Universidade de São Paulo
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Deformações integráveis do modelo sigma da supercorda em espaços semi-simétricos

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On integrable deformations of semi-symmetric space sigma-models

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*To my parents and brother:
Yoli, Hugo and Eduar.*

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Abstract

In this thesis we review some aspects of Yang-Baxter deformations of semi-symmetric space sigma models. We start by giving a short review of the sigma model description of superstrings and then we offer a self contained introduction to the Yang-Baxter deformation technique. We then show how to obtain an integrable deformation of the hybrid sigma model. Also, we show that the gravity dual of β -deformed ABJM theory can be obtained as a Yang-Baxter deformation. This is done by selecting a convenient combination of Cartan generators in order to construct an Abelian r -matrix satisfying the classical Yang-Baxter equation.

Keywords: Superstrings; Integrability; Integrable deformations; Supergravity; Yang Baxter equation; R-matrix; Hybrid sigma model.

Resumo

Nesta tese revisamos alguns aspectos das deformações de Yang-Baxter de modelos sigma em espaço semi-simétricos. Damos uma breve revisão do modelo sigma de supercordas e, em seguida, oferecemos uma introdução ao método de deformação de Yang-Baxter. Em seguida, mostramos como obter uma deformação integrável do modelo sigma híbrido. Além disso, mostramos que o dual gravitacional da teoria ABJM β -deformada pode ser obtida como uma deformação de Yang-Baxter. Isso é feito selecionando-se uma combinação conveniente de geradores de Cartan para construir uma matriz r Abelianas satisfazendo a equação clássica de Yang-Baxter.

Palabras chave: Supercordas; Integrabilidade; Deformações integráveis; Supergravidade; Equação de Yang Baxter; Matriz R; Modelo sigma híbrido.

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Chapter 1

Introduction

The gauge/gravity duality is probably the most important tool to study quantum gravity that has been developed in the last decades. The original example of the duality establishes the correspondence between Type IIB superstring theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ superconformal Yang-Mills theory in four dimensions [1]. There is no doubt that integrability has become one of the most useful tools to study both sides of the correspondence but it was not until [2] that a Lax pair for the string theory side was constructed. The $\mathbb{Z}_4$ automorphism that the algebra $\mathfrak{psu}(2, 2|4)$ admits proved to be a key ingredient to achieve this result. However, rather surprisingly, a full proof of complete classical integrability, that is, the existence of an infinite number of conserved charges in involution, was only found later [3, 4].

The property of integrability is not very common when trying to study two dimensional field theories. Furthermore, there is no systematic way to prove whether a two dimensional field theory is integrable or not. For this reason, relating the original superstring sigma model to deformed versions of it which preserve this integrable structure is a very useful exercise. A very important type of deformations are the so called β -deformations associated with strings moving in Lunin-Maldacena backgrounds [5]. The integrability of these models was shown in [6, 7]. A systematic way of constructing the gravity dual of these gauge theories using TsT transformations¹, was developed in [8].

Apart from β -deformations, two families of integrable deformations of a very different nature has been studied over the past few years, both of which have been successfully applied to the $AdS_5 \times S^5$ superstring sigma model.

The first family of integrable deformations are usually referred as λ -deformations, they

¹TsT transformations are a sequence of T-duality, shift and T-duality transformations in a given geometry. The gravity dual of some β -deformed gauge theories can be obtained using these transformations.

were first proposed in [9], where integrable deformations of the non-abelian T-dual of the principal chiral model were considered. This procedure directly deforms the action of the theory by adding a WZW-like term to the action. The generalization of this deformations to symmetric sigma models was developed in [10] and then applied to semi-symmetric sigma models, in particular $AdS_5 \times S^5$, in [11]. We will not be studying λ deformations in this thesis.

A second family of integrable deformations was recently proposed in [12, 13] for the principal chiral model on an arbitrary Lie group G . The core of these kind of deformations is the use of linear operators $R \in \text{End}(\mathfrak{g})$ which are solutions to the modified classical Yang-Baxter equation (mCYBE) on $\mathfrak{g} = \text{Lie}(G)$, which is given by

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = -c^2[X, Y], \quad (1.1)$$

for any $X, Y \in \mathfrak{g}$ and $c^2 \in \mathbb{R}$. In fact we can classify solutions of the mCYBE using the parameter c^2 as follows

$$c^2 < 0, \quad c^2 > 0, \quad c^2 = 0. \quad (1.2)$$

The deformations of the principal chiral model we discussed above, make use of solutions of the mCYBE with $c^2 < 0$, these kind of R -matrices are usually referred as non-split R -matrices. These deformations are constructed by modifying the Poisson structure of the model in such a way that the Lax pair is not changed, thus ensuring the integrability of the resulting deformed theory. The generalization of this deformations to symmetric sigma models was achieved in [14], while the extension to semi-symmetric sigma models, in particular to $AdS_5 \times S^5$, was developed in [15]. These deformations are sometimes referred as η -deformations, $\varkappa$ -deformations, q -deformations or Yang-Baxter deformations. The first denomination responds to the usual notation used for the deformation parameter, while the notation $\varkappa$ -deformations were used for the first time in [16, 17, 18] where the background fields of the deformed model were computed. The last denomination refers to the fact that the global symmetry algebra of this model was shown to be q -deformed in [14].

The case of deformations using solution of the mCYBE with $c^2 > 0$ is a bit problematic and they have received very little attention. In particular, there are no solutions to the mCYBE with $c^2 > 0$ in $\mathfrak{psu}(2, 2|4)$ [19].

Solutions to the classical Yang-Baxter equation (CYBE), that is solutions with $c^2 = 0$, have been considered for the symmetric sigma model in [20], and have been generalized to the semi-symmetric case in [21]. In comparison to the mCYBE, the method using solutions to the CYBE admit partial deformations. That is, in $AdS_5 \times S^5$ for instance, deformations of AdS_5 or S^5 can be considered separately. Furthermore, using these kind of

deformations, it is possible to show that Lunin-Maldacena backgrounds have an algebraic origin [22]². In fact, the connection between Abelian r -matrices and TsT transformations for string theory sigma models has been observed in many situations [26, 27, 28, 29] and proven recently [30]. Moreover, it has been suggested [31] and then demonstrated in general [32, 33] that Yang-Baxter deformations are equivalent to non-Abelian duality transformations for superstring sigma models.

A second example of the gauge/Gravity duality is the so called AdS_4/CFT_3 correspondence which establishes the duality between Type IIA superstring theory in $AdS_4 \times \mathbb{CP}^3$ and $\mathcal{N} = 6$ superconformal Chern-Simons-matter theory (ABJM theory) [34]. Almost all of what can be said about superstrings moving in $AdS_5 \times S^5$ can be extended to the $AdS_4 \times \mathbb{CP}^3$ counterpart. In particular, both of them admit a supercoset description with $\mathbb{Z}_4$ grading and as a consequence, the integrability discussion in both is analogous [35]. That being said, all of the analysis that lead to the formulation of Yang-Baxter deformations could in principle be extended to $AdS_4 \times \mathbb{CP}^3$. Evidence of this is the fact that TsT transformations can also be used to construct the gravity dual of β -deformations of ABJM theory in $AdS_4 \times \mathbb{CP}^3$. We will later show in this thesis that Yang-Baxter deformations can also be used to construct these deformed backgrounds, thus showing their algebraic r -matrix origin.

The usual coset description of $AdS_4 \times \mathbb{CP}^3$ is given by

$$\frac{OSp(6|4)}{SO(1,3) \times U(3)} . \quad (1.3)$$

However the $\mathbb{CP}^3$ metric that one obtains from this coset does not exactly match the standard Fubini-Study metric shown in [8, 36]. Furthermore, the parametrization using 6×6 generators of $SO(6)$ turns out to be very complicated making computations quite messy. The solution we propose [37] is to modify the coset describing the $\mathbb{CP}^3$ part of the sigma model to

$$\frac{SU(4) \times SU(2)}{U(3) \times SU(2)} . \quad (1.4)$$

This description is analogous to the one performed in the analysis of Yang-Baxter deformations of $T^{1,1}$ [38]. As we will show, it is possible to embed some of the generators of $\mathfrak{u}(3) \oplus \mathfrak{su}(2)$ in the denominator in a convenient way to define an orthogonal basis of

²More recently, the relation between solutions of the CYBE and supergravity have been studied using open-closed string maps where Yang-Baxter deformations can be generalized beyond cosets [23, 24, 25].

$\mathfrak{su}(4) \oplus \mathfrak{su}(2)$, thus describing the standard Fubini-Study metric of $\mathbb{CP}^3$.

This thesis is organized as follows. Chapter 2 is devoted to the superstring sigma model in $AdS_5 \times S^5$. We give a detailed derivation of some of the most important properties of the Green-Schwarz and pure spinor formulations. For completeness, we also offer a description of $\mathfrak{psu}(2, 2|4)$ superalgebra and its most important properties. In chapter 3 we offer a self-contained review of the methods that led to the formulation of the so called η -deformed models. We give a review of some aspects of the modern definition of classical integrability. We then show that a new type of η -deformed sigma model can be formulated in semi-symmetric spaces, namely the η -deformed hybrid sigma model, the matter part of the pure spinor sigma model. On chapter 4 we give a few examples of how to extract the deformed fields of η -deformed models. In particular, we construct Yang-Baxter deformations of the gravity dual of ABJM theory. In order to do this, we first propose a new coset construction of $\mathbb{CP}^3$ leading to a convenient parametrization that correctly reproduces the standard Fubini-Study metric on $\mathbb{CP}^3$ [37]. We present the appropriate combinations of Cartan generators that constitute the R -matrix that needs to be used in order to reproduce the well known one-parameter and three-parameter β -deformation results [8], as Yang-Baxter deformations. Our last chapter is devoted to conclusions. Appendix A reviews some important aspects of integrable field theories. Appendix B is used to give a few examples of the simplest solutions to the mCYBE and finally Appendix C establishes some conventions and reviews useful representations of various algebras that we use throughout the text.

Chapter 2

The sigma model description of superstrings

In order to have a better understanding of the *AdS/CFT* duality, it is important to study how to formulate superstrings in *AdS* backgrounds. One highly non-trivial property of *AdS* backgrounds is the so called $\mathbb{Z}_4$ gradation, this property allowed the proof of classical integrability of the superstring sigma model [2][39]. The purpose of this chapter is to give a formal definition of semi-symmetric superspaces and how to formulate the motion of superstrings on this kind of spaces.

The best reference to study superstrings in *AdS* backgrounds is the review by Mazzucato [40] which we will follow closely. The supercoset sigma model formulation of the Green-Schwarz superstring is explained in detail in [41]. On the other hand, the reader can consult [42] and [43] for a basic approach to the theory of manifolds and superalgebras respectively.

2.1 Superstrings in flat space

For the sake of completeness, we give a very short overview of the bosonic string sigma model and superstrings in flat space.

2.1.1 Bosonic strings as a non-linear sigma model

The action for bosonic strings is given by the Polyakov action which can be interpreted as proportional to the area of the two dimensional surface described by a string when

propagating in D dimensions. The Polyakov action is given by

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu. \quad (2.1)$$

The theory described by this action is relatively simple and can be solved exactly, for instance one knows that the massless spectrum contains a rank 2 symmetric tensor excitation (the graviton), a rank 2 antisymmetric excitation and a scalar, the so called dilaton [44]. A natural generalization to the Polyakov action can be obtained by considering some generic background with metric $G_{\mu\nu}(X)$. It turns out that this is equivalent to include vertex operators of the rank 2 symmetric excitations of the massless spectrum. This then leads to the question: Can we modify S_P to include all the effects of the massless excitations? The answer is found in the non-linear sigma model [45][46]

$$S = \int d^2\sigma \left[\sqrt{-g} g^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \epsilon^{ab} B_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \sqrt{-g} \Phi(X) R^{(2)}(g) \right], \quad (2.2)$$

here G, B and Φ , the fields of the massless spectrum, are just couplings. As we will see in a moment, this action can be rewritten in a way that shows that one can treat the fields of these models as points on a manifold, that is, the action shown above is just an action for embeddings of the string worldsheet to some generic non trivial background with metric G .

2.1.2 Superstrings

The Ramond-Neveu-Schwarz (RNS) formalism of the superstring is a field theory which maps a supersymmetric worldsheet to a bosonic space-time, while the Green-Schwarz formalism maps a bosonic worldsheet to a supersymmetric space-time. On one hand, the RNS action is rather easy to quantize but the quantum spectrum of the theory requires an ad-hoc prescription in order to make it supersymmetric, namely the GSO projection. Apart from that, the procedure to compute amplitudes higher than one loop is very complicated because of the lack of manifest space-time supersymmetry. On the other hand, the Green-Schwarz formalism possesses manifest space-time supersymmetry and can be easily generalized to a generic supergravity background. However, due to the nature of the constraints of the theory, the covariant quantization for this model is not known, and one is forced to use the kappa symmetry of the theory to quantize it in the light cone gauge, thus losing manifest Lorentz covariance. Furthermore, computing amplitudes in light cone gauge is extremely difficult.

The Green-Schwarz action describing a string moving on a supersymmetric flat space-time is given by:

$$S_{GS} = -\frac{1}{4\pi\alpha'} \int_{\mathcal{M}} d^2\sigma \left\{ \sqrt{-g} g^{ab} \Pi_a \cdot \Pi_b - 2\epsilon^{ab} (\bar{\theta}^1 \gamma^m \partial_a \theta^1) (\bar{\theta}^2 \gamma_m \partial_b \theta^2) + \right. \\ \left. + 2i\epsilon^{ab} \partial_a X_m (\bar{\theta}^1 \gamma^m \partial_b \theta^1 - \bar{\theta}^2 \gamma^m \partial_b \theta^2) \right\}, \quad (2.3)$$

where $\Pi_a^m = \partial_a X^m - i\delta_{AB} \bar{\theta}^A \gamma^m \partial_a \theta^B$ and $\mathcal{A}, \mathcal{B} = 1, 2$ and $m = 0, \dots, 9$. Finding an analogue to the bosonic non-linear sigma model for the action above is the task which will concern us in this chapter. On the other hand, another approach to the theory of superstrings is the so called pure spinor formalism proposed by Berkovits [47]. The pure spinor formalism possess manifest space-time supersymmetry and the covariant quantization can be achieved by means of the inclusion of a BRST charge. In this way this formalism takes the advantages of the RNS and Green-Schwarz formalisms but it does not suffer from their difficulties. The pure spinor action can be written as follows:

$$S_{PS} = \int d^2z \left(\frac{1}{2} \partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial} \theta_\alpha - \omega_\alpha \bar{\partial} \lambda^\alpha \right). \quad (2.4)$$

This action is a clever modification of the Green-Schwarz action and it is written in conformal gauge. The last term of S_{PS} represents the curved beta-gamma system which describes the pure spinor manifold. Apart from that, the bosonic complex Weyl spinor ghost $\lambda^\alpha, \alpha = 1, \dots, 16$ satisfies the pure spinor constraint

$$\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0. \quad (2.5)$$

The physical states of the theory are defined as the ghost number one cohomology of the nilpotent BRST operator

$$\mathcal{Q} = \oint dz \lambda^\alpha d_\alpha, \quad (2.6)$$

where d_α are the supersymmetric Green-Schwarz constraints [40, 48, 49].

2.2 Superstrings in curved backgrounds

As we mentioned in the last section, describing the motion of superstrings in a flat background is kind of straightforward. In fact, for instance, since the bosonic string is a two dimensional conformal field theory one can exploit the symmetries and show that this model is completely solvable. On the other hand, if one is interested in the AdS/CFT

correspondence then we should try to ask how complicated our task can get if we want to study the dynamics of superstrings when moving on nontrivial curved backgrounds. It turns out that these cases are extremely complicated and in general one does not even know how to solve the equations of motion.

The non-linear sigma model, as described in [50] and [51], can be defined as a field theory whose dynamical fields happen to be elements of some Lie group (manifold). The example of non-linear sigma model that we are interested in is precisely the motion of a string on a given group manifold. Thus, the motion of superstrings will just be described by embeddings of the string worldsheet to some group manifold, which will be referred to as the target space.

2.2.1 Green-Schwarz sigma model

To begin with, let us try to discuss the most general way to construct a superstring sigma model as a direct generalization of the flat Green-Schwarz action we described in (2.3). Let us start by considering a target superspace provided with curved supercoordinates $Z^M = \{X^m, \theta^\mu, \hat{\theta}^{\hat{\mu}}\}$,¹ where just as we considered before, the Grassmann fermionic coordinates are Majorana-Weyl spinors² and $M = (m, \mu, \hat{\mu})$; $m = 0, \dots, 9$. The Green-Schwarz Type II superstring can be naturally extended to curved backgrounds [52]

$$S_{GS} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{-g} g^{ij} G_{MN}(Z) + \epsilon^{ij} B_{NM}(Z) \right) \partial_i Z^M \partial_j Z^N. \quad (2.7)$$

The action presented above represents a generalization of the action (2.3) we considered before. The first term is analogous to the kinetic term of (2.3), while the second one corresponds to the Wess-Zumino term, G_{MN} and B_{NM} are background superfields.

As we mentioned before, the superspace can be regarded as a supermanifold, so at every point Z we can define a tangent superspace with flat metric η_{ab} and a cotangent superspace. The latter admits coordinate dual basis $\{dZ^M\}$ and orthonormal basis $\{J^{\mathcal{A}}\}$, where $\mathcal{A} = (a, \alpha, \hat{\alpha})$ with $a = 0, \dots, 9$; $\alpha, \hat{\alpha} = 1, \dots, 16$ are indices on the tangent superspace. The change of basis define the *supervielbein* $E_M^{\mathcal{A}}$ as follows

$$J^{\mathcal{A}} = E_M^{\mathcal{A}} dZ^M. \quad (2.8)$$

The superfield G_{MN} represents a generalization of the metric to the superspace, then we can write

$$G_{MN}(Z) = E_M^a(Z) E_N^b(Z) \eta_{ab}. \quad (2.9)$$

¹We will make a little change in the notation with respect to the last section, we use hats instead of bars to discriminate the chirality of the spinorial coordinates.

²Type IIB corresponds to spinors of the same chirality, while Type IIA to those of opposite chiralities.

On the other hand, in a generic supergravity background the Wess-Zumino term is given by the 2-superform

$$B = \frac{1}{2} B_{MN} dZ^M \wedge dZ^N = \frac{1}{2} B_{\mathcal{A}\mathcal{B}} J^{\mathcal{A}} \wedge J^{\mathcal{B}}, \quad (2.10)$$

where

$$B_{MN}(Z) = E_M^{\mathcal{A}}(Z) E_N^{\mathcal{B}}(Z) B_{\mathcal{A}\mathcal{B}}(Z). \quad (2.11)$$

The Wess-Zumino term can be written as an integral of the 2-superform B as follows

$$\begin{aligned} S_{WZ} &= \frac{1}{2\pi\alpha'} \int B = \frac{1}{4\pi\alpha'} \int B_{MN} dZ^M \wedge dZ^N \\ &= + \frac{1}{4\pi\alpha'} \int B_{MN} \partial_i Z^M \partial_j Z^N d\sigma^i \wedge d\sigma^j \\ &= - \frac{1}{4\pi\alpha'} \int d^2\sigma \epsilon^{ij} B_{MN} \partial_i Z^M \partial_j Z^N. \end{aligned} \quad (2.12)$$

The expression (2.8) defining the supervielbein can be written in components in the following way

$$J_i^{\mathcal{A}} = E_M^{\mathcal{A}} \partial_i Z^M, \quad (2.13)$$

so using equations (2.9), (2.12) and (2.13), the Green-Schwarz sigma model action can be written as follows

$$S_{GS} = - \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{-g} g^{ij} \eta_{ab} J_i^a J_j^b + \epsilon^{ij} B_{\mathcal{A}\mathcal{B}} J_i^{\mathcal{A}} J_j^{\mathcal{B}} \right). \quad (2.14)$$

2.2.2 Semi-symmetric superspaces

We will apply the ideas we developed in the last section by considering non-linear sigma models defined over target spaces which can be realized as coset manifolds. The most interesting way to construct manifolds with this kind of structure is the construction of the so called (semi)symmetric spaces.

Definition 2.2.1 *Let us consider some Lie group G and a subgroup H . An Homogeneous space equipped with coset topology G/H is said to be symmetric if H is the fixed point set of an automorphism σ of G of order 2. That is, at the level of the algebras:*

$$\sigma : \mathfrak{g} \rightarrow \mathfrak{g} \quad ; \quad \sigma[X, Y] = [\sigma X, \sigma Y], \quad (2.15)$$

where $\sigma^2 = I$ and $X, Y \in \mathfrak{g}$, such that the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a $\mathbb{Z}_2$ grading of $\mathfrak{g}$, that is:

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}. \quad (2.16)$$

The construction above can be directly applied to our case of interest, that is, one can identify $AdS_5 \times S^5$ as a semi-symmetric superspace. We call it *semi* because the automorphism is now of order four and as a consequence it induces a $\mathbb{Z}_4$ grading with $SO(1, 4) \times SO(5)$ as the fixed point set. We will examine this construction in some detail in the next section.

2.2.3 Basic properties of $\mathfrak{psu}(2, 2|4)$

Some Lie superalgebras can be defined in terms of matrix realizations, which are very useful when defining the real forms of them. Our definition of a supermatrix is given below

Definition 2.2.2 *Let $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ be a $\mathbb{Z}_2$ -graded vector space, where $\dim(\mathcal{V}_0) = m$ and $\dim(\mathcal{V}_1) = n$. We define a supermatrix M as an element of $End(\mathcal{V})$, that is $M : \mathcal{V} \rightarrow \mathcal{V}$ which may be decomposed as follows*

$$M \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_1 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_1 \end{pmatrix}, \quad M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}. \quad (2.17)$$

where m and n are of even grading and of dimension $m \times m$ and $n \times n$ respectively. In contrast, θ and η are of odd grading and of dimension $m \times n$ and $n \times m$ respectively.

We define the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ as the set of all supermatrices (2.17) over the field $\mathbb{C}$. The supertrace and supertranspose are defined as follows

$$\text{str} M \equiv \text{tr } m - \text{tr } n, \quad M^{st} = \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}. \quad (2.18)$$

Then, in analogy to the theory of classical Lie algebras, one can define the special unitary Lie superalgebra by taking a subalgebra of $\mathfrak{gl}(m|n)$. Let us define the matrix Lie superalgebra $\mathfrak{sl}(m|n)$ over $\mathbb{C}$ as follows

$$\mathfrak{sl}(m|n) = \{M \in \mathfrak{gl}(m|n); \quad \text{str} M = 0\}. \quad (2.19)$$

We now specialize to the case of the superalgebra $\mathfrak{sl}(4, 4)$ over the field $\mathbb{C}$. The defining representation of this superalgebra is given by a graded vector space which is spanned by 8×8 supermatrices (2.17), which are constructed in terms of 4×4 blocks, where m and n are regarded as even (bosonic) and θ and η are regarded as odd (fermionic).³ As we

³The entries of θ and η can be thought as Grassmann anti-commuting variables

established in (2.19), we define the superalgebra $\mathfrak{sl}(4, 4)$ as spanned by the matrices M with vanishing supertrace $\text{str} M \equiv \text{tr } m - \text{tr } n = 0$.

Furthermore, the superalgebra $\mathfrak{su}(2, 2|4)$ is defined as a non-compact real form of $\mathfrak{sl}(4, 4)$. We consider the so-called *Cartan involution*

$$\phi(M) \equiv M^\star = M, \quad (2.20)$$

where $M^\star$ is defined as follows

$$M^\star = -HM^\dagger H^{-1}, \quad (2.21)$$

here $M^\dagger$ stands for the adjoint of the supermatrix M defined by $M^\dagger = (M^{st})^*$. Combining equations (2.20) and (2.21) we find that a supermatrix M from $\mathfrak{su}(2, 2|4)$ satisfies the following reality condition

$$MH + HM^\dagger = 0. \quad (2.22)$$

The Hermitian matrix H is defined as follows⁴

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & 1_4 \end{pmatrix}, \quad (2.23)$$

and Σ is a 4×4 matrix defined as

$$\Sigma = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}. \quad (2.24)$$

We can find how the reality condition (2.22) acts on the block entries of M by expanding it as follows

$$\begin{pmatrix} m\Sigma & \theta \\ \eta\Sigma & n \end{pmatrix} = \begin{pmatrix} -\Sigma m^\dagger & -\Sigma \eta^\dagger \\ -\theta^\dagger & -n^\dagger \end{pmatrix}, \quad (2.25)$$

then the condition (2.22) implies

$$m^\dagger = -\Sigma m \Sigma, \quad n^\dagger = -n, \quad \eta^\dagger = -\Sigma \theta, \quad (2.26)$$

from these conditions we can easily see that the matrix blocks m and n span the unitary algebras $\mathfrak{u}(2, 2)$ and $\mathfrak{u}(4)$ respectively. From this simple analysis we can deduce that the bosonic subalgebra of $\mathfrak{su}(2, 2|4)$ is given by

$$\mathfrak{su}(2, 2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1), \quad (2.27)$$

⁴Here 1_n stands for the $n \times n$ identity matrix.

where we have added the final factor since the $\mathfrak{u}(1)$ generator $i1_4$ is also part of $\mathfrak{su}(2, 2|4)$ because it satisfies (2.22) and possess vanishing supertrace. Finally, We define the $\mathfrak{psu}(2, 2|4)$ superalgebra as a quotient algebra of $\mathfrak{su}(2, 2|4)$ over this $\mathfrak{u}(1)$ factor.

The most important property of $\mathfrak{psu}(2, 2|4)$ is that it admits a fourth order automorphism $\Omega : M \mapsto \Omega(M) \in \mathfrak{psu}(2, 2|4)$ which can be defined as follows

$$\Omega(M) = \begin{pmatrix} J m^t J & -J \theta^t J \\ J \eta^t J & J n^t J \end{pmatrix}; \quad J = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix}. \quad (2.28)$$

It is easy to check that this definition satisfies $\Omega^4(M) = M$, that is

$$\Omega^4 = I, \quad (2.29)$$

Hence, the linear map Ω has eigenvalues $\pm 1, \pm i$. Thus, if we denote $\mathcal{H}_k$ as the eigenspace associated to the eigenvalue i^k ($k = 0, 1, 2, 3$), we have

$$\Omega(H_k) = i^k H_k \quad ; \quad H_k \in \mathcal{H}_k. \quad (2.30)$$

As a consequence, we can use this automorphism to construct a decomposition that exhibits explicitly the $\mathbb{Z}_4$ grading structure of $\mathfrak{psu}(2, 2|4)$.

$$\mathfrak{psu}(2, 2|4) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3. \quad (2.31)$$

Furthermore, one can note that:

$$\Omega([H_k, H_l]) = [\Omega(H_k), \Omega(H_l)] = i^{k+l} [H_k, H_l], \quad (2.32)$$

thus:

$$[H_k, H_l] \in \mathcal{H}_{k+l \mod 4}, \quad (2.33)$$

which is compatible with the supertrace operation

$$\text{str} H_k H_l = 0 \quad , \quad \text{unless } k + l = 0 \mod 4. \quad (2.34)$$

Since we want to construct some homogeneous space that can be identified with the $AdS_5 \times S^5$ manifold, we need to study the explicit form of $\mathcal{H}_k$. By definition $\Omega(H_0) = H_0$, which applied to the block structure of the supermatrices gives

$$m^t J + J m = 0 \quad , \quad n^t J + J n = 0 \quad , \quad \theta = \eta = 0. \quad (2.35)$$

These means that $m, n \in \mathfrak{sp}(4)$, which is the Lie algebra of the symplectic group $Sp(4)$. From the discussion we had before $m \in \mathfrak{u}(2, 2)$ and $n \in \mathfrak{u}(4)$. Defining the Lie algebra $\mathfrak{usp}(n)$ of the unitary symplectic group $USp(n) \equiv SU(n) \cap Sp(n)$, we can conclude that:

$$m \in \mathfrak{usp}(2, 2) , \quad n \in \mathfrak{usp}(4). \quad (2.36)$$

Since $\theta = \eta = 0$, then $\mathcal{H}_0$ is a bosonic subalgebra that can be identified with

$$\mathcal{H}_0 = \mathfrak{usp}(2, 2) \oplus \mathfrak{usp}(4), \quad (2.37)$$

which is the Lie algebra of the subgroup $USp(2, 2) \times USp(4)$ of $PSU(2, 2|4)$. From classical group theory $USp(2, 2) \cong SO(4, 1)$ and $USp(4) \cong SO(5)$, thus the fixed point set of Ω is given by $SO(4, 1) \times SO(5)$.

In summary, $AdS_5 \times S^5$ is a semi-symmetric space with fourth order automorphism given by Ω and fixed point set $SO(4, 1) \times SO(5)$. That is, it can be described by the following coset manifold:

$$AdS_5 \times S^5 \cong \text{Bos} \left[\frac{PSU(2, 2|4)}{SO(4, 1) \times SO(5)} \right]. \quad (2.38)$$

2.3 Green-Schwarz superstring in $AdS_5 \times S^5$

Now that we know some precise properties of our target space, we need to define the analogue to the currents (2.13) and procedure to construct a candidate for the action.

2.3.1 Constructing the action

Just as we saw in the flat space case in section 2.1, the action for the Type IIB superstring propagating in $AdS_5 \times S^5$ will be given by a Wess-Zumino-Witten like sigma-model Lagrangian, where the first term corresponds to a kinetic term and the second one is obtained as the integral of a closed 3-form Ω_3 over a three dimensional manifold $\mathcal{D}$ which has the string worldsheet as its boundary.

The first step is to define the Maurer-Cartan form. Let g be an element of the supergroup $SU(2, 2|4)$, we define the Maurer-Cartan form as $A = -g^{-1}dg$, since it takes values in $\mathfrak{su}(2, 2|4)$ we can write it as follows

$$A = -g^{-1}dg = A^{(0)} + A^{(2)} + A^{(1)} + A^{(3)}, \quad (2.39)$$

where we have exhibited the $\mathbb{Z}_4$ -decomposition of A . We should observe that, by construction A satisfies the Maurer-Cartan equation $dA - A \wedge A = 0$, which in components can be written as follows⁵

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0. \quad (2.40)$$

Let us now consider local right $SO(4,1) \times SO(5)$ multiplication, that is we consider the following transformation

$$g \rightarrow gh, \quad (2.41)$$

where h belongs to $SO(4,1) \times SO(5)$. It is easy to see that under this transformation the Maurer-Cartan form transforms as follows

$$\begin{aligned} A &\rightarrow -(gh)^{-1} (dg h + g dh) \\ &\rightarrow h^{-1} A h - h^{-1} dh. \end{aligned} \quad (2.42)$$

which in $\mathbb{Z}_4$ components reads

$$A^{(1,2,3)} \rightarrow h^{-1} A^{(1,2,3)} h, \quad A^{(0)} \rightarrow h^{-1} A^{(0)} h - h^{-1} dh. \quad (2.43)$$

In order to formulate the Lagrangian, we need to make a key observation here. The transformation on $A^{(0)}$ is typical of a gauge field, so we can understand $A^{(0)}$ as the $SO(4,1) \times SO(5)$ gauge field, while $A^{(1,2,3)}$ transform according to the adjoint representation of $SO(4,1) \times SO(5)$. Furthermore, since only the components $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ undergo a similarity transformation, then any gauge invariant Lagrangian in the supercoset is given by a bilinear in the A 's which can not contain $A^{(0)}$,⁶ thus the Lagrangian should only depend on a coset element and not on the group element g .

