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Topics in gauge/gravity dualities

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Estudos na dualidade calibre/gravidade

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"There is geometry in the humming of the strings.

There is music in the spacing of the spheres."

Pythagoras

Resumo

Essa tese consiste num estudo autocontido das dualidades calibre/gravidade na linha do modelo do Klebanov-Witten. Aqui nos exploramos primeiro de um jeito razoavelmente detalhado, a conhecida dualidade do Maldacena que relaciona a teoria $\mathcal{N} = 4$ SYM em quatro dimensões com as supercordas tipo IIB no espaço $AdS_5 \times S^5$, depois de alguns preliminares necessários sobre teorias supersimétricas de calibre, onde nós mostramos em detalhe a álgebra supersimétrica e as representações para $\mathcal{N} \geq 1$ supersimetria. Nós também construímos os conhecidos supercampos que são úteis para escrever lagrangianas invariantes para teorias de calibre facilmente, e então serão úteis para construir a teoria de calibre do modelo de Klebanov-Witten. Na correspondência AdS/CFT original e as suas extensões fenomenologicamente interessantes, as Dp-branas, como soluções de supergravidade e objetos não perturbativos na teoria de cordas onde as teorias de calibre moram, são essenciais. Assim, a fim de preservar a natureza autocontida desse trabalho, nós incluímos uma breve revisão sobre teoria de supercordas dirigida a entender a necessidade de incluir esses objetos extra-dimensionais usando dualidade-T e, no limite de baixa-energia da teoria de cordas, como soluções das equações de Einstein. O primeiro clímax desse trabalho ocorre quando nós usamos tudo o que aprendemos para estabelecer a conjectura do Maldacena, a teoria de calibre $\mathcal{N} = 4$ SYM que nós estudamos no capítulo de supersimetria, morando no volume de mundo quadridimensional de uma pilha de N_c D3-branas (sim, o subscrito “c” significa cor!) em espaço plano, corresponde exatamente à teoria de supergravidade tipo IIB no espaço $AdS_5 \times S^5$. A fim de testar ela, nós identificamos simetrias e operadores com estados em ambos lados da dualidade. Mas na verdade isto corresponde à forma fraca da correspondência, porque não é possível lidar nem com a teoria de cordas nem com a teoria de calibre no limite de acoplamento forte.

O foco e motivo principal de porque nós temos que aprender as primeiras cem páginas aqui, será estender a teoria de calibre dual que estudamos em AdS/CFT, para teorias de calibre mais re-

alísticas como duais de alguma teoria de supergravidade. O modelo do Klebanov-Witten, consiste em substituir a esfera de cinco dimensões no fundo de supergravidade da teoria de supercordas tipo IIB por um espaço que é Eistein mais interessante X_5 , um espaço coset chamado $T^{1,1}$. Nós esperamos que a teoria de calibre dual que resulta é menos supersimétrica, e na verdade é $\mathcal{N} = 1$ superconforme com um conteúdo de matéria na representação bifundamental do grupo de calibre $SU(N) \times SU(N)$, e um superpotencial quártico que tem simetria global $SU(2) \times SU(2) \times U(1)$, que é precisamente a simetria do espaço coset no lado da gravidade. Mas isso não é tudo, o modelo do Klebanov-Witten estendeu a correspondência do Maldacena e encontrou como teoria dual uma teoria menos supersimétrica mas ainda conforme. A quebra da simetria conforme, proposta pelo Klebanov, Nekrasov e Tseytlin, é obtida introduzindo M D3-branas fracionais além das N D3-branas regulares. A teoria resultante é uma teoria de calibre $SU(N + M) \times SU(N)$ com $\mathcal{N} = 1$ supersimetria, não mais conforme e então um pouco mais interessante como parte da nossa cruzada para encontrar uma teoria tipo-QCD. Isso ainda não é o final, o modelo anterior sofre de uma singularidade no IR profundo, tornando inválido a descrição gravitacional. Foi conjecturado então que a dinâmica do acoplamento forte na teoria de gauge deveria de algum jeito resolver esse problema. Klebanov, de novo, e Strassler mostraram que essa conjectura foi correta, e argumentaram que o fluxo do GR é de fato uma serie infinita de transformações de dualidade de Seiberg - uma cascata - onde o numero de cores cai repetidamente de $N \rightarrow N - M$, e o grupo de calibre muda de $SU(N + M) \times SU(N)$ a $SU(N - M) \times SU(N)$. O processo pode ser repetido até o limite IV onde o grupo de calibre simplesmente torna-se $SU(M)$. Então, no final nós obtemos uma $\mathcal{N} = 1$ teoria de calibre $SU(M)$, ou seja uma teoria tipo-QCD. Então, nós dissemos que o modelo padrão mesmo pode se situar na base da cascata de dualidade.

Palavras chave correspondência AdS/CFT, modelo do Klebanov-Witten, cascatas de dualidade, teorias tipo-QCD.

Abstract

This thesis consists in a self-contained study of gauge/gravity dualities in the line of the Klebanov-Witten model. Here we explore first the known Maldacena duality that relates $\mathcal{N} = 4$ SYM theory in four dimensions to type IIB supergravity on $AdS_5 \times S^5$ in reasonable detail, after some necessary preliminaries on supersymmetric gauge theories, where we display in detail the supersymmetry algebra and representations for $\mathcal{N} \geq 1$ supersymmetry. There we also construct the so-called superfields that will be helpful to write invariant lagrangians for gauge theories readily, and then useful to construct the gauge theory side of the Klebanov-Witten model. In the original AdS/CFT correspondence and its phenomenologically interesting extensions, Dp-branes as solutions of supergravity and nonperturbative objects in string theory where gauge theory lives are crucial. So, to preserve the self-contained nature of this work, we include a brief review of superstring theory addressed to understand the need to include this higher-dimensional objects by T-duality and, at low-energy limit of the string theory, as solutions of the Einstein equations. The first climax of this work occurs when we use all we learned to establish the Maldacena conjecture, $\mathcal{N} = 4$ $SU(N_c)$ SYM theory we study in the supersymmetry chapter, living on the four-dimensional worldvolume of a stack of N_c D3-branes in a flat-space, corresponds exactly to type IIB supergravity on $AdS_5 \times S^5$. In order to “prove” it, we match symmetries and operators with states in both sides. But actually this corresponds to the weak form of the correspondence, because it is not possible to handle neither string theory or gauge theory at strong coupling.

The focus and main motive to have to learn the first hundred of pages here will be to extend the dual gauge theory we studied in AdS/CFT towards more realistic gauge theories as duals of some supergravity theory. The Klebanov-Witten model, consists in replacing the five-sphere in the gravity background of type IIB for a more interesting Einstein manifold X_5 , a coset space called $T^{1,1}$. The resulting dual gauge theory is expected to be less supersymmetric, and it is indeed $\mathcal{N} = 1$ superconformal field theory with matter content in the bifundamental representation of the

gauge group $SU(N) \times SU(N)$, and a quartic superpotential that exhibits $SU(2) \times SU(2) \times U(1)$ global symmetry, which is precisely the symmetry of the coset space in the gravity side. This is not the end of the story, the Klebanov-Witten model extended the Maldacena correspondence and found as a dual gauge theory a less supersymmetric but still conformal theory. Breaking of the conformal theory, proposed by Klebanov, Nekrasov and Tseytlin, is achieved by introducing fractional M D3-branes in addition to the N regular D3-branes. The resulting theory is an $SU(N+M) \times SU(N)$ gauge theory with $\mathcal{N} = 1$ supersymmetry, no longer conformal and then a little more interesting as a part of the crusade to find a QCD-like theory. This is still not the end, the last model suffers from a singularity in the deep IR, rendering the gravitational description invalid in that regime. It was conjectured that the strong dynamics of the gauge theory should somehow resolve this problem. Klebanov, again, and Strassler showed that this conjecture was correct, and argue that the RG flow is in fact an infinite series of Seiberg duality transformations - a cascade - in which the number of colors repeatedly drops from $N \rightarrow N - M$, so the gauge group changes from $SU(N+M) \times SU(N)$ to $SU(N-M) \times SU(N)$. This process can be repeated until the IR limit where the gauge group simply becomes $SU(M)$. So, at the end we get a $\mathcal{N} = 1$ $SU(M)$ gauge theory, a QCD-like theory. We say that the standard model itself may lie at the base of a duality cascade.

Keywords AdS/CFT correspondence, Klebanov-Witten model, duality cascades, QCD-like theories.

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Chapter 1

Introduction and Overview

String theory has been introduced in so many ways, some of them are very pedagogical and others make it look like science fiction. Indeed, string theory is still a strange way to understand nature, strings instead point-particles, higher dimensional spacetimes with exotic geometries, compactifications and also higher dimensional objects called branes arise in order to make the theory consistent. It is commonly said that string theory is one of the most powerful and complete attempts to unify, not only quantum theory with general relativity, but all forces of nature in a uniquely beautiful (mathematically elegant) description. It also has proved to be very useful in the context of the gauge/gravity dualities. They allow to have two complementary descriptions of a theory in different regimes, in other words, when we cannot deal with one side of the theory, we can use the dual formulation, which corresponds to a well-behaved theory in the opposite regime. The striking feature is that the dual formulation for a specific gauge theory, being supersymmetric, conformal or not, is actually a theory of gravity, described by some particular string theory.

String theory was born during the 60's in the framework of hadronic physics as an attempt to explain strong interactions, the force which holds protons and neutrons together inside the nucleus of an atom as well as quarks together inside the protons and neutrons. The idea was to consider the flux tubes mediating hadronic interactions as string-like structures emerging in different arrangements of mesons and baryons. Later, the idea was obscured by the formulation of the quantum chromodynamics and subsequently by the formulation of the standard model which has described successfully three of the four fundamental interactions so far. Nevertheless gravity, formulated classically by the general relativity, has been reluctant to be formulated as a quantum theory, principally because it is not renormalizable.

The interest in strings as fundamental phenomena began in the 80's. Considering the elementary particles as vibrations of one-dimensional objects, gravity arises naturally in the spectrum of the theory as the graviton state. This is significant, because general relativity arises within the framework of a consistent quantum theory. Moreover, string theory does not just contain gravity, but it comes inevitably with a large number of particles and interactions. Unfortunately, those predicted particles and interactions are far from being unique. Since string theory is actually a set of string theories which differ from each other by conditions imposed in the states, there are actually five consistent string theories as being different perturbative regimes of a still not completely uncovered bigger theory, called M-theory. In turn, these string theories are related by a chain of dualities which connect the different regimes of that M-theory.

A mind-blowing fact in string theory is that it is not consistent in all spacetime backgrounds, but only in those satisfying certain conditions. For the so-called bosonic string theory in flat spacetime, the condition is that the dimension of spacetime is 26. And its supersymmetric extension leads to a consistent theory only in ten-dimensional spacetimes. It is obvious that, at first sight, we do not trust on a theory that make sense mathematically only in spaces with more dimensions we live. But there are many ways to overcome this “drawback”, we could split the spacetime in two or more subspaces, with one of them being our familiar four-dimensional spacetime and some other compact spaces.

Every string theory has, as low-energy limit, a theory of supergravity. Such supergravities are nonrenormalizable but their classical solitonic solutions, called Dp-branes, are relevant as solutions of the full string theories. Those objects are nonperturbative and source the massless sector of closed superstring theory. Moreover by T-duality they have a dual description as the hypersurfaces in flat space where open strings attach. In this case, dynamics of a Dp-brane give rise to a gauge theory living in its worldvolume.

Juan Maldacena conjectured in 1997, a specific duality relating a gauge theory, living on the D-branes of the open string theory, with a string theory whose massless spectrum is sourced by those solitonic solutions of supergravity. The statement is that type IIB string theory, a particular type of closed superstring theory, on $AdS_5 \times S^5$ is dual to the four-dimensional $\mathcal{N} = 4$ superconformal Yang-Mills theory with $SU(N_c)$ gauge group. In other words, the closed string sector of string theory quantized on this ten-dimensional strange spacetime is conjectured to be dual to this field theory living on a stack of N_c D3-branes. This is known as holographic duality, in the sense that

the boundary of the AdS_5 space where the gauge theory lives encodes all the bulk information. Checking the correspondence includes matching correlations functions and symmetries between both theories. But for technical reasons, this duality has been more accurately tested so far in the low-energy limit of the strings side, i.e. supergravity. In this limit we can extract information about the strong coupling regime of the gauge theory side by merely performing computations in a supergravity background. This is the celebrated AdS/CFT correspondence, whose first clue was realized by 't Hooft by expanding a $U(N_c)$ gauge theory on the dimensionless parameter $1/N_c$ and taking the limit of large N_c . The Feynman diagrams can be arranged as a sum over the genus of surfaces in which the diagrams can be drawn. This is similar (dual) to the computation of string amplitudes where the sum is now over the genus of the possible worldsheets of the string. These results correspond as we mentioned to a weak form of the duality, a solution of supergravity is dual to a certain supersymmetric gauge theory at strong ('t Hooft) coupling. The fact that the AdS/CFT correspondence is a weak/strong coupling duality makes it interesting, but, at the same time, also difficult to prove (or disprove) because reliable computational techniques on the two sides of the duality do not have an overlapping domain of validity.

Then we have found a way to “complete“ a theory which is untreatable at scales when the coupling becomes strong, and perturbatives methods are not useful anymore. Indeed, someone could say that the dual theory is supersymmetric and conformal, it does not describe reality and it is actually true. $\mathcal{N} = 4$ SYM involves supersymmetry, we know that some forms of supersymmetry are being searched for by the LHC now. But those forms involve symmetries that are broken. In our gauge theory, supersymmetry is unbroken. Every particle has the same mass (zero) and charge. There are no particles like that in the standard model. Even though this theory does not describe a real system (it does not pretend to do it either), it allows to extract very useful knowledge about gauge theories. The AdS/CFT correspondence helps a lot when field theory is not well-behaved.

Extensions of the above ideas towards more realistic theories (from a phenomenological point of view) have been considered since Maldacena conjectured the original gauge/gravity duality. The reduction of the amount of supersymmetries and the breaking of the conformal invariance would lead to a more (phenomenologically) interesting statement of dualities. The final goal would be to find the stringy dual of QCD, a nonsupersymmetric, nonconformal and asymptotically free theory.

One of the first attempts to generalize the AdS/CFT conjecture was to consider spacetimes in the string theory side which break supersymmetries but not conformal invariance, conifold geometries. The Klebanov-Witten model was suggested as a way to obtain a less supersymmetric gauge theory, but it could be thought as a even less realistic theory because its gauge group is the product $SU(N) \times SU(N)$. We have gained an $\mathcal{N} = 1$ gauge theory as was expected from the stringy side. The dual gauge theory, as we said, is still conformal; so in a way to break this symmetry, Klebanov, Gubser, Nekrasov and Tseylin considered to add fractional D3-branes together the usual D3-branes on the conifold space. This is known as the Klebanov-Tseytlin model, since they developed the $\mathcal{N} = 1$ non-conformal gauge theory beyond the leading terms of a M/N expansion as was studied by Klebanov and Nekrasov. For this model, the presence of this D5-branes leads to an $SU(N + M) \times SU(N)$ gauge theory, which is no longer conformal. They get an expression for the RG and show that the obtained gauge theory becomes invalid in the IR limit.

The clue to solve this problem was to consider a symmetry between two supersymmetric theories in the IR limit, the so-called Seiberg duality. It relates, under some conditions, two different supersymmetric gauge theories when we approach to the IR limit, so by means of this duality we can cascade down the gauge group, i.e., modify it in the IR in order to obtain a QCD-like gauge group. As a consequence, all branes are replaced by flux, which geometrically translates into a deformation of the conical singularity.

Organization of the thesis

In Chapter 2, we make a self-contained review about supersymmetric field theories. Sections (2.1) and (2.2) introduce basics on algebra and representations for $\mathcal{N} = 1$ and $\mathcal{N} > 1$ supersymmetry. There we define the different multiplets, massless and massive, which are labeled by energy and mass, and contain states with different helicities and spins. Section (2.3) is about the superspace formalism, which introduces two extra fermionic coordinates, allows to define superfields as general fields containing the usual fields in an expansion in terms of those fermionic coordinates. Defining the corresponding variations together with covariant derivatives in this superspace, we can obtain a general form for variations of the content fields. Next we study in (2.3.3.1) and (2.3.3.2), chiral and vector superfields respectively, by imposing conditions as chirality and reality, the general form of a superfield will reduce to a specific kind of field with a characteristic

content. A very important part of this section will be the definition of supersymmetric field strength in (2.3.3.3) from a vector superfield. All these superfields will be included in supersymmetric lagrangian, that are expanded in terms of their component field, will give general expressions for explicitly supersymmetric lagrangians including gauge fields accompanied with their corresponding supersymmetric partners. In this chapter we dedicate a review about some basic constructions of supersymmetric lagrangians, (2.4). We introduce the Wess-Zumino model in (2.4.1), in order to include interactions in a supersymmetric model of scalars and fermions. Gauge theories are studied next in (2.4.3) as a way to include more complicated interactions mediated by gauge fields. Potential coming from the kinetic term in this theories leads to define the moduli space of vacua, in terms of the called D- and F-terms. We will be back to these issues at the end for the Klebanov-Witten model. A very important topic, in supersymmetric gauge theories is to calculate the β function, which in the case of supersymmetric theories, is exact. The NSVZ β function is briefly explained and calculated in (2.4.3.1). It will allow to define the fixed-curve of couplings in the Klebanov-Witten model later.

Section about basic conformal and superconformal symmetry is included in (2.5), since the particular case of $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions is, besides being supersymmetric, conformal. This gauge theory will be important since is dual to a supergravity theory. $\mathcal{N} = 4$ SYM is studied in (2.6) from its higher dimensional origin, an $\mathcal{N} = 1$ SYM theory in ten dimensions.

Chapter 3 is dedicated to a brief revision in string theory and supergravity. In section (3.1), we begin to study superstring theory in the RNS formalism. Covariant quantization is developed in section (3.1.2), where we establish commutations rules as usual. In section (3.1.3) we classify the spectrum in terms of sectors for open and closed strings states. The critical dimension is found by requiring that there are no states with negative norm. GSO projection allows to eliminate tachyonic states and have the same number of bosons and fermions. In the section (3.1.4) we write explicitly the spectrum for type IIB superstring theory and recognize the different field of its massless spectrum. Compactifying one of the directions in spacetime, D-branes are introduced in section (3.2) as objects required by T-duality for open strings. Extra degrees of freedom called Chan-Paton factors are studied in section (3.2.3). It leads to define gauge field on the brane worldvolume and scalars defining its position in the compactified direction. If we compactify more than one direction, we could define the so-called Dp-branes, in section (3.2.4),

we generalize the idea of gauge field on branes. Focusing in a particular type of superstring, type IIB, is section (3.2.6.1) dedicated to define this Dp-branes as charged objects in type IIB superstring theory. There we mention some about stability of branes and remind the idea of BPS states in this context. The Dirac-Born-Infeld action is introduced in section (3.2.7) in order to explain the nonperturbative nature of branes. From section (3.3, we focus in the low-energy limit of superstring theories, the called supergravity (see [1] for a classic review and [2] for a modern one). Being even more specific, section (3.3.1 study the particular case of IIB supergravity because it will be the corresponding stringy side of the duality proposed by Maldacena and later by Klebanov and Witten. In this section we establish the supersymmetric transformations for type IIB supergravity by using the inherent $SU(1,1)$ symmetry of the theory. The called Dp-branes, as solutions of supergravity will be studied in section (3.4) for type IIB. There we find the exact form of the metric of the spacetime in presence of N D3-branes, and we find its limit when we approach to the origin. The section (3.5) concentrates in a brief study of the anti de Sitter space, which is necessary because it is the near-horizon limit of the space curved by branes.

With all tools and concepts reviewed in Chapter 2 and 3, we introduce formally the AdS/CFT correspondence in chapter 4. In section (4.1) we establish the near-horizon limit of the solution for black p-branes. We obtain the product of spaces $AdS_5 \times S^5$. In section (4.1.1), we focus in the deep throat and boundary (of the throat, i.e. the region before spacetime becomes totally Minkowskian), where we appreciate the decoupling of the spectra for extreme limits of the parameters of spacetime and the string/gauge theory. This decoupling allows to understand the original Maldacena conjecture, two totally "different" theories describe the same physics at different scales. This is not the end of the story, we must prove or test this correspondence. In section (4.1.2.1) we review the matching parameters between both sides of the duality. In section (4.1.2.3, we explain how the symmetries of spacetime in the string theory side match to the symmetries of the gauge theory. Finally, in section (4.1.2.4), we show partially how observables in the gauge theory, i.e. operators, translate into fields on the boundary of the spacetime in the string theory. Since calculations to prove the equivalence between correlation functions are a little large for our purposes, we will focus on the case of the scalar field on AdS -space. Precise results for the specific gauge and string theories involved in the correspondence will be referenced.

Chapter 5 focuses in the Klebanov-Witten model, one extension of the Maldacena's idea for considering more general spaces with conical singularities. In section (5.1 we sketch the form of

the conical metric as the transverse compactified part of the ten-dimensional spacetime where type IIB superstrings live now, and mention some of its properties. There we also give some detailed calculations in presence of the five-form flux, and find some precise condition for the spinors and thus for the transverse compactified six-dimensional space. We specialize in a conical space whose basis is constructed as a coset $T^{1,1}$. This particular space is parametrized by four "fields" which define it by a equation that come from intersecting the cone and the sphere. In section (5.3, those "fields" become fields in a $\mathcal{N} = 1$ superconformal gauge theory. The gauge theory living on the worldvolume of the stack of N D3-branes will have some subtle details, one of them is that its gauge group is inherited from a theory in orbifold, as we argue in (5.3.2). That is why we review some basic concepts in orbifold in section (5.3.2.1), and focus in the field theory in the particular case of $AdS_5 \times S^5/\mathbb{Z}_2$ in (5.3.2.2). Superpotential is introduced in section (5.3.3) as a way to perform Higgs mechanism to give mass the chiral superfield, in order to have branes at generic smooth points on the conifold. In section (5.3.4) we apply the NVSZ β function to impose conformal invariance. We find the fixed surface of parameters that leave our the theory conformally invariant. We also argued the existence of exactly marginal operators associated to this surface. Finally we establish the conjecture in (5.4), and perform some tests by comparing symmetries in both the string theory and the field theory sides.

Chapter 6 we give some comments about the continuation of this extension of the Maldacena idea. The Klebanov-Tseylin and Klebanov-Strassler models which are still supersymmetric but not longer conformal. We also give some advances on non-supersymmetric theories and duality cascades without going into detail.

Chapter 2

Supersymmetric field theories

Symmetries are crucial in the description of physical systems. In the realm of particle physics symmetries are believed to permit a classification of all observed particles. A very important symmetry, which has been established both theoretical and experimentally is that of Poincaré group. Besides this fundamental symmetry there are other so-called internal symmetries (such as the $SU(3)$ symmetry in the QCD) which have also been firmly established over the last decades. In the course of time several attempts have been made to unify the spacetime symmetry of the Poincaré group with some of the internal group. Such attempts have been shown to be futile if the theory is expected to satisfy certain basic requirements. The well-known no-go theorem established by S.Coleman and J.Mandula shows that if one makes the assumptions of locality, causality, positivity of energy and finiteness of the number of particles the invariance group of the theory can at best be the direct product of the Poincaré group and the internal group, and this therefore does not offer a true unification of one group with the other [3,4]. The generators of the Poincaré group satisfy well-known commutation rules. It was realized by J.Wess and B.Zumino that if one allows also anticommutation relations of generators of supersymmetry transformations which transform bosons into fermions and viceversa, then the unification of the spacetime symmetries of the Poincaré group with this internal symmetry can be achieved. The formal proof was established by R.Haag, J.T.Lopuszański and M.F.Sohnius [5]. Supersymmetry thus arises as a symmetry which combines bosons and fermions in the same representation of the enlarged group which encompasses both the transformations of the Poincaré group and the appropriate supersymmetry transformations. So, every bosonic particle must have a fermionic partner and viceversa.

In this first chapter, we will review some interesting concepts about supersymmetry, irreducible representations to construct states and the superspace formalism to get manifest supersymmetry in our lagrangians. This chapter is based on [4] and [6]. General reviews on supersymmetry may be also found in [2, 3, 7, 8].

As an introductory chapter, that is very necessary for understanding the gauge field side of the correspondence and its generalizations to less symmetric cases, we decided to include some detailed calculations and comments that allow to explain what we need to learn and also some further topics, in a reasonable extended chapter about supersymmetry with emphasize in gauge theories at the end. In this sense, the aim of this chapter will be to give some concepts and detailed results to understand how to construct a supersymmetric gauge theory, and in particular to construct $\mathcal{N} = 4$ super Yang-Mills. This is a critical starting point that could be skipped out if the reader has a suitable background in this subject.

2.1 The supersymmetric algebra

2.1.1 Graded algebra

In order to have an extension of the Poincaré algebra, Galfand and Likhtman suggested to introduce the concept of graded algebras, considering even and odd generators in the same algebra. Then

$$\begin{aligned} [\text{even}, \text{even}] &= \{ \text{odd}, \text{odd} \} = \text{even}, \\ [\text{even}, \text{odd}] &= \text{odd}. \end{aligned}$$

For supersymmetry, generators are the Poincaré generators P^μ , $M^{\mu\nu}$ and the new spinor generators Q_α^A and $\bar{Q}_{\dot{\alpha}}^A$, where $A = 1, \dots, \mathcal{N}$. For $\mathcal{N} = 1$ we speak of a simple SUSY, and for $\mathcal{N} > 1$ of an extended SUSY. Since we added two generators, there will be new commutation rules,

$$\begin{aligned} [Q_\alpha, M^{\mu\nu}] &= (\sigma^{\mu\nu})_\alpha^\beta Q_\beta, \\ [Q_\alpha, P^\mu] &= \{Q_\alpha, Q_\beta\} = 0, \\ \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P^\mu, \end{aligned} \tag{2.1}$$

where $(\sigma^{\mu\nu})_\alpha^\beta = \frac{1}{4}[\gamma^\mu, \gamma^\nu]_\alpha^\beta$.

The first relation says that Q_α is a Weyl spinor, since it transforms in the Lorentz representation. The second commutation rule shows us that Q_α is invariant under translations. The fourth

one is very important, it indicates that the number of bosons and fermions in the theory are the same [3]. Two SUSY transformations (Q_α) are equivalent to a simple translation.

2.1.2 Casimir operators for SUSY

It is easy to notice that $[Q_\alpha, P^\mu] = 0 \Rightarrow [Q_\alpha, P^2] = 0$, so P^2 is a Casimir operator, one that commutes with all generators. Then, all the states of a supersymmetry multiplet will have the same mass, then the mass will label a massive multiplet. In the non-SUSY Poincaré group, there is another Casimir operator which follows from the squared Pauli-Ljubanski vector W_μ ,

$$W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}. \quad (2.2)$$

But in the supersymmetric case this operator is no longer a Casimir, it doesn't commute with all the fermionic generators and its eigenvalues cannot be a label for the multiplet. In other words, it could contain fields with different spins,

$$[W^2, Q_\alpha] = W^\mu (\not{P} \gamma_\mu \gamma^5 Q)_\alpha + \frac{3}{4} P^2 Q_\alpha \neq 0. \quad (2.3)$$

We define B_μ by

$$B_\mu := W_\mu + \frac{1}{8} \bar{Q} \gamma_\mu \gamma^5 Q, \quad (2.4)$$

and then,

$$C_{\mu\nu} := B_\mu P_\nu - B_\nu P_\mu. \quad (2.5)$$

This second rank tensor commutes with the supercharge Q_α . From $C_{\mu\nu}$ we can construct the Casimir operator, i.e. It commutes with all generators of the SuperPoincaré group,

$$C^2 = C_{\mu\nu} C^{\mu\nu}, \quad [C^2, \{Q_\alpha, P_\mu, M_{\mu\nu}\}] = 0. \quad (2.6)$$

2.2 Classification of irreducible representations

2.2.1 $\mathcal{N} = 1$ supersymmetry

The irreducible representations of the super Poincaré algebra are characterized by the eigenvalues of P^2 and C^2 . Let $P^2 = m^2 > 0$, and for simplicity we choose the rest frame $P_\mu = (m, \vec{0})$. Then,

$$C^2 = 2m^2 B_k B^k, \quad (2.7)$$

where $k = 1, 2, 3$. Now,

$$B_k = W_k + \frac{1}{4}X_\mu = mS_k + \frac{1}{8}\bar{Q}\gamma_k\gamma^5 Q := mJ_k. \quad (2.8)$$

Here J is called the superspin operator (we will see later that this operator has the same eigenvalue of the spin operator). We can now rewrite C^2 in terms of J_k as

$$C^2 = 2m^4 J_k J^k. \quad (2.9)$$

2.2.1.1 Massless multiplet

States of massless particles have P_μ -eigenvalues $p_\mu = (E, 0, 0, E)$. The Casimirs $C_1 = P^2$ and $C_2 = C_{\mu\nu}C^{\mu\nu}$ are zero. Consider the algebra:

$$\{Q_\alpha, \bar{Q}_\beta\} = 2(\sigma^\mu)_{\alpha\beta} P_\mu = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.10)$$

which implies that Q_2 is zero in the representation

$$\{Q_2, \bar{Q}_2\} = 0 \Rightarrow Q_2 = 0. \quad (2.11)$$

The Q_1 satisfies $\{Q_1, \bar{Q}_1\} = 4E$, so defining the creation and annihilation operators a and a^\dagger via

$$a := \frac{Q_1}{2\sqrt{E}}, \quad a^\dagger := \frac{\bar{Q}_1}{2\sqrt{E}} \Rightarrow \{a, a^\dagger\} = 1. \quad (2.12)$$

From the relation (2.1),

$$[S^3, a] = \frac{1}{2}(\sigma^3)_{11} a = \frac{1}{2}a. \quad (2.13)$$

For the massless case S^3 is the helicity¹ operator with eigenvalue λ . Choosing an eigenstate $|p^\mu, \lambda\rangle$

$$S^3(a |p^\mu, \lambda\rangle) = (\lambda - 1/2)a |p^\mu, \lambda\rangle. \quad (2.14)$$

So $a |p^\mu, \lambda\rangle$ has helicity $\lambda - 1/2$. In the same way, $a^\dagger |p^\mu, \lambda\rangle$ has helicity $\lambda + 1/2$. To build a representation, start with a vacuum state (which is not actually a physical vacuum) of minimum helicity λ , let's call $|\Omega\rangle$. Obviously $a |\Omega\rangle = 0$ and $a^\dagger a^\dagger |\Omega\rangle = 0$. Then, the whole multiplet consists of two states

$$|\Omega\rangle = |p^\mu, \lambda\rangle \quad a^\dagger |\Omega\rangle = |p^\mu, \lambda + 1/2\rangle. \quad (2.15)$$

¹Remember that the helicity is the projection of the spin along the momentum direction and allow us to label massless particles.

CPT invariance ² requires the presence of states with helicity $-\lambda$ and $-(\lambda + 1/2)$ as well,

$$|\bar{\Omega}\rangle = |p^\mu, -(\lambda + 1/2)\rangle \quad a^\dagger |\bar{\Omega}\rangle = |p^\mu, -\lambda\rangle. \quad (2.16)$$

We can construct the massless chiral multiplet by choosing the called Clifford vacuum (and its CPT conjugate) with zero helicity,

$$|\Omega_o\rangle = |0\rangle, \quad a^\dagger |\Omega_o\rangle = |+1/2\rangle, \quad (2.17)$$

$$|\bar{\Omega}_o\rangle = |-1/2\rangle, \quad a^\dagger |\bar{\Omega}_o\rangle = |0\rangle, \quad (2.18)$$

so the massless chiral multiplet corresponds to a Weyl fermion and a complex scalar. The massless vector multiplet is constructed from a vacuum state with helicity $1/2$:

$$|\Omega_o\rangle = |+1/2\rangle, \quad a^\dagger |\Omega_o\rangle = |+1\rangle, \quad (2.19)$$

$$|\bar{\Omega}_o\rangle = |-1\rangle, \quad a^\dagger |\bar{\Omega}_o\rangle = |-1/2\rangle. \quad (2.20)$$

Thus, the massless vector multiplet corresponds to a Weyl fermion and a massless spin 1 particle (gauge boson). As well as the graviton and his partner:

$$|\Omega_o\rangle = |3/2\rangle, \quad a^\dagger |\Omega_o\rangle = |+2\rangle,$$

$$|\bar{\Omega}_o\rangle = |-2\rangle, \quad a^\dagger |\bar{\Omega}_o\rangle = |-3/2\rangle. \quad (2.21)$$

These are a tensor-spinor and a tensor.

2.2.1.2 Massive multiplet

In case of $m \neq 0$, there are P^μ - eigenvalues $p^\mu = (m, 0, 0, 0)$ and Casimirs,

$$C_1 = P_\mu P^\mu \quad C_2 = C_{\mu\nu} C^{\mu\nu} = 2m^4 J_i J^i, \quad (2.22)$$

where J_i is the superspin operator. This operator also satisfies the $SU(2)$ algebra,

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad (2.23)$$

²**CPT symmetry** means that the superfield must be invariant under charge conjugation, time reversal and parity operations. Since the total CPT operation changes the sign of spin, for massless particles (labeled by helicity), invariance under CPT follows that components with helicity λ have $(-\lambda)$ -helicity partners.

so, J_i is an angular momentum operator. For J^2 , eigenvalues are $j(j+1)$, they label irreducible representations by $|m, j\rangle$. Again, the anticommutation relations for Q and \bar{Q} are the key to get the states,

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu = 4m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.24)$$

Since both Q 's have nonzero anticommutators with their \bar{Q} partner, define two sets of ladder operators:

$$a_{1,2} := \frac{Q_{1,2}}{2\sqrt{E}} \quad a_{1,2}^\dagger := \frac{\bar{Q}_{\dot{1},\dot{2}}}{2\sqrt{E}} \quad (2.25)$$

with anticommutation relations

$$\{a_p, a_q^\dagger\} = \delta_{pq}, \quad \{a_p, a_q\} = \{a_p^\dagger, a_q^\dagger\} = 0. \quad (2.26)$$

Let $|\Omega\rangle$ be the vacuum state, annihilated by $a_{1,2}$. Consequently,

$$J_k |\Omega\rangle = S_k |\Omega\rangle - \frac{1}{4m} \bar{Q}_{\dot{\alpha}} (\bar{\sigma}_k)^{\dot{\alpha}\beta} Q_\beta |\Omega\rangle \stackrel{0}{=} J_k |\Omega\rangle = S_k |\Omega\rangle, \quad (2.27)$$

i.e. for $|\Omega\rangle$ the spin number s and the superspin number j are the same. We can label the massive states by $|m, s, s_k\rangle$. From the relation

$$[M_{ij}, Q_a] = -(\sigma_{ij})_{ab} Q_b, \quad (2.28)$$

for $k=3$,

$$[S_3, Q_1] = \frac{1}{2}(\sigma_3)_{11} Q_1 = \frac{1}{2}Q_1 \quad [S_3, Q_2] = \frac{1}{2}(\sigma_3)_{22} Q_2 = -\frac{1}{2}Q_2. \quad (2.29)$$

So a_1 decreases the spin in $\frac{1}{2}$, and a_2 increases it in $\frac{1}{2}$. In other words, as we did for the massless multiplet

$$\begin{aligned} a_1 |s, s_3\rangle &= |s_3 - \frac{1}{2}\rangle & a_1^\dagger |s, s_3\rangle &= |s_3 + \frac{1}{2}\rangle, \\ a_2 |s, s_3\rangle &= |s_3 + \frac{1}{2}\rangle & a_2^\dagger |s, s_3\rangle &= |s_3 - \frac{1}{2}\rangle. \end{aligned} \quad (2.30)$$

For example, the case $S=0$, the massive chiral multiplet

$$\begin{aligned} |\Omega\rangle &= |m, j=0, p^\mu, j_3=0\rangle, \\ a_{1,2}^\dagger |\Omega\rangle &= |m, j=\frac{1}{2}, p^\mu, j_3=\pm\frac{1}{2}\rangle, \\ a_1^\dagger a_2^\dagger |\Omega\rangle &= |m, j=0, p^\mu, j_3=0\rangle = |\Omega\rangle. \end{aligned} \quad (2.31)$$

which corresponds to a Majorana fermion and a complex scalar. We say a Majorana fermion since we have a single two component fermion that we have assumed to be massive.

The massive vector multiplet is formed by starting with the spin $\frac{1}{2}$ Clifford vacuum:

$$|\Omega\rangle = |m, j = \frac{1}{2}, p^\mu, j_3 = \pm \frac{1}{2}\rangle = |\frac{1}{2}, \pm \frac{1}{2}\rangle. \quad (2.32)$$

Then,

$$\begin{aligned} a_1^\dagger |\frac{1}{2}, -\frac{1}{2}\rangle &= |0, 0\rangle, \\ a_2^\dagger |\frac{1}{2}, -\frac{1}{2}\rangle &= |1, -1\rangle, \\ a_1^\dagger |\frac{1}{2}, +\frac{1}{2}\rangle &= |1, +1\rangle, \\ a_2^\dagger |\frac{1}{2}, +\frac{1}{2}\rangle &= |0, 0\rangle. \end{aligned} \quad (2.33)$$

and,

$$\begin{aligned} a_1^\dagger a_2^\dagger |\frac{1}{2}, -\frac{1}{2}\rangle &= |\frac{1}{2}, -\frac{1}{2}\rangle, \\ a_1^\dagger a_2^\dagger |\frac{1}{2}, +\frac{1}{2}\rangle &= |\frac{1}{2}, +\frac{1}{2}\rangle. \end{aligned} \quad (2.34)$$

and no more. These correspond to two Majorana fermions, a massive vector and a real scalar.

2.2.2 $\mathcal{N} > 1$ extended SUSY

The spinor generators get an additional label $A, B = 1, \dots, \mathcal{N}$. The algebra is the same as for $\mathcal{N} = 1$ except for

$$\left\{ Q_\alpha^A, \bar{Q}_{\dot{\beta} B} \right\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta_B^A, \quad \left\{ Q_\alpha^A, Q_\beta^B \right\} = 2\sqrt{2}\epsilon_{\alpha\beta} Z^{AB}, \quad (2.35)$$

where Z^{AB} is the antisymmetric central charge commuting with all the generators.

$$[Z^{AB}, P^\mu] = [Z^{AB}, M^{\mu\nu}] = [Z^{AB}, Q_\alpha^A] = [Z^{AB}, Z^{CD}] = [Z^{AB}, T_a] = 0. \quad (2.36)$$

Then, they form an invariant subalgebra of internal symmetries. In presence of this Z^{AB} , the SUSY operators cannot be straightforwardly interpreted as creation and annihilation operators. For $Z^{AB} = 0$, we obtain the following representations [4, 6, 9].

2.2.2.1 Massless multiplet

This case is the simplest but it has very important implications. Let $p_\mu = (E, 0, 0, E)$, then

$$\{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta_B^A \quad (2.37)$$

So $Q_2^A = 0$ for all A . Immediately we see that the central charges must vanish. There are no problems to define the creation and annihilation operators to construct the massless multiplet,

$$a^A := \frac{1}{2\sqrt{E}} Q_1^A, \quad a^{A\dagger} := \frac{1}{2\sqrt{E}} \bar{Q}_1^A. \quad (2.38)$$

Starting from the vacuum $|\Omega\rangle$, with helicity λ_0 and considering antisymmetric products of k different creation operators, we get that the state

$$a^{A_1\dagger} \dots a^{A_k\dagger} |\Omega\rangle, \quad (2.39)$$

has helicity $\lambda_0 + \frac{k}{2}$. It is not difficult to see that there are actually $\binom{\mathcal{N}}{k}$ states with this helicity, by ordering creation operators. Then, the full multiplet will have

$$\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}} \quad (2.40)$$

states. Now consider the following examples:

2.2.2.2 $\mathcal{N} = 2$ vector multiplet

Starting from a state with $\lambda_0 = 0$,

$$\begin{aligned} |\Omega_0\rangle &= |0\rangle & (1), \\ a^{A\dagger} |\Omega_0\rangle &= |+\frac{1}{2}\rangle & (2), \\ a^{A\dagger} a^{B\dagger} |\Omega_0\rangle &= |+1\rangle & (1). \end{aligned} \quad (2.41)$$

Considering the CPT conjugate states, then the $\mathcal{N} = 2$ consists of a scalar, a Weyl spinor and a vector. Note that the massless this vector multiplet can be built from one $\mathcal{N} = 1$ vector multiplet and one $\mathcal{N} = 1$ chiral multiplet.

2.2.2.3 $\mathcal{N} = 2$ hypermultiplet

Starting from a state with $\lambda_0 = -\frac{1}{2}$. The states of helicity are

$$\begin{aligned} |\Omega_0\rangle &= |-\frac{1}{2}\rangle \quad (1), \\ a^{A\dagger} |\Omega_0\rangle &= |0\rangle \quad (2), \\ a^{A\dagger} a^{B\dagger} |\Omega_0\rangle &= |+\frac{1}{2}\rangle \quad (1). \end{aligned} \quad (2.42)$$

So, this multiplet is composed by a complex scalar and two Weyl spinors. Notice also that it is CPT invariant. If we are considering theories with gauge interactions then the fermions of this multiplet must be in the same gauge representation. Theories where all fermions can have mass terms are called vector-like as opposed to chiral theories where at least some of the fermions do not have gauge invariant mass terms. The structure of the $\mathcal{N} = 1$ chiral multiplet allows us to construct chiral theories. Since the standard model of particle physics is a chiral theory, it is thought that $\mathcal{N} = 1$ theories are more directly relevant to the real world, as we will see at the end of this work, when we study the known duality cascade that leads to an $\mathcal{N} = 1$ $SU(M)$ gauge theory living on branes placed at the tip of a conical singularity.

2.2.2.4 $\mathcal{N} = 4$ vector-multiplet

In this case we start with $\lambda_0 = -1$,

$$\begin{aligned} |\Omega_0\rangle &= |-1\rangle \quad (1), \\ a^{A\dagger} |\Omega_0\rangle &= |-\frac{1}{2}\rangle \quad (4), \\ a^{A\dagger} a^{B\dagger} |\Omega_0\rangle &= |0\rangle \quad (6), \\ a^{A\dagger} a^{B\dagger} a^{C\dagger} |\Omega_0\rangle &= |+\frac{1}{2}\rangle \quad (4), \\ a^{A\dagger} a^{B\dagger} a^{C\dagger} a^{D\dagger} |\Omega_0\rangle &= |+1\rangle \quad (1). \end{aligned} \quad (2.43)$$

This multiplet consists of one $\mathcal{N} = 2$ vector multiplet and two $\mathcal{N} = 2$ hypermultiplets plus their CPT conjugates. Or one $\mathcal{N} = 1$ vector and three $\mathcal{N} = 1$ chiral multiplets plus their CPT conjugates. From these results we can extract that for every multiplet,

$$\lambda_{\max} - \lambda_{\min} = \frac{\mathcal{N}}{2}. \quad (2.44)$$

We know that renormalizable theories does not contain states with $|\lambda_0| > 1$ (since gravity is not renormalizable). It implies that renormalizable supersymmetric theories will be possible when $\mathcal{N} \leq 4$. Hence $\mathcal{N} = 4$ is called the maximal supersymmetric theory.

2.2.2.5 Massive representations and BPS states

As shown by Haag, Lapuszanski and Sonnius, the superalgebra admits central extensions, called central charges. Existence of central charges do not allow to define creation and annihilation operators to construct states. Let us consider for simplicity $p_\mu = (m, 0, 0, 0)$ (the static case). So

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.45)$$

Here, the central charge will not vanish. Therefore, we have to distinguish two cases. First, when $Z_{AB} = 0$, there are $2\mathcal{N}$ creation and annihilation operators, corresponding to $Q_{1,2}^A$ and $\bar{Q}_{1,2}^A$, leading to $2^{2\mathcal{N}}$ states instead $2^\mathcal{N}$. In this case the number of states, given \mathcal{N} is larger than the massless case. When $Z_{AB} \neq 0$, we see clearly in (2.36) that is not possible to define creation and annihilation operators independently. But the central charge can be skew-diagonalized to $\mathcal{N}/2$ real eigenvalues. It is sufficient to consider the $\mathcal{N} = 2$ case,

$$\begin{aligned} \{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta_B^A, \\ \{Q_\alpha^A, Q_\beta^B\} &= 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{AB} Z. \end{aligned} \quad (2.46)$$

Central charges can be skew-diagonalized by defining,

$$\begin{aligned} A_\alpha &= \frac{1}{2}(Q_\alpha^1 + \epsilon_{\alpha\beta}\bar{Q}_\beta^2), \\ B_\alpha &= \frac{1}{2}(Q_\alpha^1 - \epsilon_{\alpha\beta}\bar{Q}_\beta^2). \end{aligned} \quad (2.47)$$

These combinations have the algebra,

$$\begin{aligned} \{A_\alpha, \bar{A}_\beta\} &= \delta_{\alpha\beta}(M + \sqrt{2}Z), \\ \{B_\alpha, \bar{B}_\beta\} &= \delta_{\alpha\beta}(M - \sqrt{2}Z). \end{aligned} \quad (2.48)$$

Where M and Z are the mass and central charge of the supermultiplet. The other commutator vanishes. Let us introduce the idea of BPS state. Since all physical states have positive definite norm, it follows that for massless states, the central charge is trivially realized ($Z = 0$). For massive states, this leads to a bound on the mass $M \geq \sqrt{2}|Z|$. When $M = \sqrt{2}|Z|$, one set of operators in the last algebra is trivially realized and the algebra resembles the massless case and the dimension of the representation is greatly reduced. For example, a reduced massive $\mathcal{N} = 2$ multiplet has the same number of states as a massless $\mathcal{N} = 2$ one. Thus the representation of

$\mathcal{N} = 2$ algebra with central charge can be classified as either long multiplets (when $M > \sqrt{2}|Z|$) or short multiplets (when $Z = \sqrt{2}|Z|$). The latter are usually known as BPS multiplet. Because the BPS condition, this state is annihilated by half of the supercharges, i.e. it is invariant under half of the supersymmetries. We will see the same concept in the context of D-branes.

