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Instituto de Física

Estudos sobre teorias quânticas de campos
integráveis em duas dimensões

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Studies in two-dimensional integrable quantum field theories

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To my family for supporting me, no matter the distance.

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Resumo

Esta dissertação de mestrado consiste de uma revisão sobre teorias quânticas de campos integráveis em duas dimensões, explorando tanto aspectos clássicos como aspectos quânticos dessas teorias munidas de infinitas cargas conservadas. Em nível clássico, consideramos uma teoria de supercampos escalares em duas dimensões com superpotencial arbitrário. Através da imposição da não produção de partículas a nível-árvore, restringimos a forma das interações admissíveis, recuperando uma extensão supersimétrica do modelo de *sinh-Gordon*, o qual é provado ser integrável não somente através da obtenção do conjunto infinito de cargas conservadas, mas também através de *S-matrix bootstrap*. Ainda no nível clássico também mostramos uma profunda relação entre as *Toda theories* e os *conformal minimal models*, a qual se estende para nível quântico onde obtemos uma família de fluxos de renormalização entre os *unitary conformal minimal models* conhecida como *staircase model*.

palavras-chaves: Teoria quântica de campos; integrabilidade; supersymmetria; duas dimensões; espalhamento

Abstract

This master thesis is an overview of integrability in two-dimensional field theories. We explore both classical and quantum aspects of these theories which are characterized by infinitely many conserved charges. At the classical level, we consider a theory of scalar superfields in two dimensions with arbitrary superpotential. By imposing no particle production in tree-level scattering, we constrain the form of the admissible interactions, recovering a supersymmetric extension of the sinh-Gordon model. This model is proven to be integrable not only by explicitly finding the infinite set of conserved charges but also via the S-matrix bootstrap. We also show a deep relation between Affine Toda theories and conformal minimal models, that extends to the quantum level, where we find a family of integrable renormalization group flows between the unitary conformal minimal models, known as the staircase model.

keywords: quantum field theory; integrability; supersymmetry; two dimensions; scattering

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Chapter 1

Introduction

The idea of constraining physical observables through a minimal set of indisputable principles is what is commonly referred to as bootstrap philosophy. It showed up first in [1], which is nowadays referred to as integrable bootstrap, and was extended to different frameworks, the most well known being the conformal bootstrap [2] and the S-matrix bootstrap [3, 4]. However, this bootstrap approach faded away during the last decades of the last century, when the focus shifted to different approaches to quantum field theories.

Recent years have witnessed a revival [5] of the old S-matrix bootstrap program, which is the attempt to solve massive quantum field theories solely from imposing basic consistency principles, such as analyticity, unitarity, and crossing symmetry of their scattering matrices.

The general approach in physics revolves around perturbation theory. One considers the problem as a solvable theory, deformed by a perturbation. The strength of this perturbation is quantified by its coupling constant λ . In weakly coupled theories ($\lambda \ll 1$), it is consistent to do an expansion in λ up to some order and analyze the problem. But this approach clearly fails for strongly coupled theories ($\lambda \gtrsim 1$) since the λ -expansion loses its meaning. An important feature of all bootstrap methods is that it can be applied regardless of the strength of the coupling constant. The bootstrap approach is often the only analytic tool available to study strongly coupled theories.

One very simple arena where this approach can be carried out is the case of two-dimensional integrable theories [6]. These theories have an infinite set of conserved charges which severely constrain their scattering behavior. Namely, these theories are bound to have no particle production, conservation of the full set of momenta and factorization of the S-matrix.

These properties allow integrable theories to be of interest in a variety of ways. In the S-matrix bootstrap, integrable models are bound to saturate unitarity for the two-particle scattering [5]. This feature also allows for the evaluation of the full quantum S-matrix in some cases.

As first observed by Zamolodchikov, integrable models are deeply related to conformal field theories [9]. This relation is also present in an algebraic way by the coset correspondence between integrable theories and conformal minimal models [10]. The minimal models $\mathcal{M}_p$ are conformal field theories with central charge $c(p) \leq$

1 and a finite number of primary operators $\phi_{r,s}$, more precisely, a finite number of irreducible representations of the Virasoro algebra.

The minimal model $\mathcal{M}_p$ perturbed by the least relevant operator $\phi_{1,3}$ can be studied via conformal perturbation theory in the limit $p \gg 1$, since in this case this operator is just slightly relevant (its conformal weight being $h \sim 1 + 1/p$). The perturbation by this operator generates a flow to a new minimal model $\mathcal{M}_{p-1}$, [11]. It turns out that this RG flow is integrable [13]. Therefore, it is possible to apply the integrable machinery [14] in the form of the Thermodynamical Bethe Ansatz (TBA). Since these integrable results do not rely on perturbation results, it is possible to show that the perturbation promotes a flow that passes close to all minimal models $\mathcal{M}_p$ until it flows towards a massive IR theory. This is known as the staircase model [14].

If one considers the superconformal minimal models $\mathcal{SM}_p$ to begin with, in the limit $p \gg 1$, conformal perturbation theory [12] also shows a RG flow connecting two superconformal minimal models. This flow is also integrable and the TBA formalism allows for the evaluation of the RG flow for all values of p giving a supersymmetric generalization of the staircase model. This generalization can be carried further. Using the coset correspondence between integrable models and CFTs it is possible to show that the staircase models can be extended even for non unitary minimal models [15, 16, 17].

Besides the usual two-dimensional theories studied in the integrable bootstrap approach, there is another interesting four-dimensional theory, the $\mathcal{N} = 4$ Super Yang-Mills. This theory has the maximal amount of supersymmetry with four pairs of supercharges and also a color group $SU(N)$. It turns out that in the planar limit (large N) the problem of evaluating some correlation functions¹ can be identified with computing the energies of a certain spin-chain [18, 19]. It turns out that this spin chain is integrable and therefore solvable. The machinery used to solve these integrable systems is the Bethe ansatz [20] which not only gives the tree-level values for these correlators but also higher loop corrections.

In conformal field theories, the symmetries impose that one-point functions must vanish (except for the identity operator). However, introducing a defect in the theory breaks the conformal symmetry allowing for non vanishing one-point functions to appear. One example of defect CFT is the $\mathcal{N} = 4$ SYM theory with a co-dimension one defect. Analogously to the non-defect case [19], the evaluation of some one-point functions can be mapped to eigenvalue problems of the integrable spin chain [21].

Perhaps the most important aspect of the study of $\mathcal{N} = 4$ SYM is related to the AdS/CFT correspondence [22]. Briefly, this conjecture states a correspondence between a conformal field theory living in the boundary with a quantum gravity theory that lives in the Anti-de Sitter bulk. More precisely, in our case, this is a correspondence between the $\mathcal{N} = 4$ SYM in the four-dimensional boundary and the type IIB string theory on the product space $AdS_5 \times S_5$ on the bulk.

The AdS/CFT correspondence is a strong/weak duality, meaning that a strongly coupled CFT in the boundary is associated with the weakly coupled theory in the bulk. And it is precisely this aspect that makes the duality difficult to explore.

¹More precisely, the problem of evaluating the anomalous dimensions of single trace operators composed of scalar fields.

However, integrability allows evaluations in the strong coupling limit, thus allowing the test of aspects of the conjecture not only at tree-level but at higher loop orders. It is also interesting to note that the results of dCFTs obtained by [21] can also be matched with string theory computations [23] testing the gauge-gravity duality in a situation where both supersymmetry and conformal symmetry are partially broken.

Therefore, it is important to look at alternative tools to find integrable models. One way of doing this is by following the bootstrap philosophy and posing the question: from a general theory with all possible interactions (respecting a given symmetry) how does integrability impose constraints in the possible interactions that can appear?

This question can be answered, in some cases, by recursively imposing the integrability constraints of absence of particle production and conservation of the full momenta set, as proposed by [6] and extended by [7, 8]. This process results in a closed formula for all the couplings of the theory which can be resummed into known potentials. For example, in theories with one scalar field one finds the sinh-Gordon and Bulloch-Dodd models, and for theories with one scalar superfield one finds the supersymmetric sinh-Gordon as the only model arising. The process of including more than one field should lead to more interesting results as observed by [7], opening up possibilities to find overlooked integrable theories.

This thesis starts by defining integrability, studying its canonical examples, and how they are connected with conformal field theories. Later on, the bootstrap ideas start to appear where we see how integrable theories emerge from the S-matrix bootstrap and how one can actually find integrable Lagrangians from the absence of particle production.

Chapter 2 introduces integrability in the context of two dimensional quantum field theories, showing not only the factorization of the S-matrix, but also the key aspect of absence of particle production and conservation of the whole set of momenta.

Chapter 3 further explores the canonical examples of integrable quantum field theories: sinh-Gordon (dual to the Thirring model) and Bulloch-Dodd. These are examples of a more general class of integrable theories, the Toda theories.

Chapter 4 relates integrable models to perturbed conformal field theories and how one can use thermodynamical Bethe ansatz to obtain an integrable RG flow that passes by the critical points associated with the unitary minimal models.

Chapter 5 treats one of the most important aspects of an integrable field theory, the two particle scattering S-matrix, studying its general properties and how the integrable models are related to the saturation of the unitary bound.

Chapter 6 follows the same bootstrap ideas, using the absence of particle production of integrable theories in different cases, to find the constraints imposed on the possible interactions of the theory.

Appendices intend to make the thesis self-consistent by presenting some aspects of conformal field theory, supersymmetry and the thermodynamical Bethe ansatz.

Chapter 2

Integrability

The S-matrix of a general relativistic quantum field theory is a complicated object. Even considering the two-dimensional case (1+1), we have extremely nontrivial structures arising. There are, however, some specific theories where the structure of the S-matrix simplifies drastically, meaning it becomes factorized [1].

The factorization of the S-matrix was first observed in the non-relativistic one dimensional scattering of particles interacting through the δ -function potential [24] and after that observed for various theories, especially in the study of solitons scattering, using inverse scattering method [25]. A common feature of all the studied systems is the presence of an infinite set of conservation laws.

The modern definition of an integrable theory is a relativistic quantum field theory possessing an infinite number of conserved charges. It is precisely these charges that oblige the factorization of the S-matrix.

To explicitly obtain this factorization as well other useful properties we begin a careful analysis of the scattering theory. In two-dimensions it is easier to adopt light-cone coordinates, that on-shell ($p^2 = m^2$) have only one free parameter (the rapidity θ), defined as

$$p = (p^0 + p^1, p^0 - p^1) = m (e^\theta, e^{-\theta}). \quad (2.1)$$

It is easy to build an n -particle asymptotic state:

$$|A(\theta_1)A(\theta_2)\dots A(\theta_n)\rangle \quad (2.2)$$

where $A(\theta_i)$ represents a particle traveling with rapidity θ_i .

The theories here considered are massive and the interactions have a short range, which allows us to properly define a wave function for a collection of particles as just free particles traveling together, except during the scattering when these wave packets overlap. This is not important for the derivation, but it plays an important role in numerous cases that uses solutions via the Bethe ansatz, as in [26] and [27].

The *in* state cannot have further interactions for $t \rightarrow -\infty$, so the particles must be velocity-ordered, meaning the fastest particle is in the right, therefore

$$|in\rangle \equiv |A(\theta_1)A(\theta_2)\dots A(\theta_n)\rangle \text{ with } \theta_1 > \theta_2 > \dots > \theta_n \quad (2.3)$$

and by the same argument the *out* state has the form

$$|out\rangle \equiv |A(\theta_1)A(\theta_2)\dots A(\theta_n)\rangle \text{ with } \theta_1 < \theta_2 < \dots < \theta_n. \quad (2.4)$$

The S-matrix provides precisely the mapping between the *in*-state and the *out*-state, that for a two-particle *in*-state can be written as

$$A(\theta_1)A(\theta_2) = \sum_{n=2}^{\infty} \sum_{\theta'_1 < \dots < \theta'_n} S(\theta_1, \theta_2; \theta'_1, \dots, \theta'_n) A(\theta'_1) \dots A(\theta'_n), \quad (2.5)$$

with the possible rapidities constrained by the conservation of energy and momentum:

$$e^{\pm\theta_1} + e^{\pm\theta_2} = e^{\pm\theta'_1} + \dots + e^{\pm\theta'_n}. \quad (2.6)$$

So far these results are completely general for a 1+1 quantum field theory, but the integrable theory has infinitely more conserved charges. These charges can be envisaged as operators transforming in higher representations of the 1+1 dimensional Lorentz group:

$$Q_s |A(\theta)\rangle = q^{(s)} e^{s\theta} |A(\theta)\rangle. \quad (2.7)$$

The operators Q_s considered here are integrals of local densities, meaning that the action in a multiparticle wave packet is additive:

$$Q_s |A(\theta_1) \dots A(\theta_n)\rangle = (q^{(s)} e^{s\theta_1} + \dots + q^{(s)} e^{s\theta_n}) |A(\theta_1) \dots A(\theta_n)\rangle. \quad (2.8)$$

These are local conserved charges that commute (they are an involution) and are simultaneously diagonalized by the basis of asymptotic states introduced before.

It is important to discuss now the Coleman-Mandula theorem [28], which states that in more than two dimensions the most general invariant group of a non trivial field theory is the product of the Poincaré group¹ and an internal symmetry group. The proof shows that an infinite number of conserved charges would imply that individual momenta are conserved and that the scattering would consist only of an exchange of quantum numbers, meaning that the S-matrix would not depend analytically on the scattering momenta. However, in two dimensions the situation changes drastically, since the scattering angle can be only zero or π and the analyticity requirement at the base of the theorem no longer exits.

Consider an amplitude of a process of n particles going into m particles. Since the charge Q_s is conserved we must have a new conservation equation

$$q^{(s)} e^{s\theta_1} + \dots + q^{(s)} e^{s\theta_n} = q^{(s)} e^{s\theta'_1} + \dots + q^{(s)} e^{s\theta'_m}, \quad (2.9)$$

that reduces to the energy momentum conservation, associated with the spin one operator ($s = 1$).

If there is a conserved charge Q_s for all s , there will be an infinitely long set of equations for the rapidities and the only way of satisfying all these equations, for generic *in*-momenta, is by the trivial solution:

$$n = m ; \quad \theta_i = \theta'_i ; \quad i = 1, \dots, n. \quad (2.10)$$

This result constrains the scattering behavior of the theory: it is not possible to have particle production, since the *in*-state and the *out*-state must have the same number of particles.

¹The case with graded Lie groups (supersymmetry) has been studied later in [29].

Furthermore, not only the energy and momentum must be conserved, but the whole set of rapidities must remain the same, meaning that the set of *in* momentum and energy is the same as the *out* set. Translating this in momentum transference during the interaction, we have also a strange behavior: the particles during a scattering can at most swap values of momenta.

Due to the conservation of infinitely many charges, we end up with no particle production and also conservation of the set of moment. What is left to see is that these charges imply factorization of the S-matrix.

Figure 2.1: Scattering of two particles going into m particles with the interaction points t_{12} where the initial particles (1 and 2) interact and t_{23} where the slowest incoming particle (2) and the fastest outgoing particle (3) interact.

We start by considering a $2 \rightarrow m$ scattering with the particles labeled as in Figure 2.1. In the far past the two incoming particles are freely propagating, which allows us to extend their world-lines up to a contact point denoted by t_{12} . By also extending the world-line of the fastest particle (labeled by 3) we also have a contact point with the slowest incoming particle (labeled by 2) denoted by t_{23} .

Since the incoming particles do not interact until their wave packets overlap, causality requires that

$$t_{12} \lesssim t_{23}, \quad (2.11)$$

since Q_s is conserved, it commutes with the S-matrix,

$$\langle out|S|in\rangle = \langle out|e^{i\alpha Q_s} S e^{-i\alpha Q_s}|in\rangle. \quad (2.12)$$

The operator $e^{-i\alpha Q_s}$ can be used to rearrange the initial and final configurations. Since the amplitude remains the same, one can use this rearrangement of particles to make the interactions happen in a two-by-two fashion. The final amplitude can be represented as a product of two-particles scattering. This can shine light over the Coleman-Mandula theorem, since in dimensions other than (1+1) it is possible to rearrange the particles in such a way that the particles never meet and the interaction never happens, implying a trivial realization of the S-matrix.

To better understand how it is possible to rearrange the particles, consider a wave function for a single-particle state

$$\psi(x) \propto \int_{-\infty}^{\infty} dp e^{-a^2(p-p_*)^2} e^{ip(x-x_*)}, \quad (2.13)$$

which can be easily interpreted as particle with momentum approximately p_* and position approximately x_* . Acting in this state with an operator like Q_s will give a momentum dependent phase $\phi(p)$ in the wave function

$$\tilde{\psi}(x) \propto \int_{-\infty}^{\infty} dp e^{-a^2(p-p_*)^2} e^{ip(x-x_*)} e^{-i\phi(p)}. \quad (2.14)$$

Taylor expanding the phase around the point p_* allows an easy interpretation of the new wave function: the particle still has approximately momentum p_* , but approximately position $x_* + \phi'(p_*)$. Note also that this wave function formulation also holds for a multiparticle state, provided that the component particles are well separated.

If we act with the operator $e^{-i\alpha Q_s}$ the phase factor is just $\phi(p) = \alpha p^s$. The usual case where we only have momenta conservation ($s = 1$) does not allow any simplification since all particles are shifted by the same amount α . However, in our theory with infinitely many conserved charges we can use this relation with $s \neq 1$ allowing the construction of a personalized shift for each particle and transforming the whole process in subsequent scattering of two particles.

One important comment here is that, although the integrability definition requires infinitely many conserved charges, to achieve factorization of the S-matrix one needs only two extra charges with higher spin, as shown by [30]. In fact, as discussed in [6], in parity symmetric theories one only needs to find one charge Q_s , because there will be another parity conjugate conserved charge Q_{-s} .

To see a more concrete example of this factorization, consider the case with three incoming wave packets, where Figure 2.2 represents the three possible cases when there is no interactions in the model.

Figure 2.2: Possible configurations of three incoming wave packets.

Switching the interaction back on again, we already understand cases (a) and (c), they are not different in any essential way, the wave packets overlap pairwise as described by a series of the two-particle S-matrix (factorization) and there is no particle production. The different case (b) in general would give something new, however, using the operator $e^{-i\alpha Q_s}$ as explained before, the three-particle S-matrix can be converted into one of the other cases.

There is even a new constraint arising from this factorization. Since it is possible to change the amplitude in case (b) into either amplitudes shown in (a) or (c), there is a consistency condition that the factorization must obey, the so called Yang-Baxter equation [31]:

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12} . \quad (2.15)$$

So far everything has been done classically, but in order for a theory be quantum integrable the conserved charges must survive quantization in the sense that the not only must stay conserved (through Ward identities), but they also need to be in convolution, meaning they commute with each other. There is no guarantee that a classically integrable system will remain integrable after quantization. This also means that the fulfillment of the Yang-Baxter equation, at the classical level, is not sufficient to claim integrability in the quantized theory.