We now proceed to construct the Green-Schwarz action on the supercoset (2.38). We consider the kinetic term for the bosonic components $A^{(2)}$, but we can not allow a kinetic term for the fermionic components because it would break kappa symmetry. In this way, the fermionic components of the Maurer-Cartan form enter through the Wess-Zumino term. The sigma model Type II superstring action on $AdS_5 \times S^5$ is given by [53]

$$S_{GS} = \int d\tau d\sigma L, \quad (2.44)$$

where L is given by

$$L = -\frac{1}{4\pi\alpha'} \left[\gamma^{\alpha\beta} \text{str} \left(A_\alpha^{(2)} A_\beta^{(2)} \right) + \kappa \epsilon^{\alpha\beta} \text{str} \left(A_\alpha^{(1)} A_\beta^{(3)} \right) \right], \quad (2.45)$$

⁵Here α and β denote worldsheet coordinates, so you should not confuse them with spinorial indices.

⁶The components $A^{(2)}$ define the supervielbeins, so we could have formulated the kinetic part of the Lagrangian by making use of (2.14).

where $\gamma^{\alpha\beta} = g^{\alpha\beta} \sqrt{-g}$ is the Weyl invariant combination of the worldsheet metric $g_{\alpha\beta}$ such that $\det \gamma = 1$. The parameter κ is a constant number which must be real in order to guarantee the reality of the Lagrangian. Furthermore, since this formulation pretends to be a good generalization of the flat Green-Schwarz superstring, we actually need to require this action to have κ -symmetry and this leaves only two possibilities $\kappa = \pm 1$, as we will see in a moment.

2.3.2 Equations of motion

Let us find the equations of motion which follow from the Lagrangian (2.45). In order to do this, we will make use of the following property

$$\text{str} \left(\Omega^k(M_1) M_2 \right) = \text{str} \left(M_1 \Omega^{4-k}(M_2) \right) . \quad (2.46)$$

So, taking the variation of the Lagrangian (2.45) we get

$$\delta L = -\frac{1}{4\pi\alpha'} \left[2\gamma^{\alpha\beta} \text{str} \left(\delta A_\alpha^{(2)} A_\beta^{(2)} \right) + \kappa \epsilon^{\alpha\beta} \text{str} \left(\delta A_\alpha^{(1)} A_\beta^{(3)} + A_\alpha^{(1)} \delta A_\beta^{(3)} \right) \right] . \quad (2.47)$$

Let us compute each of the terms of (2.47), for instance the term inside the first supertrace on the right hand side can be written as follows

$$\begin{aligned} \text{str} \left(\delta A_\alpha^{(2)} A_\beta^{(2)} \right) 1 &= \frac{1}{4} \text{str} \left(\delta A_\alpha A_\beta^{(2)} - \Omega(\delta A_\alpha) A_\beta^{(2)} + \Omega^2(\delta A_\alpha) A_\beta^{(2)} - \Omega^3(\delta A_\alpha) A_\beta^{(2)} \right) \\ &= \text{str} \left(\delta A_\alpha A_\beta^{(2)} \right) , \end{aligned} \quad (2.48)$$

where in the second line we have used

$$\begin{aligned} \text{str} \left(\Omega(\delta A_\alpha) A_\beta^{(2)} \right) &= \text{str} \left(\delta A_\alpha \Omega(A_\beta) - \delta A_\alpha A_\beta + \delta A_\alpha \Omega^3(A_\beta) - \delta A_\alpha \Omega^2(A_\beta) \right) \\ &= -\text{str} \left(\delta A_\alpha A_\beta^{(2)} \right) . \end{aligned} \quad (2.49)$$

In a similar way, each of the terms in the first line of (2.48) contributes with one factor of $\text{str}(\delta A_\alpha A_\beta^{(2)})$, proving (2.48). The second term of the right hand side of (2.47) can be found in a similar way as we show next

$$\begin{aligned} \epsilon^{\alpha\beta} \text{str} \left(\delta A_\alpha^{(1)} A_\beta^{(3)} + A_\alpha^{(1)} \delta A_\beta^{(3)} \right) &= \epsilon^{\alpha\beta} \text{str} \left(\delta A_\alpha^{(3)} A_\beta^{(1)} - \delta J_\alpha^{(1)} A_\beta^{(3)} \right) \\ &= \frac{\epsilon^{\alpha\beta}}{4} \text{str} \left(\delta A_\alpha \left(A_\beta^{(1)} - A_\beta^{(3)} \right) \right) , \end{aligned} \quad (2.50)$$

where the only computation we did is to carefully apply identity (2.46). With the help of the results (2.48) and (2.50) we can write for the variation of the Lagrangian

$$\delta L = -\text{str}(\delta A_\alpha \Lambda^\alpha) , \quad (2.51)$$

where

$$\Lambda^\alpha = \frac{1}{2\pi\alpha'} \left[\gamma^{\alpha\beta} A_\beta^{(2)} - \frac{1}{2} \kappa \epsilon^{\alpha\beta} \left(A_\beta^{(1)} - A_\beta^{(3)} \right) \right]. \quad (2.52)$$

What is more, the variation of A_α is given by

$$\delta A_\alpha = -\delta \left(g^{-1} \partial_\alpha g \right) = -g^{-1} \delta g A_\alpha - g^{-1} \partial_\alpha (\delta g), \quad (2.53)$$

plugging the result above into the variation for the Lagrangian we wrote before, we get

$$\begin{aligned} \delta L &= +\text{str} \left[g^{-1} \delta g A_\alpha \Lambda^\alpha + g^{-1} \partial_\alpha (\delta g) \Lambda^\alpha \right] \\ &= -\text{str} \left[g^{-1} \delta g \left(\partial_\alpha \Lambda^\alpha - [A_\alpha, \Lambda^\alpha] \right) \right], \end{aligned} \quad (2.54)$$

where we made use of integration by parts, and we arrived to the second line by neglecting total derivatives. Since equation (2.54) must vanish for an arbitrary group element $\mathbf{g}$ and if we consider $\partial_\alpha \Lambda^\alpha - [A_\alpha, \Lambda^\alpha]$ as an element of $\mathfrak{su}(2, 2|4)$ which contains the central element $i1$, we must require that

$$\partial_\alpha \Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = c \cdot i1 \quad (2.55)$$

where the parameter c can be found by taking the trace of both sides in the expression above. Since we defined $\mathfrak{psu}(2, 2|4)$ as the quotient algebra of $\mathfrak{su}(2, 2|4)$ over its central element, we may identify any elements proportional to the identity as zero, in this way we can write the equations of motion as follows

$$\partial_\alpha \Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = 0. \quad (2.56)$$

We can use $\mathbb{Z}_4$ decomposition and rearrange all the terms of the equation above according to its $\mathbb{Z}_4$ grading. For instance, the 0-grading term is given by

$$\gamma^{\alpha\beta} \left[A_\alpha^{(2)}, A_\beta^{(2)} \right] - \frac{1}{2} \kappa \epsilon^{\alpha\beta} \left(\left[A_\alpha^{(3)}, A_\beta^{(1)} \right] - \left[A_\alpha^{(1)}, A_\beta^{(3)} \right] \right) = 0. \quad (2.57)$$

This equation is trivial, since each of the parts of the left hand side vanishes when using the symmetry and anti-symmetry properties of $\gamma^{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ respectively leading to $0 = 0$.

The 2-grading component is given by

$$\partial_\alpha \left(\gamma^{\alpha\beta} A_\beta^{(2)} \right) - \gamma^{\alpha\beta} \left[A_\alpha^{(0)}, A_\beta^{(2)} \right] + \frac{1}{2} \kappa \epsilon^{\alpha\beta} \left(\left[A_\alpha^{(1)}, A_\beta^{(1)} \right] - \left[A_\alpha^{(3)}, A_\beta^{(3)} \right] \right). \quad (2.58)$$

The 1-grading and 3-grading components of the equations of motion are given respectively by

$$\gamma^{\alpha\beta} \left[A_\alpha^{(3)}, A_\beta^{(2)} \right] + \kappa \epsilon^{\alpha\beta} \left[A_\alpha^{(2)}, A_\beta^{(3)} \right] = 0, \quad (2.59)$$

$$\gamma^{\alpha\beta} \left[A_\alpha^{(1)}, A_\beta^{(2)} \right] + \kappa \epsilon^{\alpha\beta} \left[A_\alpha^{(2)}, A_\beta^{(1)} \right] = 0, \quad (2.60)$$

where we made use of the Maurer-Cartan equation (2.40). Let us define the tensors⁷

$$P_{\pm}^{\alpha\beta} = \frac{1}{2} (\gamma^{\alpha\beta} \pm \kappa \epsilon^{\alpha\beta}) . \quad (2.61)$$

Considering this definition, the equations of motion (2.59) and (2.60) can be written in the following way

$$P_-^{\alpha\beta} [A_{\alpha}^{(2)}, A_{\beta}^{(3)}] = 0 , \quad (2.62)$$

$$P_+^{\alpha\beta} [A_{\alpha}^{(2)}, A_{\beta}^{(1)}] = 0 . \quad (2.63)$$

Furthermore, for $\kappa = \pm 1$, the tensors $P_{\pm}^{\alpha\beta}$ are orthogonal projectors, that is they satisfy the relations

$$P_+^{\alpha\beta} + P_-^{\alpha\beta} = \gamma^{\alpha\beta} , \quad P_{\pm}^{\alpha\delta} P_{\pm\delta}^{\beta} = P_{\pm}^{\alpha\beta} , \quad P_{\pm}^{\alpha\delta} P_{\mp\delta}^{\beta} = 0 . \quad (2.64)$$

When varying the Lagrangian, we have not considered the variations of the worldsheet metric, so we can easily derive the equations of motion for the worldsheet metric by varying the Lagrangian with respect to $\gamma^{\alpha\beta}$, obtaining the so-called Virasoro constraints

$$\text{str} \left(A_{\alpha}^{(2)} A_{\beta}^{(2)} \right) - \frac{1}{2} \gamma_{\alpha\beta} \gamma^{\rho\delta} \text{str} \left(A_{\rho}^{(2)} A_{\delta}^{(2)} \right) = 0 . \quad (2.65)$$

2.3.3 Kappa symmetry

Just as in the flat superstring, the Metsaev-Tseytlin Lagrangian has κ -symmetry. Let us start by noting that the canonical form A is trivially invariant under global left $PSU(2, 2|4)$ multiplication

$$g \rightarrow g' g , \quad g' \in PSU(2, 2|4) . \quad (2.66)$$

The $PSU(2, 2|4)$ supergroup plays the role of the global symmetry group which is realized on a coset element by multiplication from the left. Similarly, we can view the kappa symmetry transformations as the right local action of $G = \exp(\epsilon)$ on the coset representative g as follows

$$\mathbf{g} \cdot G = g' h , \quad (2.67)$$

where $\epsilon = \epsilon(\tau, \sigma)$ is a local fermionic parameter taking values in $\mathfrak{psu}(2, 2|4)$ and h is a compensating element from the coset denominator $SO(4, 1) \times SO(5)$. Unlike the global

⁷These are equivalent to the projector tensors in flat space, see [54].

symmetry group case, the string action is not invariant for an arbitrary form of the ϵ parameter. However, it is possible to find a set of such parameters that allows one to remove the unphysical degrees of freedom. We now find the conditions on ϵ which guarantee the invariance of the action.

Let us now investigate how the Maurer-Cartan form transforms under this right multiplication, using (2.67) we can write

$$\begin{aligned} A = -g^{-1}dg &\rightarrow -(ge^\epsilon)^{-1}d(ge^\epsilon) \\ &= -e^{-\epsilon}g^{-1}(dge^\epsilon + ge^\epsilon d\epsilon) \\ &= e^{-\epsilon}Ae^\epsilon - d\epsilon. \end{aligned} \quad (2.68)$$

Using the Baker-Hausdorff formula we can write, up to an infinitesimal parameter ϵ , the following transformations

$$\delta_\epsilon A = -d\epsilon + [A, \epsilon]. \quad (2.69)$$

If we consider $\epsilon = \epsilon^{(1)} + \epsilon^{(3)}$, the transformations showed above can be written, upon $\mathbb{Z}_4$ decomposition, as follows

$$\delta_\epsilon (A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}) = d(\epsilon^{(1)} + \epsilon^{(3)}) + [A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}, \epsilon^{(1)} + \epsilon^{(3)}]. \quad (2.70)$$

We can rearrange the terms according to its respective gradation, if we do this we end up with the following $\mathbb{Z}_4$ relations

$$\delta_\epsilon A^{(0)} = [A^{(3)}, \epsilon^{(1)}] + [A^{(1)}, \epsilon^{(3)}], \quad (2.71)$$

$$\delta_\epsilon A^{(2)} = [A^{(1)}, \epsilon^{(1)}] + [A^{(3)}, \epsilon^{(3)}], \quad (2.72)$$

$$\delta_\epsilon A^{(1)} = -d\epsilon^{(1)} + [A^{(0)}, \epsilon^{(1)}] + [A^{(2)}, \epsilon^{(3)}], \quad (2.73)$$

$$\delta_\epsilon A^{(3)} = -d\epsilon^{(3)} + [A^{(2)}, \epsilon^{(1)}] + [A^{(0)}, \epsilon^{(3)}]. \quad (2.74)$$

We can make use of these formulae to find the variation of the Lagrangian as we do next

$$\begin{aligned} \delta_\epsilon \mathcal{L} = & -\frac{1}{4\pi\alpha'} \left[\delta_\epsilon \gamma^{\alpha\beta} \text{str} \left(A_\alpha^{(2)} A_\beta^{(2)} \right) + 2\gamma^{\alpha\beta} \text{str} \left(\delta_\epsilon A_\alpha^{(2)} A_\beta^{(2)} \right) + \right. \\ & \left. + \kappa \epsilon^{\alpha\beta} \text{str} \left(\delta_\epsilon A_\alpha^{(1)} A_\beta^{(3)} - \delta_\epsilon A_\alpha^{(3)} A_\beta^{(1)} \right) \right]. \end{aligned} \quad (2.75)$$

The variation of the terms inside the supertraces can be found easily as we show next

$$\begin{aligned} \text{STr} \left(\delta_\epsilon A_\alpha^{(2)} A_\beta^{(2)} \right) &= \text{STr} \left([A_\alpha^{(1)}, \epsilon^{(1)}] A_\beta^{(2)} + [A_\alpha^{(3)}, \epsilon^{(3)}] A_\beta^{(2)} \right) \\ &= \text{STr} \left([A_\beta^{(2)}, A_\alpha^{(1)}] \epsilon^{(1)} + [A_\beta^{(2)}, A_\alpha^{(3)}] \epsilon^{(3)} \right), \end{aligned} \quad (2.76)$$

where we made use of the $\mathbb{Z}_4$ kappa transformations together with the cyclic property of the supertrace. Similarly we can write

$$\text{STr} \left(\delta_\epsilon A_\alpha^{(1)} A_\beta^{(3)} \right) = \text{STr} \left(-\partial_\alpha \epsilon^{(1)} A_\beta^{(3)} + \left[A_\beta^{(3)}, A_\alpha^{(0)} \right] \epsilon^{(1)} + \left[A_\beta^{(3)}, A_\alpha^{(2)} \right] \epsilon^{(3)} \right), \quad (2.77)$$

$$\text{STr} \left(\delta_\epsilon A_\alpha^{(3)} A_\beta^{(1)} \right) = \text{STr} \left(-\partial_\alpha \epsilon^{(3)} A_\beta^{(1)} + \left[A_\beta^{(1)}, A_\alpha^{(2)} \right] \epsilon^{(1)} + \left[A_\beta^{(1)}, A_\alpha^{(0)} \right] \epsilon^{(3)} \right). \quad (2.78)$$

Using all of these results, we can finally write for the variation of the Lagrangian

$$\begin{aligned} \delta_\epsilon \mathcal{L} = & -\frac{1}{4\pi\alpha'} \left[\delta_\epsilon \gamma^{\alpha\beta} \text{STr} \left(A_\alpha^{(2)} A_\beta^{(2)} \right) - 2\gamma^{\alpha\beta} \text{STr} \left(\left[A_\alpha^{(1)}, A_\beta^{(2)} \right] \epsilon^{(1)} + \left[A_\alpha^{(3)}, A_\beta^{(2)} \right] \epsilon^{(3)} \right) + \right. \\ & + \kappa \epsilon^{\alpha\beta} \text{STr} \left(\partial_\alpha A_\beta^{(3)} \epsilon^{(1)} - \partial_\alpha A_\beta^{(1)} \epsilon^{(3)} + \left[A_\beta^{(3)}, A_\alpha^{(0)} \right] \epsilon^{(1)} + \left[A_\beta^{(3)}, A_\alpha^{(2)} \right] \epsilon^{(3)} + \right. \\ & \left. \left. + \left[A_\beta^{(1)}, A_\alpha^{(2)} \right] \epsilon^{(1)} + \left[A_\beta^{(1)}, A_\alpha^{(0)} \right] \epsilon^{(3)} \right) \right], \end{aligned} \quad (2.79)$$

where we performed integration by parts and we neglected the total derivatives. In order to reduce this expression, let us give a look at the Maurer-Cartan equations (2.40)

$$2\epsilon^{\alpha\beta} \partial_\alpha A_\beta = \epsilon^{\alpha\beta} [A_\alpha, A_\beta], \quad (2.80)$$

which using $\mathbb{Z}_4$ decomposition can be written as follows

$$2\epsilon^{\alpha\beta} \partial_\alpha \left(A_\beta^{(0)} + A_\beta^{(1)} + A_\beta^{(2)} + A_\beta^{(3)} \right) = \epsilon^{\alpha\beta} [A_\alpha^{(0)} + A_\alpha^{(1)} + A_\alpha^{(2)} + A_\alpha^{(3)}]. \quad (2.81)$$

Comparing the terms of grading 1 and 3 of both sides of the equation above, we end up with

$$\epsilon^{\alpha\beta} \partial_\alpha A_\beta^{(1)} = \epsilon^{\alpha\beta} [A_\alpha^{(0)}, A_\beta^{(1)}] + \epsilon^{\alpha\beta} [A_\alpha^{(2)}, A_\beta^{(3)}], \quad (2.82)$$

$$\epsilon^{\alpha\beta} \partial_\alpha A_\beta^{(3)} = \epsilon^{\alpha\beta} [A_\alpha^{(0)}, A_\beta^{(3)}] + \epsilon^{\alpha\beta} [A_\alpha^{(1)}, A_\beta^{(2)}]. \quad (2.83)$$

Using these identities we can write the variation of the Lagrangian as follows

$$\delta_\epsilon \mathcal{L} = -\frac{1}{4\pi\alpha'} \delta_\epsilon \gamma^{\alpha\beta} \text{str} \left(A_\alpha^{(2)} A_\beta^{(2)} \right) + \frac{1}{\pi\alpha'} \text{str} \left(P_+^{\alpha\beta} [A_\beta^{(1)}, A_\alpha^{(2)}] \epsilon^{(1)} + P_-^{\alpha\beta} [A_\beta^{(3)}, A_\alpha^{(2)}] \epsilon^{(3)} \right). \quad (2.84)$$

where $P_\pm^{\alpha\beta}$ are defined in (2.61). It is easy to see that the last two terms must vanish because they are simply the projections of the equations of motion we wrote in (2.62) and (2.63), whereas the first term vanishes due to the Virasoro constraints (2.65). However, for the kappa transformations to define a true symmetry, the action should be invariant

without the use of the equations of motion, therefore we need to find the appropriate transformation rule for the worldsheet metric $\delta_\epsilon \gamma^{\alpha\beta}$.

We now consider the projection $V_\pm^\alpha$ of any vector V^α as follows

$$V_\pm^\alpha = P_\pm^{\alpha\beta} V_\beta. \quad (2.85)$$

Taking into consideration the definition we wrote above, the variation of the Lagrangian takes the following form

$$\delta_\epsilon \mathcal{L} = -\frac{1}{4\pi\alpha'} \delta_\epsilon \gamma^{\alpha\beta} \text{STr} \left(A_\alpha^{(2)} A_\beta^{(2)} \right) + \frac{1}{\pi\alpha'} \text{STr} \left(\left[A_+^{(1),\alpha}, A_{\alpha,-}^{(2)} \right] \epsilon^{(1)} + \left[A_-^{(3),\alpha}, A_{\alpha,+}^{(2)} \right] \epsilon^{(3)} \right). \quad (2.86)$$

We will consider the following ansatz for the components of the fermionic parameter ϵ

$$\epsilon^{(1)} = A_{\alpha,-}^{(2)} \kappa_+^{(1),\alpha} + \kappa_+^{(1),\alpha} A_{\alpha,-}^{(2)}, \quad (2.87)$$

$$\epsilon^{(3)} = A_{\alpha,+}^{(2)} \kappa_-^{(3),\alpha} + \kappa_-^{(3),\alpha} A_{\alpha,+}^{(2)}, \quad (2.88)$$

where $\kappa_\pm^{(k),\alpha}$ are new independent parameters of kappa symmetry transformations which are homogeneous elements of gradation $k = 1, 3$ with respect to Ω . It is possible to represent the traceless even component $A^{(2)}$ as follows

$$A^{(2)} = \begin{pmatrix} m^i \gamma^i & 0 \\ 0 & n^i \gamma^i \end{pmatrix}. \quad (2.89)$$

Here γ^i are the $SO(5)$ Dirac matrices. The coefficients n^i are considered as purely imaginary while m^i are real for $i = 1, \dots, 4$ and imaginary for $i = 5$. We can further write

$$A_{\alpha,\pm}^{(2)} A_{\beta,\pm}^{(2)} = \begin{pmatrix} m_{\alpha,\pm}^i m_{\beta,\pm}^j \gamma^i \gamma^j & 0 \\ 0 & n_{\alpha,\pm}^i n_{\beta,\pm}^j \gamma^i \gamma^j \end{pmatrix}. \quad (2.90)$$

As a consequence of the property $P_\pm^{\alpha\beta} A_{\beta,\mp} = 0$ it is possible to show that the components $A_{\tau,\pm}$ and $A_{\sigma,\pm}$ must be proportional to each other, allowing us to rewrite the expression above as follows

$$\begin{aligned} A_{\alpha,\pm}^{(2)} A_{\beta,\pm}^{(2)} &= \begin{pmatrix} m_{\alpha,\pm}^i m_{\beta,\pm}^j \frac{1}{2} \gamma^i \gamma^j & 0 \\ 0 & n_{\alpha,\pm}^i n_{\beta,\pm}^j \frac{1}{2} \gamma^i \gamma^j \end{pmatrix} \\ &= \begin{pmatrix} m_{\alpha,\pm}^i m_{\beta,\pm}^i & 0 \\ 0 & n_{\alpha,\pm}^i n_{\beta,\pm}^i \end{pmatrix} \end{aligned} \quad (2.91)$$

$$= \frac{1}{8} \Upsilon \text{str} \left(A_{\alpha,\pm}^{(2)} A_{\beta,\pm}^{(2)} \right) + \frac{1}{2} (m_{\alpha,\pm}^i m_{\beta,\pm}^i + n_{\alpha,\pm}^i n_{\beta,\pm}^i) 1_8. \quad (2.92)$$

Here 1_8 is the identity matrix and Υ is the diagonal hypercharge matrix defined as $\Upsilon = (1_4, -1_4)$. As you can see in the variation (2.86), upon substitution of our ansatz (2.87) and (2.88) we will have products of $J^{(2)}$'s which according to the expression above can be expressed as a linear combination of two matrices, one of them being the identity matrix and the other being Υ . When substituting all of these results in the variation of the Lagrangian, we find that the term proportional to the identity matrix will drop out, leaving

$$\begin{aligned} \delta_\epsilon \mathcal{L} = & -\frac{1}{4\pi\alpha'} \delta_\epsilon \gamma^{\alpha\beta} \text{str} \left(A_\alpha^{(2)} A_\beta^{(2)} \right) + \frac{1}{8\pi\alpha'} \text{str} \left(A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)} \right) \text{STr} \left(\Upsilon \left[\kappa_+^{(1),\beta}, A_+^{(1),\alpha} \right] \right) + \\ & + \frac{1}{8\pi\alpha'} \text{STr} \left(A_{\alpha,+}^{(2)} A_{\beta,+}^{(2)} \right) \text{STr} \left(\Upsilon \left[\kappa_-^{(3),\beta}, A_-^{(3),\alpha} \right] \right). \end{aligned} \quad (2.93)$$

Thus, it is easy to see that for the Lagrangian to be invariant under kappa transformations, the worldsheet metric must transform as follows

$$\delta_\epsilon \gamma^{\alpha\beta} = \frac{1}{4} \text{str} \left(\Upsilon \left(\left[\kappa_+^{(1),\alpha}, A_+^{(1),\beta} \right] + \left[\kappa_+^{(1),\beta}, A_+^{(1),\alpha} \right] + \left[\kappa_-^{(3),\alpha}, A_-^{(3),\beta} \right] + \left[\kappa_-^{(3),\beta}, A_-^{(3),\alpha} \right] \right) \right). \quad (2.94)$$

Furthermore, using the fact that the insertion of the hypercharge Υ turns the supertrace into a regular trace of the matrix, and making use of the identity $P_\pm^{\alpha\gamma} P_\pm^{\beta\delta} = P_\pm^{\beta\gamma} P_\pm^{\alpha\delta}$, we have

$$\delta_\epsilon \gamma^{\alpha\beta} = \frac{1}{2} \text{tr} \left(\left[\kappa_+^{(1),\alpha}, A_+^{(1),\beta} \right] + \left[\kappa_-^{(3),\alpha}, A_-^{(3),\beta} \right] \right). \quad (2.95)$$

We need to make a very important observation here; we have considered that $P_\pm^{\alpha\beta}$ are orthogonal projectors throughout the derivation we presented here, that is, the projectors satisfy

$$P_\pm^{\alpha\gamma} g_{\beta\gamma} P_\mp^{\gamma\delta} = 0. \quad (2.96)$$

Looking at the definition of the projector operators (2.61), it is easy to note that the orthogonality condition is equivalent to require $\kappa^2 = 1$, and then the parameter κ is allowed to take values of one or minus one only.

2.4 Pure spinor superstrings in general backgrounds

We finish this chapter by giving a very short presentation of the so-called Berkovits-Howe action, which is none other than the pure spinor sigma model in curved backgrounds. We will present the action and explain the terms involved in it.

2.4.1 The Berkovits-Howe superstring

The sigma model action for Type II pure spinor superstrings in a generic supergravity background is given by [55]

$$\begin{aligned}
S = & \frac{1}{2\pi\alpha'} \int d^2z \left[\frac{1}{2} (G_{MN}(Z) + B_{MN}(Z)) \partial Z^M \bar{\partial} Z^N + \right. \\
& + (E_M^\alpha(Z) d_\alpha + \Omega_{M\alpha}^\beta(Z) \lambda^\alpha \omega_\beta) \bar{\partial} Z^M + (E_M^{\hat{\alpha}}(Z) \hat{d}_{\hat{\alpha}} + \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}}(Z) \hat{\lambda}^{\hat{\alpha}} \hat{\omega}_{\hat{\beta}}) \partial Z^M + \\
& + C_{\alpha}^{\beta\hat{\gamma}}(Z) \lambda^\alpha \omega_\beta \hat{d}_{\hat{\gamma}} + \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}(Z) \hat{\lambda}^{\hat{\alpha}} \hat{\omega}_{\hat{\beta}} d_\gamma + \\
& \left. + P^{\alpha\hat{\beta}}(Z) d_\alpha \hat{d}_{\hat{\beta}} + S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}(Z) \lambda^\alpha \omega_\beta \hat{\lambda}^{\hat{\gamma}} \hat{\omega}_{\hat{\delta}} + \frac{1}{2} \alpha' \Phi(Z) R^{(2)} \right] + S_\lambda + S_{\hat{\lambda}}. \quad (2.97)
\end{aligned}$$

Here the action is written in conformal gauge, so z and $\bar{z}$ are the worldsheet coordinates. On the other hand $M = (m, \mu, \hat{\mu})$ are curved superspace indices and $\mathcal{A} = (a, \alpha, \hat{\alpha})$ are tangent superspace indices. The independent worldsheet fields in this action are $Z^M = (X^m, \theta^\mu, \hat{\theta}^{\hat{\mu}})$ and $(d_\alpha, \hat{d}_{\hat{\alpha}})$ in the matter sector, and $(\omega_\alpha, \lambda^\alpha, \hat{\omega}_{\hat{\alpha}}, \hat{\lambda}^{\hat{\alpha}})$ in the ghost sector. If we vary the action with respect to these fields one could find their respective equations of motion.

