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Ondas Gravitacionais em Cosmologias com Decaimento do Vácuo

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Gravitational Waves in Decaying Vacuum Cosmologies

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Resumo

Na presente monografia foram estudadas em detalhe as ondas gravitacionais primordiais em cosmologias com decaimento do vácuo. Os modelos de decaimento do vácuo são uma alternativa para resolver o problema da constante cosmológica atribuindo uma dinâmica à energia do vácuo. O problema de ondas gravitacionais primordiais é discutido no âmbito de um Universo FLRW em expansão, plano, espacialmente homogêneo e isotrópico descrito pela teoria da Relatividade Geral com decaimento da densidade de energia do vácuo do tipo $\Lambda \equiv \Lambda(H)$. Dois limites particularmente interessantes de uma classe de modelos de decaimento do vácuo foram investigados. Um termo tensorial perturbativo em primeira ordem foi introduzido na métrica de FLRW, a equação de evolução das perturbações foi derivada e depois expressa em termos de uma expansão de Fourier, onde a parte dependente do tempo desacopla-se da parte espacial. A equação resultante tem a forma de um oscilador harmônico amortecido, que depende do fator de escala, que carrega todas as características cosmológicas e do decaimento do vácuo.

No primeiro modelo estudado, o decaimento do vácuo tem a forma $\Lambda \propto H^2$. A equação da onda gravitacional é estabelecida e a sua parte dependente do tempo foi resolvida analiticamente para diferentes épocas no caso de uma geometria plana. O resultado principal é que a diferença da cosmologia Λ CDM padrão (sem decaimento do vácuo): neste modelo h amplificação de ondas gravitacionais durante a era de radiação, que em teoria quântica de campos significa produção de grávitons. Esta diferença é uma assinatura clara dos modelos de decaimento do vácuo cuja eventual observação poderia dar pistas empíricas sobre o assunto. No entanto, os modos de alta frequência são amortecidos ainda mais rapidamente do que na cosmologia padrão, tanto na era da radiação e da matériavácuo. As características físicas das ondas gravitacionais, como o módulo da função de modos, espectros de potência e de densidade de energia de onda gravitacional geradas em diferentes eras cosmológicas também foram avaliadas explicitamente.

O segundo modelo estudado é um decaimento do vácuo da forma $\Lambda \propto H^3$. Este modelo leva a uma cosmologia plana não singular que é denominado completo no sentido de que a evolução cósmica ocorre entre duas eras de Sitter extremas. A particularidade que torna interessante este modelo é que a transição do início da era de Sitter era para a fase da radiação é suave evitando o graceful exit problem. A equação de onda gravitacional é derivada e sua parte dependente do tempo foi integrada numericamente num período relevante previamente delimitado. As soluções das ondas gravitacionais para as outras eras foram calculadas analiticamente. Os espectros de hoje das ondas gravitacionais foram calculados e comparados com o resultado padrão onde é assumida uma transição abrupta. Verificou-se que o fundo estocástico de ondas gravitacionais é muito semelhante ao previsto pelo modelo de concordância cósmica mais a inflação, exceto para as frequências mais altas.

Abstract

In the present monograph we study in detail the primordial gravitational waves in cosmologies with a decaying vacuum. The decaying vacuum models are an alternative to solve the cosmological constant problem attributing a dynamic to the vacuum energy. The problem of primordial gravitational waves is discussed in the framework of an expanding, flat, spatially homogeneous and isotropic FLRW Universe described by General Relativity theory with decaying vacuum energy density of the type $\Lambda \equiv \Lambda(H)$. Two particular interesting limits of a class of decaying vacuum models were investigated. A first-order tensor perturbation term was introduced to the FLRW metric, the evolution equation of the perturbations was derived and then expressed in terms of a Fourier expansion, the time-dependent part decouples from the spatial part. The resulting equation has the form of a damped harmonic oscillator which depends on the scale factor, which carries all the cosmological and decaying vacuum characteristics.

In the first model studied, the decaying vacuum has the form $\Lambda \propto H^2$. The gravitational wave equation is established and its time-dependent part has analytically been solved for different epochs in the case of a flat geometry. The main result is unlike the standard Λ CDM cosmology (no interacting vacuum): in this model there is gravitational wave amplification during the radiation era, which in quantum field theory means graviton production. This difference is a clear signature of the decaying vacuum models which a eventual observation could give empirical clues about it. However, high frequency modes are damped out even faster than in the standard cosmology, both in the radiation and matter-vacuum dominated epoch. The physical gravitational wave quantities like the modulus of the mode function, power and gravitational wave energy density spectra generated at different cosmological eras are also explicitly evaluated.

The second model studied is a decaying vacuum of the form $\Lambda \propto H^3$. This model drives a nonsingular flat cosmology which is termed complete in the sense that the cosmic evolution occurs between two extreme de Sitter stages. The particularity which makes interesting this model is that the transition from the early de Sitter era to the radiation phase is smooth avoiding the graceful exit problem. The gravitational wave equation is derived and its time-dependent part numerically integrated in a relevant period previously delimited. The gravitational wave solutions for the other eras were calculates analytically. Today's gravitational wave spectra were calculated and compared with the standard result where an abrupt transition is assumed. It is found that the stochastic background of gravitational waves is very similar to the one predicted by the cosmic concordance model plus inflation except for the higher frequencies.

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Conventions and notation

greek indices μ, ν, \dots	run over the four spacetime coordinates
latin indices i, j, \dots	run over the three spacial coordinates
-,+,+,+	metric signature
$rac{\partial}{\partial x^{\mu}}$ or ∂_{μ} or $_{,\mu}$	ordinary derivative
$ abla_{\mu} ext{ or } {}_{;\mu}$	covariant derivative
$d\eta = a^{-1}dt$	conformal time
$\frac{dx}{dt} = \dot{x}$	derivative respect the physical time
$\frac{dx}{d\eta} = x'$	derivative respect the conformal time
$c = \hbar = k_B = 1$	physical units

Other notation is introduced as needed.

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l Chapter

Introduction

1.1 Cosmology and gravitational waves

The aim of (physical) cosmology is to study the structure, origin, evolution and future of the Universe. For this task, cosmology, as the other branches of physics, needs a theoretical background supported by fundamental physical principles to construct models that reproduce and explain the nature of the Universe. On the other hand, it also needs a experimental part in order to contrast the predictions of the cosmological models with the real Universe and to guide their construction and improvement. The modern cosmology is based on two main pillars: a gravitational theory and the astronomical observations. A gravitational theory is needed because gravity is the dominant force at cosmological scales, the other forces are important at scales where some assumptions of cosmology are not valid anymore. The astronomical observations are our window to the Universe.

Thinking in the first pillar; a gravitational theory is needed to know the dynamics of the observed Universe, to predict the future evolution and to infer its past. In order to yield a complete description of the Universe, this theory must be unified with the other fundamental physical theories at some regime to describe adequately the Universe including gravity and the other forces of nature. Also, currently exist several gravitational theories such as the General Relativity, Einstein-Cartan, Brans-Dicke, bi-metric theory, Sting Theory, Quantum Loop Gravity, etc. Of all these theories the, Einstein's General Relativity is the most successful, its predictions have been confirmed in multiple times with a great accuracy and for a large number of different phenomena. For these reasons, most of the cosmological models are supported by this theory, included the so-called *Standard Cosmological Model* which at the moment is the best model of the large scale Universe, it will be discussed later in very general terms. Despite of all its success, the General Relativity has some important limitations. General Relativity is a classical theory in the sense that is not quantized making it incompatible with the quantum physics indicating that is not the final gravitational theory, but probably is the limit of a more general quantum gravitational theory at some energy scale. This means that General Relativity is not the last theoretical pillar of cosmology, but is the best we have for the time.

Now thinking of the other pillar; cosmology as a branch of physics should always have an empirical part; without this, it cannot be considered a natural science. Ideally the object of study of cosmology is the entire Universe, but due to physical limitations, we know that we only have access to a part of it: the observable Universe. And for practical reasons we can only mathematically model the large scale Universe because entering into more details makes it makes it too complex to be manipulated. We cannot manipulate or reproduce it in the lab, so cosmology needs artifices to test its predictions and to get clues to delimit and guide the construction and improvement of cosmological models. In most of cases the only way to test cosmology with nature is to look up to the sky. The direct and indirect astronomical observations are of fundamental importance. Also simulations provide useful information about the theoretical models. Observations taken of the sky give information about the present and past structure of the Universe, this with the gravitational theory is the starting point of cosmology. The astronomy had a great development in the last century with the vertiginous evolution of technology which improved the astronomical observations in a way never seen before. This gave important empirical evidences to establish cosmology as a modern science.

Astronomy is a very ancient science. At the beginning, for centuries the mankind has cultivated it for many practical activities such as making calendars, knowing when to harvest, navigation, religious celebrations, etc. Later, in addition to the practical uses, astronomy begins to be practiced for the curiosity of the unknown. Although they are closely related, cosmology can be considered a very recent science compared to astronomy. Every civilization has a cosmogony, that is, an explication of how the Universe was created and how it is. The origin of cosmogonies dates back to the origin of man, explained in a fantastic way linked with unexplained forces and gods. Differently of cosmogony, cosmology is based in physics. The cosmology origin can be dated around the 1700's when it was stated that the stars in the Milky Way form a bigger structure and there could exist other ones. The Prussian philosopher Immanuel Kant called these structures *Island Universe*. Today we know that they were referring to galaxies. The modern cosmology begins in the early 20nd century with the famous work of Albert Einstein. Using its own theory, the General Relativity, Einstein described the first cosmological mathematical model in 1915 [1]. It was a model of a static and finite Universe as he believed it was. A few years later, were found other solutions of the Einstein equations for no-static Universes by A. Friedmann, G. Lematre, H. Robertson, A. Walker and W. de Sitter [2, 3]. This opened the possibilities for many mathematical cosmological models using General Relativity. These new mathematical cosmological models needed to be confronted with astronomical observations to be tested.

Edwin Hubble in 1929 [4], gave the first empirical proof that the universe is not static, it is expanding. He discovered that galaxies are moving away from us and also found an approximate relation between the distances with the velocity with which they are moving away, this relation is nowadays termed Hubble's law. This was a great rupture with the traditional thinking that established that the universe is infinite and static, as believed some important figures of science like Newton and Einstein. The expanding Universe lead to the *standard big bang model*, which in a nutshell proposes that the Universe emerged from a very high temperature and density state, and has been expanding and cooling since then. The initial moment of the big band model is a singularity in the sense that physical quantities are infinite being meaningless, after the singularity the Universe begins to expand.

One consequence of the Universe expansion is that at some time in the past, all the matter observed today was concentrated in a state of very high density and therefore at high temperature. This high temperature of the Universe makes the electrons not to be more attached to the nucleus, being a plasma in thermodynamic equilibrium with radiation. As the Universe expanded, the temperature decreased and at some point radiation decoupled from matter, travelling freely from then on the Universe. This radiation must be permeating all the Universe having almost the same temperature, carrying important information about the physics of the decoupling time, like a footprint of the Universe of that time. Today this radiation is known as the *Cosmic Microwave Background Radiation* (CMB). Was predicted in 1948 by the Russian physicist G. Gamow with the collaboration of R. Alpher and R. Herman [5], and then observed in 1965 by A. Penzias and R. Wilson [6]. The prediction and later observation of the CMB was of fundamental importance for cosmology. It established the basis of the physical cosmology, guiding the improvement of new cosmological models. A detailed study of the CMB reveals important details of

the microphysics and the global structure of the Universe at early times. It helps to understand why the Universe is as we see it today.

A more recent and astonishing discovery is the fact that the cosmic expansion is accelerating. Studies measuring the redshift of supernovae confirmed it in 1998 [7, 8]. A uniformly accelerated expansion needs something with repulsive gravity acting in every part of the Universe. There is no known kind of energy with this characteristic, that is why this mysterious component of the Universe is called *dark energy*. The most natural candidate to be the dark energy is the cosmological constant because in the Einstein equations for cosmology it appears with a negative pressure that acts as a repulsive gravity [9].

The most important feature about of the Universe is that at large scales it is homogeneous and isotropic, this is known as the *Cosmological Principle*. It is clear that this principle is limited, at some scale are evident the inhomogeneities. Nowadays, the cosmological model supported by the theoretical frame of General Relativity, accepting the Cosmological Principle, and compatible with most of the astronomical observations is known as the Standard Cosmological Model, or The Hot Big Bang Model. In general terms the Standard Cosmological Model and the observations confirm that our Universe [10, 11, 12]:

- Is homogeneous and isotropic at scales larger than 100 Mpc.
- Expands according to the Hubble law.
- Its age is nearly 14 billion years.
- It is pervaded by the CMB with temperature $T \simeq 2.73K$.
- The composition of baryonic matter is about 75% hydrogen, 25% helium and trace amounts of heavier elements. No substantial amount of antimatter.
- Radiation and baryonic matter contribute to around the 5% of the total energy density. The rest corresponds to the dark sector; the cold dark matter is $\sim 25\%$ and dark energy $\sim 70\%$.

Some details of this model will be shown in the next chapter.

Despite the great success of this model, it has some important problems. To name some of them: the flatness problem, the horizon problem, structure-origin problem and the magnetic-monopole problem. Also, we have to add also the *Cosmological Constant Problem.* It essentially states that when we compare the value of the vacuum energy calculated in the context of the cosmology (General Relativity) with the value given by the quantum field theory we have an error of the order of 10^{120} . Many ideas and alternatives had been proposed in an attempt to solve or alleviate this and other problems of the standard cosmology [13, 14].

To solve part of these problems, Alan Guth in 1982 developed the inflation theory [15]. It states an exponential expansion of the Universe in a very brief period in its primordial stages driven by a scalar field called inflaton. This rapid expansion not only solves some problems, but it can also be the generating mechanism of primordial perturbations. This is possible because the quantum fluctuations of the vacuum (virtual pairs of particles and antiparticles constantly created and annihilated in the quantum vacuum) can be materialized by the accelerated expansion. Inflation can separate two virtual particles before they annihilate disassociating them, creating a pair of real particles. By this way the inflation offers a possible mechanism of creation of particles different of baryogenesis .

In General Relativity, we have three types of fluctuations of the gravitational field: scalar, vector and tensor. Of these, the tensor perturbations are associated with the field itself, they are "perturbations of gravity" or *gravitational waves* (GWs) which quantum interpretation are *gravitons*.

