ has dimension $-D$ and the action must be dimensionless. One can define dimensionless coupling constants by $\lambda_i = \Lambda^{N_i-D} g_i$. The *naturalness* property tells us that these dimensionless couplings should take relative values of order 1 in a natural theory. This is in contrast with some theories like EFT for few-nucleon systems at energies below the pion mass [14], where the scattering length is unnaturally large. The same happens for the EFT of two-alpha system at energies below the pion mass (see Chapter 4). This will have serious implications in calculations of physical observables. Dimensional analysis gives for the i th term in (3.1) the following expression to estimate its size,

$$\int d^D x g_i \mathcal{O}^i(x) \sim \left(\frac{E}{\Lambda} \right)^{N_i-D}. \quad (3.2)$$

Another way to construct an EFT applies when the fundamental high-energy theory is not known. One simply begins with the operator expansion (3.1), introduces all operators allowed by low-energy symmetries and introduces couplings which depend inversely on the high energy scale Λ to the power appropriate for the dimension of the operator. An example of bottom-up construction is indeed the EFT for the two-alpha system, the main subject of the present work. A complete discussion about that is addressed in section four. Another example is, in the view of many physicists, the Standard Model itself as a low-energy approximation to a more fundamental theory, such as a unified field theory or string theory.

3.3 EFT for few-nucleon systems

Quantum chromodynamics (QCD) is the theory that deals with the strong interaction among quarks and gluons [33]. At low-energy scales the confinement property forces quarks and gluons to remain bound into hadrons such as the proton, the neutron, the pion or the kaon. Hadrons are the relevant degrees of freedom in the low-energy regime of QCD. The typical scale of QCD is of the order of 1 GeV, while nucleons in nuclear matter have typical momentum much smaller than the QCD scale. In the nucleon-nucleon interaction the low scales are the nucleon momentum $p \approx 280$ MeV and the pion mass $m_\pi \approx 140$ MeV, while the high scales would be the masses of the vector mesons *e.g.*, $m_\rho \approx 700$ MeV and higher resonances.

The energy gap between the typical scales of QCD and nuclear physics allows the construction of an EFT dealing with the low-energy regime of QCD. This EFT is called Chiral perturbation theory (ChPT) and is useful to deal with the interaction of hadrons

with pions. It was first suggested by S. Weinberg [8] and systematically developed by Gasser and Leutwyler [34] (see the review [35]).

The chiral effective Lagrangian consists of a set of operators, ranked based on the number of powers of the expansion parameters p/Λ_χ and m_π/Λ_χ , where Λ_χ is the chiral symmetry breaking scale of the order of 1 GeV. These operators are consistent with the (approximate) chiral symmetry of quantum chromodynamics (QCD) as well as other symmetries, like parity and charge conjugation.

In nuclear physics, the perturbative aspect of ChPT requires changes due to the non-perturbative aspects of nuclear processes. This so-called chiral EFT was proposed by Weinberg [9] and carried out by van Kolck and others [36]

However, in Weinberg's original work the power counting scheme proposal was shown not to be consistent, encountering difficulties coming from the large scattering length in the 1S_0 and 3S_1 NN scattering amplitudes. These difficulties were outlined by Kaplan, Savage and Wise [14, 37], where they developed a technique, which we present here, for computing properties of nucleon-nucleon interactions. Similar to this technique is the approach that we want to use to describe the two-alpha system.

For energies much less than the pion mass the only relevant degree of freedom is the non-relativistic nucleon field N of mass M and the appropriate expansion parameter is p/Λ , where Λ is set by the pion mass ($\Lambda \sim m_\pi$). The effective Lagrangian for non-relativistic nucleons must obey the symmetries of the strong interactions at low energies, *i.e.* parity, time-reversal and Galilean invariance. It only contains contact interactions, and ignoring spin and isospin indices, the effective Lagrangian has the following form:

$$\mathcal{L} = N^\dagger \left(i\partial_t + \frac{\nabla^2}{2M} \right) N + C_0(N^\dagger N)^2 + \frac{C_2}{8} \left[(NN)^\dagger (N \overleftrightarrow{\nabla}^2 N) + h.c. \right] + \dots \quad (3.3)$$

with the operator

$$\overleftrightarrow{\nabla} \equiv 1/2(\overleftarrow{\nabla} - \overrightarrow{\nabla}). \quad (3.4)$$

In the unit system where $\hbar = c = 1$ the action is dimensionless. Therefore, since in the kinetic term the operators $i\partial_t$ and $\nabla^2/2M$ have mass dimension 1, the nucleon field N has dimension 3/2. We now see that the lowest dimension contribution to NN scattering at low energies would come from the leading order operator $C_0(N^\dagger N)^2$ (dimension $D = 6$), where C_0 is a coupling constant of mass dimension -2. According to Eq. (3.2) we also see that the action term

$$\int d^D x \ C_0(N^\dagger N)^2 \sim \left(\frac{p}{\Lambda} \right)^2, \quad (3.5)$$

FIGURE 3.1: Four-nucleon vertex.

while the second term:

$$\int d^D x \ C_2 (NN)^\dagger (N \overleftrightarrow{\nabla}^2 N) \sim \left(\frac{p}{\Lambda}\right)^4. \quad (3.6)$$

From these equations we see that at low energies the first one dominates, so that in the limit where the energy goes to zero, the interaction of the lowest dimension remains and one can use the following effective Lagrangian,

$$\mathcal{L} = N^\dagger \left(i\partial_t + \frac{\nabla^2}{2M} \right) N + C_0 (N^\dagger N)^2. \quad (3.7)$$

The C_0 interaction in (3.7) is non-renormalizable and correspond to a singular delta function potential. It is represented by the four nucleon vertex in Fig. 3.1.

From quantum mechanics in the limit where the energy goes to zero, due to the effective range expansion, Eq. (2.10), the S-wave partial wave amplitude depends on a single parameter, the scattering length a ,

$$T(p) = -\frac{4\pi}{M} \frac{1}{p \cot \delta(p) - ip} = -\frac{4\pi}{M} \frac{1}{-1/a - ip}, \quad (3.8)$$

where p is the relative momentum.

In quantum field theory, the amplitude T is given by the sum of Feynman diagrams. It is the sum of the four nucleon vertex, the bubble diagram in Fig. 3.2 and the multi-loop Feynman diagrams in Fig. 3.3. The rules to computing them are simple: For each vertex we have

$$V = iC_0, \quad (3.9)$$

while the nucleon propagator is

$$i\Delta(q) = \frac{i}{q_0 - \mathbf{q}^2/2M + i\epsilon}. \quad (3.10)$$

For each bubble we need to incorporate the loop integral

$$I = \int \frac{d^4 q}{(2\pi)^4} \cdot \frac{i}{E/2 - q_0 - \mathbf{q}^2/2M + i\epsilon} \cdot \frac{i}{E/2 + q_0 - \mathbf{q}^2/2M + i\epsilon}, \quad (3.11)$$

FIGURE 3.2: One-loop Feynman diagram.

FIGURE 3.3: Higher order Feynman diagrams.

where $E = p^2/M$ is the energy flowing through the diagrams and p is the magnitude of the nucleon momentum in the center of mass (c.m.) frame.

Let us start with the tree-level S-wave amplitude which comes from Fig. 3.1,

$$iT_{tree} = -iC_0. \quad (3.12)$$

The expression for the one-loop diagram (which contains two vertices and a single bubble-like topology) is written as

$$iT_{one-loop}(p) = -iC_0 I(p) iC_0. \quad (3.13)$$

Note that the scattering amplitude given by the Feynman diagrams comes with a factor -1.

At this point counting rules are necessary to estimate the importance of loop diagrams to the scattering amplitude. If a characteristic momentum Q flows through the diagram in Fig. 3.2, the spatial components q_i of the four-momentum of each internal line scale as Q . On the other hand, since the energy typically scales as $E \sim Q^2/M$, the temporal component q_0 should scale as Q^2/M . The propagator (3.10) scales as M/Q^2 and the loop integration $\int d^4q$ as $Q^5/4\pi M$, where 4π is a geometrical factor. The estimated magnitude of the one-loop correction in Fig. 2 is thus $C_0^2 M Q / 4\pi$. This will be a perturbative correction when $C_0 \sim 4\pi/M\Lambda$ and thus obtain $C_0 M Q / 4\pi < 1$. In this case, each insertion would contribute an additional power of Q/Λ to the amplitude, which is small at low energy. The situation when $C_0 M Q / 4\pi \geq 1$ makes the physics non-perturbative.

Returning to the sum of Feynman diagrams, the expressions for the multi-loops are simple. By adding them all, the full scattering amplitude is given in terms of a geometric series of the factor $iC_0 I(p)$

$$T(p) = -C_0 [1 + iC_0 I(p) + (iC_0 I(p))^2 + \dots]. \quad (3.14)$$

From Eq. (3.8) we see that the radius of convergence of a momentum expansion of $T(p)$ depends on the size of the scattering length a . For example, when the scattering length has a natural size, $|a| \sim 1/\Lambda$, the expansion parameter $ap \ll 1$ allows writing the expression for the scattering amplitude Eq. (3.8) in the form

$$T(p) = \frac{4\pi a}{M} [1 - iap + (iap)^2 - (iap)^3 + \dots], \quad (3.15)$$

Reproducing it in EFT depends on the size of the coupling constant and on the subtraction scheme used to render all the diagrams finite.