The ghost content S_λ and $S_{\hat{\lambda}}$ are given by expressions which are analogous to the ones that are used in the flat space case, that is they are given by

$$S_\lambda + S_{\hat{\lambda}} = \frac{1}{2\pi\alpha'} \int d^2z \left(\omega_\alpha \bar{\partial} \lambda^\alpha + \hat{\omega}_{\hat{\alpha}} \partial \hat{\lambda}^{\hat{\alpha}} \right). \quad (2.98)$$

If we substitute the factor $S_\lambda + S_{\hat{\lambda}}$ that we wrote above into the action (2.97) then we could define the covariant derivative on λ ($\hat{\lambda}$) as the pullback of the left moving (right moving) spin connection $\Omega_\alpha{}^\beta = dZ^M \Omega_{M\alpha}{}^\beta$ ($\hat{\Omega}_{\hat{\alpha}}{}^{\hat{\beta}} = dZ^M \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}}$) as follows

$$(\nabla \lambda)^\alpha = \partial \lambda^\alpha + \Omega_\beta{}^\alpha \lambda^\beta, \quad (\nabla \hat{\lambda})^{\hat{\alpha}} = \partial \hat{\lambda}^{\hat{\alpha}} + \hat{\Omega}_{\hat{\beta}}{}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}}. \quad (2.99)$$

The first line in the action (2.97) is just the standard Type II Green-Schwarz action we saw in (2.7), whereas the other lines are needed for BRST invariance. $R^{(2)}$ is the worldsheet curvature and G , B , P , C , $\hat{C}$, S and Φ are background superfields. G_{MN} is the background metric superfield which can be written as $G_{MN} = E_M^a E_N^b \eta_{ab}$; the background superfield B_{AB} is the superspace 2-form potential; the lowest components of $C_{\alpha}^{\beta\hat{\gamma}}$ and $\hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}$ are related to the gravitini and dilatini; the lowest component of $P^{\alpha\hat{\beta}}$ is the Ramond-Ramond bispinor field strength; $S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}$ is related to the curvature associated to the connections Ω and $\hat{\Omega}$; $E_M^{\mathcal{A}}$ are the supervielbeins and the Fradkin-Tseytlin term $\frac{1}{2} \alpha' \Phi(Z) R^{(2)}$ describes the coupling of the dilaton to the worldsheet curvature $R^{(2)}$. If

the Fradkin-Tseytlin term is omitted, (2.97) is the most general action with classical worldsheet conformal invariance and zero (left-right)-moving ghost number which can be constructed from the Type II worldsheet variables.

Berkovits and Howe showed that, up to the lowest order in α' , the conservation of the BRST currents $\bar{\partial}(\lambda^\alpha d_\alpha) = \partial(\hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}) = 0$ and nilpotency of the BRST charges, which are postulated to have the same structure as in the flat space case, that is $\mathcal{Q} \sim \oint \lambda^\alpha d_\alpha$ and $\hat{\mathcal{Q}} \sim \oint \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}$, actually imply Type II supergravity equations of motion for the background superfields.

2.4.2 The pure spinor superstring in $AdS_5 \times S^5$

We can construct the Pure Spinor superstring action in $AdS_5 \times S^5$ in a very similar way to the flat space case that is done for instance in [56]. Before doing this, let us write the Metsaev-Tseytlin action we presented in (2.44) in conformal gauge. We consider the following definitions

$$A_z^a \equiv A^{(k)}; \quad A_{\bar{z}}^{(k)} \equiv \bar{A}^{(k)}. \quad (2.100)$$

We also need to remember that the only non-vanishing components of the worldsheet metric in $z, \bar{z}$ coordinates are $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$, whereas for the Levi-Civita tensor we have $\epsilon^{z\bar{z}} = -\epsilon^{\bar{z}z} = 2i$. Using this information we can easily write

$$S_{GS} = \frac{1}{\pi\alpha'} \int d^2z \, \text{str} \left[\frac{1}{2} A^{(2)} \bar{A}^{(2)} - \frac{1}{4} (A^{(3)} \bar{A}^{(1)} - A^{(1)} \bar{A}^{(3)}) \right], \quad (2.101)$$

where we used $\kappa = -1$. We recall the relation between the Siegel and Green-Schwarz action (see [56]), but now such a relation reads

$$S_{matter} = S_{GS} + \frac{1}{\pi\alpha'} \int d^2z \, (d_\alpha \bar{A}^{\alpha(1)} + d_{\hat{\alpha}} A^{\hat{\alpha}(3)} + P^{\alpha\hat{\alpha}} d_\alpha d_{\hat{\alpha}}). \quad (2.102)$$

It is not hard to realize that the first two terms in the integral above break the kappa symmetry of the Green-Schwarz action. The third term in the integral is a coupling term to the Ramond-Ramond superfield $P^{\alpha\hat{\alpha}}$. When the Ramond-Ramond superfield is invertible, d_α and $\hat{d}_{\hat{\alpha}}$ become auxiliary fields and their equations of motion can be written as follows

$$d_\alpha = A^{\hat{\alpha}(3)} \eta_{\alpha\hat{\alpha}}, \quad \hat{d}_{\hat{\alpha}} = -\bar{A}^{\alpha(1)} \eta_{\alpha\hat{\alpha}}. \quad (2.103)$$

Furthermore, $P^{\alpha\hat{\beta}}$ can be taken as

$$P^{\alpha\hat{\beta}} = \eta^{\alpha\hat{\beta}}, \quad (2.104)$$

where the matrix $\eta^{\alpha\hat{\beta}}$ has rank sixteen and is numerically equal to the identity matrix. We can use the equations of motion (2.103) to integrate out the auxiliary fields and we obtain

$$\begin{aligned} S_{matter} &= S_{GS} + \frac{1}{\pi\alpha'} \int d^2z \operatorname{str} (A^{(3)} \bar{A}^{(1)}) \\ &= \frac{1}{\pi\alpha'} \int d^2z \operatorname{str} \left(\frac{1}{2} A^{(2)} \bar{A}^{(2)} + \frac{3}{4} A^{(3)} \bar{A}^{(1)} + \frac{1}{4} A^{(1)} \bar{A}^{(3)} \right). \end{aligned} \quad (2.105)$$

To construct the pure spinor action, we need to add a ghost term to the action (2.105). The ghost fields must take values in the fermionic eigenspaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(3)}$, hence we can define them as $\lambda^{(1)}$ and $\hat{\lambda}^{(3)}$. In flat space the momentum has opposite chirality with respect to its conjugate field, so that the coupling ghost-momentum is Lorentz invariant. Analogously in curved space we take each momentum in a different eigenspace with respect to its ghost, that is, $\omega^{(3)} \in \mathcal{H}^{(3)}$ is the conjugate momentum of $\lambda^{(1)}$ and $\hat{\omega}^{(1)} \in \mathcal{H}^{(1)}$ is the conjugate momentum of $\hat{\lambda}^{(3)}$. To construct the pure spinor ghost action in $AdS_5 \times S^5$, we have to substitute the canonical derivative with the covariant one. Since we interpreted $A^{(0)}$ as the $SO(4,1) \times SO(5)$ gauge field, it is natural to define

$$\nabla \equiv \partial + [A^{(0)}, \dots], \quad \bar{\nabla} \equiv \bar{\partial} + [\bar{A}^{(0)}, \dots]. \quad (2.106)$$

Then the pure spinor ghost action can be written in the following way

$$S_{ghost} = \frac{1}{\pi\alpha'} \int d^2z \operatorname{str} \left(\omega^{(3)} \bar{\nabla} \lambda^{(1)} + \hat{\omega}^{(1)} \nabla \hat{\lambda}^{(3)} \right). \quad (2.107)$$

Finally the pure spinor action $S_{PS} = S_{matter} + S_{ghost}$ in $AdS_5 \times S^5$ reads

$$\begin{aligned} S_{PS} &= \frac{1}{\pi\alpha'} \int d^2z \operatorname{str} \left[\frac{1}{2} A^{(2)} \bar{A}^{(2)} + \frac{3}{4} A^{(3)} \bar{A}^{(1)} + \frac{1}{4} A^{(1)} \bar{A}^{(3)} + \right. \\ &\quad \left. + \omega^{(3)} \bar{\nabla} \lambda^{(1)} + \hat{\omega}^{(1)} \nabla \hat{\lambda}^{(3)} - N \hat{N} \right]. \end{aligned} \quad (2.108)$$

Here we added a current-current coupling term in the action where N and $\hat{N}$ are the Lorentz generators in the ghost sector and are given by

$$N = -\{\omega^{(3)}, \lambda^{(1)}\}, \quad \hat{N} = -\{\hat{\omega}^{(1)}, \hat{\lambda}^{(3)}\}, \quad (2.109)$$

where $\lambda^{(1)}$ and $\hat{\lambda}^{(3)}$ satisfy the following $SO(4,1) \times SO(5)$ pure spinor constraints

$$\{\lambda^{(1)}, \lambda^{(1)}\} = \lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0, \quad \{\hat{\lambda}^{(3)}, \hat{\lambda}^{(3)}\} = \hat{\lambda}^{\hat{\alpha}} \gamma_{\hat{\alpha}\hat{\beta}}^m \hat{\lambda}^{\hat{\beta}} = 0. \quad (2.110)$$

It is important to note that the matter sector of the pure spinor superstring action (2.108), what we will call hybrid sigma model from now on, is not kappa symmetric since the Green-Schwarz action (2.44) is the unique such action. Another important difference is that the pure spinor action does not produce Virasoro constraints even though it is written in conformal gauge. In the pure spinor formalism, both the kappa symmetry and the Virasoro constraints are replaced by BRST symmetry.

2.4.3 BRST invariance

In the coset formulation of the pure spinor σ -model, BRST transformations are gauge transformations where the ghost fields act as gauge parameters. In this way, the BRST transformation of a group element can be written in the following way:

$$\mathcal{Q}(g) = g \Lambda, \quad (2.111)$$

where $\Lambda = \lambda^{(1)} + \lambda^{(3)}$ is the gauge parameter. Using these BRST transformations on the Maurer-Cartan forms A we get:

$$\mathcal{Q}(A) = \partial\Lambda + [A, \Lambda] = \partial(\lambda^{(1)} + \lambda^{(3)}) + [A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}, \lambda^{(1)} + \lambda^{(3)}], \quad (2.112)$$

which according to $\mathbb{Z}_4$ -grading components read as follows

$$\begin{aligned} \mathcal{Q}(A^{(0)}) &= [A^{(1)}, \lambda^{(3)}] + [A^{(3)}, \lambda^{(1)}], \\ \mathcal{Q}(A^{(1)}) &= \partial\lambda^{(1)} + [A^{(0)}, \lambda^{(1)}] + [A^{(2)}, \lambda^{(3)}] = \nabla\lambda^{(1)} + [A^{(2)}, \lambda^{(3)}], \\ \mathcal{Q}(A^{(2)}) &= [A^{(1)}, \lambda^{(1)}] + [A^{(3)}, \lambda^{(3)}], \\ \mathcal{Q}(A^{(3)}) &= \partial\lambda^{(3)} + [A^{(0)}, \lambda^{(3)}] + [A^{(2)}, \lambda^{(1)}] = \nabla\lambda^{(3)} + [A^{(2)}, \lambda^{(1)}]. \end{aligned} \quad (2.113)$$

Of course, just as in the flat space case, we should check that our BRST charge is nilpotent. In order to do this, let us apply our BRST transformation for the second time, we get

$$\mathcal{Q}^2(A) = \{\partial\Lambda, \Lambda\} + \{[A, \Lambda], \Lambda\}, \quad (2.114)$$

which separated in $\mathbb{Z}_4$ -grading components take the following form

$$\begin{aligned} \mathcal{Q}^2(A^{(0)}) &= -\frac{1}{2}[\{\lambda^{(1)}, \lambda^{(1)}\} + \{\lambda^{(3)}, \lambda^{(3)}\}, A^{(2)}] + \nabla\{\lambda^{(1)}, \lambda^{(3)}\}, \\ \mathcal{Q}^2(A^{(1)}) &= -\frac{1}{2}[\{\lambda^{(1)}, \lambda^{(1)}\} + \{\lambda^{(3)}, \lambda^{(3)}\}, A^{(3)}] - [\{\lambda^{(1)}, \lambda^{(3)}\}, A^{(1)}], \\ \mathcal{Q}^2(A^{(2)}) &= \frac{1}{2}\nabla\{\lambda^{(1)}, \lambda^{(1)}\} + \frac{1}{2}\{\lambda^{(3)}, \lambda^{(3)}\} - [\{\lambda^{(1)}, \lambda^{(3)}\}, A^{(2)}], \\ \mathcal{Q}^2(A^{(3)}) &= -\frac{1}{2}[\{\lambda^{(1)}, \lambda^{(1)}\} + \{\lambda^{(3)}, \lambda^{(3)}\}, A^{(1)}] - [\{\lambda^{(1)}, \lambda^{(3)}\}, A^{(3)}]. \end{aligned} \quad (2.115)$$

Imposing the pure spinor constraints in $AdS_5 \times S^5$ given in (2.110) we obtain

$$\begin{aligned}
 \mathcal{Q}^2(A^{(0)}) &= \nabla\{\lambda^{(1)}, \lambda^{(3)}\}, \\
 \mathcal{Q}^2(A^{(1)}) &= -[\{\lambda^{(1)}, \lambda^{(3)}\}, A^{(1)}], \\
 \mathcal{Q}^2(A^{(2)}) &= -[\{\lambda^{(1)}, \lambda^{(3)}\}, A^{(2)}], \\
 \mathcal{Q}^2(A^{(3)}) &= -[\{\lambda^{(1)}, \lambda^{(3)}\}, A^{(3)}],
 \end{aligned} \tag{2.116}$$

A careful look at the equations above let us realize that our BRST charge is nilpotent, that is $\mathcal{Q}^2(A^{(i)}) = 0$, up to gauge transformations.

In the next chapter we will review the integrability of both, Green-Schwarz sigma model and hybrid sigma model (matter part of the pure spinor sigma model) from the point of view of integrable deformations.

Chapter 3

Yang-Baxter deformations of semi-symmetric sigma-models

As we already mentioned in the introduction, the main advantage of the so called η -deformations is the fact that one obtains a deformed model which is known to be integrable from the very outset. The purpose of this chapter is to explain the deformation procedure. In the first part we give a short introduction to the modern algebraic approach to integrability and then we explain the deformation method applying it first to the principal chiral model and then to the Green-Schwarz sigma model. Finally, we explain how we can actually get the η -deformed version of the hybrid sigma model (matter part of the pure spinor sigma model) by writing an ansatz inspired by the Green-Schwarz deformed action.

3.1 Complete integrability

Before reviewing the deformation method, let us make some basic definitions related to classical integrability. The first convention we need to fix is the modern definition of completely integrable systems. For other equally important aspects of the modern algebraic approach to integrability, the interested reader can consult Appendix A and the useful reviews [57][58].

3.1.1 Classically integrable systems

If some Hamiltonian system can be defined on a Poisson manifold $\mathcal{M}$ (the phase space of the system) and a Hamiltonian $H \in C^\infty(\mathcal{M})$, its time evolution is given by following the

integral curves of the Hamiltonian vector field X_H on $\mathcal{M}$ corresponding to H , in fact

$$X_H(f) = \{H, f\}, \quad \forall f \in C^\infty(\mathcal{M}), \quad (3.1)$$

it follows that

$$\frac{df}{dt} = \{H, f\}. \quad (3.2)$$

This equation is a shorthand for $\left(\frac{d}{dt}\right) f(\sigma(t)) = \{H, f\}(\sigma(t))$, where $\sigma(t)$ is any integral curve of X_H . From the definition of integral curves on a manifold, we can write

$$\sigma_* \left(\frac{d}{dt} \right) f = X_H(f)(\sigma(t)) = \{H, f\}(\sigma(t)) \quad (3.3)$$

$$\implies \left(\frac{d}{dt} \right) f(\sigma(t)) = \{H, f\}(\sigma(t)). \quad (3.4)$$

We can make use of this result of differential geometry to make the following definitions on the phase space of our system.

Definition 3.1.1 *An observable variable f (function on the phase space) is said to be conserved or a constant of the motion, if and only if $\{H, f\} = 0$.*

Definition 3.1.2 *Two observables f_1 and f_2 , defined on a Poisson manifold $\mathcal{M}$, are said to be in involution if and only if $\{f_1, f_2\} = 0$.*

Now that we know how to identify conserved quantities in some generic Poisson manifold and the definition of involution, we give some general criteria to establish whether a system can be regarded as classically integrable or not

Definition 3.1.3 (Complete integrability) *The dynamical system defined on a $2m$ dimensional Poisson manifold $\mathcal{M}$ by a Hamiltonian $H \in C^\infty(\mathcal{M})$ is said to be completely integrable if there exist m independent conserved quantities $f_1, \dots, f_m \in C^\infty(\mathcal{M})$ which are in involution.*

Even though there is no systematic way of proving whether a classical system is (completely) integrable or not, the modern theory of integrability teaches us that if we can rewrite the equations of motion of the system by using a so called Lax pair, then the identification of a set of conserved quantities is always possible. We summarize this notion in the following definition

Definition 3.1.4 (Lax Pair) *A Lax pair for a Hamiltonian system on a Poisson manifold $\mathcal{M}$ is a pair (L, P) of C^∞ functions on $\mathcal{M}$ which take values in a real Lie algebra $\mathfrak{g}$, such that the equations of motion of the system are equivalent to*

$$\frac{dL}{dt} = [L, P] , \quad (3.5)$$

where $[\cdot, \cdot]$ is the Lie bracket on $\mathfrak{g}$.

As we mentioned before, even though one can not prove whether or not a system is integrable, if the equations of motion accept the so called Lax representation presented above, then we can actually produce conserved quantities from the Lax pair. For instance, let f be some polynomial function of degree n on $\mathfrak{g}$ which is invariant under the coadjoint action of $\mathfrak{g}$. Then, $f(L, \dots, L)$ is conserved. In fact

$$\begin{aligned} \frac{d}{dt} f(L, \dots, L) &= f\left(\frac{dL}{dt}, L, \dots, L\right) + \dots + f\left(L, L, \dots, \frac{dL}{dt}\right) \\ &= f([L, P], L, \dots, L) + \dots + f(L, L, \dots, [L, P]) \\ &= \text{ad}_P^*(f)(L, L, \dots, L) \\ &= 0. \end{aligned} \quad (3.6)$$

In order to produce quantities of this type that are not only conserved but are also in involution, one must assume further that:

$$\{L, L\} = [r, L_{\underline{1}} + L_{\underline{2}}] , \quad (3.7)$$

for some $r \in \mathfrak{g} \times \mathfrak{g}$. Here $L_{\underline{1}} = L \otimes 1$, $L_{\underline{2}} = 1 \otimes L$, and the Poisson brackets is extended to $\mathfrak{g}$ -valued functions in the obvious way, that is if $\{X_s\}$ is a basis for $\mathfrak{g}$ and:

$$f_1 = f_1^s X_s, \quad f_2 = f_2^s X_s; \quad f_1^s, f_2^s \in C^\infty(\mathcal{M}), \quad (3.8)$$

then:

$$\{f_1, f_2\} = \sum_{s,t} \{f_1^s, f_2^t\} X_s \otimes X_t. \quad (3.9)$$

3.1.2 The r -matrix approach to integrability

It appears that we have included condition (3.7) by hand and we have not even showed that provided that this relation holds, one can find conserved charges that are in involution. Our goal here is not to try to show rigorously that this is the case but to describe

a framework where this condition holds and we in fact end up with a completely integrable system. The machinery involved in this kind of reasoning is the so called r -matrix approach to integrability.

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and let $(\cdot, \cdot)$ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. Let $r \in \mathfrak{g} \times \mathfrak{g}$ be a skew-symmetric solution of the mCYBE. We may regard r as a linear map $r : \mathfrak{g}^* \rightarrow \mathfrak{g}$, and hence as a linear map $R : \mathfrak{g} \rightarrow \mathfrak{g}$, identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ by using the inner product $(\cdot, \cdot)$.

Proposition 3.1.1 *With the notation above, let $r \in \mathfrak{g} \times \mathfrak{g}$ be a skew-symmetric solution of the mCYBE and let us define the bracket $[\cdot, \cdot]_r$ for $X, Y \in \mathfrak{g}$ as follows:*

$$[X, Y]_r \equiv [RX, Y] + [X, RY], \quad (3.10)$$

then $[\cdot, \cdot]_r$ defines a Lie bracket on $\mathfrak{g}$.

In fact, the (modified) classical Yang-Baxter equation (mCYBE) is a sufficient condition for $[\cdot, \cdot]_r$ to define a Lie bracket on $\mathfrak{g}$. This statement is precisely what defines a classical r -matrix [59]. The pair $(\mathfrak{g}, r)$ is sometimes called a double Lie algebra.

The next theorem is usually called the main theorem for a good reason. It states that one can construct a classically integrable system from an algebraic point of view, given a solution to the mCYBE and some Lie algebra $\mathfrak{g}$, and exploiting the fact that the dual space $\mathfrak{g}^*$ is a Poisson manifold, it is always possible to show that there exist a Lax Pair and equation (3.7) holds, that is, one can construct completely integrable system from solutions of the mCYBE.

Proposition 3.1.2 (The main theorem) *Let $L : \mathfrak{g}^* \rightarrow \mathfrak{g}$ be the canonical map associated to the inner product on some Lie algebra $\mathfrak{g}$, and let $P(\xi) = \rho(dH(\xi))$, $\xi \in \mathfrak{g}^*$. If H is a smooth function on $\mathfrak{g}^*$ which is invariant under the coadjoint action of G , and $r \in \mathfrak{g} \times \mathfrak{g}$ is a skew-symmetric solution of the mCYBE, then the Hamiltonian system defined on $\mathfrak{g}^*$ with Poisson brackets $[\cdot, \cdot]_r$ and Hamiltonian H admits a Lax pair (L, P) . Moreover,*

$$\{L, L\}_r = [r, L \otimes 1 + 1 \otimes L]. \quad (3.11)$$

We will not offer a proof for this proposition because that is not something that concern us in this text, the interested reader may consult the collection of reviews [57] for a detailed proof.

In fact, this proposition as stated here is valid for what is usually called ultralocal theories. If one considers Poisson manifolds defined on affine Lie-algebras one gets a

generalization of (3.1.2) which includes terms proportional to derivatives of the delta function. Theories whose phase space is described by this kind of structure are usually called non-ultralocal.

Now that we have a formal definition of what we call integrable systems and more importantly, we know the deep relation between integrable systems and the (modified) classical Yang-Baxter equation, we can give a summary of the method of deformation employed in the so called η -deformations of sigma models.

3.2 Explaining the deformation procedure

Given an integrable sigma model one interesting question one may ask is if it is possible to formulate deformations which preserve the integrability. Such kind of integrable deformations have been the subject of intensive study, the main motivation for this of course being the *AdS/CFT* correspondence.

There are two main families of integrable deformations that one can identify, the first of them proposed by Hollowood et al. [10][11] which is based on a previous work by Sfetsos [9], adopts a method to directly deform the action of the model and even though the resulting deformed model is integrable, one needs to prove this by hand. We will not study this kind of approach in this text. The second family of integrable deformations is the one that we will review in this chapter, which are usually called Yang-Baxter deformations [60]. The name is used to reflect the fact that these type of deformations are constructed from solutions of the (modified) classical Yang-Baxter equation.

The basic principle to deform an integrable system without losing integrability is to preserve the Lax connection. As we know from the preceding section, a completely integrable model can be identified by finding a Lax pair and showing that it satisfies the following relation

$$\{L, L\} = [r, L_1 + L_2] . \quad (3.12)$$

In other words, proving that a system is completely integrable implies identifying a Lax pair and showing that it satisfies the equation above. In this sense, if one is somehow capable of deforming the original model without changing the Lax connection L , then the relation above will still hold and the deformed model will still be completely integrable.

Starting from some integrable field theory, like the principal chiral model or the non-linear sigma model, we will follow the procedure sketched below:

1. **Identify the algebraic integrable structure of the model.** For instance, the phase space of the principal chiral model or the superstring sigma model can be

identified with a loop group. The most natural algebraic setup to study classical integrability are Poisson manifolds and the simplest Poisson manifold one can think of is the dual space $\mathfrak{g}^*$ of some Lie algebra $\mathfrak{g}$. For the kind of sigma model we are interested in, we will be studying some loop algebra and its dual. As it was shown in [61], in this framework, one can prove an analogue to Proposition 3.1.2. That is, given a phase space described by a central extension of some loop algebra $\mathcal{L}\mathfrak{g}$ and a Lax connection $\mathcal{L}(\sigma, \lambda)$ (for instance non-ultralocal theories), a fairly general form of the Poisson bracket of $\mathcal{L}(\sigma, \lambda)$ with itself which will guarantee the complete integrability of the theory is ¹

$$\begin{aligned} \{\mathcal{L}_1(\sigma, \lambda), \mathcal{L}_2(\sigma', \mu)\} &= [r_{12}(\lambda, \mu), \mathcal{L}_1(\sigma, \lambda) + \mathcal{L}_2(\sigma, \mu)] \delta_{\sigma\sigma'} + \\ &+ [s_{12}(\lambda, \mu), \mathcal{L}_1(\sigma, \lambda) - \mathcal{L}_2(\sigma, \mu)] \delta_{\sigma\sigma'} + \\ &+ 2 s_{12}(\lambda, \mu) \delta'_{\sigma\sigma'}. \end{aligned} \quad (3.13)$$

2. **Identify an additional compatible Poisson structure.** The deformation of the Poisson structure that we will perform can be naively understood as the substitution of the original Poisson brackets of the theory we want to deform by a linear combination of compatible Poisson brackets, in the sense that if $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ define two different Poisson structures, they are said to be compatible if $\alpha\{\cdot, \cdot\}_1 + \beta\{\cdot, \cdot\}_2$ also defines a Poisson structure. It is precisely in this linear combination where we will include a deformation parameter ϵ . In the case of the principal chiral model (PCM), the most natural additional compatible Poisson structure we can define is the so called Faddeev-Reshetikhin Poisson structure [63] which we will explain later. On the other hand, for the superstring sigma model an analogue to the Faddeev-Reshetikhin Poisson structure can be used [64][65].

3. **Modify the inner product.** Of course the naive intuition of deforming the Poisson structure by using compatible Poisson brackets is not very accurate. The formal procedure is to deform the inner product on the loop algebra (more precisely the dual space of the loop algebra) of the phase space (which is given by a loop group) of the theory we are studying. The inner product in these kind of spaces, which for spaces with $\mathbb{Z}_n$ -gradation are called twisted loop algebras, is given by using a complex function ϕ which is usually called the twist function. The way to identify the twist function is to compute, by brute force, the exchange algebra of the model.

¹In fact, it was proven in [62][61], that this form of the exchange algebra, sometimes called r/s formalism, is a suitable generalization for theories defined on a phase that can be associated to affine Lie algebras, like the non-linear sigma model.

For instance, for the PCM, the twist function was identified in [66] and is given by:

$$\phi_{PCM}(\lambda) = -1 + \frac{1}{\lambda^2}, \quad (3.14)$$

where λ is a complex variable (the spectral parameter). Similarly, the twist function for the Superstring sigma model was found to be [67]

$$\phi_{string}(\lambda) = \frac{4z^4}{(1-z^4)^2}, \quad (3.15)$$

where z is a spectral parameter. Thus the prescription to get η -deformations is to modify the inner product by means of deforming the original twist function of the model, without changing the Lax connection or even the (twisted) loop algebra.

4. **Identify the deformed fields by using the Lax connection.** Two aspects are important when defining the deformed fields. The first one is to note that the group valued field $g \in G$ can actually be identified as a gauge parameter of the symmetry algebra in a non linear sigma model. This fact is used to extract the deformed fields from the Lax matrix by performing a gauge transformation on it. The second one relies in the fact that we need to pick a solution to the mCYBE, this will help us to define a canonical conjugate to the group valued field g . In the so called η -deformations, a non-split R -matrix is used, the so called Drinfeld-Jimbo R -matrix.
5. **Perform an inverse Legendre transform and find the deformed Lagrangian.** This last step is quite straightforward, once we have the currents expressed in terms of the deformed group valued fields we just need to take advantage of the fact that the structure of the Hamiltonian does not change. Thus, we can perform an inverse Legendre transform and write the final form of the η -deformed Lagrangian.

3.3 The η -deformed principal chiral model

The principal chiral model (PCM) will be the main example we will use to illustrate how the procedure described in Section 3.2 can be applied. The PCM on a compact Lie group G is described by the Lagrangian:

$$L_{PCM} = -\frac{1}{2} \text{tr} (A_\mu A^\mu), \quad (3.16)$$

where the currents $A_\mu = -g^{-1}\partial_\mu g = A_\mu^a t_a$ take values in the Lie algebra $\mathfrak{g}$ and t_a are Lie algebra generators. Also, the index $\mu = 0, 1$ represents the coordinates τ and σ and the elements of the Lie group $g(\tau, \sigma) \in G$ are embeddings from some worldsheet parametrized by τ and σ to the Lie group G . The currents satisfy the equations of motions and zero curvature equation listed below

$$-\partial_\tau A_0 + \partial_\sigma A_1 = 0, \quad (3.17)$$

$$\partial_\tau A_1 - \partial_\sigma A_0 - [A_0, A_1] = 0. \quad (3.18)$$

The Hamiltonian for this model is given by:

$$H_{PCM} = -\frac{1}{2} \int_{-\infty}^{+\infty} d\sigma [\text{tr}(A_0 A_0) + \text{tr}(A_1 A_1)]. \quad (3.19)$$

It is known that the integrability of the equations of motion of the PCM is encoded in the following Lax matrix

$$\mathfrak{L}(\lambda) = \frac{1}{1 - \lambda^2} (A_1 + \lambda A_0), \quad (3.20)$$

where, as discussed previously, $\mathfrak{L} \in \mathcal{L}\mathfrak{g}$. This ends our quick review of the PCM.

3.3.1 Deforming the Poisson structure

The deformations described by Vicedo et al. consist on deformations of the twist function given by the following relation:

$$\phi_\epsilon(\lambda)^{-1} \equiv \phi_{PCM}(\lambda)^{-1} + \epsilon^2 \phi_{FR}(\lambda)^{-1}, \quad (3.21)$$

here the deformation parameter ϵ is taken to be real and positive. Substituting the definitions of the twist functions ϕ_{PCM} and ϕ_{FR} ² we get the following expression for the deformed twist function:

$$\phi_\epsilon(\lambda) = \frac{1 - \lambda^2}{(1 - \epsilon^2)\lambda^2 + \epsilon^2}. \quad (3.22)$$

This deformation of the twist function results in a deformation of the inner product and, as an immediate consequence, in the deformation of the Poisson structure. The new Poisson structure is now simply given by

$$\{\cdot, \cdot\}_\epsilon = \{\cdot, \cdot\} + \epsilon^2 \{\cdot, \cdot\}_{FR}. \quad (3.23)$$

²The twist function ϕ_{FR} refers to the Faddeev-Reshetikhin procedure, where the twist function is modified as $\phi_{PCM} \rightarrow \phi_{FR} = 1$ to make the current algebra ultralocal. See for [14] a review.

The deformed Poisson brackets of the canonical fields can easily be computed now

$$\{A_{0\mathbf{1}}(\sigma), A_{0\mathbf{2}}(\sigma')\}_{FR} = -(1 - \epsilon^2) [C_{\mathbf{12}}, A_{0\mathbf{2}}(\sigma)] \delta_{\sigma\sigma'}, \quad (3.24)$$

$$\{A_{0\mathbf{1}}(\sigma), A_{1\mathbf{2}}(\sigma')\}_{FR} = -(1 - \epsilon^2) [C_{\mathbf{12}}, A_{1\mathbf{2}}(\sigma)] \delta_{\sigma\sigma'} + C_{\mathbf{12}} \delta'_{\sigma\sigma'}, \quad (3.25)$$

$$\{A_{1\mathbf{1}}(\sigma), A_{1\mathbf{2}}(\sigma')\}_{FR} = \epsilon^2 [C_{\mathbf{12}}, A_{0\mathbf{2}}(\sigma)] \delta_{\sigma\sigma'}, \quad (3.26)$$

here we used compact tensor notation, where $C_{\mathbf{12}}$ represents the tensor Casimir. It is easy to see that, when $\epsilon = 0$ we recover the original PCM Poisson brackets whereas when $\epsilon \rightarrow \infty$, the Poisson brackets are proportional to the Faddeev-Reshitikhin Poisson brackets.

A very important remark we should make here is that one can actually show that there is a direct relation between the symmetry generators of the PCM with the poles of the twist function. We will not review the deformations of the symmetry algebra in this text but it is known that when the twist function gets deformed, the symmetry algebra actually gets q -deformed and that is precisely the reason why η -deformations are sometimes also called q -deformations.

In the non-deformed case, the twist function has a double pole at $\lambda = 0$, as we will see in the next section, the group element $g \in G$ can be extracted from the expansion of the gauge transformation of the Lax matrix around this point. Thus, the poles of ϕ_ϵ will be important when defining the group valued field of the deformed model. The poles of ϕ_ϵ are located at

$$\lambda_{\pm} = \frac{i\epsilon}{\sqrt{1 - \epsilon^2}}, \quad \lambda_- = \bar{\lambda}_+. \quad (3.27)$$

For convenience, we will restrict $0 \leq \epsilon < 1$. In this way we will avoid the singularity at $\epsilon = 1$. We will give more details about this later in the text.

3.3.2 Deformed fields

Non-deformed group valued fields

The principal chiral model can also be defined in terms of the group valued fields $g \in G$ and its canonical conjugate defined as $X = gA_0g^{-1}$. As we already mentioned before, g and X can be obtained from the expansion of the Lax matrix at the poles of the twist function ϕ_{PCM} . In fact, the expansion of the Lax matrix near the double pole $\lambda = 0$ is

$$\mathfrak{L}(\lambda) = A_1 - \lambda^{-1}A_0 + \mathcal{O}(\lambda^{-2}). \quad (3.28)$$

Consider the gauge transformations of the Lax matrix $\mathfrak{L}^g(\lambda) = \partial_\sigma g g^{-1} + g\mathfrak{L}(\lambda)g^{-1}$ with the group valued field g acting as a parameter. Using the expansion of the Lax matrix

and provided that $A_1 = -g^{-1}\partial_\sigma g$ we observe that

$$\mathfrak{L}^g(\lambda) = \lambda X + \mathcal{O}(\lambda^2). \quad (3.29)$$

Thus, the group valued field g is the gauge parameter that sends $\mathfrak{L}(0)$ to 0. Furthermore, the field X corresponds to the subleading term in the expansion of $\mathfrak{L}^g(\lambda)$ at the pole. In other words we have

$$\mathfrak{L}^g(0) = 0, \quad X = \frac{d\mathfrak{L}^g}{d\lambda}(0). \quad (3.30)$$

Deformed group valued fields

Now that we know a way to identify the group valued fields from the Lax matrix, we can try to mimic this reasoning and postulate a way to extract the group valued fields from the expansion of the Lax matrix around the poles of the deformed twist function ϕ_ϵ .

The generalization of the defining condition for g is related to the symmetry algebra. Since we want the group valued fields to still take values on the compact Lie group G , one has to impose a reality condition, that is one needs to look at the real form of the complex lie algebra $\mathfrak{g}^\mathbb{C}$ and ensure that the expansion of the Lax algebra stays there. We will not look at the details of that procedure because we are mainly interested in getting the deformed action. The reader may consult the original work [14] for details.

On the other hand, we postulate the following generalization of the deformation of X in the deformed theory

$$X = \frac{i}{2\gamma} (\mathfrak{L}^g(\lambda_+) - \mathfrak{L}^g(\lambda_-)), \quad (3.31)$$

where the real normalisation factor γ will be fixed later. We know that this expression is a good generalization because in the non-deformed limit $\epsilon \rightarrow 0$ and provided that $\gamma \rightarrow \epsilon$, the expression above reduces to the derivative that we used to define X in the non-deformed case.

One more ingredient that we need to use to describe η deformations is the non-split R -matrix. As we mentioned before, this kind of deformations are typically called Yang-Baxter deformations because they make use of a solution to the mCYBE given by the so called Drinfeld-Jimbo R -matrix. Applying the non-split R -matrix (see Appendix B) we find:

$$RX = \frac{1}{2\gamma} (\mathfrak{L}^g(\lambda_+) + \mathfrak{L}^g(\lambda_-)). \quad (3.32)$$

Finally combining equations (3.31) and (3.32) and solving the system of equations that they define we get

$$\mathfrak{L}^g(\lambda_\pm) = \gamma(R \mp i)X. \quad (3.33)$$

Deformed currents

Now that we have candidates for the deformations of the group valued fields g and X , what we need to do is to relate them to the fields A_0 and A_1 . In other words, we want to get expressions for the deformations of

$$A_0 = -g^{-1}\partial_\sigma g, \quad A_1 = -g^{-1}Xg. \quad (3.34)$$

The way to achieve this is very naive. Substituting equation (3.33) into the gauge transformations of the Lax matrix we get:

$$\mathfrak{L}(\lambda_\pm) = -g^{-1}\partial_\sigma g + \gamma g^{-1}((R \pm i)X)g. \quad (3.35)$$

On the other hand, one can obtain a similar expression by evaluating the PCM Lax matrix (3.20) by hand at the poles $\lambda_\pm$, we get

$$\mathfrak{L}(\lambda_\pm) = \frac{1}{1 - \lambda_\pm^2} (A_1 + \lambda_\pm A_0) = (1 - \epsilon^2)A_1 \pm i\epsilon\sqrt{1 - \epsilon^2}A_0. \quad (3.36)$$

Comparing the two defining equations for $\mathfrak{L}(\lambda_\pm)$ given in (3.35) and (3.36) immediately yields to the desired expressions for the deformed A_0 and A_1 in terms of the group valued fields g and X

$$A_1 = -\frac{1}{1 - \epsilon^2} g^{-1}\partial_\sigma g - \epsilon\sqrt{1 - \epsilon^2} g^{-1}(RX)g, \quad (3.37)$$

$$A_0 = (1 - \epsilon^2) g^{-1}Xg. \quad (3.38)$$

where, in order to reproduce the correct deformed algebra (3.26), it turns out that we need to set

$$\gamma = -\epsilon(1 - \epsilon^2)^{3/2}. \quad (3.39)$$

Equations (3.38) are the most important consequence of deforming the Poisson structure by means of deforming the PCM twist function. What we have achieved is a family of one parameter continuous deformations of the PCM fields, and as a consequence, the Lagrangian constructed from these fields, since the Lax matrix has not been changed, will automatically be completely integrable. What remains is to use the PCM Hamiltonian and perform an inverse Legendre transform to get an expression for the deformed Lagrangian.

3.3.3 Deformed Lagrangian

As we said before, getting the deformed Lagrangian just involves performing an inverse Legendre transform. One important thing to consider before going on is that in order to preserve integrability, apart from keeping the Lax matrix unchanged through the deformation procedure, one must also ensure that the dynamics of the fields A_0 and A_1 remains the same as we vary the deformation parameter ϵ .

Since the deformed Poisson brackets do depend on the deformation parameter, this implies that, as we have already encountered (3.38), the fields will depend on the deformation parameter. In order to find the deformed Hamiltonian H_ϵ which generates the same non-deformed dynamics on the deformed fields with respect to the deformed Poisson brackets, we should solve the following equation.

$$\{H_\epsilon, \mathfrak{L}\}_\epsilon = \{H_{PCM}, \mathfrak{L}\}. \quad (3.40)$$

By using the fact that the Hamiltonian H_{PCM} has vanishing Faddeev-Reshetikhin Poisson brackets with any function of A_0, A_1 , it is easy to see that $H_\epsilon = H_{PCM}$ is also the Hamiltonian with respect to the deformed bracket.

The inverse Legendre transform can be computed as follows

$$L = \text{tr} (\partial_\tau g g^{-1} X) - h_\epsilon \quad (3.41)$$

where the Hamiltonian density $h_\epsilon = h_{PCM}$ can be extracted from (3.19). Expressing the Hamiltonian density in light cone variables given by $A_\pm = A_0 \pm A_1$ we get

$$h_\epsilon = -\frac{1}{4} \text{tr}(A_+ A_+) - \frac{1}{4} \text{tr}(A_- A_-). \quad (3.42)$$

The group valued field can be obtained by inverting (3.38), one obtains

$$X = -\frac{1}{2}(1 + \eta^2)^2 \left(\frac{1}{1 - \eta R} \partial_- g g^{-1} + \frac{1}{1 + \eta R} \partial_+ g g^{-1} \right), \quad (3.43)$$

here we have defined $\eta = \frac{\epsilon}{\sqrt{1-\epsilon^2}}$ for convenience. We use Light cone variables because the deformed currents (3.38) take a very nice look as can be seen next

$$A_\pm = \mp \frac{1}{1 - \epsilon^2} g^{-1} \partial_\sigma g + (1 - \epsilon^2) g^{-1} (1 \mp \eta R) X g. \quad (3.44)$$

The last step is to plug in all of this information to the Legendre transform. The final expression for the η -deformed Lagrangian of the principal chiral model reads

$$L = -\frac{1}{2} \text{tr} \left[\partial_+ g g^{-1} \frac{(1 + \eta^2)^2}{1 - \eta R} \partial_- g g^{-1} \right]. \quad (3.45)$$

As we may expect, in the non-deformed limit $\eta \rightarrow 0$ we recover the original non-deformed Lagrangian of the principal chiral model.

3.4 The η -deformed Green-Schwarz sigma-model

The generalization to symmetric or semi-symmetric spaces is straightforward. The interested reader can consult [14], for generalizations to symmetric sigma models and [68] for semi-symmetric sigma models. In this section we will present the action and some of its properties. In the last section we will present an alternative derivation of the action based purely on integrability arguments.

3.4.1 Deformed Lagrangian

The η -deformations of the $AdS_5 \times S^5$ superstring sigma model is given by the following expression

$$L_\epsilon = -\frac{(1+\eta^2)^2}{2(1-\eta^2)} P_-^{\alpha\beta} \text{str} (g^{-1} \partial_\alpha g dJ_\beta) , \quad (3.46)$$

here the deformed current J is given by

$$J_\alpha = \frac{1}{1-\eta R_g \circ d} (g^{-1} \partial_\alpha g) , \quad \tilde{J}_\alpha = \frac{1}{1+\eta R_g \circ \tilde{d}} (g^{-1} \partial_\alpha g) , \quad (3.47)$$

where the non-split R -matrix is given by:

$$R_g = \text{Ad}_{g^{-1}} \circ R \circ \text{Ad}_g , \quad (3.48)$$

and the combination of projectors onto the gradings of the loop algebra are given by

$$d = P_1 + \frac{2}{1-\eta^2} P_2 - P_3 , \quad \tilde{d} = -P_1 + \frac{2}{1-\eta^2} P_2 + P_3 . \quad (3.49)$$

Just as we saw for the principal chiral model, if we take the non-deformed limit $\eta \rightarrow 0$ one recovers the original Green-Schwarz superstring sigma model in $AdS_5 \times S^5$.

In the next section we offer a derivation of the deformed Lagrangian and the Lax pair of these model, thus showing its classical integrability.

3.4.2 κ -symmetry

It turns out that the η -deformed Lagrangian still has the usual kappa symmetry of the Green-Schwarz superstring sigma model. The way to show this is totally analogous to the procedure described in the non-deformed case in Chapter 2. We start by considering the action for the η -deformed $AdS_5 \times S^5$ superstring sigma model, it is given by³

$$S_\eta = -\frac{1}{4} \int d\tau d\sigma P_-^{\alpha\beta} \text{str} [A_\alpha dJ_\beta] . \quad (3.50)$$

³We are ignoring the overall normalization factor for simplicity.

Let us now consider fermionic local transformations of the group element g as follows

$$\delta g = g \epsilon, \quad \epsilon = (1 - \eta R_g) \rho^{(1)} + (1 + \eta R_g) \rho^{(3)}, \quad (3.51)$$

where the fermionic parameters $\rho^{(1)}$ and $\rho^{(3)}$ are arbitrary functions to be determined later. Notice that the symmetry has the same description as in the non-deformed case, but the transformation parameters are somewhat deformed. The variation of the current J is easy to compute, it is given by

$$\mathcal{E} = d(\partial_\alpha J_-) + \tilde{d}(\partial_\alpha \tilde{J}_+^\alpha) + [\tilde{J}_{+\alpha}, d(J_-^\alpha)] + [J_{-\alpha}, \tilde{d}(\tilde{J}_+^\alpha)], \quad (3.52)$$

where we have defined

$$J_-^\alpha = P_-^{\alpha\beta} J_\beta, \quad \tilde{J}_+^\alpha = P_+^{\alpha\beta} \hat{J}_\beta. \quad (3.53)$$

With these results in mind, we can compute the variation of the action (3.50), it is given by

$$\begin{aligned} \delta_g S_\eta &= \frac{1}{4} \int d\tau d\sigma \operatorname{str}(\epsilon \mathcal{E}) \\ &= \frac{1}{4} \int d\tau d\sigma \operatorname{str}(\rho^{(1)} P_3 (1 + \eta R_g) \mathcal{E} + \rho^{(3)} P_1 \circ (1 - \eta R_g) \mathcal{E}). \end{aligned} \quad (3.54)$$

In order to give this equation a structure that resembles the previous computation of chapter 2, we make use of the following identities

$$\begin{aligned} P_1 \circ (1 - \eta R_g)(\mathcal{E}) &= -4 \left[\tilde{J}_{+\alpha}^{(2)}, J_-^{\alpha(3)} \right] - P_1(\mathcal{Z}), \\ P_3 \circ (1 + \eta R_g)(\mathcal{E}) &= -4 \left[J_{-\alpha}^{(2)}, \tilde{J}_+^{\alpha(1)} \right] + P_3(\mathcal{Z}), \end{aligned} \quad (3.55)$$

where is the zero curvature condition of the Maurer-Cartan current A_α but written in terms of the deformed currents J_α . Using these identities, the variation of the action can be written as follows

$$\delta_g S_\eta = -2 \int d\tau d\sigma \operatorname{str} \left(\rho^{(1)} \left[J_{-\alpha}^{(2)}, \tilde{J}_+^{\alpha(1)} \right] + \rho^{(3)} \left[\tilde{J}_{+\alpha}^{(2)}, J_-^{\alpha(3)} \right] \right). \quad (3.56)$$

The equation above is analogous to the equation (2.84) of chapter 2. In a very similar to the non-deformed case, an ansatz for the transformation parameters can be taken as follows (see equations (2.87) and (2.88))

$$\rho^{(1)} = i\kappa_{+\alpha}^{(1)} J_-^{\alpha(2)} + J_-^{\alpha(2)} i\kappa_{+\alpha}^{(1)}, \quad (3.57)$$

$$\rho^{(3)} = i\kappa_{-\alpha}^{(3)} \tilde{J}_+^{\alpha(2)} + \tilde{J}_+^{\alpha(2)} i\kappa_{-\alpha}^{(3)}. \quad (3.58)$$

From now on, the computation is totally parallel to the procedure leading to equation (2.92) and (2.93). We finally obtain

$$\delta_g S_\eta = -\frac{1}{4} \int d\tau d\sigma \left(\text{str}(J_-^{\alpha(2)} J_-^{\beta(2)}) \text{str}(\Upsilon[\tilde{J}_{\alpha+}^{(1)}, i\kappa_{+\beta}^{(1)}]) + \text{str}(\tilde{J}_+^{\alpha(2)} \tilde{J}_+^{\beta(2)}) \text{str}(\Upsilon[J_{-\alpha}^{(3)}, i\kappa_{-\beta}^{(3)}]) \right), \quad (3.59)$$

where Υ is the hypercharge matrix defined in chapter 2 as $\Upsilon = \text{Diag}(1_4, -1_4)$. Just as in the non-deformed case, this expression is compensated by another term coming from the variation of the metric γ . In order to ensure κ -symmetry, this variation takes the following form

$$\delta\gamma^{\alpha\beta} = \frac{1-\eta^2}{2} \text{str} \left(\Upsilon[i\kappa_+^{\alpha(1)}, \tilde{J}_+^{\beta(1)}] + \Upsilon[i\kappa_-^{\alpha(3)}, J_-^{\beta(3)}] \right). \quad (3.60)$$

3.5 The η -deformed hybrid sigma-model

Here we will take a rather different approach to obtain the η -deformed Lagrangian. The idea relies on [39] where it was suggested that there are only two types of sigma models which admit $\mathbb{Z}_4$ decomposition and can be regarded as classically integrable; the Green-Schwarz sigma model and the hybrid sigma model (matter part of the pure spinor sigma model). What we will do is to start with the most general ansatz for an η -deformed sigma model. First of all let us remember from chapter 2, that in conformal gauge, the Green-Schwarz and Pure spinor sigma models have a very similar structure, in fact we can write them in the following compact way:

$$L = \frac{1}{2} \text{str}(A d \bar{A}), \quad \text{where } d = \begin{cases} -P^{(1)} + 2P^{(2)} + P^{(3)} & \text{for Green-Schwarz,} \\ 3P^{(1)} + 2P^{(2)} + P^{(3)} & \text{for Pure spinors.} \end{cases} \quad (3.61)$$

In the equation above, d is a linear combination of the projectors on the graded subspaces⁴.

Our ansatz for the η -deformed Lagrangian comes by generalizing the coefficients in front of the projector d in the following way

$$d = \alpha_1(\eta)P^{(1)} + \alpha_2(\eta)P^{(2)} + \alpha_3(\eta)P^{(3)}, \quad \tilde{d} = \alpha_3(\eta)P^{(1)} + \alpha_2(\eta)P^{(2)} + \alpha_1(\eta)P^{(3)}, \quad (3.62)$$

where α can depend on the deformation parameter η and $\tilde{d}$ is the transpose of d . Thus, the η -deformed Lagrangian can be written in conformal gauge in the following way

$$\begin{aligned} L_\eta &= \frac{1}{2} \text{str} \left[A d \circ \frac{1}{1 - \eta R \circ d} \bar{A} \right] \\ &= \frac{1}{2} \text{str} [A d \bar{J}] , \end{aligned} \quad (3.63)$$

⁴You should note a change in overall sign if compared to the projectors defined in (3.49), this is because we are using conformal gauge for this section.

where d depends on the deformation parameter and is given by (3.62) and we have defined the deformed current $\bar{J}$

$$\bar{J} = \mathcal{O}_-^{-1} \bar{A} = \frac{1}{1 - \eta R_g \circ d} \bar{A}, \quad J = \mathcal{O}_+^{-1} A = \frac{1}{1 + \eta R_g \circ d} A. \quad (3.64)$$

The equations of motion can be found rather easily. They can be collected in the following expression

$$\varepsilon = \bar{\partial}(\tilde{d}J) + \partial(d\bar{J}) + [\bar{J}, \tilde{d}J] + [J, d\bar{J}] = 0. \quad (3.65)$$

On the other hand, the zero curvature equation, which is satisfied by the Maurer-Cartan current A_α can be written as

$$\begin{aligned} \mathcal{Z} &= \bar{\partial}A - \partial\bar{A} + [\bar{A}, A] \\ &= \bar{\partial}J - \partial\bar{J} + [\bar{J}, J] - c^2\eta^2[d\bar{J}, \tilde{d}J] = 0, \end{aligned} \quad (3.66)$$

where we have made use of definitions (3.64), c is a complex parameter and also the mCYBE was used

$$[R(X), R(Y)] - R([R(X), Y] + [Y, R(X)]) = -c^2[X, Y]. \quad (3.67)$$

At this point the strategy should be clear. We will define an ansatz for the Lax pair and find the appropriate parameters that make it possible to obtain the equations of motion and the zero curvature condition. Before doing that, it is convenient to project the gradings of the equations written above. The grading projections of the equations of motion ε are

$$\begin{aligned} \varepsilon^{(1)} &= \alpha_3 \bar{\nabla} J^{(1)} + \alpha_1 \nabla \bar{J}^{(1)} + (\alpha_2 - \alpha_3) [\bar{J}^{(3)}, J^{(2)}] + (\alpha_1 - \alpha_2) [\bar{J}^{(2)}, J^{(3)}], \\ \varepsilon^{(2)} &= \alpha_2 (\bar{\nabla} J^{(2)} + \nabla \bar{J}^{(2)}) + (\alpha_3 - \alpha_1) [\bar{J}^{(1)}, J^{(1)}] + (\alpha_1 - \alpha_3) [\bar{J}^{(3)}, J^{(3)}], \\ \varepsilon^{(3)} &= \alpha_1 \bar{\nabla} J^{(3)} + \alpha_3 \nabla \bar{J}^{(3)} + (\alpha_3 - \alpha_2) [\bar{J}^{(2)}, J^{(1)}] + (\alpha_2 - \alpha_1) [\bar{J}^{(1)}, J^{(2)}]. \end{aligned} \quad (3.68)$$

On the other hand, the grading projections for the zero curvature condition $\mathcal{Z}$ are given by

$$\begin{aligned} \mathcal{Z}^{(0)} &= \bar{\partial}J^{(0)} - \partial\bar{J}^{(0)} + [\bar{J}^{(0)}, J^{(0)}] + (1 - c^2\eta^2\alpha_1^2) [\bar{J}^{(1)}, J^{(3)}] \\ &\quad + (1 - c^2\eta^2\alpha_2^2) [\bar{J}^{(2)}, J^{(2)}] + (1 - c^2\eta^2\alpha_3^2) [\bar{J}^{(3)}, J^{(1)}], \\ \mathcal{Z}^{(1)} &= \bar{\nabla} J^{(1)} - \nabla \bar{J}^{(1)} + (1 - c^2\eta^2\alpha_1\alpha_2) [\bar{J}^{(2)}, J^{(3)}] + (1 - c^2\eta^2\alpha_2\alpha_3) [\bar{J}^{(3)}, J^{(2)}], \\ \mathcal{Z}^{(2)} &= \bar{\nabla} J^{(2)} - \nabla \bar{J}^{(2)} + (1 - c^2\eta^2\alpha_1\alpha_3) [\bar{J}^{(1)}, J^{(1)}] + (1 - c^2\eta^2\alpha_1\alpha_3) [\bar{J}^{(3)}, J^{(3)}], \\ \mathcal{Z}^{(3)} &= \bar{\nabla} J^{(3)} - \nabla \bar{J}^{(3)} + (1 - c^2\eta^2\alpha_1\alpha_2) [\bar{J}^{(1)}, J^{(2)}] + (1 - c^2\eta^2\alpha_2\alpha_3) [\bar{J}^{(2)}, J^{(1)}]. \end{aligned} \quad (3.69)$$

As we mentioned earlier, we need to show that the equations of motion and the zero curvature condition above imply the flatness of the so called Lax pair and vice-versa. In other words, using the following ansatz for the Lax pair

$$\begin{aligned}\bar{\mathfrak{L}}(\lambda) &= \bar{J}^{(0)} + \lambda^{a_1} \beta_1 \bar{J}^{(1)} + \lambda^{a_2} \beta_2 \bar{J}^{(2)} + \lambda^{a_3} \beta_3 \bar{J}^{(3)}, \\ \mathfrak{L}(\lambda) &= J^{(0)} + \lambda^{b_1} \gamma_1 J^{(1)} + \lambda^{b_2} \gamma_2 J^{(2)} + \lambda^{b_3} \gamma_3 J^{(3)},\end{aligned}\quad (3.70)$$

where λ is the so called spectral parameter. We need to show if it is possible or not to fix every coefficient a, b, β and γ such that the flatness condition below implies the equations of motion and the zero curvature equation

$$\bar{\partial} \mathfrak{L}(\lambda) - \partial \bar{\mathfrak{L}}(\lambda) + [\bar{\mathfrak{L}}(\lambda), \mathfrak{L}(\lambda)] = 0. \quad (3.71)$$

Equation (3.71) represents a big set of equations for each of the gradings. In order to solve each of these equations let us project the flatness condition to each of the gradings just as we did with the equations of motion and the zero curvature equation

$$\begin{aligned}\text{flat}^{(0)} &= \bar{\partial} \mathfrak{L}^{(0)} - \partial \bar{\mathfrak{L}}^{(0)} + [\bar{\mathfrak{L}}^{(0)}, \mathfrak{L}^{(0)}] + [\bar{\mathfrak{L}}^{(1)}, \mathfrak{L}^{(3)}] + [\bar{\mathfrak{L}}^{(2)}, \mathfrak{L}^{(2)}] + [\bar{\mathfrak{L}}^{(3)}, \mathfrak{L}^{(1)}] \\ \text{flat}^{(1)} &= \bar{\nabla} \mathfrak{L}^{(1)} - \nabla \bar{\mathfrak{L}}^{(1)} + [\bar{\mathfrak{L}}^{(2)}, \mathfrak{L}^{(3)}] + [\bar{\mathfrak{L}}^{(3)}, \mathfrak{L}^{(2)}], \\ \text{flat}^{(2)} &= \bar{\nabla} \mathfrak{L}^{(2)} - \nabla \bar{\mathfrak{L}}^{(2)} + [\bar{\mathfrak{L}}^{(1)}, \mathfrak{L}^{(1)}] + [\bar{\mathfrak{L}}^{(3)}, \mathfrak{L}^{(3)}], \\ \text{flat}^{(3)} &= \bar{\nabla} \mathfrak{L}^{(3)} - \nabla \bar{\mathfrak{L}}^{(3)} + [\bar{\mathfrak{L}}^{(1)}, \mathfrak{L}^{(2)}] + [\bar{\mathfrak{L}}^{(2)}, \mathfrak{L}^{(1)}].\end{aligned}\quad (3.72)$$

We can now plug in our ansatz in each of the equations above, we will do this term by term to avoid confusion. As we will see, the constraints on each of the constants appearing in our ansatz will emerge naturally from requiring the holding of the equations of motion and the zero curvature condition. Let us start with the zero grading component

$$\begin{aligned}\text{flat}^{(0)} &= \bar{\partial} J^{(0)} - \partial \bar{J}^{(0)} + [\bar{J}^{(0)}, J^{(0)}] + \lambda^{a_1+b_3} \beta_1 \gamma_3 [\bar{J}^{(1)}, J^{(3)}] \\ &\quad + \lambda^{a_2+b_2} \beta_2 \gamma_2 [\bar{J}^{(2)}, J^{(2)}] + \lambda^{a_3+b_1} \beta_3 \gamma_1 [\bar{J}^{(3)}, J^{(1)}] \\ &= -(1 - c^2 \eta^2 \alpha_1^2) [\bar{J}^{(1)}, J^{(3)}] - (1 - c^2 \eta^2 \alpha_2^2) [\bar{J}^{(2)}, J^{(2)}] + (1 - c^2 \eta^2 \alpha_3^2) [\bar{J}^{(3)}, J^{(1)}] \\ &\quad + \lambda^{a_1+b_3} [\bar{J}^{(1)}, J^{(3)}] + \lambda^{a_2+b_2} [\bar{J}^{(2)}, J^{(2)}] + \lambda^{a_3+b_1} [\bar{J}^{(3)}, J^{(1)}],\end{aligned}\quad (3.73)$$

it is easy to see that for this equation to imply the 0-grading part of the zero curvature equation $\mathcal{Z}^{(0)}$, the following relations must hold

$$\begin{aligned}-(1 - c^2 \eta^2 \alpha_1^2) + \lambda^{a_1+b_3} \beta_1 \gamma_3 &= 0 &\rightarrow & a_1 = -b_3, \quad \beta_1 \gamma_3 = 1 - c^2 \eta^2 \alpha_1^2, \\ -(1 - c^2 \eta^2 \alpha_2^2) + \lambda^{a_2+b_2} \beta_2 \gamma_2 &= 0 &\rightarrow & a_2 = -b_2, \quad \beta_2 \gamma_2 = 1 - c^2 \eta^2 \alpha_2^2, \\ -(1 - c^2 \eta^2 \alpha_3^2) + \lambda^{a_3+b_1} \beta_3 \gamma_1 &= 0 &\rightarrow & a_3 = -b_1, \quad \beta_3 \gamma_1 = 1 - c^2 \eta^2 \alpha_3^2.\end{aligned}\quad (3.74)$$

Let us now examine the grading one component

$$\text{flat}^{(1)} = \lambda^{b_1} \gamma_1 \bar{\nabla} J^{(1)} - \lambda^{a_1} \beta_1 \nabla \bar{J}^{(1)} + \lambda^{a_2+b_3} \beta_2 \gamma_3 [\bar{J}^{(2)}, J^{(3)}] + \lambda^{a_3+b_2} \beta_3 \gamma_2 [\bar{J}^{(3)}, J^{(2)}]. \quad (3.75)$$

It is not so trivial to see which constraints are obtained from the relation above, so let us consider the following convenient combinations

$$\begin{aligned} \varepsilon^{(1)} + \alpha_1 \mathcal{Z}^{(1)} &= (\alpha_3 + \alpha_1) \bar{\nabla} J^{(1)} + (\alpha_2 - \alpha_3 + \alpha_1 (1 - c^2 \eta^2 \alpha_2 \alpha_3)) [\bar{J}^{(3)}, J^{(2)}] \\ &\quad + (\alpha_1 - \alpha_2 + \alpha_1 (1 - c^2 \eta^2 \alpha_1 \alpha_2)) [\bar{J}^{(2)}, J^{(3)}], \\ \varepsilon^{(1)} - \alpha_3 \mathcal{Z}^{(1)} &= (\alpha_1 + \alpha_3) \bar{\nabla} J^{(1)} + (\alpha_2 - \alpha_3 - \alpha_3 (1 - c^2 \eta^2 \alpha_2 \alpha_3)) [\bar{J}^{(3)}, J^{(2)}] \\ &\quad + (\alpha_1 - \alpha_2 - \alpha_3 (1 - c^2 \eta^2 \alpha_1 \alpha_2)) [\bar{J}^{(2)}, J^{(3)}], \end{aligned} \quad (3.76)$$

it is clear that we can solve this system of equations for the covariant derivative $\bar{\nabla} J^{(1)}$ but we need to consider two cases separately, when $\alpha_1 + \alpha_3 = 0$ and when $\alpha_1 + \alpha_3 \neq 0$.

Case 1: ($\alpha_1 + \alpha_3 \neq 0$)

Solving equations (3.76) and using equation (3.75) we get the following constraints

$$\begin{aligned} -\frac{\lambda^{b_1} \gamma_1}{\alpha_1 + \alpha_3} (\alpha_1 + \alpha_2 - \alpha_3 - c^2 \eta^2 \alpha_1 \alpha_2 \alpha_3) + \frac{\lambda^{a_1} \beta_1}{\alpha_1 + \alpha_3} (\alpha_2 - 2\alpha_3 + c^2 \eta^2 \alpha_2 \alpha_3^2) + \lambda^{a_3+b_2} \beta_3 \gamma_2 &= 0, \\ -\frac{\lambda^{b_1} \gamma_1}{\alpha_1 + \alpha_3} (2\alpha_1 - \alpha_2 - c^2 \eta^2 \alpha_1^2 \alpha_2) + \frac{\lambda^{a_1} \beta_1}{\alpha_1 + \alpha_3} (\alpha_1 - \alpha_2 - \alpha_3 + c^2 \eta^2 \alpha_1 \alpha_2 \alpha_3) + \lambda^{a_2+b_3} \beta_2 \gamma_3 &= 0, \end{aligned} \quad (3.77)$$

Case 2: ($\alpha_1 + \alpha_3 = 0$)

In this case, solving the system (3.76) is straightforward. Using equation (3.75) for $\lambda^{b_1} \gamma_1 = \lambda^{a_1} \beta_1$ we have

$$\begin{aligned} 0 &= -2\alpha_2 \lambda^{b_1} \gamma_1 (1 - c^2 \eta^2 \alpha_1^2) + \lambda^{a_2+b_3} \beta_2 \gamma_3 (2\alpha_1 + \alpha_2 + c^2 \eta^2 \alpha_1^2 \alpha_2) \\ &\quad - \lambda^{a_3+b_2} \beta_3 \gamma_2 (2\alpha_1 - \alpha_2 - c^2 \eta^2 \alpha_1^2 \alpha_2). \end{aligned} \quad (3.78)$$

Turning our attention to the grading two component we have

$$\text{flat}^{(2)} = \lambda^{b_2} \gamma_2 \bar{\nabla} J^{(2)} - \lambda^{a_2} \beta_2 \nabla \bar{J}^{(2)} + \lambda^{a_1+b_1} \beta_1 \gamma_1 [\bar{J}^{(1)}, J^{(1)}] + \lambda^{a_3+b_3} \beta_3 \gamma_3 [\bar{J}^{(3)}, J^{(3)}], \quad (3.79)$$

again in order to get useful constraints from the equation above we need to make use of the following convenient combinations

$$\begin{aligned} \varepsilon^{(2)} + \alpha_2 \mathcal{Z}^{(2)} &= 2\alpha_2 \bar{\nabla} J^{(2)} + (\alpha_3 - \alpha_1 + \alpha_2 (1 - c^2 \eta^2 \alpha_1 \alpha_3)) [\bar{J}^{(1)}, J^{(1)}] \\ &\quad + (\alpha_1 - \alpha_3 + \alpha_2 (1 - c^2 \eta^2 \alpha_1 \alpha_3)) [\bar{J}^{(3)}, J^{(3)}], \\ \varepsilon^{(2)} - \alpha_2 \mathcal{Z}^{(2)} &= 2\alpha_2 \bar{\nabla} J^{(2)} + (\alpha_3 - \alpha_1 - \alpha_2 (1 - c^2 \eta^2 \alpha_1 \alpha_3)) [\bar{J}^{(1)}, J^{(1)}] \\ &\quad + (\alpha_1 - \alpha_3 - \alpha_2 (1 - c^2 \eta^2 \alpha_1 \alpha_3)) [\bar{J}^{(3)}, J^{(3)}], \end{aligned} \quad (3.80)$$

solving the system of equations (3.80) for $\bar{\nabla} J^{(2)}$ and $\nabla \bar{J}^{(2)}$ and then using (3.79) it is easy to obtain the following constraints

$$\begin{aligned} 0 &= -\frac{\lambda^{b_2}\gamma_2}{2\alpha_2}(-\alpha_1 + \alpha_2 + \alpha_3 - c^2\eta^2\alpha_1\alpha_2\alpha_3) + \lambda^{a_1+b_1}\beta_1\gamma_1 \\ &\quad + \frac{\lambda^{a_2}\beta_2}{2\alpha_2}(-\alpha_1 - \alpha_2 + \alpha_3 + c^2\eta^2\alpha_1\alpha_2\alpha_3), \end{aligned} \quad (3.81)$$

$$\begin{aligned} 0 &= -\frac{\lambda^{b_2}\gamma_2}{2\alpha_2}(\alpha_1 + \alpha_2 - \alpha_3 - c^2\eta^2\alpha_1\alpha_2\alpha_3) + \lambda^{a_3+b_3}\beta_3\gamma_3 \\ &\quad + \frac{\lambda^{a_2}\beta_2}{2\alpha_2}(\alpha_1 - \alpha_2 - \alpha_3 + c^2\eta^2\alpha_1\alpha_2\alpha_3). \end{aligned} \quad (3.82)$$

At last, let us turn our attention to the grading three component

$$\text{flat}^{(3)} = \lambda^{b_3}\gamma_3\bar{\nabla} J^{(3)} - \lambda^{a_3}\beta_3\nabla \bar{J}^{(3)} + \lambda^{a_1+b_2}\beta_1\gamma_2[\bar{J}^{(1)}, J^{(2)}] + \lambda^{a_2+b_1}\beta_2\gamma_1[\bar{J}^{(2)}, J^{(1)}], \quad (3.83)$$

once again in order to get useful constraints from the equation above we need to make use of the following convenient combinations

$$\begin{aligned} \varepsilon^{(3)} + \alpha_3\mathcal{Z}^{(3)} &= (\alpha_1 + \alpha_3)\bar{\nabla} J^{(3)} + (2\alpha_3 - \alpha_2)[\bar{J}^{(2)}, J^{(1)}] + (\alpha_2 - \alpha_1 + \alpha_3)[\bar{J}^{(1)}, J^{(2)}], \\ \varepsilon^{(3)} - \alpha_1\mathcal{Z}^{(3)} &= (\alpha_3 + \alpha_1)\bar{\nabla} J^{(3)} + (\alpha_3 - \alpha_2 - \alpha_1)[\bar{J}^{(2)}, J^{(1)}] + (\alpha_2 - 2\alpha_1)[\bar{J}^{(1)}, J^{(2)}], \end{aligned} \quad (3.84)$$

just as we did for the grading one component, we will consider two cases separately.

Case 1: $(\alpha_1 + \alpha_3 \neq 0)$

Solving equations (3.84) and using equation (3.83) we get the following constraints

$$\begin{aligned} -\frac{\lambda^{b_3}\gamma_3}{\alpha_1 + \alpha_3}(-\alpha_2 + 2\alpha_3 - c^2\eta^2\alpha_2\alpha_3^2) + \frac{\lambda^{a_3}\beta_3}{\alpha_1 + \alpha_3}(-\alpha_1 - \alpha_2 + \alpha_3 + c^2\eta^2\alpha_1\alpha_2\alpha_3) + \lambda^{a_2+b_1}\beta_2\gamma_1 &= 0, \\ -\frac{\lambda^{b_3}\gamma_3}{\alpha_1 + \alpha_3}(-\alpha_1 + \alpha_2 + \alpha_3 - c^2\eta^2\alpha_1\alpha_2\alpha_3) + \frac{\lambda^{a_3}\beta_3}{\alpha_1 + \alpha_3}(-2\alpha_1 + \alpha_2 + c^2\eta^2\alpha_1^2\alpha_2) + \lambda^{a_1+b_2}\beta_1\gamma_2 &= 0, \end{aligned} \quad (3.85)$$

Case 2: $(\alpha_1 + \alpha_3 = 0)$

In this case, solving the system (3.84) is straightforward. Using equation (3.83) for $\lambda^{b_3}\gamma_3 = \lambda^{a_3}\beta_3$ we have

$$\begin{aligned} 0 &= -2\lambda^{b_3}\gamma_3(\alpha_2 - c^2\eta^2\alpha_1^2\alpha_2) + \lambda^{a_1+b_2}\beta_1\gamma_2(2\alpha_1 + \alpha_2 + c^2\eta^2\alpha_1^2\alpha_2) \\ &\quad + \lambda^{a_2+b_1}\beta_2\gamma_1(-2\alpha_1 + \alpha_2 + c^2\eta^2\alpha_1^2\alpha_2). \end{aligned} \quad (3.86)$$

In summary, we have a big system of equations (3.74, 3.77, 3.78, 3.81, 3.82, 3.85, 3.86) involving the parameters α, a, b, β and γ . Of course we recognize two different cases, one version of the system of equations for $\alpha_1 + \alpha_3 \neq 0$ and one for $\alpha_1 + \alpha_3 = 0$. The solution is quite straightforward and it only involves basic algebra. For this reason we only give the results.

For $\alpha_1 + \alpha_3 = 0$ we get the following Green-Schwarz d projector

$$d = \alpha_1 P_1 + \frac{2\alpha_1}{1 + c^2\eta^2\alpha_1^2} P_2 - \alpha_1 P_3, \quad (3.87)$$

and the Lax Pair takes the following form

$$\begin{aligned} \bar{\mathfrak{L}}(\lambda) &= \bar{J}^{(0)} + \lambda^{a_1} \beta_1 \bar{J}^{(1)} + \frac{\lambda^{2a_1} \beta_1^2}{1 + c^2\eta^2\alpha_1^2} \bar{J}^{(2)} + \frac{\lambda^{-a_1} (1 - c^2\eta^2\alpha_1^2)}{\beta_1} \bar{J}^{(3)}, \\ \mathfrak{L}(\lambda) &= J^{(0)} + \lambda^{a_1} \beta_1 J^{(1)} + \frac{\lambda^{-2a_1} (1 - c^2\eta^2\alpha_1^2)^2}{\beta_1^2 (1 + c^2\eta^2\alpha_1^2)} J^{(2)} + \frac{\lambda^{-a_1} (1 - c^2\eta^2\alpha_1^2)}{\beta_1} J^{(3)}. \end{aligned} \quad (3.88)$$

Provided that we make the choice ($\alpha_1 = a_1 = \beta_1 = 1$), the Lax Pair above correctly reduces to the BPR Lax pair [2] in the non-deformed limit ($\eta = 0$). What is more, after rescaling the spectral parameter as $\lambda \rightarrow \sqrt{1 + \eta^2} \lambda$, the Lax pair above exactly reproduces the η -deformed Lax pair presented in [15] given by

$$\bar{\mathfrak{L}}(\lambda) = \bar{J}^{(0)} + \lambda \sqrt{1 + \eta^2} \bar{J}^{(1)} + \frac{\lambda^2 (1 + \eta^2)}{1 - \eta^2} \bar{J}^{(2)} + \lambda^{-1} \sqrt{1 + \eta^2} \bar{J}^{(3)}, \quad (3.89)$$

$$\mathfrak{L}(\lambda) = J^{(0)} + \lambda \sqrt{1 + \eta^2} J^{(1)} + \frac{\lambda^{-2} (1 + \eta^2)}{1 - \eta^2} J^{(2)} + \lambda^{-1} \sqrt{1 + \eta^2} J^{(3)}, \quad (3.90)$$

notice that we have also made $c = -1$ in the equation above, this is the condition to have non-split solutions of the mCYBE as explained in [68].

The Lax pair of Jordanian deformations [21] is also encoded in (3.88). Jordanian deformations arise when considering solutions of the CYBE, that is for $c = 0$.

Finally, let us examine the case when $\alpha_1 + \alpha_3 \neq 0$. In summary, we get the following projector

$$d = P_1 + \frac{2}{1 + c^2\eta^2\alpha_1^2} P_2 + \frac{3 + c^2\eta^2\alpha_1^2}{1 + 3c^2\eta^2\alpha_1^2} P_3. \quad (3.91)$$

This projector reduces to the hybrid sigma model projector in the non-deformed limit. The Lax pair takes the following form

$$\begin{aligned} \bar{\mathfrak{L}}(\lambda) &= \bar{J}^{(0)} + \lambda^{a_1} (1 + 3c^2\eta^2\alpha_1^2) \bar{J}^{(1)} + \frac{\lambda^{2a_1} (1 + 3c^2\eta^2\alpha_1^2)}{1 + c^2\eta^2\alpha_1^2} \bar{J}^{(2)} + \lambda^{3a_1} \bar{J}^{(3)}, \\ \mathfrak{L}(\lambda) &= J^{(0)} + \frac{\lambda^{-3a_1} (1 - c^2\eta^2\alpha_1^2)^3}{(1 + 3c^2\eta^2\alpha_1^2)^2} J^{(1)} + \frac{\lambda^{-2a_1} (1 - c^2\eta^2\alpha_1^2)^2}{(1 + 3c^2\eta^2\alpha_1^2)(1 + c^2\eta^2\alpha_1^2)} J^{(2)} + \frac{\lambda^{-a_1} (1 - c^2\eta^2\alpha_1^2)}{(1 + 3c^2\eta^2\alpha_1^2)} J^{(3)}. \end{aligned} \quad (3.92)$$

The Lax pair shown above reduces to a very simple expression for Jordanian deformations of the hybrid sigma model ($c = 0$) which as expected coincides with the expected Lax pair for the undeformed model found in [69]. It is given by

$$\bar{\mathfrak{L}}(\lambda) = \bar{J}^{(0)} + \lambda \bar{J}^{(1)} + \lambda^2 \bar{J}^{(2)} + \lambda^3 \bar{J}^{(3)}, \quad (3.93)$$

$$\mathfrak{L}(\lambda) = J^{(0)} + \lambda^{-3} J^{(1)} + \lambda^{-2} J^{(2)} + \lambda^{-1} J^{(3)}. \quad (3.94)$$

To summarize our results, we have shown that there are only two possible integrable deformations (Jordanian) of a semi-symmetric sigma model, the Green-Schwarz sigma model and the hybrid sigma model. We can write their Lagrangian in the following compact way

$$L = \frac{1}{2} \text{str} \left[A d \circ \frac{1}{1 - \eta R_g \circ d} \bar{A} \right], \quad (3.95)$$

where

$$d = \begin{cases} -P^{(1)} + 2P^{(2)} + P^{(3)} & \text{for Green Schwarz,} \\ 3P^{(1)} + 2P^{(2)} + P^{(3)} & \text{for Hybrid.} \end{cases} \quad (3.96)$$

In the next chapter we will give a prescription to extract bosonic fields from η -deformed models. In particular we will see how the gravity dual of β -deformed theories can be obtained using Yang-Baxter deformations, that is, we will offer examples of a fundamental correspondence between TsT transformations and solutions to the classical Yang-Baxter equation.

Chapter 4

Super coset construction of η -deformed sigma models

In this final chapter we offer various examples of how to compute the background fields arising from the deformed superstring Lagrangian. In the first part we explicitly compute the background fields of $(AdS_2 \times S^2)_\eta$ by closely following the method described in [18]. These fields were previously obtained by consistent truncation in [17]. In the final part we present some examples of integrable deformations that make use of r -matrices which are solutions of the CYBE, the Lunin-Maldacena computation from [22] is reproduced in detail and we present a construction of β -deformations of $AdS_4 \times \mathbb{CP}^3$ from the CYBE [37].

4.1 Coset construction of $(AdS_2 \times S^2)_\eta$

The purpose of this section is to set up the procedure to extract the background fields arising from the deformed Lagrangian (3.46). For the sake of simplicity, we will only consider bosonic deformations, that is, we will switch off the fermions¹. In this way, the procedure to find the deformed metric and antisymmetric (B) field is described in detail.

¹Obtaining the deformed R-R fluxes is straightforward but the computations are a bit lengthy as shown in [18]. Since the objective of this chapter is to reformulate some TsT backgrounds as Yang-Baxter deformations, fermions will be ignored.

4.1.1 Coset parametrization for $AdS_2 \times S^2$

Our starting point is the η -deformed Lagrangian presented in chapter 3

$$L_\eta = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{str} \left(A_\alpha d \circ \frac{1}{1 - \eta R_g \circ d} A_\beta \right). \quad (4.1)$$

As we mentioned in chapter 3, this Lagrangian describes the η -deformations of a sigma model formulated in a supercoset G/H possessing $\mathbb{Z}_4$ symmetry, that is, there exists a fourth-order automorphism Ω such that the lie algebra $\mathfrak{g} = \text{Lie}(G)$ admits, as a vector space, the following decomposition

$$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(3)}, \quad (4.2)$$

where $\mathfrak{g}^{(0)}$ is isomorphic to $\text{Lie}(H)$.

There are quite a few examples of sigma models that are described by cosets with these kind of symmetry. The main example of course being type IIB superstrings moving in $AdS_5 \times S^5$ which can be described using the supercoset $PSU(2, 2|4)/SO(1, 4) \times SO(5)$. Even though the original deformations [15],[68] were constructed for this space, the same results apply to similar sigma models such as $AdS_2 \times S^2$ described by the supercoset $PSU(1, 1|2)/SO(1, 1) \times SO(2)$, also $AdS_3 \times S^3$ described by the coset $SO(2, 2) \times SO(4)/SO(2, 1) \times SO(3)$ and finally type IIA superstrings moving in $AdS_4 \times \mathbb{CP}^3$ described by the supercoset $OSp(4|6)/SO(3, 1) \times U(3)$. Taking this into account, all the results that we will obtain in this section apply for any deformed semi-symmetric space sigma model². Of course the difference between each of the cases remains in the selection of the r -matrix and the coset parametrization being used.

Let us start by considering η -deformations of the $AdS_2 \times S^2$ sigma model. We choose to write the coset representative $g \in SO(1, 2) \times SO(3)/SO(1, 1) \times SO(2)$ as follows

$$g = g_b g_f; \quad g_f = \exp \chi \quad \chi = Q^I \theta_I, \quad (4.3)$$

where Q^I denotes the odd generators and θ_I parametrize the fermionic coordinates. The purely bosonic element g_b is given by

$$g_b = \begin{pmatrix} g_a & \mathbf{0}_{2 \times 2} \\ \mathbf{0}_{2 \times 2} & g_s \end{pmatrix}, \quad (4.4)$$

²Actually, as we will see later, the case of $AdS_4 \times \mathbb{CP}^3$ is a bit subtle and requires some modifications. We will discuss that in the last section.

here g_a parametrizes the AdS coordinates and g_s parametrizes the sphere. The explicit form of these matrices is given below

$$g_a = e^{\frac{1}{2}it\sigma_3} \begin{pmatrix} \rho_+ & i\rho_- \\ -i\rho_- & \rho_+ \end{pmatrix}, \quad g_s = e^{\frac{1}{2}i\varphi\sigma_3} \begin{pmatrix} r_+ & r_- \\ -r_- & r_+ \end{pmatrix}, \quad (4.5)$$

where $\rho_\pm$ and $r_\pm$ are defined in function of the AdS and sphere coordinates ρ and r respectively as follows

$$\rho_\pm = \frac{1}{\sqrt{2}} \sqrt{\sqrt{\rho^2 + 1} \pm 1}, \quad r_\pm = \frac{1}{\sqrt{2}} \sqrt{1 \pm \sqrt{1 - r^2}}. \quad (4.6)$$

It is easy to show that this parametrization correctly reproduces the usual $AdS_2 \times S^2$ metric and a vanishing antisymmetric field

$$ds^2 = -(1 + \rho^2) dt^2 + \frac{d\rho^2}{(1 + \rho^2)} + (1 - r^2) d\varphi^2 + \frac{dr^2}{(1 - r^2)}, \quad B = 0. \quad (4.7)$$

4.1.2 Expanding the Maurer-Cartan current

The strategy to find the explicit structure of the background fields contained in (4.1) is to expand the left invariant Maurer-Cartan current in terms of the Lie algebra basis in the following way

$$\begin{aligned} A_\alpha = g^{-1} \partial_\alpha g &= e_\alpha^m P_m + \frac{1}{2} \omega_\alpha^{mn} J_{mn} - Q^J D^{IJ} \theta_J \\ &\quad - \frac{i}{2} \hat{\theta}_I \gamma^m D^{IJ} P_m - \frac{1}{4} \epsilon^{IJ} \bar{\theta}_I \gamma^{mn} J_{mn} D^{JK} \theta_K. \end{aligned} \quad (4.8)$$

Here e_α^m and ω_α^{mn} denote the vielbeins and spin connections of $AdS_2 \times S^2$ respectively and the generators P_m, J_{mn} and Q_I are listed in Appendix C. Since we are interested in extracting the metric and NS-NS flux only, from now on we will switch off the fermions ($g_f = 1$), thus the bosonic Maurer-Cartan current is only given by

$$A_\alpha^b = g_b^{-1} \partial_\alpha g_b = e_\alpha^m P_m + \frac{1}{2} \omega_\alpha^{mn} J_{mn}, \quad (4.9)$$

The action of the projector $d = P_1 + \frac{2P_2}{1-\eta^2} - P_3$ on each of the bosonic generators (P_m, J_{mn}) can easily be computed as follows

$$\begin{aligned} d(J_{mn}) &= 0, \\ d(P_m) &= \frac{2}{1 - \eta^2} P_m. \end{aligned} \quad (4.10)$$

We now need to find the action of the linear operator R_g in each of the generators and replace all of this information in (4.1). We do exactly that in the following section.

4.1.3 Deformation matrix

The R -matrix that we will use in this part of our work is the one described in Appendix B as Drinfeld-Jimbo R -matrix. Thus R is defined as an endomorphism that solves the modified classical Yang-Baxter equation. By construction, this operator multiplies by $-i$, generators associated with positive roots, $+i$ on generators associated with negative roots, and 0 on Cartan generators [15]. The action of R on a matrix M can be summarized as follows

$$R(M)_{ij} = -i \varepsilon_{ij} M_{ij}, \quad \varepsilon_{ij} = \begin{cases} 1 & \text{if } i < j \\ 0 & \text{if } i = j \\ -1 & \text{if } i > j \end{cases}. \quad (4.11)$$

We now consider the action of the operator $R_{g_b} = \text{Ad}_{g_b^{-1}} \circ R \circ \text{Ad}_{g_b}$ on each of the generators. Schematically, this operator acts as follows

$$R_{g_b}(P_m) = \lambda_m^n P_n + \frac{1}{2} \lambda_m^{np} J_{np}, \quad (4.12)$$

$$R_{g_b}(J_{mn}) = \lambda_{mn}^p P_p + \frac{1}{2} \lambda_{mn}^{pq} J_{pq}. \quad (4.13)$$

All of the coefficients shown in the equation above can be obtained by direct computation using a convenient coset parametrization and a matrix representation for the generators. For $AdS_2 \times S^2$ we use the basis shown in Appendix C and the coset representative defined in (4.4). We list all the non-vanishing elements below

$$\begin{aligned} \lambda_0^1 &= \lambda_1^0 = \rho, & \lambda_2^3 &= -\lambda_3^2 = -r, \\ \lambda_1^{01} &= 2\sqrt{1+\rho^2}, & \lambda_3^{23} &= -2\sqrt{1-r^2}, \\ \lambda_{01}^1 &= -\sqrt{1+\rho^2}, & \lambda_{23}^3 &= \sqrt{1-r^2}, \end{aligned} \quad (4.14)$$

Of course, the operator $\mathcal{O}$, which is the one that modifies the Lagrangian, is related to the inverse of this coefficients as follows

$$\mathcal{O}^{-1} = \frac{1}{1 - \frac{2\eta}{1-\eta^2} R_{g_b}}. \quad (4.15)$$

Schematically, this operator acts on each of the Lie algebra generators as follows

$$\mathcal{O}^{-1}(P_m) = k_m^n P_n + w_m^{np} J_{np}. \quad (4.16)$$

Notice that we are ignoring the action of $\mathcal{O}^{-1}$ on the zeroth-graded generators J_{mn} because they will be projected out during the computation so they do not contribute to the

Lagrangian. Furthermore, combining the action of R_{g_b} and the action of $\mathcal{O}^{-1}$ together with the definition (4.15), the following equation can be deduced

$$k_m^n \left(\delta_n^p - \frac{2\eta}{1-\eta^2} \lambda_n^p \right) P_p = P_m, \quad (4.17)$$

in matrix notation this equation can be written as follows

$$k = \left(I - \frac{2\eta}{1-\eta^2} \lambda \right)^{-1}. \quad (4.18)$$

For the $AdS_2 \times S^2$ space we show the non-vanishing elements of the deformation matrix k in the following list

$$\begin{aligned} k_0^0 = k_1^1 &= \frac{1}{1 - \varkappa^2 \rho^2}, & k_2^2 = k_3^3 &= \frac{1}{1 + \varkappa^2 r^2}, \\ k_0^1 = k_1^0 &= \frac{\varkappa \rho}{1 - \varkappa^2 \rho^2}, & k_2^3 = -k_3^2 &= \frac{-\varkappa r}{1 + \varkappa^2 r^2}. \end{aligned} \quad (4.19)$$

4.1.4 Metric and B-field for $(AdS_2 \times S^2)_\eta$

In order to extract the metric, let us make use of equations (4.9) and (4.19) in the η -deformed Lagrangian. We get the following result

$$\begin{aligned} L_{def} &= -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{str} \left[\left(e_\alpha^m P_m + \frac{1}{2} \omega_\alpha^{mn} J_{mn} \right) \mathcal{O}^{-1} \left(e_\beta^p P_p + \frac{1}{2} \omega_\beta^{pq} J_{pq} \right) \right] \\ &= -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) e_\alpha^m e_\beta^m \text{str} (P_m \mathcal{O}^{-1} P_p) \\ &= -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) e_\alpha^m e_\beta^m k_{pm} \\ &= -\frac{1}{2} [\gamma^{\alpha\beta} e_M^m e_N^n k_{(mn)} \partial_\alpha X^M \partial_\beta X^N - \epsilon_{\alpha\beta} e_M^m e_N^n k_{[mn]} \partial_\alpha X^M \partial_\beta X^N]. \end{aligned} \quad (4.20)$$

Here, in the last line, we have made use of the usual relations for the vielbeins $e_\alpha^m = e_M^m \partial_\alpha X^M$ and X^M represent the collection of target space variables (ρ, t, r, φ) . Furthermore, we have separated the Lagrangian using the symmetrized matrix $k_{(mn)}$ and anti-symmetrized matrix $k_{[mn]}$. These two terms are related to the metric and NS-NS

field respectively which can be extracted with ease. They are given by

$$d s^2_{(AdS_2)_\eta} = -\frac{(1+\rho^2)}{1-\kappa^2\rho^2} dt^2 + \frac{d\rho^2}{(1+\rho^2)(1-\kappa^2\rho^2)}, \quad (4.21)$$

$$d s^2_{(S^2)_\eta} = \frac{(1-r^2)}{1+\kappa^2r^2} d\varphi^2 + \frac{dr^2}{(1-r^2)(1+\kappa^2r^2)}, \quad (4.22)$$

$$B_{(AdS_2)_\eta} = \left(\frac{\kappa\rho}{1-\kappa^2\rho^2} \right) dt \wedge d\rho, \quad (4.23)$$

$$B_{(S^2)_\eta} = -\left(\frac{\kappa r}{1+\kappa^2r^2} \right) d\varphi \wedge dr. \quad (4.24)$$

Here we have conveniently redefined our deformation parameter as

$$\kappa = \frac{2\eta}{1-\eta^2}. \quad (4.25)$$

As expected, the η -deformed metric and antisymmetric fields reduce to the non-deformed ones in the non-deformed limit $\kappa \rightarrow 0$. We should also point out here that the contributions of the NS-NS field components $B_{t\rho}$ and $B_{\varphi r}$ to the lagrangian turn out to be total derivatives, and hence can be ignored without loss of generality³.

4.2 Lunin-Maldacena backgrounds from the CYBE

In this section we will review, as a preparation for the final section, the construction of γ -deformations of $AdS_5 \times S^5$ from the classical Yang-Baxter equation. As we did in the $AdS_2 \times S^2$ case, for simplicity, we will switch off the fermions because we are only interested in computing the metric and NS-NS antisymmetric field.

4.2.1 A parametrization for S^5

This section is concerned with deformations of the 5-sphere only so we will leave the AdS part of the Lagrangian completely untouched by the r -matrix. This kind of separation can only be done if dealing with solutions of the classical Yang-Baxter equation. Solutions of the modified classical Yang-Baxter equation (as the one we used in the previous section) do not admit this kind of separation [38].

³This observation is not relevant for us but it is important when trying to study classical string solutions in $(AdS_2 \times S^2)_\eta$. Check for instance [70],[71]

As we have discussed in chapter 2, the 5-sphere can be described as a symmetric space by the following coset

$$S^5 = \frac{SO(6)}{SO(5)}. \quad (4.26)$$

A convenient basis to describe this coset can be obtained by noting that the basis describing $\mathfrak{so}(6)$, given by

$$\mathfrak{so}(6) \cong \mathfrak{su}(4) = \text{span}_{\mathbb{R}}\{J_{mn}, J_{m6}\}; \quad m = 1, \dots, 6, \quad (4.27)$$

contains a basis of $\mathfrak{so}(5)$ as a sub algebra. That is, the elements

$$J_{mn} = \frac{1}{4} [\gamma_m, \gamma_n]; \quad (m, n = 1, \dots, 5), \quad (4.28)$$

generate $\mathfrak{so}(5)$. The γ matrices used in the equations above are defined in Appendix C and J_{m6} are defined as follows

$$J_{m6} = -J_{6m} = \frac{i}{2} \gamma_m. \quad (4.29)$$

In summary. the coset space described in (4.26) is spanned by a basis of γ_m matrices as shown below

$$\frac{\mathfrak{su}(4)}{\mathfrak{so}(5)} = \text{span}_{\mathbb{R}} \left\{ \frac{i}{2} \gamma_m \right\}; \quad m = 1, \dots, 5. \quad (4.30)$$

A convenient parametrization for a generic coset element $g \in SU(4)/SO(5)$ can be chosen in the following way

$$g = \Lambda(\phi_1, \phi_2, \phi_3) \Xi(\xi) \hat{g}_r(r), \quad (4.31)$$

where the Λ matrix is defined in terms of the Cartan generators (h_1, h_2, h_3) as follows

$$\Lambda(\phi_1, \phi_2, \phi_3) = \exp \left[\frac{i}{2} (\phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3) \right],$$

In this representation, the Cartan generators are given by diagonal matrices that can be written in the following way

$$h_1 = \text{diag}(-1, 1, -1, 1), \quad h_2 = \text{diag}(-1, 1, 1, -1), \quad h_3 = \text{diag}(1, 1, -1, -1). \quad (4.32)$$

The other two matrices appearing in the coset representative are given by

$$\Xi(\xi) = \begin{pmatrix} \cos \frac{\xi}{2} & \sin \frac{\xi}{2} & 0 & 0 \\ -\sin \frac{\xi}{2} & \cos \frac{\xi}{2} & 0 & 0 \\ 0 & 0 & \cos \frac{\xi}{2} & -\sin \frac{\xi}{2} \\ 0 & 0 & \sin \frac{\xi}{2} & \cos \frac{\xi}{2} \end{pmatrix}, \quad \hat{g}_r(r) = \begin{pmatrix} \cos \frac{r}{2} & 0 & 0 & i \sin \frac{r}{2} \\ 0 & \cos \frac{r}{2} & -i \sin \frac{r}{2} & 0 \\ 0 & -i \sin \frac{r}{2} & \cos \frac{r}{2} & 0 \\ i \sin \frac{r}{2} & 0 & 0 & \cos \frac{r}{2} \end{pmatrix}, \quad (4.33)$$

These are nothing but exponentiations of convenient γ matrices. It is easy to show that using the Maurer-Cartan one form, $A = g^{-1}dg$, the coset representative presented above exactly reproduces the S^5 metric in global coordinates.

4.2.2 Abelian r -matrices

As we have seen in chapter 3, the most important element of this η -deformations is the linear operator R satisfying the modified classical Yang-Baxter equation. In the last section we used a very simple kind of R -matrix called Drinfeld-Jimbo R -matrix. These kind of matrices solve the mCYBE and were used to compute the $(AdS_5 \times S^5)_\eta$ background fields in [18].

In this section we consider η -deformations that make use of solutions of the CYBE rather than the mCYBE. The deformed Lagrangian is basically the same and can be written as follows [20]

$$L = -\frac{1}{4}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{str} \left(A_\alpha d \circ \frac{1}{1 - \eta R_g \circ d} A_\beta \right). \quad (4.34)$$

As usual, the Maurer-Cartan currents are defined as

$$A_\alpha = g^{-1} \partial_\alpha g, \quad (4.35)$$

The main difference with the η -deformations described in Chapter 3 is that the operator d does not get deformed, so it just take the following form

$$d = P_1 + 2P_2 - P_3, \quad (4.36)$$

where as usual, the operators P_i ($i = 0, 1, 2, 3$) project to the corresponding $\mathbb{Z}_4$ -gradations.

In contrast to our previous section, where we made use of Drinfeld-Jimbo r -matrices, here we will define the R operator by using the following construction

$$r_{Abe} = \sum_{i \neq j} \mu_{ij} h_i \wedge h_j, \quad (4.37)$$

where h_i are Cartan generators and $\mu_{ij} = -\mu_{ji}$ are arbitrary parameters. We refer to the r -matrices of this type as Abelian r -matrices. Since they are constructed out of Cartan generators they trivially satisfy the CYBE.

4.2.3 One-parameter deformation

The simplest example we can think of for an Abelian r -matrix is given by the following combination of Cartan generators

$$r_{Abe}^\mu = \mu h_1 \wedge h_2. \quad (4.38)$$

Here μ is an arbitrary parameter and the Cartan generators are given in (4.32). The action of the associated R -operator on a generic Lie algebra element X can be easily written as

$$R_{Abe}^\mu(X) = \mu [h_1 \text{tr}(h_2 X) - h_2 \text{tr}(h_1 X)]. \quad (4.39)$$

As we saw in our previous section for $AdS_2 \times S^2$, the deformation matrix λ_m^n can easily be computed once the action of the R operator and a coset representative are known. Using the definition

$$R_g = \text{Ad}_{g^{-1}} \circ R_{Abe}^\mu \circ \text{Ad}_g, \quad (4.40)$$

together with the coset representative g given in (4.31), we can compute the only non-vanishing elements of λ_m^n . They are given by

$$\lambda_2^4 = -\lambda_4^2 = -2\mu \sin^2 r \sin 2\xi. \quad (4.41)$$

Now, using the equation (4.18) to invert this matrix, we can compute the non-vanishing elements of k_m^n . We list them below

$$\begin{aligned} k_2^2 &= \frac{1}{1 + 16\mu^2 \eta^2 \sin^4 r \sin^2 2\xi}, & k_2^4 &= -\frac{4\mu\eta \sin^2 r \sin 2\xi}{1 + 16\mu^2 \eta^2 \sin^4 r \sin^2 2\xi}, \\ k_4^4 &= \frac{1}{1 + 16\mu^2 \eta^2 \sin^4 r \sin^2 2\xi}, & k_4^2 &= \frac{4\mu\eta \sin^2 r \sin 2\xi}{1 + 16\mu^2 \eta^2 \sin^4 r \sin^2 2\xi}, \\ k_1^1 &= k_3^3 = k_5^5 = 1. \end{aligned} \quad (4.42)$$

The symmetric part L_G and antisymmetric part L_B of the deformed Lagrangian can easily be obtained by using the vielbeins of S^5 and the equation (4.20). They are given by

$$\begin{aligned} L_G &= -\frac{\gamma^{\alpha\beta}}{2} \left[\partial_\alpha r \partial_\beta r + \sin^2 r \partial_\alpha \xi \partial_\beta \xi + \cos^2 r \partial_\alpha \phi_3 \partial_\beta \phi_3 + \right. \\ &\quad \left. + \frac{\sin^2 r}{1 + 16\mu^2 \eta^2 \sin^4 r \sin^2 2\xi} (\cos^2 \xi \partial_\alpha \phi_1 \partial_\beta \phi_1 + \sin^2 \xi \partial_\alpha \phi_2 \partial_\beta \phi_2) \right], \end{aligned} \quad (4.43)$$

$$L_B = \frac{2\eta\mu \sin^4 r \sin^2 2\xi}{1 + 16\mu^2 \eta^2 \sin^4 r \sin^2 2\xi} \epsilon^{\alpha\beta} \partial_\alpha \phi_1 \partial_\beta \phi_2. \quad (4.44)$$

Provided that we impose the following relation

$$\mu = \frac{\hat{\gamma}_3}{8\eta}. \quad (4.45)$$

The metric and NS-NS form exactly agree with the results obtained via TsT transformations in [8]⁴. Thus the r -matrix (4.38) represent the algebraic origin of the TsT transformations used to obtain the one-parameter γ -deformations of S^5 .

4.2.4 Three-parameter deformation

The three-parameter deformation is obtained in the same way as the previous case. To reproduce the desired deformed fields, the following r -matrix needs to be used

$$r_{Abe}^{(\mu_1, \mu_2, \mu_3)} = \mu_1 h_2 \wedge h_3 + \mu_2 h_3 \wedge h_1 + \mu_3 h_1 \wedge h_2. \quad (4.46)$$

Here μ_i are arbitrary parameters and h_i are Cartan generators given in (4.32). The computation is completely parallel to the previous case so let us present the non-vanishing components of the matrix λ_m^n , they are given by

$$\begin{aligned} \lambda_4^2 &= -\lambda_2^4 &= 2 \sin^2 r \sin 2\xi \mu_3, \\ \lambda_5^2 &= -\lambda_2^5 &= 2 \cos \xi \sin 2r \mu_2, \\ \lambda_5^4 &= -\lambda_4^5 &= -2 \sin 2r \sin \xi \mu_1 \end{aligned} \quad (4.47)$$

These coefficients can be used to compute the non-vanishing components of k_m^n , we list them below

$$\begin{aligned} k_2^2 &= \mathcal{M} (1 + 16\eta^2 \sin^2 2r \sin^2 \xi \mu_1^2), \\ k_4^2 &= \mathcal{M} [4\eta \sin 2\xi (2\eta \sin^2 2r \mu_1 \mu_2 + \sin^2 r \mu_3)], \\ k_5^2 &= \mathcal{M} [4\eta \cos \xi \sin 2r (\mu_2 - 8\eta \sin^2 r \sin^2 \xi \mu_1 \mu_3)], \\ k_2^4 &= \mathcal{M} [4\eta \sin 2\xi (2\eta \sin^2 2r \mu_1 \mu_2 - \sin^2 r \mu_3)], \\ k_4^4 &= \mathcal{M} (1 + 16\eta^2 \cos^2 \xi \sin^2 2r \mu_2^2), \\ k_5^4 &= \mathcal{M} [4\eta \sin 2r \sin \xi (\mu_1 + 8\eta \cos^2 \xi \sin^2 r \mu_2 \mu_3)], \\ k_2^5 &= \mathcal{M} [4\eta \cos \xi \sin 2r (\mu_2 + 8\eta \sin^2 r \sin^2 \xi \mu_1 \mu_3)], \\ k_4^5 &= \mathcal{M} [4\eta \sin 2r \sin \xi (\mu_1 - 8\eta \cos^2 \xi \sin^2 r \mu_2 \mu_3)], \\ k_5^5 &= \mathcal{M} (1 + 16\eta^2 \sin^4 r \sin^2 2\xi \mu_3^2), \\ k_1^1 &= k_3^3 = 1, \end{aligned} \quad (4.48)$$

⁴In fact, to obtain the exact structure, we need to use global coordinates in the traditional γ -deformed metric shown in [72].

where we have conveniently defined the multiplicative factor $\mathcal{M}$ as

$$\mathcal{M}^{-1} = 1 + 16\eta^2 \sin^2 2r (\mu_1^2 \sin^2 \xi + \mu_2^2 \cos^2 \xi) + 16\mu_3^2 \eta^2 \sin^4 r \sin^2 2\xi. \quad (4.49)$$

Finally, we can make use of these coefficients to compute the symmetric and antisymmetric part of the deformed Lagrangian. Provided that we make the following identifications

$$\mu_1 = \frac{\hat{\gamma}_1}{8\eta}, \quad \mu_2 = \frac{\hat{\gamma}_2}{8\eta}, \quad \mu_3 = \frac{\hat{\gamma}_3}{8\eta}, \quad (4.50)$$

and after a very tedious but straightforward computation we get to the following result

$$\begin{aligned} L_G = & -\frac{\gamma^{\alpha\beta}}{2} \left[\sin^2 r \partial_\alpha r \partial_\beta r + (\cos r \sin \xi \partial_\alpha r + \sin r \cos \xi \partial_\alpha \xi)(\cos r \sin \xi \partial_\beta r + \sin r \cos \xi \partial_\beta \xi) \right. \\ & + (\cos r \cos \xi \partial_\alpha r - \sin r \sin \xi \partial_\alpha \xi)(\cos r \cos \xi \partial_\beta r - \sin r \sin \xi \partial_\beta \xi) \\ & + \mathcal{M} \left[\sin^2 r (\cos^2 \xi \partial_\alpha \phi_1 \partial_\beta \phi_1 + \sin^2 \xi \partial_\alpha \phi_2 \partial_\beta \phi_2) + \cos^2 r \partial_\alpha \phi_3 \partial_\beta \phi_3 \right. \\ & \left. \left. + \cos^2 r \sin^4 r \cos^2 \xi \sin^2 \xi \left(\sum_i \hat{\gamma}_i \partial_\alpha \phi_i \right) \left(\sum_j \hat{\gamma}_j \partial_\beta \phi_j \right) \right] \right], \end{aligned} \quad (4.51)$$

$$\begin{aligned} L_B = & \epsilon^{\alpha\beta} \mathcal{M} \left[\hat{\gamma}_3 \sin^4 r \sin^2 \xi \cos^2 \xi \partial_\alpha \phi_1 \partial_\beta \phi_2 \right. \\ & \left. + \sin^2 r \cos^2 r (\hat{\gamma}_1 \sin^2 \xi \partial_\alpha \phi_2 \partial_\beta \phi_3 + \hat{\gamma}_2 \cos^2 \xi \partial_\alpha \phi_3 \partial_\beta \phi_1) \right]. \end{aligned} \quad (4.52)$$

Once again these deformed fields are in complete agreement with the results found via TsT transformations in [8]. Furthermore, as it is easy to check, the one-parameter deformation is recovered by making the identification $\hat{\gamma}_1 = \hat{\gamma}_2 = 0$ and $\hat{\gamma}_3 = \hat{\gamma}$ as expected.

4.3 Coset construction of $\mathbb{CP}^3$

There are only two superstring sigma models with vanishing β -function in ten dimensional semi-symmetric spaces, $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ [73]. Like $AdS_5 \times S^5$, $AdS_4 \times \mathbb{CP}^3$ is classically integrable and admits a coset formulation [35, 74]. In this section we will briefly review some of its main properties.

The usual coset description of $AdS_4 \times \mathbb{CP}^3$ is given by

$$\frac{OSp(2, 2|6)}{SO(1, 3) \times U(3)}. \quad (4.53)$$

However, the $\mathbb{CP}^3$ metric that is obtained from this coset does not exactly match the standard Fubini-Study metric [36]. Furthermore, the parametrization using generators of

$SO(6)$ which are 6×6 matrices turns out to be very complicated making computations quite messy. The solution proposed in this section is to modify the coset describing the $\mathbb{CP}^3$ part of the sigma model to

$$\frac{SU(4) \times SU(2)}{U(3) \times SU(2)}. \quad (4.54)$$

This description is analogous to the one performed in the analysis of Yang-Baxter deformations of $T^{1,1}$ [38]. As we will show, it is possible to embed some of the generators of $\mathfrak{u}(3) \oplus \mathfrak{su}(2)$ into $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$, giving rise to an orthogonal basis that can be used to get the standard Fubini-Study metric of $\mathbb{CP}^3$.

4.3.1 Superstrings in $AdS_4 \times \mathbb{CP}^3$

Type IIA superstrings moving in $AdS_4 \times \mathbb{CP}^3$ can be described by a sigma model formulated in the coset (4.53). Its Lagrangian is a straightforward extension of the Metsaev-Tseytlin Lagrangian and it is given by [35]

$$\mathcal{L} = -\frac{R^2}{4\pi\alpha'} \text{str} \left[\gamma^{\alpha\beta} A_\alpha^{(2)} A_\beta^{(2)} - \epsilon^{\alpha\beta} A_\alpha^{(1)} A_\beta^{(3)} \right], \quad (4.55)$$

where α, β are world-sheet indices and the metric $\gamma^{\alpha\beta}$ and antisymmetric tensor $\epsilon^{\alpha\beta}$ are normalized as $\gamma^{\alpha\beta} = \text{diag}(-1, 1)$ and $\epsilon^{\tau\sigma} = 1$. The radius of $\mathbb{CP}^3$ is R and $A_\alpha = g^{-1} \partial_\alpha g$, with $g \in OSp(2, 2|6)$, is the usual Maurer-Cartan current.

One important and subtle aspect of this formulation is that the supercoset (4.53) only has 24 fermionic directions which is the number of supersymmetries preserved by the background. As we saw in chapter 2, in ten dimensions the Green-Schwarz superstring is built using two Majorana-Weyl fermions with a total of 32 degrees of freedom. Thus, the coset description of $AdS_4 \times \mathbb{CP}^3$ is missing 8 fermions and, therefore, lacks the correct amount of supersymmetry of usual GS superstrings. We can go around this problem arguing that the 8 missing fermions are part of the 16 fermionic degrees of freedom that are removed by κ -symmetry. In this way one can think of the coset (4.53) as a model with partially fixed κ -symmetry [35].

Let us consider the geometrical structure of $AdS_4 \times \mathbb{CP}^3$. The metric can be split in two terms

$$ds^2 = R^2 \left[\frac{1}{4} ds_{AdS_4}^2 + ds_{\mathbb{CP}^3}^2 \right], \quad (4.56)$$

with the factor $1/4$ being due to the fact that the radius of $\mathbb{CP}^3$ is twice the radius of AdS_4 . In global coordinates the metric for AdS_4 reads

$$ds_{AdS_4}^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (4.57)$$

where $-\infty < t < \infty$, $0 \leq \rho < \infty$, $0 \leq \theta < \pi$ and $0 \leq \varphi < 2\pi$. The standard Fubini-Study metric on $\mathbb{CP}^3$ is

$$ds_{\mathbb{CP}^3}^2 = d\xi^2 + \frac{1}{4} \cos^2 \xi (d\theta_1^2 + \sin^2 \theta_1 d\varphi_1^2) + \frac{1}{4} \sin^2 \xi (d\theta_2^2 + \sin^2 \theta_2 d\varphi_2^2) + \cos^2 \xi \sin^2 \xi \left(\frac{1}{2} \cos \theta_1 d\varphi_1 - \frac{1}{2} \cos \theta_2 d\varphi_2 + d\psi \right)^2, \quad (4.58)$$

where (θ_1, φ_1) and (θ_2, φ_2) parametrize two spheres, the angle $0 \leq \xi < \pi/2$ determines their radii, and the angle $0 \leq \psi < 2\pi$ corresponds to a $U(1)$ isometry.

4.3.2 Supercoset construction of $\mathbb{CP}^3$

As we have discussed earlier, the usual coset representation of $\mathbb{CP}^3$ is not convenient to formulate a Yang-Baxter sigma model. Instead let us consider the following coset

$$\mathbb{CP}^3 = \frac{SU(4) \times SU(2)}{U(3) \times SU(2)}, \quad (4.59)$$

The basis of $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$ that we will consider is composed of $\mathfrak{su}(4)$ generators denoted by $\mathbf{L}_m$ ($m = 1, \dots, 15$) and $\mathfrak{su}(2)$ generators denoted by $\mathbf{M}_a$ ($a = 1, 2, 3$). As we have discussed, the usual construction of $\mathbb{CP}^3$ as a coset involves a representation of the generators as 6×6 bosonic matrices. Instead of that we will choose $(4|2) \times (4|2)$ supermatrices in the fundamental representation of $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$ ⁵

$$\mathbf{L}_m = -\frac{i}{2} \left(\begin{array}{c|c} \lambda_m & \mathbf{0}_{4 \times 2} \\ \hline \mathbf{0}_{2 \times 4} & \mathbf{0}_{2 \times 2} \end{array} \right), \quad \mathbf{M}_a = -\frac{i}{2} \left(\begin{array}{c|c} \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 2} \\ \hline \mathbf{0}_{2 \times 4} & \sigma_a \end{array} \right), \quad (4.60)$$

where λ_m are the conventional 4×4 $\mathfrak{su}(4)$ generators generalizing the Gell-Mann matrices of $\mathfrak{su}(3)$ ⁶, while σ_a are the usual 2×2 Pauli matrices.

The commutation relations and the supertraces can be deduced from the usual $\mathfrak{su}(4)$ and $\mathfrak{su}(2)$ counterparts and they are given by

$$[\mathbf{L}_m, \mathbf{L}_n] = f_{mn}^{p} \mathbf{L}_p, \quad [\mathbf{M}_a, \mathbf{M}_b] = \epsilon_{ab}^{c} \mathbf{M}_c, \quad (4.61)$$

$$\text{str}(\mathbf{L}_m \mathbf{L}_n) = -\frac{1}{2} \delta_{mn}, \quad \text{str}(\mathbf{M}_a \mathbf{M}_b) = \frac{1}{2} \delta_{ab}, \quad (4.62)$$

where f_{ab}^{c} are the structure constants of $\mathfrak{su}(4)$ while for $\mathfrak{su}(2)$ we normalize $\epsilon_{123} = 1$. We notice that the the Cartan generators of $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$ are given by $\mathbf{L}_3, \mathbf{L}_8, \mathbf{L}_{15}$ and $\mathbf{M}_3$. We will make use of these generators to construct r -matrices in the next section.

⁵We are using bold letters to denote generators to avoid confusion with some operators defined later.

⁶For a complete review of the representations of $\mathfrak{su}(4)$ see [75].

We will now show that our coset construction (4.59) does indeed reproduce the $\mathbb{CP}^3$ metric. We first redefine some of the generators of the basis of $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$. The following embedding of the generators $\mathbf{L}_6 \subset \mathfrak{u}(3)$ and $\mathbf{M}_1 \subset \mathfrak{su}(2)$ into the numerator of (4.59) turn out to be very convenient

$$\mathbf{T}_1 = \mathbf{L}_6 - \mathbf{L}_9, \quad \mathbf{T}_2 = \mathbf{L}_6 + \mathbf{L}_9 + 2\mathbf{M}_1. \quad (4.63)$$

Here, $\mathbf{T}_2$ have been defined conveniently for our basis to exactly reproduce the $\mathbb{CP}^3$ metric. The orthogonal basis for $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$ resulting after this embedding is given by the following 18 generators

$$\mathfrak{su}(4) \oplus \mathfrak{su}(2) = \text{span}_{\mathbb{R}}\{\mathbf{L}_{m'}, \mathbf{M}_2, \mathbf{M}_3, \mathbf{T}_1, \mathbf{T}_2, \mathbf{H}\}, \quad (4.64)$$

where $\mathbf{L}_{m'}$ runs through all the basis of $\mathfrak{su}(4)$ except for $\mathbf{L}_6$ and $\mathbf{L}_9$. Also $\mathbf{H}$ is defined as

$$\mathbf{H} = \mathbf{L}_6 + \mathbf{L}_9 + \mathbf{M}_1, \quad (4.65)$$

and its norm is given by

$$\text{str}(\mathbf{H}\mathbf{H}) = -\frac{1}{2}. \quad (4.66)$$

With these redefinitions, and in order to write an appropriate coset parametrization of (4.59), we define the following orthogonal basis

$$\frac{\mathfrak{su}(4) \oplus \mathfrak{su}(2)}{\mathfrak{u}(3) \oplus \mathfrak{su}(2)} = \text{span}_{\mathbb{R}}\{\mathbf{P}_m\}, \quad m = 1, \dots, 6, \quad (4.67)$$

where, for the sake of simplicity, we have renamed the set of generators of the basis as

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{L}_{11}, & \mathbf{P}_2 &= \mathbf{L}_{12}, & \mathbf{P}_3 &= \mathbf{L}_{13}, \\ \mathbf{P}_4 &= \mathbf{L}_{14}, & \mathbf{P}_5 &= \mathbf{H}, & \mathbf{P}_6 &= \mathbf{L}_{10}. \end{aligned} \quad (4.68)$$

Using (4.62) and (4.66), the following relation can be obtained

$$\text{str}(\mathbf{P}_m \mathbf{P}_n) = -\frac{1}{2}\delta_{mn}. \quad (4.69)$$

As you can see in (4.67), the $\mathfrak{u}(3)$ generators, namely $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4, \mathbf{L}_5, \mathbf{L}_7, \mathbf{L}_8, \mathbf{T}_1, \mathbf{L}_{15}$ and the $\mathfrak{su}(2)$ generators $\mathbf{T}_2, \mathbf{M}_2, \mathbf{M}_3$ are not part of the quotient space. Thus (4.67) describes a consistent parametrization of the symmetric space $\mathbb{CP}^3$.

We can now write an appropriate coset representative that allows us to reproduce the desired $\mathbb{CP}^3$ metric. It is given by

$$g = \exp[\varphi_1 \mathbf{L}_3 + \varphi_2 \mathbf{L} - \psi \mathbf{M}_3] \exp[\theta_1 \mathbf{L}_2 + (\theta_2 + \pi) \mathbf{L}_{14}] \exp[(2\rho + \pi)(\mathbf{L}_{10} + \mathbf{M}_2)], \quad (4.70)$$

where $\mathbf{L}$ is a linear combination of $\mathbf{L}_8$ and $\mathbf{L}_{15}$

$$\mathbf{L} = -\frac{\sqrt{3}}{3}\mathbf{L}_8 + \frac{\sqrt{6}}{3}\mathbf{L}_{15}. \quad (4.71)$$

Then we can write a Lie algebra element of $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$ by using the Maurer-Cartan one form $A = g^{-1}dg$.

In order to write the metric we need to project out all of the components proportional to generators of $\mathfrak{u}(3) \oplus \mathfrak{su}(2)$ in the Maurer-Cartan form. In order to do this, let us define the following projector

$$\begin{aligned} P(A) &= A + 2 \left[\sum_i \mathbf{L}_i \text{str}(\mathbf{L}_i A) - \frac{\mathbf{T}_1}{2} \text{str}(\mathbf{T}_1 A) + \frac{\mathbf{T}_2}{2} \text{str}(\mathbf{T}_2 A) \right. \\ &\quad \left. - \mathbf{M}_2 \text{str}(\mathbf{M}_2 A) - \mathbf{M}_3 \text{str}(\mathbf{M}_3 A) \right] \\ &= \cos \xi \sin \theta_1 d\varphi_1 \mathbf{P}_1 + \cos \xi d\theta_1 \mathbf{P}_2 - \sin \theta_2 \sin \xi d\varphi_2 \mathbf{P}_3 \\ &\quad - \sin \xi d\theta_2 \mathbf{P}_4 - \sin \xi \cos \xi (\cos \theta_1 d\varphi_1 - \cos \theta_2 d\varphi_2 + 2d\psi) \mathbf{P}_5 + 2d\xi \mathbf{P}_6, \end{aligned} \quad (4.72)$$

where in the first line the summation is over $i = 1, 2, 3, 4, 5, 7, 8, 15$. Then the metric for $\mathbb{CP}^3$ (4.58) can be obtained from the following symmetric Lagrangian

$$\mathcal{L} = -\frac{1}{2} \gamma^{\alpha\beta} \text{str} [A_\alpha P(A_\beta)]. \quad (4.73)$$

4.4 Yang-Baxter deformations of $AdS_4 \times \mathbb{CP}^3$

In this section we will use Yang-Baxter deformations to obtain a one parameter and a three parameters deformation of the $AdS_4 \times \mathbb{CP}^3$ sigma model [37].

4.4.1 Yang-Baxter sigma model for $\mathbb{CP}^3$

It is important to stress that we will consider deformations of the $\mathbb{CP}^3$ part of the Lagrangian only since the AdS_4 part will remain undeformed⁷. The deformation for the bosonic part of the Lagrangian based on the classical Yang-Baxter equation is

$$\mathcal{L}_\eta = -\frac{1}{2} (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{str} (A_\alpha P \circ \mathcal{O}^{-1} A_\beta), \quad (4.74)$$

where the operator $\mathcal{O}$, that depends on the deformation parameter η , is given by

$$\mathcal{O}^{-1} = \frac{1}{1 - 2\eta R_g \circ P}. \quad (4.75)$$

⁷This is not possible for an r -matrix solving the mCYBE [38].

The projector P is that in (4.72) and R_g is given by

$$R_g(X) = \text{Ad}_{g^{-1}} \circ R \circ \text{Ad}_g(X), \quad (4.76)$$

where g is a coset representative, and the linear operator R is an R -matrix satisfying the classical Yang-Baxter equation. This operator can also be written in tensorial notation as

$$R(X) = \text{str}_2[r(1 \otimes X)] = \sum_i [a_i \text{str}(b_i X) - b_i \text{str}(a_i X)], \quad (4.77)$$

where a_i and b_i denote elements of the basis for $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$ and the r -matrix is constructed using these generators as

$$r = \sum_i a_i \wedge b_i = \sum_i (a_i \otimes b_i - b_i \otimes a_i). \quad (4.78)$$

In order to extract the background fields from the Lagrangian (4.74) we need to find the action of R_g on each generator of the basis (4.67). This can be written as

$$R_g(\mathbf{P}_m) = \Lambda_m^n \mathbf{P}_n + \dots, \quad (4.79)$$

where $\mathbf{P}_m$ are defined in (4.68) and the dots stand for generators of $\mathfrak{u}(3) \oplus \mathfrak{su}(2)$ which can be ignored because they will be projected out by P .

The action of the operator (4.75) can also be computed in a similar way

$$\mathcal{O}^{-1}(\mathbf{P}_m) = K_m^n \mathbf{P}_n + \dots, \quad (4.80)$$

where again the dots represent terms that will be projected out. Combining equations (4.79) and (4.80) it is easy to find the relation between the coefficients Λ_m^n and K_m^n

$$\mathbf{P}_m = (K_m^n \mathbf{P}_n - 2\eta K_m^n \Lambda_n^p \mathbf{P}_p), \quad (4.81)$$

which can be solved for K_m^n , in matrix notation we can write

$$\mathbf{K} = (\mathbf{I} - 2\eta \mathbf{\Lambda})^{-1}. \quad (4.82)$$

The Maurer-Cartan current A_α can be expanded in the basis (4.64). However, as we have discussed in section 4.3, the operator P will project the current to the basis (4.67) as follows⁸

$$P(A_\alpha) = E_\alpha^m \mathbf{P}_m. \quad (4.83)$$

⁸We should stress that despite of the notation being used E_α^m does not represent the usual vielbein of $\mathbb{CP}^3$ but rather the coefficients in front of each of the generators in the third and fourth lines of (4.72).

Combining (4.80) and (4.83) in the Lagrangian (4.74) we get

$$\mathcal{L}_\eta = \frac{1}{4} \left(\gamma^{\alpha\beta} K_{(mn)} E_\alpha^m E_\beta^n - \epsilon^{\alpha\beta} K_{[mn]} E_\alpha^m E_\beta^n \right), \quad (4.84)$$

where the coefficients $K_{(mn)}$ and $K_{[mn]}$ are the symmetric and antisymmetric parts of the matrix (4.82). The metric and the antisymmetric field can easily be read off from (4.84).

4.4.2 One-parameter deformation

Finally, the last step is the choice of an r -matrix. The most natural choice are the so called Abelian r -matrices which are basically constructed out of Cartan generators. There are four Cartan generators in $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$, that is $\mathbf{L}_3, \mathbf{L}_8, \mathbf{L}_{15}$ and $\mathbf{M}_3$. In order to get a single parameter TsT transformation of $\mathbb{CP}^3$ we make the following choice

$$r = \mu \mathbf{L}_3 \wedge \mathbf{L}, \quad (4.85)$$

where $\mathbf{L}$ has been defined in (4.71) and μ is an arbitrary parameter. The action of the operator R on each of the generators (4.67) is given by (4.79). Using the coset parametrization (4.70) together with definitions (4.76) and (4.77) it can be shown that the only non-vanishing components of the deformation matrix Λ_m^n are

$$\begin{aligned} \Lambda_1^3 = -\Lambda_3^1 &= -\frac{1}{2} \mu \sin \theta_1 \sin \theta_2 \sin \xi \cos \xi, \\ \Lambda_1^5 = -\Lambda_5^1 &= -\frac{1}{2} \mu \cos \theta_2 \cos^2 \xi \sin \theta_1 \sin \xi, \\ \Lambda_3^5 = -\Lambda_5^3 &= \frac{1}{2} \mu \cos \theta_1 \cos \xi \sin \theta_2 \sin^2 \xi. \end{aligned} \quad (4.86)$$

The elements of the matrix K_m^n can then be found by using (4.82), we get

$$\begin{aligned} K_1^1 &= \frac{1 + \mu^2 \eta^2 \cos^2 \theta_1 \cos^2 \xi \sin^2 \theta_2 \sin^4 \xi}{1 + \mu^2 \eta^2 f \sin^2 \xi \cos^2 \xi}, \\ K_3^1 &= \frac{\mu^2 \eta^2 \cos \theta_1 \cos \theta_2 \cos^3 \xi \sin \theta_1 \sin \theta_2 \sin^3 \xi + \frac{1}{2} \mu \eta \sin \theta_1 \sin \theta_2 \sin 2\xi}{1 + \mu^2 \eta^2 f \sin^2 \xi \cos^2 \xi}, \\ K_5^1 &= \frac{\mu \eta \cos \theta_2 \cos^2 \xi \sin \theta_1 \sin \xi - \frac{1}{2} \mu^2 \eta^2 \cos \theta_1 \cos \xi \sin \theta_1 \sin^2 \theta_2 \sin^2 \xi \sin 2\xi}{1 + \mu^2 \eta^2 f \sin^2 \xi \cos^2 \xi}, \\ K_1^3 &= \frac{\mu^2 \eta^2 \cos \theta_1 \cos \theta_2 \cos^3 \xi \sin \theta_1 \sin \theta_2 \sin^3 \xi - \frac{1}{2} \mu \eta \sin \theta_1 \sin \theta_2 \sin 2\xi}{1 + \mu^2 \eta^2 f \sin^2 \xi \cos^2 \xi}, \\ K_3^3 &= \frac{1 + \mu^2 \eta^2 \cos^2 \theta_2 \cos^4 \xi \sin^2 \theta_1 \sin^2 \xi}{1 + \mu^2 \eta^2 f \sin^2 \xi \cos^2 \xi}, \end{aligned}$$

$$\begin{aligned}
K_5^3 &= \frac{-\mu\eta \cos \theta_1 \cos \xi \sin \theta_2 \sin^2 \xi - \frac{1}{2}\mu^2\eta^2 \cos \theta_2 \cos^2 \xi \sin^2 \theta_1 \sin \theta_2 \sin \xi \sin 2\xi}{1 + \mu^2\eta^2 f \sin^2 \xi \cos^2 \xi}, \\
K_1^5 &= \frac{-\mu\eta \cos \theta_2 \cos^2 \xi \sin \theta_1 \sin \xi - \frac{1}{2}\mu^2\eta^2 \cos \theta_1 \cos \xi \sin \theta_1 \sin^2 \theta_2 \sin^2 \xi \sin 2\xi}{1 + \mu^2\eta^2 f \sin^2 \xi \cos^2 \xi}, \\
K_3^5 &= \frac{\mu\eta \cos \theta_1 \cos \xi \sin \theta_2 \sin^2 \xi - \frac{1}{2}\mu^2\eta^2 \cos \theta_2 \cos^2 \xi \sin^2 \theta_1 \sin \theta_2 \sin \xi \sin 2\xi}{1 + \mu^2\eta^2 f \sin^2 \xi \cos^2 \xi}, \\
K_5^5 &= \frac{1 + \frac{1}{4}\mu^2\eta^2 \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 2\xi}{1 + \mu^2\eta^2 f \sin^2 \xi \cos^2 \xi}, \\
K_2^2 &= K_4^4 = K_6^6 = 1,
\end{aligned} \tag{4.87}$$

where

$$f(\theta_1, \theta_2) = \sin^2 \theta_1 \sin^2 \theta_2 + \cos^2 \xi \sin^2 \theta_1 \cos^2 \theta_2 + \sin^2 \xi \sin^2 \theta_2 \cos^2 \theta_1. \tag{4.88}$$

We now use these results to compute the symmetric $\mathcal{L}_{sym}$ and antisymmetric $\mathcal{L}_{anti}$ parts of the deformed Lagrangian (4.84)

$$\begin{aligned}
\mathcal{L}_{sym} &= \gamma^{\alpha\beta} \mathcal{M} \left[\frac{1}{4} \cos^2 \xi (\mathcal{M}^{-1} \partial_\alpha \theta_1 \partial_\beta \theta_1 + \sin^2 \theta_1 \partial_\alpha \varphi_1 \partial_\beta \varphi_1) \right. \\
&\quad + \frac{1}{4} \sin^2 \xi (\mathcal{M}^{-1} \partial_\alpha \theta_2 \partial_\beta \theta_2 + \sin^2 \theta_2 \partial_\alpha \varphi_2 \partial_\beta \varphi_2) \\
&\quad + \mu^2 \eta^2 \sin^4 \xi \cos^4 \xi \sin^2 \theta_1 \sin^2 \theta_2 \partial_\alpha \psi \partial_\beta \psi \\
&\quad + \cos^2 \xi \sin^2 \xi \left(\partial_\alpha \psi + \frac{1}{2} \cos \theta_1 \partial_\alpha \varphi_1 - \frac{1}{2} \cos \theta_2 \partial_\alpha \varphi_2 \right) \\
&\quad \left. \times \left(\partial_\beta \psi + \frac{1}{2} \cos \theta_1 \partial_\beta \varphi_1 - \frac{1}{2} \cos \theta_2 \partial_\beta \varphi_2 \right) + \mathcal{M}^{-1} \partial_\alpha \xi \partial_\beta \xi \right]. \tag{4.89}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{anti} &= -2 \epsilon^{\alpha\beta} \mu\eta \mathcal{M} \sin^2 \xi \cos^2 \xi \left(\frac{1}{2} \cos^2 \xi \sin^2 \theta_1 \cos \theta_2 \partial_\alpha \psi \partial_\beta \varphi_1 + \right. \\
&\quad \left. + \frac{1}{2} \sin^2 \xi \sin^2 \theta_2 \cos \theta_1 \partial_\alpha \psi \partial_\beta \varphi_2 + \frac{1}{4} f \partial_\alpha \varphi_1 \partial_\beta \varphi_2 \right), \tag{4.90}
\end{aligned}$$

where $\mathcal{M}$ is given by

$$\mathcal{M} = \frac{1}{1 + \mu^2 \eta^2 f \sin^2 \xi \cos^2 \xi}. \tag{4.91}$$

To compare our deformations to single parametric TsT transformations we need to make the following identification

$$\mu\eta = \hat{\gamma}, \tag{4.92}$$

It is now straightforward to read the metric and antisymmetric fields from the above expressions, we get

$$\begin{aligned} ds_{\mathbb{CP}^3_\gamma}^2 = & d\xi^2 + \frac{1}{4} \cos^2 \xi (d\theta_1^2 + \mathcal{M} \sin^2 \theta_1 d\varphi_1^2) + \frac{1}{4} \sin^2 \xi (d\theta_2^2 + \mathcal{M} \sin^2 \theta_2 d\varphi_2^2) + \\ & + \mathcal{M} \cos^2 \xi \sin^2 \xi \left(d\psi + \frac{1}{2} \cos \theta_1 d\varphi_1 - \frac{1}{2} \cos \theta_2 d\varphi_2 \right)^2 + \\ & + \hat{\gamma}^2 \mathcal{M} \sin^4 \xi \cos^4 \xi \sin^2 \theta_1 \sin^2 \theta_2 d\psi^2, \end{aligned} \quad (4.93)$$

$$\begin{aligned} B = & -\hat{\gamma} \mathcal{M} \sin^2 \xi \cos^2 \xi \left(\frac{1}{2} \cos^2 \xi \sin^2 \theta_1 \cos \theta_2 d\psi \wedge d\varphi_1 + \right. \\ & \left. + \frac{1}{2} \sin^2 \xi \sin^2 \theta_2 \cos \theta_1 d\psi \wedge d\varphi_2 + \frac{1}{4} f d\varphi_1 \wedge d\varphi_2 \right). \end{aligned} \quad (4.94)$$

These results are in complete agreement with the one-parameter γ -deformations of $AdS_4 \times \mathbb{CP}^3$ presented in [8] showing that they have an algebraic origin given by the Abelian R -matrix (4.85).

4.4.3 Three-parameter deformation

As we will show now, three-parameter γ -deformations can be obtained as a straightforward generalization of the computation presented for the one-parameter case⁹. Let us make the following choice for the r -matrix

$$r = \mu_1 \mathbf{L} \wedge \mathbf{M}_3 + \mu_2 \mathbf{L}_3 \wedge \mathbf{M}_3 + \mu_3 \mathbf{L}_3 \wedge \mathbf{L}, \quad (4.95)$$

where $\mathbf{L}$ is a linear combination of Cartan generators defined in (4.71) and μ_1, μ_2 and μ_3 are arbitrary parameters. The non-vanishing components of the deformation matrix Λ_m^n are now

$$\begin{aligned} \Lambda_1^3 = -\Lambda_3^1 &= -\frac{1}{2} \mu_3 \sin \theta_1 \sin \theta_2 \sin \xi \cos \xi, \\ \Lambda_1^5 = -\Lambda_5^1 &= -\frac{1}{2} (2\mu_2 + \mu_3 \cos \theta_2) \cos^2 \xi \sin \theta_1 \sin \xi, \\ \Lambda_3^5 = -\Lambda_5^3 &= \frac{1}{2} (2\mu_1 + \mu_3 \cos \theta_1) \cos \xi \sin \theta_2 \sin^2 \xi, \end{aligned} \quad (4.96)$$

⁹To avoid confusion, what we will do here is to show that three parameter TsT transformations of $AdS_4 \times \mathbb{CP}^3$ can be obtained from Yang-Baxter deformations, which involve a single deformation parameter

and the non-vanishing elements of K_m^n are

$$\begin{aligned}
K_1^1 &= \mathcal{M} [1 + \eta^2 (2\mu_1 + \mu_3 \cos \theta_1)^2 \cos^2 \xi \sin^2 \theta_2 \sin^4 \xi] , \\
K_3^1 &= \mathcal{M} [-\eta^2 (2\mu_1 + \mu_3 \cos \theta_1) (2\mu_2 + \mu_3 \cos \theta_2) \cos^3 \xi \sin \theta_1 \sin \theta_2 \sin^3 \xi + \\
&\quad + \frac{1}{2} \mu_3 \eta \sin \theta_1 \sin \theta_2 \sin 2\xi] , \\
K_5^1 &= \mathcal{M} [\eta (2\mu_2 + \mu_3 \cos \theta_2) \cos^2 \xi \sin \theta_1 \sin \xi + \\
&\quad - \frac{1}{2} \mu_3 \eta^2 (2\mu_1 + \mu_3 \cos \theta_1) \cos \xi \sin \theta_1 \sin^2 \theta_2 \sin^2 \xi \sin 2\xi] , \\
K_1^3 &= \mathcal{M} [\eta^2 (2\mu_1 + \mu_3 \cos \theta_1) (2\mu_2 + \mu_3 \cos \theta_2) \cos^3 \xi \sin \theta_1 \sin \theta_2 \sin^3 \xi + \\
&\quad - \frac{1}{2} \mu_3 \eta \sin \theta_1 \sin \theta_2 \sin 2\xi] , \\
K_3^3 &= \mathcal{M} [1 + \eta^2 (2\mu_2 + \mu_3 \cos \theta_2)^2 \cos^4 \xi \sin^2 \theta_1 \sin^2 \xi] , \\
K_5^3 &= \mathcal{M} [-\eta (2\mu_1 + \mu_3 \cos \theta_1) \cos \xi \sin \theta_2 \sin^2 \xi \\
&\quad - \frac{1}{2} \mu_3 \eta^2 (2\mu_2 + \mu_3 \cos \theta_2) \cos^2 \xi \sin^2 \theta_1 \sin \theta_2 \sin \xi \sin 2\xi] , \\
K_1^5 &= \mathcal{M} [-\eta (2\mu_2 + \mu_3 \cos \theta_2) \cos^2 \xi \sin \theta_1 \sin \xi + \\
&\quad - \frac{1}{2} \mu_3 \eta^2 (2\mu_1 + \mu_3 \cos \theta_1) \cos \xi \sin \theta_1 \sin^2 \theta_2 \sin^2 \xi \sin 2\xi] , \\
K_3^5 &= \mathcal{M} [\eta (2\mu_1 + \mu_3 \cos \theta_1) \cos \xi \sin \theta_2 \sin^2 \xi \\
&\quad - \frac{1}{2} \mu_3 \eta^2 (2\mu_2 + \mu_3 \cos \theta_2) \cos^2 \xi \sin^2 \theta_1 \sin \theta_2 \sin \xi \sin 2\xi] , \\
K_5^5 &= \mathcal{M} [1 + \frac{1}{4} \mu_3^2 \eta^2 \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 2\xi] , \\
K_2^2 &= K_4^4 = K_6^6 = 1 ,
\end{aligned} \tag{4.97}$$

where

$$\begin{aligned}
\mathcal{M}^{-1} &= 1 + \eta^2 \cos^2 \xi \sin^2 \xi \left[\mu_3^2 \sin^2 \theta_1 \sin^2 \theta_2 + (2\mu_2 + \mu_3 \cos \theta_2)^2 \cos^2 \xi \sin^2 \theta_1 + \right. \\
&\quad \left. + (2\mu_1 + \mu_3 \cos \theta_1)^2 \sin^2 \xi \sin^2 \theta_2 \right] .
\end{aligned} \tag{4.98}$$

Using these results in (4.84) we can find the symmetric and antisymmetric parts of the deformed Lagrangian

$$\begin{aligned}
L_{sym} &= \gamma^{\alpha\beta} \mathcal{M} \left[\frac{1}{4} \cos^2 \xi (\mathcal{M}^{-1} \partial_\alpha \theta_1 \partial_\beta \theta_1 + \sin^2 \theta_1 \partial_\alpha \varphi_1 \partial_\beta \varphi_1) \right. \\
&\quad \left. + \frac{1}{4} \sin^2 \xi (\mathcal{M}^{-1} \partial_\alpha \theta_2 \partial_\beta \theta_2 + \sin^2 \theta_2 \partial_\alpha \varphi_2 \partial_\beta \varphi_2) \right]
\end{aligned}$$

$$\begin{aligned}
& + \cos^2 \xi \sin^2 \xi \left(\partial_\alpha \psi + \frac{1}{2} \cos \theta_1 \partial_\alpha \varphi_1 - \frac{1}{2} \cos \theta_2 \partial_\alpha \varphi_2 \right) \\
& \times \left(\partial_\beta \psi + \frac{1}{2} \cos \theta_1 \partial_\beta \varphi_1 - \frac{1}{2} \cos \theta_2 \partial_\beta \varphi_2 \right) + \mathcal{M}^{-1} \partial_\alpha \xi \partial_\beta \xi \\
& + \eta^2 \sin^4 \xi \cos^4 \xi \sin^2 \theta_1 \sin^2 \theta_2 (-\mu_1 \partial_\alpha \varphi_1 + \mu_2 \partial_\alpha \varphi_2 + \mu_3 \partial_\alpha \psi) \\
& \times (-\mu_1 \partial_\beta \varphi_1 + \mu_2 \partial_\beta \varphi_2 + \mu_3 \partial_\beta \psi) \Big]. \tag{4.99}
\end{aligned}$$

$$\begin{aligned}
L_{anti} = & -2 \epsilon^{\alpha\beta} \eta \mathcal{M} \sin^2 \xi \cos^2 \xi \left[\frac{1}{2} (2\mu_2 + \mu_3 \cos \theta_2) \cos^2 \xi \sin^2 \theta_1 \partial_\alpha \psi \partial_\beta \varphi_1 \right. \\
& + \frac{1}{2} (2\mu_1 + \mu_3 \cos \theta_1) \sin^2 \xi \sin^2 \theta_2 \partial_\alpha \psi \partial_\beta \varphi_2 \\
& + \frac{1}{4} \left(\mu_3 \sin^2 \theta_1 \sin^2 \theta_2 + (2\mu_2 + \mu_3 \cos \theta_2) \cos^2 \xi \sin^2 \theta_1 \cos \theta_2 \right. \\
& \left. \left. (2\mu_1 + \mu_3 \cos \theta_1) \sin^2 \xi \sin^2 \theta_2 \cos \theta_1 \right) \partial_\alpha \varphi_1 \partial_\beta \varphi_2 \right]. \tag{4.100}
\end{aligned}$$

To compare our deformations to three parameter TsT transformations we need to make the following identifications

$$\mu_1 \eta = -\hat{\gamma}_1, \quad \mu_2 \eta = \hat{\gamma}_2, \quad \mu_3 \eta = \hat{\gamma}_3, \tag{4.101}$$

It is now straightforward to read the metric and antisymmetric fields from the above expressions, we get

$$\begin{aligned}
ds_{\mathbb{CP}^3_{\hat{\gamma}}}^2 = & d\xi^2 + \frac{1}{4} \cos^2 \xi (d\theta_1^2 + \mathcal{M} \sin^2 \theta_1 d\varphi_1^2) + \frac{1}{4} \sin^2 \xi (d\theta_2^2 + \mathcal{M} \sin^2 \theta_2 d\varphi_2^2) + \\
& + \mathcal{M} \cos^2 \xi \sin^2 \xi \left(d\psi + \frac{1}{2} \cos \theta_1 d\varphi_1 - \frac{1}{2} \cos \theta_2 d\varphi_2 \right)^2 + \\
& + \mathcal{M} \sin^4 \xi \cos^4 \xi \sin^2 \theta_1 \sin^2 \theta_2 (\hat{\gamma}_1 d\varphi_1 + \hat{\gamma}_2 d\varphi_2 + \hat{\gamma}_3 d\psi)^2, \tag{4.102}
\end{aligned}$$

$$\begin{aligned}
B = & -\mathcal{M} \sin^2 \xi \cos^2 \xi \left[\frac{1}{2} (2\hat{\gamma}_2 + \hat{\gamma}_3 \cos \theta_2) \cos^2 \xi \sin^2 \theta_1 d\psi \wedge d\varphi_1 + \right. \\
& + \frac{1}{2} (-2\hat{\gamma}_1 + \hat{\gamma}_3 \cos \theta_1) \sin^2 \xi \sin^2 \theta_2 d\psi \wedge d\varphi_2 + \\
& + \frac{1}{4} \left(\hat{\gamma}_3 \sin^2 \theta_1 \sin^2 \theta_2 + (2\hat{\gamma}_2 + \hat{\gamma}_3 \cos \theta_2) \cos^2 \xi \sin^2 \theta_1 \cos \theta_2 + \right. \\
& \left. \left. (-2\hat{\gamma}_1 + \hat{\gamma}_3 \cos \theta_1) \sin^2 \xi \sin^2 \theta_2 \cos \theta_1 \right) d\varphi_1 \wedge d\varphi_2 \right]. \tag{4.103}
\end{aligned}$$

These results are in complete agreement with three-parameter β -deformations of $AdS_4 \times \mathbb{CP}^3$ which were obtained using TsT transformations [8]. As a consequence we have demonstrated that three-parameter γ -deformations arise naturally from Yang-Baxter deformations¹⁰, showing that they also have an algebraic origin given by the r -matrix (4.95).

¹⁰Where these deformations reduce to the single parameter β deformations in the case $\hat{\gamma}_2 = \hat{\gamma}_3 = 0$, $\hat{\gamma}_3 = \gamma$ as expected.

Chapter 5

Concluding remarks

We conclude this thesis by offering a brief summary of our main results and by establishing some future perspectives.

The main objective of this thesis was to study one of the most interesting techniques that has been developed to construct deformations of integrable field theories. The so called Yang-Baxter deformations of sigma models are a powerful technique that makes use of endomorphisms R of a Lie algebra which are solutions of the modified classical Yang-Baxter equation.

In chapter 2 we gave a self-contained introduction to the sigma model formulation of superstrings. The most important feature of these models is that they can be realized as supercoset sigma models with a defining algebraic property, the isometry algebra of the Metsaev-Tseytlin formulation in various supercosets accept a fourth order automorphism that can be used to define a semi-symmetric space. This property is of crucial importance to prove that these kind of sigma models is classically integrable. In fact, the only two formulations one can use in a semi-symmetric space, which are integrable, are the Green-Schwarz action and the hybrid action (matter part of the pure spinor action).

We devoted chapter 3 to review the basic concepts that led to the original construction of Yang-Baxter sigma models. In the first part we summarized the deformation procedure and then we gave a detailed derivation of the η -deformed action for the principal chiral model. We also listed a few properties of the deformed Green-Schwarz sigma model in $AdS_5 \times S^5$. Most importantly, this deformed sigma model preserves a deformed version of κ -symmetry observed in the non-deformed case. However, the most important section of chapter 3 is the one devoted to Yang-Baxter deformations of the hybrid sigma model. We considered an ansatz representing the most general Yang-Baxter sigma model in a semi-symmetric space and we proved that requiring integrability imposes a set of constraints.

Remarkably, we managed to explicitly show, constructing the Lax pair of this model, that the only possible η -deformed models in a semi-symmetric space, which are integrable, are the Green-Schwarz deformed sigma model and a new kind of η -deformed hybrid sigma model.

Chapter 4 was devoted to give a series of examples of Yang-Baxter deformations. Most importantly, we considered TsT transformations of the gravity dual of ABJM theory as Yang-Baxter deformations. In order to do this, we proposed a new supercoset description of $\mathbb{CP}^3$. The main advantage of this new parametrization is that it leads to the standard Fubini-Study metric of $\mathbb{CP}^3$ and makes Yang-Baxter deformations a lot easier to compute.

It was shown in previous works, that TsT transformations have a fundamental relation to Yang-Baxter sigma models, for γ -deformed $AdS_5 \times S_5$ in [22] and for $AdS_5 \times T^{1,1}$ in [38]. We managed to show that it is possible to consider Yang Baxter deformations using a convenient combination of Cartan generators to construct an Abelian r -matrix that exactly reproduces the metric and NS-NS two form of the gravity dual of β -deformed ABJM theory [37], previously computed using TsT transformations in [8]. Our deformations of $AdS_4 \times \mathbb{CP}^3$ can be regarded as a new example of the relation between TsT transformations and solutions of the classical Yang-Baxter equation, sometimes referred as the gravity/CYBE correspondence.

One direct generalization of the work presented here could be to compute the deformations of the two-form (F_2) and four-form (F_4) Ramond-Ramond fluxes, that appear in γ -deformed $AdS_4 \times \mathbb{CP}^3$, using Yang-Baxter deformations. However, to the best of our knowledge, TsT transformations for fermionic fields have not been reproduced using Yang-Baxter deformations yet. We hope that the simplified parametrization for $\mathbb{CP}^3$ given in this thesis, can offer some light to solve this problem.

The procedure developed in this thesis can also be applied to other situations. As it is well known, Yang-Baxter deformations lead to supergravity backgrounds only if the associated r -matrix is unimodular [19]. Abelian r -matrices trivially satisfy the unimodularity condition and they are related to TsT transformations [30]. On the other hand, non-Abelian r -matrices may not be unimodular and Yang-Baxter deformations related to unimodular non-Abelian r -matrices have been interpreted as the gravity dual of some non-commutative gauge theory [76, 77]. The gravitational duals of non-commutative and dipole deformations of ABJM theory found in [8] can, in principle, be obtained as Yang-Baxter deformations. In particular, an extension of our coset parametrization could be built to include deformations of the AdS_4 part in order to compute dipole deformations.

Finally, the non-deformed limit of η -models is given by the standard supercoset sigma model, while the non-deformed limit of λ -models is given by the non-abelian T-dual of

the latter. This relation clearly suggests that the η -model and λ -model must be in some sense dual to each other. Some progress in this direction have been achieved in [78], where an analytic continuation between the metric and B field of both models was presented. The construction of generalized λ -models [79, 80] also tries to address this issue but a definitive solution is still unknown.

Appendix A

Algebraic structure of integrable field theories

The phase space of a sigma model is given by a loop group. We will use these appendix to make some definitions related to the integrable structures of these kind of theories.

A.1 R -matrices associated to loop algebras

The Lax connection takes values on the space of loops of the algebra associated the the phase space. Thus, to compute the exchange algebra, we need to define a suitable Poisson bracket on this kind of spaces. We will see that the most consistent way to do this is by means of the Kostant-Kirillov R -bracket on some central extension of the current algebra associated to the phase space.

Definition A.1.1 (Loop algebras) *Let $\mathfrak{g}$ be a classical Lie algebra. We define the loop algebra $\mathcal{L}\mathfrak{g}$ as the algebra of formal Laurent polynomials in the complex variable λ with coefficients in $\mathfrak{g}$, that is*

$$\mathcal{L}\mathfrak{g} = \mathfrak{g}[\lambda, \lambda^{-1}] \equiv \left\{ X(\lambda) = \sum_i x_i \lambda^i; \quad x_i \in \mathfrak{g} \right\}. \quad (\text{A.1})$$

The commutator that turns $\mathcal{L}\mathfrak{g}$ into an (affine) Lie algebra is given by

$$[x\lambda^i, y\lambda^j] = [x, y]\lambda^{i+j}, \quad (\text{A.2})$$

where of course $[\cdot, \cdot]$ is the Lie bracket associated to $\mathfrak{g}$. This definition implies that the loop algebra $\mathcal{L}\mathfrak{g}$ has a natural grading by powers of λ , that is

$$\mathcal{L}\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g} \lambda^i. \quad (\text{A.3})$$

Following this decomposition, let us denote by $\mathcal{L}\mathfrak{g}_+ \equiv \mathfrak{g}[\lambda]$, the subalgebra of formal Laurent series in λ and by $\mathcal{L}\mathfrak{g}_- \equiv \lambda^{-1}\mathfrak{g}[\lambda^{-1}]$ that of polynomials in λ^{-1} .

One suitable property of Lie algebras is that they accept decompositions that can be used to construct the so called split R -matrices. Given $X(\lambda) \in \mathcal{L}\mathfrak{g}$ one can define the complementary projections $\pi_+ X(\lambda) \in \mathcal{L}\mathfrak{g}_+$ and $\pi_- X(\lambda) \in \mathcal{L}\mathfrak{g}_-$, in this way loop algebras can always be decomposed into a direct sum of algebras

$$\mathcal{L}\mathfrak{g} = \mathcal{L}\mathfrak{g}_+ \oplus \mathcal{L}\mathfrak{g}_-. \quad (\text{A.4})$$

We can use this decomposition to introduce the so called Adler-Kostant-Symes split R -matrix. The corresponding R -matrix $R \in \text{End}(\mathcal{L}\mathfrak{g})$ with respect to decomposition (A.4) reads

$$R \equiv \pi_+ - \pi_-, \quad (\text{A.5})$$

It is quite straightforward to prove that such a construction is a solution to the modified classical Yang-Baxter equation (see Appendix B).

Let $\langle \cdot, \cdot \rangle$ define a non-degenerate invariant bilinear form in $\mathfrak{g}$. We may use this to define an inner product on $\mathcal{L}\mathfrak{g}$ as follows

$$(X, Y) \equiv \oint \frac{d\lambda}{2\pi\lambda} \langle X(\lambda), Y(\lambda) \rangle, \quad (\text{A.6})$$

which as usual may be used to identify the dual space $\mathcal{L}\mathfrak{g}^*$. In models where we can exploit the symmetry provided by an n -th order automorphism Ω , one can construct a subalgebra of $\mathcal{L}\mathfrak{g}$ which consists of invariants under Ω .

Definition A.1.2 (Twisted loop algebras) *Let $\Omega \in \mathfrak{g}$ be an automorphism of $\mathfrak{g}$ of finite order n . We promote Ω to act on $\mathcal{L}\mathfrak{g}$ according to $\hat{\Omega} : \mathcal{L}\mathfrak{g} \rightarrow \mathcal{L}\mathfrak{g}$, where $\hat{\Omega}(X)(\lambda) = \Omega(X(-i\lambda))$. We define the so called twisted loop algebra $\mathcal{L}\mathfrak{g}^\Omega$ as follows*

$$\mathcal{L}\mathfrak{g}^\Omega = \left\{ X \in \mathcal{L}\mathfrak{g}; \quad \hat{\Omega}(X) = X \right\}. \quad (\text{A.7})$$

Analogously to what we did previously, one can set $\mathcal{L}\mathfrak{g}_\pm^\Omega = \mathcal{L}\mathfrak{g}^\Omega \cap \mathcal{L}\mathfrak{g}_\pm$ and then the decomposition of loop algebras reads

$$\mathcal{L}\mathfrak{g}^\Omega = \mathcal{L}\mathfrak{g}_+^\Omega \oplus \mathcal{L}\mathfrak{g}_-^\Omega. \quad (\text{A.8})$$

Thus, one can define a split R -matrix $R \in \text{End}(\mathcal{L}\mathfrak{g}^\Omega)$ for this decomposition too. The inner product in $\mathcal{L}\mathfrak{g}^\Omega$, which receives the especial name of twisted inner product, can be defined as follows

$$(X, Y)_\phi \equiv \oint \frac{d\lambda}{2\pi i \lambda} \phi(\lambda) \langle X(\lambda), Y(\lambda) \rangle, \quad (\text{A.9})$$

here $\phi(\lambda)$ is a complex function called the twist function, which can be showed to be precisely the same twist function as in chapter 3.

A.2 Centrally extended current algebra

In order to obtain zero curvature equations which constitute the main ingredient in any two dimensional integrable field theory, the relevant algebra to consider is a central extension of the current algebra of $\mathcal{L}\mathfrak{g}^\Omega$.

Definition A.2.1 (Current algebra) *The current algebra $\mathfrak{G}$ of $\mathcal{L}\mathfrak{g}^\Omega$ is defined as the set of smooth loops in $\mathcal{L}\mathfrak{g}^\Omega$, that is: $\mathfrak{G} \equiv C^\infty(S^1, \mathcal{L}\mathfrak{g}^\Omega)$.*

The decomposition we introduced for $\mathcal{L}\mathfrak{g}$ and $\mathcal{L}\mathfrak{g}^\Omega$ can also be lifted to the current algebra. Setting $\mathfrak{G}_\pm \equiv C^\infty(S^1, \mathcal{L}\mathfrak{g}_\pm^\Omega)$, we have

$$\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-. \quad (\text{A.10})$$

On the other hand, the inner product on $\mathfrak{G}$ will be defined by using the following non-degenerate invariant bilinear form

$$((\mathfrak{X}, \mathfrak{Y}))_\phi \equiv \int_{S^1} d\sigma (\mathfrak{X}(\sigma), \mathfrak{Y}(\sigma))_\phi, \quad (\text{A.11})$$

where $\sigma \in S^1$ and $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{G}$. Analogously to $\mathcal{L}\mathfrak{g}^*$ and $\mathcal{L}\mathfrak{g}^{\Omega*}$, this bilinear form can be used to identify the dual space $\mathfrak{G}^*$.

The current algebra $\mathfrak{G}$ admits an important central extension $\hat{\mathfrak{G}}$ defined with the help of the 2-cocycle

$$\omega(\mathfrak{X}, \mathfrak{Y}) \equiv \int_{S^1} d\sigma (\mathfrak{X}(\sigma), \partial_\sigma \mathfrak{Y}(\sigma))_\phi. \quad (\text{A.12})$$

To be precise, $\hat{\mathfrak{G}}$ is defined as a vector space according to $\mathfrak{G} \oplus \mathbb{C}$ and equipped with the Lie brackets

$$[(\mathfrak{X}, a), (\mathfrak{Y}, b)] \equiv ([\mathfrak{X}, \mathfrak{Y}], \omega(\mathfrak{X}, \mathfrak{Y})), \quad (\text{A.13})$$

where $a, b \in \mathbb{C}$. The fact that this definition does not depend on a or b reflects the fact that the extension is indeed central. On the other hand we can extend the inner product of $\mathfrak{G}$ to $\hat{\mathfrak{G}}$ as follows

$$(((\mathfrak{X}, a), (\mathfrak{Y}, b)))_{\phi} \equiv ((\mathfrak{X}, \mathfrak{Y}))_{\phi} + ab. \quad (\text{A.14})$$

Once again, one can use this inner product to identify the dual space $\hat{\mathfrak{G}}^*$. As it is very well known, the dual space of a Lie algebra can be given a Poisson structure turning it into a Poisson manifold by means of the so called Kostant-Kirillov bracket (see Appendix B). Thus the Lie bracket (A.13) on $\hat{\mathfrak{G}}$ naturally induces a Poisson structure on $\hat{\mathfrak{G}}^*$. Given $f \in C^\infty(\hat{\mathfrak{G}}^*)$ and defining its derivative $\hat{d}f = (df, c) \in \hat{\mathfrak{G}}$ at $(\mathfrak{L}, a)$ as:

$$(((\mathfrak{X}, b)\hat{d}f))_{\phi} = \left. \frac{d}{dt} \right|_{t=0} f((\mathfrak{L}, a) + t(\mathfrak{X}, b)), \quad \forall (\mathfrak{X}, b) \in \hat{\mathfrak{G}}^*, \quad (\text{A.15})$$

then for any two functions $f, g \in C^\infty(\hat{\mathfrak{G}}^*)$ we set the Kostant-Kirillov brackets as

$$\{f, g\}(\hat{\mathfrak{L}}, a) = ((\hat{\mathfrak{L}}, [\hat{d}f, \hat{d}g]))_{\phi}, \quad \hat{\mathfrak{L}} = (\mathfrak{L}, a). \quad (\text{A.16})$$

However, it can be checked that this bracket is degenerate. In order to circumvent this difficulty one uses the previously defined split R -matrix (A.5) defined originally as $R \in \text{End}(\mathcal{L}\mathfrak{g}^\Omega)$ and extend it to the space $\hat{\mathfrak{G}}$ thus defining an R -matrix $R \in \text{End}(\hat{\mathfrak{G}})$ that can be used to define an additional Lie bracket on $\hat{\mathfrak{G}}$ which will naturally induce a second Kostant-Kirillov bracket on $\hat{\mathfrak{G}}^*$. Thus, defining $(R\mathfrak{X})(\sigma) = R(\mathfrak{X}(\sigma))$, where $R \in \text{End}(\mathcal{L}\mathfrak{g}^\Omega)$, one can set the following bracket

$$[\mathfrak{X}, \mathfrak{Y}]_R \equiv [R\mathfrak{X}, \mathfrak{Y}] + [\mathfrak{X}, R\mathfrak{Y}]. \quad (\text{A.17})$$

This is of course a Lie bracket because R is a solution to the modified classical Yang-Baxter equation. We then further extend the R -matrix to $\text{End}(\hat{\mathfrak{G}})$ simply as $R(\mathfrak{X}, c) = (R\mathfrak{X}, c)$, thus the corresponding Lie bracket induced by R on $\hat{\mathfrak{G}}$ can be defined as:

$$[(\mathfrak{X}, a), (\mathfrak{Y}, b)]_R = [R(\mathfrak{X}, a), (\mathfrak{Y}, b)] + [(\mathfrak{X}, a), R(\mathfrak{Y}, b)] = ([\mathfrak{X}, \mathfrak{Y}]_R, \omega_R(\mathfrak{X}, \mathfrak{Y})), \quad (\text{A.18})$$

where $\omega_R(\mathfrak{X}, \mathfrak{Y}) \equiv \omega(R\mathfrak{X}, \mathfrak{Y}) + \omega(\mathfrak{X}, R\mathfrak{Y})$. Finally the Kostant-Kirillov R-bracket on $\hat{\mathfrak{G}}^*$ will be given by

$$\{f, g\}_R(\hat{\mathfrak{L}}) \equiv ((\hat{\mathfrak{L}}, [\hat{d}f, \hat{d}g]_R))_{\phi}. \quad (\text{A.19})$$

Using the explicit form of the R -bracket (A.18), one can show that the Kostant-Kirillov bracket of two functions f, g can be written in the following more familiar way

$$\{f, g\}_R(\hat{\mathfrak{L}}) = \int_{S^1} d\sigma ([R_{\underline{12}}, \mathfrak{L} \otimes 1] - [R_{\underline{12}}^*, 1 \otimes \mathfrak{L}], df \otimes dg)_{\phi} + (R_{\underline{12}} + R_{\underline{12}}^*, df \otimes \partial_{\sigma} dg)_{\phi}. \quad (\text{A.20})$$

A.3 The r/s formalism

After all of this algebraic mess one may wonder how can we use all of this formalism to compute for instance the Poisson brackets of a field theory or the classical exchange algebra. Since the Kostant-Kirillov R-brackets are defined for functions $f, g \in C^\infty(\hat{\mathfrak{G}}^*)$, if we take $X, Y \in \mathcal{L}\mathfrak{g}^\Omega$ and set $\mathfrak{X} = X \cdot \delta_{\sigma_1}$ and $\mathfrak{Y} = Y \cdot \delta_{\sigma_2}$ where δ_σ denote the Dirac delta function based at the point $\sigma \in S^1$, we can define the following linear functions

$$f : \hat{\mathfrak{L}} \mapsto ((\hat{\mathfrak{L}}, (\mathfrak{X}, 0)))_\phi = (\mathfrak{L}(\sigma_1), X)_\phi, \quad f : \hat{\mathfrak{L}} \mapsto ((\hat{\mathfrak{L}}, (\mathfrak{Y}, 0)))_\phi = (\mathfrak{L}(\sigma_2), Y)_\phi. \quad (\text{A.21})$$

Using these functions on the brackets (A.20) and evaluating the integral at the right hand side as well as using the notation $\mathfrak{L}_1 = \mathfrak{L} \otimes 1$, $\mathfrak{L}_2 = 1 \otimes \mathfrak{L}$, we obtain

$$\{\mathfrak{L}_1, \mathfrak{L}_2\}_R = [R_{12}, \mathfrak{L}_1] \delta_{\sigma_1 \sigma_2} - [R_{12}^*, \mathfrak{L}_2] \delta_{\sigma_1 \sigma_2} + (R_{12} + R_{12}^*) \delta'_{\sigma_1 \sigma_2}, \quad (\text{A.22})$$

where $\delta_{\sigma_1 \sigma_2} = \delta_{\sigma_2}(\sigma_1)$ and $\delta'_{\sigma_1 \sigma_2} = \delta'_{\sigma_2}(\sigma_1)$. Comparing this bracket to the standard r/s algebra given in [61] we see that they are equivalent if we make the following identifications

$$r = \frac{1}{2}(R - R^*), \quad s = -\frac{1}{2}(R + R^*). \quad (\text{A.23})$$

In particular, the condition $s = 0$ for ultralocality is equivalent to the skew-symmetry $R^* = -R$ of the R -matrix. The non-ultralocality of the non-linear sigma model can therefore be attributed to the twisted inner product, which results in the non skew-symmetry of the R -matrix.

Equation (A.22) is precisely the algebra we showed in equation (3.13). The classical exchange algebra emerges naturally from the algebraic structure of integrable field theories defined on twisted loop algebras, in other words this is the generalization of Proposition 3.1.2. This is not the only result we can get from this algebraic approach, the Poisson brackets of all the fields of a given integrable field theory can be constructed with ease by using the Kostant-Kirillov R-bracket, we only need to make suitable definitions of the fields. However, the most important consequence of this analysis is to highlight the fact that the Poisson structure of integrable field theories is profoundly related to the inner product and the R -matrix. Thus, if one wants to modify the Poisson structure one is forced to modify the inner product, that is, one is forced to modify the twist function.

An immediate application of this reasoning is the procedure developed by Faddeev and Reshitikhin. In this procedure, the Poisson structure was modified to get rid off the non-ultralocalities of the exchange algebra of the $SU(2)$ principal chiral model. By following the formalism presented here, this procedure is equivalent to a suitable modification of

the twist function to get a skew-symmetric R -matrix. Thus, this formalism is the most suitable to modify the Poisson structure of a given integrable field theory , for instance when trying to perform η -deformations.

Appendix B

Solutions of the classical Yang-Baxter equation

In this appendix we will give a very short review on classical r -matrices and the relevant solutions to the (modified) classical Yang-Baxter equation that we have used in the text.

B.1 Classical r -matrix

Let $\mathfrak{g}$ be a Lie algebra. A classical r -matrix $r = \sum_i a_i \otimes b_i$ is defined as a solution to the classical Yang-Baxter equation (CYBE)

$$[r_{\underline{12}}, r_{\underline{23}}] + [r_{\underline{12}}, r_{\underline{13}}] + [r_{\underline{13}}, r_{\underline{23}}] = 0, \quad (\text{B.1})$$

where, as in the entirety of the text, we are using tensor notation $r_{\underline{12}} = r \otimes 1$, $r_{\underline{23}} = 1 \otimes r$ and $r_{\underline{13}} = \sum_i a_i \otimes 1 \otimes b_i$. The name to this equation comes from the fact that it is regarded as a classical analogue of the quantum Yang-Baxter equation (YBE)

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \quad (\text{B.2})$$

When there exist a non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$, one can introduce a linear operator denoted by $R \in \text{End}(\mathfrak{g})$ in the following way

$$R(x) = \langle r, x \rangle_2 = \sum_i a_i \langle b_i, x \rangle, \quad \forall x \in \mathfrak{g}, \quad (\text{B.3})$$

here the subindex in $\langle r, x \rangle_2$ stands for the inner product over the second entry of the tensor. If the classical r -matrix happens to be skew-symmetric, that is $r_{\underline{12}} = -r_{\underline{21}}$, the CYBE can be equivalently written as follows

$$[R(x), R(y)] - R([R(x), y] + [x, R(y)]) = 0, \quad \forall x, y \in \mathfrak{g}. \quad (\text{B.4})$$

A very important generalization of the CYBE is the so called modified classical Yang-Baxter equation (mCYBE), which can be written in the following way

$$[R(x), R(y)] - R([R(x), y] + [x, R(y)]) = -c^2 [x, y], \quad \forall x, y \in \mathfrak{g}, \quad (\text{B.5})$$

which includes a parameter $c \in \mathbb{C}$. As it can be easily seen, the mCYBE reduces to the CYBE in the limit $c \rightarrow 0$. When $\mathfrak{g}$ is a real Lie algebra, the possible non-zero parameters are $c = 1$ or $c = i$ up to rescaling of the R operator as $R \rightarrow |c| R$. Depending on these values, the associated R operators are classified as ($c = 1$) *split* and *non-split* ($c = i$) type, respectively.

B.2 Lie algebra splittings

Historically, the first example of an R -matrix (operator) comes from a Lie algebra splitting. If $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra and $\mathfrak{g}$ is written as a vector space direct sum $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ of two Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ of $\mathfrak{g}$, then $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ is called a Lie algebra splitting. Let us define two projection maps $P_+ : \mathfrak{g} \rightarrow \mathfrak{g}_+$ and $P_- : \mathfrak{g} \rightarrow \mathfrak{g}_-$. An important consequence is that a Lie algebra splitting leads to an R -matrix for $\mathfrak{g}$.

Proposition B.2.1 *Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a Lie algebra splitting, with corresponding projections $P_{\pm} : \mathfrak{g} \rightarrow \mathfrak{g}_{\pm}$. The involutive endomorphism $R \equiv P_+ - P_-$ of $\mathfrak{g}$ is a solution of the mCYBE with $c = 1$ (split), namely*

$$[Rx, Ry] - R([Rx, y] + [x, Ry]) = -[x, y], \quad (\text{B.6})$$

for all $x, y \in \mathfrak{g}$. That is, R is an R -matrix for $\mathfrak{g}$.

Proof: Since this proof is quite straightforward let us present it here. For x, y we have:

$$[Rx, Ry] + [x, y] = 2([x_+, y_+] + [x_-, y_-]), \quad (\text{B.7})$$

here we have used the shorthand notation $P_{\pm}(x) = x_{\pm}$ and $P_{\pm}(y) = y_{\pm}$. On the other hand we have

$$R([Rx, y] + [x, Ry]) = R(2[x_+, y_+] - 2[x_-, y_-]) = 2([x_+, y_+] + [x_-, y_-]), \quad (\text{B.8})$$

where we have used in the last step the fact that $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$. Comparing the two results above prove that R satisfies the mCYBE. $\square$

B.3 Drinfeld-Jimbo R -matrix

Let us take a particular example here. Let $\mathfrak{g} = \mathfrak{gl}(N)$ generated by E_{ij} with $i, j = 1, \dots, N$ which satisfy the following relations

$$[E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}. \quad (\text{B.9})$$

For $\mathfrak{gl}$, the bilinear form that we will use will just be the trace in the fundamental representation, that is

$$\langle E_{ij}E_{kl} \rangle = \text{tr}(e_{ij}e_{kl}) = \delta_{ij}\delta_{kl}, \quad (\text{B.10})$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

A well known solution to the mCYBE is the so called Drinfeld-Jimbo solution given by

$$r_{DJ} = c \sum_{1 \leq i < j \leq N} E_{ij} \wedge E_{ji}, \quad (\text{B.11})$$

with $c \in \mathbb{C} \setminus \{0\}$, and the usual wedge product is given by $A \wedge B = A \otimes B - B \otimes A$. The associated linear R -matrix is given by

$$R_{DJ}(E_{ij}) = \begin{cases} +cE_{ij}, & i < j \\ 0, & i = j \\ -cE_{ij}, & i > j. \end{cases} \quad (\text{B.12})$$

The R -matrix written above with $c = i$ (non-split) is the one used to describe η -deformations of the principal chiral model and the superstring sigma model. More generally, the Drinfeld-Jimbo R -matrix is determined by fixing a borel subalgebra of $\mathfrak{g}$.

Appendix C

Conventions and useful representations

C.1 Matrix realization of $\mathfrak{psu}(1, 1|2)$

The superalgebra $\mathfrak{sl}(2|2)$ can be represented by 4×4 matrices M , provided that they satisfy the condition:

$$\text{str } M = 0, \quad (\text{C.1})$$

that is, taking m_{ij} as 2×2 block matrices, the $\mathfrak{sl}(2|2)$ superalgebra can be thought as generated by the 4×4 matrices:

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix} \iff \text{str } M = \text{tr } m - \text{tr } n = 0, \quad (\text{C.2})$$

the diagonal blocks are defined as even whereas the off-diagonal ones are odd. The algebra $\mathfrak{su}(2|2)$ is a real form of $\mathfrak{sl}(2|2)$, and it can be obtained by using the following reality condition:

$$M^\dagger H + H M = 0, \quad (\text{C.3})$$

where $H = \text{diag}(\sigma_3, \mathbf{1}_2)$.

Automorphism

The fourth order automorphism Ω acts in the following way

$$\Omega(M) = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}^\Omega = - \begin{pmatrix} K^{-1}m^t K & -K^{-1}\eta^t K \\ K^{-1}\theta^t K & K^{-1}n^t K \end{pmatrix}, \quad (\text{C.4})$$

where the supertranspose is defined as usual

$$M^{st} \equiv \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}, \quad (\text{C.5})$$

The matrix K is just the Pauli matrix σ_3

Even Generators

We denote P_m, J_{mn} as the bosonic generators of the algebra, here $m = 0, 1$ are for AdS_2 and $m, n = 2, 3$ for S^2 . We define them as follows:

$$\check{P}_m = \begin{pmatrix} -\frac{1}{2}\check{\gamma}_m & 0_2 \\ 0_2 & 0_2 \end{pmatrix}, \quad \check{J}_{mn} = \begin{pmatrix} -\frac{1}{2}\check{\gamma}_{mn} & 0_2 \\ 0_2 & 0_2 \end{pmatrix}, \quad (\text{C.6})$$

$$\hat{P}_m = \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & \frac{i}{2}\hat{\gamma}_m \end{pmatrix}, \quad \hat{J}_m = \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & \frac{1}{2}\hat{\gamma}_m \end{pmatrix}, \quad (\text{C.7})$$

where we define the $\hat{\gamma}$ in the AdS_2 and S^2 directions as follows

$$AdS_2 : \quad \check{\gamma}_0 = i\sigma_3 \quad ; \quad \check{\gamma}_1 = \sigma_2 \quad (\text{C.8})$$

$$S^2 : \quad \hat{\gamma}_2 = -\sigma_3 \quad ; \quad \hat{\gamma}_3 = -\sigma_2. \quad (\text{C.9})$$

Also, the generators of the denominator subgroups are constructed of $\hat{\gamma}$ matrix commutators as we show below

$$\check{\gamma}_{mn} = \frac{1}{2} [\check{\gamma}_m, \check{\gamma}_n] \quad ; \quad \hat{\gamma}_{mn} = \frac{1}{2} [\hat{\gamma}_m, \hat{\gamma}_n] \quad (\text{C.10})$$

In this representation, the generator P_m are elements of the grading 2 subalgebra while the generators J_{mn} are element of grading 0 subalgebra.

Odd Generators

We use matrices $Q^{I\check{\alpha}\hat{\alpha}}$, where $I = 1, 2$ and $\check{\alpha}, \hat{\alpha} = 1, 2$ are spinor indices in AdS_2 and S^2 corresponding to the fundamental representation of $\mathfrak{su}(1, 1)$ and $\mathfrak{su}(2)$ respectively

$$Q^{I\check{\alpha}\hat{\alpha}} = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 0_2 & m^{I\check{\alpha}\hat{\alpha}} \\ -\sigma_3 (m^{I\check{\alpha}\hat{\alpha}})^\dagger \sigma_3 & 0_2 \end{pmatrix} \quad (\text{C.11})$$

where

$$m^{1\check{\alpha}\hat{\alpha}} = -e^{+i\pi/4} (-1)^{\hat{\alpha}} u^{\check{\alpha}\hat{\alpha}}, \quad m^{2\check{\alpha}\hat{\alpha}} = e^{-i\pi/4} (-1)^{\hat{\alpha}} u^{\check{\alpha}\hat{\alpha}}. \quad (\text{C.12})$$

Here $u^{\check{\alpha}\hat{\alpha}}$ are 2×2 matrices with zero everywhere except at the position $(\check{\alpha}, \hat{\alpha})$ where the matrix element is equal to one.

Commutation relations for $\mathfrak{su}(1, 1|2)$

The commutation relations can directly be read off by computing the matrix multiplication, they are give by

$$[\check{P}_m, \check{P}_n] = \check{J}_{mn} \quad , \quad [\hat{P}_m, \hat{P}_n] = -\hat{J}_{mn} \quad (\text{C.13})$$

$$[\check{P}_m, \check{J}_{np}] = \eta_{m[n} \check{P}_{p]} \quad , \quad [\hat{P}_m, \hat{J}_{np}] = \eta_{m[n} \hat{P}_{p]} \quad (\text{C.14})$$

$$[Q^I \theta_I, P_m] = -\frac{i}{2} \epsilon^{IJ} Q^J \gamma_m \theta_I \quad , \quad [Q^I \theta_I, J_{mn}] = -\frac{1}{2} \delta^{IJ} Q^J \gamma_{mn} \theta_I \quad (\text{C.15})$$

$$[Q^I \lambda_I, Q^J \psi_J] = i \delta^{IJ} \bar{\lambda}_I \gamma^m \psi_J P_m - \epsilon^{IJ} (\gamma^{01} J_{01} - \gamma^{23} J_{23}) \psi_J - \frac{i}{2} \delta^{IJ} \bar{\lambda}_I \psi_J 1_{4 \times 4} \quad , \quad (\text{C.16})$$

where we have conveniently introduced the 4×4 matrices γ defined as

$$\gamma_m = \check{\gamma}_m \otimes 1_{2 \times 2} \quad ; \quad m = 0, 1 \quad \quad \gamma_m = 1_{2 \times 2} \otimes i \hat{\gamma}_m \quad ; \quad m = 2, 3, \quad (\text{C.17})$$

$$\gamma_{mn} = \check{\gamma}_{mn} \otimes 1_{2 \times 2} \quad ; \quad m = 0, 1 \quad \quad \gamma_{mn} = 1_{2 \times 2} \otimes i \hat{\gamma}_{mn} \quad ; \quad m = 2, 3, \quad (\text{C.18})$$

To get $\mathfrak{psu}(1, 1|2)$ from $\mathfrak{su}(1, 1|2)$ one simply needs to project out the generator proportional to the identity.

Supertraces

In the representation for $\mathfrak{su}(1, 1|2)$ presented in this text, the following bilinear form, induced by the supertrace, can be defined

$$\text{str}[J_{01} J_{01}] = \frac{1}{2} \quad , \quad \text{str}[P_m P_n] = \frac{1}{2} \eta_{mn} \quad , \quad (\text{C.19})$$

$$\text{str}[J_{23} J_{23}] = \frac{1}{2} \quad , \quad \text{str}[Q^I \lambda_I Q^J \psi_J] = -\epsilon^{IJ} \bar{\lambda}_I \psi_J = -\epsilon^{IJ} \bar{\psi}_J \lambda_I \quad . \quad (\text{C.20})$$

C.2 Matrix realization of $\mathfrak{su}(2, 2|4)$

The matrix realization for this superalgebra is very similar to the one we described in the first part of this appendix. In fact, some of the properties of $\mathfrak{psu}(2, 2|4)$, like the fourth order automorphism, have already been studied in chapter 2. Here we present the explicit matrix realization that was used in the computation of γ -deformations of $AdS_5 \times S^5$ in chapter 4.

An element of $\mathfrak{su}(2, 2|4)$ can be defined using 8×8 supermatrices given by

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix} \quad , \quad (\text{C.21})$$

where m and n are 4×4 bosonic even matrices, while θ and η are 4×4 odd matrices describing fermionic elements. Let us introduce an explicit basis for the bosonic part of the superalgebra $\mathfrak{su}(2, 2|4)$. Let us start by defining the 8×8 Dirac matrices as follows

$$\begin{aligned} \gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & \gamma^2 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, & \gamma^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ \gamma^4 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & i & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & \gamma^5 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \Sigma, \end{aligned} \quad (\text{C.22})$$

which satisfy the $\mathfrak{so}(5)$ Clifford algebra relations

$$\{\gamma^i, \gamma^j\} = \delta^{ij}, \quad i, j = 1, \dots, 5. \quad (\text{C.23})$$

It can be easily proven that $\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4$. Since these matrices are Hermitian $(\gamma^i)^* = (\gamma^i)^t$, the anti-Hermitian matrices constructed as $i\gamma^i$ belong to $\mathfrak{su}(4)$. As we know, we can construct the spinor representation of $\mathfrak{so}(5)$ by constructing the generators $n^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$ satisfying the relations

$$[n^{ij}, n^{kl}] = \delta^{jk}n^{il} - \delta^{ik}n^{jl} - \delta^{jl}n^{ik} + \delta^{il}n^{jk}, \quad n^{ij} = -n^{ji}. \quad (\text{C.24})$$

The Weyl spinor representation $\mathfrak{so}(6) \sim \mathfrak{su}(4)$ can be found by adding $n^{i6} = \frac{i}{2}\gamma^i$ to the set of generators, satisfying the same relations (C.24), but now $i, j = 1, \dots, 6$.

On the other hand, the set $\{i\gamma^5, \gamma^i\}$ with $i = 1, \dots, 4$ generates the Clifford algebra for $\mathfrak{so}(4, 1)$. Analogously, if we introduce $\gamma^0 = i\gamma^5$, then the generators $m^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$, with $i, j = 0, \dots, 4$ satisfy the $\mathfrak{so}(4, 1)$ relations

$$[m^{ij}, m^{kl}] = \eta^{jk}m^{il} - \eta^{ik}m^{jl} - \eta^{jl}m^{ik} + \eta^{il}m^{jk}, \quad m^{ij} = -m^{ji}, \quad (\text{C.25})$$

which compared to (C.24) involve $\eta = \text{diag}(-1, 1, 1, 1)$ instead of the Kronecker delta. We can enlarge the set of generators by defining $m^{i5} = \frac{1}{2}\gamma^i$, where $i = 0, \dots, 4$, thus obtaining a realization of $\mathfrak{so}(4, 2) \sim \mathfrak{su}(2, 2)$, with the same algebra as in (C.25), but now with the metric $\eta = \text{diag}(-1, 1, 1, 1, 1, -1)$ and $i, j = 0, \dots, 5$.

In summary, the bosonic algebras $\mathfrak{su}(2, 2)$ and $\mathfrak{su}(4)$ can be thought as spanned by the basis

$$\begin{aligned} \mathfrak{su}(4) \quad \text{spanned by} \quad & \left\{ \frac{i}{2}\gamma^i, \frac{1}{4}[\gamma^i, \gamma^j] \right\}, & i, j &= 1, \dots, 5, \\ \mathfrak{su}(2, 2) \quad \text{spanned by} \quad & \left\{ \frac{1}{2}\gamma^i, \frac{i}{2}\gamma^5, \frac{1}{4}[\gamma^i, \gamma^j], \frac{i}{4}[\gamma^5, \gamma^j] \right\}, & i, j &= 1, \dots, 4. \end{aligned} \quad (\text{C.26})$$

If we add the $\mathfrak{u}(1)$ generator $i1$, the full set of generators provides an explicit basis for the bosonic subalgebra of $\mathfrak{su}(2, 2|4)$.

C.3 Usual parametrization of $\mathbb{CP}^3$

The usual parametrization of $\mathbb{CP}^3$ can be given in terms of the following six-parametric matrix

$$Y = \begin{pmatrix} 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ 0 & 0 & y_2 & -y_1 & y_4 & -y_3 \\ -y_1 & -y_2 & 0 & 0 & y_5 & y_6 \\ -y_2 & y_1 & 0 & 0 & y_6 & -y_5 \\ -y_3 & -y_4 & -y_5 & -y_6 & 0 & 0 \\ -y_4 & y_3 & -y_6 & y_5 & 0 & 0 \end{pmatrix}. \quad (\text{C.27})$$

This matrix can be obtained by considering the following linear combination

$$Y = \sum_i y_i T_i, \quad (\text{C.28})$$

where the six matrices T_i appearing in the equation above are Lie algebra generators of $\mathfrak{so}(6)$ along the $\mathbb{CP}^3$ directions which can be written as follows

$$T_1 = E_{13} - E_{31} - E_{24} + E_{42}, \quad T_2 = E_{14} - E_{41} + E_{23} - E_{32}, \quad (\text{C.29})$$

$$T_3 = E_{15} - E_{51} - E_{26} + E_{62}, \quad T_4 = E_{16} - E_{61} + E_{25} - E_{52}, \quad (\text{C.30})$$

$$T_5 = E_{35} - E_{53} - E_{46} + E_{64}, \quad T_6 = E_{36} - E_{63} + E_{45} - E_{54}. \quad (\text{C.31})$$

Here E_{ij} denote matrices with a unit entry in the i^{th} row and j^{th} column, all other entries being zero. The matrices T_i are normalized as follows

$$\text{tr}(T_i T_j) = -4 \delta_{ij}. \quad (\text{C.32})$$

The nine generators of $\mathfrak{u}(3)$ inside $\mathfrak{so}(6)$ can be written as $[T_i, T_j]$. Of course only nine of these combinations are linearly independent.

Quite remarkably the matrix (C.27) obeys the following identities

$$Y^3 = -\rho^2 Y, \quad \rho^2 = \sum_i^6 y_i^2 = 1. \quad (\text{C.33})$$

Thus, a Lie algebra element A parametrizing the coset space $AdS_4 \times \mathbb{CP}^3$ can be represented in the following way

$$A = \begin{pmatrix} x_\mu \Gamma^\mu & 0 \\ 0 & Y \end{pmatrix}, \quad Y = \sum_i y_i T_i. \quad (\text{C.34})$$

Here Γ^u are gamma matrices satisfying the Clifford algebra of $\mathfrak{so}(3, 1)$.

In particular, an $SO(6)$ matrix parametrizing the coset space $SO(6)/U(3)$, and therefore $\mathbb{CP}^3$, can be obtained by exponentiating the generic element (C.27). The matrix exponential can be computed by using the identities (C.33), and gives a coset representative of the following form

$$g = e^Y = I + \frac{\sin \rho}{\rho} Y + \frac{1 - \cos \rho}{\rho^2} Y^2. \quad (\text{C.35})$$

Even though this coset element does not look that complicated, it does not lead to a convenient form for the Fubini-Study metric of $\mathbb{CP}^3$. Furthermore, using this parametrization makes computations a lot more lengthy.

C.4 A basis for $\mathfrak{su}(4)$

A basis for $\mathfrak{su}(4)$ can be constructed in terms of anti-Hermitian 4×4 matrices. A direct generalization of the well known Gell-Mann matrices is given below

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_9 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \lambda_{10} &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \lambda_{11} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \lambda_{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \quad (\text{C.36})$$

The first 8 matrices shown above represent a basis for $\mathfrak{su}(3) \subset \mathfrak{su}(4)$. Furthermore, these matrices are orthogonal and satisfy the following relation

$$\text{tr}(\lambda_m \lambda_n) = 2 \delta_{mn}, \quad m = 1, \dots, 15, \quad (\text{C.37})$$

The commutation relations are given by

$$[\lambda_m, \lambda_n] = 2i f_{mn}^p \lambda_p, \quad (\text{C.38})$$

The list of non-vanishing structure constants can be consulted in an appropriate textbook like [75]. In this representation, the Cartan generators are given by λ_3, λ_8 and λ_{15} , that is, those are the only generators that commute mutually

$$[\lambda_3, \lambda_8] = [\lambda_3, \lambda_{15}] = [\lambda_8, \lambda_{15}] = 0. \quad (\text{C.39})$$

Therefore the Lie algebra $\mathfrak{su}(4)$ has rank 3. Looking at the matrix representation, the Lie algebra $\mathfrak{su}(3)$ is a subalgebra of $\mathfrak{su}(4)$. Generally, the algebras $\mathfrak{su}(N')$ with $N' < N$ are subalgebras of $\mathfrak{su}(N)$.

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