2.3 $\mathcal{N} = 1$ superspace formalism

In order to construct supersymmetric lagrangians, we need a formalism in which supersymmetry is manifest. It means that SUSY arises naturally in the theory together the other transformations. Then we need to extend the Minkowski space to a space that allows us to include SUSY manifestly, a superspace. Then, if we formulate the theory in terms of the two-components Weyl spinor formalism, these new extra coordinates will be,

$$\{\theta_A\}_{A=1,2}, \quad \{\bar{\theta}_{\dot{B}}\}_{\dot{B}=1,2}. \quad (2.49)$$

With these Grassmann coordinates we can transform the graded Lie algebra (involving both commutators and anticommutators) into a regular Lie algebra (which involves only commutators) by writing the elements of the spinor sector of the algebra,

$$\theta^A Q_A, \quad \bar{\theta}_{\dot{A}} \bar{Q}^{\dot{A}}. \quad (2.50)$$

The anticommutation relations of the two-component Weyl spinors are given by

$$\{\theta_A, \theta_B\} = \{\bar{\theta}_{\dot{A}}, \bar{\theta}_{\dot{B}}\} = \{\theta_A, \bar{\theta}_{\dot{B}}\} = 0, \quad (2.51)$$

and an element of superspace is given by the supercoordinate $(x_\mu, \theta_A, \bar{\theta}_{\dot{B}})$. Then, with these properties the relevant part of the super Poincaré algebra in the Weyl formalism can be rewritten as

$$[\theta^A Q_A, \bar{\theta}_{\dot{B}} \bar{Q}^{\dot{B}}] = 2\theta^A \sigma^\mu_{A\dot{B}} \bar{\theta}^{\dot{B}} P_\mu, \quad (2.52)$$

$$[\theta^A Q_A, \theta^B Q_B] = [\bar{\theta}_{\dot{A}} \bar{Q}^{\dot{A}}, \bar{\theta}_{\dot{B}} \bar{Q}^{\dot{B}}] = 0. \quad (2.53)$$

2.3.1 SUSY transformations

In this formalism, the spacetime is essentially eight-dimensional, being parametrized by x_μ and θ_A and $\bar{\theta}_{\dot{A}}$. A simple way to obtain the group action on this homogeneous space is to define the

unitary operators. Then we define translations on this superspace by

$$\begin{aligned} x_\mu &\rightarrow x_\mu + i\theta\sigma_\mu\bar{\epsilon} - i\epsilon\sigma_\mu\bar{\theta}, \\ \theta &\rightarrow \theta + \epsilon, \quad \bar{\theta} \rightarrow \bar{\theta} + \bar{\epsilon}. \end{aligned} \quad (2.54)$$

Next we will get the infinitesimal SUSY transformation. For Φ ,

$$\begin{aligned} \delta_s\Phi &= \Phi(x, \theta, \bar{\theta}) + i(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta})\partial_\mu\Phi(x, \theta, \bar{\theta}) + \epsilon\frac{\partial}{\partial\theta}\Phi(x, \theta, \bar{\theta}) + \\ &\quad \bar{\epsilon}\frac{\partial}{\partial\bar{\theta}}\Phi(x, \theta, \bar{\theta}) + \dots - \Phi(x, \theta, \bar{\theta}). \end{aligned} \quad (2.55)$$

Considering expression for the variation of a field generated by some operator (Q in this case)

$$\delta_s\Phi = i\epsilon Q\Phi(x, \theta, \bar{\theta}) + i\bar{\epsilon}\bar{Q}\Phi(x, \theta, \bar{\theta}). \quad (2.56)$$

Comparing the two expressions for $\delta_s\Phi$ we obtain the following differential operator representation of the generators,

$$\begin{aligned} Q_A &= -i\partial_A - (\sigma^\mu\bar{\theta})_A\partial_\mu, \\ \bar{Q}_{\dot{A}} &= -i\bar{\partial}_{\dot{A}} - (\bar{\sigma}^\mu\theta)^{\dot{A}}\partial_\mu. \end{aligned} \quad (2.57)$$

For reasons of consistency, these operators must satisfy the same algebra we assumed before. And they do,

$$\{Q_A, Q_B\} = 0, \quad \{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = 0, \quad (2.58)$$

which is not difficult to be proved. And

$$\{Q_A, \bar{Q}_{\dot{B}}\} = 2\sigma^\mu_{A\dot{B}}P_\mu. \quad (2.59)$$

Covariant derivatives are derivatives which are useful in the construction of manifestly supersymmetric lagrangians. So we define them as

$$\begin{aligned} D_A &:= \partial_A + i\sigma^\mu_{A\dot{B}}\bar{\theta}^{\dot{B}}\partial_\mu, \\ \bar{D}_{\dot{A}} &:= -\bar{\partial}_{\dot{A}} - i\theta^B\sigma^\mu_{B\dot{A}}\partial_\mu. \end{aligned} \quad (2.60)$$

These operators are invariant under SUSY transformations. In other words,

$$\begin{aligned} \{D_A, Q_B\} &= \{D_A, \bar{Q}^{\dot{B}}\} = 0, \\ \{\bar{D}_{\dot{A}}, Q_B\} &= \{\bar{D}_{\dot{A}}, \bar{Q}^{\dot{B}}\} = 0, \end{aligned} \quad (2.61)$$

derivatives obey the following algebra,

$$\begin{aligned}\{D_A, D_B\} &= \{\bar{D}_{\dot{A}}, \bar{D}_{\dot{B}}\} = 0, \\ \{D_A, \bar{D}_{\dot{B}}\} &= 2\sigma_{A\dot{B}}^\mu P_\mu.\end{aligned}\tag{2.62}$$

2.3.2 Components fields

For a general scalar superfield $\Phi(x, \theta, \bar{\theta})$, one can make an expansion in powers of θ and $\bar{\theta}$ with a finite number of nonzero terms,

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) &= f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + (\theta\theta)m(x) + (\bar{\theta}\bar{\theta})n(x) + (\theta\sigma^\mu\bar{\theta})V_\mu(x) \\ &\quad + (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\psi(x) + (\theta\theta)(\bar{\theta}\bar{\theta})d(x),\end{aligned}\tag{2.63}$$

where $f(x)$, $\phi(x)$, $\bar{\chi}(x)$, $m(x)$, $n(x)$, $V_\mu(x)$, $\bar{\lambda}(x)$, $\psi(x)$, $d(x)$ are called component fields. Notice also why this expansion stops, it is because of the anticommuting nature of θ and $\bar{\theta}$ that only allow us to have terms until $(\theta\theta)(\bar{\theta}\bar{\theta})$. The transformation law for superfields is defined as

$$\begin{aligned}\delta\Phi(x, \theta, \bar{\theta}) &= \delta f(x) + \theta\delta\phi(x) + \bar{\theta}\delta\bar{\chi}(x) + (\theta\theta)\delta m(x) + (\bar{\theta}\bar{\theta})\delta n(x) \\ &\quad + (\theta\sigma^\mu\bar{\theta})\delta V_\mu(x) + (\theta\theta)\bar{\theta}\delta\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\delta\psi(x) \\ &\quad + (\theta\theta)(\bar{\theta}\bar{\theta})\delta d(x).\end{aligned}\tag{2.64}$$

On the other hand, remember the transformation we found before,

$$\delta\Phi(x, \theta, \bar{\theta}) = [\epsilon\partial + \bar{\epsilon}\bar{\partial} + i\theta\sigma^\mu\bar{\epsilon}\partial_\mu - i\epsilon\sigma^\mu\bar{\theta}\partial_\mu] \Phi(x, \theta, \bar{\theta}).\tag{2.65}$$

Then, acting this variation operation over the expansion we wrote,

$$\begin{aligned}\delta_s\Phi(x, \theta, \bar{\theta}) &= \{ \epsilon\partial + \bar{\epsilon}\bar{\partial} + i\theta\sigma^\mu\bar{\epsilon}\partial_\mu - i\epsilon\sigma^\mu\bar{\theta}\partial_\mu \} \times \{ f(x) + \theta\phi(x) \\ &\quad + \bar{\theta}\bar{\chi}(x) + (\theta\theta)m(x) + (\bar{\theta}\bar{\theta})n(x) + (\theta\sigma^\mu\bar{\theta})V_\mu(x) + (\theta\theta)\bar{\theta}\bar{\lambda}(x) \\ &\quad + (\bar{\theta}\bar{\theta})\theta\psi(x) + (\theta\theta)(\bar{\theta}\bar{\theta})d(x) \}.\end{aligned}\tag{2.66}$$

Multiplying out this yields

$$\begin{aligned}
\delta_s \Phi &= \epsilon \phi + \bar{\phi} \bar{\chi} \\
&+ \theta \{ 2\epsilon m + i(\sigma^\mu \bar{\epsilon}) \partial_\mu f + (\sigma^\mu \bar{\epsilon}) V_\mu \} \\
&+ \{ 2\bar{\epsilon} n + i(\epsilon \sigma^\mu) \partial_\mu f - (\epsilon \sigma^\mu) V_\mu \} \bar{\theta} \\
&+ (\theta \theta) \left\{ \bar{\epsilon} \bar{\lambda} - \frac{i}{2} \partial_\mu \phi \sigma^\mu \bar{\epsilon} \right\} \\
&+ (\bar{\theta} \bar{\theta}) \left\{ \epsilon \psi + \frac{i}{2} \epsilon \sigma^\mu \partial_\mu \bar{\chi} \right\} \\
&+ (\theta \sigma^\mu \bar{\theta}) \left\{ \epsilon \sigma_\mu \bar{\lambda} + \psi \sigma_\mu \bar{\epsilon} + \frac{i}{2} \epsilon \partial_\mu \phi - \frac{i}{2} \partial_\mu \bar{\chi} \bar{\epsilon} \right\} \\
&+ (\theta \theta) \left\{ 2\bar{\epsilon} d + \frac{i}{2} \bar{\epsilon} \partial^\mu V_\mu + i(\epsilon \sigma^\mu) \partial_\mu m \right\} \bar{\theta} \\
&+ (\bar{\theta} \bar{\theta}) \theta \left\{ 2\epsilon d - \frac{i}{2} \epsilon \partial^\mu V_\mu + i(\sigma^\mu \bar{\epsilon}) \right\} \\
&+ (\theta \theta) (\bar{\theta} \bar{\theta}) \frac{i}{2} \left\{ \partial_\mu \psi \sigma^\mu \bar{\epsilon} + \epsilon \sigma^\mu \partial_\mu \bar{\lambda} \right\}. \tag{2.67}
\end{aligned}$$

Comparing this result with the transformation law written before in (2.64), we obtain the transformation laws for each field,

$$\begin{aligned}
\delta_s f &= \epsilon \phi + \bar{\phi} \bar{\chi}, \\
\delta_s \phi &= 2\epsilon m + i(\sigma^\mu \bar{\epsilon}) \partial_\mu f + (\sigma^\mu \bar{\epsilon}) V_\mu, \\
\delta_s \bar{\chi} &= 2\bar{\epsilon} n + i(\epsilon \sigma^\mu) \partial_\mu f - (\epsilon \sigma^\mu) V_\mu, \\
\delta_s m &= \bar{\epsilon} \bar{\lambda} - \frac{i}{2} \partial_\mu \phi \sigma^\mu \bar{\epsilon}, \\
\delta_s n &= \epsilon \psi + \frac{i}{2} \epsilon \sigma^\mu \partial_\mu \bar{\chi}, \\
\delta_s V_\mu &= \epsilon \sigma_\mu \bar{\lambda} + \psi \sigma_\mu \bar{\epsilon} + \frac{i}{2} \epsilon \partial_\mu \phi - \frac{i}{2} \partial_\mu \bar{\chi} \bar{\epsilon}, \\
\delta_s \bar{\lambda} &= 2\bar{\epsilon} d + \frac{i}{2} \bar{\epsilon} \partial^\mu V_\mu + i(\epsilon \sigma^\mu) \partial_\mu m, \\
\delta_s \psi &= 2\epsilon d - \frac{i}{2} \epsilon \partial^\mu V_\mu + i(\sigma^\mu \bar{\epsilon}), \\
\delta_s d &= \frac{i}{2} \left\{ \partial_\mu \psi \sigma^\mu \bar{\epsilon} + \epsilon \sigma^\mu \partial_\mu \bar{\lambda} \right\}, \\
&= \frac{i}{2} \partial_\mu \left\{ \psi \sigma^\mu \bar{\epsilon} + \epsilon \sigma^\mu \bar{\lambda} \right\}. \tag{2.68}
\end{aligned}$$

Notice that $\delta_s d$ is a total derivative (the significance of this result will be seen later).

2.3.3 Irreducible representations of superfields

Since the component fields of our superfield transform independently we can eliminate some of them by imposing constraints, leading to a smaller superfield. There are some important conditions to get those irreducible superfields.

Depending of the condition we impose we can define different irreducible representations.

2.3.3.1 Chiral superfields

Superfields which satisfy either of the constraints,

$$\bar{D}\Phi = 0, \quad D\Phi = 0, \quad (2.69)$$

are called chiral or scalar superfields. A superfield Φ which satisfies $\bar{D}\Phi = 0$ is called left-handed chiral superfield, and if it satisfies $D\Phi = 0$, is called right-handed chiral superfield. Let us consider the left-handed chiral superfield. The constraint,

$$\bar{D}_{\dot{A}}\Phi(x, \theta, \bar{\theta}) = 0, \quad (2.70)$$

where $\bar{D}_{\dot{A}} = -\bar{\partial}_{\dot{A}} - i\theta^B\sigma_{B\dot{A}}^{\mu}\partial_{\mu}$,

$$y^{\mu} := x^{\mu} + i\theta\sigma^{\mu}\bar{\theta}. \quad (2.71)$$

The constraint for chiral superfields reduces to $\bar{\partial}_{\dot{A}}\Phi = 0$. For a superfield $\Phi(y, \theta)$,

$$\Phi(y, \theta) = A(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y). \quad (2.72)$$

Where $A(y)$ and $F(y)$ are complex scalars. Back to the variable x , the superfield can be expanded by using (2.71) as

$$\begin{aligned} \Phi(x^{\mu}, \theta) &= A(x) + i(\theta\sigma^{\mu}\bar{\theta})\partial_{\mu}A(x) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^2A + \sqrt{2}\theta\psi \\ &\quad - \frac{i}{\sqrt{2}}(\theta\theta)\partial_{\mu}\psi\sigma^{\mu}\bar{\theta} + (\theta\theta)F(x). \end{aligned} \quad (2.73)$$

In the same way, we can impose other condition:

$$D_A\Phi^{\dagger}(x, \theta, \bar{\theta}) = 0, \quad (2.74)$$

and introduce the new variables,

$$z^{\mu} := x^{\mu} - i\theta\sigma^{\mu}\bar{\theta}, \quad (2.75)$$

and $D_A = \partial_A + i\sigma_{A\dot{B}}^\mu \bar{\theta}^{\dot{B}} \partial_\mu$. Working out in the same way we did before we will get,

$$\begin{aligned} \Phi^\dagger(x, \theta, \bar{\theta}) &= A^*(x) - i(\theta\sigma^\mu\bar{\theta})\partial_\mu A^*(x) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^2 A^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) \\ &\quad - \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})(\theta\sigma^\mu\partial_\mu\bar{\psi}(x)) + (\bar{\theta}\bar{\theta})F^*(x). \end{aligned} \quad (2.76)$$

This is called left-handed chiral superfield. From the general expression of the supersymmetry transformation (2.68), we can deduce the transformation laws for these superfields. For the components of the left-handed chiral one we obtain,

$$\begin{aligned} \delta_s A(y) &= \sqrt{2}\alpha\psi(y), \\ \delta_s \psi_A(y) &= \sqrt{2}\alpha_A F(y) + i\sqrt{2}\sigma_{A\dot{B}}^\mu \bar{\alpha}^{\dot{B}} \partial_\mu A(y), \\ \delta_s F(y) &= -i\sqrt{2}\partial_\mu\psi(y)\sigma^\mu\bar{\alpha}. \end{aligned} \quad (2.77)$$

It is evident that these component fields constitute an irreducible representation of the supersymmetry algebra since they transform into themselves. It is also important to notice in the third transformation that δF is proportional to the field equation of ψ . So the algebra closes on- and off-shell. The fact that F transforms as a total derivative allows us to get a supersymmetry invariant quantity whose variation can be integrated out.

In order to construct supersymmetric lagrangians we need the product of a left- and a right-handed chiral superfield. This product will not be neither chiral nor antichiral. Consider a set of fields,

$$\begin{aligned} \Phi^\dagger(z) &= A^*(z) + \sqrt{2}\bar{\theta}\bar{\psi}(z) + (\bar{\theta}\bar{\theta})F^*(z), \\ \Phi(y) &= A(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y), \end{aligned} \quad (2.78)$$

then,

$$\begin{aligned} \Phi^\dagger\Phi(x) &= A^*(x)A(x) + \sqrt{2}\theta\psi(x)A^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x)A(x) + (\theta\theta)A^*(x)F(x) + (\bar{\theta}\bar{\theta})F^*(x)A(x) \\ &\quad + 2\bar{\theta}\bar{\psi}(x)\theta\psi(x) + i(\theta\sigma^\mu\bar{\theta})[\partial_\mu A(x)A^*(x) - \partial_\mu A^*(x)A(x)] \\ &\quad - \sqrt{2}(\theta\theta)\bar{\theta}_A \left\{ \frac{i}{2}\sigma_{A\dot{B}}^\mu \epsilon^{\dot{B}\dot{A}} (\psi^{\dot{A}}(x)\partial_\mu A^*(x) - A^*(x)\partial_\mu\psi^{\dot{A}}(x)) + \bar{\psi}^{\dot{A}}(x)F(x) \right\} \\ &\quad + \sqrt{2}(\bar{\theta}\bar{\theta})\theta^A \left\{ -\frac{i}{2}\sigma_{A\dot{B}}^\mu (\bar{\psi}^{\dot{B}}(x)\partial_\mu A(x) - A(x)\partial_\mu\bar{\psi}^{\dot{B}}(x)) + \psi_A F^*(x) \right\} \\ &\quad + (\theta\theta)(\bar{\theta}\bar{\theta}) \left\{ \frac{1}{2}\partial_\mu A^*(x)\partial^\mu A(x) - \frac{1}{4}A^*(x)\partial^2 A(x) - \frac{1}{4}A(x)\partial^2 A^*(x) + \frac{i}{2}\partial_\mu\psi(x)\sigma^\mu\bar{\psi}(x) \right. \\ &\quad \left. - \frac{i}{2}\psi(x)\sigma^\mu\partial_\mu\bar{\psi}(x) + F^*(x)F(x) \right\}. \end{aligned}$$

The last term $(\theta\theta)(\bar{\theta}\bar{\theta})N(x)$ can be written easily in a more suggestive form,

$$N(x) = -A(x)^* \partial^2 A(x) + i \partial_\mu \bar{\psi}(x) \bar{\sigma}^\mu \psi(x) + |F_i(x)|^2 + \text{T.Der.} \quad (2.79)$$

2.3.3.2 Vector superfields

They are superfields that satisfy the reality condition,

$$V(x, \theta, \bar{\theta}) = V^\dagger(x, \theta, \bar{\theta}). \quad (2.80)$$

The superfield is as usual defined by its power series expansion in θ and $\bar{\theta}$,

$$\begin{aligned} V(x, \theta, \bar{\theta}) = & C(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + (\theta\theta)M(x) + (\bar{\theta}\bar{\theta})N(x) + (\theta\sigma^\mu\bar{\theta})V_\mu(x) \\ & + (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\psi(x) + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x). \end{aligned} \quad (2.81)$$

The hermitian conjugate is

$$\begin{aligned} V^\dagger(x, \theta, \bar{\theta}) = & C^*(x) + \bar{\theta}\bar{\phi}(x) + \theta\chi(x) + (\bar{\theta}\bar{\theta})M^*(x) + (\theta\theta)N^*(x) + (\theta\sigma^\mu\bar{\theta})V_\mu^*(x) \\ & + (\bar{\theta}\bar{\theta})\theta\lambda(x) + (\theta\theta)\bar{\theta}\bar{\psi}(x) + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D^*(x). \end{aligned} \quad (2.82)$$

And the reality condition constraints the expansion to be

$$\begin{aligned} V(x, \theta, \bar{\theta}) = & C(x) + \theta\phi(x) + \bar{\theta}\bar{\phi}(x) + (\theta\theta)M(x) + (\bar{\theta}\bar{\theta})M^*(x) + (\theta\sigma^\mu\bar{\theta})V_\mu(x) \\ & + (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\lambda(x) + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x). \end{aligned} \quad (2.83)$$

where $M(x), D(x)$ and $C(x)$ are scalar fields, $\lambda(x)$ and $\phi(x)$ are spinor fields, and $V_\mu(x)$ is a vector field. The vector field lends its name to the entire multiplet $V(x, \theta, \bar{\theta})$. We already know how the $(\theta\theta)(\bar{\theta}\bar{\theta})$ -component transforms (as we see in (2.68)), thus for our case we will get,

$$\delta D(x) = \frac{i}{2}(\partial_\mu \lambda(x) \sigma^\mu \bar{\alpha} - \partial_\mu \bar{\lambda}(x) \bar{\sigma}^\mu \alpha), \quad (2.84)$$

which is a total derivative, and a good candidate for a supersymmetric lagrangian. An important example of a vector superfield is the sum of a left- and right-handed chiral superfield. We can do this since $(\Phi + \Phi^\dagger)^\dagger = \Phi + \Phi^\dagger$, which is the reality condition we imposed before. Expanding this sum in terms of component fields,

$$\begin{aligned} \Phi + \Phi^\dagger = & A(x) + A^*(x) + \sqrt{2}\theta\psi(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + (\theta\theta)F(x) + (\bar{\theta}\bar{\theta})F^*(x) \\ & + i(\theta\sigma^\mu\bar{\theta})\partial_\mu(A(x) - A^*(x)) - \frac{i}{\sqrt{2}}(\theta\theta)\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) - \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})\theta\sigma^\mu\partial_\mu\bar{\psi}(x) \\ & - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^2(A(x) + A^*(x)). \end{aligned} \quad (2.85)$$

Now, we could modify some components of the vector superfield in order they are invariant under gauge transformations. This is achieved by making the following replacements,

$$\begin{aligned}\lambda(x) &\rightarrow \lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\phi}(x), \\ D(x) &\rightarrow D(x) - \frac{1}{4}\partial^2 C(x).\end{aligned}\tag{2.86}$$

With these changes,

$$\begin{aligned}V(x, \theta, \bar{\theta}) &= C(x) + \theta\phi(x) + \bar{\theta}\bar{\phi}(x) + (\theta\theta)M(x) + (\bar{\theta}\bar{\theta})M^*(x) + (\theta\sigma^\mu\bar{\theta})V_\mu(x) \\ &\quad + (\theta\theta)\bar{\theta}[\bar{\lambda}(x) - \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\phi] + (\bar{\theta}\bar{\theta})\theta[\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\phi}] \\ &\quad + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})(D(x) - \frac{1}{4}\partial^2 C(x)).\end{aligned}\tag{2.87}$$

The following transformation for the vector superfield, V ³,

$$V(x, \theta, \bar{\theta}) \rightarrow V'(x, \theta, \bar{\theta}) = V(x, \theta, \bar{\theta}) + \Phi(x, \theta, \bar{\theta}) + \Phi^\dagger(x, \theta, \bar{\theta}).\tag{2.88}$$

The transformed superfield then has the following form,

$$\begin{aligned}V'(x, \theta, \bar{\theta}) &= C(x) + A(x) + A^*(x) + \theta[\phi(x) + \sqrt{2}\psi(x)] + \bar{\theta}[\bar{\phi}(x) + \sqrt{2}\bar{\psi}(x)] \\ &\quad + (\theta\theta)[M(x) + F(x)] + (\bar{\theta}\bar{\theta})[M^*(x) + F^*(x)] \\ &\quad + (\theta\sigma^\mu\bar{\theta})[V_\mu(x) + i\partial_\mu(A(x) - A^*(x))] \\ &\quad + (\theta\theta)\bar{\theta}[\bar{\lambda}(x) - \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\{\phi(x) + \sqrt{2}\psi(x)\}] + (\bar{\theta}\bar{\theta})\theta[\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\{\bar{\phi} + \sqrt{2}\bar{\psi}\}] \\ &\quad + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})[D(x) - \frac{1}{4}\partial^2\{C(x) + A(x) + A^*(x)\}].\end{aligned}\tag{2.89}$$

³If we choose $\Phi = i\Lambda$, the gauge transformation will be

$$V(x, \theta, \bar{\theta}) \rightarrow V'(x, \theta, \bar{\theta}) = V(x, \theta, \bar{\theta}) + i\{\Lambda(x, \theta, \bar{\theta}) - \Lambda^\dagger(x, \theta, \bar{\theta})\},$$

which came from applying an $U(1)$ transformation $U = e^{iq\Lambda(x, \theta, \bar{\theta})}$,

$$e^{qV'(x, \theta, \bar{\theta})} = e^{-iq\Lambda^\dagger(x, \theta, \bar{\theta})} e^{qV(x, \theta, \bar{\theta})} e^{iq\Lambda(x, \theta, \bar{\theta})}.$$

So, this is an Abelian gauge transformation. We will see later that it is obviously possible to do non-Abelian transformations.

Comparing (2.87) and (2.89), we obtain the transformations for each component of the superfield,

$$\begin{aligned}
\delta C(x) &= A(x) + A^*(x), \\
\delta\phi(x) &= \sqrt{2}\psi(x), \\
\delta M(x) &= F(x), \\
\delta V_\mu(x) &= i\partial_\mu\{A(x) - A^*(x)\}, \\
\delta\lambda(x) &= 0, \\
\delta D(x) &= 0.
\end{aligned} \tag{2.90}$$

We see that λ and D are invariant under (2.88). We also observe that $\delta V_\mu(x) = i\partial_\mu\{A(x) - A^*(x)\}$, which corresponds to an Abelian gauge super-transformation. For vector gauge fields we usually define the field strength, $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. The transformation (2.88) is called a the supersymmetric extension of a gauge transformation. Since the $(\theta\theta)(\bar{\theta}\bar{\theta})$ term is a good candidate for a supersymmetric lagrangian as we mentioned above, we see that the invariance of $D(x)$ under (2.88) implies the invariance of this lagrangian under supersymmetry gauge transformations. We also could choose a particular gauge, looking at the Φ superfield, in order to reduce the components of the vector superfield. Hence we choose the so-called Wess-Zumino gauge [4, 9],

$$\begin{aligned}
V_{WZ}(x, \theta, \bar{\theta}) &= (\theta\sigma^\mu\bar{\theta})[V_\mu(x) - 2\partial_\mu\text{Im} A] + (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\lambda(x) \\
&\quad + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x).
\end{aligned} \tag{2.91}$$

Here V_μ is the gauge field and λ is its supersymmetric partner. $D(x)$ is the so-called auxiliary field, and its significance will become clear later. Note that in the last expression $\text{Im}(A)$ is not defined. We say then that the Wess-Zumino gauge does not fix the gauge freedom completely [4].

Now, we will see why this gauge is useful to construct lagrangians in supersymmetric gauge theories. Let us calculate powers of V_{WZ} by taking into account the fact that $\theta^3 = 0$. By taking $\text{Im}(A) = 0$, we get

$$\begin{aligned}
V_{WZ}(x, \theta, \bar{\theta}) &= (\theta\sigma^\mu\bar{\theta})V_\mu(x) + (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\lambda(x) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x), \\
V_{WZ}^2(x, \theta, \bar{\theta}) &= \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})V_\mu(x)V^\mu(x), \\
V_{WZ}^3(x, \theta, \bar{\theta}) &= 0.
\end{aligned} \tag{2.92}$$

These properties make this gauge convenient. The expression $\exp\{V\}$ acquire a simple form

$$\exp\{V\} = 1 + V + \frac{1}{2}V^2.$$

Then,

$$\begin{aligned} \exp\{V\} = & 1 + (\theta\sigma^\mu\bar{\theta})V_\mu(x) + (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\lambda(x) \\ & + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})(D(x) + \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})V_\mu(x)V^\mu(x)). \end{aligned} \quad (2.93)$$

2.3.3.3 The supersymmetric field strength

In a supersymmetric theory, the supercovariant field strength of an arbitrary vector superfield, $V(x, \theta, \bar{\theta})$ is defined by

$$W_A := -\frac{1}{4}(\bar{D}\bar{D})D_A V(x, \theta, \bar{\theta}), \quad (2.94)$$

$$\bar{W}_{\dot{A}} := -\frac{1}{4}(DD)\bar{D}_{\dot{A}} V(x, \theta, \bar{\theta}), \quad (2.95)$$

where W_A is a left-handed chiral superfield and $\bar{W}_{\dot{A}}$ is right-handed. We express this statement as

$$\bar{D}_{\dot{A}}W_A = 0, \quad D_A\bar{W}_{\dot{A}} = 0, \quad (2.96)$$

which is easy to prove by using the definitions. As field strength, these fields are invariant under supersymmetric gauge transformations. The next step is to expand W_A in the Wess-Zumino gauge. In this gauge,

$$W_A(x, \theta, \bar{\theta}) = -\frac{1}{4}(\bar{D}\bar{D})D_A V_{WZ}(x, \theta, \bar{\theta}), \quad (2.97)$$

where the covariant derivative was defined in (2.60). It is more convenient to use new variables, which are just the translated x^μ , (2.71) and (2.75). The vector superfield, in the Wess-Zumino gauge, acquires a little different form

$$\begin{aligned} V_{WZ}(y, \theta, \bar{\theta}) = & (\theta\sigma^\mu\bar{\theta})V_\mu(y) + (\theta\theta)\bar{\theta}\bar{\lambda}(y) + (\bar{\theta}\bar{\theta})\theta\lambda(y) \\ & + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta}) \left[D(y) - \frac{i}{2}\partial_\mu V^\mu(y) \right] = V_{WZ}(z, \theta, \bar{\theta}), \end{aligned} \quad (2.98)$$

and which are equivalent to $V_{WZ}(x, \theta, \bar{\theta})$. In these coordinates, the covariant derivatives are,

$$\begin{aligned} D_A(y, \theta, \bar{\theta}) &= \partial_A + 2i\sigma_{AB}^\mu \bar{\theta}^{\dot{B}} \partial_\mu, \\ \bar{D}_{\dot{A}}(y, \theta, \bar{\theta}) &= -\partial_{\dot{A}}, \\ \bar{D}^{\dot{A}}(y, \theta, \bar{\theta}) &= \partial^{\dot{A}}. \end{aligned} \quad (2.99)$$

Thus, the supersymmetric field strength is “easily” obtained,

$$\begin{aligned} W_A(y, \theta, \bar{\theta}) &= -\frac{1}{4} \bar{D}_{\dot{A}} \bar{D}^{\dot{A}} D_A V_{WZ}(y, \theta, \bar{\theta}) \\ &= \frac{1}{4} \partial_{\dot{A}} \partial^{\dot{A}} D_A \{ (\theta \sigma^\mu \bar{\theta}) V_\mu(y) + (\theta \theta) \bar{\theta} \bar{\lambda}(y) + (\bar{\theta} \bar{\theta}) \theta \lambda(y) \\ &\quad + \frac{1}{2} (\theta \theta) (\bar{\theta} \bar{\theta}) [D(y) - \frac{i}{2} \partial_\mu V^\mu(y)] \}. \end{aligned}$$

After some work with Grassmann variables, we obtain

$$W_A(y, \theta, \bar{\theta}) = \lambda_A(y) + \theta_A D(y) + (\sigma^{\mu\nu} \theta)_A F_{\mu\nu}(y) - i(\theta \theta) \sigma_{A\dot{B}}^\mu D_\mu \bar{\lambda}^{\dot{B}}(y), \quad (2.100)$$

where $F_{\mu\nu}$ is the non-supersymmetric field strength for V_μ . The component expansion of $\bar{W}_{\dot{A}}$ can be obtained in a similar way, but by using z^μ (2.75) instead of x^μ . It is given by

$$\bar{W}_{\dot{A}}(z, \theta, \bar{\theta}) = \bar{\lambda}_{\dot{A}}(z) + \bar{\theta}_{\dot{A}} - \epsilon_{\dot{A}\dot{B}} (\bar{\sigma}^{\mu\nu} \bar{\theta})^{\dot{B}} F_{\mu\nu}(z) + i(\bar{\theta} \bar{\theta}) (\partial_\mu \lambda(z) \sigma^\mu)_{\dot{A}}. \quad (2.101)$$

These are the supersymmetric field strength tensors. They also obey

$$\bar{D}_{\dot{A}} \bar{W}^{\dot{A}} = D^A W_A. \quad (2.102)$$

2.4 Supersymmetric lagrangians and gauge theories

In the last section we introduced basic concepts of supersymmetry in order to have a good background for the following sections and the next chapter. Now, we would like to construct consistent supersymmetric theories. The simplest model that is manifestly supersymmetric was proposed by Wess and Zumino [3, 4, 6]. Their method considers auxiliary fields that allow us to write the transformations of fields under SUSY in a way that is independent of the interactions in the model, and also does not require field equations. The superspace formalism we studied before, allows to compact in a supermultiplet all the field in the theory including the auxiliary ones.

2.4.1 The Wess-Zumino model: free and interacting

Recall the vector superfield (2.83), its last component, D , is the coefficient in front of $(\theta \theta)(\bar{\theta} \bar{\theta})$ transforms as a total derivative. Then we can use this fact to construct a suitable lagrangian. The action,

$$S = \int d^4x d^2\theta d^2\bar{\theta} V(x, \theta, \bar{\theta}), \quad (2.103)$$

is invariant under supersymmetry. We saw before that $\Phi^\dagger\Phi$, the product of a chiral and antichiral, is a vector superfield. This is given by (2.79). Since only the term with $(\theta\theta)(\bar{\theta}\bar{\theta})$ survives the integration and looks like a lagrangian, this product can be written as

$$\Phi^\dagger\Phi = \dots + (\theta\theta)(\bar{\theta}\bar{\theta})\{\partial_\mu\phi^\dagger\partial^\mu\phi + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + |F|^2\}. \quad (2.104)$$

Then, the lagrangian will be

$$\begin{aligned} \mathcal{L} &= \int d^2\theta d^2\bar{\theta} \Phi^\dagger\Phi \\ &= \int d^2\theta d^2\bar{\theta} (\theta\theta)(\bar{\theta}\bar{\theta})\{\partial_\mu\phi^\dagger\partial^\mu\phi + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + |F|^2\} \\ &= \partial_\mu\phi^\dagger\partial^\mu\phi + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + |F|^2. \end{aligned} \quad (2.105)$$

The action is

$$S_{\text{kin}} = \int d^4x \{\partial_\mu\phi^\dagger\partial^\mu\phi + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + |F|^2\}, \quad (2.106)$$

where ϕ is a complex scalar, ψ is a Weyl spinor and F is auxiliary field that is needed for matching degrees of freedom and closing the supersymmetry algebra off-shell. Note that the field equation of this field is $F = 0$. The supersymmetry transformations are

$$\begin{aligned} \delta\psi_A &= -i(\sigma^\mu\bar{\epsilon})_A\partial_\mu\phi + \epsilon F, \\ \delta F &= -i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi, \\ \delta\phi &= \bar{\epsilon}\psi. \end{aligned} \quad (2.107)$$

The invariance of the action under these transformations is easily showed. It is possible to show that this algebra closes off-shell (i.e. without using the equations of motion). It is proved by calculating the commutator of two transformations

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]X = i(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1)\partial_\mu X, \quad (2.108)$$

where X are all the components of the off-shell multiplet.

The superpotential

Now we will consider the interacting case. By definition, the potential terms are those with no derivatives. In general, they are quadratic or of higher order in the component fields. For example, the mass terms, for both bosons and fermions, are quadratic. So, we can consider a

function $\mathcal{W}(\Phi)$ of the chiral field, called superpotential. This function will be a polynomial of Φ . Since the argument of \mathcal{W} is a chiral superfield, the \mathcal{W} will be chiral too. So the \mathcal{W} term in the action will be

$$S_{\text{int}} = \int d^4x \{d^2\theta \mathcal{W}(\Phi) + d^2\bar{\theta} \bar{\mathcal{W}}(\bar{\Phi})\}, \quad (2.109)$$

which is superinvariant. Note that the superpotential must have dimension $[\text{mass}]^3$, then for renormalization grounds, in four spacetime dimensions, \mathcal{W} must be at most cubic in Φ [9], since the measure of the integral has dimension $[\text{mass}]^{-3}$ and the chiral superfield has $[\text{mass}]^1$ by construction [4].

The interacting model

The interacting Wess-Zumino model contains one chiral superfield, its complex conjugate and the potential terms (interactions). The action is

$$S_{WZ} = \int d^4x d^2\theta d^2\bar{\theta} \Phi^\dagger \Phi + \int d^4x \{d^2\theta \mathcal{W}(\Phi) + d^2\bar{\theta} \bar{\mathcal{W}}(\bar{\Phi})\}. \quad (2.110)$$

The holomorphic function $\mathcal{W}(\Phi)$ must be viewed as a generic superpotential. In order to have terms that interact with the component fields of Φ , the superpotential lagrangian must be

$$\mathcal{L}_{\text{int}} = F \mathcal{W}'(\phi) - \frac{1}{2} \mathcal{W}''(\phi) \psi \psi + \text{h.c.} \quad (2.111)$$

This form is justified from the fact that, \mathcal{W} has mass dimension 3, the auxiliary field F has mass dimension 2 and, the Weyl spinor ψ and the scalar ϕ have mass dimensions 3/2 and 1, respectively. The action of the interactive Wess-Zumino model is

$$S_{WZ} = \int d^4x \{ \partial_\mu \phi^\dagger \partial^\mu \phi + i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi + |F|^2 + (F \mathcal{W}'(\phi) - \frac{1}{2} \mathcal{W}''(\phi) \psi \psi + \text{h.c.}) \}. \quad (2.112)$$

From (2.111) we can eliminate the auxiliary field F on-shell. Its equation of motion is

$$F^* = -\frac{\partial \mathcal{W}}{\partial \phi}. \quad (2.113)$$

Hence the lagrangian without auxiliary field is

$$S_{WZ} = \int d^4x \{ \partial_\mu \phi^\dagger \partial^\mu \phi + i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - |\mathcal{W}'|^2 - \frac{1}{2} \{ \mathcal{W}''(\phi) \psi \psi + \text{h.c.} \} \}. \quad (2.114)$$

Note that the scalar potential describing the self-interaction of ϕ is

$$V(\phi) = |\mathcal{W}'|^2. \quad (2.115)$$

This expression implies that the potential energy is positive definite. Unlike non-supersymmetric theories, supersymmetric theories can have vacuum degeneracy without spontaneous symmetry breaking. Then, when $V = 0$ we get the so-called vacuum manifold. A silly example could be, when \mathcal{W} vanishes, any coordinate independent field ϕ_0 can serve as a vacuum. Then, the vacuum manifold is the one-dimensional complex manifold $C^1 = \{\phi_0\}$. The continuous degeneracy is due to the absence of potential energy, while the kinetic energy vanishes for any constant ϕ_0 . We will see later more interesting examples of moduli spaces of vacua when we study specific theories.

2.4.2 R-symmetry

Remember that $[T_a, T_b] = iC_{abc}T_c$, where T_a represents a generator of some group of symmetry G . Let G be an internal symmetry group, then define the R-symmetry $H \subset G$ to be the set of G elements that do not commute with the SUSY generators,

$$[Q_\alpha^A, T_a] = S_a^A{}_B Q_\alpha^B \neq 0 \quad T_a \in H. \quad (2.116)$$

We say that R-symmetry “rotates” the supersymmetry generators without changing the supersymmetry algebra. If $Z^{AB} = 0$, then the R-symmetry is $H = U(\mathcal{N})$, and for $Z^{AB} \neq 0$ it will be a subgroup $H \subset U(\mathcal{N})$. In the case of $\mathcal{N} = 4$ super Yang-Mills, we should have $U(4)$ supersymmetry, but actually have $SU(4)$ as we will see later.

R-charges

Since R-symmetry does not commute with supersymmetry, the component fields of the chiral superfield do not carry the same R-charge. We will call the R-charge of the lowest component field of a given superfield, the R-charge of the superfield. From (2.116) we say that for a general chiral multiplet containing a scalar ϕ and a fermion ψ , the following relation for the R-charges holds,

$$R_\psi = R_\phi - 1. \quad (2.117)$$

Now, a way to establish fixed values for these R-charges, we need to assume that the superpotential has R-charge 2. Since the superpotential is at most of the form Φ^3 , we deduce that the R-charge of the scalar field is $2/3$. And by using (2.117), the R-charge of the fermion field will be $-1/3$.

2.4.3 Supersymmetric gauge theories

This section will summarize what we have learned before but applied to gauge theories. As usual, we begin considering abelian gauge theories. Remember the interacting Wess-Zumino model (2.114) (we could have more than one fields). The kinetic term of the WZ model is invariant under global phase transformations, as is easy to see. But the superpotential part must satisfy some special requirements that are also easy to find. When we go towards gauge transformation, this kinetic term must change in order to be invariant under local transformations. Remember our study before about vector superfields. Under super gauge transformations, a vector superfield V transforms as (2.88) (see also the footnote under that expression). So, as the kinetic term is actually a vector superfield, we can define an invariant vector superfield, and thus an gauge invariant kinetic term, as

$$\mathcal{L} = \Phi_i^\dagger(\theta, \bar{\theta}, x) e^{qV(\theta, \bar{\theta}, x)} \Phi_i(\theta, \bar{\theta}, x). \quad (2.118)$$

Thus, the local $U(1)$ invariant Wess-Zumino model (written as superfields like in (2.110)), plus the dynamics of the gauge fields coming from the squared field strength, will have the following form

$$S = \int d^4x d^2\theta d^2\bar{\theta} \{ \Phi^\dagger e^{qV} \Phi + \mathcal{W} \delta^2(\bar{\theta}) + \bar{\mathcal{W}} \delta^2(\theta) + \frac{1}{4} W^A W_A \delta^2(\bar{\theta}) + \frac{1}{4} \bar{W}_{\dot{A}} \bar{W}^{\dot{A}} \delta^2(\theta) \}, \quad (2.119)$$

where,

$$S_{\text{gauge}} = \int d^4x d^2\theta d^2\bar{\theta} \{ \frac{1}{4} W^A W_A \delta^2(\bar{\theta}) + \frac{1}{4} \bar{W}_{\dot{A}} \bar{W}^{\dot{A}} \delta^2(\theta) \}, \quad (2.120)$$

represents the dynamical part of the gauge field, just like the non-supersymmetric gauge theory. Besides being gauge invariant, the kinetic term in (2.119) contains the supersymmetric generalization of the minimal coupling in ordinary gauge theories. Consider the expansion of e^{qV} (2.93) and the $(\theta\theta)(\bar{\theta}\bar{\theta})$ -term (2.79) in the product $\Phi^\dagger\Phi$, then

$$\begin{aligned} \Phi^\dagger e^{qV} \Phi &= \{ \Phi^\dagger \Phi + q(\theta\sigma^\mu\bar{\theta}) V_\mu \Phi^\dagger \Phi + q(\theta\theta)\bar{\theta}\bar{\lambda}\Phi^\dagger\Phi + q(\bar{\theta}\bar{\theta})\theta\lambda\Phi^\dagger\Phi \\ &\quad + \frac{q}{2}(\theta\theta)(\bar{\theta}\bar{\theta})(D + \frac{q}{4}V^2)\Phi^\dagger\Phi \}_{|(\theta\theta)(\bar{\theta}\bar{\theta})}, \end{aligned} \quad (2.121)$$

where it is obvious that we must consider only the $(\theta\theta)(\bar{\theta}\bar{\theta})$ -term of the expansion. So,

$$\Phi^\dagger e^{qV} \Phi = |F|^2 + i(D_\mu^* \bar{\psi}) \bar{\sigma}^\mu \psi + |D_\mu A|^2 - \frac{q}{\sqrt{2}}(\bar{\lambda}\bar{\psi}A + \lambda\psi A^*) + qD|A|^2, \quad (2.122)$$

where we have defined the covariant derivative as

$$D_\mu := \partial_\mu - iqV(x). \quad (2.123)$$

Thus, the kinetic term considers the supersymmetric form of the minimal coupling as we mentioned before. Now, if we expand the lagrangian in (2.120), considering (2.100) and 2.101), we will get

$$\begin{aligned} W^A W_A|_{(\theta\theta)} &= \{ D^2 - i \lambda \sigma^\mu \partial_\mu \bar{\lambda} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \frac{i}{2} F_{\mu\nu} F^{\mu\nu*} \}, \\ \bar{W}_{\dot{A}} \bar{W}^{\dot{A}}|_{(\bar{\theta}\bar{\theta})} &= \{ D^2 + i \partial_\mu \lambda \sigma^\mu \bar{\lambda} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} F_{\mu\nu} F^{\mu\nu*} \}. \end{aligned} \quad (2.124)$$

Replacing these results in 2.120) we obtain

$$S_{\text{gauge}} = \int d^4x \{ \frac{1}{2} D^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \lambda \sigma^\mu \partial_\mu \bar{\lambda} \}. \quad (2.125)$$

Then the scalar potential will acquire a new form,

$$V(\phi^*, \phi) = |F|^2 + \frac{1}{2} D^2. \quad (2.126)$$

In order for the vacuum to preserve supersymmetry, we must have $V = 0$ and hence $F = 0$ and $D = 0$ independently. The field equation for F is $F = 0$, but for D , it is

$$D = -q|\phi|^2. \quad (2.127)$$

Then, if we consider a set of scalar fields, say $\{\phi_i\}$, with $U(1)$ charges q_i . Vanishing the potential will give an equation that defines the vacuum manifold,

$$\sum_i q_i |\phi_i|^2 = 0. \quad (2.128)$$

We will be back to this definition later in order to define the conifold as the vacuum manifold in the Klebanov-Witten model.

