Universidade de São Paulo
Instituto de Física

Quantificadores de Contextualidade construídos a partir de funções rendimento

Tiago da Costa Santos

Orientadora: Profa. Dra. Bárbara Lopes Amaral

Dissertação de mestrado apresentada ao Instituto de Física da Universidade de São Paulo, como requisito parcial para a obtenção do título de Mestre(a) em Ciências.

Profa. Dra. Bárbara Lopes Amaral - Orientadora (USP)
Prof. Dr. Fernando Roberto de Luna Parisio Filho (UFPE)
Prof. Dr. Rafael Chaves Souto Araújo (UFRN)

São Paulo

2022

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Santos, Tiago da Costa

Quantificadores de contextualidade construídos a partir de funções de rendimento. São Paulo, 2022.

Dissertação (Mestrado) – Universidade de São Paulo. Instituto de Física. Depto. de Física Matemática.

Orientador(a): Profa. Dra. Bárbara Lopes Amaral

Área de Concentração: Fundamentos da Teoria Quântica

Unitermos: 1. Teoria Quântica; 2. Contextualidade; 3. Teoria de Recursos

USP/IF/SBI-016/2022

University of São Paulo
Physics Institute

Contextuality Quantifiers built from cost and yield functions

Tiago da Costa Santos

Supervisor: Prof. Dr. Bárbara Lopes Amaral

Dissertation submitted to the Physics Institute of the University of São Paulo in partial fulfillment of the requirements for the degree of Master of Science.

Examining Committee:
Prof. Dr. Bárbara Lopes Amaral - Supervisor (Physics Institute - USP)
Prof. Dr. Fernando Roberto de Luna Parisio Filho (UFPE)
Prof. Dr. Rafael Chaves Souto Araújo (UFRN)

São Paulo
2022

A todos que até aqui me ajudaram, em especial meus pais, Margarete e Niraildo, minha orientadora e a incrível Jojo.

Acknowledgements

This project was supported financially by brazilian funding agency Conselho Nacional de Desenvolvimento Científica e Tecnológico (CNPQ) - Regular program, grant number $131630/2019 - 9$.

Abstract

Resource theories constitute a powerful theoretical framework and a tool that captures, in an abstract structure, pragmatic aspects of the most varied theories and processes. For physical theories, while this framework deals directly with questions about the concrete possibilities of carrying out tasks and processes, resource theories also make it possible to recast these already established theories on a new language, providing not only new perspectives on the potential of physical phenomena as valuable resources for technological development, for example, but they also provide insights into the very foundations of these theories. In this work, we will investigate some properties of a resource theory for quantum contextuality, an essential characteristic of quantum phenomena that ensures the impossibility of interpreting the results of quantum measurements as revealing properties that are independent of the set of measurements being made. We will present the resource theory to be studied and investigate certain global properties of this theory using tools and methods that, although already developed and studied by the community in other resource theories, had not yet been used to characterize resource theories of contextuality. In particular, we will use the so called cost and yield monotones, making use of their power in the study of resource theories for nonlocality, in an attempt to extend these results to this more general class of phenomena, contextuality.

Keywords: resource theory, quantum physics, contextuality, nonlocality, monotones, global properties.

Resumo

Teorias de recurso constituem um poderoso formalismo teórico e uma ferramenta que captura, em uma estrutura abstrata, aspectos pragmáticos das mais variadas teorias e processos. Para teorias físicas, ao mesmo tempo em que esse formalismo lida diretamente com questões sobre as possibilidades concretas de realização de tarefas e processos, as teorias de recurso também possibilitam recolocar essas teorias já estabelecidas em uma nova linguagem, fornecendo não somente novos olhares sobre o potencial de fenômenos físicos enquanto valiosos recursos para o avanço tecnológico, por exemplo, mas também fornecem intuições sobre os próprios fundamentos dessas teorias. Nesse trabalho, investigaremos algumas propriedades de uma teoria de recursos para contextualidade quântica, característica essencial de fenômenos quânticos que assegura a impossibilidade de se interpretar os resultados de medições quânticas como reveladoras de propriedades que independem do conjunto de medições sendo feitas. Apresentaremos a teoria de recurso a ser estudada e investigaremos certas propriedades globais dessa teoria com o uso de ferramentas e métodos que, embora já desenvolvidos e estudados pela comunidade em relação a outras teorias de recurso, não haviam ainda sido utilizados para caracterizar teorias de recurso de contextualidade. Em particular, utilizaremos os chamados quantificadores de custo e ganho, bem como seu uso no estudo de teorias de recurso para não-localidade, na tentativa de estender esses resultados para essa classe mais geral de fenômenos, a contextualidade.

Palavras chave: teoria de recursos, física quântica, contextualidade, não-localidade, monótonos, propriedades globais.

Sumário

Capítulo 1

Introductory Considerations and Notions

With the advent of quantum information theory, which brought to physics techniques and methods from computer science, the laws of physics began to be probed through new sets of questions. In particular, there arose an interest in finding out what is possible within a theory given a set of resources and operations, that is, what the theory allows one to actually perform (Horodecki and Oppenheim, 2013). It is in this spirit that one can distinguish two traditions of theory-building: a *dynamicist* tradition, in which the aim is to describe and predict the behaviour of physical systems in the absence of intervention and regardless of knowledge about them; and a *pragmatic* tradition, in which one is trying to understand and describe how and how much can a physical system be known and controlled through human intervention (Coecke et al., 2016). It is important to stress that this distinction amongst traditions of theory development in no way should be taken to mean that each and every theory of physics belongs exclusively to one or the other of these realms. The distinction should only illuminate the different kinds of interests that can drive the direction of scientific development *in each field*.

Now, one of the ways in which this pragmatic perspective has come to be formalized by the community is through the so called **resource theories**. A resource theory is a framework that aims for characterization of physical states and processes in terms of availability, quantification and interconversion of resourceful objects (Coecke et al., 2016). In such a framework a chosen property is treated as an operational resource (Amaral, 2019) and physical phenomena are studied in order to better leverage this specific resource. In a resource theory one tries to answer questions such as: Which (resourceful) objects can be converted into which other ones and what are the ways in which a conversion can be made? What is the rate at which arbitrarily many copies of one object can be converted into arbitrarily many copies of another? Can a catalyst help in making an impossible transformation possible? And other related questions.

Before actually getting into the formal structure, it can be illuminating to illustrate the kind of theory we are talking about with some examples, which will also help us to gain some intuition and motivation about the particular ways in which the framework can be formalized.

Two good examples of scientific fields that have more of a pragmatic flavour, or at least had in their beginnings, are Thermodynamics and Chemistry. Both began as endeavours to determine and better understand the ways in which resourceful systems and materials could be transformed and used for one's advantage. Alchemy sought to transform basic metals into nobler ones, and one of the endeavors that marked the early days of thermodynamics was the study of thermal nonequilibrium and its resourcefulness for doing useful work. Even today, after so much development in both fields, this perspective still drives much of the interest from the community (Horodecki and Oppenheim, 2013).

Hence, the main concepts behind this kind of approach are resourceful objects and advantageous transformations amongst these objects. There are many more examples of resource theories and they need not to be extremely practical in purpose or scope. By abstracting the framework one may begin to cast many areas of science in this language and interesting ways of understanding these fields begin to emerge. Even mathematics can be seen as a resource theory in which the resourceful objects are mathematical propositions and the transformations are mathematical proofs, understood as sequences of inference rules (Coecke et al., 2016).

A particularly important (for us) class of resource theories are the *quantum resource theories*, resource theories defined in terms of quantum states, processes, protocols and concepts. Quantum resource theories are an example of a particular way to arrive at a particular resource theory from a theory of physics. In it we have a set of processes — state preparations, transformations or measurements, for example — and we divide this set into costly implementable processes and freely implementable ones. Assuming unlimited availability of elements in the free subset of elements in the theory, one can then study the *structure* that is induced on the costly set, considered then the resourceful objects. This kind of resource theory then is specified by a chosen class of operations, which in the case of quantum a resource theory is a restriction on the set of all quantum operations that can be implemented. Given this restriction, some quantum states will not be accessible from some fixed initial state and thus become resourceful states which could be harnessed by some agent to reach and end not possible via only the free set (Horodecki and Oppenheim, 2013).

An example of a quantum resource theory is the **resource theory of entanglement**: If we choose the restrict two or more parties to classical communication and local quantum operations (LOCC), entangled states become resourceful. And thus the full set of quantum states gets separated between the free set of separable states and the costly set of entangled ones. Given access to the free set (separable states), one cannot achieve an entangled state by LOCC. Moreover, access to entangled states allows one to perform tasks such as quantum teleportation that were not possible only via LOCC and the free set of states (Horodecki and Oppenheim, 2013). There are yet many more examples of the use of resource theoretic framework in quantum information theory and other areas of physics, such as in the study of asymmetry and quantum reference frames, quantum thermodynamics, quantum coherence and superposition, non-Gaussianity and non-Markovianity (Chitambar and Gour, 2019). Furthermore, it has proven advantageous to recast even more foundational concepts of quantum theory, as contextuality and Bell nonlocality, in resource theoretic frameworks.

Amongst the advantages of casting a quantum property in resource theory language, we can cite (Chitambar and Gour, 2019):

- Resource theories are particularly fitting for restricting our attention to operations and procedures that reflect current experimental capabilities, as generally one can associate a particular resource theory to any specific experiment by taking as free operations only those that can be performed within the limitations of the experimental setup available. Thus, such theory is precisely concerned with the particular tasks that can be done with the setup;

- Resource theories provide a means of rigorously comparing the *quantity* of resource present in quantum states or channels. As by construction the amount of resource held by an object is at least equal to the amount in another if one can transform the former into the latter by a free operation in the given theory, by studying the interconversion relations in a theory together with the possibilities of quantification, one is able to establish a pre-order on the set of objects within the theory. This ordering structure offers a lot of insight into the role that the property investigated as a resource plays within the bigger theory as a whole. This particular perspective is a great part of this work;

- Resource theory allows one to better analyze how and what fundamental processes are responsible for a certain phenomenon. By considering the particular restrictions on the set of operations, one can point out, in a systematic manner, what are the physical requirements for performing a specific task. Interestingly, this can lead one to better consider resource trade-offs through decomposing a certain task in terms of free operations and resource consumption. In certain situations it might be really advantageous to know if by making use of more free objects one can lessen resource consumption;

- Because the same framework is applicable to really diverse properties, by studying one property of interest within a particular resource theory one can be actually doing much more as it might lead to identification of structures and applications that are common to resource theories in general. As an example we note that "elegant solutions to the problem of entanglement reversibility emerge when drawing resource-theoretic connections to thermodynamics".