During inflation the primordial tensor perturbations are generated in principle in all wavelengths and due the cosmic expansion they are stretched abruptly. Perturbations with wavelengths greater that the Hubble radius "freeze" (do not interact with the gravitational field) until the Universe expands sufficiently and enter into its casual horizon. With this, all tensor perturbations at the end of inflation have almost the same amplitude, we can think this as the probability of creation of particles in inflation is the same in all the points of the Universe.

One mechanism of creation of particles in an asymptotic state Universe is studied by the quantum field theory in curved spacetimes [16, 17]. In this context the basic principle is that the concept of vacuum energy in curved spacetimes is not well defined for all spacetimes [18]. This fact is of great importance in the study of creation of gravitons due to the cosmic expansion. Another important point is that these primordial quantum perturbations are not restricted to the inflation mechanism [19]. The inflation is a sufficient condition for generate them, but not a necessary condition. All that is necessary are the tensor perturbations over a classical background metric and a variable gravitational field of an expanding Universe.

These tensor perturbations of the gravitational field created en the very early Universe are commonly named in the specialized literature as "primordial GWs" or "relic gravitons". Its study is fundamental for cosmology of the early Universe because they have traveled freely and unperturbed since the moment they were created. This happens because of the weak interaction of gravity with the other forces of nature. On one side, the CMB is a picture of the Universe at a time about 400,000 years after the Big Bang. Studies about nucleosynthesis [20] of how the primordial hydrogen, helium, deuterium, and lithium were created reveal the conditions of the Universe few minutes after the Big Bang. On the other hand, primordial GWs were produced at times earlier than $\sim 10^{-24}$ seconds after the Big Bang, their spectrum is the footprint of the very (very!) early Universe. Its observation would give invaluable information about the primordial times, setting a new comprehension of the Cosmos. It also would be a test of inflationary theories and high energy physics, providing information about the fundamental interactions of physics at the characteristic energies of that epoch, which are far higher than we can reach with current accelerators. The observation of primordial GWs would undoubtedly be one of the most important measurements astronomy could make.

The sources of GWs can be of different natures, from quantum vacuum fluctuations of the early Universe as said before, but also from astronomical sources as inspiral black hole binaries, supernovae, pulsars among others. These different sources have their own GW characteristics and their possible observation will be a new and useful tool for astronomy and cosmology. The interaction between a GW with matter is very small compared to the interaction between electromagnetic radiation and matter. Because of that, the GW signal from a source would travel as if there was nothing in its way, without absorption and dispersion as if occurs with electromagnetic radiation. They would reach the detector bringing important information about its source, which could not be obtained via common light observations. From electromagnetic radiation we get almost all astronomical data cosmology. The measurement of distances and velocities of objects at cosmological distances are of fundamental importance to infer other important cosmological parameters. For that reason electromagnetic radiation based detectors need to be more and more potent in issue to observe more distant objects with more accuracy. This leads to the necessity to construct devices that demand more technology and money. An alternative is given by GW measurements of cosmological parameters. Observations of massive black hole coalescence at cosmological distances could determinate the distance to the source with a better accuracy than the usual methods since this system is like a standard siren. If the wave amplitude, frequency, and chirp rate of the binary can be measured, then its luminosity distance can be inferred [21]. Intermediate mass ratio inspirals, systems that correspond to the inspirals of intermediate mass black holes of mass ~ $10^3 - 10^4 M_{\odot}$, should provide an accurate determination of the Hubble Constant [22]. Also with GW astronomy it could be characterized the evolution of the dark energy, which usually is described inserting a parameter ω in the equation of state of dark energy, $p = \omega \rho$. If $\omega = -1$, then the dark energy is equivalent to a cosmological constant [23] and the energy density will be the same at all epochs. If $\omega > -1$, the dark energy is an evolving field whose energy density diminishes with time. According with [24], GW measurements have the potential to measure ω with an accuracy of the 10% (for advanced ground-based detectors) and around 4% (for space-based detectors) [25].

For these reasons it is important the study of GWs of primordial or astrophysical origin from the theoretical framework, clarifying its origin and nature from the fundamental theories of gravitation, developing different mathematical techniques that allow detailed models depending on the GW source and numerical simulations that yield an idea of how it would be possible to observe them. And its observational search, developing methods and technology for a direct or indirect detection, building new and better detectors than the current ones and look for novel mechanisms not previously used to facilitate its observation.

1.2 Cosmology with decaying vacuum

In cosmology the cosmological constant, usually denoted by Λ , commonly is associated with the the vacuum energy. It was introduced by Albert Einstein in his General Relativity field equations to hold back gravity and achieve a static universe, which was the most accepted view of the Universe at the time and for him. Einstein abandoned it after Hubble's 1929 discovery about the expanding away of distant galaxies [4], implying an overall expanding Universe. This discovery is consistent with the cosmological solutions of the Einstein equations found by Friedmann. Later, Einstein accepted the validity of the expansion of the Universe, he removed the cosmological constant term of the equations calling it the "biggest blunder" of his life. Surprisingly decades later of the elimination of the cosmological constant, the discovery of cosmic acceleration in 1998 by Riess, Perlmutter and Schmidt [7, 8] have revived the need for a non-zero cosmological constant, this time to add a small acceleration to the cosmic expansion.

The Einstein's field equations with cosmological constant Λ are:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \qquad (1.1)$$

where the Ricci tensor $R_{\mu\nu}$ and the metric tensor $g_{\mu\nu}$ describe the geometric structure of the spacetime, the energy-momentum tensor $T_{\mu\nu}$ describes the matter and energy, and G is Newton's gravitational constant. When Λ is zero, (1.1) reduces to the original field equation of General Relativity. When $T_{\mu\nu}$ is zero, the field equation describes empty space, i.e. the vacuum. One important feature of Λ in its original conception, is that it is not dynamic, the vacuum energy does not depend on time nor on space. A positive vacuum energy density resulting from the cosmological constant implies a negative pressure, $p_{\Lambda} = -\rho_{\Lambda}$ [26]. If the energy density is positive, the associated negative pressure will drive an accelerated expansion of the universe as observed. This is the basic motivation to associate the cosmological constant with dark energy. The present Standard Cosmological Model with a cosmological constant included is in very good agreement with the available astronomical observations giving confidence to the introduction of Λ into the relativistic cosmology.

Nevertheless, when the quantum vacuum energy density is introduced important problems appear. Probably, the most important one is the so-called Cosmological Constant Problem [14] mentioned quickly in the previous section. Also we have the *Cosmic Coincidence Problem*; that in few words its asks why the matter density, which is a time varying quantity, is now so close to the dark energy density which is constant. If dark energy begins to dominate too early in the Universe history, entities such as galaxies (and consequently life) would never have had a chance to form, a matter dominated era is needed for structure formation. In fact, it becomes relevant exactly at the present time. This astounding "coincidence" is one of the greatest mysteries of contemporary cosmology.

Notice that the Cosmological Constant Problem concerns into the two main areas of modern physics, General Relativity and Quantum Field theory. The great difference between the vacuum energy density estimated in each of these two theories clearly shows that there is an inconsistency in the definition of the vacuum state. Despite the great simplifications made to do the calculation, it seems unlikely that a new calculation taking account more details of the physics can make compatible both calculations. The origin of the Cosmological Constant Problem is deeper than the simple calculations, it is at the foundations of the definition of vacuum state. There are several ways of trying to solve this problem and the solution probably comes from a quantum theory of gravity.

One alternative is to consider a time dependence of Λ . This kind of models was studied even before the discovery of the accelerating Universe [27]. Its principal motivations are the Cosmological Constant Problem together with problems about the age of the Universe, accelerating cosmological models and the Cosmic Coincidence Problem. A dynamical vacuum allows to evolve from a state of high energy in the early Universe as expected by Grand Unified Theories and then decay various steps as the Universe cools down due to its own expansion reaching today the very small value as observed. Thus, in principle, the Cosmological Constant Problem could be solved by this way. But things are not so simple, it is necessary to investigate whether a $\Lambda(t)$ is compatible with both the theoretical cosmology as well as with astronomical observations.

The first functional forms of $\Lambda(t)$ were phenomenological, proposed with free parameters adjusted to fit with observational data. Despite the efforts, the precise functional form still unknown and it is worth mentioning that the most usual critique to these $\Lambda(t)$ scenarios is that in order to establish a model and study its observational and theoretical predictions, we need first to specify a phenomenological time-dependence for Λ . Are approaches of the $\Lambda(t)$ models from the Quantum Field Theory in the context of the renormalization group. These dynamical vacuum energy models emphasize the evolution of the vacuum energy as a function of the Hubble function, i.e. $\Lambda(t) = \Lambda(H(t))$, namely functions containing powers of H and including also an additive constant term. These proposals were confronted with observational data like supernovae data in [28], and later on with the modern observations of supernovae, baryonic acoustic oscillations, CMB and structure formation in [29, 30, 31]. These models face the Cosmic Coincidence Problem and some aspects of the Cosmological Constant Problem [32].

This new scenario enriches the cosmology, solving and/or alleviating important cosmological problems, but also opens up new horizons. As a fundamental piece the modern theoretical physics, the vacuum as physical entity must be studied exhaustively. Great theoretical efforts have been made to address this new paradigm of the energy of the vacuum with observations. But it still needs to be investigated in depth from various physical viewpoints. And one of those points is to investigate in what way it affects the production of GWs in the early universe. Primordial GWs originated from small perturbations due the vacuum quantum fluctuations contain information both from cosmological models as well as from the model of the vacuum energy. So, direct or indirect observations of the GW spectra could give proofs about the $\Lambda(t)$ models.

1.3 Outline of the thesis

In this work, the main focus is the study of the primordial GWs for decaying vacuum cosmologies. The two principal physical foundations on which it is based are the theory of GWs in the General Relativity context and cosmological models with decaying vacuum. The first three chapters are devoted to showing an overview of these themes. In chapters four and five are described specific themes in the study of the primordial GWs in decaying vacuum cosmologies, which are the cosmological tensor perturbations mathematical background and the $\Lambda(t)$ models. The next two chapters, six and seven, present the new results obtained during the PhD. First, it was calculated the modulus of the mode functions, the power spectrum and energy density of the primordial GWs for different cosmological eras with a $\Lambda \propto H^2$ model. Then it was studied the case $\Lambda \propto H^3$, which gives an interesting non-singular cosmological model where the transition from early inflation to the standard radiation phase is smooth having no exit problem. Finally, in chapter eight it is discussed the perspectives and conclusions about the results obtained. This work is structured as follows:

The chapter 2 is devoted to GWs and its importance in cosmology, beginning with an overview of the physics of GWs as well as its possible detection with current detectors. Continuing with the basics of the gravitational weak-field formalism showing how the linearized Einstein's field equations give in a natural way, a wave equation for small perturbations. The next section we will show the plane-wave solution of the wave equation obtained and its effects on matter. In the last two sections of this chapter will be dedicated to the called stochastic GW background focusing on the primordial gravitational waves and its importance in cosmology.

In chapter 3 we present a brief but self-consistent review of the Standard Cosmological Model, arguing the principal motivations and physical foundations of this model, summarizing its main physical and dynamical features. It will be described the characteristics of the main phases of the evolution of the Universe as a function of the characteristic energy in each cosmic era. We will show the advantages of the cosmological model as well as its main problems. Also, how these problems of the Standard Cosmological Model motivated the proposal of the inflationary paradigm, how does it works and how inflation solves in a simple and elegant way these problems. Finally, we will mention how the cosmologicalconstant term of the Einstein's field equations can be interpreted as the vacuum energy of the Universe, and also why this leads to the Cosmological Constant Problem when we compare the vacuum of the Universe in the context of cosmology with the vacuum calculated in the context of the quantum field theory.

The fourth chapter is devoted to the physical and mathematical motivations of the cosmological tensor perturbations. This chapter begins with an introduction to the classical theory of tensor perturbations in cosmology and it explains the phenomenon of superadiabatic amplification of GWs. It will be also introduced quantum tensor fluctuations, deriving the particle (graviton) creation and annihilation operators and showing its physical justification. The Bogoliubov transformations are presented showing the mechanism of production of gravitons due to transitions between different vacuum states. And finally we will define the power and energy density spectra and their importance in the primordial GW study.

Next, in chapter 5 presents the generalities of the time-dependent cosmological constant models, $\Lambda(t)$. First it is presented the motivation of the proposal of this family of models which come from the Cosmological Constant Problem, and how it solves or alleviates it. Then it is presented the phenomenological arguments that give the functional form of $\Lambda(H)$, to then be justified by the renormalization group formalism. At the end of this chapter it is shown an example of a $\Lambda(H)$ cosmology for early and late times.

The chapter 6 presents the first general results as well as the calculations and analysis. It begins with the introduction of a particular $\Lambda \propto H^2$ model in the Friedmann equations to then calculate the scale factor in its general form. Using the equations of state for each cosmological era it was calculated the scale factor for each of them and it was used the continuity conditions to obtain the explicit expression of the scale factor. Done this, we have an equation of the modes function for each cosmological era. Then detailed calculations of the modes function are shown as well as the calculation of its integration constants, finally obtaining an explicit solution for the era of inflation, the radiation and matter. The modulus of the mode functions, power spectrum and energy density spectrum were calculated and plotted for different values of decaying vacuum parameter β . The main results of this part have been submitted for publication [33].

The second part of the results are shown in chapter 7. A $\Lambda \propto H^3$ model is used, which is a nonsingular flat cosmology with the characteristic that the cosmic evolution occurs between two extreme de Sitter stages (early and late time de Sitter phases). The GW equation is derived and solved, the generated spectrum of GWs is compared with the standard calculations (no-decaying vacuum). It is found that the spectrum is very similar to the one predicted by the cosmic concordance model plus inflation except for high frequencies. The first results of this part have been submitted for publication [34].

Finally in chapter 8, we outline the main conclusions obtained at the moment and establish the future perspectives of the present work. There are presented the limitations and improvements of the model as well the next steps to be done.

Chapter 2

Gravitational waves and cosmology

2.1 General Relativity

The first mathematical theory of gravity was formulated by Isaac Newton in 1687, included in his magnanimous book "Philosophiæ Naturalis Principia Mathematica". Newton's Law of Universal Gravitation states that any two bodies in the Universe attract each other with a force that is directly proportional to the product of its masses and inversely proportional to the square of the distance between them. The interaction between two bodies is an instantaneous action at a distance, if one body moves the other "feels" immediately the change in the force, avoiding the existence of an "interaction messenger" as occurs in the classical electromagnetism where, if a charge moves, it produces an electromagnetic wave that travels in space to find another charge producing the effect of the electromagnetic force. Consequently, Newton's Law of Universal Gravitation does not predict GWs. The instantaneous action at distance of Newton's gravity is completely incompatible with the Special Relativity whose basis lies in establishing the speed of light as a physical limit. Therefore, it is essential to a gravitational theory to be the generalization of the Newton's Law of Universal Gravitation and compatible with Special Relativity. Today we know this theory as General Relativity.