3.3.1 Systems with scattering length of natural size

For the perturbative situation $a \sim 1/\Lambda$, the scenario is simple. In order to reproduce the momentum expansion Eq. (3.15), one can use the minimal subtraction (MS) scheme, the appropriate scheme to absorb the infinities that arise in perturbative calculations beyond leading order [38, 39]. Using dimensional regularization, one has for the one-loop integral in Eq. (3.11) the following expression (see Appendix A)

$$I(p) = -iM(-ME - i\epsilon)^{(D-3)/2} \Gamma\left(\frac{3-D}{2}\right) \frac{(\mu/2)^{4-D}}{(4\pi)^{(D-1)/2}}, \quad (3.16)$$

where μ is the renormalization mass and D is the dimensionality of the space-time. The MS scheme amounts to subtracting any $1/(D-4)$ pole before taking the $D \rightarrow 4$ limit. The integral Eq. (3.16) does not exhibit any such poles and so the result is simply

$$I_{MS}(p) = \left(\frac{M}{4\pi}\right) p \quad (3.17)$$

Since there are no poles at $D = 4$ in the MS scheme the coefficient C_0 is independent on the renormalization scale μ . Comparing Eqs. (3.14) and (3.15) we find for the coupling of the effective theory

$$C_0 = -\frac{4\pi a}{M}. \quad (3.18)$$

In this scheme $C_0 I(p) \sim p/\Lambda$ and the effective field theory is thus completely perturbative. The perturbative sum of Feynman graphs thus corresponds to a Taylor expansion of the scattering amplitude.

The MS scheme is appropriate for a perturbative renormalization, it is the case when the scattering length has a natural size. However, when this is unnaturally large, a non-perturbative renormalization is required. This situation is discussed in the next section.

3.3.2 Systems with large scattering length

Previously in Section 2.4 we have already mentioned systems with large scattering length. The proton-neutron system is the best known example with this condition. In the 1S_0 channel the scattering length $a_s \approx -23.7$ fm is much larger than the natural length scale of the system, $\ell_\pi \approx 1.4$ fm.

For these kind of systems, due to the large value of the scattering length, in the limit of zero energy, the absolute value $|ap|$ is no longer the expansion parameter. Consequently, the scattering amplitude Eq. (3.8) can no longer be written as a perturbative Taylor series. Instead, this corresponds to a non-perturbative situation, where the Feynman diagrams must be considered to all orders in the loop expansion. The fact that Eq. (3.14) forms a geometric series allows one to perform the sum, leading to

$$T(p) = -\frac{C_0}{1 - iC_0I(p)}. \quad (3.19)$$

Regarding renormalization, the large value of the scattering length turns the situation non-perturbative forcing one to look for a non-perturbative renormalization. It was shown in [40] that, in a non-perturbative situation, the MS scheme in dimensional regularization fails to reproduce the correct functional form of the scattering amplitude. The reason for this failure is known—contrary to perturbative renormalization, in the non-perturbative regime power divergences from loop integrals are crucial in driving the renormalization flow of the coupling constants. The usual dimensional regularization ignores all but the log-divergences [40].

A consistent non-perturbative renormalization was introduced by Kaplan, Savage and Wise [37], the so-called power-divergences subtraction (PDS) scheme. This involves subtracting from the dimensionally regulated loop integral not only the $1/(D-4)$ poles, but also poles in lower dimensions. This would make possible to recover the desired μ scale from the loop integral. To see that, let us apply it to the regulated integral (3.16). It has no pole at $D = 4$, but it does have a pole at $D = 3$, coming from the gamma function. This pole is related to the ultraviolet linear divergence present in the loop integration. So, in the $D \rightarrow 3$ limit we have the pole

$$\delta I = -i \frac{M\mu}{4\pi(3-D)}, \quad (3.20)$$

and then the subtracted integral, back to the $D \rightarrow 4$ limit, is

$$I_{PDS}(p) = \frac{M}{4\pi}(p - i\mu). \quad (3.21)$$

In this way, we have recovered the μ -dependence from the loop integrals. Putting this into the Eq. (3.19) we have

$$T(p) = -\frac{1}{1/C_0 - i(M/4\pi)(p - i\mu)}. \quad (3.22)$$

The above amplitude must be independent of the arbitrary parameter μ . This requirement strongly affects the values of the coupling constant whose dependence on μ is determined by the renormalization group equations, where the physical parameter a enters as a boundary condition.

In this case, we can obtain the μ -dependence of C_0 simply by comparing the amplitudes (3.22) and (3.8),

$$C_0(\mu) = \frac{4\pi}{M} \left(\frac{1}{\mu - 1/a} \right). \quad (3.23)$$

When only the lowest order C_0 interaction is included in the effective Lagrangian we see that there is no contribution to the effective range r_0 . One should be able to include corrections to the scattering amplitude by including higher order interactions in the effective theory, therefore improving the accuracy of the calculation.

3.3.3 Effective-range corrections

From quantum mechanics, the low-energy scattering amplitude is parametrized in terms of the scattering length a and the effective range r_0 . Treating the latter as a small correction, the amplitude has the following momentum expansion

$$T(p) = -\frac{4\pi}{M} \frac{1}{-1/a + r_0 p^2/2 - ip} = -\frac{4\pi}{M} \frac{1}{-1/a - ip} \left[1 - \frac{r_0/2}{-1/a - ip} p^2 + O(p^4/\Lambda^4) \right]. \quad (3.24)$$

The goal in this case is to show how a similar correction is obtained from EFT.

We saw that the leading C_0 term has dimension $D = 6$ and dominates in the limit where the energy goes to zero. Its contribution to the amplitude scales as p^{-1} and correspond to the expression (3.22). According to (3.2) the next to leading order operator

$$\mathcal{L}_2 = \frac{C_2}{8} (NN)^\dagger (N \overleftrightarrow{\nabla}^2 N) + h.c., \quad (3.25)$$

has dimension $D = 8$ and, for a natural behavior of the C_2 coupling, must be treated in first order of perturbation theory. This is equivalent to assume a natural size for the effective range, $r_0 \sim 1/\Lambda$. Its contribution is given by the sum of Feynman diagrams shown in Figure 3.4. Feynman rules give for the C_2 interaction the corresponding vertex

FIGURE 3.4: The next-to-leading order diagrams. The dotted vertices correspond to the C_0 interaction, while the square vertices correspond to the C_2 interaction.

$V_2 = i(C_2/2)(p^2 + p'^2)$, where $\mathbf{p}$ and $\mathbf{p}'$ are the incoming and outgoing momentum, respectively.

The first diagram shown in Figure 3.4A corresponds to a four nucleon vertex similar to Figure 3.1,

$$i\delta T^{(a)}(p) = -iC_2 p^2. \quad (3.26)$$

The contributions of the next two chains of bubble diagrams, Figures 3.4B and 3.4C, are the same, leading to

$$i\delta T^{(b+c)}(p) = -\frac{iC_0 iC_2}{1 - iC_0 I_0(p)} [p^2 I_0(p) + I_2(p)], \quad (3.27)$$

where $I_2(p)$ is defined using the more general loop integral

$$\begin{aligned} I_{2m}(p) &= (\mu/2)^{4-D} \int \frac{d^D q}{(2\pi)^D} q^{2n} \frac{i}{E/2 - q_0 - \mathbf{q}^2/M + i\epsilon} \cdot \frac{i}{E/2 + q_0 - \mathbf{q}^2/M + i\epsilon} \\ &= -iM(ME)^n (-ME - i\epsilon)^{(D-3)/2} \Gamma\left(\frac{3-D}{2}\right) \frac{(\mu/2)^{4-D}}{(4\pi)^{(D-1)/2}}. \end{aligned} \quad (3.28)$$

The contribution of all diagrams in Figure 3.4D is

$$i\delta T^{(d)}(p) = -\frac{(iC_0)^2 iC_2}{[1 - iC_0 I_0(p)]^2} I_0(p) I_2(p). \quad (3.29)$$

The sub-leading contribution to the scattering amplitude comes from the sum of these three partial results,

$$\delta T(p) = -\frac{C_2}{[1 - iC_0 I_0(p)]^2} [p^2 - iC_0(p^2 I_0(p) - I_2(p))]. \quad (3.30)$$

To assure a correct renormalization the divergent integrals I_0 and I_2 must be regularized within the PDS scheme. After that, I_0 corresponds to Eq. (3.21) and the new divergent integral I_2 becomes

$$I_2(p) = p^2 \frac{M}{4\pi} (p - i\mu). \quad (3.31)$$

Therefore, using PDS, the factor $(p^2 I_0 - I_2)$ in the numerator of (3.30) vanishes and the sub-leading contribution becomes

$$\delta T(p) = -\frac{C_2 p^2}{[1 - i(C_0 M/4\pi)(p - i\mu)]^2}. \quad (3.32)$$

Once again δT is independent of the renormalization mass μ and the μ -dependence of the couplings is determined by this fact. We now expect that the leading and sub-leading contributions to the scattering amplitude, Eqs. (3.22) and (3.32) respectively, allow us to reproduce the expansion (3.25). Putting them together we have

$$\begin{aligned} T(p) &= T_0 \left[1 + \frac{\delta T}{T_0} \right] \\ &= -\frac{4\pi}{M} \frac{1}{(4\pi/MC_0) - \mu - ip} \left[1 + \frac{C_2 p^2}{C_0 [1 + (MC_0/4\pi)(\mu + ip)]} \right]. \end{aligned} \quad (3.33)$$

Comparing it with the expansion (3.25) we obtain for C_0 basically the same expression (3.23), whereas for the new coupling constant C_2 , the μ -dependence comes with a dependence on the effective range r_0 ,

$$C_2(\mu) = \frac{4\pi}{M} \left(\frac{1}{\mu - 1/a} \right)^2 \frac{r_0}{2}. \quad (3.34)$$

3.3.4 Coulomb corrections

In the previous section we saw the corrections to the amplitude (3.22) that arise when considering the higher order C_2 interaction in first order of perturbation theory. Now, we want to show how to include electromagnetic interactions for cases where the scattered particles are charged. We follow the analysis made by Kong and Ravndal [15] for the proton-proton system.