So, given this brief discussion, let us now proceed and describe the mathematical language of the framework of resource theories, introduce and discuss the tools that were relevant for this work.

1.1 Introducing the Framework

We begin by describing the basic mathematical elements of a resource theory. There are different formulations oriented towards different aspects of resource theories, but the line we will follow is mainly in line with works as (Coecke et al., 2016), (Duarte and Amaral, 2018), (Gallego and Aolita, 2017) and (Amaral, 2019). For this, let us set some of the basic ingredients of general resource theories:

1. A set $\mathcal{U}$ of mathematical *objects* that may contain the resource under consideration, together with a subset $\mathcal{F} \subset \mathcal{U}$ whose elements are those which are going to be considered freely available, called *free objects*. The rationale is that in a resource theory, there is a special class of objects that are considered to be freely accessible, in the sense that the objects that are not free are those that contain the property considered to be a resource for the theory. Those nonfree objects thus have some value or cost associated with this resourcefulness. It is in this sense that free objects are considered freely available and non-resourceful;

2. A set $\mathcal{T}$ of transformations between objects, transformations that can be freely constructed or implemented, that is, without any cost, called *free transformations*. The notation $A \to B$, in which $A, B \in \mathcal{F}$, denotes that there is a free transformation $F \in \mathcal{T}$ such that $F(A) = B$ and will be used when the specific transformation is not important, but only it's existence. In terms of defining the free transformations, if the free objects are fixed, a transformation F is a free transformation when, for every free object A, the resulting object $B = F(A)$ is also a free object;

3. The possibility of combining objects and transformations through binary relations among them. If A and B are objects of the theory, the composite object regarding both is denoted by $A \otimes B$. In a similar manner, if we have two transformations F and G, we consider the composite transformation $F \otimes G$ as performing the two transformations in parallel, so that if $F(A) = B$ and $G(C) = D$, then $(F \otimes G)(A \otimes B) = C \otimes D$;

4. We assume also the existence of a trivial object I that, when combined with any object is equivalent to the object itself. This trivial object also relates to the definition of free objects and transformations in the following way: if the set of free operations is fixed, an object A is free if there exists a free transformation F with $F(I) = A$.

Thus we come to a definition of a resource theory, in terms of the elements described above, as follows:

Definition 1. *A resource theory is defined by the tuple* $(\mathcal{U}, \mathcal{F}, \mathcal{T}, \otimes)$*, in which* $\mathcal{U}$ *consists of the set of objects to which the theory refers,* $\mathcal{F} \subset \mathcal{U}$ *is the set of free objects of the theory,* $\mathcal{T}$ *is a set of free transformations acting on the objects and a binary operation* $\otimes$ *that allows parallel combinations of objects and operations.*

For the sake of the mathematically oriented reader I would like to mention that, as (Coecke et al., 2016) discusses, this formalization can be summed up by stating that objects and free transformations in a resource theory are, respectively, objects and morphisms in a symmetric monoidal category. In fact, the author of that work also states that "the difference between a resource theory and a symmetric monoidal category is not a mathematical one, but rather one of interpretational nature", that "a particular symmetric monoidal category is called a resource theory whenever one wants to think of its objects as resourceful and its morphisms as transformations or conversions between these resourceful objects". And since this is the essence of this work, we will not pursue this abstract categorization further and refer the reader to references as (Coecke et al., 2016) and (Fritz, 2017).

The pre-order of objects

Given the basic formulation of a resource theory and its ingredients, we now introduce the idea of the *pre-order* of objects in a resource theory. This idea is intimately connected to the matter of interconversion amongst objects and provides a very natural way of characterizing a given resource theory in terms of a kind of internal structure, the structure of possible interconversions induced by the set of free operations. This idea lies in the heart of this work and we will explore it further.

The idea is that in a resource theory sometimes one is not particularly interested in the particular process by which an object conversion occurs, but rather the important question is whether this conversion is possible within the theory. That is, given objects $A, B \in \mathcal{U}$, is there a transformation $A \to B$, i.e., an $F \in \mathcal{T}$ with $F(A) = B$? The focus now is specifically on how a certain choice of free operations organizes and structures the objects in your theory. But what kind of structures can emerge from such a question?

We begin with some intuitive facts about the question of interconversibility. Firstly, since free operations are those that can be done at no cost, it is fairly intuitive that doing nothing is a free operation, that is, for every object A, we have $A \to A$. Secondly, the possibility of freely implementing sequential composition of free operations is also reasonable. By definition, being able of getting from A to B and from B to C at no cost implies being able of getting from A to C at no cost. In other words, we have $A \to B, B \to C \implies A \to C$.

These basic facts make of this interconversion relation a *preordering* among the objects of a resource theory, meaning a binary relation that is *reflexive* and *transitive*, and following standard notation we write $A \succeq B$ whenever $A \to B$ in a resource theory, the "$\succeq$"relation thus defines a preorder among the objects.

Now, even though this resulting ordered structure is closely related to the specific set $\mathcal{T}$ of free transformations, being actually induced by it, once this set of interconversion relations structure is given, one can "forget"the transformations that gave rise to the ordering structure and consider only questions about the induced structure itself. In this spirit, one can speak of (Coecke et al., 2016) **theories of resource convertibility**, defined exclusively by the a set of objects equipped with a preorder and another binary relation:

Definition 2. *Given a resource theory $\mathcal{R} = (\mathcal{U}, \mathcal{F}, \mathcal{T}, \otimes)$, the **theory of resource convertibility** associated with $\mathcal{R}$ is the tuple $\tilde{\mathcal{R}} = (\mathcal{U}, \mathcal{F}, \otimes, \succeq)$, in which $\succeq$ is the preorder relation induced on the objects by the set of free operations.*

The distinction between a resource theory, as defined by a choice of free operations, and the resource theory of convertibility induced by it is interesting not only from a mathematical point of view, i.e., as two different structures, but is may also be a helpful tool when comparing different choices of free operations when looking for the best way to construct a theory of resources upon a given physical phenomena, for example. Throughout this work the concept of resource interconversibility will be in the focus of our discussions, developments and results, but since we will deal with one specific choice of free operations for a resource theory of Contextuality, we feel free to not make further reference to the distinction between a resource theory proper and the associated theory of resource interconversibility. That is, from here on we will deal with the resource theory of Contextuality defined by a choice of free operations and the resource theory of convertibility induced by it as one single general entity, to which we simply refer as *resource theory of contextuality.*

We also want to mention that there is a lot of richness on the monoidal structures of resource theories, that is, on the structures induced by the binary operations ($\otimes$) defined on objects and transformations. Different forms of parallel and sequential composition give rise to resource theories with different features, such as the possibility or impossibility of catalysis, exemplified in (Coecke et al., 2016) with two resource theories that differ only in their monoidal structure. This work, nonetheless, does not explore the monoidal structure of the resource theory with which we deal, such that for the practical purposes of our investigations, the following definitions of a resource theory and a theory of resource convertibility will suffice:

Definition 3. *We redefine a* ***resource theory*** *to be the reduced tuple* $\mathcal{R} = (\mathcal{U}, \mathcal{F}, \mathcal{T})$ *of aforementioned elements together with the induced* ***theory of resource convertibility*** *redefined as* $\tilde{\mathcal{R}} = (\mathcal{U}, \mathcal{F}, \succeq)$*.*

The specific resource theory to be studied throughout this work will be defined below. Meanwhile, we turn to some other important aspects of resource theory.

Monotones

One of the most important aspects regarding resource theories has to do with coming up with a way of actually quantifying the amount of resource contained in the objects of your theory. Thus we come to the idea of a *resource monotone*, also called a *quantifier* or for short *monotone.*

Definition 4. *Let* $(\mathcal{U}, \mathcal{F}, \succeq)$ *be a resource theory. We define a* ***resource monotone*** *as a function defined on the set of objects, that preserves the the pre-order structure, that is, for all* $A, B \in \mathcal{U}$*,*

$$M : \mathcal{U} \to \bar{\mathbb{R}} \text{ such that } B \preceq A \implies M(B) \leq M(A), \tag{1.1}$$

in which $\bar{\mathbb{R}}$ *means the set of extended real numbers* $\mathbb{R} \cup \{-\infty, \infty\}$*.*

Thus a monotone function plays the role of numerically measuring the amount of available resource in different objects. These important functions, because of their order-preserving property, end up giving us insightful information about the resource theories.

When comparing the value of a resource monotone for two different objects, this function can be interpreted as saying something along the lines of "*at least in one way of seeing things, this object has more (or less, or an equal amount of) resource than this other one*", and I state it with this caveat of "*one way of seeing things*" because one can actually define many different resource monotones, in fact it is desirable to work with various monotones as different resource monotones may be useful to study different aspects of a resource theory. Along this work, we will indeed work with various monotones and make explicit use of this fact.

It is worth mentioning again that the pre-order structure of objects in a resource theory is of a more fundamental nature than any single resource monotone. A resource monotone captures certain aspects of the pre-order by assigning numerical values that help one to better visualize these aspects, but unless the preorder is a total order (all its elements are comparable), it can never contain the total information available in the pre-order (Amaral, 2019). In fact, even though there were early works in which one of the goals of researchers developing a resource theoretic approach was to look for what would be *the* correct or better resource monotone, this is no longer the case. The contemporary view is as stated above, in general a pre-order is the fundamental object, with any particular resource monotone being a sort of coarse-grained description of the whole.

One might at this point be tempted to question the usefulness of worrying about resource monotones. If they provide only an incomplete description of the total information available at the preorder, what does one gain with their use, if anything at all? Before answering such question, let us reflect about the endeavor of pre-order characterization. As already stated, the general goal is to characterize the structure induced on the objects by a particular choice of free operations in a given resource theory. Now, an actually crucial part of such endeavor is, given two arbitrary objects, to know whether there exists a free operation taking one to another, and generally speaking it turns out that such a question may be hard to answer directly, but fortunately it can be answered algorithmically, as it will be demonstrated in this work. However, there still remain questions of a different scope. The authors in (Gonda and Spekkens, 2019), for example, introduce certain properties that they call *global structures* of the pre-order, and those are precisely the properties in which the aforementioned algorithm does not do the job completely, and resource monotones come in handy for better characterization of such properties. Part of this work is exactly trying to answer questions of this nature for a resource theory of contextuality.