Chapter 3

Examples of integrable theories

As discussed above, integrability is present in various system, from simple one dimensional spin chains, passing through string sigma models, to $\mathcal{N} = 4$ SYM in the planar limit. In this section, we shift the attention to theories with only one scalar field, the Sinh-Gordon and Bullough-Dodd models.

These models can be viewed as perturbed conformal field theories. They match the conformal minimal models if the coupling constants are analytically continued to complex values. In fact these two theories are part of a more general class of integrable theories, called Toda field theories [32], which will be briefly discussed in the last section.

Sinh-Gordon

The Sinh-Gordon action is given by

$$S_{ShG} = \int d^2x \left(\frac{1}{16\pi} (\partial_\mu \phi)^2 + \frac{\mu^2}{g^2} \cosh \left(\frac{g}{\sqrt{8\pi}} \phi(x) \right) \right), \quad (3.1)$$

which has a discrete symmetry $\mathbb{Z}_2$, that is, the Lagrangian is invariant under the transformation $\phi \rightarrow -\phi$.

Perhaps the more interesting way of thinking about this action. Is considering a conformal action S_0 as the Liouville action, see e.g [33], perturbed by S_I , with a deformation given by a relevant vertex operator

$$S_0 = \int d^2x \left(\frac{1}{16\pi} (\partial_\mu \phi)^2 + \frac{\mu^2}{2g^2} e^{\frac{g}{\sqrt{8\pi}} \phi} \right), \quad S_I = \int d^2x e^{-\frac{g}{\sqrt{8\pi}} \phi}. \quad (3.2)$$

The Liouville theory is classically conformal invariant, but the correct quantization is achieved by introducing a charge at infinity

$$Q = \frac{1}{2} \left(\frac{g}{\sqrt{8\pi}} + \frac{\sqrt{8\pi}}{g} \right), \quad (3.3)$$

which changes the UV central charge

$$c_{uv} = 1 + 24Q^2. \quad (3.4)$$

This charge at infinity also changes the conformal weight of the vertex operators, the plus exponential in S_0 acquires a conformal weight equals one, and that is precisely what allows the action of the Liouville theory to be conformally invariant at the quantum level.

Just to be complete, one can also write the Liouville theory with the minus exponential inside S_0 and the plus exponential as the relevant operator, in that case the charge at infinity has a different expression but qualitatively the results are the same.

In order to get the usual equations of motion for the field $\phi(x)$, one needs to rescale the field: $\phi(x) \rightarrow \frac{g}{\sqrt{8\pi}}\phi(x)$ (take also $\mu = 1$) and switch to light-cone coordinates $\sigma = \frac{1}{2}(x - t)$, $\tau = \frac{1}{2}(x + t)$, obtaining

$$\partial_\sigma \partial_\tau \phi = \sinh \phi. \quad (3.5)$$

A conserved charge $\mathcal{Q}_s$ will exist if there is a conserved current $\partial_\mu J_s^\mu = 0$. In light cone coordinates one can define:

$$J_s^0 = T_{s+1} + \Theta_{s-1} \quad \text{and} \quad J_s^1 = T_{s+1} - \Theta_{s-1}, \quad (3.6)$$

where $T_s[\phi]$ and $\Theta_s[\phi]$ are densities that satisfy the conservation condition in light-cone coordinates

$$\frac{\partial}{\partial \sigma} T_{s+1}[\phi] = \frac{\partial}{\partial \tau} \Theta_{s-1}[\phi]. \quad (3.7)$$

The index s refers to the spin of this conserved current J_s and by construction the charge

$$\mathcal{Q}_s = \int dx J_s^0 = \int (T_{s+1}[\phi] d\tau + \Theta_{s-1}[\phi] d\sigma) \quad (3.8)$$

is conserved, since by the equation of the conserved current (3.7), it satisfies

$$\frac{\partial}{\partial t} \mathcal{Q}_s = 0. \quad (3.9)$$

To obtain these charges it is common to use an auxiliary field $\hat{\phi}(\sigma, \tau)$ that is the solution of the Bäcklund transformations

$$\begin{aligned} \partial_\sigma(\hat{\phi} - \phi) &= 2\epsilon \sinh\left(\frac{\hat{\phi} + \phi}{2}\right), \\ \partial_\tau(\hat{\phi} - \phi) &= \frac{2}{\epsilon} \sinh\left(\frac{\hat{\phi} - \phi}{2}\right). \end{aligned} \quad (3.10)$$

If ϕ is a solution of the equation of motion (3.5), then the Bäcklund transformations (3.10) provide another solution:

$$\begin{aligned} \partial_\tau \partial_\sigma(\hat{\phi} - \phi) &= \partial_\tau \left(2\epsilon \sinh\left(\frac{\hat{\phi} + \phi}{2}\right) \right) = \epsilon \cosh\left(\frac{\hat{\phi} + \phi}{2}\right) \partial_\tau(\hat{\phi} + \phi) \\ &= 2 \sinh\left(\frac{\hat{\phi} + \phi}{2}\right) \cosh\left(\frac{\hat{\phi} + \phi}{2}\right) = \sinh \hat{\phi} - \sinh \phi. \end{aligned} \quad (3.11)$$

We can power expand the field $\hat{\phi}$ in terms of the parameter ϵ

$$\hat{\phi} = \sum \phi^{(n)} \epsilon^n. \quad (3.12)$$

To obtain $\phi^{(n)}$ one can plug this expansion in (3.10) and compare the powers in ϵ , obtaining

$$\phi^{(0)} = \phi, \quad \phi^{(1)} = 2\partial_\tau \phi, \quad \phi^{(2)} = 2\partial_\tau^2 \phi, \quad \phi^{(3)} = 2(\partial_\tau^3 \phi) - \frac{(\partial_\tau \phi)^3}{3}, \quad \dots \quad (3.13)$$

These solutions give the expression for the densities

$$T_2 = \frac{1}{2}(\partial_\tau \phi)^2, \quad T_4 = 2(\partial_\tau^2 \phi)^2 + 2(\partial_\tau \phi)(\partial_\tau^3 \phi), \quad \dots \quad (3.14)$$

With the expression for T_s , it is trivial to obtain the expression for the other density Θ_s via the current conservation (3.7) and then obtain the expression for the conserved charge via (3.8).

The densities written above are the ones with s even, because whenever we have a density with odd spin, the charge can be written as total derivative and therefore the conservation happens trivially, not imposing constraints in the theory.

The key result of [6]. Is that this set of infinite classically conserved charges, can be modified a little to remain not only conserved at the quantum level but also an involution, commuting with each other

$$[\mathcal{Q}_s, \mathcal{Q}_l] = 0. \quad (3.15)$$

This means that the integrability aspects of this model, such as no particle production, conservation of the entire set of momenta and factorization of the S-matrix, still persist even at the quantum level.

Sine-Gordon

To obtain another famous integrable model, one just needs to consider the analytic continuation of the coupling $g \rightarrow ig$, that transforms the Sinh-Gordon model in the Sine-Gordon model, with action given by

$$S_{SG} = \int d^2x \left(\frac{1}{16\pi} (\partial_\mu \phi)^2 - \frac{\mu^2}{g^2} \cos \left(\frac{g}{\sqrt{8\pi}} \phi(x) \right) \right). \quad (3.16)$$

It is also useful to think about this theory as a perturbed Liouville theory. Now the charge at infinity that is needed to close conformal symmetry, at the quantum level, gets a factor of i that changes the central charge of the UV theory in the following way

$$c_{uv} = 1 - 24Q^2, \quad Q = \frac{1}{2} \left(\frac{g}{\sqrt{8\pi}} - \frac{\sqrt{8\pi}}{g} \right). \quad (3.17)$$

Here there is the first indication of how closely integrable theories are related to conformal field theories, specially the minimal models, because it is possible to choose the coupling g of the Sine-Gordon theory in order to get a central charge correspondent to one of the minimal models.¹ Under these conditions the relevant

¹Only possible in this context, because the complex coupling allows the central charge to be less than one, which is a property of the minimal models.

operator that appears perturbing the Liouville action corresponds to the operator $\Phi_{1,3}$ of the minimal models.

This model is also integrable, since the analytic continuation of the coupling does not change the discussion about obtaining the conserved charges. Furthermore, the classically conserved charges can also be defined in the quantum context, resulting in integrability even at the quantum level.

We put the study of this model in stand by to comment about the seemingly unrelated Thirring model, to show that this fermionic model is deeply related to the Sine-Gordon theory.

Thirring model

The Thirring model is a purely fermionic model consisting of a massless Dirac fermion and a quartic current-current interaction:

$$\mathcal{L}_{Th} = \bar{\psi} i \not{\partial} \psi + \frac{1}{2} \lambda \bar{\psi} \gamma^\mu \psi \bar{\psi} \gamma_\mu \psi. \quad (3.18)$$

The equations of motion for this field can be written as

$$i \gamma^\mu \partial_\mu \psi + \lambda j^\mu \gamma_\mu \psi = 0, \quad (3.19)$$

where

$$j^\mu = \bar{\psi} \gamma^\mu \psi \quad (3.20)$$

is the conserved current, which can be written as a gradient of a scalar field φ

$$\partial_\mu j^\mu = 0 \Rightarrow j_\mu = -\frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \varphi. \quad (3.21)$$

Following [34], one can construct an one-parameter family of solutions for the equation of motion

$$\psi(x) = e^{i(\hat{\alpha}\tilde{\varphi} + \hat{\beta}\gamma_5\tilde{\varphi})} \psi^{(0)}(x), \quad (3.22)$$

where $\tilde{\varphi}$ is the dual² of φ . This is a one-parameter family of solutions, because $\hat{\alpha}$ and $\hat{\beta}$ are related to the coupling constant λ through

$$\hat{\alpha} - \hat{\beta} = -\frac{\lambda}{\sqrt{\pi}}. \quad (3.23)$$

The spinor $\psi^{(0)}$ is the canonical solution for the Dirac equation

$$\psi^{(0)}(x) = \sqrt{\frac{\mu}{2\pi}} \begin{pmatrix} : e^{i\sqrt{\pi}(\varphi + \tilde{\varphi})} : \\ : e^{i\sqrt{\pi}(\varphi - \tilde{\varphi})} : \end{pmatrix}, \quad (3.24)$$

with the parameter μ as the IR regulator of the mass associated with the field φ .

Plugging this canonical expression for $\psi^{(0)}$ in the family of solutions (3.22), we obtain

$$\psi(x) = \sqrt{\frac{\mu}{2\pi}} : e^{i(\alpha\tilde{\varphi} + \beta\gamma_5\varphi)} : \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (3.25)$$

²Dual means: $\partial_\mu \varphi = \epsilon_{\mu\nu} \partial^\nu \varphi$

where $\alpha = \hat{\alpha} + \sqrt{\pi}$ and $\beta = \hat{\beta} + \sqrt{\pi}$ define the spin s of the fermionic field via

$$s = \frac{\alpha\beta}{2\pi}. \quad (3.26)$$

Following [35], one can obtain correlation functions of the fermionic field in terms of correlation functions of the scalar field, for example³ one can identify the following normal order products

$$\begin{aligned} N[\bar{\psi}\psi] &= \frac{\mu}{2\pi} : \cos 2\beta\tilde{\varphi} : \\ N[\bar{\psi}\gamma^\mu\partial_\mu\psi] &= \frac{\beta^2}{2\pi}(\partial_\mu\tilde{\varphi})^2. \end{aligned} \quad (3.27)$$

With this in mind we can shift to the more enlightening massive Thirring model, with Lagrangian given by

$$\mathcal{L}_{mTh} = \bar{\psi}i\cancel{\partial}\psi - m\bar{\psi}\psi + \frac{1}{2}\lambda\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi. \quad (3.28)$$

One can think of such model as a perturbative expansion in the mass, meaning one must compute a correlator of an arbitrary number of mass terms

$$G(x_1, \dots, x_n) \equiv \frac{(-im)^n}{n!} \langle 0 | \bar{\psi}\psi(x_1) \dots \bar{\psi}\psi(x_n) | 0 \rangle_{Th}. \quad (3.29)$$

Using the identification (3.27), it is possible to rewrite such correlator as

$$G(x_1, \dots, x_n) = \frac{(-im)^n}{n!} \langle 0 | : \cos 2\beta\varphi(x_1) : \dots : \cos 2\beta\varphi(x_n) : | 0 \rangle. \quad (3.30)$$

This is precisely the correlator one expects to appear in the bosonic Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{m^2}{4\beta^2}(\cos 2\beta\varphi - 1). \quad (3.31)$$

Therefore we have a duality between the massive Thirring model and the Sine-Gordon model

$$\mathcal{L}_{mTh} = \bar{\psi}(i\cancel{\partial} - m)\psi + \frac{1}{2}g(\bar{\psi}\gamma^\mu\psi)^2 \sim \mathcal{L}_{SG} = \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{m^2}{4\beta^2}(\cos 2\beta\varphi - 1), \quad (3.32)$$

provided that the couplings (for the spin $s = \frac{1}{2}$ case) are related through

$$\beta^2 = \frac{\pi}{1 - \frac{\lambda}{\pi}}. \quad (3.33)$$

This Lagrangian is the same as the one associated with the action (3.16) via the identification $\phi \rightarrow \sqrt{8\pi}\varphi$ and $g \rightarrow 2\beta$.

³Here we have a notation change, the current is redefined as $j_\mu = -\frac{\beta}{\pi}\epsilon_{\mu\nu}\partial^\nu\varphi$, implying in a change of the relation between α, β and λ to $\alpha - \beta = -\frac{\beta\lambda}{\pi}$.

In the same way that the Sine-Gordon model presents infinitely many conserved charges at both classical and quantum level, the fermionic Thirring model also possesses an infinite number of conservation laws [36] that are still conserved at the quantum level as one should expect from the duality between such models.

The insights the Thirring model provides continue further. Since the fermions are expressed as a exponential of bosons one can expect a relation between the behavior of the fermions in the Thirring model with the behavior of the solitons in the bosonic Sine-Gordon model. This relation can be explicitly shown in [35] where there is a identification between the realization of the soliton operator with the fermionic operator expressed in (3.25).

In conclusion, the Sine-Gordon theory is an integrable theory of bosons, that [37] describes fermion-antifermion bound states, but also has multisolitonic solutions (dual to the fermions) that also have a purely elastic scattering process.

Bullogh-Dodd

This theory, although less known, is also a canonical example of integrable theory. Its action is given by

$$S_{BD} = \int d^2x \left(\frac{1}{16\pi} (\partial_\mu \phi)^2 + \frac{\mu^2}{6g^2} \left(2e^{\frac{g}{\sqrt{8\pi}}\phi} + e^{-2\frac{g}{\sqrt{8\pi}}\phi} \right) \right), \quad (3.34)$$

which can be also understood as a perturbation of the Liouville theory

$$S_0 = \int d^2x \left(\frac{1}{16\pi} (\partial_\mu \phi)^2 + \frac{\mu^2}{3g^2} e^{\frac{g}{\sqrt{8\pi}}\phi} \right), \quad (3.35)$$

by the relevant operator $e^{-2\frac{g}{\sqrt{8\pi}}\phi}$.

This will follow the same ideas, the addition of a charge at infinity, and also the change in the central charge for a value greater than one.

Via the analytic continuation $g \rightarrow ig$ the central charge can become less than one. In fact, it is again possible to fine tune the coupling g to obtain the central charges of the minimal models. Under this circumstances the relevant operator becomes the field $\Phi_{2,1}$ of the minimal models.

The problem of such naive change is that, contrary to the Sinh-Gordon model, this change in the coupling makes the action a complex object, and more analysis in this case is needed to truly see that the resulting action is indeed a well-defined object. However, in both cases the resulting theory has an infinite set of conserved charges characterizing an integrable theory.

Toda theories

The Sinh-Gordon, Bullogh-Dodd and their analytic continuations to complex couplings are all examples of a more general class of theories called Toda theories, first studied in [32].

These theories can be constructed⁴ with simply laced Lie algebras⁵ A_n , D_n and E_n .

The Toda field theory is defined through a Lie algebra G of rank r , that defines an action composed by r bosonic fields, arranged in the vector form $\phi = (\phi_1, \dots, \phi_r)$, given by

$$S = \int d^2x \left(\frac{1}{8\pi} (\partial_\mu \phi)^2 + \frac{\mu^2}{16\pi\beta^2} \sum_{n=1}^{r+1} q_n (e^{(\beta\alpha_n \cdot \phi)} - 1) \right), \quad (3.36)$$

where μ^2 and β are real parameters.

The set $\{\alpha_i\}_{n=1}$ is given by the simple roots of G , with norm equals to two, and the set of integers numbers $\{q_n\}$ is related to the definition of the maximal root, via

$$\alpha_{r+1} = - \sum_{n=1}^r q_n \alpha_n. \quad (3.37)$$

Following the same lines as before, one can look at this action as a sort of generalized Liouville action, given by

$$S_0 = \int d^2x \left(\frac{1}{8\pi} (\partial_\mu \phi)^2 + \frac{\mu^2}{16\pi\beta^2} \sum_{n=1}^r q_n (e^{(\beta\alpha_n \cdot \phi)} - 1) \right), \quad (3.38)$$

with the term associated with the maximal root is the relevant operator

$$S_I = \int d^2x (e^{(\beta\alpha_{r+1} \cdot \phi)} - 1), \quad (3.39)$$

where the convention of imposing $q_{r+1} = 1$ was used. Note also that the very definition of the integers q_n gives rise to

$$\sum_{n=1}^{r+1} q_n \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{r+1} q_n \equiv h \quad (3.40)$$

In order to have a UV theory that is quantically conformal invariant, it is needed to introduce a set of charges at infinity which changes the central charge of the background theory to

$$c_{\text{uv}} = r[1 + h(h+1)(\beta + 1/\beta)^2]. \quad (3.41)$$

Toda field theories posses an infinite set of conserved currents, at both classical and quantum level, resulting in an elastic scattering for the constituent particles of the theory.

One important last comment about the Toda theories is the analytic continuation $\beta \rightarrow i\beta$. Results in well-defined theories, with charges at infinty that will change

⁴It will not be necessary in this work, but the more formal way to obtain such theories is by folding Dynkin diagrams of simply laced Lie algebras [38]

⁵The first two have an easy representation in terms of matrix representations: $A_n = SU(n+1)$ and $D_n = SO(2n)$.

the expression (3.41), allowing to obtain background central charges with values less than the rank r of the algebra.

By choosing the following values of coupling

$$\beta^2 = \frac{p}{1+p} \quad \text{with } p = k+h, k+h+1, \dots, \quad (3.42)$$

the central charge becomes

$$c = r \left(1 - \frac{h(h+1)}{p(p+1)} \right), \quad (3.43)$$

that corresponds to the diagonal coset, of the form presented in (A.30) with $k_2 = 1$,

$$\frac{\mathcal{G}^{(1)} \times \mathcal{G}^{(k)}}{\mathcal{G}^{(k+1)}}, \quad (3.44)$$

which is precisely the same coset form as the conformal minimal models. Clearly there is a deeper relation between conformal field theories and integrable theories, that is discussed in the next chapter.

Due to later appearance and also to illustrate how the previous cases can emerge from the Toda theory definition, let us rediscover the sinh-Gordon model through (3.36). Consider the simplest simply laced Lie group $\mathcal{G} = A_1 = SU(2)$ which possess rank $r = 1$. The simple roots with norm equals two are just $\alpha_1 = \sqrt{2}$ and $\alpha_2 = -\sqrt{2}$. The integer $q_2 = 1$ is already fixed by definition and using (3.40) we find $q_1 = 1$. Plugging these values into the action (3.36) and rescaling the field $\phi \rightarrow \phi/\sqrt{2}$ and the coupling $\beta \rightarrow g/\sqrt{8\pi}$ results in the action (3.1) of the sinh-Gordon model.

Chapter 4

CFT and integrability

The past chapter made it clear that there is a relation between conformal field theories (which are reviewed in Appendix A) and integrable theories in two dimensions. Furthermore, for imaginary couplings of Toda theories, there is an even stronger relation that involves the families of minimal models.

An important question that arises from this relation is how one can establish that some deformation of a conformal field theory give rise to an integrable field theory in the IR.

Let's consider a minimal model $\mathcal{M}_{p,q}$ that is deformed by a relevant primary operator $\Phi_{lk}(z, \bar{z}) \equiv \phi_{lk}(z)\bar{\phi}_{lk}(\bar{z})$, with anomalous dimension $x = 2\delta < 2$. The action of this theory is given by

$$S = S_0 + \lambda \int \Phi_{lk}(z, \bar{z}) d^2z, \quad (4.1)$$

where S_0 is the conformal invariant action of the minimal model.

Consider also the following OPE

$$C_{s+1}(z)\Phi_{lk}(z, \bar{z}) \sim \sum_{n=2} \frac{d_{lk}^{(n)}}{(z-w)^n} \Phi_{lk}^{(n)}(w, \bar{w}) + \frac{1}{z-w} B_{lk}(w, \bar{w}), \quad (4.2)$$

where C_{s+1} is a local spin $s+1$ conserved current, $\bar{\partial}C_{s+1} = 0$, and both $\Phi_{lk}^{(n)}$ and B_{lk} are descendant fields of Φ_{lk} , multiplied by the structure constants $d_{lk}^{(n)}$ of the OPE.

The Ward identity for this current can be written as

$$\langle C_{s+1}(z, \bar{z}) \dots \rangle = \langle C_{s+1}(z) \dots \rangle_0 + \lambda \int dw d\bar{w} \langle C_{s+1}(z) \Phi_{lk}(w, \bar{w}) \dots \rangle_0 + \mathcal{O}(\lambda^2). \quad (4.3)$$

Substituting the OPE of the conserved current in the conformal Ward identity, to first order in λ , yields the following

$$\bar{\partial}C_{s+1}(z, \bar{z}) = \lambda(B_{lk}(z, \bar{z}) - d_{lk}^{(2)} \partial \Phi_{lk}^{(2)}). \quad (4.4)$$

The result is simple: the existence or not of a conserved charge in the IR (away from the critical point). Depends solely on whether B_{lk} is a total derivative with respect to z . If this happens, as discussed in Chapter 2, there will be a higher spin

conserved charge and its parity conjugate pair, that will lead to a factorization of the S-matrix.

For example, take the energy momentum tensor $C_2 = T$, then

$$B_{lk} = \partial\Phi_{lk}(z, \bar{z}) \quad \text{and} \quad d_{lk}^{(2)}\partial\Phi_{lk}^{(2)} = \delta\partial\Phi_{lk}(z, \bar{z}). \quad (4.5)$$

There is also have the analogous relation for $\{z, \bar{z}\}$ of equation (3.7), that gives

$$\bar{\partial}T = -\frac{1}{4}\partial\Theta \Rightarrow \Theta = -4\lambda(1 - \delta)\Phi_{lk}(z, \bar{z}), \quad (4.6)$$

and, in analogy with (3.8), the conserved charge is

$$\mathcal{Q}_1 = \int \left(Tdz + \frac{1}{4}\Theta d\bar{z} \right). \quad (4.7)$$

To conclude this discussion, let us consider a less trivial example. The minimal model $\mathcal{M}_{4,5}$ is the universality class of the tricritical Ising model, but it has more symmetry, in fact it is the first model of the superconformal series. We take as C_s the spin $s = \frac{3}{2}$ supercurrent $G_{\frac{3}{2}}$ and the relevant operator that deform this superconformal action is $\Phi_{\frac{3}{5}, \frac{3}{5}}$.

The OPE of this current with the relevant operator yields

$$G(z)\Phi_{\frac{3}{5}, \frac{3}{5}}(w, \bar{w}) \sim \frac{\frac{1}{5}\Phi_{\frac{1}{10}, \frac{3}{5}}(w, \bar{w})}{(z - w)^2} + \frac{\partial_w \Phi_{\frac{1}{10}, \frac{3}{5}}}{z - w}, \quad (4.8)$$

which leads to the conservation

$$\bar{\partial}G(z, \bar{z}) = \partial\bar{\Psi}(z, \bar{z}) \Rightarrow \bar{\Psi}(z, \bar{z}) = \frac{4}{5}\lambda\Phi_{\frac{1}{10}, \frac{3}{5}}(z, \bar{z}). \quad (4.9)$$

The conserved charge associated with this conserved current has spin $s = 1/2$ and can be written as

$$\mathcal{Q}_{\frac{1}{2}} = \int (Gdz + \bar{\Psi}d\bar{z}). \quad (4.10)$$

It is not difficult to rewrite this expression to get

$$\mathcal{Q}_{\frac{1}{2}}^2 = 2(E + P), \quad (4.11)$$

where E and P are the components of the 4-momentum. Analogously one can analyze the supercurrent $\bar{G}_{\frac{3}{2}}$ and obtain a similar conserved charge $\bar{\mathcal{Q}}_{\frac{1}{2}}$ with the following properties

$$\mathcal{Q}_{\frac{1}{2}}^2 = 2(E - P), \quad (4.12)$$

and

$$\mathcal{Q}_{\frac{1}{2}}\bar{\mathcal{Q}}_{\frac{1}{2}} + \bar{\mathcal{Q}}_{\frac{1}{2}}\mathcal{Q}_{\frac{1}{2}} = \mathcal{T}, \quad (4.13)$$

where $\mathcal{T}$ is a topological charge, which characterizes the solitonic solutions that interpolate the degenerate vacua of this theory. Analogously to the sinh-Gordon case, the integrability of this theory implies an elastic scattering behavior for the solitons.

The tricritical Ising model, perturbed by this operator with $\lambda < 0$, goes into a phase with degenerate vacua. Therefore, the system acquires solitonic solutions that interpolate between the vacua and are precisely characterized by the topological charge $\mathcal{T}$. The scattering of such solitons is elastic, granting the integrable behavior of this theory, that furthermore has the charges $\{\mathcal{Q}_{\frac{1}{2}}, \bar{\mathcal{Q}}_{\frac{1}{2}}, P, \bar{P}\}$ responsible to generate a global supersymmetry in the IR.

Staircase models

Another interesting example of integrable theories are the so-called staircase models [14]. They put together not only the ideas of CFT and integrability, but also RG flows and factorization of the S-matrix, by using the thermodynamic Bethe ansatz, [26, 27].

The starting object is the S-matrix of the two particle scattering of what Zamolodchikov calls a relativistic factorized scattering theory (RFST). This theory has the property of having a factorized S-matrix just like an integrable field theory, but the more general name is used, because it is possible to define a physical¹ S-matrix that does not have a well-defined (or well understood) field content behind it.

The S-matrix considered in this case is

$$S(\theta) = \frac{\sinh \theta - i \cosh \theta_0}{\sinh \theta + i \cosh \theta_0}, \quad (4.14)$$

where θ is the difference between rapidities of the two incoming particles and θ_0 is a real parameter that take the theory away from the usual S-matrix of the Sine-Gordon theory. This is the first contact with the S-matrix of an integrable theory, but the extensive discussion about its properties and expressions will be discussed in the next chapter.

We take this two-dimensional theory of one massive scalar field and put it in the surface of a flat torus of with geodesic circles of circumference R and L , as shown in Figure 4.1.

Figure 4.1: Flat torus generated by two orthogonal geodesic circles of circumference R along the latitude and L along the longitude.

In configuration space we can have free regions of N particles denoted by

$$\{x_{i_1}, x_{i_2}, \dots, x_{i_N}\} \text{ where } x_{i_1} < x_{i_2} < \dots < x_{i_N}, \quad (4.15)$$

¹Physical in the sense that it respects both unitarity and crossing symmetry, which will be discussed in the next chapter.

with the particles spaced enough, in order to be non interacting, $|x_{i_j} - x_{i_k}| \ll R_{\text{int}}$. In this regime a wave function, denoted by $\Psi(x_{i_1}, \dots, x_{i_N})$, is a well-defined² object.

As special of RFST, the number of particles must be the same in all free regions, because the *in* state and the *out* state must have the same number of particles. Furthermore, the full set of momenta in each free region must be the same. Therefore, just as in the spin chain case [39], the only thing that can happen is a swapping between to particles

$$\{x_{i_1}, \dots, x_{i_p}, x_{i_{p+1}}, \dots, x_{i_N}\} \rightarrow \{x_{i_1}, \dots, x_{i_{p+1}}, x_{i_p}, \dots, x_{i_N}\}. \quad (4.16)$$

Each region will have a well defined wave function, namely the Bethe wave function, but the key property of RFST is that the two wave functions can be matched via the S-matrix of the two particle process

$$\Psi(x_{i_1}, \dots, x_{i_{p+1}}, x_{i_p}, \dots, x_{i_N}) = S_{p,p+1} \Psi(x_{i_1}, \dots, x_{i_p}, x_{i_{p+1}}, \dots, x_{i_N}). \quad (4.17)$$

From this behavior, it is possible to obtain the Bethe equations, by using the same argument of spin chains. Take the i -particle and move it around the free region, interacting pairwise with all the other particles and coming back to the initial configuration via the periodicity of the boundary conditions. After all this process the wave function remains the same, implying that

$$e^{ip_i L} \prod_{j \neq i} S_{i,j} = 1. \quad (4.18)$$

The first term $e^{ip_i L}$ is nothing more than taking the i -particle with momentum p_i and taking it around L , but each time the i -particle interacts with the j -particle we get also a factor of S_{ij} . After one complete circle the sum of all this contributions must not alter the wave function.

This gives a set of N transcendental equations that select the admissible values of rapidities of each particles and the allowed free regions, which is difficult to solve for arbitrary values of N .

As explicitly shown in Appendix B, in the thermodynamical limit ($L \rightarrow \infty$) it is possible to change the discrete behavior of the N transcendental equations (4.18) to a solvable continuum. It is also possible to define thermodynamical quantities, such as the entropy, free energy and Casimir's energy and use the minimization of free energy to obtain another equation for the system. This pair of equations lead to the Thermodynamical Bethe equations

$$Rm \cosh \theta = \epsilon(\theta) + \int \varphi(\theta - \theta') \log(1 + e^{-\epsilon(\theta')}) \frac{d\theta'}{2\pi}, \quad (4.19)$$

$$E(R) = -m \int \cosh \theta \log(1 + e^{\epsilon(\theta)}) \frac{d\theta}{2\pi}. \quad (4.20)$$

Using the results of [40], we can modify the second equation to

$$c(R) = \frac{6R}{\pi} m \int \cosh \theta \log(1 + e^{\epsilon(\theta)}) \frac{d\theta}{2\pi}, \quad (4.21)$$

²No real/virtual particle creation, the particles move freely without the presence of off-mass-shell effects

where $c(R)$ is a function that for small R is identical to the central charge of the background conformal field theory. It is analogous to the c -function, which has the characteristics of been monotonically decreasing and is identical to the central charge of the conformal field theory at the critical points [41].

As explicitly shown in [14] in the thermodynamical limit, one can numerically solve the equation (4.19) obtaining the pseudoenergy $\varepsilon(\theta)$. This result can now be used in the second thermodynamical Bethe equation (4.20) yielding the scaling function $c(R)$. Furthermore, this particular c -function has the “staircase” behavior, shown in Figure 4.2, that gives the name to the model.

Figure 4.2: Behavior of the central charge of the background theory as a function of the radius R , with θ_0 being the parameter that tells how close the flow passes to the fixed points. The fixed points are the values of central charges for the minimal models of the $\mathcal{G} = A_1 = SU(2)$ series, where the notation $\mathcal{M}_{p+1,p} \equiv \mathcal{M}_p$ was adopted. Taken from [14].

The plateaus that appear in Figure 4.2, are located precisely at the values of central charges associated with the minimal models, so how can one interpret such result?

In a previous paper [13], Zamolodchikov proves that the tricritical Ising model ($\mathcal{M}_4$) perturbed by the relevant operator $\Phi_{1,3}$ leaves this fixed point and flows towards the Ising model ($\mathcal{M}_3$) attracted by the other operator $\Phi_{3,1}$, as shown in Figure 4.2.

The staircase models represent a generalization of such flow, in fact when $R \rightarrow 0$ we have the maximum value of the central charge $c_{uv} = 1$, for the minimal model $\mathcal{M}_\infty$. As R gets bigger one can imagine this as a RG flow, promoted by $\Phi_{1,3}$, that visits all the minimal models of the $\mathcal{G} = A_1$ series up the one with smallest central charge, the Ising model $\mathcal{M}_3$. From this one, the perturbation of theory by the relevant operator generates a theory with massive IR behavior.

The Goddard-Kent-Olive coset construction [10] allows us to start from a simply laced Lie algebra $\mathcal{G} = A, D$ or E , and define a set of $W_{\mathcal{G}}$ -minimal models, as discussed

in the Appendix A, via the coset

$$\frac{\mathcal{G}^{(1)} \times \mathcal{G}^{(l)}}{\mathcal{G}^{(1+l)}}. \quad (4.22)$$

The staircase model obtained in [14] are the series associated with taking $\mathcal{G} = A_1 = SU(2)$ and this is no surprise, since the starting S-matrix presented in (4.14) is an analytic continuation of the sinh-Gordon S-matrix, which, as discussed before, is a Toda theory associated with the simply laced Lie algebra $\mathcal{G} = A_1 = SU(2)$.

This suggests a generalization of the possible staircase models that one can discover. For example, one could start with analytic continuations of the D -series Toda theories and this might give a staircase model associated with a series of W_D minimal models. In fact, this is even more general, as shown in [15, 16], there are models given by the coset

$$\frac{\mathcal{G}^{(k)} \times \mathcal{G}^{(l)}}{\mathcal{G}^{(k+l)}}, \quad (4.23)$$

that for each k form a series of minimal models.

In these cases the relevant operator is no longer the $\Phi_{1,3}$ but $\Phi_{\text{id},\text{id},\text{adj}}$, where each index label the representation of $\mathcal{G}^{(k)}$, $\mathcal{G}^{(l)}$ and $\mathcal{G}^{(k+l)}$ respectively. This relevant operator, under the same circumstances³ creates a RG flow of staircase behavior that leaps between this $W_{\mathcal{G}}$ of minimal models, with the difference that the leap is exactly given by k , meaning that if the theory starts from the l -minimal model the promoted RG flow will have a new plateau at the $(l - k)$ -minimal model.

This short digression on staircase models. Proves an interesting feature of integrable theories, or more generally RFST, that in modern days can even be understood in more primordial ways. For example, [17] obtain a function for the central charge of the background theory, with the same physical content as the one obtained via TBA, but using only properties of the c-function and form factors.

³In the thermodynamic limit of $L \gg 1$ and small radius $R \ll 1$.

Chapter 5

The S-matrix

Since the S-matrix of integrable theories factorize, the objective of finding a full quantum S-matrix for the two-particle scattering is equivalent to solving the theory. There is no doubt that this two-particle S-matrix is the most important object of an integrable theory. We begin by studying general properties of the S-matrix and then shift to a more general question of S-matrix bootstrap, which results in the appearance of integrable models and their S-matrices.

General properties of the S-matrix

Reducing the expression (2.5) to the case with only two outgoing particles, we get the two-particle S-matrix

$$A_i(\theta_1)A_j(\theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2)A_l(\theta_2)A_j(\theta_1), \quad (5.1)$$

with $\theta_1 > \theta_2$ to correct represent the *in* and *out* states, as presented in the left part of Figure 5.2. The relation has been generalized to multiple particles labeled by the indices in the states, with the condition that repeated indices are summed over. Since Lorentz boosts ($s = 1$ conserved charges) shift the rapidities by a constant, the S-matrix can only depend on the difference between rapidities.

In an integrable theory, the infinite conserved charges further restrict the S-matrix, meaning that the matrix element S_{ij}^{kl} can be only non zero if the particles with the same rapidities have the same mass ($m_i = m_k$ and $m_j = m_l$). More generally, one can also require C, P and T invariance

$$\begin{aligned} C : S_{ij}^{kl}(\theta) &= S_{\bar{i}\bar{j}}^{\bar{k}\bar{l}}(\theta), \\ P : S_{ij}^{kl}(\theta) &= S_{ji}^{lk}(\theta), \\ T : S_{ij}^{kl}(\theta) &= S_{lk}^{ji}(\theta), \end{aligned} \quad (5.2)$$

where the indices with a bar represent a possible operation of conjugation in the particle label.

In two dimensions, only one of the Mandelstan variables is independent, which is usually taken to be s , whose square root is the center of mass energy. In terms of the rapidities

$$s = m_i^2 + m_j^2 + 2m_i m_j \cosh(\theta_1 - \theta_2). \quad (5.3)$$

For a physical process, $\theta = \theta_1 - \theta_2$ is real. Physical processes have s real and $s > (m_i + m_j)^2$, which is intuitively what one would expect, since $\sqrt{s}$ is the energy of the center of mas: two-particle production must start for values of energy bigger than the values of the rest energy of the two incoming particles, the so-called "two-particle" threshold.

One can continue s to the complex plane, obtaining a complex function $S(s)$ that presents branch cuts, see Figure 5.1, and the following properties:

- i. S is a singlevalued, meromorphic function with cuts in the regions $s < (m_i - m_j)^2$ and $s > (m_i + m_j)^2$ of the real axis.
- ii. The physical values of S are found just above the right-hand cut (C region indicated in Figure 5.1) and the correspondent Riemann surface is called the physical sheet.
- iii. S is real analytic, meaning it takes complex conjugated values to complex conjugated points:

$$S_{ij}^{kl}(s^*) = (S_{ij}^{kl}(s))^* . \quad (5.4)$$

- iv. In the region between the cuts $(m_i - m_j)^2 < s < (m_i + m_j)^2$, s is real.

Figure 5.1: s -plane with the branch cuts, zeros and poles of the function $S(s)$. Taken from [6].

The values in the physical sheet, just above the right hand cut are denoted by s^+ and can be written as $s^+ = s + i0$ with $s > (m_i + m_j)^2$. Unitarity requires that for these physical values $S(s^+)S^\dagger(s^+) = 1$, which must be understood as a matrix equation that sums over the asymptotic states between S and $S^\dagger$. As s grows, the scattering energy grows as well, crossing more particles threshold, which give a series of new branch cuts at the real axis for the function S . These new thresholds bring processes with more than two particles into the sum (they are of the type $2 \rightarrow m$). As special of integrable theories the amplitudes of these processes must be zero, which simplifies drastically the unitarity condition

$$S_{ij}^{kl}(s^+) (S_{kl}^{nm}(s^+))^* = \delta_i^n \delta_j^m , \quad (5.5)$$

which can be rewritten, through property (iii), as

$$S_{ij}^{kl}(s^+) S_{kl}^{nm}(s^-) = \delta_i^n \delta_j^m , \quad (5.6)$$

with $s^- = s - i0$, just below the right-hand cut (region B of Figure 5.1). This expression shows the need for a branch starting at the two-particle threshold and going rightwards.

It can also be proven. That this branch cut is of square-root type. Let $S_\gamma(s)$ be the analytic continuation of $S(s)$ once anticlockwise around the branch point $s = (m_i + m_j)^2$. Unitarity requires that $S(s^+)S_\gamma(s^+) = 1$, which can also be analytically continued for not only the physical values s^+ , but for all values of s ,

$$S(s)S_\gamma(s) = 1 \Rightarrow S_\gamma(s) = S^{-1}(s). \quad (5.7)$$

For a point below the physical cut, s^- , we have

$$S_\gamma(s^-) = S^{-1}(s^-) = S(s^+), \quad (5.8)$$

where in the last equality, the unitarity of S has been used. But now $S_\gamma(s^-)$ is the analytic continuation of $S(s^+)$ twice around the right-hand branch point. Going twice around this branch point gets us back to where we started, so the singularity is a square-root type.

The left-hand part of the s -plane can be understood through crossing symmetry. In the right part of Figure 5.2, the incoming j particle is crossed to be outgoing and the outgoing particle l is crossed to incoming, resulting in the amplitude for another physical process, with incoming particles $i, \bar{l}$ and outgoing particles $k, \bar{j}$.

Figure 5.2: In the right part the s -channel scattering, in the right part the particles j and l are crossed to result in the t -channel scattering, presented in the right.

In the crossed amplitude, the forward momentum is not s but rather t . Moreover, $p_3 = p_2$ meaning it is possible to relate t and s :

$$t = (p_1 - p_3)^2 = (p_1 - p_2)^2 = 2p_1^2 + 2p_2^2 - (p_1 + p_2)^2 = 2m_i^2 + 2m_j^2 - s. \quad (5.9)$$

Crossing symmetry states that the amplitude of this crossed process can be obtained through an analytic continuation of initial amplitude into a region of the s plane where t becomes the physical object, that is t real and $t > (m_i + m_j)^2$, so

$$S_{ij}^{kl}(s^+) = S_{i\bar{l}}^{k\bar{j}}(2m_i^2 + 2m_j^2 - s^+). \quad (5.10)$$

The physical values of t are in the region below the right-hand branch cut (region A), because the minus sign with respect to s inverts the sign of the complex part.

Even though the left-hand branch point is of square-root type, the Riemann surface has more than two sheets. One can connect the left-hand cut with a different sheet found through the right-hand cut and this connection continues, connecting up, down, left and right cuts of the s -plane, therefore the typical $S(s)$ lives in a infinite cover of the physical sheet.

This picture can be simplified [42], by switching from Mandelstan variables back to the rapidity difference θ

$$\theta = \cosh^{-1} \left(\frac{s - m_i^2 - m_j^2}{2m_i m_j} \right), \quad (5.11)$$

which maps the physical sheet in the region

$$0 \leq \text{Im } \theta \leq \pi \quad (5.12)$$

of the θ -plane, called the physical strip, as shown in Figure 5.3.

Figure 5.3: The process of mapping from s to θ can be seen as opening up the cuts and rotating the plane of $\pi/2$ clockwise. It is precisely this opening together with rotation that takes the overlapped branch cuts in the s plane and smears them around the θ -plane.

Since the cuts are opened up, $S(\theta)$ is analytic at the images 0 and $i\pi$ of the physical branch points. Furthermore, $S(\theta)$ is also analytical at the images $in\pi$ of the other unphysical branch points on all other unphysical sheets, meaning that these other sheets are mapped onto a succession of unphysical strips

$$n\pi \leq \text{Im } \theta(n+1)\theta \quad n = 1, 2, \dots \quad (5.13)$$

The previous relations can now be formally translated for $S(\theta)$ yielding

$$\begin{aligned} \text{Real analyticity: } & S(\theta) \text{ is real for } \theta \text{ purely imaginary.} , \\ \text{Unitarity: } & S_{ij}^{nm}(\theta) S_{nm}^{kl}(-\theta) = \delta_i^k \delta_j^l , \\ \text{Crossing: } & S_{ij}^{kl}(\theta) = S_{i\bar{l}}^{k\bar{j}}(i\pi - \theta) . \end{aligned} \quad (5.14)$$

Treating S in the θ -plane allows us to analytically continue the unitarity and crossing conditions to the whole θ -plane. Moreover, it also provides a more formal form for the Yang-Baxter equation

$$S_{ij}^{\beta\alpha}(\theta_{12}) S_{\beta k}^{m\gamma}(\theta_{13}) S_{\alpha\gamma}^{ml}(\theta_{23}) = S_{jk}^{\beta\gamma}(\theta_{23}) S_{i\gamma}^{\alpha l}(\theta_{13}) S_{\alpha\beta}^{nm}(\theta_{12}), \quad (5.15)$$

where $\theta_{ab} = \theta_a - \theta_b$ and θ_1, θ_2 and θ_3 are the rapidities of the particles i, j , and k , respectively.

S-matrix Bootstrap

The object of study will be the S-matrix, of a two dimensional quantum field theory associated with the elastics scattering of the two lightest neutral particles in the theory.¹

This S-matrix, denoted here by $S(s)$ must be unitary and crossing symmetric, that can be written simply in this case as

$$\begin{aligned} \text{Crossing:} \quad S(s) &= S(4m^2 - s), \\ \text{Unitarity:} \quad |S(s)|^2 &\leq 1, \quad s \geq 4m^2. \end{aligned} \tag{5.16}$$

Until this point the discussion about the poles of the S-matrix has been avoided. Between the s -channel two-particle production threshold ($s = 4m^2$) and the t -channel two particle production threshold ($s = 0$) is where the action takes place. The S-matrix presents poles in this region associated with single-particle asymptotic states (bound states). In the eyes of the S-matrix, these bound states are as fundamental as the original particles of the theory, and we will denote both as particles.

Due to crossing symmetry, the poles of the S-matrix come always in pairs:

$$S \sim -\mathcal{J}_k \frac{g_k^2}{s - m_k^2} \quad \text{and} \quad S \sim -\mathcal{J}_k \frac{g_k^2}{4m^2 - s - m_k^2}, \tag{5.17}$$

the first one corresponding to the production of a particle (bound state) of mass m_k in the s -channel and the other corresponding to the production of the same particle in the t -channel. It is always possible to distinguish the channel associated with a pole, since in a unitary theory g_k^2 is positive, a s -channel pole has negative residue (in s) while the t -channel pole has a positive residue (also in s).

The prefactor $\mathcal{J}_k$, given by

$$\mathcal{J}_k = \frac{m^4}{2m_k \sqrt{4m^2 - m_k^2}}, \tag{5.18}$$

is responsible to normalize g_k^2 to be the residue of the invariant matrix element T , which is related to S via the subtraction of the identity together with the Jacobians associated with the renormalization of delta functions

$$\mathbf{S} = \mathbf{1} \times S(s) = \mathbf{1} + i(2\pi)^2 \delta^{(2)}(P) T. \tag{5.19}$$

The identity² $\mathbf{1} = (2\pi)^2 4E_1 E_2 \delta(\vec{k}_1 - \vec{k}_3) \delta(\vec{k}_2 - \vec{k}_4) + (\vec{k}_1 \longleftrightarrow \vec{k}_2)$ corresponds to disconnected contributions, while T is associated to connected contributions.

¹This process is simpler since it is free of Coleman-Thun singularities [43].

²In order to be clear, here we denoted the spatial momentum by $\vec{k}$, even though it is just a number, to differentiate it from the 2-momentum k

The delta function in (5.19) gives the energy-momentum conservation $\delta^{(2)}(P) = \delta^{(2)}(k_1 + k_2 - k_3 - k_4)$, on the support of the solution $\vec{k}_1 = \vec{k}_3$ and $\vec{k}_2 = \vec{k}_4$,

$$4E_1 E_2 (\delta(\vec{k}_1 - \vec{k}_3) \delta(\vec{k}_2 - \vec{k}_4) + (\vec{k}_1 \longleftrightarrow \vec{k}_2)) = 2\sqrt{s}\sqrt{s - 4m^2} \delta^{(2)}(P). \quad (5.20)$$

This Jacobian relating the delta functions is precisely the one appearing in the definition of $\mathcal{J}_k$ at the pole $s = m_k^2$. In the numerator of $\mathcal{J}_k$ there is also the term m^4 whose only function is to leave g_k dimensionless, meaning now that the couplings g_k are measured in units of the external mass m .

With the ingredients in hand, it is now easy to pose a question. Suppose the theory has a definite spectrum of stable particles $m_1 < m_2 < \dots m_N$ which appears as poles in the region $s = (0, 4m^2)$, note we cannot have $m_k > 2m$. What is the maximum possible value of the coupling to the lightest exchange particle (g_1) compatible with such spectrum,

$$g_1^{\max} \equiv \max_{\text{fixed } m_k} g_1 = ? \quad (5.21)$$

A big value for the coupling with the particle of mass m_1 means a huge attractive force mediated by the particle m_1 between the external particles, if we keep increasing this coupling the this force is such that new bound states must show up, contradicting the spectrum took as input. Thus, there must be a finite upper bound for g_1 .

Figure 5.4: Maximum coupling g_1 between two external particles of mass m and one particle of mass m_1 , with the spectrum composed only by the external particles and this bound state of mass m_1 . The red dots are the numerical values and the solid line is an analytic curve. Taken from [5].

Numerical bootstrap. The details of the numerical approach are outlined in part II of [5], but the idea behind is simple. Given an initial spectrum, find the

maximum value of g_1 under the conditions (5.16), that is the S-matrix is unitary and crossing symmetric.³

The simplest case is when there is only one extra particle of mass m_1 that couples with the external particles of mass m . Using this bootstrap ideas one can obtain the maximum value of coupling between this particles, $g_1^{\max}$ for each value of m_1/m , as presented in Figure 5.4.

The spike in Figure 5.4 happens when $m_1 \rightarrow \sqrt{2}m$, but this is precisely the point where the s -channel pole and t -channel have opposite values canceling each other, therefore there is no way to bound the residue at this point.

There is also a symmetry $g_1^{\max}(m_1^2) = g_1^{\max}(4m^2 - m_1^2)$ in Figure 5.4. This can be understood as a further application of crossing symmetry, since any solution of the problem with $m_1 > \sqrt{2}m$ can be transformed into a solution of the problem with $m_1 < \sqrt{2}m$ via the re-interpretation of which is the s -channel pole and the t -channel pole, which can be easily done by multiplying the whole S-matrix by -1.

The tail of Figure 5.4 is associated with a bound state of almost twice the mass of the external particles $m_1 \rightarrow 2m$, and this represents a weakly coupled theory studied in [44], which is intuitively correct since a stronger coupling would decrease the mass of the bound state.

Finally, the most important aspect of the S-matrices with maximal residues is that they saturate unitarity at all values $s > 4m^2$. But this implies no particle production, which can be easily seen by writing unitarity as

$$\sum_n |S_{2 \rightarrow n}(s)|^2 = 1 \Rightarrow |S_{2 \rightarrow 2}(s)|^2 = 1 - \sum_{n \neq 2} |S_{2 \rightarrow n}(s)|^2, \quad s > 4m^2, \quad (5.22)$$

and observing that the fact that our $2 \rightarrow 2$ S-matrix is already one, implies that all the other processes must be zero.

The absence of particle production is the characteristics of integrable models. In fact, when $m_1 > \sqrt{2}m$, the S-matrix with a single bound state is the Sine-Gordon S-matrix describing the scattering of the two lightest breathers⁴ in the theory. This S-matrix, first obtained in [45] has the form

$$S_{SG}(s) = \frac{\sqrt{s}\sqrt{4m^2 - s} + m_1\sqrt{4m^2 - m_1^2}}{\sqrt{s}\sqrt{4m^2 - s} - m_1\sqrt{4m^2 - m_1^2}}. \quad (5.23)$$

There is no unitary relativistic quantum field theory in two dimensions whose S-matrix element has a single pole at $s = m_1^2 > 2m^2$ and bigger residue (coupling) than the Sine-Gordon theory.

Analytical bootstrap. Here we focus in the saturation of unitarity, which can be regarded as integrability and how the expressions of the S-matrices of this integrable model emerge.

³Due to the way of writing the poles in crossing symmetric way, the starting S-matrix is already crossing symmetric and this condition does not need to be imposed during the numerical calculations.

⁴This is state constructed by a soliton and an anti-soliton that can be regarded as the fundamental stable particles of the theory.

As discussed before, it is better to express the S-matrix in terms of the rapidity θ ,

$$s = 4m^2 \cosh^2(\theta/2). \quad (5.24)$$

In terms of θ the crossing symmetry and unitarity for an arbitrary theory have the form

$$S(\theta) = S(i\pi - \theta), \quad S(\theta = i0)S(\theta + i0) = f(\theta), \quad (5.25)$$

where $f \in [0, 1]$ is the constraint of unitarity for physical values, that is $\theta \in \mathbb{R}$. A solution for (5.25) can always be written as

$$S(\theta) = S_{\text{CDD}}(\theta) \exp \left(- \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi i} \frac{\log f(\theta')}{\sinh(\theta - \theta' + i0)} \right), \quad (5.26)$$

where S_{CDD} is the solution that saturates unitarity ($f = 1$), obtained by Castillejo-Dalitz-Dyson in [46]. Note also that the exponential that carries the f dependence has no poles (or zeros) in the physical strip, meaning that this physical information is all inside S_{CDD} .

Since f is an even function it is possible to symmetrize the integral in the minimal solution in the physical strip,

$$S(it) = S_{\text{CDD}}(it) \exp \left(\int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \underbrace{\frac{\sin(t) \cosh(\theta')}{|\sinh(it - \theta')|^2}}_{\text{positive for } t \in [0, \pi]} \underbrace{\log f(\theta')}_{\text{negative}} \right), \quad (5.27)$$

the exponential will always decrease the magnitude of the S-matrix in the physical strip, unless $f = 1$. Since the goal is to maximize the residue of the S-matrix in precisely this region the optimal solution will have $f = 1$, saturating unitarity and obtaining S-matrices associated with integrable models.

These S-matrices are given only by the CDD-term, which solves the equations (5.25) with $f = 1$,

$$S_{\text{CDD}}(\theta) = S_{\text{CDD}}(i\pi - \theta), \quad S_{\text{CDD}}(\theta)S_{\text{CDD}}(-\theta) = 1, \quad (5.28)$$

there are infinitely many solutions to this homogeneous problem, which can be constructed via the multiplication of CDD-factors [47],

$$S_{\text{CDD}}(\theta) = \prod_j [\alpha_j], \quad [\alpha] \equiv \frac{\sinh(\theta) + i \sinh(\alpha)}{\sinh(\theta) - i \sinh(\alpha)}, \quad (5.29)$$

with α taken to be in the physical strip, $\text{Re } \alpha \in [-\pi, \pi]$. Depending on these values of α the CDD-factor has a very different behavior.

The one that matters for us the most is called CDD-pole and happens when α is real and $\alpha \in [0, \pi]$. This CDD-factor will have a pole at $\theta = i\alpha$ in the physical strip. Since the physical strip is associated with the region $s \in [0, 4m^2]$ this pole represents the bound state formed by the external particles.

In our case we have just one bound state of mass m_1 , so using the relation (5.24) the bound state at $s = m_1^2$ is related to a CDD-factor $[\alpha_1]$ given by,

$$m_1^2 = 4 \cosh^2(i\alpha_1/2), \quad (5.30)$$

so the S-matrix that saturates unitarity is simply

$$S_{\text{CDD}} = [\alpha_1] = \frac{\sinh(\theta) + i \sinh(\alpha_1)}{\sinh(\theta) - i \sinh(\alpha_1)}, \quad (5.31)$$

which is the usual form of write the S-matrix for the Sine-Gordon model, with bound state given in (5.30).

There are other types of CDD factors, denominated CDD zeros and CDD resonances, that become important when considering an initial spectrum composed of more than one bound state. One example of such theory is studied in part II of [5], where the bootstrap approach leads to the integrable model of the Ising model with magnetic field.

In the case with just one bound state one discovers the Sine-Gordon model as the integrable theory that saturates the boundary, but could it be that there is another integrable theory that somehow is overlooked in this S-matrix bootstrap approach? This motivates another method for searching integrable theories, the no particle production approach, that is presented in the next chapter.

Chapter 6

Integrable models from no particle production

Continuing with the bootstrap philosophy, one can imagine of "rediscovering" the existence of such integrable theories by imposing the absence of particle production in scattering processes and thus checking which kind of interactions are admissible [7]. We¹ will carry out this procedure for a simple supersymmetric model in two dimensions, the scalar $\mathcal{N} = (1, 1)$ superfield. Writing a generic superpotential for this field and imposing that no particles be produced introduces a recursion relation among the couplings of generic vertices and allows to recover the $\mathcal{N} = 1$ supersymmetric sinh-Gordon model, which is known to be integrable from the explicit construction of infinite conserved charges [48, 49, 50, 51].²

Our starting point is an $\mathcal{N} = (1, 1)$ scalar superfield Φ in two dimensions with generic superpotential $W(\Phi)$. This superfield consists of a real scalar field $\phi(x)$, a two-component Majorana spinor $\psi(x)$, and an auxiliary field $F(x)$; see e.g. [52]. Basic notions of supersymmetry are reviewed in Appendix C. Which also contains the derivation of the two dimensional on-shell Lagrangian³ written in component fields as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 + \bar{\psi} i \not{\partial} \psi - M \bar{\psi} \psi - \sum_{n=3}^{\infty} \frac{\Lambda_n}{n!} \phi^n - \sum_{n=3}^{\infty} \frac{\lambda_n}{(n-2)!} \phi^{n-2} \bar{\psi} \psi, \quad (6.1)$$

with scalar couplings

$$\Lambda_n = Mn\lambda_n + \frac{n!}{2} \sum_{i=3}^n \sum_{j=3}^n \frac{\lambda_i}{(i-1)!} \frac{\lambda_j}{(j-1)!} \delta_{i+j-2,n}, \quad (6.2)$$

that are the coefficients of the power expansion of the superpotential:

$$W(\phi) = \frac{1}{2} M \phi^2 + \sum_{n=3}^{\infty} \frac{\lambda_n}{n!} \phi^n. \quad (6.3)$$

¹Chapter based in the published article [8].

²Absence of particle production in the bosonic sine-Gordon model was proven in [53].

³The auxiliary field F is removed through its equation of motion.

We now consider tree-level scattering and impose that no particles be produced in these processes. This will introduce recursion relations in the couplings, which can be solved to find the generic expression for λ_n and the potential in (6.3). It is sufficient to look at scattering amplitudes with just two initial states: $\mathcal{M}_{2 \rightarrow n}$. Our convention is that all particles are taken to be incoming, with the understanding that in the end all but two particles are crossed to outgoing.

It is convenient to go to light-cone coordinates and write the momentum of the i -th particle $(p_i^+, p_i^-) = (p_i^0 + p_i^1, p_i^0 - p_i^1)$ in terms of a single real parameter a_i as

$$p_i = (Ma_i, M/a_i), \quad (6.4)$$

which guarantees $p_i^+ p_i^- = M^2$. From now on we set $M = 1$ without loss of generality. Conservation of energy and momentum is expressed as

$$\sum_i a_i = \sum_i \frac{1}{a_i} = 0. \quad (6.5)$$

The scalar and fermion propagators carrying momentum $p = \sum_i p_i$ are given, respectively, by

$$G(p) = \frac{1}{p^2 - 1} = \frac{1}{(\sum_i a_i) \left(\sum_j 1/a_j \right) - 1}, \quad (6.6)$$

and

$$\not{G}(p) \equiv \frac{\not{p} + 1}{p^2 - 1} = G(p) \begin{pmatrix} 1 & -i \sum_i 1/a_i \\ i \sum_i a_i & 1 \end{pmatrix}. \quad (6.7)$$

The bosonic sector

Let us focus for the moment on the purely bosonic case, which is also studied in [7]. The simplest scattering process with particle production is $\mathcal{M}_{2b \rightarrow 3b}$, which is evaluated at tree level by computing the diagrams in Fig. 6.1.

Figure 6.1: Tree-level diagrams contributing to $\mathcal{M}_{2b \rightarrow 3b}$.

The key observation [6] is that if one could set the sum of the first two diagrams in the figure to be constant, for arbitrary configurations of momenta obeying (6.5), these first two diagrams could be eliminated by the third one, by simply tuning Λ_5 to minus that constant. One can check that this strategy can indeed be implemented! Imposing $\Lambda_4 = 3\Lambda_3^2$ guarantees that the sum of the first two diagrams be a constant, which happens to be equal to $-5\Lambda_3^3$. This process of particle creation can then be eliminated by tuning $\Lambda_5 = 5\Lambda_3^3$.

Of course, the next step would be to generalize this procedure for amplitudes with an arbitrary number of final bosons, in order to find expressions for all the couplings in terms of Λ_3 . This can be done recursively, using the factorization of the

amplitudes and a clever choice of momenta. Note in fact that all tree-level diagrams for n particles, except for the constant one equal to Λ_n , can be factorized in a left blob and a right blob connected by a propagator

$$G_{L \rightarrow R} = \frac{1}{\left(\sum_{i \in L} a_i\right) \left(\sum_{j \in R} 1/a_j\right) - 1}. \quad (6.8)$$

One can also show from the analyticity properties of the amplitudes [7] that the sum of all diagrams remains constant, even for a generic number of external particles. This allows to pick a convenient choice of momenta which simplifies the evaluation of the recursive relations among the Λ_n 's. This convenient choice turns out to be [7]

$$a = \{a_1(x), 1, x, x^2, \dots, x^{n-3}, a_n(x)\}, \quad (6.9)$$

with $a_i(x)$ and $a_n(x)$ being determined by (6.5). In the limit of $x \rightarrow \infty$,

$$a_1(x) = -1 + \mathcal{O}(1/x), \quad a_n(x) = -x^{n-3} (1 + \mathcal{O}(1/x)), \quad (6.10)$$

and one can check that

$$G_{L \rightarrow R} = \begin{cases} -1, & \text{if } a_j^L = (a_1, a_2, \dots, a_k), a_j^R = (a_{k+1}, a_{k+2}, \dots, a_n) \\ 0, & \text{otherwise} \end{cases}, \quad (6.11)$$

that is, only the diagrams where the particles are ordered contribute. To see this, it is sufficient to evaluate a few cases. For example, the set $a_j^L = \{a_2, a_3\}$ gives a vanishing propagator and it is easy to see that any other set of two or more momenta not including a_1 goes to zero as well. Similarly, the set $a_j^L = \{a_1, a_{i \geq 3}\}$ also gives zero. On the other hand, the set $a_j^L = \{a_1, a_2\}$ yields -1 . By adding $a_{i \geq 4}$ to this set, we get zero again. The only case left to analyze is if an ordered set of any size continues to converge to -1 , which it does.

An important consequence of (6.11) is that only ordered line-type diagrams survive, as shown pictorially in Fig. 6.2 for the case of six particles.

Figure 6.2: This diagram seems to be non-vanishing since it is ordered for the cuts A and B . However, for the cut C it is $a_j^L = \{a_3, a_4\}$, which implies that this diagram does in fact vanish. In general, only (ordered) line-type diagrams survive the large x limit.

After these considerations, one is ready to compute the amplitude $\mathcal{M}_{2b \rightarrow (n-2)b}$, schematically shown in Fig. 6.3. It is useful to start thinking about the diagram that

Figure 6.3: The $\mathcal{M}_{2b \rightarrow (n-2)b}$ scattering. The sum is over all ordered line-typed diagrams.

has Λ_{n-k} as its rightmost vertex. This diagram factorizes into the form $\mathcal{M}_{2b \rightarrow kb} \cdot \Lambda_{n-k}$. Imposing $\mathcal{M}_{2b \rightarrow kb}$ to vanish, one gets the recursion relation

$$\mathcal{M}_{2b \rightarrow (n-2)b} = -\Lambda_3 \Lambda_{n-1} + (\Lambda_3)^2 \Lambda_{n-2} - \Lambda_4 \Lambda_{n-2} + \Lambda_n = 0. \quad (6.12)$$

This can be solved by setting $\Lambda_n = \gamma^n$, writing a generic combination of the two roots, and imposing consistency with $\Lambda_4 = 3\Lambda_3^2$. One finds

$$\Lambda_n = \frac{\Lambda_3^{n-2}}{6} (2(-1)^n + 2^n). \quad (6.13)$$

Plugging this in the potential in (6.1) and resumming, one ends up with the Bullough-Dodd potential [54, 32, 55], a known integrable model in two dimensions:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{6\Lambda_3^2} (2e^{-\Lambda_3 \phi} + e^{2\Lambda_3 \phi} - 3). \quad (6.14)$$

One can repeat this exercise, imposing extra symmetries, for example invariance under $\phi \rightarrow -\phi$. Odd powers of the expansion must vanish, so that $\Lambda_{\text{odd}} = 0$. The amplitude $\mathcal{M}_{2b \rightarrow 3b}$ is now trivially zero, the first non-trivial process being $\mathcal{M}_{2b \rightarrow 4b}$, shown in Fig. 6.4. This amplitude vanishes provided $\Lambda_6 = \Lambda_4^2$. Going through the

Figure 6.4: Tree-level diagrams contributing to $\mathcal{M}_{2b \rightarrow 4b}$ in a $\mathbb{Z}_2$ -invariant theory.

same steps as above for generic diagrams, one finds the recursion

$$-\Lambda_{n-2} \Lambda_4 + \Lambda_n = 0, \quad (6.15)$$

which is solved by

$$\Lambda_n = \Lambda_4^{\frac{n-2}{2}}. \quad (6.16)$$

Resumming the potential results in the sinh-Gordon potential

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{\Lambda_4} (\cosh(\sqrt{\Lambda_4} \phi) - 1), \quad (6.17)$$

another known integrable model in two dimensions.

Figure 6.5: Diagrams contributing to the process $\bar{\psi}\psi \rightarrow \phi\phi\phi\phi$. Solid lines represent bosons and dashed lines represent fermions.

Including the fermions

We now move on to the supersymmetric case. We can extend the considerations above to the fermionic propagator in (6.7). In the limit $x \rightarrow \infty$ for the momenta in (6.9), this is such that

$$\mathcal{G}_{L \rightarrow R}(p_1, \dots) \rightarrow \begin{pmatrix} -1 & 0 \\ -ix^{\max\{i \in L\}-2} & -1 \end{pmatrix}, \quad (6.18)$$

when the particles are ordered and

$$\mathcal{G}_{L \rightarrow R}(p_1, \dots) \rightarrow q, \quad (6.19)$$

with $q = (-1, 0)$, otherwise. Ordered diagrams involving $\mathcal{G}_{L \rightarrow R}$ do not appear at tree level, so that (6.18) will not be necessary.

Initially, we restrict our attention to theories preserving a $\mathbb{Z}_2$ -symmetry. These are invariant under $\phi \rightarrow -\phi$ and have $\lambda_{\text{odd}} = 0$, which implies $\Lambda_{\text{odd}} = 0$. The processes which involve bosons only are the same as the real scalar theories studied above. One finds the recursion relation (6.16) for the scalar couplings, which, using equation (6.2), translates into an identical relation for the λ_n 's appearing in the superpotential

$$\lambda_n = \lambda_4^{\frac{n-2}{2}}. \quad (6.20)$$

Performing the sum, one finds that

$$W(\phi) = \frac{1}{\lambda_4} \left(\cosh \sqrt{\lambda_4} \phi - 1 \right), \quad (6.21)$$

which is the superpotential of the $\mathcal{N} = 1$ supersymmetric sinh-Gordon model [48, 49, 50, 51].

In order to complete the derivation, one must prove that this particular choice of couplings implies no particle production in the processes which include fermions.

2 fermions into 4 bosons. First, we consider the simplest possible scattering process where particle production could occur, namely $\bar{\psi}\psi \rightarrow \phi^4$, and show that the corresponding amplitude $\mathcal{M}_{2f \rightarrow 4b}$ is zero, independently of the choice of momenta. The three diagrams in Fig. 6.5 contribute to this process. Their evaluation is simplified enormously by considering the choice of momenta in (6.9) above.

The process has only two fermions, with associated momenta p_1 and p_2 . Since the diagrams are evaluated at tree level, the fermionic propagator will never be ordered.

Whenever a fermionic propagator appears in a diagram, it can be replaced by $\not{q}$. The amplitude $\mathcal{M}_{2f \rightarrow 4b}$ will never display any divergence in this limit: the fermionic propagator cannot diverge and the bosonic propagator goes to zero or -1 . If any term of a given diagram goes to zero, it can be discarded from the very beginning.

In the limit $x \rightarrow \infty$, we evaluate the three classes of diagrams in Fig. 6.5 as

$$\begin{aligned} (a) & \rightarrow \bar{v}_2(6\lambda_4^2 \not{q})u_1 \\ (b) & \rightarrow \bar{v}_2(-\lambda_4\Lambda_4)u_1 = \bar{v}_2(-4\lambda_4^2)u_1 \\ (c) & \rightarrow \bar{v}_2(\lambda_6)u_1 \end{aligned} \quad (6.22)$$

Here we employ the usual conventions for the external fermionic particles: u ($\bar{u}$) denotes the initial (final) fermions, whereas v ($\bar{v}$) is for the final (initial) antifermions.

Diagram (c) is trivially evaluated, diagram (b) has only one non-vanishing configuration, whereas diagram (a) has six contributing configurations, resulting in the amplitude

$$\mathcal{M}_{2f \rightarrow 4b} \rightarrow \bar{v}_2(\lambda_4^2 6\not{q} + \lambda_6 - 4\lambda_4^2)u_1 = \lambda_4^2 \bar{v}_2(6\not{q} + \beta)u_1, \quad (6.23)$$

with $\beta = \frac{1}{\lambda_4^2}(\lambda_6 - 4\lambda_4^2)$. In order for the average modulus square $|\overline{\mathcal{M}_{2f \rightarrow 4b}}|^2$ to vanish, it must be $\beta = -3$, which gives the correct constraint for the couplings: $\lambda_6 = \lambda_4^2$, consistently with (6.20).

2 fermions into n bosons. The result above can be generalized to a general process involving two fermions going into n bosons: $\psi\bar{\psi} \rightarrow \phi^n$, where n is even by symmetry. All the diagrams are divided into three groups, schematically shown in Fig. 6.6.

Figure 6.6: Diagrams contributing to the process $\psi\bar{\psi} \rightarrow \phi^n$.

Diagrams in group 1 are essentially bosonic diagrams with an extra fermionic vertex attached to the first boson. They will always be vanishing, by the constraint

imposed on bosonic diagrams above. Group 3 consists of two diagrams, which can be trivially evaluated to

$$\bar{v}_2(\lambda_{n+2})u_1 + \bar{v}_2(\lambda_n(-1)\Lambda_4)u_1 = \bar{v}_2(\lambda_{n+2} - 4\lambda_4^{n/2})u_1. \quad (6.24)$$

Finally, group 2 is divided into three types of diagrams, which we dub of the *A*-type, *B*-type, and *C*-type. *A*-type diagrams have two or more fermionic propagators that will not be ordered, implying that the only surviving contribution will be proportional to $\not{q}\not{q}\dots\not{q}$. In the evaluation of $|\overline{\mathcal{M}}_{2f \rightarrow nb}|^2$ however, anything with at least two $\not{q}$'s will be proportional to $q \cdot q = 0$. Therefore, these diagrams do not contribute.

Figure 6.7: *B*-type diagrams for $n = 6, 8$ and also a generalization for any n . The numbers between the external legs represent the number of bosons in the vertex.

B-type diagrams are detailed in Fig. 6.7 and start appearing when $n = 6$. They have only one fermionic propagator. Looking for example at the case $n = 8$, it is easy to see that a cancellation takes place between the last terms due to the constraint imposed on the bosonic scattering, $\Lambda_6 = \Lambda_4^2$. This can be checked to happen also at arbitrary n , resulting in an effective expression given in the last line of Fig. 6.7. The factor of 2 comes from an exchange of the fermions that can always be made in this kind of diagrams. Anything with a different arrangement of the right-most legs will contribute to the bosonic scattering (of $n - 2$ bosons), already shown to be zero. In the end, the contribution of these diagrams to the amplitude is

$$B(n) \rightarrow 2 \sum_{\substack{i=2 \\ i \text{ even}}}^{n-4} \lambda_{i+2} \lambda_{n-i} (-1) \Lambda_4 \bar{v}_2 b(n, i) \not{q} u_1. \quad (6.25)$$

The minus sign comes from the ordered bosonic propagator and $b(n, i)$ is the number of permutations which leave the last three legs with momenta $(n+2, n+1, n)$ and give an ordered bosonic propagator. For each particular configuration, a propagator will depend on the i -th momentum, but since the last three must be fixed to get a non-zero contribution, $b(n, i)$ is the ways we can arrange $n - 3$ objects into i slots,

without worrying about their order, so that

$$B(n) \rightarrow -8\lambda_4^{n/2} \sum_{\substack{i=2 \\ i \text{ even}}}^{n-4} \bar{v}_2 \frac{(n-3)!}{i!(n-3-i)!} \not{q} u_1. \quad (6.26)$$

Figure 6.8: C -type diagrams for $n = 4, 6, 8$ and also a generalization for any n . The numbers between the external legs represent the number of bosons in the vertex.

The general form of a C -type diagram is shown in Fig. 6.8, remembering that only line-type diagrams will have an ordered bosonic propagator, potentially contributing to the amplitude. Each fermionic propagator can be split into a diagonal and off-diagonal part. Since the momenta are always out of order, the diagonal part will go to zero, and the off-diagonal one will go to $\not{q}$. This will happen for every permutation in the diagram, each one contributing in the same manner. The expression for $C(n)$ can be written as follows

$$C(n) \rightarrow \sum_{\substack{i=2 \\ i \text{ even}}}^{n-2} \lambda_{i+2} \lambda_{n-i+2} \bar{v}_2 c(n, i) \not{q} u_1. \quad (6.27)$$

The number of permutations $c(n, i)$ is associated with the diagram of i bosonic legs in the first vertex and $n-i$ bosonic legs in the second vertex. The amplitude associated with one configuration has a propagator that depends on the i -th momentum, so that $c(n, i)$ is the number of ways we can arrange n objects into i slots, irrespectively of their order:

$$C(n) \rightarrow \lambda_4^{n/2} \sum_{\substack{i=2 \\ i \text{ even}}}^{n-2} \bar{v}_2 \frac{n!}{i!(n-i)!} \not{q} u_1 \quad (6.28)$$

To write this expression we used a recursion, by assuming that the relation is satisfied to λ_n in order to check if the result is correct for λ_{n+2} .

Summing B -type and C -type diagrams, we end up with the full contribution to the group 2 diagrams

$$B(n) + C(n) = \lambda_4^{n/2} \bar{v}_2 \alpha \not{q} u_1, \quad (6.29)$$

where

$$\alpha = \sum_{\substack{i=2 \\ i \text{ even}}}^{n-2} \frac{n!}{i!(n-i)!} - 8 \sum_{\substack{i=2 \\ i \text{ even}}}^{n-4} \frac{(n-3)!}{i!(n-3-i)!} = 6. \quad (6.30)$$

The amplitude of the process of two fermions going to n bosons is given by the sum of these groups:

$$\mathcal{M}_{2f \rightarrow nb} = \bar{v}_2 \left(\lambda_4^{n/2} 6\not{q} + \lambda_{n+2} - 4\lambda_4^{n/2} \right) u_1 = \lambda_4^{n/2} \bar{v}_2 (6\not{q} + \beta) u_1, \quad (6.31)$$

where

$$\beta = \frac{1}{\lambda_4^{n/2}} (\lambda_{n+2} - 4\lambda_4^{n/2}). \quad (6.32)$$

This amplitude is zero only if $\beta = -3$, which implies the same recursive relation as in (6.20). Note that for the fermions the cancellation is at the level of the squared amplitude, not just at the level of the sum of the diagrams, as in the bosonic case.

2 bosons into n fermions The next class of diagrams left to be checked is two bosons going into four or more fermions. As explicitly show in [51], the process of two bosons going into four fermions is given by one diagram that vanishes trivially. To extend this to n fermions, we need to pick the set of momenta (6.9) and note that in the large x limit we have

$$|\overline{\mathcal{M}_{2b \rightarrow nf}}|^2 \sim x^{(n-2)! - ((n-2)!)^2} \rightarrow 0. \quad (6.33)$$

This happens just by the conservations laws (6.5) and does not impose new constraints on the coupling constants.

2 particles into n particles Finally, with the ingredients derived so far, it is possible to show that all other processes in the theory involving two initial particles and more than two final particles are zero. To show this, one can think recursively and use the following facts:

- (i) One can flip signs in the set of momenta in (6.9) and exchange what we call ingoing and outgoing particles.
- (ii) In the limit of x going to infinity, only line-type diagrams will contribute and one can factorize the new cases into the cases which have already been shown to vanish.

Since we showed that $\psi\bar{\psi} \rightarrow \phi^4$ has vanishing amplitude, by (i) we are automatically guaranteed that $\phi\phi \rightarrow \phi^2\psi\bar{\psi}$ also vanishes, just like $\phi\psi \rightarrow \phi^3\bar{\psi}$, and so on. There is only one 8-particle process that is not related by (i) to old cases. Working in the x going to infinity limit, it is possible, however, to factorize this diagram into terms always containing some vanishing $\mathcal{M}_{2 \rightarrow 4}$. One can go on recursively to higher point amplitudes, which will all vanish without imposing new constraints on the λ_n 's.

In conclusion, we see that the choice of couplings in (6.20) guarantees that no particle are produced in any process involving either bosons or fermions. The $\mathcal{N} = 1$ supersymmetric sinh-Gordon potential (and its analytic continuation to sine-Gordon)

Figure 6.9: Diagrams contributing to the process $\bar{\psi}\psi \rightarrow \phi\phi\phi$.

is then the only supersymmetric model of a single scalar superfield with two supercharges and a $\mathbb{Z}_2$ -symmetry that exists in two dimensions and does not produce particles. It would be interesting to extend this analysis to theories with more than one superfield and also to $\mathcal{N} = (2, 2)$ multiplets.

We relax now the assumption of $\mathbb{Z}_2$ -invariance. We discover that particle production does necessarily take place and apparently no supersymmetric extension of the Bullough-Dodd model exists. First of all, making use of (6.2) in (6.13), one can compute the first few λ_n 's:

$$\lambda_4 = 6\lambda_3^2, \quad \lambda_5 = 15\lambda_3^3. \quad (6.34)$$

In the fermionic sector, the simplest process is two fermions going into three bosons: $\bar{\psi}\psi \rightarrow \phi^3$. There are six diagrams, contributing to the amplitude of this process, as shown in Fig. 6.9. In the limit of interest for the momenta, all diagrams can be easily evaluated

$$\begin{aligned} \text{Group 1} &\rightarrow \bar{v}_2 (\lambda_3(-1)\Lambda_3(-1)\Lambda_3) + \lambda_3(-1)\Lambda_4 u_1 = \bar{v}_2 (-18\lambda_3^3) u_1 \\ \text{Group 2} &\rightarrow 2\bar{v}_2 (3\lambda_3\lambda_4\not{q}) u_1 = \bar{v}_2 (36\lambda_3^3\not{q}) u_1 \\ \text{Group 3} &\rightarrow \bar{v}_2 (\lambda_4(-1)\Lambda_3 + \lambda_5) u_1 = \bar{v}_2 (\lambda_5 - 18\lambda_3^3) u_1 \end{aligned} \quad (6.35)$$

The amplitude of this process is

$$\mathcal{M}_{2f \rightarrow 3b} \rightarrow \bar{v}_2 (36\lambda_3^3\not{q} + \lambda_5 - 36\lambda_3^3) u_1 = 6\lambda_3^3\bar{v}_2 (6\not{q} + \beta) u_1, \quad (6.36)$$

where

$$\beta = \frac{1}{6\lambda_3^3}(\lambda_5 - 36\lambda_3^3). \quad (6.37)$$

This amplitude will be zero when $\beta = -3$, or

$$\lambda_5 = 18\lambda_3^3, \quad (6.38)$$

which is a different choice than the one in the bosonic case in (6.34). Therefore, the bosonic and fermionic processes cannot be set to zero simultaneously with the same choice of couplings.

More than one scalar field

In the recent work [7], the limit 6.9 provided a tool to extend the recursion relation (6.12) to the case with more than one real scalar field.

The most general Lagrangian for a theory with N real scalar fields is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_\alpha)(\partial^\mu \phi_\alpha) - \frac{1}{2}m_\alpha^2 \phi_\alpha^2 - \sum_{n=3}^{\infty} \frac{v_n^{\alpha_1 \dots \alpha_n}}{n!} \phi_{\alpha_1} \dots \phi_{\alpha_n}, \quad (6.39)$$

where ϕ_α , $\alpha = 1, \dots, N$ are the real scalar fields and the repeated indices are summed over. The coupling constants $v_n^{\alpha_1 \dots \alpha_n}$ are totally symmetric tensors of rank n .

This addition of fields does not change the fact that in the limit (6.9) only the ordered type of diagrams will survive, leading to the exact same derivation for the recursion relations, with the only difference being the need to sum over the particle species in the internal propagators.

Therefore, the generalization of the recursion relation (6.12) is⁴

$$\tilde{v}_n^{\alpha_1 \dots \alpha_n} = \tilde{v}_{n-1}^{\alpha_1 \dots \alpha_{n-2} \beta} \tilde{v}_3^{\beta \alpha_{n-1} \alpha_n} + \tilde{v}_{n-2}^{\alpha_1 \dots \alpha_{n-3} \beta} \left(\tilde{v}_4^{\beta \alpha_{n-2} \alpha_{n-1} \alpha_n} - \tilde{v}_3^{\beta \alpha_{n-2} \gamma} \tilde{v}_3^{\gamma \alpha_{n-1} \alpha_n} \right) \quad (6.40)$$

which determines $\tilde{v}_n^{\alpha_1 \dots \alpha_n}$ in terms of $\tilde{v}_3^{\alpha\beta\gamma}$ and $\tilde{v}_4^{\alpha\beta\gamma\delta}$. Since the coupling constants are totally symmetric tensors, the recursion relation imposes direct constraints in $\tilde{v}_3^{\alpha\beta\gamma}$ and $\tilde{v}_4^{\alpha\beta\gamma\delta}$. For example, taking $N = 2$ and imposing

$$\begin{aligned} \tilde{v}_5^{11122} &= \tilde{v}_5^{11221} = \tilde{v}_5^{11212}, \\ \tilde{v}_5^{11112} &= \tilde{v}_5^{11121} = \tilde{v}_5^{11211}, \end{aligned} \quad (6.41)$$

allows us to completely fix the quartic coupling in terms of the cubic coupling:

$$\begin{aligned} \tilde{v}_4^{1111} &= \hat{v}_4^{1111} + \tilde{v}_3^{111} \tilde{v}_3^{111} - \tilde{v}_3^{111} \tilde{v}_3^{122} + 2\tilde{v}_3^{112} \tilde{v}_3^{112} + \tilde{v}_3^{122} \tilde{v}_3^{122} - \tilde{v}_3^{112} \tilde{v}_3^{222}, \\ \tilde{v}_4^{1112} &= \hat{v}_4^{1112} + \tilde{v}_3^{111} \tilde{v}_3^{112} + \tilde{v}_3^{112} \tilde{v}_3^{122}, \\ \tilde{v}_4^{1122} &= \hat{v}_4^{1122} + \tilde{v}_3^{112} \tilde{v}_3^{112} + \tilde{v}_3^{122} \tilde{v}_3^{122}, \\ \tilde{v}_4^{1222} &= \hat{v}_4^{1222} + \tilde{v}_3^{112} \tilde{v}_3^{122} + \tilde{v}_3^{122} \tilde{v}_3^{222}, \\ \tilde{v}_4^{2222} &= \hat{v}_4^{2222} + \tilde{v}_3^{112} \tilde{v}_3^{112} - \tilde{v}_3^{111} \tilde{v}_3^{122} + 2\tilde{v}_3^{122} \tilde{v}_3^{122} - \tilde{v}_3^{112} \tilde{v}_3^{222} + \tilde{v}_3^{222} \tilde{v}_3^{222}, \end{aligned} \quad (6.42)$$

where $\hat{v}_4^{\alpha\beta\gamma\delta}$ is the solution of the following linear homogeneous system

$$\begin{pmatrix} \tilde{v}_3^{112} & \tilde{v}_3^{122} - \tilde{v}_3^{111} & -\tilde{v}_3^{112} & 0 & 0 \\ -\tilde{v}_3^{122} & 2\tilde{v}_3^{112} - \tilde{v}_3^{222} & 2\tilde{v}_3^{122} - \tilde{v}_3^{111} & -\tilde{v}_3^{112} & 0 \\ \tilde{v}_3^{122} & \tilde{v}_3^{222} - \tilde{v}_3^{112} & -\tilde{v}_3^{122} & 0 & 0 \\ 0 & \tilde{v}_3^{122} & \tilde{v}_3^{222} - \tilde{v}_3^{112} & -\tilde{v}_3^{122} & 0 \\ 0 & -\tilde{v}_3^{122} & 2\tilde{v}_3^{112} - \tilde{v}_3^{222} & 2\tilde{v}_3^{122} - \tilde{v}_3^{111} & -\tilde{v}_3^{112} \\ 0 & 0 & \tilde{v}_3^{122} & \tilde{v}_3^{222} - \tilde{v}_3^{112} & \tilde{v}_3^{122} \end{pmatrix} \cdot \begin{pmatrix} \hat{v}_4^{1111} \\ \hat{v}_4^{1112} \\ \hat{v}_4^{1122} \\ \hat{v}_4^{1222} \\ \hat{v}_4^{2222} \end{pmatrix} = 0. \quad (6.43)$$

⁴The tilde means we are normalizing the coupling constants by the masses: $\tilde{v}_n^{\alpha_1 \dots \alpha_n} = \frac{v_n^{\alpha_1 \dots \alpha_n}}{(m_{\alpha_1} \dots m_{\alpha_n})}$.

If this matrix has rank five the system admits only the trivial solution $\hat{v}_4^{\alpha\beta\gamma\delta} = 0$, but it can happen, due a fine tuning of the cubic couplings that this matrix becomes degenerate allowing for nontrivial solutions for $\hat{v}_4^{\alpha\beta\gamma\delta} = 0$.

Besides this constraints and recursion realations, one needs a seed, that is provided by imposing no particle production for the first non trivial process: the scattering $2 \rightarrow 3$ for the theory with no additional symmetries and the process $2 \rightarrow 4$ for the $\mathbb{Z}_2$ -symmetric case. By doing that in the case where $N = 2$ and the matrix has rank five, [7] recovers the six Toda field theories with two particles [56].

The cases with lower rank found by [7] represent decouple pairs of Bullough-Dodd and Sinh-Gordon theories, and definitely more look must be done in this non-maximal rank cases, perhaps finding overlooked integrable models. There is also the possibility of going to $N > 2$, perhaps in this case also there is theories that are not in the form (3.36) of the Toda theories. A more immediate work could be done by combining the results of multiparticles Lagrangian [7] with supersymmetric Lagrangian [8] to analyze supersymmetric models with $\mathcal{N} > 1$.

Chapter 7

Conclusion

The definition of Toda theories and their integrability [32, 55] are algebraic conditions ultimately linked with the properties of the simply laced Lie algebras. The study of [7] suggests a correspondence between the recursion relations obtained and the algebraic constraints of Toda theories.

This correspondence is associated with the rank of the matrix (6.43). In the maximal rank, it seems to exist a one-to-one correspondence between these two integrability conditions. Perhaps a study in the algebraic side of Toda theories could show that in fact the recursion relation and the algebraic constraints are equivalent. However, in the non maximal rank case, this equivalence is not clear. In principle, one can potentially obtain a tensorial form of a Taylor expansion that cannot be resummed into the form of a Toda theory, therefore finding a new integrable theory.

Nonetheless, the already known integrable theories can be studied even further combining the aspects presented here. For example, it is possible to extend the staircase model to the supersymmetric case.

In the non-supersymmetric case, the sine-Gordon theory could be fine tuned to possess not only the central charge of the unitary minimal models but also the same coset representation. Using an analytic continuation of the S-matrix of this integrable theory it was possible to extract the staircase behavior via TBA. Analogously, in the $\mathcal{N} = 1$ case, the supersymmetric sine-Gordon theory can be fine tuned to match the central charge and coset representation of the $\mathcal{N} = 1$ superconformal minimal models [10].

Since this supersymmetric theory is integrable, it is possible to use the TBA and obtain a supersymmetric extension of the staircase model [57, 58]. Differently from the non-supersymmetric case, the perturbed $\mathcal{N} = 1$ minimal models ($\mathcal{SM}_p$) present a RG flow that jumps by an interval of two: $\mathcal{SM}_p \rightarrow \mathcal{SM}_{p-2}$. This result respects the generalization discussed in [16], since the superconformal minimal models are obtained by setting $k = 2$ in the coset representation (4.23).

Although not explored in this work, there is also the $\mathcal{N} = 2$ supersymmetric extension of the sine-Gordon model. This theory is related with the $\mathcal{N} = 2$ superconformal minimal models in the same way as before. Furthermore, the integrability of this theory allows for the use of the TBA to obtain the $\mathcal{N} = 2$ supersymmetric generalization of the staircase models [59].

The study of the perturbations of the minimal models relies on integrability, in

both non-supersymmetric and supersymmetric cases. Integrable perturbations allow for the use of the TBA to obtain the behavior of the RG flows in the vicinity of all minimal models. In the $\mathcal{N} = 1$ case, the amount of symmetries constrains the possible RG flows of the superconformal minimal models and allows for the study of non-integrable perturbations of these conformal theories. These yields a complete map for the supersymmetric flows of these theories [58]. This mapping could be the first step to understand the flows in the non-supersymmetric case.

In order to apply the Wilsonian ideas of RG flow one needs to use a Lagrangian description of the theories. This can be achieved via the Landau-Ginzburg formulation [60], where the minimal models appear as fixed points of a scalar Lagrangian and the RG flow can be obtained by standard perturbation around these points. This Lagrangian formulation of the (super)conformal minimal models is ideal to apply the topological ideas discussed in [61, 62].

In the space of quantum field theories one can define closed regions that define a topological object called Conley index, [62]. This topological index is associated with the number of regions of outgoing flows and the number of fix points inside of the region, but alone this is not enough to completely define what happens inside of the given region. In the arena of unitary minimal models we know not only the number of fixed points but also their location.

This framework could play an important role in the quest of mapping the RG flows of the unitary minimal models. For example, we could take a region around one minimal model and study the known RG flows that leave it. And by evaluating the Conley index one could understand, at least qualitatively what the unknown flows must do. By doing the same analysis in another region with another minimal model inside it, one could hope to obtain more information about this unknown flow. Perhaps this process can be done recursively imposing consistency conditions in the flows, that maybe are powerful enough to completely map the flows of the conformal minimal models.

Appendix A

Aspects of two-dimensional conformal field theories

Conformal invariance in two dimension is much more powerful than in other dimensions and can be simply understood as a combination of the usual Poincaré invariance with scale invariance [63], meaning that the system remains the same under rotations, translations, boosts and any rescaling of the coordinates.

Switching to the usual complex coordinates $\{z, \bar{z}\}$ one can find that the conformal symmetry in two dimensions corresponds to all sets of holomorphic transformations of z and antiholomorphic transformations of $\bar{z}$. This will generate an infinite dimensional algebra that in the quantum context must be central extended [2], to the so called Virasoro algebra

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\ [L_n, \bar{L}_m] &= 0, \end{aligned} \tag{A.1}$$

where c is the central charge. Since the holomorphic and antiholomorphic parts of the algebra decouple, it is sufficient to look only at holomorphic representations of this algebra.

It is common to choose the operator L_0 to be diagonal in the representation space, so it follows trivially from the algebra that L_m with $m > 0$ is a lowering operator. With the concept of lowering operator, it is easy to define a highest-weight state $|h\rangle$ in the following way

$$\begin{aligned} L_0|h\rangle &= h|h\rangle, \\ L_m|h\rangle &= 0, \text{ for all } m > 0. \end{aligned} \tag{A.2}$$

In the same way, the operators L_{-m} with $m > 0$ are raising operators, that can create what is called a descendant state, by applying multiples rising operators to a given highest-weight state. For example

$$|h'\rangle \equiv L_{-m_1} \dots L_{-m_n}|h\rangle, \tag{A.3}$$

with weight increased by

$$L_0|h'\rangle = h'|h'\rangle \quad \text{where } h' = h + m_1 + \dots + m_n. \quad (\text{A.4})$$

The number $l = m_1 + \dots + m_n$ is called level of the descendant $|h'\rangle$.

One can think of a highest-weight state as a starting point, from where the successive application of raising operators generates a tower of descendants, denominated Verma module.

On this Verma module we define a inner product, using the definition of Hermitian conjugate: $L_m^\dagger = L_{-m}$. This inner product also defines the norm of a state. For example, the state $|h'\rangle$ of (A.3) has norm (squared) given by

$$\langle h'|h'\rangle = \langle h|L_{m_n} \dots L_{m_1} L_{-m_1} \dots L_{-m_n}|h\rangle. \quad (\text{A.5})$$

Null-vectors. The representation of the Virasoro algebra can be reducible, meaning that can exist submodules inside this Verma module that by themselves are a full-fledged representation of the Virasoro algebra. One can imagine this sub modules to be generated by a highest-weight state $|\tilde{h}\rangle$, such that $L_m|\tilde{h}\rangle = 0$ for $m > 0$, although this state is also descendant from another highest-weight state, that is, it can be written in the form (A.3).

These states $|\tilde{h}\rangle$ are orthogonal to the whole Verma module

$$\langle \tilde{h}|h'\rangle = \langle \tilde{h}|L_{-m_1} \dots L_{-m_n}|h\rangle = \langle h|L_{m_n} \dots L_{m_1}|\tilde{h}\rangle^* = 0, \quad (\text{A.6})$$

and are called null-vectors, because in particular not only they have zero norm, $\langle \tilde{h}|\tilde{h}\rangle = 0$, but all the descendants have it as well, since their norm ends up in an expression that is proportional to $\langle \tilde{h}|\tilde{h}\rangle$.

Unitarity. A representation of the Virasoro algebra is unitary if there is no state with negative norm. Since the explicitly value of the inner product depends on the weight h and is related, through the algebra, to the central charge c , the positivity of the norm will impose constrains on the possible values of the pair (h, c) .

For example, take the norm of $L_{-m}|h\rangle$:

$$\begin{aligned} \langle h|L_m L_{-m}|h\rangle &= \langle h|(L_{-m}L_m + 2mL_0 + \frac{c}{12}m(m^2 - 1))|h\rangle \\ &= (2mh + \frac{c}{12}m(m^2 - 1))\langle h|h\rangle. \end{aligned} \quad (\text{A.7})$$

The unitarity condition requires this norm to be positive, but that already implies in two constrains: if $c < 0$ one can increase m to a sufficient large number that makes the norm negative and second, taking $n = 1$ in the above expression implies that the conformal weight must be positive, in order to have a positive norm.

To generalize these unitarity constrains, it is useful to define the Gram matrix of inner products between the basis states

$$M_{i,j} = \langle h_i|h_j\rangle, \quad (\text{A.8})$$

where $|h_k\rangle$ are the basis states. This matrix is block diagonal, with each block $M^{(l)}$ corresponding to states of level l .

A generic state can be written in terms of the basis $|v\rangle = \sum v_i |h_i\rangle$ and its norm, in matrix form, is

$$\langle v|v\rangle = v^\dagger M v, \quad (\text{A.9})$$

since the Gram matrix is Hermitian, it can be diagonalized in Λ by a unitary matrix U : $M = U \Lambda U^\dagger$, and

$$\langle v|v\rangle = v^\dagger U \Lambda U^\dagger v = \alpha^\dagger \Lambda \alpha = \sum \Lambda_i |\alpha_i|^2, \quad (\text{A.10})$$

where $\alpha = U^\dagger v$. Consequently, there will be states with negative norm if M has one or more negative eigenvalues. Furthermore, if the eigenvalues are zero, that means we have a vanishing norm for the state, characterizing a null-vector and a reducible Verma module.

Due the block structure of the Gram matrix, it is possible to analyze separately the block matrices $M^{(l)}$:

$$\begin{aligned} M^{(0)} &= \langle h|h\rangle = 1 \\ M^{(1)} &= \langle h|L_1 L_{-1} h\rangle = 2h \\ M^{(2)} &= \begin{pmatrix} \langle h|L_1 L_1 L_{-1} L_{-1}|h\rangle & \langle h|L_1 L_1 L_{-2}|h\rangle \\ \langle h|L_2 L_{-1} L_{-1}|h\rangle & \langle h|L_2 L_{-2}|h\rangle \end{pmatrix} = \\ &= \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h + c/2 \end{pmatrix} \end{aligned} \quad (\text{A.11})$$

From $M^{(0)}$ there is no unitarity condition, but in $M^{(1)}$ we recover the constraint that $h > 0$. For $M^{(2)}$, it is necessary to evaluate the determinant

$$\det M^{(2)} = 32(h - h_{1,1})(h - h_{1,2})(h - h_{2,1}), \quad (\text{A.12})$$

where

$$\begin{aligned} h_{1,1} &= 0 \\ h_{1,2} &= \frac{1}{16}(5 - c - \sqrt{(1-c)(25-c)}) \\ h_{2,1} &= \frac{1}{16}(5 - c + \sqrt{(1-c)(25-c)}). \end{aligned} \quad (\text{A.13})$$

Since the product of the two eigenvalues of $M^{(2)}$ is equal to its determinant, the value of $\det M^{(2)}$ must be positive in order to have a unitary conformal theory. This process allows, at each level, the exclusion of non-unitary regions of the plane (c, h) , by tracing curves where $\det M^{(2)} = 0$, that is when $h = h_{1,1}, h_{1,2}, h_{2,1}$, as portrayed in part (a) of Figure A.1.

The good news is that there is a general formula [64] for the determinant of the Gram matrix, the Kac determinant:

$$\det M^{(l)} = \alpha_l \prod_{\substack{r,s \geq 1 \\ rs \leq l}} (h - h_{r,s}(c))^{p(l-rs)}, \quad (\text{A.14})$$

Figure A.1: The vanishing curves $\mathcal{C}_{r,s}$ in the (c, h) plane, where (a), (b) and (c) represent these vanishing curves at the levels 2, 3 and 4. The curves with lower levels appear also with the higher level curves, with the values of r and s indicated near the curves when they first appear. The shaded areas correspond to manifestly nonunitary theories.

where $p(l - rs)$ is the number of partitions of the integer $(l - rs)$ and α_l is just a positive constant independent of h and c

$$\alpha_l = \prod_{\substack{r,s \geq 1 \\ rs \leq l}} ((2r)^s s!)^{p(l-rs)-p(l-r(s+1))}. \quad (\text{A.15})$$

The function $h_{r,s}$ can be written as

$$h_{r,s}(c) = \frac{c-1}{24} + \frac{1}{4} \left(r \frac{\sqrt{1-c} + \sqrt{25-c}}{\sqrt{24}} + s \frac{\sqrt{1-c} - \sqrt{25-c}}{\sqrt{24}} \right). \quad (\text{A.16})$$

In the (c, h) plane, the Kac determinant vanishes along the curves $h = h_{r,s}(c)$. This is explicitly shown for the levels $l = 2, 3$ in Figure A.1. These vanishing curves $\mathcal{C}_{r,s}$ are precisely the ones that characterize the appearance of null-vectors and also define the boundary in this (c, h) plane between unitary and non-unitary theories.

Figure A.2: Vanishing curves at level $l = 10$, with the shaded area representing nonunitary theories.

Unitary representations with $c > 1$. As Figure A.2 suggests, and shown in [65], the vanishing curves $\mathcal{C}_{r,s}$ do not enter the region $R = \{c > 1, h > 0\}$.

When for a given level l the value $|h|$ is much bigger than $\max |h_{r,s}|$, then $\det M^{(l)} \sim \alpha_l h^r$, for some positive r . Since α_l is positive, the Kac determinant is also positive in this limit. However, none of the roots of $h_{r,s}$ lies in the region R , so the Kac determinant must stay positive inside the entire region.

Since the Kac determinant is positive, the number of negative eigenvalues must be an even number. Since the number of negative eigenvalues changes only across the curves $\mathcal{C}_{r,s}$, this number must stay the same in the region R , and as shown in [65] it is zero.

In conclusion, all the representations with $c > 1$ and $h > 0$ are unitary.

Unitary representations with $c < 1$. As presented in Figure A.1, the regions inside the vanishing curves $\mathcal{C}_{r,s}$ have states of negative norm and are associated with non-unitary representations. In order for a representation to be unitary, it cannot lie inside these vanishing curves for all levels. As illustrated in Figure A.2 and shown in [65], the vanishing curves fill all the region $S = c < 1, h > 0$ so the only place a unitary region can appear in S is precisely on the intersection of the vanishing curves, as shown by the dots in Figure A.1.

With this result a sufficient condition for unitarity is established inside the region S : only the following discrete values of representations

$$\begin{aligned} c &= 1 - \frac{6}{m(m+1)} \\ h_{r,s} &= \frac{((m+1)r-ms)^2-1}{4m(m+1)}, \end{aligned} \tag{A.17}$$

where

$$1 \leq r < m, \quad 1 \leq s < r \quad \text{and} \quad m = 3, 4, \dots, \tag{A.18}$$

can be unitary representations in the region S . In fact these are all the unitarity representations that exist in this region and the proof will come after the general comment about minimal models, where we discuss the coset construction.

Minimal models. Going back to the Kac determinant and looking at the case where $h = h_{r,s}$, there exists a null-vector at level $l = rs$. Since there is a map between states and operators, the appearance of a null vector imposes a constraint [64] in the operator algebra,

$$\phi_{r_1, s_1} \times \phi_{r_2, s_2} = \sum_{\substack{k=r_1+r_2-1 \\ k=1+|r_1-r_2| \\ k+r_1+r_2=1 \pmod{2}}}^{k=r_1+r_2-1} \sum_{\substack{l=s_1+s_2-1 \\ l=1+|s_1-s_2| \\ k+s_1+s_2=1 \pmod{2}}}^{l=s_1+s_2-1} \phi_{k_l}, \tag{A.19}$$

where $\phi_{r,s}$ is the primary field operator of dimension $h_{r,s}$, and in the summation the variable k and l are incremented by 2.

The minimal models are the representations that satisfy these constraints in the operator algebra, that, following [65] can be translated in the conditions

$$\begin{aligned} c &= 1 - 6 \frac{(p-p')^2}{pp'} \\ h_{r,s} &= \frac{(pr-p's)^2 - (p-p')^2}{4pp'}, \end{aligned} \tag{A.20}$$

where

$$1 \leq r < p' \text{ and } 1 \leq s < p. \quad (\text{A.21})$$

For each pair of (p, p') there is a minimal model denoted by $\mathcal{M}(p, p')$ that is only unitary if $p = m + 1$ and $p' = m$ with $m = 2, 3, \dots$. Under these conditions the general expression for the minimal models (A.20) coincides with the unitary case (A.17).

The canonical examples of unitary minimal models are the famous Ising model $\mathcal{M}(4, 3)$ with $c = 1/2$, the Tricritical Ising model $\mathcal{M}(5, 4)$ with $c = 7/10$, and the Three-State Potts model $\mathcal{M}(6, 5)$ with $c = 4/5$; see e.g. [65].

The coset construction. Consider an affine Lie group $\mathcal{G}$ with algebra $\hat{\mathfrak{g}}$ and let $\hat{\mathfrak{p}}$ be a subalgebra of $\hat{\mathfrak{g}}$. We denote by L_m^g and L_m^p the Virasoro modes obtained respectively for $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{p}}$ through the Sugawara construction [10].

The $\hat{\mathfrak{p}}$ generators $\tilde{J}_n^a$ are linear combinations of the generators of $\hat{\mathfrak{g}}$ J_n^a :

$$\tilde{J}_n^a = \sum_b m_{a,b} J_n^b. \quad (\text{A.22})$$

Since the Virasoro modes and the generators of $\hat{\mathfrak{g}}$ are related through

$$[L_m^g, J_n^a] = -n J_n^a, \quad (\text{A.23})$$

it is possible to multiply by $m_{a,b}$ and sum over b to obtain the generators of $\hat{\mathfrak{p}}$

$$[L_m^g, \tilde{J}_n^a] = -n \tilde{J}_n^a. \quad (\text{A.24})$$

The generators of $\hat{\mathfrak{p}}$ are also related to their Virasoro modes through an analogous expression to (A.23)

$$[L_m^p, \tilde{J}_n^a] = -n \tilde{J}_n^a. \quad (\text{A.25})$$

Therefore

$$[L_m^g - L_m^p, \tilde{J}_n^a] = 0, \quad (\text{A.26})$$

which implies in

$$[L_m^g - L_m^p, L_n^p] = 0 \Rightarrow [L_m^g, L_n^p] = [L_m^p, L_n^p]. \quad (\text{A.27})$$

Defining $L_m^{(g/p)} \equiv L_m^g - L_m^p$ it is trivial to obtain the commutation relation

$$\begin{aligned} [L_m^{(g/p)}, L_n^{(g/p)}] &= [L_m^g, L_n^g] - [L_m^p, L_n^p] \\ &= (m - n) L_{m+n}^{(g/p)} + \frac{c(\hat{\mathfrak{g}}) - c(\hat{\mathfrak{p}})}{12} m(m^2 - 1) \delta_{m+n, 0}, \end{aligned} \quad (\text{A.28})$$

which shows that $L_m^{(g/p)}$ satisfies the Virasoro algebra. Its central charge is the central charge of its constituents

$$c(\hat{\mathfrak{g}}/\hat{\mathfrak{p}}) = \frac{k \dim g}{k + g} - \frac{\chi_e k \dim p}{\chi_e k + p}, \quad (\text{A.29})$$

where k is the level of $\hat{\mathfrak{g}}$ and $\chi_e k$ is the level of $\hat{\mathfrak{p}}$, with χ_e being the embedding index. This is called GKO coset construction [10].

The most common encountered coset forms are the diagonal coset models:

$$\frac{\hat{\mathfrak{g}}_{k_1} \oplus \hat{\mathfrak{g}}_{k_2}}{\hat{\mathfrak{g}}_{k_1+k_2}}. \quad (\text{A.30})$$

This does not change the derivation of the central charge, since the direct sum will imply only in a simple sum of the generator of each group, with the central charge being

$$c = \dim g \left(\frac{k_1}{k_1 + g} + \frac{k_2}{k_2 + g} - \frac{k_1 + k_2}{k_1 + k_2 + g} \right). \quad (\text{A.31})$$

If we consider the affine Lie algebra $\hat{\mathfrak{g}} = \hat{\mathfrak{su}}(2)$ and the diagonal coset

$$\frac{\hat{\mathfrak{su}}(2)_k \oplus \hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_{k+1}}, \quad (\text{A.32})$$

the central charge, by simply substitution in the expression (A.31), is

$$c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)}. \quad (\text{A.33})$$

Making the identification $k+2 = m$ one rediscovers the expression in (A.17) for the central charge unitary minimal models.

Moreover, the unitary minimal models can be described by the coset (A.32) since not only the central charge is the same, but as explicitly shown in [10], the primary fields of the minimal models appear in the character decomposition of this coset.

It is important to comment that although the unitary minimal models mentioned earlier $\mathcal{M}(4,3)$, $\mathcal{M}(5,4)$ and $\mathcal{M}(6,5)$ can be written in the coset form (A.32), the coset representation is not unique, meaning that they can be expressed as another diagonal coset form.

Appendix B

Thermodynamical Bethe Ansatz

It follows below a short derivation, based on [26, 27], of how one (in the thermodynamical limit) can obtain the scaling function $c(R)$ that is associated with the central charge of the background field theory and has the staircase behavior shown in Figure 4.2.

Before going any further, let us focus on the theory given by (4.14). It is a bosonic theory, but it is of fermionic type: $S(0) = -1$. A quick look at equation (4.17) shows that exchanging two particles with the same rapidity yields in the an antisymmetric wave function, which is not a bosonic behavior! Therefore, this theory possess selection rules, namely, two particles cannot have the same rapidity.

We can quantize the quantum field theory on the torus in two different ways. The first way is to put the states in the R -direction (defining a Hilbert space denoted by $\mathcal{R}$) and use the L -direction as time evolution. The other way is to define the states in the L -direction (now the Hilbert space is denoted by $\mathcal{L}$) and associate the R -direction with time evolution. Let us call these two quantizations RL and LR respectively.

In RL quantization the partition function is

$$Z[R, L] = \text{Tr}_{\mathcal{R}} e^{-LH_{\mathcal{L}}}, \quad (\text{B.1})$$

where the subscript means that the states are traced over $\mathcal{R}$. In the thermodynamical limit ($L \rightarrow \infty$), only the ground state of the Hamiltonian will survive

$$Z[R, L] \rightarrow e^{-LE(R)}. \quad (\text{B.2})$$

The quantity $E(R)$ is the Casimir of the system on $\mathcal{R}$ at temperature $1/L$.

Analogously in LR quantization the partition function is

$$Z[R, L] = \text{Tr}_{\mathcal{L}} e^{-RH_{\mathcal{R}}}, \quad (\text{B.3})$$

and for $L \rightarrow \infty$ we have

$$Z[R, L] \rightarrow e^{-RLf(R)}, \quad (\text{B.4})$$

where $f(R)$ is the bulk free energy density on $\mathcal{L}$ at temperature $1/R$, which follows from the very definition of free energy.

Since the quantization of the theory should not change physical aspects like the partition function, one can simply match the expressions (B.2) and (B.4) finding

$$E(R) = Rf(R). \quad (\text{B.5})$$

Besides this relation, the system must also obey the boundary conditions imposed by the torus configuration. As done in Chapter 4, it is possible to use the periodic boundary conditions in the RL quantization to obtain the Bethe equation,

$$e^{ip_i L} \prod_{j \neq i} S_{i,j} = 1 \quad (\text{B.6})$$

which now must be rewritten in the thermodynamical limit.

The first step is to write explicitly p in terms of the rapidity θ and define the phase $\exp i\gamma(\theta) = S(\theta)$, yielding

$$mL \sinh(\theta_i) + \sum_{j \neq i} \gamma(\theta_i - \theta_j) = 2\pi n_i. \quad (\text{B.7})$$

In the thermodynamical limit the number of particles N also goes to infinity, maintaining the ratio L/N fixed. Therefore, the spectrum of rapidity solutions to (B.7) becomes dense, meaning the separation of two adjacent values is of order $(\theta_i - \theta_{i+1} \sim 1/mL)$.

It is convenient to define a rapidity density $\rho_r(\theta)$ as just the number of particles in the interval $\theta, \theta + \Delta\theta$ divided by $L\Delta\theta$. With this definition the sum on (B.7) can now be expressed as an integral.

$$mL \sinh(\theta_i) + \int \gamma(\theta_i - \theta') \rho_r(\theta') d\theta' = 2\pi n_i. \quad (\text{B.8})$$

Each time that n_i is a set of admissible quantum numbers, the corresponding solution θ_i is said to be root and the density of these solutions around the value θ is the previously defined $\rho_r(\theta)$. However, due to the selection rules, there are solutions θ_i associated with integers $\tilde{n}_i$ that do not correspond to admissible quantum numbers. They are called holes and their density around θ is denoted by $\rho_h(\theta)$.

The possibility of having these two kinds of solutions is due to two circumstances. The first one is that

$$2\pi J(\theta) = mL \sinh(\theta) + \int \gamma(\theta - \theta') \rho_r(\theta') d\theta' \quad (\text{B.9})$$

is monotonically increasing. So the possible n_i must be increasing, but due the selection rules, it is possible to have missing elements in this sequence. Therefore, in the thermodynamical limit there are both densities of roots and holes. The total density $\rho(\theta)$ of the occupied and empty levels is equal to the derivative of the function $J(\theta)$

$$\rho(\theta) = \rho_r(\theta) + \rho_h(\theta) = \frac{dJ(\theta)}{d\theta}. \quad (\text{B.10})$$

We thus arrive at the first thermodynamical Bethe equation

$$\rho(\theta) = mL \cosh(\theta) + \int \varphi(\theta - \theta') \rho_r(\theta') d\theta', \quad (\text{B.11})$$

where

$$\varphi(\theta) = \frac{d\gamma(\theta)}{d\theta} = -i \frac{d}{d\theta} \log S(\theta) \quad (\text{B.12})$$

is called kernel and contains all the information of the integrable theory S -matrix.

The problem needs at least another equation to be solvable, since the two unknown densities had to be introduced. This new equation comes from working in the other quantization in the thermodynamical limit. In this limit we have that the free energy is given by the minimization condition:

$$F = U - TS \Rightarrow Lf(R) = H_{\mathcal{L}} - \frac{1}{R}\mathcal{S}. \quad (\text{B.13})$$

Let us analyze this relation. The first term is the Hamiltonian in the LR , quantization which can be trivially written in terms of the density of roots

$$H_{\mathcal{L}} = \int m \cosh \theta \rho_r(\theta) d\theta. \quad (\text{B.14})$$

The second term is the entropy. In the thermodynamical limit there is $N \sim L\rho(\theta)\Delta\theta$ of levels in each interval $\Delta\theta$ and $n \sim L\rho_r(\theta)\Delta\theta$ particles distributed among them. In this fermionic type theory, the number of different ways of distributing the particles among these levels is given by

$$\Omega = \frac{(N+n-1)!}{(N-1)!n!} = \frac{[L(\rho(\theta) + \rho_r(\theta) - 1)\Delta\theta]!}{[L(\rho(\theta) - 1)\Delta\theta]![L\rho_r(\theta)\Delta\theta]!}. \quad (\text{B.15})$$

Using $\mathcal{S} = \log(\Omega)$ we obtain

$$\mathcal{S} = \int d\theta (\rho \log \rho - \rho_r \log \rho_r - (\rho - \rho_r) \log(\rho - \rho_r)). \quad (\text{B.16})$$

To derive the thermodynamics of the system it is necessary to minimize the free energy with respect to the densities ρ and ρ_r , subject to the constraint (B.11). This can be solved by introducing the pseudo-energy $\epsilon(\theta)$, defined as

$$e^{-\epsilon} \equiv \frac{\rho_r}{\rho - \rho_r}. \quad (\text{B.17})$$

In this notation, the extremum condition takes the form

$$Rm \cosh \theta = \epsilon(\theta) + \int \varphi(\theta - \theta') \log(1 + e^{-\epsilon(\theta')}) \frac{d\theta'}{2\pi}. \quad (\text{B.18})$$

It turns out that the densities ρ and ρ_r enter the expression for the extremal free energy f only in the ratio ρ/ρ_r , giving simply

$$f(R) = \frac{-m}{R} \int \cosh \theta \log(1 + e^{-\epsilon(\theta)}) \frac{d\theta}{2\pi}. \quad (\text{B.19})$$

Here we can use the relation (B.5) to write it in terms of the Casimir energy $E(R)$ giving the second thermodynamical Bethe equation

$$E(R) = -m \int \cosh \theta \log(1 + e^{-\epsilon(\theta)}) \frac{d\theta}{2\pi}. \quad (\text{B.20})$$

Appendix C

Aspects of supersymmetry

Supersymmetry can be intuitively understood as the invariance of a system under the exchange of bosons and fermions. It is, however, more than that, it is a spacetime symmetry achieved by an enhancement of the Poincaré algebra (with generators P^μ and $M^{\mu\nu}$), through a graded Lie algebra [66]. This means we add spinor generators Q_α^A and $\bar{Q}_{\dot{\alpha}A}$, called supercharges, where $A = 1, 2, \dots, \mathcal{N}$ is the number of supercharge pairs.

A graded algebra treats differently operators depending on their bosonic or fermionic nature:

$$\mathcal{O}_a \mathcal{O}_b - (-1)^{\eta_a \eta_b} \mathcal{O}_b \mathcal{O}_a = i C_{ab}^e \mathcal{O}_e, \quad (\text{C.1})$$

where

$$\eta_a = \begin{cases} 0, & \text{if } \mathcal{O}_a \text{ is a bosonic generator} \\ 1, & \text{if } \mathcal{O}_a \text{ is a fermionic generator.} \end{cases} \quad (\text{C.2})$$

The full supersymmetry algebra can be easily obtained. For example, the commutator of the supercharges with $M^{\mu\nu}$ is the usual commutator for a spinor, since by construction these supercharges must behave as spinors under Lorentz transformations. The other commutators (and anticommutators) are obtained through index analysis together with Jacobi identity,[67],

$$\begin{aligned} [Q_\alpha^A, M^{\mu\nu}] &= (\sigma^{\mu\nu})_\alpha^\beta Q_\beta^A, & [Q_\alpha^A, P^\mu] &= 0, \\ \{Q_\alpha^A, Q_\beta^B\} &= \epsilon_{\alpha\beta} Z^{AB}, & \{Q_\alpha^A, Q_{\dot{\alpha}B}\} &= 2(\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu \delta_B^A, \end{aligned} \quad (\text{C.3})$$

with analogous relations for the $\bar{Q}$ supercharge. The central charges Z^{AB} commute with all generators and are antisymmetric in the exchange of the indices, implying that for $\mathcal{N} = 1$ there is no central extension of the superalgebra.

There can also be an internal symmetry of the system, denoted by R -symmetry that for the case with $Z^{AB} = 0$ is just $H = U(\mathcal{N})$, but for the central extended superalgebra will be a subgroup of H . It is the very existence of supercharges that changes dramatically the study of extended superalgebras, but from this point forward it will only be considered the supersymmetric case with one pair of supercharges: $\mathcal{N} = 1$.

Superspace formalism. With supersymmetry it is very natural to expand the concept of spacetime to the concept of superspace. This formalism, first proposed

by [68], has a very intuitive motivation: in the same way that the energy-momentum tensor generates translations in ordinary space time, the supercharges together with the energy-momentum tensor will generate translations in this superspace.

This realization of the supersymmetry is achieved by enlarging the spacetime with four anticommuting (Grassmann) variables θ^α and $\bar{\theta}^{\dot{\alpha}}$, that give the fermionic dimensions of the superspace. More formally one can define the superspace as the coset between the super Poincaré group and the Lorentz group.

A finite element of the super Poincaré group can be easily written¹ as

$$g(x^\mu, \theta, \bar{\theta}) = e^{i(\theta Q + \bar{\theta} \bar{Q} - x^\mu P_\mu)}, \quad (\text{C.4})$$

and by considering the action of this group elements in the superspace

$$\int \cosh \theta \log(1 + e^{\epsilon(\theta)} \frac{d\theta}{2\pi} \{x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}\}), \quad (\text{C.5})$$

it easily follows that

$$g(x^\mu, \theta, \bar{\theta})g(a^\mu, \epsilon, \bar{\epsilon}) = g(x^\mu + a^\mu + i\epsilon\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\epsilon}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}), \quad (\text{C.6})$$

meaning that the coordinates of the superspace transform, under supersymmetric transformations, as

$$\{x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}\} \rightarrow \{x^\mu + a^\mu + i\epsilon\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\epsilon}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}\}. \quad (\text{C.7})$$

This has a linear behavior, corroborating to the interpretation of supersymmetric transformations as just translations in superspace.

In conventional field theory after the proper definition of spacetime, one defines the functions that take values in the spacetime coordinates, in other words the fields of the theory, that can be scalar, spinor, vector, tensor, etc. function of the coordinates $\{x^\mu\}$. In superspace formalism is also possible to define a superfield [69], which is nothing more than a scalar, spinor, vector, etc. function that takes values in the coordinates of the superspace $\{x^\mu, \theta, \bar{\theta}\}$.

Due the anticommuting properties of the Grassmann variables it is possible to expand this general superfield in θ and $\bar{\theta}$, generating a finite number of terms with the coefficients of this expansion being the conventional fields (called component fields) in the spacetime.

The most general superfield is

$$\begin{aligned} S(x^\mu, \theta, \bar{\theta}) = & \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + (\theta\theta)F(x) + (\bar{\theta}\bar{\theta})G(x) + \\ & + (\theta\sigma^\mu\bar{\theta})V_\mu(x) + (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\rho(x) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x). \end{aligned} \quad (\text{C.8})$$

The translation in the superspace (C.7) implies the following representations for the operators Q , $\bar{Q}$ and P in superspace,

$$\begin{aligned} \mathcal{Q}_\alpha &= -i\partial_\alpha - (\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \\ \bar{\mathcal{Q}}_{\dot{\alpha}} &= +i\bar{\partial}_{\dot{\alpha}} + \theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu, \end{aligned} \quad (\text{C.9})$$

$$\mathcal{P}_\mu \int \cosh \theta \log(1 + e^{\epsilon(\theta)} \frac{d\theta}{2\pi}) = -i\partial_\mu.$$

¹Notation: $\theta Q \equiv \theta^\alpha Q_\alpha$ and $\bar{\theta} \bar{Q} \equiv \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}$

In order to obtain how the superfield $S(x^\mu, \theta, \bar{\theta})$ transforms under supersymmetric transformations, we first treat it as a field operator

$$S(x^\mu, \theta, \bar{\theta}) \mapsto e^{-i(\epsilon Q + \bar{\epsilon} \bar{Q})} S e^{i(\epsilon Q + \bar{\epsilon} \bar{Q})}, \quad (\text{C.10})$$

and secondly as a Hilbert vector

$$S(x^\mu, \theta, \bar{\theta}) \mapsto e^{-i(\epsilon Q + \bar{\epsilon} \bar{Q})} S(x^\mu, \theta, \bar{\theta}) = S(x^\mu + a^\mu + i\epsilon\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\epsilon}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}). \quad (\text{C.11})$$

Comparing both expressions (to first order in the parameters $\epsilon, \bar{\epsilon}$), one finds the transformation rule for the superfield

$$i(\epsilon Q + \bar{\epsilon} \bar{Q}) S = \delta S. \quad (\text{C.12})$$

By using this expression, together with (C.9), one can verify explicitly the transformation of each component field of S under supersymmetric transformation.

Superfields are linear representations of the superalgebra, however, these representations are highly reducible, meaning that it is possible to eliminate some component fields by imposing covariant constraints, that is, constraints that do not restrain the x -dependence of the component fields.

If S_1 and S_2 are superfields, that is, they transform like (C.12), it is trivial to check that the product $S_1 S_2$ is also a superfield. Although $\partial_\mu S$ is also a superfield, the derivative with respect to the Grassmann variables $\partial_\alpha S$ is not, so it is possible to define a covariant derivative:

$$\mathcal{D}_\alpha \equiv \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu, \quad \bar{\mathcal{D}}_{\dot{\alpha}} \equiv -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta (\sigma^\mu)_{\beta\dot{\alpha}} \partial_\mu \quad (\text{C.13})$$

which anticommute with both $\mathcal{Q}$ and $\bar{\mathcal{Q}}$. Therefore

$$[\mathcal{D}_\alpha, \epsilon Q + \bar{\epsilon} \bar{Q}] = 0, \quad (\text{C.14})$$

then $\mathcal{D}_\alpha S$ is a superfield.

These covariant derivatives (or superderivatives) allow us to get irreducible representations of supersymmetry by imposing constraints on the superfields. The one we are going to treat more carefully is the chiral (or antichiral) superfield constraints [70],

$$\bar{\mathcal{D}}_{\dot{\alpha}} S = 0 \quad \text{or} \quad \mathcal{D}_\alpha \bar{S} = 0 \quad (\text{C.15})$$

The chiral constraint creates an irreducible representation, known as chiral superfield, with the following expression

$$\begin{aligned} \Phi(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) &= \phi(x) + \sqrt{2}\theta\psi(x) + (\theta\theta)F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \\ &\quad - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\phi(x). \end{aligned} \quad (\text{C.16})$$

where ϕ is a complex scalar field, ψ is the usual spinor field with spin $s = 1/2$ and F is an auxiliary field that is discussed below.

As already mentioned, (C.9) together with (C.12) can be used to obtain the way each component field transforms under supersymmetric transformations. For the

case of the chiral superfield, the transformations under supersymmetric transformations are given by

$$\begin{aligned}\delta\phi &= \sqrt{2}\epsilon\psi, \\ \delta\psi &= i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu\phi(x) + \sqrt{2}\epsilon F, \\ \delta F &= i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi(x).\end{aligned}\tag{C.17}$$

An important aspect that need to be noticed here is that the field F transforms as a total derivative, which is special for the case of chiral fields. For any superfield that can be constructed the field D that appears in (C.8) will also transform as a total derivative, and it is precisely this behavior under supersymmetric transformations that gives an easy way to define a supersymmetric Lagrangian.

Supersymmetric Lagrangian. The most general supersymmetric invariant Lagrangian for a chiral superfield can be written as:

$$\mathcal{L} = K(\Phi, \Phi^\dagger)|_D + (W(\Phi)|_F + \text{h.c.}),\tag{C.18}$$

where K and W are respectively denominated Kähler potential and super potential, see e.g. [69].

The subscript D in K means that from the resulting superfield of the Kähler potential, we select only the D -term (i.e. the term accompanied² by $(\theta\theta)$ as seen in (C.8)) and analogous the subscript F represents taking only the F -term of the superfields inside the super potential.

The final Lagrangian will have only D - and F -terms, but this means that under supersymmetric transformations the variation of the Lagrangian will be proportional to a total derivative, $\delta\mathcal{L} \propto \partial_\mu(\dots)$, and the action is invariant under supersymmetry!

With a supersymmetric invariant Lagrangian in hands, the goal is to write this in the usual component fields. In order to do so, one needs to analyze the dimensions of the fields in the theory:

- The superfield has the same dimensions as the scalar field: $[\Phi] = [\phi] = 1$.
- Since $[\psi] = \frac{3}{2}$ it follows from (C.16) that $[\theta] = -\frac{1}{2}$.
- With the Grassmann dimensions in hands, (C.16) implies that $[F] = 2$.

In order to have $[\mathcal{L}] = 4$ we must have

- $[K_D] \leq 4$ in the Kähler potential $K = \dots + (\theta\theta)(\bar{\theta}\bar{\theta})K_D$.
- $[W_F] \leq 4$ in the super potential $W = \dots + (\theta\theta)W_F$.

Therefore $[K] \leq 2$ and $[W_F] \leq 3$.

The only possible term for the Kähler potential is $\Phi\Phi^\dagger$, leading to the general expressions for K and W :

$$K = \Phi\Phi^\dagger \quad \text{and} \quad W = \alpha + \beta_\Phi + \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3.\tag{C.19}$$

²This process of taking a term from the superfield can be more formally expressed as integrals in the Grassmann variables.

Written in components, this is the well known Wess-Zumino model:

$$\mathcal{L} = \partial_\mu \phi^* \partial_\mu \phi - i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi + F F^* + \left(\frac{\partial W}{\partial \phi} F + \text{h.c.} \right) - \frac{1}{2} \left(\frac{\partial^2 W}{\partial \phi^2} \psi \psi + \text{h.c.} \right) \quad (\text{C.20})$$

with the superpotential given by (C.19). Note also that this requirement in the superpotential is a way to impose that the final theory has renormalizable interactions, as in principle one could write any holomorphic function of the chiral fields for the superpotential.

A quick look at the Wess-Zumino Lagrangian shows that the F field is a non-dynamical entity being nothing more than an auxiliary field. It arises solely to close supersymmetry off-shell [69].

To study on-shell supersymmetry one must solve the equations of motion for this field

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial F} = 0 &\Rightarrow F^* = -\frac{\partial W}{\partial \phi}, \\ \frac{\partial \mathcal{L}}{\partial F^*} = 0 &\Rightarrow F = -\frac{\partial W^*}{\partial \phi^*}, \end{aligned} \quad (\text{C.21})$$

and substitute back in the Lagrangian. This gives

$$\mathcal{L}_{\text{on-shell}} = \partial_\mu \phi^* \partial_\mu \phi - i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - \left| \frac{\partial W}{\partial \phi} \right|^2 - \frac{1}{2} \left(\frac{\partial^2 W}{\partial \phi^2} \psi \psi + \text{h.c.} \right), \quad (\text{C.22})$$

the on-shell Lagrangian for the Wess-Zumino model.

Our goal is to write the most general $\mathcal{N} = 1$ supersymmetric invariant Lagrangian composed by one chiral superfield in two dimensions, using the concepts and machinery developed in the usual case of a four dimensional spacetime.

Supersymmetry in two dimensions. In two dimensions there is no genuine spinor, because there is no notion of spatial rotations. Nonetheless spin can be introduced, in a similar way to the usual, by imposing that bosons are quantized by imposing a quantization condition on the canonical commutators, while for the fermions the quantization condition is imposed on the anticommutators.

In two dimensions, one can require the spinors to be both chiral and Majorana simultaneously, giving rise to the existence of a number of exotic supersymmetries. This will not be considered in this case, in fact we will deal with the simplest case: $\mathcal{N} = 1$. This theory has two real supercharges, one left-handed and the other right-handed (that is why this supersymmetry is also denoted by $\mathcal{N} = (1, 1)$).

The superspace is obtained by including a two-component real Grassmann variable $\theta_\alpha = \{\theta_1, \theta_2\}$. We take the gamma matrices as $\gamma^\mu = \{\sigma_2, -i\sigma_1\}$.

The irreducible representation of the superfield in this case is even simpler than the chiral superfield studied before. It is known as the real superfield and has the form

$$\Phi(x, \theta) = \phi(x) + \bar{\theta} \psi(x) + \frac{1}{2} \theta \bar{\theta} F, \quad (\text{C.23})$$

where $\bar{\theta} = \theta \gamma^0$. This superfield consists of a real scalar field $\phi(x)$, a two-component Majorana spinor $\psi(x)$, and an auxiliary field F , see e.g. [52].

Following the same procedure as described before, one can obtain the off-shell supersymmetric Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\bar{\psi}i\not{\partial}\psi + \frac{1}{2}F^2 + W'(\phi)F - \frac{1}{2}W''\bar{\psi}\psi. \quad (\text{C.24})$$

Note that this Lagrangian has the same structure as the one presented in (C.20).

Again, we focus on the on-shell realization of supersymmetry, eliminating the auxiliary field via its equation of motion, which sets $F = -W'(\phi)$ and gives

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(W'(\phi))^2 - \frac{1}{2}W''\bar{\psi}\psi. \quad (\text{C.25})$$

The most different aspect from the four-dimensional theory is that the scalar field in two dimensions has dimension zero, which does not impose any renormizability constrain for the superpotential!

In order to represent the most general function $W(\phi)$, one has to use a power expansion

$$W(\phi) = \frac{1}{2}M\phi^2 + \sum_{n=3}^{\infty} \frac{\lambda_n}{n!} \phi^n, \quad (\text{C.26})$$

where the linear term was intentionally excluded to avoid a spontaneous breaking of supersymmetry.

Substituting the power expansion (C.26) in the on-shell Lagrangian (C.25), results in the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \bar{\psi}i\not{\partial}\psi - M\bar{\psi}\psi - \sum_{n=3}^{\infty} \frac{\Lambda_n}{n!} \phi^n - \sum_{n=3}^{\infty} \frac{\lambda_n}{(n-2)!} \phi^{n-2} \bar{\psi}\psi \quad (\text{C.27})$$

with scalar couplings

$$\Lambda_n = Mn\lambda_n + \frac{n!}{2} \sum_{i=3}^n \sum_{j=3}^n \frac{\lambda_i}{(i-1)!} \frac{\lambda_j}{(j-1)!} \delta_{i+j-2,n}. \quad (\text{C.28})$$

To obtain the canonical form for the fermionic propagators we have rescaled the fermionic fields: $\psi \rightarrow \sqrt{2}\psi$ and $\bar{\psi} \rightarrow \sqrt{2}\bar{\psi}$. This is the Lagrangian used as a starting point in chapter 6.

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