The General Relativity theory is the geometric theory of gravitation published by Albert Einstein in 1915 [1] and the current description of gravitation in modern physics. General Relativity generalizes Einstein's Special Relativity and Newton's Law of Universal Gravitation, providing a unified description of gravity as a geometric property of space and time as an unique entity, or spacetime. Basically states it relation between the curvature, the geometric properties of the spacetime, and the energy and momentum of its material content. As John's Wheeler famous quote says [35]: "Matter tells spacetime how to curve, and spacetime tells matter how to move."

The relation is specified by the Einstein equations which is a system of ten non-linear partial differential equations.

Some of the predictions of General Relativity differ significantly from those of classical physics and sometimes are counterintuitive. Examples of such differences include gravitational time dilation, gravitational lenses (the bending of light due to massive bodies which, for instance, multiple images of the same distant astronomical object may visible in the sky), the gravitational redshift of light, black holes (regions of space in which space and time are distorted in such a way that nothing, not even light, can escape) and the GWs (ripples of spacetime propagating with the speed of light). All these predictions have been confirmed by direct and indirect observations and experiments. However, unanswered questions remain, the most fundamental being how General Relativity can be reconciled with the laws of quantum physics to produce a complete and self-consistent theory of quantum gravity.

The General Relativity mathematical basis is the Riemannian geometry and all its terms are expressed in this context. The Einstein's field equations are (without cosmological constant):

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$
 (2.1)

On the left-hand side there is a specific divergence-free combination of the Ricci tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$. The term $G_{\mu\nu}$ is called the *Einstein tensor* which is symmetric and contains the geometrical information of the spacetime. On the right-hand side, $T_{\mu\nu}$ is the energy-momentum tensor that represent the material content (anything that is not the gravitational field) of the system. By construction of the Einstein tensor gives $\nabla_{\nu}G^{\mu\nu} = 0$ that implies that the energy-momentum tensor satisfies a conservation law

$$\nabla_{\nu}T^{\mu\nu} = 0. \tag{2.2}$$

For details about the General Relativity theory there exists many textbooks in the specialized literature, to cite some of them [12, 36, 37, 38]. General Relativity will be our gravitation theoretical background in this work.

2.2 General idea of gravitational waves

A naive physically definition of GWs can be they are small ripples propagating as waves on a certain background spacetime metric, like waves in the ocean. They are waves of curvature carrying energy and momentum travelling at the speed of light. They are a direct prediction of the General Relativity, but also are contemplated in other gravity metric theories.

They are produced by several phenomena like the accelerated movement of an object with mass, collisions of localized massive objects like black holes and/or neutron stars, non-spherical gravitational collapse, binary quasi Keplerian systems, amplification of vacuum fluctuations, decay of cosmic strings, etc. In fact, any accelerated mass emit GWs in the same way that an accelerated charged particle emits electromagnetic radiation. Even the movement of the wings of a small bird emit (astonishingly very weak) GWs.

Its effect is to stretch and contract the matter configuration, making its detection apparently seem pretty easy. But its detection is not trivial because GWs interact with all the test particles at the same time making very difficult to define a reference frame to measure this interaction, unlike with electromagnetic waves where uncharged particles can be used as a reference point to measure the relative motion of charged particles. The electromagnetic waves are very easy to detect and observe; our eyes, radio antennas, photographic cameras do this all the time. The reason is because radio waves interact very strongly with charged particles, for example with the free electrons in an antenna. Unlike radio waves, GWs interact with all forms of matter, both charged and neutral, but its interaction is far weaker than any electromagnetic force. For example, a supernova explosion in our own galaxy would be a strong source of gravitational radiation (GWs), yet a 1 km ring would deform no more than a one thousandth the size of an atomic nucleus. This makes the direct detection of GWs extremely challenging.

As early as 1776, Laplace searching for an explanation for the observed decrease of the Moon's orbital period with respect to ancient eclipse observations, had considered the possibility of an orbital damping force arising from a finite speed of propagation of gravity. Later in 1908, Poincaré anticipated the idea of GW, suggesting that planetary orbits must slowly lose energy through emission of waves in the gravitational field.

But the history of the GWs began in 1916, when Einstein realized the propagation

effects of the finite velocity in the gravitational equations and predicted the existence of wavelike solutions of the linearized vacuum field equations. He found some solutions with longitudinal components that do not transport energy and others with transverse components that do. Eddington in 1922, without assuming propagation at the speed of light, discovered that the velocity of the solutions with longitudinal components had a coordinate dependence, demonstrating that the only speed of propagation relevant to the GWs is the speed of light.

In 1936 Einstein and Nathan Rosen publish a controversial paper entitled "Do Gravitational Waves exist?", and their answer was no. They had found an exact solution to the field equations of General Relativity, which described plane GWs, but they necessarily had to introduce singularities into the components of the metric which describes it. They supposed they had shown that regular periodic wavelike solutions to the equations were impossible. Their result is incorrect, Howard Percy Robertson found that the singularity could be avoided by constructing a cylindrical wave solution [39]. Regarding the existence of gravitational radiation, Feynman in 1957 argued that a particle lying in a stick would be moved back and forth against the stick by a passing GW, the friction would generate heat, making this a mechanism to extract energy from it. Furthermore, he argued that any system which could be an absorber of GWs, could also be an emitter. For these reasons, he expected GWs to exist.

In the decade of the 1960s Joseph Weber of the University of Maryland designed the first GW detector, being the pioneer in this field. It consists of a solid bar of metal isolated from outside vibrations. It strains in space due to a passing GW that excites the bar's resonant frequency and could thus be amplified to detectable levels. He announced in 1969 that he had detected GWs. The trouble was that calculations suggested that Weber was seeing far too much gravitational radiation. According to the accepted theory, with his calculated sensitivity, he should have seen nothing. A number of groups tried to repeat his experiment and they eventually concluded that Weber was wrong.

Then the works of Bondi [40] in the 60s, gave a robust framework about GWs. In 1970 Burke and Thorne derived the quadrupole formula for the emission from binary systems. Then the discovery of the first binary pulsar by Hulse and Taylor in 1974 revolutionized the research in the field, providing the first indirect test of strong field effects of General Relativity and energy loss of a system by GW.
The second generation of GW detectors is like Weber's except that they are cooled to liquid-helium temperatures. This design of detector dominated the field from 1975 until the early 1990s. Once more, the consensus is that they have still detected nothing. The third generation of detectors was developed in the 1990s, using a different technique. They use interferometry: carefully controlled beams of laser light are directed along long arms and reflected back to the origin. The way the light beams interfere with each other reveals any comparative changes in the arm's length during the passage of the light.

The first decade of the 2000s, the GW research developed intensely both theoretically and observationally. Numerical simulations of GWs from a large number of sources, both localized (collisions between black holes and/or neutron stars, supernovae, nonaxisymmetric collapse etc.) and non localized (stochastic GW background and primordial GWs) have become of great relevance in the field. Also, the sensibility of the detector have increased considerably: a fourth generation of space-based interferometers was planned (that has not been possible due to funding problems). In 2014 it was announced the first indirect observation of primordial GWs looking their imprint in the CMB polarization [41]. This announcement caused great excitement in the specialized community, detailed studies showed that actually the galactic dust effect was observed [42].

The first indirect observational evidence of the existence of GWs came in 1974 when Richard Hulse and Joseph Taylor using the Arecibo 305m antenna discovered the binary pulsar PSR1913 + 16 [43]. They were awarded with the 1993 Nobel Prize in Physics for their discovery. They detected a pulsed radio emission for a rapidly rotating, highly magnetized neutron star. After timing the pulses for some time, they noticed that there was a systematic variation in the arrival time of the pulses. Such behavior is predicted if the pulsar was in orbit around another (neutron) star. Observations have shown that the pulsars orbits are gradually contracting. Consequently, there must be an energy-loss mechanism which can be explained by gravitational-radiation emission. The orbit of the binary system has decayed since its discovery in precise agreement with the gravitationalradiation energy-loss calculations: the ratio of observed to predicted rate of orbital decay is 0.997 ± 0.002 [44].

Subsequently, many other binary pulsars have been observed, all fitting very well with the GW predictions [45]. These observations showed the consequences of gravitational radiation, but do not allow us to measure the GWs themselves. The next goal is a direct observation. Nowadays large international collaborations are working hard for it, nearly a century after its theoretical prediction. For a detailed review see [25, 46] and references therein.

2.3 Gravitational wave detectors

We will briefly summarize in this section the GW detectors. For a detailed review about this theme, see [47]. The experimental search for direct observation of gravitational radiation begun only in the 1960s with the pioneering work of Weber using resonant bar detectors. Modern forms of the Weber bar operate with some improvements like cryogenically cooled, with superconducting quantum interference devices to detect vibration. Weber bars are not sensitive enough to detect anything but extreme powerful GWs [48]. Five cryogenic resonant-bar detectors are in operation since 1990: ALLEGRO, AURIGA, EXPLORER, NAUTILUS and NIOBE.

MiniGRAIL is another type of detector: it is a spherical GW antenna using the same principle of the Weber bars [49]. It is based at Leiden University, consist in a 1150 kg sphere cryogenically cooled to 20mK. The spherical configuration allows for equal sensitivity in all directions, and is somewhat experimentally simpler than larger linear devices requiring high vacuum. Events are detected by measuring deformation of the detector sphere. MiniGRAIL is highly sensitive in the 24 kHz range, suitable for detecting GWs from rotating neutron star instabilities or small black hole mergers. A similar detector named "Mario Schenberg" is currently operating at the University of São Paulo, which will strongly increase the chances of detection by looking at coincidences [50].

A more sensitive type of detector uses laser interferometry to measure the motion between separated free masses induced by gravitational radiation. This allows the masses to be separated by large distances, increasing the signal. A further advantage is that it is sensitive to a wide range of frequencies (not just those near a resonance as is the case of Weber bars). Ground-based laser interferometers GW detectors are now operating in a few countries. In Japan with the detector TAMA300 [51], in the United States the Laser Interferometer Gravitational wave Observatory (LIGOs) [52], the German-British GEO600 [53] and the Italo-French VIRGO [54]. The Laser Interferometer Space Antenna (LISA) was planned by the European Space Agency (ESA) and the NASA. However, on April 8, 2011, the NASA announced that it would likely be unable to continue its LISA partnership with the ESA, due to funding limitations. The ESA began a full revision of the mission's concept and renamed it as the New (or Next) Gravitational wave Observatory (NGO).

Laser interferometer GW detectors are L-shaped instruments that consist in two kilometer-size arms. Laser beams are bounced back and forth along the two arms, being reflected by mirrors at the ends. These mirrors are suspended by wires, approximating free-falling masses. The reflected beams are recombined and their interference pattern is monitored. A GW passing through the detector causes dilatation on the arms and shift the interference pattern. The amplitude of the displacement will be extremely small in comparison with the arm's length, like the size of an atom's nucleus. The spectrum of these displacements carries the physical information of the gravitational radiation of the source. Unlike optical and radio telescopes, which monitor only a tiny portion of the sky at any one time, GW detectors are sensitive to radiation coming from a wide region of the sky. In fact, an interferometer can monitor more than 40% of the sky at any one time. A world network of just four or five detectors could cover the entire sky.

Despite all these advances and efforts of the different types GW detectors, a direct observation is still not confirmed yet. Even with a sensitivity equivalent of measuring the distance to the nearest star to the accuracy of the width of a human hair, recognizing GW signals will have to be aided by a clear understanding of detector performance and background noise, accurate source modelling and robust data processing techniques.

2.4 Wave equation from weak field theory

The non-linear nature of the gravity described by the Einstein equations makes it hard to distinguish GWs from other curvature contributions of other physical sources and also makes difficult to find wave solutions. For this reason is useful to look for solutions describing waves carrying a small amount of energy and momentum, and in a very good approximation do not interfere with its own propagation neither with the background. This approach is the so-called weak field approximation. Basically the idea is to make an expansion in the deviations of an unperturbed, non-radiative configuration and linearizing the Einstein equations in vacuum. This formalism was first developed by Lifshitz in 1946 [55] and for a more detailed and extensive description of the GW physics, see [46].

Sufficiently far away from any source of curvature the spacetime can be considered flat, that is, as the Minkowski spacetime. So that, a weak gravitational field could be said that is a spacetime almost flat. By this way we can express the metric tensor for the weak field as Minkowski spacetime plus a small contribution in the following way

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad |h_{\alpha\beta}| \ll \eta_{\alpha\beta}. \tag{2.3}$$

One important aspect to consider now are the *Lorentz transformations*, which in Special Relativity shows the way to change from one reference frame to another, they are given by

$$(\Lambda^{\overline{\alpha}}{}_{\beta}) = \begin{pmatrix} \gamma & -v\gamma & 0 & 0\\ -v\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1 - v^2}}.$$
 (2.4)

For a point of the spacetime acts as, $x^{\overline{\alpha}} = \Lambda^{\overline{\alpha}}_{\beta} x^{\beta}$, and for the Minkowski metric $\Lambda^{\mu}_{\overline{\alpha}} \Lambda^{\nu}_{\overline{\beta}} \eta_{\mu\nu} = \eta_{\overline{\alpha}\overline{\beta}}$. If we change our reference frame we notice that the metric obeys

$$g_{\overline{\mu}\overline{\nu}} = \Lambda^{\alpha}_{\overline{\mu}}\Lambda^{\beta}_{\overline{\nu}}g_{\alpha\beta} = \Lambda^{\alpha}_{\overline{\mu}}\Lambda^{\beta}_{\overline{\nu}}(\eta_{\alpha\beta} + h_{\alpha\beta}) = \eta_{\overline{\mu}\overline{\nu}} + \Lambda^{\alpha}_{\overline{\mu}}\Lambda^{\beta}_{\overline{\nu}}h_{\alpha\beta} = \eta_{\overline{\mu}\overline{\nu}} + h_{\overline{\mu}\overline{\nu}}, \qquad (2.5)$$

where we have defined $h_{\overline{\mu}\overline{\nu}} \equiv \Lambda^{\alpha}_{\overline{\mu}} \Lambda^{\beta}_{\overline{\nu}} h_{\alpha\beta}$. So we can work as if we have a flat spacetime with an additional term $h_{\mu\nu}$ defined in it, and use the Minkowski metric $\eta_{\mu\nu}$ to rise and lower the indices. The unique exception to this rule is the own metric tensor $g_{\mu\nu}$, because the additional term has to be considered at first order approximation. We need to know explicitly the form of $g^{\mu\nu}$, defining $g^{\mu\nu} = \eta^{\mu\nu} + \gamma^{\mu\nu}$, where $\gamma^{\mu\nu}$ is also a small perturbation. To know who is $\gamma^{\mu\nu}$ we use the fact that $g^{\mu\nu}$ is by definition the inverse of $g_{\mu\nu}$, thus from $g^{\mu\alpha}g_{\nu\alpha} = \delta^{\mu}_{\nu}$:

$$g_{\alpha\beta} g^{\alpha\beta} \sim \delta^{\alpha}_{\alpha} = (\eta_{\alpha\beta} + h_{\alpha\beta})(\eta^{\alpha\beta} + \gamma^{\alpha\beta}) = \eta_{\alpha\beta}\eta^{\alpha\beta} + \eta_{\alpha\beta}\gamma^{\alpha\beta} + \eta^{\alpha\beta}h_{\alpha\beta} + h_{\alpha\beta}\gamma^{\alpha\beta},$$

as $\eta_{\alpha\beta}\eta^{\alpha\beta} \sim \delta^{\alpha}_{\alpha}$ and if we consider only first order terms $h_{\alpha\beta}\gamma^{\alpha\beta} = 0$, so we have

$$\eta_{\alpha\beta}\gamma^{\alpha\beta} + \eta^{\alpha\beta}h_{\alpha\beta} = 0$$

$$\Rightarrow \quad \gamma^{\alpha\beta} = -h^{\alpha\beta}$$

$$\Rightarrow \quad g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}.$$
(2.6)