The non-abelian case

As usual in gauge theories, the internal degrees of freedom are associated with hermitean generators T^a , allow to have internal components for the vector field and its gauge transformations,

$$V = V^a T^a, \quad \Lambda = \Lambda^a T^a. \quad (2.129)$$

Then, unlike of the abelian case, where we only have one generator (the unit operator), the non-abelian symmetry allows to “rotate” fields in an internal space of freedom. That generator satisfies a Lie Algebra,

$$[T^a, T^a] = i f^{abc} T^c. \quad (2.130)$$

We want to keep the kinetic term $\Phi^\dagger e^{qV} \Phi$, invariant under gauge transformation $\Phi \rightarrow e^{iq\Lambda} \Phi$ but the noncommutative nature of Λ and V enforces a nonlinear transformation law $V \rightarrow V'$,

$$\begin{aligned} e^{V'} &= e^{-i\Lambda^\dagger} e^V e^{i\Lambda} \\ &= \exp\left\{V + i(\Lambda - \Lambda^\dagger) + \frac{1}{2}i \left[V, \Lambda + \Lambda^\dagger \right] + \dots\right\}. \end{aligned}$$

Then we can deduce the transformation for the vector superfield V ,

$$V' = V + i(\Lambda - \Lambda^\dagger) + \frac{1}{2}i \left[V, \Lambda + \Lambda^\dagger \right] + \dots$$

An important fact is that this transformation above is independent of the particular representation of the gauge group, since the last transformation can be expressed as $V' = V'^a T^a$. The field strength we defined before in (2.94) and (2.95) must be redefined to be invariant under non-abelian gauge transformations. An appropriate definition is

$$\begin{aligned} W_A &:= -\frac{1}{4}(\bar{D}\bar{D})e^{-gV} D_A e^{gV}, \\ \bar{W}_{\dot{A}} &:= -\frac{1}{4}(DD)e^{-gV} \bar{D}_{\dot{A}} e^{gV}. \end{aligned} \quad (2.131)$$

Expanding W_A , we can see how much it has changed. In the same way we calculated W_A (2.100) for the abelian case, we get

$$W_A^a = \lambda_A^a + \theta_A D^a + (\sigma^{\mu\nu}\theta)_A F_{\mu\nu}^a - i(\theta\theta)\sigma_{A\dot{B}}^\mu D_\mu \bar{\lambda}^{\dot{B}a}, \quad (2.132)$$

where,

$$F_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + g f^{abc} V_\mu^b V_\nu^c, \quad (2.133)$$

$$D_\mu \bar{\lambda}^a = \partial_\mu \bar{\lambda}^a + g V_\mu^b \bar{\lambda}^c f^{abc}. \quad (2.134)$$

If we replace $W_A^a T^a$ in 2.120 with coupling g ,

$$S_{\text{gauge}} = \int d^4x d^2\theta d^2\bar{\theta} \left\{ \frac{1}{4g^2} W^A W_A \delta^2(\bar{\theta}) + \frac{1}{4g^2} \bar{W}_{\dot{A}} \bar{W}^{\dot{A}} \delta^2(\theta) \right\} \quad (2.135)$$

we get,

$$S_{\text{gauge}} = \int d^4x \left\{ \frac{1}{2g^2} D^a D^a - \frac{1}{4g^2} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{g^2} i\lambda^a \sigma^\mu D_\mu \bar{\lambda}^a \right\}. \quad (2.136)$$

Note that with this convention, the vector fields A_μ and the gaugino (gluino) field λ_α contain the coupling constant and hence, in accordance with the non-abelian gauge symmetry are not renormalized.

The kinetic term for non-abelian gauges theories is the same we developed in eq. (2.121). The scalar potential will be

$$V(A^*, A) = |F|^2 + \frac{1}{2g^2} D^a D^a. \quad (2.137)$$

2.4.3.1 The NVSZ β function for $\mathcal{N} = 1$ SUSY gauge theories

As we know, the β -function in quantum field theory encodes information about the behavior of the couplings of the theory with the energy scale. It is defined by [10]

$$\beta(g) = \frac{\partial g}{\partial \log \mu}, \quad (2.138)$$

for a theory with coupling g and scale μ . This function can be positive, negative or zero. The $\beta = 0$ case is special, since it says that the theory is conformal.

For supersymmetric field theories, that β -function can be also calculated. Actually, for a general supersymmetric gauge theories, the β -function can be calculated exactly. That is what V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov did in 1986 by using instanton methods for a $SU(N)$ gauge group [11, 12]. Let us review some facts about it.

The bare lagrangian of an $\mathcal{N} = 1$ supersymmetric gauge theory with generic matter content is given by [13–15],

$$\mathcal{L} = \int d^2\theta \frac{1}{4g^2} W^A W_A + \text{h.c.} + \int d^2\theta \sum_i \Phi_i^\dagger e^{V_i} \Phi_i + \text{h.c.}, \quad (2.139)$$

where W^A is the supersymmetric field strength and $\{\Phi_i\}$ is a set of scalar superfields. The gauge coupling is complexified as

$$\frac{1}{g^2(M)} \rightarrow \frac{1}{g_h^2(M)} = \frac{1}{g^2(M)} + i \frac{\theta(M)}{8\pi^2} = \frac{\tau(M)}{4\pi}, \quad (2.140)$$

where $g(M)$ and $\theta(M)$ stands for the bare coupling constant and vacuum parameter, M being the ultraviolet cutoff. By generalized nonrenormalization theorem⁴, the effective Lagrangian at scale μ (one loop corrections only) takes the form

$$\mathcal{L} = \int d^2\theta \frac{1}{4} \left(\frac{1}{g^2} + \frac{b_0}{8\pi^2} \log \frac{M}{\mu} \right) W^A W_A + \text{h.c.} + \int d^2\theta \sum_i Z_i(\mu, M) \Phi_i^\dagger e^{V_i} \Phi_i + \text{h.c.}, \quad (2.141)$$

⁴The nonrenormalization theorems state that the superpotential is not renormalizable, i.e. it does not receive quantum corrections. Furthermore, the gauge field term is renormalized only through the gauge coupling τ , such that its dependence is preserved. Moreover, the Adler-Bardeen theorem ensures that τ receives quantum contributions only at first order, i.e. through 1-loop graphs. The kinetic term on the other hand does receive renormalization at perturbative level.

where $b_0 = -3N_c + \sum_i T_{Fi}$. The factor $Z_i(\mu, M)$ in front of the kinetic term represents the change of the parameters with the scale μ .

In order to work with canonically normalized matter fields, we need to make the standard change of variable $\Phi_i = Z_i^{-1/2} \Phi_i^{(R)}$, but the measure in the path integral, $\mathcal{D}\Phi_i \mathcal{D}\Phi_i^\dagger$, is not invariant under this change, and there is an anomalous Jacobian [16]. It is exactly known and cutoff independent,

$$\mathcal{D}(Z_i^{-1/2} \Phi_i) \mathcal{D}(Z_i^{-1/2} \Phi_i^\dagger) = \mathcal{D}\Phi_i \mathcal{D}\Phi_i^\dagger e^{-\sum_i \frac{T_F}{32\pi^2} \log Z_i(\mu, M) \int d^4x d^2\theta W^A W_A + \text{h.c.}} \quad (2.142)$$

Then, at scale μ , the renormalized lagrangian is

$$\mathcal{L} = \int d^2\theta \frac{1}{4g_h^2} W^A W_A + \text{h.c.} + \int d^2\theta \sum_i \Phi_i^{\dagger(R)} e^{V_i} \Phi_i^{(R)} + \text{h.c.}, \quad (2.143)$$

where,

$$\frac{1}{g_h^2} \equiv \frac{1}{g^2} + \frac{b_0}{8\pi^2} \log \frac{M}{\mu} - 2 \sum_i \frac{T_F}{16\pi^2} \log Z_i(\mu, M). \quad (2.144)$$

The beta function can be calculated as the real part of

$$\frac{1}{\mu} \frac{\partial}{\partial \mu} \left(\frac{1}{g_h^2} \right) = -2 \frac{1}{\mu} \frac{1}{g_h^3} \frac{\partial}{\partial \mu} g_h \quad \rightarrow \quad \beta_h(\mu) = -\frac{1}{2} g_h^3 \frac{1}{\mu} \frac{\partial}{\partial \mu} \left(\frac{1}{g_h^2} \right),$$

to obtain

$$\beta_h(\mu) = -\frac{g_h^3}{8\pi^2} \left(\frac{3}{2} N_c - \frac{1}{2} \sum_i T_{Fi} (1 - \gamma_i) \right), \quad (2.145)$$

where $\gamma_i = -\frac{1}{\mu} \frac{\partial}{\partial \mu} Z_i(\mu, M)$ is the anomalous dimension. The ‘‘holomorphic’’ coupling constant $g_h(\mu)$ is well-behaved as $M \rightarrow \infty$ with μ finite.

Remember that the gauge term in (2.143) is given by (2.136) is not normalized yet. One would naively think that canonical normalization can be easily achieved through the following scaling redefinition $V = g_c V_c$ (and so its components given in (2.91)). The analogous Jacobian coming from this redefinition is

$$\mathcal{D}(g_c V_c) = \mathcal{D}V_c e^{\frac{N_c}{32\pi^2} \log g_c^2 \int d^4x d^2\theta W^A W_A + \text{h.c.}} \quad (2.146)$$

and as consequence, it leads to the change of the coupling constant coming from canonical normalization,

$$\text{Re} \frac{1}{g_h^2} = \frac{1}{g_c^2} + \frac{N_c}{8\pi^2} \log g_c^2, \quad (2.147)$$

as a way to reconcile holomorphy and renormalization. Then, the NSVZ beta function follows from (2.145) and (2.147) [6, 13–15],

$$\beta_c(g) = -\frac{g_c^3}{8\pi^2} \frac{\frac{3}{2}N_c - \frac{1}{2}\sum_i T_{Fi}(1 - \gamma_i)}{1 - N_c g_c^2 / 8\pi^2}. \quad (2.148)$$

As we will see later, a conformal gauge theory will impose an equation for the γ_i 's, the so-called critical surface which is parametrized by the coupling of the theory.

In the calculations above there are some subtle details connecting exact and perturbative results. The so-called anomaly puzzle in SUSY gauge theories arises as follows. In $\mathcal{N} = 1$ SYM theory, the R-symmetry, which is just $U(1)$, has an anomaly [13]. This anomaly is known to be exactly of 1-loop order because of the Adler-Bardeen theorem (see [11, 13] and the original reference therein), implying that its trace (the trace anomaly) is exhausted also at 1-loop. On the other hand, this trace anomaly, which in turns represents the trace of the stress tensor, is proportional to the β function and receives higher-order contributions. Then, apparently, the β function is exactly of 1-loop order. However, explicit perturbative calculations show that there are higher-order corrections. This is the puzzle.

Shifman and Vainshtein presented the solution to this puzzle [11]. They distinguished between the wilsonian holomorphic gauge coupling g_h appearing in the effective action, which runs at 1-loop, and the physical coupling g_c , which receives higher-order corrections. This distinction led to the NSVZ β function.

2.5 Conformal and superconformal symmetry in d dimensions

An interesting generalization of Poincaré symmetry is the addition of scale invariance, linking physics at different scales. Theories that are invariant under both conformal and Poincaré transformations are called conformal field theories. These extra symmetries allow to get information and even solve the theory, just exploiting the consequences of the symmetries.

We will see that something special occurs when the dimension of the spacetime is $d = 2$. So, let us introduce first the conformal group and its respective algebra for any d . There are many good textbooks on conformal field theory, in this section we recommend [17–19]. There are also other standard references in the context of string theory as [20–22]

2.5.1 The conformal group and the conformal algebra

Conformal transformations are those that preserve the angle between two lines. Mathematically, the conformal transformations preserve the form of the metric up to an arbitrary scale factor. In other words,

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho} \partial x'^{\sigma}}{\partial x^{\mu} \partial x^{\nu}} = \Lambda(x) \eta_{\mu\nu}, \quad (2.149)$$

where the positive function $\Lambda(x)$ is called the scale factor. Note also we are dealing with flat spaces. Now, let us study infinitesimal coordinate transformations up to first order

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2). \quad (2.150)$$

Then, following (2.149), we get

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho} \partial x'^{\sigma}}{\partial x^{\mu} \partial x^{\nu}} = \eta_{\mu\nu} + (\partial_{\mu} \epsilon^{\nu} + \partial_{\nu} \epsilon^{\mu}) + \mathcal{O}(\epsilon^2). \quad (2.151)$$

The second term can be written as

$$\partial_{\mu} \epsilon_{\nu} + \partial_{\nu} \epsilon_{\mu} = K(x) \eta_{\mu\nu}, \quad (2.152)$$

where $K(x)$ is some function that allow to perturb the flat metric η . This function can be obtained easily, by tracing the last equation out

$$K(x) = \frac{2}{d} \partial^{\mu} \epsilon_{\mu}. \quad (2.153)$$

So, the condition for the infinitesimal transformation is

$$\partial_{\mu} \epsilon_{\nu} + \partial_{\nu} \epsilon_{\mu} = K(x) \eta_{\mu\nu} = \frac{2}{d} \partial \cdot \epsilon \eta_{\mu\nu}. \quad (2.154)$$

The scale factor Λ is

$$\Lambda(x) = 1 + \frac{2}{d} \partial^{\mu} \epsilon_{\mu} + \mathcal{O}(\epsilon^2). \quad (2.155)$$

The infinitesimal coordinate transformation (2.150) must contain the conformal and the Poincaré transformations. It is possible to show that the infinitesimal function $\epsilon^{\mu}(x)$ is at most quadratic in x , and so we can make the following ansatz

$$\epsilon_{\mu}(x) = a_{\mu} + b_{\mu\nu} x^{\nu} + c_{\mu\nu\rho} x^{\nu} x^{\rho}, \quad (2.156)$$

where a_{μ} , $b_{\mu\nu}$ and $c_{\mu\nu\rho}$ are small constants. The explicit for of these constants can be easily obtained by taking derivatives of (2.154) and replacing our ansatz in them. After some calculations,

- $b_{\mu\nu} = \alpha \eta_{\mu\nu} + m_{\mu\nu}$ where α is some scale, and $m_{\mu\nu}$ represents infinitesimal rotations.
- $c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu$ where $b_\mu = \frac{1}{d} c^\rho_{\rho\mu}$.

Thus, ϵ will have the final form,

$$\epsilon_\mu(x) = a_\mu + \alpha x_\mu + m_{\mu\nu} x^\nu + 2(b \cdot x)x_\mu - (x \cdot x)b_\mu. \quad (2.157)$$

The last two terms are referred to Special Conformal Transformations (SCT). The finite form of this transformation is

$$x^\mu \rightarrow x'^\mu = \frac{x^\mu - (x \cdot x)b^\mu}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)}. \quad (2.158)$$

The whole finite conformal transformations and corresponding generators can be summarized as

Translation	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
Dilation	$x'^\mu = \alpha x^\mu$	$D = -ix^\mu \partial_\mu$
Rotation	$x'^\mu = M^\mu_\nu x^\nu$	$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
SCT	$x'^\mu = \frac{x^\mu - (x \cdot x)b^\mu}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)}$	$K_\mu = 2(b \cdot x)x_\mu - (x \cdot x)b_\mu$

Table 2.1: Conformal transformations

It is important to note that the SCT can be thought as a sequence of inversion, translation and inversion transformations,

$$\frac{x'^\mu}{x' \cdot x'} = \frac{x^\mu}{x \cdot x} - b^\mu \quad (2.159)$$

or,

$$x'^\mu = \frac{\frac{x^\mu}{x \cdot x} - b^\mu}{\left(\frac{x^\mu}{x \cdot x} - b^\mu\right)^2}.$$

Notice also that the Special Conformal Transformations are not globally well defined when $x^\mu = \frac{1}{b \cdot b} b^\mu$, so

$$1 - 2(b \cdot x) + (b \cdot b)(x \cdot x) = 0.$$

The transformations in Table 2.1 obey the following algebra,

$$\begin{aligned}
[D, P_\mu] &= i P_\mu, \\
[D, L_{\mu\nu}] &= 0, \\
[D, K_\mu] &= -i K_\mu, \\
[K_\mu, P_\nu] &= 2i \{\eta_{\mu\nu} D - L_{\mu\nu}\}, \\
[K_\rho, L_{\mu\nu}] &= i \{\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu\}, \\
[P_\rho, L_{\mu\nu}] &= i \{\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu\}, \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i \{\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}\}.
\end{aligned} \tag{2.160}$$

which is known as Conformal Algebra. The number of generators, the dimension of the algebra, is easily determined as

$$N = d + 1 + \frac{d(d-1)}{2} + d = \frac{(d+2)(d+1)}{2}.$$

In order to put the above rules into a simpler form, we define the following generators

$$\begin{aligned}
J_{\mu,\nu} &= L_{\mu\nu}, \\
J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_{m\mu}), \\
J_{-1,0} &= D, \\
J_{0,\mu} &= \frac{1}{2}(P_\mu + K_{m\mu}),
\end{aligned} \tag{2.161}$$

where $J_{m,n} = -J_{n,m}$ with $m, n = \{-1, 0, 1, \dots, d-1\}$. These generators satisfy the following commutation relations,

$$[J_{mn}, J_{rs}] = i \{\eta_{ms} J_{nr} + \eta_{nr} J_{ms} - \eta_{mr} J_{ns} - \eta_{ns} J_{mr}\}. \tag{2.162}$$

It is worth to see that the dimension of this algebra is the same that we found before, since we just redefined the generators. But the algebra (2.162) with $\frac{(d+2)(d+1)}{2}$ generators, represents the $SO(d+1, 1)$ algebra⁵. Then, we conclude that the conformal algebra is isomorphic to $SO(d+1, 1)$.

An important property in conformal theories is that the energy-momentum tensor is traceless for arbitrary d . This is easy to show by writing the scaling of the metric as

$$g_{\mu\nu}(x) \rightarrow e^{\sigma(x)} g_{\mu\nu}(x). \tag{2.163}$$

⁵For Euclidean d -dimensional space $\mathbb{R}^{d,0}$, the metric η we used above was $\eta_{mn} = \text{diag}(1, 1, \dots, 1)$. If we consider Minkowski space, the metric will be $\eta_{mn} = \text{diag}(-1, 1, \dots, 1)$, and the conformal group $SO(d, 2)$ [17].

Then $\delta g_{\mu\nu} = \sigma(x)g_{\mu\nu}$. The energy-momentum tensor is defined as usual by

$$T_{\mu\nu}(x) = -\frac{2}{\sqrt{g}} \frac{\delta W[g]}{\delta g^{\mu\nu}(x)}. \quad (2.164)$$

Invariance under scaling leads to

$$0 = \delta W[g] = -\frac{1}{2} \int d^d x \sqrt{g} T_{\mu}{}^{\mu} \sigma(x). \quad (2.165)$$

Hence $T_{\mu}{}^{\mu} = 0$. This is one of the most important constraints in order to consider a classical field to be conformal.

Conformal anomaly

As we know from quantum field theory, anomalies represent the breaking of classical symmetries. Conformal symmetry is sensitive to quantization, and in the context of string theory, cancellation of this anomaly in the Polyakov formalism leads to the critical dimension (see [21]).

2.5.1.1 Conformal symmetry in $d = 2$

There are very interesting features of the conformal group in two dimensions [17, 19, 23]. The condition for infinitesimal conformal invariance (2.154) reduces to the usual Cauchy-Riemann conditions of the complex analysis,

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1, \quad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0. \quad (2.166)$$

Then, it is natural to work in complex coordinates,

$$z = x^0 + i x^1, \quad \bar{z} = x^0 - i x^1. \quad (2.167)$$

An infinitesimal conformal transformation will acquire the following form

$$z \rightarrow f(z), \quad (2.168)$$

where $f(z) = z + \epsilon(z)$, since $\epsilon(z) = \epsilon^0 + i \epsilon^1$ must be holomorphic because of (2.166). The metric transforms as

$$ds^2 = dzd\bar{z} \rightarrow \underbrace{\frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial \bar{z}}}_{\left| \frac{\partial f}{\partial z} \right|^2 \text{ scale factor}} dzd\bar{z}. \quad (2.169)$$

The function $\epsilon(z)$ is holomorphic, but it can be promoted to a meromorphic function having isolated singularities outside the open set where the function is holomorphic. Then it is possible

to perform a Laurent expansion of $\epsilon(z)$ around $z = 0$. So the conformal transformation can be written as

$$z' = z + \epsilon(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n (-z^{n+1}), \quad (2.170)$$

$$\bar{z}' = \bar{z} + \bar{\epsilon}(z) = \bar{z} + \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n (-\bar{z}^{n+1}), \quad (2.171)$$

where ϵ_n and $\bar{\epsilon}_n$ are infinitesimal constant parameters. It is possible to note, by making an infinitesimal transformation of any field, that the generators of transformations for each n are

$$\begin{aligned} l_n &= -z^{n+1} \partial_z, \\ \bar{l}_n &= -\bar{z}^{n+1} \partial_{\bar{z}}. \end{aligned} \quad (2.172)$$

These generator obey a particular conformal algebra,

$$\begin{aligned} [l_m, l_n] &= (m - n) l_{m+n}, \\ [\bar{l}_m, \bar{l}_n] &= (m - n) \bar{l}_{m+n}, \\ [l_m, \bar{l}_n] &= 0, \end{aligned} \quad (2.173)$$

which is infinite dimensional since $n \in \mathbb{Z}$. Since the l_n 's and the \bar{l}_n 's satisfy independent algebras (Witt algebras), it is customary to treat z and \bar{z} as independent variables. So, actually we are considering \mathbb{C}^2 instead \mathbb{C} . Despite the algebra is infinite, there are only three of those generator that close to form the algebra for the global conformal transformations we saw before. In order to do that we need to compactify \mathbb{R}^2 into a Riemann sphere $\mathbb{C} \cup \{\infty\}$ ⁶. Then, the globally defined conformal transformations on this compactified space is generated by $\{l_{-1}, l_0, l_1\}$.

The operator $l_{-1} = -\partial_z$, generates translations $z \rightarrow z + b$. The operator l_0 and \bar{l}_0 form together dilations and rotations as $l_0 + \bar{l}_0 = -r \partial_r$ and $i(l_0 - \bar{l}_0) = -\partial_\phi$, respectively. In order to have these expressions we expressed the z variable in polar coordinates. The SCT's are generated by l_1 . Then, the conformal transformations has the following form

$$z \rightarrow \frac{az + b}{cz + d}, \quad (2.174)$$

with $a, b, c, d \in \mathbb{C}$. And, in order for this transformation to be invertible ($z \rightarrow z' \rightarrow z$), these constants must satisfy $ad - bc = 1$. The last expression is also invariant under \mathbb{Z}_2 . Then,

⁶Note that we are focusing only in l_n , one copy of the Witt algebra.

in order to “factorize out” this symmetry, we need to divide the $SL(2, \mathbb{C})$ (symmetry group generated by $l_{-1,0,1}$) by this group. The conformal group then is known as the Möbius group [17] $SL(2, \mathbb{C})/\mathbb{Z}_2 = SO(3, 1)$.

As we will see in the next chapter, the Witt algebra admits a central extension, i.e. an extra term proportional to the unit. This is known as the Virasoro algebra.

2.5.2 Constraints of conformal invariance in d dimensions

Back to arbitrary d , we will study how this new symmetry constraints the theory, in particular, the correlation functions.

2.5.2.1 Representations of the conformal group

The conformal group of spacetime consists of coordinate transformations that include as we saw, dilations, special conformal transformations and the inhomogeneous Lorentz transformations whose generators satisfy the algebra (2.160). The subgroup of conformal transformations that leaves $x = 0$ invariant is given by special conformal transformations, dilations and homogeneous Lorentz transformations. From the algebra (2.160) one finds that the Lie algebra of this subgroup is isomorphic to a Poincaré algebra + dilations [24],

$$(SO(3, 1) \otimes \{D\}) \otimes T_4.$$

The four-dimensional translation subgroup T_4 corresponds to the special conformal transformations, and $SO(3, 1)$ is the spin part of the Lorentz group. Then, given a representation of the “little” group, we can now determine the complete action of the generators of the whole conformal group on the field φ as follow. On the field φ we have, for every element X of the conformal algebra,

$$X\varphi(x) = \exp(-i P_\mu x^\mu) X' \varphi(0), \quad (2.175)$$

where,

$$\begin{aligned} X' &= \exp(+i P_\mu x^\mu) X \exp(-i P_\mu x^\mu), \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} x^{\mu_1} \dots x^{\mu_n} [P_{\mu_1}, [\dots [P_{\mu_n}, X] \dots]], \end{aligned} \quad (2.176)$$

by using the Hausdorff formula. The sum on the RHS of (2.176) is actually finite. From the commutations rules (2.160) it is found by inspection that there are most three non-vanishing

terms in the sum. Evaluating the finite commutators. For example, for $X = K_\mu$ we obtain

$$\exp(+i P_\mu x^\mu) K_\alpha \exp(-i P_\mu x^\mu) = K_\alpha - 2x^\beta (g_{\alpha\beta} D + M_{\alpha\beta}) + 2x_\alpha x^\beta P_\beta - x^2 P_\alpha. \quad (2.177)$$

Thus, we can apply this operation to deduce the action of K_μ , D and $M_{\mu\nu}$ on $\varphi(x)$, since the action on $\varphi(0)$ is known by hypothesis; e.g. $K_\mu \varphi(0) = \kappa_\mu \varphi(0)$. The final results are

$$\begin{aligned} P_\mu \varphi(x) &= i \partial \varphi(x), \\ M_{\mu\nu} \varphi(x) &= \{ i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu} \} \varphi(x), \\ D \varphi(x) &= \{ i x_\nu \partial^\nu + \delta \} \varphi, \\ K_\mu \varphi(x) &= \{ i(2x_\mu x_\nu \partial^\nu - x^2 \partial_\mu - 2i x^\nu [g_{\mu\nu} \Delta + \Sigma_{\mu\nu}]) + \kappa_{\mu\nu} \} \varphi(x), \end{aligned} \quad (2.178)$$

where the matrices $\Sigma_{\mu\nu}$, Δ and κ_μ are the infinitesimal generator corresponding to Lorentz transformations, dilations and special conformal transformations, respectively, and satisfy (2.160). So, we have shown that all field theoretically admissible representation of the conformal algebra are induced by a representation of the algebra of the little group that leaves the point $x = 0$ invariant.

2.5.2.2 Conformal invariance, correlation functions and OPE's

It is worth to show how the conformal symmetry allows to “solve” part of the theory without reference to an action, i.e. to find the correlation functions of the theory, up to structure constants depending of the theory. If we assume conformal invariance in a theory, at quantum level, those correlation functions must be related by that symmetry. This property is precisely that we will exploit to anticipate its spacetime dependence. The two-point function for scalar fields,

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{1}{Z} \int [d\phi] \phi_1(x_1) \phi_2(x_2) \exp\{-S[\phi]\}. \quad (2.179)$$

We assume that, according to

$$\langle \phi_1(x'_1) \phi_2(x'_2) \rangle = \langle \mathcal{F}(\phi_1(x_1)) \mathcal{F}(\phi_2(x_2)) \rangle, \quad (2.180)$$

where \mathcal{F} describes the functional change of the field under some transformation, the two-point function can be then written as

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\Delta_1/d} \left| \frac{\partial x'}{\partial x} \right|_{x=x_2}^{\Delta_2/d} \langle \phi_1(x'_1) \phi_2(x'_2) \rangle, \quad (2.181)$$

where the scalar fields are chosen to be quasi-primary⁷, i.e. they transform as

$$\phi(x) \rightarrow \left| \frac{\partial x'}{\partial x} \right|^{\Delta_1/d} \phi(x'). \quad (2.182)$$

Under scaling $x \rightarrow \lambda x$, then $\left| \frac{\partial x'}{\partial x} \right|^{\Delta/d} = \lambda^\Delta$. For (2.181) we get

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\phi_2(\lambda x_2) \rangle. \quad (2.183)$$

If we impose also invariance under rotation and translation, the two-point function will be

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \quad (2.184)$$

The finite special conformal transformation was given in Table(2.1), then,

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{(1 - 2b \cdot x + b^2 x^2)^d} = \frac{1}{\gamma^d}.$$

Replacing in (2.181), and requiring invariance as

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1} \gamma_2^{\Delta_2}} \frac{(\gamma_1 \gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \quad (2.185)$$

we deduce that two quasi-primary fields are correlated only if they have the same scaling dimension

$$\Delta_1 = \Delta_2 = \Delta. \quad (2.186)$$

Finally, the two-point function for scalar fields with scalar dimension Δ is

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta}}. \quad (2.187)$$

The three-point function is found performing the same analysis. Rotation, translation and dilation invariance require that the three-point function has the following form

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{C_{123}}{|x_1 - x_2|^a |x_2 - x_3|^b |x_1 - x_3|^c}, \quad (2.188)$$

where $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$, which come from the invariance under dilations. Invariance under special conformal transformations allows to find a , b and c as functions of the scaling dimensions Δ_i ,

$$\frac{C_{123}}{x_{12}^a x_{23}^b x_{13}^c} = \frac{1}{\gamma^{\Delta_1} \gamma^{\Delta_2} \gamma^{\Delta_3}} \frac{(\gamma_1 \gamma_2)^{a/2} (\gamma_2 \gamma_3)^{b/2} (\gamma_1 \gamma_3)^{c/2}}{x_{12}^a x_{23}^b x_{13}^c}, \quad (2.189)$$

⁷The rest of fields can be expressed as linear combinations of quasi-primary fields and its derivatives.

by requiring invariance we get a system of equations

$$\begin{aligned} 2\Delta_1 &= a + c \\ 2\Delta_2 &= a + b \\ 2\Delta_3 &= b + c \end{aligned} \tag{2.190}$$

Then, we can solve them

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3 \\ b &= \Delta_2 + \Delta_3 - \Delta_1 \\ c &= \Delta_3 + \Delta_1 - \Delta_2 \end{aligned} \tag{2.191}$$

The three-point function then can be written as

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1 + \Delta_2 - \Delta_3} |x_2 - x_3|^{\Delta_2 + \Delta_3 - \Delta_1} |x_1 - x_3|^{\Delta_3 + \Delta_1 - \Delta_2}}. \tag{2.192}$$

The structure constant C_{123} will depend of the theory. The four-point function is not possible to write explicitly by conformal invariance as we were doing above. Instead, we could define the so-called harmonic ratios as

$$r = \frac{x_{12}x_{34}}{x_{13}x_{24}} \quad \text{and} \quad s = \frac{x_{12}x_{34}}{x_{23}x_{14}}, \tag{2.193}$$

which are conformally invariant. Then, the n -point function, for $n \geq 4$, may have an arbitrary dependence on these harmonic ratios as, e.g. for $n = 4$

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = f(r, s) \prod_{i < j}^4 x_{ij}^{\Delta/3 - \Delta_i - \Delta_j}, \tag{2.194}$$

where $\Delta = \sum_{i=1}^4 \Delta_i$. The function $f(r, s)$ may be singular only when two of the four points coincide, so

$$r = 0, s = 1, \quad r = 1, s = 0, \quad r = s = \infty, \tag{2.195}$$

for all the other choices of (r, s) the function f must have finite values. The correlation function must also respect crossing symmetry if some of the fields in the correlation function are equal,

$$\begin{aligned} \text{if } \Delta_1 &= \Delta_3 & f(r, s) &= f(s, r), \\ \text{if } \Delta_1 &= \Delta_2 & f\left(\frac{r}{s}, \frac{1}{s}\right) &= f(r, s), \\ \text{if } \Delta_1 &= \Delta_4 & f\left(\frac{1}{r}, \frac{s}{r}\right) &= f(r, s). \end{aligned} \tag{2.196}$$

We can also use the two- and three-point functions as fundamental objects in order to construct correlation functions for $n \geq 4$ by bringing two operators to the same point and defining an Operator Product Expansion (OPE).

Let us review how this OPE can reduce products of fields inside correlators [17, 18, 21]. Given a correlation function in a d -dimensional field theory

$$\langle A(x)B(y)\phi_2(x_2)\dots\phi_n(x_n)\rangle. \quad (2.197)$$

Then, we will be interested in its behavior when $x \rightarrow y$, i.e. when the operators $A(x)$ and $B(y)$ approach one another. Wilson was who has the idea to replace the product $A(x)B(y)$ by

$$A(x)B(y) = \sum_l C_l(x-y)O^l(y), \quad (2.198)$$

where O^l are a set of local operator and C_l are their coefficients which depend on only $x-y$. OPE's are valid only within the correlation function and as such operators which appear in them must be regarded as being normal ordered. If the expansion (2.198) holds, the four-point correlation function can be expanded as follows

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_k C_{12}^k(x_1-x_2)\langle\phi_k(x_2)\phi_3(x_3)\phi_4(x_4)\rangle. \quad (2.199)$$

Then, we can construct a four-point function from combinations of three-point functions, that we can write explicitly from the theory. Now, let us consider the supersymmetric extension of the conformal group.

2.5.3 The superconformal algebra and its representations

We saw that one generalization of the Poincaré algebra was adding supersymmetry, which includes additional fermionic operators Q . It is interesting to ask whether supersymmetry and the conformal group can be joined together to form the largest possible simple algebra including the Poincaré group. In addition to the generators of the conformal group, given in (2.160), and the supersymmetry, (2.1), the called superconformal algebra include two other types of generators. There are fermionic generators (one for each supersymmetry generator) coming from the commutator of K_μ with Q , and there are (sometimes) R-symmetry generators forming some Lie Algebra, which appear as the anticommutator of Q with S . The full classification of the superconformal

algebra was given by Nahm [25]. The superconformal algebras, in addition to (2.1), read as⁸

$$\begin{aligned}
\{Q_\alpha^a, Q_\beta^b\} &= \{S_{\alpha a}, S_{\beta b}\} = \{Q_\alpha^a, \bar{S}_\beta^b\} = 0, \\
\{Q_\alpha^a, \bar{Q}_{\dot{\beta} b}\} &= 2\sigma_{\alpha\dot{\beta}}^\mu P^\mu \delta_b^a, \\
\{S_\alpha^a, \bar{S}_{\dot{\beta} b}\} &= 2\sigma_{\alpha\dot{\beta}}^\mu K^\mu \delta_b^a, \\
\{Q_\alpha^a, S_{\beta b}\} &= \epsilon_{\alpha\beta} \delta_b^a (D + T^a{}_b) + \frac{1}{2} \delta_b^a M_{\mu\nu} \sigma_{\alpha\beta}^{\mu\nu}.
\end{aligned} \tag{2.200}$$

In general, the generator of a superalgebra can be considered to be matrix elements in a vector space with both commuting and anticommuting components. The matrices take the form

$$\begin{pmatrix} \text{conformal group} & Q, \bar{S} \\ \bar{Q}, S & \text{R-symmetry} \end{pmatrix}$$

The generators of the bosonic subalgebra appear in the diagonal blocks. All structure relations are straightforward from this representation.

2.6 $\mathcal{N} = 4$ $SU(N)$ superconformal Yang Mills theory in $d = 4$

Four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is a very special quantum field theory. Its action was given for the first time in [26, 27]. As we saw before in (2.43), $\mathcal{N} = 4$ supersymmetry contains a gauge field A_μ , four Weyl fermions λ_α^a , and six real scalars Φ_m . Let us see, how this theory and its corresponding lagrangian are obtained by dimensional reduction of a ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory on a T^6 (a six-dimensional torus). This latter contains a gauge field A_M and a Weyl spinor λ_α , as can be seen in our treatment for $\mathcal{N} = 1$ with field content given in (2.19) and (2.20).

We begin with a ten dimensional $\mathcal{N} = 1$ super Yang Mills theory, whose lagrangian is (see (2.136)) [7, 26, 28]

$$\mathcal{L}_{10} = \text{tr} \left\{ -\frac{1}{2g_{YM}^2} F_{MN} F^{MN} - \frac{\theta}{8\pi^2} F_{MN} \tilde{F}^{MN} + i \frac{1}{g_{YM}^2} \bar{\lambda} \Gamma^M D_M \lambda \right\}, \tag{2.201}$$

where λ is the gaugino field and Γ is the set of 32×32 Gamma matrices in ten dimensions. They satisfy the algebra,

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}, \tag{2.202}$$

⁸ The exact form of the commutation relations is different for different dimensions and for different symmetry groups. Here, we will focus in $d = 4$.

where $M, N = 0, 1, \dots, 9$, and,

$$\tilde{F}_{MN} = \frac{1}{2} \epsilon_{MNAB} F^{AB}. \quad (2.203)$$

In order to get a theory in four dimensions, we need to compactify six of the ten dimensions. We choose the torus $T^6 = S_1 \times \dots \times S_1$ (six factors), and split the spacetime as $M^{10} = M^4 \times T^6$. We also assume that the fields depend only on the four-dimensional coordinates. Then, gauge field will split in $A_\mu(x_0, \dots, x_3)$, where $\mu = 0, \dots, 3$, and $\Phi_m(x_0, \dots, x_3)$, where $m = 1, \dots, 6$. The gaugino $\lambda(x_0, \dots, x_3)$ is a 16-component chiral spinor of $SO(1, 9)$, which is broken to $SO(1, 3) \times SO(6)$ because of the compactification. We have considered a Majorana-Weyl spinor **16**, which in turns decomposes as (see [22])

$$\mathbf{16} \rightarrow (\mathbf{2}, \mathbf{4}) + (\bar{\mathbf{2}}, \bar{\mathbf{4}}), \quad (2.204)$$

where **2** and $\bar{\mathbf{2}}$ are Weyl spinors living in $SO(1, 3)$, and **4** and $\bar{\mathbf{4}}$ corresponding to spinors in $SO(6) \cong SU(4)$. In $d = 10$, Gamma matrices also split as

$$\Gamma_\mu = \gamma_\mu \otimes 1_{8 \times 8}, \quad (\mu = 0, \dots, 3), \quad \Gamma_m = \gamma_5 \otimes \gamma_m \quad (m = 1, \dots, 6).$$

Here, γ_m is the set of 8×8 Gamma matrices in six dimensions, and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$.

In this representation λ in ten dimensions could be written as two complex spinors according to (2.204) by

$$\lambda_\alpha^a, \bar{\lambda}_{\dot{\alpha}b} \quad (\alpha, \dot{\alpha} = 1, 2, \quad a, b = 1, \dots, 4). \quad (2.205)$$

So the algebra of supersymmetry generators will be

$$\{ Q_\alpha^a, \bar{Q}_{\dot{\beta}b} \} = 2 \sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta_b^a, \quad (2.206)$$

which is the algebra for $\mathcal{N} = 4$ supersymmetry. Now, let us see how the lagrangian (2.201) changes under dimensional reduction. Remember that we did the fields in $\mathcal{N} = 1$, dependent only on coordinates on M^4 . The field strength F_{MN} will split as

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{YM} [A_\mu, A_\nu], \\ F_{\mu m} &= \partial_\mu \Phi_m - ig_{YM} [A_\mu, \Phi_m] \equiv D_\mu \Phi_m, \\ F_{mn} &= -ig_{YM} [\Phi_m, \Phi_n]. \end{aligned} \quad (2.207)$$

Then the first term in (2.201) can be written by

$$F_{MN} F^{MN} = F_{\mu\nu} F^{\mu\nu} + 2 D_\mu \Phi_m D^\mu \Phi_m - g_{YM}^2 [\Phi_m, \Phi_n]^2. \quad (2.208)$$

The kinetic term for the spinor λ in ten dimensions will read as

$$i\bar{\lambda}\Gamma^M D_M\lambda = i\bar{\lambda}\Gamma^\mu D_\mu\lambda + g_{YM}\bar{\lambda}\Gamma^m [\Phi_m, \lambda], \quad (2.209)$$

and,

$$F_{MN}\tilde{F}^{MN} = F_{\mu\nu}\tilde{F}^{\mu\nu}. \quad (2.210)$$

Here we have considered the fact that the only $SU(4)$ invariant tensor in four dimensions is $\epsilon_{\mu\nu\rho\sigma}$.

Then we obtain an $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions [6, 28–30],

$$\begin{aligned} \mathcal{L}_4 = & \text{tr}\left\{-\frac{1}{2g_{YM}^2}F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{g_{YM}^2}D_\mu\Phi_m D^\mu\Phi_m + \frac{1}{2}\sum_{m,n}[\Phi_m, \Phi_n]^2\right. \\ & \left.+ i\frac{1}{g_{YM}^2}\bar{\lambda}\Gamma^\mu D_\mu\lambda + \frac{1}{g_{YM}}\bar{\lambda}\Gamma^m [\Phi_m, \lambda]\right\}. \end{aligned} \quad (2.211)$$

Rescaling the fields as $(A_\mu, \Phi_i, \lambda) \rightarrow g_{YM}(A_\mu, \Phi_i, \lambda)$ we get,

$$\begin{aligned} \mathcal{L}_4 = & \text{tr}\left\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - D_\mu\Phi_m D^\mu\Phi_m + \frac{g_{YM}^2}{2}\sum_{m,n}[\Phi_m, \Phi_n]^2\right. \\ & \left.+ i\bar{\lambda}\Gamma^\mu D_\mu\lambda + g_{YM}^2\bar{\lambda}\Gamma^m [\Phi_m, \lambda]\right\}. \end{aligned} \quad (2.212)$$

2.6.1 Symmetries of $\mathcal{N} = 4$ SYM

If we make explicit the $SU(4)$ indexes, we can show that the terms involving Weyl spinors are invariant under $SU(4)$, and since the supersymmetry operators Q and \bar{Q} are also Weyl spinors, this group also rotate them. So the $SU(4)$ group is actually the R-symmetry of $\mathcal{N} = 4$ super Yang-Mills in four dimensions. But there is a subtle detail, we know that the $\mathcal{N} = 4$ SUSY algebra has actually a $U(4)$ R-symmetry given by the action on the generators Q , $Q_a \rightarrow U_a^b Q_b$, where U_a^b is a $U(4)$ matrix. In four dimensions $\mathcal{N} = 4$ is special, the diagonal part of the R-symmetry generator decouples from the algebra, as can be seen explicitly from the commutators in [28, 31].

The second term in (2.211) has an very special meaning, it allows to violate CP invariance in order to include magnetic monopoles (dyons) with non-integer magnetic charges without spoiling renormalizability [29, 32]. We are not to deal with this topic here.

In addition, there are two more ways to formulate $\mathcal{N} = 4$ SYM. First in terms of $\mathcal{N} = 1$ superfields, one vector V and three chiral Φ superfields all in the adjoint representation of the gauge group. And it is also worth to consider $\mathcal{N} = 2$ multiplets (a vector and a hypermultiplet), which are useful in instanton calculus.

Classically the action (2.211) is scale invariant. This may be seen by assigning the standard mass dimensions to the fields and couplings,

$$[A_\mu] = [\Phi] = 1, \quad [\lambda] = \frac{3}{2}, \quad [g] = 0. \quad (2.213)$$

Then all terms in the lagrangian are of dimension 4, from which scale invariance follows. Then our action (2.211), possesses together with $SU(4)_R$ and $\mathcal{N} = 4$ supersymmetry, conformal invariance. Moreover, the gauge theory is in general $SU(N)$, since we can give internal degrees of freedom to the gauge field, $A_\mu \rightarrow A_\mu^a$. We could see that the global invariance group is actually larger. If we calculate the corresponding energy-momentum tensor of (2.211) or (2.212) as usual by means of

$$T^{\mu\nu} = \partial^\nu X \frac{\partial \mathcal{L}}{\partial \partial_\mu X} - g^{\mu\nu} \mathcal{L}, \quad (2.214)$$

where $X = (A_\mu, \Phi_i, \lambda)$. We easily show that $T^\mu{}_\mu = 0$. Then, $\mathcal{N} = 4$ SYM is classically conformal. For this reason, the theory is invariant under the full conformal group, that in Minkowski four-dimensional space is $SO(2,4) \simeq SU(2,2)$. Moreover, we know that combination of conformal invariance with supersymmetry gives rise to a more general and extended symmetry group. Our theory thus exhibits superconformal symmetry, with four supersymmetry generators. This symmetry group is denoted as $PSU(2,2|4)$, with subgroups

- four dimensional conformal transformations $SO(2,4)$,
- maximally extended global supersymmetry,
- and $SU(4)$ R-symmetry.

To summarize, at the classical level we are dealing with a four-dimensional interacting gauge theory with a huge global invariance. This is the most constrained interacting theory without gravity in four dimensions.