Some of the global properties of pre-order of resource objects to which we will give some attention are: the **height** of a pre-order (cardinality of the largest chain in it), its **width** (cardinality of the largest antichain), whether or not it is **totally ordered** (all its elements are comparable), whether or not it is **weak** (transitivity of the incomparability relation), and whether or not it is **locally finite** (finite number of inequivalent elements between any two ordered elements). Therefore, in general, as it will be in our case, the effort in constructing and investigating resource monotones does pay off and ends up being a crucial part of developing useful resource theories. In fact, in terms of actual practical applications of resource monotones, we cite (Duarte et al., 2018), which shows that contextuality monotones can be used to study geometrical aspects of particular sets of possible behaviours inside and outside the quantum set, as well as (Amaral, 2019) and (Chitambar and Gour, 2019), that present and discuss different monotones and their applicability.

Now, even though resource theory is part of the core concept of this work, this is not a work on resource theories in general, we will work with a specific resource theory, a resource theory of Quantum Contextuality as defined in (Amaral, 2019), thus we now proceed to briefly introduce the notion.

Capítulo 2

A Resource Theory of Contextuality

Quantum Contextuality, in this work simply referred to as contextuality, as is the standard in the literature, is the name given to an important and central property displayed by quantum systems. This property, in simple terms, has to do with the impossibility of thinking about statistical results of measurements in quantum systems as revealing pre-existing objective properties of that system, meaning properties or results to those measurements which are independent of the actual set of measurements one chooses to make on the given system (Budroni et al., 2021), (Kochen and Specker, 1975).

In terms of resourceful aspects of contextuality, we may cite (Amaral, 2019) that quantum contextuality is a necessary resource for universal computing, in what is called magic state distillation (Howard et al., 2014), in measured based quantum computation (Delfosse et al., 2015), (Raussendorf, 2013) and in computational models of qubits (Bermejo-Vega et al., 2017). Contextuality can also be a resource in cryptography tasks, as it can used to certify the true generation of random numbers (Um et al., 2013).

Thus, we proceed to the actual formal framework of the resource theory of contextuality that we are going to use. It will be given by a specific choice of free operations, the *noncontextual wirings*. The objects we consider are families of probability distributions on measurement scenarios, we call those *behaviours*. After presenting the basic language and ingredients of the framework, we detour a little to present the reader to an auxiliary resource theory from which we will adapt many concepts, the resource theory of Bell nonlocality given by choice of free operations to be the set of Local Operations and Shared Randomness. We then show how our general framework of contextuality resource theory is a somewhat natural generalization of the Bell nonlocality resource theory framework, an idea that will prove to be really useful and fruitful throughout this work. In this work we shall not go into the conceptual frameworks of neither contextuality nor Bell nonlocality, that is, we will not properly present the details of the formulations of contextuality and Bell nonlocality as physical theories, but rather we shall mainly stick to the resource theory frameworks of each of these phenomena. For those readers interested the theories of Bell nonlocality and Contextuality in themselves, there are excellent introductory resources and reviews, amongst which we cite (Nielsen and Chuang, 2002), (Amaral et al., 2011), (Brunner et al., 2014) and (Budroni et al., 2021). Let us now proceed with some basic ingredients needed for a resource theory of contextuality.

The Language and Its Ingredients

Definition 5. *Following (Amaral et al., 2018), (Amaral, 2019) and (Amaral and Cunha, 2018) we define a compatibility scenario by a triple* $\Upsilon := (\mathcal{M}, \mathcal{C}, \mathcal{O})$*, where* $\mathcal{O}$ *is a finite set (representing the number of outputs we want our measurements to have, whose cardinality is equivalent to the number of outputs of our measurements, say),* $\mathcal{M}$ *is a finite set (of measurements) of random variables in* $(\mathcal{O}, \mathcal{P}(\mathcal{O}))$*, in which* $\mathcal{P}(\mathcal{O})$ *is the set of subsets of* $\mathcal{O}$*, and* $\mathcal{C}$ *is a family of subsets of* $\mathcal{M}$*. The elements* $\gamma \in \mathcal{C}$ *are the contexts of measurements in our scenario.*

An important property is that for each context γ, the set of all possible outcomes for the joint measurement of the properties in γ is the set $\mathcal{O}^\gamma$, that is, each measurement in γ can give as result $|\mathcal{O}|$ different outputs. When we jointly perform the measurements of γ, our output is encoded in a vector $\mathbf{s} \in \mathcal{O}^\gamma$.

Behaviours

The main ingredient of our theory, for now, is what we call a behaviour.

Definition 6. *Given a scenario* $(\mathcal{M}, \mathcal{C}, \mathcal{O})$*, a* ***behaviour*** B *for this scenario is a family of probability distributions over* $\mathcal{O}^\gamma$*, one for each context* $\gamma \in \mathcal{C}$*,*

$$B = \left\{ p_\gamma : \mathcal{O}^\gamma \to [0,1] \middle| \sum_{\boldsymbol{s} \in \mathcal{O}^\gamma} p_\gamma(\boldsymbol{s}) = 1, \gamma \in \mathcal{C} \right\} \tag{2.1}$$

To each context γ and output $\mathbf{s} \in \mathcal{O}^\gamma$ the behaviour gives the probability $p_\gamma(\mathbf{s})$ of obtaining this output.

We also call a behaviour a **box**, as a way of creating a mental picture, where we imagine the elements of $\mathcal{M}$ as buttons of the box, and, for each measurement, we imagine the box having $|\mathcal{O}|$ output lights that inform us the result of the (joint) measurement.

An important idea that arises when talking about behaviours is that they may or may not satisfy what we call the *non-disturbance condition*, that we state as follows: a behaviour is non-disturbing if, given two contexts γ and γ', the marginal for their intersection is well defined. If we have, for example, $\gamma = \{x, y\}$ and $\gamma' = \{y, z\}$, the non-disturbance condition implies:

$$\sum_a p_{x,y}(a, b) = \sum_c p_{y,z}(b, c) \tag{2.2}$$

This defines an important set of behaviours (of theories, in a sense): the *non-disturbance set* $\mathbf{ND}(\Upsilon)$ is the set of behaviours that satisfy the non-disturbance condition for any intersection of contexts in the scenario. In the context of Quantum Theory, this condition ensures that our measurements are actually well defined, irrespective of context of measurement.

Another important idea for contextuality is the possibility of assigning a single probability distribution on the whole set $\mathcal{O}^{\mathcal{M}}$, we call this probability distribution $p_{\mathcal{M}} : \mathcal{O}^{\mathcal{M}} \to [0, 1]$ a *global section* for the scenario. We say that $p_{\mathcal{M}}$ is a global section for a scenario B if, in each context, the marginal probability distributions coincide with the ones given by B. The behaviours that have a global section are called *non-contextual*, the set of non-contextual behaviours is denoted by $\mathbf{NC}(\Upsilon)$.

When a behaviour is non-contextual, i.e. when it has a global section, all probabilities can be written as

$$p_\gamma(\mathbf{s}) = \sum_\lambda p(\lambda) \prod_{\gamma_i \in \gamma} p_{\gamma_i}(s_i). \tag{2.3}$$

These are the classical behaviours, in which the probability distributions of different measurements are independent of each other.

Pre-processing and post-processing operations

To define the free operations of our resource theory, we begin by defining certain general operations that take behaviours (our objects) in a given scenario into others.

One of the basic operations is the operation of *pre-processing* a behaviour. We introduce a new scenario $\Upsilon_{PRE} = (\mathcal{M}_{PRE}, \mathcal{C}_{PRE}, \mathcal{O}_{PRE})$, with new measurements, contexts and outputs, and a new non-contextual behaviour B_{PRE} associated with it. We associate each output of B_{PRE} with an input of B, in such a way that every output configuration of B_{PRE} defines a possible input configuration in B, that is, associated with every output $\mathbf{r} \in \mathcal{O}_{PRE}$, we have a possible context $\gamma(\mathbf{r}) \in \mathcal{C}_{PRE}$.

With this, we define a new behaviour $\mathcal{W}_{PRE}(B)$ given by

$$p_\beta(\mathbf{s}) = \sum_{\mathbf{r}} p_\beta(\mathbf{r}) p_{\gamma(\mathbf{r})}(\mathbf{s}), \tag{2.4}$$

where the sum runs over all outputs $\mathbf{r}$ associated with the context β in B_{PRE}.

In the same manner, we can define the *post-processing* of a behaviour. We again introduce $\Upsilon_{POS} = (\mathcal{M}_{POS}, \mathcal{C}_{POS}, \mathcal{O}_{POS})$ together with a non-contextual behaviour B_{POS}. The same association is made between outputs $\mathbf{s} \in \mathcal{O}$ and contexts $\delta(\mathbf{s}) \in \mathcal{M}_{POS}$. The new behaviour obtained $\mathcal{W}_{POS}(B)$ is given by

$$p_{\gamma}(\mathbf{t}) = \sum_{\mathbf{s}} p_{\gamma}(\mathbf{s}) p_{\delta(\mathbf{s})}(\mathbf{t}) \tag{2.5}$$

Non-contextual wiring

With these tools in hand, we can define a *non-contextual wiring*. We start with an arbitrary behaviour B and compose it with a pre-processing B_{PRE} and a pos-processing B_{POS}, along with one additional possibility that the probabilities of B_{POS} may also depend on the inputs and outputs of B_{PRE}. With this additional freedom, the probabilities of B_{POS} are of the form $p_{\delta}(\mathbf{t}|\beta, \mathbf{r})$, but since we want to guarantee that there is no contextuality generated by the processing itself, as done in (Amaral et al., 2018), we demand that

$$p_{\delta}(\mathbf{t}|\beta, \mathbf{r}) = \sum_{\phi} p(\phi) \prod_{i} p_{\delta_i}(t_i|\beta_i, r_i, \phi). \tag{2.6}$$

Following the constructions, we get at a final scenario $(\mathcal{M}_{PRE}, \mathcal{C}_{PRE}, \mathcal{O}_{POS})$ with an associated behaviour $\mathcal{W}_{NC}(B)$, given by

$$p_{\beta}(\mathbf{t}) = \sum_{\mathbf{r},\mathbf{s}} p_{\beta}(\mathbf{r}) p_{\gamma(\mathbf{r})}(\mathbf{s}) p_{\delta(\mathbf{s})}(\mathbf{t}) \tag{2.7}$$

This particular class of operations, henceforth referred to as **NCW**, constitutes the free operations of our resource theory. We present here two important results derived in (Amaral, 2019) about this class of operations:

- The non-disturbing class of behaviours **ND** is closed under **NCW**.
- The non-contextual class of behaviours **NC** is closed under **NCW**.