We can calculate the Riemann and Ricci tensors in this approximation substituting $g_{\mu\nu}$

and $g^{\mu\nu}$ in its definitions. Considering only first order terms, for the Riemann tensor we have:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2} (\partial_{\beta}\partial_{\mu}g_{\alpha\nu} + \partial_{\alpha}\partial_{\nu}g_{\beta\mu} - \partial_{\beta}\partial_{\nu}g_{\alpha\mu} - \partial_{\alpha}\partial_{\mu}g_{\beta\nu}) = \frac{1}{2} (\partial_{\beta}\partial_{\mu}h_{\alpha\nu} + \partial_{\alpha}\partial_{\nu}h_{\beta\mu} - \partial_{\beta}\partial_{\nu}h_{\alpha\mu} - \partial_{\alpha}\partial_{\mu}h_{\beta\nu}), \qquad (2.7)$$

and for the Ricci tensor

$$R^{\alpha}{}_{\mu\alpha\nu} \equiv R_{\mu\nu} = \frac{1}{2} (\partial^{\alpha}\partial_{\mu}h_{\nu\alpha} + \partial^{\alpha}\partial_{\nu}h_{\mu\alpha} - \partial_{\mu}\partial_{\nu}h - \partial_{\alpha}\partial^{\alpha}h_{\mu\nu}), \qquad (2.8)$$

where $h \equiv h^{\alpha}_{\alpha}$ is the trace of $h_{\mu\nu}$, and we denote $\partial^{\alpha} = \eta^{\alpha\beta}\partial_{\beta}$. The calculus of the Einstein tensor in the weak field approximation can be done in a similar way, but is more convenient to define before the *trace-reverse tensor*

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{\eta_{\mu\nu}h}{2}, \qquad (2.9)$$

and write the Einstein tensor in terms of it, this follows

$$G_{\mu\nu} = \frac{1}{2} (\partial^{\alpha} \partial_{\mu} \overline{h}_{\nu\alpha} + \partial^{\alpha} \partial_{\nu} \overline{h}_{\mu\alpha} - \partial_{\alpha} \partial^{\alpha} \overline{h}_{\mu\nu} - \eta_{\mu\nu} \partial^{\alpha} \partial^{\beta} \overline{h}_{\alpha\beta}), \qquad (2.10)$$

using (2.1) we obtain

$$\partial^{\alpha}\partial_{\mu}\overline{h}_{\nu\alpha} + \partial^{\alpha}\partial_{\nu}\overline{h}_{\mu\alpha} - \partial_{\alpha}\partial^{\alpha}\overline{h}_{\mu\nu} - \eta_{\mu\nu}\partial^{\alpha}\partial^{\beta}\overline{h}_{\alpha\beta} = 16\pi T_{\mu\nu}.$$
 (2.11)

These last equations can been simplified considering an arbitrary small change of coordinates $x^{\overline{\mu}} \to x^{\mu} + \xi^{\mu}$, where ξ^{μ} is a small vector in the sense of $|\partial_{\nu}\xi^{\mu}| \ll 1$. The Jacobian matrix is

$$\Lambda^{\overline{\mu}}_{\nu} = \partial_{\nu} x^{\overline{\mu}} = \delta^{\mu}_{\nu} + \partial_{\nu} \xi^{\mu}, \qquad (2.12)$$

and the inverse (at first order)

$$\Lambda^{\nu}_{\overline{\mu}} = \delta^{\nu}_{\mu} - \partial_{\mu}\xi^{\nu}. \tag{2.13}$$

The first order transformation of the metric $g_{\overline{\mu}\overline{\nu}}$ is

$$g_{\overline{\mu}\overline{\nu}} = \Lambda^{\alpha}_{\overline{\mu}}\Lambda^{\beta}_{\overline{\nu}}g_{\alpha\beta} = (\delta^{\alpha}_{\mu} - \partial_{\mu}\xi^{\alpha})(\delta^{\beta}_{\nu} - \partial_{\nu}\xi^{\beta})(\eta_{\alpha\beta} + h_{\alpha\beta})$$

$$= \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\eta_{\alpha\beta} + \partial_{\mu}\xi^{\alpha}\partial_{\nu}\xi^{\beta}\eta_{\alpha\beta} - \partial_{\nu}\xi^{\beta}\delta^{\alpha}_{\mu}\eta_{\alpha\beta} - \partial_{\mu}\xi^{\alpha}\delta^{\beta}_{\nu}\eta_{\alpha\beta}$$

$$+ \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}h_{\alpha\beta} + \partial_{\mu}\xi^{\alpha}\partial_{\nu}\xi^{\beta}h_{\alpha\beta} - \partial_{\nu}\xi^{\beta}\delta^{\alpha}_{\mu}h_{\alpha\beta} - \partial_{\mu}\xi^{\alpha}\delta^{\beta}_{\nu}h_{\alpha\beta},$$

simplifying and considering only first order terms, finally we have

$$g_{\overline{\mu\nu}} = \eta_{\mu\nu} + h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}.$$
(2.14)

As it is only a coordinate transformation for arbitrary, but small changes of ξ^{μ} , the physical nature will be not affected by the coordinate transformation. This is known as a *gauge transformation*. Therefore we can make the following change without affecting the physical description

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu},$$
 (2.15)

and in terms of $\overline{h}_{\mu\nu}$ and $\overline{h}^{\mu\nu}$,

$$\overline{h}_{\mu\nu} \to \overline{h}_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} + \eta_{\mu\nu}\partial_{\alpha}\xi^{\alpha}, \qquad (2.16)$$

$$\overline{h}^{\mu\nu} \to \overline{h}^{\mu\nu} - \partial^{\mu}\xi^{\nu} - \partial^{\nu}\xi^{\mu} + \eta^{\mu\nu}\partial_{\alpha}\xi^{\alpha}.$$
(2.17)

We can use now the gauge freedom to simplify the (2.11) equations choosing an arbitrary vector ξ^{β} which satisfy the condition $\partial_{\alpha}\partial^{\alpha}\xi^{\beta} = \partial_{\alpha}\overline{h}^{\alpha\beta}$. Always this last equation can be solved in a simple way starting with the fact that is a wave equation for ξ^{β} with a source $\partial_{\alpha}\overline{h}^{\alpha\beta}$. Differentiating the equation (2.17) and using the last relations we find

$$\partial_{\nu}\overline{h}^{\nu\mu} \rightarrow \partial_{\nu}(\overline{h}^{\mu\nu} - \partial^{\mu}\xi^{\nu} - \partial^{\nu}\xi^{\mu} + \eta^{\mu\nu}\partial_{\alpha}\xi^{\alpha})$$

$$= \partial_{\nu}\overline{h}^{\mu\nu} - \partial_{\nu}\partial^{\mu}\xi^{\nu} - \partial_{\nu}\partial^{\nu}\xi^{\mu} + \partial^{\mu}\partial_{\alpha}\xi^{\alpha}$$

$$= \partial_{\nu}\overline{h}^{\mu\nu} - \partial_{\nu}\overline{h}^{\mu\nu} - \partial_{\nu}\partial^{\mu}\xi^{\nu} + \partial^{\mu}\partial_{\nu}\xi^{\nu} = 0.$$

$$(2.18)$$

This means that we can always find a gauge such that the divergence of the perturbations is zero, $\partial^{\nu} \overline{h}_{\mu\nu} = 0$. This is known as the *Lorentz gauge*¹ [36]. We will assume that we are in this gauge, in which the Einstein equations (2.11) reduce to

¹Alternatively, this gauge is called Einstein, de Donder or harmonic gauge.

$$\Box \overline{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \qquad (2.19)$$

where \Box is the wave operator in flat spacetime. For the vacuum case $(T_{\mu\nu} = 0)$ we have the vacuum wave equation in a flat spacetime

$$\Box \overline{h}_{\mu\nu} = (-\partial_t^2 + \nabla^2) \overline{h}_{\mu\nu} = 0.$$
(2.20)

This is simply the wave equation whose propagation velocity is the speed of light (c = 1). In other words, small tensor perturbations of the metric behave as waves propagating in the spacetime with the speed of light, showing that linearized General Relativity does predict the existence of GWs.

2.5 Gravitational plane waves

The simplest solution for the equation (2.20) is a plane wave solution of the form

$$\overline{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_{\alpha}x^{\alpha}), \qquad (2.21)$$

where $A_{\mu\nu}$ is the amplitude tensor and k_{α} the wave vector. Substituting (2.21) into the vacuum wave equation (2.20) we have

$$k_{\mu}k^{\mu}\overline{h}_{\mu\nu} = 0, \qquad (2.22)$$

as $\bar{h}_{\mu\nu}$ is arbitrary, we have that $k_{\mu}k^{\mu} = 0$. This implies that k^{μ} must be a null vector, which corroborates that the GW propagates with the speed of light. Using this solution in the Lorentz gauge $(\partial_{\nu}\bar{h}_{\mu\nu} = 0)$ we find

$$A^{\alpha\beta}k_{\beta} = 0, \qquad (2.23)$$

this condition shows that the amplitude tensor is perpendicular to the wave vector. In our case of a four dimensional spacetime, the amplitude tensor $A^{\alpha\beta}$ has $2^4 = 16$ components, of which six components are dependent of the others because $h_{\mu\nu}$ is symmetric and the orthogonality condition in (2.23) shows that other four are dependent too, resulting at least six components. However, we have to considerate another gauge freedom left within the Lorentz gauge. When we impose the Lorentz gauge choosing a specific gauge transformation vector ξ^{α} , in fact we are only restricting the value of $\Box \xi^{\alpha} = 0$, then we can add to ξ^{α} any other vector ζ^{α} only if also fulfills $\Box \zeta^{\alpha} = 0$ without changing anything. In

particular, if we choose $\zeta^{\alpha} = B^{\alpha} \exp(ik_{\nu}x^{\nu})$ with k_{ν} being the aforementioned wave vector and B^{α} an arbitrary constant vector, we then have four extra gauge degrees of freedom, reducing into two independent components of $A^{\alpha\beta}$.

In this way we can choose B^{α} and any unitary time-like vector u^{ν} (in particular a constant 4-velocity) in an appropriate way to impose the further conditions to the amplitude tensor

$$A^{\mu}{}_{\mu} = 0, \quad A_{\mu\nu}u^{\nu} = 0. \tag{2.24}$$

The first condition indicates that A^{μ} is traceless, meaning that in this gauge $h_{\mu\nu} = \overline{h}_{\mu\nu}$. The second condition says that it is orthogonal to the 4-velocity u^{ν} . Now, if we are in a particular reference system where $u^{\nu} = (1, 0, 0, 0)$, then the conditions (2.23) and(2.24) imply

$$A_{\mu 0} = 0, \quad \sum_{j} A_{ij} k_j = 0, \quad \sum_{j} A_{jj} = 0.$$
 (2.25)

Finally $A_{\mu\nu}$ and consequently $h_{\mu\nu}$ is traceless since its trace clearly vanishes, and transverse because it is both purely spatial and orthogonal to its own direction of propagation. Any tensor that satisfies these conditions is called a *transverse traceless tensor*. Without loosing in generality, we can assume that the plane wave propagates along the z axis, thus $A_{\mu\nu}$ has the form

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^+ & A^{\times} & 0 \\ 0 & A^{\times} & -A^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
(2.26)

where A^+ and A^{\times} are the independent components.

Given the solution for GW plane waves, the next step is to know its physical effect on matter. To do this, we use the amplitude tensor of the form (2.26) for a GW that then at certain moment interacts with a free particle, writing the geodesic equation

$$\frac{du^{\alpha}}{d\tau} + \Gamma^{\alpha}_{\mu\nu}u^{\mu}u^{\nu} = 0.$$
(2.27)

Supposing that the particle is at rest $u^{\alpha} = (1, 0, 0, 0)$, the last equation reduces to

$$\frac{du^{\alpha}}{d\tau}|_{t=t_0} = -\Gamma^{\alpha}_{tt} = \frac{\eta^{\alpha\beta}}{2} (2\partial_t h_{t\beta} - \partial_\beta h_{tt}).$$
(2.28)

The last equality holds because we are considering the the weak field approximation. Nevertheless, in the traceless-transverse gauge we have $h_{\mu 0} = 0$, so the acceleration is null, i.e. the GW does not affect the particle. This can lead to the wrong conclusion that GWs are pure gauge effects and are therefore not physical. However, at "rest" means being constant in its coordinate position, and all we have shown is that the particle remains at the same coordinate location but coordinates are just labels. In the traceless-transverse gauge we have attached our coordinate labels to freely falling particles, so we see them keeping constant coordinate position.