Conformal quantum symmetry

We ask whether or not this theory preserves its symmetries, in particular the conformal, at the quantum level. Due to its relation with Poincaré invariance, supersymmetry cannot be anomalous, so it is preserved also at quantum level. As we saw in section (2.4.3.1), NSVZ β function at one loop for $SU(N)$ SYM in the $\mathcal{N} = 1$ language is proportional to $(3T(G) - \sum_i T_{Fi})$. Since, from the

section (2.2.2.4), a $\mathcal{N} = 4$ vector consists of a $\mathcal{N} = 1$ vector and three $\mathcal{N} = 1$ chiral superfields, we say that the NVSZ β function vanishes to all orders. Thus, it is conformal invariant.

This theory gives rise to 32 supercharges (since in any conformal theory, the supersymmetries are doubled). Then we say that the theory is maximally supersymmetric. It also displays a (strong/weak) S-duality under which the complexified coupling constant (as given in (2.140)),

$$\tau_{YM} = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g_{YM}^2}, \quad (2.215)$$

transforms into $-1/\tau_{YM}$ ⁹. This combines with shifts in θ_{YM} to complete the group $SL(2, \mathbb{Z})$ (see [6, 9]), as we will see later, this symmetry exists also in type IIB SUGRA, and motivates one test for the AdS/CFT correspondence.

2.6.2 Spectrum of $\mathcal{N} = 4$ SYM

Now we are interested in analysing more conformal properties in this theory. For this reason, a classification of the local operators of the theory in the superconformal phase is a good starting point.

2.6.2.1 Superconformal multiplets of local operators

In this section we construct local, gauge invariant operators in the theory that are polynomial in the fundamental fields. This restriction stems from the fact that such operators must have definite dimensions. The properties of the adjoint representation of $SU(N)$ determine that such operators necessarily involve traces of products of fields.

The canonical field Φ_m , λ_a and A_μ have unrenormalized dimensions, given by 1, 3/2 and 1 respectively. But gauge invariant operators will be constructed rather from the gauge covariant objects Φ_m , λ_a and $F_{\mu\nu}$ and also the covariant derivative D_μ , whose dimensions are

$$[\Phi_m] = [D_\mu] = 1 \quad [F_{\mu\nu}] = 2 \quad [\lambda_a] = \frac{3}{2}. \quad (2.216)$$

If we ignore for now the renormalization effects of composite operators, we see that all operator dimensions will be positive and that the number of operators whose dimension is less than a given number is then finite.

⁹A particular case may be seen for $\theta_{YM} = 0$, where the map for g_{YM} is $g_{YM} \rightarrow 1/g_{YM}$. As we will see later, this will correspond to the map $\lambda \rightarrow N^2/\lambda$ in the 't Hooft coupling. We say that a weak (strong) coupled theory has been mapped into a strong (weak) coupled one. This statement will be important in Chapter 4.

Now we introduce the idea of superconformal primary operator [7], since the conformal supercharges S given in (2.200) have dimension $-1/2$ (as is easy to see since K_μ has dimension -1), successive application of S to any operator of definite dimension will lead eventually to an operator with dimension 0, otherwise we would start generating operators with negative dimension, which are not allowed in unitary representations. Therefore, we define a superconformal primary operator \mathcal{O} to be a non-vanishing operator such that,

$$[S, \mathcal{O}]_{\pm} = 0, \quad \mathcal{O} \neq 0. \quad (2.217)$$

We could define a superconformal primary operator equivalently as the one with lowest dimension in a given superconformal multiplet or representation. Now, let us define the superconformal descendant operator, \mathcal{O} , as

$$\mathcal{O} = [Q, \mathcal{O}']_{\pm}. \quad (2.218)$$

where \mathcal{O}' is a well-defined local polynomial gauge invariant operator. Then we say that \mathcal{O} is a descendant of \mathcal{O}' , and belongs together \mathcal{O} to the same superconformal multiplet. From (2.218) we say that their conformal weights are related by $\Delta_{\mathcal{O}} = \Delta_{\mathcal{O}'} + 1/2$, where Δ , the conformal dimension, is eigenvalue of the dilation operator and can be read off from the two-point function, as we learn in (2.187)¹⁰

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle \sim \frac{1}{x^{2\Delta}}. \quad (2.219)$$

The operator \mathcal{O} can never be a superconformal primary operator because there is in the same multiplet at least one operator \mathcal{O}' operator with lower dimension than \mathcal{O} . Thus, all local gauge-invariant operators which are not superconformal primaries can be derived from one of the superconformal primaries by successive

$$\mathcal{O}'(x) = [Q, [Q, \dots, [Q, \mathcal{O}(x)]]]_{\pm}. \quad (2.220)$$

Such operators are called superconformal descendants of \mathcal{O}' . As a result, in a given irreducible superconformal multiplet, there is only one superconformal primary operator (the one of lowest dimension) and all others are superconformal descendants of this one.

In $\mathcal{N} = 4$ SYM, we construct superconformal primary operators by using the fact that a superconformal primary operator cannot be a Q-commutator of another operator, since they

¹⁰In general Δ depends on the 't Hooft coupling, i.e. $\Delta = \Delta_0 + \gamma(\lambda)$, where Δ_0 is the classical dimension and γ is called the anomalous dimension.

are descendants. Thus, by commuting each canonical field in $\mathcal{N} = 4$ SYM, we conclude that superconformal primary fields in $\mathcal{N} = 4$ SYM can be formed only by products (neither derivatives nor commutators) of scalars Φ_m , since they do not come from Q-commutations of the field in the theory. Moreover, invariance under gauge transformations leads to the simplest form for these superconformal primary fields, the single trace operators,

$$\mathcal{O}(x) = \text{str} \Phi^{\{i_1} \Phi^{i_2} \dots \Phi^{i_n\}}, \quad (2.221)$$

where $i_k, k = 1, \dots, n$ stands for the $SO(6)_R$ fundamental representation indexes. Here, “str” denotes symmetrized trace and $\{ \}$ stands for the traceless (with respect to the $SO(6)$ indexes) part. The operator in (2.221) is totally symmetric in the $SO(6)_R$ -indices i_k and in general they transform under a reducible representation (namely the symmetrized product of n -fundamentals) and irreducible operators may be obtained by isolating the traces over the $SO(6)_R$ indexes. Since $\text{tr} \Phi^i = 0$, the simplest operators are

$$\begin{aligned} \sum_i \text{tr} \Phi^i \Phi^i &\sim \text{Konishi multiplet,} \\ \text{tr} \Phi^i \Phi^j &\sim \text{SUGRA multiplet.} \end{aligned} \quad (2.222)$$

2.6.2.2 $\mathcal{N} = 4$ chiral or BPS multiplets of operators

The representations of the superconformal algebra $PSU(2, 2|4)$ may be labeled by the quantum numbers of the bosonic subgroup as

$$\begin{aligned} SO(1, 3) \times SO(1, 1) \times SU(4)_R, \\ (j_1, j_2) \quad \Delta \quad [r_1, r_2, r_3], \end{aligned} \quad (2.223)$$

where $j_{1,2}$ are spin labels, Δ is the positive dimension and $[r_1, r_2, r_3]$ are the Dynkin labels of the irreducible representations of $SU(4)_R$. In unitary representations, the dimension Δ of the superconformal primary operators are bounded from below by the spin and $SU(4)_R$ quantum numbers [7, 8]. Since in $\mathcal{N} = 4$ SYM these operators are products of scalars, the spin quantum number vanishes, and the dimension is bounded from below by the $SU(4)_R$ quantum numbers only. Now, some of these bounded dimensions will correspond to the so-called discrete series of representations, for which at least one supercharge Q (or some combination of Q 's) commutes with the primary operator. Such representations are shortened and usually referred to as chiral

multiplets or BPS multiplets. These kind of operators are protected from quantum corrections and then do not renormalize. Hence for chiral primaries, $\Delta = \Delta_0$ at any order.

Depending on the supercharges that annihilate the primary operator, BPS operators are called 1/2 BPS (8 supercharges), 1/4 BPS (4 supercharges), and 1/8 BPS (2 supercharges). If all 16 supercharges annihilate the primary operator, the representation is one-dimensional and moreover trivial, i.e. the identity operator.

An example of a 1/2 BPS operator is $\mathcal{O}_k \sim \text{str} \Phi^{i_1} \Phi^{i_2} \dots \Phi^{i_k}$ (the normalization constant is fixed by the two-point function). It has dimension k and transforms in the $[0, k, 0]$ representation under $SU(4)_R$. These operators will correspond to modes of scalar fields in supergravity, as we will see later.

Chapter 3

Superstrings and supergravity

String theory was born out of attempts to understand the strong interactions (see [33] for a short and understandable review, and also the classic historical introduction given in [34] and [21] are recommendable). But soon it became in one of the most successful theoretical descriptions that include gravity in a natural way. In principle, string theory was proposed as a theory of bosonic strings which was consistent only in 26 dimensions, but supersymmetry led to a theory that includes fermions in ten dimensions, a superstring theory.

Moreover, bosonic and supersymmetric string theories include, because of T-duality, a variety of higher-dimensional and nonperturbative objects called D-branes. Which in the supersymmetric case, are charged and source fields analogous to the electromagnetic field sourced by charged point particles. Then, we could imagine those D-branes as generalization of those point-like sources that also allow to attain open strings in them. Thus, superstrings and D-branes together give a consistent theory that includes gravity plus other fields.

It is also possible to see how gravity emerge from local supersymmetry and gauge theory, noting that actually the first one is an example of gauge theory, that contains the frame field describing the graviton, plus a specific number of vector-spinor fields, whose quanta are the gravitinos, the supersymmetric partners of the graviton. This supersymmetric theory of gravity coming from local supersymmetry is known as supergravity (see [2, 3]). It can be seen also as an effective theory coming from the superstring theory at low energy limit, then it is very nonlinear and interacting (see the classical textbook [35]).

For our purposes we will focus in a specific supergravity theory, called Type IIB which consists of the massless content of Type IIB superstring theory [20, 34]. With that matter and gauge

content we can solve Einstein equations in some cases as we will see later.

This chapter is equally important as the last one. Here we describe the gravity side of the AdS/CFT correspondence, type IIB supergravity, as the low-energy limit of type IIB superstring theory. As was mentioned in the introduction, we review with some detail, superstring theory in order to understand how the massless field content we will consider later in the context of the correspondence, is obtained from covariant quantization [20, 34].

As was pointed out also in the first chapter, this is a revision that we included in order to give a self-contained work that the interested reader could take almost without outer references (we recommend obviously some classical textbooks and important papers that give support our topics). So, if the reader consider necessary, here we give a concise overview on superstring theory addressed to its low-energy limit, supergravity that will be recalled when we establish the correspondence between results in this chapter and those we studied at the end of the last one. Confident and experienced readers could skip the firsts sections where, we practically summarize what is usually explained in textbooks, to go to the end where we analyze the results and give extended calculations on supergravity.

3.1 A brief review on superstring theory

Strings are one-dimensional extended objects which move in a d -dimensional target spacetime. As they move they span a two-dimensional surface Σ , which is referred to as the world-sheet. Analogously to the point-particle description, the action for a string is proportional to its proper area is called the Nambu-Goto action (see [36] for a detailed review),

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}. \quad (3.1)$$

Bosonic string theory in flat Minkowski spacetime as described by the CFT theory (3.1) has a number of drawbacks. Most importantly, its spectrum contains tachyons, i.e. a state of negative mass, which means that flat space is an unstable vacuum of bosonic string theory. Moreover, the perturbative spectrum of bosonic string theory contains no fermionic states. Even though this is not an inconsistency of the theory in itself, it shows that bosonic theory cannot be provide a description of the fermions.

Hence, we will try to modify the string action (3.1) in such a way that the theory can still be quantized perturbatively and be stable. There are two basic approaches to face these ideas [34]:

- The Ramond-Neveu-Schwarz (RNS) formulation, and
- The Green-Schwarz (GS) formalism.

Which differ in where the supersymmetry is manifested, either on the string world-sheet or on the Minkowski space-time.

This section is a quick review on string theory, based on some classical textbook as [20,21,34,35] and [22]. We also recommend some modern complete and pedagogical textbooks that contain almost all we need to learn about string theory as [37] and [38].

3.1.1 The RNS string

The bosonic string action (3.1) is invariant under d -dimensional Poincaré, diffeomorphisms and Weyl transformations, and needs to be supplemented by Virasoro constraints, which come from the vanishing of the energy-momentum tensor. This is a free theory in two dimensions. In order to include fermions on the world-sheet¹ we could think of another free theory for fermions. They will be new degrees of freedom of the world-sheet space where the string can propagate. One can consider d Majorana fermions (real spinors) that belong to the vector representation of the Lorentz group $SO(d-1,1)$. So, the desired action will be the Dirac action for d free massless fermions. Then, the total action (in the Polyakov formalism [20,21,36]) is

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu), \quad (3.2)$$

where ρ^α ($\alpha = 1, 2$) represent the two-dimensional Dirac matrices

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.3)$$

which satisfy the Dirac algebra,

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}. \quad (3.4)$$

These ψ_μ are Grassmann numbers, then $\{\psi^\mu, \psi^\nu\} = 0$. In this representation the spinor ψ^μ has two components (two chiralities) ψ_A^μ , $A = \pm$

$$\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix} \quad (3.5)$$

¹In the GS formalism, fermions are on the spacetime.

Majorana spinors obey a number of important identities that do not hold for Dirac spinors. For example, for Majorana fermions, $\bar{\psi}$ is simply $\psi^T \rho^0$ (we do not need to take the complex conjugate since ψ is real). Another property is that, for χ and ψ , $\bar{\chi}\psi = \chi\bar{\psi}$. Remember we wrote an action for a two-dimensional field theory (on the world-sheet), not a field theory in spacetime; then ψ_A^μ transforms as a spinor under transformations of the two-dimensional world-sheet, and as a $SO(d-1, 1)$ vector in the spacetime. We say that the Lorentz group $SO(d-1, 1)$ is then an internal symmetry from the world-sheet point of view.

In the bosonic string case there is an important problem coming from the commutation relation of the bosonic coordinates,

$$[X^\mu(\sigma), \dot{X}^\nu(\sigma')] = i\pi\eta^{\mu\nu}\delta(\sigma - \sigma'). \quad (3.6)$$

This metric is not positive, and lead us to negative norm states or ghosts. The Virasoro algebra allows us to get rid of those ghosts in the critical dimension $d = 26$, and we found an extra symmetry, the conformal symmetry. Now, for our supersymmetric action, we have to face the precisely analogous question. For the fermionic fields satisfy the quantum version of the equal-time Poisson brackets for Grassmann variables,

$$\{\psi_A^\mu(\sigma), \psi_B^\nu(\sigma)\} = \pi\eta^{\mu\nu}\delta_{AB}\delta(\sigma - \sigma'). \quad (3.7)$$

The familiar problem appears again. Since $\eta^{00} = -1$, the timelike fermions $\psi_A^0(\sigma)$ create negative norm states, just like the timelike bosons $X^0(\sigma)$. Then, to solve this problem in the superstring case we have to find a new symmetry and new constraints that can do for fermions what the Virasoro conditions do for bosons. In our theory which is supersymmetric, the new emerging symmetry will be then superconformal.

3.1.1.1 Global worldsheet supersymmetry

Let ϵ represent a constant (in σ and τ) anticommuting infinitesimal Majorana spinor. The usual supersymmetry transformations which mix bosonic and fermionic coordinates are

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \quad (3.8)$$

$$\delta\psi^\mu = \rho^\alpha\partial_\alpha X^\mu\epsilon. \quad (3.9)$$

The action must be invariant under these transformations, this is easy to prove. The result is

$$\delta S = -\frac{1}{2}\int d^2\sigma\partial_\alpha(\bar{\psi}^\mu\rho^\alpha\delta\psi_\mu). \quad (3.10)$$

That is a total derivative and vanishes as usual. We also need to show that the commutator of two supersymmetry transformations gives a spatial translation. Then,

$$[\delta_1, \delta_2]X^\mu = 2\bar{\epsilon}_2\rho^\alpha\epsilon_1\partial_\alpha X^\mu. \quad (3.11)$$

In the same way, for ψ

$$[\delta_1, \delta_2]\psi^\mu = 2\bar{\epsilon}_2\rho^\alpha\epsilon_1\partial_\alpha\psi_\mu + \bar{\epsilon}_1\rho_\beta\epsilon_2\rho^\beta\rho^\alpha\partial_\alpha\psi^\mu. \quad (3.12)$$

Here we have an important detail, to close the algebra one needs to use the fact that ψ obeys the Dirac equation, $\rho^\alpha\partial_\alpha\psi^\mu = 0$. In this case, algebra closes on-shell. Later, we will look for that the algebra closes without using the field equations. Then, on-shell we get

$$[\delta_1, \delta_2]\psi^\mu = 2\bar{\epsilon}_2\rho^\alpha\epsilon_1\partial_\alpha\psi^\mu. \quad (3.13)$$

It is important to emphasize that these are translations of the string world-sheet, since the supersymmetry was imposed on the world-sheet.

Supercurrent and energy-momentum tensor

Because of the Weyl invariance in the free bosonic string theory, the energy-momentum tensor is traceless. For the supersymmetric case, $T_{\alpha\beta}$ is traceless too. Let $T_{\alpha\beta}$ be the energy-momentum tensor,

$$T_{\alpha\beta} = \partial_\alpha X_\mu\partial_\beta X^\mu + \frac{1}{4}\bar{\psi}^\mu\rho_\alpha\partial_\beta\psi_\mu + \frac{1}{4}\bar{\psi}^\mu\rho_\beta\partial_\alpha\psi_\mu - \frac{1}{2}h_{\alpha\beta}\text{tr}T_{\alpha\beta}. \quad (3.14)$$

It is not difficult to prove that it is conserved while the field equations are satisfied, and $T_\alpha^\alpha = 0$. This means that the two-dimensional supersymmetric field theory on the world-sheet is conformal.

3.1.2 Mode expansion and quantization

The fermion equation of motion we can obtain from the last action is $\rho^\alpha\partial_\alpha\psi = 0$, a two-dimensional Dirac equation, which must be supplemented by boundary conditions. Considering the basis we choose before, the equation decomposes as

$$\left(\frac{\partial}{\partial\sigma} \pm \frac{\partial}{\partial\tau}\right)\psi_\mp^\mu = 0, \quad (3.15)$$

where ψ was expressed as in (3.5). Thus ψ_-^μ and ψ_+^μ describe right- and left-moving movers, respectively. The two-dimensional Dirac equation can be written in a form that makes manifest

the decoupling of the modes by introducing light-cone coordinates on the world-sheet $\sigma^\pm = \tau \pm \sigma$ and $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. So the fermionic action can be expressed as

$$S_F = \frac{i}{\pi} \int d^2\sigma \{ \psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+ \}. \quad (3.16)$$

The equations of motion are easily obtained as

$$\partial_+ \psi_-^\mu = \partial_- \psi_+^\mu = 0. \quad (3.17)$$

The supersymmetry transformations in light-cone coordinates are

$$\delta \psi_\mp^\mu = \mp 2 \partial_\mp X^\mu \epsilon_\pm. \quad (3.18)$$

Thus, the bosonic and fermionic coordinates are related by

$$\partial_\pm \psi_\mp^\mu = \partial_\pm \partial_\mp X^\mu. \quad (3.19)$$

The latter results mean that the positive- and negative-chirality modes decouple. Then, the world-sheet supersymmetry current and energy-momentum tensor can be written in terms of positive- and negative-chirality modes. The commutation rules are

$$\begin{aligned} \{ \psi_+^\mu(\sigma), \psi_+^\nu(\sigma') \} &= \{ \psi_-^\mu(\sigma), \psi_-^\nu(\sigma') \} = \pi \eta^{\mu\nu} \delta(\sigma - \sigma'), \\ [\partial_\pm X^\mu(\sigma), \partial_\pm X^\nu(\sigma')] &= \pm i \frac{\pi}{2} \eta^{\mu\nu} \delta(\sigma - \sigma'), \\ \{ \psi_+^\mu, \psi_-^\nu \} &= [\partial_+ X^\mu, \partial_- X^\nu] = 0. \end{aligned} \quad (3.20)$$

3.1.3 Boundary conditions and analysis of the spectrum

In the bosonic string theory, the spacetime coordinate X^μ obeys a free wave equation in the world-sheet coordinates. Then, we could expand this field on the world-sheet in different mode series. These series satisfy boundary conditions corresponding to either closed or open strings². Recall the fermionic lagrangian (3.16), when the field equations for fermions (3.17) are satisfied, the boundary terms in the variation of the action must also vanish. For open strings,

$$\delta S \sim \int d\tau \{ (\psi_- \delta \psi_- - \psi_+ \delta \psi_+)_{\sigma=0} - (\psi_- \delta \psi_- - \psi_+ \delta \psi_+)_{\sigma=\pi} \}. \quad (3.22)$$

²The modes expansion for X is

$$X^\mu = x^\mu + l_s^2 p^\mu + l_s \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\tau} \cos(m\sigma), \quad (3.21)$$

where l_s is the length of the string, x^μ is the position of the center of mass of the string and p^μ is the momentum.

So, the vanishing of δS implies that

$$\psi_- \delta \psi_- = \psi_+ \delta \psi_+. \quad (3.23)$$

The last equation is satisfied by making (in particular, for $\sigma = 0$),

$$\psi_+(0, \tau) = \pm \psi_-(0, \tau). \quad (3.24)$$

The relative sign allows us to consider two cases for expanding the fermionic variables:

Ramond boundary conditions (R)

In this case we choose $\psi_+(\pi, \tau) = \psi_-(\pi, \tau)$ (for $\sigma = \pi$). The mode expansion of the fermionic field in the R-sector is

$$\psi_{\pm}^{\mu}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^{\mu} e^{-in(\tau \pm \sigma)}. \quad (3.25)$$

Neveu-Schwarz boundary conditions (NS)

In this case we choose $\psi_+(\pi, \tau) = -\psi_-(\pi, \tau)$ (for $\sigma = \pi$). The mode expansion of the fermionic field in the NS-sector is

$$\psi_{\pm}^{\mu}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^{\mu} e^{-ir(\tau \pm \sigma)}. \quad (3.26)$$

For closed strings the surface terms are given in (3.22) but with limits in σ and $\sigma + \pi$. Then, vanishing of them implies that,

$$\psi_{\pm}(\sigma, \tau) = \pm \psi_{\pm}(\sigma + \pi, \tau), \quad (3.27)$$

Here the boundary conditions are periodic and antiperiodic for each component of ψ separately.

Thus, we have the so-called sectors of closed strings:

- For the right-movers,

$$\begin{aligned} \text{R} : \quad \psi_-^{\mu} &= \sum_{n \in \mathbb{Z}} d_n^{\mu} e^{-2in(\tau - \sigma)}, \\ \text{NS} : \quad \psi_-^{\mu} &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^{\mu} e^{-2ir(\tau - \sigma)}. \end{aligned} \quad (3.28)$$

- For the left-movers,

$$\begin{aligned} \text{R} : \quad \psi_+^\mu &= \sum_{n \in \mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)}, \\ \text{NS} : \quad \psi_+^\mu &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)}. \end{aligned} \quad (3.29)$$

Different pairings of left- and right-moving modes give the four sectors for closed strings:

- R – R

$$\psi_-^\mu = \sum_{n \in \mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)}, \quad \psi_+^\mu = \sum_{n \in \mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)},$$

- R – NS

$$\psi_-^\mu = \sum_{n \in \mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)}, \quad \psi_+^\mu = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)},$$

- NS – R

$$\psi_-^\mu = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\mu e^{-2ir(\tau-\sigma)}, \quad \psi_+^\mu = \sum_{n \in \mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)},$$

- NS – NS

$$\psi_-^\mu = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\mu e^{-2ir(\tau-\sigma)}, \quad \psi_+^\mu = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)}. \quad (3.30)$$

where R-R and NS-NS are called bosonic, and R-NS and NS-R, fermionic [20]. Using the commutation rule (3.20), we can find a relation for the modes in the expansion:

$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu} \delta_{m+n,0}, \quad (3.31)$$

with $m, n \in \mathbb{Z}^3$. And

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu} \delta_{r+s,0}, \quad (3.32)$$

³This relation for zero modes is identical to the Dirac algebra,

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}.$$

Then we say that the ground states in the R -sector must furnish a representation of this algebra. Hence, we can write the action of d_0^μ on a state $|a\rangle$, where a is a spinor index, as

$$d_0^\mu |a\rangle = \frac{1}{\sqrt{2}} \Gamma_{ab}^\mu |b\rangle.$$

with $r, s \in (\mathbb{Z} + \frac{1}{2})$. For completeness sake, we recall the commutation rule for the bosonic coordinates which came from the mode expansion of X

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}. \quad (3.33)$$

Back to the anticommutation relations, notice that for $\mu, \nu = 0$, the right-hand side in (3.31) and (3.32) have negative sign, which implies that they give rise to negative-norm states, just like the time components of the bosonic modes.

The oscillator ground state is defined by

$$\alpha_m^\mu |0\rangle_{\text{R}} = d_m^\mu |0\rangle_{\text{R}} = 0 \quad \text{for } m > 0, \quad (3.34)$$

in the R-sector, or

$$\alpha_m^\mu |0\rangle_{\text{NS}} = b_r^\mu |0\rangle_{\text{NS}} = 0 \quad \text{for } m, r > 0, \quad (3.35)$$

for the NS-sector. The zero-frequency part of the Virasoro constraints give the mass-shell condition,

$$\alpha' M^2 = N + C, \quad (3.36)$$

where C is a constant we will define later. The number operators in terms of the bosonic and fermionic modes in each sector is, in the R-sector,

$$N = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{m=1}^{\infty} m d_{-m} \cdot d_m, \quad (3.37)$$

and

$$N = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r} \cdot b_r, \quad (3.38)$$

for the NS-sector. When the mass operator (3.36) is applied to the two ground states,

- R-sector

$$\alpha' M^2 |0\rangle_{\text{R}} = C |0\rangle_{\text{R}}, \quad (3.39)$$

- NS-sector

$$\alpha' M^2 |0\rangle_{\text{NS}} = C |0\rangle_{\text{NS}}. \quad (3.40)$$

So, this constant is the minimum mass value for both ground states. But, are these ground states unique?. The answer is no, for the R ground state. In the case with half-integer modes it is possible to choose a unique (nondegenerate) ground state, which may therefore be identified as

a spin zero state $|0\rangle_{\text{NS}}$. In the case with integer modes this is not possible because the mode d_0 , can be applied on $|0\rangle_{\text{R}}$ without changing the ground mass. So, in this the ground state is degenerate.

The d_0^μ and d_0^ν modes obey the algebra (3.31), which is just the Dirac algebra for $d_0^\mu = \Gamma^\mu/\sqrt{2}$. Hence, the set of ground states in the R sector must be an irreducible representation of $SO(d-1, 1)$, a set of fermionic states.

The super-Virasoro operators are given by the modes of $T_{\alpha\beta}$ and J_α ⁴. For open strings,

$$\begin{aligned} L_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} d\sigma e^{im\sigma} T_{++} = L_m^{(b)} + L_m^{(f)} & m \in \mathbb{Z}, \\ F_m &= \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} d\sigma e^{im\sigma} J_+ = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot d_{m+n} & m \in \mathbb{Z}, \\ G_r &= \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} d\sigma e^{ir\sigma} J_+ = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot b_{r+n} & r \in \mathbb{Z} + \frac{1}{2}. \end{aligned} \quad (3.41)$$

where,

$$L_m^{(b)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_{-n} \cdot \alpha_{m+n} :, \quad (3.42)$$

for the bosonic modes. There are two options for the fermionic modes,

- R-sector

$$L_m^{(f)} = \frac{1}{2} \sum_{n \in \mathbb{Z} + \frac{1}{2}} (n + \frac{m}{2}) : d_{-r} \cdot d_{m+n} :, \quad (3.43)$$

- NS-sector

$$L_m^{(f)} = \frac{1}{2} \sum_{n \in \mathbb{Z} + \frac{1}{2}} (r + \frac{m}{2}) : b_{-r} \cdot b_{m+n} :, \quad (3.44)$$

where $: :$ is for normal ordering, which means that the creation operators appear to the right of the annihilation operators.

⁴These expressions can be obtained by inserting the mode expansions of X and ψ given in (3.21), (3.25) and (3.26) in the expressions for the components of the energy-momentum tensor and the supercurrent,

$$\begin{aligned} J_\pm &= \psi_\pm^\mu \partial_\pm X_\mu, \\ T_{++} &= \partial_+ X \cdot \partial_+ X + \frac{i}{2} \psi_+ \cdot \partial_+ \psi_+, \\ T_{--} &= \partial_- X \cdot \partial_- X + \frac{i}{2} \psi_- \cdot \partial_- \psi_-. \end{aligned}$$

An important generator will be the L_0 ,

$$L_0 = \frac{\alpha_0^2}{2} + \sum_{n=1}^{\infty} : \alpha_{-n} \cdot \alpha_n : + L_0^{(f)} = \frac{\alpha_0^2}{2} + N, \quad (3.45)$$

where $L_0^{(f)}$ has two options,

$$L_0^{(f)} = \sum_{r=\frac{1}{2}}^{\infty} r : b_{-r} \cdot b_r :,$$

or,

$$L_0^{(f)} = \sum_{n=1}^{\infty} n : d_{-n} \cdot d_n : .$$

3.1.3.1 Super-Virasoro algebra for open strings

In the R-sector, the algebra is

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{D}{8}m^3\delta_{m+n,0}, \quad (3.46)$$

$$[L_m, F_n] = \left(\frac{m}{2} - n\right)F_{m+n}, \quad (3.47)$$

$$\{F_m, F_n\} = 2L_{m+n} + \frac{D}{2}m^2\delta_{m+n,0}. \quad (3.48)$$

And, in the NS-sector, the algebra is

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{D}{8}m(m^2 - 1)\delta_{m+n,0}, \quad (3.49)$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r}, \quad (3.50)$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}. \quad (3.51)$$

When quantizing the RNS string one can only require that the positive modes of the Virasoro generators annihilate the physical state. Let $|\phi\rangle$, in the NS-sector. A physical state is one that is annihilated by half of the Virasoro generators⁵,

$$\begin{aligned} J_+ \sim G_r|\phi\rangle &= 0 & r > 0, \\ T_{++} \sim L_m|\phi\rangle &= 0 & m > 0. \end{aligned} \quad (3.52)$$

Together with the mass-shell condition ⁶,

$$(L_0 - a_{\text{NS}})|\phi\rangle = 0. \quad (3.53)$$

⁵This statement come from the Gupta-Bleuler quantization in electrodynamics, the requirement that only the positive-frequency states annihilate the physical state, which is very useful to eliminate the two non-physical degrees of freedom of the photon field.

⁶The usual mass-shell condition is $p^2 + M^2 = 0$. Since $L_0 = \alpha' p^2 + N$ and $\alpha' M^2 = N - a$ (here we changed $C = -a$); (3.53) and (3.55) represent the mass-shell condition for $|\phi\rangle$.

Similarly, in the R-sector last conditions are,

$$\begin{aligned} J_+ \sim F_n |\phi\rangle &= 0 & n \geq 0, \\ T_{++} \sim L_m |\phi\rangle &= 0 & m > 0, \end{aligned} \tag{3.54}$$

and the mass-shell one,

$$(L_0 - a_R) |\phi\rangle = 0. \tag{3.55}$$

From (3.48), for $m, n = 0$, we get

$$L_0 = F_0^2 \Rightarrow a_R = 0.$$

It is worth to notice that in the NS–algebra, which is infinite, exists a closed superalgebra. In (3.51) the anomaly term vanishes when $r = \pm\frac{1}{2}$, then the commutator in (3.50) vanishes for $m = \pm 1$. But the algebra closed also when $m = 0$. The anomaly term in (3.49) vanishes with these values of m . Hence, $G_{\pm\frac{1}{2}}$, $L_{\pm 1}$ and L_0 form a closed algebra (no anomaly terms) which is called $SU(1, 1|1)$.

3.1.3.2 The critical dimension

An important issue is to determine what is the critical dimension where this superstring theory works as a consistent description of nature. In a string theory (whether bosonic or supersymmetric) we could have negative-norm states. A clue to notice that our quantum theory is not correct at all, is when we promote the position and momentum to operators, the commutation rules (3.20) are proportional to $\eta^{\mu\nu}$, so when $\mu, \nu = 0$, a minus sign appears. The spectrum of the theory is free of these states only for certain values of the dimension of spacetime, d .

In string theory, the space of states with positive and negative norm, is divided by a huge number of zero-norm states. In order to determine the value of d , the strategy is to look for zero-norm states that satisfy the physical-state conditions.

A state $|\psi\rangle$ is called spurious if it satisfies the mass-shell condition and is orthogonal to all physical states, so it decouples from all physical processes,

$$(L_0 - a) |\psi\rangle = 0 \quad \text{and} \quad \langle \phi | \psi \rangle = 0, \tag{3.56}$$

where $|\phi\rangle$ represents any physical state in both sectors of the theory, and a is $a_R = 0$ or $a_{NS} = 1/2$. The dimension of flat-spacetime that allow the presence of these non-physical states is $d = 10$.

Spurious states appear before that negative-norm states appear, that is why the value $d = 10$ for the dimension of spacetime is called critical.

There is another way to restrict the dimensionality of the string theory. In the bosonic case, the Weyl anomaly coming from the nonvanishing of the trace of the energy-momentum tensor, leads to consider in the Polyakov formalism to cancel the central charge appearing as proportional to the anomalous trace. In the supersymmetric case we have a similar situation involving the central charge, that by including the anticommuting scalar ghosts, leads to an expression for the total central charge, whose vanishing results in $d = 10$ for an anomaly-free theory (see for example [21, 22], for detailed calculations).

3.1.3.3 The spectrum of the open string

We found that $a_{\text{NS}} = \frac{1}{2}$ in the NS-sector. By acting the mass operator on the ground state $|0\rangle_{\text{NS}}$ we get,

$$\begin{aligned}\alpha' M^2 |0\rangle_{\text{NS}} &= \left(\sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r} \cdot b_r - \frac{1}{2} \right) |0\rangle_{\text{NS}} = -\frac{1}{2} |0\rangle_{\text{NS}}, \\ M^2 |0\rangle_{\text{NS}} &= -\frac{1}{2\alpha'} |0\rangle_{\text{NS}} < 0.\end{aligned}\tag{3.57}$$

Thus, the ground state in this sector is a tachyon, and must be eliminated. The first excited state can be obtained by acting a rising operator $b_{\frac{1}{2}}^i$ ⁷ on the ground state, $b_{-\frac{1}{2}}^i |0\rangle_{\text{NS}}$, which is a vector with eight transverse components. The the mass will be calculated and gives,

$$\begin{aligned}\alpha' M^2 b_{-\frac{1}{2}}^i |0\rangle &= \left(\sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r} \cdot b_r - \frac{1}{2} \right) b_{-\frac{1}{2}}^i |0\rangle. \\ M^2 (b_{-\frac{1}{2}}^i |0\rangle) &= 0.\end{aligned}\tag{3.58}$$

We could apply another rising operator α_{-1} , but the mass of $\alpha_{-1} |0\rangle_{\text{NS}}$ is greater than zero ($M^2 = \frac{1}{2\alpha'}$). That is the reason why the first excited state is built by acting with a $b_{-\frac{1}{2}}^i$ operator. The we got a massless spacetime vector with eight polarizations.

In the R-sector, the mass-shell condition is,

$$\begin{aligned}\alpha' M^2 |0\rangle_{\text{R}} &= \left(\sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{m=1}^{\infty} m d_{-m} \cdot d_m \right) |0\rangle_{\text{R}}, \\ M^2 |0\rangle_{\text{R}} &= 0.\end{aligned}\tag{3.59}$$

⁷We only consider the transverse coordinates because of the light-cone gauge, that allows us to eliminate two of ten degrees of freedom.

Then, the ground state $|0\rangle_R$ is massless. Recall that since there is a degenerate set of ground states, so the solutions to (3.59) is not unique. We saw before that this ground state belongs to the spinor representation of $SO(9,1)$ or $Spin(9,1)$. In ten dimensions, $SO(9,1)$ spinors have 32 complex components, but can be restricted by Majorana and Weyl conditions. The Majorana condition impose reality, then we have 32 real components (16 complex). And the Weyl condition⁸ selects one chirality (since one could imagine that both chiralities are allowed), hence the spinor has eight components at the end, and is called a Majorana-Weyl spinor in ten-dimensions. So, the minimal possibility for a Ramond ground state has eight degrees of freedom corresponding to an irreducible spinor of $Spin(8)$.

3.1.3.4 GSO projection, eliminating the tachyon state

From our results above, we can realize that the RNS string has several problems. First, the NS-sector contains a tachyon (see (3.57)). Second, there is no spacetime supersymmetry, tachyon is a scalar which has not fermionic partner. The number of degrees of freedom in the ground state is not the same in both sectors, we have a scalar and a spinor with eight components.

A way to turn the RNS formulation into a consistent theory was proposed by Gliozzi, Scherk and Olive in 1976. It consists in truncating (or projecting) the spectrum in a very specific way that eliminates the tachyon to match the number of bosonic and fermionic states in the spectrum. This method is called GSO projection (see [34]).

Let us define the G -parity operator that in the NS-sector is given by

$$G = (-1)^{F+1} = (-1)^{\sum_{r=\frac{1}{2}}^{\infty} b_{-r}^i b_r^{i+1}}, \quad (3.60)$$

where F is the number of b -modes, which is the world-sheet fermion number. So this operator determines whether a state has an even or an odd number of world-sheet fermion excitations. In

⁸Whenever d is even one can define a matrix, analogous to γ_5 in four dimensions, that can be used to define chirality of spinors in d dimensions. For our case, in $d = 10$, we introduce

$$\Gamma_{11} = \Gamma^0 \Gamma^1 \dots \Gamma^9,$$

which satisfies Γ_{11}, Γ_μ and (Γ_{11}^2) . Spinors ψ that satisfy $\Gamma_{11}\psi = +\psi$ or $\Gamma_{11}\psi = -\psi$ are called spinor of positive and negative chirality, respectively. Then, the operator $(1 \pm \Gamma_{11})/2$ projects one of the chiralities. A spinor which is eigenspinor of this operator, having a definite chirality, is called a Weyl spinor. And, the restriction to spinors of one chirality or the other is called Weyl condition.

the R-sector the corresponding definition is

$$G = \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} d_{-n}^i d_n^i}. \quad (3.61)$$

The GSO projection consists of keeping only the states with $G = 1$ in the NS-sector, then

$$(-1)^F = -1, \quad (3.62)$$

whilst the states with negative G-parity should be eliminated. Then, states constructed with an even number of b -oscillator excitations will not be considered. In the R-sector, truncation will depend of the chirality of the spinor ground state. Then, GSO projection eliminates the tachyon from the spectrum of open strings, leaving to the massless vector as the ground state of the NS-sector. Hence the bosonic and fermionic degrees of freedom match nicely. This is a clue that the spectrum could be spacetime supersymmetric after GSO projection.

It is possible to show that the GSO condition, artificial in a way, allows us to have equal number of boson and fermion states at each massive level. Then, the GSO projection indeed leads to a supersymmetric spectrum.

3.1.4 The closed string spectrum, type IIA and IIB superstrings

Recall that for closed strings, we need to consider left- and right- movers. There are four sectors: R-R, R-NS, NS-R and NS-NS. The tachyon was eliminated by GSO in the NS sector, by projecting onto states with positive G-parity. As we said before, in the R-sector one can project onto states with positive or negative G-parity depending on the chirality of the ground state on which the states are built. Depending on whether the G-parity of the left- and right-moving states in the R-sector, is the same or opposite, we can formulate two types of superstring theories, called type IIA and type IIB.

For our purposes, type IIB superstring theory will be a central matter. In this theory the left- and right-moving R-sector ground states have the same chirality, which by definition is positive. Therefore, the two R-sectors have the same G-parity. We can use our results for open strings to obtain the massless ground states for type IIB closed strings, just taking products of the ground

states for open strings. Thus, the massless spectrum is

$$\begin{aligned}
& |+\rangle_{\text{R}} \otimes |+\rangle_{\text{R}} \quad , \\
& \tilde{b}_{-\frac{1}{2}}^i |0\rangle_{\text{NS}} \otimes b_{-\frac{1}{2}}^j |0\rangle_{\text{NS}} \quad , \\
& \tilde{b}_{-\frac{1}{2}}^i |0\rangle_{\text{NS}} \otimes |+\rangle_{\text{R}} \quad , \\
& |+\rangle_{\text{R}} \otimes b_{-\frac{1}{2}}^i |0\rangle_{\text{NS}} \quad .
\end{aligned} \tag{3.63}$$

One can notice that each sector have $\mathbf{8} \times \mathbf{8}$ physical states. The other superstring theory is called type IIA. Here, the left- and right-moving R-sector ground states have opposite chirality, since that is the other option. Then, the massless spectrum is given by

$$\begin{aligned}
& |-\rangle_{\text{R}} \otimes |+\rangle_{\text{R}} \quad , \\
& \tilde{b}_{-\frac{1}{2}}^i |0\rangle_{\text{NS}} \otimes b_{-\frac{1}{2}}^j |0\rangle_{\text{NS}} \quad , \\
& \tilde{b}_{-\frac{1}{2}}^i |0\rangle_{\text{NS}} \otimes |+\rangle_{\text{R}} \quad , \\
& |-\rangle_{\text{R}} \otimes b_{-\frac{1}{2}}^i |0\rangle_{\text{NS}} \quad .
\end{aligned} \tag{3.64}$$

Both theories contain two Majorana-Weyl spinors (gravitinos), therefore they form $\mathcal{N} = 2$ supersymmetric theories. Let us name the massless states for both. Remember that each sector must have **64** states ⁹

- NS – NS Type IIA-IIB: A scalar (dilaton), an antisymmetric two-form gauge field and a symmetric traceless rank-two tensor (graviton). They have 1, 28 and 35 degrees of freedom respectively.
- NS – R and R – NS Type IIA-IIB: spin 3/2 gravitino (56 states) and a spin 1/2 fermion called the dilatino (8 states).
- R – R
 - Type IIA: A one-form (vector) gauge field (8 states) and a three-form gauge field (56 states).
 - Type IIB: A zero-form (scalar) gauge field (1 state), a two-form gauge field (28 states) and a four-form gauge field with a self-dual field strength (35 states).

⁹This matter content comes from the counting of states of each term in the product of representations (see [34] and [22]).

3.2 T-duality and D-branes

We have discussed some details about superstrings, but that is not the whole story. There exist higher-dimensional and nonperturbative objects called D-branes where open strings end. In principle, it is natural to postulate those objects, open strings must be attached somewhere. But there is a stronger way to motivate their necessary existence in the theory, T-duality. In order to achieve those concepts, we need to establish the idea of T-duality first for closed bosonic strings, and then continue to open and supersymmetric strings.

3.2.1 T-duality for closed bosonic strings

Remember the mode expansion for closed strings,

$$\begin{aligned} X_R^\mu &= \frac{1}{2}x^\mu + \alpha'p^\mu(\tau - \sigma) + \sqrt{\frac{\alpha'}{2}}i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)}, \\ X_L^\mu &= \frac{1}{2}x^\mu + \alpha'p^\mu(\tau + \sigma) + \sqrt{\frac{\alpha'}{2}}i \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)}. \end{aligned} \quad (3.65)$$

Now, assume that the spacetime is $M_{25} \times S_1$, where we have compactified X^{25} in a circle S_1 of radius R , and M_{25} is for 25-dimensional Minkowski spacetime.

To describe a closed bosonic string in this compactified theory, one takes periodic boundary conditions for X^{25} as

$$X^{25}(\sigma + \pi, \tau) = X^{25}(\sigma, \tau) + 2\pi WR \quad \text{where } W \in \mathbb{Z}, \quad (3.66)$$

where K is called the winding number, which indicates the number of times the string winds around the circle. X^{25} can be split into left- and right-movers as

$$X^{25}(\sigma, \tau) = X_L^{25}(\sigma + \tau) + X_R^{25}(\sigma - \tau). \quad (3.67)$$

In order to include this compactification, the expansions for the left- and right-movers must change as

$$\begin{aligned} X_R^{25} &= \frac{1}{2}x^{25} + \alpha' \frac{K}{R}(\tau - \sigma) - WR(\tau - \sigma) + \dots, \\ X_L^{25} &= \frac{1}{2}x^{25} + \alpha' \frac{K}{R}(\tau + \sigma) + WR(\tau + \sigma) + \dots, \end{aligned} \quad (3.68)$$

where "... " refers to the oscillator terms. Here K is an integer number coming called Kaluza-Klein excitation number, and came from the quantization of the linear momentum p^{25} in the compactified direction, $p^{25} = K/R$.

We can set the zero-modes α_0^{25} and $\tilde{\alpha}_0^{25}$ to be

$$\begin{aligned}\sqrt{2\alpha'}\alpha_0^{25} &= \alpha'\frac{K}{R} - WR, \\ \sqrt{2\alpha'}\tilde{\alpha}_0^{25} &= \alpha'\frac{K}{R} + WR.\end{aligned}\tag{3.69}$$

The 26-dimensional mass squared is given by

$$\begin{aligned}\frac{1}{2}\alpha'M^2 &= (\alpha_0^{25})^2 + 2N_R - 2, \\ &= \frac{1}{2}\alpha'\left[\left(\frac{K}{R}\right)^2 + \left(\frac{WR}{\alpha'}\right)^2\right] - WK + 2N_R - 2, \\ &= \frac{1}{2}\alpha'\left[\left(\frac{K}{R}\right)^2 + \left(\frac{WR}{\alpha'}\right)^2\right] + N - 2,\end{aligned}\tag{3.70}$$

where we have defined,

$$\begin{aligned}N_R - N_L &= WK, \\ N_R + N_L &= N.\end{aligned}\tag{3.71}$$

The usual level matching condition that in closed bosonic string theory $N_R = N_L$ is modified for closed strings with both nonzero winding number W and Kaluza-Klein momentum K .