FIGURE 3.5: Lowest order Coulomb correction to the four-nucleon vertex.

Electromagnetic interactions are included by minimal substitution, $\partial_\mu \rightarrow \partial_\mu + ieA_\mu$, where A_μ is the electromagnetic four-potential and e is the electric charge. An appropriate gauge choice is essential for a straightforward treatment. We choose the Coulomb gauge, defined by the gauge condition $\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0$, in order to allow a separation between Coulomb and transverse radiative photons. We start from the effective Lagrangian (3.7), in this section renamed to $\mathcal{L}_0$. Changing the derivatives according to the minimal substitution and adding the electromagnetic Lagrangian $\mathcal{L}_0^\gamma$ (see Ref. [41] for scalar electrodynamics) yield

$$\mathcal{L} = \mathcal{L}_0(N) + \mathcal{L}_0^\gamma(A) + \mathcal{L}_{int}, \quad (3.35)$$

where

$$\mathcal{L}_{int} = -eA_0(N^\dagger N) + i\frac{e}{M}N^\dagger(\mathbf{A} \cdot \nabla N) - \frac{e^2}{2M}\mathbf{A}^2(N^\dagger N). \quad (3.36)$$

The first term of $\mathcal{L}_{int}$ corresponds to the interaction among nucleons and Coulomb photons coupling through the electric charge. The second and third terms correspond to the interaction among nucleons and transverse photons coupling additionally through the proton velocity and the electric charge respectively. In comparison to the Coulomb photons, the effects of the transverse photons are negligible in both NN [15] and $\alpha\alpha$ [12] scattering.

As result, Kong and Ravndal found that each photon exchange is proportional to the Sommerfeld parameter $\eta = k_C/p$, where k_C is Coulomb scale. For instance, the Coulomb correction for the four-nucleon vertex shown in Fig. 3.1 is given by the Feynman diagram in Fig. 3.5. Counting rules give for this term

$$\delta T(p) = C_0 \int \frac{d^3q}{(2\pi)^3} \frac{e^2}{q^2 + \lambda^2} \frac{M}{p^2 - (\mathbf{p} - \mathbf{q})^2 + i\epsilon} \sim C_0 \frac{k_C}{p} \quad (3.37)$$

where $\lambda \rightarrow 0$ is the photon mass which acts as an infrared regulator. For one more Coulomb photon exchange in the four-nucleon vertex, counting rules give a contribution of the order $C_0\eta^2$. Thus, for momentum $p \leq k_C$ the Sommerfeld parameter $\eta = k_C/p \geq 1$, and the Coulomb repulsion must be included in a non-perturbative way.

In non-perturbative cases, the Coulomb propagator (2.21) results from the infinite sum

FIGURE 3.6: Coulomb propagator as a infinite sum of Coulomb photon exchanges [15].

FIGURE 3.7: Coulomb-distorted Feynman diagrams. The shaded bubble represents a infinite sum of non-perturbative Coulomb contributions.

of Feynman diagrams shown in Figure 3.6, with zero, one, two, etc., photon exchanges [15]. Such diagrams result from the iteration of the integral equation

$$\begin{aligned}\hat{G}_C^{(\pm)} &= \hat{G}_0^{(\pm)} + \hat{G}_0^{(\pm)} \hat{V}_C \hat{G}_C^{(\pm)} \\ &= \hat{G}_0^{(\pm)} + \hat{G}_0^{(\pm)} \hat{V}_C \hat{G}_0^{(\pm)} + \hat{G}_0^{(\pm)} \hat{V}_C \hat{G}_0^{(\pm)} + \dots\end{aligned}\quad (3.38)$$

On the other hand, the two potential formalism gives for T_{SC} the expression (2.41), which can be expressed as the diagrammatic form shown in Figure 3.7, where the shaded bubbles represents the same infinite sum of non-perturbative Coulomb contributions of Figure 3.6. In the $\alpha\alpha$ system the Sommerfeld parameter is large, therefore, the Coulomb repulsion must be included in a non-perturbative way.

The Two-alpha-particle System

The alpha particle, the nucleus of the helium (^4He) atom, is made out of two protons and two neutrons. Its ground state has a zero total angular momentum $J^\pi = 0^+$, and it can be a positive-parity mixture of three 1S_0 , six 3P_0 and five 5D_0 orthogonal states [42]. It must be clear that, at low energies, the S -wave is the dominant part of the wave function, with a small D -wave and almost negligible P -wave contributions. Then at low energies, the alpha particle is essentially in S -wave and its space wave function is symmetric under the interchange of either two protons, or two neutrons. In the ground state, the constituents of the alpha particle are strongly bound as shown in Figure 4.1. It shows the average binding energy per nucleon of common isotopes, where for the ^4He the energy is relatively high.

FIGURE 4.2: Energy levels of ${}^4\text{He}$ are plotted on a vertical scale giving the c.m. energy, in MeV, relative to its ground state. Horizontal lines representing the levels are labeled by the level energies and values of total angular momentum, parity, and isospin (J^π, I) [44].

In addition to the ground state, Figure 4.2 shows the excited states of the ${}^4\text{He}$. The first three $I = 0$ states, 0^+ , 0^- and 2^- , are observed to have energies above 20 MeV in the center of mass frame relative to its ground state.

The present amount of ${}^4\text{He}$ in the universe is mostly attributed to the Big Bang nucleosynthesis, the process by which the first nuclei were formed about three minutes after the Big Bang. It was then created the hydrogen and helium to be the content of the first stars. With the formation of the stars, the creation of ${}^4\text{He}$ continues to day through hydrogen fusion. Also in stars, other heavier nuclei are formed from preexisting hydrogen and helium nuclei. Besides nucleosynthesis alpha particles may emerge from alpha decay [20] of heavy radioactive nuclei. This decay is favorable for nuclei of mass number A above 191.

The strong binding of alpha particles and the fact that they may emerge from a heavy nucleus led some investigators to conjecture that alpha particles also exist as stable substructures inside these heavy nuclei before they decay. They suggested that the binding energies of some $Z = N$ (that is, equal number of protons and neutrons), with Z even, may be described by a simple model with an integer number of alpha particles. Although this is generally disputed (alpha particles can not maintain their identity for a very long time inside condensed nuclear matter), Wheeler [45] and others spoke in terms of relative average lifetimes at which alpha particles maintain their identities, at least as far as the low excited states of the nucleus are concerned.

In a modern approach, EFT has been used within the same spirit. It has been used to deal with halo nuclear states [46, 47]: a nucleus consisting of a tightly bound core

and one or more weakly bound (valence) nucleons. In a first approximation, the core is treated as an explicit degree of freedom and the EFT is written in terms of contact interactions between the valence nucleons and the core. Other effects like the size and shape of the core are encapsulated in a derivative expansion of local operators. Systems like the ${}^7\text{Be}$ core with a weakly bound proton is considered a halo nuclear state forming the ${}^8\text{B}$ nucleus [48].

As in the past, alpha particles have been received special attention. In halo nuclear states, like neutron-alpha ($n\alpha$) and proton-alpha ($p\alpha$) systems, alpha particles are considered as a core whenever the energy of the valence nucleons is smaller compared with the excitation energy of the alpha particles. The nuclear interaction between nucleons and alpha particles have been studied separately in neutron-alpha [30, 49] and proton-alpha [50] scattering, while the $\alpha\alpha$ interaction has been studied by Higa, Hammer, and van Kolck [12]. These interactions are important input to systems with more than two alpha particles in multi-body calculations, like the triple-alpha (3α) reaction describing the formation of ${}^{12}\text{C}$ via the Hoyle state.

As in [12], we work on the problem of the $\alpha\alpha$ system readdressing the scattering observables and its low-energy resonance identified as the ${}^8\text{Be}$ ground state. This, is a $(0^+, 0)$ state and has a c.m. energy $E_R \approx 0.1$ MeV above the $\alpha\alpha$ threshold (the threshold for break-up into two alpha particles), with a narrow decay width of $\Gamma \approx 6$ eV.

In Section 4.2, we present the strong effective Lagrangian with momentum-dependent contact interactions and discuss how electromagnetic interactions are included. In Section 4.3, we compute the $\alpha\alpha$ amplitude to match the amplitude under the effective-range parametrization. The experimental situation is discussed in Section 4.4. Finally, the analysis of the Wigner bound is addressed in Section 4.5.

4.2 EFT with Coulomb interactions

The energy of the ${}^8\text{Be}$ ground state $E_R = 0.1$ MeV is determined from alpha-alpha scattering across the resonance region, and is much smaller than the alpha-particle excitation energy $E_x \approx 20$ MeV. Thus, an EFT may be constructed to calculate observables at momentum around the resonance region. The low-momentum scale is set by the energy of such process, $Q_L \sim k_R = \sqrt{m_\alpha E_R} \approx 20$ MeV, where we used the mass of the ${}^4\text{He}$, $m_\alpha \approx 3.7$ GeV, while the breakdown momentum scale Q_H is established by the first internal degrees of freedom that appear within the alpha particle. These include nucleons at momentum $\sqrt{m_N E_x} \approx 140$ MeV, and pions at momentum of the order of the pion mass

$m_\pi \approx 140$ MeV. Thus, an estimation is that this scale is $Q_H \sim \sqrt{m_N E_x} \sim m_\pi \approx 140$ MeV.

At energies below the pion mass, each alpha particle may be represented by a scalar-isoscalar field Φ . As before, other effects like nucleus deformation are encapsulated in a derivative expansion. This EFT provides a controlled expansion of observables, where the small parameter is given by the ratio between the low- and high-momentum scales $Q_L/Q_H \sim 1/7$.

Far below the alpha excitation level, the interactions between two alphas are only in the S -wave channel. Thus, the proposed EFT for alpha particles interacting through contact interactions has the following strong effective Lagrangian:

$$\mathcal{L} = \Phi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m_\alpha} \right) \Phi + C_0 (\Phi^\dagger \Phi)^2 + \frac{C_2}{8} \left[(\Phi \Phi)^\dagger \left(\Phi \left(\overleftrightarrow{\nabla} \right)^2 \Phi \right) + h.c. \right] + \dots, \quad (4.1)$$

where C_0 and C_2 are coupling constants. The ellipsis represent higher derivative operators.

The difference from Kong and Ravndal's work [15] is due to the existence of the low-energy resonance in the $\alpha\alpha$ system. To observe that, the two coupling constants must be considered in leading order [12]. This is different from [15], where the authors considered C_0 as leading order and C_2 in first order of perturbation theory.

As in Chapter 3, the Coulomb repulsion comes in a non-perturbative way. The charge of each alpha particle is $Z_\alpha = 2$ and the reduced mass is $m_r = m_\alpha/2$, with $m_\alpha = 3.7$ GeV. The Coulomb momentum scale is $k_C = Z_\alpha^2 \alpha_{em} m_r \approx 60$ MeV. At momentum k smaller than k_C the Sommerfeld parameter $\eta = k_C/k > 1$ and the Coulomb repulsion must be included in a non-perturbative manner. After minimal substitution, the Coulomb gauge choice allows to separate the Coulomb and transverse photons. Neglecting the higher-order effects of the latter [12, 15], only the Coulomb repulsion plus the strong interaction appear in the equations of the two-potential formalism developed in Section 2.3. Accordingly, the T -matrix element can be written as the sum of two parts

$$T = T_C + T_{SC}, \quad (4.2)$$

where T_C and T_{SC} are the pure-Coulomb and Coulomb-modified strong scattering amplitude, respectively.

4.3 Calculation of the scattering amplitude

Here, we present the amplitude T_{SC} derived from our EFT leaving the details in the Appendices B and C. The resulting T_{SC} has the same form as the parametrized formula (2.47). The corresponding expression for the $K_{SC}(p^2)$ is

$$K_{SC}(p^2) = -\frac{4\pi}{m_\alpha} \left[\frac{(1 + \frac{C_2}{2} I_3)^2}{\left[C_0 - 2C_2(k_C^2 + k_C\mu) - (\frac{C_2}{2})^2 I_5 \right] + \left[C_2 + (\frac{C_2}{2})^2 I_3 \right] p^2} - I_1 \right], \quad (4.3)$$

where μ is the renormalization scale and I_1 , I_3 and I_5 are divergent integrals (labeled by its degree of divergence) defined by the general expression,

$$I_n = m_\alpha \int \frac{d^3q}{(2\pi)^3} \frac{2\pi\eta_q}{e^{2\pi\eta_q} - 1} q^{n-3}. \quad (4.4)$$

In order to do a consistent matching with the effective-range parameters (and thus a correct renormalization of the theory), the right side of (4.3) must be expanded. The only way to do that seems to be exploiting the I_3 and I_5 divergences. Since I_5 is more divergent than I_3 , it is possible the construction of an expansion parameter. However, to do that, we need to evaluate the scale μ at infinity, in the regularized expressions for I_3 and I_5 , Eqs. (C.46) and (C.51) respectively. Assuming that this is possible, we obtain the renormalization conditions

$$\frac{m_\alpha}{4\pi a_0} = \frac{(1 + \frac{C_2}{2} I_3)^2}{\left[C_0 - 2C_2(k_C^2 + k_C\mu) - (\frac{C_2}{2})^2 I_5 \right]} - I_1, \quad (4.5)$$

$$\frac{m_\alpha r_0}{8\pi} = \left(\frac{m_\alpha}{4\pi a} + I_1 \right)^2 \left(\frac{1}{I_3} - \frac{1}{I_3(1 + C_2 I_3/2)^2} \right). \quad (4.6)$$

These equations close the calculation of the $\alpha\alpha$ amplitude. The Coulomb-modified effective-range parameters fix our EFT parameters and allows our comparison with the experimental data.

4.4 Comparison to data

In this Section we address the experimental situation regarding the scattering of alpha particles at low energies and we present the theoretical phase shift derived from the theory.

	a_0 (10^3 fm)	r_0 (fm)	$\mathcal{P}_0$ (fm^3)
LO	-1.8	1.083	-
NLO	-1.92 ± 0.09	1.098 ± 0.005	-1.46 ± 0.08

TABLE 4.1: Coulomb-modified effective-range parameters determined by Higa *et al.* [12] in LO and NLO.

FIGURE 4.3: S-wave phase shift δ_0 as function of the laboratory energy E_{Lab} . The solid and dashed line represent the EFT results in LO and NLO, respectively. While the solid circles with error bars represent the experimental phase shift [51].

In [12] were derived the Coulomb-modified effective-range parameters. Using the position of the poles the authors computed a_0 and r_0 at leading order (LO). At next to leading order (NLO), the shape parameter $\mathcal{P}_0$ was determined from a global χ^2 -fit to data [12]. Table 4.1 shows these parameters in LO and NLO. Here we use these parameters to compute the phase shift through the parametrization of the amplitude (2.42)

$$T_{SC}(p) = -\frac{4\pi}{m_\alpha} \frac{e^{2i\sigma_0}}{k(\cot \delta_0 - i)} = -\frac{4\pi}{m_\alpha} \frac{C_\eta^2 e^{2i\sigma_0}}{K_{SC}(p^2) - 2k_C H(\eta)}, \quad (4.7)$$

where

$$K_{SC}(p^2) = -\frac{1}{a_0} + \frac{1}{2}r_0 p^2 - \frac{1}{4}\mathcal{P}_0 p^4 \dots \quad (4.8)$$

Figure 4.3 shows the experimental phase shift fitted by the LO and NLO curves. The first one matches the data around the resonance region, but above 1 MeV this moves away. The NLO curve reaches better results. The low predictive power of the LO curve in comparison to the NLO curve are in line with the theoretical error expected.

4.5 Analysis of the Wigner bound

In this Section, we present the analysis of the Wigner bound to this specific system. In Section 4.3 we presented the resulting expression for the $\alpha\alpha$ amplitude computed from

our EFT and the renormalization conditions (4.5) and (4.6). To obtain such conditions, we look in a formal way the limit μ to infinity.

In this limit, given the leading behavior of (4.4), that is, $I_n \sim \mu^n$, we have

$$\frac{(I_1)^2}{I_3} \sim \frac{1}{\mu}, \quad (4.9)$$

and thus the first term on the second parenthesis of Eq. (4.6) vanishes. Consequently, in this limit

$$\frac{m_\alpha r_0}{8\pi} \rightarrow -\frac{1}{I_3} \left(\frac{I_1}{1 + C_2 I_3/2} \right)^2, \quad (4.10)$$

which means that $r_0 \leq 0$, independent of the value of C_2 as long as C_2 is a real number. Even when this restriction was found for the $\mu \rightarrow \infty$ limit, the r_0 sign remains unchanged for other values of μ because of its assumed μ -independent. A similar result concerning the negative value of r_0 was found first by Phillips *et al.*[40]. Using the same strong potential but leaving out the electromagnetic interactions, the authors studied the scattering of two identical bosons. They obtained similar renormalization conditions and, like us, the effective range parameter proved to be negative.

The negative value of the effective range can be related to the Wigner's causality bound [13] which says that, in cases of zero range potentials, the effective range should be negative, $r_0 < 0$. Wigner derived this fundamental rule based on the principle of causality, the statement saying that the scattering wave cannot leave the scattering center before the incident wave reaches it.

The result regarding $r_0 < 0$ seems to contradict the positive r_0 derived from the experiment (see Table 4.1). However, we must not forget that the renormalization conditions and the resulting negative effective range arise from the evaluation of the scale μ at infinity, which is not entirely clear. In this sense, we can not take this result as a fact. Instead, we should look for other value of the renormalization scale μ at which we obtain a consistent renormalization condition and a positive effective range. For instance, the renormalized coupling $C_2(\mu)$ starts developing an imaginary part for relatively low values of μ , $\mu \sim 100$ MeV. That may be an indication that one should restrict μ to a certain range. Physically, that amounts to take small, but finite size of the $\alpha\alpha$ interaction. A similar conclusion, though made in a wave-function language, was done in [52]. Concerning the Wigner bound part, our study indicates that the question is still open and deserves further investigations.

Chapter 5

Conclusions

In this thesis we deal with the problem of two interacting alpha particles, which are under the combined influence of the electromagnetic and strong forces. To handle such system, we highlight selected important aspects from quantum mechanics and quantum field theory. We start with a review of the general theory of elastic scattering, with emphasis on processes at relatively small energies. We address the main aspects necessary to construct an effective field theory. Furthermore, we present specific examples of how these ideas apply in nuclear physics.

In order to explore the low-energy features of two alpha particles, we propose an effective field theory in which the only degrees of freedom are the alpha particles themselves. The propose theory was provided with an effective Lagrangian which consists of a derivative series of local operators representing the strong interactions of two alpha particles. The effects of the electromagnetic interactions have also been included. The goals were, to describe the low-energy side of the scattering, with emphasis on the resonance of two alphas corresponding to the ground state of Beryllium-8, the intermediate state in the triple-alpha reaction leading to the ^{12}C formation. Our EFT amplitude with momentum-dependent interactions shows convergence to scattering data in a similar way as in the previous work [12].

We have taken into account only the first two lowest-order operators of the derivative series to construct the effective Lagrangian when looking for a non-perturbative renormalization for the respective coupling constant. To carry out the renormalization, it was required the computation of the $\alpha\alpha$ amplitude to match the parametrized formula which is written in terms of the effective-range parameters. These latter can be determined from a fit to scattering data. However, a naive analysis showed that the effective

range parameter should be negative, which is incompatible with its positive experimental value. A more careful study shows that $C_2(\mu)$ develops an imaginary part around $\mu \sim 100$ MeV, which may invalidate the previous naive analysis.

The results outlined in the previous section suggest that care should be taken while taking the limit $\mu \rightarrow \infty$ in the renormalization conditions, and the issue of proper non-perturbative renormalization conditions for EFT with Coulomb forces is still an open question.

Appendix A

Dimensional regularization

The loop integral (3.11) has divergences that we need to regularize. Here, we calculate this integral using the well known dimensional regularization scheme. The idea is to compute the Feynman diagrams as an analytic function of D , the space-time dimension. Integrating out the temporal coordinate we obtain

$$\begin{aligned} I(p) &= \int \frac{d^4 q}{(2\pi)^4} \frac{i}{E/2 - q_0 - \mathbf{q}^2/2M + i\epsilon} \cdot \frac{i}{E/2 + q_0 - \mathbf{q}^2/2M + i\epsilon} \\ &= M \int \frac{d^3 q}{(2\pi)^3} \frac{-i}{q^2 - ME - i\epsilon}. \end{aligned} \quad (\text{A.1})$$

The remaining integral shows a linearly surface divergence which should be seen explicitly after regularization. Then, using dimensional regularization it is written as

$$I(p) = M(\mu/2)^{4-D} \int \frac{d^{D-1} q}{(2\pi)^{D-1}} \frac{-i}{q^2 - ME - i\epsilon}. \quad (\text{A.2})$$

For this, we can use the formula

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2m}}{(\ell^2 + \Delta)^n} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(\frac{D}{2} + m)\Gamma(n - \frac{D}{2} - m)}{\Gamma(\frac{D}{2})\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{D}{2} - m}. \quad (\text{A.3})$$

It will be useful to consider the simplest case $n = 1$

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2m}}{(\ell^2 + \Delta)} = \frac{1}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \left(\frac{1}{\Delta}\right)^{1 - \frac{D}{2} - m}. \quad (\text{A.4})$$

With this, the regulated integral (A.1) becomes

$$I(p) = -iM(-ME - i\epsilon)^{(D-3)/2} \Gamma\left(\frac{3-D}{2}\right) \frac{(\mu/2)^{4-D}}{(4\pi)^{(D-1)/2}}. \quad (\text{A.5})$$

At $D = 4$ we do not find any pole. This drawback arises from the use of dimensional regularization in this type of integrals. To see that, let us analyze the formula (A.3). To resolve the more general integral on the left side of the Eq. (A.3) we use the integral formulation of the Beta function

$$B(x, y) = \int_0^1 \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (\text{A.6})$$

with $x = d/2 + m$ and $y = n - d/2 - m$. However, this integral formulation can be used consistently only when $\Re[x] > 0$ and $\Re[y] > 0$. In a four-dimensional space with $m = 0$ and $n = 2$, $x > 0$ and in the $y \rightarrow 0$ limit the integral formulation of the beta function is still valid. So that, when $d \rightarrow 4^-$ we see quickly that the integral on the left side of (A.3) has a logarithmic surface divergence, and that the gamma function on the right side goes to $\Gamma(0^+)$. This pole corresponds to the logarithmic divergence in the momentum integral.

Now let us consider $m = 2$ and like before $n = 2$. The momentum integral on the left side of (A.3) has a quadratic divergence while on the right side, the gamma function goes to $\Gamma(-1)$. As before, we could say that this pole corresponds to the quadratic divergence in the momentum integral. Or more generally, we could say that the isolated poles of the gamma function at $z = 0, -1, -2, \dots$, correspond to the logarithmic, quadratic, quartic, and so on, divergences in the momentum integral respectively.

However, dimensional regularization are designed to retain only the logarithmic divergences. For example, the previous quadratic divergence was related to the gamma function evaluated at $z = -1$. Moreover, the formula $z\Gamma(z) = \Gamma(z+1)$ relates both poles, $\Gamma(0)$ and $\Gamma(-1)$ by a simple constant. With this, we can always express any pole of the gamma function as a constant times $\Gamma(0)$, the pole corresponding to a logarithmic divergence. Consequently, for quadratic, quartic and in general for even surface divergences, the resulting regulated integral retain only the logarithmic divergence.

Looking carefully, for divergences greater than the logarithmic one, dimensional regularization simply change the extra internal momenta by the external momenta. This is clear if we regularize the more divergent loop integral (3.28),

$$\begin{aligned} I_{2m}(p) &= \int \frac{d^4 q}{(2\pi)^4} q^{2m} \frac{i}{E/2 - q_0 - \mathbf{q}^2/2M + i\epsilon} \cdot \frac{i}{E/2 + q_0 - \mathbf{q}^2/2M + i\epsilon} \\ &= M \int \frac{d^3 q}{(2\pi)^3} q^{2m} \frac{-i}{q^2 - ME - i\epsilon}. \end{aligned} \quad (\text{A.7})$$

Using the formula (A.4) we obtain

$$I_{2m}(p) = -iM(ME)^m (-ME - i\epsilon)^{(D-3)/2} \Gamma\left(\frac{3-D}{2}\right) \frac{(\mu/2)^{4-D}}{(4\pi)^{(D-1)/2}}. \quad (\text{A.8})$$

Note that the only difference with (A.5) is the factor $(ME)^m = p^{2m}$. Thus, dimensional regularization effectively changed the internal momenta q^{2m} by the external momentum p^{2m} .

All these properties appear when we force the formulas to be valid outside of their domains. For example, it is the case when we used the formulas to obtain $\Gamma(-1)$ for the quadratic divergent integral.

In our case, this formulation leads to some undesirable issues. When we evaluate the formula (A.4) at an odd dimension no poles appear. For example, when we evaluate the regulated expression (A.5) at $D = 4$, we are using the formulas for the odd dimension $D - 1$, and in the right side no poles appear. This shows explicitly why dimensional regularization is not the appropriate scheme to deal with linearly or in general with odd surface divergences. Even when we can associate a pole for even divergences, we can not do the same for odd surface divergences.

The form of the non-relativistic propagators like (3.10) generates odd power divergences after the q_0 integration. Dimensional regularization fails when attempting to regularize this type of integral. The same applies for loop integrals when the Coulomb interaction is switched on.

Appendix B

The Coulomb modified scattering amplitude

Here, we calculate the Coulomb modified part of the scattering amplitude Eq. (4.2), using the expression given by the Eq. (2.41),

$$T_{SC}(\mathbf{p}, \mathbf{p}') = \sum_{n=0}^{\infty} \langle \chi_{\mathbf{p}}^{(-)} | \hat{V}_S (\hat{G}_C^{(+)} \hat{V}_S)^n | \chi_{\mathbf{p}'}^{(+)} \rangle. \quad (\text{B.1})$$

The states $|\chi_{\mathbf{p}}^{(+/-)}\rangle$ are the (outgoing/incoming) Coulomb state of momentum $\mathbf{p}$ defined in the Eqs. (2.18) and (2.19),

$$\chi_{\mathbf{p}}^{(+)}(\mathbf{r}) = e^{-\frac{1}{2}\pi\eta}\Gamma(1+i\eta) M(-i\eta, 1, i\mathbf{p}\mathbf{r} - i\mathbf{p} \cdot \mathbf{r}) e^{i\mathbf{p} \cdot \mathbf{r}}, \quad (\text{B.2})$$

$$\chi_{\mathbf{p}}^{(-)}(\mathbf{r}) = e^{-\frac{1}{2}\pi\eta}\Gamma(1-i\eta) M(i\eta, 1, -i\mathbf{p}\mathbf{r} - i\mathbf{p} \cdot \mathbf{r}) e^{i\mathbf{p} \cdot \mathbf{r}}. \quad (\text{B.3})$$

The operator $\hat{G}_C^{(+)} = 1/(E_p - \hat{H}_0 - \hat{V}_C + i\epsilon)$ is the Coulomb propagator, whose spectral representation in terms of the Coulomb wave functions is

$$\hat{G}_C^{(+)}(E) = 2\mu \int \frac{d^3q}{(2\pi)^3} \frac{|\chi_{\mathbf{q}}^{(+)}\rangle \langle \chi_{\mathbf{q}}^{(+)}|}{2\mu E - \mathbf{q}^2 + i\epsilon}. \quad (\text{B.4})$$

The other ingredient in the calculation of the amplitude is the short-range local operator $\hat{V}_S$ derived from the effective Lagrangian Eq. (4.1), with C_0 and C_2 at the same leading order. In momentum space it is given by

$$\langle \mathbf{q} | \hat{V}_S | \mathbf{p} \rangle = C_0 + \frac{C_2}{2}(q^2 + p^2). \quad (\text{B.5})$$

In order to calculate the expression for the modified scattering amplitude, we need to insert the Eq. (B.5) into Eq. (B.1) in a separable way, *i.e.*;

$$\langle \mathbf{q} | \hat{V}_S | \mathbf{p} \rangle = \sum_{i=1}^3 f_i(q) g_i(p) \quad (\text{B.6})$$

where $f_i(q)$ and $g_i(p)$ are the components of the following F and G vectors respectively,

$$F = \begin{pmatrix} \sqrt{C_0} \\ \sqrt{C_2/2} \, q^2 \\ \sqrt{C_2/2} \end{pmatrix}, \quad G = \begin{pmatrix} \sqrt{C_0} \\ \sqrt{C_2/2} \\ \sqrt{C_2/2} \, p^2 \end{pmatrix}. \quad (\text{B.7})$$

With this, it is possible to rewrite the Eq. (B.1) in a matricial way

$$T_{SC}(\mathbf{p}, \mathbf{p}') = A^T \cdot [1 + M + M^2 + \dots] \cdot B. \quad (\text{B.8})$$

The A and B factors are the three-component vectors

$$A = \begin{pmatrix} \sqrt{C_0} \psi_0^{(-)*}(p) \\ \sqrt{C_2/2} \psi_2^{(-)*}(p) \\ \sqrt{C_2/2} \psi_0^{(-)*}(p) \end{pmatrix}, \quad B = \begin{pmatrix} \sqrt{C_0} \psi_0^{(+)}(p') \\ \sqrt{C_2/2} \psi_0^{(+)}(p') \\ \sqrt{C_2/2} \psi_2^{(+)}(p') \end{pmatrix}, \quad (\text{B.9})$$

whose components are in terms of the functions

$$\psi_{2n}^{(-)*}(p) = \int \frac{d^3 q}{(2\pi)^3} \chi_{\mathbf{p}}^{(-)*}(\mathbf{q}) q^{2n}, \quad (\text{B.10})$$

and

$$\psi_{2n}^{(+)}(p) = \int \frac{d^3 q}{(2\pi)^3} \chi_{\mathbf{p}'}^{(+)}(\mathbf{q}) q^{2n}. \quad (\text{B.11})$$

The sequence in the brackets of Eq. (B.8) is a geometric series generated by M , where M is the square matrix whose components are

$$M_{ij} = 2\mu \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} g_i(k) \frac{\chi_{\mathbf{q}}^{(+)}(\mathbf{k}) \chi_{\mathbf{q}}^{(+)*}(\mathbf{k}')}{2\mu E - q^2 + i\epsilon} f_j(k'). \quad (\text{B.12})$$

Assuming convergence, the next step is to resolve the geometric series generated by M

$$\sum_{k=0}^{\infty} M^k = \frac{1}{1 - M}. \quad (\text{B.13})$$

This means that the series equates the inverse matrix of $D = 1 - M$.

Furthermore, looking for the Coulomb wave function, we can simplify our expressions with the relation

$$\psi_{2n}^{(-)*} = \psi_{2n}^{(+)} \quad (\text{B.14})$$

Combining the resulting D^{-1} matrix with the above relation, we obtain the following preliminary expression;

$$T_{SC}(\mathbf{p}, \mathbf{p}') = \frac{C_0 \psi_0^{(+)^2} + C_2 \psi_0^{(+)} \psi_2^{(+)} + \left(\frac{C_2}{2}\right)^2 \left[\psi_2^{(+)^2} J_0 - 2 \psi_0^{(+)} \psi_2^{(+)} J_2 + \psi_0^{(+)^2} J_4 \right]}{1 - C_0 J_0 - C_2 J_2 + \left(\frac{C_2}{2}\right)^2 [J_2^2 - J_0 J_4]} \quad (\text{B.15})$$

where J_0 , J_2 and J_4 are the following momentum-dependent integral terms,

$$J_0(p) = m_\alpha \int \frac{d^3 q}{(2\pi)^3} \frac{\psi_0^{(+)}(q) \psi_0^{(+)*}(q)}{p^2 - q^2 + i\epsilon}, \quad (\text{B.16})$$

$$J_2(p) = m_\alpha \int \frac{d^3 q}{(2\pi)^3} \frac{\psi_2^{(+)}(q) \psi_0^{(+)*}(q)}{p^2 - q^2 + i\epsilon}, \quad (\text{B.17})$$

and

$$J_4(p) = m_\alpha \int \frac{d^3 q}{(2\pi)^3} \frac{\psi_2^{(+)}(q) \psi_2^{(+)*}(q)}{p^2 - q^2 + i\epsilon}. \quad (\text{B.18})$$

All of them are ultraviolet divergent and must be regularized. This is done in Appendix C.

Appendix C

Divergent integrals and dimensional regularization

In this Appendix we regulate the integrals (B.16) to (B.18). Since these calculations are very extensive, here we present the main steps of this calculation. Explanations are given in order to alleviate the work for the reader.

Let us start by J_0 ,

$$J_0(p) = m_\alpha \int \frac{d^3 q}{(2\pi)^3} \frac{\psi_0^{(+)}(q) \psi_0^{(+)*}(q)}{p^2 - q^2 + i\epsilon}. \quad (\text{C.1})$$

To solve this, first we need to solve $\psi_0^{(+)}(q)$ defined in (B.11). Using dimensional regularization we write

$$\psi_0^{(+)}(q) = \left(\frac{\mu}{2}\right)^\epsilon \int \frac{d^D k}{(2\pi)^D} \chi_q^{(+)}(\mathbf{k}), \quad (\text{C.2})$$

where μ is the renormalization scale and $\epsilon = 3 - D$. The integrand corresponds to the Fourier-transformed Coulomb wave function

$$\chi_q^{(+)}(\mathbf{k}) = \int d^3 r \chi_q^{(+)}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}. \quad (\text{C.3})$$

To carry out the integration we use the partial-wave expansion of the Coulomb wave function Eq (2.25),

$$\chi_q^{(+)}(\mathbf{r}) = \sum_{l=0}^{\infty} i^l (2l+1) e^{i\sigma_l(q)} R_l^+(qr) P_l(\cos\theta). \quad (\text{C.4})$$

Using spherical coordinates $\mathbf{k} \cdot \mathbf{r} = kr[\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')]$ and thus

$$\chi_q^{(+)}(\mathbf{k}) = \sum_{l=0}^{\infty} i^l (2l+1) e^{i\sigma_l(q)} \int d^3r R_l^+(qr) P_l(\cos \theta) e^{-ikr[\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')]}, \quad (\text{C.5})$$

where the prime indicates the set for the momentum coordinate $\mathbf{k}$. Without losing generality we choose $\phi' = 0$. Thus, the integral over ϕ gives

$$\int_0^{2\pi} d\phi e^{-ikr \sin \theta \sin \theta' \cos \phi} = J_0(-kr \sin \theta \sin \theta'), \quad (\text{C.6})$$

where $J_0(-kr \sin \theta \sin \theta')$ is the Bessel function of the first kind. For the integral over θ we have

$$\int_0^\pi d\theta \sin \theta P_l(\cos \theta) J_0(-kr \sin \theta \sin \theta') e^{-ikr \cos \theta \cos \theta'} = i^l \sqrt{\frac{2\pi}{-kr}} P_l(\cos \theta') J_{l+\frac{1}{2}}(-kr). \quad (\text{C.7})$$

Then, a partial result for the Fourier transformed Coulomb wave function is

$$\chi_q^{(+)}(\mathbf{k}) = 2\pi \sum_{l=0}^{\infty} i^l (2l+1) e^{i\sigma_l(q)} \int_0^\infty dr r^2 R_l^+(qr) i^l \sqrt{\frac{2\pi}{-kr}} P_l(\cos \theta') J_{l+\frac{1}{2}}(-kr). \quad (\text{C.8})$$

Putting this into Eq (C.2) and commuting the integrations

$$\begin{aligned} \psi_0^{(+)}(q) &= 2\pi \sum_{l=0}^{\infty} i^l (2l+1) \left(\frac{\mu}{2}\right)^\epsilon e^{i\sigma_l(q)} \int \frac{d^d k}{(2\pi)^d} \int_0^\infty dr r^2 R_l^+(qr) i^l \sqrt{\frac{2\pi}{-kr}} P_l(\cos \theta') J_{l+\frac{1}{2}}(-kr) \\ &= \frac{2\pi}{(2\pi)^d} \sum_{l=0}^{\infty} i^l (2l+1) \left(\frac{\mu}{2}\right)^\epsilon e^{i\sigma_l(q)} \int dr r^2 R_l^+(qr) i^l \int_0^\infty dk k^{2-\epsilon} \sqrt{\frac{2\pi}{-kr}} J_{l+\frac{1}{2}}(-kr) \\ &\quad \times \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \sin \theta' P_l(\cos \theta'). \end{aligned} \quad (\text{C.9})$$

The integration over θ' vanishes for $l \neq 0$ and for $l = 0$ it gives a factor 2. The integration over ϕ' gives a factor 2π . After that, for the integration in k we use the definition of the spherical Bessel functions $j_l(z)$ for $l = 0$

$$j_0(z) = \sqrt{\frac{\pi}{2z}} J_{\frac{1}{2}}(z), \quad (\text{C.10})$$

which is an even function allowing us to change the sign of the factor $-kr$ and thus use the result

$$\int_0^\infty dk k^{\frac{3}{2}-\epsilon} J_{\frac{1}{2}}(kr) = 2^{\frac{3}{2}-\epsilon} r^{-\frac{5}{2}+\epsilon} \frac{\Gamma\left(\frac{3}{2}-\frac{\epsilon}{2}\right)}{\Gamma\left(\frac{\epsilon}{2}\right)}. \quad (\text{C.11})$$

Then, for the integration over the radial coordinate we recall the expression for the radial wave function $R_0^+(qr)$, which is given by the Eq. (2.27)

$$R_0^+(qr) = C_\eta e^{iqr} {}_1F_1(1+i\eta, 2, -2iqr), \quad (\text{C.12})$$

and the following formula involving hypergeometric functions,

$$\int_0^\infty r^\nu e^{-\mu r} {}_1F_1(a, b, qr) = \Gamma(\nu+1) \mu^{-\nu-1} {}_2F_1(a, \nu+1, b; p/\mu). \quad (\text{C.13})$$

When all the integrations are performed we obtain

$$\psi_0^{(+)}(q) = 2^{3-\epsilon} (2\pi)^{2-d} \sqrt{\pi} \left(\frac{\mu}{2}\right)^\epsilon e^{i\sigma_0} \frac{\Gamma(\frac{3}{2} - \frac{\epsilon}{2})}{\Gamma(\frac{\epsilon}{2})} C_\eta \Gamma(\epsilon) (-iq)^{-\epsilon} {}_2F_1(1+i\eta, \epsilon, 2; 2). \quad (\text{C.14})$$

Using the facts ${}_2F_1(1+i\eta, 0, 2; 2) = 1$ and $\Gamma(\epsilon)/\Gamma(\epsilon/2) = 1/2$ we finally have, in the limit $\epsilon \rightarrow 0$,

$$\psi_0^{(+)}(q) = C_\eta e^{i\sigma_0} \quad (\text{C.15})$$

We now proceed to calculate $J_0(p)$. According to Eqs. (C.1) and (C.15)

$$J_0(p) = m_\alpha \int \frac{d^3q}{(2\pi)^3} \frac{2\pi\eta_q}{e^{2\pi\eta_q} - 1} \frac{1}{p^2 - q^2 + i\epsilon}. \quad (\text{C.16})$$

This integral can be solved writing it in two parts, a finite and a divergent part,

$$J_0(p) = m_\alpha \int \frac{d^3q}{(2\pi)^3} \frac{2\pi\eta_q}{e^{2\pi\eta_q} - 1} \frac{1}{p^2 - q^2 + i\epsilon} \left[\frac{p^2}{q^2} - \frac{p^2 - q^2}{q^2} \right] = J_0^{fin} + J_0^{div}. \quad (\text{C.17})$$

For the finite part

$$J_0^{fin} = m_\alpha \int \frac{d^3q}{(2\pi)^3} \frac{2\pi\eta_q}{e^{2\pi\eta_q} - 1} \frac{1}{p^2 - q^2 + i\epsilon} \frac{p^2}{q^2}, \quad (\text{C.18})$$

we perform the substitution $x = A/q$, where $A = 2\pi k_C$, and we use the fact

$$\lim_{\epsilon \rightarrow 0^+} \left[-\frac{p^2}{A^2} - i\epsilon \right] = \lim_{\xi \rightarrow 0^+} \frac{p^2}{A^2} e^{-i(\pi-\xi)} = \lim_{\xi \rightarrow 0^+} \left[\frac{A}{p} e^{i(\pi-\xi)/2} \right]^{-2} = (2\pi i\eta_q) = b^{-2}, \quad (\text{C.19})$$

to rewrite it in the form

$$J_0^{fin} = -\frac{m_\alpha A}{2\pi^2} \int_0^\infty \frac{xdx}{(e^x - 1)(x^2 + b^2)}. \quad (\text{C.20})$$

The integral over x gives

$$\int_0^\infty \frac{xdx}{(e^x - 1)(x^2 + b^2)} = \frac{1}{2} \left[\ln \left(\frac{b}{2\pi} \right) - \frac{\pi}{b} - \psi \left(\frac{b}{2\pi} \right) \right], \quad (\text{C.21})$$

where the ψ -function is the logarithmic derivative of the Γ -function. Thus, the finite part becomes

$$J_0^{fin} = -\frac{m_\alpha A}{4\pi^2} \left[\chi(i\eta_p) + \frac{1}{2i\eta_p} - \ln(i\eta_p) \right] = -\frac{m_\alpha k_C}{2\pi} H(\eta_p). \quad (C.22)$$

For the divergent part we perform dimensional regularization and PDS subtraction in order to take into account extra divergences besides the logarithmic one. Using the same substitution $x = A/q$ and integrating over the angles we obtain

$$\begin{aligned} J_0^{div} &= -m_\alpha \left(\frac{\mu}{2} \right)^{3-D} \int \frac{d^D q}{(2\pi)^D} \frac{2\pi\eta_q}{e^{2\pi\eta_q} - 1} \frac{1}{q^2} \\ &= -\frac{m_\alpha \mu^3}{4A^2} \left(\frac{A^2}{\pi\mu^2} \right)^{D/2} \frac{1}{\Gamma(n/2)} \int_0^\infty \frac{x^{2-D}}{e^x - 1} dx. \end{aligned} \quad (C.23)$$

For the integral over x we use the formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx, \quad (C.24)$$

involving the gamma function $\Gamma(s)$ and the Riemann zeta function $\zeta(s)$, giving for the divergent part of J_0 the following expression

$$J_0^{div} = -\frac{m_\alpha \mu^3}{4A^2} \left(\frac{A^2}{\pi\mu^2} \right)^{D/2} \frac{1}{\Gamma(D/2)} \Gamma(3-D) \zeta(3-D). \quad (C.25)$$

If we evaluate the above expression at $D = 3$ the zeta function becomes finite $\zeta(0) = -1/2$, but we gain a pole from the gamma function $\Gamma(0) = \infty$. This pole corresponds to the logarithmic divergence present in the integral J_0 . However, this is not the only pole that we find. For $D = 2$ the gamma function is finite but the zeta function diverges $\zeta(1) = \infty$. This pole corresponds to the linear divergence also present in J_0 . The logarithmic divergence is obtained by expanding around $\epsilon = 3 - D$, when ϵ goes to zero.

$$J_0^{div} = \frac{m_\alpha k_C}{2\pi} \left[\frac{1}{\epsilon} + \ln \left(\frac{\mu\sqrt{\pi}}{2k_C} \right) + 1 - \frac{3}{2}\gamma_E \right]. \quad (C.26)$$

where $\gamma_E = 0.5772\dots$ is the Euler's constant.

Expanding around $\epsilon = 2 - D$ we recover the linear divergence part from (C.18). For this we obtain

$$\delta J_0^{div} = \frac{m_\alpha \mu}{4\pi} \frac{1}{D - 2}. \quad (C.27)$$

PDS regularization scheme tells us that this contribution should be subtracted from the result (C.19) for the $D \rightarrow 3$ limit. In this limit J_0 takes the form

$$J_0^{div} = \frac{m_\alpha k_C}{2\pi} \left[\frac{1}{\epsilon} + \ln \left(\frac{\mu\sqrt{\pi}}{2k_C} \right) + 1 - \frac{3}{2}\gamma_E \right] - \frac{m_\alpha\mu}{4\pi}. \quad (C.28)$$

Finally, from (C.22) and (C.28) we obtain

$$J_0(p) = \frac{m_\alpha}{4\pi} \left\{ 2k_C \left[\frac{1}{\epsilon} + \ln \left(\frac{\mu\sqrt{\pi}}{2k_C} \right) + 1 - \frac{3}{2}\gamma_E \right] - \mu - 2k_C H(\eta_p) \right\}. \quad (C.29)$$

The next integral to solve is (B.17),

$$J_2(p) = m_\alpha \int \frac{d^3q}{(2\pi)^3} \frac{\psi_2^{(+)}(q)\psi_0^{(+)*}(q)}{p^2 - q^2 + i\epsilon}, \quad (C.30)$$

which it is required to know $\psi_2^{(+)}(q)$ defined in (B.11) for $n = 1$. For this, we write

$$\begin{aligned} \psi_2^{(+)}(q) &= \left(\frac{\mu}{2} \right)^\epsilon \int \frac{d^D k}{(2\pi)^D} \chi_{\mathbf{q}}^{(+)}(\mathbf{k}) k^2 \\ &= \frac{2\pi}{(2\pi)^D} \sum_{l=0}^{\infty} i^l (2l+1) \left(\frac{\mu}{2} \right)^\epsilon e^{i\sigma_l(q)} \int_0^\infty dr r^2 R_l^+(qr) i^l \int_0^\infty dk k^{4-\epsilon} \sqrt{\frac{2\pi}{-kr}} J_{l+\frac{1}{2}}(-kr) \\ &\quad \times \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \sin \theta' P_l(\cos \theta'). \end{aligned} \quad (C.31)$$

This expression is similar to the Eq. (C.9) for $\psi_0^{(+)}(q)$, except for the exponent 4 in $k^{4-\epsilon}$ just after the k -integral. As before, the integral over the angles will give us a factor of 4π while cancels all terms different from $l = 0$. After that, in order to use the formula (C.11) to perform the integral over k , we must use the definition of the spherical Bessel function Eq. (C.10) to change the sign of the factor $-kr$ and use the properties of the sine function to replace two k -powers in the integrand for two derivatives on r ,

$$k^2 j_0(kr) = -\frac{1}{r} \left(\frac{d^2}{dr^2} \right) r j_0(kr). \quad (C.32)$$

Then, we leave the derivatives on r out of the k -integral. After carrying out the integration over k using (C.11), we apply the r derivatives to the resulting expression. Finally, the integration over r is performed using the formula (C.13). Thus we obtain

$$\psi_2^{(+)}(q) = 2^{4-\epsilon} (2\pi)^{2-D} \sqrt{\pi} \left(\frac{\mu}{2} \right)^\epsilon e^{i\sigma_0} \frac{\Gamma\left(\frac{5}{2} - \frac{\epsilon}{2}\right)}{\Gamma\left(\frac{\epsilon}{2}\right)} C_\eta \Gamma(-1+\epsilon) (-iq)^{2-\epsilon} {}_2F_1(1+i\eta, -2+\epsilon, 2; 2). \quad (C.33)$$

In the $\epsilon \rightarrow 0$ limit the divergent terms cancel each other. So, using ${}_2F_1(1+i\eta, -2, 2; 2) = 1/3 - 2\eta^2/3$ we have

$$\psi_2^{(+)}(q) = C_\eta e^{i\sigma_0}(q^2 - 2k_C^2). \quad (\text{C.34})$$

However, there is a pole for smaller dimensions. Rewriting (C.33) for $\epsilon = 2 - D$

$$\psi_2^{(+)}(q) = 2^{3-\epsilon}(2\pi)^{2-D}\sqrt{\pi}\left(\frac{\mu}{2}\right)^{1+\epsilon} e^{i\sigma_0} \frac{\Gamma(2-\frac{\epsilon}{2})}{\Gamma(\frac{1}{2}+\frac{\epsilon}{2})} C_\eta \Gamma(\epsilon)(-iq)^{1-\epsilon} {}_2F_1(1+i\eta, -1+\epsilon, 2; 2). \quad (\text{C.35})$$

Now, in the $\epsilon \rightarrow 0$ limit we use $\Gamma(\epsilon) = 1/\epsilon$ and ${}_2F_1(1+i\eta, -1, 2; 2) = -i\eta$ and then this pole becomes

$$\delta\psi_2^{(+)}(q) = C_\eta e^{i\sigma_0} \frac{2k_C\mu}{D-2}. \quad (\text{C.36})$$

Returning to the dimension $D = 3$, and subtracting (C.36) from (C.34) one has

$$\psi_2^{(+)}(q) = C_\eta e^{i\sigma_0}(q^2 - 2k_C^2 - 2k_C\mu). \quad (\text{C.37})$$

Note that, an useful relation is given by the Eqs. (C.15) and (C.37)

$$\psi_2^{(+)}(q) = \psi_0^{(+)}(q)(q^2 - 2k_C^2 - 2k_C\mu). \quad (\text{C.38})$$

Now we proceed to calculate $J_2(p)$. According to the Eq. (B.16) and the relation (C.36), $J_2(p)$ may be written as

$$J_2(p) = (p^2 - 2k_C^2 - 2k_C\mu)J_0(p) - I_3, \quad (\text{C.39})$$

where I_3 is a number given by the following integral

$$I_3 = m_\alpha \int \frac{d^3q}{(2\pi)^3} \frac{2\pi\eta_q}{e^{2\pi\eta_q} - 1}. \quad (\text{C.40})$$

This is similar to the integral (C.23). Dimensional regularization gives for this integral, as well as (C.23), a result in terms of the gamma and zeta functions,

$$\begin{aligned} I_3 &= \frac{m_\alpha \mu^3}{4} \left(\frac{A^2}{\pi\mu^2}\right)^{D/2} \frac{1}{\Gamma(D/2)} \int_0^\infty \frac{x^{-D}}{e^x - 1} dx \\ &= \frac{m_\alpha \mu^3}{4} \left(\frac{A^2}{\pi\mu^2}\right)^{D/2} \frac{1}{\Gamma(D/2)} \Gamma(1-D)\zeta(1-D). \end{aligned} \quad (\text{C.41})$$

As we see, if we evaluate the expression for I_3 at $D = 3$ we gain a pole from the gamma function. However, the zeta function becomes zero. Therefore, expanding around

$\epsilon = 3 - D$, when ϵ goes to zero, we obtain

$$I_3 = 2\pi m_\alpha k_C^3 \zeta'(-2). \quad (\text{C.42})$$

Now, looking again for (C.41) we see that it has a PDS pole at $D = 2$ from the gamma function. This pole gives

$$I_{3,1} = -\pi m_\alpha k_C^2 \frac{\mu}{12}. \quad (\text{C.43})$$

For $D = 1$, the gamma function in (C.41) diverges, so that the pole in this case is

$$I_{3,2} = m_\alpha k_C \frac{\mu^2}{8}. \quad (\text{C.44})$$

For $n = 0$, the remaining pole is

$$I_{3,3} = -m_\alpha \frac{\mu^3}{8}. \quad (\text{C.45})$$

Subtracting these poles from (C.42), PDS gives for I_3 the following expression

$$I_3 = m_\alpha \left(2\pi \zeta'(-2) k_C^3 + \frac{\pi}{12} k_C^2 \mu - \frac{1}{8} k_C \mu^2 + \frac{\mu^3}{8} \right). \quad (\text{C.46})$$

With this we have completed the calculation of $J_2(p)$.

To compute the final integral Eq. (B.18), we apply for this the same steps used to calculate the above integrals. First, we rewrite it in terms of the less-divergent integrals J_0 , J_2 and I_3 , and in terms of the more divergent integral,

$$I_5 = m_\alpha \int \frac{d^3 q}{(2\pi)^3} \frac{2\pi \eta_q}{e^{2\pi \eta_q} - 1} q^2. \quad (\text{C.47})$$

With this, the integral J_4 can be expressed as

$$J_4(p) = (p^2 - 2k_C^2 - 2k_C \mu) J_2(p) + (2k_C^2 + 2k_C \mu) I_3 - I_5, \quad (\text{C.48})$$

Applying dimensional regularization as before, the new integral term I_5 becomes

$$\begin{aligned} I_5 &= \frac{m_\alpha \mu^3 A^2}{4} \left(\frac{A^2}{\pi \mu^2} \right)^{D/2} \frac{1}{\Gamma(D/2)} \int_0^\infty \frac{x^{-2-D}}{e^x - 1} dx \\ &= \frac{m_\alpha \mu^3}{4} \left(\frac{A^2}{\pi \mu^2} \right)^{D/2} \frac{1}{\Gamma(D/2)} \Gamma(-1-D) \zeta(-1-D). \end{aligned} \quad (\text{C.49})$$

Expanding around $\epsilon = 3 - D$ no pole is observed, so that

$$I_5 = \frac{2}{3}\pi^3 m_\alpha \zeta'(-4) k_C^5. \quad (\text{C.50})$$

Including the PDS $D = 2, 1, 0, -1, -2$ poles I_5 becomes

$$I_5 = m_\alpha \left(\frac{2}{3}\pi^3 \zeta'(-4) k_C^5 - \frac{4\pi^3 k_C^4 \mu}{3} - \frac{\pi^2 \zeta'(-2) k_C^3 \mu^2}{4} + \frac{\pi^2 k_C^2 \mu^3}{24} + \frac{\pi^2 k_C \mu^4}{32} - \frac{\pi \mu^5}{16} \right) \quad (\text{C.51})$$

Thus, we have completed the calculations for the final divergent integral defined in (B.18). To do that, we have used the PDS regularization scheme, in order to recover divergences beyond the logarithmic one.

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