To know what really is happening, we have to calculate the tidal forces. For this, consider the *geodesic deviation* which describes the acceleration between nearby geodesics, with tangent vectors u^{α} separated by χ^{α}

$$a^{\alpha} \equiv u^{\mu} \partial_{\mu} (u^{\nu} \partial_{\nu} \chi^{\alpha}) = -R^{\alpha}{}_{\mu\beta\nu} u^{\mu} u^{\nu} \chi^{\beta}, \qquad (2.29)$$

Set a plane wave traveling along the z direction. From (2.7), the non-zero independent components of the weak field limit of the Riemann tensor at the traceless-transverse gauge are

$$R_{\mu x\nu x} = -\frac{1}{2}\partial_{\mu}\partial_{\nu}h_{xx}, \quad R_{\mu y\nu y} = -\frac{1}{2}\partial_{\mu}\partial_{\nu}h_{yy}, \quad R_{\mu x\nu y} = -\frac{1}{2}\partial_{\mu}\partial_{\nu}h_{xy}, \quad (2.30)$$

where μ and ν only take the values (t, z). Now, consider two free particles at rest between each other separated by χ^{α} and one of them at the origin of the reference frame $u^{\mu} = (1, 0, 0, 0)$, at first order the equation (2.29) reduces to

$$a^{\alpha} = \frac{d^2}{dt^2} \chi^{\alpha} = -R^{\alpha}{}_{t\beta t} \chi^{\beta}, \qquad (2.31)$$

explicitly we have:

$$a^{x} = \frac{1}{2} \left(\frac{d^{2}h_{xx}}{dt^{2}} \chi^{x} + \frac{d^{2}h_{xy}}{dt^{2}} \chi^{y} \right) = -\frac{\omega^{2}}{2} (A^{+}\chi^{x} + A^{\times}\chi^{y}) e^{ik_{\alpha}x^{\alpha}}, \qquad (2.32)$$

$$a^{y} = \frac{1}{2} \left(\frac{d^{2}h_{xy}}{dt^{2}} \chi^{x} + \frac{d^{2}h_{xx}}{dt^{2}} \chi^{y} \right) = -\frac{\omega^{2}}{2} (A^{\times} \chi^{x} - A^{+} \chi^{y}) e^{ik_{\alpha}x^{\alpha}}, \qquad (2.33)$$

where ω is the temporal component of k^{α} .

From this we see that GWs have two independent polarizations, the first one corresponds to $A^+ \neq 0$ and $A^{\times} = 0$ and is called "plus" polarization, while the second one corresponds to $A^+ = 0$ and $A^{\times} \neq 0$ and is called "cross" polarization.

The effect of GWs over matter is to stretch and contract the spacetime (i.e. zoom in and zoom out the two particles) in an oscillatory way. To exemplify it lets consider a ring of particles in rest in the x - y plane as is schematized in 2.5a and suppose that a GW pass through it in the z direction. If is a plus-polarized GW, the ring will oscillate elongating and compressing alternatively along the x and y directions (in terms of its relative proper distance to the center) as is shown in figure 2.5b. Now if the GW is cross-polarized it will produce elongations and compressions along the diagonal directions as figure 2.5c shows. Note that the GW polarizations are rotated 45° respect to the other, this contrasts with the electromagnetic waves that are rotated 90° [56].



Figure 2.1: a) A ring of free particles. b) Deformations of the ring produced by pluspolarized GW traveling in the direction of the z axis. c) Deformations of the ring produced by cross-polarized GW traveling in the direction of the z axis.

2.6 Stochastic gravitational wave background

The stochastic GW background is formed by a continuous superposition of GWs that fills all the Universe, acting as a noise in the background metric, arising from an extremely large number of unresolved, independent and uncorrelated events of gravitational radiation emission. The term stochastic in this sense refers to that although ideally we could study eachone of the GW sources, the superposition of all waves from all sources results in an indeterminate background where it is only possible to obtain global information of the system. These sources are unresolved in the following sense: if we study an optical source using a telescope with a certain angular resolution, then details of the source can be resolved if the angular resolution of the telescope is smaller than the angular size of the features or objects being studied.

For example, in the case of the LIGO experiment and similar detectors, the angular size of the antenna pattern is of order 90°. Hence almost any source is unresolved. When many of such sources are present, even if they are pointlike, the resulting signal has a stochastic nature [57]. The principal features of the stochastic GW background are that it is isotropic, stationary, and Gaussian; therefore its main property will be its frequency power spectrum. The stochastic background of GWs is mainly formed by two kinds of sources: the random superposition of localized astronomical sources and the GWs produced in the early Universe.

From the first kind of source, there are at least three main astronomical GW sources: (1) coalescing binary systems, composed of neutron stars and/or black holes, (2) pulsars (and other periodic sources), and (3) supernovae (and other transient or burst sources). These localized astrophysical sources are processes that took place recently in the Universe history, around the past several billion years. The second kind of source are the GWs produced during the early Universe phase transitions, decaying of cosmic strings and the amplification of the quantum vacuum fluctuations of the metric due to the interaction with the background curvature. They are the so-called *primordial or relic gravitational waves*. In the standard approach the creation of these gravitational waves took place very shortly after the Big Bang.

The detectable signal of the stochastic GW background mixes these two principal sources. But because of their different physical origin, data should be separated into the astronomical and early the Universe part. The main physical difference between them is the characteristic frequency: the first kind of source is in general in a low-frequency regime; the second one is on the other band, the high-frequency regime. In the specialized literature the term stochastic GW background refers sometimes indistinctly for the continuous superposition of both astronomical and early Universe sources, and other times for only the early Universe source part. To avoid confusions in this work we will use 'stochastic GW background' for the complete source background and 'primordial GWs' for the quantum vacuum origin GWs created in the early Universe.

2.7 Primordial gravitational waves and its importance

The amplification of quantum vacuum fluctuations due the interaction with the gravitational field is named in the specialized literature as the creation of primordial (relic) GWs (gravitons). The primordial GW background is the gravitational analogous of the CMB, forms an isotropic stochastic background with a non-thermal distribution. Currently, our most detailed image of the early Universe comes from the CMB photons, which decoupled from matter about 10^5 years after the Big Bang, and gives us an accurate picture of the Universe at this time. Continuing this analogy, because GWs interact very weakly with matter and other fields only at energies of the order of the Planck energy ($\sim 10^{19}$ GeV), the small scale perturbations of the gravitational field must have decoupled from the dynamics of the Universe at extremely early times [58], and it offers a unique way to probe the structure and evolution of the very early Universe. Primordial GWs will carry information of the Universe as it was at about 10^{-22} seconds after the Big Bang.

The detailed study of the primordial background of GWs began with the works of Grishchuck [59]. The basic idea is the following: the Heisenberg's Uncertainty Principle implies that the vacuum is not a static entity, since it is always creating and annihilating virtual particles (this could be interpreted as quantum vacuum fluctuations). Taking this idea we can think that in the early Universe (also today) existed fluctuations of the background metric interpreted as small perturbations of it. As the metric is a tensor of second rank, it admits scalar, vector and tensor (of second rank) perturbations, and as Lifshitz demonstrated [55] in the decade of 1940's at first order of approximation these three kind of perturbations in the metric are independent. The scalar perturbations are associated with energy density fluctuations; the vector perturbations to rotations, and the tensor perturbations, to GWs. So, the quantum vacuum fluctuations are the origin of tensor perturbation of the background metric of the Universe which leads to the production of GWs.

In Minkowski spacetime, these fluctuations have a minor effect because the spacetime is static, it does not change in time, implying that the quantum vacuum state can be defined unambiguously. But in general, curved spacetimes there is not a clear definition of the quantum (vacuum) state [16, 17] and this is the basic principle of the gravitational wave amplification or in other words, the graviton production. In a Universe described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric (which is an exact solution of Einstein's field equations of General Relativity and describes a homogeneous, isotropic expanding or contracting universe) the cosmic expansion stretches these small tensor perturbations created in the early Universe to cosmological scales. If the process is adiabatic (without transfer of energy from the gravitational potential to the perturbations) these are redshifted as the Universe expands being completely negligible today. But there are mechanisms, like inflation, that makes this process no more adiabatic. In this case the gravitational field of the background acts as a "pump field" transferring energy to the GWs being amplified. This process was named by Griskchuk as *superadiabatic amplification*.

From the point of view of the quantum field theory, the superadiabatic amplification is interpreted as creation of particles due the cosmic expansion. This is a consequence of the non-conformally invariant character of the perturbation equation in curved spacetimes [60]. The GWs now seen as gravitons, are created because the definition of a particle state at some time of the Universe evolution not necessarily have to be the same for a later time. Hence it admits the possibility that the vacuum state at certain moment can be different at another stage, which means that a zero-particle state becomes a multi-particle state. Summarizing, the vacuum fluctuations creates tensor perturbations that are amplified by the cosmic expansion defining its own vacuum state and the passage of one vacuum to another creates particles.

Inflationary models are attractive scenarios for the early Universe and are highly predictive. They also provide a mechanism for producing initial density perturbations large enough and with the appropriate power spectrum to evolve into galaxies as the Universe expands. These perturbations are accompanied by GWs that travel through the Universe, redshifting in the same way that photons do. Today, these perturbations should form a random background of gravitational radiation. The huge expansion associated with inflation puts energy into these quantum fluctuations, converting them into real GWs. The primordial GW spectrum produced by these models are largely independent of the inflationary mechanism. In the first approximation, the graviton spectrum produced by an inflationary stage is entirely independent of the mechanism that produces the inflation. The only inputs which are needed to find the graviton spectrum are the classical metric of the spacetime, and the initial quantum state of the gravitational perturbations. No detailed knowledge about of the classical metric and the initial quantum state is needed. There could also be a thermal background under certain circumstances. If inflation did not occur, but at the Planck time there was some kind of equipartition between gravitational degrees of freedom and other fields, then there would also be a thermal background of gravitational waves at a temperature similar to that of the CMB. But this radiation would have such a high frequency that it would not be detectable by any known or proposed technique. If inflation occurred, it would have redshifted this background down to undetectable frequencies.

For complementary and more detailed information about the primordial background of relic gravitons see the review [61].

Chapter 3

The Standard Cosmological Model

3.1 Large scale Universe

As mentioned before, the fundamental theoretical background of the Standard Cosmological Model is the Einstein's General Relativity theory. Is a geometric theory of gravitation that generalizes Special Relativity and Newton's law of universal gravitation, providing a unified description of gravity as a geometric property of the spacetime. It states the relation between the geometry of the spacetime with the energy and momentum of whatever matter and radiation are present. The relation is specified by the Einstein equations that is a system of ten non-linear partial differential

Because of the difficulty in solving the Einstein equations in most of the general cases, some extra conditions and/or symmetries must be imposed in order to simplify the problem. So that, in cosmology it is imposed the so-called *Cosmological Principle* that states that at large scales the Universe is spatially homogeneous and isotropic. For homogeneity we understand that all the spatial points of the Universe are equivalent, i.e. invariant under arbitrary translations. For isotropy we understand that all the directions from every spatial point of the universe are equivalent i.e. invariant under arbitrary rotations. These two concepts jointly state that all the positions in the Universe are equivalent, there is not a privileged position in the Universe. Here on Earth or in any other part of the cosmos we would see the same dynamics and configuration.

Clearly the Cosmological Principle is only valid at some large scale limit. It is evident that at smaller scales the Universe has an inhomogeneous structure with galaxies, planets, life, students, etc. But, the assumption of a homogeneous and isotropic Universe at large scales is well supported with observations (like Sloan Digital Sky Survey) [62, 63] that show that the Universe is almost the same in scales greater than 100Mpc. This is also corroborated by the observations of the CMB [64], who show that the Universe temperature is the same in all directions except small deviations.

The Cosmological Principle refers only to the spatial part of the Universe. In relation to the time coordinate, General Relativity states that there is not a privileged reference frame which implies that there is not an absolute time. This means that the temporal evolution of the Universe is different for observers with different reference frames. This generates an ambiguity because the description of the Universe for different observers mounted in different reference frames are completely equivalent. There is not a privileged description of the evolution of the Universe.

To alleviate this problem, Hermann Weyl in 1923 [65] proposed that there is a special class of observers associated with the expansion of the Universe. These observers are called *comoving observers* because they move accompanying the expansion of the Universe excluding every relative movement, so they are at rest in relation to the background that fulfill the Universe. This background can be considered as a perfect fluid because if we consider two different separated small elements of this "fluid", their geodesics do not intersect. The elements of this background are in time geodesic congruence, they come from a point in the infinite past diverging to the infinite future. This assumption that the background can be considered as a perfect fluid is very important because it allows us to describe the Universe content in a general and well defined way.

According to General Relativity, the geometry of the spacetime is determined by its material content. And because we had assumed that the perfect fluid that fulfills the Universe is also homogenous and isotropic, the spatial geometry of the Universe must be homogenous and isotropic in a global sense. The metric with these characteristics was first described by A. Friedmann in 1922 [2], parallel also proposed by G. Lemaître and then, H. Robertson and A. Walker proved that is unique. This is the so called FLRW metric, it is an exact solution of Einstein's field equations and describes a homogenous, isotropic expanding or contracting universe, its line element is

$$ds^{2} = g_{\mu\nu}dx^{\mu}dx^{\nu} = -dt^{2} + a(t)\left[\frac{dr^{2}}{1 - kr^{2}} + r^{2}(d\theta^{2} + \sin^{2}\theta \, d\varphi^{2})\right].$$
 (3.1)

The term a(t) is called the *scale factor* and contains the information on how the universe expands. Each point $x^{\mu} = (t, r, \theta, \varphi)$, is defined by a temporal variable t called *cosmic time* or *proper time* which is the time measured by comoving observers and the spatial part (r, θ, φ) called *comoving spherical coordinates*. We can see that the metric tensor $g_{\mu\nu}$ is diagonal

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & \frac{a^2(t)}{1-kr^2} & 0 & 0\\ 0 & 0 & a^2(t)r^2 & 0\\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{pmatrix}.$$
 (3.2)

The Cosmological Principle also implies that the metric must have the same constant curvature in every point of the Universe. There are only three three-dimensional spaces that satisfy this condition [12], they are:

- For k = 0 we have the flat space \mathbb{E}^3 . Solution for a flat Universe.
- For k = 1 we have the three-dimensional sphere \mathbb{S}^3 . Solution for a closed Universe.
- For k = -1 we have the three-dimensional hyperbolic space $\mathbb{S}^{2,1}$. Solution for an open Universe.

The Einstein equations were not used in deriving the general form for the FLRW metric, only the geometric properties of homogeneity and isotropy. It establishes the global structure of the spacetime, but still does not say anything about its evolution; these geometric assumptions determine the left-side of (1.1). To know how it evolves, it is necessary to know the nature of the material content of the Universe, i.e. to determine the energy-momentum tensor of the right-side of the Einstein equations. We have anticipated that it is a perfect fluid, but not of what kind. The clue is to look at the sky and see the dynamic of the cosmos to infer of what it is made.