Now, let us establish the duality. If we interchange simultaneously $W \leftrightarrow K$ and $R \leftrightarrow \alpha'/R$. Equation (3.70) is invariant under this change, which is called T-duality. It suggest a physical equivalence¹⁰ between compactifying on a circle of radius R and on a circle of radius $\tilde{R} = \alpha'/R$.

As $R \rightarrow \infty$, all states with $W \neq 0$ become infinitely massive, while the $W = 0$ states for all values of K go over to a continuum. As $R \rightarrow 0$, all states with $K \neq 0$ become infinitely massive, and when $K = 0$, we get a continuum of states, since it is very “cheap” to wind around a small circle. Then, we say that, in this limit, the compactified dimension reappears as a dual theory with $R \rightarrow \infty$. This is one piece of evidence for the idea that there is a minimum length in string theory, the self-dual radius $R = \sqrt{\alpha'} = l_s$ with mass

$$\alpha'M^2 = K^2 + W^2 + 2(N_R + N_L) - 4.\tag{3.72}$$

T-duality can also be expressed as

$$\alpha_0^{25} \rightarrow -\alpha_0^{25}, \quad \tilde{\alpha}_0^{25} \rightarrow \tilde{\alpha}_0^{25},\tag{3.73}$$

¹⁰This physical equivalence is a clear indication that ordinary and intuition can break down in string theory at the string scale.

as we can see from (3.69). Then we say that T-duality is a left-handed parity transformation¹¹. Actually, it is not just the zero mode, but the entire right-moving part of the compact coordinate that flips sign under T-duality,

$$X_R^{25} \rightarrow -X_R^{25}, \quad X_L^{25} \rightarrow X_L^{25}. \quad (3.74)$$

Then, X^{25} given in (3.67) transforms as

$$X^{25}(\sigma, \tau) \rightarrow \tilde{X}^{25} = X_L^{25}(\sigma + \tau) - X_R^{25}(\sigma - \tau), \quad (3.75)$$

which has an expansion

$$X^{25}(\sigma, \tau) \rightarrow \tilde{X}^{25} = \tilde{x} + 2\alpha' \frac{K}{R} \sigma + 2WR\tau + \dots \quad (3.76)$$

From we can read off that the periodicity in the dual circle is $2\pi\tilde{R}$, and its conjugate momentum $\tilde{p}^{25} = WR/\alpha' = W/\tilde{R}$

3.2.2 T-duality for open bosonic strings

The action for bosonic strings was given at the beginning of the chapter in (3.1), in conformal gauge as

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \eta_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu. \quad (3.77)$$

As we know, variation of this action gives the equations of motion and a boundary term, which must vanish,

$$\delta S = -\frac{1}{2\pi\alpha'} \int d\tau \eta_{\mu\nu} (\partial_\sigma X^\mu \delta X^\nu) \Big|_{\sigma=0}^{\sigma=\pi} = 0. \quad (3.78)$$

The only choice of boundary condition that is compatible with Poincaré invariance is Neumann boundary conditions for all X^μ ,

$$\partial_\sigma X^\mu(\sigma, \tau) = 0, \quad \text{for } \sigma = 0, \pi. \quad (3.79)$$

The expansion for open strings is also known,

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + \sqrt{2\alpha'} i \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\tau} \cos m\sigma. \quad (3.80)$$

¹¹Actually, this was a choice because there are some references where T-duality is a right-handed transformation (see [21, 39] and [22]). That came from the definition of the left- and right-movers.

Again, it convenient to split this expansion into left- and right- movers,

$$\begin{aligned} X_R^\mu &= \frac{1}{2}x^\mu + \alpha'p^\mu(\tau - \sigma) + \sqrt{\frac{\alpha'}{2}}i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau - \sigma)}, \\ X_L^\mu &= \frac{1}{2}x^\mu + \alpha'p^\mu(\tau + \sigma) + \sqrt{\frac{\alpha'}{2}}i \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in(\tau + \sigma)}. \end{aligned} \quad (3.81)$$

If we compactify, once again, on a circle and carrying out T-duality to get,

$$X_R^{25} \rightarrow -X_R^{25}, \quad X_L^{25} \rightarrow X_L^{25}, \quad (3.82)$$

as before. The dual coordinate $\tilde{X}^{25} = X_L^{25} - X_R^{25}$ reads,

$$\tilde{X}^{25} = \tilde{x}^{25} + 2\alpha'p^{25}\sigma + \sqrt{2\alpha'} \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\tau} \sin m\sigma. \quad (3.83)$$

Notice that the T-dual coordinate has not oscillator terms at $\sigma = 0, \pi$, then, its position is fixed. Remember that we begin with Neumann boundary condition in the 25 direction. So now, for \tilde{X}^{25} we actually have Dirichlet boundary conditions (then we say that T-duality maps Neumann to Dirichlet boundary conditions and viceversa). The difference,

$$\tilde{X}^{25}(\pi, \tau) - \tilde{X}^{25}(0, \tau) = 2\pi K \tilde{R}, \quad (3.84)$$

allows to say that the endpoints of the open strings are fixed in those points, and are free to move in the other directions,

$$\begin{aligned} \tilde{X}^{25}(0, \tau) &= \tilde{x}^{25}, \\ \tilde{X}^{25}(\pi, \tau) &= \tilde{x}^{25} + 2\pi K \tilde{R}. \end{aligned} \quad (3.85)$$

Then, there exists a hyperplane described by the other 25 remaining coordinates where those endpoints lie (see [40]). Observe also that this string wraps the dual circle K times. This winding mode is then topologically stable, since the end points of the string are fixed by the Dirichlet boundary conditions. We say that the string cannot unwind without breaking.

3.2.3 T-duality with Chan-Paton factors and Wilson lines

We can endow the string endpoints with non-dynamical degrees of freedom without spoiling neither spacetime Poincaré nor conformal invariance. These are called Chan-Paton degrees of freedom [39], and can be seen by defining a basis to write a string state $|k\rangle$ with momentum k as

$$|k\rangle = \sum_{i,j=1}^N |k; ij\rangle \lambda_{ij}, \quad (3.86)$$

where λ_{ij} is a $N \times N$ matrix and $|k; ij\rangle$ are called the Chan-Paton factors. Those matrices must be invariant under global $U(N)$ in order to have invariant amplitudes. Then every state in the open bosonic string spectrum has now an additional N^2 multiplicity. For the case of oriented strings, the two endpoints are distinguished, so it make sense to associate the fundamental representation \mathbf{N} with the $\sigma = 0$ end and the antifundamental representation $\bar{\mathbf{N}}$ with $\sigma = \pi$ end the bifundamental representation $(\mathbf{N}, \bar{\mathbf{N}})$ of $U(N)$.

Now let us introduce the idea of Wilson line, or more precisely a Wilson loop, as a gauge invariant object defined for the 25-direction by

$$W_C = \exp \left(i \oint_C dX^{25} A_{25} \right). \quad (3.87)$$

Then we define a pure gauge which breaks $U(N) \rightarrow U(1)^N$ as

$$A_{25} = \text{diag}\{\theta_1, \dots, \theta_N\}/2\pi R = -i\Lambda^{-1} \partial_{25} \Lambda, \quad (3.88)$$

where,

$$\Lambda = \text{diag} \left\{ e^{iX^{25}\theta_1/2\pi R}, \dots, e^{iX^{25}\theta_N/2\pi R} \right\}. \quad (3.89)$$

Then the Wilson line will be

$$W_C = \text{diag} \left\{ e^{i\theta_1}, \dots, e^{i\theta_N} \right\}. \quad (3.90)$$

The gauge transformation is not periodic, it will acquire a phase W_C ,

$$\Lambda(X^{25} + 2\pi R) = W_C \Lambda(X^{25}), \quad (3.91)$$

under $X^{25} \rightarrow X^{25} + 2\pi R$. Now, let us see how T-duality acts on Chan-Paton factors. Since λ_{ij} matrices are invariant under $U(N)$, in particular under $U(1)^N$, the transformed state $|k'\rangle$ is given by

$$|k'\rangle = \sum_{i,j=1}^N |k; ij\rangle e^{-i\frac{X^{25}}{2\pi R}(\theta_i - \theta_j)} \lambda_{ij}, \quad (3.92)$$

and under $X^{25} \rightarrow X^{25} + 2\pi R$ we finally obtain the string state,

$$|k'\rangle_C = \sum_{i,j=1}^N e^{i(\theta_i - \theta_j)} |k; ij\rangle e^{-i\frac{X^{25}}{2\pi R}(\theta_i - \theta_j)} \lambda_{ij}. \quad (3.93)$$

Furthermore, if we write $|k; ij\rangle = e^{iX^{25}p^{25}} |ij\rangle$,

$$|k'\rangle_C = \sum_{i,j=1}^N e^{j(\theta_j - \theta_i)} |ij\rangle e^{iX^{25}(p^{25} + \frac{1}{2\pi R}(\theta_j - \theta_i))} \lambda_{ij}, \quad (3.94)$$

then we can read off that the state $|ij\rangle$ picks up a phase $e^{i(\theta_j - \theta_i)}$ and the corresponding momentum will be given by

$$p_{ij}^{25} = \frac{K}{R} + \frac{1}{2\pi R}(\theta_j - \theta_i) = \frac{[2\pi K + (\theta_j - \theta_i)]\tilde{R}}{2\pi\alpha'}. \quad (3.95)$$

The Dirichlet boundary condition corresponding to $|ij\rangle$ is

$$\tilde{X}^{25}(\pi, \tau) - \tilde{X}^{25}(0, \tau) = [2\pi K + (\theta_j - \theta_i)]\tilde{R}. \quad (3.96)$$

Since the momentum is dual to the winding number, we expect the field in the T-dual description to have fractional winding number, which means that their endpoints are no longer on the same hyperplane. Moreover, note that $[2\pi K + (\theta_j - \theta_i)]\tilde{R}$ is the minimum length of a string winding between two of N D-branes placed in $\theta_i\tilde{R}$ and $\theta_j\tilde{R}$.

3.2.4 Dp-branes

Let us compactify a set of directions, say $\{X^{p+1}, \dots, X^{25}\}$, instead of only one. We can use T-duality to find that the open string endpoints lie on N $(p+1)$ -dimensional hyperplanes. That hyperplane is the worldvolume of a p -dimensional extended object called Dirichlet p -brane, or Dp-brane for short.

Dp-branes are in fact not rigid in spacetime. They are dynamical, and can fluctuate both in shape and position. That is what will lead us to a description in terms of gauge fields on the brane, and scalars as its positions.

We firstly look at the mass spectrum,

$$\begin{aligned} M_{ij}^2 &= (p^{25})^2 + \frac{1}{\alpha'}(N-1), \\ &= \left[\frac{[2\pi K + (\theta_j - \theta_i)]\tilde{R}}{2\pi\alpha'} \right]^2 + \frac{1}{\alpha'}(N-1). \end{aligned} \quad (3.97)$$

Let us focus in string states with $K = 0$ (non-winding) and $N = 1$ ($\alpha_{-1}^i |k; ij\rangle$), then mass spectrum will be proportional to the distance between hyperplanes i and j as

$$M_{ij} = \frac{|\theta_i - \theta_j|\tilde{R}}{2\pi\alpha'} \quad (3.98)$$

Thus, generically massless states only arise for non-winding open strings, i.e. open strings that not necessarily ends in the same point of the brane, whose ends lie on the same D-brane $i = j$.

We can see that there are two such types of states,

- $\alpha_{-1}^\mu |k; ii\rangle$: These states correspond to a gauge field $A_\mu(X^a)$ on the D-brane with $p + 1$ coordinates tangent to the hyperplane, where $\mu, a = 0, 1, \dots, p$ are coordinates on the D-brane worldvolume.
- $\alpha_{-1}^m |k; ii\rangle$: These states correspond to massless scalar fields $\phi^m(X^a)$, $m = p + 1, \dots, 25$ which give the transverse position of the D-brane in the compact directions.

Then, a flat hyperplane in spacetime can be described by open string states corresponding to gauge fields. This gives a remarkable description of these objects in terms of gauge theory and conversely, we could learn much about gauge theories from D-branes dynamics and string theories, such as their strong coupling expansions as we will see later.

3.2.5 Non-abelian gauge symmetry and Higgs fields

Suppose that $k \leq N$ D-branes coincide, say

$$\theta_1 = \theta_2 = \dots = \theta_k = \theta. \quad (3.99)$$

From (3.98), it follows that $M_{ij} = 0$ for $1 \leq i, j \leq k$. The new massless states can arise because of open strings can now attain a vanishing length. Then, there will be $k \times k$ massless vectors forming an adjoint representation of a $U(k)$ gauge group. This coincident limit corresponds to the Wilson line,

$$W = \text{diag}\{ e^{i\theta} \mathbf{1}_{k \times k}, e^{i\theta_{k+1}}, \dots, e^{i\theta_N} \}, \quad (3.100)$$

where $\mathbf{1}_{k \times k}$ denotes the $k \times k$ identity matrix. The background field configuration leaves unbroken a $U(k) \subset U(N)$ subgroup, acting on the upper $k \times k$ block of W in (3.100) [41]. Thus the D-brane worldvolume carries a $U(k)$ gauge field $\alpha_{-1}^\mu |ij\rangle \leftrightarrow A^\mu(X^a)_{ij}$, and, at the same time, a set of k^2 massless scalar fields $\alpha_{-1}^m |ij\rangle \leftrightarrow \Phi^m(X^a)_{ij}$, where the subscript $ij = 1, \dots, k$ represents the corresponding field for a coincident pair of D-branes placed in $\theta_i = \theta_j$. Then, the k D-branes positions in spacetime are promoted to a set of matrices $\Phi^m(X^a)$ in the adjoint representation of the unbroken $U(k)$ gauge group.

Generally, the gauge symmetry of the theory is broken to the subgroup of $U(N)$ which commutes with the Wilson line W . If all D-branes coincide, we say that W belongs to the center of the Chan-Paton gauge group¹², and then we recover the original $U(N)$ gauge symmetry. Since Φ^m is

¹²The center of a group G is the set of elements that commute with every element of G . The center is a subgroup of G , which by definition is abelian and contains the identity.

a Hermitian matrix, we can diagonalize it by a $U(N)$ gauge transformation as $U^\dagger \Phi^m U = \Phi_{\text{diag}}^m$.

The diagonal elements of Φ_{diag}^m describe the classical positions of the D-branes, in other words, the ground state of the system of N D-branes. The $N \times N$ unitary matrices U describe the fluctuations U_{ij} , $i \neq j$ about classical spacetime. Thus, the off-diagonal elements Φ_{ij} , $i \neq j$ may act as Higgs fields for the symmetry breaking mechanism. We will see this later specifically for the Klebanov-Witten model.

3.2.6 T-duality for type II superstring theories

We can generalize our results to superstrings. Let us consider the effects of T-duality on the closed, oriented type II theories. Remember that under T-duality the bosonic coordinates change as (3.74). Then, if we compactify X^9 on a circle of radius R we will get the same result since the expansions are the same,

$$X_R^9 \rightarrow -X_R^9, \quad X_L^9 \rightarrow X_L^9. \quad (3.101)$$

Supersymmetry requires the world-sheet fermion ψ^9 to transform in the same way as its bosonic partner, then

$$\psi_R^9 \rightarrow -\psi_R^9, \quad \psi_L^9 \rightarrow \psi_L^9. \quad (3.102)$$

Remind the mode expansions for the fermion field given in (3.28) and (3.29). In particular, the zero mode of ψ_R^9 in the Ramond sector transforms as

$$d_0^9 \rightarrow -d_0^9. \quad (3.103)$$

Then, under T-duality Γ^9 will transform as $\Gamma^9 \rightarrow -\Gamma^9$. There is also defined the matrix $\Gamma^{11} = \Gamma_0 \Gamma_1 \dots \Gamma_9$ which projects the chirality of the Ramond ground states, which also flips sign under T-duality.

We conclude that T-duality transforms type IIA (same chirality) superstrings compactified on a circle of radius R into type IIB (opposite chirality) superstrings compactified on a circle of radius \tilde{R} . Then in this sense the type IIA and the IIB theories are duals under T-duality ([42–45] are recommendable.).

3.2.6.1 D-branes in type IIB superstring theories

There are some new properties that these branes will have in order to ensure its stability, they must carry a conserved charge, just like point particles carry electric (or magnetic) charge.

p-forms and charged p-branes

Let us remember the ordinary electromagnetism, where charged point particles moving in some background, source an electromagnetic field. This field is described by a 1-form, $A_{(1)}$. This field couples naturally to the worldline of the particle γ in the action,

$$S_P = -m \int_{\gamma} d\tau + q \int_{\gamma(1)} A_{(1)}.$$

And, if we also consider that electromagnetic field as part of the matter content of the system, we must to include its kinetic term, which is, as usual, written by means of the field strength $F_{(2)} = dA_{(1)}$, as F^2 . The equation of motion of electromagnetism are the Maxwell equations, written in a compact form, they read as

$$d * F_{(d-2)} = *J_{(d-3)e}, \quad dF_{(2)} = *J_{(1)m}.$$

The first equation contains the dual field strength, that in a d-dimensional spacetime, is a (d-2)-form. We also have included a magnetic current J_m , which is a (d-3)-form.

Now, Maxwell equations, written in the differential form language, do not allude neither to the rank of the forms nor the dimension of the spacetime, then we can generalize the theory by making F an (p+2)-form, then A will be an (p+1)-form sourced by a p-dimensional object, which will be called p-brane.

Following the last discussion, the (p+1)-field $A_{(p+1)}$ will couple to the worldvolume of the p-brane,

$$\int_{V_{p+1}} A_{(p+1)}.$$

Locally we could define the a (d-p-2)-form, $F_{(d-p-2)}$, which is dual to $F_{(p+2)}$. This dual form comes from $*F = d * A$, where $A_{(d-p-3)}$ is called the dualized potential sourced by a "magnetically" charged (d-p-4)-dimensional object. This potential couples to the worldvolume of this "magnetic" object as

$$\int_{V_{(d-p-3)}} *A_{(d-p-3)}.$$

Gauss law allows to comment about how the electric and magnetic charges are distributed in those higher dimensional objects,

$$Q_e = \int_{\partial V_{d-p-1}} *F, \tag{3.104}$$

$$Q_m = \int_{\partial V_{p+1}} F. \tag{3.105}$$

Thus, we have extended the ideas of the electromagnetic theory, firstly considering "magnetic" charges and then, higher dimensional objects, p -branes, that source electric and magnetic potentials which in turn defines field strengths F and $*F$. These p -branes are electrically and magnetically charged, and as special case, some of them are charged electrically and magnetically at the same time when

$$p = \frac{d-2}{2}. \quad (3.106)$$

D-branes as BPS states

Remember what we learned in the chapter about supersymmetry, a BPS state is an invariant state under a half of the supersymmetry, as we have mentioned before in (2.48). Let us see how this concept arises in the context of D-branes. Far from the D-brane we see only the closed string spectrum, with two gravitinos. However, worldsheet boundary conditions (R and NS) reflect the right-moving into the left-moving, so only one linear combination of the two supercharges is a good symmetry of the full state (containing both R and NS states). In other words, in the type II theory with D-branes, half of the supersymmetries of the bulk theory are broken, this is a BPS state [46].

Stable D-branes

Let us focus first in the massless R-R sector of type IIB superstrings given in (3.64), where we have obtained as gauge fields, a zero-form, a two-form and a four-form. According to our generalized idea of charged objects, if type IIB has electrically and magnetically charged Dp -branes, there will exist Dp -branes for $p = -1, 1, 3, 5, 7$. The $D(-1)$ -brane is known as a Dirichlet instanton, the $D1$ -brane is a one-dimensional object, similar to a string but electrically charged. It is coupled magnetically to a $D5$ -brane. The $D3$ -brane is an special case, an both object electrically and magnetically charged at the same time. The $D3$ -brane source the four-form of the massless spectrum, which in turns has as field strength a five-form $F_{(5)}$ which is self-dual, $F_{(5)} = *F_{(5)}$.

In type IIA, we know from (3.63) that the massless spectrum is given by a one-form, a three-form and a five-form. Then Dp -branes in this theory will exist for $p = 0, 2, 4, 6$.

We conclude that stable D-branes arise for type IIA with p even, and with p odd for type IIB superstrings.

But why do we say that those higher dimensional objects are stable?

BPS objects are stable because they are the lightest objects with given values of certain charges. So there exists no potential final state that would be lighter.

3.2.7 The Dirac-Born-Infeld action and the nonperturbative nature of branes

We mentioned that D-branes are nonperturbative and dynamical objects in string theory. Then it is natural to try to write the corresponding action for them.

Since Dp-branes source $(p+1)$ -forms that, just like electromagnetism of point particles, have kinetic terms described by $(p+2)$ -field strengths, we could generalize the Maxwell theory in order to introduce them as kinetic terms in an action. If we consider a Dp-brane, then there will be a F_{ab} field strength of the gauge fields A_a living on its worldvolume,

$$F_{ab} = \partial_a A_b - \partial_b A_a + \dots \quad (3.107)$$

In 1989, Leigh proposed a σ -model for Dp-branes, in which one introduces coordinates ξ_a , $a = 0, \dots, p$ on the Dp-brane worldvolume. The action, called Dirac-Born-Infeld (DBI) action [43–45]¹³, for a Dp-brane is

$$S_p = -T_p \int d^{p+1}\xi \sqrt{\det(G_{ab} + k F_{ab})}, \quad (3.108)$$

where $k = 2\pi\alpha'$. We also have the pulled back the ten dimensional metric $G_{\mu\nu}$ to a $(p+1)$ -dimensional one.

Now, tension of branes, T_p , depends of the inverse of the string coupling g_s . It can be seen by noticing that the DBI action contains the usual kinetic term for the field strength of the gauge field A_a living in the D9-brane worldvolume. Maxwell action for a gauge field in $p+1$ dimensions is

$$S_{\text{YM}_{p+1}} = -\frac{1}{4g_{YM}^2} \int d^{p+1}\xi F_{ab} F^{ab}, \quad (3.109)$$

where g is the corresponding coupling to open strings. Then we can compare this action with the

¹³The Born-Infeld action was suggested as a way to generalize Maxwell theory in order to eliminate the infinite self-energy of a charged point particle.

analogous term coming from the expansion of the DBI action and find that¹⁴,

$$T_p \sim \frac{1}{g_{YM}^2} \sim \frac{1}{g_s}. \quad (3.110)$$

The last result allows to say that Dp-branes are indeed nonperturbative objects, since they become heavy at weak coupling. We will clarify better some details about this weak-coupling limit in the next section.

3.3 Low-energy effective actions

The low-energy string effective action describes the low-energy dynamics of a given string theory. Here, low-energy means energies lower than the relevant energy scale, the string mass. Then, at low-energies only the massless modes are relevant and their dynamics is described by the so-called supergravity [2].

Exact effective actions for massless fields are very complicated and non-local. Even in field theory, integrating out a massive field in the whole action, introduces non-locality. But, in the extreme low-energy limit, the leading terms in the effective action can be constructed just from invariance principles (gauge and supersymmetry) [35] as we will do below.

3.3.1 Type IIB supergravity

The massless spectrum of type IIB superstring theory was given in the last section, where we found that in terms of transverse $SO(8)$ representations, the physical content can be written by [34],¹⁵

$$(\mathbf{8}_v + \mathbf{8}_c) \otimes (\mathbf{8}_v + \mathbf{8}_c) = (\mathbf{1} + \mathbf{1} + \mathbf{28} + \mathbf{35}_v + \mathbf{28} + \mathbf{56}_c)_B + (\mathbf{8}_s + \mathbf{8}_s + \mathbf{56}_s + \mathbf{56}_s)_F, \quad (3.111)$$

¹⁴The relation between open- and closed-string couplings comes from the vertex operators for open- and closed-strings. The first one acts on the boundary, and the second on the interior of the worldsheet. Then we could say that g_s is the coupling of strings to the two-dimensional worldsheet.

¹⁵In order to write this expression, we need to review a little about group theory and representations. The product in (3.111) can be written as

$$(\mathbf{8}_v + \mathbf{8}_c) \otimes (\mathbf{8}_v + \mathbf{8}_c) = U^i \otimes V^j + U^i \otimes \psi_\beta + \chi_\alpha \otimes V^i + \chi_\alpha \otimes \psi_\beta,$$

where U^i and V^j are vector of $SO(8)$, and, χ_α and ψ_β correspond to Majorana-Weyl spinors in $SO(8)$,

$$|i, s\rangle \Gamma_{ss'}^i,$$

that transform among themselves under $SO(8)$ in the $\mathbf{8}'$ representation with opposite chirality. There are 56 degrees

where, in the bosonic sector, the $\mathbf{1}$'s are scalars (ϕ, τ), the $\mathbf{28}$'s are two second-rank antisymmetric tensors (A_{MN}, B_{MN}), $\mathbf{35}_v$ is the graviton (g_{MN}) and $\mathbf{56}_c$ is four-rank antisymmetric self-dual tensor (A_{MNPQ}). In the fermionic sector we have that the $\mathbf{8}_s$'s are a pair of Majorana-Weyl spinors (λ, ν) and the $\mathbf{56}_s$'s are the two gravitinos (ψ_M, φ_M), two vector-spinor quantities which are the supersymmetric partners of the graviton. Then there are 128 bosonic states and 128 fermionic states. The self-duality of the $SO(8)$ representation $\mathbf{35}_c$ is reflected in the free covariant theory by the $SO(9,1)$ self-duality of the field strength $F_{MNPQR} = 5\partial_{[M}A_{NPQR]}$. Because of the self-duality of the field strength, there will be some problems constructing an action for the theory and then the field equations coming from it (see however [47–49], and also [22] for some constructions). But there is a way to obtain the field equations without varying a covariant action, and we will follow that route. The key to do this is to identify the symmetries of the theory and to work out the symmetry transformations formulas of the fields.

In supersymmetry, as we saw before, the algebra has two ways to close, on-shell and off-shell. The first one means that we must consider the field equations in order to close the algebra, in the second we usually need to include auxiliary fields to close it. Then, if we know the supersymmetry variations, the commutator of two local supersymmetry transformations corresponds to a combination of the local symmetry of the theory, which can be general coordinates, local Lorentz, local supersymmetry or gauge transformations. Since we do not have any auxiliary field, the algebra (commutation of two variation) must close on-shell. It means that in the process of constructing supersymmetry transformations of the fields with a consistent algebra one can actually deduce some of the fields equations (see [3] for a revision on Noether method on/off-shell closing of SUSY algebra).

In addition to the local symmetries, the theory possesses a global $SU(1,1)^{16}$ invariance. Once this fact is recognized it is not very difficult to derive the supersymmetry transformations formulas of each of the fields. Let us see briefly how that symmetry helps to find the variation of the content fields, and finally its field equations (see for that [35, 50–52] and [53] for a modern review.).

of freedom left. They form an irreducible representation too. Thus, the product $U^i \otimes \psi_\alpha$ splits as $\mathbf{8}'_s$ and $\mathbf{56}_s$. The same for the third term,

The fourth term, $\chi_\alpha \otimes \psi_\beta$, splits as

$$8_s \times 8_s = [0] + [2] + [4]_\pm,$$

where $[4]_\pm$ will depends of chiralities (see [22] for a brief review on product representations),

¹⁶This group is the noncompact form of $SU(2)$, isomorphic to $SL(2, \mathbb{R})$, the conformal group in two dimensions.

We can arrange the two scalars of the theory as a $SU(1, 1)$ (see [51, 53]),

$$V = \begin{pmatrix} V_-^1 & V_+^1 \\ V_-^2 & V_+^2 \end{pmatrix} \quad (3.112)$$

where the V_{\pm}^{α} 's contain an auxiliary field, and \pm labels the $U(1)$ charges¹⁷. And satisfying,

$$\det V = \epsilon_{\alpha\beta} V_-^{\alpha} V_+^{\beta} = 1. \quad (3.113)$$

From the left-invariant form

$$V^{-1} \partial_M V = \begin{pmatrix} -iQ_M & P_M \\ P_M^* & +iQ_M \end{pmatrix} \quad (3.114)$$

we read off two important quantities. An $U(1)$ -covariant quantity¹⁸

$$P_M = -\epsilon_{\alpha\beta} V_+^{\alpha} \partial_M V_+^{\beta}, \quad (3.115)$$

which has charge 2 under $U(1)$; and the connection¹⁹

$$Q_M = -i\epsilon_{\alpha\beta} V_-^{\alpha} \partial_M V_+^{\beta}. \quad (3.116)$$

The two-forms are collected in an $SU(1, 1)$ doublet A_{MN}^{α} satisfying the constraint $A_{MN}^1 = A_{MN}^{2*}$.

Its corresponding field strength,

$$F_{MNP}^{\alpha} = 3 \partial_{[M} A_{NP]}^{\alpha}, \quad (3.117)$$

is invariant respect to the gauge transformations,

$$\delta A_{MN}^{\alpha} = 2 \partial_{[M} \Lambda_{N]}^{\alpha}. \quad (3.118)$$

¹⁷Each arrangement of scalars V_{\pm}^{α} transform, under local $U(1)$, as

$$\delta V_{\pm}^{\alpha} = \pm i \Sigma V_{\pm}^{\alpha}.$$

So, scalars of the theory as described by the matrix U have symmetry $SU(1, 1)/U(1)$.

¹⁸The local $U(1)$ transformations of some field Φ with infinitesimal parameter $\Sigma(x)$ are written as

$$\delta_{\Sigma} \Phi = iq \Sigma \Phi.$$

Then q is called the charge.

¹⁹Or gauge field. It means that the field transform as the derivative of the parameter, i.e.

$$\delta Q_M = \partial_M \Sigma.$$

The four-form is invariant under $SU(1, 1)$, and varies as

$$\delta A_{MNPQ} = 4\partial_{[M}\Lambda_{NPQ]} - \frac{i}{4}\epsilon_{\alpha\beta}\Lambda_{[M}^{\alpha}F_{NPQ]}^{\beta}, \quad (3.119)$$

under local gauge transformations, so that the gauge invariant five-form field strength is

$$F_{MNPQR} = 5\partial_{[M}A_{NPQR]} + \frac{5i}{4}\epsilon_{\alpha\beta}A_{[MN}^{\alpha}F_{PQR]}^{\beta}. \quad (3.120)$$

This five-form satisfies the self-duality condition,

$$F^{MNPQR} = \frac{1}{5!}\epsilon^{MNPQRSTUVW}F_{STUVW}. \quad (3.121)$$

The $SU(1, 1)$ doublet of $U(1)$ neutral fields strengths (3.117) can be replaced by an equivalent expression that is an $SU(1, 1)$ singlet with $U(1)$ -charge $U = 1$,

$$G_{MNP} = -\epsilon_{\alpha\beta}V_{+}^{\alpha}F_{MNP}^{\beta}. \quad (3.122)$$

By requiring the closure of the algebra on the bosonic fields, it is possible to deduce the supersymmetric transformations of each field,

$$\begin{aligned} \delta e_M^N &= -2\kappa\text{Im}(\bar{\epsilon}\Gamma^N\psi_M), \\ \delta V_{+}^{\alpha} &= \kappa V_{-}^{\alpha}\bar{\epsilon}^*\lambda, \quad \delta V_{-}^{\alpha} = \kappa V_{+}^{\alpha}\bar{\epsilon}\lambda^*, \\ \delta A_{MN}^{\alpha} &= V_{+}^{\alpha}\bar{\epsilon}^*\Gamma_{MN}\lambda^* + V_{-}^{\alpha}\bar{\epsilon}\Gamma_{MN}\lambda + 4iV_{+}^{\alpha}\bar{\epsilon}\Gamma_{[M}\psi_{N]}^* + 4iV_{-}^{\alpha}\bar{\epsilon}^*\Gamma_{[M}\psi_N], \\ \delta A_{MNPQ} &= 2\text{Re}(\bar{\epsilon}\Gamma_{[MNP}\psi_Q]) - \frac{3}{8}i\kappa\epsilon_{\alpha\beta}A_{[MN}^{\alpha}\delta A_{PQ]}^{\beta}, \\ \delta\lambda &= \frac{i}{\kappa}\Gamma^M\epsilon^*\hat{P}_M - \frac{1}{24}i\Gamma^{MNP}\epsilon\hat{G}_{MNP}, \\ \delta\psi_M &= \frac{1}{\kappa}D_M\epsilon + \frac{i}{480}\gamma^{NLPQR}\Gamma_M\epsilon\hat{F}_{NLPQR} + \frac{1}{96}(\Gamma_M^{NLP}\hat{G}_{NLP} - 9\Gamma^{PQ}\hat{G}_{MPQ})\epsilon^* \\ &\quad + \text{fermions}^2. \end{aligned} \quad (3.123)$$

The first variation corresponds to the usual supersymmetric transformation of the vierbein e_M^N , the second line contains the supersymmetric variations of the scalars, which are written having the correct $SU(1, 1) \times U(1)$ properties. The supersymmetry transformations of the gauge fields A_{MN}^{α} and δA_{MNPQ} , the third and fourth lines, can be obtained so that the commutator of two supersymmetry transformations give rise to a consistent set of gauge transformations with special parameters. The last two variations, corresponding to the supersymmetric variations of the dilatino and the gravitino, are calculated in the same way the gauge fields transformations were done.

In general, the commutator of two local supersymmetry transformations gives all six types of local symmetry transformations [52].

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta(\xi) + \delta(\iota) + \delta(\epsilon) + \delta(\Lambda) + \delta(\Lambda\dots) + \delta(\Sigma) \quad (3.124)$$

where ξ refers to general coordinate transformations, ι is for local Lorentz, ϵ is for local supersymmetry, Λ and $\Lambda\dots$ correspond to local gauge transformations, and Σ parametrizes local $U(1)$ transformations.

Furthermore, supercharges were defined as

$$\begin{aligned} \hat{P}_M &= P_M + \text{fermions}, \\ \hat{G}_{MNP} &= G_{MNP} + \text{fermions}, \\ \hat{F}_{MNPQR} &= F_{MNPQR} + \text{fermions}. \end{aligned} \quad (3.125)$$

And also, the supercovariant derivative,

$$D_M \epsilon = (\partial_M + \frac{1}{4} \omega_M^{AB} \Gamma_{AB} - \frac{1}{2} i Q_M) \epsilon, \quad (3.126)$$

where the last term was added since Q_M transforms as a gauge field. The spin connection ω_M^{AB} contains a combination of bilinear products of fermions.

There is no auxiliary field in type IIB supergravity to close the supersymmetry algebra (3.124) off-shell. This fact can be turned to advantage as we mentioned before. In the case of e_M^N , V_\pm^α , and A_{MN}^α , closure of the algebra is achieved without invoking equation of motion. However, for A_{MNPQ} , it is necessary to use its field equation, the self duality condition of the five-form F_{MNPQR} . The equations of motion can be deduced from the variation of the Fermi fields λ and ψ_M . Two of them, which we will consider later, are

$$\begin{aligned} R_{MN} &= P_M P_N^* + P_M^* P_N + \frac{1}{6} \kappa^2 F_{ABCDM} F^{ABCD}_N \\ &\quad + \frac{1}{8} \kappa^2 (G_M^{PQ} G_{NPQ}^* + G_M^{*PQ} G_{NPQ} - \frac{1}{6} g_{MN} G^{PQR} G_{PQR}^*), \end{aligned} \quad (3.127)$$

$$F_{MNPQR} = \tilde{F}_{MNPQR}. \quad (3.128)$$

It is interesting to investigate the field equations of this theory for the solution that describes compactified spaces as will be seen later.

3.4 p-branes and black p-branes

In the last section we have arrived to a supersymmetric theory describing a gravity as the low-energy limit of a particular superstring theory. But solving the whole theory, considering all the fields of supergravity, is practically impossible. In order to find solutions to the field equations, we motivate the existence of some higher dimensional objects that source those fields [46].

In d dimensions, the spacetime can be decomposed as $SO(1, p) \times SO(d-p)$, where the $SO(d-p)$ factor came from the fact that the brane can rotate in the transverse direction. Hence, we can write the spacetime metric h_{MN} that respects that symmetry (Schwarzschild-like) as (see [54] and [55] for general solutions)

$$ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)} dy^m dy^m \delta_{mn}, \quad (m, n = p+1, \dots, d-1)$$

$$(\mu, \nu = 0, \dots, p), \quad (3.129)$$

where $r = \sqrt{y^m y_m}$ is the isotropic radial coordinate in the transverse space. Since the metric components depend only on r , translational invariance in the worldvolume directions x^μ and $SO(d-p)$ symmetry in the transverse directions y^m is guaranteed.

Now, general solutions for any values of d and p are beyond of the scope of this work, but they can be found in [54, 56–58]. So we will focus in finding classical solution for $d = 10$ and $p = 3$. Then, the object to be studied will be a three-brane in a ten-dimensional spacetime.

Following the discussion about generalized electromagnetism and the ansatz for the brane, we conclude that the three-brane will source a four-form $A_{(4)}$, which in turn have as field strength a five form $F_{(5)}$. A special feature of this field strength is that it is a self-dual form, i.e. its Hodge dual (see [59] for studying some about differential geometry) \tilde{F} (or $*F$) is

$$\tilde{F}^{MNPQR} = \frac{1}{5!} \epsilon^{MNPQRSTUUVW} F_{STUUVW} = F^{MNPQR}. \quad (3.130)$$

For $p = 3$, $d = 10$, the ansatz of the brane (3.129) is [60, 61]

$$ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)} dy^m dy^m \delta_{mn}, \quad (m, n = 4, \dots, 9)$$

$$(\mu, \nu = 0, \dots, 3), \quad (3.131)$$

where δ_{mn} corresponds to the transversal flat space $\delta_{mn} = \text{diag}\{1, 1, 1, 1, 1, 1\}$.

3.4.1 Type IIB solution

Let us consider type IIB string theory in ten dimensions, and look for a black hole solution carrying electric/magnetic charge with respect to the R-R four-form A_4 . The R-R charges can be quantized according (3.104) as

$$\int_{S_5} *F_{(5)} \sim N, \quad (3.132)$$

where we have chosen the R-R charge as $Q_e = N$.

Now, we will try to solve the field equations in order to find the expressions for A and B in (3.131). As first step, we assume that the only matter content in the type IIB SUGRA consists of the graviton and the four-form appearing as a five-form in the supersymmetry transformations. Then, we set the fermions vanish in the theory. In (3.123),

$$\delta\lambda = \frac{i}{\kappa} \Gamma^M \epsilon^* \hat{P}_M = 0, \quad (3.133)$$

$$\delta\psi_M = \frac{1}{\kappa} D_M \epsilon + \frac{i}{480} \Gamma^{NPQRS} \Gamma_M \epsilon F_{NPQRS} = 0, \quad (3.134)$$

with

$$D_M \epsilon = (\partial_M + \frac{1}{4} \omega_M^{AB} \gamma_{AB}) \epsilon, \quad (3.135)$$

are known as the supersymmetry conditions. (3.133) and (3.134) allow to obtain the Killing spinors of the theory and then, the number of preserved supersymmetries in the background.

The equation (3.133) says that the dilatino field λ is actually just a constant. Then, we must solve (3.134). First, remember that the five-form is the field-strength of a four-form $A_{(4)}$. An ansatz for this form can be thought of as proportional to the worldvolume of the brane, M_4 , [54, 61, 62]

$$A_{\mu\nu\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma} e^{C(r)}, \quad (3.136)$$

where $e^{C(r)}$ allows to have $SO(6)$ symmetry, corresponding to the symmetry of the transverse space in (3.131). The five-form field strength was defined in (3.120), and for the case of three-branes and the field content mentioned above, $F_{(5)}$ is defined by

$$F_{MNPQR} = 5 \partial_{[M} A_{NPQR]}. \quad (3.137)$$

For our ansatz we obtain,

$$F_{\mu\nu\rho\sigma m} = -\frac{1}{4h} \epsilon_{\mu\nu\rho\sigma} \partial_m e^{C(r)} = e^{-4A(r)} \epsilon_{\mu\nu\rho\sigma} \partial_m e^{C(r)}, \quad (3.138)$$

where 4h is the determinant of the four-dimensional metric h_{MN} .

Back to the ansatz for the brane in (3.131), we obtain the spin connections in this background,

$$\begin{aligned} e^{\underline{\mu}} &= e^{A(r)} dx^{\mu}, \\ e^{\underline{m}} &= e^{B(r)} e^m. \end{aligned} \quad (3.139)$$

The torsion-free condition again allows us to obtain the spin-connections for the metric.

For $M = \underline{\mu}$,

$$de^{\underline{\mu}} + \omega^{\underline{\mu}}{}^m_m \wedge e^m + \omega^{\underline{\mu}}{}^{\nu}{}_{\nu} \wedge e^{\nu} = 0.$$

Then,

$$\omega^{\underline{\mu}}{}^{\nu}{}_{\nu} = 0, \quad \omega^{\underline{\mu}}{}^m_m = \partial_m A(r) e^{\underline{\mu}}. \quad (3.140)$$

In the same way for the $M = \underline{m}$,

$$de^{\underline{m}} + \omega^{\underline{m}}{}^n_n \wedge e^n + \omega^{\underline{m}}{}^{\underline{\mu}}{}_{\underline{\mu}} \wedge e^{\underline{\mu}} = 0, \quad (3.141)$$

we get,

$$\omega^{\underline{m}}{}^n_n = \partial_n B(r) e^{\underline{m}} = \omega^{\underline{m}}{}^n_n. \quad (3.142)$$

We also need to split the coordinates and the Γ matrices, since we have assumed that the ten-dimensional spacetime is of the form $M_4 \times B_6$, where B_6 is an Euclidean six-dimensional space with $SO(6)$ symmetry. The coordinates split as $x^M = (x^{\mu}, x^m)$, where $\mu = 0, \dots, 3$ and $m = 4, \dots, 9$. Then, the Γ matrices split as

$$\Gamma^{\mu} = \gamma^{\mu} \otimes \mathbb{I} \quad (3.143)$$

$$\Gamma^m = \gamma^5 \otimes \gamma^m, \quad (3.144)$$

where γ^{μ} and γ^m are $SO(1,3)$ and $SO(6)$ gamma matrices respectively. These are constant matrices in the tangent space $\eta_{MN} = \{-1, +1, \dots, +1\}$ and satisfy,

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}. \quad (3.145)$$

We recall that,

$$\Gamma_{AB\dots C} = \Gamma_{[A} \Gamma_{B\dots C]}, \quad (3.146)$$

is the antisymmetrized product of Γ matrices. We also define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (3.147)$$

so $(\gamma^5)^2 = 1$, and

$$\gamma^7 = -i\gamma^4\gamma^5\gamma^6\gamma^7\gamma^8\gamma^9, \quad (3.148)$$

so that $(\gamma^7)^2 = 1$. Thus, $\Gamma^{11} = \gamma^5 \otimes \gamma^7$. The most general spinor consistent with $SO(1,3) \times SO(6)$ takes the form,

$$\varepsilon(x^M) = \epsilon \otimes \eta, \quad (3.149)$$

where ε is a spinor of $SO(1,3)$ which may be further decomposed into chiral eigenstates by using the projection operators $1 \pm \gamma^5$, and η is an $SO(6)$ spinor which may be also decomposed into chiral eigenstates via the projection operator $1 \pm \gamma^7$.

A final assumption to solve (3.134) is $\frac{1}{\kappa} = 4$, then

$$D_M \epsilon + \frac{i}{4 \times 480} \Gamma^{NPQRS} \Gamma_M \epsilon F_{NPQRS} = 0. \quad (3.150)$$

The covariant derivatives in the last expression can be written, by using (3.140) and (3.142),

$$\begin{aligned} D_\mu \epsilon &= \partial_\mu \epsilon + \frac{1}{2} \partial_m A(r) \Gamma_\mu \Gamma^m \epsilon, \\ D_n \epsilon &= \partial_n \epsilon + \frac{1}{2} \partial_n B(r) + \frac{1}{2} \partial_m B(r) \Gamma^m \Gamma_n \epsilon, \end{aligned} \quad (3.151)$$

Now we need to work in the second term of (5.22). The contraction of the antisymmetrized product of Γ matrices and the five-form $F^{(5)}$ can be written as

$$\Gamma^{NPQRS} F_{NPQRS} = \Gamma^{\mu\nu\rho\sigma m} F_{\mu\nu\rho\sigma m} + \Gamma^{nlpqr} F_{nlpqr}.$$

Apparently we need to know F_{nlpqr} , but it is not necessary. We can use the self-duality (3.130) condition and the formula

$$\Gamma^{M_1 \dots M_N} = -\frac{1}{(10-N)!} (-1)^{\frac{N(N-1)}{2}} \frac{\epsilon^{M_1 \dots M_{10}}}{\sqrt{-h}} \Gamma_{M_{N+1} \dots M_{10}} \Gamma_{11}, \quad (3.152)$$

to obtain,

$$\Gamma^{NPQRS} F_{NPQRS} = \Gamma^{\mu\nu\rho\sigma m} F_{\mu\nu\rho\sigma m} (1 - \Gamma^{11}). \quad (3.153)$$

For $M = \mu$, (3.150) reads as

$$\partial_\mu \epsilon + \frac{1}{4} \omega_{\mu AB} \Gamma^{AB} \epsilon + \frac{i}{4 \times 480} \Gamma^{NPQRS} F_{NPQRS} \Gamma_\mu \epsilon = 0. \quad (3.154)$$

Now we can include our last results to get,

$$\partial_\mu \epsilon + \frac{1}{2} \partial_m A(r) \Gamma_\mu \Gamma^m \epsilon + \frac{i}{4 \times 480} \Gamma^{\mu\nu\rho\sigma m} F_{\mu\nu\rho\sigma m} (1 - \Gamma^{11}) \Gamma_\mu \epsilon = 0. \quad (3.155)$$

If we choose the chirality of ϵ to be positive, i.e. $\Gamma_{11}\epsilon = \epsilon$, the ansatz (3.138), and our splitting of Gamma matrices (3.143) and (3.144), we obtain

$$\begin{aligned}\partial_\mu\epsilon + \frac{1}{2}\partial_m A(r)\Gamma_\mu\Gamma^m\epsilon + \frac{i}{8 \times 120}\Gamma^{\mu\nu\rho\sigma m}F_{\mu\nu\rho\sigma m}\Gamma_\mu\epsilon &= 0, \\ \partial_\mu\epsilon + \frac{1}{2}\gamma_\mu \otimes \gamma^m \{\gamma_5\partial_m A - \frac{1}{4}e^{-4A}\partial_m e^C\}\epsilon &= 0.\end{aligned}\quad (3.156)$$

For $M = m$, (3.150) acquire the form

$$\begin{aligned}\partial_m\epsilon + \frac{1}{2}\partial_m B(r) + \frac{1}{2}\partial_n B(r)\Gamma^n\Gamma_m\epsilon + \frac{i}{8 \times 120}\Gamma^{\mu\nu\rho\sigma n}F_{\mu\nu\rho\sigma n}\Gamma_m\epsilon &= 0, \\ \partial_n\epsilon + \frac{1}{2}\partial_m B\epsilon - \frac{1}{2}\gamma_5 \otimes \gamma^n \gamma_m \{\gamma_5\partial_n B + \frac{1}{4}e^{-4A}\partial_m e^{-C}\} &= 0.\end{aligned}\quad (3.157)$$

Let us assume a form for the spinor (3.149) to be $\epsilon = e^{A(r)/2}\epsilon_0 \otimes \eta_0$, where ϵ_0 and η_0 are constant spinors in M_4 and B_6 respectively. Then we obtain some extra relations for the functions $A(r)$, $B(r)$ and $C(r)$,

$$C = 4A, \quad B = -A. \quad (3.158)$$

The ansatz for the brane (3.131) will be now,

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(r)}\delta_{mn}dx^m dx^n, \quad (3.159)$$

and the ansatz for the five-form (3.138),

$$F_{\mu\nu\rho\sigma m} = -\frac{1}{4h}\epsilon_{\mu\nu\rho\sigma}\partial_m e^{C(r)} = e^{-4A(r)}\epsilon_{\mu\nu\rho\sigma}\partial_m e^{4A(r)}. \quad (3.160)$$

The explicit form of $A(r)$ can be obtained by solving the equations of motion (3.127) for our case, in other words, just metric and the five-form are turned on. Then,

$$R_{\mu\nu} = \frac{1}{96}F_{\mu\rho\sigma\lambda m}F_\nu^{\rho\sigma\lambda m}. \quad (3.161)$$

Since $R = \frac{1}{4}F^2 = 0$, (3.161) reduces to

$$e^{8A(r)}\partial^m\partial_m e^{-4A(r)} = 0, \quad (3.162)$$

A is a function of $r = \sqrt{x^m x^m}$ only, we need to write the last equation in this coordinates. If we go to ‘‘spherical’’ coordinates in this six-dimensional space δ_{mn} , we get the laplacian

$$r^{-3}\partial_r \left(r^5 \partial_r e^{-4A(r)} \right) = 0, \quad (3.163)$$

which is easy to solve to obtain

$$e^{-4A(r)} = 1 + \frac{|Q|}{r^4}. \quad (3.164)$$

The ten-dimensional metric (3.159) takes the form

$$ds^2 = \left(1 + \frac{|Q|}{r^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{|Q|}{r^4}\right)^{1/2} \delta_{mn} dx^m dx^n. \quad (3.165)$$

The near-horizon geometry at $r \rightarrow 0$ turns out to be

$$ds_h^2 = \frac{r^2}{\sqrt{Q}} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{\sqrt{Q}}{r^2} dr^2 + \sqrt{Q} d\Omega_5, \quad (3.166)$$

where $d\Omega_5$ is the volume element of the five-dimensional sphere at fixed r in the transversal six-dimensional space, and $|Q|$ is an integration constant. We say that the near-horizon geometry of D3-branes placed in a ten-dimensional spacetime $M_4 \times B_6$ looks like $AdS_5 \times S^5$.

3.5 Anti-De Sitter space

As was mentioned above, in the limit in which $r \rightarrow 0$ the black p-brane solution (3.165) becomes into (3.166). This spacetime is a product of a five-dimensional space with constant negative curvature called Anti De Sitter space [2, 30, 63–65]. This space can be expressed in terms as the six-dimensional embedding of a hyperboloid (with AdS radius L),

$$-X_0^2 - X_5^2 + \sum_{i=1}^4 X_i^2 = -L^2, \quad (3.167)$$

into a flat six-dimensional $\mathbb{R}^{2,4}$ space,

$$ds^2 = -dX_0^2 - dX_5^2 + \sum_{i=1}^4 dX_i^2. \quad (3.168)$$

By construction, the space has the isometry $SO(2,4)$, and it is homogeneous and isotropic²⁰. Eq. (3.167) can be solved by setting,

$$\begin{aligned} X_0 &= L \cosh \rho \cos \tau, & X_5 &= L \cosh \rho \sin \tau, \\ X_i &= L \sinh \rho \Omega_i \quad (i = 1, \dots, 4; \sum_i \Omega_i^2 = 1). \end{aligned} \quad (3.169)$$

²⁰Homogeneity is the statement that the metric is the same throughout the manifold \mathcal{M} . In other words, given any two points p and q in \mathcal{M} , there is an isometry that takes p into q . Isotropy applies at some specific point in the manifold, and states that the space looks the same no matter in what direction you look. There is no necessary relationship between homogeneity and isotropy.

where Ω_i parametrizes an S^3 . Substituting this into (3.168), we obtain the metric on AdS_5 ,

$$ds^2 = L^2(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega^2). \quad (3.170)$$

By taking $0 \leq \rho$ and $0 \leq \tau < 2\pi$ the solution (3.169) covers the entire hyperboloid once. Therefore, (τ, ρ, Ω) are called the global coordinates of AdS . The metric behaves near $\rho = 0$ as

$$ds^2 \approx L^2(-d\tau^2 + d\rho^2 + \rho^2 d\Omega^2), \quad (3.171)$$

where $d\Omega^2$ is the volume element of an S^3 . The hyperboloid (3.171) in this limit has the topology of $S^1 \times \mathbb{R}^4$, with S^1 being the timelike curves in the τ directions. To prevent inconsistencies concerning causality, AdS_5 is therefore regarded as the causal spacetime obtained by unwrapping S^1 , taking $-\infty < \tau < \infty$ without any identification.

The isometry group $SO(2, 4)$ of AdS_5 has a maximal compact subgroup $SO(2) \times SO(4)$, the former generating translations in τ , the latter rotating the X_i 's of the S^3 .

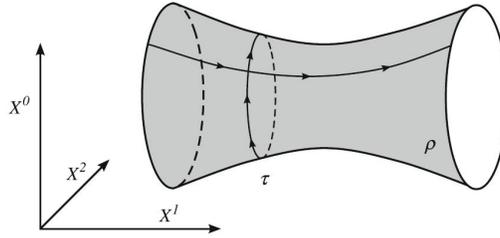


Figure 3.1: AdS_3 spacetime as a hyperboloid with closed timelike curves (source: Strings and Fundamental Physics [29]).

To study the causal structure of AdS_5 in more detail, it is convenient to introduce a new coordinate θ related to ρ by $\tan \theta = \sinh \rho$ ($0 \leq \theta < \pi/2$). Then the metric (3.170) takes the form

$$ds^2 = \frac{L^2}{\cos^2 \theta}(-d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega^2), \quad (3.172)$$

by a conformal rescaling, the causal structure of the spacetime does not change. Multiplying the metric by $L^{-2} \cos^2 \theta$,

$$ds'^2 = -d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega^2, \quad (3.173)$$

which, at fixed τ , is known as the Einstein static universe $\mathbb{R} \times S^2$. However, since $0 \leq \theta < \pi/2$, this metric covers only half of $\mathbb{R} \times S^2$. Namely, AdS_5 can be conformally mapped into one-half

of the four-dimensional Einstein static universe; the spacelike hypersurface of constant τ is a four-dimensional hemisphere. The equator at $\theta = \pi/2$ is a boundary of the spacetime ($\rho \rightarrow \infty$) with the topology of S^3 as shown below (for AdS_3)

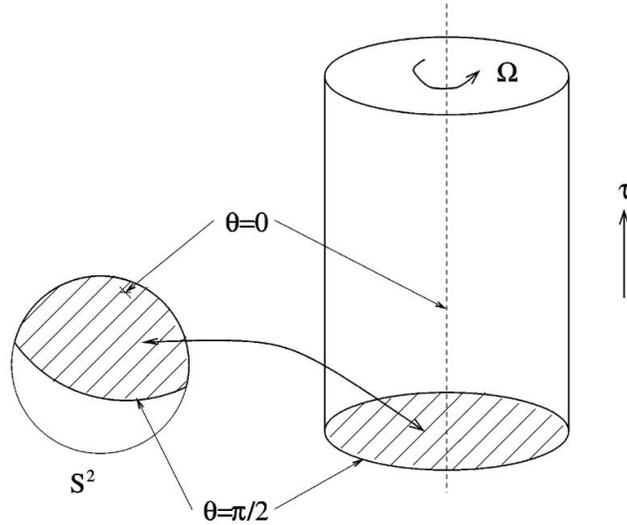


Figure 3.2: AdS_3 can be conformally mapped into one-half of $\mathbb{R} \times S^2$ (figure taken from [65]).

In addition to the global parametrization (3.169) of AdS , there is another set of coordinates (u, t, x) where $0 < u, x \in \mathbb{R}^3$ which will be useful later. They are defined by

$$\begin{aligned} X_0 &= \frac{1}{2u}(1 + u^2(L^2 + \vec{x}^2 - t^2)), & X_5 &= Lut, \\ X_i &= L u x_i, \quad (i = 1, \dots, 3), \\ X_4 &= \frac{1}{2u}(1 + u^2(L^2 + \vec{x}^2 - t^2)). \end{aligned} \tag{3.174}$$

These coordinates cover one-half of the hyperboloid (3.167). Substituting them into (3.168), we obtain another form of the AdS_5 metric

$$ds^2 = L^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2) \right). \tag{3.175}$$

The set of coordinates (u, t, x) are called the Poincaré patch. In this form of the metric, the subgroups of $SO(2, 4)$: $ISO(1, 3)$ (Poincaré transformations on (t, x)) and $SO(1, 1)$

$$(t, x, u) \rightarrow (ct, cx, c^{-1}u), \quad c > 0, \tag{3.176}$$

is manifest.

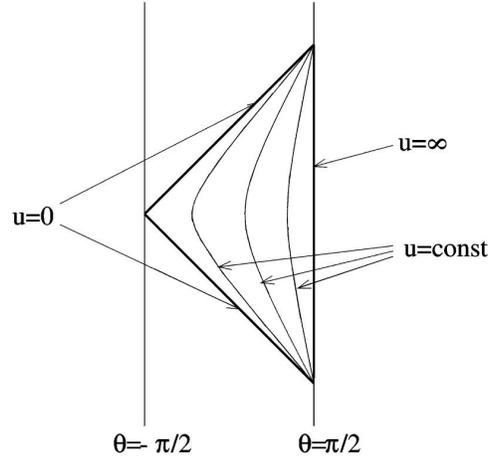


Figure 3.3: Poincaré patch, AdS_2 can be conformally mapped into $\mathbb{R} \times [-\pi/2, \pi/2]$. The (u, t) coordinates cover the triangular region. Light rays propagating in this plane can reach the boundary and bounce back in finite time, whereas massive particles moving along geodesic cannot (figure taken from [65]).

The boundary at $u \rightarrow \infty$ can be better analyzed in terms of a new variable y ,

$$y := \frac{1}{u} \Rightarrow ds^2 = \frac{L^2}{y^2} (dy^2 - dt^2 + d\vec{x}^2), \quad (3.177)$$

After a conformal rescaling by y^2 , we obtain

$$ds^2 = dy^2 - dt^2 + d\vec{x}^2. \quad (3.178)$$

When we fix $y \rightarrow 0$ ($u \rightarrow \infty$), we get the Minkowski four-dimensional spacetime.

Chapter 4

AdS/CFT correspondence

String theory emerged as an attempt to explain strong interactions, and later it evolved into a proposal for a theory able to incorporate all forces of nature into a coherent framework (as we saw before, it contains gravity!). Gauge theories are the core of the standard model, and they have been successful so far. Then, a theory that describes nature in some sense, must “intersect” our current physics in some regime. That is why one can realize that there exists a special limit in string theories where we will get gauge theories. We could invert the assertion to say that, hidden within every gauge theory is a theory of quantum gravity. Then, we have a duality in words. Two totally different theories match in some special regimes, a theory with gravity will correspond to a gauge theory (without gravity). But, why do we need another formulation of the same thing? It is well-known that gauge theories become untreatable when the coupling becomes stronger. Since normally quantum field theory calculations are done by means of perturbation theory, there is no way to solve the strongly coupled theory. We cannot achieve the whole description of nature with gauge theories. Duality, in turns, allows to go further but not from the same description of nature. The idea in dualities, is to find a theory that “completes” that strong coupling regime (see [64–67] and also [7] for some reviews).

We can see the general idea of this statement in fig.(4.1)

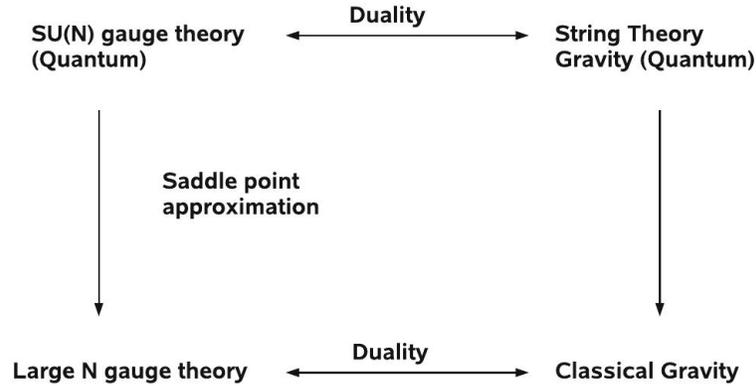


Figure 4.1: General structure of the AdS/CFT and the limit involved.

At the end of the nineties, J. Maldacena proposed the exact and complete equivalence between a specific superstring theory in a specific background, and a specific supersymmetric non-abelian gauge theory in four dimensions. This ideas can be written in a simple and short expression (as was first introduced by Maldacena in [68]),

$$D = 4, \mathcal{N} = 4, SU(N) \text{ Yang-Mills} \equiv \text{Type IIB superstrings on } AdS_5 \times S^5.$$

The left-hand side corresponds to a supersymmetric non-abelian gauge theory, which obviously does not describe some realistic interactions, and the right-hand side is a specific superstring theory we studied before. In spite of our standard model of particles is neither conformal nor supersymmetric, this duality is useful because it is a huge step forward to describe the whole coupling regime of couplings of any gauge theory. Unfortunately, it is very hard to prove the exact equivalence between type IIB superstrings on this special background, and this supersymmetric gauge theory. In the gravity side (strings) there are almost the same problems we found in gauge theories, the UV regime of the string theory, i.e. the nonperturbative formulation of superstrings, is not very well understood. Even at string tree level, we do not know how to solve it. Then, we are forced to accept a weaker form of the correspondence. The strong coupling limit of the supersymmetric gauge theory is dual to type IIB supergravity, the low-energy limit of type IIB superstring theory. We say that the AdS/CFT correspondence is a strong/weak coupling duality. There are arguments that not will be discussed in this work pointing towards the fact that Maldacena conjecture might be stronger, a complete duality. We then might state three different versions of the conjecture which, in increasing level of conservatism, can be named as the strong,

the mild and the weak. They are described below, where the appropriate limits are assumed on the parameters of the theory are also mentioned [69, 70].

Gravity side	Gauge theory side	
Type IIB string theory on $AdS_5 \times S^5$ $\forall g_s, g_s N_c$	$\mathcal{N} = 4 SU(N_c)$ super Yang-Mills theory	Strong*
Classical type IIB strings on $AdS_5 \times S^5$ $g_s \rightarrow 0, g_s N_c$ fixed	't Hooft large N_c limit of $\mathcal{N} = 4 SU(N_c)$ SYM $N_c \rightarrow \infty, \lambda = g_{YM}^2 N_c$ fixed	Mild*
Classical type IIB supergravity on $AdS_5 \times S^5$ $g_s \rightarrow 0, g_s N_c \rightarrow \infty$	Large 't Hooft coupling limit of $\mathcal{N} = 4 SU(N_c)$ SYM $N_c \rightarrow \infty, \lambda \rightarrow \infty$	Weak

Table 4.1: Three versions of the Maldacena conjecture: the strong, the mild and the weak (source: J.Edelstein and R.Portugues [69]). The (*) was put because we actually do not have evidence for these forms of the correspondence [71].

So, actually we do not have a formal correspondence, but two complementary descriptions of the same “thing”. In the following we will try to explain how this duality is formulated. It is time to apply all we have learned before.

4.1 The Maldacena limit

The spacetime metric of N_c ¹ coincident D3-branes given in (3.165) as

$$ds^2 = \left(1 + \frac{|Q|}{r^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{|Q|}{r^4}\right)^{1/2} \delta_{mn} dx^m dx^n, \quad (4.1)$$

and,

$$\begin{aligned} |Q| = R^4 &= 4\pi g_s N_c \alpha'^2 \\ &= 4\pi g_s N_c l_s^4 \Rightarrow \left(\frac{R}{l_s}\right)^4 = 4\pi g_s N_c, \end{aligned} \quad (4.2)$$

¹The subscript in N_c is because of the number of branes will characterize the rank of gauge group in their worldvolume.

where the radius R of the D3-brane. As $r \gg R$, we recover flat spacetime \mathbb{R}^{10} . When $r < R$, the geometry is often referred to as the throat and would at first appear to be singular $r \ll R$. If we redefine $u \equiv R/r$ as we did in section (3.5), the throat limit with this redefinition is

$$ds^2 = R^2 \left(\frac{1}{u^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\Omega_5^2 \right). \quad (4.3)$$

So the geometry close the stack on branes ($r \rightarrow 0, u \rightarrow \infty$) is regular and highly symmetrical. And as we saw in section (3.4.1), this geometry is $AdS_5 \times S^5$.

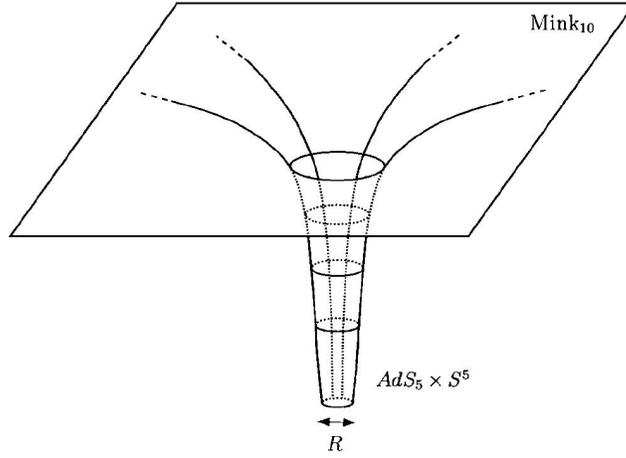


Figure 4.2: The geometry of the black D3-brane solution. Far away we have ten-dimensional flat space. Near horizon we have a throat $AdS_5 \times S^5$ (source of the picture: [72]).

4.1.1 The decoupling limit

An important property of the metric (4.1) is its nonconstant redshift factor coming from the g_{00} term of the metric [29],

$$g_{00} = \left(1 + \frac{|Q|}{r^4} \right)^{-1/4},$$

whose near-horizon limit is

$$\left(1 + \frac{R^4}{r^4} \right)^{-1/4} = \begin{cases} \sim 1 & : \text{large } r \\ \sim r/R & : \text{small } r \end{cases} \quad (4.4)$$

The energy E_p of an object measured by an observer at constant position r differs from the energy E_i of the same object measured by an observer at infinity

$$\left(1 + \frac{R^4}{r^4} \right)^{-1/4} E_p = E_i. \quad (4.5)$$

When the object approaches to the throat $r \rightarrow 0$, it appears to have lower and lower energy to the observer at infinity. This gives another geometric notion of low energy regime. Then, we have to distinguish two kinds of low-energy excitations (as seen from $r \rightarrow \infty$)

- particles approaching to the throat $r \rightarrow 0$,
- and massless particles propagating in the “bulk“ (away from $r = 0$).

Their excitations decouple from each other in the low-energy limit, in other words, bulk massless particles decouple from the near horizon region around $r \rightarrow 0$. On the other hand, excitations close to $r = 0$ are trapped by the gravitational potential to the $AdS_5 \times S^5$ region. Thus we have two decoupled actions in the low-energy limit, supergravity of massless particles in flat space and supergravity (of arbitrarily massive modes) in the $AdS_5 \times S^5$ region.

Now, let us study some approximations in terms of the parameters involved in R (see for example [65]). In (4.2), if we fix $g_s N_c \sim R^4/l_s^4 \ll 1$, then the description in terms of open strings attached to heavy (fixed in flat space) branes and closed strings is reliable². When $g_s N_c \sim R^4/l_s^4 \gg 1$, classical gravity becomes a good description, since the radius of curvature R of AdS and of S^5 becomes large compared to the string length. A way to understand this limit is to think that D3-branes produce backreaction in M_{10} , so they could be considered as black threebranes whose solution was found in section (3.4.1) for the type IIB case.

These limits are not connected continuously, in other words, we cannot get into the gravity regime, $R/l_s \gg 1$, by taking N_c small and g_s very large, since in that case the D1-brane or D-string becomes light and renders the the gravity approximately invalid as we see from (3.108). Another way to see this is to notice that,

$$\frac{R^4}{l_P^4} = \frac{4\pi g_s l_s^4 N_c}{g_s l_s^4} = 4\pi N_c \sim N_c.$$

So, it is always necessary, but not sufficient, to have large N_c in order to have weakly coupled supergravity description.

Now we are ready to motivate the AdS/CFT correspondence (in its weak version), by considering excitations around the ground state in the two descriptions above and taking a low-energy or decoupling limit. As we said, in the first description, the excitations of the system consist of open and closed strings. At low-energy we may focus on the light (massless) degrees of freedom.

²Notice that we have assumed $g_s < 1$, in order to have a perturbatively well-behaved theory.

Let us be more specific. In the first description, when $g_s N_c \ll 1$, consider a system of N_c D3-branes. As we saw, the dynamics on the D3-branes is a $U(N_c)$ gauge theory³ in four-dimensions. Furthermore, this theory will be actually $\mathcal{N} = 4$ supersymmetric, since it is conformal and is also the most general renormalizable consistent possibility as we learned from section (2.6). We saw in (3.2.7) that the DBI action for a D3-brane contains a Yang-Mills term which couples as $\sim 1/g_s$, so we could relate $g_{YM}^2 \sim g_s$. As this gauge theory comes from open strings with N_c Chan-Paton factors, ending on the D3-branes, the string coupling g_s gets dressed by a factor N_c , hence the effective loop expansion parameter will be $g_s N_c$. On the other hand, quantization of the closed string modes leads to a massless graviton supermultiplet plus a tower of massive string states, all of them propagating in flat ten-dimensional spacetime. The strength of interactions of closed strings modes is controlled by the Newton's constant G_{10} which in turns is related to g_s as $G_{10} \sim g_s^2 l_s^8$, so the closed string coupling g_s scales (at an energy E) as $G_{10} E^8$. So, the strength of interaction of closed strings with matter vanishes at low-energy and so in this limit, closed strings become non-interacting, which is essentially the statement that gravity is infrared free. Moreover, interactions between closed and open strings⁴ are also controlled by the same parameter, since gravity couples universally to all forms of matter (including the gauge field on the brane). Therefore, at low-energies, open strings decouple from closed strings, i.e. gauge theory decouples from gravity,

$$S = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{int}} \Rightarrow S_{\text{brane}} + S_{\text{bulk}}, \quad (4.6)$$

³ $U(N_c)$ can be split as $U(N_c) \rightarrow U(1) \times SU(N_c)$. The $U(1)$ factor can be shown to decouple so that one is eventually left with $SU(N_c)$.

⁴The interaction terms between closed and open strings are proportional to g_s , so they vanish at low-energy.

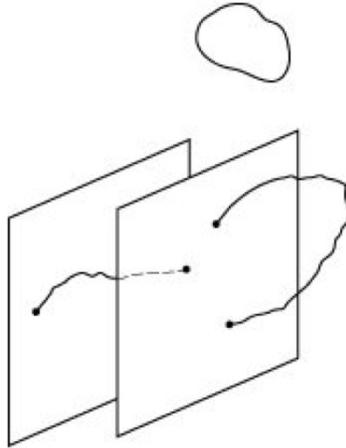


Figure 4.3: When $g_s N_c$ is very small, gravity (closed strings) on the bulk and the gauge theory (open strings) on the branes decouple.

where S_{brane} is four-dimensional $\mathcal{N} = 4$ super Yang-Mills with gauge group $SU(N_c)$ with coupling $\lambda = g_s N_c$, and S_{bulk} is ten-dimensional type IIB supergravity.

Let us examine the same limit in the second description, type IIB closed strings on $AdS_5 \times S^5$. At the beginning of this section, we examine some interesting properties of the solution for black threebranes in (4.1). If we apply them to type IIB strings on this spacetime, we see that as seen from infinity, only the massless spectrum of type IIB strings will be considered at low-energies. In fact, closed strings in the flat space have in this limit very large wavelengths and do not “see” the throat. In the throat region, however, the whole tower of massive string excitations survives. As we focus on low-energies, these modes become supported deeper and deeper in the throat as so they decouple from the massless states in the outer Minkowskian region. We conclude that at low-energies the second description of the system reduces to interacting closed strings on $AdS_5 \times S^5$ plus free gravity on flat ten-dimensional spacetime.

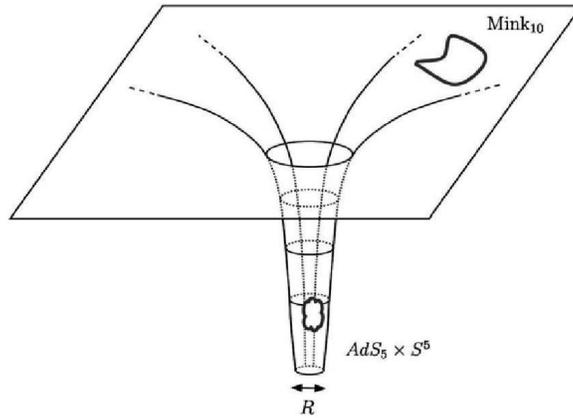


Figure 4.4: When $g_s N_c$ is very large, gravity on in the throat decouple to gravity on the flat space (figure taken from [72]).

	$r \rightarrow \infty$	$r \rightarrow 0$
D-brane ($g_s N_c \ll 1$)	free gravity	$\mathcal{N} = 4$ $d = 4$ $SU(N_c)$ SYM
Black brane ($g_s N_c \gg 1$)	free gravity	Type IIB SUGRA on $AdS_5 \times S^5$

Table 4.2: Decoupling limits when $r \rightarrow \infty$. Descriptions in the throat ($r \rightarrow 0$) for $g_s N_c \ll 1$ and $g_s N_c \gg 1$.

4.1.2 Maldacena’s conjecture

Comparing the results of the low-energy limits in the table (4.1.1), it is ”reasonable“ to conjecture that $\mathcal{N} = 4$ $SU(N_c)$ SYM in four dimensions and type IIB string theory on $AdS_5 \times S^5$ are two apparently different descriptions of the same underlying physics, and we will say that the two theories are ”dual“ to each other (see [68])⁵.

This is a striking statement, but actually we are not able to prove it. As we said before, at present there is no consistent nonperturbative formulation of string theory yet, in particular not in curved spacetime. Let us focus instead in the weak formulation of the correspondence, the equivalence between $\mathcal{N} = 4$ $SU(N_c)$ SYM in four dimensions and Type IIB SUGRA on $AdS_5 \times S^5$.

⁵Notice we still have a $SU(N)$ gauge theory, the $U(1)$ extra factor describes the collective motion of the stack of branes, so it can be decoupled not only in the IR limit [64]

4.1.2.1 Matching parameters

Let us examine more closely the parameters that enter in the definition of each theory, and the map between them. In the gauge theory side, the gauge symmetry is specified by the rank of the group, N_c , and also the 't Hooft coupling constant, $\lambda = g_{YM}^2 N_c$. The string theory is determined by the string coupling constant, g_s , and by the size of the AdS_5 and S^5 spaces. Both spaces are maximally symmetric and are completely characterized by a single scale, their radius of curvature. Remind that $R^4/l_s^4 \sim g_s N_c$ so $R^4/l_s^4 \sim \lambda$. This means that the so-called α' -expansion (since R is fixed) on the string side, which controls corrections associated to the finite size of the string (when $l_s \ll R$) as compared to the size of the spacetime it propagate in, corresponds to the strong coupling $\lambda \gg 1$. Hence, it is necessary, in order to supergravity limit to be a good approximation, we must have actually $\lambda \rightarrow \infty^6$ in order to ensure the additional degrees of freedom such as D-strings, whose tension as $1/g_s$, remains heavy and not propagate.

The string coupling is related to the gauge theory parameters through,

$$g_s \sim g_{YM}^2 \sim \frac{\lambda}{N_c},$$

which means that, for a fixed size of $AdS_5 \times S^5$ geometry (i.e. for fixed λ), the string loop expansion corresponds precisely to the $1/N_c$ expansion in the gauge theory.

The above statements imply that the correspondence in this AdS/CFT conjecture holds in the $N_c \rightarrow \infty$ limit at large and fixed 't Hooft coupling λ .

Let us give some words about this $1/N_c$ expansion in the gauge theory side. Large N_c gauge theory amplitudes have a convenient topological expansion in terms of powers of N_c . In this picture Feynman diagrams can be viewed as two-dimensional surfaces and assigned Euler characteristic number, $\chi = 2 - 2g$, for closed oriented surfaces,

$$\sim \sum_g N_c^{2-2g},$$

where g is the genus (the number of handles) of the surface. In the large N_c limit we see that any computation will be dominated by the surface of maximal χ or minimal genus (sphere or equivalently, a plane). All these planar diagrams will give a contribution of order N_c^2 , while all other diagrams will be suppressed by powers of $1/N_c^2$.

⁶Note, however, that this condition is not sufficient: it must be supplemented by the requirement that $g_s \rightarrow 0$ (which then implies that $N_c \rightarrow \infty$).

4.1.2.2 Holographic duality

The metric on AdS_5 , in the so-called Poincaré match was written in (3.166) as

$$ds_h^2 = \frac{r^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^2} dr^2. \quad (4.7)$$

The coordinates x^μ may be thought of as the coordinates along the worldvolume of the brane, and hence they can be identified with the gauge theory coordinates. The coordinate r together those of S^5 , describe the transverse directions to the brane. As $r \rightarrow \infty$, we approach to the so-called "boundary" of AdS_5 , ∂AdS_5 . Actually, this is not a boundary in the topological sense but in the conformal sense of the word.

As we know $\mathcal{N} = 4$ SYM is a conformal field theory, in this side, the transformation $x \rightarrow \Lambda x$ is a symmetry of the theory. At the same time, in the gravity side, this transformation is also a symmetry. In (4.7), the rescalings $r \rightarrow r/\Lambda$ plus $x \rightarrow \Lambda x$ leave the metric invariant.

Let us focus in the limits of Λ (see [72] also for some advances on QCD). When $\Lambda \gg 1$, in the gauge theory side, we get physics at large distances, i.e., at small energies (IR limit). This limit is associated with the near-horizon limit, $r \rightarrow 0$. Whereas, when $\Lambda \ll 1$ we have physics at short distances, which means large energies (UV limit). We conclude that the energy in the gauge theory side is associated with the radial coordinate in gravity side.

Scale Factor	Gauge Theory	AdS_5 space
$\Lambda \gg 1$	$E \ll 1$	$r \rightarrow 0$
$\Lambda \ll 1$	$E \gg 1$	$r \rightarrow \infty$

Table 4.3: Correspondence between the energy in the gauge theory and the radial distance in the AdS_5 space.

In other words, r is identified with the renormalization group (RG) scale in the gauge theory

$$E \sim r \quad (4.8)$$

We know that $\mathcal{N} = 4$ SYM is a conformal field theory in the UV limit, and since this limit corresponds to $r \rightarrow \infty$ in the AdS_5 space, we say that this gauge theory resides at the boundary of AdS_5 . The correspondence between string theory in AdS space and super Yang-Mills theory is an example of the holographic principle according to which a quantum theory with gravity must be describable by a boundary theory. According to this principle, a macroscopic region of

space and everything inside it can be represented by a boundary theory living on the boundary of this region. A question emerges: where are the D3-brane located in the AdS space then? At the beginning we said that they are at the horizon $r = 0$. But what we found is that the place of the D-branes depends on what frequency (energy) we consider. In other words the location of the D-branes is given by the RG, at low-energies we see that branes are at $r = 0$ but as the frequency increases, the brane appears to move toward the boundary at $r \rightarrow \infty$. This is known as the UV-IR connection [73] (see also [74, 75] for excellent revisions on holography).

4.1.2.3 Matching symmetries

Now, let us study in a little more detail the global symmetries of the both sides of the correspondence. The $\mathcal{N} = 4$ SYM is invariant under $\text{Conf}(1, 3) \times SU(4)$, where $\text{Conf}(1, 3)$ is the conformal group in four dimensions, which contains the Poincaré group, the dilation symmetry and special conformal transformations. The second factor is the R-symmetry of the theory. The theory is also invariant under the 16 supersymmetries, the fermionic superpartners of the translation generators, as well as under 16 special conformal transformations, the fermionic superpartners of the special conformal symmetry generators. On the other hand, in the string theory side, excitations extend all the way to the boundary where the gauge theory lives in. When this space is written as a hypersurface embedded in $\mathbb{R}^{2,4}$, as in (3.167), it is easy to see that it possesses an $SO(2, 4)$ isometry group. The remaining S^5 factor has obviously a $SO(6)$ isometry. Now, here comes the magic matching of symmetries, the $SO(2, 4)$ symmetry of AdS_5 is isomorphic to the conformal group $\text{Conf}(1, 3)$, and the second group, $SO(6)$, corresponds to the R-symmetry group in the gauge theory side, $SU(4)$.

We therefore conclude that the global symmetries are the same on the both sides of the duality.

4.1.2.4 Field-operator map

We will now make more precise the conjecture relating our gauge theory on the boundary of AdS to supergravity (and then string theory) in the bulk. We will make an ansatz whose justification, initially, is to combine ingredients of both sides in a natural way.

Then the aim of this section is to work out the relation between observables of both equivalent theories, in other words, to work out “the dictionary” between representations of the gauge theory and the string theory side, operators and field respectively. We will relate them in the bosonic

subgroup of the superconformal group, $SO(2,4) \times SO(6)$. This matching provides a one-to-one map between gauge invariant operators in $\mathcal{N} = 4$ SYM and classical fields in type IIB on $AdS_5 \times S^5$.

As a conformal theory, $\mathcal{N} = 4$ does not allow to define asymptotic states [65], so if we say spectrum of the theory, we refer instead to the collection of gauge invariant operators $\mathcal{O}(x)$ that are polynomial in the field of the theory (the gauge invariant condition is respected by taking traces over the gauge group, as we saw in section (2.6.2)).

Consider a general n -point function of composite regularized gauge invariant operators $\mathcal{O}_k(x)$ (1/2 BPS operators for our $\mathcal{N} = 4$ SYM theory, or 1/4 BPS [76])

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n) \rangle = \frac{\delta^n \ln Z[J]}{\delta J_1(x_1)\delta J_2(x_2)\dots\delta J_n(x_n)} \Big|_{J_i=0}, \quad (4.9)$$

where,

$$Z[J] := \left\langle \exp \left(- \int d^D x \mathcal{L}_J \right) \right\rangle, \quad (4.10)$$

and \mathcal{L}_J is the lagrangian of a given QFT with added source term coupled to a basis $\{\mathcal{O}_i\}$ of gauge invariant local operators

$$\mathcal{L}_J = \mathcal{L} + \sum_i J_i(x)\mathcal{O}_i(x), \quad (4.11)$$

as usual. Then for our gauge theory we write

$$Z_{\mathcal{N}=4 \text{ SYM}}[J] = \left\langle \exp \left(- \int d^4 x J_k(x)\mathcal{O}_k(x) \right) \right\rangle. \quad (4.12)$$

As we learned, this theory lives in ∂AdS_5 (the boundary of AdS_5) as a strong coupled gauge theory. In the gravity side, consider, for sake of simplicity, a scalar field with mass m on $\mathcal{M} = AdS_{p+1}$. The action is

$$S = \frac{1}{2} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} (g^{AB} \partial_A \Phi \partial_B \Phi + m^2 \Phi). \quad (4.13)$$

This leads to the Klein-Gordon equation in $d+1$ dimensions

$$(-\Delta + m^2)\Phi = 0, \quad \Delta = \frac{1}{\sqrt{-g}} \partial_A (\sqrt{-g} g^{AB} \partial_B) \quad (4.14)$$

with the metric in (3.177), we obtain

$$\begin{aligned} \Delta &= \frac{z^2}{R^2} (z^{d-1} \partial_z (z^{-d+1} \partial_z) + f \partial^2) \\ &= \frac{z^2}{R^2} (\partial_z^2 - (d-1)z^{-1} \partial_z + \partial_\mu \partial^\mu). \end{aligned} \quad (4.15)$$

There is also a boundary term coming from integrating by parts (4.13)

$$S_b = \frac{1}{2} \int_{\mathcal{M}} d^{d+1}x \partial_A (\sqrt{-g} g^{AB} \Phi \partial_B \Phi) = \frac{1}{2} \int_{\partial\mathcal{M}} d^d y \sqrt{\gamma} \Phi n^z \partial_z \Phi \Big|_{z=\varepsilon}^{z=\infty}, \quad (4.16)$$

where $n^z = z/R$ is the vector normal to $\partial\mathcal{M}$, γ is the induced metric and y the coordinates on the surface. We have introduced a cutoff because the induced metric diverges for $z = 0$, so $\gamma_\varepsilon = \frac{R^2}{\varepsilon^2} \eta$. If we “assume“ an exponential decay for Φ at $z \rightarrow \infty$, then

$$S_b = \frac{1}{2} \int_{\mathcal{M}} d^{d+1}x \partial_A (\sqrt{-g} g^{AB} \Phi \partial_B \Phi) = \frac{1}{2} \int_{\partial\mathcal{M}} d^d y \sqrt{\gamma} \Phi n^z \partial_z \Phi \Big|_{z=\varepsilon}, \quad (4.17)$$

where actually $n^z = \varepsilon$. Now we want to solve (4.14). Let us write the field Φ as

$$\Phi(z, x) = f(z)\phi(x). \quad (4.18)$$

Then (4.14) becomes

$$-\frac{z^2}{R^2} (z^{d-1} \partial_z (z^{-d+1} \partial_z f) \phi + f \partial^2 \phi) + m^2 f \phi = 0, \quad (4.19)$$

where $\partial^2 = \partial_\mu \partial^\mu$. Performing separation of variables

$$-\frac{z^{d-1}}{f} \partial_z (z^{-d+1} \partial_z f) + \frac{m^2 R^2}{z^2} = \frac{\partial^2 \phi}{\phi} = -k^2, \quad (4.20)$$

where k^2 is the norm of a d -dimensional vector k^μ . This gives

$$-z^{d+1} \partial_z (z^{-d+1} \partial_z f_k) + (m^2 R^2 + k^2 z^2) f_k = 0, \quad (4.21a)$$

$$\partial^2 \phi_k + k^2 \phi_k = 0. \quad (4.21b)$$

The solution can be written as

$$\Phi(z, x) = \int d^d k f_k(z) \phi_k(x), \quad (4.22)$$

since one get modes depending on k_μ , the full solution will be obtained by superposing all of them⁷. Solving (4.21b) we obtain plane-waves

$$\phi(x) = \frac{1}{(2\pi)^d} e^{ik_\mu x^\mu}. \quad (4.23)$$

⁷Before solving the equations, let us summarize some consequences of the k^2 sign:

- $k^2 = \mu^2 > 0$, the momentum is off-shell. The z solution is real exponential. μ cannot be considered as the mass.
- $k^2 = -\mu^2 < 0$, the momentum satisfies the usual on-shell mass condition and so μ would be indeed the mass in x -space (M^4).

So let us "sum" the modes,

$$\Phi(z, x) = \int \frac{d^d k}{(2\pi)^d} f_k(z) e^{ik_\mu x^\mu}. \quad (4.24)$$

The z -equation can be written as

$$z^2 f_k'' - (d-1)z f_k' - (z^2 k^2 + m^2 R^2) f_k = 0. \quad (4.25)$$

This is almost the modified Bessel equation. Let us do a clever change of variable, $f_k = z^{d/2} g_k$ to get

$$(kz)^2 g_k'' + (kz) g_k' - \left\{ \frac{d^2}{4} + m^2 R^2 + (kz)^2 \right\} g_k = 0, \quad (4.26)$$

which is the modified Bessel equation (see [77] for an excellent handbook of mathematical functions). Here $g_k'' = \partial^2 g_k / \partial^2(kz)$ and $g_k' = \partial g_k / \partial(kz)$. The solutions to (4.26) can be written in terms of the so-called modified Bessel functions, $\mathcal{K}_\nu(kz)$ and $\mathcal{I}_\nu(kz)$,

$$f_k(z) = z^{d/2} [a_k \mathcal{K}_\nu(kz) + b_k \mathcal{I}_\nu(kz)], \quad (4.27)$$

where $\mathcal{K}_\nu(kz)$ and $\mathcal{I}_\nu(kz)$ are the modified Bessel functions of first and second kind respectively. We have defined the parameter ν as

$$\nu = \sqrt{\frac{d^2}{4} + m^2 R^2} \quad (4.28)$$

Now we have to impose that solutions are regular everywhere, and more specifically for $z \rightarrow \infty$ ($r \rightarrow 0$, the deep interior of AdS -space). Modified Bessel functions are well known. So in this limit the modified Bessel functions behave as

$$\mathcal{K}_\nu(kz) \sim e^{-kz}, \quad \mathcal{I}_\nu(kz) \sim e^{kz}. \quad (4.29)$$

We see that $\mathcal{I}_\nu(kz)$ diverges, so $b_k = 0$ and then,

$$f_k(z) = a_k (kz)^{d/2} \mathcal{K}_\nu(kz). \quad (4.30)$$

Using the asymptotic form of $\mathcal{K}_\nu(kz)$ when $z \rightarrow 0$ (near the boundary of AdS -space), we find

$$f_k(z) \approx a_k (kz)^{d/2} \left[\frac{\Gamma(\nu)}{2} \left(\frac{2}{kz} \right)^\nu + \frac{\Gamma(-\nu)}{2} \left(\frac{2}{kz} \right)^{-\nu} \right]. \quad (4.31)$$

Let us express these results as

$$f_k(z) \approx \phi_0(k) z^{\Delta_-} + \phi_1(k) z^{\Delta_+}, \quad (4.32)$$

where we have defined

$$\phi_0(k) = a_k 2^{\nu-1} \Gamma(\nu) k^{\Delta_-}, \quad \phi_1(k) = a_k 2^{-\nu-1} \Gamma(-\nu) k^{\Delta_+}. \quad (4.33)$$

In position space (4.32) becomes

$$\phi(z, x) \approx \phi_0(x) z^{\Delta_-} + \phi_1(x) z^{\Delta_+}. \quad (4.34)$$

The “scaling” components Δ_{\pm} are defined as

$$\Delta_{\pm} = \frac{d}{2} \pm \nu = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2} \Rightarrow \Delta_+ + \Delta_- = d. \quad (4.35)$$

We can redefine

$$\Delta_+ = h, \quad \Delta_- = d - h. \quad (4.36)$$

Notice also that the positivity of the square-root gives the Breitenlohner-Freedman bound [6, 20, 39],

$$m^2 R^2 > -\frac{d^2}{4}. \quad (4.37)$$

Now we study in more detail the asymptotic form (4.32) of the solution. Since $\Delta_+ > 0$, the solution z^{Δ_+} is a normalizable solution and corresponds to a bulk excitation which decays at the boundary ($z \rightarrow 0$). But the other solution, z^{Δ_-} does not decay since $\Delta_- < 0$. Then we say that it defines a non-normalizable⁸ field on the boundary,

$$\phi_0(k) = \lim_{z \rightarrow 0} z^{-\Delta_-} f_k(z) = \lim_{z \rightarrow 0} z^{h-d} f_k(z) \Rightarrow \phi_0(x) = \lim_{z \rightarrow 0} z^{h-d} \phi(z, x). \quad (4.38)$$

In principle one asks only for normalizable modes, since they correspond to modes which propagate in the bulk. On the other side, non-renormalizable modes are necessary to specify boundary conditions, and then these modes do not fluctuate and they provide the classical background on which the normalizable modes propagate. Thus we should not throw away them.

It is also possible to verify that ϕ_0 is an operator with conformal dimension $\Delta_- = h - d$. Under scaling $z \rightarrow \lambda z$,

$$\begin{aligned} \phi_0(\lambda x) &= \lim_{z \rightarrow 0} z^{h-d} \phi(z, \lambda x) = \lambda^{h-d} \lim_{z \rightarrow 0} (\lambda^{-1} z)^{h-d} \phi(z, \lambda x) \\ &= \lambda^{h-d} \lim_{z' \rightarrow 0} (z')^{h-d} \phi(\lambda z', \lambda x) = \lambda^{h-d} \lim_{z' \rightarrow 0} (z')^{h-d} \phi(z', x), \end{aligned}$$

⁸For non-normalizable fields, it is not possible to construct the corresponding Hilbert space, \mathcal{H} of the theory

where we have done $\phi(\lambda z', \lambda x) = \phi(z', x)$. This leads to

$$\phi_0(\lambda x) = \lambda^{d-h} \phi_0(x). \quad (4.39)$$

So we will label $\phi_0(x)$ as $\phi_{d-h}(x)$, and define a “source“ on the boundary,

$$J(k) \equiv \phi_{d-h}(k) = \lim_{z \rightarrow 0} z^{h-d} \phi(k, z). \quad (4.40)$$

In this sense we could relabel our solution in (4.34) and write the “bulk field“ as

$$\phi(z, x) \approx \phi_{d-h}(x) z^{d-h} + \phi_h(x) z^h, \quad (4.41)$$

where $\phi_h(x) = \phi_1(x)$ will correspond to a field in the boundary with scaling dimension h , which in the gauge theory side will be a local operator $\mathcal{O}_h(x)$

$$\langle \mathcal{O}_h(k) \rangle \equiv \phi_h(k) = \lim_{z \rightarrow 0} z^h \phi(k, z). \quad (4.42)$$

If we insert (4.41) in (4.21a) we obtain an important relation between h and the mass of the scalar field in AdS -space

$$h(h-d) = m^2 R^2. \quad (4.43)$$

In order to see that $\phi_h(x)$ corresponds to an operator in the gauge theory, and $\phi_{d-h}(x)$ is a source on the boundary, we establish the equivalence between partition functions of both sides of the correspondence (see [63, 78, 79])

$$Z_{\text{QFT}}^d[J] \equiv Z_{\text{QG}}^{d+1}[\phi] \Big|_{\phi_{\partial\mathcal{M}}=J}, \quad (4.44)$$

which translates all the ideas we explained before, a quantum field theory in d -dimensions lives on the boundary of a string theory (quantum theory of gravity) in $d+1$ -dimensions. But since we are considering the Maldacena limit, i.e. $g_s N_c \gg 1$, we have actually

$$\left\langle \exp \left(- \int d^4x J(x) \mathcal{O}_h(x) \right) \right\rangle \approx e^{-I_{\text{SUGRA}}} \Big|_{\phi_{\partial\mathcal{M}}=J}, \quad (4.45)$$

where \mathcal{O}_h is a local operator in the CFT (single-trace operator in $\mathcal{N} = 4$ SYM) and I_{SUGRA} is the effective action in the bulk of AdS . Under scale transformations, this operator transforms as

$$\mathcal{O}(x) \rightarrow \lambda^h \mathcal{O}(\lambda x), \quad (4.46)$$

where we intentionally write the conformal dimension of \mathcal{O} as h . Then, for our scalar action (4.13),

$$\langle \mathcal{O}_h(x) \rangle = \frac{\delta}{\delta \phi_{d-h}(x')} \left(-S[\phi(z, x)] \right) \Big|_{z \rightarrow 0}. \quad (4.47)$$

As we saw before, the scalar field has a boundary behavior given in (4.41). Since the first term dominates over the second near $z = 0$, the construction of $\phi(z, x)$ from $\phi_{d-h}(x)$ is accomplished with the help of the boundary-to-bulk propagator,

$$\phi(z, x) = \int d^d x' K(z, x; x') \phi_{d-h}(x'), \quad (4.48)$$

where,

$$K(z, x; x') = C_h \frac{z^h}{(z^2 + |x - x'|^2)^h}. \quad (4.49)$$

So we write the bulk field propagated from the boundary as (see for example [63, 80])

$$\phi(z, x) = C_h \int d^d x' \frac{z^h}{(z^2 + |x - x'|^2)^h} \phi_{d-h}(x'). \quad (4.50)$$

Here C_h is a constant chosen by boundary requirements. Now, let us compute the r.h.s of (4.47).

Variation on-shell of the action (4.13) leads to

$$\delta S = C_h h \int_{\partial \mathcal{M}} d^d x d^d x' \frac{\phi_{d-h} \delta \phi_{d-h}}{|x - x'|^{2h}}, \quad (4.51)$$

where we have taken the limit $z \rightarrow 0$ and suppressed terms of order z^{h+1} . Then we obtain

$$\langle \mathcal{O}_h(x) \rangle = -C_h h \int_{\partial \mathcal{M}} d^d x d^d x' \frac{\phi_{d-h}}{|x - x'|^{2h}}, \quad (4.52)$$

which is in agreement with the expansion of [81]

$$\langle \mathcal{O}_h(x) \rangle = \langle \mathcal{O}_h(x) e^{-\int d^d x' \mathcal{O}_h(x') \phi_{d-h}(x')} \rangle \sim \int d^d x' \langle \mathcal{O}_h(x) \mathcal{O}_h(x') \rangle \phi_{d-h}(x'), \quad (4.53)$$

and since $\langle \mathcal{O}_h(x) \mathcal{O}_h(x') \rangle \sim \frac{1}{|x - x'|^{2h}}$ (see (2.187)). We say that ϕ_{d-h} is indeed the source of an operator \mathcal{O}_h on the boundary. We also notice from (4.41) and (4.50) when $z \rightarrow 0$ that

$$\phi_h(x) = C_h \int_{\partial \mathcal{M}} d^d x d^d x' \frac{\phi_{d-h}}{|x - x'|^{2h}} \sim \langle \mathcal{O}_h(x) \rangle. \quad (4.54)$$

Then we confirm that ϕ_h corresponds to the expectation value of and operator $\mathcal{O}_h(x)$ with conformal dimension h .

The mapping between correlation functions in the case of $\mathcal{N} = 4$ SYM and the SUGRA dynamics is given as follow: The generating functional for correlators of single-trace operators \mathcal{O}_h given in (2.221) is related to the source fields ϕ_{d-h} . The boundary values of these supergravity fields at the four-dimensional boundary of the five-dimensional AdS space become the sources for the gauge theory operators (see [6, 7, 65] for exact results).

The complete correspondence between the representations of $PSU(2, 2|4)$ on both sides of the correspondence is given in [7]. The whole mapping of the descendants states is also given in [7], and developed totally in [82, 83]. Since here those calculations are very extensive for the scope of this work, we only reference them.

Chapter 5

The Klebanov-Witten model (KW)

This is the central chapter, here we will study a different form of the usual gauge/gravity duality established by Maldacena, proposed by I.Klebanov and E.Witten in 1998 [84]. Let us think instead of the S^5 (which is Ricci-Flat) factor in the $AdS_5 \times S^5$ background for the type IIB superstring theory we studied before, in a special five-dimensional compact space X_5 with some interesting properties that will be mentioned later.

Our task is to know what is the new dual field theory, how many supersymmetries are preserved and what are the new symmetry groups that both the gravity theory and the gauge theory share. Recall that for the $AdS_5 \times S^5$ case, the dual field theory was $\mathcal{N} = 4$ $SU(N)$ gauge theory in four dimensions. The number of conserved supercharges in this theory was determined by the number of preserved components of the Killing spinor equation on S^5 , namely 32. But when we consider another less symmetric transverse space, this number will be reduced.

5.1 Branes at conical singularities

We will consider parallel Dp-branes near a conical singularity, a point on an n -dimensional manifold Y_n where the metric can locally be put in the form [61, 84]

$$h_{mn}dx^m dx^n = dr^2 + r^2 g_{ij} dx^i dx^j \quad (i, j = 1, \dots, n-1), \quad (5.1)$$

g_{ij} is the metric on an $(n-1)$ -dimensional manifold X_{n-1} ; the point $r=0$ is a singularity unless X_{n-1} is a round sphere. Let us clear up some facts about it. First, the last comment could be applied to a two-dimensional plane. In polar coordinates the angular variable θ runs from 0 to 2π , for constant radius, this will be a circle S^1 . But, what happens if we impose that θ runs

to some $\theta_0 < 2\pi$, and identify $\theta = 0$ with θ_0 ?. It is not difficult to imagine that this will be a two-dimensional cone, with the apex in $r = 0$. Thus, the basic property of the metric h_{mn} , which makes it conelike, is that there is a group of diffeomorphisms of Y_n that rescale the metric (make the cone larger or smaller); the group is $r \rightarrow tr$, with $t > 0$. Back to our two-dimensional example, the base of the cone will be an S_2 circle. Then, we call Y_n , a cone over X_{n-1} .

For both our example and the Y_n space, the point $r = 0$ will be a singularity since the geodesic curves cannot be continued any further than $r = 0$. That is what is called a geodesically incomplete singularity.

A crucial ingredient in identifying two theories, is to understand what symmetries they exhibit. Thus, we must compute precisely how many supersymmetries, if any, are preserved by the background we have just introduced. As we mentioned above, the duality between type IIB strings on $AdS_5 \times S_5$ and the dual $N = 4$ SYM is naturally generalized to dualities between backgrounds of the form $AdS_5 \times X_5$ and other conformal gauge theories (see [85–87] for some detailed papers about extensions of AdS/CFT correspondence to supersymmetric theories with less than maximal supersymmetry). This space X_5 is required to be a positively curved Einstein manifold. Steps to follow are the same as we performed in the chapter 3.

For $n > 2$, the condition that Y_n is Ricci-Flat is that X_{n-1} is an Einstein manifold of positive curvature. Let us see some results:

For h_{ij} , the $(n - 1)$ -dimensional part of h_{mn} , we have

$$R_{ij}(h) = R_{ij}(g) + \partial_r \Gamma_{ij}^r + \Gamma_{kr}^k \Gamma_{ij}^r - \Gamma_{jk}^r \Gamma_{ri}^k - \Gamma_{jr}^k \Gamma_{ki}^r. \quad (5.2)$$

Solving the Christoffel symbols for g_{ij}^1 , we get

$$R_{ij}(h) = R_{ij}(g) - (n - 2)g_{ij}. \quad (5.3)$$

Now, if we set $R_{ij}(h) = 0$ (Ricci-flatness condition), then

$$R_{ij}(h) = (n - 2)g_{ij}. \quad (5.4)$$

Thus, for $n > 2$, g_{ij} is an Einstein metric.

Now we focus in the case that $n = 6$. We take spacetime to be $M_4 \times Y_6$, with M_4 being a four-dimensional Minkowski space and the Y_6 space is the above one. Then, we consider N

¹Christoffel symbols: $\Gamma_{ij}^r = -rg_{ij}$, $\Gamma_{kr}^k = \frac{1}{r}$, $\Gamma_{ij}^k(h) = \Gamma_{ij}^k(g)$.

parallel D3-branes on $M_4 \times P$, where P is the conical singularity of Y_6 . The ten-dimensional spacetime has the metric [84]

$$ds^2 = H^{-1/2}(r)\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(r)h_{mn}dx^m dx^n, \quad (5.5)$$

where,

$$H(r) = 1 + \frac{|Q|}{r^4}, \quad |Q| \sim g_s N \alpha'^2. \quad (5.6)$$

If we want to know how many components of a spinor in this space will survive parallel transport, we need to solve an equation defining Killing spinors in h_{mn} ,

$$\nabla_m \eta = 0, \quad (5.7)$$

or by definition,

$$(\partial_m + \frac{1}{4}\omega_{mab}\Gamma^{ab})\eta = 0, \quad (5.8)$$

here ω_{mab} are the spin-connections, which allow us to define a covariant spinor over Y_6 . Let us evaluate this expression. First, we have to write the vierbeins for the metric h_{mn}

$$e^r = dr, \quad e^{\underline{i}} = r e^i. \quad (5.9)$$

The torsion-free condition allows us to find the connection forms. First for the r -coordinate,

$$de^r + \omega_i^r \wedge e^i = 0,$$

then,

$$\omega_i^r \wedge e^i = 0 \rightarrow \omega_i^r \sim e_i. \quad (5.10)$$

For the \underline{i} -coordinate, the torsion-free condition has more terms,

$$de^{\underline{i}} + \omega_{\underline{j}}^{\underline{i}} \wedge e^{\underline{j}} + \omega_r^{\underline{i}} \wedge e^r = 0,$$

which yields,

$$\omega_r^{\underline{i}} = e^i \quad \omega_{\underline{j}}^{\underline{i}} = \omega^i_j. \quad (5.11)$$

The Killing spinor equation for the r -variable can be expanded in the following form,

$$(\partial_r + \frac{1}{4}\omega_{rij}\Gamma^{ij})\eta = 0,$$

and the more interesting Killing spinor equation for the i -variable,

$$(\partial_i + \frac{1}{4}\omega_{ijk}\Gamma^{jk} + \frac{1}{2}\omega_{ijr}\Gamma^{jr})\eta = 0.$$

Replacing the values of the ω 's, we get²

$$(\partial_i + \frac{1}{4}\omega_{ijk}\Gamma^{jk} + \frac{1}{2}\Gamma_i^r)\eta = 0, \quad (5.12)$$

which is the same Killing spinor equation coming from Type IIB strings on $AdS_5 \times X_5$, including the effects of the five-form.

Type IIB Superstrings with R-R Flux

We can check the last assertion writing down the supersymmetric transformations of the dilatino and gravitino in a pure bosonic background (no fermions, so $\delta(bosons) = 0$ at the first order formulation [51]). We are also considering only the effect of in (3.123)³. Then

$$\delta\lambda = 0 \quad (5.13)$$

This equation says that the dilaton ϕ is constant. And,

$$\delta\psi_M = D_M\epsilon + \frac{i}{4 \times 480}\Gamma^{NPQRS}F_{NPQRS}\Gamma_M\epsilon = 0, \quad (5.14)$$

where,

$$D_M\epsilon = (\partial_M + \frac{1}{4}\omega_{MAB}\Gamma^{AB})\epsilon, \quad (5.15)$$

and ϵ is a complex Weyl spinor ($\Gamma^{11}\epsilon = \epsilon$). The five-form F_{NPQRS} satisfies (3.130),

$$\sqrt{-h}F_{MNPQR} = \frac{1}{5!}\varepsilon_{MNPQRSTUVW}F^{STUVW}. \quad (5.16)$$

The first possibility for $A_{(4)}$ is to support a four-dimensional worldvolume. This is what we shall call the elementary or electric ansatz (known as Freund-Rubin type ansatz in [62]),

$$A_{\mu\nu\rho\sigma} \sim \varepsilon_{\mu\nu\rho\sigma}. \quad (5.17)$$

We need to have $SO(1,3) \times SO(6)$ symmetry. Hence, in order to guarantee the symmetry, the four-form field acquire the following form⁴,

$$A_{\mu\nu\rho\sigma} = -\frac{1}{4h}\varepsilon_{\mu\nu\rho\sigma}e^{C(r)}. \quad (5.18)$$

²Notice that these Γ -matrices are not constants. But we could use \underline{i} -coordinates in the tangent space where the Γ -matrices are actually constants.

³Remember that the five-form $F_{(5)}$ ($F_{(5)} = dA_{(4)}$) is the field strength of a three-dimensional object, in the same way that the usual two-form $F_{(2)}$ is the field strength of a zero-dimensional object, a point particle.

⁴This form was assumed by M.J.Duff and J.X.Lu (1991) in [60].

Notice that the factor r -dependent manifests $SO(6)$ invariance. The five-form, is then

$$F_{\mu\nu\rho\sigma m} = -\frac{1}{4h}\epsilon_{\mu\nu\rho\sigma}\partial_m e^{C(r)} = e^{-4A(r)}\epsilon_{\mu\nu\rho\sigma}\partial_m e^{C(r)}. \quad (5.19)$$

Where 4h is the determinant of the four-dimensional metric in (5.21). This field strength satisfies the self-duality condition,

$$F_{MNPQR} = *F_{MNPQR}. \quad (5.20)$$

And the metric can be thought as we motivate before, but in more intuitive form,

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}h_{mn}dx^m dx^n. \quad (5.21)$$

This metric has some subtle details. It is not the same that (3.129), the metric h_{mn} has not $SO(6)$ symmetry, but its tangent metric.

Now, we can to solve (5.14) to know how many Killing spinors survive or are conserved by supersymmetry,

$$\partial_M \epsilon + \frac{1}{4}\omega_{MAB}\Gamma^{AB}\epsilon + \frac{i}{4 \times 480}\Gamma^{NPQRS}F_{NPQRS}\Gamma_M\epsilon = 0. \quad (5.22)$$

We split the coordinates as $x^M = (x^\mu, x^m)$, and the Γ matrices as we did before in (3.143) and (3.144), where $\mu = 0, \dots, 3$ and $m = 4, \dots, 9$, and the Γ matrices split as

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \mathbb{I}, \\ \Gamma^m &= \gamma^5 \otimes \gamma^m. \end{aligned} \quad (5.23)$$

In our background (5.21), the vierbeins are

$$\begin{aligned} e^\mu &= e^{A(r)}dx^\mu \\ e^m &= e^{B(r)}e^m \end{aligned} \quad (5.24)$$

The torsion-free condition again allows us to obtain the spin-connections for the metric.

For $M = \underline{\mu}$,

$$de^\mu + \omega^\mu_m \wedge e^m + \omega^\mu_\nu \wedge e^\nu = 0.$$

Then,

$$\omega^\mu_\nu = 0, \quad \omega^\mu_m = \partial_m A(r)e^\mu.$$

In the same way for the $M = \underline{m}$,

$$de^m + \omega^m_n \wedge e^n + \omega^m_\mu \wedge e^\mu = 0, \quad (5.25)$$

we get,

$$\omega_{\underline{n}}^m = \partial_n B(r) e^m + \omega_n^m.$$

Hence, the covariant derivatives (5.15) are,

$$D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{2} \partial_m A(r) \Gamma_\mu \Gamma^m \epsilon, \quad (5.26)$$

$$\begin{aligned} D_n \epsilon &= \nabla_n \epsilon + \frac{1}{2} \partial_m B(r) \Gamma_n \Gamma^m \epsilon \\ &= \nabla_n \epsilon + \frac{1}{2} \partial_n B(r) + \frac{1}{2} \partial_m B(r) \Gamma^m \Gamma_n \epsilon. \end{aligned} \quad (5.27)$$

The second term in (5.22) satisfies,

$$\Gamma^{NPQRS} F_{NPQRS} = \Gamma^{\mu\nu\rho\sigma m} F_{\mu\nu\rho\sigma m} (1 - \Gamma^{11}). \quad (5.28)$$

So, (5.22), for $M = \mu$ is

$$\partial_\mu \epsilon + \frac{1}{4} \omega_{\mu AB} \Gamma^{AB} \epsilon + \frac{i}{4 \times 480} \Gamma^{NPQRS} F_{NPQRS} \Gamma_\mu \epsilon = 0. \quad (5.29)$$

By inserting (5.26) and (5.28) we obtain,

$$\partial_\mu \epsilon + \frac{1}{2} \partial_m A(r) \Gamma_\mu \Gamma^m \epsilon + \frac{i}{4 \times 480} \Gamma^{\mu\nu\rho\sigma m} F_{\mu\nu\rho\sigma m} (1 - \Gamma^{11}) \Gamma_\mu \epsilon = 0. \quad (5.30)$$

Using (5.19), (5.23) and the fact that $\Gamma_{11} \epsilon = \epsilon$, it is not difficult to get that

$$\partial_\mu \epsilon + \frac{1}{2} \gamma_\mu \otimes \gamma^n \{ \gamma_5 \partial_n A - \frac{1}{4} e^{-4A} \partial_n e^C \} \epsilon = 0. \quad (5.31)$$

In the same way, for $M = m$, by using (5.27) and (5.28),

$$\nabla_n \epsilon + \frac{1}{2} \partial_m B \epsilon - \frac{1}{2} \gamma_5 \otimes \gamma^n \gamma_m \{ \gamma_5 \partial_n B + \frac{1}{4} e^{-4A} \partial_m e^{-C} \} = 0. \quad (5.32)$$

In order to solve these equations, let us assume a form for ϵ to be $\epsilon = e^{A/2} \varepsilon \otimes \eta$. Where ε and η , are chiral spinors, then

$$(1 - \gamma_5) \varepsilon = (1 - \gamma_7) \eta = 0 \quad (5.33)$$

The equation (5.32) is reduced to

$$\nabla_n \eta = 0, \quad (5.34)$$

a covariant spinor in B^6 . Hence,

$$C = 4A, \quad B = -A. \quad (5.35)$$

The five-form (5.19) and the ten-dimensional metric (5.21) will have the following form,

$$F_{\mu\nu\rho\sigma m} = \epsilon_{\mu\nu\rho\sigma} \partial_m C(r), \quad (5.36)$$

and,

$$ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{-2A(r)} h_{mn} dx^m dx^n. \quad (5.37)$$

Let us extract a little more information about $M_4 \times Y_6$ from (5.34). The integrability condition guarantees the existence of a spinor, and is written as

$$[\nabla_n, \nabla_m] \eta = \frac{1}{4} R_{mnpq} \Gamma^{pq} \eta = 0. \quad (5.38)$$

Then,

$$\Gamma^n \Gamma^{pq} R_{mnpq} \eta = 2g^{nq} \Gamma^p R_{mnpq} \eta = 0 \quad \rightarrow R_{mn} = 0. \quad (5.39)$$

Hence, the six-dimensional internal space must be Ricci-flat. The exact form of the function $A(r)$ will come by solving the field equations. Fortunately we have already found the solution for the corresponding Einstein equations with this matter content. Obviously we are dealing with a different spacetime, but it is Ricci-flat too. Then we have, in the same way that (3.165),

$$ds^2 = \left(1 + \frac{|Q|}{r^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{|Q|}{r^4}\right)^{1/2} \underbrace{h_{mn} dx^m dx^n}_{SO(6)}. \quad (5.40)$$

This h_{mn} space (B_6) has symmetry $SO(6)$ which is covered by $SU(4)$. A real spinor on such a manifold has eight components, but they can be decomposed into two irreducible $SU(4)$ representations (see [88] for general surveys on field theories with 16 supercharges),

$$\mathbf{8} = \mathbf{4} \oplus \bar{\mathbf{4}}. \quad (5.41)$$

A spinor that is covariantly constant remains unchanged after being parallel transported around a closed curve. The existence of such spinor is required if some supersymmetry is to remain unbroken. The largest subgroup of $SU(4)$ for which a spinor of definite chirality can be invariant is $SU(3)$, which is called the holonomy group. We have deduced two more properties of the internal spacetime, it is Ricci-flat and has $SU(3)$ holonomy, which identify B^6 as a Calabi-Yau manifold [59]. Thus, the equation (5.27) has as solution, a spinor of eight-components. In other words, the internal space preserves 1/4 of the supersymmetries.

These eight conserved spinors allow us to conclude that the dual theory will be $\mathcal{N} = 1$ superconformal in four dimensions, since this theory has eight generators, four for supersymmetry and four for conformal transformations.

Back to the specific form to h_{mn} , given in (5.1). The metric for a ten-dimensional spacetime, in presence of N D3-branes sourcing an $A_{(4)}$ gauge field with a $F_{(5)}$ form as field strength, and with a conelike tranverse space read

$$ds^2 = \left(1 + \frac{|Q|}{r^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{|Q|}{r^4}\right)^{1/2} (dr^2 + r^2 g_{ij} dx^i dx^j), \quad (5.42)$$

where $|Q| \sim N$. The near-horizon limit ($r \rightarrow 0$) of the spacetime (5.42) is

$$ds_h^2 = \frac{r^2}{\sqrt{Q}} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{\sqrt{Q}}{r^2} dr^2 + \sqrt{Q} g_{ij} dx^i dx^j, \quad (5.43)$$

and, thus, it is $AdS_5 \times X_5$. With the metric (5.1), (5.34) turns out to be

$$\nabla_n \eta = \left(\partial_i + \frac{1}{4} \omega_{ijk} \Gamma^{jk} + \frac{1}{2} \Gamma_i^r\right) \eta = 0, \quad (5.44)$$

which is the same equation (5.12).

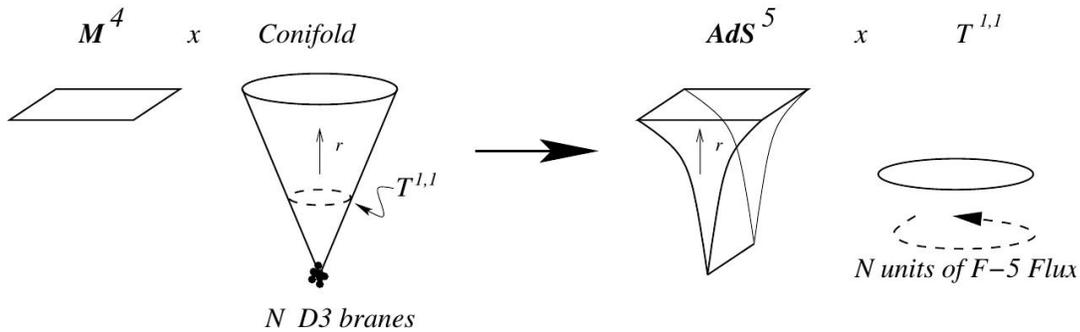


Figure 5.1: In the Maldacena limit, D3-branes on M^4 space times the conifold are replaced with an $AdS^5 \times T^{1,1}$ space with F_5 flux on it (source of the figure, [89]).

5.2 A special case: $T^{1,1}$

We could consider the simplest case for a non-trivial (non Euclidean) five-dimensional space, the base of the cone X_5 (see Fig(5.1)). We can define the conifold, a complex manifold in \mathbb{C}^4 as⁵ [90],

$$\mathcal{C} : \sum_{A=1}^4 (z^A)^2 = 0. \quad (5.45)$$

⁵A conifold could be thought as a manifold near a singularity of some manifold that not necessarily has to be non-compact.

This equation describes a surface which is smooth apart from the point $z^A = 0$, which is a double point singularity⁶. Note also that if z^A solves (5.45) then so does λz^A , so the surface is made up of complex lines through the origin and is therefore a cone.

We define an everywhere nonzero holomorphic three-form Ω as⁷ [35],

$$\Omega = \frac{dz_2 \wedge dz_3 \wedge dz_4}{z_1}. \quad (5.47)$$

The product $\Omega \wedge \bar{\Omega}$ gives the volume form on the Calabi-Yau. The holomorphic form should also transform as the volume form of the superspace $d^2\theta$. If we assign charge one to the coordinates z under the $U(1)$ symmetry, then Ω will have charge two under those transformations. Thus, it is natural to identify this $U(1)$ symmetry with the R-symmetry, as we will see in short.

The base of the cone can be found intersecting (5.45) with a sphere of radius 1 in $\mathbb{C}^4 = \mathbb{R}^8$,

$$\sum_{A=1}^4 |z^A|^2 = 1. \quad (5.48)$$

The group $SO(4)$ acts transitively on this intersection. So the base of the conifold will have this symmetry⁸. There is also a $U(1) \subset SO(4)$ which leaves invariant any point of the intersection.

⁶The apex or node is a double point, i.e. a singularity for which \mathcal{C} and $d\mathcal{C}$ but for which the matrix of second derivatives is nondegenerate.

⁷In general, given a single homogeneous polynomial P of degree k in the $N + 1$ complex variables z_1, \dots, z_{N+1} . The equation $P = 0$ defines a hypersurface Q in CP^N (the complex projective space). There is a well defined holomorphic form for certain k and N . In the region of Q in which $z_{N+1} \neq 0$, we could reparametrize Q with coordinates $x_a = z_a/z_{N+1}$, $a = 1, \dots, N$. The $(N - 1)$ -form,

$$\Omega = dx^1 \wedge dx^2 \wedge \dots \wedge dx^N / \frac{\partial P}{\partial x^N}. \quad (5.46)$$

By P , here mean $P(x_1, \dots, x_N, 1)$. At first sight it seems that the definition of Ω depends on arbitrarily singling out x^N . Actually not, we should remember that on Q , the polynomial P is a constant on Q , so $dP = 0$ on Q . This allows us to write Ω as,

$$\Omega = (-1)^{M-m} dx^1 \wedge dx^1 \wedge \dots \wedge d\hat{x}^m \wedge \dots \wedge dx^N / \frac{\partial P}{\partial x^m},$$

where $d\hat{x}^m$ means that dx^m is not considered, since it was single out. This form shows that the apparent singularity in the first expression, is harmless as long as $z_{N+1} \neq 0$.

⁸Formally we say that the base of the conifold has the topology $S_2 \times S_3$. It can be easily elucidated by separating the complex coordinates into its real and imaginary parts, $z = x + iy$. Then (5.45) and (5.48) becomes

$$x \cdot x = \frac{1}{2}, \quad y \cdot y = \frac{1}{2}, \quad x \cdot y = 0.$$

The first of these defines an S_3 with radius $1/\sqrt{2}$. The other two equations define an S_2 “fiber” over S_3 . Since all “bundles” over S_3 are trivial, we have that the base X_5 has the topology of $S_2 \times S_3$. Then we say that \mathcal{C} is a cone over $S_2 \times S_3$.

Hence, in order to consider only non-identifiable point, we divide by this $U(1)$, so the base will be the coset $SO(4)/U(1) = (SU(2) \times SU(2))/U(1)$. We can clarify more this coset by making a simple change of variables,

$$\begin{aligned} z'_1 &= z_1 + iz_2, & z'_2 &= z_1 - iz_2, \\ z'_3 &= z_4 - iz_3, & z'_4 &= -z_4 - iz_3, \end{aligned} \quad (5.49)$$

so the conifold \mathcal{C} is defined by

$$z_1 z_2 - z_3 z_4 = 0. \quad (5.50)$$

This equation can be “solved” by writing,

$$z_1 = A_1 B_1, \quad z_2 = A_2 B_2, \quad z_3 = A_1 B_2, \quad z_4 = A_2 B_1, \quad (5.51)$$

where A_k and B_l ($k, l = 1, 2$) are variables that satisfy,

$$(|A_1|^2 + |A_2|^2)(|B_1|^2 + |B_2|^2) = 1, \quad (5.52)$$

in the intersection with the four-sphere. Notice that we obtain the same z_i if we do,

$$A_k \rightarrow \lambda A_k, \quad B_l \rightarrow \lambda^{-1} B_l, \quad k, l = 1, 2, \quad (5.53)$$

with $\lambda \in \mathbb{C}^*$. We can choose λ in order to have,

$$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 = 1. \quad (5.54)$$

There is an extra $U(1)$ symmetry (which is the $U(1)_B$ in the field theory), which “rotates” around the base,

$$A_k \rightarrow e^{i\alpha} A_k, \quad B_l \rightarrow e^{i\alpha} B_l. \quad (5.55)$$

Back to (5.52), we see that this is the product $S^3 \times S^3 = SU(2) \times SU(2)$. The base of the conifold X_5 is obtained by dividing by this $U(1)$,

$$X_5 = \frac{SU(2) \times SU(2)}{U(1)}, \quad (5.56)$$

which is called $T^{1,1}$ (see [62,90] for detailed studies about this space), a coset space that belongs to a family of Sasaki-Einstein manifolds labeled by $Y^{p,q}$ (or $T^{p,q}$), originally considered by Romans in the context of Kaluza-Klein supergravity [62] (see [91–93] for some important papers about this countably infinite class of Sasaki-Einstein manifolds and their corresponding AdS/CFT duals,

and also [94] for a very good work about these general backgrounds). In terms of this variables, the base of the cone can be written as,

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0. \quad (5.57)$$

The metric on $T^{1,1}$ was found by P.Candelas and X. de la Ossa (see [90]),

$$ds_5^2 = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6} \sum_{i=1}^2 [d\theta_i^2 + \sin^2\theta_i d\phi_i^2], \quad (5.58)$$

where the angular coordinates were introduced as follows. One can parameterize two $SU(2)$ by two set of Euler angles $\{\psi_i, \theta_i, \phi_i\}$, $i = 1, 2$. Factorization by a $U(1)$ in this case means that we need to identify the two ψ angles. Then $\psi \in [0, 4\pi]$.

5.3 Constructing the dual field theory

5.3.1 $\mathcal{N} = 1$ Superconformal field theory

Remember the conifold parametrization (5.51), the idea is to find a field theory whose vacuum manifold, i.e. $V = 0$, gives us that definition. Then we consider a $U(1)$ gauge theory with $\mathcal{N} = 1$ supersymmetry, and field content $\{A_k, B_l\}$, where $k, l = 1, 2$ [84]. These superfields are chiral and have $U(1)_B$ charges 1 and -1 , respectively.

If we are not considering interactions, the F -term will vanish (no superpotential), so the potential will only contain the called D -term. As we saw in the first chapter, vanishing of the potential will yield not point but a curve or in general, a vacuum manifold (just like in (2.128)) [6],

$$D = |A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0. \quad (5.59)$$

But there is a subtle symmetry in the last expression. Each field can be rotated by an $U(1)$ phase, giving the same expression. So, in order to get rid of this “overcounting”, we need to divide by the gauge group. Then we get what we want, the vacuum manifold is actually the definition of conifold.

As was mentioned before, this theory should be $\mathcal{N} = 1$ superconformal, in order to have the same number of conserved supercharges than that were found by analyzing the geometry of the conifold.

To describe parallel $D3$ -branes on $M_4 \times \mathcal{C}$, we actually need to introduce a second $U(1)$, so the gauge group will be $U(1) \times U(1)$ in each brane (see [84, 95, 96]). The chiral superfields A_i

and B_j , will have opposite $U(1)$ charges, $(1, -1)$ and $(-1, 1)$. But why do we need to introduce a second $U(1)$ in the gauge group?

In order to answer this question, we need to do a short study in orbifolds.

5.3.2 Strings on orbifolds, a review. The flux to $AdS_5 \times T^{1,1}$: A slight digression

The field theory living on $D3$ -branes placed in conical singularities is related with theories on $D3$ -branes at orbifold singularities. In this sense, the field theory will inherit some properties of the gauge group from the orbifold to conifold theory. That is why we need to do a short review on orbifolds in order to understand the reasons for introducing the second $U(1)$ gauge group when we consider parallel branes in $M_4 \times \mathcal{C}$. Next, we will do a brief and qualitative review on orbifold compactifications, in particular type IIB on $AdS_5 \times S_5/\mathbb{Z}_2$, whose dual field theory flows to the field theory coming from the coset space $T^{1,1}$.

5.3.2.1 The idea of orbifold

With the purpose of understanding at least superficially the emergence of this second gauge group, we need firstly to acquire some basics on orbifolds (we recommend [20, 39, 59]). Since it is not a principal issue here, some mathematical details will not be given.

Let M be a smooth manifold and let G be a discrete symmetry group of M . Then the quotient space M/G is called an orbifold. Then, a point of this quotient space belongs to an orbit. If the manifold has some points that are invariant under G , the quotient space will have singularities in those points. At nonsingular points, the quotient space is well-defined and indistinguishable from the original manifold M . So, it is natural to define local structures coming from the smooth manifold to its quotient. Literature gives a lot of illustrative examples, but there is one in particular that will be useful later, the noncompact orbifold \mathbb{C}/\mathbb{Z}_2 . This orbifold is defined by identifying z and $-z$ in \mathbb{C} , the resulting space looks like a cone, which is smooth except at the origin (conical singularity). The base of the cone is a circle with a deficit angle equal to π . Note that this conelike orbifold is not a manifold, because of the singularity at the origin.

5.3.2.2 Orbifolds of type IIB on $AdS_5 \times S_5/\mathbb{Z}_2$, breaking $\mathcal{N} = 4$ to $\mathcal{N} = 2$

Four-dimensional conformal field theories were provided as nonperturbative definitions of type IIB string theory in orbifold backgrounds with 0, 8 and 16 supercharges [97–99]. Then, the

strategy is to study orbifolds of type IIB on $AdS_5 \times S_5$ which preserve the AdS_5 structure but break some of the supersymmetries, in other words, that act only over the S_5 factor. Thus, the dual field theory on the $D3$ -brane will have that superconformal symmetry translated from the symmetry group of the AdS space. The type IIB string theory has constant coupling, that is the reason why we need to have as dual field theory, a SCFT.

One might think that a physical theory defined on an orbifold space should be singular, since orbifold indeed has singularities as we mentioned. Interestingly, these theories are nonsingular and well behaved because of the presence of the so-called twisted sector [100–102].

Now, let us consider the $AdS_5 \times S_5/\mathbb{Z}_2$ case. In general, we take N type IIB $D3$ -branes at the \mathbb{Z}_k orbifold singularity of an A_{k-1} ALE (asymptotically locally Euclidean) space^{9,10} [103, 104]. Then, we can construct the worldvolume theory by taking kN $D3$ -branes on the covering space and doing a \mathbb{Z}_k projection on both the worldvolume fields and the Chan-Paton factors. This translates into a \mathbb{Z}_k orbifold of $AdS_5 \times S_5$ type IIB supergravity solution. The \mathbb{Z}_k acts only on the S_5 factor, leaving the AdS_5 untouched.

S. Kachru and E. Silverstein [97] proposed that a $\mathcal{N} = 2$ SCFT can be constructed by studying $D3$ -branes at orbifolds singularities of the form \mathbb{R}^4/Γ , where $\Gamma \subset SU(2)$ is a discrete group (see also [98]) and this \mathbb{R}^4 is the four-dimensional space contained in the \mathbb{R}^6 , transversal to the $D3$ -branes worldvolume. But before taking the scale limit, the $D3$ -branes sit at a point in this \mathbb{R}^6 . The orbifold ($\Gamma = \mathbb{Z}_2$) fixes a plane in \mathbb{R}^6 , which intersects the S^5 in an big “singular circle“ $S^1 \subset S^5$ [105, 106] when,

$$(x^6, x^7, x^8, x^9) \rightarrow -(x^6, x^7, x^8, x^9). \quad (5.60)$$

Then, in \mathbb{R}^6 , exists a circle $(x^4)^2 + (x^5)^2 = 1$, when $x^6 = x^7 = x^8 = x^9 = 0$ [106]. This is the fixed circle where are localized the so-called twisted states, propagating in the six-dimensional space-time $AdS_5 \times S^1$. This orbifold operation leads to the breaking of supersymmetries, from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ (see [97–99, 105]). The \mathbb{Z}_2 breaks the $SO(6)$ down to $SO(4) \times SO(2)$, where the $SO(2)$ factor corresponds to the fixed circle. Since

$$SO(6) \equiv SU(4)_R \rightarrow SO(4) \times SO(2) \rightarrow \Gamma \times SU(2)_R \times U(1)_R, \quad (5.61)$$

⁹Example: An obvious generalization of the conelike orbifold, is the orbifold \mathbb{C}/\mathbb{Z}_N , where the group is generated by a rotation by $2\pi/N$. This type of singularity is called A_N singularity, and is included in the more general ADE classification of singularities.

¹⁰An ALE space or gravitational instanton \mathcal{M}_4 is a four-manifold with hyperkähler metric asymptotic to \mathbb{R}^4/Γ , where $\Gamma \in SU(2)$ is a discrete group.

where we see clearly the breaking of $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$.

The presence of the unchanged AdS_5 factor in spacetime means we will have an $SO(4,2)$ conformal invariance after the ‘‘orbifolding’’. The gauge group will be $U(N)^k$, and the matter content will be in the

$$(N, \bar{N}, 1, \dots, 1) \oplus (1, N, \bar{N}, \dots, 1) \oplus \dots \oplus (1, \dots, 1, N, \bar{N}), \quad (5.62)$$

representation [97, 99, 103, 107]. There are k terms containing $2N$ fundamental hypermultiplets each. A simple form to understand the last statement is by means of introducing accompanying image D3-branes placed on the S^1 fixed circle, in order to have an \mathbb{Z}_k invariant transverse space. There will be open strings whose endpoints are placed only on the D3-brane, only on its images, and on both. Correspondingly, the massless spectrum of this orbifold will be the one given in (5.62).

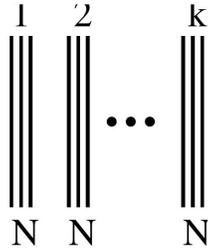


Figure 5.2: Under orbifold action, the kN D3-branes regroup in k stacks of N D3-branes each (figure taken from [30]).

The $\mathcal{N} = 2$ gauge theory is superconformal, as expected, and have a fixed surface parametrized by the gauge coupling of each factor. If the bare gauge couplings of all the $U(1)$ theories are equal one can set them equal to the string coupling.

In the near horizon limit the field theory becomes the $SU(N)^k$ with the same matter content, since the diagonal $U(1)$ factors are IR-free. This theory is dual to type IIB superstring theory on $AdS_5 \times S_5/\mathbb{Z}_k$, where the \mathbb{Z}_k action leaves fixed an S_1 inside the S_5 sphere. The $SO(4,2)$ isometry of AdS_5 is not broken and corresponds to the conformal symmetry of the SCFT on the D3-brane worldvolume.

Now, for $k = 2$, the gauge group on N D3-branes will be $U(N) \times U(N)$, with matter in the $(N, \bar{N}) \oplus (\bar{N}, N)$ bifundamental representation. In the IR limit, the gauge group becomes $SU(N) \times SU(N)$, since the $U(1)$ is free and becomes a global $U(1)_B$ baryonic symmetry. This

gauge theory is dual to type IIB string theory on $AdS_5 \times S^5/\mathbb{Z}_2$.

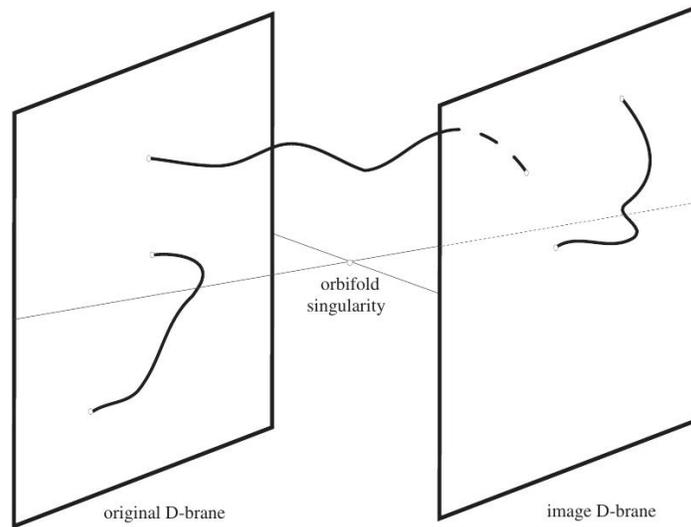


Figure 5.3: D-branes in presence of orbifold singularities and their image. There are open strings whose endpoints are on the D3-branes, on its image, and on both. The resulting gauge group is doubled to $U(1) \times U(1)$ (source of figure, [69]).

5.3.2.3 Blowing up S^5/\mathbb{Z}_2 , breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$

The orbifold theory we have just reviewed, inherits its gauge group to our $T^{1,1}$ theory by blowing up the orbifold singularity of the factor S^5/\mathbb{Z}_2 . We explain briefly the method given in [84], where one can see that the blowing up of the singularity of S^5/\mathbb{Z}_2 corresponds exactly to $S^2 \times S^3$, which is precisely our $T^{1,1}$ space. The blow up consists in replacing the singular point of the orbifold S^5/\mathbb{Z}_2 by a copy of S^2 in such way a way that we can map the regular points of $S^5/\mathbb{Z}_2 \equiv \mathbb{R}P^3$ to S^2 . Mathematically, S^5/\mathbb{Z}_2 defines a fiber over S^2 . And the inverse map from S^2 to S^5/\mathbb{Z}_2 defines a copy of S^3 . Hence, we say that the blow up of S^5/\mathbb{Z}_2 is an S^3 bundle over S^2 or, as was argued in [90], an $S^3 \times S^2$. Now, if we recall our $T^{1,1} = (SU(2) \times SU(2))/U(1)$, one can say (see again [84]) that it is a fiber bundle over S^2 . Let us see. If we forget for a moment, say, the second $SU(2)$, $T^{1,1}$ maps to $SU(2)/U(1) = S^2$. So our $T^{1,1}$ is also a fiber bundle over S^2 . The fiber, connecting $T^{1,1}$ and the base manifold S^2 is the remaining $SU(2)$ which is isomorphic topologically to S^3 . Then we say that $T^{1,1}$ is an S^3 bundle over S^2 .

In other words, this means that when we approach $r \rightarrow 0$ (IR limit) the S^5/\mathbb{Z}_2 looks like the

$T^{1,1}$, then the theory on $AdS_5 \times S^5/\mathbb{Z}_2$ flows to $AdS_5 \times T^{1,1}$.

Remarkably, an interpolating Calabi-Yau solution between $\mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$ and the conifold, and then the holographic Klebanov-Witten RG flow between the two CFT's exists [108], from the orbifold theory at infinity to the cone over $T^{1,1}$ in the interior.

The blow up of the singularity breaks down $\mathcal{N} = 2$ to $\mathcal{N} = 1$ when $r \rightarrow 0$ [65, 84, 106, 109, 110]. This result can be seen by perturbing an $\mathcal{N} = 2$ superpotential with a \mathbb{Z}_2 term to produce the $\mathcal{N} = 1$ quartic superpotential of the Klebanov-Witten model. We say that the \mathbb{Z}_2 orbifold with relevant perturbation flows to the $T^{1,1}$ model associated with the conifold [84, 111, 112].

We have just understood that the gauge group of our dual field theory was inherited from a different theory, dual to strings on an orbifold background. Now, it is time to consider the presence of interactions or contact terms involving our chiral superfields.

5.3.3 The superpotential

The theory does not admit a renormalizable superpotential¹¹, so as a first guess we suppose that it vanishes. If we assume that the chiral multiplets A_k, B_l are $N \times N$ matrices that in some basis are also diagonal with distinct eigenvalues, one finds a family of vacua parametrized by the positions of N D3-branes at distinct points on the conifold, as [113]

$$|A_1^i|^2 + |A_2^i|^2 - |B_1^i|^2 - |B_2^i|^2 = 0, \quad i = 1, \dots, N. \quad (5.63)$$

Then the group will be broken down to $U(1)^N$, one factor of $U(1)$ for each D3-brane. If we place the D3-branes at generic smooth points in \mathcal{C} , there will arise massive modes for the chiral multiplets as non-diagonal modes that also describe the motion of the branes in the space, transversal to the conifold. In order to make massive those modes, we need to introduce a superpotential which, by means of Higgs mechanism, gives mass to the non-diagonal components of A_k, B_l .

We are dealing with a $U(N) \times U(N)$ gauge theory, since the $U(1)_B$ decouples at the low-energy limit (because its β function is positive), we have a $SU(N) \times SU(N)$ gauge theory coupled to two chiral superfields, A_i , in the (N, \bar{N}) bifundamental representation and two chiral superfields, B_i , in the (\bar{N}, N) bifundamental representation of the gauge group. The A 's transform as a doublet under one of the global $SU(N)$'s, while the B 's transform as a doublet under the other $SU(N)$, as has been motivated from (5.51). As we know, the defining equation of the manifold

¹¹This is because of nonrenormalization theorems which say that couplings in the superpotential are not renormalized at any order in perturbation theory.

is related to the moduli space of vacua of the gauge theory. The anomaly-free¹² $U(1)_R$ of the $\mathcal{N} = 1$ superconformal algebra is realized as a common phase rotation of A_k and B_l (or of the four coordinates z , $z = e^{i\alpha}z$, as we saw in the definition of the conifold (5.45) and the holomorphic form in (5.47)). Both A_k and B_l have charge $1/2$ under $U(1)_R$ in order to cancel the anomaly¹³.

Then, we try to construct a superpotential that satisfies the $SU(2) \times SU(2) \times U(1)_B \times U(1)_R$ symmetry of the chiral fields. The simplest choice is

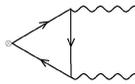
$$W = \frac{\lambda}{2} \epsilon^{ij} \epsilon^{kl} \text{tr} A_i B_k A_j B_l, \quad (5.64)$$

where λ is dimensionless, so its classical β function is zero. Since the chiral fields carry R-charge $1/2$, W has R-charge 2. Then we say that the superpotential is a non renormalizable marginal¹⁴ perturbation of the free conformal theory.

The vacuum configuration, $|W'|^2 = 0$, in some basis, has a family of vacua which describes the motion of the N D3-branes on \mathcal{C} , and then the gauge group is broken to $U(1)^N$. And in general, the off-diagonal fields will receive mass from the superpotential by Higgs mechanism.

We summarize the representations of the chiral superfields under the gauge and global symmetry group in Table 5.3.3.

¹²An anomalous symmetry in QFT corresponds to a symmetry which is satisfied in the classical theory, but is violated by quantum effects (divergence of its current is not zero anymore). Then, a symmetry which is not modified by quantum effects is called anomaly-free. It is also known that the anomalies come from the “triangle graph” of fermions, with all the fermions of the theory appearing in the loop, the R-current inserted at the cross, and two gluons coming out as is depicted below



The contribution from each fermion is proportional to its R-charge times the index of the $SU(N)$ representation. For our case, summing over the “gluino” and fermions gives [6],

$$1 \times T(\text{Ad}) + \left(-\frac{1}{2}\right) \times T(\square) 2F = 0.$$

If we evaluate our $SU(N)$ theory, with $2N$ flavors, we conclude that the assignment of R-charges leads to an anomaly-free R-symmetry.

¹³ We define R-symmetry so that chiral superspace coordinates have charge one. Then, gluino fields λ_A in W_A have also charge one.

¹⁴In a CFT, there is a special class of operators, say \mathcal{O} , known as marginal operators whose scaling dimension is 2. For such an operator, the density $\mathcal{O} dzd\bar{z}$ is conformally invariant. If we perturb our action by $g \int \mathcal{O} dzd\bar{z}$, we can expect that the theory remains conformally invariant, at least classically.

	$SU(N) \times SU(N)$	$SU(2)_A$	$SU(2)_B$	$U(1)_R$	$U(1)_B$
A	(N, \bar{N})	2	1	1/2	1
B	(\bar{N}, N)	1	2	1/2	-1

Table 5.1: Field content and symmetries of the Klebanov-Witten model with massless flavors.

This field content is usually summarized in the “quiver diagram”,

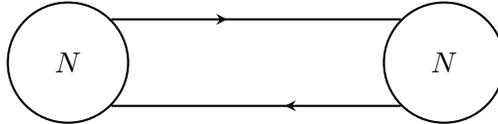


Figure 5.4: Quiver diagram for the Klebanov-Witten theory. Circles represent gauge groups (and vector superfields), arrows represent bifundamental chiral superfields.

where circles represent gauge groups and arrows represent bifundamental fields (fundamental under the gauge group the arrow is exiting from, antifundamental under the gauge group the arrow is entering).

5.3.4 Conformal invariance

We can study the behavior of our $SU(N) \times SU(N)$ gauge theory looking at from one of the $SU(N)$'s point of view. Indeed, the $SU(N) \times SU(N)$ looks like a supersymmetric $SU(N)$ gauge theory with $2N$ flavors (copies of $N \oplus \bar{N}$) as seen from one of the $SU(N)$'s. Now, let us see how the coupling behaves in this theory by studying its β function.

For a supersymmetric $SU(N_c)$ gauge theory, the β function can be obtained from (2.148). N.Seiberg argued [114] in 1995 that there exists a fixed point in the range $\frac{3}{2}N_c < N_f < 3N_c$. The same behavior is then expected for our $SU(N) \times SU(N)$ gauge theory, where $N_c = N$ and $N_f = 2N$.

The superpotential (5.64) will inherit its marginal nature in the infrared theory, as a marginal perturbation. Let us impose now the conditions of conformal invariance for our $SU(N) \times SU(N)$ gauge theory with the superpotential (5.64). From the vanishing of the NSVZ β function (2.148), we get

$$\beta = 3N - 2N(1 - \gamma_{A_1} - \gamma_{A_2} - \gamma_{B_1} - \gamma_{B_2}) = 0. \quad (5.65)$$

Because of the $SU(2) \times SU(2)$ of the superpotential, the anomalous dimension satisfy,

$$\gamma_A = \gamma_{A_1} = \gamma_{A_2}, \quad \gamma_B = \gamma_{B_1} = \gamma_{B_2}, \quad (5.66)$$

If we apply the vanishing of the NVSZ β function to either of the two $SU(N)$'s with matter in the adjoint representation, we find the condition for the anomalous dimensions

$$\gamma_A(g_1, g_2, \lambda) + \gamma_B(g_1, g_2, \lambda) + \frac{1}{2} = 0, \quad (5.67)$$

where g_1 and g_2 are the couplings of the two gauge factors which appear in front of the corresponding kinetic terms of the gauge field. The scale invariance of the superpotential leads to the same condition. The equation (5.67) gives a critical surface for the couplings. But, if we impose the symmetry under interchange of the two $SU(N)$'s, then the fixed surface degenerates into a fixed line with $g_1 = g_2 = g$,

$$\gamma_A(g, \lambda) + \gamma_B(g, \lambda) + \frac{1}{2} = 0. \quad (5.68)$$

We have generated a line of fixed points or a fixed line from adding a marginal perturbation [115], then we say that the superpotential is actually an exactly marginal operator¹⁵ [116]. But actually there are two exactly marginal operators. We can see prove this statement from the fixed surface (5.67). This is a two-dimensional surface in the three-dimensional space of couplings. The surface can be parameterized by two couplings-exactly marginal couplings, which in general has a complicated relation to the original couplings. Then, corresponding to these couplings, there are two exactly marginal operators. In the other hand, we know that the superpotential is exactly marginal, and in the limit $g_1 = g_2$, the critical surface becomes a critical line. Therefore the second marginal coupling must be a function of the difference of the two couplings, more general an antisymmetric combination, that corresponds to the difference of the kinetic terms. So, since the coupling λ is untouched in the parameterization, the difference between the kinetic energies of the two $SU(N)$'s corresponds also to an exactly marginal operator.

In 1995, R.G.Leigh and M.J.Strassler showed that manifolds of fixed lines, which are generated by exactly marginal operators, are common in $\mathcal{N} = 1$ supersymmetric gauge theories.

¹⁵Quantum theory typically spoils conformal invariance. When it persists after adding the marginal perturbation, we say that \mathcal{O} is an exactly marginal perturbation. Perturbing by \mathcal{O} we obtain a continuous family of CFT's parameterized by g .

5.4 The Klebanov-Witten conjecture

We finally arrive to the conjecture, proposed by I.Klebanov and E.Witten in 1999. The $SU(N) \times SU(N)$ SCFT with the superpotential (5.64) is dual to type IIB string theory on $AdS_5 \times T^{1,1}$, with N units of R-R flux on $T^{1,1}$. Unlike the original conjecture, the Klebanov-Witten's conjecture could be tested by comparing symmetries between both sides, superstrings and SCFT.

Let us give some usual tests of the duality:

5.4.1 Comparison of R-symmetries

Remember the $U(1)$ symmetry that rotates the conifold coordinates by $z_i \rightarrow e^{i\phi} z$. Under this transformation, the holomorphic three-form (5.46) transforms as $\Omega \rightarrow e^{2i\phi} \Omega$. The existence of covariantly constant spinors on the manifold implies the existence of covariantly constant p-forms given by

$$\Omega_{i_1 \dots i_p} = \epsilon^T \gamma_{i_1 \dots i_p} \epsilon, \quad (5.69)$$

where $\gamma_{i_1 \dots i_p}$ is the antisymmetrized product of gamma matrices on the six-dimensional space. Thus, these spinors transform as $\Omega^{1/2}$, and then the chiral superspace coordinates transform as $e^{i\phi}$. If we set $\phi = \pi$, it gives an element of the R-symmetry that acts on the conifold coordinates by $z_i \rightarrow -z_i$, and on the chiral superspace coordinates θ by $\theta \rightarrow -\theta$. In the gauge theory, since A_k and B_l have R-charge 1/2, under $\theta \rightarrow -\theta$, they transform as $A_k \rightarrow iA_k$ and $B_l \rightarrow iB_l$. This agrees with the definitions (5.51), which for the dual field theory reads as $z = \text{tr} AB$ (traced because of the gauge d.o.f's), so $z \rightarrow -z$ in terms of the fields in the gauge theory, since the chiral superfields must have R-charge 1/2 to cancel the anomaly of the R-symmetry.

5.4.2 Global structure of the symmetry group

Remember that $SU(2) \times SU(2)$ is a symmetry of the superpotential, with one $SU(2)$ acting on A_k , and one on B_l . Actually, this symmetry must be $(SU(2) \times SU(2))/\mathbb{Z}_2$ where \mathbb{Z}_2 is the diagonal subgroup of the product of centers of the two $SU(2)$ 's¹⁶, i.e. $A_k \rightarrow -A_k$, $B_l \rightarrow B_l$ and $A_k \rightarrow A_k$, $B_l \rightarrow -B_l$ are equivalent. This statement is easily seen from the form of the superpotential (5.64). Another way to see this equivalence is by means of the gauge group $U(N) \times U(N)$. It contains a $U(1)_B$ subgroup that acts by $A_k \rightarrow e^{i\alpha} A_k$, $B_l \rightarrow e^{-i\alpha} B_l$. Setting $\alpha = \pi$ we get the

¹⁶See [117] for more details.

gauge transformation $A_k \rightarrow -A_k$, $B_l \rightarrow -B_l$, so the transformations $A \rightarrow -A$, $B \rightarrow B$ and $A \rightarrow A$, $B \rightarrow -B$ are indeed gauge equivalent to each other.

Then, the group acting properly is $(SU(2) \times SU(2))/\mathbb{Z}_2$, which is isomorphic to $SO(4)$ ¹⁷. But this is precisely the global symmetry group of the conifold.

5.4.3 Reflection

Invariance under $SO(4)$ rotations of the conifold, can be extended to $O(4)$, as the conifold is invariant under $z_4 \rightarrow -z_4$ with other coordinates invariant, in other words, under reflection. This transformation changes the sign of the holomorphic three-form in (5.46), $\Omega \rightarrow -\Omega$. It leads to the transformation, $\theta \rightarrow i\theta$, of the chiral superspace coordinates. Since this transformation came from an special case of R-transformation of z_4 , we say that $\theta \rightarrow i\theta$ is also an R-symmetry.

In the gauge theory side, there is a \mathbb{Z}_2 symmetry coming from the $SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$ symmetry of the conifold. It translates into the exchange symmetry between the A 's and B 's. But as A transforms under $U(N) \times U(N)$ as (N, \bar{N}) as while B transforms as (\bar{N}, N) , the exchange of A and B in the geometry corresponds to an exchange of the chiral fields accompanied by either exchange of the two factors of the gauge group or charge conjugation that exchanges the N and \bar{N} representations as can be seen from the definition of the conifold in terms of the gauge fields as the vanishing D-term in (5.59).

Υ -symmetry

There are some subtle details about the R-symmetry, $\theta \rightarrow i\theta$, from the point of view of field theory. The superpotential (5.64) is odd under exchange of $A_k \leftrightarrow B_k$. So, we must find an odd transformation that accompanies this R-symmetry in order to the superpotential is even. Such a transformation, if exists, is not unique since one could always multiply it by an ordinary symmetry of the theory. Then we could fix that symmetry by asking for a symmetry that leave fixed the lowest components of the superfields A and B .

The required symmetry is actually the naive R-symmetry that acts as we saw before, $\theta \rightarrow i\theta$. It transforms the gluino in the vector superfield as $\lambda \rightarrow i\lambda$, as we can see from (2.91). For the fermionic component ψ of the chiral superfields (see (2.73)), the transformation that leaves it invariant will be $\psi \rightarrow -i\psi$. We will call this transformation Υ . This symmetry is in agreement

¹⁷This \mathbb{Z}_2 corresponds to the reflection of the $SU(2)$'s that give the same $SO(4)$ rotation.

with the reflection of the conifold, which in turns corresponds to the exchange of A and B . Hence, the superpotential is odd under Υ , so the product of this R- and the Υ -symmetries lead to an even symmetry of the superpotential.

Possible anomalies in this new symmetry are not meaningful. In their paper [84], Klebanov and Witten argued that the relevant combined symmetry, Υ and the exchange of the two gauge group factors, do not have any anomaly.

5.4.4 Center of $SL(2, \mathbb{Z})$

The center element

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (5.70)$$

of the symmetry group of $SL(2, \mathbb{Z})$, that is a symmetry group of type IIB superstring theory, will be called w -symmetry. Then, this symmetry acts trivially on the coupling (and theta angle) of type IIB superstring theory on the conifold as long as the NS NS two-form vanish, as we have assumed so far.

This transformation is equivalent to $\Omega(-1)^{F_L}$, where Ω is the exchange of the left- and right-movers on the string worldsheet, and $(-1)^{F_L}$ multiplies left-moving worldsheet fermions by -1 [84]. If we write Q_L and Q_R for superscharges that come from left- and right-movers, then Ω acts by $Q_L \leftrightarrow Q_R$, and $(-1)^{F_L}$ by $Q_L \rightarrow -Q_L$, $Q_R \rightarrow Q_R$. Hence our w transformation acts by $Q_L \rightarrow Q_R$ and $Q_R \rightarrow -Q_L$. The unbroken supersymmetries of the spacetime are linear combinations of the form $\epsilon_L Q_L + \epsilon_R Q_R$, where $\epsilon_R = \Gamma_{0123} \epsilon_L$. Then w acts on $\epsilon_{A,B}$ by $\epsilon_L \rightarrow \epsilon_R$, $\epsilon_R \rightarrow -\epsilon_L$, or equivalently $\epsilon_L \rightarrow \Gamma_{0123} \epsilon_L$. Here Γ_{0123} ($\Gamma_{0123} = -i\Gamma_5$) acts on spinors of positive (negative) chirality as $i(-i)$. So w is an R-symmetry, since it acts on chiral superspace by $\theta \rightarrow i\theta$.

From the field theory point of view, we expect that w to act on the gauge group by charge conjugation as we saw for the reflection symmetry. But, different from reflection, this symmetry only exchange the two factors in the gauge group. So we identify w with a transformation that exchanges the two factors of the gauge group, in such a way that the N of the first $U(N)$ is exchanged with the \bar{N} of the second (and \bar{N} of the first is exchanged with N of the second) while mapping A to A and B to B .

This transformation can be seen as transposing the matrix form of A and B , which lead to a changing of sign of the superpotential. To compensate for this, we must include the action of Υ , which we encountered in analyzing the reflection $z_4 \rightarrow -z_4$.

5.4.5 Chiral operators (the field-operator map)

As a final argument in favor of the duality one could discuss the chiral operators of the field theory, namely the gauge invariant operators which have the lowest possible conformal dimension for a given R-charge. We assigned the R-charge $1/2$ to each of the A 's and B 's. Thus, the lowest possible R-charge for a gauge invariant operator is 1. We construct the corresponding chiral operator as

$$\mathrm{tr} A_k B_l. \quad (5.71)$$

It has dimension $3/2$ (since the product of them must have dimension 3), and they transform as $(2, 2)$ under $SU(2) \times SU(2)$ (vector). In general, we find that a chiral operator of positive R-charge n and dimension $3n/2$ are of the form,

$$C_L^{k_1 \dots k_n} C_R^{l_1 \dots l_n} \mathrm{tr} A_{k_1} B_{l_1} \dots A_{k_n} B_{l_n}, \quad (5.72)$$

where C_L and C_R are completely symmetric tensors. It can be seen from the equation for the critical points of the superpotential (5.64)

$$B_1 A_k B_2 = B_2 A_k B_1, \quad A_1 B_l A_2 = A_2 B_l A_1. \quad (5.73)$$

Now, remember that $A_K B_l$ was defined as the original conifold coordinate z . Then the operator (5.72) will represent traceless polynomials in z of order n .

The chiral operators (5.72) are analogous to the traceless symmetric polynomials (2.221) in $\mathcal{N} = 4$ SYM. On the SUGRA side, such operators corresponds to modes of h^α_α (the trace of the metric on the compact manifold) and the four-form gauge potential on S^5 which are described by scalar spherical harmonics (see [7, 82]). Thus, we expect that the spectrum of the chiral operators (5.72) should coincide with the spectrum of scalar spherical harmonics on $T^{1,1}$.

As we said in Chapter 4, some results about this map between fields and operators go further the scope of this work, and give the here will be a little hard. That is why we reference again to very complete papers about it that also include general results on coset spaces as [85, 95, 96, 111, 118, 119].

Chapter 6

Some comments on extensions of the Klebanov-Witten model: Towards “more realistic” gauge theories

If we think that gauge theories arising from string theories should finally have something to do with the physical world, we might be interested in seeing if some construction of strings and D-branes can give rise to a gauge theory with more “realistic” properties with respect to ones we have studied in the previous chapters.

But, what do we mean precisely by “realistic”? As we have said, the standard model of particle physics is described (in part) by a gauge theory that is neither supersymmetric nor conformal, but by QCD, which is an $SU(3)$ gauge theory which becomes strongly coupled at low-energies (see [70] for a comparison between SYM and QCD). Then it would be interesting to find a good description of this theory in the regime where the original variables of QCD (the quarks and gluons) become strongly coupled. For describing nature it would be more interesting to understand various strong coupling properties of QCD, such as confinement and chiral symmetry breaking. That is what we call a realistic gauge theory, and since QCD has shown to be useful to explain several experimental data, the aim since the original Maldacena conjecture was proposed was to find the gravitational description of the QCD [69, 112, 120]. This crusade is still ongoing.

Even though $\mathcal{N} = 4$ SYM differs in many ways from realistic theories such as QCD, it is important from theoretical perspective to understand its structure in detail. The same is true

for more general classes of supersymmetric field theories, as they can provide models where our intuitions about different phenomena can be scrutinized in detail while retaining analytic computational power. In this sense, while it is weakly coupled at high energies, but strongly coupled at low-energies developing holographic techniques for a wide range of theories is essential.

As we saw in Chapter 4, the theoretical motivation for the AdS/CFT correspondence arises by comparing the low-energy limit of a stack of N coincident D3-branes in flat space with the corresponding back-reacted gravitational solution [68]. On the one hand, such limit leads to $\mathcal{N} = 4$ SYM as the theory describing the brane dynamics. On the other hand, one finds a black p-brane supergravity background whose near horizon geometry is precisely $AdS_5 \times S^5$. Following the same reasoning as in [68], a considerable number of dualities have been discovered by studying the dynamics of D-branes in backgrounds with richer structure. A well known example is that we studied in detail in Chapter 5, D3-branes placed near conical singularities [84] whose field theoretic description turns out to be dual to string theory on $AdS_5 \times X_5$, for $X_5 = T^{1,1}$. The field theory is an $\mathcal{N} = 1$ $SU(N) \times SU(N)$ SCFT with a quadratic superpotential that exhibits $SU(2) \times SU(2) \times U(1)$ global symmetry (just as $T^{1,1}$).

In this chapter we review, without much detail, some progress in this line. The first step was given by I.Klebanov, S.S.Gubser, N.Nekrasov and A.A.Tseytlin in [121–123]. In those papers the Klebanov-Witten model is perturbed by M fractional D3-branes placed at the conifold singularity in order to break conformal symmetry.

6.1 Klebanov-Tseytlin (KT): Non-conformal field theory

Breaking of conformal symmetry by adding fractional branes to the background was proposed by S.S.Gubser and I.Klebanov in [121] inspired by a work of E.Witten [124] that suggested to construct baryon vertex in the gauge theory side (since it is an $SU(N)$ gauge theory, rather than $U(N)$)¹ of the AdS/CFT correspondence by wrapping a D5-brane over S^5 . He argued that if external quarks are regarded as endpoints of strings in AdS space, then one can construct a baryon vertex connecting N external quarks placed at the boundary points x_1, \dots, x_N , by an N^{th}

¹Finding a baryonic vertex in the $\mathcal{N} = 4$ theory does not mean that the theory has baryonic particles, or operators. Baryonic particles would appear in a theory that has dynamical quark fields, i.e. fields transforming in the fundamental representation of $SU(N)$. In their absence, we get only baryonic vertex, a gauge invariant configuration of N external charges.

order antisymmetric tensor of $SU(N)$. The configuration can be depicted as below in figure 6.1.

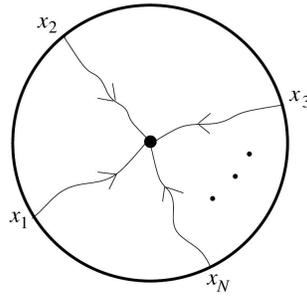


Figure 6.1: N elementary strings attached to N points $x_1 \dots x_N$ on the boundary of AdS space and joining at a baryon vertex in the interior (this image was taken from [124])

Then, if the N strings are joined at the baryon vertex in AdS_5 and their ends are in $S^5 \times R$ boundary, we could say that those strings are attached to a D5-brane wrapped on S^5 , and the N endpoints are on its worldvolume. In [124] was argued that this D5-brane is actually the baryon vertex.

Witten also studied the effect of adding D3-branes in AdS_5 , which was understood as a domain wall in AdS_5 , i.e. the flux of the R-R F_5 field strength jumps by one unit when the D3-brane wall is crossed.

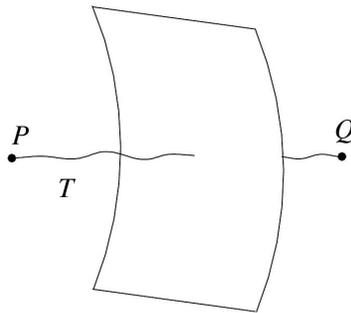


Figure 6.2: A domain wall D3-brane separates the flux units of the F_5 by one (figure taken from [124]).

Since this flux is related to the number of colors in the dual gauge theory, the effect in it is to decrease the gauge group from, say, $SU(N + 1)$ (in P) to $SU(N)$ (in Q).

As we learned in the last chapter, the coset space $T^{1,1}$ can be thought topologically as $S^2 \times S^3$ [90] and represented below

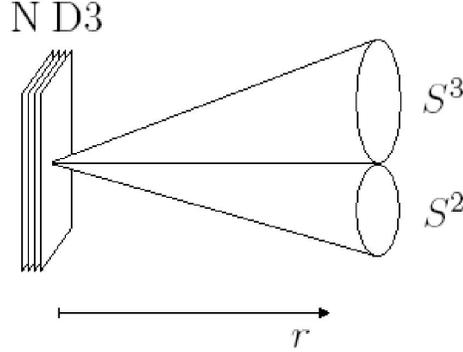


Figure 6.3: In the Klebanov-Witten solution, only a stack of N parallel D3-branes located at the tip of the conifold with base $T^{1,1} = S^2 \times S^3$ (figure taken from [125]).

Now, in [121] it is argued that wrapping D3-branes over the 3-cycle of $T^{1,1}$ is identified with baryon-like operators A^N and B^N in the field theory, where the indexes of $SU(N)$ groups are fully antisymmetrized. It is also argued that a domain wall made out of a D5-brane wrapped over a 2-cycle of $T^{1,1}$ separates a $SU(N) \times SU(N)$ gauge theory from a $SU(N+1) \times SU(N)$ gauge theory. In this sense, a D5-brane does not change the nature of the matter field, so A_k and B_l are still bifundamentals.

With these results in mind, I. Klebanov and N. Nekrasov considered in [122] to add M fractional D3-branes or equivalently M D5-branes wrapped on the S^2 of $T^{1,1}$ [126]. This equivalence can be easily understood in the inverse direction, a D5-brane when wrapped on the S^2 looks like a D3-brane as seen from the four-dimensional worldvolume point of view. The difference between these fractional D3-branes and the regular ones, is that the first cannot move off as the regular D3-branes, since there is no moduli space for them [37, 39].

Since the M D5-branes do not lose their nature when wrapping, they source a F_7 , and from the Gauss law we write the “electric charge” to be,

$$\int_{S^3} *F_7 = \int_{S^3} F_3 \sim M. \quad (6.1)$$

The presence of these D5-branes also turns on a NS-NS B_2 two-form [22], whose flux goes through the S^2 ,

$$\int_{S^2} B_2, \quad (6.2)$$

since $\tilde{F}_5 = F_5 + B_2 \wedge F_3$. In other words, presence of D5-branes lead to a nonzero F_3 which in turn leads to the existence of a NS-NS B_2 flux in order to close the supersymmetry transformations.

If M is fixed as $N \rightarrow \infty$, then the back-reaction of F_3 on the metric and the F_5 background may be ignored to leading order in N , since their contributions comes in at order $(N/M)^2$ [122]. However, we still have the NS-NS two-form. Then, adding D5-branes wrapped on the S^2 breaks the symmetry between two $SU(N)$ factors in the gauge group, which now, based on [121, 124], becomes $SU(N + M) \times SU(N)$, as we mentioned. The chiral superfields A_k and B_l are now in the $(N + M, \bar{N})$ and $(\bar{N} + \bar{M}, N)$ representations. The space of vacua will still be given by (5.59), which tells that the D3-branes are classically constrained to move on the conifold as before.

With these results, the NSVZ β function can be calculated from (2.148) to be proportional to

$$\begin{aligned}\beta(g_1) &\sim 3(N + M) - 2N(1 - \gamma_A - \gamma_B), \\ \beta(g_2) &\sim 3N - 2(N + M)(1 - \gamma_A - \gamma_B),\end{aligned}\tag{6.3}$$

where g_1 and g_2 are the corresponding coupling constants for the two gauge groups, and γ_A and γ_B are the anomalous dimensions of operators A and B . Now, if $M = 0$ (if $\gamma_A = \gamma_B = 1/4$) then the theory is conformal again. At nonzero M , however, it is impossible to make both β functions vanish (even if we allow to the anomalous dimensions of A and B to be different) and then the difference of the inverse squared gauge couplings $g_1^{-2} - g_2^{-2}$ undergoes a logarithmic RG flow.

The solution we have described above was valid only in the limit $M/N \rightarrow 0$, and did not take into account the back-reaction from the metric and the five-form. In [123], I.Klebanov and A.A.Tseytlin got the complete solution of supergravity equations (see also [113, 125] for concise revisions), and found the supersymmetric RG flow at all scales, however, since the singular conifold has no preferred scale, the warp factor is not cut off at small r , and at some point there will be a naked singularity. The model can be illustrated as

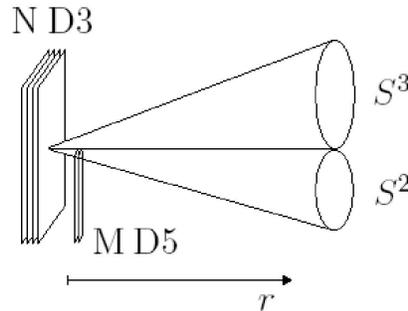


Figure 6.4: In the Klebanov-Tseytlin solution, M parallel D5-branes are placed on the top of the N D3-branes on the collapsed S^2 of $T^{1,1}$ (figure taken from [125]).

An exact supergravity solution was found for the three- and five-form fluxes, where D5-branes source M units of fluxes of F_3 . If we assume a ten-dimensional metric $4 + 6$ as in (5.21), in this case we get [94, 123, 125]

$$e^{-4A(r)} \sim \frac{\pi g_s}{r^4} \left(N + \frac{3g_s M^2}{2\pi} \ln r/r_0 + \frac{3g_s M^2}{8\pi} \right), \quad (6.4)$$

where r_0 is a UV scale and,

$$\mathcal{K}(r) = \int_{T^{1,1}} F_5 \sim N + \frac{3g_s M^2}{2\pi} \ln r/r_0, \quad (6.5)$$

represents the effective flux of F_5 through $T^{1,1}$. Notice that when $M = 0$ we recover the solution when $r \rightarrow 0$ in the Klebanov-Witten model in the same way we did it gauge theory side. This flux may completely disappear at some value of $r = r_{N/M}$ given by

$$r_{N/M} = r_0 \exp\left(-\frac{2\pi N}{3g_s M^2}\right). \quad (6.6)$$

This behavior is related to the fact that $\int_{S^2} B_2$ is no longer a periodic variable in the supergravity solution once the M fractional D3-branes are introduced [111, 127]. Klebanov and Tseytlin noticed that the RG flow described by this background seems to enjoy a cascade, where $\mathcal{K}(r) \rightarrow \mathcal{K}(r') = \mathcal{K}(r) - M$ as $r \rightarrow r' = \exp\left(-\frac{2\pi}{3g_s M}\right) r$. This comes from the non-conservation of the \tilde{F}_5 flux $d\tilde{F}_5 = H_3 \wedge F_3 \neq 0$. Given a radius r_0 the flux units of F_5 corresponds to N , we can see easily that when we move to r_1 ,

$$r_1 = r_0 \exp\left(-\frac{2\pi}{3g_s M}\right), \quad (6.7)$$

the flux units changes from $N \rightarrow N - M$ [111, 127, 128]. In general, the flux units will changes from $N \rightarrow N - kM$, with $k \in \mathbb{Z}$. Then

$$r_k = r_0 \exp\left(-\frac{2\pi k}{3g_s M}\right). \quad (6.8)$$

Thus we say that the continuous logarithmic variation of $\mathcal{K}(r)$ is related to a continuous reduction in the number of degrees of freedom as the theory flows to the IR.

The metric has also a naked singularity of repulson-type at $r = r_s$, where $\exp(-4A(r)) = 0$. If we write,

$$e^{-4A(r)} = \frac{L^4}{r^4} \ln(r/r_s), \quad L^2 \sim M. \quad (6.9)$$

At the radial position where the effective flux vanishes (and the warp factor changes sign), gravity becomes repulsive [129]. The singularity of the supergravity solution appears due to the fact that

there are M units of F_3 flux through the S^3 . The flux does not depend on the radius of the sphere, therefore, to maintain the constant flux (M), the energy density F_3^2 must become infinite at zero radius (when S^3 shrinks to zero). Now, the metric (5.21) looks like [94, 111, 127, 130]

$$ds^2 = \frac{r^2}{L^2 \sqrt{\ln r/r_s}} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2 \sqrt{\ln r/r_s}}{r^2} dr^2 + L^2 \sqrt{\ln r/r_s} ds_{T^{1,1}}^2. \quad (6.10)$$

From the gauge theory point of view, since $r \sim \Lambda$. When we flow to the IR limit ($r \rightarrow 0$), the F_5 flux reduces to zero and eventually becomes negative. So, the “cascade” downwards must stop, since both the gravity and field theory sides breaks down. This solution cannot be used to extract IR properties of the dual field theory, and should be modified to resolve the naked singularity in the metric.

Thus, once the effect of the M fractional branes is accounted for, the supergravity metric can no longer be $AdS_5 \times S^5$. The dual field theory will not inherit conformal symmetry from the geometry of the space. In the gravity side, the F_5 flux will not be neither conserved nor quantized. Then we see that the fractional branes introduce a logarithmic running of the couplings, and thereby explicitly breaking conformality.

6.2 Klebanov-Strassler (KS): The duality cascade

The background introduced by Klebanov and Tseytlin in [123] becomes invalid in the deep IR. It was conjectured that the strong dynamics of the gauge theory should somehow resolve this singularity. In a seminal paper [130], I.Klebanov and M.Strassler showed that this conjecture indeed is correct. Their solution not only provides a mechanism to resolve and avoid the singularity for $r < \tilde{r}$, but also gives a geometrical realization of confinement, as would be like to obtain from QCD-like theories! Let us explore briefly how this occurs.

In the gauge theory side, the central idea is based on the Seiberg duality [114, 131], that relates two different gauge theories in the IR limit. Given an $\mathcal{N} = 1$ $SU(N)$ gauge theory, with F of the fundamental representation N , and F flavors of the antifundamental representation \bar{N} , strongly coupled. There exists a $\mathcal{N} = 1$ $SU(F - N)$ gauge theory being also asymptotically free in the UV. And flowing to a non-trivial IR fixed point, just like its cousin when $3N/2 < F < 3N$, the conformal window that we mentioned in the last chapter. Then for our $SU(N + M) \times SU(N)$ gauge theory with $2N$ flavors for the first factor, the Seiberg duality gives,

$$SU(N + M) \times SU(N) \rightarrow SU(N) \times SU(N - M) \rightarrow SU(N - M) \times SU(N - 2M) \rightarrow \dots, \quad (6.11)$$

where the first factor of each gauge group would get strongly coupled faster in the IR than the second one. So the second factor on the left will not govern the IR behavior, but it has $2N$ flavors (N from A_1 and B_1 , and N from A_2 and B_2). We can see this by vanishing (6.3), from we find the anomalous dimensions for each gauge factor,

$$\begin{aligned}\gamma_1 &= -\frac{1}{2} - \frac{3M}{2N} < 0, \\ \gamma_2 &= -\frac{1}{2} + \frac{3M}{2(M+N)} > 0.\end{aligned}\tag{6.12}$$

Notice also the deviation of the values for the γ 's due to the presence of the M wrapped D5-branes. In general the transition from the k^{th} of the cascade to the $(k+1)^{\text{th}}$ is

$$SU(N+M-(k-1)M) \times SU(N-(k-1)M) \rightarrow SU(N-(k-1)M) \times SU(N-kM),\tag{6.13}$$

for $k = (1, 2, \dots)$. If we calculate the symmetrized (i.e. averaged) anomalous dimension for the first term of the cascade $SU(N+M) \times SU(N)$ and the k^{th} term can be read off from (6.3) [132],

$$\gamma = -\frac{1}{2} - \frac{3M^2}{4N(N+M)} \rightarrow -\frac{1}{2} - \frac{3M^2}{4(N-kM)(N-(k-1)M)}.\tag{6.14}$$

Notice that as $k \rightarrow \infty$, the theory comes closer and closer to the Klebanov-Witten model ($\gamma = -1/2$), and our understanding of the latter should prove conceptually and technically useful. The cascade should stop at the point, where $N - kM$ becomes zero or negative.

Let us study an special case, $N = kM$. So the cascade will end as

$$\dots \rightarrow SU(M) \times SU(0) = SU(M).\tag{6.15}$$

We reached pure $\mathcal{N} = 1$ $SU(M)$ SYM theory with no flavors, whereupon we get confinement, chiral symmetry breaking, etc [89]. The inverse cascade never stops. In a sense, the UV limit is a $SU(\infty) \times SU(\infty)$ gauge theory [133].

From the supergravity point of view, at the end of the cascade we will have the well-known resolution of the geometry. This changes the compactification geometry from that of the conifold to the deformed conifold with a finite size of S^3 in the IR. In this limit, both B_2 and F_5 go to zero at $r = 0$ while F_3 remains non-zero and is spread over the S^3 . This is understood as there are no regular D3-branes, and the D5-branes wrapping the collapsed (as $r = 0$) S^2 are smeared or dissolved over the S^3 , leaving no branes remaining in the IR limit.

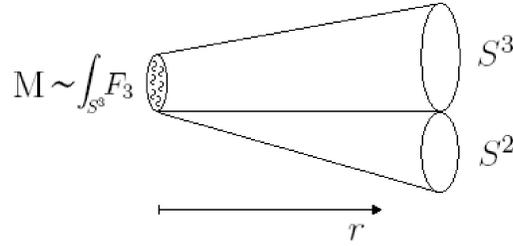


Figure 6.5: At the end of the duality cascade, all the D3- and D5-branes have been dissolved into fluxes on the geometry, resulting in a finite size S^3 at the tip of the conifold (figure taken from [125]).

Let us summarize it, the full Klebanov-Strassler solution interpolates the Klebanov-Tseytlin solution in the UV, where branes wrap the conifold as a superstring solution and generate the gauge group on their worldvolume, and the deformed solution in the IR limit, where branes dissolve into the geometry, reducing the gauge group and leaving us with a pure supergravity solution without branes as a source.

6.3 Pando Zayas-Tseytlin (PT): The resolved conifold

It is of obvious interest to explore further the class of backgrounds which have similar D3-branes on conifold-type structure. There are two natural ways to smooth the singularity in the apex of the conifold at $r = 0$: Making the S^3 finite sized (deforming) [123], or making the S^2 finite sized (resolving) [134].

Similarly to the work of Klebanov and Strassler [130], Pando Zayas and Tseytlin [134] showed a way to complement the deformed conifold in [130], exploring a similar case for the S^2 factor of the base $T^{1,1}$. The type IIB supergravity solution they found coincides with the original background of [123] for large r but has somewhat different (though still singular) small r (IR) behavior. To make short the idea, Pando Zayas and Tseytlin wrapped M D5-branes on a non-singular S^2 , as

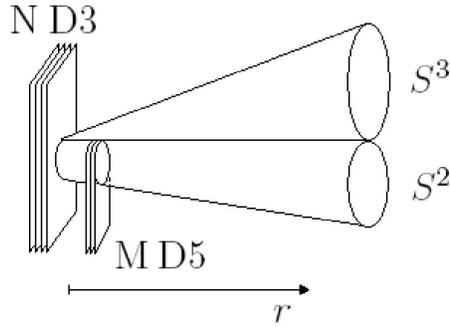


Figure 6.6: The PT solution is a natural extension of the KT solution, based on the KS one, where in this case, the non-singular S^2 is wrapped by D5-branes (fractional D3-branes) (figure taken from [125]).

The asymmetry in the resolved geometry plays a crucial role as it determines the asymmetry in the flux on the S^2 and is the source of supersymmetry breaking.

All three geometries we have studied, the KW solution in Chapter 5, KS and PT solutions, share the base $S^2 \times S^3$. So we can begin with the KS (the deformed) solution at the bottom of the cascade, pass through the singular solution of the KW solution, to attain the PT (the resolved) solution by blowing up the S^2 . This is known as conifold transition [135]. The geometric transition between conifold geometries is an example of a string theory duality between compactifications on different geometrical backgrounds.

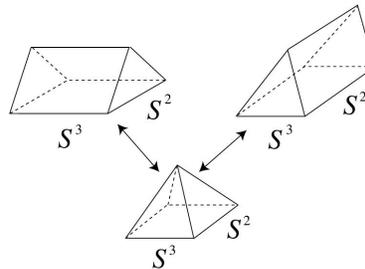


Figure 6.7: The conifold transition. All three geometries share the same base $S^2 \times S^3$, but the S^3 of the deformed conifold remains finite. In the conifold transition it can be shrunk to zero size to give the singular conifold, from which blowing up the S^2 gives the resolved conifold (this figure was taken from [20]).

So far we have considered theories without flavors. It is interesting to generalize these pure “glue” theory to a gauge theory with flavor degrees of freedom. In the gravity side this can be done by adding D7-branes that fill the four dimensional space tangent to the D3-branes, and

wrap four dimensions in the conifold. Since we are not going to discuss these results here, we recommend some interesting further references in this subject in [136–142].

Conclusions

In this thesis we have studied in reasonable detail some general aspects of the original AdS/CFT correspondence, addressed towards extensions that can be phenomenologically interesting, in order to obtain the gravitational dual of QCD-like theories that are neither conformal nor maximally supersymmetric as was $\mathcal{N} = 4$ SYM.

As was mentioned at the beginning, this work intended to be self-contained so we included some reviews on supersymmetric gauge theories and superstring theories. These previous chapters are very important since in there we explained for example, irreducible representations for $\mathcal{N} \geq 1$ supersymmetry that allowed to construct the field content for supersymmetric theories, and the superspace formalism that simplifies the construction of supersymmetric lagrangians. We also studied conformal symmetries (and their corresponding supersymmetric extension), in order to develop at the end the $\mathcal{N} = 4$ SYM theory, which is the superconformal field theory dual by means of AdS/CFT to a particular supergravity theory, type IIB on $AdS_5 \times S^5$. Because of this, we needed to include a concise chapter on superstring theory focused at the end on its low-energy limit, supergravity. There we understood the need to consider new higher-dimensional objects called Dp-branes where open strings can end, and also how a gauge theory lives on its worldvolume. We also solved with detail the Einstein equations for type IIB supergravity in presence of 5-form RR flux and give some basics on anti-de Sitter space as a way to complete the necessary background of the gravitational side of the correspondence.

The Maldacena correspondence proposed the total equivalence between $\mathcal{N} = 4$ SYM in four dimensions and type IIB string theory, not only supergravity. This complete duality has not been proved so far, since the nonperturbative side of string theory is still not calculated.

Forty-one years after the discovery of the asymptotic freedom, QCD, the theory of the interactions between quarks and gluons, remains a challenge. There is no analytic, truly systematic methods with which to analyze its nonperturbative limit. The aim of this work was to study,

in the spirit of the AdS/CFT, models that could give some light in this way. The Klebanov-Witten model was one of the firsts attempts that considered a less symmetric transversal space, a conifold \mathcal{C} , that also has a conical singularity. Placing N D3-branes at the tip of this space led to a supersymmetric $\mathcal{N} = 1$ field theory with gauge group $SU(N) \times SU(N)$ living on their worldvolume. This gauge group was inherited from the orbifold theory. In the stringy side this corresponded to a solution of type IIB supergravity on $AdS_5 \times X_5$, where X_5 is the base of the conifold, that in general was not S^5 , as in the original Maldacena’s work. As the maximally supersymmetric case, the gauge theory inherited the conformal symmetry of the AdS part, so the field theory on the worldvolume was superconformal. The five-dimensional space, X_5 , was the coset $T^{1,1}$, that topologically can be expressed as $S^2 \times S^3$. This way to write the space allowed to modify the model by adding M fractional D3-branes (which actually were D5-branes wrapped over the S^2), the called Klebanov-Tseytlin model broke the conformal symmetry of the gauge theory to $SU(N + M) \times SU(N)$. Since the variable r of the AdS part of the ten-dimensional metric corresponds to the energy scale, and since in this model the metric has a logarithmic dependence of r in the warp-factor, the IR limit of the gauge theory was not well defined in its dual description. So we broke conformal symmetry but did not obtain a suitable description in this limit. This apparent singularity was solved in the Klebanov-Strassler model, by performing consecutive Seiberg duality transformations that reduce the gauge group when r became smaller. This process in the field theory could be understood as a decreasing of the number of regular D3-branes. In the supergravity side, we had a decreasing of the flux units of the NS-NS two-form as r increases. This duality, known as duality cascade, led to an $\mathcal{N} = 1$ $SU(M)$ gauge theory without flavors.

What happened at the end of the cascade was best illustrated from the gauge theory side. In the UV, D5-branes wrapped the conifold as a superstring solution and generate the gauge group on their worldvolume. As we moved towards the deep IR, the D3- and D5-branes dissolved into the geometry, deforming the conifold, reducing the gauge group. Then, at the end of the cascade we got a finite size S^3 , there is no longer a singularity.

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