2.1 A Resource Theory of Bell Nonlocality

Now, in order to characterize the properties of the resource theory defined above, we will make use of some of the the methods and results presented and obtained in (Wolfe et al., 2020), so let us introduce some of the discussions therein. That work deals exclusively with Bell scenarios, and because of this we momentarily adopt an alternative notation for the probability distributions of our behaviours. In the notation we adopt we divide our systems in two parts, or *wings*, A and B, and label the **settings variables** S (left wing) and T (right wing), while the **outcome variables** are denoted X (left wing) and Y (right wing).

Sometimes, one is interested on problems involving the cardinality of the systems. For the operations of a resource theory such as the one to be defined, the cardinality of the sets of a behaviour can be used to classify the transformations. This technical issue must be taken into account, for many of the results obtained in (Wolfe et al., 2020) are statements regarding transformations among behaviours of the same cardinality. Some of our results will also have to consider this technical aspect. Therefore, we briefly mention the following definition,

Definition 7. *The* ***type*** *of a box is defined as a* $2 \times n$ *matrix,* n *being the number of wings, so that, for a two-wing scenario, the type is given by* $\begin{pmatrix} |X| & |Y| \\ |S| & |T| \end{pmatrix}$.

The CHSH scenario, for example, a typical scenario used in discussions of Bell nonlocality phenomena, representing two parts, agents or laboratories, each of them having access to two possible measurements to perform, with two outcomes each, has type $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$.

The set of objects of the resource theory presented in (Wolfe et al., 2020) are given by conditional probabilities distributions $P_{XY|ST}$. The set of *free objects*, that is, classical probabilities distributions (free of nonlocality), are given by distributions that can be modelled in the form

$$P_{XY|ST}(xy|st) = \sum_{\lambda_A \lambda_B} P_{X|S\Lambda_A}(x|s\lambda_A) P_{Y|T\Lambda_B}(y|t\lambda_B) P_{\Lambda_A\Lambda_B}(\lambda_a\lambda_b). \tag{2.8}$$

The set of *free operations* of this resource theory represent the process of embedding the systems in circuits that are composed of box-type processes that are classical and that respect the causal structure of the scenario. These processes can be very general, and the constrains one imposes on such operations will determine the kind process, the kind of theory, one wishes to consider. On their work, the chosen free operations are those realizable by classical probability theory, those of the form

$$P_{XY|ST} \to P_{X'Y'|S'T'}(x'y'st|xys't') = \sum_{\lambda_A \lambda_B} P_{X'S|XS'\Lambda_A}(x's|xs'\lambda_A) P_{Y'T|YT'\Lambda_B}(y't|yt'\lambda_B) P_{\Lambda_A\Lambda_B}(\lambda_A\lambda_B) \tag{2.9}$$

The set of operations that can written in this form is called Local Operations and Shared Randomness (LOSR). In (Schmid et al., 2020), the authors show that the most appropriate and prominent set of operations to characterize and quantify the nonclassicality of Bell scenarios is not Local Operations and Classical Communication (LOCC), but instead the set LOSR. Therefore, we adopt this paradigm in order to extend the results of a resource theory of Bell nonlocality to a resource theory of contextuality.

Now, since quantum nonlocality is in a sense a particular instance of contextuality, there is a sense in which there is a relationship between both frameworks presented above. More precisely, we now know that non-contextual wiring is the appropriate generalization of an LOSR operation for contextuality scenarios in general, (Horodecki et al., 2015) (Amaral, 2019), (Gallego and Aolita, 2017), (Amaral et al., 2018). Therefore, we now turn to the extension of some concepts present in this resource theory of Bell nonlocality onto a non-contextual wirings language. Let us begin by working out the definitions of locally deterministic operations and local symmetry operations. These definitions, although they might seem artificial at first, will actually be quite important in questions of interconversibility of objects in our resource theory, as we will see ahead. We will first review the definition of such operations for a Bell nonlocality framework and then try to extend them to our more general contextuality framework.

Locally deterministic operations

For Bell nonlocality, an LOSR operation is called *locally deterministic* and is said to be in the **LDO** set if and only if the conditional probabilities $P_{X'Y'ST|XYS'T'}$ that define the transformation take values in $\{0, 1\}$ for all values of X', Y', S, T, X, Y, S' and T'. When this happens, the operation factorizes in the following form

$$P^{det}_{X'Y'ST|XYS'T'} = P^{det}_{X'S|XS'} P^{det}_{Y'T|YT'}. \tag{2.10}$$

Because of the *no-retrocausation* conditions, we have that the deterministic operations must have the form

$$\begin{aligned} P^{det}_{X'S|XS'} &= \delta_{S,f_A(S')}\delta_{X',g_A(X,S')} \\ P^{det}_{Y'T|YT'} &= \delta_{T,f_B(T')}\delta_{Y',g_B(Y,T')} \end{aligned} \tag{2.11}$$

Local symmetry operations

Inside the **LDO** set, we have operations that are not only deterministic, but also invertible, the **LSO** set. This kind of operation is achieved when the functions f_A and g_A defined above are such that $P^{det}_{X'S|XS'}$ becomes an invertible map from (X, S') to (X', S), and the functions f_B, g_B are such that $P^{det}_{Y'T|YT'}$ becomes an invertible map from (Y, T') to (Y', T).

In this way, we have a final operation $P^{sym}_{X'Y'ST|XYS'T'}$ of the form

$$P^{sym}_{X'Y'ST|XYS'T'} = P^{sym}_{X'S|XS'} P^{sym}_{Y'T|YT'}. \tag{2.12}$$

Finally, we turn to the main objects of our resource theory.

2.2 Extending these operations to the Contextuality framework

What we are after is a way of generalizing these notions to general contextuality beyond Bell nonlocality. Bearing in mind that our contextuality transformations are the non-contextual wirings acting on behaviours, a first attempt to generalize **LDO** to wirings could be to investigate the separability of the wings of the processing operations. For example, writing the pre-processing as

$$p_\beta(r) = \sum_\phi p(\phi) \prod_i p_{\beta_i}(r_i|\phi), \tag{2.13}$$

we see that demanding that p_{β_i} takes values in $\{0,1\}$ for each wing β_i guarantees that the final form of such a probability distribution is a product of the form

$$p^{Det}_\beta(r) = \prod_i p^{Det}_{\beta_i}(r_i) = \prod_i \delta_{r_i, f_i(\beta_i)}. \tag{2.14}$$

For each wing of the context β, the function $f_i(\beta_i)$ effectively associates a measurement β_i with an output r_i, the one for which $p_{\beta_i}(r_i) = 1$. Hence, in a deterministic processing, each context selects a unique output string r.

This allows us to formulate a definition for a deterministic non-contextual wiring operation. Given a behaviour $\mathbf{B}$, we say that a non-contextual wiring operation is deterministic when both processings (pre and pos) are deterministic. Then, the behaviour $\mathcal{W}^{Det}(B)$ is given by

$$p^{Det}_\beta(\mathbf{t}) = \sum_{\mathbf{r},\mathbf{s}} p^{Det}_\beta(\mathbf{r}) p_{\gamma(\mathbf{r})}(\mathbf{s}) p^{Det}_{\delta(\mathbf{s})}(\mathbf{t}), \tag{2.15}$$

where the product $p^{Det}_\beta(\mathbf{r})p^{Det}_{\delta(\mathbf{s})}(\mathbf{t})$ factorizes as

$$p^{Det}_\beta(\mathbf{r})p^{Det}_{\delta(\mathbf{s})}(\mathbf{t}) = \prod_i \delta_{r_i, f_i(\beta_i)} \delta_{t_i, g_i(\delta_i)} \tag{2.16}$$

In this language, symmetry operations can be formulated as deterministic operations for which the families of functions $\{f_i\}$ and $\{g_i\}$ define one-to-one maps between contexts and outputs.

Cardinality of NCW Operations

Just as can be done for **LOSR** operations in Bell-type scenarios, we can define the ***type*** of a box in terms of the cardinalities of its input and output variables. Since we are dealing with a scenario $\Upsilon = (\mathcal{M}, \mathcal{C}, \mathcal{O})$, the number of input and output variables are fixed by $|\mathcal{C}|$, cardinalities of input variables are fixed by $|\mathcal{M}|$, and cardinalities of output variables are fixed by $|\mathcal{O}|$. Now, as our **NCW** operations take boxes in $\Upsilon = (\mathcal{M}, \mathcal{C}, \mathcal{O})$ to boxes in $\Upsilon_{\mathcal{W}} = (\mathcal{M}_{PRE}, \mathcal{C}_{PRE}, \mathcal{O}_{POS})$, we define the ***type of an operation*** $\mathcal{W}$ as $[\mathcal{W}] \doteq [B] \to [\mathcal{W}(B)]$. The set of all operations of type $[B_1] \to [B_2]$ is denoted by $\underset{[B_1]\to[B_2]}{\mathbf{NCW}}$. As mentioned above, some of our results will be type-specific, meaning results concerning type-preserving operations.

Convexity of Operations

Now, another important technical aspect when trying to define a resource theory has to do with the convexity of the chosen set of free operations. Convexity is a desirable property for your operations since they usually represent the choice of deciding what transformation to implement probabilistically, like through tossing a coin to decide which amongst two different operations to implement.

Given that this is a desirable feature, we can ask ourselves how to implement such a restriction or in what sense can we talk of convexity of the set of free operations. Well, let us begin by considering a general free operation as defined in the last section and write it as

$$p_\beta(t) = \sum_{r,s} p_{\beta,\delta}(r,t) p_{\gamma(r)}(s), \tag{2.17}$$

where we write $p_{\beta,\delta}(r,t) \doteq p_\beta(r) p_{\delta(s)}(t)$. We want to ask in what sense or circumstance can we talk about a convex sum of two such operations, say $p^{(0)}_{\beta,\delta}(r,t)$ and $p^{(1)}_{\beta,\delta}(r,t)$.

As mentioned above, such convex mixings are usually understood in a probabilistic sense, kind of tossing a coin to decide which operation is to be implemented overall. Because of this, a way of incorporating this notion into the formalism is by making use of the random variables already present in the operations.

Imagine we want to represent a convex mixing where $p^{(0)}_{\beta,\delta}(r,t)$ is implemented with probability α, and $p^{(1)}_{\beta,\delta}(r,t)$ with probability $1-\alpha$. What we do is to sample from a new binary probability distribution $p(\Lambda)$, in which $p(\Lambda = 0) = \alpha$, $p(\Lambda = 1) = 1 - \alpha$, such that $\Lambda = 0$ results in $p^{(0)}_{\beta,\delta}$ being implemented, while $\Lambda = 1$ results in $p^{(1)}_{\beta,\delta}$ being implemented.

Formally, we want to implement

$$\sum_{\Lambda} p(\Lambda) p^{(\Lambda)}_{\beta\to\delta}(r\to t) = \sum_{\Lambda} p(\Lambda) \sum_{\lambda} p(\lambda|\Lambda) \prod_{i} p_{\beta_i}(r_i|\lambda) p_{\delta_i}(t_i|\beta_i, r_i, \lambda), \tag{2.18}$$

where the extra superscript (Λ) in $p^{(\Lambda)}_{\beta,\delta}(r,t)$ denotes the dependence of the operation on the initial sampling over $p(\Lambda)$ through the explicit dependence of $p(\lambda|\Lambda)$ on the variable Λ. Now, defining $\tilde{p}(\lambda) \doteq \sum_{\Lambda} p(\lambda|\Lambda)p(\Lambda)$ allows us to write

$$\sum_{\Lambda} p(\Lambda) p^{(\Lambda)}_{\beta,\delta}(r,t) = \sum_{\lambda} \tilde{p}(\lambda) \prod_{i} p_{\beta_i}(r_i|\lambda) p_{\delta_i}(t_i|\beta_i, r_i, \lambda), \tag{2.19}$$

which is the standard form of a non-contextual wiring operation. That is, general convex mixings can be naturally incorporated in the formalism, which is surely desirable. When the set of free operations defining a resource theory is convex, we say that the resource theory is a *convex resource theory.* An thus we have just derived an important technical result:

- ***The set of free operations given by noncontextual wirings is a convex set. Thus, a resource theory of Contextuality defined by this set of free operations is a convex resource theory.***

Now, recalling the discussion on deterministic operations, the convexity of our set of operations gives us another powerful result. Notice that in a general operation with respective contexts β, δ, etc, we have that for each particular measurement in each context $\beta_i, \delta_i, ...$, we can generally write

$$p_{\beta_i,\delta_i}(r_i, t_i|\lambda) = \sum_{\Lambda_i} p^{Det(\Lambda_i)}_{\beta_i,\delta_i}(r_i, t_i) p(\Lambda_i|\lambda), \tag{2.20}$$

so that

$$\begin{aligned} p_{\beta,\delta}(r,t) &= \sum_{\lambda} p(\lambda) \prod_{i}^{|\beta|} \sum_{\Lambda_i} p^{Det(\Lambda_i)}_{\beta_i,\delta_i}(r_i, t_i) p(\Lambda_i|\lambda) \\ &= \sum_{\Lambda_1 ... \Lambda_{|\beta|}} \prod_{i} p^{Det(\Lambda_i)}_{\beta_i,\delta_i}(r_i, t_i) \sum_{\lambda} \prod_{j} p(\Lambda_j|\lambda) p(\lambda) \\ &= \sum_{\vec{\Lambda}} P(\vec{\Lambda}) \prod_{i} p^{Det(\Lambda_i)}_{\beta_i,\delta_i}(r_i, t_i), \end{aligned} \tag{2.21}$$

in which $\vec{\Lambda} \doteq (\Lambda_1, ..., \Lambda_{|\beta|})$ and $P(\vec{\Lambda}) \doteq \sum_{\lambda} \prod_i p(\Lambda_i|\lambda)p(\lambda)$. That is, any operation $p_{\beta,\delta}(r,t)$ is a convex combination of products of deterministic operations, and so from considerations of convexity we arrive at another important result, a sort of generalization of Fine's theorem regarding local distributions (Fine, 1982):

- ***In our resource theory of contextuality, the free operations of a given type form a polytope whose vertices are precisely the locally deterministic operations of that type.***

For further reference on aspects of convexity in resource theories, specifically a quantum theory framework, the reader can check the discussions about types of quantum resource theories in (Chitambar and Gour, 2019).

Capítulo 3

Investigating the Global Properties of the Pre-order of Objects

Here we finally delve into the main quest of this work, coming up with a path to explore the possibilities of interconversion between the objects of our resource theory of Contextuality, the behaviours introduced on last chapter. We begin by retaking some concepts regarding objects interconversibility, resource monotones and introduce some useful nomenclature.

Given two behaviours B_1 and B_2, we say that B_1 can be *converted* to B_2 if there is a free operation $\mathcal{W} \in \mathbf{NCW}$ such that $B_2 = \mathcal{W}(B_1)$, in which case we write $B_1 \to B_2$. If no such operation exists, we write $B_1 \nrightarrow B_2$. It is worth mentioning that if there are two free operations, one taking B_1 to B_2 and another taking B_2 to B_3, the composite operation from B_1 to B_3 is also free. That is, the conversion relation defined above is transitive.

In terms of possible relations among two resources, we define:

- B_1 is *strictly above* B_2 when $B_1 \to B_2$ and $B_2 \nrightarrow B_1$.
- B_1 is *strictly below* B_2 when $B_1 \nrightarrow B_2$ and $B_2 \to B_1$.
- B_1 is *equivalent* B_2 when $B_1 \to B_2$ and $B_2 \to B_1$.
- B_1 is *incomparable* B_2 when $B_1 \nrightarrow B_2$ and $B_2 \nrightarrow B_1$.

Monotones

We recall that resource monotones are useful quantifiers and try to show that they help us to understand the underlying structures of a resource theory. Remember that a monotone is a resource quantifier M such that $B_1 \to B_2 \implies M(B_1) \geq M(B_2)$.

Whenever we have a monotone M and two resources that satisfy $M(B_1) < M(B_2)$, we say that M is a *witness* to the fact that $B_1 \nrightarrow B_2$.

An interesting case arises when we have incomparable objects in our theory. Although it is true that $M(B_1) < M(B_2) \implies B_1 \nrightarrow B_2$, we can ask ourselves about the status of the inverse implication, namely, is it the case that a relation like $B_1 \nrightarrow B_2$ implies the existence of a monotone $\tilde{M}$ such that $\tilde{M}(B_1) < \tilde{M}(B_2)$ holds? If so, the incomparibility of B_1 and B_2 would imply that there are at least two monotones, say M_1 and M_2, such that $M_1(B_1) < M_1(B_2)$ while $M_2(B_2) < M_1(B_1)$, motivating a sort of refinement in our characterization.

Motivated by the question above, we introduce the notion of a *complete family of monotones*, a family which better characterizes the pre-order of objects. A family of $\{M_i\}_i$ of monotones is said to be *complete* when, for any M_i, the following equivalence holds,

$$\forall B_1, B_2 : B_1 \to B_2 \iff M_i(B_1) \geq M_i(B_2). \tag{3.1}$$

With such a construction, we actually achieve a correspondence between the structures imposed on our theory by the free operations and by certain families of monotones.

Cost and Yield monotones

There is a useful construction that can be formulated in general resource theories, from which we get two useful quantities regarding resourcefulness of objects.

Given any subset $S \subseteq \mathcal{O}$ of objects in a resource theory and a function $f : S \to \mathbb{R}$ from this set to real numbers, we define the **yield** and **cost** relative to S and f as

$$Y_{(S|f)}(a) \doteq \max_{\tilde{a} \in S} \{f(\tilde{a}),\ \textit{such that } a \to \tilde{a}\}, \tag{3.2}$$

$$C_{(S|f)}(a) \doteq \min_{\tilde{a} \in S} \{f(\tilde{a}),\ \textit{such that } \tilde{a} \to a\}. \tag{3.3}$$

Moreover, if there is no such object $\tilde{a}$ such that $a \to \tilde{a}$ ($\tilde{a} \to a$), the yield (cost) is set to $-\infty$ ($+\infty$).

In other words, basically, what $Y_{(S|f)}(a)$ gives is the value of f for the most resourceful object in S that can be freely obtained from a. On the other hand, $C_{(S|f)}(a)$ gives the value of f for the least resourceful object in S from which one can freely obtain a.

Back to the question of Convexity

As mentioned, one of the goals of a resource theory description is the complete characterization of the pre-order of objects, i.e., actually knowing which of the four possible interconversion relations holds for each pair of objects. The result obtained above about the decomposition of non-contextual free operations into combinations of extremal deterministic ones proves to be actually pretty useful in this characterization.

Let $\mathcal{P}^{NCW}_{[B_2]}(B_1)$ denote the set of behaviours of type $[B_2]$ that can be obtained by general non-contextual wirings from B_1, and $\mathcal{V}^{Det}_{[B_2]}(B_1)$ denote the set of behaviours of type $[B_2]$ obtained from B_1 now through deterministic non-contextual wirings (defined in eqs. (2.15) and (2.16)).

The finite cardinality of the set $\mathcal{V}^{Det}_{[B_2]}(B_1)$ and the existence of a polytope of free operations can be nicely summed up and expressed in the following result:

$$\mathcal{P}^{NCW}_{[B_2]}(B_1) = \mathbf{Conv}\left(\mathcal{V}^{Det}_{[B_2]}(B_1)\right), \tag{3.4}$$

where $\mathbf{Conv}\left(\mathcal{V}^{Det}_{[B_2]}(B_1)\right)$ is the convex hull of the discrete set $\mathcal{V}^{Det}_{[B_2]}(B_1)$.

This statement is equivalent to

$$B_1 \to B_2 \iff B_2 \in \mathbf{Conv}\left(\mathcal{V}^{Det}_{[B_2]}(B_1)\right), \tag{3.5}$$

which is a very important result, since it actually allows one to check if $B_1 \to B_2$ through an efficient algorithm. It is one of the goals of coming up with a resource theory. To check if $B_1 \to B_2$ holds, one has only to determine all the deterministic operations that take behaviours of type $[B_1]$ to behaviours of type $[B_2]$ (which are finite in number), compute the image of B_1 under these deterministic operations and then determine whether B_2 can be obtained through a convex combination of these images of B_1. The answer to last step can be decided with the use of linear programming (Wolfe et al., 2020).

Now, although the result just obtained is indeed a useful result, reducing greatly the number of operations needed to know if $B_1 \to B_2$, in order to fully characterize the pre-order of objects through this method, one would need to apply the linear program to every pair of objects, which is not a practical necessity. Another alternative used to characterize the pre-order is by the use of monotones, and even though in principle we could use the linear program defined above to generate a complete set of monotones[1], a full characterization still requires a monotone to be defined and the linear program to be applied to every pair of objects.

[1]Given a set of behaviours $\mathcal{S}$, for each $B \in \mathcal{S}$, define $M_B(B') = 1$, **i.f.f.** $B' \to B$, and $M_B(B') = 0$ otherwise.

The desirable situation to be achieved is the full characterization of the pre-order through a finite number of resource monotones. In order to try to achieve this goal for our resource theory of contextuality, we define and investigate some properties to be characterized in our pre-order.

Global Properties that characterize a pre-order

- When the pre-order is such that every pair of objects is either strictly ordered or equivalent, the set of objects is said to be **totally pre-ordered**.
- When there are incomparable objects in the pre-order, if the incomparability relation is transitive among the objects, we say that the pre-order is **weak**.
- A **chain** is a subset of objects in which *every pair of elements is strictly ordered*. The **height** of the pre-order is the cardinality of the largest chain in this pre-order. Likewise, an **antichain** is a subset of elements in which *every pair of elements is incomparable*. The **width** of the pre-order is the cardinality of the largest antichain contained in the pre-order.
- We say that an object B_2 lies in the **interval** of objects B_1 and B_3 **i.f.f.** both $B_1 \to B_2$ and $B_2 \to B_3$ hold. If the number of equivalence classes in the interval between a pair of objects is *finite* for every pair of inequivalent objects, we say that the pre-order is **locally finite**, otherwise it is said to be **locally infinite**.

Our objective now is the characterization of our resource theory of contextuality in terms of these global properties, in the hope that such characterization can be achieved through something less than the need to perform some task on every pair of resources.

3.1 Monotones, the path to characterize the pre-order

The monotones construction we will use is partially based on the notion of maximally violating behaviours (Amaral et al., 2018), much like **PR**-boxes work for Bell-like scenarios (Popescu and Rohrlich, 1994). Because of this, despite our efforts to be as general as possible, we shall have to focus on a specific type of contextuality scenario, the n-cycle, since for this class of behaviours, both noncontextuality inequalities and their quantum violations are known.

The n-cycle scenario consists of n dichotomic measurements X_i and a set of contexts of the form $\{X_i, X_{i+1}\}$ modulo n. There is a lot to be said about this kind of scenario, but for now let us focus on necessary considerations to define the monotones that will be used henceforth.

The n-cycle noncontextual inequalities

The most general objects we will work with in this scenario are the non-disturbing behaviours. This set of behaviours define a polytope whose facets are defined by the following positivity constrain inequalities (Araújo et al., 2013)

$$\begin{cases} 4p(++|X_iX_{i+1}) = 1 + \langle X_i\rangle + \langle X_{i+1}\rangle + \langle X_iX_{i+1}\rangle \geq 0, \\ 4p(+-|X_iX_{i+1}) = 1 + \langle X_i\rangle - \langle X_{i+1}\rangle - \langle X_iX_{i+1}\rangle \geq 0, \\ 4p(-+|X_iX_{i+1}) = 1 - \langle X_i\rangle + \langle X_{i+1}\rangle - \langle X_iX_{i+1}\rangle \geq 0, \\ 4p(--|X_iX_{i+1}) = 1 - \langle X_i\rangle - \langle X_{i+1}\rangle + \langle X_iX_{i+1}\rangle \geq 0, \end{cases} \tag{3.6}$$

which are expressed in terms of components of the vector of correlations for simplicity. Every behaviour we work with has to satisfy this set of equations.

Another important class of inequalities that we are going to use is the set of inequalities defining the noncontextual polytope of behaviours. These are of the form (Araújo et al., 2013)

$$\Omega_k = \sum_{i=0}^{n-1} \gamma_i \langle X_iX_{i+1}\rangle \leq n-2, \tag{3.7}$$

with each value of k being associated with a particular choice of values for $\gamma_i \in \{-1,-1\}$ such that the number of $\gamma_i = -1$ is odd.

A particularly important feature of such inequalities for the constructions and results that will follow is the fact that when we speak of different noncontextual inequalities in a given scenario, their respective regions of strict violation are non-intersecting, i.e., for a contextual behaviour B there is a unique k for which $\Omega_k(B) > n-2$.

Hence, for a given scenario (a choice of n), one does not have a single noncontextual inequality, but many such inequalities, and each of these inequalities then defines an associated functional Ω_k. If the distinction amongst the noncontextual inequalities of a given scenario is unimportant, we shall drop the label k and refer to a general Ω function for simplicity.

Fortunately, for this specific class of scenarios, not only the form of the inequalities is known but also the value of the associated maximal quantum violations (which are called *Tsirelson bounds*). This problem has interesting connections to graph theory methods, which proved to be fruitful tools for exploring geometric problems of contextuality theory (for discussions of such ideas, see (Amaral and Cunha, 2017) and (Amaral and Cunha, 2018)). The value for the maximal violations are given by (Araújo et al., 2013)

$$\Omega_Q = \begin{cases} \frac{3n\cos\left(\frac{\pi}{n}\right)-n}{1+\cos\left(\frac{\pi}{n}\right)} & \text{for odd } n, \\ n\cos\left(\frac{\pi}{n}\right) & \text{for even } n. \end{cases} \tag{3.8}$$

Behaviours for which the value of the Ω function is larger than Ω_Q will be of interest to us, specifically those that *maximally* violate a noncontextual inequality. This maximal violation can be understood and quantified as follows: since, by construction, we have $|\gamma_i| = 1$, and the correlations obey $|\langle X_i, X_{i+1}\rangle| \leq 1$, it follows that in a general n-cycle scenario, the highest value that the function Ω can have is $\Omega_{\mathrm{PR}} = n$. We call B_{PR} the behaviours for which $\Omega(B_{\mathrm{PR}}) = \Omega_{\mathrm{PR}}$. We say that such behaviours are *PR-like* by direct analogy with the so called *PR-boxes* defined in the CHSH-scenario as being exactly the behaviours that maximally violate a given Bell-inequality[2] (Brunner et al., 2014).

3.1.1 Two useful Cost and Yield monotones

With these concepts at hand, we are ready to define the functions we will use to characterize the preorder of objects.

Monotone 1: We call M_Ω the yield of a behaviour B with respect to the set $\mathbf{ND}(\mathcal{N})$ of general non-disturbing behaviours in the n-cycle, as measured by the function Ω, that is,

$$M_\Omega(B) \doteq Y_{(\mathbf{ND}(\mathcal{N})|\Omega)}(B) = \max_{B^* \in \mathbf{ND}(\mathcal{N})} \{\Omega(B^*),\ \textit{such that } B \to B^*\}. \tag{3.9}$$

We notice that, since the maximization is being carried over the whole set of non-disturbing behaviours, regardless of B, any noncontextual behaviour can be freely generated after discarding B. In particular, one can always freely choose a behaviour B^* with $\Omega(B^*) = n-2$, the highest value a noncontextual behaviour can achieve. Hence, the maximum value of Ω is never less than $n-2$.

To define the second monotone that we are going to use, we need to define some objects. Firstly, we want to define a behaviour given by a mixture of a *PR*-like behaviour B_{PR} and the maximally mixed behaviour $B_\varnothing$, which have equal probabilities for all outputs,

$$B = xB_{\mathrm{PR}} + (1-x)B_\varnothing,\ \textit{with } 0 < x < 1, \tag{3.10}$$

such that $\Omega(B) = n-2$, that is, we want this behaviour to be in the boundary of the noncontextual set. For this, since $\Omega(B_{\mathrm{PR}}) = n$ and $\Omega(B_\varnothing) = 0$, we have simply to choose $x = \frac{n-2}{n}$. We call such behaviour $B_{\mathbf{NPR}}$ (Noisy-PR),

$$B_{\mathrm{NPR}} \doteq \frac{(n-2)}{n} B_{\mathrm{PR}} + \frac{2}{n} B_\varnothing. \tag{3.11}$$

[2]One may verify that the CHSH-scenario is a particular case of the n-cycle scenario, namely the case $n = 4$. Accordingly, well established properties of the CHSH scenario can be recovered from the general n-cycle properties stated above by setting $n = 4$.

As a side note, this construction seems to me very interesting and even suggestive. What we did by finding the suited weight for our convex mix was basically quantifying the amount of noise one has to mix to a PR-like behaviour for making it noncontextual. A very interesting feature of the result obtained is how this quantity scales with n, the size of the system. In particular, as n grows, we notice that the amount of noise necessary gets smaller and smaller, going to zero in the limiting case. Since $PR-like$ behaviours lie on the boundary of the non-disturbing polytope and the behaviour B_{NPR} lies on the boundary of the noncontextual polytope, it seems that this may represent a measure of the volume that the noncontextual set occupies in the full non-disturbing set, in particular how it scales with n, *i.e.,* in the limit, the noncontextual set fills in the whole non-disturbing set. Furthermore, knowledge of the quantum, classical and maximal bounds for other scenarios allows one to use this construction to estimate all such relative volumes (classical to quantum, classical to non-disturbing and quantum to non-disturbing) and quantify the scaling of these volumes. The authors in (Duarte et al., 2018) employ some different techniques to study this kind of geometric characterization of Bell nonlocality phenomena and nonlocal correlations.

Now, returning to our discussion, with this behaviour, we define a one-parameter family of mixtures of B_{NPR} and B_{PR}:

$$\mathcal{F}_{\text{NPR}} \doteq \{F(\alpha) : \alpha \in [0,1]\}, \tag{3.12}$$

where $F(\alpha) = \alpha B_{\text{PR}} + (1-\alpha) B_{\text{NPR}}$. Thus, α interpolates between a noncontextual behaviour and a maximal violating one (alternatively, α interpolates between the boundaries of the respective polytopes).

Finally, with these tools we can define our second monotone.

Monotone 2: We call M_{NPR} the cost of a behaviour B with respect to the subset $\mathcal{F}_{\text{NPR}}$ of non-disturbing behaviours in the n-cycle, as measured by the function Ω, that is,

$$M_{\text{NPR}}(B) \doteq C_{(\mathcal{F}_{\text{NPR}}|\Omega)}(B) = \min_{B^* \in \mathcal{F}_{\text{NPR}}} \{\Omega(B^*),\ \textit{such that } B^* \to B\}, \tag{3.13}$$

such that if there is no $B^* \in \mathcal{F}_{\text{NPR}}$ for which $B^* \to B$, we define this cost to be infinite.

Now, we have

$$\begin{aligned} \Omega(F(\alpha)) &= \alpha\Omega(B_{\text{PR}}) + (1-\alpha)\Omega(B_{\text{NPR}}) \\ &= \alpha n + (1-\alpha)(n-2) \\ &= n + 2(\alpha - 1). \end{aligned} \tag{3.14}$$

With this, Ω defines a bijection between points (behaviours) on the line $\mathcal{F}_{\text{NPR}}$ and real numbers, leading us to a powerful equivalence: minimizing Ω over $B^* \in \mathcal{F}_{\text{NPR}}$ such that

$B^* \to B$ amounts to minimizing the quantity $n+2(\alpha-1)$ under the constrain $F(\alpha) \to B$, that is,

$$M_{\mathrm{NPR}}(B) = \min_{\alpha\in[0,1]} \{n + 2(\alpha - 1),\ \textit{such that } F(\alpha) \to B\}. \tag{3.15}$$

Evaluating the monotones

With these at hand, we now proceed to find closed form expressions for M_Ω and M_{NPR}.

Beginning with M_Ω, we already know that $M_\Omega(B) \geq n - 2$, where the inequality is saturated by any $B \in \mathbf{NC}(\mathcal{N})$, therefore it remains to evaluate the monotone for B nonfree. As already mentioned, for a given nonfree B, there is a unique noncontextual inequality associated with a functional Ω_k for which $\Omega_k(B) > n - 2$.

With the fact that every nonfree B can be uniquely decomposed into a *PR*-like behaviour for Ω_k, with $\Omega_k(B_{\mathrm{PR},k}) = n$, and a free behaviour $B_{f,k}$, with $\Omega_k(B_{f,k}) = n - 2$, such that $B = \lambda B_{\mathrm{PR},k} + (1-\lambda)B_{f,k}$, we have that such a decomposition gives $\Omega_k(B) = \lambda n + (1-\lambda)(n-2)$.

Now, consider a general noncontextual operation $\mathcal{W}$, the decomposition above gives

$$\Omega_k\Big(\mathcal{W}(B)\Big) = \lambda\Omega_k\Big(\mathcal{W}(B_{\mathrm{PR},k})\Big) + (1-\lambda)\Omega_k\Big(\mathcal{W}(B_{f,k})\Big), \tag{3.16}$$

and since we have $\Omega_k\Big(\mathcal{W}(B_{\mathrm{NPR},k})\Big) \leq n$, and $\Omega_k\Big(\mathcal{W}(B_{f,k})\Big) \leq n-2$, it follows that

$$\Omega_k\Big(\mathcal{W}(B)\Big) \leq \lambda n + (1-\lambda)(n-2) = \Omega_k(B). \tag{3.17}$$

What this means is that for a given nonfree B with respect to a funcion Ω_k, the maximum value of $\Omega_k(B^*)$ for which $B \to B^*$ is the value $\Omega_k(B)$ itself, from which we conclude that

$$M_\Omega(B) = \begin{cases} n-2, \text{ for } B \text{ free}, \\ \Omega_k(B), \text{ for } B \text{ nonfree}. \end{cases} \tag{3.18}$$

Turning our attentions to M_{NPR}, the evaluation is not as straightforward as it was for M_Ω. First, recall the definition of the behaviours $F(\alpha)$. Now, considering the whole set of non-disturbing behaviours $\mathbf{ND}(\mathcal{N})$, let us define the set $\mathbf{B}_b$ of behaviours B_b lying on the boundary of the noncontextual polytope ($\Omega(B_b) = n-2$). Let us also define the smaller set $\mathbf{B}_{bb}$ of behaviours B_{bb} that both saturate Ω and also lie on the boundary of $\mathbf{ND}(\mathcal{N})$. We have, by construction $\mathbf{B}_{bb} \subseteq \mathbf{B}_b$.

Beginning with the case of a non-contextual behaviour B, since it is a free object, carrying the minimization in the definition of M_{NPR} amounts to simply looking for the lowest value of $\Omega(B^*)$ for $B^* = F(\alpha)$ for some α. The result obtained above actually assures us that this is achieved by the minimum of $n+2(\alpha-1)$, which is $n-2$, for $\alpha = 0$.

Now, for the case of nonfree behaviours, it will be useful to prove the following auxiliary result: for any $B : \Omega(B) \geq n-2$, the minimization to be carried is equivalent to the following ones,

$$\min_{\alpha}\Big\{\Omega(F(\alpha)) \mid F(\alpha) \to B\Big\}, \tag{3.19}$$

$$\min_{\alpha}\Big\{\Omega(F(\alpha)) \mid \exists\gamma \geq 0, \exists\tilde{B}_b \in \mathbf{B}_b,\ \text{with } B = \gamma\tilde{B}_b + (1-\gamma)F(\alpha)\Big\}, \tag{3.20}$$

$$\begin{cases} \text{if } B \in \mathcal{F}_{\text{NPR}} : \Omega(B),\ \text{else} \\ \text{if } B \notin \mathcal{F}_{\text{NPR}} : n+2(\alpha-1),\ \text{with } \alpha,\gamma \geq 0,\ \text{and } \tilde{B}_{bb} \in \mathbf{B}_{bb} \text{ all} \\ \text{uniquely defining the decomposition } B = \gamma\tilde{B}_{bb} + (1-\gamma)F(\alpha). \end{cases} \tag{3.21}$$

The first of these quantities is explicitly equivalent to the definition of M_{NPR} given in section (3.15) and is taken as the starting point.

For proving the equivalence between (3.19) and (3.20), we deal separately with the case in which $B \in \mathcal{F}_{\text{NPR}}$ and the case $B \notin \mathcal{F}_{\text{NPR}}$. If $B \in \mathcal{F}_{\text{NPR}}$, then $F(\alpha) \to B$ implies that B is lower on the chain $\mathcal{F}_{\text{NPR}}$, *i.e.*, one can go freely from $F(\alpha)$ to B by mixing $F(\alpha)$ with $B_{\text{NPR}} \in \mathbf{B}_b$, giving us eq. (3.20).

If, on the other hand, $B \notin \mathcal{F}_{\text{NPR}}$, we first recall that

$$F(\alpha) \to B \iff B \in \mathbf{Conv}\left(\mathcal{V}^{Det}_{[B]}\big(F(\alpha)\big)\right). \tag{3.22}$$

Now, in order to verify that this implies that B can be generated by mixing $F(\alpha)$ with a behaviour in $\mathbf{B}_b$, we use the notion of a *screening-off inequality.* We say that an inequality $f(B) \geq y$ screens-off the set of behaviours of a fixed type that satisfies it **if** the set of behaviours that saturate the inequality is free. This notion is pretty useful when it comes to evaluating statements about behaviour convertibility. As an example, consider the question of whether $B_2 \to B_1$. If B_1 happens to lie in a screened-off region, the statement that B_1 is in the convex-hull of images of B_2 under deterministic operations of type $[B_1]$ becomes equivalent to saying that B_1 actually lies in a smaller set: the convex-hull of the images of B_2 under deterministic operations of type $[B_1]$ *that are properly inside the screened-off region* plus one boundary point, that is

$$\begin{aligned} f(B) \geq y \text{ screens-off } [B_1] \implies B_2 \to B_1 \textbf{ iff. } \exists\tilde{B}^b : f(\tilde{B}^b) = y,\ \text{and} \\ B_1 \in \mathbf{Conv}\left(\tilde{B}^b, \mathcal{V}^{Det}_{[B_1]}\big(B_2\big)\right) \cap \{B : f(B) > y\}\Big). \end{aligned} \tag{3.23}$$

Therefore, in general, knowledge of such a screening-off inequalities allows for convertibility statements regarding screened-off points to be decided through sampling from smaller sets.

For our purposes, the equivalence above is exactly what we need in order to derive eq. (3.20). The screening-off inequality to be considered is naturally $\Omega(B) \geq n-2$, with the associated *screening-off region* being the nonfree set, with boundary set $\mathbf{B}^b$; the convertibility statement under question being $F(\alpha) \to B$; the result just obtained gives us that for any nonfree B,

$$\begin{aligned} F(\alpha) \to B \textbf{ iff. } & \exists \tilde{B}_b \in \mathbf{B}_b, \\ & \text{with } B \in \mathbf{Conv}\left(\tilde{B}^b, \mathcal{V}^{Det}_{[B]}\Big(F(\alpha)\Big) \cap \{B' : \Omega(B') > n-2\}\right). \end{aligned} \tag{3.24}$$

Now, it turns out that the only image of $F(\alpha)$ under a deterministic operation that lies in the region screened-off by our inequality is $F(\alpha)$ itself (Wolfe et al., 2020), from which we conclude our desired equivalence

$$F(\alpha) \to B \textbf{ iff. } \exists\gamma \geq 0, \exists\tilde{B}_b \in \mathbf{B}_b, \text{ with } B = \gamma\tilde{B}_b + (1-\gamma)F(\alpha), \tag{3.25}$$

i.e., (3.19) and (3.20) are equivalent.

Next, we notice that what we have in (3.20) is a minimization to be carried under the constraint of α being such that $B = \gamma\tilde{B}^b + (1-\gamma)F(\alpha)$, and that such a problem could in principle be recast as the following constrained optimization to be carried:

$$\begin{aligned} \min_{0\leq\alpha\leq 1} \Omega\Big(F(\alpha)\Big), \text{ such that } \tilde{B}^b \doteq \frac{B-(1-\gamma)F(\alpha)}{\gamma}, \\ \text{under the constrain that all probabilities of } \tilde{B}^b \text{ are} \\ \text{nonnegative, with } \gamma \text{ being an implicit function of } \alpha \text{ given by} \\ \Omega\Big(\tilde{B}^b\Big) = \frac{\Omega(B)-(1-\gamma)\Omega\Big(F(\alpha)\Big)}{\gamma} = n-2. \end{aligned}$$

And here, we use the argument given in section $B.1$ of (Wolfe et al., 2020), which says that *essentially, this is a constrained optimization problem with a linear objective subject to one linear constraint; namely, that the* ***smallest*** *conditional probability in* $\tilde{B}_b$ *is nonnegative. For such optimization problems, it is always the case that the objective is maximized when the constraint is not merely satisfied but saturated, that is, the optimal* α *arises for some unique* $\boldsymbol{B}_b \ni \tilde{B}_b = \tilde{B}_{bb} \in \boldsymbol{B}_{bb}$ *where the smallest conditional probability is precisely zero.* This argument plus some of the facts established in the preceding paragraphs give us the equivalence between (3.20) and (3.21).

From the considerations above, we finally obtain the desired closed-form expression for M_{NPR}: if B is free, $M_{\text{NPR}}(B) = n - 2$; for a nonfree B, there is one noncontextual inequality and associated function Ω for which $\Omega(B) > n - 2$. Within this region, if $B \in \mathcal{F}_{\text{NPR}}$, then $M_{\text{NPR}}(B) = \Omega(B)$; if $B \notin \mathcal{F}_{\text{NPR}}$, we have

$$\begin{aligned} M_{\text{NPR}}(B) = n + 2(\alpha - 1),\ &\text{where } \alpha \text{ is such that} \\ &B = \gamma \tilde{B}_{bb} + (1-\gamma) F(\alpha),\ \text{with } F(\alpha) \in \mathcal{F}_{\text{NPR}} \text{ and } \tilde{B}_{bb} \in \mathbf{B}_{bb}. \end{aligned} \tag{3.26}$$

3.2 Characterizing the pre-order

Obtaining closed-form expressions for both M_Ω and M_{NPR} allows us to properly characterize the pre-order of objects of our resource theory. For this, we introduce the following construction of two-parameter families of behaviours

$$\begin{aligned} \mathcal{B}_{(*)} \doteq \Big\{ B(\alpha, \gamma) : \alpha \in [0,1], \gamma \in [0,1] \Big\},\ &\text{where } B(\alpha,\gamma) \doteq \gamma B^*_{bb} + (1-\gamma) F(\alpha), \\ &B^*_{bb} \text{ is a choice of behaviour in } \mathbf{B}_{bb},\ \text{and } F(\alpha) \in \mathcal{F}_{\text{NPR}}. \end{aligned} \tag{3.27}$$

Each choice of $B^*_{bb} \in \mathbf{B}_{bb}$ defines a family, hence the subscript in $\mathcal{B}_{(*)}$. Moreover, each such family is also given by the convex-hull of B^*_{bb} and the chain $\mathcal{F}_{\text{NPR}}$, that is, $\mathcal{B}_{(*)} = \mathbf{Conv}\left(\left\{ B^*_{bb}, B_{\text{PR}}, B_{\text{NPR}} \right\}\right)$.

In terms of our monotones, starting with with M_{NPR}, for general (α, γ), we have

$$M_{\text{NPR}}\Big(B(\alpha,\gamma)\Big) = n + 2(\alpha - 1). \tag{3.28}$$

For M_Ω, since $\Omega\Big(B(\alpha,\gamma)\Big) \geq n - 2$ for any (α, γ), we have

$$M_\Omega\Big(B(\alpha,\gamma)\Big) = \Omega\Big(B(\alpha,\gamma)\Big), \tag{3.29}$$

where, recalling that $\Omega(B^*_{bb}) = \Omega(B_{NPR}) = n - 2$ and $\Omega(B_{\text{PR}}) = n$, we get

$$M_\Omega\Big(B(\alpha,\gamma)\Big) = n + 2\alpha(1-\gamma) - 2. \tag{3.30}$$

Turning now to the proper characterization of the pre-order in terms of the properties introduced in section 4.3, we can say:

(1) Without even considering the whole set $\mathbf{ND}(\mathcal{N})$, but looking only at the chain $\mathcal{F}_{\text{NPR}}$, one sees that between any two given behaviours $F(\alpha_1)$ and $F(\alpha_2)$ there are infinite inequivalent objects, for α runs continuously. Furthermore, since the chain is strictly ordered, each pair of inequivalent objects defines a unique equivalence class. We have,

then, an infinite number of such equivalence classes of inequivalent objects between any two behaviours, and hence the chain - and consequently the whole set - is ***locally infinite***.

(2) We also demonstrate that there are incomparable resources in the pre-order. For this, consider the following two objects in $\mathcal{B}_{(*)}$: $B_1 \doteq B\,(0,0)$ and $B_2 \doteq B\left(1, \frac{1}{2}\right)$. These behaviours are incomparable, as witnessed by our two monotones:

$$\begin{aligned} &M_{\mathrm{NPR}}(B_1) = n-2 < M_{\mathrm{NPR}}(B_2) = n, \text{ and} \\ &M_{\Omega}(B_1) = n-2 > M_{\Omega}(B_2) = n-3, \end{aligned} \tag{3.31}$$

which allows us to conclude that ***the pre-order is not totally pre-ordered***.

(3) Next, consider the following three behaviours: $B_1 \doteq B\,(0,0)$, $B_2 \doteq B\left(\frac{1}{2}, \frac{1}{2}\right)$ and $B_3 \doteq B\left(\frac{1}{2}, \frac{3}{4}\right)$. By the same reasoning applied in *(2)*, one may verify that we have $B_1 \not\leftrightarrow B_2$ and $B_1 \not\leftrightarrow B_3$. But since $\frac{1}{2}B^*_{bb} + \frac{1}{2}B_2 = B_3$ and B^*_{bb} is free, we have that $B_2 \to B_3$, what shows that the incomparability relation is not transitive, hence ***the pre-order is not weak***.

(4) Recalling that the height of the pre-order is the cardinality of the largest chain contained therein, since we know that $\mathcal{F}_{\mathrm{NPR}}$ is a chain with infinite elements, we conclude that ***the height of the pre-order is infinite***. To investigate the width of the pre-order, consider the set of points $\{B(x,x) \mid \frac{1}{2} \leq x \leq 1\}$, the line segment between points $B\left(\frac{1}{2}, \frac{1}{2}\right)$ and $B\,(1,1)$. By inspection, we notice that within this region the function M_{NPR} is strictly increasing, while M_{Ω} is strictly decreasing, *i.e.*, the pair of monotones witness the incomparability of every pair of objects in this line segment. Hence this line segment constitutes an antichain, and since here there is also an infinite number of incomparable points amongst each other, by the same logic applied to the height of the pre-order, we conclude that ***the width of the pre-order is also infinite***.

Capítulo 4

Conclusion

Hence, we come to the ending of this work. We introduced the resource theory framework in general terms and introduced the resource theory of contextuality given by choosing the noncontextual wirings operations as the free operations of the theory. Then, we quickly explored a resource theory of Bell nonlocality in order to introduce some of the concepts we later tried to extend to our resource theory of contextuality, explicitly showing how the free operations of this resource theory of contextuality are in a way a somewhat natural generalization of LOSR operations for Bell nonlocality. Thus, extending some results known for a specific scenario in the Bell nonlocality framework, we were able to show that not only our set of free operations forms a polytope, but also were able to provide a algorithmic solution to the problem of deciding whether an arbitrary object of the resource theory can be converted into another. We proceeded then to explore the global properties of the preorder of objects of our resource theory, and since the methods for exploring these properties were also known for this specific case in the resource theory of Bell nonlocality, the effort we spent relating both resource theories paid off. We managed to extend without much difficulty the definitions of proper families of useful resource quantifiers, in particular cost and yield monotones, to a whole class of measurement scenarios in the more general contextuality framework, which we then explored and were able to characterize, leveraging these families of monotones to finally derive the global properties of the resource theory of contextuality.

Interestingly, all of the investigated global properties ended up behaving in the general contextuality framework exactly as they behaved in the nonlocality framework. Just as happened in the resource theory of nonlocality given by LOSR operations, in the resource theory of contextuality defined by noncontextual wirings as free operations, the induced preorder of objects is locally infinite, not totally pre-ordered, is not weak, and both its height and its width are infinite. These results reveal a great structural similarity between both frameworks. At least in terms of what these properties tell us about the nature of the respective resource theories, it seems that a significant part of how both resource theories are constituted as structures is indeed present in both, which might suggest that one does not lose so much in looking at how nonlocality works as a resource instead of looking at how contextuality constitutes itself as a resource, or conversely, it might suggest that results on how Bell nonlocality behaves as a resource might be actually telling us more than we might think at first. Bell nonlocality as a resource has been widely studied and there are many results that characterize what are its capabilities in terms of what are features of such resource theory. Relating some of these properties to our framework, meaning to properties that are a direct consequence or that can be shown to be exclusively related to the global properties we presented, one has a direct insight into the possibilities of resource usage for a wider class of phenomena.

Yet, there still remain questions to be answered, even related to the results we obtained. This is because when defining the monotones in terms of violations of Bell-type noncontextual inequalities, we had to restrict our discussion to the $n-$cycle scenario, for which we have explicit inequalities and bounds for their violation. Therefore, all of our results are actually bound to this limitation, this is the most general type of scenario we were able to work with, but it is far from everything, and more general scenarios still need to be probed. Yet, we feel that, since contextuality is linked with the presence of a cycle in the measurement scenario, there is something fundamental in these results, but nonetheless the question remains open. Furthermore, there are many directions of inquiry one could take at this point.

One of them has to do with the completeness of the monotones M_Ω and M_{NPR}. Since these monotones are not complete within the Bell nonlocality framework, they are also not complete in the general contextuality framework, but we can ask what and how many are the monotones needed to completely characterize our objects.

One can also wonder about the role that symmetry operations play in the kinds of structures we investigated. We explored the role of deterministic operations and in particular showed how, for reasons of convexity, they form the backbone of the structure of our free transformations set. But what about the specific set of symmetry operations? What is the relationship between them and the polytope of free operations? What kind of structure do them induce on the preorder of objects?

Finally, a last direction that can be pursued by further research is how to incorporate within our framework the possibility of not only single-shot operations, but also multi-copy, asymptotic and indeterministic conversions, for example, which could be interesting especially in regards to experimental settings.

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