3.2 The Universe dynamics

Probably the main discovery about our Universe is that it is expanding. This means that the Universe is not static, it is dynamic, breaking a millennial paradigm of a static Universe. The cosmic expansion was discovered by the American astronomer Edwin Hubble in the year of 1929 [4] when he was investigating an especial type of variable stars called cepheids outside of the Milky Way. The conclusion of its work was that the other galaxies are moving away from us in a simple linear relation

$$v = H_0 l, \tag{3.3}$$

where v is the recessional velocity, typically expressed in km s⁻¹; H_0 is Hubble's constant and corresponds to the value of the Hubble parameter taken at the time of observation, its measured value is $H_0 = 67.8 \pm 0.9$ km s⁻¹ Mpc⁻¹ taken from the Planck satellite [66]; and l is the proper distance (which can change over time, unlike the comoving distance which is constant plus peculiar velocities) from the galaxy to the observer, measured in mega parsecs (approx. 1 Mpc = 3.2616×10^6 ly = 3.0857×10^{22} m). Is important to mention that this linear relation is only valid for the present time.

The description of the dynamics of the Universe only considering the Hubble law is incomplete, we need to know the evolution of the scale factor. For this, we first use the Cosmological Principle assuming that the matter is homogeneously and isotropically distributed and use the Weyl's postulate, which states that in good approximation on large scales the Universe can be considered filled with a perfect fluid. In general the energy-momentum of a perfect fluid can be written as

$$T^{\mu}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + p\delta^{\mu}_{\nu}, \qquad (3.4)$$

characterized by the energy density ρ , pressure p and 4-velocity u^{α} of the observer. The equation $p = p(\rho)$ depends on the properties of the fluid and must be specified. In many cases is assumed $p = \omega \rho$, where ω is a constant. In the comoving reference frame the 4-velocity is constant given by $u_{\mu} = (-1, 0, 0, 0)$, in this way the energy-momentum tensor becomes diagonal

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0\\ 0 & p & 0 & 0\\ 0 & 0 & p & 0\\ 0 & 0 & 0 & p \end{pmatrix}.$$
 (3.5)

As this fluid is homogeneous and isotropic the density ρ and the pressure p depend only on time. Now we must determine the functions a(t) and $\rho(t)$ to determine the dynamics of the Universe. Inserting the FLRW metric (3.1) and the perfect fluid energy-momentum tensor (3.4) into the Einstein equations (1.1), we obtain

$$8\pi G\rho + \Lambda = 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2}, \qquad (3.6)$$

$$8\pi Gp - \Lambda = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - 3\frac{k}{a^2}, \qquad (3.7)$$

where now $\rho = \sum \rho_n$ and $p = \sum p_n$ are the sum of the densities of energy and pressures of all the material components of the Universe. These two last equations are known as the *Friedmann equations*. Taking the temporal derivative of (3.6) and using (3.7), we obtain a continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{3.8}$$

This last equation can be derived also using the energy conservation law $\nabla_{\nu}T^{\mu\nu} = 0$. Dividing (3.6) by the square of the Hubble parameter, $H = \dot{a}/a$, this equation can be written in the following way

$$1 = \Omega - \frac{k}{a^2 H^2},\tag{3.9}$$

where $\Omega = \sum_i \Omega_i = \sum_i \rho_i / \rho_{\text{crit}}$ is the sum of all the relative densities of each component, and

$$\rho_{\rm crit} = \frac{3H^2}{8\pi G},\tag{3.10}$$

is known as the cosmological critical density which is the density of a flat Universe. The numerical value of $\rho_{\rm crit}$ can be computed using $H_0 = 67.3$ km s⁻¹ Mpc⁻¹, giving 9.212×10^{-26} g cm⁻³ = 3.968×10^{-7} eV⁴. Rewriting we have $\Omega = 1 + k/a^2 H^2 = 1 + \Omega_k$, this expression relates the material content of the universe with the curvature. In this way we can see that we have three possibilities

- $\Omega > 1$ if k = 1, closed Universe.
- $\Omega = 1$ if k = 0, flat Universe.
- $\Omega < 1$ if k = -1, open Universe.

Recent observations show that $|\Omega_k| < 0.005$ [66], meaning that we live in an almost flat Universe. There is not a mechanism in the Standard Cosmological Model which explain why the Universe has this particular value of Ω . There are some paradigms out of the Standard Cosmological Model, like inflation, which solve these and other problems of the model. This will be discussed afterwards. The two Friedmann equations (3.6, 3.7) and the continuity equation (3.8) are a system of the dynamics of a(t), $\rho(t)$ and p(t). As the continuity equation was derived from the first two Friedmann equations, an extra equation is needed to complete the system. All the spacetime symmetries were used, so we must use some property of the fluid which fills the spacetime. We are going to impose a relation between the energy density and the pressure in the following simple way

$$p(t) = \omega \rho(t), \tag{3.11}$$

this relation is a particular type of equation of state of the cosmic fluid. Each component of the cosmic fluid contributes in a different way to the evolution of the Universe depending of the value of its particular ω , which in general can be supposed to be constant. For baryonic and dark matter the value of the parameter is $\omega = 0$, that is, is a fluid without pressure. For radiation we have $\omega = 1/3$. For the dark energy (cosmological constant) we have $\omega = -1$.

With the equation of state the system of equations is complete and can be solved. Assuming that the constituents of the Universe does not interact one to the other, the continuity equation (3.8) can be integrated for each fluid.

$$\rho(t) = \rho_0 a(t)^{-3(1+\omega)} \tag{3.12}$$

where ρ_0 is the value of the energy density today. With this solution we can know some characteristics of each material fluid of the Universe.

- Matter: This includes non-relativistic baryonic matter and dark matter. This fluid has neglige or no pressure p = 0. The energy density has the form $\rho = \rho_0^{\text{mat}} a^{-3}$. Notice that the energy density of matter is inversely proportional to the physical volume of the Universe.
- Radiation: Here we consider all massless relativistic particles. The equation of state is p = ρ/3, so ρ = ρ₀^{rad}a⁻⁴. We see that the radiation decays faster than the matter. This implies that in the early Universe was dominant and then at some moment of the evolution of the Universe became to be matter dominated.
- Cosmological Constant: The equation of state is $p = -\rho$, so $\rho = \rho_0^{\Lambda}$. The energy density of this fluid is constant, as it does not decay, meaning that it was negligible in the very early universe but it will be dominant in the future.

We have solved for the energy density, now let's look at the scale factor. Assuming ω constant and considering a flat Universe (k = 0), substituting (3.12) in (3.6) we have

$$a(t) = a_0 t^{2/3(1+\omega)}. (3.13)$$

This solution shows that the evolution of the Universe depends intimately on its material content. Considering that each fluid dominates over the other ones in some epoch of the Universe, we can divide the evolution of the scale factor in different eras:

$$a(t) = \begin{cases} a_0^{\text{mat}} t^{2/3}, & \text{matter} \\ a_0^{\text{rad}} t^{1/3}, & \text{radiation} \\ a_0^{\Lambda} e^{H_{\Lambda} t}, & \text{cosmological constant} \end{cases}$$
(3.14)

where $H_{\Lambda} = \sqrt{8\pi G \rho_0^{\Lambda}/3}$. Taking the second derivative to calculate the acceleration, it is easy to notice that for matter and radiation it is negative, indicating a decelerated expansion. Contrary, for the case of cosmological constant the second derivative is positive showing an accelerated exponential expansion; this case is called *de Sitter Universe*.

To complement, it will be defined an important and useful concept in cosmology: the *horizon*. The General Relativity states the speed of light is constant and it is a physical limit. This fact determines the accessible (observable) Universe by a determined observer, this limit is known as horizon. It can be defined two types of horizons: the *particle horizon* and the *event horizon*.

Particle Horizon

The particle horizon can be defined as the maximum distance that a particle can travel. In a null trajectory (for an example, a photon) from the beginning of the Universe to an observer. If we are the observers, this horizon delimits the portion of the Universe accessible to us, our observable Universe. Assuming that the photon does not suffer any kind of dispersion or absorption, the maximum distance that a photon can travel from an initial time $t_i = 0$ to an observer at a time t, the particle horizon is defined as

$$d_p(t) = a(t) \int_0^t \frac{dt}{a(t)}.$$
 (3.15)

The events out of this horizon are not causally connected with the point from which is defined the horizon. From the equations (3.14) and (3.15) we can determine the particle horizon for each material component of the Universe. For a matter-dominated Universe we have $\chi_p = 3t$, for radiation-dominated $\chi_p = (3/2)t$ and for cosmological constant dominated $\chi_p = (e^{H_{\Lambda}t} - 1)/H_{\Lambda}$.

Event Horizon

The event horizon is defined as the distance that a photon can travel from an observer to the end of the Universe in the infinite future, defines the volume of spacetime that contains the events that could be observed in the future. Again, assuming no dispersion and absorption of photons, the comoving distance that a photon can travel from an observer at the time t to a final time $t_f \to \infty$ is given by

$$d_e(t) = a(t) \int_t^\infty \frac{dt}{a(t)},$$
(3.16)

For a matter dominated or radiation dominated Universe there is not event horizon because $\chi_e \to \infty$. For a cosmological constant dominated Universe an event horizon exists and is $\chi_p = H_{\Lambda}^{-1}$, which is the Hubble radius of this kind of Universe. In this case any event that occurs at distances greater than the Hubble radius (event horizon) can never be seen neither affect the future of the observer.

3.3 A brief (thermal) history of the Universe

Briefly it will be shown the chronology of the Universe according to the Standard Cosmological Model in complement with some theories and hypothesis about the very early Universe where the General Relativity doesn't work. The history of the Universe is divided into different eras describing the main physical features at each time associated with its typical energy. The origin of times is unknown, somewhere in the past, namely the Big Bang in this model. This will be our start point which we will associate to t = 0.

Planck era. Up to 10^{-43} s after the Big Bang, $T \sim 10^{19}$ GeV.

In the Planck era the temperature was so high that the four fundamental forces (electromagnetism, gravitation, weak nuclear interaction and strong nuclear interaction) were unified into a single fundamental force. Little is understood about physics at this temperature, different hypothesis propose different scenarios. The Big Bang cosmology, based only on General Relativity, predicts a gravitational singularity before this time and it is expected to break down due to quantum effects. Because of the small scale of the Universe in this era, quantum effect must dominate and General Relativity can not be trusted anymore. A quantum gravity theory would eventually provide a better understanding of this era. This era is not described by the Standard Cosmological Model.

Grand Unification era. $10^{-43} - 10^{-36}$ s, $T \sim 10$ TeV -10^{19} GeV.

As the Universe expands and the temperature decreases, it crosses transition temperatures at which the fundamental forces separate from each other. The Grand Unification era begins when gravitation separates from the other three forces of nature. The nongravitational physics in this epoch would be described by a so-called Grand Unified Theory (GUT) and there is no reason to expect that nonperturbative quantum gravity plays a significant role below 10¹⁹ GeV therefore General Relativity can be used to describe the dynamics of the Universe [10]. This era ends when the strong force decouples from the electroweak force. This transition should produce magnetic monopoles in large quantities, which are not observed. As we will see later the magnetic monopoles abundance is one problem in the standard cosmology.

Inflationary era. $10^{-14} - 10^{-36}$ s, $T \sim 10^{14}$ GeV.

In this era occurs an accelerated expansion of the Universe produced by a hypothetical scalar field called the inflaton. This accelerated expansion makes the Universe more homogeneous and it could be an explanation of why we see the Universe today with this characteristic. When the Inflationary era ends, the inflaton field decays into ordinary particles seen today, this process is called reheating. It is expected that this era occurs somewhere near the Grand Unification era. Is not derived from the Standard Cosmological Model, it was introduced to solve the magnetic monopole problem by Alan Guth [15]. The original version of inflation had several problems that made it to be discarded, but inspired new versions of inflation that are surprisingly successful solving some cosmological problem besides the monopole problem and also explain naturally some properties of the Universe today. We will see more details of inflation afterwards.

Electroweak symmetry breaking and quark era. $10^{-10}-10^{-14}$ s, $T \sim 200$ MeV-10 TeV.

As the Universe's temperature falls below a certain very high energy level some symmetries of the nature forces begins to break down. The strong force is the first to separate, leaving the electroweak force. Then it is believed that the Higgs field spontaneously acquires its vacuum expectation value breaking electroweak gauge symmetry. The gauge W and Z bosons of the weak force and the photon of the electromagnetic force became different separating these two forces. During the quark era the Universe was filled with a dense, hot quark-gluon plasma, containing quarks, leptons and their antiparticles. Collisions between them were too energetic to allow quarks confine it into mesons or baryons. At the end of this epoch, the fundamental interactions of gravitation, electromagnetism, the strong interaction and the weak interaction have now taken their present forms, and fundamental particles have mass, but the temperature of the universe is still too high to allow quarks to bind together to form hadrons.

Hadron era. $10^{-10} - 1$ s, $T \sim 1 - 2$ MeV.

In this era the quark-gluon plasma cools forming hadrons like protons and neutrons. At approximately 1s after the Big Bang, neutrinos decouple and begin traveling freely through space. This cosmic neutrino background, while unlikely to ever be observed in detail since the neutrino cross section is very low, it is analogous to the CMB that was emitted much later.

Lepton era. 1 - 200s, $T \sim 0.5$ MeV.

The majority of hadrons and anti-hadrons annihilate each other at the end of the hadron Era, leaving leptons and anti-leptons dominating. Approximately 10s after the Big Bang, the temperature of the Universe falls to the point where new lepton/anti-lepton pairs are no more created so leptons and anti-leptons begin to annihilation leaving a small residue of leptons and creating a big amount of photons. After most leptons and anti-leptons annihilate at the end of the lepton era, the energy of the universe was dominated by photons. These photons interact with other protons, electrons and hadrons, being in thermal equilibrium the next $\sim 300,000$ years.

Nucleosynthesis era. 3-20 minutes after the Big Bang, $T \sim 0.05$ MeV.

The temperature falls to the point where protons and neutrons begin to combine, forming atomic nuclei of light elements like hydrogen H, deuterium H^2 , tritium H^3 , helium He^4 and traces of others, like lithium. The expected abundances of these elements given by the Standard Cosmological Model are in very good agreement with the observations.

Radiation Era. From 20 minutes to $\sim 300,000$ years, $T \sim eV$.

Before the formation of the light elements the Universe was dominated by the radiation. In this era, matter and radiation were in thermal equilibrium. Photons were continuously scattered by electrons and nuclei, making the Universe opaque. As the Universe expands and cools, the energy was not high enough to maintain the equilibrium, allowing electrons bind to the protons and nuclei forming atoms and ending the photon scattering. This is known as recombination. At the end of this era radiation and matter are not in equilibrium anymore and the photons decouple and can travel freely making the Universe transparent. The last moment where photons and matter interact is called *last scattering surface* and the residual radiation that travels free in all the Universe is the CMB. This radiation carries important information about the physical conditions of the Universe at that time. The detailed study of the CMB first initiated by COBE [67] and then by WMAP [68] was of great importance for the understanding of the Universe and a key piece to consolidate the Standard Cosmological Model. The present Planck mission [69, 70] has provided detailed data with without precedent from the CMB with important information of the early universe. Its study is not complete and promises a high precision cosmology in the not too distant future.

Matter Era. ~ 300,000 - 13.9 billion years (today), $T \sim 10^{-4} \text{eV}$.

After the decoupling of the radiation, the matter component dominates. Galaxies and their clusters are formed from the small initial inhomogeneities. As a result of gravitational collapse, stars are formed giving up thermonuclear reactions which at the end of the life of some stars produce all the rest of heavy elements allowing them the formation of more complex structures like life. Besides the ordinary baryonic matter (well known from particle physics), observational evidence shows the existence of another noninteracting kind of matter called *dark matter* [71]. Dark matter only have gravitational interaction and it is the most important massive part of galaxies dominating their gravitational dynamics. The nature of dark matter is a mystery for today's particle physics.

Dark Energy Era. Future.

The discovery that today the Universe is expanding in an accelerated way brought new challenges to cosmology [8]. There is not a clear explanation how is the mechanism that underlies the accelerated expansion and there exist several attempts to explain this phenomenon. The fact is the existence of a gravitational repulsive component, the dark energy, that permeates all the Universe [72]. The nature of dark energy is almost unknown, there exists various theoretical alternative models, but the most popular is make the cosmological constant [23]. In this model the energy density of the dark energy is constant $\rho_{\Lambda} = constant$, whereas the energy density of the radiation an matter depends of the scale factor ($\rho_{\rm rad} \propto a^{-4}$ and $\rho_{\rm mat} \propto a^{-3}$), so as the Universe expands the energy densities of those diminish. At some time in the Universe evolution, the scale factor will be sufficiently big making $\rho_{\rm rad}$ and $\rho_{\rm mat}$ smaller than ρ_{Λ} . At this moment, as the Universe expands, the dark energy begins to dominate.

3.4 Problems of the Standard Cosmological Model

The success of the Standard Cosmological Model is supported by a great number of astronomical observations and clearly derives from the Einstein's General Relativity. Is undoubtedly a very useful and successful model of our Universe, but it is not exempt of some important problems when it is studied in more detail. This means that is still incomplete and it must be extended. One cause of these problems is that we still looking for a theory that include quantum physics and gravitation together, others come from our ignorance of the initial conditions of the early Universe and the complete material component understanding (dark matter and energy). Of these problems, probably the most concerning one is the Cosmological Constant Problem in which is intimately related with the quantum field theory. About this problem we will discuss later.

Some of the other problems are related mainly to the conditions of the early Universe. To name some of these problems, we have: the flatness problem, the horizon problem, structure origin problem and the magnetic monopoles problem. These will be resumed briefly.

Flatness problem

The observational data indicate that the Universe is approximately flat. This means that the energy density is equal to the critical density. In a Universe with spatial curvature filled with radiation and matter, the value $\Omega \cong 1$ is an unstable point, any small variation of Ω evolves to a closed or open Universe. Let's consider the initial conditions needed to have this particular case. First rewrite the equation (3.9) in the following way

$$(\Omega - 1)a^2H^2 = k. (3.17)$$

Knowing that the curvature parameter k is a constant, it means that the left-side of the equation must be also for any time. We know that today $(\Omega_0 - 1)$ is of the order of the unity. Now calculating for the matter dominated Universe $(\Omega - 1) \propto a$, for the radiation dominated Universe $(\Omega - 1) \propto a^2$, and making extrapolation that the Einstein equations

are valid at times near the Planck Era; comparing these quantities we have

$$\frac{\Omega - 1|_{t=t_{\rm Pl}}}{|\Omega_0 - 1|} \approx \frac{a_{\rm Pl}^2}{a_0^2} \approx \frac{T_0^2}{T_{\rm Pl}^2}.$$
(3.18)

Remembering that the temperature of the Universe in the Planck era is $T_{\rm Pl} \sim 10^{19} \text{GeV}$ and the temperature of the Universe today is $T_0 \sim 10^{-13} \text{GeV}$, we have that the ratio of them is of the order of $(T_0/T_{\rm Pl})^2 = 10^{-64}$. If we make the same calculation for the time of the nucleosynthesis in which the temperature is pretty much lower around of $T_{\rm NS} = 0.05 \text{MeV}$, the ratio is $(T_0/T_{\rm NS})^2 = 10^{-16}$. With this we can see that if we have the condition $|\Omega_0 - 1| \sim \mathcal{O}(1)$, the term $|\Omega - 1|$ for the early Universe must be very small, very close to zero. Any small variation of this quantity drives unavoidably to a very different evolution of the Universe today. This extremely sensitive adjust is known as the *fine tuning problem*.

Horizon problem

One of the most important observational facts of cosmology is the great homogeneity of the temperature of the CMB. It is almost the same in any part of the sky except small deviations. However, taking account the concept of particle horizon, causally disconnected regions in the past not necessarily should have the same temperature today. It seems natural to think that if they have the same temperature these regions, it is because they were in causal contact at some time in the past to achieve the thermal equilibrium. Another possibility is to think that each region was "born" coincidentally with the same temperature, but this does not seem very convincing. To see this more clearly, let's compare the particle horizon at the time of the decoupling of the photons t_{dec} , with the matter with the distance that the light travel from the last scattering surface to us. As $a(t) \propto t^{2/3}$ in the phase where the Universe is almost dominated by matter, the reason is given by

$$\frac{\chi(t_{\rm dec})}{\chi(t_i = t_{\rm dec}, t_0)} = \frac{H^{-1}}{\chi(t_i = t_{\rm dec}, t_0)} \simeq \left(\frac{t_{\rm dec}}{t_0}\right)^{1/3} \simeq 10^{-2}.$$
(3.19)

Now we can project any comoving distance l in the las scattering surface and obtain the angle in the sky which it corresponds. This is given by

$$\theta = \frac{l}{\eta_0 - \eta_{\text{dec}}}, \quad \eta(t) = \int^t \frac{dt}{a(t)}, \quad (3.20)$$

using this, with the result of the equation (3.19) we notice that it corresponds to $\sim 1^{\circ}$. This

means that at the decoupling time, the angle between two causally connected regions are approximately of one degree. We can calculate the number of regions causally connected that are contained in the volume formed the last scattering surface.

$$N = \frac{4\pi D^3/3}{4\pi (D\theta/3)^3} = \frac{8}{\theta^3} \sim 10^6.$$
(3.21)

Now, the question is how all this 10^6 regions can have almost the same temperature without having any causal contact between them in the past.

The origin of the large scale structure problem

As it has been said before, probably the most remarkable characteristic of our Universe is that at large scale it is homogeneous and isotropic. But an evident characteristic is that at smaller scales there exist structures like galaxies, stars, life, scientist, etc. So, a natural question is how all these structures (inhomogeneities) were formed from a homogeneous and isotropic Universe. In the CMB small deviations in the temperature are observed that can be related to the structure formation. This can be explained by the well-known classical perturbation theory which states that in an expanding Universe, initial energy density perturbations could form large scale structures by gravitational collapse. The problem is that there is no mechanism within the Big Bang theory to account for these initial perturbations. One fact obtained by analysing the CMB is that the spectrum of these perturbations is a nearly scale invariant, so we need a theory that does not postulate these initial perturbations and also adjusts the initial conditions to be almost scale invariant.

Magnetic monopoles problem

The cosmic expansion implies that in the very early Universe the temperature was extremely high and then the Universe cools down by its own expansion. This fall down, from big temperatures to low ones, makes that some symmetries of the fields break down. The GUT states that in the period when the Universe cools down from very high temperatures, it may manifest topological defects in the form of magnetic monopoles with a big production rate. The theory predicts also that the masses of this magnetic monopoles are very big, around 10¹⁶ times the proton mass. One possible consequence of the presence of this great number of massive particles is the gravitational collapse of the Universe, which not occurred. Another possibility is that if there is great abundance of these heavy particles (but no so much to collapse the Universe), they should have been observed by particle

detectors, but today no magnetic monopole has been detected. This magnetic monopoles must have been diluted in the early Universe in order to have very low densities today to make them practically unobservable.

3.5 Inflation

As it has been shown in the previous section, the Standard Cosmological Model describes very successfully the evolution of the Universe but it does not say anything about the origin of the Universe and presents important problems concerning the initial conditions. In the last section some of the most important problems of the model were summarized. Inflation is a paradign that solves some of the problems of the Standard Cosmological Model.

In 1979, Alexei Starobinsky proposed that the early universe went through an inflationary de Sitter era [73]. This resolved the cosmology problems and led to specific predictions for the corrections to the CMB. Few later, in an attempt to solve the magnetic monopoles problem, Alan Guth formulated the first inflation theory in 1981 [15]. Today this first theory is known as the "old inflation". From the particle physics viewpoint, he studied the properties of the Great Unification theories noticing that the theory predicts the creation of a big number of magnetic monopoles. To solve this problem in this context, he proposed of a scalar field in the early Universe which make the Universe enter into a period of a very accelerated expansion that dilutes the created magnetic monopoles. The responsible scalar field for inflation is called *inflaton*.

In other words, the Universe was in a false vacuum with an energy density very high where the scalar field was at a local minimum of the potential, acting as a cosmological constant. At some time inflation ends due to the scalar field goes to the global minimum of the potential via quantum tunneling, which corresponds to the state of the Universe at its real vacuum. Guth realized that his model could also solve other problems of the Standard Cosmological Model. Despite seemed very promising, Guth's model has some problems, the most important of them is that old inflation cannot occur in the whole Universe whose consequence is an inhomogeneous and anisotropic Universe. This problem is known as the *graceful exit problem*.

The old inflation was substituted by the "new inflation" formulated by Andre Linde [74] and parallel by Andreas Albrecht and Paul Steinhardt [75]. In this new version of

inflation, the Universe expands almost exponentially in a phase that the inflaton has a slow-roll into the direction to the minimum of the potential and when this roll is slower than that the expansion of the Universe, inflation occurs.

This model is capable of amplifying the quantum fluctuations of the inflaton that later was shown that are the seeds of the large scale structures. There exist several inflationary models proposed in the literature that solve the graceful exit problem, to name some of them: chaotic inflation, natural inflation, hybrid inflation, etc. The inflationary models not only solve the magnetic monopole problem, they also solve the other problems mentioned before in a very elegant way. Despite its great success, the inflation theory presents some theoretical problems like the origin of the inflaton from the particle physics.

Let's explain briefly how the inflation works. Let's suppose that the Universe is in a phase where the particle horizon (physical scales) expands faster than the Hubble radius H^{-1} , that is, $d_p(t_i = t_{\text{dec}}, t_0) > H^{-1}$, where $d_p(t_i = t_{\text{dec}}, t_0) \sim a$. With this we have the following condition

$$\frac{d}{dt}\left(\frac{a}{H^{-1}}\right) = \frac{d}{dt}\left(a\frac{da/dt}{a}\right) = \ddot{a} > 0.$$
(3.22)

This condition is very important giving us a geometrical interpretation of the accelerated expansion. It means that if the comoving Hubble radius $(aH)^{-1}$ is decreasing, then the comoving Hubble parameter $\mathcal{H} = aH$ must increase in the inflationary phase. This characteristic is quite important. The decrease of this parameter indicates that after this accelerated expansion phase, our accessible Universe is smaller than it was at the beginning of the inflation, in that way far away regions of the today Universe were causally connected before the accelerated expansion.

We can define what is necessary for inflation to occur before the radiation era. Using the last equation (3.22) in the Friedmann equation (3.7), we find a necessary condition for inflation to occur

$$\rho + 3p < 0, \tag{3.23}$$

i.e. $\omega < -1/3$. This particular condition is for a repulsive gravitation. There is no known kind of matter or radiation which satisfy this special condition. This is one important problem that the inflationary paradigm faces. We can consider the particular case $\omega = -1$ which corresponds to cosmological constant case. This choice gives a de Sitter Universe, whose scale factor is given by the equation (3.14)

$$a = a_i e^{H_I(t - t_i)}. (3.24)$$

Notice that $H_I = \dot{a}/a$ is constant. Here we used the assumption that the beginning of the inflation is the Planck time, $t_i = t_{\text{Pl}}$.

As it was said above, inflation was conceived as a period of accelerated expansion where the Universe is in a false vacuum state and the energy of this vacuum acts as a cosmological constant. However, the cosmological constant does not explain very well the inflation because we know that inflation had a beginning and an end. This makes that the vacuum energy dynamic, so we need to specify a dynamic physical component that makes gravity repulsive for some time. We had said that no known matter nor radiation has the property of having negative pressure. This condition can be obtained introducing a scalar field that in some circumstances can mimic the cosmological constant and present a dynamic behavior.

Scalar fields were quite studied by the quantum field theory before being introduced into cosmology. They describe a particle field of spin 0. We need to calculate the energymomentum tensor of the inflaton through its action functional to describe the dynamics of inflation. Let's begin defining a real scalar field minimally coupled to gravity with a functional action given by [10, 17]:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \phi^{,\mu} \phi_{,\mu} + V(\phi) \right], \qquad (3.25)$$

in which \mathcal{L} is the Lagrangian density, g is the determinant of the metric $g_{\mu\nu}$ and $V(\varphi)$ is the scalar field potential. The functional form of the potential determines the inflation model in which we are working. Varying the action in relation of the metric we can obtain the expression of the energy-momentum tensor

$$T^{\mu}_{\nu} = \varphi^{,\sigma}\varphi_{,\sigma} - \left[\frac{1}{2}\varphi^{,\alpha}\varphi_{,\alpha} + V(\varphi)\right]\delta^{\mu}_{\nu}.$$
(3.26)

Defining the velocity

$$u^{\gamma} \equiv \frac{\varphi}{\sqrt{\varphi^{,\alpha}\varphi_{,\alpha}}},\tag{3.27}$$

we can rewrite the energy-momentum tensor as one of a perfect fluid, using the Friedmann equations we obtain the expressions for the energy density and for the pressure

$$\rho = \frac{1}{2}\varphi^{,\alpha}\varphi_{,\alpha} + V(\varphi) = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \qquad (3.28)$$

$$p = \frac{1}{2}\varphi^{,\alpha}\varphi_{,\alpha} - V(\varphi) = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \qquad (3.29)$$

the last equalities are because $\varphi = \varphi(t)$. Let's derive the evolution equation of the scalar field φ from the action

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi}(\varphi) = 0. \tag{3.30}$$

This is the Klein-Gordon equation for this particular case where $\sqrt{-g} = a^3(t)$. Now substituting the expressions of the energy density and pressure of the scalar field in the Freidmann equation (3.6) with no curvature term (k = 0), we have

$$H^{2} = \frac{8\pi G}{3} \left(\frac{1}{2} \dot{\varphi}^{2} + V(\varphi) \right).$$
 (3.31)

Now we have to determine the conditions that the scalar field must satisfy to produce the inflation. The scalar field that satisfies these conditions is what we call a inflaton. We have shown that a necessary condition is $p < -\rho/3$. We choose the particular case $p \simeq -\rho$. By the definitions of ρ and p, we notice that

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi),$$

$$p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi),$$

$$\dot{\varphi} \ll V(\varphi) \implies p = -\rho.$$
(3.32)

The potential always dominate, this is known as the slow-roll condition. With this we can reduce the equations (3.30) and (3.31) to

$$H^2 \simeq \frac{8\pi G}{3} V(\varphi), \tag{3.33}$$

$$3H\dot{\varphi} \simeq -V_{,\varphi}(\varphi).$$
 (3.34)

We have considered too that the acceleration of the field is very small compared with the velocity, $\ddot{\varphi} \ll 3h\dot{\varphi}$. These conditions for the scalar field restricts the form of the potential.

Warm inflation

During inflation all the material component of the Universe, except the scalar field (inflaton), is redshifted to extremely low densities. Also this expansion supercooled the Universe into relative low temperatures that maintained during the inflationary phase. So, at the end of inflation the Universe was in a stage of very low energy density and very low temperature which is a very different view of what we expect from the early Universe. CMB observations contradict this view, so we need a mechanism to bring back a hot and dense Universe. This is the motivation of the reheating of the Universe.

The reaheating is a process that occurs immediately after inflation, where the energy density of the inflaton decays filling the Universe with the Standard Model particles. Once the slow-roll conditions break down, the scalar field rolls down and oscillates at the bottom of the potential decaying into conventional matter and radiation. The first formulations of reheating [13] added a phenomenological decay term, then this was constrained to be very small and making the reheating very inefficient. These formulations allowed a very big redshifting after the end of inflation and before the Universe returned to thermal equilibrium, hence the reheat temperature would be lower, by several orders of magnitude, than suggested by the energy density at the end of inflation [76].

Then in 1994 Kofman, Linde and Starobinsky [77] improved the model. They suggested that the inflaton decay can undergo broad parametric resonance, with an extremely efficient transfer of energy from the coherent oscillations of the inflaton field. This initial transfer has been named preheating. With such an efficient start to the reheating process, it now appears possible that the reheating epoch may be very short indeed and hence that most of the energy density in the inflaton field at the end of inflation may be available for conversion into a thermalized form. However the ideas of the background physics of the reheating and preheating are not well known. Particle physics beyond the Standard Model and inflation have to be studied more deeply in an attempt to understand these processes.

Now let's show how does inflation works solving the problems of the Standard Cosmological Model stated above [78]. The principal feature of inflation is an accelerated expansion:

if
$$\ddot{a} > 0 \Rightarrow \frac{d}{dt} \left(\frac{1}{aH}\right) = \frac{d}{dt} \left(\frac{1}{H}\right) < 0$$
 (3.35)

This last relation says that the Hubble radius, as measured in comoving coordinates

 $\mathcal{H}^{-1} = (aH)^{-1}$, decreases during inflation. At any other time, the comoving Hubble radius increases. This has an important geometrical meaning, although typically the expansion of the Universe is very rapid, the crucial characteristic scale of the Universe is actually becoming smaller when measured relative to that expansion [76]. This is the starting point for solving the problems.

Flatness problem solution.

In the equation of the curvature parameter, $k/(aH)^2 = \Omega - 1$, the term $(aH)^2$ grows during inflation. If it does it fast enough, Ω rapidly approaches unity, i.e. we have a flat Universe. After the inflationary period ends, the evolution of the Universe is described by the Standard Cosmological Model and $|\Omega - 1|$ begins to increase. If the inflation makes Ω very close to one, it could still be of the order unity even in the present time as the observations show.

Horizon problem solution.

Since the scale factor could evolve as a power law, $a \propto t^p$ with p > 1, during inflation [79], the physical wavelength $a\lambda$ grows faster than the Hubble radius $H^{-1} \propto t$. Therefore the physical wavelength goes outside the Hubble radius during inflation. This means that regions causally connected are stretched on scales much larger than the Hubble radius. This solves the horizon problem. After inflation, the Hubble radius begins to grow faster than the physical wavelength in the radiation and matter eras. In order to solve the horizon problem, it is required that the following condition is satisfied for the comoving particle horizon:

$$\int_{t_i}^{t_{\text{dec}}} \frac{dt}{a(t)} \gg \int_{t_{\text{dec}}}^{t_0} \frac{dt}{a(t)}$$
(3.36)

The comoving distance that photons can travel before decoupling must be much larger than that after the decoupling. It is achieved when the universe expands $\sim e^{70}$ times during inflation [79].

The origin of the large scale structure problem solution.

The fact that the comoving Hubble radius decreases during inflation makes it possible to generate the nearly scale-invariant density perturbations on large scales. Since the scales of perturbations are within the Hubble radius in the early stage of inflation, causal physics works to generate small quantum fluctuations. After a scale is pushed outside the Hubble radius during inflation, the first horizon crossing, the perturbations can be described as classical. When the inflationary period ends, the evolution of the universe is followed by the Standard Cosmological Model where the comoving Hubble radius begins to increase. Then the scales of the perturbations cross inside the Hubble radius again, the second horizon crossing, after which causality works. The small perturbations created during inflation appear as large-scale perturbations after the second horizon crossing. This is how inflation naturally provides a mechanism that generates the seeds of density perturbations observed in the CMB anisotropies today [78].

Magnetic monopoles problem solution.

During the inflationary phase, the energy density of the Universe decreases very slowly. For example, when the universe evolves as $a \propto t^p$ with p > 1, we have $H \propto t^{-1} \propto a^{-1/p}$ and $\rho \propto^{-2/p}$. Meanwhile the energy density of massive particles decreases much faster ($\sim a^{-3}$), these particles are redshifted away during inflation, solving the magnetic monopole problem. We do not have to be worried about the case where these unwanted particles are produced after inflation because in the process of reheating followed by inflation, the energy of the Universe can be transferred to radiation or other light particles expected for the Standard Cosmological Model.

3.6 Vacuum energy and the Cosmological Constant Problem

The cosmological constant Λ in the Einstein equations (1.1) is a parameter with dimensions of $(length)^2$. In the General Relativity context, there is not a preferred length scale that Λ might have. However, in the particle physics context, the cosmological constant turns out to be a measure of the state of lowest energy, that is, the energy density of the vacuum. Although we cannot calculate the vacuum energy with confidence, this identification allow to estimate the contributions of the cosmological constant to the energy density of the Universe [26].

To see this more clearly, let's consider a single scalar field ϕ , with potential energy $V(\phi)$. The action can be written

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \phi^{,\mu} \phi_{,\mu} - V(\phi) \right], \qquad (3.37)$$

and the corresponding energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{2}\phi_{,\mu}\phi_{,\nu} + \frac{1}{2}(\phi^{,\alpha}\phi_{,\alpha})g_{\mu\nu} + V(\phi)g_{\mu\nu}.$$
(3.38)

If it exists, the lowest energy density will be the one in which there is no contribution from kinetic or gradient energy, i.e. $\phi_{,\mu} = 0$, in this case the energy-momentum tensor reduces to $T_{\mu\nu} = V(\phi_0)g_{\mu\nu}$, where ϕ_0 is the value of ϕ which minimizes $V(\phi)$. There is no reason in principle why $V(\phi_0)$ should vanish. Taking account these considerations, the vacuum energy-momentum tensor can be written

$$T_{\mu\nu}^{\rm vac} = p^{\rm vac} g_{\mu\nu}, \tag{3.39}$$

with p^{vac} in given by $V(\phi_0)$. This form is the only Lorentz-invariant form for $T^{\text{vac}}_{\mu\nu}$ [23]. The vacuum can therefore be thought of as a perfect fluid described by (3.4), with

$$p^{\rm vac} = -\rho^{\rm vac}.\tag{3.40}$$

The effect of an energy-momentum tensor of the form (3.39) is equivalent to the the cosmological constant $\Lambda g_{\mu\nu}$ term in (1.1) just moving it from the left-hand side to the right-hand side and setting

$$\rho^{\rm vac} = \rho_{\Lambda} = \frac{\Lambda}{8\pi G}.$$
(3.41)

This equivalence is the origin of the identification of the cosmological constant with the energy of the vacuum. In the literature usually the terms "vacuum energy" and "cosmological constant" are used as synonyms. It is not necessary to introduce a scalar field to obtain the non-zero vacuum energy. The action for General Relativity in the presence of a "bare" cosmological constant Λ_0 is

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda_0).$$
 (3.42)

Extremizing this action leads to the equations (1.1). Thus, the cosmological constant can be thought of as simply a constant term in the Lagrange density of the theory. Classically, the effective cosmological constant is the sum of a bare term Λ_0 and the potential energy $V(\phi)$, where the latter may change with time as the Universe evolves. Quantum mechanics adds another contribution from the zero-point energies associated with vacuum fluctuations: the quantum vacuum fluctuations.

To exemplify the contributions of the quantum vacuum fluctuations, consider a simple harmonic oscillator in a one-dimensional potential of the form $V(x) = \frac{1}{2}\omega^2 x^2$. Classically,
the "vacuum" of this system is the state in which the particle is motionless and at the minimum of the potential x = 0, where the energy vanishes. On the side of quantum mechanics, the Heisenberg's uncertainty principle forbids the particle to have a defined position and momentum in the same state, making the calculations easy to show that the lowest energy state has an energy $E_0 = \frac{1}{2}\hbar\omega$.

An analogous situation holds in quantum field theory. A free quantum field can be thought of as a collection of an infinite number of harmonic oscillators in the momentum space. Formally, the zero-point energy of such an infinite collection will be infinite [14]. If very high-momentum modes are discarded imposing an ultraviolet momentum cutoff k_{max} , the resulting energy density is of the form

$$\rho_{\Lambda} \sim \hbar k_{\rm max}^4. \tag{3.43}$$

In the absence of gravity this energy has no effect, and is traditionally discarded by a process known as "normal-ordering". However, gravity does exist, and the actual value of the vacuum energy has important consequences. Vacuum fluctuations already been observed, as evidenced by the Casimir effect. The vacuum energy density seems to be the sum of a number of apparently disparate contributions: the potential energies from scalar fields, the quantum vacuum fluctuations of each field theory and bare cosmological constant.

In the first two cases the energy density can be roughly estimated. In the electroweak model, the phases of broken and unbroken symmetry are distinguished by a potential energy difference of approximately $M_{\rm EW} \sim 200$ GeV. The expected contribution to the vacuum energy in this case is

$$\rho_{\Lambda}^{\rm EW} \sim (200 \,{\rm GeV})^4 \sim 3 \times 10^{47} \,{\rm erg/cm}^3.$$
(3.44)

In the case of the energy difference between the symmetric and broken phases in the quantum chromodynamics (QCD), the energy scale is $M_{\rm QCD} \sim 0.3$ GeV, so the corresponding contribution to the vacuum energy is of the order

$$\rho_{\Lambda}^{\text{QCD}} \sim (0.3 \,\text{GeV})^4 \sim 1.6 \times 10^{36} \,\text{erg/cm}^3.$$
(3.45)

For the GUT case the contribution is of order $M_{\rm GUT} \sim 10^{16}$ GeV. In the case of vacuum fluctuations, the cutoff is chosen at the energy above which the field theory doesn't hold anymore. Assuming that we can use the quantum field theory at Planck scales (M_{Pl} =

 $(8\pi G)^{1/2} \sim 10^{18} GeV$), the contribution is of the order

$$\rho_{\Lambda}^{\rm Pl} \sim (10^{18} \,{\rm GeV})^4 \sim 2 \times 10^{110} \,{\rm erg/cm}^3.$$
(3.46)

Quantum field theory may fail at some specific scale. But we can have an idea of the magnitude of the zero-point contributions of known quantum fields. On the opposite side cosmological observations show that

$$|\rho_{\Lambda}^{(\text{obs})}| \le (10^{-12} \,\text{GeV})^4 \sim 2 \times 10^{-10} \,\text{erg/cm}^3.$$
 (3.47)

Incredible much (much!) smaller than any of the individual effects listed above. The ratio of (3.45) to (3.46) is the origin of the famous discrepancy of 120 orders of magnitude between the theoretical and observational values of the cosmological constant [14]. There is no obstacle to imagining that all these apparently unrelated contributions listed above add together with different signs to produce a net cosmological constant consistent with the limit (3.47). Is a possibility, but seem to be implausible. There is no known special symmetry which could enforce a vanishing vacuum energy while remaining consistent with the known laws of physics [23]. This riddle is the Cosmological Constant Problem, one of the most significant unsolved problems in fundamental physics.

Chapter

Cosmological tensor perturbations

4.1 Classical cosmological perturbation theory

We have assumed that we live in an approximately homogeneous and isotropic Universe. Our observable part of the Universe includes an immense number of galaxies and various types of radiation. On the scales of galaxies and their clusters, matter is concentrated in well discernible systems and in smaller scales the inhomogeneities of the Universe are even more evident. However, on larger scales accessible for observations (≤ 100 Mpc), the distributions of matter, its velocity, and accompanying gravitational field do not show any significantly preferred positions or directions.

The photons of the last scattering surface have traveled toward us for 10 billion years and, yet, the temperature of the microwave radiation in different directions on the sky is remarkably the same. Most convincingly, this is demonstrated by the fact that the measured large angular scale anisotropies of the CMB temperature $\delta T/T$ have the level of only 10^{-5} - 10^{-6} [69]. So, when we work with homogeneous isotropic cosmological models plus small perturbations, this is not just a mathematical simplification, this is a reflection of observational data about the real world.[80].

As we have stated before, there are three ways in which one can perturb a homogeneous and isotropic distribution of matter and fields. These are the scalar, vector and tensor perturbations. In the context of cosmology using the FRW metric (3.1), the perturbations have the following features: First, we can compress matter in various places perturbing its mass density, velocity and the accompanying gravitational field. This is called cosmological density perturbations. