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Teoria Quântica em uma Subvariedade

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Quantum Theory on a Submanifold

CLASSICAL, QUANTUM AND THERMAL EFFECTS

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*“Un soñador es, por definición, un
desmemoriado.”*

— Eugenio Dors.

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Resumo

O problema de como descrever a dinâmica efetiva de partículas ou campos confinados a um espaço-tempo de baixa dimensão é de interesse em muitas áreas da física. Vários métodos foram propostos na literatura para atacar este problema. Recentemente, foram relatadas algumas evidências experimentais que são consistentes com a chamada *abordagem de potencial confinante*. À luz destes resultados, o objetivo desta tese é o de construir uma teoria quântica para partículas confinadas em uma subvariedade, imersa em um espaço-tempo plano, empregando o cenário conceitual da abordagem de potencial confinante. A tese está dividida em duas partes. A primeira parte é dedicada exclusivamente ao estudo da mecânica quântica em uma subvariedade. Para esta finalidade, deduzimos as equações efetivas de Schrödinger, Dirac e Klein-Gordon, em uma subvariedade curva, na presença de um campo electromagnético externo. Examinamos as características singulares destas equações e apresentamos algumas aplicações em física da matéria condensada. Na segunda parte, partimos da mecânica quântica na subvariedade e então formulamos a teoria quântica de campos (TQC) na subvariedade. Mostramos que a TQC “livre” em uma subvariedade pode ser representada esquematicamente como uma teoria de campos livres, em um fundo curvo, mais um potencial escalar e um campo externo $SO(n - m)$ de Yang-Mills. Calculamos para essa teoria a ação efetiva em ordem de um laço para bósons e férmions a temperatura e potencial químico não nulos, em todas as ordens, usando a expansão de Seely-DeWitt. Para teorias com interações, demonstramos que a teoria conhecida como eletrodinâmica quântica reduzida ($RQED_{d_\gamma, d_e}$) pode ser recuperada a partir da abordagem de potencial confinante. Para uma teoria bidimensional, propomos uma grande classe de extensões do modelo de Schwinger, em que a interação entre férmions vai além do potencial linear. Demonstramos que, notavelmente, essas extensões são exatamente solúveis para férmions sem massa e que não há geração de massa dinâmica para os férmions. Além disso, construímos uma nova família de teorias que podem ser exatamente bosonizadas. Também mostramos que $RQED_{4,2}$ tem as características necessárias para ser identificada como uma teoria de campo efetivo para fios de grafeno. Finalmente, estudamos o efeito da interação de campos magnéticos e pseudomagnéticos no grafeno. Para este sistema, calculamos o condensado fermiônico, a densidade de carga induzida de vácuo, a ação efetiva em ordem de um laço e a magnetização. Demonstramos que a presença de um campo pseudo-magnético diferente de zero torna possível observar experimentalmente uma densidade de carga induzida de vácuo. Além disso, vamos mostrar que é possível ter controle sobre a magnetização, bem como a massa dinâmica para cada vale no grafeno, quer seja por deformações ou variando o campo magnético aplicado.

Palavras-Chave: Potencial confinante, Subvariedade curva, Eletrodinâmica quântica reduzida, Modelo de Schwinger, Grafeno.

Abstract

The problem of how to describe the effective dynamics of particles or fields confined to a lower dimensional curved space-time is of interest in many areas of physics. In the literature, several methods have been proposed to attack this problem. Recently it has been reported some experimental evidences that are consistent with the so-called *confining potential approach*. In light of these results, the purpose of this thesis is to construct a quantum theory for particles confined in a submanifold of a flat space-time employing the theoretical framework of the confining potential approach. The thesis is divided into two parts. The first one is dedicated exclusively to the study of quantum mechanics on a submanifold. For this purpose, we derive the *effective* Schrödinger, Dirac and Klein-Gordon equations on a curved submanifold, in the presence of an external electromagnetic field. We examine the singular features of these equations and present some applications to condensed matter. In the second part of this thesis, we start from the quantum mechanics on the submanifold and then we formulate the quantum field theory (QFT) on the submanifold. We will show that the “free” QFT on a submanifold can be schematically represented as a quantum theory of free fields on a curved background plus a scalar potential and an external $SO(n - m)$ Yang-Mills field. For this theory, we compute the one-loop effective action for scalars and fermions at finite temperature and chemical potential to all orders using the Seely-DeWitt expansion. For interacting theories, we will prove that the theory known as reduced quantum electrodynamics ($RQED_{d_\gamma, d_e}$) can be recovered from the confining potential approach. For a two-dimensional theory, we propose a large class of extensions of the Schwinger model, in which the interaction between fermions goes beyond the linear potential. We demonstrate that, remarkably, these extensions are exactly solvable for massless fermions and that there is no dynamical mass generation for the fermions. Furthermore, we construct a new family of exactly bosonized theories. We also show that $RQED_{4,2}$ has the necessary features to be identified as an effective field theory for graphene wires. Finally, we study the effect of an interplay of real and pseudomagnetic fields in graphene. We compute the fermion condensate, the induced vacuum charge density, the one-loop effective action and the magnetization, for this system. We will show that the presence of a non-zero pseudomagnetic field makes it possible, experimentally, to observe an induced vacuum charge density. Moreover, we will show that it is possible to have control over the magnetization as well as the dynamical mass for each valley in graphene, by straining or varying the applied magnetic field.

Keywords: Confining potential, Curved submanifold, Reduced quantum electrodynamics, Schwinger model, Graphene.

Publications

A large portion of this thesis based on the following publications:

- F. T. Brandt and **J. A. Sánchez-Monroy**. Induced magnetic moment for a spinless charged particle in the thin-layer approach. *Europhysics Letters*, **111**, (2015) 67004.
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- F. T. Brandt and **J. A. Sánchez-Monroy**. Absence of effects of an in-plane magnetic field in a quasi two-dimensional electron system. (Working paper).
- **J. A. Sánchez-Monroy**. Gauge Invariant Extensions of the Massless and Massive Schwinger Model. (Working paper).
- **J. A. Sánchez-Monroy** and C. J. Quimbay. Fermion Condensate and Magnetization in Graphene: Interplay Between Real and Pseudomagnetic Fields. (Working paper).

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CHAPTER 1

INTRODUCTION

“We are blind people, imagining what we don’t see.”

— Alan Lightman.

1.1 Lagrangian Approach vs. Newtonian Approach

In classical mechanics the motion of a particle constrained to a submanifold of the configuration space¹ is usually treated with two different approaches, which we refer to as *Newtonian* and *Lagrangian approach*. In the *Newtonian approach* the particle lives in a n -dimensional space and, in the absence of external forces, it can move in all directions. In this approach, the constraints are introduced by a set of forces, which maintain its velocity oriented along a preselected range of directions, in such a way that the particle can only move in a m -dimensional space [1]. In the *Lagrangian approach*, the constraint is introduced from the beginning through the generalized coordinates. For subsequent calculations, it is not necessary to make any mention to the space in which the space is supposed to be embedded. These two approaches are found to be equivalent for holonomic constraints [1, 2], the choice between one or the other turns out to be a matter of convenience.

In quantum mechanics *a priori* the two approaches are not necessarily equivalents. Therefore, we may ask two questions: First, do Lagrangian and Newtonian approaches represent the same physical system at the quantum level? Second, are these two approaches (mathematically) equivalents at the quantum level? To address these two questions, let us note that thanks to the Heisenberg uncertainty principle, in the Newtonian approach, the wave function cannot be completely confined in the m -dimensional space, which is seen as a *submanifold* of a larger space. On the other hand, since in the Lagrangian approach the m -dimensional space is all accessible space, which can be seen as

¹Constraints that restrict the configuration space are called *holonomic constraints* [1].

a *manifold* which need not be embedded in any space, the wave function is fully within the m -dimensional space, Fig 1.1. Therefore, the answer to the first question is no. One can also realize that in the Lagrangian approach, the dimensional reduction procedure² is before the quantization. On the contrary, in the Newtonian approach, the system is quantized before of dimensional reduction³.

Figure 1.1: Probability density for a particle confined in a surface. a) In the Newtonian approach part of the probability density live outside of the surface. b) In the Lagrangian approach the probability density is fully within surface.

Although these two approaches do not represent the same physical system, it can happen that are mathematically equivalent, as in classical mechanics. The problem of the quantum dynamics of a particle constrained to a m -dimensional submanifold has been extensively studied in the literature and have been the subject of controversy for decades⁴. In most of these proposals, the quantization is performed so that the result depends only on the intrinsic properties of the submanifold and we can forget all the

²In this case the procedure by which one arrive at the Lagrangian in generalized coordinates.

³In this case the dimensional reduction is perform by introducing a potential.

⁴Usually this problem is referred to as quantization with constraints.

properties related to external space (Lagrangian approach). In general, these schemes of quantization suffer from ordering ambiguities and the literature has many examples where the Schrödinger equation in a curved space differs by a term proportional to the scalar curvature [3–6]. As it has been pointed out by Maraner [7], the common feature of these methods is that the constraints are implemented without explicit reference to the physical mechanism (assuming that there is one) that produces the confinement of the particles on the submanifold. Following Ref. [7], let us distinguish between two kinds of constraints: the *formal constraints* which appear in the formulation of a dynamical theory and *real constraints* which are produced by a real potential confining the motion of the system to a submanifold of its configuration space. The latter are the relevant constraints in the Newtonian approach.

The use of real constraints was considered in the seventies by Jensen and Koppe [8] and independently by Costa⁵ [9] and Switkes *et al.* [10]. This proposal is known as *the thin layer quantization* (also known as *confining potential approach*⁶). The two remarkable features of this method are that the quantization is performed in a flat space, where there are no ordering ambiguities related to the scalar curvature; and that the real constraints are introduced by a potential V_c satisfying the following three conditions [7–9]:

1. V_c has a deep minimum relative to the submanifold,
2. V_c depends only on coordinates which are normal to the submanifold,
3. V_c depends on a dimensionless parameter ϵ in such a way that as $\epsilon \rightarrow 0$, the potential goes to infinity outside of the submanifold.

Since the third condition is *unrealistic* and incompatible with the Heisenberg uncertainty principle, it is more appropriate to consider a perturbative expansion in ϵ which can be understood as an *effective theory* on the submanifold. In general, the zeroth-order term is interpreted as strong fluctuations of the system in directions normal to the submanifold; the next non-vanishing term is interpreted as the Hamiltonian describing the effective dynamics on a submanifold and the remaining contributions take into account further interactions between normal and tangent degrees of freedom [7, 11, 12]. Notably, it was shown that the effective dynamics on a submanifold depends on a curvature-induced quantum potential (the so-called *geometric potential*) that is a function of the intrinsic and extrinsic curvatures of the submanifold [7–9, 11]. Therefore, one can conclude that the Newtonian and the Lagrangian approaches are not equivalent at the quantum level.

From the experimental point of view, Szameit *et al.* in 2010 found experimentally a (classical) optical analogue of the geometric potential⁷ [13]. Furthermore, Onoe *et al.* reported in 2012 [14] the first experimental evidence of the geometric potential in the

⁵As pointed out by Costa, initially, the idea of real constraints was unjustly neglected. However, as we see the idea is not only mathematically consistent but also experimentally tested.

⁶A more appropriate name when the dimension of the submanifold is not equal two.

⁷Although the interpretation of the uncertainty principle is correct, the only strictly necessary to have an effect like the geometric potential is to start with a wave equation rather than of an equation for point particles.

electronic properties of a quantum system, such as a Tomononaga-Luttinger liquid. These discoveries open a new observational window to study how the geometrical effects affect, or not, the properties of confined quantum systems, such as those studied in condensed matter.

Motivated by the mathematical consistency of the method and experimental evidence, we follow in this thesis the Newtonian approach (quantization on submanifolds) rather than the Lagrangian approach (quantization in manifolds)⁸.

1.2 The program: Quantum Theory on a Submanifold

In the usual route to study physics, one begins formulating the classical mechanics, for particles and fields, then go to the non-relativistic quantum mechanics and from there one constructs the relativistic quantum mechanics. Using the latter one builds the quantum theory of free fields and then goes to a quantum theory of interacting fields. In each of these stages, one can use thermodynamics to build statistical mechanics, either, classical statistical mechanics, quantum statistical mechanics or thermal field theory. Typically in an undergraduate course, these issues are covered for a flat space-time. The same program can be carried, with slightly more difficulty, in a curved space-time (a curved manifold). Inspired by this route, in this thesis, we carry the equivalent program when the space-time is a submanifold rather than a manifold. Namely, we will study the problem of particles and fields, when they initially live in a flat space-time and are then confined to a lower dimension curved space-time. Since classical mechanics in a submanifold is well known, we only devote an appendix F regarding it.

This thesis will be divided into two parts. The first one dedicated exclusively to consider the non-relativistic and relativistic quantum mechanics on a submanifold. In the second one, we will then consider quantum field theory (QFT) in a submanifold, along with some related problems. During the thesis the philosophy will be considered, as much as possible, the most general scenario from the theoretical viewpoint. Before detailing the structure of the thesis, let us review the current status of the theoretical developments of the confining potential approach in quantum mechanics.

1.2.1 Part I: Quantum Mechanics

Current Status of the Confining Potential Approach

In non-relativistic quantum mechanics the confining potential approach has been used mainly in the context of condensed matter physics, in order to study the quantum dynamics of a particle confined to a low-dimensional space [7, 10, 11, 15–32]. For example, it

⁸One can think that in the Newtonian approach the curved manifold is considered as a submanifold in a flat ambient space of higher dimension, this higher dimensional space is the complete universe, while in the Lagrangian approach the curved manifold is the universe and it is not embedded in an ambient space.

has been studied the influence of the geometric potential in the electronic properties and bound-state formation in curved nanostructures [22], geometry-induced charge separation on helicoidal ribbon [20], mechanical-quantum-bit states [23], effects of periodic curvature on the electrical resistivity of corrugated semiconductor films [24, 25], curvature-induced p-n junctions in bilayer graphene [32], distorted quantum rings [10, 28, 30, 33] as well as geometry-driven shift in the Tomonaga-Luttinger liquid [26].

The confining potential approach has been generalized to codimensions⁹ higher than one [7, 15–17, 34]. There a geometry-induced Yang-Mills field emerge if the the low-dimensional space has a geometric torsion [7, 11, 15–17, 34]. The geometry-induced Yang-Mills field emerges because the orbital angular momentum of the normal coordinates to the low-dimensional space couples with the geometric torsion. Physical consequences of geometry-induced Yang-Mills fields have been explored in the last decade [17, 35–37]. It was studied, as the geometric torsion inherent in the quantum ring induced a quantum phase shift [35], persistent current flow [36] and an Aharonov-Bohm-like conductance oscillation [36, 37].

Furthermore, the approach has been extended to include external electromagnetic fields [27, 29, 31] and has also been applied to the study of Maxwell [13, 38] and Pauli equations [39, 40] as well as relativistic quantum mechanics [12, 41–50]. For the Klein-Gordon equation was first studied for the case of a 4D submanifold embedded in the flat 5D space [41]. A more general case was studied in Ref. [12], where the motivation for applying the confining potential approach to the Klein-Gordon equation was the study of the dynamics of test particles confined to a four-dimensional submanifold (brane) embedded in a higher dimensional flat manifold (bulk) at the quantum level. For the Dirac equation works have focused on studying particles confined to (1 + 1)- [42, 44, 44, 51] and (2 + 1)-dimensional submanifold¹⁰ [43, 46, 50]. The Dirac equation in the presence of external electromagnetic field in a (2 + 1)-dimensional submanifold embedded in a (3 + 1)-dimensional space-time was studied in Ref. [49]. The generalization to higher dimensions in the absence of electromagnetic field was studied in Ref. [48].

Part I: Structure

We started the first part of this thesis illustrating the confining potential approach in the case of a non-relativistic particle confined to planar curve, Ch. 2. The overall structure of the method is then summarized in a “*recipe*”, sec. 2.1.1. Thus, we will apply the method to describe quantum rings with arbitrary shape, emphasizing the effects of the geometric potential, sec. 2.2. In particular, we consider a family of quantum rings with discrete rotational symmetries, cyclic symmetries (C_n), sec. 2.2.2. In Ch. 3, we will deduce the effective Schrödinger equation to spinless particles confined on a m -dimensional submanifold in the presence of an electromagnetic field. Besides the already known effects reported in the literature, such as the geometric potential and the induced Yang-Mills field; we find a new quantum potential, when the codimension is greater than one. We

⁹The codimension is the difference between the dimension of the high-dimensional space and the dimension of the low-dimensional space.

¹⁰In these works the codimension was equal to one.

will show that this can be interpreted as an induced Zeeman interaction, sec. 3.1.3. Additional applications to quantum rings in the presence of magnetic field are presented in sec. 3.2 as well as some remarks about in-plane magnetic field in quasi two-dimensional electron systems, sec. 3.3. After that, we will study the relativistic quantum mechanics on a submanifold. The case of the Klein-Gordon equation will be treated in the Appendix C. Finally, the Dirac equation on a submanifold will be study in Ch. 4. We will show that this equation deserves a full review with respect to what previously done in the literature. A new Dirac equation in a submanifold will be present in this chapter.

1.2.2 Part II: Quantum Field Theory

In part two of this thesis, we will deal with of quantum field theory in a submanifold. Although, as we will show, this theory differs from the widely studied quantum field theory in curved space-time¹¹ (*i.e.* QFT on a manifold), they share many common features. Hence, QFT on a manifold will be a guide for us. Moreover, although the descriptions are not identical, a quantum system on a curved submanifold could help understand and simulate some aspects of quantum field theory in a curved space-time.

The structure of this part of the thesis will be divided into three chapters

QFT of “Free” field and Thermal QFT

The natural way to construct a quantum field theory is to start with free fields. The idea is to take the shortest path and build from the field equations already obtained the classical action, with this action, then build the Feynman path integral. As we will show in Ch. 5 the “Free” QFT on a submanifold can be understood as the theory of a field in a curved background in the presence of a scalar potential and an external geometrically induced Yang-Mills field. Although there are plenty of questions that one might ask and many methods that might be used, we will take advantage of the formalism of the effective action [53–55]. In particular, we will compute the one-loop effective action at finite temperature and chemical potential effective action for scalars and fermions in the background fields aforementioned, sec 5.3.

Interacting QFT on a submanifold

The next step is of course the study of interacting quantum field theories on a submanifold, Ch. 6. Here, while the matter fields (scalars or fermions) are confined to a submanifold, the electromagnetic field propagating in the flat higher dimensional manifold. After presenting the Feynman path integral for such theories, we restrict ourselves to study fermions confined to a flat submanifold. In this case a significant progress has been made in the literature at zero [56, 56–65] and finite temperature [66, 67]. The theory in which the photon propagates in a d_γ -dimensional flat space-time and the fermion fields are localized on a d_γ -dimensional flat space-time is known as *reduced quantum electrodynamics* (RQED $_{d_\gamma, d_e}$), also called *pseudo quantum electrodynamics* [56, 57, 60, 68–70].

¹¹About QFT in curved space-time there are excellent books on literature [52–54].

We will show that there is a profound relationship between $\text{RQED}_{d_\gamma, d_e}$ and the confining potential approach, secs. 6.2 and 6.3. Furthermore, that $\text{RQED}_{d_\gamma, d_e}$ can be obtained from the confining potential approach (first principles) allowing to determine clearly the assumptions on which $\text{RQED}_{d_\gamma, d_e}$ is physically well-defined.

The distinctive feature of $\text{RQED}_{d_\gamma, d_e}$ is that the kinetic term of this gauge theory is a pseudo-differential operator. We will study some properties of these pseudo-differential operators, and we will propose a large class of *exotic* gauge invariant Lagrangians. Focusing on two-dimensional theories, in sec 6.4 we will find that remarkably these *exotic* Lagrangians are all exactly solvable for massless fermions, extending the result obtained by Julian Schwinger in his seminal article: “*Gauge invariance and mass II*” [71]. There he shows that the two-dimensional massless quantum electrodynamics (QED) is exactly solvable. QED_{1+1} has been extensively used to study many aspects of quantum field theory [72–77] and as a toy model in quantum chromodynamics [78–83] and condensed matter [84]. A mass term for the fermions can be introduced and we will show that the exotic family of models can be exactly bosonized¹². Finally, we will return to $\text{RQED}_{4,2}$ because as we will discuss this theory can be used as an effective field theory of relativistic quantum wires (*e.g.*, graphene wires).

Interplay Between Real and Pseudomagnetic Fields

Although the last chapter of the thesis is independent of the previous results, this chapter can be seen as complementary. Before explaining that let us make a short preamble.

Graphene is a material consisting of a single layer of graphite, in which the carbon atoms are arranged in a two-dimensional honeycomb lattice. Graphene exhibits a diverse range of interesting features among which is the anomalous quantum Hall effect [87], a record high Young’s modulus [88] as well as very high thermal conductivity [89]. It has been shown that the band structure of a flat sample of graphene can be modeled by a $2 + 1$ -dimensions massless Dirac Hamiltonian, with minimal coupling to the (external) electromagnetic field, which can be derived from the tight-binding model in the continuum approximation [90]. In that Hamiltonian the speed of light is replaced by the Fermi velocity v_f , which is about 300 times smaller than the speed of light, and the pseudo-spin play the role of the spin.

When the sample of graphene presents; ripples, lattice distortions, strains or curvature, there are two main approaches to model these [91, 92]. One model is based on a combination of tight-binding and elasticity theory [93] and the other model is based on the formulation of quantum field theory in curved space [94]. In the latter, it is assumed that the dynamics of carriers in graphene in low energies is described by Dirac equation in a $(2 + 1)$ -dimensional curved space-time. In the other one, it has been shown that, for example, strains cause alterations in graphene dispersion relation, which mathematically are equivalent to an “elastic” vector field, which is known as *pseudomagnetic field* [95]. Originally was observed that strain induces a strong gauge field that effectively act as a

¹²Bosonization is a highly useful method in which a fermionic theory is mapped into a bosonic one [78, 85, 86].

uniform (pseudo)magnetic field exceeding 10T [96]. More recently, it was experimentally reported a pseudomagnetic field greater than 300 teslas, which opens the door to previously inaccessible high magnetic field regimes [97]. Furthermore, the mechanical control over the electronic structure of graphene has been explored as a potential approach to so-called “strain engineering” [97, 98].

There is a general understanding that the two aforementioned approach presented discrepancies [92]. The most evident of these discrepancies is due to in-plane distortions. While for the elasticity approach one has a non-zero pseudomagnetic field, since in-plane distortions do not induce intrinsic curvature, the QFT approach does not produce any observable effect. Although the two approaches discussed overlook the fact that the carriers are intrinsically three-dimensional objects, and in principle, the effects of extrinsic curvature could have observable consequences. The fact is that even considering this, in-plane distortions do not induce intrinsic or extrinsic curvature. Therefore, in this case, to examine the effects of extrinsic curvature do not provide any additional effect. Thus, it is clear that the elasticity approach is capturing other physical contributions to the possible effects of curvatures.

In chapter 7, in order to study the interplay between real and pseudomagnetic fields, we will study a $2 + 1$ -dimensional Dirac Hamiltonian in the presence of magnetic and pseudomagnetic fields. In sec. 7.2.1, we will discuss some aspects of the symmetries of the irreducible and reducible representation of the γ -matrices. In sec. 7.4, we will compute the fermionic condensate and the vacuum current as well as we will discuss various aspects of the magnetic catalysis and the dynamical mass generation. Finally, we will compute the one-loop effective action with which we can calculate the magnetization, sec 7.5.

Part I

**Quantum Mechanics on a
Submanifold**

CHAPTER 2

QUANTUM PARTICLE CONSTRAINED TO A PLANAR CURVE

*"No problem is too small or too trivial if we
can really do something about it."*

— Richard P. Feynman.

To illustrate the confining potential approach, we will start with the simple case of a non-relativistic spinless particle confined to a planar curve. We show in detail how to obtain the one-dimensional effective Schrödinger equation to a planar curve starting from the two-dimensional Schrödinger equation. Moreover, we apply this equation to model one-dimensional quantum rings.

2.1 Effective Schrödinger Equation to a Plane Curve

Consider a non-relativistic particle that can initially move freely in $\mathbb{R}^2$, so, it can be described by the Schrödinger equation in two dimensions

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(x_1, x_2, t) = i\hbar\frac{\partial}{\partial t}\Psi(x_1, x_2, t), \quad (2.1)$$

with ∇^2 is the Laplacian in two-dimensions (2D). Let C be a curve in $\mathbb{R}^2$ given by the parametric form

$$\mathbf{r} = \mathbf{r}(s) = (x_1(s), x_2(s)), \quad (2.2)$$

where s is a arc-length parametrization¹. Assuming that C admits a tubular neighborhood such that

$$\mathbf{R}(s, y) = \mathbf{r}(s) + y\mathbf{n}(s), \quad (2.4)$$

defines an arbitrary point in the vicinity of C . Here, $\mathbf{n}(s)$ denotes the unit normal of

Figure 2.1: Adapted coordinate system for a planar curve C .

C in s and y is the distance between C and a point Q in $\mathbb{R}^2$, Fig. 2.1. With the aim of confining the particle in a curve, we follow the so-called thin-layer approach² (also known as confining potential approach). In this approach confinement is produced through a potential V_c . In the orthogonal curvilinear coordinates (s, y) the Schrödinger equation reads

$$H\Psi(s, y, t) = -\frac{\hbar^2}{2m}\square_{2D}\Psi(s, y, t) + V_c(y)\Psi(s, y, t) = i\hbar\frac{\partial}{\partial t}\Psi(s, y, t), \quad (2.5)$$

where $\square_{2D}$ is the Laplace-Beltrami operator in 2D. We denote a tangent vector to the curve C as $\mathbf{t} = d\mathbf{r}/ds$. By definition the vector $\mathbf{n}$ is such that $\mathbf{t} \cdot \mathbf{n} = 0$. The (extrinsic) curvature α is given by $\frac{d\mathbf{t}}{ds} \cdot \mathbf{n} = \alpha$, therefore, the following relations hold [99]

$$\frac{d\mathbf{t}}{ds} = \alpha\mathbf{n}, \quad \frac{d\mathbf{n}}{ds} = -\alpha\mathbf{t}. \quad (2.6)$$

¹The *arc-length* of a curve γ starting at the point $\gamma(u_0)$ is the function $s(u)$ given by

$$s(u) = \int_{u_0}^u \|\dot{\gamma}(u')\| du'. \quad (2.3)$$

²The one-dimensional effective Schrödinger was obtained initially in [8–10]. Here, however, we will proceed with a slightly more formal approach following Ref. [16].

Using Eq (2.6) one can calculate the 2D metric explicitly

$$G_{11} = \left(\frac{\partial \mathbf{R}}{\partial s} \right)^2 = \left(\frac{\partial \mathbf{r}}{\partial s} + y \frac{\partial \mathbf{n}}{\partial s} \right)^2 = (1 - y\alpha)^2, \quad (2.7)$$

$$G_{22} = \left(\frac{\partial \mathbf{R}}{\partial y} \right)^2 = \mathbf{n}^2 = 1, \quad (2.8)$$

$$G_{12} = G_{21} = \frac{\partial \mathbf{R}}{\partial s} \cdot \frac{\partial \mathbf{R}}{\partial y} = 0, \quad (2.9)$$

where we used that $\mathbf{t} \cdot \mathbf{t} = 1$. Denoting $|G| = \det(G_{ij})$, we have that $|G|^{1/2} = (1 - y\alpha)$. Thus, the Laplace-Beltrami operator can be write as

$$\begin{aligned} \square_{2D} &= \frac{1}{|G|^{1/2}} \partial_i (G^{ij} |G|^{1/2} \partial_j) \\ &= G^{-1/2} \partial_s (G^{-1/2} \partial_s) + G^{-1/2} \partial_y (G^{1/2} \partial_y). \end{aligned} \quad (2.10)$$

The normalization condition in curvilinear coordinates is

$$\int |\Psi|^2 dx_1 dx_2 = \int |\Psi|^2 G^{1/2} ds dy = \int \left(\int |\Psi|^2 G^{1/2} dy \right) ds = 1. \quad (2.11)$$

Our aim now is to separate the Schrödinger equation and obtain an effective Schrödinger equation to the curve, so we introduce a new wave-function $\chi(s, y, t) = \chi_t(s, t) \chi_n(y, t)$, where the subscripts t and n stand for tangent and normal degrees of freedom. To keep a probabilistic interpretation along the curve; the old wave-function is related with the new wave-function by a “rescale”

$$\chi(s, y, t) \equiv |G|^{\frac{1}{4}} \Psi(s, y, t) = (1 - y\alpha)^{1/2} \Psi(s, y, t). \quad (2.12)$$

Thus, the normalization condition implies

$$\begin{aligned} 1 &= \int \left(\int |\Psi|^2 G^{\frac{1}{2}} dy \right) ds = \int \left(\int |\chi|^2 dy \right) ds \\ &= \int \left(\int |\chi_n(y, t)|^2 dy \right) |\chi_t(s, t)|^2 ds = \int |\chi_t(s, t)|^2 ds, \end{aligned}$$

where was chosen without loss of generality that $\int |\chi_n(y)|^2 dy = 1$. Substituting the rescaled wave-function in the Schrödinger equation we obtain

$$\overbrace{\left[|G|^{\frac{1}{4}} H |G|^{-\frac{1}{4}} \right]}^{\mathcal{H}} \chi(s, y, t) = i \frac{\partial \chi(s, y, t)}{\partial t} \quad (2.13)$$

where we have define $\mathcal{H}$ as the “rescaled” Hamiltonian. Three conditions are required for the confining potential V_c [7, 9]: (i) V_c has a deep minimum relative to the curve; (ii) V_c depends only on the coordinate y , and (iii) V_c depends on a dimensionless parameter ϵ in such a way that as $\epsilon \rightarrow 0$ the potential goes to infinity outside of the curve. Since

from the experimental point of view it is *unrealistic* take $\epsilon \rightarrow 0$ (and incompatible with the Heisenberg uncertainty principle), it is more appropriate to consider a perturbative expansion in ϵ which can be understood as an *effective theory* on the curve. Expanding V_c as a power series in the y about its minimum and assuming that it is symmetric in y up to quadratic order [11], we arrive at³

$$V_c(y) = \frac{1}{2\epsilon^4} m\omega^2 y^2 + O(y^3). \quad (2.14)$$

The idea is to use the dimensionless parameter ϵ as a perturbative parameter [7, 11], so, let us rescale the normal coordinate by $y \rightarrow \epsilon y$. Then, using Eq. (2.10) we expand the Hamiltonian $\mathcal{H}$ up to order ϵ^2 , after some algebra we arrive at

$$\epsilon^2 \mathcal{H} = \mathcal{H}^{(0)} + \epsilon \mathcal{H}^{(1)} + \epsilon^2 \mathcal{H}^{(2)} + \dots \quad (2.15)$$

where the zero, first and second order terms are respectively

$$\mathcal{H}^{(0)} = -\frac{\hbar^2}{2m} \partial_y^2 + \frac{1}{2} m\omega y^2, \quad (2.16)$$

$$\mathcal{H}^{(1)} = 0, \quad (2.17)$$

$$\mathcal{H}^{(2)} = -\frac{\hbar^2}{2m} \partial_s^2 - \frac{\hbar^2}{8m} \alpha(s)^2. \quad (2.18)$$

By setting $\chi(s, y, t) = \chi_t(s, t) \chi_n(y, t)$, one can be separated the above equation noting that $V_c(y)$ is not time-dependent and so one can assume a solution of the form

$$\chi_n(y, t) = e^{-\frac{i\bar{E}t}{\hbar}} \bar{\chi}_n(y), \quad (2.19)$$

The usual procedure of separation of variables yields

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \chi_n(y)}{\partial y^2} + V(y) \bar{\chi}_n(y) = \bar{E} \bar{\chi}_n(y), \quad (2.20)$$

$$-\frac{\hbar^2}{2m} \frac{\partial}{\partial s} \left(\frac{\partial \chi_t(s, t)}{\partial s} \right) - \frac{\hbar^2 \alpha(s)^2}{8m} \chi_t(s, t) + \bar{E} \chi_t(s, t) = i\hbar \frac{\partial \chi_t(s, t)}{\partial t}, \quad (2.21)$$

in the last equation, the energy $\bar{E}$ can be omitted since it does not affect the dynamics of $\chi_t(s, t)$. Eq. (2.21) is the 1D effective Schrödinger equation to planar curve in the arc-length parametrization. Since in general, it is not possible to find analytically the arc-length, let us write (2.21) for an arbitrary parametrization x

$$-\frac{\hbar^2}{2m} g^{-1/2} \frac{\partial}{\partial x} \left(g^{-1/2} \frac{\partial \chi_t(x, t)}{\partial x} \right) - \frac{\hbar^2 \alpha(x)^2}{8m} \chi_t(x, t) = i\hbar \frac{\partial \chi_t(x, t)}{\partial t}, \quad (2.22)$$

where $g^{-1/2} = \frac{\partial x}{\partial s}$ and now $\alpha(x) = \frac{d\mathbf{t}}{dx} \cdot \mathbf{n}$. Therefore, Eq. (2.22) is the *effective Schrödinger* equation for a particle constrained to a planar curve. The first term in the right-hand side of Eq. (2.22) is the Laplace-Beltrami operator in a curve and the second term is a *quantum potential* that depend on the extrinsic properties of the curve. This term is known as the *geometric potential*. Note that now the normalizing condition for $\chi_t(x)$ is

$$\int_C |\chi_t(x)|^2 g^{1/2} dx = 1. \quad (2.23)$$

³Asymmetric confining potentials was considered in [7, 16].

2.1.1 The Recipe

For the following chapters let us provide a “recipe” to the confining potential approach:

- Consider a *field* (Ψ) and a *differential operator* (O) in manifold.
- Write this operator in an adapted coordinate system.
- Rescale the field and the differential operator as

$$\chi \equiv \frac{|G|^{1/4}}{|g|^{1/4}} \Psi, \quad (2.24)$$

$$\mathcal{O} \equiv \left(\frac{|G|^{1/4}}{|g|^{1/4}} \right) O \left(\frac{|g|^{1/4}}{|G|^{1/4}} \right). \quad (2.25)$$

- Insert a dimensionless parameter ϵ to rescale the normal coordinates, y^i , as $y^i \rightarrow \epsilon y^i$.
- Expand the rescaled differential operator in a perturbative expansion on ϵ .
- Take the first terms, use a product solution and the normalization condition to decoupling the tangent and the normal degrees of freedom.
- Congratulations, we arrive to a *field* and a *differential operator* on a submanifold.

2.2 Energy Spectrum in Distorted Quantum Rings

In this section, we will use the 1D effective Schrödinger equation to a planar curve as a simple model to describe 1D quantum rings⁴ (QR).

2.2.1 Quantum Rings without Geometric Potential

To better understand the effect of the geometric potential, let us study the time-independent Schrödinger equation solution without this term. Because the scalar curvature (or Gaussian curvature) is zero in 1D, any closed curve differs from the perfect circular ring only by its extrinsic properties. Therefore, any closed curve is described by the Hamiltonian describing a perfect circular QR. Explicitly, in this case, the spectrum and eigenfunctions are

$$E_l = 2 \frac{\hbar^2 \pi^2}{m p^2} l^2, \quad l = 0, \pm 1, \dots, \quad (2.26)$$

and

$$\psi_l(s) = \frac{1}{\sqrt{p}} \exp \left(\frac{2\pi i l s}{p} \right), \quad l = 0, \pm 1, \dots, \quad (2.27)$$

respectively. As it is well established, for $l > 0$ all energy levels are degenerate. Thus, a model for 1DQR without a geometric potential is inadequate⁵ [100–102].

⁴As we see in sec. 3.1.3 the effects of the third dimension will not be relevant in for what follows.

⁵In the particular case of a perfect circular quantum ring of radius R , the extrinsic curvature is $\alpha^2 = 1/R^2$. Therefore, in this case, the geometric potential is only an irrelevant shift in all energy levels.

2.2.2 Symmetric Deformations

The inclusion of geometric potential will lead to changes in the energy spectrum to observe such effects consider a planar closed curve, with parametrization $\mathbf{r}(\varphi) = \{x_1(\varphi), x_2(\varphi)\}$, where without loss of generality, we take $0 \leq \varphi \leq 2\pi$. In this parametrization Eq. (2.22) is rewritten as

$$E\chi(\varphi) = -\frac{\hbar^2}{2mg^{1/2}} \frac{\partial}{\partial \varphi} \left(\frac{1}{g^{1/2}} \frac{\partial \chi(\varphi)}{\partial \varphi} \right) - \frac{\hbar^2 \alpha(\varphi)^2}{8m} \chi(\varphi), \quad (2.28)$$

where $g^{1/2}(\varphi) = \frac{ds(\varphi)}{d\varphi} = \left\| \frac{d\mathbf{r}(\varphi)}{d\varphi} \right\|$ ⁶.

Theoretical efforts have been focused on how deformations and external fields modify the energy spectrum of one-dimensional [28, 33, 103] and two-dimensional QRs [28, 33, 100, 104–113]. The study of deformed QRs has been concentrated mainly on elliptical deformations [104, 107, 109, 112–116], Cassini ovals [117] and limaçon-shaped [107]. For instance, Switkes *et al.* studied the effects of the effective potential in a pseudo elliptical ring [10]. In this geometry, the extrinsic curvature is piecewise constant, and so the effective potential is analogous to the Kronig-Penney Model in a finite interval. This kind of effective potential also appears in distorted QRs with single [28, 33], double [28] and multiple distortions [30]. Here, inspired by on the classic vibrational modes of a ring, we consider the following family of curves with non-constant extrinsic curvature,

$$\mathbf{r}(\varphi) = R\{(1 + \lambda \cos n\varphi) \cos \varphi, (1 + \lambda \cos n\varphi) \sin \varphi\}, \quad (2.29)$$

where $0 \leq \lambda < 1$ and $n \in \mathbb{N}$. Henceforth, we refer to the family of curves $\mathbf{r}_n(\varphi)$ as resonant quantum rings (RQR). For $n \geq 2$, QRs described by the curve $\mathbf{r}_n(\varphi)$ preserve the discrete symmetry C_n like the n -th classical vibrational mode and hence, we will call it C_n RQR. Whereas, for $n = 1$ we have limaçon-shaped rings. The parametrization in Eq. (2.29) allows us to introduce in a simple way a family of symmetric deformation. As an example, in Fig. 2.2 we present the C_3 and C_4 RQRs geometries for different values of λ . When λ tends to zero, all RQRs tend to have a circle geometry.

Introducing the parametrization (2.29) into Eq. (2.28) leads to

$$-A\chi''(\varphi) + B\chi'(\varphi) - V_g\chi(\varphi) = E\chi(\varphi), \quad (2.30)$$

where

$$A = \frac{\hbar^2}{mR^2 (2 - \lambda^2 (n^2 - 1) \cos(2n\varphi) + \lambda^2 (n^2 + 1) + 4\lambda \cos(n\varphi))}, \quad (2.31)$$

$$B = \frac{2\hbar^2 \lambda n \sin(n\varphi) (\lambda (n^2 - 1) \cos(n\varphi) - 1)}{mR^2 (2 - \lambda^2 (n^2 - 1) \cos(2n\varphi) + \lambda^2 (n^2 + 1) + 4\lambda \cos(n\varphi))^2}, \quad (2.32)$$

$$V_g = \frac{(\lambda^2 - \lambda^2 (n^2 - 1) \cos(2n\varphi) + 3\lambda^2 n^2 + 2\lambda (n^2 + 2) \cos(n\varphi) + 2)^2}{4R^2 (\lambda^2 n^2 \sin^2(n\varphi) + (\lambda \cos(n\varphi) + 1)^2)^3}. \quad (2.33)$$

⁶We must take care that $g^{1/2}$ and ∂_φ do not commute to avoid erroneous conclusions [100–102].

Figure 2.2: Different values of λ for the resonant quantum ring, a) $n = 3$ and b) $n = 4$.

Figure 2.3: Geometric potential divide by $\mathcal{E} = \hbar^2/(mR^2)$ for three different values of λ . (a) $n = 3$ and (b) $n = 4$.

In an arbitrary C_n RQR there are n petal-shaped curves. Because $\alpha = \partial_\varphi^2 \mathbf{r}_n \cdot \mathbf{n}$ gives the extrinsic curvature, with $\mathbf{n}$ being an orthonormal vector to the curve, there are $2n$

points in which $|\alpha|$ is maximum, n of them in the middle of each petal-shaped curve and n between the petal-shaped curves. At these points, the geometric potential, Eq. (2.33), is minimum, as showed in Fig. 2.3. Hence, the geometric potential can be identified as a periodic quantum well, having $2n$ minimum (n of them absolute and n of them relative). The well depth can be control by modifying n and λ , the well depth increases when n and λ are increased, as shown in Figs. 2.3a and 2.3b.

In order to make explicit the symmetry breaking effect arising from the geometric potential, we calculate the QR energy spectrum. For arbitrary values of λ , we expand the Hamiltonian in a Fourier basis $\phi_l = \frac{1}{\sqrt{2\pi R}} \exp(il\varphi)$ and, using a numerical diagonalization method we compute the eigenfunctions and eigenvectors. We find that for the C_n symmetry: *i*) when n is an even number, the degeneracy is removed only for the energy levels whose index m is a multiple of $n/2$ and, *ii*) when n is an odd number, the degeneracy of the energy levels whose index m is a multiple of n is removed. This behavior suggests that the symmetry breaking generates a selection rule, allowing the prediction of the energy levels that remain degenerate. As an example of the proposed selection rule, in Figs. (2.4)a (2.4)b and (2.4)c we show the energy spectrum for C_2 , C_3 and C_4 RQR, respectively. One can verify that for $C_{n=2}$, the degeneracy is removed for all energy levels ($n/2 = 1$). Whereas, for $C_{n=4}$, the degeneracy is removed only in the energy levels whose index is an even number ($n/2 = 2$). On the contrary, for $C_{n=3}$, the degeneracy is removed only in the energy levels whose index is a multiple of 3.

As λ increases, there is an accidental degeneration (see the inset in Fig. 2.4a), which appears as a consequence of the variation of the shape and the position of the absolute minimum of the geometric potential. Additionally, for large values of λ the energy levels are grouped, tending asymptotically to the same energy value but never touch each other. Each group is formed by n energy levels regardless of whether the n is even or odd.

2.2.3 Asymmetric Deformations

Asymmetric deformations can be introduced by interpolating symmetric curves describing different RQRs. Let us consider two QRs preserving the C_q and C_p rotation symmetry, respectively. We define the asymmetric geometry in terms of the parameter λ , where $\lambda = c$ ($\lambda = 0$) stand for the C_q (C_p) RQR geometry,

$$\mathbf{r}_{p,q}(\varphi) = \{f_{p,q}(\varphi)R \cos(\varphi), f_{p,q}(\varphi)R \sin(\varphi)\}, \quad (2.34)$$

where

$$f_{p,q}(\varphi) = (c - \lambda) \cos(p\varphi) + \frac{1}{2}\lambda \cos(q\varphi) + 1. \quad (2.35)$$

In a “superposition” of $\mathbf{r}_p(\varphi)$ and $\mathbf{r}_q(\varphi)$ curves, C_m rotation symmetries are preserved, where m is a common divisor of p and q . However, the symmetry group is defined by the greatest common divisor M . Hence, by analogy with symmetry deformation, we define the same selection rule in terms of the symmetry C_M . As an example in Fig. 2.5a, we show the energy spectrum for $p = 4$, $q = 8$ and $c = 0.3$. In that case, C_4 is the symmetry associated with the greatest common divisor. Thus, in the same way that a C_4 RQR

spectrum, the degeneracy in the energy levels $\pm 2, \pm 4, \pm 6, \dots$ is removed, whereas the energy states $\pm 1, \pm 3, \pm 5, \dots$ remain degenerate. A natural consequence of the proposed rule is that when the greatest common divisor of p and q is 1, the symmetry of the parametrization is just C_1 and every state could be nondegenerate. For instance, in Fig. 2.5b, we present the case where $p = 3$, $q = 5$ and $c = 0.45$, one can verify that for $0 < \lambda < 0.45$ all the states are nondegenerate. In that case the parametrization in Eq. (2.34) leads to an asymmetry QR. In light of our results, it is convenient to enunciate the selection rule for a given symmetric deformation C_n as follows:

- If the quantum rings has a symmetry C_n ; this *protects* the degeneration of all states that are not multiples of n if n is odd, and all states that are not multiples of $n/2$ if n is even.

This simple rule explains all the observed spectra considered here and is consistent with previous results in 1D [28, 30, 107, 118] and 2D [109, 113, 117] QRs with cyclic symmetries.

2.2.4 The Effects of Cyclic Symmetries in Perturbation Theory

Through a simple analysis based on perturbation theory and symmetry arguments, one can understand the origin of the proposed rule defining the index of the states preserving degeneracy. Let us carry out a perturbative analysis of Eq. (2.28) for systems preserving the discrete symmetry C_q . The unperturbed Hamiltonian is

$$H_0 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial s^2}, \quad (2.36)$$

and the perturbation is

$$V(s) = -\frac{\hbar^2}{8m} \alpha(s)^2. \quad (2.37)$$

The energy spectrum and the eigenfunction of $E^{(0)}\psi^{(0)} = H_0\psi^{(0)}$ are described by Eqs. (2.26) and (2.27), respectively. As we mentioned above, except for the ground state all states are doubly degenerate. The first order correction of the eigenvalues is calculated as roots of a secular equation [119]

$$\det(V_{l',l} - E^{(1)}\delta_{l'l}) = 0, \quad (2.38)$$

where $l = \pm l'$ and

$$V_{l',l} = \frac{1}{p} \int_0^p V(s) e^{\frac{2\pi i s(l-l')}{p}} ds. \quad (2.39)$$

It is clear that $V_{l,l} = V_{-l,-l}$. If the potential is invariant under C_q , *i.e.* $V(s) = V(s + p/q)$, the transformation $s \rightarrow s + p/q$ yields

$$\begin{aligned} V_{l',l} &= \frac{1}{p} \int_0^p V(s + p/q) e^{\frac{2\pi i s(l-l')}{p} + \frac{2\pi i (l-l')}{q}} ds \\ &= e^{\frac{2\pi i (l-l')}{q}} V_{l',l}. \end{aligned} \quad (2.40)$$

The above equation implies that $V_{l',l} = 0$ or $e^{\frac{2\pi i(l-l')}{q}} = 1$. Thus, for $l = l'$, the exponential term is one, which leaves us with the case $l = -l'$, in which $V_{-l,l}$ is different from zero only if there is an integer, ω , such that $l = q\omega/2$. For q odd Eq. (2.40) is not satisfied if l is not a multiple of q . On the other hand, for q even it is not satisfied if l is not a multiple of $q/2$. Hence, in these two cases the solutions of the secular equation, (2.38), are $E_{\pm l}^{(1)} = V_{l,l}$, then, these levels are still degenerate.

The second order correction to the eigenvalues that remains degenerate can be computed as roots of a secular equation [119]

$$\det \left(\sum_{k \neq l, -l} \frac{V_{l',k} V_{k,l}}{E_l^{(0)} - E_k^{(0)}} - E^{(2)} \delta_{l'l} \right) = 0. \quad (2.41)$$

For the diagonal elements, using $V_{l,k} = V_{k,l}^* = V_{-k,-l}$ and $E_k^{(0)} = E_{-k}^{(0)}$, one can prove that

$$\sum_{k \neq l, -l} \frac{V_{l,k} V_{k,l}}{E_l^{(0)} - E_k^{(0)}} = \sum_{k \neq l, -l} \frac{V_{-l,k} V_{k,-l}}{E_l^{(0)} - E_k^{(0)}}. \quad (2.42)$$

In the non-diagonal elements, we have the product $V_{-l,k} V_{k,l}$ with $k \neq l, -l$. In a similar way as in the first-order perturbation theory, we obtain that $V_{-l,k}$ is different from zero only if there is an integer ω_1 such that $l + k = q\omega_1$ and $V_{k,-l}$ is different from zero only if there is an integer ω_2 such that $l - k = q\omega_2$. However, these conditions can not be satisfied simultaneously if l is not a multiple of q ($q/2$) for q odd (even). Therefore, the energy levels whose index l is not a multiple of q ($q/2$) are protected by the C_q symmetry, as found in the first-order perturbation theory.

Although the proof presented here is up to second order, we conjecture that this could be carried to all orders in degenerate perturbation theory. Of course, this conjecture is in agreement with our numerical simulations and previous works [28, 30, 107, 109, 113, 117, 118].

Figure 2.4: Energy spectrum for RQR versus deformation ($\mathcal{E} = \hbar^2/(mR^2)$): a) $n = 2$, b) $n = 3$ and c) $n = 4$.

Figure 2.5: Energy spectrum for a quantum ring geometry arising from the superposition of $C_{p=4}$ and $C_{q=8}$ symmetries for $c = 0.3$ (a) and, $C_{p=3}$ and $C_{q=5}$ symmetries for $c = 0.45$ (b).

CHAPTER 3

NON-RELATIVISTIC QUANTUM MECHANICS ON A SUBMANIFOLD

In this chapter, we study the effective Schrödinger equation to spinless particles confined on a m -dimensional submanifold¹, in the presence of an electromagnetic field, which propagates into a n -dimensional Euclidean space. In the absence of an electromagnetic field, this has been just carried in [7, 11, 16]. In the presence of an electromagnetic field, the following two cases have been studied: (i) a particle in a 2D curved surface embedded in a 3D space [29, 31] and (ii) a particle in a curve embedded in a plane [27]. After obtaining a general expression for the effective Schrödinger equation, we discuss its features and present some examples.

3.1 Effective Schrödinger Equation

In the appendix A, we present a set of definitions and conventions of differential geometry of pseudo-Riemannian submanifold that we use in what follows. However, unlike the rest of the thesis, in this chapter we consider the non-relativistic case, and therefore, the particle will be confined in a m -dimensional Riemannian submanifold (not a pseudo-Riemannian submanifold) of an Euclidean space. Thus, some remarks are required.

We employ Latin indices $A, B, \dots$, running from 1 to n , for the coordinates on the Euclidean space, Greek indices $\mu, \nu, \dots$, running from 1 to m , for coordinates on $\mathcal{M}$ (the Riemannian submanifold) and Latin indices $i, j, \dots$, running from $m + 1$ to n for the normal coordinates. An *adapted coordinates frame* $\mathcal{F}$ may be introduced by using a general system of coordinates x^μ on $\mathcal{M}$ and a set of distances y^i ($i = 1, \dots, p = n - m$) from $\mathcal{M}$ in directions normal to $\mathcal{M}$. We assume that $\mathcal{M}$ has a tubular neighborhood, such that [7, 11]

$$\mathbf{R}(x^\mu, y^i) = \mathbf{r}(x^\mu) + y^j \mathbf{n}_j(x^\mu), \quad (3.1)$$

¹A low-dimensional space. For instance a curve or a surface.

Figure 3.1: Adapted coordinate frame.

defines an arbitrary point in the vicinity of $\mathcal{M}$, here, the $\mathbf{n}_j$ are orthonormal vectors to $\mathcal{M}$ in every point $x \in \mathcal{M}$, Fig. 3.1 If $\mathbf{t}_\mu = \frac{\partial \mathbf{r}(x)}{\partial x^\mu}$ are m tangent vectors on $\mathcal{M}$, the first fundamental form (metric), the second fundamental form (extrinsic curvature) and the normal fundamental form (geometric or extrinsic torsion) are respectively,

$$g_{\mu\nu} \equiv \mathbf{t}_\mu \cdot \mathbf{t}_\nu, \quad (3.2)$$

$$\alpha_{i\mu\nu} \equiv -\mathbf{t}_\mu \cdot \partial_\nu \hat{\mathbf{n}}_i, \quad (3.3)$$

$$\beta_{\mu i}^{k} \eta_{kj} \equiv \hat{\mathbf{n}}_i \cdot \partial_\mu \hat{\mathbf{n}}_j. \quad (3.4)$$

The metric for the Euclidean space is given by

$$G_{AB} = \begin{pmatrix} \gamma_{\mu\nu} + y^k y^l \beta_{\mu k}^{h} \beta_{\nu l}^{m} \delta_{hm} & -y^k \beta_{\mu k}^{h} \delta_{ih} \\ -y^k \beta_{\nu k}^{h} \delta_{jh} & \delta_{ij} \end{pmatrix} \quad (3.5)$$

where $\gamma_{\mu\nu}$ is given by

$$\gamma_{\mu\nu} = g_{\mu\nu} - 2y^k \alpha_{k\mu\nu} + y^k y^l \alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{l\sigma\nu}. \quad (3.6)$$

Denoting the determinants of G_{AB} and $\gamma_{\mu\nu}$ by $|G|$ and $|\gamma|$, respectively, it is possible to show that $|G| = |\gamma|$. The inverse of the metric tensor G_{AB} can be calculated exactly, and is given by

$$G^{AB} = \begin{pmatrix} \lambda^{\mu\nu} & \lambda^{\mu\sigma} y^k \beta_{\sigma k}^{j} \\ \lambda^{\nu\sigma} y^k \beta_{\sigma k}^{i} & \delta^{ij} + y^k y^l \beta_{\sigma k}^{i} \beta_{\rho l}^{j} \lambda^{\sigma\rho} \end{pmatrix}, \quad (3.7)$$

where $\lambda^{\mu\nu} \equiv (\gamma^{-1})_{\mu\nu}$ is the inverse of $\gamma_{\mu\nu}$ [11]. We employ a gauge covariant derivative $\nabla_J = \partial_J - \frac{ie}{\hbar} A_J$, where A_J are the components of the (n) -vector potential $\mathbf{A}$. This

covariant derivative can be extended to $(n+1)$ -dimensions defining a gauge covariant derivative for the time variable as $\nabla_0 = \partial_t - ieA_0/\hbar$, with $A_0 = -\phi$ and where ϕ is the scalar potential. Thus, the Schrödinger equation in n spatial dimensions in the presence of the electromagnetic field is

$$i\hbar\nabla_0\Psi = H\Psi = \left[-\frac{\hbar^2}{2m|G|^{1/2}}\nabla_A|G|^{1/2}G^{AB}\nabla_B + V_c(y) \right] \Psi. \quad (3.8)$$

Now, following our recipe, sec. 2.1.1, we rescale the Schrödinger field Ψ and the operator $\Delta = \frac{\hbar^2}{|G|^{1/2}}\nabla_A|G|^{1/2}G^{AB}\nabla_B$ to

$$\chi(x, y) \equiv \frac{|G|^{1/4}}{|g|^{1/4}}\Psi(x, y), \quad (3.9)$$

$$\square \equiv \left(\frac{|G|^{1/4}}{|g|^{1/4}} \right) \Delta \left(\frac{|g|^{1/4}}{|G|^{1/4}} \right). \quad (3.10)$$

Using Eq. (3.9) the normalization condition can be written as

$$\int d^m|G|^{1/2}|\Psi|^2 = \int d^m x d^p y |G|^{1/2}|\Psi|^2 = \int d^m x |g|^{1/2} \left[\int d^p y |\chi|^2 \right] = 1, \quad (3.11)$$

so, $\int d^p y |\chi|^2$ describes a density the probability on a submanifold. The rescaled operator $\square$ has four terms and is given by

$$\begin{aligned} \square\chi &= \frac{\hbar^2}{|G|^{1/4}g^{1/4}}\partial_A \left(|G|^{1/2}G^{AB}\partial_B \frac{|g|^{1/4}}{|G|^{1/4}}\chi \right) - \frac{ie\hbar}{|G|^{1/2}}\partial_A(|G|^{1/2}G^{AB}A_B)\chi \\ &\quad - 2ie\hbar \left(\frac{|G|^{1/4}}{|g|^{1/4}} \right) A_A G^{AB} \partial_B \left(\frac{|g|^{1/4}}{|G|^{1/4}}\chi \right) - e^2 G^{AB} A_A A_B \chi. \end{aligned} \quad (3.12)$$

In appendix B.1, we expand this operator to obtain

$$\begin{aligned} -\square\chi &= -\frac{\hbar^2}{|\gamma|^{1/4}}\partial_i \left(|\gamma|^{1/2}\partial^i \frac{1}{|\gamma|^{1/4}} \right) \chi - \frac{\hbar^2}{|g|^{1/4}|\gamma|^{1/4}} \left[\hat{\partial}_\mu \lambda^{\mu\nu} |\gamma|^{1/2} \hat{\partial}_\nu \right] \frac{|g|^{1/4}}{|\gamma|^{1/4}} \chi \\ &\quad + ie\hbar \left[\frac{1}{|\gamma|^{1/2}}\partial^i (|\gamma|^{1/2}A_i) + \frac{1}{|\gamma|^{1/2}}y^k \beta_{\sigma k}^i \partial_i (|\gamma|^{1/2}y^l \beta_{\rho l}^j \lambda^{\sigma\rho} A_j) \right. \\ &\quad \left. + \frac{1}{|\gamma|^{1/2}}y^k \beta_{\sigma k}^i \partial_i (|\gamma|^{1/2}\lambda^{\nu\sigma} A_\nu) \right. \\ &\quad \left. + \frac{1}{|\gamma|^{1/2}}\partial_\mu (|\gamma|^{1/2}\lambda^{\mu\nu} A_\nu) + \frac{1}{|\gamma|^{1/2}}\partial_\mu (|\gamma|^{1/2}\lambda^{\mu\sigma} y^k \beta_{\sigma k}^j A_j) \right] \chi \\ &\quad + 2ie\hbar \left[\left(\frac{|g|^{1/4}}{|g|^{1/4}} \right) A_i \partial^i \left(\frac{|g|^{1/4}}{|g|^{1/4}}\chi \right) + \left(\frac{|g|^{1/4}}{|g|^{1/4}} \right) A_i \lambda^{\nu\sigma} y^k \beta_{\sigma k}^i \partial_\nu \left(\frac{|g|^{1/4}}{|g|^{1/4}}\chi \right) \right. \\ &\quad \left. + \left(\frac{|g|^{1/4}}{|g|^{1/4}} \right) A_\mu \lambda^{\mu\nu} \partial_\nu \left(\frac{|g|^{1/4}}{|g|^{1/4}}\chi \right) + \left(\frac{|g|^{1/4}}{|g|^{1/4}} \right) A_\mu \lambda^{\mu\sigma} y^k \beta_{\sigma k}^j \partial_j \left(\frac{|g|^{1/4}}{|g|^{1/4}}\chi \right) \right] \\ &\quad + e^2 \left[A_i A^i + A_i \lambda^{\nu\sigma} y^k \beta_{\sigma k}^i A_\nu + A_\mu \lambda^{\mu\nu} A_\nu + A_\mu \lambda^{\mu\sigma} y^k \beta_{\sigma k}^j A_j \right] \chi, \end{aligned} \quad (3.13)$$

where $\hat{\partial}_\mu \equiv \partial_\mu + \frac{1}{2\hbar}i\beta_\mu^{ij}L_{ij}$ and $L_{ij} = i\hbar(y_j\partial_i - y_i\partial_j)$ are the angular momentum operators in the space normal to $\mathcal{M}$.

3.1.1 Perturbation Theory

We now take into account the explicit conditions satisfied by the potential $V(y)$. It has a deep minimum relative to the submanifold, depends only on the normal coordinates to the submanifold as well as on a parameter ϵ such that when $\epsilon \rightarrow 0$ the potential goes to infinity outside the submanifold [7, 9]. These conditions are satisfied by

$$V(y) = \frac{m\omega^2}{2\epsilon^4} y^i y_i + O(y^3). \quad (3.14)$$

Following Refs. [7, 11], let us rescale the normal coordinates by $y \rightarrow \epsilon y$. Expanding $\square$ in powers of ϵ yields

$$\epsilon^2 \square = \square^{(0)} + \epsilon \square^{(1)} + \epsilon^2 \square^{(2)} + \epsilon^3 \square^{(3)} + \dots \quad (3.15)$$

Since that $A_B(x^\mu, \epsilon y^i)$ is now dependent on ϵ its Taylor expansion is given by

$$A_B(x^\mu, \epsilon y^i) = A_B(x^\mu, 0) + \epsilon y^j \left. \frac{\partial A_B(x^\mu, y^i)}{\partial y^j} \right|_{y=0} + \dots \quad (3.16)$$

Using Eqs. (B.7) and (B.8), we obtain the following contributions to the zero, first² and second order terms

$$\square^{(0)} = \hbar^2 \partial_i \partial^i, \quad (3.17)$$

$$\square^{(1)} = -2ie\hbar \bar{A}_i \partial^i, \quad (3.18)$$

$$\begin{aligned} \square^{(2)} \chi = & \frac{\hbar^2}{|g|^{1/2}} \left[\hat{\partial}_\mu g^{\mu\nu} |g|^{1/2} \hat{\partial}_\nu \right] \chi + 2V_G m \chi - 2ie\hbar \bar{A}_\mu g^{\mu\nu} \partial_\nu \chi - \frac{ie\hbar}{|g|^{1/2}} \partial_\mu \left(|g|^{1/2} g^{\mu\nu} \bar{A}_\nu \right) \chi \\ & - 2ie\hbar \bar{A}_\mu g^{\mu\sigma} y^k \beta_{\sigma k}^j \partial_j \chi - e^2 (\bar{A}_i \bar{A}^i \chi + \bar{A}_\mu g^{\mu\nu} \bar{A}_\nu) \chi - ie\hbar \frac{\partial \bar{A}_i}{\partial y_i} \chi - 2ie\hbar \frac{\partial \bar{A}_i}{\partial y^j} y^j \partial^i \chi, \end{aligned} \quad (3.19)$$

where V_G is a *geometric potential* given by³

$$V_G = -\frac{\hbar^2}{8m} g^{\nu\mu} g^{\rho\sigma} \left(\alpha_{k\mu\nu} \alpha_{\rho\sigma}^k - 2\alpha_{k\mu\sigma} \alpha_{\rho\nu}^k \right). \quad (3.20)$$

Therefore, the Schrödinger equation can be rewritten as

$$i\hbar \frac{\partial \chi(x, y, t)}{\partial t} = \left[-\frac{\square^{(0)}}{2m} + \frac{m\omega^2}{2} y^i y_i - \epsilon \frac{\square^{(1)}}{2m} - \epsilon^2 \frac{\square^{(2)}}{2m} + O(\epsilon^3) \right] \chi(x, y, t). \quad (3.21)$$

²The notations $\bar{A}_B = A_B(x^\mu, 0)$ and $\frac{\partial^n \bar{A}_B}{\partial (y^j)^n} = \frac{\partial^n A_B(x^\mu, y^i)}{\partial (y^j)^n} \Big|_{y=0}$ have been introduced.

³We have used

$$\begin{aligned} & \frac{1}{|\gamma|^{1/4}} \partial_i \left(|\gamma|^{1/2} \partial^i \frac{\chi}{|\gamma|^{1/4}} \right) = \partial_i \partial^i \chi + \epsilon^2 \chi \left(\frac{3}{16} \frac{\partial_i |\gamma| \partial^i |\gamma|}{|\gamma|^2} - \frac{1}{4} \frac{\partial_i \partial^i |\gamma|}{|\gamma|} \right) \\ & \approx \partial_i \partial^i \chi - \epsilon^2 \frac{1}{4} g^{\nu\mu} g^{\rho\sigma} \left(\alpha_{k\mu\nu} \alpha_{\rho\sigma}^k - 2\alpha_{k\mu\sigma} \alpha_{\rho\nu}^k \right) \chi + O(\epsilon^3). \end{aligned}$$

To obtain an effective dynamics on the submanifold, it is necessary to “freeze” the normal degrees of freedom [7, 11]. However, the first-order term (Eq. (3.18)) couples the dynamics of the tangent and normal degrees of freedom through the normal components of the electromagnetic potential evaluated at y^j equal to zero ($\bar{A}_i = A_i(x^\mu, 0)$). When the codimension is equal to one, it is always possible to choose a gauge in such a way that $A_{n-1}(x^\mu, y) = 0$, similarly to what was done in Ref [29]. In the general case, we see that it is only necessary that $A_i(x^\mu, 0) = 0$ in order to decouple the dynamics. In the following, we will show that there is always a gauge transformation that allows us to decouple the equations. The gauge transformation $A'_B = A_B + \partial_B \gamma$ such that $\bar{A}'_i = 0$ is

$$\gamma(x^\mu, y^j) = -A_i(x^\mu, 0)y^i. \quad (3.22)$$

Although there is no gauge transformation which eliminates the term $T_{kj} \equiv \frac{\partial A_k(x^\mu, y^i)}{\partial y^j} \Big|_{y=0}$ (except for a codimension equal to one) it is possible to eliminate the symmetric part of T_{ij} ,

$$\gamma'(x^\mu, y^j) = -y^k y^j \frac{T_{kj}}{2}, \quad (3.23)$$

whereby

$$A'_i = A_i + \partial_i \gamma' = -\frac{y^j}{2} \bar{F}_{ij} + O(y^2), \quad (3.24)$$

where $\bar{F}_{ij} = F_{ij}|_{y=0}$. Thus, without loss of generality, in what follows we will set $\bar{A}_i = 0$ and assume that T_{ij} is purely antisymmetric. With these conditions Eq. (3.19) can be written as

$$\begin{aligned} \square^{(2)} \chi &= \frac{\hbar^2}{|g|^{1/2}} \left[\left(\partial_\mu + \frac{1}{2\hbar} i \beta_\mu^{ij} L_{ij} - \frac{ie}{\hbar} \bar{A}_\mu \right) g^{\mu\nu} |g|^{1/2} \left(\partial_\nu + \frac{1}{2\hbar} i \beta_\nu^{ij} L_{ij} - \frac{ie}{\hbar} \bar{A}_\nu \right) \right] \chi \\ &\quad + 2V_G m \chi + e \bar{F}^{ij} L_{ij} \chi. \end{aligned} \quad (3.25)$$

Let us now consider the separated solution⁴

$$\chi(x, y, t) = \sum_\beta \varphi_\beta(x, t) \psi_\beta(y, t), \quad (3.27)$$

where the index β runs over the degeneracy that exists in the spectrum of the normal degrees of freedom [11] (this degeneracy is due to the invariance of $\mathcal{H}^{(0)} = -\frac{\square^{(0)}}{2m}$ under

⁴Strictly, we should write the solution (3.27) as

$$\chi^l(x, y, t) = \sum_\beta \varphi_\beta^l(x, t) \psi_\beta^l(y, t), \quad (3.26)$$

to emphasize that the system is an individual energy eigenstates (λ_l).

rotations of the coordinates y^i). The normalization conditions (3.11) yields

$$\begin{aligned} \int d^m x |g|^{1/2} \left[\int d^p y |\chi|^2 \right] &= \sum_{\beta, \alpha} \int d^m x |g|^{1/2} \varphi_\beta(x, t) \varphi_\alpha^*(x, t) \left[\int d^p y \psi_\beta(y, t) \psi_\alpha^*(y, t) \right] \\ &= \sum_{\beta} \int d^m x |g|^{1/2} |\varphi_\beta(x, t)|^2 \\ &= \sum_{\beta} (\varphi_\beta, \varphi_\beta) = 1, \end{aligned} \quad (3.28)$$

where we have taken into account, without loss of generality, that $\int d^p y \psi_\beta^*(y) \psi_\alpha(y) = \delta_{\beta\alpha}$. Note that φ_β form a multiplet, $\vec{\varphi} = (\varphi_1, \dots, \varphi_\beta)$, with dimension d_e that depends on the degeneracy that exists in the spectrum of the normal degrees of freedom. One can define a scalar product for the multiplets as $\langle \vec{\varphi} | \vec{\varphi}' \rangle \equiv \sum_{\beta} (\varphi_\beta, \varphi'_\beta)$. Thus, the normalization condition (3.28) can be rewritten as $\langle \vec{\varphi} | \vec{\varphi} \rangle = 1$.

Now, inserting (3.27) into Eq. (3.21), multiplying by $\psi_\alpha^*(y)$ and integrating in $\int d^p y$ we obtain

$$\frac{1}{2m} \sum_{\beta} \int d^p y \psi_\alpha^*(y) \square^{(2)} \psi_\beta(y) \varphi_\beta(x) + \sum_{\beta} \bar{\lambda}_{\alpha\beta} \varphi_\beta(x) = 0,$$

with

$$\bar{\lambda}_{\alpha\beta} = \int d^p y \psi_\alpha^*(y) \left[-\frac{\hbar^2}{2m} \partial_i \partial^i + \frac{m\omega^2}{2\epsilon^4} y^i y_i \right] \psi_\beta(y). \quad (3.29)$$

In the above equation have rescaled the coordinate $y \rightarrow y/\epsilon$ again. Since the normal degrees of freedom are in an individual energy eigenstates λ_l , the eigenvalue equation is

$$\mathcal{H}^{(0)} \psi_\beta^l(y) = \left[-\frac{\hbar^2}{2m} \partial_i \partial^i + \frac{m\omega^2}{2\epsilon^4} y^i y_i \right] \psi_\beta^l(y) = \bar{\lambda} \psi_\beta^l(y). \quad (3.30)$$

This equation describes a system of $p = n - m$ decoupled quantum harmonic oscillators. Thus, the spectrum is $\bar{\lambda}_l = \hbar \beta_l \omega (l+1/2)/\epsilon^2$, with $\beta = \beta_l = \frac{(p+l-1)!}{l!(p-1)!}$, and the eigenfunctions are

$$\psi_\beta^l(y) = \tilde{\psi}_\beta^{(l,1)}(y^1) \dots \tilde{\psi}_{\beta_l}^{(l,p)}(y^p), \quad (3.31)$$

with

$$\tilde{\psi}_\beta^{(l,i)}(y^i) = \left(\frac{\omega}{\epsilon^2 \pi \hbar} \right)^{1/4} \frac{e^{-\frac{\omega}{2\epsilon^4 \hbar} (y^i)^2}}{(2^l l!)^{1/2}} H_l \left(\frac{m^{1/4} \omega^{1/2}}{\epsilon \hbar^{1/2}} y^i \right). \quad (3.32)$$

The difference between two eigenvalues increases as $\epsilon \rightarrow 0$ to the point that the particle is increasingly confined close to the submanifold. One can show that for symmetric confining potentials, $\square^{(2)}$ can not produce transitions between the energy levels of the normal degrees of freedom. Then the system is block diagonal, with each block built with

the subspace spanned by ψ_α . Thus, the effective Schrödinger equation for a particle in an electromagnetic field on a submanifold is

$$i\hbar \frac{\partial \varphi(x, t)}{\partial t} = \frac{-\hbar^2}{2m|g|^{1/2}} \left[\left(\partial_\mu - i \frac{\mathcal{B}_\mu}{\hbar} - \frac{ie}{\hbar} \bar{A}_\mu \right) g^{\mu\nu} |g|^{1/2} \left(\partial_\nu - i \frac{\mathcal{B}_\nu}{\hbar} - \frac{ie}{\hbar} \bar{A}_\nu \right) \right] \varphi(x, t) \\ + \underbrace{\frac{\hbar^2}{8m} g^{\nu\mu} g^{\rho\sigma} \left(\alpha_{k\mu\nu} \alpha^k_{\rho\sigma} - 2\alpha_{k\mu\sigma} \alpha^k_{\rho\nu} \right)}_{V_G} \varphi(x, t) - \frac{e}{2m} \bar{F}^{ij} \mathcal{L}_{ij} \varphi(x, t) - e \bar{A}_0 \varphi(x, t), \quad (3.33)$$

where $\mathcal{B}_\mu = -\frac{1}{2} \beta_\mu^{ij} (\mathcal{L}_{ij})$,

$$(\mathcal{L}_{ij})_{\alpha\beta} \equiv \int d^p y (\psi_\alpha(y))^* L_{ij} \psi_\beta(y), \quad (3.34)$$

and an irrelevant shift in the energy was eliminated, since it has no physical effect. In the special cases when $A_\mu = 0$ and $A_i = 0$, it reproduces what was found in [7, 11]. Note that the metric tensor on the submanifold, $g_{\mu\nu}$, has emerged naturally. This can be considered a classical effect since this also appears in the classical formulation of the problem.

Let us compare the effective Schrödinger equation on a submanifold with the Schrödinger equation in the same space, but now it is seen as a manifold

$$i\hbar \frac{\partial \phi(x, t)}{\partial t} = \frac{-\hbar^2}{2m|g|^{1/2}} \left[\left(\partial_\mu - \frac{ie}{\hbar} \bar{A}_\mu \right) g^{\mu\nu} |g|^{1/2} \left(\partial_\nu - \frac{ie}{\hbar} \bar{A}_\nu \right) \right] \phi(x, t) \\ - e \bar{A}_0 \phi(x, t) + \frac{\xi \hbar^2}{2m} R \phi(x, t), \quad (3.35)$$

where R is the scalar of curvature⁵ of $\mathcal{M}$. Comparing Eq. (3.33) with Eq. (3.35), one can identify four main differences. First, the extrinsic torsion plays no role in (3.35). Second, Eq. (3.33) depends on the extrinsic properties of the submanifold via V_G , as well as the extrinsic torsion. Third, in Eq. (3.33) $\vec{\phi}$ is a multiplet whose dimension depends on the degeneracy that exists in the spectrum of the normal degrees of freedom. Fourth, in Eq. (3.33) there is a coupling with $\bar{F}^{ij}$. In conclusion, these equations do not describe the same physical system. In the following, we discuss in detail the features of Eq. (3.33).

3.1.2 Geometric Potential and Geometrically-induced Yang-Mills field

The term V_G is known in the literature as *geometric potential* and have information about the intrinsic and extrinsic properties of the submanifold. When $n = 3$ and $m = 1$, we recover the result of last chapter, and when $n = 3$ and $m = 2$, we obtain [9]

$$V_G = -\frac{\hbar^2}{2m} (H^2 - K), \quad (3.36)$$

⁵The scalar of curvature is an intrinsic property of $\mathcal{M}$, so it just depends on g .

where H is the *mean curvature* and K is the *Gaussian curvature* defined by

$$H \equiv -\frac{1}{2}\text{tr}(\alpha_{3\mu}{}^\nu) = -\frac{1}{2}\alpha_{3\mu}{}^\mu, \quad K \equiv \det(\alpha_{3\mu}{}^\nu). \quad (3.37)$$

In general the geometric potential implies that the quantum dynamics is affected by changes in the extrinsic properties of the submanifold.

In addition to the geometric potential, a *geometry-induced Yang-Mills field* ($\mathcal{B}_\mu$) emerge when the low-dimensional space has a geometric torsion [7, 11, 15–17, 34]. The geometry-induced Yang-Mills field emerges because the orbital angular momentum of the normal coordinates to the submanifold couples with the geometric torsion. Therefore, this geometry-induced Yang-Mills field is only non-zero if the normal field, $\psi_\beta^l(y)$, lies in a degenerate, non-trivial representation of $SO(p = n - m)$, *i.e.* the potential appears when the field is in an excited state of the normal coordinates, $l \neq 0$. Additionally, this can only exist if the codimension is greater than one.

Let us discuss the case $p = 2$. In this case $\beta_l = l + 1$, so, the scalar field appears as a singlet, doublet, triplet..., for $l = 0, l = 1, l = 2...$, respectively. From the theory of 2D-harmonic oscillator, one knows that the angular momentum is zero for the singlet, $\hbar$ and $-\hbar$ for the doublet, $2\hbar, 0$ and $-2\hbar$ for the triplet, etc.... Thus, for an “inner” observer (an observer who lives on a submanifold) each representation contains a spectrum of particles with a quantized normal angular momentum, for which one might say that looks like an “*induced intrinsic angular momentum*”, where the operator $\mathcal{L}_{ij}$ plays the role of a “*induced spin operator*”. Although this is not, strictly speaking, a real spin, let us show that it shares several additional features with the real spin.

3.1.3 Induced Zeeman Coupling

It is well known that a particle with intrinsic angular momentum $\mathbf{S}$ (spin), in the presence of a magnetic field, interacts with the magnetic field through a Zeeman coupling

$$g_s \frac{\mu}{\hbar} \mathbf{B} \cdot \mathbf{S} = \frac{\mathbf{B} \cdot \mu_{\mathbf{s}}}{\hbar}, \quad (3.38)$$

where the Bohr magneton is $\mu = -\frac{e\hbar}{2m}$, g_s the Landé factor and μ_s the intrinsic magnetic moment. The Zeeman coupling naturally appears in the Pauli equation or through the non-relativistic limit of the Dirac equation [120]. It is clear that for spinless particles this term should no longer appear. However, Eq. (3.33) shows that the “*induced intrinsic angular momentum*” also couples with the normal components of the electromagnetic tensor,

$$\bar{F}^{ij} = \left(\partial^i A^j - \partial^j A^i \right) \Big|_{y=0}. \quad (3.39)$$

Since, it is clear that $\bar{F}^{ij}$ is a magnetic field generated by the normal components, A^i ($i, j = 1, \dots, n - m$) and given that, for an inner observer, $\mathcal{L}_{ij}$ is perceived as an “*induced spin operator*”, one can interpret the term $\frac{e}{2} \bar{F}^{ij} \mathcal{L}_{ij}$ as an *induced Zeeman coupling*, where $\bar{\mu}_{ij} = -\frac{e}{2} \mathcal{L}_{ij}$ would be identified as an *induced magnetic moment tensor*. For example,

when the codimension is two $\mathcal{L}_{ij}$ has only one independent component and it can be written as

$$-\frac{e}{2m}\bar{F}^{ij}\mathcal{L}_{ij} = -\frac{e}{m}\bar{F}^{12}\mathcal{L}_{12} = \frac{2}{\hbar}B_s\mu\mathcal{S}, \quad (3.40)$$

with $\mu = -e\hbar/(2m)$, $B_s = \bar{F}^{12}$ a magnetic field and defining $\mathcal{S} = \mathcal{L}_{12}$ as the only independent component of the matrix $\mathcal{L}_{ij}$. In this case, one can interpret an *induced magnetic moment*

$$\mu_s = -\frac{2}{\hbar}\mu\mathcal{S}, \quad (3.41)$$

with a induced Landé g -factor $g_s = 2$. Therefore, the induced Zeeman coupling reinforces the idea that from the point of view of an “inner” observer, the orbital angular momentum of the normal coordinates is perceived as an “intrinsic angular momentum”.

Finally, let us remark that the induced Zeeman coupling is a new quantum potential, which is independent of the curvature and torsion, as a consequence it is present even in flat submanifolds⁶. In particular, these effects must be considered when working on systems such as quantum rings or quantum wires⁷.

3.2 Quantum Rings and Quantum Wires in the Presence of Magnetic Field

In this section we will consider the particular cases of space and plane curves ($n = 3$ and $m = 1$). The relevant effective Schrödinger equation for this case is

$$i\hbar\frac{\partial\vec{\varphi}(s,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\left(\partial_s - \frac{i\tau\mathcal{S}}{\hbar} - \frac{ie}{\hbar}\bar{A}_s\right)^2 - e\bar{A}_0 - \frac{\hbar^2}{8m}\alpha_k\alpha^k + \frac{2}{\hbar}B_s\mu\mathcal{S} \right]\vec{\varphi}(s,t), \quad (3.42)$$

where $B_s = \bar{F}^{12}$ the magnetic field tangent to $\mathcal{C}$ and $\mathcal{S} = \mathcal{L}_{12}$. Here, we have used that β_{ij} is antisymmetric to define the *torsion* $\tau \equiv \hat{\mathbf{n}}_2 \cdot \partial_s \hat{\mathbf{n}}_1 = \beta_{21}$ as the only independent component of β_{ij}

3.2.1 Example 1: Untwisted Quantum Ring

Let us show a simple example, a ring crossed by an infinite wire carrying current I . The magnetic field lines are tangential to the ring with an intensity of $B_s = \frac{\mu_0 I}{2\pi R}$, with

⁶This extends previous effects found in the literature due to dimensional reduction, in particular, the results of Refs. [27, 29] where only were studied cases when the dimension is reduced by one.

⁷ Since the ground state of the $\mathcal{H}^{(0)}$ has zero angular momentum, it is necessary prepared the system in an excited state of $\mathcal{H}^{(0)}$ if one want to see associated effects to the angular momentum of the normal degrees of freedom.

R the radius of the ring. The effective Schrödinger equation (3.42) yields the following equation

$$E\bar{\varphi}^{\mathcal{E}} = \left[-\mathbf{1}\frac{\hbar^2}{2m}\partial_s^2 - \mathbf{1}\frac{\hbar^2}{8m}\alpha_k\alpha^k + \frac{2}{\hbar}B_s\mu\mathcal{S} \right] \bar{\varphi}^{\mathcal{E}}, \quad (3.43)$$

where we put the subscript $\mathcal{E}$ again to remember that this state depends on the energy levels of $\mathcal{H}^{(0)}$ and with $\mathbf{1}$ the $d_e \times d_e$ identity matrix. If the system is prepared in the ground state of $\mathcal{H}^{(0)}$, the angular momentum in the normal coordinates ($\mathcal{S}$) is zero. Thus,

$$E_l = \frac{\hbar^2}{2mR^2}l^2 - \frac{\hbar^2}{8mR^2}, \quad l = 0, \pm 1, \pm 2, \dots, \quad (3.44)$$

note that except for the ground state all levels has a 2-fold degeneracy. Now, for a system prepared in the first excited state of $\mathcal{H}^{(0)}$, one has that $\vec{\varphi} = (\varphi_1, \varphi_2)$ due to that $\mathcal{H}^{(0)}$ is doubly degenerate. One calculate that in this level $\mathcal{S} = \text{diag}(\hbar, -\hbar)$, so

$$E_l = \frac{\hbar^2}{2mR^2}l^2 - \frac{\hbar^2}{8mR^2} \pm 2B_s\mu, \quad l = 0, \pm 1, \pm 2, \dots, \quad (3.45)$$

with $+$ for φ_1 and $-$ for φ_2 . When $B_s \neq 0$ there is a split in the spectrum, otherwise, the ground state is 2-fold degenerate and all other states are 4-fold degenerate.

3.2.2 Example 2: Helix

Let us consider particles confined in a helix parameterized as

$$\vec{r} = \left(\frac{R \cos s}{\sqrt{R^2 + c^2}}, \frac{R \sin s}{\sqrt{R^2 + c^2}}, c \right),$$

where s is the arc-length and c is a constant. As in the last example, the helix is crossed by an infinite wire carrying current I . One can show that the time-independent Schrödinger equation is given by

$$E\bar{\varphi}^{\mathcal{E}} = \left[-\frac{\hbar^2}{2m} \left(\partial_s - \frac{ic\mathcal{S}}{(R^2 + c^2)\hbar} + \frac{iec\mu_0 I \ln(R/R_0)}{2\pi\hbar\sqrt{R^2 + c^2}} \right)^2 - \frac{\hbar^2 R^2}{8m(R^2 + c^2)^2} + \frac{2}{\hbar} \frac{\mu_0 I}{2\pi\sqrt{R^2 + c^2}} \mu \mathcal{S} \right] \bar{\varphi}^{\mathcal{E}}.$$

If the system is in the first excited state of $\mathcal{H}^{(0)}$, the energy the spectrum is continuous and is given by

$$E_p = \frac{1}{2m} \left(p_s \pm \frac{c\hbar}{(R^2 + c^2)} - \frac{ec\mu_0 I \ln(R/R_0)}{2\pi\sqrt{R^2 + c^2}} \right)^2 - \frac{\hbar^2 R^2}{8m(R^2 + c^2)^2} \pm \frac{\mu_0 I}{\pi\sqrt{R^2 + c^2}} \mu \quad (3.46)$$

with $+$ for φ_1 and $-$ for φ_2 . The above example clearly illustrates that, in general, is important to consider the effect of torsion, curvature and the induced Zeeman coupling in the dynamics of particles in one-dimensional systems.

3.2.3 Example 3: Aharonov-Bohm Oscillations

Charged particles confined in a ringlike structure in the presence of an external magnetic field⁸ manifest a topological quantum interference phenomenon known as Aharonov-Bohm (AB) effect [121]. The signatures of this phenomena is the oscillatory behavior of the energies levels as a function of the magnetic flux and the occurrence of persistent currents, with period $\Phi_0 = hc/e$ [122, 123]. These phenomena have been experimentally reported in the mesoscopic regime in metallic [124, 125] and semiconducting rings [126, 127], containing many electrons, as well as in self-assembled quantum rings with few electrons [128, 129]. Because of the presence of AB oscillations, persistent currents and Berry phase [130–132], quantum rings have been the subject of many experimental and theoretical studies.

In sec. 2.2.2 we studied the spectrum of quantum rings over deformation with a cyclic symmetry C_n . Now we will study its spectrum in the presence of an external perpendicular magnetic field. In the presence of electromagnetic fields, the effective dynamics of a charged particle in the arc-length parametrization is given by

$$E\tilde{\psi}(s) = \left[-\frac{\hbar^2}{2m} \left(\frac{\partial}{\partial s} - \frac{ie}{c\hbar} \bar{A}_s \right)^2 - e\bar{A}_0 - \frac{\hbar^2}{8m} \alpha(s)^2 \right] \tilde{\psi}(s). \quad (3.47)$$

When $\bar{A}_0 = 0$, the electromagnetic field can be eliminated of Eq. (3.47), by the following gauge transformation [28, 33]

$$\tilde{\psi}(s) = \exp \left[\frac{ie}{\hbar c} \int_0^s \bar{A}_{s'} ds' \right] \psi(s), \quad (3.48)$$

so,

$$E\psi(s) = -\frac{\hbar^2}{2m} \partial_s^2 \psi(s) - \frac{\hbar^2}{8m} \alpha(s)^2 \psi(s). \quad (3.49)$$

Nevertheless, the boundary conditions still depend on A_s . For the untransformed wave function $\tilde{\psi}(s)$ the boundary conditions demand continuity in the wave function ($\tilde{\psi}(0) = \tilde{\psi}(p)$) and in the derivative ($\partial_s \tilde{\psi}(0) = \partial_s \tilde{\psi}(p)$), where p is the perimeter of the curve. Thus, these conditions lead to [28, 33]

$$\psi(0) = e^{2\pi i \frac{\Phi}{\Phi_0}} \psi(p), \quad \partial_s \psi(0) = e^{2\pi i \frac{\Phi}{\Phi_0}} \partial_s \psi(p), \quad (3.50)$$

where Φ is the magnetic flux through the distorted ring and $\Phi_0 = \frac{hc}{e}$ is the quantum of magnetic flux. Using that $g(0) = g(2\pi)$, one can prove that in a general parametrization θ , the boundary conditions remain unchanged.

For our parametrization, Eq. (2.29), we carried out some numerical simulations for different values of n and λ . In particular, Fig. 3.2 show the cases of $n = 3$ with $\lambda = 0.4$ and $n = 4$ with $\lambda = 0.3$. In general, the spectrum exhibits level crossings at half-integer and integer multiples of Φ_0 as in the perfect circular ring. More importantly, we find that for rings with a cyclic symmetry C_n , the Aharonov-Bohm oscillations are split into sets of n -states. This property is compatible with previous simulations where the system had a cyclic symmetry [28, 30, 107, 118], even if the system is a 2DQR [109, 113, 117].

⁸Perpendicular to the plane of the ring.

Figure 3.2: Energy spectrum vs Φ/Φ_0 for a) $n = 3$ and $\lambda = 0.4$, b) $n = 4$ and $\lambda = 0.3$.

3.3 Absence of Effects of an In-Plane Magnetic Field in a q2DES

Quasi-two-dimensional electron system (q2DES) in the presence of a magnetic field has been a topic extensively studied. Exceptional attention has been paid to perpendicular magnetic fields, which leads to the integer [133] and fractional quantum Hall effects [134, 135]. Although less attention has been given in the recent decades to in-plane or tilted magnetic fields, researchers have shown an increased interest (see, for instance, [109, 111, 136–144]). For example, in Refs. [109, 111, 139, 140] the authors, neglected the Zeeman

term, reported a modification in the spectrum of energy as well as a squeeze of the electron wavefunction in two-dimensional quantum rings for an in-plane magnetic field. It was also reported an increased threshold frequency of absorption in a two-dimensional quantum dot when the magnetic field increases regardless of the direction [145]. However, the above effects appear to contradict the widely recognised fact that the energy-level structure of a q2DES in the presence of a tilted magnetic field ($B_T = \sqrt{B_\perp^2 + B_\parallel^2}$) and confined in a narrow quantum well is mainly controlled by the well width (ℓ_c) [146], in such a way that, as long as $\ell_B \gg \ell_c$, where ℓ_B is the magnetic length, the motion in the plane is quantized into Landau levels with the cyclotron frequency determined *only* by the perpendicular component of B [146]. Thus, the in-plane magnetic field couples to a q2DES through the electron spin, adding to the Zeeman energy produced by the perpendicular magnetic field, and via the finite thickness of the one-particle wave function in the direction normal to the plane [138].

For all the above, the aim of this section is to clarify how an in-plane magnetic field (not) couples to an effective equation of a q2DES, from the point of view of the thin-layer approach.

3.3.1 Non-Relativistic q2DES

Let us recall the effective Schrödinger equation in the presence of the electromagnetic field up second-order in ϵ , when the submanifold is a plane⁹ ($\bar{A}_\mu = A_\mu|_{z=0}$ and $\bar{\phi} = \phi|_{z=0}$),

$$\epsilon^2 i\hbar \frac{\partial \Psi(t, \vec{R})}{\partial t} = \overbrace{\left[-\frac{\hbar^2}{2m} \partial_z^2 + \frac{1}{2} m \omega^2 z^2 \right]}^{\mathcal{H}^{(0)}} \Psi(t, \vec{R}) + \epsilon^2 \overbrace{\left[-\frac{\hbar^2}{2m} (\partial_\mu - \frac{ie}{\hbar} \bar{A}_\mu)^2 + e\bar{\phi} \right]}^{\mathcal{H}^{(2)}} \Psi(t, \vec{R}) \quad (3.51)$$

In the absence of a coupling term the separation of variables, $\Psi(t, \vec{R}) = \varphi(z)\psi(t, x, y)$, is justified. As before, the zeroth-order term leads to $\mathcal{H}^{(0)}\varphi(z) = \mathcal{E}_0\varphi(z)$, and the next non-zero term leads to the effective Schrödinger equation for a (quasi)-two-dimensional system¹⁰

$$i\hbar \frac{\partial \psi(t, x, y)}{\partial t} = \left[-\frac{\hbar^2}{2m} (\partial_\mu - \frac{ie}{\hbar} \bar{A}_\mu)^2 + e\bar{\phi} \right] \psi(t, x, y). \quad (3.52)$$

Although Eq. (3.52) might be affected by the presence of an in-plane magnetic field, this does not happen because the potential $\bar{A}_\mu$ is a function only of x and y and thus it can just generate magnetic fields perpendicular to the plane¹¹.

In order to clarify the physics of our approximation, let us consider for the sake of simplicity the case of a constant in-plane magnetic field in the y direction. Consistent with $A_z = 0$, we choose the in-plane magnetic vector potential as $\vec{A}_{in} = (B_\parallel z, 0, 0)$. In

⁹The coordinates in $\mathbb{R}^3$ are $\vec{R} = (x, y, z)$, with z the coordinate to be confined.

¹⁰Obviously, for a plane the Gaussian and the mean curvature are zero.

¹¹Note that Eq. (3.52) is invariant under the gauge transformation $A'_\mu = A_\mu + \partial_\mu \gamma(t, x, y)$.

this case it is easy to prove that if one considers all orders in ϵ , one obtain¹²

$$\begin{aligned} \epsilon^2 i\hbar \frac{\partial \Psi(t, \vec{R})}{\partial t} &= \left[-\frac{\hbar^2}{2m} \partial_z^2 + \frac{1}{2} m \omega^2 z^2 \right] \Psi(t, \vec{R}) \\ &+ \epsilon^2 \left[-\frac{\hbar^2}{2m} \left((\partial_x - \frac{ie}{\hbar} B_{\parallel} \epsilon z)^2 + \partial_y^2 \right) \right] \Psi(t, \vec{R}). \end{aligned} \quad (3.53)$$

If $\ell_{B_{\parallel}} \gg \ell_c$, where $\ell_{B_{\parallel}} = \sqrt{\hbar/(eB_{\parallel})}$ is the magnetic length and $\ell_c = \epsilon \sqrt{2\hbar/(m\omega)}$ is the confinement width, we can “freeze” the normal degree of freedom in the ground state of $\mathcal{H}^{(0)}$. Thus, the perturbation in Eq. (3.53) is computed as

$$\mathcal{H}_{eff} = \langle \varphi_0^*(z) | \left[-\frac{\hbar^2}{2m} \left((\partial_x - \frac{ie}{\hbar} B_{\parallel} \epsilon z)^2 + \partial_y^2 \right) \right] | \varphi_0(z) \rangle \times \psi(t, x, y), \quad (3.54)$$

where $\mathcal{H}_{eff}$ is the effective Hamiltonian for the q2DES. Because $\varphi_0(z) = \varphi_0(-z)$, the terms proportional to z are zero, *i.e.* $\langle \varphi_0^*(z) | z | \varphi_0(z) \rangle = 0$, and $\langle \varphi_0^*(z) | z^2 | \varphi_0(z) \rangle = \ell_c^2$, so, we arrive at

$$\mathcal{H}_{eff} = -\frac{\hbar^2 (\partial_x^2 + \partial_y^2)}{2m} \psi(t, x, y) + \frac{\ell_c^2 e^2}{2m} B_{\parallel}^2 \psi(t, x, y). \quad (3.55)$$

Thus, the terms beyond $\epsilon^2 \mathcal{H}^{(2)}$ only produce an irrelevant shift in all energy levels. This is true as long as $\ell_{B_{\parallel}} \gg \ell_c$. On the other hand, if $\ell_{B_{\parallel}} \sim \ell_c$ the energy spectrum is modified because the spatial confinement and the magnetic field compete. Therefore, one cannot consider $\mathcal{H}^{(2)}$ and the higher-order terms as perturbations, so, the terms proportional to z and z^2 have to be taken into account to calculate $\varphi_0(z)$, and $\langle \varphi_0^*(z) | z | \varphi_0(z) \rangle = 0$ is not longer valid. In summary, if $\ell_{B_{\parallel}} \sim \ell_c$, the quasi two-dimensional approximation breaks.

In Refs. [109, 139] another gauge was chosen, $\vec{A} = (0, 0, yB)$, leading to the emergence of the term $\epsilon \mathcal{H}^{(1)} = \epsilon y_{\parallel} \hat{p}_z / m$, it was assumed that $\langle \varphi_0^*(z) | p_z | \varphi_0(z) \rangle = 0$ to separate variables, which leads to erroneous conclusions¹³.

One can explain the absence of the in-plane magnetic field effects from a classical point of view. The in-plane magnetic field tends to produce circular motion out of the plane with a cyclotron radius $r_c \sim \ell_b$, if $\ell_{B_{\parallel}} \gg \ell_c$ the confining potential is strong enough to prevent any trajectory out of the plane. However, when $\ell_{B_{\parallel}} \sim \ell_c$ (or $\ell_{B_{\parallel}} \ll \ell_c$), the electrons move under the competing influence of the Lorentz force and the force due to the confining potential, allowing trajectories out of the plane and within the confinement width.

In the case where the tilted magnetic field is constant, and the confining potential is parabolic, the exact solution of the 3D-Schrödinger equation is known [147, 148]. Choosing $\vec{A} = \{-yB_{\perp} + zB_{\parallel}, 0, 0\}$, one can show that the spectrum of Eq. (3.8) is

$$E_{n,m} = \hbar\omega_1 \left(n + \frac{1}{2} \right) + \hbar\omega_2 \left(m + \frac{1}{2} \right), \quad (3.56)$$

¹²Assuming that the confining potential is only quadratic.

¹³If one assumes that the effects of $\epsilon y_{\parallel} \hat{p}_z / m$ is negligible, necessarily also must be $B_{\parallel}^2 y^2 / (2m)$, so that the approximation will be consistent. Consequently, the in-plane magnetic field disappears from the effective equation for the q2DES.

where

$$\omega_{1,2} = \frac{1}{2} \left(\sqrt{\omega_y^2 + (\omega_z + \omega)^2} \mp \sqrt{\omega_y^2 + (\omega_z - \omega)^2} \right) \quad (3.57)$$

with $\omega_y = |B_{||}|/m$ and $\omega_z = |B_{\perp}|/m$. Our approximation is valid when $\omega \gg \omega_y, \omega_z$, thus, in this limit, we obtain that $\omega_1 \approx \omega_z$ and $\omega_2 \approx \omega$ [136]. Thus, the subband of interest is

$$E_{n,0} \approx \hbar \omega_z \left(n + \frac{1}{2} \right) + \hbar \frac{\omega}{2}, \quad (3.58)$$

which is consistent with that for a confinement width the spectrum only depends on the perpendicular magnetic field. It should be noted that if one take the next order term for the frequencies

$$\omega_1 \approx \omega_z \left(1 - \frac{\omega_y^2}{2\omega^2} \right), \quad \text{and} \quad \omega_2 \approx \omega \left(1 + \frac{\omega_y^2}{2\omega^2} \right), \quad (3.59)$$

then, it is clear that there is an interplay between the in-plane magnetic field and the confining potential. Hence, any effect of the in-plane magnetic field appears in the form ω_y/ω , this contrary to what can be found in Refs. [109, 111, 139, 140, 145].

3.3.2 Pauli Equation

We may ask if a (quasi)-two-dimensional system never “feel” an in-plane magnetic field. The answer depends on whether the particle has spin, since if the particle has spin, a Zeeman coupling can be introduced. As it is well known, for a charged spin-1/2 particle the Zeeman coupling is the lowest-order relativistic correction. With this correction, the resulting equation is referred to as Pauli equation

$$i\hbar \frac{\partial \Psi(t, \vec{R})}{\partial t} = \left[-\frac{\hbar^2}{2m} \left(\nabla - \frac{ie}{\hbar} \vec{A}(t, \vec{r}) \right)^2 + e\phi(t, \vec{R}) - \frac{e}{m} \vec{B} \cdot \vec{S} + V_c(z) \right] \Psi(t, \vec{R}),$$

where now the wave function contains two components, $\Psi^T = (\Psi_+, \Psi_-)$, and $\vec{S} = \hbar \vec{\sigma}/2$ is the spin operator, with $\vec{\sigma} = \{\sigma^1, \sigma^2, \sigma^3\}$. Proceeding as before, we arrive at

$$i\hbar \frac{\partial \psi(t, x, y)}{\partial t} = \left[-\frac{\hbar^2}{2m} \left(\partial_\mu - \frac{ie}{\hbar} \bar{A}_\mu \right)^2 + e\bar{\phi} - \frac{e}{m} \bar{B}_z S_z - \frac{e}{m} \vec{\bar{B}}_{||} \cdot \vec{S}_{||} \right] \psi(t, x, y), \quad (3.60)$$

where $B_z(t, x, y, z=0) = \bar{B}_z(t, x, y)$, $\vec{S}_{||} = \frac{\hbar}{2} \{\sigma^1, \sigma^2\}$ and $\vec{B}_{||}(t, x, y, z=0) = \vec{\bar{B}}_{||}(t, x, y)$, with $\vec{\bar{B}}_{||}$ the in-plane magnetic field. Hence, using the thin-layer approach it is clear that the spin coupled with the in-plane magnetic field, as is well-known [138].

Finally, the conclusions presented in this section can be extrapolated to the case of a curved surface (see Eq. (3.33)) implying that to this system the tangential magnetic field is negligible, and a particle on a curved surface only coupled to the tangential magnetic field through the spin.

CHAPTER 4

RELATIVISTIC QUANTUM MECHANICS ON A SUBMANIFOLD: DIRAC FIELD

"Science is the belief in the ignorance of experts."

— Richard P. Feynman.

In this chapter, we study the Dirac equation on a submanifold in the confining potential approach. When a submanifold is a surface, the effective Dirac equation was already considered long ago by Burgess and Jensen [43], for surfaces with zero intrinsic curvature. This result has been used in the literature for more than two decades¹ [42–47, 49, 50, 149]. However, as we see the Dirac equation on a surface needs to be reconsidered. For this reason, we shall proceed first the case of a surface and then will continue with the more general case.

4.1 Dirac Equation on a Surface

In this section we consider a 1/2-spin particle confined in a surface S . In this particular case, $\mathcal{N}$ is a four-dimensional Minkowski space-time and $\mathcal{M}$ is a three-dimensional pseudo-Riemannian submanifold. We employ Latin indices $A, B, \dots$, running from 0 to 3 and Greek indices $\mu, \nu, \dots$, running from 0 to 2. The adapted coordinates system is (q^μ, q^3) , with $q^\mu = (q^0, q^1, q^2)$. Here q^3 is the coordinate to be confined. The metric G_{AB} in the adapted coordinates system can write as²

$$G_{AB} = \begin{pmatrix} \gamma_{\mu\nu} & 0 \\ 0 & \eta_{33} \end{pmatrix}, \quad (4.1)$$

¹An exception is found in the article of Maraner and Pachos, [48], where the effective Dirac equation on the surface, up to first-order in ϵ , appears without the geometric potential. However, the authors do not acknowledge any disagreement with the previous literature, nor show the explicit calculations that would explain why the geometric potential is absent.

²As in this case the codimension is one the extrinsic torsion is zero.

with $\eta_{33} = \eta^{33} = -1$ and where

$$\gamma_{\mu\nu} = g_{\mu\nu} - 2q^3\alpha_{3\mu\nu} + (q^3)^2\alpha_{3\mu\rho}g^{\rho\sigma}\alpha_{3\sigma\nu}. \quad (4.2)$$

The extrinsic curvature $\alpha_{3\mu\nu}$ is defined such that

$$\alpha_{3\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha_{3ij} \end{pmatrix}, \quad (4.3)$$

and we also demand that $g_{\mu\nu}$ and $\alpha_{3\mu\nu}$ do not depend on the time (q^0).

In Appendix D, we introduce a *non-coordinate basis* for a n -dimensional pseudo-Riemannian Manifold. Let us particularize what has been done there to the case considered here. The *vierbeins*³ E_I^A are chosen such that $\eta_{IJ} = E_I^A E_J^B G_{AB}$ and $G_{AB} = \eta_{IJ} E_A^I E_B^J$, where E_A^I is the inverse of E_I^A . In our case one can choose the *vierbein* as⁴

$$E_A^I = \begin{pmatrix} E_\mu^a & E_\mu^3 \\ E_3^a & E_3^3 \end{pmatrix} = \begin{pmatrix} E_\mu^a & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.4)$$

where the *dreibein* E_μ^a was chosen such that $\gamma_{\mu\nu} = \eta_{ab} E_\mu^a E_\nu^b$, with $\eta_{ab} = \text{diag}(1, -1, -1)$. Now, the inverse of E_A^I is [48]

$$E_I^A = \begin{pmatrix} E_a^\mu & E_a^3 \\ E_3^\mu & E_3^3 \end{pmatrix} = \begin{pmatrix} E_a^\mu & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.5)$$

where the *dreibein* E_a^μ was chosen such that $E_a^\mu E_\nu^a = \delta_\nu^\mu$ and $E_a^\mu E_\mu^b = \delta_a^b$. Let us rescale the normal coordinates by $q^3 \rightarrow \epsilon q^3$ and use ϵ as a perturbation parameter. Using Eq. (D.5) it is easy to prove that

$$E_\mu^a = e_\mu^a + \epsilon q^3 \alpha_{3\mu}{}^\nu e_\nu^a, \quad (4.6)$$

where e_μ^a were chosen in such a way that⁵ $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$, $e_a^\mu e_\nu^a = \delta_\nu^\mu$ and $e_a^\mu e_\mu^b = \delta_a^b$. Finally, the inverse of E_μ^a , up to first-order in ϵ , is given by

$$E_a^\mu = e_a^\mu - \epsilon q^3 \alpha_{3\nu}{}^\mu e_a^\nu + O(\epsilon^2). \quad (4.7)$$

4.1.1 Thin-layer Method for the Dirac Equation

The four-dimensional Dirac equation in curvilinear coordinates is given by [54]

$$\mathcal{D}\Psi = \left[i\hbar \gamma^A \left(\partial_A - \bar{\Omega}_A \right) - mc - V_c(q^3) \right] \Psi = 0, \quad (4.8)$$

³The coefficients E_I^A are called the vielbeins if the space is n dimensional, vierbeins in four dimensions and dreibein in three dimensions.

⁴Latin indices $a, b, \dots$ run from 0 to 2.

⁵The vielbeins e_μ^a are rotations and have to be square matrices. The choice made in Ref. [50] is not consistent with this fact and, therefore, cannot be a non-coordinate basis. This is one of the reasons why the results of [50] do not agree with the results to be presented here.

where the confining potential is given by a 4×4 matrix-value function $V_c(q^3)$. The γ -matrices generate a Clifford algebra

$$\{\underline{\gamma}^A, \underline{\gamma}^B\} = 2G^{AB}, \quad (4.9)$$

where the underline is used to distinguish between the space-dependent γ -matrices and the constant γ -matrices defined by $\underline{\gamma}^A = \underline{\gamma}^A(q^C) = E_I^A(q^C)\gamma^I$. The n -dimensional irreducible representation of the γ -matrices is $2^{\lfloor \frac{n}{2} \rfloor} \times 2^{\lfloor \frac{n}{2} \rfloor}$ dimensional matrices. The spinorial affine connections $\bar{\Omega}_A$ are given by [54]

$$\bar{\Omega}_A = -\frac{1}{4}\gamma^I\gamma^J\eta_{JK}E_I^B(\partial_A E_B^K - \bar{\Gamma}_{AB}^C E_C^K). \quad (4.10)$$

Since we are looking for an effective dynamics on the surface, S , we rescale the field Ψ as well as the Dirac operator $\mathcal{D}$ [7, 11, 12, 48]

$$\chi(q^A) \equiv \frac{|G|^{1/4}}{|g|^{1/4}}\Psi(q^\mu, q^3), \quad (4.11)$$

$$D \equiv \left(\frac{|G|^{1/4}}{|g|^{1/4}}\right)\mathcal{D}\left(\frac{|g|^{1/4}}{|G|^{1/4}}\right). \quad (4.12)$$

Thus, the normalization condition for the Dirac field in the adapted coordinates frame is

$$(\Psi, \Psi) = \int dq^1 dq^2 dq^3 |G|^{1/2} |\Psi|^2 = \int dq^1 dq^2 |g|^{1/2} \left(\int dq^3 |\chi|^2 \right) = 1. \quad (4.13)$$

Therefore, it is clear from Eq. (4.13) that $\int dq^3 |\chi|^2$ has a probabilistic interpretation on the surface.

The next step is to rescale the normal coordinates $q^3 \rightarrow \epsilon q^3$ and expand the Dirac operator in a perturbative expansion keeping terms of order ϵ . For this purpose let us calculate the spin connection in the adapted coordinates frame. One can show that the $(3+1)$ -dimensional Christoffel symbols in the adapted coordinates frame, to zero-order in ϵ , are given by Eq. (E.16) to Eq. (E.21)

$$\begin{aligned} \bar{\Gamma}_{\mu\nu}^\sigma &= \Gamma_{\mu\nu}^\sigma, & \bar{\Gamma}_{\mu\nu}^3 &= \eta^{33}\alpha_{3\mu\nu} \equiv \alpha_{\mu\nu}^3, & \bar{\Gamma}_{3\mu}^\beta &= \alpha_{3\mu}^\beta, \\ \bar{\Gamma}_{33}^\beta &= 0, & \bar{\Gamma}_{\mu 3}^3 &= 0, & \bar{\Gamma}_{33}^3 &= 0, \end{aligned}$$

where $\Gamma_{\mu\nu,\beta} = \frac{1}{2}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})$ are the $(2+1)$ -dimensional curved space-time Christoffel symbols.⁶

Let us compute the spinor connection retaining terms of zero-order ϵ . For a general index A , we obtain

$$-\bar{\Omega}_A = \frac{1}{4}\gamma^a\gamma^b\eta_{bc}e_a^\beta(\partial_A E_\beta^c - \bar{\Gamma}_{A\beta}^\nu e_\nu^c) - \frac{1}{4}\gamma^a\gamma^3\eta_{33}e_a^\beta\bar{\Gamma}_{A\beta}^3 - \frac{1}{4}\gamma^3\gamma^a\eta_{ac}\bar{\Gamma}_{A3}^\nu e_\nu^c + O(\epsilon). \quad (4.14)$$

⁶Note that in our case only the space is curved.

For $A = \mu$ with $(\mu = 0, 1, 2)$, Eq. (4.14) yields

$$-\bar{\Omega}_\mu = -\Omega_\mu + \frac{1}{2}\gamma^a\gamma^3\eta_{ac}\alpha_{3\mu}{}^\nu e_\nu^c + O(\epsilon), \quad (4.15)$$

where we have used that $\gamma^a\gamma^3 = -\gamma^3\gamma^a$ and that

$$\begin{aligned} \eta_{33}e_a^\beta\eta^{33}\alpha_{3\mu\beta} &= -\eta_{33}\eta^{33}\alpha_{3\mu}{}^\nu g_{\nu\beta}e_a^\beta \\ &= -\alpha_{3\mu}{}^\nu\eta_{cb}e_\nu^ce_\beta^be_a^\beta \\ &= -\alpha_{3\mu}{}^\nu\eta_{cb}e_\nu^c\delta_a^b \\ &= -\alpha_{3\mu}{}^\nu\eta_{ca}e_\nu^c, \end{aligned} \quad (4.16)$$

where in the second and third line of Eq. (4.16) we have used the relations $g_{\mu\nu} = \eta_{ab}e_\mu^ae_\nu^b$ and $e_a^\mu e_\mu^b = \delta_a^b$. In Eq. (4.15), the Ω_μ are the spinorial affine connections associated to $g_{\mu\nu}$. The second term of the right-hand side of Eq. (4.15) appears in the Dirac equation multiplied by the space-dependent γ -matrices, so that

$$\begin{aligned} E_I^\mu\gamma^I\frac{1}{2}\gamma^a\gamma^3\eta_{ac}\alpha_{3\mu}{}^\nu e_\nu^c &= e_b^\mu\gamma^b\frac{1}{2}\gamma^a\gamma^3\eta_{ac}\alpha_{3\mu}{}^\nu e_\nu^c + O(\epsilon) \\ &= -\frac{1}{2}\underline{\gamma}^\mu\underline{\gamma}^\nu\alpha_{3\mu\nu}\gamma^3 + O(\epsilon) \\ &= -\frac{1}{4}\left([\underline{\gamma}^\mu, \underline{\gamma}^\nu] + \{\underline{\gamma}^\mu, \underline{\gamma}^\nu\}\right)\alpha_{3\mu\nu}\gamma^3 + O(\epsilon) \\ &= -\frac{1}{4}(2g^{\mu\nu})\alpha_{3\mu\nu}\gamma^3 + O(\epsilon) \\ &= \frac{1}{2}\alpha_{3\mu}{}^\mu\gamma^3 + O(\epsilon), \end{aligned} \quad (4.17)$$

where in the second last line we used that $\alpha_{3\mu\nu} = \alpha_{3\nu\mu}$ and that $\{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} = 2g^{\mu\nu}$. For $A = 3$, Eq. (4.14) yields

$$-\bar{\Omega}_3 = \frac{1}{4}\gamma^a\gamma^b\eta_{bc}e_a^\sigma(\alpha_{3\sigma}{}^\nu e_\nu^c - \alpha_{3\sigma}{}^\nu e_\nu^c) + O(\epsilon) = 0, \quad (4.18)$$

where we used that to zero-order in ϵ , $\partial_3 E_\sigma^c = \alpha_{3\sigma}{}^\nu e_\nu^c$, $\bar{\Gamma}_{3\sigma}^\nu = \alpha_{3\sigma}{}^\nu$, $\bar{\Gamma}_{\beta 3}^3 = 0$ and $\bar{\Gamma}_{33}^\nu = 0$.

The mean and Gaussian curvatures are defined by

$$H \equiv -\frac{1}{2}\text{tr}(\alpha_{3\mu}{}^\nu) = -\frac{1}{2}\alpha_{3\mu}{}^\mu, \quad K \equiv \det(\alpha_{3\mu}{}^\nu), \quad (4.19)$$

respectively. Thus, inserting (4.15) and (4.18) into $\mathcal{D}$, the Dirac operator to zero-order in ϵ can be written as

$$\mathcal{D} = i\hbar e_a^\mu\gamma^a(\partial_\mu - \Omega_\mu) - i\hbar H\gamma^3 - mc + i\frac{\hbar}{\epsilon}\gamma^3\partial_3 - V_c(\epsilon q^3) + O(\epsilon). \quad (4.20)$$

Note that our derivation is general and does not make any assumption on the value of the intrinsic curvature. Now, if the potential V_c satisfies the three conditions mentioned

above, its explicit form does not affect subsequent calculations.⁷ In what follows we will only need the following property

$$V_c(\epsilon q^3) = \frac{\tilde{V}(q^3)}{\epsilon} + O(\epsilon^0), \quad (4.21)$$

where $\tilde{V}(q^3)$ is independent of ϵ . In order to calculate the rescaled Dirac operator (4.12), we first consider the term

$$\begin{aligned} \left(\frac{|G|^{1/4}}{|g|^{1/4}} \right) \gamma^3 \partial_3 \left(\frac{|g|^{1/4}}{|G|^{1/4}} \right) \chi &= \gamma^3 \partial_3 \chi + (|G|^{1/4} \gamma^3 \partial_3 |G|^{-1/4}) \chi \\ &= \gamma^3 \partial_3 \chi + \frac{1}{2} \epsilon H \gamma^3 \chi + O(\epsilon^2). \end{aligned}$$

Then, the rescaled Dirac operator to zero-order in ϵ can be written as

$$D = i\hbar e_a^\mu \gamma^a (\partial_\mu - \Omega_\mu) - i\hbar H \gamma^3 + i\hbar H \gamma^3 - mc + \frac{i\hbar}{\epsilon} \gamma^3 \partial_3 - \frac{\tilde{V}(q^3)}{\epsilon}. \quad (4.22)$$

Multiplying by ϵ and simplifying yields

$$\epsilon D = i\epsilon \hbar e_a^\mu \gamma^a (\partial_\mu - \Omega_\mu) - mc + i\hbar \gamma^3 \partial_3 - \tilde{V}(q^3). \quad (4.23)$$

Notice that the term $i\hbar H \gamma^3$ coming from the rescaling of the wave function exactly cancels the term $-i\hbar H \gamma^3$ coming from the calculation of spin connection. Thus, the Dirac equation to first-order in ϵ can be rewritten as

$$\left[i\hbar \gamma^3 \partial_3 + \tilde{V} + i\epsilon \hbar e_a^\mu \gamma^a (\partial_\mu - \Omega_\mu) - \epsilon mc \right] \chi = 0, \quad (4.24)$$

which, being independent of H , has lost any information about the extrinsic properties of the surface. Eq. (4.24) coincides with the result recently found by Maraner and Pachos [48]⁸. The next step is “freezing” the motion normal to the surface in a ground state of the confining potential [7]. Although this is an easy task for the Schrödinger or Maxwell equations, the spinorial character of the Dirac equation makes it difficult to perform the separation of variables. Thus, let us first consider the case of zero Gaussian curvature. It is well known that for any Riemann surface, it is possible to choose an “isothermal” coordinate system in which the metric is conformally related to the Euclidean metric everywhere [150]. Then, the metric g_{ij} can be always written as

$$g_{ij} = e^{-2\beta(q_i)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.25)$$

⁷When second-order equations are considered it is required that the confining potential has a deep minimum relative to the surface. However, since the Dirac equation can be viewed as a coupled system of first-order equations, we require that $W_\pm = V_c^2 \pm \partial_3 V_c$ has a deep minimum relative to the surface. This choice is appropriate, since the Dirac equation can usually be decoupled in terms of a system of second-order equations where $W_\pm$ plays the role of the potential.

⁸However, the authors do not acknowledge any disagreement with the previous literature [42, 43], nor show the explicit calculations that would explain why the geometric potential is absent.

where g_{ij} is the metric in the surface and is related with $g_{\mu\nu}$ by⁹

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -g_{ij} \end{pmatrix}. \quad (4.28)$$

In addition, when the surface has zero Gaussian curvature there exists a coordinate transformation, which reduces the metric to the Euclidean form *i.e.* $g_{ij} = \text{diag}(1, 1)$. Thus, it is clear that all the spinorial affine connections are zero ($\bar{\Omega}_A = 0$) and that $e_a^\mu = \delta_a^\mu$. Therefore, in this particular case we obtain

$$[i\hbar\gamma^3\partial_3 + \tilde{V}(q^3) + i\epsilon\hbar\gamma^\mu\partial_\mu - \epsilon mc]\chi = 0, \quad (4.29)$$

which has solutions of the form $\chi(q^A) = \xi(q^\mu)\varrho(q^3)$, with $\varrho(q^3)$ a spinor and $\xi(q^\mu) = e^{-iq^\mu p_\mu/\hbar}$. This allows us to decouple the equation (see below) to obtain

$$[i\hbar\gamma^\mu\partial_\mu - mc]\tilde{\xi}(q^\mu) = 0. \quad (4.30)$$

The above equation differs from that found in Ref. [43] on the same assumptions¹⁰

$$[i\hbar e_a^\mu \gamma^a (\partial_\mu - \tilde{\Omega}_\mu) - mc + \gamma^3 H]\chi = 0. \quad (4.31)$$

Let us consider a more general case, for which the Gaussian curvature is not necessarily zero. We expect that the probability of finding a particle at the surface has a maximum at $q^3 = 0$, and this is more pronounced when ϵ is smaller (for example, a Gaussian centered on $q^3 = 0$). Thus, we assume a product solution of the form

$$\chi(q^A) = \tilde{\chi}(q^A)e^{-\frac{iq^0\mathcal{E}}{c\hbar\epsilon}} = e^{-\frac{iq^0\mathcal{E}}{c\hbar\epsilon}} \begin{pmatrix} \phi_1(q^3)\psi_1(q^\mu) \\ \phi_2(q^3)\psi_2(q^\mu) \end{pmatrix}, \quad (4.32)$$

where $\phi_1(q^3)$ and $\phi_2(q^3)$ are functions, $\psi_1(q^\mu)$ and $\psi_2(q^\mu)$ are bispinors and, without loss of generality, $\int dq^3 |\phi_1(q^3)|^2 = \int dq^3 |\phi_2(q^3)|^2 = 1$. Choosing the following explicit representation of the γ -matrices

$$\gamma^a = \begin{pmatrix} \Gamma^a & 0 \\ 0 & -\Gamma^a \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i\mathbb{I}_{2\times 2} \\ i\mathbb{I}_{2\times 2} & 0 \end{pmatrix}, \quad (4.33)$$

⁹ In the adapted coordinate frame any point in $\mathbb{R}^3$, in a sufficiently small neighborhood of S , is given by

$$\mathbf{R}(q^1, q^2, q^3) = \mathbf{r}(q^1, q^2) + q^3 \mathbf{n}(q^1, q^2), \quad (4.26)$$

where $\mathbf{r}(q^1, q^2)$ is the parametric equations for the surface (S) and $\mathbf{n}(q^1, q^2)$ is an orthonormal vector field to S . Thus, the metric tensor in the surface is given by

$$g_{ij} = \frac{\partial \mathbf{R}}{\partial q^i} \cdot \frac{\partial \mathbf{R}}{\partial q^j}, \quad (4.27)$$

with $i, j = 1, 2$.

¹⁰The sign of the mean curvature depends on the choice of the unit normal vector and has no intrinsic meaning, Eq. (4.31) is not necessarily invariant to the choice of the normal vector, which leads to ambiguity. Although it is sometimes overlooked, we point out that the original deduction in Ref. [43] was performed only for surfaces with zero Gaussian curvature. Moreover, as we will show it is not clear from previous investigations [42–47, 49, 50] what would be the exact meaning of $\tilde{\Omega}_\mu$.

and taking into account that $g_{\mu\nu} = \text{diag}(1, -g_{ij})$ and so $e_0^\mu = \delta_0^\mu$ and $e_a^0 = \delta_a^0$. After some algebra we obtain

$$\left[i\hbar\gamma^3\partial_3 + \tilde{V}(q^3) + \gamma^0\frac{\mathcal{E}}{c} \right] \tilde{\chi} + \epsilon [i\hbar e_a^\mu \gamma^a (\partial_\mu - \Omega_\mu) - mc] \tilde{\chi} = 0. \quad (4.34)$$

Note that the first line in (4.34) is equivalent to a one-dimensional Dirac equation in the presence of a potential $\tilde{V}$ in a 4×4 representation. Demanding that $\mathcal{E}$, ϕ_1 and ϕ_2 solve this one-dimensional Dirac equation, namely

$$\left[i\hbar\gamma^3\partial_3 + \tilde{V}(q^3) + \gamma^0\frac{\mathcal{E}}{c} \right] \tilde{\chi} = 0 \quad (4.35)$$

and noting that $\nabla_\mu = \partial_\mu - \Omega_\mu$ is block diagonal, because $\gamma^\mu\gamma^\nu$ is block diagonal, one obtains that

$$[i\hbar e_a^\mu \Gamma^a (\partial_\mu - \tilde{\Omega}_\mu) - mc] \phi_1(q^3) \psi_1(q^\mu) = 0, \quad (4.36)$$

$$[-i\hbar e_a^\mu \Gamma^a (\partial_\mu - \tilde{\Omega}_\mu) - mc] \phi_2(q^3) \psi_2(q^\mu) = 0, \quad (4.37)$$

multiplying (4.36) by $\phi_1^*(q^3)$ and (4.37) by $\phi_2^*(q^3)$ and integrating in $\int dq^3$ we obtain the following effective Dirac equation on a surface

$$[i\hbar e_a^\mu \gamma^a (\partial_\mu - \Omega_\mu) - mc + O(\epsilon)] \psi(q^\mu) = 0, \quad (4.38)$$

where we have rewritten the two equations again in a representation 4×4 , with $\psi(q^\mu)^T = (\psi_1(q^\mu), \psi_2(q^\mu))$. The normalization condition, (4.13), now reads as

$$\begin{aligned} (\Psi, \Psi) &= \int dq^1 dq^2 |g|^{1/2} dq^3 |\chi|^2 \\ &= \int dq^1 dq^2 |g|^{1/2} dq^3 (|\phi_1(q^3)|^2 |\psi_1(q^\mu)|^2 + |\phi_2(q^3)|^2 |\psi_2(q^\mu)|^2) \\ &= \int dq^1 dq^2 |g|^{1/2} |\psi(q^\mu)|^2 = 1. \end{aligned} \quad (4.39)$$

Hence, $\psi(q^\mu)$ has a probabilistic interpretation on the surface. In the appendix (G), we show an example where the confining potential is a scalar linear potential,¹¹ $\tilde{V}(q^3) = kq^3 \mathbb{I}_{4 \times 4}$. This particular case justifies certain assumptions made previously in the literature [46, 50] and leads to the natural physical picture, a Gaussian probability distribution with a maximum at $q^3 = 0$.

Now, comparing the effective Dirac equation on the curved surface, (4.38), with the usual $(2+1)$ -dimensional Dirac equation in a curved space, (4.8), it should be noted that although both have the same form, the γ -matrices in (4.38), with the index a running from 0 to 2, are not a irreducible representation of the γ -matrices in $2+1$ dimensions. One can choose the γ -matrices in Eq. (4.38) as (4.33), such that the matrices Γ^a and $-\Gamma^a$ are the two inequivalent irreducible representations of the γ -matrices in $2+1$ dimensions.

¹¹Since that the potential is scalar, there will be no creation of particle pairs, in contrast with a electrostatic potential [43, 50].

One can show that, while one of the representations describes a fermion with spin up and an antifermion with spin down, the other describes a fermion with spin down and an antifermion with spin up [151]. Moreover, notice that in Eq. (4.38), Ω_μ are the spin connections associated to S , the same spin connections that appear in the Dirac equation if S is seen not as a *submanifold* but as a *manifold* itself.

Although the effective dynamics considered here is exact to order ϵ (see Eq. (4.38)), the complete dynamics contains arbitrary powers of ϵ (which can be neglected depending on the experimental setup). This is true for any system that starts from three dimensions and wherein its dynamics is confined in a curved surface.

4.1.2 Comparison with Previous Results

Unexpectedly, in contrast to the equations of Schrödinger, Pauli, Maxwell and Klein–Gordon on a surface, a geometric potential is absent in the effective Dirac equation on a surface, given by Eq. (4.38). For more than two decades, there has been some misunderstanding since it is often claimed that the Dirac equation on a Hypersurface feels a geometric potential [42–47, 49, 50]. The source of this confusion lies in the seminal article of Burgess and Jensen [43].¹² In this reference, the spin connection in a (3+1)-dimensional space–time was calculated in the adapted coordinate frame (Eq. (35) in [43]) and this showed a dependence of the extrinsic curvature of the surface. However, inexplicably, the final result, Eq. (41) in [43], ignores this contribution and assumes that the spin connections, $\tilde{\Omega}$, in a (3+1)-dimensional space–time projected onto the surface¹³ are equal to the spin connections on the surface, Ω . As we shown, in Eq. (4.15), this cannot be true, even for surfaces with zero intrinsic curvature.

In order to emphasize this point, it is instructive to consider a counterexample, with zero intrinsic curvature, of the references [42, 43]. Let us consider the (3+1)-Dirac operator in cylindrical coordinates (t, r, θ, z)

$$D = i\hbar \left(\gamma^0 \partial_t + \gamma^1 \left(\partial_r + \frac{1}{2r} \right) + \frac{\gamma^2}{r} \partial_\theta + \gamma^3 \partial_z \right), \quad (4.40)$$

the perturbative expansion around the cylinder or radius R (eliminating terms of order $\frac{1}{\epsilon}$) is

$$D = i\hbar \left(\gamma^0 \partial_t + \frac{\gamma^2}{R} \partial_\theta + \gamma^3 \partial_z \right) - i\hbar \gamma^1 H, \quad (4.41)$$

where $H = -1/(2R) = -\alpha_{3\mu}^\mu/2$ is the mean curvature. Comparing this with the (2+1)-Dirac equation in a cylinder of radius R

$$\bar{D} = i\hbar \left(\bar{\gamma}^0 \partial_t + \frac{\bar{\gamma}^2}{R} \partial_\theta + \bar{\gamma}^3 \partial_z \right), \quad (4.42)$$

¹²In Ref. [42] the spin connections were wrongly calculated, for a Dirac equation on an elastic rod embedded into (2+1)-dimensional space–time, leading to the same erroneous conclusion.

¹³By projected, we mean that a perturbative expansion around the surface is made and the first term is taken.

we conclude that it differs by a term proportional to the mean curvature, as derived in (4.20). This example illustrates the difference between calculating the spin connection in $2 + 1$ dimensions and calculate it in a $3 + 1$ dimensional flat space-time and then project it onto a surface.

4.1.3 Non-relativistic Limit

Note that up to this point all relations are valid even for massless particles. Nevertheless, in the following we restrict ourselves to massive particles. Let us consider the non-relativistic limit of the Eq. (4.38). In this limit, the decoupling of particles and antiparticles can be straightforwardly obtained using the following representation of the γ -matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_{2 \times 2} & 0 \\ 0 & -\mathbb{I}_{2 \times 2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (4.43)$$

with $i = 1, 2$. Using Eq. (4.28) the Eq. (4.38) can be rewritten as

$$i\hbar\partial_0 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -i\hbar \begin{pmatrix} 0 & \sigma^i e_i^j \\ \sigma^i e_i^j & 0 \end{pmatrix} \nabla_j \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + cm \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix},$$

where $\nabla_i = \partial_i - \Omega_i$ and $g_{ij} = e_i^k e_j^l \delta_{kl}$. Note that ∇_i is block diagonal because $\gamma^i \gamma^j$ is block diagonal, thus, we define $\bar{\nabla}_i$ by means of

$$\nabla_i = \begin{pmatrix} \bar{\nabla}_i & 0 \\ 0 & \bar{\nabla}_i \end{pmatrix}. \quad (4.44)$$

Now, proceeding in the usual way [152] and using that

$$(\sigma^i e_i^j \bar{\nabla}_i)^2 = g^{ij} \bar{\nabla}_i \bar{\nabla}_j - \frac{1}{4} R \quad (4.45)$$

where R is the scalar of curvature of the 2D surface, we obtain

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m|g|^{1/2}} \partial_i g^{ij} |g|^{1/2} \partial_j \psi + \frac{\hbar^2}{8m} R \psi, \quad (4.46)$$

which is the usual Schrödinger equation in a curved surface, with the potential $\hbar^2 R/(8m)$, “inherited” from the Dirac equation, note that the Gaussian curvature is related to the scalar of curvature in 2D through $K = \frac{R}{2}$. On the other hand, if we take the non-relativistic limit of Eq. (4.8) and then the thin-layer method is implemented, we obtain (see. Eq. (3.36))

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m|g|^{1/2}} \partial_i g^{ij} |g|^{1/2} \partial_j \psi - \frac{\hbar^2}{2m} (H^2 - K) \psi, \quad (4.47)$$

which is the effective Schrödinger equation in a curved surface. At this point two questions arise. Firstly, why a geometric potential, that depends on the extrinsic curvature,

appears in the Schrödinger, Maxwell and Klein–Gordon equations but not so in the Dirac equation? Second, why the non-relativistic limit depends on whether this is taken before or after implementing the thin-layer method? To answer these two questions, it is necessary to realize that the geometric potential in the effective Schrödinger equation on a surface comes from the second derivative term

$$\frac{\partial_3(|G|^{1/2}\partial^3(|G|^{-1/4}\psi))}{|G|^{1/4}} \approx \partial_3\partial^3\psi + \frac{\epsilon^2}{2}(H^2 - K)\psi,$$

and that the geometric potential in the effective Schrödinger equation on a surface comes from a term of order ϵ^2 in thin-layer method (see Chapter 3).¹⁴ Note that there are no terms of order ϵ that contribute to the geometric potential in the effective Schrödinger equation. Hence, since in the effective Dirac equation we retain only terms of order ϵ and they do not lead to the geometric potential, when one takes the non-relativistic limit the correct geometric potential will not appear. This is so, because the only contributions that we are considering to the geometric potential, in the non-relativistic limit, come from terms of order ϵ in the effective Dirac equation. However, the product of order ϵ^0 terms with the ones of order ϵ^2 (which we are not taking into account) yields terms of order ϵ^2 when we take the non-relativistic limit. Therefore, we are losing relevant information for the non-relativistic limit when we remove terms of order ϵ^2 in the effective Dirac equation.

If we retain terms of order ϵ^2 the Dirac equation, these terms could be identified as a geometric potential, and it is clear that the non-relativistic limit (up to order ϵ^2) will contain the usual geometric potential shown in Eq. (4.47), plus possible relativistic corrections. However, one can show that if retain terms of order ϵ^2 in the Dirac equation they will lead to terms of the form $F(q^\mu, q^3)$ that couples the dynamics of the normal and tangent degrees of freedom. Thus, we cannot decouple the dynamics and obtain an effective Dirac equation on the surface that depends only on the coordinates q^μ , which was our initial goal. In addition, since in general the spinor components are coupled in the Dirac equation and the usual methods to solve require that the “square” of the Dirac equation is taken, it is clear that the “square” does not commute with the thin-layer method for the same reason that the method of thin-layer does not commute with the non-relativistic limit. Therefore, although Eq. (4.38) is exact up to order ϵ , this equation loses relevant information contained in the terms of order ϵ^2 , either when the Dirac equation is solved or when the non-relativistic limit is taken.

Recovering the Non-Relativistic Limit of the Dirac Equation

Although the non-relativistic is recovered when second-order terms are retained, this would preclude the decoupling of the normal and tangential degrees of freedom. Therefore, here we propose to add *a posteriori* a term to Eq. (4.38) such that one can recover the non-relativistic limit. This is justified by the above discussion and also phenomenologically, since the effects of the geometric potential have already been observed experimentally in Tomononaga–Luttinger liquids [14]. Additionally, our proposal is consistent with Pauli and Klein–Gordon equations on a surface.

¹⁴The same is true for the Maxwell and Klein–Gordon equations.

First, let us note that the Pauli equation in the absence of electromagnetic fields and in the presence of a confining potential is

$$\begin{aligned} i\hbar \frac{\partial \Phi(\mathbf{x}, t)}{\partial t} &= -\frac{\hbar^2 (\boldsymbol{\sigma} \cdot \nabla)^2}{2m} \Phi(\mathbf{x}, t) + V_c(\mathbf{x}) \Phi(\mathbf{x}, t) \\ &= -\frac{\hbar^2}{2m} \nabla^2 \Phi(\mathbf{x}, t) + V_c(\mathbf{x}, z) \Phi(\mathbf{x}, t), \end{aligned} \quad (4.48)$$

where $\mathbf{x} = (x, y, z)$ and $\Phi(\mathbf{x}, t)^T = (\Phi_1(\mathbf{x}, t), \Phi_2(\mathbf{x}, t))$. Thus, one can apply the thin-layer method to the Pauli equation as if it was simply two Schrödinger equations. Therefore, at least in the non-relativistic approximation, the spin plays no role in the thin-layer method.

Let us now address the problem concerning the kind of coupling to be expected in the Dirac equation. In order to do that, we consider the effective Klein–Gordon equation on a surface [12]

$$\left[\frac{\hbar^2 \partial_\mu g^{\mu\nu} |g|^{1/2} \partial_\nu}{2|g|^{1/2}} + \frac{m^2 c^2}{2} - \frac{\hbar^2}{2} (H^2 - K) \right] \omega = 0, \quad (4.49)$$

where the geometric potential couples like a scalar potential and not, for instance, as an electromagnetic potential A_0 . It is natural to suggest that the term we need to introduce in the Dirac equation should couple as a scalar potential. Therefore, we propose to introduce a scalar potential of the form

$$V_G = -\frac{\hbar^2}{2mc} \left(H^2 - \frac{K}{2} \right). \quad (4.50)$$

Therefore, the effective Dirac equation on a surface reads

$$\left[i\hbar e_a^\mu \gamma^a (\partial_\mu - \Omega_\mu) - mc + \frac{\hbar^2}{2mc} \left(H^2 - \frac{K}{2} \right) \right] \psi = 0. \quad (4.51)$$

This last equation correctly leads to Eq. (4.47) in the non-relativistic limit. Note that, the geometric potential that we have introduced is suppressed with respect to the other terms since it is proportional to $\hbar^2$. Furthermore, the potential is gauge invariant and invariant under general coordinate transformations on the surface.

4.1.4 Relativistic Quasi Two-Dimensional Electron System

The aim of this section is two-fold. On the one hand, we will complement the argument of the previous section in a case where it is straightforward to calculate the terms of ϵ^2 , to show explicitly that if one takes into account these terms the non-relativistic limit and the confining potential approach commute. On the other hand, we want to extend the conclusions of sec. 3.3 to the relativistic case. The latter is motivated by the recent proposal that in graphene it could be possible to mimic the so-called *chiral magnetic effect* (CME)¹⁵ using an in-plane magnetic field [153, 154].

¹⁵Because the pseudo-spin is the participant, rather than the actual spin, this effect has been called *pseudo chiral magnetic effect* (PCME) [153, 154]. CME is the phenomenon of electric charge separation

Although the effective Dirac equation on a submanifold in the presence of a electromagnetic field will be discussed in detail in sec. 4.2, here, we present the case of the Dirac equation on a flat surface in the presence of an electromagnetic field. The (3+1)-dimensional Dirac equation in the presence of an external electromagnetic field is given by ($J = 0, \dots, 3$)

$$\left[i\hbar\gamma^J \left(\partial_J - \frac{ie}{\hbar}\bar{A}_J \right) - mc - V_c(z) \right] \Psi(t, \vec{r}) = 0, \quad (4.52)$$

where the γ -matrices are

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_{2 \times 2} & 0 \\ 0 & -\mathbb{I}_{2 \times 2} \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \quad (4.53)$$

As in the previous section, we rescale the z coordinates by $z \rightarrow \epsilon z$ and perform a perturbative expansion in ϵ , so, to first-order in ϵ

$$\left[i\hbar\gamma^3\partial_z - \tilde{V}_c(z) \right] \Psi + \epsilon \left[i\hbar\gamma^\mu \left(\partial_\mu - \frac{ie}{\hbar}\bar{A}_\mu \right) - mc + e\bar{A}_z\gamma^3 \right] \Psi = 0, \quad (4.54)$$

where $\bar{A}_\mu = A_\mu(t, x, y, z = 0)$, with $\mu = 0, 1, 2$. As before, we assume that $V_c(\epsilon z) = \tilde{V}(z)/\epsilon + O(\epsilon^1)$, with $\tilde{V}_c(z)$ independent of ϵ and removing the term $\bar{A}_z\gamma^3$ by a gauge transformation. Thus, performing a separation of variables, we obtain an effective Dirac equation for the (quasi)-two dimensional system

$$\left[i\hbar\gamma^\mu \left(\partial_\mu - \frac{ie}{\hbar}\bar{A}_\mu(t, x, y) \right) - mc \right] \bar{\psi}(t, x, y) = 0. \quad (4.55)$$

One might expect that, if we take the non-relativistic limit of Eq. (4.55) should arrive at the Eq. (3.60), however, this is not the case. In order to identify the source of lack of commutation between the non-relativistic limit and the thin-layer method, let us take the non-relativistic limit of Eq. (4.55), with $\bar{\psi}^T = (\psi, \varphi)$

$$i\hbar \left(\partial_0 - \frac{ie}{\hbar}\bar{A}_0 \right) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = i\hbar \begin{pmatrix} 0 & -\vec{\sigma}_\parallel \cdot \nabla \\ -\vec{\sigma}_\parallel \cdot \nabla & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix} + cm \begin{pmatrix} \psi \\ -\varphi \end{pmatrix},$$

where $\nabla_i = \partial_i - \frac{ie}{\hbar}\bar{A}_i$, with $i = 1, 2$, and $\vec{\sigma}_\parallel = \{\sigma^1, \sigma^2\}$. Using $[\nabla_i, \nabla_j] = -\frac{ie}{\hbar}\bar{F}_{ij}(t, x, y)$ and proceeding in the usual way [152], we obtain¹⁶

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \left(\partial_\mu - \frac{ie}{\hbar}\bar{A}_\mu \right)^2 \psi - eA_0\psi - i\frac{e\hbar}{8m}\bar{F}_{ij}[\sigma^i, \sigma^j]\psi. \quad (4.56)$$

along the external magnetic field that is induced by the chirality imbalance [155–157]. In quantum chromodynamics (QCD) a nonzero chiral charge density could appear in the quark-gluon-plasma phase producing a chirality imbalance [158]. In graphene, the proposed scenario [153, 154] involves the generation of the two different effective mass for the charge carriers from different Dirac points of the Brillouin zone, eg. by distorted graphene and, more importantly, the inclusion of an in-plane magnetic field that couples via $e\bar{\Psi}A_z\gamma^3\Psi$, *i.e.* through a minimal coupling with the electromagnetic field.

¹⁶The same result is obtained if one take the non-relativistic limit of (4.52) and then separate variables.

With some algebra the last term in (4.56) can be written as $e\bar{B}_z S_z/m$. Comparing Eqs. (4.55) and (3.60), we see that differ by the term $-e\vec{B}_{||} \cdot \vec{S}_{||}/m$. Now, keeping second-order terms in ϵ , yields

$$0 = \left[i\hbar\gamma^3\partial_z - \tilde{V}_c(z) \right] \Psi + \epsilon \left[i\hbar\gamma^\mu \left(\partial_\mu - \frac{ie}{\hbar}\bar{A}_\mu + \epsilon z(\partial_z A_\mu|_{y=0}) \right) - mc \right] \Psi. \quad (4.57)$$

One can show that the non-relativistic limit of (4.57), up to second-order in ϵ , is just (3.52). Therefore, as we argue above this lack of commutation is owing to that in the effective Dirac equation we retain only terms of order ϵ . Although the non-relativistic is recovered when second-order terms are retained, we are interested in an effective Dirac equation that just depends on the coordinates on the plane. Therefore, we can proceed in an alternative way and add the term, $-e\hbar\vec{B}_{||} \cdot (\gamma^0\vec{\gamma}_{||})/(2m)$, with $\vec{\gamma}_{||} = \{\gamma^1, \gamma^2\}$, *a posteriori*. Whereby the effective Dirac equation on the plane is

$$\left[i\hbar\gamma^\mu \left(\partial_\mu - \frac{ie}{\hbar}\bar{A}_\mu \right) - mc - \frac{e\hbar}{2m}\vec{B}_{||} \cdot (\gamma^0\vec{\gamma}_{||}) \right] \psi = 0, \quad (4.58)$$

which just depends on the coordinates on the plane and correctly leads to Eq. (3.52) in the non-relativistic limit, apart from that this equation is invariant under the gauge transformations $\bar{A}'_\mu(t, x, y) = \bar{A}_\mu(t, x, y) + \partial_\mu\gamma(t, x, y)$. Note that Eq. (4.58) only reproduces the first relativistic correction, additional terms need to be included if one wants to recover other terms in the series of relativistic corrections.

Monolayer Graphene

As is well known, the low-energy excitations in graphene are described for the $(2+1)$ -dimensional Dirac equation where the speed of light is replaced by the Fermi velocity v_f and the pseudo-spin play the role of the spin. Although in the previous section, we consider the Dirac equation for a particle confined to a plane system, it is important to note that graphene is not obtained from a relativistic system which is confined to the plane. Instead of one starts from a two-dimensional non-relativistic Hamiltonian for electrons on the honeycomb lattice and then using the tight-binding approach, in the continuum limit, one obtains that the low-energy excitations in graphene are described by a Dirac Hamiltonian. From this point of view, one can argue that from the beginning, the dynamics in the layer has been decoupled from the dynamics of the normal coordinate. Therefore, the effect of an in-plane magnetic field can be analyzed from a “three-dimensional” non-relativistic Hamiltonian for electrons on the honeycomb lattice, where the potential produced by the hexagonal arrangements of carbon atoms play the role of the confining potential (V_c). Thus, the arguments are the same as in Sec. (3.3.1), so, as long as $\ell_B \gg \ell_c$ (where ℓ_c is the thickness of graphene) the effects of the in-plane magnetic vector potential is negligible, owing to the one atom thickness of graphene this condition is quite realistic. Therefore, the proposal of Mizher *et.al.* [153, 154] of introduce of an in-plane magnetic field that couples via $e\bar{\Psi}A_z\gamma^3\Psi$ is not consistent, and the so-called pseudo chiral magnetic effect is not feasible. What is more, it has been experimentally shown that in monolayer graphene, an in-plane magnetic field only interacts with electrons via the Zeeman coupling [159–161].

4.2 Dirac Equation for a Particle in an Electromagnetic Field on a Submanifold

In this section, we will find the effective dynamics for a spin-1/2 charged particle constrained to a submanifold of a flat space-time. The conventions for this section is given in Appendix A. The Dirac equation for a particle in an electromagnetic field A_A in a curvilinear coordinates or curved space-time is given by [54]

$$\mathcal{D}\Psi = i\hbar\underline{\gamma}^A \left(\partial_A - \bar{\Omega}_A - \frac{ie}{\hbar} A_A \right) \Psi - (mc + V_c(y))\Psi = 0. \quad (4.59)$$

where $V_c(y)$ is a confining potential that depends only on the normal coordinates y^i . The γ -matrices generate a Clifford algebra $\{\underline{\gamma}^A, \underline{\gamma}^B\} = 2G^{AB}$, where the underline is used to distinguish between the space-dependent γ -matrices and the constant γ -matrices related by $\underline{\gamma}^A = \underline{\gamma}^A(x^\mu, y^i) = E_I^A(x^\mu, y^i)\gamma^I$, as before. They are all $2^{\lfloor \frac{n}{2} \rfloor} \times 2^{\lfloor \frac{n}{2} \rfloor}$ dimensional matrices for irreducible representations. The spinorial affine connections $\bar{\Omega}_A$ is given by [54]

$$\bar{\Omega}_A = -\frac{1}{4}\gamma^I\gamma^J\eta_{JK}E_I^B(\partial_A E_B^K - \Gamma_{AB}^C E_C^K). \quad (4.60)$$

The normalization condition for the Dirac field in the adapted coordinates frame is

$$(\Psi, \Psi) = \int d^{m-1}x dy |G|^{1/2} |\Psi|^2 = 1. \quad (4.61)$$

The first step for obtaining the Dirac equation on a submanifold is to rescale the field ψ and the Dirac operator $\mathcal{D}$ to

$$\chi(x, y) \equiv \frac{|G|^{1/4}}{|g|^{1/4}} \Psi(x, y), \quad (4.62)$$

$$D \equiv \left(\frac{|G|^{1/4}}{|g|^{1/4}} \right) \mathcal{D} \left(\frac{|g|^{1/4}}{|G|^{1/4}} \right). \quad (4.63)$$

The next step is to rescale the normal coordinates $y \rightarrow \epsilon y$ and expand the Dirac operator in a perturbative expansion keeping terms of order ϵ , so that

$$\epsilon^2 D = D^{(0)} + \epsilon D^{(1)} + \epsilon^2 D^{(2)} + \dots \quad (4.64)$$

Before being able to proceed in this way let us calculate the spin connection in the adapted coordinates frame.

4.2.1 Spin Connections in the Adapted Coordinates Frame

One can show that the Christoffel symbols in the adapted coordinates frame up to zero-order in ϵ are given by Eq. (E.16) to Eq. (E.21)

$$\begin{aligned} \bar{\Gamma}_{\mu\nu}^\sigma &= \Gamma_{\mu\nu}^\sigma, & \bar{\Gamma}_{\mu\nu}^i &= \alpha_{\mu\nu}^i, & \bar{\Gamma}_{j\mu}^\beta &= \alpha_{j\mu}^\beta, \\ \bar{\Gamma}_{ij}^\beta &= 0, & \bar{\Gamma}_{\mu j}^i &= -\beta_{\mu j}^i, & \bar{\Gamma}_{ij}^m &= 0, \end{aligned} \quad (4.65)$$

where $\Gamma_{\mu\nu,\beta} = \frac{1}{2}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})$ are the Christoffel symbols on the submanifold, $\mathcal{M}$. Let us compute the spinor connection retaining terms of zero-order in ϵ . For a general index A , we obtain

$$\begin{aligned} -\bar{\Omega}_A &= \frac{1}{4}\gamma^a\gamma^b\eta_{bc}e_a^\beta(\partial_A E_\beta^c - \bar{\Gamma}_{A\beta}^\nu e_\nu^c) + \frac{1}{4}\gamma^a\gamma^m\eta_{mn}e_a^\beta(\partial_A E_\beta^n - \bar{\Gamma}_{A\beta}^n) \\ &\quad - \frac{1}{4}\gamma^n\gamma^a\eta_{ac}\bar{\Gamma}_{An}^\nu e_\nu^c - \frac{1}{4}\gamma^i\gamma^m\eta_{mp}\bar{\Gamma}_{Ai}^p + O(\epsilon), \end{aligned} \quad (4.66)$$

For $A = \mu$ with $(\mu = 0, 1, \dots, m-1)$, Eq. (4.14) yields

$$-\bar{\Omega}_\mu = -\Omega_\mu - \frac{1}{4}\gamma^a\gamma^m\eta_{mn}e_a^\beta\alpha_{\mu\beta}^n - \frac{1}{4}\gamma^m\gamma^a\eta_{ac}\alpha_{m\mu}^\nu e_\nu^c + \frac{1}{4}\gamma^i\gamma^m\eta_{mp}\beta_{\mu i}^p + O(\epsilon), \quad (4.67)$$

where Ω_μ are the spinorial affine connections associated to $g_{\mu\nu}$. Using that $\eta_{ac}\alpha_{m\mu}^\nu e_\nu^c = -\eta_{nm}e_a^\beta\alpha_{\mu\beta}^n$, Eq. (4.67) can be simplified to

$$-\bar{\Omega}_\mu = -\Omega_\mu + \frac{1}{2}\gamma^a\gamma^m\eta_{ac}\alpha_{m\mu}^\nu e_\nu^c + \frac{1}{8}[\gamma^i, \gamma^m]\beta_{\mu i}^m + O(\epsilon), \quad (4.68)$$

where we are taking into account that $\gamma^a\gamma^m = -\gamma^m\gamma^a$. The second term of the right-hand side of Eq. (4.68) appear in the Dirac equation multiplied by the space-dependent γ -matrices, so that

$$\begin{aligned} E_I^\mu\gamma^I\frac{1}{2}\gamma^a\gamma^m\eta_{ac}\alpha_{m\mu}^\nu e_\nu^c &= e_b^\mu\gamma^b\frac{1}{2}\gamma^a\gamma^m\eta_{ac}\alpha_{m\mu}^\nu e_\nu^c + O(\epsilon) \\ &= -\frac{1}{2}\underline{\gamma}^\mu\underline{\gamma}^\nu\alpha_{m\mu\nu}\gamma^m + O(\epsilon) \\ &= \frac{1}{2}\alpha_{m\mu}^\mu\gamma^m + O(\epsilon). \end{aligned} \quad (4.69)$$

For $A = j$ with $(j = m, m+1, \dots, n)$, Eq. (4.66) yields

$$-\bar{\Omega}_j = \frac{1}{4}\gamma^a\gamma^b\eta_{bc}e_a^\sigma(\alpha_{j\sigma}^\nu e_\nu^c - \alpha_{j\sigma}^\nu e_\nu^c) + \frac{1}{4}\gamma^a\gamma^m\eta_{mn}e_a^\sigma(-\beta_{\sigma j}^n + \beta_{\sigma j}^n) + O(\epsilon) = 0. \quad (4.70)$$

Inserting (4.68) and (4.70) into $\mathcal{D}$, we obtain the following expression for the Dirac operator to zero-order in ϵ

$$\begin{aligned} &\mathcal{D} + mc + V_c(y) \\ &= i\hbar\underline{\gamma}(x)^A\left(\partial_A - \bar{\Omega}_A - \frac{ie}{\hbar}A_A\right) \\ &= i\hbar e_a^\mu(x)\gamma^a\left(\partial_\mu - \Omega_\mu + \frac{1}{8}[\gamma^i, \gamma^m]\beta_{\mu i}^m + y^k\beta_{\mu k}^j\partial_j - \frac{ie}{\hbar}\bar{A}_\mu\right) \\ &\quad + i\frac{\hbar}{2}\alpha_{m\mu}^\mu\gamma^m + i\frac{\hbar}{\epsilon}\gamma^i\partial_i + \gamma^ie\bar{A}_i \\ &= i\hbar e_a^\mu(x)\gamma^a\left(\partial_\mu - \Omega_\mu + \frac{i}{2\hbar}J^{im}\beta_{\mu i}^m - \frac{ie}{\hbar}\bar{A}_\mu\right) + \frac{i\hbar}{2}\alpha_{m\mu}^\mu\gamma^m + \frac{i\hbar}{\epsilon}\gamma^i\partial_i + e\gamma^i\bar{A}_i, \end{aligned} \quad (4.71)$$

where A_B has been expanded in a Taylor series and, without loss of generality, we will set $\bar{A}_i = A_i(x^\mu, 0) = 0$. In Eq. (4.71) $J^{ij} = L^{ij} + \Sigma^{ij}$ are the total angular momentum operators in the space normal to $\mathcal{M}$, $L^{ij} = i\hbar(y^j\partial^i - y^i\partial^j)$ is the orbital angular momentum and $\Sigma^{ij} = -i\frac{\hbar}{4}[\gamma^i, \gamma^m]$ the spin angular momentum.

4.2.2 Perturbation Theory

We now take into account the explicit conditions for the potential $V(y)$, and as before, we assume that

$$V_c(\epsilon y) = \frac{\tilde{V}(y)}{\epsilon} + O(\epsilon^0), \quad (4.72)$$

where $\tilde{V}(y)$ is independent of ϵ . In order to calculate the rescaled Dirac operator (4.63), we first consider the term

$$\begin{aligned} \left(\frac{|G|^{1/4}}{|g|^{1/4}} \right) \gamma^i \partial_i \left(\frac{|g|^{1/4}}{|G|^{1/4}} \right) \chi &= \gamma^i \partial_i \chi + (|G|^{1/4} \gamma^i \partial_i |G|^{-1/4}) \chi \\ &= \gamma^i \partial_i \chi - \frac{1}{2} \epsilon \alpha_{i\mu}{}^\mu \gamma^i \chi + O(\epsilon), \end{aligned}$$

so that, the rescaled Dirac operator can be written as

$$\begin{aligned} D + mc + V_c &= i\hbar e_a^\mu(x) \gamma^a \left(\partial_\mu - \Omega_\mu + \frac{i}{2\hbar} J^{im} \beta_{\mu im} - \frac{ie}{\hbar} \bar{A}_\mu \right) \\ &\quad + \frac{i\hbar}{2} \alpha_{m\mu}{}^\mu \gamma^m - \frac{i\hbar}{2} \alpha_{i\mu}{}^\mu \gamma^i + \frac{i\hbar}{\epsilon} \gamma^i \partial_i. \end{aligned}$$

Multiplying by ϵ and simplifying yields

$$\epsilon D + = i\hbar \gamma^i \partial_i + \epsilon i\hbar e_a^\mu(x) \gamma^a \left(\partial_\mu - \Omega_\mu + \frac{i}{2\hbar} J^{im} \beta_{\mu im} - \frac{ie}{\hbar} \bar{A}_\mu \right).$$

As in the case of the Dirac operator on the surface (4.23), in the general case one can observe that the term $\frac{i\hbar}{2} \alpha_{i\mu}{}^\mu \gamma^i$, coming from the rescale of the wave function, exactly cancels the term $\frac{i\hbar}{2} \alpha_{m\mu}{}^\mu \gamma^m$, coming from the calculation of spin connection. Comparing with Eq. (4.64) we conclude that

$$D^{(0)} = i\hbar \gamma^i \partial_i, \quad (4.73)$$

$$D^{(1)} = \epsilon i\hbar e_a^\mu(x) \gamma^a \left(\partial_\mu - \Omega_\mu + \frac{i}{2\hbar} J^{im} \beta_{\mu im} - \frac{ie}{\hbar} \bar{A}_\mu \right). \quad (4.74)$$

Therefore, the Dirac equation can be rewritten as

$$\left[i\hbar \frac{\gamma^i}{\epsilon} \partial_i - \frac{V_c(y)}{\epsilon} + i\hbar e_a^\mu(x) \gamma^a \left(\partial_\mu - \Omega_\mu + \frac{i}{2\hbar} J^{im} \beta_{\mu im} - \frac{ie}{\hbar} \bar{A}_\mu \right) - mc \right] \chi(x, y) = 0. \quad (4.75)$$

Our goal now is to separate the Dirac equation in the normal and tangent degrees of freedom. Before that let us choose the representation of γ matrices in d -dimensions.

4.2.3 γ -Matrices for the Dirac Equation on a Submanifold

The γ -matrices in (4.75) with indices a are irreducible representations in a higher n -dimensional space-time $\mathcal{N}$. Whereas, in general, they are not irreducible representations in

$\dim(\mathcal{N})$	$\dim(\mathcal{M})$	γ^a
$n = 2d$	$m = 2g$	$\mathbb{I} \otimes \bar{\gamma}^a$
$n = 2d + 1$	$m = 2g$	$\mathbb{I} \otimes \bar{\gamma}^a$
$n = 2d + 1$	$m = 2g + 1$	$\mathbb{I} \otimes \bar{\gamma}^a$
$n = 2d$	$m = 2g + 1$	$\sigma^3 \otimes \mathbb{I} \otimes \bar{\gamma}^a$

Table 4.1: Four possibilities for the embedding. The γ^a and $\bar{\gamma}^a$ are matrices of the irreducible representations of $\mathcal{N}$ and $\mathcal{M}$, respectively, with $a = 0, 1, \dots, m - 1$. $\mathbb{I}$ denotes the $2^{\lfloor \frac{n-m}{2} \rfloor} \times 2^{\lfloor \frac{n-m}{2} \rfloor}$ identity matrix. When n is odd there are two equivalent representations each of which are mapped to one of the inequivalent representations of $\mathcal{M}$, if m is odd, or to the same representation if m is even. Curiously, when n is even and m is odd the matrices γ^μ are built from the two inequivalent irreducible representations of $\mathcal{M}$.

a low m -dimensional space-time $\mathcal{M}$. In order to explore all possibilities, let us construct inductively a representation of γ -matrices for a n -dimensional space-time. The Pauli matrices may be defined by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.76)$$

We start in $n = 1 + 1$, where

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad (4.77)$$

for odd dimensions ($n = 2d + 1$), there are two inequivalent irreducible representations for the Γ -matrices which can be chosen as

$$\Gamma^\alpha = \pm \gamma^\alpha, \quad \Gamma^{2d} = \begin{pmatrix} 0 & \pm I_{2^{d-1}} \\ \mp I_{2^{d-1}} & 0 \end{pmatrix}, \quad (4.78)$$

where γ^α are the matrices in $2d$ -dimensions and $I_{2^{d-1}}$ is the $2^{d-1} \times 2^{d-1}$ identity matrix. Now, for even dimensions ($n = 2d + 2$), there is only one irreducible representation

$$\gamma^\alpha = \sigma^3 \otimes \Gamma^\alpha = \begin{pmatrix} \Gamma^\alpha & 0 \\ 0 & -\Gamma^\alpha \end{pmatrix}, \quad (4.79)$$

$$\gamma^{2d+1} = i\sigma^1 \otimes I_{2^d} = \begin{pmatrix} 0 & iI_{2^d} \\ iI_{2^d} & 0 \end{pmatrix}. \quad (4.80)$$

In our problem the dimension of $\mathcal{M}$ is always less than $\mathcal{N}$ ($n < m$). Therefore, we choose that the representation γ^μ of $\mathcal{M}$ ($\mu = 0, \dots, m - 1$) are among the first m matrices γ^α (or Γ^α) depending if n is odd or even. There will be four possibilities depending if the dimensions of $\mathcal{N}$ and $\mathcal{M}$ are odd or even. As shown in Tab. 4.1, the γ^a are not irreducible representations of $\mathcal{M}$ (except when n is odd and $n = m + 1$), but they are built with copies of the irreducible representations of $\mathcal{M}$.

4.2.4 Effective Dirac Equation on a Submanifold

The separation of variable for spinors is harder than in the case of scalars. For this purpose, we propose a product solution of the form¹⁷

$$\chi(x, y) = \sum_l \sum_\beta \phi_{\beta,l}(y) \psi_{\beta,l}(x), \quad (4.82)$$

where the index β runs over the degeneracy that exists in the spectrum of the normal degrees of freedom and l runs over the number of copies of the irreducible representations of $\mathcal{M}$ required to write the first m matrices of the irreducible representations of $\mathcal{N}$ ¹⁸. In order to find a effective Dirac equation in the submanifold, we assume that

$$\left[i\hbar \frac{\gamma^i}{\epsilon} \partial_i - \frac{V_c(y)}{\epsilon} - mc \right] \sum_l \sum_\beta \phi_{\beta,l}(y) \psi_{\beta,l}(x) = \bar{m}c \sum_l \sum_\beta \phi_{\beta,l}(y) \psi_{\beta,l}(x), \quad (4.83)$$

inserting Eq. (4.83) into Eq. (4.75), yields

$$\sum_l \sum_\beta \left[i\hbar e_a^\mu(x) \gamma^a \left(\partial_\mu - \Omega_\mu + \frac{i}{2\hbar} J^{im} \beta_{\mu im} - \frac{ie}{\hbar} \bar{A}_\mu \right) - \bar{m}c \right] \phi_{\beta,l}(y) \psi_{\beta,l}(x) = 0. \quad (4.84)$$

The normalization condition can be rewritten as

$$\begin{aligned} (\Psi, \Psi) &= \int d^{m-1}x d^p y |G|^{1/2} |\Psi|^2 \\ &= \sum_{l,r} \sum_{\beta,\alpha} \left(\int d^p y \phi_{\beta,l}^* \phi_{\alpha,r} \right) \int d^{m-1}x |g|^{1/2} \psi_{\beta,l}^\dagger \psi_{\alpha,r} \\ &= \sum_l \sum_\beta \int d^{m-1}x |g|^{1/2} \psi_{\beta,l}^\dagger(x) \psi_{\beta,l}(x) \\ &= \sum_l \sum_\beta (\psi_{\beta,l}, \psi_{\beta,l}) = 1, \end{aligned}$$

where we have taken into account, without loss of generality, that $\int d^p y \phi_{\beta,l}^*(y) \phi_{\alpha,r}(y) = \delta_{\beta\alpha} \delta_{lr}$. Multiplying Eq. (4.84) by $\phi_{\alpha,r}^*(y)$ and integrating in $\int d^p y$ we obtain that the

¹⁷ In the context of Brane gravity, the problem of separation the variables of a higher-dimensional space-time to a lower-dimensional space-time has already been considered in two case [162–166]. The first one when going a five dimensional space-time to a four dimensional [162, 163, 166]. In that scenario the Dirac spinor was written as

$$\Psi(x^\mu, y) = U_L(y) \psi_L(x^\mu) + U_R(y) \psi_R(x^\mu), \quad (4.81)$$

where $\psi_{L/R}(x^\mu)$ are spinors, while $U_{L/R}(y)$ are functions. The second one when going a six dimensional space-time to a four dimensional [164, 165]. In this case, before to assume a solution analogous to Eq. (4.81), the condition $\Psi_8 = (\Psi_4, 0)^T$ was imposed, where Ψ_8 is a eight-component spinor and Ψ_4 is a four-component spinor. Unfortunately this assumption rules out a number of existing solutions. Here we go beyond this assumption.

¹⁸As before, the system is an individual energy eigenstates of the normal coordinates.

effective Dirac equation in the presence of an external field on a submanifold is given by¹⁹

$$\left[i\hbar e_a^\mu(x) \gamma^a \left(\partial_\mu - \Omega_\mu - \frac{ie}{\hbar} \bar{A}_\mu \right) - \bar{m}c \right] \psi_{\alpha,r}(x) = \sum_l \sum_\beta i\hbar e_a^\mu(x) \gamma^a \frac{i}{\hbar} (\mathcal{G}_\mu)_{\alpha\beta}^{rl} \psi_{\beta,l}(x), \quad (4.86)$$

where $\mathcal{G}_\mu = -\frac{1}{2}\beta_{\mu im}(\mathcal{J}^{im})$ and

$$(\mathcal{J}^{im})_{\alpha\beta}^{rl} \equiv \int d^p y (\phi_{\alpha,r}(y))^\dagger J^{im} \phi_{\beta,l}(y). \quad (4.87)$$

In Eq. (4.86) the term proportional to $\mathcal{G}_\mu$ was explicitly separated since it is the only non-diagonal in β and l . Thus, for the first three cases in Tab. 4.1, we can write the γ -matrices as $\mathbb{I} \otimes \bar{\gamma}^a$. Therefore, we have $2^{\lfloor \frac{n-m}{2} \rfloor}$ equivalent copies of the Dirac equation that one can decouple²⁰, if the space-time is torsion-free. For the latter case of Tab. 4.1 it always appears unexpectedly the two inequivalent irreducible representations of $\mathcal{M}$ and for torsion-free, one can decouple the Dirac equation in two nonequivalent copies.

Finally, it should be noted that again there is also no commutativity between the non-relativistic limit and the perturbative expansion up to zero-order in ϵ . However, one can show that, in the absence of an electromagnetic field, adding the following scalar potential recovers the non-relativistic limit²¹

$$V_{GD} = \frac{\hbar^2}{8mc} \left(R - \alpha_{\mu}^{\kappa} \alpha_{\kappa\nu}^{\mu} \right). \quad (4.88)$$

4.3 “Gravity”, Torsion and Integrability Conditions

In this section, we will briefly discuss some aspects of the Gauss-Codazzi, Codazzi-Mainardi and Ricci integrability conditions to emphasize some geometric aspects of the theory.

4.3.1 Gauss-Codazzi, Codazzi-Mainardi and Ricci Integrability Conditions

The integrability conditions for the existence of the embedding of the submanifold $\mathcal{M}$ in the pseudo-Riemannian manifold $\mathcal{N}$ are derived requiring the commutation of second

¹⁹Eq. (4.86) can be rewritten schematically as

$$\left[i\hbar e_a^\mu(x) \gamma^a \left(\partial_\mu - \Omega_\mu - \frac{i}{\hbar} \mathcal{G}_\mu - \frac{ie}{\hbar} \bar{A}_\mu \right) - \bar{m}c \right] \vec{\psi} = 0. \quad (4.85)$$

²⁰By equivalent we mean that they satisfy the same Dirac equation.

²¹Strictly, this is only valid if space is torsion-free since it also contributes to the first relativistic correction. Moreover, when the electromagnetic field is included, one can show that in the non-relativistic limit the induced Zeeman coupling has also disappeared. The reason for this is the same as before, and in this case, one could also propose a posteriori a term to recover it, without having to calculate terms of second-order in ϵ . In the same way as was done in Secs. 4.1.3 and 4.1.3.

partial derivatives of the tangent and normal vectors [167, 168], namely

$$\partial_\sigma \partial_\mu \mathbf{t}_\nu - \partial_\mu \partial_\sigma \mathbf{t}_\nu = 0, \quad (4.89)$$

$$\partial_\nu \partial_\mu \hat{\mathbf{n}}_i - \partial_\mu \partial_\nu \hat{\mathbf{n}}_i = 0. \quad (4.90)$$

Using the *Gauss formula* and the *Weingarten Equations* given by (see Appendix A)

$$\partial_\mu \mathbf{t}_\nu = \Gamma_{\mu\nu}^\rho \mathbf{t}_\rho + \alpha_{\mu\nu}^i \hat{\mathbf{n}}_i, \quad (4.91)$$

$$\partial_\mu \hat{\mathbf{n}}_i = \alpha_{i\mu}^\nu \mathbf{t}_\nu - \beta_{\mu i}^k \hat{\mathbf{n}}_k, \quad (4.92)$$

respectively, one obtain the so-called *Gauss-Codazzi*, *Codazzi-Mainardi*, and *Ricci* equations,

$$R_{\nu\sigma\mu}^\rho = \alpha_{\sigma\nu}^k g^{\rho\lambda} \alpha_{k\mu\lambda} - \alpha_{\mu\nu}^k g^{\rho\lambda} \alpha_{k\sigma\lambda}, \quad (4.93)$$

$$\partial_\sigma \alpha_{\mu\nu}^i + \Gamma_{\mu\nu}^\lambda \alpha_{\sigma\lambda}^i + \beta_{\sigma}^i{}_k \alpha_{\mu\nu}^k = \partial_\mu \alpha_{\sigma\nu}^i + \Gamma_{\sigma\nu}^\lambda \alpha_{\mu\lambda}^i + \beta_{\mu}^i{}_k \alpha_{\sigma\nu}^k, \quad (4.94)$$

$$f_{\mu\nu i}{}^k = \alpha_{i\mu\lambda} g^{\lambda\rho} \alpha_{\nu\rho}^k - \alpha_{i\nu\lambda} g^{\lambda\rho} \alpha_{\mu\rho}^k, \quad (4.95)$$

respectively. $R_{\nu\sigma\mu}^\rho$ is the *Riemann-Christoffel curvature tensor* in the submanifold. The *normal curvature*, associated with the normal connection $\beta_{\mu ij}$, is denoted by $f_{\mu\nu kj}$. Explicitly

$$f_{\mu\nu kj} = (\partial_\mu \beta_{\nu kj} - \partial_\nu \beta_{\mu kj} + [\beta_\mu, \beta_\nu]_{kj}). \quad (4.96)$$

Note that the normal curvature is anti-symmetric under the permutations $(\mu \leftrightarrow \nu)$ and $(i \leftrightarrow k)$.

The Gauss-Codazzi (4.93), Codazzi-Mainardi (4.94) and Ricci (4.95) equations for the embedding resemble “*dynamical equations*” for the metric $g_{\mu\nu}$, the extrinsic curvature $\alpha_{k\mu\nu}$ and the normal fundamental form $\beta_{\mu ij}$, respectively.

4.3.2 “Einstein Field Equations” and “Gravity”

Contracting the indices ρ and σ in the Gauss-Codazzi relation (4.93) yields

$$\begin{aligned} R_{\nu\mu} = R_{\nu\rho\mu}^\rho &= \alpha_{\mu\nu}^k g^{\rho\lambda} \alpha_{k\rho\lambda} - \alpha_{\rho\nu}^k g^{\rho\lambda} \alpha_{k\mu\lambda} \\ &= -\alpha_{\mu\nu}^k \alpha_{k\lambda}^\lambda - \alpha_{\rho\nu}^k g^{\rho\lambda} \alpha_{k\mu\lambda}. \end{aligned} \quad (4.97)$$

Performing another contraction,

$$R - \alpha_{\nu}^k{}^\nu \alpha_{k\lambda}^\lambda + \alpha_{\rho\nu}^k \alpha_k^{\rho\nu} = 0. \quad (4.98)$$

This equation is called the scalar Gauss relation. Combining the Eq. (4.97) with Eq. (4.98) yields

$$R_{\nu\mu} - \frac{1}{2} R g_{\nu\mu} = \tilde{T}_{\nu\mu}, \quad (4.99)$$

where

$$\begin{aligned}\tilde{T}_{\nu\mu} = & -\alpha_{\mu\nu}^k \alpha_{k\lambda}^\lambda - \alpha_{\rho\nu}^k g^{\rho\lambda} \alpha_{k\mu\lambda} \\ & -\frac{1}{2} g_{\nu\mu} \alpha_{\beta}^k \alpha_{k\lambda}^\lambda + \frac{1}{2} g_{\nu\mu} \alpha_{\rho\beta}^k \alpha_k^{\rho\beta}.\end{aligned}\quad (4.100)$$

Although (4.99) resembles the *Einstein field equations*, $\tilde{T}_{\nu\mu}$ tells us that matter does not curve the space-time and that the “sources” of (intrinsic) curvature are only the extrinsic curvatures. In addition, it is important to note that the dynamical variables on $\mathcal{M}$ are the embedding functions $\mathcal{Y}^A(x^\mu)$ rather than the metric $g_{\mu\nu}$ or the second fundamental form itself. Nevertheless, the effective dynamics of the spinless particles (C.11), 1/2-spin particles (4.86) and classical particles (F.12) on a submanifold, enables us to interpret the curvature on a submanifold as an “*induced-gravity*” due to dimensional reduction. In other words, an “inner” observer on a submanifold feels a curved space. Of course, it is understood as a geometric effect where the equations of motion of the “*gravitational*” field (4.93) have a classic origin²² and never needs to be quantized. This is in contrast to a true gravitational effect.

4.3.3 Geometry-Induced Yang-Mills Field

The Ricci integrability conditions Eq. (4.95) tell us that the normal curvature is completely determined by the induced metric $g_{\mu\nu}$ and the second fundamental form $\alpha_{i\mu\nu}$ ²³. Since the normal fundamental form $\beta_{\mu ij}$ represents the connection induced by $\mathcal{N}$ on the normal bundle of $\mathcal{M}$, $N\mathcal{M}$, one can introduce a $SO(n-m)$ covariant derivative on the submanifold, $\tilde{\nabla}_\mu$, defined as

$$\tilde{\nabla}_\mu \phi_i = \partial_\mu \phi_i + \beta_{\mu i}^j \phi_j, \quad (4.101)$$

with ϕ^i an arbitrary normal scalar and, as usual, the connection satisfies $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] \phi_i = f_{\mu\nu}^k \phi_k$. Since that the choice of the tangent and orthonormal vectors is not unique, one can always perform local rotations in the orthonormal vectors $\hat{\mathbf{n}}_i$. Under a local rotation the *extrinsic torsion* transform as $\hat{\mathbf{n}}_i \rightarrow \mathcal{R}_i^j \hat{\mathbf{n}}_j$, where $\mathcal{R}$ is an element of the vector representation of $SO(n-m)$. Thus, the *extrinsic torsion* transform as

$$\begin{aligned}\beta_{\mu i}^j & \rightarrow \mathcal{R}_i^k \hat{\mathbf{n}}_k \cdot \partial_\mu (\mathcal{R}_l^j \hat{\mathbf{n}}^l) \\ & = \mathcal{R}_i^k \hat{\mathbf{n}}_k \cdot (\partial_\mu \hat{\mathbf{n}}^l) \mathcal{R}_l^j + \mathcal{R}_i^k \hat{\mathbf{n}}_k \cdot \hat{\mathbf{n}}^l \partial_\mu \mathcal{R}_l^j \\ & = \mathcal{R}_i^k \beta_{\mu k}^l \mathcal{R}_l^j + \mathcal{R}_i^k \delta_k^l \partial_\mu \mathcal{R}_l^j \\ & = \mathcal{R}_i^k \beta_{\mu k}^l (\mathcal{R}^T)_l^j + \mathcal{R}_i^k \partial_\mu (\mathcal{R}^T)_k^j,\end{aligned}$$

so, one can define a matrix β_μ such that

$$\beta_\mu \rightarrow \mathcal{R} \beta_\mu \mathcal{R}^T + \mathcal{R} \partial_\mu \mathcal{R}^T, \quad (4.102)$$

²²By classic origin, we mean the experimentalist in the classical world who modifies the confining potential $V(y)$.

²³In the case of codimension equal to one the Ricci equation does not exist.

therefore, β_μ transforms as a *gauge field* and an observer on the submanifold will see the rotational freedom of the normal vector fields as gauge transformations, with a gauge group $SO(n - m)$ [15, 16, 48]. The second fundamental form transform as

$$\begin{aligned}\alpha_{i\mu\nu} &\rightarrow -\mathbf{t}_\mu \cdot \partial_\nu (\mathcal{R}_i^k \hat{\mathbf{n}}_k) \\ &= -\mathcal{R}_i^k \mathbf{t}_\mu \cdot \partial_\nu \hat{\mathbf{n}}_k - \partial_\nu \mathcal{R}_i^k \mathbf{t}_\mu \cdot \hat{\mathbf{n}}_k \\ &= -\mathcal{R}_i^k \mathbf{t}_\mu \cdot \partial_\nu \hat{\mathbf{n}}_k \\ &= \mathcal{R}_i^k \alpha_{k\mu\nu},\end{aligned}$$

so, one can define a vector $\alpha_{\mu\nu}$ such that $\alpha_{\mu\nu} \rightarrow \mathcal{R}\alpha_{\mu\nu}$. Thus, $\alpha_{i\mu\nu}$ transform as the components of a $SO(n - m)$ vector [7]. With these properties of transformation one can show that the integrability conditions (4.93), (4.94) and (4.95) are $SO(N)$ -gauge invariants.

Since that the orbital angular momentum operators satisfy the commutation relation

$$[L_{ij}, L_{km}] = i(\delta_{ik}L_{jm} + \delta_{jm}L_{ik} - \delta_{im}L_{jk} - \delta_{jk}L_{im}), \quad (4.103)$$

and this is the Lie algebra of the group $SO(n - m)$ then L_{ij} are the generators the (self-representation) of the group $SO(n - m)$ [169]. Since the induced Yang-Mills field, $\mathcal{B}_\mu$ introduced in the effective Klein-Gordon and effective Schrödinger equations appear through the product $\beta_\mu^{ij} L_{ij}$ this gauge field couple in the standard form (*i.e.* the gauge field contracted with the group generators). On the other hand, the induced Yang-Mills field $\mathcal{C}_\mu$ couples in the effective Dirac equation as $\beta_\mu^{ij} (L_{ij} + \Sigma_{ij})$. Since Σ_{ij} are the generators of connected Lie group G_N (the covering group of $SO(n - m)$ [169]) the gauge fields couples the internal and external degrees of freedom.

On the other hand, on $\mathcal{M}$ there is a natural invariant which is quadratic in the normal curvature: the $SO(N)$ Yang-Mills functional,

$$I_0[\mathcal{Y}^A] = \int dV f_{\mu\nu}^{ij} f_{ij}^{\mu\nu}. \quad (4.104)$$

Although, it resembles an $SO(N)$ Yang-Mills theory on $\mathcal{M}$, this is not a genuine Yang-Mills theory, because the dynamical variables on $\mathcal{M}$ are the embedding functions $\mathcal{Y}^A$, instead of the connection itself $\beta_{\mu ij}$ [170]. The theory should be considered, more appropriately, as an *induced Yang-Mills theory* [16, 170]. Indeed, it has been shown in Ref. [48] the possibility of geometrically induced gauge structures as a feature of *dimensional reduction*. Since the dimensional reduction breaking of higher-dimensional general covariance and then the residual internal coordinates, and reference frame transformations are always perceived by effective low-dimensional observers (for example, observers on the brane) as gauge transformations [48]. Finally, note that while the Ricci integrability conditions Eq. (4.95) are of the first order in derivatives of the connection $\beta_{\mu ij}$, it is of second order in derivatives of the embedding functions $\mathcal{Y}^A$ [170]. Similarly to the induced-gravitational field, this induced Yang-Mills field is a classical field, which never needs to be quantized.

Part II

Quantum Field Theory on a Submanifold

CHAPTER 5

QUANTUM FIELD THEORY ON A SUBMANIFOLD: “FREE” FIELDS

In this chapter, based on relativistic quantum mechanics on a submanifold, we formulate the quantum theory of “free” fields on a curved submanifold. After this we introduce the concept of one-loop effective action at finite temperature and chemical potential. Thus, we compute this one-loop effective action for scalars and fermions in a background field. The problem is solved in a way that can be applied both to fields in manifolds as well as submanifolds.

5.1 Free QFT on a Submanifold

From the effective field equations obtained in previous chapters, one can recognize the action for the fields on a submanifold. For a scalar field the action which can be derived the field equation (C.11), with zero external electromagnetic fields, is

$$S[\phi] = - \int d^m x \left[\vec{\phi}^*(x) \frac{\hbar^2}{|g|^{1/2}} \left(\partial_\mu - i \frac{\mathcal{B}_\mu}{\hbar} \right) g^{\mu\nu} |g|^{1/2} \left(\partial_\nu - i \frac{\mathcal{B}_\nu}{\hbar} \right) \vec{\phi}(x) + \bar{m}^2 c^2 |\vec{\phi}(x)|^2 + V_G |\vec{\phi}(x)|^2 \right], \quad (5.1)$$

with $V_G = -\hbar^2 g^{\nu\mu} g^{\rho\sigma} (\alpha_{k\mu\nu} \alpha_{\rho\sigma}^k - 2\alpha_{k\mu\sigma} \alpha_{\rho\nu}^k) / 8$. On the other hand, for a Dirac field the action which can be derived from the field equation (4.86), with zero external electromagnetic fields and including the correction to the geometric potential¹ (V_{GD}), is

$$S[\psi] = \int d^m x \vec{\psi} \left[i \hbar e_a^\mu(x) \gamma^a \left(\partial_\mu - \Omega_\mu - \frac{i}{\hbar} \mathcal{G}_\mu \right) - \bar{m}c - V_{GD} \right] \vec{\psi}. \quad (5.2)$$

where $V_{GD} = \hbar^2 (R - \alpha_\mu^k{}^\mu \alpha_{k\nu}{}^\nu) / (8mc)$.

¹As mentioned in the last chapter this potential can be included a posteriori to recover the non-relativistic limit.

One can write the Feynman path integral formula for these two actions which will be the formulation of a quantum field theory on a submanifold. Ignoring the torsion and the geometric potential, the actions (5.1) and (5.2) describes a quantum field theory of scalar and fermion in a curved background, respectively. Schematically one could represent a quantum field theory on a submanifold as

$$\begin{aligned} \text{Free QFT on a manifold} + \text{geometric potential} + \text{External } SO(n-m) \text{ gauge field} \\ = \\ \text{"Free" QFT on a submanifold.} \end{aligned}$$

In other words, the “free” fields on a submanifold will feel not only the intrinsic curvature but also the effects of the torsion and the extrinsic curvature. The only case in which the descriptions of a free field on a manifold and a submanifold are equal is when space-time is flat. Finally, although the descriptions are not identical, a quantum system on a curved submanifold could help understand and simulate some aspects of quantum field theory in a curved space-time.

5.2 Effective Action

Quantum field theory (QFT) can be obtained from classical field theory by replacing the classical field variables with appropriate operators obeying certain commutation relations. In practice, there are very few realistic models for which one can solve the theory exactly. Thus, it is convenient to use some approximation schemes to compute physical quantities. The more traditional path concentrates on the calculation of scattering amplitudes and cross-sections perturbatively. An alternative approach to obtain non-perturbative results is by using the formalism of the *effective action*. Let us outline the method [55], we will consider initially a free field whose action functional, in the presence of an external source J , is given by

$$S[\phi]_J = \int d^n x \left(-\frac{1}{2} \phi(x) (\square_x + m^2 - i\epsilon) \phi(x) + J(x) \phi(x) \right). \quad (5.3)$$

The equation of motion for the field operator ϕ that follows from the action functional is $(\square_x + m^2 - i\epsilon) \phi(x) = J(x)$. One can rewritten (5.3) as

$$S[\phi]_J = \int d^n x d^n x' \phi(x) A(x, x') \phi(x') + \int d^n x J(x) \phi(x), \quad (5.4)$$

with $A(x, x') = (\square_x + m^2 - i\epsilon) \delta^n(x - x')$, which can formally be treated as a symmetric matrix K with continuous indices x and x' , having the properties [52]

$$\int d^n z A(x, z)^{1/2} A(z, x')^{1/2} = A(x, x'), \quad (5.5)$$

$$\int d^n z A(x, z) G_F(z, x') = (\square_x + m^2 - i\epsilon) G_F(x, x') = \delta(x - x'), \quad (5.6)$$

where G_F is the Feynman green function. Thus, one can solve the equation of motion as

$$\phi(x) = \int d^n z G_F(x, z) J(z). \quad (5.7)$$

The fundamental object in QFT is the vacuum-to-vacuum transition amplitude $Z[j] = \langle \text{out} | \text{in} \rangle$, *i.e.* the amplitude for the system to go from the initial vacuum $|\text{in}\rangle$ in the infinite past to the final vacuum $\langle \text{out} |$ in the infinite future. The vacuum-to-vacuum transition amplitude is defined by

$$Z[J] = \int \mathcal{D}\phi \exp(iS[\phi]/\hbar). \quad (5.8)$$

One can use the Schwinger variational principle to examine the variation of $Z[0]$ [55]

$$\delta \langle \text{out} | \text{in} \rangle = \frac{i}{\hbar} \delta \langle \text{out} | \delta S | \text{in} \rangle. \quad (5.9)$$

Introducing the notation $Z[j] = \langle \text{out} | \text{in} \rangle = \exp\left(\frac{i}{\hbar} W[j]\right)$, we find that

$$\begin{aligned} \frac{\delta}{\delta J} \langle \text{out} | \text{in} \rangle &= \frac{i}{\hbar} \langle \text{out} | \phi | \text{in} \rangle \\ &= \frac{i}{\hbar} \frac{\delta W[J]}{\delta J} \exp\left(\frac{i}{\hbar} W[J]\right), \end{aligned} \quad (5.10)$$

so,

$$\begin{aligned} \langle \phi \rangle &\equiv \frac{\langle \text{out} | \phi | \text{in} \rangle}{\langle \text{out} | \text{in} \rangle} \\ &= \frac{\delta W[J]}{\delta J} = \int d^n z G_F(x, z) J(z). \end{aligned} \quad (5.11)$$

Integrating this equation we arrive at

$$W[J] = \frac{1}{2} \int d^n x d^n x' J(x) G_F(x, x') J(x') + W[0], \quad (5.12)$$

with $W[0]$ independent of the source J . Applying the Schwinger action principle, Eq. (5.9), with respect to $A(x, x')$, one can prove that [54, 55]

$$W[0] = \frac{i\hbar}{2} \log \det \left(l^2 A(x, x') \right), \quad (5.13)$$

where we have introduced an arbitrary constant l^2 to keep the argument of the logarithm dimensionless. Now, if we define a background field $\bar{\phi}$ by

$$\bar{\phi} = \langle \phi \rangle = \left. \frac{\delta W[J]}{\delta J} \right|_{J_{\bar{\phi}}},$$

and perform the Legendre transform of $W[J]$,

$$\Gamma[\bar{\phi}] = W[J_{\bar{\phi}}] - J_{\bar{\phi}} \bar{\phi}, \quad (5.14)$$

one can arrive at [55]

$$\Gamma[\bar{\phi}] = S[\bar{\phi}]_{J=0} + \frac{i\hbar}{2} \log \det (l^2 A(x, x')). \quad (5.15)$$

The $\Gamma[\bar{\phi}]$ is referred as the *effective action*². Because the field equation for $\bar{\phi}$ include quantum corrections to the classical field equation, we can recognise $\Gamma(\bar{\phi})$ as being comprised of the classical action plus all of its quantum corrections³ [55]. The variation derivative of $\Gamma(\bar{\phi})$ with respect $\bar{\phi}$ is

$$\frac{\delta \Gamma[\bar{\phi}]}{\delta \bar{\phi}} = \int d^n x \left[\frac{\delta J_{\bar{\phi}}}{\delta \bar{\phi}} \frac{\delta W[J_{\bar{\phi}}]}{\delta J_{\bar{\phi}}} - \frac{\delta J_{\bar{\phi}}}{\delta \bar{\phi}} \bar{\phi} \right] - J_{\bar{\phi}} = -J_{\bar{\phi}}. \quad (5.16)$$

Thus, in the absence of sources, the effective action is an extremum

$$\frac{\delta \Gamma[\bar{\phi}]}{\delta \bar{\phi}} = 0. \quad (5.17)$$

This may be compared with the classical field equations, which are obtained by requiring that the action be stationary. Therefore, Eq. (5.17) may be regarded as the equation of motion for the external field $\bar{\phi}$, taking quantum correction into account [171].

Let us obtain an alternative expression to the effective action, which later will be very useful for calculating thermal properties in QFT. Consider $W[0]$ over a small change the operator A to $A + \delta A$, then the change in $W[0]$, to first order in δA , is

$$\begin{aligned} \delta W &= \frac{i\hbar}{2} \delta \log \det (l^2 A) \\ &= \frac{i\hbar}{2} \delta \text{tr} \log (l^2 A) \\ &= \frac{i\hbar}{2} \text{tr}(\delta A) A^{-1}, \end{aligned} \quad (5.18)$$

where in the second line we use that $\log \det = \text{tr} \log$. Assuming the A have a small negative imaginary part by virtue of Feynman boundary conditions, we may therefore write [54]

$$A^{-1} = - \int_0^\infty ds e^{-isA}. \quad (5.19)$$

Hence, Eq. (5.18) can be rewritten as

$$\delta W = - \frac{i\hbar}{2} \delta \left(\int_0^\infty \frac{ds}{s} \text{tr} e^{-isA} \right), \quad (5.20)$$

²Because $\Gamma[\bar{\phi}] = S[\bar{\phi}]_{J=0} + W[0]$, is also usual to refer to $W[0]$ as effective action.

³The effective potential $V(\langle\phi\rangle)$ is defined as (minus) the effective action per unit volume for constant expectation value $\langle\phi\rangle$.

so,

$$W = -\frac{i\hbar}{2} \int_0^\infty \frac{ds}{s} \text{tr} e^{-isA}. \quad (5.21)$$

In a Euclidean space-time, many of the operators appearing in QFT become elliptic rather than hyperbolic. From the mathematical point of view, elliptic operators on Riemannian manifolds have the desirable property of having a spectrum bounded from below. In the following, we are interested in thermal properties of systems where the background fields are static. Consequently, we will work in a Riemannian space-time. Because the action does not explicitly depend on time, always exists a local coordinate system in which the metric is time-independent with $(t = -i\tau)$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00}(\mathbf{x}) d\tau^2 + g_{ij}(\mathbf{x}) dx^i dx^j. \quad (5.22)$$

In Riemannian space we define W by

$$Z[j] = \exp\left(-\frac{1}{\hbar} W[j]\right) = \int \mathcal{D}\phi \exp(-S_E[\phi]/\hbar), \quad (5.23)$$

where S_E is the action in the Euclidean space-time⁴. Similar to what was done before one can prove that

$$W[0] = \frac{\hbar}{2} \log \det (l^2 A_E(x, x')) = -\frac{\hbar}{2} \int_0^\infty \frac{ds}{s} \text{tr} e^{-sA_E}. \quad (5.24)$$

Now we will show that $\text{tr} e^{-sA_E}$ can be expressed through the trace of the so-called *heat-kernel*⁵. Give a operator $\hat{A}_E$ with eigenvalues λ_n and a complete set of eigenvectors $|\psi_n\rangle$,

$$e^{-s\hat{A}_E} = \sum_n e^{-s\lambda_n} |\psi_n\rangle \langle \psi_n|. \quad (5.25)$$

Thus, if $K(s) \equiv e^{-s\hat{A}_E}$ we obtain

$$\left(\frac{\partial}{\partial s} + \hat{A}_E\right) K(s) = 0, \quad (5.26)$$

and

$$K(s \rightarrow 0) = \sum_n |\psi_n\rangle \langle \psi_n| = \hat{1}, \quad (5.27)$$

which is the definition of the heat kernel, Eqs. (H.3) and (H.4), in operator form. Then, its trace is

$$\begin{aligned} \text{tr} K(s) &= \int d^d x |g|^{1/2} \text{tr}' K(s|x, x) \\ &= \text{tr} e^{-s\hat{A}_E} = \text{tr} \sum_n e^{-s\lambda_n}, \end{aligned} \quad (5.28)$$

⁴Here, we are abusing the language when referring to the action or the space-time as “Euclidean”; it is in fact curved.

⁵In the Appendix H, we some basic definitions of the heat kernel

where in the first line we used a coordinate basis, *i.e.* $K(s|x, x') = \langle x|K(s)|x'\rangle$, the tr' refers to a trace on other indices, such spinorial indices, we omit the $(')$ hereinafter. Finally, Eq. (5.24) can be written as

$$W = -\frac{\hbar}{2} \int d^d x |g|^{1/2} \int_0^\infty \frac{ds}{s} \text{tr} K(s|x, x). \quad (5.29)$$

5.3 One-Loop Effective Action at Finite Temperature and Chemical Potential

In the imaginary time formalism [172], it is well known that a QFT in a Euclidean space-time, which is periodic (or antiperiodic) in imaginary time τ with a period $\beta = 1/T$, amounts to a thermal field theory with temperature T . Thus, the problem of finding *Grand canonical potential* Ω^β , and so all thermodynamic properties, of the system in equilibrium reduces to the computation of the effective action W , in this Euclidean QFT [54].

The partition function is defined by

$$Z = e^{\beta\Omega(T, \mu)} = \text{Tr} e^{-\beta(H - \mu N)} \quad (5.30)$$

where H is the Hamiltonian, μ is the chemical potential and N is the number operator. One can write the Feynman path integral formula for the trace in Eq. (5.30) and perform the momentum integrations. The result is [172]

$$Z = \oint \mathcal{D}\phi \exp \left(\int_0^\beta d\tau \int d^{d-1}x |g|^{1/2} [\mathcal{L}_E + \mu \mathcal{N}] \right), \quad (5.31)$$

with $N = \int d^{d-1}x |g|^{1/2} \mathcal{N}$, $\mathcal{L}_E = L(t = -i\tau)$ and $\mathcal{L}$ the Lagrangian density. The symbol $\oint$ means that the integration over the field is constrained in such a way that $\phi(\mathbf{x}, 0) = \pm \phi(\mathbf{x}, \beta)$, with $(+)$ for bosons and $(-)$ for fermions. The Grand canonical potential is⁶ ($\hbar = 1$)

$$\begin{aligned} \Omega(T, \mu) &= -\frac{1}{\beta} \log Z = \frac{1}{\beta} W_{\beta, \mu} \\ &= -\frac{1}{2\beta} \int d^d x |g|^{1/2} \int_0^\infty \frac{ds}{s} K_{\beta, \mu}(s|x, x). \end{aligned} \quad (5.33)$$

where we have used Eq. (5.23) in the first line, and Eq. (5.29) in the second line. Here, $K_{\beta, \mu}(s)$ is the heat kernel with the boundary condition (for bosons) [52, 173]

$$K_{\beta, \mu}(s|\tau, \mathbf{x}; \tau' \mathbf{x}') = e^{-\mu n \beta} K_{\beta, \mu}(s|\tau, \mathbf{x}; \tau' + n\beta, \mathbf{x}'), \quad (5.34)$$

with $n \in \mathbb{Z}$.

⁶In the following, we only work with ultrastatic spacetimes

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = d\tau^2 + g_{ij}(\mathbf{x}) dx^i dx^j. \quad (5.32)$$

5.3.1 Scalar Field

Let us recall the equation of motion of a “free” scalar particle on a curved submanifold, Eq. (C.11)

$$\frac{\hbar^2}{2|g|^{1/2}} \left[\bar{\nabla}_\mu g^{\mu\nu} |g|^{1/2} \bar{\nabla}_\nu \right] \vec{\phi}(x) + \left(\frac{\bar{m}^2 c^2}{2} + V_G \right) \vec{\phi}(x) = 0, \quad (5.35)$$

where $\bar{\nabla}_\mu = (\partial_\mu - i \frac{\mathcal{B}_\mu}{\hbar})$. Here, the metric tensor g and the field $\mathcal{B}_\mu$ will be considered as background fields⁷. It is convenient to write the geometric potential in the form

$$V_G(x) \equiv \frac{1}{2}(\xi R(x) - P(x)), \quad (5.36)$$

whereby, we can introduce an operator $F(\bar{\nabla})$ defined by (in what follows, we use for simplicity, $\hbar = 1$ and $c = 1$)

$$\begin{aligned} F(\bar{\nabla}) \vec{\phi}(x) &= (\square - \bar{m}^2 + P(x) - \xi R) \vec{\phi}(x) = 0 \\ &= (\partial_\tau^2 + \square_{\mathbf{x}} - \bar{m}^2 + P(x) - \xi R) \vec{\phi}(x) = 0. \end{aligned} \quad (5.37)$$

We have used that the metric is ultrastatic so that $\square = \partial_\tau^2 + \square_{\mathbf{x}}$. The thermal heat kernel at finite chemical potential $K_{\beta,\mu}$ can be expressed as an infinite sum of zero temperature vacuum heat kernels $(K(s|x, x'))$ ⁸ [52, 174]

$$K_{\beta,\mu}(s|\tau, \mathbf{x}; \tau', \mathbf{x}') = \sum_{n=-\infty}^{\infty} e^{-\mu n \beta} K(s|\tau, \mathbf{x}; \tau' + \beta n, \mathbf{x}'), \quad (5.38)$$

It is convenient to factorize the zero-temperature heat kernel according to [174]

$$\begin{aligned} K(s|\tau, \mathbf{x}; \tau', \mathbf{x}') &= \mathcal{K}(s|\tau, \tau') K^{d-1}(s|\mathbf{x}, \mathbf{x}') \\ &= \frac{1}{(4\pi s)^{1/2}} \exp\left(\frac{-(\tau - \tau')^2}{4s}\right) K^{d-1}(s|\mathbf{x}, \mathbf{x}'), \end{aligned} \quad (5.39)$$

where $\mathcal{K}(s|\tau, \tau')$ is the temporal part and $K^{d-1}(s|\mathbf{x}, \mathbf{x}')$ is the spatial part of K . In the second line of (5.39), we have used the solution of

$$\left(\frac{\partial}{\partial s} - \frac{\partial^2}{\partial \tau^2} \right) \mathcal{K}(s|\tau, \tau') = 0. \quad (5.40)$$

The trace of the $K_{\beta,\mu}$ can be rewritten using the *Jacobi theta functions*

$$\theta_3(a, b) = \sum_{n=-\infty}^{\infty} e^{2ina} b^{n^2}, \quad (5.41)$$

⁷Below the field $\mathcal{B}_\mu$ can be identified as the geometrically induced Yang-Mills field as well as an external gauge field.

⁸ $K(s|x, x')$ is solution of $[\frac{d}{ds} - F(\bar{\nabla})] K(s|x, x') = 0$ with $K(s \rightarrow 0|x, x') = \delta(x, x')$.

so,

$$\mathrm{tr} K_{\beta, \mu}(s) = \mathrm{tr} \sum_{n=-\infty}^{\infty} \frac{1}{(4\pi s)^{1/2}} e^{-\mu n \beta} \int_0^{\beta} d\tau e^{\frac{-\beta^2 n^2}{4s}} \int d^{d-1} x |g|^{1/2} K^{d-1}(s|\mathbf{x}, \mathbf{x}) \quad (5.42)$$

$$= \frac{\beta}{(4\pi s)^{1/2}} \theta_3 \left(\frac{i\mu\beta}{2}, e^{\frac{-\beta^2}{4s}} \right) \int d^{d-1} x |g|^{1/2} \mathrm{tr} K^{d-1}(s|\mathbf{x}, \mathbf{x}). \quad (5.43)$$

Using

$$\theta_3(a, b) = 1 + 2 \sum_{n=1}^{\infty} b^{n^2} \cos(2na), \quad (5.44)$$

we arrive at

$$\mathrm{tr} K_{\beta, \mu}(s) = \frac{\beta}{(4\pi s)^{1/2}} \left[1 + 2 \sum_{n=1}^{\infty} e^{\frac{-\beta^2 n^2}{4s}} \cosh(\mu\beta n) \right] \int d^{d-1} x |g|^{1/2} \mathrm{tr} K^{d-1}(s|\mathbf{x}, \mathbf{x}). \quad (5.45)$$

It is well-known that the ultraviolet divergences are temperature independent⁹. Thus, the divergent part of the trace comes exclusively from the zero-temperature contribution. Moreover, because we are only interested in the thermal properties, it is possible to drop the vacuum mode ($n = 0$) in the infinite sum, which corresponds to the zero temperature effective action. Thus

$$\begin{aligned} \bar{\Omega}_B(T, \mu) &= \Omega_B(T, \mu) - \Omega_B(T = 0, \mu) \\ &= - \int d^{d-1} x |g|^{1/2} \left[\int_0^{\infty} \frac{ds}{s} \frac{1}{(4\pi s)^{1/2}} \sum_{n=1}^{\infty} e^{\frac{-\beta^2 n^2}{4s}} \cosh(\mu\beta n) \mathrm{tr} K^{d-1}(s|\mathbf{x}, \mathbf{x}) \right], \end{aligned} \quad (5.46)$$

is the temperature dependent part, which is ultraviolet finite. Using the asymptotic expansion (H.6), which is referred to as the *Seely-DeWitt expansion*, we arrive at¹⁰

$$\bar{\Omega}_B(T, \mu) = - \int d^{d-1} x |g|^{1/2} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \cosh(\mu\beta n) \mathrm{tr} a_l(\mathbf{x}) \left[\int_0^{\infty} ds \frac{s^{l-1-\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}} e^{\frac{-\beta^2 n^2}{4s}} e^{-m^2 s} \right]. \quad (5.47)$$

To compute $\bar{\Omega}_B$ we need to integrate from $s = 0$ to $s = \infty$. Since the asymptotic expansion will be truncated in some N , one could argue that it should not be used to calculate $\bar{\Omega}_B$ ¹¹. However, let us note that at least for massive theories the main contributions to the integral in s comes from values of s near to 0. Moreover, noting that s has dimension x^2 , small values of s correspond to short distances (local effects) [53]. Thus, although the Seely-DeWitt expansion might fail to capture the infrared (large s) regime for massless

⁹Nevertheless, infrared divergences may appear at finite temperature.

¹⁰With the help of Eq. (H.2), it is easy to show that the inclusion of a mass term only leads to an exponential factor in the heat kernel.

¹¹Besides strictly the asymptotic expansion is only defined for s near 0.

theories (non-local effects) [53, 175, 176], this expansion capture properly the ultraviolet behaviour of QFT [175, 177].

Alternatively to the Seely-DeWitt expansion we could use the so-called *covariant perturbation theory*, which is an expansion in the curvatures¹² [175, 176, 178, 179]. Although this expansion is valid uniformly for all s , and hence is appropriate for the infrared regime, the expansion is valid only in an asymptotically flat Riemannian spacetime [175, 176]. Furthermore, the calculations beyond the first order are considerably more complicated. Notably, up to first order the two expansions match, so at least until these order the calculations are justified, even for massless theories. In what follows, we present an expression to all orders using the Seely-DeWitt expansion, and then work with this expression only first order.

In Eq. (5.47) the integration with respect to S may be performed at once giving

$$\int_0^\infty ds \frac{s^{l-1-\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}} e^{\frac{-\beta^2 n^2}{4s}} e^{-m^2 s} = \frac{2^{-\frac{d}{2}-l+1}}{\pi^{\frac{d}{2}}} \left(\frac{m}{\beta n}\right)^{\frac{d}{2}-l} K_{\frac{d}{2}-l}[mn\beta], \quad (5.48)$$

where $K_\alpha[z]$ are the modified Bessel function of the second kind. Thus, the Grand canonical potential at finite temperature and chemical potential is

$$\bar{\Omega}_B(T, \mu) = - \int d^{d-1}x |g|^{1/2} \sum_{n=1}^\infty \sum_{l=0}^\infty \frac{2^{-\frac{d}{2}-l+1}}{\pi^{\frac{d}{2}}} \left(\frac{m}{\beta n}\right)^{\frac{d}{2}-l} K_{\frac{d}{2}-l}[mn\beta] \cosh(\mu\beta n) \text{tr } a_l(\mathbf{x}). \quad (5.49)$$

Then, up to first-order the Grand canonical potential is

$$\begin{aligned} \bar{\Omega}_B(T, \mu) = - \int d^{d-1}x |g|^{1/2} \sum_{n=1}^\infty \left[\frac{m^{\frac{d}{2}} K_{\frac{d}{2}}[mn\beta]}{2^{\frac{d}{2}-1} n^{\frac{d}{2}} \pi^{\frac{d}{2}} \beta^{d/2}} \text{tr } \hat{1} \right. \\ \left. + \frac{m^{\frac{d}{2}-1} K_{\frac{d}{2}-1}[mn\beta]}{n^{\frac{d}{2}-1} (2\pi)^{\frac{d}{2}} \beta^{\frac{d}{2}-1}} \text{tr } \left(P + \left(\frac{1}{6} - \xi\right) R \right) \right] \cosh(\mu\beta n). \quad (5.50) \end{aligned}$$

The zero-order term is equivalent to the contribution of a free boson gas in d -dimensions at finite temperature and chemical potential. In Appendix I, we present an alternative derivation for this contribution (see Eq. (I.8)). The first-order term in (5.50) takes into account the effects of the curvatures. In the case of a submanifold, it is sufficient to note that $P - (\xi - 1/6)R = -2V_G + R/6$ (see Eq. (5.36)), to identify the first contribution of the geometric potential. The contributions of the geometrically induced Yang-Mills field (or an external gauge field) $\mathcal{B}_\mu$ will appear only in the second-order terms (see Eq. H.8).

Let us now consider some limit cases. For $d = 4$, Eq. (5.50) yields

$$\bar{\Omega}_B^{d=4}(T, \mu) = - \int d^3x |g|^{1/2} \sum_{n=1}^\infty \left[\frac{m^2 K_2[mn\beta]}{2n^2 \pi^2 \beta^2} \text{tr } \hat{1} + \frac{m K_1[mn\beta]}{4n \pi^2 \beta} \text{tr } \mathcal{V} \right] \cosh(\mu\beta n)$$

¹²By curvatures here we mean $R^{\mu,\nu}$, P and $\Omega^{\mu\nu}$.

where $\mathcal{V} = \left(P + \left(\frac{1}{6} - \xi\right) R\right)$. This expression is in agreement with Ref. [180]¹³.

In the massless case, and remembering that $|\mu| \leq m$, the series can be exactly summed. Thus, Eq. (5.50) simplifies to

$$\bar{\Omega}_B(T, 0)^{m=0} = - \int d^{d-1}x |g|^{1/2} \left[\frac{\zeta[d]}{\pi^{\frac{d}{2}} \beta^d} \Gamma\left[\frac{d}{2}\right] \text{tr } \hat{1} + \frac{\zeta[d-2]}{4\beta^{d-2} \pi^{\frac{d}{2}}} \Gamma\left(\frac{d-2}{2}\right) \text{tr } \mathcal{V} \right], \quad (5.51)$$

where $\zeta[x]$ is the Riemann-zeta function. For $d = 2$ the second term in Eq. (5.51) is (infrared) divergent. This had been noticed before (see. Ref. [175, 179])¹⁴. For $d = 4$ Eq. (5.51) simplifies to

$$\bar{\Omega}_B(T, 0)^{m=0, d=4} = - \int d^3x |g|^{1/2} \left[\frac{\pi^2}{90\beta^4} \text{tr } \hat{1} + \frac{1}{24\beta^2} \text{tr } \mathcal{V} \right], \quad (5.52)$$

in agreement with [175].

5.3.2 Spinor Field

Consider a spinor field governed by the equation

$$D\psi(x) = \left[i\hbar \underline{\gamma}^\mu(x) (\partial_\mu - \Omega_\mu - i\mathcal{G}_\mu) - m \right] \psi(x) = 0, \quad (5.53)$$

where the $\underline{\gamma}^A$ are the space-dependent γ -matrices, Ω_μ the spinorial affine connections and $\mathcal{G}$ is a Yang-Mills field, which may be related to the torsion or simply be an external field¹⁵. Again, the space-time is ultrastatic, so, the equation is time-independent. One can show, analogously to what has been done before, that the effective action for spinor fields is given by (see sec. 5.7 in ref. [54])

$$W[0] = -i\hbar \log \det(lD). \quad (5.54)$$

Let us define

$$\tilde{D} = i\hbar \underline{\gamma}^\mu(x) (\partial_\mu - \Omega_\mu - i\mathcal{G}_\mu) + m, \quad (5.55)$$

and also a matrix γ_* such that $\{\gamma^\mu, \gamma_*\} = 0$ and $\gamma_*^2 = 1$. The latter is always possible for irreducible representations in an even-dimensional space-time. If the space-time is odd-dimensional, it is assumed that we are working on a non-reducible representation¹⁶. Using

¹³However we noticed a discrepancy with Eq. (59) in [176].

¹⁴ While the Seely-DeWitt expansion suffers infrared divergences in the massless case, the covariant perturbation theory remains (infrared) finite for $d > 2$ [175].

¹⁵As was discussed before, in the case of a submanifold, Eq. (5.53) is exact to first-order in ϵ . However, this equation loses some information about extrinsic effects which will be ignored from now on.

¹⁶In fact this is the case of physical interest. For example, in a system such as graphene, where the space-time is three dimensional, one works in a non-reducible representation due to the presence of the two valleys.

γ_* , it is straightforward to prove that

$$\begin{aligned}\det(lD) &= \det(lD\gamma_*^2) \\ &= \det(\gamma_* l\tilde{D}\gamma_*) \\ &= \det(\tilde{D}).\end{aligned}\tag{5.56}$$

Therefore, the effective action can be rewritten as [54, 181]

$$\begin{aligned}W[0] &= -\frac{i}{2}\hbar \log \det(lD)^2 \\ &= -\frac{i}{2}\hbar \log \det(l^2 D\tilde{D}).\end{aligned}\tag{5.57}$$

Now, with $D = \underline{\gamma}^\nu \bar{\nabla}_\mu$, we obtain

$$\begin{aligned}\underline{\gamma}^\mu \bar{\nabla}_\mu \underline{\gamma}^\nu \bar{\nabla}_\nu &= \underline{\gamma}^\mu \underline{\gamma}^\nu \bar{\nabla}_\mu \bar{\nabla}_\nu \\ &= \frac{1}{2}(\{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} + [\underline{\gamma}^\mu, \underline{\gamma}^\nu]) \frac{1}{2}(\{\bar{\nabla}_\mu, \bar{\nabla}_\nu\} + [\bar{\nabla}_\mu, \bar{\nabla}_\nu]) \\ &= g^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu + \frac{1}{4}[\underline{\gamma}^\mu, \underline{\gamma}^\nu][\bar{\nabla}_\mu, \bar{\nabla}_\nu],\end{aligned}\tag{5.58}$$

where in the first line we used that $\bar{\nabla}_\mu \underline{\gamma}^\nu = 0$ [54]. Using that [54]

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu] = -\frac{1}{8}R_{\mu\nu\alpha\beta}[\underline{\gamma}^\alpha, \underline{\gamma}^\beta] + i\Omega_{\mu\nu},\tag{5.59}$$

and

$$\underline{\gamma}^\mu \underline{\gamma}^\nu \underline{\gamma}^\alpha \underline{\gamma}^\beta R_{\mu\nu\alpha\beta} = -2R,\tag{5.60}$$

where $\Omega_{\mu\nu} = \partial_\mu \mathcal{G}_\nu - \partial_\nu \mathcal{G}_\mu - [\mathcal{G}_\mu, \mathcal{G}_\nu]$, we arrive at

$$D\tilde{D} = g^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu + \frac{R}{4} + \frac{1}{2}\Omega_{\mu\nu}\sigma^{\mu\nu} + m^2,\tag{5.61}$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\underline{\gamma}^\mu, \underline{\gamma}^\nu]$. The above equation has the same form that Eq. (5.37). Then, we can proceed as before, but now taking into account anti-periodic boundary conditions, which lead to the image sum [52, 175, 176, 181]

$$K_{\beta,\mu}^F(s|\tau, \mathbf{x}; \tau' + \beta n, \mathbf{x}') = \sum_{n=-\infty}^{\infty} (-1)^n e^{\mu n \beta} K(s|\tau, \mathbf{x}; \tau' + \beta n, \mathbf{x}').\tag{5.62}$$

Thus, the trace of the $K_{\beta,\mu}^F$ is given by

$$\begin{aligned}\text{tr} K_{\beta,\mu}(s) &= \text{tr} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(4\pi s)^{1/2}} e^{\mu n \beta} \int_0^\beta d\tau e^{\frac{-\beta^2 n^2}{4s}} \int d^{d-1}x |g|^{1/2} K^{d-1}(s|\mathbf{x}, \mathbf{x}) \\ &= \frac{\beta}{(4\pi s)^{1/2}} \theta_4\left(\frac{i\mu\beta}{2}, e^{\frac{-\beta^2}{4s}}\right) \int d^{d-1}x |g|^{1/2} \text{tr} K^{d-1}(s|\mathbf{x}, \mathbf{x}) \\ &= \frac{\beta}{(4\pi s)^{1/2}} \left[1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{\frac{-\beta^2 n^2}{4s}} \cosh(\mu\beta n)\right] \int d^{d-1}x |g|^{1/2} \text{tr} K^{d-1}(s|\mathbf{x}, \mathbf{x}),\end{aligned}\tag{5.63}$$

where we have introduced the *Jacobi theta functions* θ_4 ,

$$\theta_4(a, b) = \sum_{n=-\infty}^{\infty} (-1)^n e^{2ina} b^{n^2} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n b^{n^2} \cos(2na). \quad (5.64)$$

As before, we are concerned only with the temperature dependent part, so, the Grand canonical potential is given by

$$\begin{aligned} \bar{\Omega}_F(T, \mu) &= \Omega_F(T, \mu) - \Omega_F(T = 0, \mu) \\ &= \int d^{d-1}x |g|^{1/2} \left[\int_0^\infty \frac{ds}{s} \frac{1}{(4\pi s)^{1/2}} \sum_{n=1}^{\infty} (-1)^n e^{-\frac{\beta^2 n^2}{4s}} \cosh(\mu\beta n) \text{tr} K^{d-1}(s|\mathbf{x}, \mathbf{x}) \right]. \end{aligned} \quad (5.65)$$

Introducing the Seely-DeWitt expansion and proceeding as before (from Eq. (5.47) to Eq. (5.49)), we arrived at

$$\bar{\Omega}_F(T, \mu) = \int d^{d-1}x |g|^{1/2} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (-1)^n \frac{2^{-\frac{d}{2}-l+1}}{\pi^{\frac{d}{2}}} \left(\frac{m}{\beta n} \right)^{\frac{d}{2}-l} K_{\frac{d}{2}-l}[mn\beta] \cosh(\mu\beta n) \text{tr} a_l(\mathbf{x}), \quad (5.66)$$

Then, up to first-order the Grand canonical potential is

$$\begin{aligned} \bar{\Omega}_F(T, \mu) &= \int d^{d-1}x |g|^{1/2} \sum_{n=1}^{\infty} (-1)^n \left[\frac{m^{\frac{d}{2}} K_{\frac{d}{2}}[mn\beta]}{2^{\frac{d}{2}-1} n^{\frac{d}{2}} \pi^{\frac{d}{2}} \beta^{d/2}} \text{tr} \hat{1} \right. \\ &\quad \left. + \frac{m^{\frac{d}{2}-1} K_{\frac{d}{2}-1}[mn\beta]}{n^{\frac{d}{2}-1} (2\pi)^{\frac{d}{2}} \beta^{\frac{d}{2}-1}} \text{tr} \mathcal{V}^F \right] \cosh(\mu\beta n), \end{aligned} \quad (5.67)$$

with $\mathcal{V}^F = \left(-\frac{1}{12}R - \frac{1}{2}\Omega_{\mu\nu}\sigma^{\mu\nu}\right)$. The zero-order term is equivalent to the contribution of a free fermion gas in d -dimensions at finite temperature and chemical potential, see Appendix I. The first-order term takes into account the effect of the intrinsic curvature (R) and the geometrically induced Yang-Mills field (or an external gauge field). The latter in contrast to the bosonic case where these contributions appear only second-order.

In the massless case the series can be summed fully¹⁷,

$$\begin{aligned} \bar{\Omega}_F^{m=0}(T, \mu) &= \int d^{d-1}x |g|^{1/2} \left[\frac{1}{2} \pi^{-\frac{d}{2}} \beta^{-d} \Gamma\left(\frac{d}{2}\right) \left[\text{Li}_d(-e^{-\beta\mu}) + \text{Li}_d(-e^{\beta\mu}) \right] \text{tr} \hat{1} \right. \\ &\quad \left. + \frac{2^{3-\frac{d}{2}}}{\pi} \beta^{2-d} \Gamma\left(\frac{d-2}{2}\right) \left[\text{Li}_{d-2}(-e^{-\beta\mu}) + \text{Li}_{d-2}(-e^{\beta\mu}) \right] \text{tr} \mathcal{V}^F \right]. \end{aligned} \quad (5.68)$$

It should be noted that for even dimension $\left(\text{Li}_d(-e^{-\beta\mu}) + \text{Li}_d(-e^{\beta\mu})\right)$ is a real polynomial function of β and μ .

¹⁷Unlike the bosonic case there is no restriction on the chemical potential

CHAPTER 6

INTERACTING QUANTUM FIELD THEORY ON A SUBMANIFOLD

*"...in five minutes you will say that it is all so
absurdly simple."*

— Arthur Conan Doyle. *Sherlock Holmes.*
The Adventure of the Dancing Men.

In this chapter we study of interacting quantum field theories on a submanifold. We present the Feynman path integral of a theory where the matter fields (scalars or fermions) are confined to a curved submanifold and the electromagnetic field propagating in the flat higher dimensional manifold. In the particular case of a flat submanifold, we discuss the relationship between $\text{RQED}_{d_\gamma, d_e}$ and the confining potential approach. We show that $\text{RQED}_{d_\gamma, d_e}$ can be obtained from first principles. Some aspects of pseudo-differential operators are discussed. Moreover, we propose a large class of gauge invariant extensions of the Schwinger model with exotic gauge field Lagrangian, in which the interaction between fermions goes beyond the linear potential of the Schwinger model. Employing the Fujikawa's path integral technique, we demonstrate that these extensions are exactly solvable for massless fermions. Since these models can be exactly bosonized, in the quadratic approximation, bosonic degrees of freedom can be integrated out to obtain an effective Lagrangian for the gauge field. We find that the spectrum of the gauge field is not trivially modified. We illustrate our proposal with some examples. In particular, the $\text{RQED}_{4,2}$ which can be identified as an effective field theory for graphene wires.

6.1 QFT on a Curved Submanifold

Having presented a theory of “free” fields on a submanifold, we will formulate the QFT, for scalar and fermions, on a submanifold with external fields propagating in the flat higher dimensional manifold. An inspection of Eqs. (C.11), (4.86), (5.1) and (5.2)

allows us to conclude that the action describing such theory is¹

$$S[\phi] = - \int d^m x \left[\vec{\phi}^*(x) \frac{\hbar^2}{|g|^{1/2}} \left(\partial_\mu - i \frac{\mathcal{B}_\mu}{\hbar} - \frac{ie}{\hbar} \bar{A}_\mu \right) g^{\mu\nu} |g|^{1/2} \left(\partial_\nu - i \frac{\mathcal{B}_\nu}{\hbar} - \frac{ie}{\hbar} \bar{A}_\nu \right) \vec{\phi}(x) \right. \\ \left. + \bar{m}^2 c^2 |\vec{\phi}(x)|^2 + V_G |\vec{\phi}(x)|^2 \right] - \int d^n x \frac{1}{4} F_{AB} F^{AB}, \quad (6.1)$$

and²

$$S[\psi] = \int d^m x \bar{\psi} \left[i \hbar e_a^\mu(x) \gamma^a \left(\partial_\mu - \Omega_\mu - \frac{i}{\hbar} \mathcal{G}_\mu - \frac{ie}{\hbar} \bar{A}_\mu \right) - \bar{m}c - V_{GD} \right] \psi \\ - \int d^n X \frac{1}{4} F_{AB} F^{AB}, \quad (6.2)$$

for scalars and fermions, respectively. The main feature of this theory, regarding theories usually considered, is that integration in action for matter fields is in m dimensions and for the electromagnetic field is in n dimensions. The latter is because the electromagnetic field is not confined to the submanifold and can propagate in n -dimensional space-time.

When the electromagnetic field is an external field, and the interaction between the particles of matter is neglected, one can use some of the methods presented in the last chapter. However, in the most general case where the matter fields and the electromagnetic field do not propagate in the same dimensions, the calculations are more complicated. For the above and also motivated by systems like quantum wires and graphene in the remainder of the thesis, we restrict ourselves to the case of a flat submanifold.

6.2 Reduced Quantum Electrodynamics

In the nineties, Marino [56], motivated by the quantum Hall effect [182] and by the high-temperature superconductivity [183] as two-dimensional phenomena, observed that although the electrons are confined to a plane the electromagnetic field through which they interact is not. He then proposes a theory which is known as *reduced quantum electrodynamics* (RQED_{4,3}) (also called *pseudo quantum electrodynamics*). In this theory, the electrostatic potential between two electrons on a plane is the usual $1/r$ Coulomb potential instead of a logarithmic potential which is distinctive of quantum electrodynamics in $2 + 1$ dimensions (QED₂₊₁).

In general one can define RQED _{d_γ, d_e} as a theory where the photon propagates in d_γ dimensions and the fermion fields are localized on d_e dimensions [56, 57, 60, 68–70]. Let us review the deduction of RQED _{d_γ, d_e} [56] and relate it to the confining potential approach. We start from the Lagrangian density of QED in n dimensions

$$\mathcal{L}_{QED_n} = -\frac{1}{4} F_{AB} F^{AB} - e j_n^B A_B + \mathcal{L}_{GF} + \mathcal{L}_m, \quad (6.3)$$

¹Again Latin indices $A, B, \dots$, running from 0 to $n - 1$, for the coordinates on $\mathcal{N}$ and Greek indices $\mu, \nu, \dots$, running from 0 to $m - 1$, for coordinates on $\mathcal{M}$.

²Let us remember that in our notation, $X^A = (x^\mu, y^i)$ and $\bar{A}_B = A_B(x^\mu, 0)$.

where $\mathcal{L}_m$ is the Lagrangian density for the matter fields, $\mathcal{L}_{GF} = -(\partial_B A^B)/(2\xi)$ is the gauge fixing term and j_n^B the n -dimensional matter current. The generating functional for the photon can be written as

$$Z^{EM} = \int DA_B \exp \left\{ i \int d^n X \left[-\frac{1}{4} F_{AB} F^{AB} - e j_n^B A_B + \mathcal{L}_{GF} \right] \right\}, \quad (6.4)$$

One can obtain an effective interaction of matter integrating over A_B , namely

$$\begin{aligned} Z^{EM} &= N \int DA_B \exp \left\{ i \int d^n X \left(\frac{1}{2} A_A \left[\square G^{AB} - \left(\frac{1}{\xi} - 1 \right) \partial^A \partial^B \right] A_B - e j_n^B A_B \right) \right\} \\ &= N \exp \left\{ i \int d^n X d^n X' \left(\frac{1}{2} j_n^A(X) \mathfrak{G}_{AB}(X - X') j_n^B(X') \right) \right\}. \end{aligned} \quad (6.5)$$

In the first line we perform an integration by parts of Eq. (6.4) and in the second line we used a Gaussian integration (see Ch. 4 in [184]). In Eq. (6.5), $\mathfrak{G}_{AB}(X - X')$ is the n -dimensional photon propagator, which in the momentum space is given by

$$\mathfrak{G}_{AB}(p) = -\frac{1}{p^2} \left[G_{AB} - (1 - \xi) \frac{p_A p_B}{p^2} \right]. \quad (6.6)$$

Due to current conservation, the only term of the photon propagator that contributes to (6.5) is the first one. Now the proposal of RQED is to take the current as ($p = n - m$)

$$j_n^B(x, y) = \begin{cases} j_m^B(x) \delta^p(y), & B = 0, \dots, m-1, \\ 0, & B = m, \dots, n-1, \end{cases} \quad (6.7)$$

where $j_m^\nu(x)$ is the m -dimensional current. Note that for fermionic matter the n -dimensional current is

$$j_n^B = \bar{\Psi}(x, y) \gamma^B \Psi(x, y), \quad (6.8)$$

as a result of $j_n^{B=\mu} = j_m^\mu(x) \delta^p(y)$. Thus, we obtain

$$\Psi(x, y) = \psi(x) \sqrt{\delta^p(y)}, \quad (6.9)$$

where $\psi(x)$ is a spinor that only depends on the coordinates of the submanifold. Eq. (6.9) is an idealization. As we have already discussed, it violates the Heisenberg uncertainty principle. Moreover, the condition $j_n^{B=i} = 0$ implies that

$$\bar{\psi}(x) \gamma^i \psi(x) = 0, \quad (6.10)$$

which is a restriction to the solutions of the submanifold. From first principles, one could apply the confining potential approach up to first-order in ϵ , obtaining that

$$j_n^B(x, y) = j_m^B(x) f(y), \quad (6.11)$$

where $f(y)$ has a maximum in the submanifold³. However, because the submanifold is flat $f(y) = \delta^p(y)$ does not cause problems⁴. Returning to Eq. (6.4) the coupling $j_n^B A_B$ can be rewritten as

$$\begin{aligned} j_n^B A_B &= j_n^\mu(x, y) A_\mu(x, y) + j_n^i(x, y) A_i(x, y) \\ &= j_m^\mu(x) \delta^p(y) A_\mu(x, y) + j_m^i(x) \delta^p A_i(x, y) \\ &= j_m^\mu(x) \delta^p(y) \bar{A}_\mu(x) + j_m^i(x) \delta^p \bar{A}_i(x) \\ &= j_m^\mu(x) \delta^p(y) \bar{A}_\mu(x), \end{aligned} \quad (6.12)$$

with $\bar{A}_B(x) = A_B(x, y = 0)$ and where it has been used that $\bar{A}_i(x)$ can always be removed by a gauge transformation. In conclusion the coupling $j_n^i A_i$ disappears when one uses the confining potential approach up to first-order in ϵ . Hence, Eq. (6.5) can be rewritten as

$$\begin{aligned} Z^{EM} &= N \exp \left\{ i \int d^m x d^p y d^m x' d^p y' \left(\frac{1}{2} j_n^\mu(x, y) \mathfrak{G}_{\mu\nu}(x - x', y - y') j_n^\nu(x', y') \right) \delta(y) \delta(y') \right\} \\ &= N \exp \left\{ i \int d^m x d^m x' \left(\frac{1}{2} j_m^\mu(x) \mathfrak{G}_{\mu\nu}(x - x', 0) j_m^\nu(x') \right) \right\}, \end{aligned} \quad (6.13)$$

where $\mathfrak{G}_{\mu\nu}(x - x', 0)$ represents the Green function in coordinate space with the coordinates y^i taken as zero. The problem now is to find a operator in m dimensions that leads us to the Green function, $\mathfrak{G}_{\mu\nu}(x - x', 0)$. This will be the subject of the next section.

6.3 Dimensional Reduction

Dimensional reduction refers to the approach that enables to build theories in low dimensional space-time starting from higher dimensional theories. Usually this procedure is applied dropping out extra coordinates at the Lagrangian level, such that the differential operator appearing in the Lagrangian, (*i.e.* the kinetic energy), remains unchanged. Consequently, the Green function in momentum space preserves its form, but in configuration space drastically changes [65]. Clearly, the Green function of a differential operator, for instance the D'Alembertian, depend on the number of dimensions of space-time. As a result, the interactions between particles in a dimensionally reduced theory is modified in configuration space [65].

Here, we use an alternative dimensional reduction prescription which changes the Green function in momentum space and the kinetic energy operator in the Lagrangian. However, it preserves the higher dimensional dynamics in the restricted lower dimensional space-time [56, 65]. We will proceed in a slightly more general form than it had been considered before [56, 62–65].

Consider the pseudo-differential operator $L(\square)$, its Green function equation is

$$L(\square)G(X - X'; n, L) = \delta(X - X'), \quad (6.14)$$

³This fact was also pointed out by Nieves in Ref. [66].

⁴It is also assumed that we are only in the ground state of the normal coordinates, since otherwise it would appear to codimension greater than two a induced Zeeman coupling.

where n is the dimension of the space-time, $\square = \partial_{X^0}^2 - \partial_{\mathbf{X}}^2$ and the notation $G(X - X'; n, L)$ highlight the fact that G is the Green function of the operator L . The Green function in momentum space is

$$G(X; n, L) = \frac{1}{(2\pi)^{n/2}} \int d^n K \tilde{G}(K; n, L) e^{-iK_A X^A}. \quad (6.15)$$

Taking the Fourier transform in Minkowski space and using that

$$\delta(X) = \frac{1}{(2\pi)^m} \int d^m K e^{-iK_A X^A}, \quad (6.16)$$

we arrive at

$$\tilde{G}(K; n, L) = \frac{1}{(2\pi)^{n/2}} \frac{1}{L(-(K^0)^2 + \mathbf{K}^2)}. \quad (6.17)$$

Thus, the Green function in configuration space is given by

$$G(X; n, L) = \frac{1}{(2\pi)^n} \int d^n K \frac{e^{iK_A X^A}}{L(-(K^0)^2 + \mathbf{K}^2)}. \quad (6.18)$$

Now, our dimensional reduction prescription is take zero $p = n - m$ coordinates in the Green function. To this end, we separate the coordinates as $X^A = (x^\mu, y^i) = (x^0, \mathbf{x}, \mathbf{y})$ and take $y = 0$. The resulting Green function is

$$G(x, y = 0; n, L) = \frac{1}{(2\pi)^m} \int d^m k \left(\int \frac{d^p k_y}{(2\pi)^{n-m}} \frac{1}{L(-(k^0)^2 + \mathbf{k}^2 + \mathbf{k}_y^2)} \right) e^{ik_\mu x^\mu}, \quad (6.19)$$

where $K^A = (k^\mu, k_y^i) = (k^0, \mathbf{k}, \mathbf{k}_y)$. We may ask if there exist a operator $T(\square)$ in m dimensions such that its Green function is (6.19)? If $T(\square)$ is other differential operator in m dimensions, its Green function satisfies

$$G(x; m, T) = \frac{1}{(2\pi)^m} \int d^m k \frac{e^{ik_\mu x^\mu}}{T(-(k^0)^2 + \mathbf{k}^2)}. \quad (6.20)$$

Then, comparing (6.19) and (6.20) it is clear that T in momentum space is given by

$$T(-(k^0)^2 + \mathbf{k}^2) = \left(\int \frac{d^p k_y}{(2\pi)^{n-m}} \frac{1}{L(-(k^0)^2 + \mathbf{k}^2 + \mathbf{k}_y^2)} \right)^{-1}. \quad (6.21)$$

Although this is a very general relation, unfortunately the integral will not always be well defined and may need some kind of regularization. It should be pointed out that in Eq. (6.21) the ϵ -prescription must be used, *i.e.* $k^2 = (k^0)^2 - \mathbf{k}^2 \rightarrow k^2 + i\epsilon$.

6.3.1 D'Alembertian

In particular, here we are interested in the D'Alembertian operator, which it appears in the usual kinetic term of scalar and vector fields. For the sake of simplicity, let us start with the case of codimension one ($p = 1$)

$$\begin{aligned} T(-(k^0)^2 + \mathbf{k}^2) &= \left(\int \frac{dk_y}{(2\pi)^{n-m}} \frac{1}{(-(k^0)^2 + \mathbf{k}^2 + \mathbf{k}_y^2)} \right)^{-1} \\ &= 2(-(k^0)^2 + \mathbf{k}^2)^{1/2}, \end{aligned} \quad (6.22)$$

which is equivalent to the operator

$$T(\square) = 2(\partial_{x_0}^2 - \partial_{\mathbf{x}}^2)^{1/2} = 2\square^{1/2}, \quad (6.23)$$

in configuration space. When the codimension is greater than one, we can use the hyperspherical coordinates system to the k_y^i momentums. The hyperspherical coordinates system are defined by

$$k_y^i = \rho_y \prod_{j=1}^{p-i} \sin \theta_j \cos \theta_{p-i+1}, \quad i = 1, \dots, p = n - m, \quad (6.24)$$

where $|\mathbf{k}_y| = \rho_y$ and $\theta_p = 0$. After integration in angular coordinates we arrive at

$$\begin{aligned} T(-(k^0)^2 + \mathbf{k}^2) &= \frac{2^{p-1} \Gamma\left(\frac{p}{2}\right)}{\pi^{-\frac{p}{2}}} \left(\int_0^\infty d\rho_y \frac{\rho_y^{p-1}}{(-(k^0)^2 + \mathbf{k}^2 + \rho_y^2)} \right)^{-1} \\ &= \frac{2^p \pi^{\frac{p}{2}}}{\Gamma\left(1 - \frac{p}{2}\right)} (-(k^0)^2 + \mathbf{k}^2)^{1-\frac{p}{2}} \\ &\rightarrow \frac{2^p \pi^{\frac{p}{2}}}{\Gamma\left(1 - \frac{p}{2}\right)} (-(k^0)^2 + \mathbf{k}^2 - i\epsilon)^{1-\frac{p}{2}}, \end{aligned} \quad (6.25)$$

which is only well defined when p is odd, because the gamma function, $\Gamma(z)$, has simple poles in $z = 0, -1, -2, -3, \dots$. Eq (6.25) is equivalent to the operator

$$T(\square) = \frac{2^p \pi^{\frac{p}{2}}}{\Gamma\left(1 - \frac{p}{2}\right)} (\square)^{1-\frac{p}{2}}. \quad (6.26)$$

An alternative way to obtain Eq. (6.25) is compute explicitly the Green's functions of the pseudo-differential operator $\square^\alpha$ in m dimensions and compare these with the Green's functions of the differential operator $\square$ in n dimensions. In the appendix J, we present the Feynman, advanced and retarded Green's functions of theses operators using analytic continuation.

For our purposes, we will be interested in codimension two, so, we perform integration in Eq. (6.25) but now we introduce a regularization function

$$f_r(\rho_y) = e^{-\rho_y^2/\Lambda^2}, \quad (6.27)$$

which depends on a cutoff scale $\Lambda \rightarrow \infty$, thus

$$\begin{aligned} T(-(k^0)^2 + \mathbf{k}^2) &= 2\pi \left(\int_0^\infty d\rho_y \frac{\rho_y e^{-b^2 \rho_y^2}}{(-(k^0)^2 + \mathbf{k}^2 + \rho_y^2)} \right)^{-1} \\ &\approx -\frac{4\pi}{\log\left(\frac{(-(k^0)^2 + \mathbf{k}^2)^2}{\Lambda^2}\right)} \\ &\rightarrow -\frac{4\pi}{\log\left(\frac{(-(k^0)^2 + \mathbf{k}^2 - i\epsilon)^2}{\Lambda^2}\right)} \end{aligned} \quad (6.28)$$

which is equivalent to the operator

$$T(\square) = -\frac{4\pi}{\log\left(\frac{\square}{\Lambda^2}\right)}. \quad (6.29)$$

In conclusion, we find the operator that generates the Green function that appears in Eq. (6.13). Moreover, we can then reverse the sequence of steps that led from (6.4) to (6.5) and get an effective QED on the flat submanifold (which it is none other than RQED [56, 57, 60, 68–70])

$$\mathcal{L}_{RQED} = \begin{cases} -\frac{1}{4} \frac{2^p \pi^{\frac{p}{2}}}{\Gamma(1-\frac{p}{2})} F_{\mu\nu}(\square)^{-\frac{p}{2}} F^{\mu\nu} - e j_m^\mu A_\nu + \mathcal{L}_{GF} + \mathcal{L}_m & p \neq 2, 4, 6, \dots, \\ F_{\mu\nu} \frac{\pi}{\square \log(\frac{\square}{\Lambda^2})} F^{\mu\nu} - e j_m^\mu A_\nu + \mathcal{L}_{GF} + \mathcal{L}_m, & p = 2, \end{cases} \quad (6.30)$$

The advantage of this theory is that the Lagrangian depends only on coordinates on the submanifold. One may then ask what is the corresponding operator for the case of a curved submanifold? The first thing we may notice is that, in this case, the initial operator also depends on the coordinates and not just its derivatives. Then the method used in this section becomes complicated. Naively, one might think that $\square$ should be replaced by the Laplace-Beltrami operator for the metric $g_{\mu\nu}$. However, the Laplace-Beltrami operator carries only intrinsic information of the submanifold, which is insufficient to calculate the real distance between two points of the submanifold⁵, through which the electromagnetic field propagates. On the basis of this argument, the required operator must depend at least $\alpha_{\mu\nu}^k$ and β_μ^{ij} . Nevertheless, we believe that because $\alpha_{\mu\nu}^k$ and β_μ^{ij} are local quantities, it is actually impossible in the general case to build the required operator. In the next section, we will concentrate in a two-dimensional flat submanifold.

6.4 Gauge Invariant Extensions of the Massless and Massive Schwinger Model

The Schwinger model is the two-dimensional quantum electrodynamics (QED₁₊₁) of massless fermions. This model is exactly solvable [71, 185] and suffers from the so-called

⁵The distance measured from the high dimension space $\mathcal{N}$.

axial anomaly [72–74], i.e. the axial-vector current is not conserved at the quantum level [186]. As a result of the anomaly, the chiral symmetry is dynamically broken [74–76], and the photon becomes massive [71]. If a fermion has mass, the chiral symmetry is violated by the mass term, and the axial-vector current fails to be conserved even at the classical level. Although the massive Schwinger model is no longer exactly solvable (unless the electric charge goes to zero, i.e. in the free Dirac theory), many features can be studied by using bosonization [77–79, 81, 187]. Bosonization is a highly useful method in which a fermionic theory is mapped into a bosonic one. [78, 85, 86].

Since QED_{1+1} exhibits many analogies with four-dimensional quantum chromodynamics (QCD_{3+1}), such as chiral symmetry breaking, axial anomaly, instantons and topological θ -vacuum, it has been widely used as a toy model to study screening and confinement [78–83]. As is well known, one obtains screening in the massless case, but confinement in the massive particle case. Schwinger model has also been studied in the context of condensed matter [84]. Recently, an effective field theory (EFT) was employed to study the quantum electrodynamics of electrons and holes in a thin graphene wire [188]. It was concluded that the low-energy dynamics of electrons and holes is well described by the (one-flavor) Schwinger model [188]. However, there are three remarks for which the Schwinger model cannot describe graphene wires:

- (i) The electric charge in QED_{1+1} is a dimensional parameter, while the real electric charge, the physical one, is a dimensionless parameter.
- (ii) The electrons in a graphene wire interact through the usual $1/r$ potential, inherited from QED_{3+1} , in contrast with QED_{1+1} for which the (classical) potential is linear.
- (iii) The Hamiltonian of graphene, ignoring real spin, consists of 4×4 matrices taking into account both two sublattices and two valleys, so, one can expect that the resulting Hamiltonian also consists of 4×4 matrices. Below, we will see that this leads to a theory of two flavors.

We may ask if there exist a gauge invariant generalization of the Schwinger model within which the fermions interact via a wide range of potentials beyond a purely linear potential. One possibility is to modify the kinetic term in the gauge field Lagrangian

$$\mathcal{L}_G = -\frac{1}{4}F_{\mu\nu}f(\square)F^{\mu\nu} - ej^\mu A_\mu, \quad (6.31)$$

since this changes the “Maxwell’s equations”,

$$\partial_\mu f(\square)F^{\mu\nu} = ej^\nu, \quad (6.32)$$

where $f(\square)$ can be a pseudo-differential operator⁶. Within this family of extensions, we have several well-known particular cases, such as the Lee-Wick QED [189], Podolsky electrodynamics [190, 191] and also the *reduced quantum electrodynamics* ($\text{RQED}_{d_\gamma, d_e}$) [56, 57, 60, 68–70].

⁶Because the pseudo-differential operator is defined via the Fourier transform (see sec. 6.3), the property $\partial_\mu f(\square)F^{\mu\nu} = f(\square)\partial_\mu F^{\mu\nu}$ is satisfied.

In what follows we will study what happens with this type of Lagrangian quantum level⁷.

6.4.1 Massless Generalization of the Schwinger Model

The generating functional of N_f -flavor massless Schwinger model with the exotic kinetic term for the gauge field, Eq. (6.31), is

$$Z_{TOT} = \int DA_\mu D\bar{\psi} D\psi \exp \left(i \int d^2x \left[\sum_{j=1}^{N_f} \overbrace{i\bar{\psi}_j(\not{\partial} - ie\not{A})\psi_j}^{\mathcal{L}_{F,j}} - \frac{1}{4} F_{\mu\nu} f(\square) F^{\mu\nu} + \mathcal{L}_{GF} \right] \right), \quad (6.33)$$

where $\mathcal{L}_{GF}$ is a gauge-fixing term. In two dimensions, the gauge field can be expressed as

$$A_\mu = \partial_\mu \sigma + \epsilon_{\mu\nu} \partial^\nu \eta, \quad (6.34)$$

where $\epsilon_{\mu\nu}$ is the antisymmetric two-dimensional Levi-Civita tensor, with $\epsilon^{01} = 1$. Therefore, in the Landau gauge ($\partial^\nu A_\nu = 0$), we can write $A_\mu = \epsilon_{\mu\nu} \partial^\nu \eta$ [74]. Now, let us employ the Fujikawa's path integral technique [193, 194], which has been widely used in the Schwinger model [195], and perform a local chiral transformation of the form

$$\psi_j(x) = e^{ie\gamma_5\alpha(x)}\psi_j(x), \quad (6.35)$$

$$\bar{\psi}_j(x) = \bar{\psi}'_j(x)e^{ie\gamma_5\alpha(x)}, \quad (6.36)$$

then the fermion Lagrangian density becomes⁸

$$\begin{aligned} \mathcal{L}_{F,j} &= i\bar{\psi}'_j(x)e^{ie\gamma_5\alpha(x)}(\not{\partial} - ie\not{A})e^{ie\gamma_5\alpha(x)}\psi_j(x) \\ &= i\bar{\psi}'_j(x)e^{ie\gamma_5\alpha(x)}\gamma^\mu(\partial - ie\gamma_5\eta(x))e^{ie\gamma_5\alpha(x)}\psi_j(x) \end{aligned} \quad (6.37)$$

$$= i\bar{\psi}'_j(x)\gamma^\mu(\partial + ie\gamma_5\partial_\mu\alpha(x) - ie\gamma_5\partial_\mu\eta(x))\psi_j(x). \quad (6.38)$$

Moreover, one can show that the measure $D\bar{\psi}D\psi$ transforms as [195] (see also [74] for an alternative deduction)

$$D\bar{\psi}_j D\psi_j = D\bar{\psi}'_j D\psi'_j \exp \left[-i \frac{e^2}{\pi} \int d^2x \alpha(x) \partial_\mu \partial^\mu \left(\frac{1}{2} \alpha(x) - \eta(x) \right) \right]. \quad (6.39)$$

Therefore, if we choose $\alpha(x) = \eta(x)$, the generating functional for the fermions becomes

$$Z = \int D\bar{\psi}' D\psi' \exp \left(i \int d^2x \left[\sum_{j=1}^{N_f} \bar{\psi}'_j i \not{\partial} \psi'_j + N_f \frac{e^2}{2\pi} A_\mu A^\mu \right] \right), \quad (6.40)$$

where we used that $A_\mu A^\mu = \eta \partial_\mu \partial^\mu \eta(x)$. Finally, introducing Eq. (6.40) into (6.33), we obtain

$$Z_{TOT} = \int dA_\mu D\bar{\psi}' D\psi' \exp \left(i \int d^2x \mathcal{L}_{eff} \right), \quad (6.41)$$

⁷See for instance [192], for the quantization of non-local kinetic terms in the Lagrangian.

⁸In two dimensions one can prove that $\gamma_5 \gamma^\mu = \epsilon^{\mu\nu} \gamma_\nu$ and that $\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \gamma_5 \epsilon^{\mu\nu}$.

where

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}f(\square)F^{\mu\nu} + N_f \frac{e^2}{2\pi} A_\mu A^\mu + i \sum_{j=1}^{N_f} \bar{\psi}'_j \not{\partial} \psi'_j. \quad (6.42)$$

Remarkably, as in the Schwinger model, the fermions, and the gauge field decouple completely, and the theory is *exactly solvable*. The resulting effective Lagrangian density looks like the one for a massive gauge particle. However, due to the exotic kinetic term, this can only be concluded after studying the real part of the poles in the propagator of the gauge field in each particular case. In the Landau gauge ($\partial^\nu A_\nu = 0$) this is an easy task, hence the propagator in the momentum space can be written as

$$\Pi_{\mu\nu}(p^2) = -\frac{\eta_{\mu\nu}}{p^2 f(-p^2) - N_f \frac{e^2}{\pi}}. \quad (6.43)$$

Although the relation between the term $N_f \frac{e^2}{2\pi} A_\mu A^\mu$ and the axial anomaly is known, let us compute the expectation value of the vector current in the presence of an external field A_μ . Focusing only on the generating functional for fermions

$$Z = \int d\bar{\psi} d\psi \exp \left(i \int d^2x \sum_{j=1}^{N_f} \bar{\psi}_j (i\not{\partial} - e\not{A}) \psi_j \right), \quad (6.44)$$

the expectation value of the vector current can be computed as (see, e.g., Ref. [74])

$$\begin{aligned} \langle j^\mu(x) \rangle &= \frac{e}{2} \langle [\bar{\psi}(x), \gamma^\mu \psi(x)] \rangle \\ &= \frac{1}{iZ} \frac{\delta Z}{\delta A_\mu(x)} \\ &= N_f \frac{e^2}{\pi} (\eta^{\mu\nu} - \partial^\mu \square^{-1} \partial^\nu) A_\nu. \end{aligned} \quad (6.45)$$

Since in two dimensions $j_5^\mu = \epsilon^{\mu\nu} j_\nu$, the term $N_f \frac{e^2}{2\pi} A_\mu A^\mu$ is connected with the axial anomaly [72, 73], it can be concluded that the axial anomaly is independent of the gauge field Lagrangian. One can integrate (6.45) and introduce the result in (6.33) to obtain

$$Z_{TOT} = \int dA_\mu \exp \left(i \int d^2x \mathcal{L}_{eff} \right), \quad (6.46)$$

where

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}f(\square)F^{\mu\nu} + N_f \frac{e^2}{2\pi} A_\mu (\eta^{\mu\nu} - \partial^\mu \square^{-1} \partial^\nu) A_\nu + \mathcal{L}_{GF}, \quad (6.47)$$

which is an effective action for the gauge fields in a general gauge⁹.

Note that the dimensions of the gauge field and the electric charge depend on the kinetic term. In particular for RQED_{d_γ,2} we have ($c = \hbar = 1$) [68]

$$[A_\mu] = \frac{d_\gamma}{2} - 1, \quad [e] = \frac{4 - d_\gamma}{2}. \quad (6.48)$$

⁹Note that in Eqs. (6.46) and (6.47) does not appear $\bar{\psi}$ and ψ , since that they have been integrated out.

Theory	$f(\Box)$	m_{ph}^2
QED ₁₊₁	1	$N_f \frac{e_{2D}^2}{\pi}$
Lee-Wick QED	$1 + \frac{\Box}{M^2}$	$\frac{1}{2} \left(M^2 \pm M \sqrt{M^2 - 4N_f e_{2D}^2 / \pi} \right)$
RQED _{4,2}	$-\frac{4\pi}{\Box \log(\Box/\Lambda^2)}$	$\Lambda^2 \exp \left(\frac{4\pi^2}{N_f e_{4D}^2} \right)$

Table 6.1: Dynamical mass for the gauge field with massless fermions.

In the following, we denote $e \equiv e_{d_\gamma D}$ to emphasize that e depend on the space-time dimensions.

Using Eq. (6.43) we find the poles for the gauge field for the generalized electrodynamics; the results for three examples are summarized in Table 6.1. Let us discuss these results.

QED₁₊₁: As it is well known the photon becomes massive [71].

Lee-Wick QED₁₊₁: In the absence of fermion fields, this electrodynamics contains two poles, one corresponding to the massless photon, and the other corresponding to a massive Lee-Wick photon ($p^2 = M^2$) [196]. In the presence of massless fermion fields, the theory contains two massive poles¹⁰.

RQED_{4,2}: Since there are no dimensional parameters in massless QED₃₊₁, if the dimensions in which fermions are localized reduces to two dimensions ($3 + 1 \rightarrow 1 + 1$), one would expect that no dimensional parameters appear in RQED_{4,2}. However, the free propagator of RQED_{4,2} in the momentum space is indeed only well-defined as a regularized object, and so a cutoff Λ is introduced. On the contrary, in the coordinate space, the free propagator is well-defined even if the cutoff is sent to infinity. Thus, a dynamical mass for the photon can be induced only through the mechanism of dimensional transmutation [57] (or equivalently, a dynamical generation of scale in a seemingly scale-free theory [197]). Thus, the dynamical mass for the photon can be only proportional to Λ . One can show that $1/\Lambda$ characterizes the lateral confinement size (i.e., the radius of the cylindrical section of the quantum wire) [198–201]¹¹.

Now, let us note that in RQED_{4,2}, the classical interaction between the fermions is the usual Coulomb potential ($\propto 1/r$), the electric charge is a dimensionless parameter (the physical charge) and RQED_{4,2} is naturally defined as a theory of two flavors¹². Conse-

¹⁰One must take the restriction $M^2 \geq 4N_f e_{2D}^2 / \pi$.

¹¹See Eq. (6.27).

¹²For RQED _{$d_\gamma, 2$} the γ -matrices are $2^{\lfloor \frac{d_\gamma}{2} \rfloor} \times 2^{\lfloor \frac{d_\gamma}{2} \rfloor}$ dimensional matrices and there is one irreducible representation if d_γ is even and two if d_γ is odd. Therefore, one can consider that RQED _{$d_\gamma, 2$} is built by copies of the irreducible representation of the γ -matrices in $1 + 1$ dimensions and that the number of copies represents the number of flavors. For example, in three dimensions there are two 2×2 irreducible representations, so RQED_{3,2} can be defined just as a theory of n flavors if just one irreducible representation is used. Or as a theory of $2 \times n$ flavors, if the two irreducible representations are used (with n a natural number). Now, in four dimensions there is just one 4×4 irreducible representation, so, RQED_{4,2}

quently, $\text{RQED}_{4,2}$ is indeed an effective Lagrangian for electrons in a graphene wire with one flavor for each valley. Surprisingly, the theory of graphene wires is exactly soluble for massless carriers.

It should be pointed out that, in general, there is no dynamical mass generation for the fermions, because as we saw the photon decouples in the model.

6.4.2 Bosonization

Let us consider N_f Dirac fields with an identical mass m . The Lagrangian density for the fermions is now

$$\mathcal{L}_f = i \sum_{j=1}^{N_f} \bar{\psi}_j (\not{\partial} - m - ie\not{A}) \psi_j. \quad (6.49)$$

One can turn from fermionic field variables $\bar{\psi}_j, \psi_j$ into bosonic ones ϕ_j using bosonization rules [78, 85]

$$\mathcal{L} = -\frac{1}{2} F f(\square) F + N_m \sum_{j=1}^{N_f} \left(\frac{1}{2} (\partial_\mu \phi_j)^2 + \frac{e}{\sqrt{\pi}} F \phi_j + cm^2 \cos(2\sqrt{\pi} \phi_j) \right), \quad (6.50)$$

where, $F = F_{01}$ (the only nonzero component), $c = e^\gamma/(2\pi)$, with $\gamma = 0.5774$ the Euler constant, and N_m denotes normal ordering with respect to the fermion mass. The theory is quadratic in F , so, one can integrate out F to obtain

$$\mathcal{L} = N_m \sum_{j=1}^{N_f} \left(\frac{1}{2} (\partial_\mu \phi_j)^2 - \frac{e^2}{2\pi} \phi_j \frac{1}{f(\square)} \phi_j + cm^2 \cos(2\sqrt{\pi} \phi_j) \right). \quad (6.51)$$

The second term on the r.h.s. carries all the information about how the bosons(fermions) interact with the gauge field. For QED_{1+1} this term is just a mass term for the scalar field [78, 81, 187, 202], while in the general case this term leads to a non-local scalar field theory. Significantly, Eq. (6.51) is a new family of bosonized theories in $1+1$ dimensions. In particular, Eq. (6.51) with $f(\square) = -\frac{4\pi}{\square \log(\square/\Lambda^2)}$ is the bosonized theory for graphene wires.

In the following, however, we will focus on studying the gauge field. To this purpose, we integrate out ϕ_j in Eq. (6.50). If $m = 0$, this can be done exactly since the Lagrangian density is quadratic in ϕ_j

$$\mathcal{L}_{eff} = -\frac{1}{2} F f(\square) F - N_f \frac{e^2}{2\pi} F \frac{1}{\square} F, \quad (6.52)$$

which is equivalent to Eq. (6.47) in the Landau gauge.

In the general case ($m \neq 0$), we take the usual approach and Eq. (6.50) is expanded in powers of $2\sqrt{\pi} \phi_j$ (see, Ref. [81, 83] for more details). Taking only quadratic terms and integrating out ϕ_j , we arrive at

$$\mathcal{L}_{eff} = -\frac{1}{2} F f(\square) F - N_f \frac{e^2}{2\pi} F \frac{1}{\square + 4\pi cm^2} F, \quad (6.53)$$

can be defined just as a theory of $N_f = 2 \times n$ flavors.

Theory	m_{ph}^2
QED ₁₊₁	0, $m_1^2 + N_f \frac{e_{2D}^2}{\pi}$
Lee-Wick	0, $\frac{1}{2} \left(M^2 + m_1^2 \pm \sqrt{(M^2 - m_1^2)^2 - 4N_f e_{2D}^2 / \pi} \right)$
RQED _{4,2}	$-\frac{4m_1^2 \pi^2}{N_f e_{4D}^2} W(y)^{-1}$

Table 6.2: Dynamical mass for the gauge field for massive fermions ($m_1^2 = 4\pi c m^2$).

for the other terms, one can show that expansion $\mathcal{L}_{eff}$ is a power expansion in $(eF)^2$ [81]. In the quadratic approximation, one can obtain the gauge field propagator

$$\Pi_{\mu\nu}(p^2) = -\frac{\eta_{\mu\nu}(p^2 - 4\pi c m^2)}{p^2 f(-p^2)(p^2 - 4\pi c m^2) - N_f \frac{e^2}{\pi} p^2}. \quad (6.54)$$

Thus, the poles of the propagator are the solutions of the equation

$$p^2 \left(f(-p^2)(p^2 - 4\pi c m^2)\pi - N_f e^2 \right) = 0. \quad (6.55)$$

In general, there is more than one solution. Although it seems that there is always a massless mode, since $f(-p^2)$ can be singular in $p = 0$, this is not the general case. The poles are calculated for some particular cases and shown in Table 6.2. Let us discuss these results.

QED₁₊₁: In the absence of fermion mass, there is only one massive pole (see Table 6.1). As one can see in Table 6.2, the mass of the fermion does not prevent a dynamical mass generation for the gauge field, as is sometimes claimed [203]. But rather it enables the existence of two poles, one massless and one massive ($m_{ph}^2 = 4\pi c m^2 + N_f \frac{e_{2D}^2}{\pi}$). Now, we have an alternative point of view to understand the confinement in QED₁₊₁: for $m = 0$ the gauge field is massive and, so, the field is only screened; while for $m \neq 0$ we have one massive pole (screening), but surprisingly also we recover the massless pole, and the classical result prevails (confinement).

Lee-Wick QED₁₊₁: We recover the massless pole (as in QED₁₊₁) and we also have two massive poles.

RQED_{4,2}: In this case, we have one massive pole. In Table 6.1, $W(y)$ is the Lambert function, with

$$y = -\frac{4m_1^2 \pi^2}{N_f e_{4D}^2 \Lambda^2} \exp\left(-\frac{4\pi^2}{N_f e_{4D}^2}\right).$$

In conclusion, the proposal of this section extends the set of exactly solvable models and allows construct a new family of theories that can be exactly bosonized.

CHAPTER 7

FERMION CONDENSATE AND MAGNETIZATION IN GRAPHENE: INTERPLAY BETWEEN REAL AND PSEUDOMAGNETIC FIELDS

*"An error does not become truth by reason or
multiplied propagation, nor does truth become
error because nobody sees it."*

— Mahatma Gandhi.

In this chapter, we will study a $(2 + 1)$ -dimensional Dirac Hamiltonian in the presence of magnetic and pseudomagnetic fields. After discussing some questions about the Hamiltonian and its symmetries, we introduce the so-called Furry picture. Then, we compute the fermionic condensate and the vacuum current, which as discussed below have unique features. In addition, we will discuss some aspects of the magnetic catalysis and the dynamical mass generation. Finally, we will calculate the one-loop effective action and the magnetization for this system.

7.1 Magnetic Catalysis

Chiral symmetry-breaking in $(2 + 1)$ - and $(3 + 1)$ -dimensional theories has been a subject of intense scrutiny over the past two decades [151, 204–224]. In the presence of an uniform magnetic field, the appearance of a nonvanishing chiral condensate $\langle \bar{\Psi}\Psi \rangle \neq 0$, in the limit $m \rightarrow 0$, produce spontaneous chiral symmetry breaking [204–206]. For example [204, 207], in the Nambu-Jona-Lasinio (NJL) model, the spontaneous symmetry breaking occurs when the coupling constant exceeds some critical value, *i.e.* when $\lambda > \lambda_c$. With an external uniform magnetic field $\lambda_c \rightarrow 0$, independent of the intensity of the magnetic field B , hence, the magnetic field is a strong *catalyst* of chiral symmetry breaking (see Ref. [225] for review).

The exact expression for a fermion propagator in an external magnetic field in $3 + 1$ dimensions was found for the first time by Schwinger, using the proper-time formalism, in Ref. [226]. For $2 + 1$ dimensions the fermion propagator was presented, in the momentum representation, by Gusynin *et al.* [204]. Using the expression of the fermion propagator in the presence of an uniform magnetic field, the vacuum condensate has been computed in the reducible representation (four-component spinor) [204, 205, 208]. One can extend directly this method for the two inequivalent irreducible representation of the Dirac matrices and compute the vacuum condensate. On the other hand, Das and Hott introduced an alternative derivation of the magnetic condensate at zero-temperature which was easily extended to the finite temperature case [227, 228]. In this derivation, the condensate is computed starting from the quantization of the solutions of the Dirac equation in the presence of a magnetic field, the so-called Furry picture. This method has been used to calculate the magnetic vacuum condensate in $3 + 1$ dimensions [229], as in the presence of parity violating mass terms [229] and at finite temperature [213, 217, 227, 228]. As we will see, in contrast with the condensate via the fermion propagator, the calculation of the condensate in the Furry picture does not refer to the behavior of the propagator in equal times.

In the next section we will introduce the Lagrangian for graphene in the presence of magnetic and pseudomagnetic fields. Then, we will use the Furry picture to calculate the vacuum condensate for this system.

7.2 Dirac Hamiltonian for Graphene

The Dirac Hamiltonian in the presence of real (A) and pseudo (a) magnetic potentials is [230]¹

$$H_D[A, a] = \Gamma^0 \Gamma^i (p_i - eA_i - ea_i) + \Gamma^0 m, \quad (7.1)$$

where m is a mass term², $a_i = a_i^{35} \Gamma^{35}$, $\Gamma^{35} = i\Gamma^3 \Gamma^5$. This 4×4 Hamiltonian acts on the four-component “spinor”, $\psi^T = (\psi_A^K, \psi_B^K, \psi_A^{K'}, \psi_B^{K'})$, where the components taking into account both two *valleys* ($\vec{K}$ and $\vec{K}'$) and the two *sublattices* (A y B) [233], the quantum number associated with sublattices is usually referred to as *pseudospin*. If one want to include the real spin, the spinor will have eight-components and the Dirac Hamiltonian will be $H_{D(8 \times 8)} = I_2 \otimes H_D[A, a]$ [231]. For subsequent calculations, it is sufficient to consider $H_D[A, a]$, given that include the real spin only increase the degeneration of the

¹The electric charge e is multiplying the term a_i only for dimensional reasons, strictly one could write $eA_i = \tilde{A}_i$ to emphasize that it is independent of e .

²It turns out that one can identified a_i^{35} as one component of a non-Abelian $SU(2)$ gauge field within the low-energy theory of graphene [231, 232]. The other two components of this non-Abelian $SU(2)$ gauge field are proportional to Γ^3 and Γ^5 , since they are off diagonal in valley index mixing the two inequivalent valleys [231, 232]. In this case, the pseudo-gauge potential is $a_i = a_i^3 \Gamma^3 + a_i^5 \Gamma^5 + a_i^{35} \Gamma^{35}$. Assuming a smooth enough deformation in the graphene sheet, one can keep only the component a_i^{35} , which does not mix the two inequivalent valleys [231]. Hence, Eq. (7.1) captured the physics of low-energy strained graphene.

Landau levels by two ($g_s = 2$). As a matter of convenience, here, we choose the Γ -matrices as [234]

$$\begin{aligned}\Gamma^0 &= \sigma^3 \otimes \sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, & \Gamma^1 &= \sigma^3 \otimes i\sigma^1 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \\ \Gamma^2 &= \sigma^3 \otimes i\sigma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, & \Gamma^3 &= i\sigma^1 \otimes I_{2 \times 2} = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}, \\ \Gamma^5 &= -\sigma^2 \otimes I_{2 \times 2} = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, & \Gamma^{35} &= i\sigma^3 \otimes I_{2 \times 2} = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix},\end{aligned}\quad (7.2)$$

so that $(\Gamma^3)^2 = -1$ and $(\Gamma^5)^2 = 1$, Γ^3 and Γ^5 anticommute with Γ^μ , while Γ^{35} commutes with Γ^μ and anticommutes with Γ^3 and Γ^5 . Note that Γ^μ ($\mu = 0, 1, 2$) are block-diagonal, where each block is one of two inequivalent irreducible representations of the Clifford algebra in $2 + 1$ dimensions. For odd dimensions there are two inequivalent irreducible representations of the Dirac matrices, that we denote as $\mathcal{R}_1$ and $\mathcal{R}_2$. Here, the two inequivalent representations were chosen as γ^μ and $-\gamma^\mu$ for $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, where

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2. \quad (7.3)$$

Given that there is no intervalley coupling, we can rewrite Dirac Hamiltonian as $H_D = H_+[A, a] \oplus H_-[A, a]$, thus

$$H_\pm = i\sigma^3 \sigma^i (p_i - eA_i \mp ea_i^{35}) \pm \sigma^3 m, \quad (7.4)$$

where H_+ and H_- represent the Hamiltonian near of valley $\vec{K}$ (representation $\mathcal{R}_1$) and $\vec{K}'$ (representation $\mathcal{R}_2$), respectively. H_+ acts in a two-component spinor that describes a fermion with pseudospin up and an antifermion with pseudospin down, while H_- acts in a two-component spinor that describes a fermion with pseudospin down and an antifermion with pseudospin up. Thus, we arrive at two decoupled Dirac equations in $(2 + 1)$ -dimensions ($\hbar = c = 1$)

$$i \frac{\partial \psi(x, y, t)}{\partial t} = H_\pm \psi(x, y, t). \quad (7.5)$$

Finally, one can write the Lagrangian density for this system as the sum of two Lagrangian densities one for each valley

$$\mathcal{L} = \overbrace{i\bar{\psi}^K \gamma^\mu D_\mu^+ \psi^K - m\bar{\psi}^K \psi^K}^{\mathcal{L}_+} + \overbrace{i\bar{\psi}^{K'} \gamma^\mu D_\mu^- \psi^{K'} + m\bar{\psi}^{K'} \psi^{K'}}^{\mathcal{L}_-}, \quad (7.6)$$

where $D_i^\pm = \partial_i - ieA_i \pm iea_i^{35}$, which can be interpreted as a system describing two species of two-component spinors, one with mass $+m$ and coupled to $eA_i - ea_i^{35}$ and the other with mass $-m$ and coupled to $eA_i + ea_i^{35}$.

7.2.1 Symmetries in the Irreducible and Reducible Representations

Irreducible representations: For irreducible representations it is possible to define the parity ($\mathcal{P}$), charge conjugation ($\mathcal{C}$), and time-reversal ($\mathcal{T}$) transformations as following

$$\mathcal{P}\psi(t, x, y)\mathcal{P}^{-1} = -i\gamma^1\psi(t, -x, y), \quad (7.7)$$

$$\mathcal{T}\psi(t, x, y)\mathcal{T}^{-1} = -i\gamma^2\psi(-t, x, y), \quad (7.8)$$

$$\mathcal{C}\psi(t, x, y)\mathcal{C}^{-1} = -\gamma^2(\bar{\psi}(t, x, y))^T, \quad (7.9)$$

where $\mathcal{P}$ and $\mathcal{C}$ are unitary operators and $\mathcal{T}$ is an anti-linear [235], *i.e.* $\mathcal{T}(c\text{-number})\mathcal{T}^{-1} = (c\text{-number})^*$. One can check that the mass terms in the Dirac Lagrangian is not invariant under $\mathcal{P}$ or $\mathcal{T}$. However, the combined transformation $\mathcal{PT}$ leaves the mass terms invariant, so $\mathcal{CPT}$ is a symmetry of the Dirac Lagrangian [236]. Because the γ^μ form three 2×2 matrices and no other matrix anticommutes with them, the chiral symmetry cannot be defined for irreducible representations.

Reducible representation: For a reducible representation let us take the four-component spinor, $\psi^T = (\psi^K, \psi^{K'})^T$. As it has been pointed out in Refs. [234, 237], because the free Lagrangian uses only three Dirac matrices, parity, charge conjugation and time-reversal transformations can be implemented by more than one operator

$$\mathcal{P}\psi(t, \mathbf{r})\mathcal{P}^{-1} = P_j\psi(t, \mathbf{r}'), \quad (7.10)$$

$$\mathcal{T}\psi(t, \mathbf{r})\mathcal{T}^{-1} = T_j\psi(-t, \mathbf{r}), \quad (7.11)$$

$$\mathcal{C}\psi(t, \mathbf{r})\mathcal{C}^{-1} = C_j(\bar{\psi}(t, \mathbf{r}))^T, \quad (7.12)$$

$$\mathbf{r} = (x, y), \quad \mathbf{r}' = (-x, y),$$

where

$$P_1 = -i\Gamma^1\Gamma^3, \quad P_2 = -\Gamma^1\Gamma^5, \quad (7.13)$$

$$T_1 = -\Gamma^2\Gamma^3, \quad T_2 = -i\Gamma^2\Gamma^5, \quad (7.14)$$

$$C_1 = -i\Gamma^0\Gamma^1, \quad C_2 = -\Gamma^2. \quad (7.15)$$

In Tab. 7.1 we present the transformation properties of some bilinears under $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{T}$ transformations. Using these properties one can easily prove that the massive (or massless) Dirac Lagrangian in $2 + 1$ dimensions for the reducible representation is invariant under $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{T}$, regardless of which transformation is used, *i.e.* the Dirac Lagrangian is invariant under P_1 and P_2 , C_1 and C_2 , T_1 and T_2 , or even a linear combination of this operator could be used, with some restrictions [237]. In our Lagrangian only appears the bilinears $\bar{\psi}\psi$, $\bar{\psi}\Gamma^\mu\psi$ and $\bar{\psi}\Gamma^\mu\Gamma^{35}\psi$, which are independent of j . Therefore, any of the operators P_j , C_j and T_j can be used to implement $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{T}$, respectively. Surprisingly, the transformation properties of $\bar{\psi}\Gamma^3\psi$, $\bar{\psi}\Gamma^5\psi$, $\bar{\psi}\Gamma^\mu\Gamma^3\psi$ and $\bar{\psi}\Gamma^\mu\Gamma^5\psi$ depends on j . As a result, we find two non-equivalent realizations of parity, charge conjugation and time-reversal, which is unusual. This kind of terms appears for example when a non-Abelian

$SU(2)$ gauge field is introduced in graphene [231,232]. In what follows we will be interested only in the Lagrangian density (7.6).

	P_j	T_j	C_j
$\bar{\psi}\psi(t, \mathbf{r})$	$\bar{\psi}\psi(t, \mathbf{r}')$	$\bar{\psi}\psi(-t, \mathbf{r})$	$\bar{\psi}\psi(t, \mathbf{r})$
$\bar{\psi}\Gamma^\mu\psi(t, \mathbf{r})$	$\bar{\psi}\tilde{\Gamma}^\mu\psi(t, \mathbf{r}')$	$\bar{\psi}\tilde{\Gamma}^\mu\psi(-t, \mathbf{r})$	$-\bar{\psi}\tilde{\Gamma}^\mu\psi(t, \mathbf{r})$
$\bar{\psi}\Gamma^\mu\Gamma^{35}\psi(t, \mathbf{r})$	$-\bar{\psi}\tilde{\Gamma}^\mu\Gamma^{35}\psi(t, \mathbf{r}')$	$-\bar{\psi}\tilde{\Gamma}^\mu\Gamma^{35}\psi(-t, \mathbf{r})$	$\bar{\psi}\tilde{\Gamma}^\mu\Gamma^{35}\psi(t, \mathbf{r})$
$\bar{\psi}\Gamma^3\psi(t, \mathbf{r})$	$(-1)^j\bar{\psi}\Gamma^3\psi(t, \mathbf{r}')$	$(-1)^{j+1}\bar{\psi}\Gamma^3\psi(-t, \mathbf{r})$	$(-1)^{j+1}\bar{\psi}\Gamma^3\psi(t, \mathbf{r})$
$\bar{\psi}\Gamma^5\psi(t, \mathbf{r})$	$(-1)^{j+1}\bar{\psi}\Gamma^5\psi(t, \mathbf{r}')$	$(-1)^j\bar{\psi}\Gamma^5\psi(-t, \mathbf{r})$	$(-1)^j\bar{\psi}\Gamma^5\psi(t, \mathbf{r})$
$\bar{\psi}\Gamma^\mu\Gamma^3\psi(t, \mathbf{r})$	$(-1)^j\bar{\psi}\tilde{\Gamma}^\mu\Gamma^3\psi(t, \mathbf{r}')$	$(-1)^{j+1}\bar{\psi}\tilde{\Gamma}^\mu\Gamma^3\psi(-t, \mathbf{r})$	$(-1)^{j+1}\bar{\psi}\tilde{\Gamma}^\mu\Gamma^3\psi(t, \mathbf{r})$
$\bar{\psi}\Gamma^\mu\Gamma^5\psi(t, \mathbf{r})$	$(-1)^{j+1}\bar{\psi}\tilde{\Gamma}^\mu\Gamma^5\psi(t, \mathbf{r}')$	$(-1)^j\bar{\psi}\tilde{\Gamma}^\mu\Gamma^5\psi(-t, \mathbf{r})$	$(-1)^j\bar{\psi}\tilde{\Gamma}^\mu\Gamma^5\psi(t, \mathbf{r})$

Table 7.1: $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{T}$ transformation properties of some bilinears, here $\tilde{\Gamma}^\mu = \{\Gamma^0, -\Gamma^1, \Gamma^2\}$ and $\bar{\Gamma}^\mu = \{\Gamma^0, -\Gamma^i\}$.

It should be noted that in the literature one can find two different transformations which are defined as time reversal. One, $\hat{T}$, which acts as $\hat{T}\psi(t, x, y)\hat{T}^{-1} = \hat{T}\psi^*(-t, \mathbf{r})$. The other, T , which is the one considered here, Eq. (7.11), and is referred as Wigner time reversal. The latter transformation was defined consistently with what has been done in four dimensions by Weinberg (Ch. 5. in [171]), and Peskin and Schroeder (Ch. 3. in [235]). Moreover, the one that concerns the $\mathcal{CPT}$ theorem is T (for a detailed discussion, see sec. 11.6. in [238]).

The transformation properties of the electromagnetic potential are [236]

$$\begin{aligned}
\mathcal{P}A_0(t, x, y)\mathcal{P}^{-1} &= A_0(t, -x, y), \\
\mathcal{P}A_1(t, x, y)\mathcal{P}^{-1} &= -A_1(t, -x, y), \\
\mathcal{P}A_2(t, x, y)\mathcal{P}^{-1} &= A_2(t, -x, y), \\
\mathcal{T}A_0(t, x, y)\mathcal{T}^{-1} &= A_0(-t, x, y), \\
\mathcal{T}\vec{A}(t, x, y)\mathcal{T}^{-1} &= -\vec{A}(-t, x, y), \\
\mathcal{C}A_\mu(t, x, y)\mathcal{C}^{-1} &= -A_\mu(t, x, y),
\end{aligned} \tag{7.16}$$

which leave the Lagrangian invariant. For pseudo-magnetic potential we should have

$$\begin{aligned}
\mathcal{P}a_1^{35}(t, x, y)\mathcal{P}^{-1} &= a_1^{35}(t, -x, y), \\
\mathcal{P}a_2^{35}(t, x, y)\mathcal{P}^{-1} &= -a_2^{35}(t, -x, y), \\
\mathcal{T}\vec{a}^{35}(t, x, y)\mathcal{T}^{-1} &= \vec{a}^{35}(-t, x, y), \\
\mathcal{C}\vec{a}^{35}(t, x, y)\mathcal{C}^{-1} &= \vec{a}^{35}(t, x, y).
\end{aligned} \tag{7.17}$$

so that the interaction $\bar{\psi}\Gamma^i\Gamma^{35}a_i^{35}\psi$ in the Lagrangian is invariant.

For reducible representations, the transformation $\psi \rightarrow e^{i\alpha_\mu\tilde{\sigma}^\mu}\psi$ leaves invariant the kinetic term, where the generators $\tilde{\sigma}^\mu = \sigma^\mu \otimes I_{2 \times 2} = \{I_{4 \times 4}, -i\Gamma^3, -\Gamma^5, -i\Gamma^{35}\}$ are the

generators of a global $U(2)$ symmetry, with $\sigma^0 \equiv I_{2 \times 2}$ and the α_μ constants. The mass term $m\bar{\psi}\psi$ breaks this global symmetry down to $U(1) \times U(1)$ symmetry, with the generators $I_{4 \times 4}$ and $-i\Gamma^{35}$ [228]. When the mass vanishes, the quantum corrections generate a vacuum expectation value of $\bar{\psi}\psi$ (to be precise $[\bar{\psi}, \psi]/2$), then, the symmetry would have broken down to $U(1) \times U(1)$.

7.3 Furry Picture

In this section we will generalize what has been done in Refs. [210, 227] for a Dirac equation in the presence of a real magnetic field to the case in which the Dirac equation is in the presence of real and pseudo magnetic fields. In a static background gauge fields, the Dirac equation (7.5) can be rewritten as

$$\begin{pmatrix} E \mp m & -(D_1^\pm - iD_2^\pm) \\ (D_1^\pm + iD_2^\pm) & E \pm m \end{pmatrix} \psi = 0. \quad (7.18)$$

There are two possible solutions depending on the threshold states ($|E| = \pm m$), the positive-energy solutions ($\psi^{(+)}$) are

$$\psi_{\pm,1}^{(+)} = e^{-i|E|t} \sqrt{\frac{|E| \pm m}{2|E|}} \begin{pmatrix} f \\ -\frac{D_1^\pm + iD_2^\pm}{|E| \pm m} f \end{pmatrix}, \quad \text{or} \quad \psi_{\pm,2}^{(+)} = e^{-i|E|t} \sqrt{\frac{|E| \mp m}{2|E|}} \begin{pmatrix} \frac{D_1^\pm - iD_2^\pm}{|E| \mp m} g \\ g \end{pmatrix}, \quad (7.19)$$

where $\psi_{+,i}^{(+)}$ refers to the positive-energy solution in the representation $\mathcal{R}_1$ and $\psi_{-,i}^{(+)}$ refers to the positive-energy solution in the representation $\mathcal{R}_2$. The negative-energy solutions ($\psi^{(-)}$) are

$$\psi_{\pm,1}^{(-)} = e^{+i|E|t} \sqrt{\frac{|E| \mp m}{2|E|}} \begin{pmatrix} f \\ -\frac{D_1^\pm + iD_2^\pm}{|E| \mp m} f \end{pmatrix}, \quad \text{or} \quad \psi_{\pm,2}^{(-)} = e^{+i|E|t} \sqrt{\frac{|E| \pm m}{2|E|}} \begin{pmatrix} \frac{D_1^\pm - iD_2^\pm}{|E| \pm m} g \\ g \end{pmatrix}, \quad (7.20)$$

where f and g are two functions such that

$$-(D_1^\pm - iD_2^\pm)(D_1^\pm + iD_2^\pm)f = (E^2 - m^2)f \quad (7.21)$$

$$-(D_1^\pm + iD_2^\pm)(D_1^\pm - iD_2^\pm)g = (E^2 - m^2)g. \quad (7.22)$$

Note that the threshold states $|E| = m$ and $|E| = -m$, must be specified separately. When $|E| = m$ is a positive (negative) energy solution, the negative (positive) energy threshold is excluded, because of the factor $1/\sqrt{|E| - m}$ [210]. For example, for the valley $\vec{K}$, or equivalently the representation $\mathcal{R}_1$, the positive-energy solutions for $|E| = m > 0$ and $|E| = m < 0$ are respectively

$$\psi_{+,1}^{(+0)} = e^{-i|m|t} \begin{pmatrix} f^{(0)} \\ 0 \end{pmatrix}, \quad \psi_{+,2}^{(+0)} = e^{-i|m|t} \begin{pmatrix} 0 \\ g^{(0)} \end{pmatrix}, \quad (7.23)$$

where $f^{(0)}(x, y)$ satisfies the first-order threshold equation

$$(D_1^+ + iD_2^+)f^{(0)} = 0, \quad (7.24)$$

and $g^{(0)}(x, y)$ satisfies

$$(D_1^+ - iD_2^+)g^{(0)} = 0. \quad (7.25)$$

It turns out that if the solutions of (7.24) are normalizable, then the solutions of (7.25) are not, and viceversa [210, 239]. Now, in the absence of pseudo-magnetic field ($a_i^{35} = 0$), one has that $D = D^+ = D^-$. Thus, if $\psi_{+,1}^{(+0)}$ ($\psi_{+,2}^{(+0)}$) is a positive-energy solution for the valley $\vec{K}$ then the valley $\vec{K}'$ only has the negative-energy solution $\psi_{-,2}^{(-0)}$ ($\psi_{-,1}^{(-0)}$). Which leads to the well-known asymmetry in the spectrum of the states. Remarkably this does not necessarily happens when there is a pseudo-magnetic field, since $D^+ \neq D^-$, if the solutions of (7.24) are normalisable this does not imply that the solutions of $(D_1^+ - iD_2^+)g^{(0)} = 0$ are not. Therefore, it can happen that both valleys have positive (or negative) energy states simultaneously. The Appendix (K) illustrates this point in the case of constant real and pseudo magnetic fields.

One can calculate the vacuum condensate $\langle \bar{\psi}\psi \rangle$ (pairing between fermions and antifermions in the vacuum) in $2 + 1$ dimensions by expanding out the fermion field in a complete orthonormal set of the positive- and negative-energy ($i = \mathcal{R}_1, \mathcal{R}_2$)

$$\Psi_i(\vec{x}, t) = \sum_n \sum_p [a_{i,n,p} \psi_{i,n,p}^{(+)} + b_{i,n,p}^\dagger \psi_{i,n,p}^{(-)}]. \quad (7.26)$$

The solutions are labeled by two quantum numbers (n, p) , in which the label n refers to the eigenvalue E_n , while the label p distinguishes between degenerate states. In general both n and p may take discrete and/or continuous values [210]. The $a_{i,n,p}$ and $b_{i,n,p}^\dagger$ are the fermion annihilation operator and antifermion creation operator, respectively, which obey the anticommutation relations

$$\{a_{i,n,p}, a_{j,n',p'}^\dagger\} = \{b_{i,n,p}, b_{j,n',p'}^\dagger\} = \delta_{ij} \bar{\delta}_{nn'} \bar{\delta}_{pp'}, \quad (7.27)$$

where $\bar{\delta}_{\alpha,\alpha'}$, is the Kronecker delta if α takes discrete values or is the Dirac delta if it takes continuous values. Using the commutation relations (7.27) the vacuum expectation value $\langle \bar{\Psi}_i \Psi_i \rangle$ can be written as

$$\langle 0 | \bar{\Psi}_i(x) \Psi_i(x) | 0 \rangle \equiv \text{tr} \langle \bar{\Psi}_{i,\alpha} \Psi_{i,\beta} \rangle = \sum_n \sum_p \bar{\psi}_{i,n,p}^{(-)}(x) \psi_{i,n,p}^{(-)}(x), \quad (7.28)$$

i.e the fermion condensate is a sum over occupied negative-energy states. The tr is over the spinorial indices $\{\alpha, \beta\}$. Let us also write the condensate $\langle \Psi_i \bar{\Psi}_i \rangle$, which will be relevant to what follows, as

$$\langle 0 | \Psi_i(x) \bar{\Psi}_i(x) | 0 \rangle \equiv \text{tr} \langle \Psi_{i,\beta} \bar{\Psi}_{i,\alpha} \rangle = \sum_n \sum_p \bar{\psi}_{i,n,p}^{(+)}(x) \psi_{i,n,p}^{(+)}(x), \quad (7.29)$$

i.e this condensate is a sum over occupied positive-energy states.

In the Appendix L, we compare the calculation of fermionic condensate in the Furry picture with the method via fermion propagator. We prove that the trace of the fermion propagator evaluated in equal times and positions must be understood as the expectation value of the commutator of two field operators, the Schwinger's choice [226], which is equivalent to perform the coincidence limit symmetrically on the time coordinate [240], *i.e.*

$$\frac{1}{2}\langle 0 | [\bar{\Psi}_i, \Psi_i] | 0 \rangle = -\text{tr} S_F(x, x) \equiv - \left(\text{tr} \lim_{\substack{x_0 \rightarrow y_0^+ \\ \mathbf{x} \rightarrow \mathbf{y}}} S_F(x, y) - \text{tr} \lim_{\substack{x_0 \rightarrow y_0^- \\ \mathbf{x} \rightarrow \mathbf{y}}} S_F(x, y) \right). \quad (7.30)$$

In addition, we argue that this must be the order parameter for chiral (or parity) symmetry breaking and not the fermionic condensate as is often asserted in the literature.

7.4 Fermion Condensate, Dynamical Mass and Vacuum Current

7.4.1 Fermion Condensate

Let us consider the case where the real (B) and pseudo (b) magnetic fields are constants, in this case we choose $\vec{A} = (0, Bx)$ and $\vec{a}^{35} = (0, bx)$. The explicit solution can be found in the Appendix (K). In similar way to as was done in appendix L.1, we compute the commutator of the condensates in the two irreducible representations for arbitrary values of m , eB and eb

$$\begin{aligned} \frac{1}{2}\langle [\bar{\Psi}_{\pm}, \Psi_{\pm}] \rangle_B &= -\text{sgn}(m) \frac{|eB \pm eb|}{4\pi} - \frac{|eB \pm eb|}{2\pi} \sum_{n=1}^{\infty} \frac{m}{|E_n^{\pm}|} \\ &= \text{sgn}(m) \frac{|eB \pm eb|}{4\pi} - m \frac{\sqrt{2|eB \pm eb|}}{4\pi} \zeta\left(\frac{1}{2}, \frac{m^2}{2|eB \pm eb|}\right) \end{aligned} \quad (7.31)$$

with $|E_n^{\pm}| = \sqrt{m^2 + 2|eB \pm eb|n}$ and where the $(-)$ refers to the representation $\mathcal{R}_1$ (valley $\vec{K}$) and the $(+)$ refers to the representation $\mathcal{R}_2$ (valley $\vec{K}'$). Here $\zeta(s, q)$ is the Hurwitz zeta function defined by

$$\zeta(s, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s}. \quad (7.32)$$

Using the integral representation of the Hurwitz zeta function [241]

$$\zeta(s, q) = \frac{q^{-s}}{2} + \frac{2^{s-1}}{\Gamma(s)} \int_0^{\infty} dt e^{-2qt} t^{s-1} \left(\coth(t) - \frac{1}{t} \right) + \frac{q^{1-s}}{s-1}, \quad (7.33)$$

we arrive at [242]

$$\int_0^{\infty} dt e^{-m^2 t} t^{-\frac{1}{2}} |\omega| \coth(|\omega|t) = (2|w|\pi)^{\frac{1}{2}} \left(\zeta\left(\frac{1}{2}, \frac{m^2}{2|w|}\right) - \frac{|w|^{\frac{1}{2}}}{2^{\frac{1}{2}}|m|} \right). \quad (7.34)$$

Thus, we can rewrite Eq. (7.31) as

$$\frac{1}{2}\langle[\bar{\Psi}_{\pm}, \Psi_{\pm}]\rangle_{B,b} = -\frac{m}{4\pi^{\frac{3}{2}}}\int_0^{\infty} dt e^{-m^2 t} t^{-\frac{1}{2}} |eB \pm eb| \coth(|eB \pm eb|t). \quad (7.35)$$

In order to calculate the purely magnetic field effect, we need to subtract the zero-field part [223, 242]

$$\frac{1}{2}\langle[\bar{\Psi}_{\pm}, \Psi_{\pm}]\rangle_{0,0} = -\frac{m}{4\pi^{\frac{3}{2}}}\int_0^{\infty} dt e^{-m^2 t} t^{-\frac{3}{2}} = \frac{m^2}{2\pi}, \quad (7.36)$$

where we used the ϵ -integration technique to regularize the integral [242]. Thus, we introduce the parameters $\mu^{\pm}$ as

$$\begin{aligned} \mu^{\pm} &= \frac{1}{2}\langle[\bar{\Psi}_{\pm}, \Psi_{\pm}]\rangle_{B,b} - \frac{1}{2}\langle[\bar{\Psi}_{\pm}, \Psi_{\pm}]\rangle_{0,0} \\ &= -\frac{m}{4\pi^{\frac{3}{2}}}\int_0^{\infty} dt e^{-m^2 t} t^{-\frac{1}{2}} \left(|eB \pm eb| \coth(|eB \pm eb|t) - \frac{1}{t} \right). \end{aligned} \quad (7.37)$$

Henceforth, we will refer to $\mu^{\pm}$ as the c-condensate. One can show that there is no critical value of the fields in which the $\mu^{\pm}/m$ changes sign. Notably, if $B \neq 0$ or $b \neq 0$, the values of the $\mu^{\pm}$ for each valley are different and when $|B| = |b|$ one of the two is zero. Finally, let us take the limit $m \rightarrow 0$

$$\mu^{\pm} = -\text{sgn}(m) \frac{|eB \pm eb|}{2\pi}. \quad (7.38)$$

As we will see below this is related to a breaking of chiral, parity and time reversal symmetry.

7.4.2 Dynamical Mass

As we mentioned in the Appendix L.3, the dynamical mass generation has already been investigated in QED₂₊₁ [204, 214, 215, 218, 243, 244]. In particular, the dynamical mass with a two-component fermion for uniform magnetic field, in the so-called constant-mass approximation, is [218]

$$m_{dyn} = 2\alpha W\left(\frac{e^{-\gamma_E/2}\sqrt{2|eB|}}{2\alpha}\right), \quad (7.39)$$

with γ_E the Euler constant, $\alpha = e^2/(4\pi)$ and $W(x)$ the Lambert W function. In the above formula, it is necessary that $|eB| \gg m_{dyn}^2$ for consistency. On the other hand, for weak magnetic field the dynamical mass have a quadratic behavior, $m_{dyn} = m_0 + m_2(eB)^2 + \dots$ [245]. In general the dynamical mass should be given by $m_{dyn} = g(|eB|)$, where g is a general function, which has as its limit the aforementioned cases. It is straightforward to

extend this result to include the pseudomagnetic field. Hence, the dynamical mass would read as $m_{dyn} = g(|eB \pm eb|)$, for example, when $|eB \pm eb| \gg m_{dyn}^2$ we obtain

$$m_{dyn}^{\pm} = 2\alpha W \left(\frac{e^{-\gamma_E/2} \sqrt{2|e(B \pm b)|}}{2\alpha} \right). \quad (7.40)$$

Therefore, we find that in general the fermion dynamical mass is different for each valley, m^+ for $\vec{K}'$ and m^- for $\vec{K}$. This is not surprising since the c-condensates, μ^+ and μ^- , are different if a pseudomagnetic field is included. Note that when $B = b$ (or $B = -b$) one of the c-condensates is zero and the dynamical mass of this valley will be independent of B and b . In conclusion, the interplay between the real and pseudomagnetic fields allow us to modify the condensate.

Because the mass term in irreducible (reducible) representations breaks parity and time reversal (chiral) symmetries. When $b = 0$, the c-condensate, $\mu = \mu^+ = \mu^-$ is the order parameter of the dynamical parity and time-reversal symmetry breaking, for irreducible representations, and the dynamical chiral symmetry breaking, for reducible representations. For a non-zero pseudomagnetic field the dynamical symmetry breaking is not only of the chiral symmetry but also of the parity and time-reversal symmetry. The reason for this is that by including pseudomagnetic field, the dynamical mass is different for each valley generating a mass term which also breaks parity and time-reversal.

7.4.3 Induced Vacuum Current

Let us next calculate the vacuum current. As in the case of the fermionic condensate, the formula $-\text{tr}\gamma^\mu S(x, x) = \langle 0 | j^\mu | 0 \rangle$, also deserves to be revised. The current must be understood as

$$j^\mu \rightarrow: j^\mu(x) := e\bar{\Psi}(x)\gamma^\mu\Psi(x) := \frac{e}{2}[\bar{\Psi}(x), \gamma^\mu\Psi(x)] = \frac{1}{2}\gamma_{\alpha\alpha}^\mu[\bar{\Psi}_\alpha(x), \Psi_\alpha(x)], \quad (7.41)$$

which is the correct definition of the current operator, where it has been subtracted an infinite charge of the vacuum state [240]. Analogously to what was done before, we can find that in the Furry picture the vacuum expectation value of the current operator is

$$\begin{aligned} \langle 0 | : j_i^\mu(x) : | 0 \rangle &= \frac{e}{2} \langle 0 | [\bar{\Psi}_i(x), \gamma^\mu\Psi_i(x)] | 0 \rangle \\ &= \frac{e}{2} \sum_n \sum_p \left(\bar{\psi}_{i,n,p}^{(-)}(x) \gamma^\mu \psi_{i,n,p}^{(-)}(x) - \bar{\psi}_{i,n,p}^{(+)}(x) \gamma^\mu \psi_{i,n,p}^{(+)}(x) \right). \end{aligned} \quad (7.42)$$

For constant real and pseudo magnetic fields, using the orthogonality of Hermite polynomials, one can show that

$$\langle 0 | : j_\pm^i(x) : | 0 \rangle = 0, \quad (i = 1, 2), \quad (7.43)$$

i.e the induced current vanishes. A nonvanishing vacuum current would arise in the presence of an external electric field [246]. On the other hand, the induced charge density

is given by

$$\langle 0 | \rho(x)_\pm | 0 \rangle = \langle 0 | : j_\pm^0(x) : | 0 \rangle = \pm \frac{\text{sgn}(m)}{4\pi} e^2 (B \pm b). \quad (7.44)$$

Curiously, in contrast to $\mu^\pm$ the induced charge density only receives contributions from the lowest Landau level (LLL), even if $m \neq 0$. Furthermore, the total induced charge density is

$$\begin{aligned} \langle 0 | \rho(x) | 0 \rangle &= \langle 0 | \rho(x)_+ | 0 \rangle + \langle 0 | \rho(x)_- | 0 \rangle = \frac{\text{sgn}(m)}{4\pi} e^2 (B + b) - \frac{\text{sgn}(m)}{4\pi} e^2 (B - b) \\ &= \frac{\text{sgn}(m)}{2\pi} e^2 b, \end{aligned} \quad (7.45)$$

what makes this observable to non-zero pseudomagnetic field. This is remarkable since it is not possible in the case of a pure magnetic field [247]. Eqs. (7.43) and (7.44) are related with the Chern-Simons relation, which in the case of zero pseudomagnetic field reads

$$\langle 0 | : j_\pm^\mu(x) : | 0 \rangle = \pm \frac{e^2}{8\pi} \text{sgn}(m) \epsilon^{\mu\nu\lambda} F_{\nu\lambda}, \quad (7.46)$$

from which it is clear that when the two representations are present, the vacuum expectation value of the total current is always zero [247].

7.5 One-Loop Effective Action and Magnetization

In this section we will compute the effective action and the magnetization in the presence of uniform real and pseudomagnetic fields. Consider the fermionic part of the generating functional for each valley

$$Z_\pm = e^{iW_\pm} = \int \mathcal{D}\bar{\Psi}_\pm \mathcal{D}\Psi_\pm \exp \left(i \int d^3x [\bar{\Psi}_\pm (i\not{\partial} + e\not{A} \mp e\not{\phi}^{35} - m) \Psi_\pm] \right), \quad (7.47)$$

and let us introduce the one-loop effective Lagrangian $\mathcal{L}_\pm^{(1)}$ via $\ln Z_\pm = i \int d^3x \mathcal{L}_\pm^{(1)}(x)$. In the presence of a static background field the one-loop effective action is proportional to the vacuum energy (Ch. 16. in [171]). The vacuum energy of the Dirac energy field can be computed using the formula [248]

$$E_{\text{vac}} = \frac{1}{2} \left(- \sum_{E_n > 0} E_n + \sum_{E_n < 0} E_n \right), \quad (7.48)$$

which depends upon the zero-point energies of both positive- and negative-energy states³. In our case, it is straightforward to obtain the density vacuum energy for each valley

$$\begin{aligned}\mathcal{E}_{\text{vac}}^{\pm}(B, b) &= \frac{E_{\text{vac}}^{\pm}}{A} = -\frac{|eB \pm eb|}{4\pi}|m| - \frac{|eB \pm eb|}{2\pi} \sum_{n=1}^{\infty} \sqrt{m^2 + 2|eB \pm eb|n} \\ &= \frac{|eB \pm eb|}{4\pi}|m| - \frac{|eB \pm eb|^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi} \zeta\left(-\frac{1}{2}, \frac{m^2}{2|eB \pm eb|}\right),\end{aligned}\quad (7.49)$$

where we have used that the Landau degeneracy per unit area is $|eB \pm eb|/(2\pi)$. Using Eq. (7.33), we obtain that the density vacuum energy can be rewritten as

$$\mathcal{E}_{\text{vac}}^{\pm}(B, b) = \frac{|m|^3}{6\pi} + \frac{1}{8\pi} \int_0^{\infty} \frac{dt}{\pi^{1/2}} e^{-m^2 t} t^{-\frac{5}{2}} (|eB \pm eb|t \coth(|eB \pm eb|t) - 1). \quad (7.50)$$

Then, we subtract the zero-field part to obtain the one-loop effective Lagrangian

$$\begin{aligned}\mathcal{L}_{\pm}^{(1)} &= -(\mathcal{E}_{\text{vac}}^{\pm}(B, b) - \mathcal{E}_{\text{vac}}^{\pm}(0, 0)) \\ &= -\frac{1}{8\pi} \int_0^{\infty} \frac{dt}{\pi^{1/2}} e^{-m^2 t} t^{-\frac{5}{2}} (|eB \pm eb|t \coth(|eB \pm eb|t) - 1).\end{aligned}\quad (7.51)$$

For $b = 0$, the result is in agreement with what was found in Ref. [218, 246, 250]. In particular when $m \rightarrow 0$, we arrive at

$$\mathcal{L}_{\pm, m=0}^{(1)} = \frac{|eB \pm eb|^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi} \zeta\left(-\frac{1}{2}\right). \quad (7.52)$$

Here $\zeta(x)$ is the Riemann-zeta function. One can compute the magnetization for each valley ($M_{\pm}$) employing the one-loop effective Lagrangian, namely $M_{\pm} = \frac{\partial \mathcal{L}_{\pm}^{(1)}}{\partial B}$. A straightforward calculation gives

$$M_{\pm} = -\frac{e}{8\pi} \int_0^{\infty} \frac{dt}{\pi^{1/2}} e^{-m^2 t} t^{-\frac{3}{2}} \left(\coth(|eB \pm eb|t) - \frac{|eB \pm eb|t}{\sinh^2(|eB \pm eb|t)} \right) \text{sgn}(eB \pm eb), \quad (7.53)$$

which can also be written as

$$\begin{aligned}M_{\pm} &= \left[-\frac{e|m|}{4\pi} - \frac{em^2}{\sqrt{32}\pi|eB \pm eb|^{1/2}} \zeta\left(\frac{1}{2}, \frac{m^2}{2|eB \pm eb|}\right) \right. \\ &\quad \left. + 3e \frac{|eB \pm eb|^{\frac{1}{2}}}{\sqrt{8}\pi} \zeta\left(-\frac{1}{2}, \frac{m^2}{2|eB \pm eb|}\right) \right] \text{sgn}(eB \pm eb).\end{aligned}\quad (7.54)$$

³Provided that for each eigenvalue E_n there is an eigenvalue $-E_n$ the two sums in Eq. (7.48) reduces to the sum over the Dirac sea

$$E_{\text{vac}} = \sum_{E_n < 0} E_n.$$

This equation is always satisfied by a charge conjugation invariant background, however this is not our case. The use of this equation in a magnetic field background has led to erroneous conclusions in Ref. [217, 249].

For $b = 0$, the result is in agreement with Refs. [218, 251]. In the limit $m \rightarrow 0$ the magnetization is

$$M_{\pm} = 3e \frac{|eB \pm eb|^{\frac{1}{2}}}{\sqrt{8\pi}} \zeta\left(-\frac{1}{2}\right) \text{sgn}(eB \pm eb), \quad (7.55)$$

for each valley. Notably, the value and sign of magnetization are different for each valley. Thus, they can be modified by strain or varying the applied magnetic field. In particular, the magnetization of one valley could be zero while the other does not. Finally, it should be noted that Eq. (7.55) also corresponds to the dominant term in the strong field expansion.

The problem of describing the dynamics of particles confined to a space of dimensionality less than its configuration space, although well understood in classical mechanics, in quantum systems is a matter of controversy. The two main strategies adopted to address this issue are distinguished by the kinds of constraints that are employed. On the one hand, there are formal constraints which are implemented without explicit reference to the physical mechanism that produces the confinement. On the other hand, the real constraints in which the fundamental hypothesis is that the confinement is caused by a potential. In analogy with classical mechanics, formal constraints and real constraints are within what has been called the Lagrangian and the Newtonian approaches, respectively.

Although in classical mechanics, these approaches are equivalent, at the quantum level neither represent the same physical system nor are equivalents mathematically. The first is due to the Newtonian approach the wave function cannot be completely confined in the lower dimensional space (the submanifold) in accordance with the Heisenberg uncertainty principle. The second is because, in the Lagrangian approach the result depends just on the intrinsic properties of the submanifold, while in the Newtonian approach depends both on the intrinsic as well as the extrinsic properties of the submanifold. From the experimental point of view only the Newtonian approach is supported by recent experimental evidence [13, 14]. It opens the door to a new class of phenomena that would not be possible if confined systems were unaffected by the extrinsic properties of space where they are.

In the present thesis, inspired by the way in which quantum theory is formulated from quantum mechanics to QFT, we undertook the program to formulate and extend, what we call the *quantum theory on a submanifold*. To this end, the thesis was divided into two parts, one devoted to quantum mechanics on a submanifold and the other to QFT on a submanifold.

The first result of this thesis has been to obtain the *effective* Schrödinger, Dirac and Klein-Gordon equations on a m -dimensional curved submanifold of a flat n -dimensional space-time, in the presence of an external electromagnetic field, Eqs. (3.33), (4.86) and

(C.11) respectively. In this case, which would be the most general case, this had not previously been made for any of the above equations. Compared with the classic case, Eq. (F.12), the first feature found is that the metric tensor on the submanifold, $g_{\mu\nu}$, emerges as a *classical effect*, *i.e.* appears both at the classical and at the quantum level. Moreover, the metric tensor has its origin in a geometric effect instead of a gravitational effect. On the other hand, in the effective Schrödinger and Klein-Gordon equations, appear as quantum effects a geometric potential and the geometrically-induced Yang-Mills field. Both effects have been reported in the literature and have the distinctive feature that depends on extrinsic properties of the submanifold.

When the codimension is greater than one, and the particle is in the presence of an external electromagnetic field, we report a new quantum potential, which is independent of the curvature and torsion, as a consequence, it is present even in flat submanifolds. This new quantum potential could have significant effects on systems such as quantum rings or quantum wires.

The effects mentioned are also present in the effective Dirac equation on the submanifold. However, the presence of the geometrical potential and the induced Zeeman coupling are obscured by the fact that the confining potential approach up to first-order ϵ loses relevant information for the effective dynamics of relativistic 1/2-spin particles on the submanifold. This fact implies that the non-relativistic limit of the Dirac equation does not commute with the confining potential approach. Although this problem can be overcome when second-order terms are retained in the perturbative expansion, this would preclude the decoupling of the normal and tangential degrees of freedom. To address this problem, we propose to introduce a first-order term which rescues the geometrical potential in the non-relativistic limit. Thus, we have obtained a new effective Dirac equation on the submanifold which is consistent with the Schrödinger, Pauli and Klein-Gordon equations on a submanifold.

A number of sections of this thesis have been devoted to clarifying or correct previous calculation found in the literature which leads to misleading conclusions. For example, the need to include the geometrical potential as an essential ingredient in the description of one-dimensional quantum rings as well as a misconception related to the geometric potential in the Dirac equation. Moreover, we show that effects due to an in-plane magnetic field reported previously in the literature for two-dimensional quantum rings, two-dimensional quantum dots as well as the so-called pseudo chiral magnetic effect, are fictitious.

Within the framework of the confining potential formalisms, there are some additional applications to condensed matter that are being developed and therefore have not been presented here. One of them refers to a deduction of Rashba and Dresselhaus spin-orbit couplings for curved wires and surfaces. Another one refers to applying the confining potential approach to the 3D-Gross-Pitaevskii equation to study one- and two-dimensional trapped Bose gas and consequently understand the Bose-Einstein condensation in low-dimensional curved systems.

One might be tempted to conclude that the Dirac equation that will be obtained in sec 4.1 with the confining potential approach could be applied directly to graphene and thereby obtain an equation for curved graphene. However, it should be noted that the

$(2+1)$ -dimensional Dirac equation describing the graphene is not derived from a relativistic system which is confined to the plane. Instead of one starts from a two-dimensional non-relativistic Hamiltonian for electrons on the honeycomb lattice and then using the tight-binding model; in the continuum limit, one obtains that the low-energy excitations in graphene are described by a Dirac Hamiltonian. Therefore, the effect of the curvature in graphene must be analyzed starting from a “three-dimensional” non-relativistic Hamiltonian for electrons on the honeycomb lattice, where the potential produced by the *curved* hexagonal arrangements of carbon atoms play the role of the confining potential (V_c) and then if apply the tight-binding model. We believe that this method could lead to additional effects, apart from the previously discussed pseudomagnetic fields. This deserves further investigation.

If one extends the confining potential approach to study a curved pseudo-Riemannian submanifold (brane) embedded in a curved pseudo-Riemannian manifold (bulk), one can study the quantum dynamics of particles in the context of brane gravity. In this scenario, the idea is that while gravity is free to propagate in the bulk, matter fields are confined to the brane. We have already explored this extension, however present this here it goes beyond the goals of this thesis.

In the second part of this thesis, we have formulated the QFT in a curved submanifold. This has been possible since one can recognize the action for the fields on a submanifold from the effective field equations. Then, one can write the Feynman path integral for these two actions. In the absence of external fields, the (“free”) QFT on a submanifold can be schematically represent as a QFT of free field on a curved background plus a scalar geometric potential and an external $SO(n - m)$ Yang-Mills field.

For the theory described above, we compute the one-loop effective action for scalars and fermions at finite temperature and chemical potential to all orders using the Seely-DeWitt expansion and up to first order in the covariant perturbation theory, just because up to that order, the two expansions match. The limits of validity of these expansions were discussed as well as some limits cases. With this expansion, we try to expose how geometric effects (extrinsic and intrinsic properties of the submanifold) modify the thermodynamic properties of the system. As expected, the largest deviations from the flat case occur for large curvature. In the case of manifolds with the same intrinsic curvature, the thermal properties are identical. However, since to particles on a submanifold its thermal properties depend on the extrinsic curvature, this does not occur. Moreover, for fermions, we find that the first-order term takes into account the effect of the geometrically induced Yang-Mills field as well as an external gauge field, in contrast to the bosonic case where these contributions appear only up to second-order. A perspective of this part of the work is compute the one-loop effective action beyond the first-order in the covariant perturbation theory. In addition would be interesting to obtain the high temperature expansion in the more general case.

After some general considerations on a theory of interacting fields in a submanifold, we focus on the case of fermions on flat submanifolds, with an electromagnetic field propagating in the flat higher dimensional manifold. We showed that using the confining potential the theory known as reduce quantum electrodynamics can be recovered. The

advantage of this theory is that the Lagrangian depends only on coordinates on the submanifold, the price to pay is that the kinetic term for the electromagnetic field is now a pseudo-differential operator. In this thesis, we have explored two alternative forms to deduce this result. In particular, we have extended some results about Feynman, retarded and advanced Green's functions of pseudo-differential operators.

For a two-dimensional theory, we propose a large class of gauge invariant extensions of the Schwinger model with an exotic gauge field Lagrangian, in which the interaction between fermions goes beyond the linear potential of the Schwinger model. Employing the Fujikawa's path integral technique, we demonstrate that notably these extensions are exactly solvable for massless fermions. Therefore, the proposal extends the set of known exactly solvable models. Moreover, we have shown that in general there is no dynamical mass generation for the fermions, because as was discussed the photon decouples in the model.

For massive fermions, we construct a new family of exactly bosonized theories. In the quadratic approximation, we showed that bosonic degrees of freedom could be integrated out to obtain an effective Lagrangian for the gauge field. Then, we find that the spectrum of the gauge field is not trivially modified. We illustrate our proposal with some examples. In particular, we highlight the RQED_{4,2} which has the necessary features to be identified as an effective field theory for graphene wires. It would certainly be interesting to investigate screening, confinement and thermal effects in future research in this new family of theories.

In the last chapter of the thesis, we have examined a Dirac Hamiltonian in $2 + 1$ dimensions. The first thing we noticed is that exist two non-equivalent realizations of parity, charge conjugation, and time-reversal, for reducible representation, which is unusual. When the system is in the presence of a magnetic field, we have compared the calculation of fermionic condensate in the Furry picture with the method via fermion propagator, Appendix L. Notably, we show that the method via the fermion propagator does not calculate the vacuum expectation value of the product of two field operators, but rather a vacuum expectation value of the commutator of two field operators. Furthermore, we argued that the commutator is the order parameter is for chiral (or parity) symmetry breaking and not the magnetic condensate as is often claimed in the literature.

Thus, we studied the system in the presence of real and a pseudomagnetic fields. We computed the condensate for this further configuration and show that this condensate is related to a breaking of chiral, parity and time reversal symmetry. We also computed the *induced vacuum charge density* and show that the presence of a non-zero pseudomagnetic field makes it possible experimentally observe this charge density, what is unexpected since it is not feasible in the case of a real magnetic field [247].

Finally, we have shown that the interplay between real and pseudomagnetic fields enables us to control the magnetization as well as the dynamical mass for each valley. In other words, one can modify by straining or varying the applied magnetic field the magnetization and the dynamical mass for each valley, which could be a new tool for the so called *valleytronics*.

Appendices

APPENDIX A

DIFFERENTIAL GEOMETRY OF SUBMANIFOLDS

Let $(\mathcal{N}, G)$ be an n -dimensional flat pseudo-Riemannian manifold and $(\mathcal{M}, g)$ be a m -dimensional submanifold of the pseudo-Riemannian manifold $\mathcal{N}$. We employ Latin indices $A, B, \dots$, running from 0 to $n-1$, for the coordinates on $\mathcal{N}$, Greek indices $\mu, \nu, \dots$, running from 0 to $m-1$, for coordinates on $\mathcal{M}$ and Latin indices $i, j, \dots$, running from m to $n-1$ for the normal coordinates. An *adapted coordinates frame* $\mathcal{F}$ may be introduced by using a general system of coordinates x^μ on $\mathcal{M}$ and a set of distances y^i ($i = 1, \dots, p = n-m$) from $\mathcal{M}$ in directions normal to $\mathcal{M}$. In a sufficiently small neighborhood of $\mathcal{M}$, the Cartesian coordinates, $\mathcal{Z}^A$, of a point in $\mathcal{N}$ can be linked to the adapted coordinates frame by [7, 11]

$$\mathcal{Z}^A(x^\mu, y^i) = \mathcal{Y}^A(x^\mu) + y^i \mathcal{N}_i^A(x^\mu), \quad (\text{A.1})$$

where $\mathcal{Y}^A(x^\mu)$ are the Cartesian coordinates of a point in $\mathcal{M}$ and $\mathcal{N}_i^A$ are the components of the $(n-m)$ orthonormal vectors $\mathbf{n}_j$ to $\mathcal{M}$ in every point $x \in \mathcal{M}$. The metric in $\mathcal{F}$ is given by

$$G_{AB} = \partial_A \mathcal{Z}^C \partial_B \mathcal{Z}^D \bar{\eta}_{CD}, \quad (\text{A.2})$$

where derivatives are taken with respect to adapted frame coordinates and $\bar{\eta}_{CD}$ is the Minkowski metric in $\mathcal{N}$. The constrained submanifold is identified with the (smooth) submanifold $\mathcal{M}$ isometrically embedded in $\mathcal{N}$ by a map $\mathcal{Y} : \mathcal{M} \rightarrow \mathcal{N}$ such that [252]

$$G_{AB} \mathcal{Y}_{,\mu}^A \mathcal{Y}_{,\nu}^B = g_{\mu\nu}, \quad (\text{A.3})$$

$$G_{AB} \mathcal{Y}_{,\mu}^A \mathcal{N}_j^B = 0, \quad (\text{A.4})$$

$$G_{AB} \mathcal{N}_i^A \mathcal{N}_j^B = \eta_{ij} = s_i \delta_{ij}; \quad s_i = \pm 1, \quad (\text{A.5})$$

where $\mathcal{Y}_{,\nu}^A(x^\mu) = \frac{\partial \mathcal{Y}^A}{\partial x^\mu} = t_\mu^A$ are the components of the m tangent vectors $\mathbf{t}_\mu$ on $\mathcal{M}$. The first fundamental form (metric), the second fundamental form (extrinsic curvature), the

normal fundamental form (geometric or extrinsic torsion) and the Christoffel symbols are respectively,

$$g_{\mu\nu} \equiv \mathbf{t}_\mu \cdot \mathbf{t}_\nu, \quad (\text{A.6})$$

$$\alpha_{i\mu\nu} \equiv -\mathbf{t}_\mu \cdot \partial_\nu \hat{\mathbf{n}}_i, \quad (\text{A.7})$$

$$\beta_{\mu i}^k \eta_{kj} \equiv \hat{\mathbf{n}}_i \cdot \partial_\mu \hat{\mathbf{n}}_j, \quad (\text{A.8})$$

$$\Gamma_{\mu\nu}^\rho \equiv \frac{1}{2} g^{\rho\sigma} (\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\nu\mu}). \quad (\text{A.9})$$

Using the above definitions in the Eqs. (A.3), (A.4) and (A.5) the embedding conditions can be rewritten in terms of the tangent and normal vectors as

$$\mathbf{t}_\mu \cdot \mathbf{t}_\nu = G_{AB} \mathcal{Y}_{,\mu}^A \mathcal{Y}_{,\nu}^B = g_{\mu\nu}, \quad (\text{A.10})$$

$$\mathbf{t}_\mu \cdot \hat{\mathbf{n}}_j = G_{AB} \mathcal{Y}_{,\mu}^A \mathcal{N}_j^B = 0, \quad (\text{A.11})$$

$$\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j = G_{AB} \mathcal{N}_i^A \mathcal{N}_j^B = \eta_{ij} = s_i \delta_{ij}. \quad (\text{A.12})$$

The embedding of $\mathcal{M}$ in R^n is completely characterized by the assignment of the first, second and normal fundamental forms on $\mathcal{M}$ [253]. The signature chosen for the metric G_{AB} is $(1, n-1)$ *i.e.* $(+1, -1, \dots, -1)$. The submanifold is such that $\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j = -\delta_{ij}$ (*i.e.* $s_i = -1$), therefore the signature for $g_{\mu\nu}$ is $(1, n-m-1)$. The m vectors $\mathbf{t}_\mu$ and the $n-m$ vectors $\hat{\mathbf{n}}_i$ are linearly independent, so it is expected that the derivative of any of these vectors can be expressed as a linear combination of them. Indeed, these relationships are known as the *Gauss formula* and the *Weingarten Equations*, respectively given by

$$\partial_\mu \mathbf{t}_\nu = \Gamma_{\mu\nu}^\rho \mathbf{t}_\rho + \alpha_{\mu\nu}^i \hat{\mathbf{n}}_i, \quad (\text{A.13})$$

$$\partial_\mu \hat{\mathbf{n}}_i = \alpha_{i\mu}^\nu \mathbf{t}_\nu - \beta_{\mu i}^k \hat{\mathbf{n}}_k. \quad (\text{A.14})$$

A.1 Gauss Formula

The Gauss formula is

$$\partial_\mu \mathbf{t}_\nu = \Gamma_{\mu\nu}^\rho \mathbf{t}_\rho + \alpha_{\mu\nu}^i \hat{\mathbf{n}}_i, \quad (\text{A.15})$$

Proof. Following Refs. [167, 168], we evaluate the ansatz

$$\partial_\mu \mathbf{t}_\nu = a_{\mu\nu}^\rho \mathbf{t}_\rho + b_{\mu\nu}^i \hat{\mathbf{n}}_i, \quad (\text{A.16})$$

the functions $a_{\mu\nu}^\rho$ and $b_{\mu\nu}^i$ will be determined. First, we perform the dot product between our ansatz and $\hat{\mathbf{n}}_j$ and using the definition of the second fundamental form, to obtain

$$\alpha_{j\mu\nu} = \partial_\nu \mathbf{t}_\mu \cdot \hat{\mathbf{n}}_j = a_{\mu\nu}^\rho \mathbf{t}_\rho \cdot \hat{\mathbf{n}}_j + b_{\mu\nu}^i \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j = b_{\mu\nu}^i \eta_{ij}.$$

Defining $\alpha_{\mu\nu}^k := \alpha_{j\mu\nu} \eta^{jk}$, one have that

$$\alpha_{\mu\nu}^k = b_{\mu\nu}^i \eta_{ij} \eta^{jk} = b_{\mu\nu}^i \delta_i^k = b_{\mu\nu}^k. \quad (\text{A.17})$$

To compute $a_{\mu\nu}^\rho$ we perform the dot product between our ansatz and $\mathbf{t}_\sigma$, so

$$\partial_\nu \mathbf{t}_\mu \cdot \mathbf{t}_\sigma = a_{\mu\nu}^\rho \mathbf{t}_\rho \cdot \mathbf{t}_\sigma + b_{\mu\nu}^i \hat{\mathbf{n}}_i \cdot \mathbf{t}_\sigma = a_{\mu\nu}^\rho g_{\rho\sigma} := a_{\mu\nu,\sigma}. \quad (\text{A.18})$$

Note that $a_{\mu\nu,\sigma} = a_{\nu\mu,\sigma}$, we can written

$$a_{\mu\nu,\sigma} = \partial_\nu \mathbf{t}_\mu \cdot \mathbf{t}_\sigma = \partial_\nu (\mathbf{t}_\mu \cdot \mathbf{t}_\sigma) - \mathbf{t}_\mu \cdot \partial_\nu \mathbf{t}_\sigma = \partial_\nu g_{\mu\sigma} - a_{\sigma\nu,\mu}, \quad (\text{A.19})$$

so that $a_{\mu\nu,\sigma} + a_{\sigma\nu,\mu} = \partial_\nu g_{\mu\sigma}$, therefore

$$a_{\mu\nu,\sigma} = \frac{1}{2} (\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\nu\mu}), \quad (\text{A.20})$$

and finally, we obtain

$$a_{\mu\nu}^\rho = \frac{1}{2} g_{\rho\sigma} (\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\nu\mu}) = \Gamma_{\mu\nu}^\rho. \quad (\text{A.21})$$

A.2 The Weingarten Equations

The Weingarten Equations are

$$\partial_\mu \hat{\mathbf{n}}_i = \alpha_{i\mu}^\nu \mathbf{t}_\nu - \beta_{\mu i}^k \hat{\mathbf{n}}_k, \quad (\text{A.22})$$

Proof. Again following Refs. [167, 168], we evaluate the ansatz

$$\partial_\mu \hat{\mathbf{n}}_i = c_{i\mu}^\nu \mathbf{t}_\nu - d_{\mu i}^k \hat{\mathbf{n}}_k. \quad (\text{A.23})$$

To compute $c_{i\mu}^\nu$ we perform the dot product between our ansatz and $\hat{\mathbf{n}}_j$. Thus, using the definition of the normal fundamental form, we have

$$-\beta_{\mu ij} = \partial_\mu \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j = c_{i\mu}^\nu \mathbf{t}_\nu \cdot \hat{\mathbf{n}}_j - d_{\mu i}^k \hat{\mathbf{n}}_k \cdot \hat{\mathbf{n}}_j = -d_{\mu i}^k \eta_{kj}, \quad (\text{A.24})$$

hence $\beta_{\mu i}^k = \beta_{\mu ij} \eta^{jk} = d_{\mu i}^k$. To compute $c_{\mu\nu}^\rho$ we perform the dot product between our ansatz and $\mathbf{t}_\sigma$

$$-\alpha_{i\sigma\mu} = \partial_\mu \hat{\mathbf{n}}_i \cdot \mathbf{t}_\sigma = c_{i\mu}^\nu \mathbf{t}_\nu \cdot \mathbf{t}_\sigma - d_{\mu i}^k \hat{\mathbf{n}}_k \cdot \mathbf{t}_\sigma = c_{i\mu}^\nu g_{\nu\sigma}. \quad (\text{A.25})$$

Then

$$-\alpha_{i\sigma\mu} g^{\sigma\rho} = c_{i\mu}^\nu g_{\nu\sigma} g^{\sigma\rho} = c_{i\mu}^\nu \delta_\nu^\rho = c_{i\mu}^\rho := \alpha_{i\mu}^\rho. \quad (\text{A.26})$$

A.3 Properties

Proposition: The normal fundamental form is antisymmetric and the first fundamental form is symmetric

i) $\beta_{\mu ij} = -\beta_{\mu ji}$,

ii) $\alpha_{i\mu\nu} = \alpha_{i\nu\mu}$.

Proof. i) Using the Eq. (A.12), we have

$$\partial_\mu(\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j) = \partial_\mu(\eta_{ij}) = 0. \quad (\text{A.27})$$

On the other hand

$$\partial_\mu(\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j) = (\partial_\mu \hat{\mathbf{n}}_i) \cdot \hat{\mathbf{n}}_j + \hat{\mathbf{n}}_i \cdot \partial_\mu \hat{\mathbf{n}}_j = \beta_{\mu ji} + \beta_{\mu ij}, \quad (\text{A.28})$$

then $\beta_{\mu ij} = -\beta_{\mu ji}$.

ii) Note that of Eq. (A.12), $\hat{\mathbf{n}}_i$ is perpendicular to $\mathbf{t}_\nu$, then

$$\frac{\partial(\hat{\mathbf{n}}_i \cdot \mathbf{t}_\nu)}{\partial x^\mu} = \partial_\mu \hat{\mathbf{n}}_i \cdot \mathbf{t}_\nu + \hat{\mathbf{n}}_i \cdot \partial_\mu \mathbf{t}_\nu = 0, \quad (\text{A.29})$$

and for definition

$$\alpha_{i\mu\nu} = -\mathbf{t}_\mu \cdot \partial_\nu \hat{\mathbf{n}}_i = \hat{\mathbf{n}}_i \cdot \partial_\nu \mathbf{t}_\mu = \hat{\mathbf{n}}_i \cdot \partial_\mu \mathbf{t}_\nu = -\mathbf{t}_\nu \cdot \partial_\mu \hat{\mathbf{n}}_i = \alpha_{i\nu\mu}, \quad (\text{A.30})$$

thus $\alpha_{i\mu\nu} = \alpha_{i\nu\mu}$ ¹.

A.4 The Metric in the Adapted Coordinates Frame

The metric G_{AB} in the adapted coordinates frame $\{x^\mu, y^i\}$ can be calculated using the Eqs. (A.2), (A.13) and (A.14), yielding

$$G_{AB} = \begin{pmatrix} \gamma_{\mu\nu} + y^k y^l \beta_{\mu k}^h \beta_{\nu l}^m \eta_{hm} & -y^k \beta_{\mu k}^h \eta_{ih} \\ -y^k \beta_{\nu k}^h \delta_{jh} & \eta_{ij} \end{pmatrix} \quad (\text{A.31})$$

where $\gamma_{\mu\nu}$ is given by

$$\gamma_{\mu\nu} = g_{\mu\nu} - 2y^k \alpha_{k\mu\nu} + y^k y^l \alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{l\sigma\nu}. \quad (\text{A.32})$$

Denoting the determinants of G_{AB} and $\gamma_{\mu\nu}$ by $|G|$ and $|\gamma|$, respectively, it is possible to show that $|G| = |\gamma|$. The inverse of the metric tensor G_{AB} can be calculated exactly, and is given by

$$G^{AB} = \begin{pmatrix} \lambda^{\mu\nu} & \lambda^{\mu\sigma} y^k \beta_{\sigma k}^j \\ \lambda^{\nu\sigma} y^k \beta_{\sigma k}^i & \eta^{ij} + y^k y^l \beta_{\sigma k}^i \beta_{\rho l}^j \lambda^{\sigma\rho} \end{pmatrix}, \quad (\text{A.33})$$

where $\lambda^{\mu\nu} \equiv (\gamma^{-1})_{\mu\nu}$ is the inverse of $\gamma_{\mu\nu}$ [11].

¹From linear algebra, $\alpha_{i\mu\nu} = \alpha_{i\nu\mu}$ implies that the second fundamental form is diagonalizable with real eigenvalues. These eigenvalues, $k_1(x), \dots, k_m(x)$ are called the *principal curvatures* of $\mathcal{M}$ at the point x .

APPENDIX B

SOME DERIVATIONS

B.1 Rescaled operator

In this appendix we write explicitly the rescaled operator $\square$. This operator has four terms and it is explicitly given by

$$\begin{aligned}
 \square\chi &= \frac{\hbar^2}{|G|^{1/4}g^{1/4}}\partial_A\left(|G|^{1/2}G^{AB}\partial_B\frac{|g|^{1/4}}{|G|^{1/4}}\chi\right) - \frac{ie\hbar}{|G|^{1/2}}\partial_A(|G|^{1/2}G^{AB}A_B)\chi \\
 &\quad - 2ie\hbar\left(\frac{|G|^{1/4}}{|g|^{1/4}}\right)A_AG^{AB}\partial_B\left(\frac{|g|^{1/4}}{|G|^{1/4}}\chi\right) - e^2G^{AB}A_AA_B\chi \\
 &\equiv \hbar^2(I) - ie\hbar(II) - 2ie\hbar(III) - e^2(IV),
 \end{aligned} \tag{B.1}$$

where

$$\begin{aligned}
 (I) &= \left[\frac{1}{|G|^{1/4}g^{1/4}}\partial_A\left(|G|^{1/2}G^{AB}\partial_B\frac{|g|^{1/4}}{|G|^{1/4}}\chi\right)\right] \\
 &= \frac{1}{|\gamma|^{1/4}}\partial_i\left(|\gamma|^{1/2}\eta^{ij}\partial_j\frac{1}{|\gamma|^{1/4}}\right) + \frac{1}{|g|^{1/4}|\gamma|^{1/4}}\times \\
 &\quad \left[\partial_i|\gamma|^{1/2}y^ky^l\beta_{\sigma k}^i\beta_{\rho l}^j\lambda^{\sigma\rho}\partial_j + \partial_i\lambda^{\nu\rho}y^k\beta_{\rho k}^i|\gamma|^{1/2}\partial_\nu\right. \\
 &\quad \left.+ \partial_\mu\lambda^{\mu\nu}|\gamma|^{1/2}\partial_\nu + \partial_\mu\lambda^{\mu\rho}y^k\beta_{\rho k}^j|\gamma|^{1/2}\partial_j\right]\frac{|g|^{1/4}}{|\gamma|^{1/4}}\chi.
 \end{aligned}$$

This expression can be simplified by taking into account that

$$\begin{aligned}
 \partial_i|\gamma|^{1/2}y^ky^l\beta_{\sigma k}^i\beta_{\rho l}^j\lambda^{\sigma\rho}\partial_j &= y^k\beta_{\sigma k}^i\partial_i|\gamma|^{1/2}y^l\beta_{\rho l}^j\lambda^{\sigma\rho}\partial_j + |\gamma|^{1/2}\eta_i^ky^l\beta_{\sigma k}^i\beta_{\rho l}^j\lambda^{\sigma\rho}\partial_j \\
 &= y^k\beta_{\sigma k}^i\partial_i|\gamma|^{1/2}y^l\beta_{\rho l}^j\lambda^{\sigma\rho}\partial_j,
 \end{aligned}$$

where in the last equality we have used $\beta^i_{i\mu} = 0$. Therefore,

$$(I) = \frac{1}{|\gamma|^{1/4}} \partial_i \left(|\gamma|^{1/2} \partial_i \frac{1}{|\gamma|^{1/4}} \right) \chi \\ + \frac{1}{|g|^{1/4} |\gamma|^{1/4}} \left[(\partial_\mu + y^k \beta_{\mu k}^i \partial_i) \lambda^{\mu\nu} |\gamma|^{1/2} (\partial_\nu + y^l \beta_{\nu l}^j \partial_j) \right] \frac{|g|^{1/4}}{|\gamma|^{1/4}} \chi.$$

Introducing $\hat{\partial}_\mu \equiv \partial_\mu + \frac{1}{2\hbar} i \beta_\mu^{ij} L_{ij}$, where the $L_{ij} = i\hbar(y_j \partial_i - y_i \partial_j)$ are the angular momentum operators in the space normal to $\mathcal{M}$, one can rewrite (I) as¹

$$(I) = \frac{1}{|\gamma|^{1/4}} \partial_i \left(|\gamma|^{1/2} \partial^i \frac{1}{|\gamma|^{1/4}} \right) \chi + \frac{1}{|g|^{1/4} |\gamma|^{1/4}} \left[\hat{\partial}_\mu \lambda^{\mu\nu} |\gamma|^{1/2} \hat{\partial}_\nu \right] \frac{|g|^{1/4}}{|\gamma|^{1/4}} \chi. \quad (\text{B.2})$$

Inserting the metric G^{AB} in the other three terms yields

$$(II) = \frac{1}{|G|^{1/2}} \partial_A (\sqrt{G} G^{AB} A_B) \chi \\ = \left[\frac{1}{|\gamma|^{1/2}} \partial^i (|\gamma|^{1/2} A_i) + \frac{1}{|\gamma|^{1/2}} y^k \beta_{\sigma k}^i \partial_i (|\gamma|^{1/2} y^l \beta_{\rho l}^j \lambda^{\sigma\rho} A_j) \right. \\ \left. + \frac{1}{|\gamma|^{1/2}} y^k \beta_{\sigma k}^i \partial_i (|\gamma|^{1/2} \lambda^{\nu\sigma} A_\nu) \right. \\ \left. + \frac{1}{|\gamma|^{1/2}} \partial_\mu (|\gamma|^{1/2} \lambda^{\mu\nu} A_\nu) + \frac{1}{|\gamma|^{1/2}} \partial_\mu (|\gamma|^{1/2} \lambda^{\mu\sigma} y^k \beta_{\sigma k}^j A_j) \right] \chi, \quad (\text{B.3})$$

$$(III) = \left(\frac{|G|^{1/4}}{|g|^{1/4}} \right) A_A G^{AB} \partial_B \left(\frac{|g|^{1/4}}{|G|^{1/4}} \chi \right) \\ = \left[\left(\frac{|\gamma|^{1/4}}{|g|^{1/4}} \right) A_i \partial^i \left(\frac{|g|^{1/4}}{|\gamma|^{1/4}} \chi \right) + \left(\frac{|\gamma|^{1/4}}{|g|^{1/4}} \right) A_i \lambda^{\nu\sigma} y^k \beta_{\sigma k}^i \partial_\nu \left(\frac{|g|^{1/4}}{|\gamma|^{1/4}} \chi \right) \right. \\ \left. + \left(\frac{|\gamma|^{1/4}}{|g|^{1/4}} \right) A_\mu \lambda^{\mu\nu} \partial_\nu \left(\frac{|g|^{1/4}}{|\gamma|^{1/4}} \chi \right) + \left(\frac{|\gamma|^{1/4}}{|g|^{1/4}} \right) A_\mu \lambda^{\mu\sigma} y^k \beta_{\sigma k}^j \partial_j \left(\frac{|g|^{1/4}}{|\gamma|^{1/4}} \chi \right) \right], \quad (\text{B.4})$$

$$(IV) = A_A G^{AB} A_B \chi \\ = \left[A_i A^i + A_i \lambda^{\nu\sigma} y^k \beta_{\sigma k}^i A_\nu + A_\mu \lambda^{\mu\nu} A_\nu + A_\mu \lambda^{\mu\sigma} y^k \beta_{\sigma k}^j A_j \right] \chi. \quad (\text{B.5})$$

Finally, inserting Eqs. (B.2), (B.3), (B.4) and (B.5) into (B.1) and simplifying we arrive at Eq. (3.13).

B.2 Some computations for perturbation theory

We will derive some useful relations. First, remembering that $\gamma_{\mu\nu}$ in the rescaled coordinates is

$$\gamma_{\mu\nu} = g_{\mu\nu} - 2\epsilon y^k \alpha_{k\mu\nu} + \epsilon^2 y^k y^l \alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{l\sigma\nu}, \quad (\text{B.6})$$

¹Note that $y^k \beta_{\mu k}^i \partial_i = y_k \beta_\mu^{ki} \partial_i = \frac{1}{2} (\beta_\mu^{ki} y_k \partial_i + \beta_\mu^{ik} y_i \partial_k) = \frac{1}{2} \beta_\mu^{ki} (y_k \partial_i - y_i \partial_k)$.

and the tensor $\lambda^{\mu\nu}$ is define as the inverse of $\gamma_{\mu\nu}$ i.e. $\gamma_{\sigma\mu}\lambda^{\mu\nu} = \delta_\sigma^\nu$. With this last relation is easy to prove that an expansion in ϵ for $\lambda^{\mu\nu}$ is

$$\lambda^{\mu\nu} = g^{\mu\nu} - 2\epsilon y^k \alpha_k^{\mu\nu} + 3\epsilon^2 y^k y^l \alpha_k^{\mu\sigma} g_{\sigma\rho} \alpha_l^{\rho\nu} + O(\epsilon^3). \quad (\text{B.7})$$

Using that the $\log \det(M) = \text{Tr} \log(M)$, for $M = I + A$ we obtain

$$\begin{aligned} \det(I + A) &= \exp \text{Tr} \log(I + A) \\ &\approx \exp \text{Tr} \left(A - \frac{A^2}{2} + \frac{A^3}{3} + \dots \right) \\ &\approx \exp \left(\text{Tr} A - \frac{\text{Tr}(A^2)}{2} + \frac{\text{Tr}(A^3)}{3} + \dots \right) \\ &\approx 1 + \text{Tr} A - \frac{\text{Tr}(A^2)}{2} + \frac{(\text{Tr} A)^2}{2} + \dots, \end{aligned}$$

so, one can calculate the determinant of $\gamma_{\mu\nu} g^{\nu\beta}$ in a perturbation series

$$\begin{aligned} \det(\gamma_{\mu\nu} g^{\nu\beta}) &= \det(\delta_\mu^\beta - 2\epsilon y^k \alpha_{k\mu\nu} g^{\nu\beta} + \epsilon^2 y^k y^l \alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{l\sigma\nu} g^{\nu\beta}) \\ &\approx 1 - 2\epsilon y^k \alpha_{k\mu\nu} g^{\nu\mu} - \epsilon^2 y^k y^l \alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{l\sigma\nu} g^{\nu\mu} + 2\epsilon^2 y^k y^l \alpha_{k\mu\nu} g^{\nu\mu} \alpha_{l\sigma\rho} g^{\rho\sigma} + \dots \end{aligned}$$

Using that $\det(AB) = \det(A) \det(B)$ and $\det(A) = 1/\det(A^{-1})$, we obtain

$$\begin{aligned} |\gamma| \equiv \det(\gamma_{\mu\nu}) &\approx |g| - 2\epsilon |g| y^k \alpha_{k\mu\nu} g^{\nu\mu} - \epsilon^2 |g| y^k y^l \alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{l\sigma\nu} g^{\nu\mu} \\ &\quad + 2\epsilon^2 |g| y^k y^l \alpha_{k\mu\nu} g^{\nu\mu} \alpha_{l\sigma\rho} g^{\rho\sigma} + O(\epsilon^3). \end{aligned} \quad (\text{B.8})$$

APPENDIX C

RELATIVISTIC QUANTUM MECHANICS ON A SUBMANIFOLD: SCALAR FIELD

We employ Latin indices $A, B, \dots$, running from 0 to $n - 1$, for the coordinates on $\mathcal{N}$, Greek indices $\mu, \nu, \dots$, running from 0 to $m - 1$, for coordinates on $\mathcal{M}$ and Latin indices $i, j, \dots$, running from m to $n - 1$ for the normal coordinates, other conventions for this section are given in Appendix A.

The starting point is the Klein-Gordon equation in $\mathcal{N}$

$$\left[\frac{\hbar^2}{2|G|^{1/2}} \nabla_A |G|^{1/2} G^{AB} \nabla_B \right] \Psi(x, y) + \left[\frac{m^2 c^2}{2} + V_c(y) \right] \Psi(x, y) = 0,$$

equation where $\nabla_B = \partial_B - \frac{ie}{\hbar} A_B$, e is the particle's charge, A_B is an Abelian gauge field, $V_c(y)$ is the confinement potential and the metric given by Eq. (A.31). The first step for obtaining the Klein-Gordon equation in a brane is to rescale the field Ψ and the operator $\Delta = \frac{\hbar^2}{|G|^{1/2}} \nabla_A |G|^{1/2} G^{AB} \nabla_B$ to

$$\chi(x, y) \equiv \frac{|G|^{1/4}}{|g|^{1/4}} \Psi(x, y), \quad (\text{C.1})$$

$$\square \equiv \left(\frac{|G|^{1/4}}{|g|^{1/4}} \right) \Delta \left(\frac{|g|^{1/4}}{|G|^{1/4}} \right). \quad (\text{C.2})$$

In the non-relativistic case the rescaling enables us to define a density probability on a brane (see below). Here the rescaled field can be justified *a posteriori*, since Eq. (C.1) decouples the dynamics of the tangent and the normal degrees of freedom. However, it will be necessary to introduce a normalization condition for the scalar field. The appropriate¹ (conserved) normalization condition for the scalar field is defined through

¹In Ref. [12, 254] the authors used that the field satisfies a normalization condition,

$$\int d^m x d^p y |G|^{1/2} |\Psi|^2 = 1, \quad (\text{C.3})$$

which it is inadequate for a relativistic scalar field.

the Klein-Gordon scalar product in Cartesian coordinates as

$$(\Psi, \Psi) = i \int d^{m-1}z |G|^{1/2} G^{0B} (\Psi^* \partial_B \Psi - \Psi \partial_B \Psi^*) = 1, \quad (\text{C.4})$$

where the integral is performed over a constant x_0 hypersurface [54]. Using Eq.(C.1) one can rewrite Eq. (C.4) as

$$(\Psi, \Psi) = i \int d^{m-1}x d^p y |g|^{1/2} G^{0B} (\chi^* \partial_B \chi - \chi \partial_B \chi^*) = 1. \quad (\text{C.5})$$

Now, the rescaled operator $\square$ is explicitly given by

$$\begin{aligned} \square \chi = & \frac{\hbar^2}{|G|^{1/4} g^{1/4}} \partial_A \left(|G|^{1/2} G^{AB} \partial_B \frac{|g|^{1/4}}{|G|^{1/4}} \chi \right) - \frac{ie\hbar}{|G|^{1/2}} \partial_A (|G|^{1/2} G^{AB} A_B) \chi \\ & - 2ie\hbar \left(\frac{|G|^{1/4}}{|g|^{1/4}} \right) A_A G^{AB} \partial_B \left(\frac{|g|^{1/4}}{|G|^{1/4}} \chi \right) - e^2 G^{AB} A_A A_B \chi, \end{aligned} \quad (\text{C.6})$$

which have the same form that Eq. (3.12) in the non-relativistic case. The only difference with the non-relativistic case is that now the Latin indices $A, B, \dots$ and the Greek indices $\mu, \nu, \dots$ included the time (x^0). In complete correspondence with what was made of Eqs. (3.12) to (3.25), (i) We write the operator $\square$ in the adapted coordinates frame, (ii) we rescale the normal coordinates $y \rightarrow \epsilon y$, (iii) we expanded the $\square$ in a perturbative expansion keeping terms up to the order ϵ^2 , and finally, (iv) we perform, without loss of generality, a gauge transformation such that $\bar{A}_i = 0$ and $T_{kj} \equiv \frac{\partial A_k(x^\mu, y^i)}{\partial y^j} \Big|_{y=0}$ will be totally antisymmetric. Thus, we arrive at

$$\begin{aligned} \square^{(2)} \chi = & \frac{\hbar^2}{|g|^{1/2}} \left[\left(\partial_\mu + \frac{1}{2\hbar} i \beta_\mu^{ij} L_{ij} - \frac{ie}{\hbar} \bar{A}_\mu \right) g^{\mu\nu} |g|^{1/2} \left(\partial_\nu + \frac{1}{2\hbar} i \beta_\nu^{ij} L_{ij} - \frac{ie}{\hbar} \bar{A}_\nu \right) \right] \chi \\ & + 2V_G \chi + e \bar{F}^{ij} L_{ij} \chi. \end{aligned} \quad (\text{C.7})$$

Let us now consider the separated solution

$$\chi(x, y) = \sum_\beta \phi_\beta(x) \psi_\beta(y), \quad (\text{C.8})$$

where the index β runs over the degeneracy that exists in the spectrum of the normal degrees of freedom [11, 12]. Inserting Eq. (C.8) into the (conserved) norm of the scalar field (C.5) yields

$$\begin{aligned} (\Psi, \Psi) = i \sum_{\beta, \alpha} \int d^{m-1}x |g|^{1/2} & \left(G^{0\mu} \left[\psi_\beta^*(y) \psi_\alpha(y) \left(\phi_\beta^*(x) \partial_\mu \phi_\alpha(x) - \phi_\alpha(x) \partial_\mu \phi_\beta^*(x) \right) \right] \right. \\ & \left. + G^{0i} \left[\phi_\beta^*(x) \phi_\alpha(x) \left(\psi_\beta^*(y) \partial_i \psi_\alpha(y) - \psi_\alpha(y) \partial_i \psi_\beta^*(y) \right) \right] \right). \end{aligned}$$

Since only the space in the submanifold is curved, and not the time, the metric satisfies

$G^{0\mu} = g^{0\mu 2}$ and $G^{0i} = 0$. Thereby, we obtain

$$\begin{aligned}
(\Psi, \Psi) &= i \sum_{\beta, \alpha} \left(\int d^p y \psi_\beta^*(y) \psi_\alpha(y) \right) \int d^{m-1} x |g|^{1/2} g^{0\mu} \left(\phi_\beta^*(x) \partial_\mu \phi_\alpha(x) - \phi_\alpha(x) \partial_\mu \phi_\beta^*(x) \right) \\
&= i \sum_{\beta} \int d^{m-1} x |g|^{1/2} g^{0\mu} \left(\phi_\beta^*(x) \partial_\mu \phi_\alpha(x) - \phi_\alpha(x) \partial_\mu \phi_\beta^*(x) \right) \\
&= \sum_{\beta} (\phi_\beta, \phi_\beta) = 1,
\end{aligned} \tag{C.9}$$

where we have taken into account, without loss of generality, that $\int d^p y \psi_\beta^*(y) \psi_\alpha(y) = \delta_{\beta\alpha}$ and ϕ_β form a multiplet, $\vec{\phi} = (\phi_1, \dots, \phi_\beta)$, with dimension d_e that depends on the degeneracy that exists in the spectrum of the normal degrees of freedom.

Now, inserting (C.8) into Eq. (C.6), multiplying by $\psi_\alpha^*(y)$ and integrating in $\int d^p y$ we obtain

$$\frac{1}{2} \sum_{\beta} \int d^p y \psi_\alpha^*(y) \square^{(2)} \psi_\beta(y) \phi_\beta(x) + \frac{m^2 c^2}{2} \phi_\alpha(x) + \sum_{\beta} \bar{\lambda}_{\alpha\beta} \phi_\beta(x) = 0,$$

with

$$\bar{\lambda}_{\alpha\beta} = \int d^p y \psi_\alpha^*(y) \left[-\frac{\hbar^2}{2} \partial_i \partial^i + \frac{k}{2\epsilon^4} y^i y_i \right] \psi_\beta(y). \tag{C.10}$$

Thus, the Klein-Gordon equation in the presence of an external field on a submanifold is

$$\begin{aligned}
\frac{\hbar^2}{2|g|^{1/2}} \left[\left(\partial_\mu - i \frac{\mathcal{B}_\mu}{\hbar} - \frac{ie}{\hbar} \bar{A}_\mu \right) g^{\mu\nu} |g|^{1/2} \left(\partial_\nu - i \frac{\mathcal{B}_\nu}{\hbar} - \frac{ie}{\hbar} \bar{A}_\nu \right) \right] \vec{\phi}(x) \\
+ \left(\frac{\bar{m}^2 c^2}{2} + V_G + \frac{e}{2} \bar{F}^{ij} \mathcal{L}_{ij} \right) \vec{\phi}(x) = 0,
\end{aligned} \tag{C.11}$$

with

$$V_G = -\frac{\hbar^2}{8} g^{\nu\mu} g^{\rho\sigma} \left(\alpha_{k\mu\nu} \alpha_{\rho\sigma}^k - 2 \alpha_{k\mu\sigma} \alpha_{\rho\nu}^k \right), \tag{C.12}$$

and where $\mathcal{B}_\mu = -\frac{1}{2} \beta_\mu^{ij} (\mathcal{L}_{ij})$, $(\mathcal{L}_{ij})_{\alpha\beta} \equiv \int d^p y (\psi_\alpha(y))^* L_{ij} \psi_\beta(y)$ and $\bar{m}^2 = m^2 + \frac{2\bar{\lambda}_l}{c^2 \epsilon^2}$. In the special cases when $A_\mu = 0$, $A_i = 0$ and $\xi = 0$ it reproduces what was found in [12, 254].

²Strictly it is always possible in our case choose $g^{0\mu} = 0$, for $\mu \neq 0$.

APPENDIX D

NON-COORDINATE BASES IN THE ADAPTED COORDINATES FRAME

In a coordinate basis the tangent space $T_p\mathcal{M}$ at a point p is spanned by the partial derivatives of the coordinates at p , $\{E_A\} = \{\partial/\partial_A\}$, thereby a basis of the cotangent space $T_p^*\mathcal{M}$ is spanned by $\{E^A\} = \{dx^A\}$. In a *non-coordinate bases* $\{\hat{E}_I\}$ the frame is obtained by a rotation of the basis $\{E_A\}$ preserving the orientation¹,

$$\hat{E}_I = E_I^A \partial_A, \quad (\text{D.1})$$

the coefficients E_I^A are called the *vielbeins*², if the dimension of the manifold is n the vielbeins form a $n \times n$ invertible matrix. The frame is such that $\{\hat{E}_I\}$ is orthonormal with respect to G , so that [255]

$$\eta_{IJ} = E_I^A E_J^B G_{AB}. \quad (\text{D.2})$$

Then, it follows that

$$G_{AB} = \eta_{IJ} E_A^I E_B^J, \quad (\text{D.3})$$

where E_A^I is the inverse of E_I^A , so that $E_I^A E_B^I = \delta_B^A$ and $E_I^A E_A^J = \delta_I^J$. From the metric (A.31) one can show that the vielbeins that satisfy (D.2) and (D.3) can be chosen as³ [48]

$$E_A^I = \begin{pmatrix} E_\mu^a & E_\mu^m \\ E_i^a & E_i^m \end{pmatrix} = \begin{pmatrix} \tilde{e}_\mu^a & -y^k \beta_{\mu k}^m \\ 0 & \delta_i^m \end{pmatrix}, \quad (\text{D.4})$$

where $\tilde{e}_\mu^a$ was chosen such that

$$\gamma_{\mu\nu} = \eta_{ab} \tilde{e}_\mu^a \tilde{e}_\nu^b. \quad (\text{D.5})$$

¹Latin indices $I, J, \dots$, running from 0 to $n - 1$.

²If the dimension of the manifold is n the vielbeins form a $n \times n$ invertible matrix [255].

³Latin indices $a, b, \dots$, running from 0 to m and indices $m, n, \dots$, running from $m + 1$ to n .

In the present work it is convenient to decompose the Minkowski metric on $\mathcal{M}$ as

$$\eta_{AB} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{ij} \end{pmatrix}. \quad (\text{D.6})$$

Now, the inverse of E_A^I is [48]

$$E_I^A = \begin{pmatrix} E_a^\mu & E_a^j \\ E_m^\mu & E_m^i \end{pmatrix} = \begin{pmatrix} \tilde{e}_a^\mu & \tilde{e}_a^\nu y^k \beta_{\nu k}^j \\ 0 & \delta_m^i \end{pmatrix}, \quad (\text{D.7})$$

where $\tilde{e}_a^\mu$ was chosen such that $\tilde{e}_a^\mu \tilde{e}_\nu^a = \delta_\nu^\mu$ and $\tilde{e}_a^\mu \tilde{e}_\mu^b = \delta_a^b$.

Let us now rescale the normal coordinates by $y \rightarrow \epsilon y$ and consider the perturbative expansion in powers of ϵ . It is easy to prove from Eq. (D.5) that

$$\tilde{e}_\mu^a = e_\mu^a + \epsilon y^k \alpha_{k\mu}{}^\nu e_\nu^a, \quad (\text{D.8})$$

where e_μ^a was chosen such that

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \quad e_a^\mu e_\nu^a = \delta_\nu^\mu, \quad e_a^\mu e_\mu^b = \delta_a^b. \quad (\text{D.9})$$

To first order in ϵ , the calculation of $\tilde{e}_\mu^a$ yields

$$\tilde{e}_a^\mu = e_a^\mu - \epsilon y^k \alpha_{k\mu}{}^\nu e_\nu^a + O(\epsilon^2). \quad (\text{D.10})$$

APPENDIX E

CHRISTOFFEL SYMBOLS

The Christoffel symbols of the first kind are defined by

$$\bar{\Gamma}_{AB,C} = \frac{1}{2} (\partial_A G_{BC} + \partial_B G_{CA} - \partial_C G_{AB}), \quad (\text{E.1})$$

where it is clear that their are symmetric *i.e* $\bar{\Gamma}_{AB,C} = \bar{\Gamma}_{BA,C}$. Thus, the Christoffel symbols of the first kind for $\mathcal{N}$ in the adapted coordinates frame are

$$\begin{aligned} \bar{\Gamma}_{\mu\nu,\beta} &= \frac{1}{2} (\partial_\mu G_{\nu\beta} + \partial_\nu G_{\beta\mu} - \partial_\beta G_{\mu\nu}) \\ &= \Gamma_{\mu\nu,\beta} - y^k (\partial_\mu \alpha_{k\nu\beta} + \partial_\nu \alpha_{k\beta\mu} - \partial_\beta \alpha_{k\mu\nu}) \\ &\quad + \frac{1}{2} y^k y^l [\partial_\mu (\alpha_{k\nu\rho} g^{\rho\sigma} \alpha_{l\sigma\beta} + \beta_{\nu k}^h \beta_{\beta l}^m \eta_{hm}) + \partial_\nu (\alpha_{k\beta\rho} g^{\rho\sigma} \alpha_{l\sigma\mu} + \beta_{\beta k}^h \beta_{\mu l}^m \eta_{hm}) \\ &\quad - \partial_\beta (\alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{l\sigma\nu} + \beta_{\mu k}^h \beta_{\nu l}^m \eta_{hm})], \end{aligned} \quad (\text{E.2})$$

with $\Gamma_{\mu\nu,\beta} = \frac{1}{2} (\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})$ the Christoffel symbols of the first kind on $\mathcal{M}$.

$$\begin{aligned} \bar{\Gamma}_{\mu\nu,j} &= \frac{1}{2} (\partial_\mu G_{\nu j} + \partial_\nu G_{j\mu} - \partial_j G_{\mu\nu}) \\ &= \alpha_{j\mu\nu} - \frac{1}{2} y^k (\alpha_{j\mu\rho} g^{\rho\sigma} \alpha_{k\sigma\nu} + \alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{j\sigma\nu} + \partial_\mu \beta_{\nu k}^m \eta_{mj} + \partial_\nu \beta_{\mu k}^m \eta_{mj} \\ &\quad + \beta_{\mu j}^h \beta_{\nu k}^m \eta_{hm} + \beta_{\mu k}^h \beta_{\nu j}^m \eta_{hm}), \end{aligned} \quad (\text{E.3})$$

$$\begin{aligned} \bar{\Gamma}_{\mu j,\nu} &= \frac{1}{2} (\partial_\mu G_{j\nu} + \partial_j G_{\nu\mu} - \partial_\nu G_{\mu j}) \\ &= -\alpha_{j\mu\nu} + \frac{1}{2} y^k (\alpha_{j\mu\rho} g^{\rho\sigma} \alpha_{k\sigma\nu} + \alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{j\sigma\nu} + \partial_\nu \beta_{\mu k}^m \eta_{mj} - \partial_\mu \beta_{\nu k}^m \eta_{mj} \\ &\quad + \beta_{\mu j}^h \beta_{\nu k}^m \eta_{hm} + \beta_{\mu k}^h \beta_{\nu j}^m \eta_{hm}), \end{aligned} \quad (\text{E.4})$$

$$\begin{aligned}
\bar{\Gamma}_{ij,\nu} &= \frac{1}{2} (\partial_i G_{j\nu} + \partial_j G_{\nu i} - \partial_\nu G_{ij}) \\
&= -\frac{1}{2} (\partial_i (y^k \beta_{\nu k}^m \eta_{mj}) + \partial_j (y^k \beta_{\nu k}^m \eta_{mi}) \\
&= -\frac{1}{2} (\beta_{\nu ij} + \beta_{\nu ji}) = 0,
\end{aligned} \tag{E.5}$$

$$\begin{aligned}
\bar{\Gamma}_{\mu j,i} &= \frac{1}{2} (\partial_\mu G_{ji} + \partial_j G_{i\mu} - \partial_i G_{\mu j}) \\
&= -\frac{1}{2} (\partial_j (y^k \beta_{\mu k}^m \eta_{mi}) - \partial_i (y^k \beta_{\mu k}^m \eta_{mj})) \\
&= -\frac{1}{2} (\beta_{\mu j}^m \eta_{mi} - \beta_{\mu i}^m \eta_{mj}) = \beta_{\mu i}^m \eta_{mj},
\end{aligned} \tag{E.6}$$

$$\bar{\Gamma}_{kj,i} = \frac{1}{2} (\partial_k G_{ji} + \partial_j G_{ik} - \partial_i G_{kj}) = 0. \tag{E.7}$$

The inverse metric is

$$G^{AB} = \begin{pmatrix} \lambda^{\mu\nu} & \lambda^{\mu\sigma} y^k \beta_{\sigma k}^j \\ \lambda^{\nu\sigma} y^k \beta_{\sigma k}^i & \eta^{ij} + y^k y^l \beta_{\sigma k}^i \beta_{\rho l}^j \lambda^{\sigma\rho} \end{pmatrix}. \tag{E.8}$$

The Christoffel symbols of the second kind are

$$\bar{\Gamma}_{AB}^D \equiv G^{CD} \bar{\Gamma}_{AB,C} = \frac{1}{2} G^{CD} (\partial_A G_{BC} + \partial_B G_{CA} - \partial_C G_{AB}). \tag{E.9}$$

$$\begin{aligned}
\bar{\Gamma}_{\mu\nu}^\sigma &= G^{\sigma\beta} \bar{\Gamma}_{\mu\nu,\beta} + G^{\sigma j} \bar{\Gamma}_{\mu\nu,j} \\
&= \lambda^{\sigma\beta} (\Gamma_{\mu\nu,\beta} - y^k (\partial_\mu \alpha_{k\nu\beta} + \partial_\nu \alpha_{k\beta\mu} - \partial_\beta \alpha_{k\mu\nu}) \\
&\quad + y^k y^l \left[\frac{1}{2} \partial_\mu (\alpha_{k\nu\rho} g^{\rho\sigma} \alpha_{l\sigma\beta} + \beta_{\nu k}^h \beta_{\beta l}^m \eta_{hm}) \right. \\
&\quad \left. + \frac{1}{2} \partial_\nu (\alpha_{k\beta\rho} g^{\rho\sigma} \alpha_{l\sigma\mu} + \beta_{\beta k}^h \beta_{\mu l}^m \eta_{hm}) - \frac{1}{2} \partial_\beta (\alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{l\sigma\nu} + \beta_{\mu k}^h \beta_{\nu l}^m \eta_{hm}) \right] \\
&\quad + \lambda^{\sigma\rho} y^l \beta_{\rho l}^j \left[\alpha_{j\mu\nu} - \frac{1}{2} y^k (\alpha_{j\mu\rho} g^{\rho\sigma} \alpha_{k\sigma\nu} + \alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{j\sigma\nu} \right. \\
&\quad \left. + \partial_\mu \beta_{\nu k}^m \eta_{mj} + \partial_\nu \beta_{\mu k}^m \eta_{mj} + \beta_{\mu j}^h \beta_{\nu k}^m \eta_{hm} + \beta_{\mu k}^h \beta_{\nu j}^m \eta_{hm}) \right],
\end{aligned} \tag{E.10}$$

$$\begin{aligned}
\bar{\Gamma}_{\mu\nu}^i &= G^{i\beta} \bar{\Gamma}_{\mu\nu,\beta} + G^{ij} \bar{\Gamma}_{\mu\nu,j} \\
&= \lambda^{\beta\rho} y^l \beta_{\rho l}^i (\Gamma_{\mu\nu,\beta} - y^k (\partial_\mu \alpha_{k\nu\beta} + \partial_\nu \alpha_{k\beta\mu} - \partial_\beta \alpha_{k\mu\nu}) \\
&\quad + \frac{1}{2} y^k y^l [\partial_\mu (\alpha_{k\nu\rho} g^{\rho\sigma} \alpha_{l\sigma\beta} + \beta_{\nu k}^h \beta_{\beta l}^m \eta_{hm}) \\
&\quad + \partial_\nu (\alpha_{k\beta\rho} g^{\rho\sigma} \alpha_{l\sigma\mu} + \beta_{\beta k}^h \beta_{\mu l}^m \eta_{hm}) - \partial_\beta (\alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{l\sigma\nu} + \beta_{\mu k}^h \beta_{\nu l}^m \eta_{hm})] \\
&\quad + (\eta^{ij} + y^n y^l \beta_{\sigma n}^i \beta_{\rho l}^j \lambda^{\sigma\rho}) \left[\alpha_{j\mu\nu} - \frac{1}{2} y^k (\alpha_{j\mu\rho} g^{\rho\sigma} \alpha_{k\sigma\nu} + \alpha_{k\mu\rho} g^{\rho\sigma} \alpha_{j\sigma\nu} \right. \\
&\quad \left. + \partial_\mu \beta_{\nu k}^m \eta_{mj} + \partial_\nu \beta_{\mu k}^m \eta_{mj} + \beta_{\mu j}^h \beta_{\nu k}^m \eta_{hm} + \beta_{\mu k}^h \beta_{\nu j}^m \eta_{hm}) \right],
\end{aligned} \tag{E.11}$$

$$\begin{aligned}
\bar{\Gamma}_{\mu j}^{\beta} &= G^{\beta\nu}\bar{\Gamma}_{\mu j,\nu} + G^{\beta i}\bar{\Gamma}_{\mu j,i} \\
&= \lambda^{\beta\nu} \left[-\alpha_{j\mu\nu} + \frac{1}{2}y^k (\alpha_{j\mu\rho}g^{\rho\sigma}\alpha_{k\sigma\nu} + \alpha_{k\mu\rho}g^{\rho\sigma}\alpha_{j\sigma\nu} \right. \\
&\quad \left. + \partial_\nu\beta_{\mu k}^m\eta_{mj} - \partial_\mu\beta_{\nu k}^m\eta_{mj} + \beta_{\mu j}^h\beta_{\nu k}^m\eta_{hm} + \beta_{\mu k}^h\beta_{\nu j}^m\eta_{hm} \right) \\
&\quad \left. + \lambda^{\beta\nu}y^k\beta_{\nu k}^i\beta_{\mu i}^m\eta_{mj} \right] \\
&= \lambda^{\beta\nu} \left[-\alpha_{j\mu\nu} + \frac{1}{2}y^k (\alpha_{j\mu\rho}g^{\rho\sigma}\alpha_{k\sigma\nu} + \alpha_{k\mu\rho}g^{\rho\sigma}\alpha_{j\sigma\nu} \right. \\
&\quad \left. + \partial_\nu\beta_{\mu k}^m\eta_{mj} - \partial_\mu\beta_{\nu k}^m\eta_{mj} - \beta_{\mu j}^h\beta_{\nu k}^m\eta_{hm} + \beta_{\mu k}^h\beta_{\nu j}^m\eta_{hm} \right) \\
&= \lambda^{\beta\nu} \left[-\alpha_{j\mu\nu} + \frac{1}{2}y^k (\alpha_{j\mu\rho}g^{\rho\sigma}\alpha_{k\sigma\nu} + \alpha_{k\mu\rho}g^{\rho\sigma}\alpha_{j\sigma\nu} + f_{\nu\mu k}^m\eta_{mj}) \right], \quad (E.12)
\end{aligned}$$

$$\bar{\Gamma}_{ij}^{\beta} = G^{\beta\nu}\bar{\Gamma}_{ij,\nu} + G^{\beta k}\bar{\Gamma}_{ij,k} = 0, \quad (E.13)$$

$$\begin{aligned}
\bar{\Gamma}_{\mu j}^i &= G^{i\nu}\bar{\Gamma}_{\mu j,\nu} + G^{im}\bar{\Gamma}_{\mu j,m} \\
&= \lambda^{\nu\sigma}y^l\beta_{\sigma l}^i \left[-\alpha_{j\mu\nu} + \frac{1}{2}y^k (\alpha_{j\mu\rho}g^{\rho\sigma}\alpha_{k\sigma\nu} + \alpha_{k\mu\rho}g^{\rho\sigma}\alpha_{j\sigma\nu} \right. \\
&\quad \left. + \partial_\nu\beta_{\mu k}^m\eta_{mj} - \partial_\mu\beta_{\nu k}^m\eta_{mj} + \beta_{\mu j}^h\beta_{\nu k}^m\eta_{hm} + \beta_{\mu k}^h\beta_{\nu j}^m\eta_{hm} \right) \\
&\quad \left. + (\eta^{im} + y^k y^l\beta_{\sigma k}^i\beta_{\rho l}^m\lambda^{\sigma\rho})\beta_{\mu m}^n\eta_{nj} \right] \\
&= -\beta_{\mu j}^i + \lambda^{\nu\sigma}y^l\beta_{\sigma l}^i \left[-\alpha_{j\mu\nu} + \frac{1}{2}y^k (\alpha_{j\mu\rho}g^{\rho\sigma}\alpha_{k\sigma\nu} + \alpha_{k\mu\rho}g^{\rho\sigma}\alpha_{j\sigma\nu} + f_{\nu\mu k}^m\eta_{mj}) \right], \quad (E.14)
\end{aligned}$$

$$\bar{\Gamma}_{ij}^m = G^{m\nu}\bar{\Gamma}_{ij,\nu} + G^{mn}\bar{\Gamma}_{ij,n} = 0, \quad (E.15)$$

Let us consider only zero-order in y^k .

$$\bar{\Gamma}_{\mu\nu}^{\sigma} = \Gamma_{\mu\nu}^{\sigma}, \quad (E.16)$$

$$\bar{\Gamma}_{\mu\nu}^i = \alpha_{\mu\nu}^i, \quad (E.17)$$

$$\bar{\Gamma}_{\mu j}^{\beta} = -g^{\beta\nu}\alpha_{j\mu\nu} = \alpha_{j\mu}^{\beta}, \quad (E.18)$$

$$\bar{\Gamma}_{ij}^{\beta} = 0, \quad (E.19)$$

$$\bar{\Gamma}_{\mu j}^i = -\beta_{\mu j}^i, \quad (E.20)$$

$$\bar{\Gamma}_{ij}^m = 0. \quad (E.21)$$

APPENDIX F

CLASSICAL MECHANICS ON A SUBMANIFOLD

As was mentioned in the introduction, in classical mechanics the motion of a particle constrained to a submanifold of the configuration space is usually treated in two different approaches, the *Newtonian* and *Lagrangian approach*. In this appendix we will deduce the equations of motion in the Newtonian approach in two different forms. First, we will use holonomic constraints (Lagrange multipliers) and then the classic version of the confining potential approach.

F.1 Lagrange Multipliers

A classical system constrained to a submanifold can be described (in the adapted coordinates frame) using the following Lagrangian

$$L = \frac{1}{2} G_{AB} \dot{X}^A \dot{X}^B - \lambda_i X^i, \quad (\text{F.1})$$

where $X^A = (x^\mu, y^i)$ and the λ_i are the Lagrange multipliers. The dynamics will be determined by the Euler-Lagrange equation

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{X}^C} \right) - \frac{\partial L}{\partial X^C} = 0, \quad (\text{F.2})$$

which yields the following set of coupled equations:

$$\ddot{X}^\mu + \bar{\Gamma}_{AB}^\mu \dot{X}^A \dot{X}^B = 0, \quad (\text{F.3})$$

$$\ddot{X}^i + \bar{\Gamma}_{AB}^i \dot{X}^A \dot{X}^B = G^{ij} \lambda_j, \quad (\text{F.4})$$

$$X^i = 0. \quad (\text{F.5})$$

The last equation implies that $\dot{X}^i = 0$ and $\ddot{X}^i = 0$, inserting these conditions into (F.3) and (F.4) yields to the well-know geodesic equation

$$\ddot{x}^\mu + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta = 0, \quad (\text{F.6})$$

and an equation for the Lagrange multipliers

$$\alpha^i_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \eta^{ij} \lambda_j. \quad (\text{F.7})$$

F.2 Confining Potential Approach in Classical Mechanics

Now, it is well-known that holonomic constraints in classical mechanics represent an *idealization* of a very strong force that restricts the configuration space to a submanifold [1, 256]. In analogy with what was done in the quantum case, we apply the confining potential approach to classical mechanics, which is nothing more than an extension of what has been previously performed in [1, 2, 256]. So, we introduce a potential in the Lagrangian that produces a force which tends to confine the particle on the submanifold [1, 2]. In the adapted coordinates frame the Lagrangian and the Hamiltonian are given respectively by

$$L = \frac{1}{2} \gamma_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} \eta_{ij} (\dot{y}^i - y^k \beta_{\mu k}^i \dot{x}^\mu) (\dot{y}^j - y^l \beta_{\nu l}^j \dot{x}^\nu) - \frac{K}{2\epsilon} (y_i y^i) \quad (\text{F.8})$$

and

$$H = \frac{1}{2} \gamma^{\mu\nu} (p_\mu + y^k \beta_{\mu k}^i \pi_i) (p_\nu + y^l \beta_{\nu l}^j \pi_j) + \frac{1}{2} \eta^{ij} \pi_i \pi_j + \frac{K}{2\epsilon} (y_i y^i), \quad (\text{F.9})$$

where $\pi_i = \frac{\partial L}{\partial \dot{y}^i}$ and $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$ are the canonically conjugate momentum to y^i and x^μ respectively. The Euler-Lagrange equation yields

$$\ddot{x}^\lambda + \bar{\Gamma}_{\mu\nu}^\lambda \dot{x}^\mu \dot{x}^\nu + 2\bar{\Gamma}_{\mu j}^\lambda \dot{x}^\mu \dot{y}^j + \bar{\Gamma}_{ij}^\lambda \dot{y}^i \dot{y}^j = 0, \quad (\text{F.10})$$

$$\ddot{y}^k + \bar{\Gamma}_{\mu\nu}^k \dot{x}^\mu \dot{x}^\nu + 2\bar{\Gamma}_{\mu j}^k \dot{x}^\mu \dot{y}^j + \bar{\Gamma}_{ij}^k \dot{y}^i \dot{y}^j = -\frac{K}{\epsilon} y^k. \quad (\text{F.11})$$

In the limit of an infinite force ($\epsilon \rightarrow 0$) the potential energy grows and the coordinates y^i is increasingly close to $y_i = 0$. In this limit, the last term in (F.11) dominates and, with the initial condition $y_i \approx 0$, the particle remains at all times there. A rescaling in the normal coordinates ($y^i \rightarrow \epsilon y^i$) allows a perturbation expansion in ϵ , so that Eq. (F.11) can be rewritten as

$$\ddot{x}^\lambda + \Gamma_{\mu\nu}^\lambda \dot{x}^\mu \dot{x}^\nu + O(\epsilon) = 0. \quad (\text{F.12})$$

Let us note the absence of any term associated with torsion. If we compare this with the quantum case, we see that in classical physics, extrinsic torsion plays no role. This is because under the rescaling of normal coordinates the angular momentum operator $L_{ij} = i\hbar(y_j \frac{\partial}{\partial y^i} - y_i \frac{\partial}{\partial y^j})$ is invariant at quantum level. However, it transforms as $L_{ij} = y_j \pi_i - y_i \pi_j \rightarrow \epsilon^2 L_{ij}$ at classical level. The equation of motion obtained by applying the confining potential approach to classical mechanics, Eq. (F.12), is in agreement with the conventional treatment of holonomic constraints, Eq. (F.6), confirming that the holonomic constraints also could have been defined as the limiting case of a system with a large potential energy. The conventional result of classical mechanics, as well as our result, differs from that found in Ref. [48] because proper rescaling of the π_i was not taken into account.

APPENDIX G

DIRAC EQUATION WITH LINEAR POTENTIAL

Let us consider the time independent Dirac equation with a linear potential,¹ $V_c(q^3) = kq^3/\epsilon$,

$$\mathcal{E}\varphi = H_0\varphi = c\gamma^0\gamma^3p_3\varphi + \gamma^0\frac{kq^3}{\epsilon}\varphi \quad (\text{G.1})$$

where φ is a spinor and $p_3 = -i\hbar\partial_{q^3}$. Using the representation of the γ -matrices given by (4.33), yields

$$\mathcal{E}\varphi_1 = icp_3\varphi_3 + \frac{ck}{\epsilon}q^3\varphi_1, \quad (\text{G.2})$$

$$\mathcal{E}\varphi_2 = -icp_3\varphi_4 - \frac{ck}{\epsilon}q^3\varphi_2, \quad (\text{G.3})$$

$$\mathcal{E}\varphi_3 = -icp_3\varphi_1 - \frac{ck}{\epsilon}q^3\varphi_3, \quad (\text{G.4})$$

$$\mathcal{E}\varphi_4 = icp_3\varphi_2 + \frac{ck}{\epsilon}q^3\varphi_4. \quad (\text{G.5})$$

After some algebra we obtain

$$-\hbar^2\frac{\partial^2\varphi_1}{\partial(q^3)^2} + \frac{(kq^3)^2}{(\epsilon)^2}\varphi_1 = \frac{\mathcal{E}^2}{c^2}\varphi_1 + \frac{\hbar k}{\epsilon}\varphi_3, \quad (\text{G.6})$$

$$-\hbar^2\frac{\partial^2\varphi_3}{\partial(q^3)^2} + \frac{(kq^3)^2}{(\epsilon)^2}\varphi_3 = \frac{\mathcal{E}^2}{c^2}\varphi_3 + \frac{\hbar k}{\epsilon}\varphi_1, \quad (\text{G.7})$$

$$-\hbar^2\frac{\partial^2\varphi_2}{\partial(q^3)^2} + \frac{(kq^3)^2}{(\epsilon)^2}\varphi_2 = \frac{\mathcal{E}^2}{c^2}\varphi_2 + \frac{\hbar k}{\epsilon}\varphi_4, \quad (\text{G.8})$$

$$-\hbar^2\frac{\partial^2\varphi_4}{\partial(q^3)^2} + \frac{(kq^3)^2}{(\epsilon)^2}\varphi_4 = \frac{\mathcal{E}^2}{c^2}\varphi_4 + \frac{\hbar k}{\epsilon}\varphi_2. \quad (\text{G.9})$$

¹Note that, we have rescaled the coordinate $q^3 \rightarrow q^3/\epsilon$ in order to make explicit the behavior of the spinors when $\epsilon \rightarrow 0$.

Adding and subtracting equations (G.6) and (G.7), we obtain the following differential equations for $f = \varphi_1 + \varphi_3$ and $g = \varphi_1 - \varphi_3$

$$-\hbar^2 \frac{\partial^2 f}{\partial (q^3)^2} + \frac{(kq^3)^2}{(\epsilon)^2} f = \frac{\mathcal{E}^2}{c^2} f + \frac{\hbar k}{\epsilon} f, \quad (\text{G.10})$$

$$-\hbar^2 \frac{\partial^2 g}{\partial (q^3)^2} + \frac{(kq^3)^2}{(\epsilon)^2} g = \frac{\mathcal{E}^2}{c^2} g - \frac{\hbar k}{\epsilon} g \quad (\text{G.11})$$

Equations (G.10) and (G.11) are the shifted Schrödinger equations for the harmonic oscillator. Therefore the spectrum and the solutions are respectively

$$\mathcal{E}_n = \pm \sqrt{\frac{2nc\hbar k}{\epsilon}}, \quad n = 0, 1, \dots, \quad (\text{G.12})$$

and

$$f = \left(\frac{a}{\pi}\right)^{1/4} H_n(a^{1/2}q^3) \exp\left(-\frac{a(q^3)^2}{2}\right), \quad (\text{G.13})$$

$$g = \left(\frac{a}{\pi}\right)^{1/4} H_{n-1}(a^{1/2}q^3) \exp\left(-\frac{a(q^3)^2}{2}\right) \quad (\text{G.14})$$

with $a = \frac{k}{c\hbar}$ and $H_{-1} = 0$ (notice that the ground state was lost in previous works [257, 258]). Returning to the functions φ_1 and φ_3 , and proceeding in the same way for (G.8) and (G.9) we obtain for the ground state ($\mathcal{E}_0 = 0$)

$$\varphi_0 = \left(\frac{a}{\pi}\right)^{1/4} \exp\left(-\frac{a(q^3)^2}{2}\right) \omega, \quad (\text{G.15})$$

where ω is a spinor independent of q^3 and with $\omega^\dagger \omega = 1$. It should be pointed out that ω can depend on the coordinates q^μ and be solution of Eq. 4.34, for instance, for zero Gaussian curvature, we can demand that $\omega \propto e^{-iq^\mu p_\mu/\hbar}$ which does not affect the condition $\omega^\dagger \omega = 1$. On the other hand, note that

$$|\varphi_0|^2 = \varphi_0^\dagger \varphi_0 = \left(\frac{a}{\pi}\right)^{1/2} \exp(-a(q^3)^2) \quad (\text{G.16})$$

implies that the probability distribution is a Gaussian with a maximum at $q^3 = 0$. Therefore, we can interpret that the particle is confined in a neighborhood of the surface (see Ref. [50] for a discussion). Moreover, when $\epsilon \rightarrow 0$ we have $a \rightarrow \infty$, so

$$\lim_{a \rightarrow \infty} |\varphi_0|^2 = \delta(q^3). \quad (\text{G.17})$$

Thus for this limit, the particle is completely confined on the surface. In this case, the spinor will be proportional to $\sqrt{\delta(q^3)}$ as was assumed in [46]. Clearly, this case is only an idealization because the particles would become purely two-dimensional objects, and the uncertainty principle would be violated.

APPENDIX H

HEAT KERNEL

For a given elliptic, second-order differential operator $\hat{L}$ in d dimension, one can define an important object known as the heat operator $e^{-\hat{L}s}$ by means the so-called *heat equation* [259]

$$(\partial_s + \hat{L}_x)u(x; s) = 0, \quad s > 0, \quad (\text{H.1})$$

with the initial condition $u(x; s \rightarrow 0) = f(x)$. One also can introduced the so-called *heat kernel* $K(s|x, x')$, with

$$u(x; s) = e^{-\hat{L}s}f(x) = \int d^d x' |g|^{1/2} K(s|x, x') f(x'). \quad (\text{H.2})$$

It is straightforward to prove that the equation and the initial condition for the heat kernel are

$$(\partial_s + \hat{L}_x)K(s|x, x') = 0 \quad \text{for} \quad t > 0, \quad (\text{H.3})$$

and

$$K(s \rightarrow 0|x, x') = \delta(x, x') \equiv |g|^{-1/2} \delta(x - x'), \quad (\text{H.4})$$

respectively [259, 260].

Locally (*e.i.* for x near y and t near 0) for a space that has no boundary, there is an asymptotic expansion

$$K(s|x, x') \sim \frac{e^{-\sigma^2(x, x')/(4s)}}{(4\pi s)^{d/2}} \sum_{n=0}^{\infty} a_n(x, x') s^n, \quad (\text{H.5})$$

where $\sigma^2(x, x')$ is the geodesic distance between two close points x and x' . In the following, we will need to compute only the diagonal matrix elements of the Heat kernel

$$K(s|x, x) \sim \frac{1}{(4\pi s)^{d/2}} \sum_{n=0}^{\infty} a_n(x) s^n, \quad (\text{H.6})$$

where the $a_n(x) \equiv a_n(x, x') \equiv [a_n]$ depends locally on $|g|$, $g^{\mu\nu}$, B_μ and V . For instance, the first three coefficients $[a_n]$ for the second-order differential operator of the form

$$L_E \phi = \frac{1}{|g|^{1/2}} (-i\partial_\mu - B_\mu) [|g|^{1/2} g^{\mu\nu} (-i\partial_\nu - B_\nu) \phi] + V \phi, \quad (\text{H.7})$$

are

$$[a_0] = 1, \quad (\text{H.8})$$

$$[a_1] = \frac{1}{6} R - V, \quad (\text{H.9})$$

$$\begin{aligned} [a_2] = & \frac{1}{2} V^2 - \frac{1}{6} \nabla^\mu \nabla_\mu V - \frac{1}{12} \Omega_{\mu\nu} \Omega^{\mu\nu} - \frac{1}{6} R V + \frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} \\ & + \frac{1}{180} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \frac{1}{30} \nabla^\mu \nabla_\mu R, \end{aligned} \quad (\text{H.10})$$

where $i\Omega_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$ and $\nabla_\mu = \partial_\mu - iB_\mu$ [260].

APPENDIX

THERMODYNAMICS IN D -DIMENSIONS

Let us start with the logarithm of the partition function in d -dimensions [172]

$$\ln Z_s = f(\mu) + sg_d V \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left[\beta\omega + \ln \left(1 + se^{-\beta(\omega-\mu)} \right) + \ln \left(1 + se^{-\beta(\omega+\mu)} \right) \right] \quad (\text{I.1})$$

where μ is the chemical potential, V is the volume of a d -dimensional box, $\omega = (\mathbf{p}^2 + m^2)^{1/2}$, g_s counts the number of degrees of freedom, and $s = -1$ for bosons and $s = +1$ for fermions. The function $f(\mu)$ is related with the Bose-Einstein condensation. The charge density or number density, assumed positive, is given by

$$n_s = \frac{T}{V} \frac{\partial \ln Z_s}{\partial \mu}, \quad (\text{I.2})$$

so, omitting the function $f(\mu)$, the number density is¹

$$\begin{aligned} n_s &= g_d \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left[\frac{1}{e^{\beta(\omega-\mu)} + s} - \frac{1}{e^{\beta(\omega+\mu)} + s} \right] \\ &= g_d \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \int_0^\infty dp p^{d-2} \left[\frac{1}{e^{\beta(\omega-\mu)} + s} - \frac{1}{e^{\beta(\omega+\mu)} + s} \right], \end{aligned} \quad (\text{I.3})$$

with $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$. Expanding in a geometric series

$$\begin{aligned} \frac{1}{e^{\beta(\omega\pm\mu)} + s} &= \frac{e^{-\beta(\omega\pm\mu)}}{1 + se^{-\beta(\omega\pm\mu)}} \\ &= e^{-\beta(\omega\pm\mu)} \sum_{n=0}^{\infty} s^n e^{-n\beta(\omega\pm\mu)} \\ &= \sum_{n=1}^{\infty} s^{n-1} e^{-n\beta\omega} e^{\mp n\beta\mu}, \end{aligned} \quad (\text{I.4})$$

¹Assuming a positive value for the number of particles, the chemical potential is such that $|\mu| \leq m$.

the number density can be rewritten as

$$\begin{aligned} n_s &= g_d \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \int_0^{\infty} dp p^{d-2} s^{n-1} e^{-n\beta\omega} [e^{n\beta\mu} - e^{-n\beta\mu}] \\ &= 2g_d V \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \int_0^{\infty} dp p^{d-2} s^{n-1} e^{-n\beta\omega} \sinh(n\beta\mu). \end{aligned} \quad (\text{I.5})$$

The integral over p can be solved

$$\begin{aligned} \int_0^{\infty} dp p^{d-2} e^{-n\beta(p^2+m^2)^{1/2}} &= m^{d-1} \int_1^{\infty} dt (t^2-1)^{d/2-3/2} t e^{-n\beta m t} \\ &= \frac{2^{\frac{d}{2}-1} \Gamma\left(\frac{d-1}{2}\right) (\beta m n)^{1-\frac{d}{2}} K_{\frac{d}{2}}(mn\beta)}{\sqrt{\pi}}, \end{aligned} \quad (\text{I.6})$$

where we have used the substitution $t = (1 + p^2/m^2)^{1/2}$. Thus, simplifying one can arrive at

$$n_s = g_d m^{d/2} 2^{2-\frac{d}{2}} \pi^{-\frac{d}{2}} \sum_{n=1}^{\infty} s^{n-1} (\beta n)^{1-\frac{d}{2}} K_{\frac{d}{2}}(mn\beta) \sinh(n\beta\mu). \quad (\text{I.7})$$

Integrating this equation, except terms that do not depend on the chemical potential, we obtain the Helmholtz free energy

$$F = -\beta \ln Z_s = -g_d V m^{d/2} 2^{2-\frac{d}{2}} \pi^{-\frac{d}{2}} \sum_{n=1}^{\infty} s^{n-1} (\beta n)^{-\frac{d}{2}} K_{\frac{d}{2}}(mn\beta) \cosh(n\beta\mu). \quad (\text{I.8})$$

APPENDIX J

GREEN'S FUNCTIONS: ANALYTIC CONTINUATION

In this appendix, we will compute the Feynman, retarded and advanced Green's functions¹ for the pseudo-differential operator $\square^\alpha$. As we will show the Green's functions in Minkowski space-time can be obtained from the Euclidean Green's functions by analytic continuation [262].

Consider a pseudo-differential wave equation in a m -dimensional space-time for a scalar field

$$\square^\alpha \phi(x^\mu) = (\partial_{x^0}^2 - \partial_{\mathbf{x}}^2)^\alpha \phi(x^\mu) = 0, \quad (\text{J.1})$$

where $x_\mu = (x^0, \mathbf{x})$. Thus, the Feynman propagator in Minkowski space-time solve the inhomogeneous wave equation with a delta function as the source term

$$(\partial_{x^0}^2 - \partial_{\mathbf{x}}^2)^\alpha G_F(x - x'; m, \alpha) = \delta(x - x'). \quad (\text{J.2})$$

with $\delta(x - x') = \delta(x^0 - x'^0) \delta(\mathbf{x} - \mathbf{x}')$. The Minkowski space-time is related to the Euclidean space by the Wick rotation

$$x_E \equiv (x^4, \mathbf{x}) = (ix^0, \mathbf{x}), \quad (\text{J.3})$$

$$x_E^2 = (x^4)^2 + \mathbf{x}^2 = -(x^0)^2 + \mathbf{x}^2 + i\epsilon = -x^2 + i\epsilon, \quad (\text{J.4})$$

which determine the Feynman Green's function as the analytic continuation of the Euclidean Green's function (we revise later this point). After the Wick rotation we obtain, with $x' = 0$,

$$(-\nabla^2)^\alpha G_E(x; m, \alpha) = (-\partial_{x^4}^2 - \partial_{\mathbf{x}}^2)^\alpha G_E(x_E; m, \alpha) = (-1)^\alpha (\nabla^2)^\alpha G_E(x; m, \alpha) = \delta(x_E). \quad (\text{J.5})$$

¹ As it is well known there are various types of propagator (or Green's functions) depending on the contour chosen when the integration over momenta is taken [63, 261]

where $\delta(x) = i\delta(x_E)$ and $G_F(x; m, \alpha) = iG_E(x_E; m, \alpha)|_{x_E=(ix^0, \mathbf{x})}$. Taking the Fourier transform in Minkowski space

$$G_E(x_E; m, \alpha) = \frac{1}{(2\pi)^{m/2}} \int d^m k \tilde{G}(k_E; m, \alpha) e^{ik_E x_E}, \quad (\text{J.6})$$

and using that

$$\delta(x_E) = \frac{1}{(2\pi)^m} \int d^m k e^{ik_E x_E}, \quad (\text{J.7})$$

we obtain that $\tilde{G}(k_E; m, \alpha) = ((2\pi)^{m/2} (k_E^2)^\alpha)^{-1}$, so,

$$G_E(x_E; m, \alpha) = \frac{1}{(2\pi)^m} \int d^m k \frac{e^{ik_E x_E}}{(k_E^2)^\alpha}, \quad (\text{J.8})$$

This is especially useful since in the Euclidean space k_E^2 is a positive quantity and the problem of the integration contour does not arise [262]. To integrate we can use the hyperspherical coordinates system defined by

$$x_k = \rho \prod_{j=1}^{m-k} \sin \theta_j \cos \theta_{m-k+1}, \quad k = 1, \dots, m \quad (\text{J.9})$$

with $\theta_m = 0$. We choose the k_E such that $k_E x_E = |k| \rho \cos \theta_1$, with $|x_E| = \rho$, then

$$\begin{aligned} G_E(x_E; m, \alpha) &= \frac{1}{(2\pi)^m} \int \frac{e^{i|k|\rho \cos \theta_1}}{(k^2)^\alpha} k^{m-1} \sin^{m-2} \theta_1 \dots \sin \theta_{m-2} dk d\theta_1 \dots d\theta_{m-1} \\ &= \Omega_{m-1} \frac{1}{(2\pi)^m} \int_0^\pi \int_0^\infty \frac{e^{i|k|\rho \cos \theta_1}}{(k^2)^\alpha} k^{m-1} \sin^{m-2} \theta_1 dk d\theta_1, \end{aligned} \quad (\text{J.10})$$

with $\Omega_m = 2\pi^{m/2}/\Gamma(m/2)$. Using that [263]

$$\int_0^\pi e^{i|k|\rho \cos \theta_1} \sin^{m-2} \theta_1 d\theta_1 = \sqrt{\pi} \left(\frac{2}{k\rho} \right)^{\frac{m}{2}-1} \Gamma\left(\frac{m-1}{2}\right) J_{\frac{m}{2}-1}(k\rho), \quad (\text{J.11})$$

and

$$\int_0^\infty k^\mu J_\nu(\rho k) dk = 2^\mu \rho^{-\mu-1} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}, \quad (\text{J.12})$$

where $J_\nu(z)$ are the Bessel functions of the first kind, we arrive at

$$G_E(x_E; m, \alpha) = \frac{1}{4^\alpha \pi^{\frac{m}{2}}} \frac{1}{\rho^{m-2\alpha}} \frac{\Gamma\left(\frac{m}{2} - \alpha\right)}{\Gamma(\alpha)}, \quad m > 2\alpha. \quad (\text{J.13})$$

Consequently, the Feynman Green's function is ($m > 2\alpha$)

$$G_F(x; m, \alpha) = iG_E(x_E; m, \alpha)|_{x_E=(ix^0, \mathbf{x})} = \frac{i}{4^\alpha \pi^{\frac{m}{2}}} \frac{1}{(-x^2 + i\epsilon)^{\frac{m}{2} - \alpha}} \frac{\Gamma\left(\frac{m}{2} - \alpha\right)}{\Gamma(\alpha)}, \quad (\text{J.14})$$

with $\alpha = 1$ and $m = 4$ we reproduce the usual four-dimensional Feynman Green's function (Eq. 3.125 in [264]), and with $m = 3$, we reproduce the Feynman Green's function obtained in Ref. [63]. In the latter the Feynman Green's function were obtained taking appropriately the contour integration and adopting the Feynman prescription [63], Eq. (J.14) is the generalization of this result to m dimensions.

Demanding that the Green's function of the operator $\eta \square^\alpha$ in m dimensions is equal to the Green's function of the operator $\square$ in n dimensions, taking zero $p = n - m$ coordinates, *i.e.* $G_E(x_E; m, \alpha)/\eta = G_E(x_E, y = 0; n, 1)$, we can conclude that

$$\alpha = 1 - \frac{p}{2} \quad \text{and} \quad \eta = \frac{2^p \pi^{\frac{p}{2}}}{\Gamma\left(1 - \frac{p}{2}\right)}. \quad (\text{J.15})$$

Which is an alternative derivation of Eq. (6.25) by explicit calculation of the Green's functions. In the following let us compute the retarded and the advanced propagators which reads [262]

$$G_{A/R}(x; m, \alpha) = 2\theta(\mp x^0) \text{Re } G_F(x; m, \alpha), \quad (\text{J.16})$$

where the minus sign is for the advanced propagator and the plus sign is for the retarded propagator. Thus

$$G_{A/R}(x; m, \alpha) = 2\theta(\mp x^0) \text{Re } iG_E(x_E; m, \alpha)|_{x_E=(ix^0, \mathbf{x})} \quad (\text{J.17})$$

$$= \frac{2\theta(\mp x^0)}{4^\alpha \pi^{\frac{m}{2}}} \frac{\Gamma\left(\frac{m}{2} - \alpha\right)}{\Gamma(\alpha)} \text{Re } \frac{i}{(-x^2 + i\epsilon)^{\frac{m-2\alpha}{2}}}, \quad m > 2\alpha. \quad (\text{J.18})$$

Let us focus on the term $\text{Re } 1/(-x^2 + i\epsilon)^{m-2\alpha}$. We need to consider two cases separately.

Even case: If $m - 2\alpha = 2q - 2$ with $q \in \mathbb{Z}$, then $m > 2\alpha \Rightarrow n > 1$

$$\frac{1}{(-x^2 + i\epsilon)^{\frac{m-2\alpha}{2}}} = \frac{1}{(-x^2 + i\epsilon)^{q-1}}. \quad (\text{J.19})$$

With some effort it is possible to prove the following identity

$$\Gamma(q-1) \frac{1}{(-x^2 + i\epsilon)^{q-1}} = \left(-\frac{1}{2r} \frac{\partial}{\partial r}\right)^{q-2} \frac{1}{(-x^2 + i\epsilon)}, \quad (\text{J.20})$$

with $x^2 = (x^0)^2 - r^2$. Besides using that [261, 265]

$$-\frac{1}{x^2 - i\epsilon} = -\mathcal{P} \frac{1}{x^2} - i\pi \delta(x^2) \quad (\text{J.21})$$

where $\mathcal{P}$ denotes the principal value, we obtain

$$G_{A/R}(x; m, \alpha) = \frac{2\theta(\mp x^0)}{4^\alpha \pi^\alpha \Gamma(\alpha)} \left(-\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{\frac{m}{2} - \alpha - 1} \delta(x^2), \quad m > 2\alpha, \quad (\text{J.22})$$

which can be rewritten taking into account that

$$\delta(x^2) = \delta((x^0)^2 - r^2) = \frac{1}{2r} [\delta(x^0 - r) + \delta(x^0 + r)], \quad r > 0, \quad (\text{J.23})$$

from which we obtain

$$G_A(x; m, \alpha) = \frac{1}{4^\alpha \pi^\alpha \Gamma(\alpha)} \left(-\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{\frac{m}{2} - \alpha - 1} \frac{\delta(x^0 + r)}{r}, \quad m > 2\alpha \quad (\text{J.24})$$

$$G_R(x; m, \alpha) = \frac{1}{4^\alpha \pi^\alpha \Gamma(\alpha)} \left(-\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{\frac{m}{2} - \alpha - 1} \frac{\delta(x^0 - r)}{r}, \quad m > 2\alpha \quad (\text{J.25})$$

In the case of $\alpha = 1$ these results are in agreement with the Green's functions for the m -Dimensional wave equation, with m even [266].

Odd case: If $m - 2\alpha = 2q - 1$ with $q \in \mathbb{Z}$, then $m > 2\alpha \Rightarrow q > 1/2$, in this case

$$\frac{1}{(-x^2 + i\epsilon)^{\frac{m-2\alpha}{2}}} = \frac{1}{(-x^2 + i\epsilon)^{q-\frac{1}{2}}}. \quad (\text{J.26})$$

With some effort it is possible to prove the following identity

$$\frac{1 \times 3 \times \cdots \times (m - 2\alpha - 2)}{2^{q-1}} \frac{1}{(-x^2 + i\epsilon)^{q-\frac{1}{2}}} = \left(-\frac{1}{2r} \frac{\partial}{\partial r} \right)^{q-1} \frac{1}{(-x^2 + i\epsilon)^{1/2}}. \quad (\text{J.27})$$

Here we can use the integral formula [265]

$$(z \pm i\epsilon)^\lambda = |z|^\lambda e^{\pm i\pi\lambda} \theta(-z) + (z)^\lambda \theta(z), \quad \lambda > -1 \quad (\text{J.28})$$

which in particular gives

$$(-x^2 + i\epsilon)^{-1/2} = -i|x^2|^{-1/2} \theta(x^2) + (-x^2)^{-1/2} \theta(-x^2), \quad \lambda > -1. \quad (\text{J.29})$$

Inserting Eqs. (J.27) and (J.29) into Eq. (J.17), we obtain for $m > 2\alpha$

$$\begin{aligned} G_{A/R}(x; m, \alpha) &= \frac{2\theta(\mp x^0)}{4^\alpha \pi^{\alpha+1/2} \Gamma(\alpha)} \frac{2^{\frac{m-2\alpha-1}{2}} \Gamma\left(\frac{m}{2} - \alpha\right)}{1 \times 3 \times \cdots \times (m - 2\alpha - 2)} \left(-\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{\frac{m-2\alpha-1}{2}} \frac{\theta(x^2)}{|x^2|^{1/2}}, \\ &= \frac{2}{4^\alpha \pi^{\alpha+1/2} \Gamma(\alpha)} \frac{2^{\frac{m-2\alpha-1}{2}} \Gamma\left(\frac{m}{2} - \alpha\right)}{1 \times 3 \times \cdots \times (m - 2\alpha - 2)} \\ &\quad \times \left(-\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{\frac{m-2\alpha-1}{2}} \frac{\theta(\mp x^0 - r)}{|(x^0)^2 - r^2|^{1/2}}. \end{aligned} \quad (\text{J.30})$$

in the last line we have employed the identity $\theta(\mp x^0)\theta(x^2) = \theta(\mp x^0 - r)$. In the case of $\alpha = 1$ we arrive at

$$G_{A/R}(x; m, 1) = \frac{1}{2\pi} \left(-\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{\frac{m-3}{2}} \frac{\theta(\mp x^0 - r)}{|(x^0)^2 - r^2|^{1/2}}, \quad (\text{J.31})$$

this results are in agreement with the Green's functions for the m -Dimensional wave equation with m odd [267, 268] (we noticed a misprint in the results presented in [266]). Finally, we consider the case $m = 2\alpha$, returning to the integral (J.12) and introducing a scale $\tilde{\rho}_0$

$$\int_{1/\tilde{\rho}_0}^{\infty} k^{-m/2} J_{m/2-1}(\rho k) dk = -\frac{2^{-\frac{m}{2}-1} m r^{\frac{m}{2}-1}}{\Gamma\left(\frac{m}{2} + 1\right)} \log\left(\frac{\rho}{\rho_0}\right) \quad (\text{J.32})$$

where ρ_0 and $\tilde{\rho}_0$ are related by an irrelevant constant. Thus, the Euclidean Green's function is

$$G_E(x_E; m, \alpha = m/2) = -\frac{2^{-m-1} \pi^{-m/2} m}{\Gamma\left(\frac{m}{2} + 1\right)} \log\left(\frac{x_E^2}{\rho_0^2}\right). \quad (\text{J.33})$$

Using the identity [261, 265]

$$\text{Log}(z \pm i\epsilon) = \text{Log}|z| \pm i\pi\theta(-z), \quad (\text{J.34})$$

we obtain

$$G_{A/R}(x; m, \alpha) = -\frac{2^{-m} \pi^{-m/2} m}{\Gamma\left(\frac{m}{2} + 1\right)} \theta(\mp x^0) \text{Re } i\text{Log}\left(-\frac{x^2}{\rho_0^2} + i\epsilon\right) \quad (\text{J.35})$$

$$= -\frac{2^{-m} \pi^{-m/2} m}{\Gamma\left(\frac{m}{2} + 1\right)} \theta(\mp x^0) \text{Re} \left[i\text{Log}\left(\left|\frac{x^2}{\rho_0^2}\right|\right) - \pi\theta(x^2) \right] \quad (\text{J.36})$$

$$= \frac{2^{-m} \pi^{-m/2} m}{\Gamma\left(\frac{m}{2} + 1\right)} \theta(\mp x^0) \theta(x^2) \quad (\text{J.37})$$

$$= \frac{2^{-m} \pi^{1-m/2} m}{\Gamma\left(\frac{m}{2} + 1\right)} \theta(\mp x^0 - r), \quad (\text{J.38})$$

the case of $\alpha = 1$ implies that $m = 2$, so,

$$G_{A/R}(x; 2, \alpha) = \frac{1}{2} \theta(\mp x^0 - r), \quad (\text{J.39})$$

which is the well known Green's function in two dimensions [267].

APPENDIX K

CONSTANT REAL AND PSEUDO MAGNETIC FIELDS

In the presence of a constant real (B) and pseudo (b) magnetic fields, one can choose the real potential and the pseudo-potential as $\vec{A} = (0, Bx)$ and $\vec{a}^{35} = (0, bx)$, respectively. For the valley $\vec{K}$ and with $m \geq 0$, it is straightforward to find the spectrum and the solutions of Eq. (7.5). However, we need to find them independently for the cases $e(B - b) > 0$ and $e(B - b) < 0$. For the first case $e(B - b) > 0$, the spectrum and solutions are given by

$$E_n^- = \pm \sqrt{m^2 + 2|eB - eb|n}, \quad (\text{K.1})$$

$$\psi_K^{(+)} = N_n^- e^{-i|E_n^-|t + ipy} \begin{pmatrix} (|E_n^-| + m)I^-(n, p, x) \\ -\sqrt{2|eB - eb|n}I^-(n - 1, p, x) \end{pmatrix}, \quad (\text{K.2})$$

$$\psi_K^{(-)} = N_n^- e^{i|E_n^-|t - ipy} \begin{pmatrix} \sqrt{2|eB - eb|n}I^-(n, -p, x) \\ (|E_n^-| + m)I^-(n - 1, -p, x) \end{pmatrix}, \quad (\text{K.3})$$

where

$$N_n^\pm = \frac{1}{\sqrt{2|E_n^\pm|(|E_n^\pm| + m)}}, \quad (\text{K.4})$$

$$I^\pm(n, x, p) = \frac{|eB \pm eb|^{\frac{1}{4}}}{\sqrt{2^n n! \pi^{\frac{1}{4}}}} H_n \left[\sqrt{|eB \pm eb|} \left(x - \frac{p}{|eB \pm eb|} \right) \right] \times e^{-\frac{|eB \pm eb|}{2} \left(x - \frac{p}{|eB \pm eb|} \right)^2} \quad (\text{K.5})$$

with $I(n = -1, p, x) = 0$ and $E_n^+ = \pm \sqrt{m^2 + 2|eB + eb|n}$. Note that the lowest Landau level (LLL), describes particle (fermion) states with energy $E_0 = m \geq 0$. In order to find the solutions for the case $e(B - b) < 0$, one can use the charge-conjugate operator, Eq. (7.9), $\mathcal{C}\psi = -\gamma^2(\bar{\psi})^T = \sigma_1\psi^*$ [269], so, the solutions are given by

$$\psi_K^{(+)} = -N_n^- e^{-i|E_n^-|t + ipy} \begin{pmatrix} (|E_n^-| + m)I^-(n - 1, p, x) \\ \sqrt{2|eB - eb|n}I^-(n, p, x) \end{pmatrix}, \quad (\text{K.6})$$

$$\psi_K^{(-)} = -N_n^- e^{i|E_n^-|t - ipy} \begin{pmatrix} -\sqrt{2|eB - eb|n} I^-(n-1, -p, x) \\ (|E_n^-| + m) I^-(n, -p, x) \end{pmatrix}, \quad (\text{K.7})$$

Now the LLL, describes hole (antifermion) states.

For the valley $\vec{K}'$, one can find the solution noting that the change of representation is equivalent to the change $m \rightarrow -m$ and $B - b \rightarrow B + b$, in the solutions (K.2), (K.3), (K.6) and (K.7). For the point $\vec{K}'$, the LLL, describes an hole (antifermion) state for $e(B + b) > 0$ and a particle (fermion) state for $e(B + b) < 0$. Hence, we can have a fermion (antifermion) state in the LLL simultaneously to $\vec{K}$ and $\vec{K}'$ if the condition $eb < eB < -eb$ ($eb > eB > -eb$) is satisfied.

APPENDIX L

MAGNETIC CONDENSATE: COMPARING THE FURRY PICTURE WITH THE METHOD VIA FERMION PROPAGATOR

In this Appendix, we will compare the calculation of fermionic condensate in the Furry picture with the method via fermion propagator. Thus, we show that the method via the fermion propagator does not calculate the vacuum expectation value of the product of two field operators (magnetic condensate), but rather a vacuum expectation value of the commutator of two field operators. Finally, we argue that the commutator is the order parameter for chiral (or parity) symmetry breaking and not the magnetic condensate as is often asserted in the literature.

L.1 Magnetic Condensate Via Furry Picture

Let us consider the fermionic condensate via de Furry picture in a constant background magnetic field. The explicit solution can be found in the Appendix (K), taking $b = 0$. Inserting Eq. (K.3) into Eq. (7.28) we obtain that for the irreducible representation $\mathcal{R}_1$ (valley $\vec{K}$), for $eB > 0$ and $m > 0$, the fermion condensate is

$$\begin{aligned} \langle 0 | \bar{\Psi}_K \Psi_K | 0 \rangle &= \sum_{n=0}^{\infty} \int dp \frac{N_n^2}{2\pi} \left[2|eB|nI(n, -p, x)^2 - (|E_n| + m)^2 I(n-1, -p, x)^2 \right] \\ &= -\frac{|eB|}{2\pi} \sum_{n=1}^{\infty} \frac{m}{|E_n|}, \end{aligned} \quad (\text{L.1})$$

with $|E_n| = \sqrt{m^2 + 2e|B|n}$ and where we use that

$$\int dp I(n-1, p, x)^2 = \begin{cases} 0 & \text{if } n = 0, \\ |eB| & \text{if } n > 0, \end{cases} \quad (\text{L.2})$$

and that for $n = 0$ all the terms are zero. Now, for $eB < 0$ inserting Eq. (K.7) into Eq. (7.28) yields

$$\begin{aligned}\langle 0|\bar{\Psi}_K\Psi_K|0\rangle &= \sum_{n=0}^{\infty} \int dp \frac{N_n^2}{2\pi} \left[2|eB|nI(n-1, -p, x)^2 - (|E_n| + m)^2 I(n, -p, x)^2 \right] \\ &= -\frac{|eB|}{2\pi} - \frac{|eB|}{2\pi} \sum_{n=1}^{\infty} \frac{m}{|E_n|},\end{aligned}\quad (\text{L.3})$$

it was clear now that $n = 0$ contribute to the condensate. In a similar way the condensates in the irreducible representation $\mathcal{R}_2$ (valley $\vec{K}'$) are

$$\langle 0|\bar{\Psi}_{K'}\Psi_{K'}|0\rangle = -\frac{|eB|}{2\pi} - \frac{|eB|}{2\pi} \sum_{n=1}^{\infty} \frac{m}{|E_n|}, \quad (\text{L.4})$$

$$\langle 0|\bar{\Psi}_{K'}\Psi_{K'}|0\rangle = -\frac{|eB|}{2\pi} \sum_{n=1}^{\infty} \frac{m}{|E_n|}, \quad (\text{L.5})$$

for $eB > 0$ and $eB < 0$, respectively. In the fermion condensates the term $\sum_{n=1}^{\infty} \frac{1}{|E_n|}$ is in general divergent. However, the results are understood by means of an appropriate analytic continuation.

Eqs. (L.1) and (L.4) are in agreement with what was found in Ref. [217]. When $m < 0$ one just need to exchange results in Eq. (L.1) with the one of Eq. (L.3), and Eq. (L.4) with Eq. (L.5). In the limit $m \rightarrow 0^+$, *i.e.* for massless fermions, the fermionic condensates are

$$\langle 0|\bar{\Psi}_K\Psi_K|0\rangle = \begin{cases} -|eB|/(2\pi) & \text{if } eB < 0, \\ 0 & \text{if } eB > 0, \end{cases} \quad (\text{L.6})$$

and

$$\langle 0|\bar{\Psi}_{K'}\Psi_{K'}|0\rangle = \begin{cases} 0 & \text{if } eB < 0, \\ -|eB|/(2\pi) & \text{if } eB > 0. \end{cases} \quad (\text{L.7})$$

Notably, only if the LLL has negative energy states there is a non-vanished magnetic condensate. The condensate in the 4×4 reducible representation is simply the sum of the irreducible representations

$$\langle 0|\bar{\Psi}\Psi|0\rangle = \langle 0|\bar{\Psi}_K\Psi_K|0\rangle + \langle 0|\bar{\Psi}_{K'}\Psi_{K'}|0\rangle = -\frac{|eB|}{2\pi}, \quad (\text{L.8})$$

which is in agreement with the results of Refs. [210, 215, 217, 227–229].

L.2 Magnetic Condensate Via the Fermion Propagator

Now we consider the usual way to calculate the vacuum condensate using the fermion propagator [204]

$$S_F(x, y) = \langle 0|T[\Psi(x)\bar{\Psi}(y)]|0\rangle, \quad (\text{L.9})$$

where T is the time-ordering operator

$$\langle 0|T[\Psi(x)\bar{\Psi}(y)]|0\rangle = \theta(x_0 - y_0)\langle 0|\Psi(x)\bar{\Psi}(y)|0\rangle - \theta(y_0 - x_0)\langle 0|\bar{\Psi}(y)\Psi(x)|0\rangle, \quad (\text{L.10})$$

being $\theta(x)$ the Heaviside step function. In the presence of a magnetic field, $S_F(x, y)$ can be calculated by using the Schwinger (proper time) approach [226]

$$S_F(x, y) = \exp \left[\int_x^y dx_\mu A_{ext}^\mu \right] \tilde{S}(x - y), \quad (\text{L.11})$$

where the integral is calculated along the straight line, and the Fourier transform of $S_F(x)$ (in Euclidean space) is ($k_3 = -ik_0$)

$$\begin{aligned} \tilde{S}(k) &= -i \int_0^\infty ds \exp \left[-s \left(m^2 + k_3^2 + \mathbf{k}^2 \frac{\tanh(eBs)}{eBs} \right) \right] \\ &\quad \times (-k_\mu \gamma_\mu + m\mathbf{1} - i(k_2 \gamma_2 - k_1 \gamma_2) \tanh(eBs)) (1 - i\gamma_1 \gamma_2 \tanh(eBs)), \end{aligned} \quad (\text{L.12})$$

with $\mathbf{1}$ and γ^μ the 4×4 (2×2) identity matrix and the 4×4 (2×2) γ -matrices in the reducible (irreducible) representations. The magnetic condensate has been calculated as [204, 205, 208]

$$\langle 0|\bar{\Psi}(x)\Psi(x)|0\rangle^S = -\lim_{x \rightarrow y} \text{tr} S_F(x, y), \quad (\text{L.13})$$

where the superscript S emphasizes that the condensate is computed via the propagator. For the reducible representations, in the limit $m \rightarrow 0$, the condensate is [204]

$$\langle 0|\bar{\Psi}(x)\Psi(x)|0\rangle^S = -\frac{|eB|}{2\pi}, \quad (\text{L.14})$$

which is in agreement with Eq. (L.8). However, for the irreducible representations the magnetic condensate is

$$\langle 0|\bar{\Psi}_{K,K'}(x)\Psi_{K,K'}(x)|0\rangle^S = -\frac{|eB|}{4\pi}, \quad (\text{L.15})$$

which is the same for both representations and different from the one what was calculated previously, Eqs. (L.6) and (L.7). To explain why the condensates are different, let us critically review the calculation through propagator¹. If one part of the definition of the propagator Eqs. (L.9) and (L.10), it is clear that $\lim_{x \rightarrow y} \text{tr} S(x, y)$ depends on how one takes the limit, namely

$$\text{tr} \lim_{\substack{x_0 \rightarrow y_0^+ \\ \mathbf{x} \rightarrow \mathbf{y}}} S_F(x, y) = \langle 0|\Psi_i(x)\bar{\Psi}_i(x)|0\rangle^S, \quad (\text{L.16})$$

$$\text{tr} \lim_{\substack{x_0 \rightarrow y_0^- \\ \mathbf{x} \rightarrow \mathbf{y}}} S_F(x, y) = -\langle 0|\bar{\Psi}_i(x)\Psi_i(x)|0\rangle^S. \quad (\text{L.17})$$

¹ Using the so-called Ritus eigenfunctions method, has recently been found Eq. (L.15), however, this method also uses the propagator to calculate the condensate.

Using Eq. (7.29) one can show that, in the limit $m \rightarrow 0^+$, the fermion condensates $\langle \Psi_i \bar{\Psi}_i \rangle$ for the irreducible representations are

$$\langle 0 | \Psi_K(x) \bar{\Psi}_K(x) | 0 \rangle = \begin{cases} 0 & \text{if } eB < 0, \\ |eB|/(2\pi) & \text{if } eB > 0, \end{cases} \quad (\text{L.18})$$

$$\langle 0 | \Psi_{K'}(x) \bar{\Psi}_{K'}(x) | 0 \rangle = \begin{cases} |eB|/(2\pi) & \text{if } eB < 0, \\ 0 & \text{if } eB > 0, \end{cases} \quad (\text{L.19})$$

and

$$\langle 0 | \Psi(x) \bar{\Psi}(x) | 0 \rangle = \langle 0 | \Psi_K(x) \bar{\Psi}_K(x) | 0 \rangle + \langle 0 | \Psi_{K'}(x) \bar{\Psi}_{K'}(x) | 0 \rangle = \frac{|eB|}{2\pi}, \quad (\text{L.20})$$

for the reducible representation. It is clear that for irreducible representations neither $\langle \bar{\Psi}_{K,K'} \Psi_{K,K'} \rangle$ nor $\langle \Psi_{K,K'} \bar{\Psi}_{K,K'} \rangle$ coincide with $\langle \bar{\Psi}_{K,K'} \bar{\Psi}_{K,K'} \rangle^S$. The calculation of the condensate in Refs. [204, 205], was actually calculated by taking $x = y$. Formally this will lead to calculate the propagator in $x_0 = y_0$ depending on $\theta(0)$. In the literature there is not a consensus of what should be the value of $\theta(0)$. For instance, $\theta(0)$ can be 1, 0 or 1/2 (page 24 in [270]). However, the only consistent value with Eqs. (L.6), (L.7), (L.8), (L.18), (L.19) and (L.20) is the value of 1/2, since

$$\text{tr} S_F(x, x) = \frac{\langle 0 | \Psi(x) \bar{\Psi}(x) | 0 \rangle}{2} - \frac{\langle 0 | \bar{\Psi}(x) \Psi(x) | 0 \rangle}{2} = \frac{1}{2} \langle 0 | [\Psi(x), \bar{\Psi}(x)] | 0 \rangle \quad (\text{L.21})$$

so,

$$\langle 0 | \bar{\Psi}(x) \Psi(x) | 0 \rangle = \langle 0 | \Psi(x) \bar{\Psi}(x) | 0 \rangle - 2 \text{tr} S_F(x, x), \quad (\text{L.22})$$

which gives us the correct value for all condensates. In fact this last choice was the Schwinger's choice: “the average of the forms obtained by letting y approach x from the future, and from the past” [226]. Therefore, the Heaviside step function in the fermion propagator should be understood as

$$\theta(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/2 & \text{if } t = 0 \\ 1 & \text{if } t > 0. \end{cases} \quad (\text{L.23})$$

In summary, $-\text{tr} S_F(x, x) \neq \langle 0 | \bar{\Psi}(x) \Psi(x) | 0 \rangle$. We believe that this has not been notice before since, in particular, when the condensate is computed in $2 + 1$ dimensions, in a reducible representation, the two values match, which as we showed is just a coincidence².

²One can show that in $3 + 1$ dimensions the two values also match, see Ref. [207] for the calculation via fermion propagator and reference [229] for the calculation in the Furry picture.

L.3 Order Parameter

It is widely claimed in the literature that by the reducible representations the fermion condensate ($\langle\bar{\Psi}\Psi\rangle$) is the order parameter of dynamical chiral symmetry breaking [204–206, 210, 227]. However, as we have just shown, a part of the literature calculates $\langle\bar{\Psi}\Psi\rangle$ and another part calculates $\langle[\bar{\Psi}, \Psi]\rangle/2$ as the order parameter, always assuming that is being calculated $\langle\bar{\Psi}\Psi\rangle$. Thus, we may ask what is the order parameter for chiral (or parity) symmetry breaking. Since in reducible representations the two values match, we will use the irreducible representations to solve this puzzle.

Although in irreducible representations we have not a chiral symmetry, as we mentioned above, the mass term breaks parity and time-reversal symmetry. If the fermion condensate is the order parameter of dynamical parity and time-reversal symmetry breaking, Eqs. (L.6) and (L.7) would lead us to conclude that these broken symmetries depend on the orientation of the magnetic field. Moreover, since the fermionic condensate is zero for $eB > 0$ ($eB < 0$) in the representation $\mathcal{R}_1$ ($\mathcal{R}_2$) there would not be a dynamical symmetry breaking in this case and therefore would not have a magnetically induced mass. However, if the order parameter is the commutator, which is independent of the orientation of the magnetic field and the representation, we would obtain a magnetically induced mass for $eB \neq 0$.

In QED₂₊₁ the dynamical mass generation has already been investigated [204, 214, 215, 218, 243, 244]. In particular, the dynamical mass generation with a two-component fermion was examined [218, 243]. From the results of these works it can be concluded that there is no evidence that the dynamical mass depends on the orientation of the magnetic field. Moreover, under some assumptions an explicit formula was found for the dynamical mass when it is much smaller than the magnetic field [218]. One can show that the dynamic mass found is independent of the orientation and the representation of the γ -matrices. Therefore, the order parameter for parity (or chiral) symmetry breaking is not the vacuum expectation value of the product of two field operators (magnetic condensate), but rather a vacuum expectation value of the commutator of two field operators. The latter is equivalent to the trace of the fermion propagator evaluated in equal times and positions.

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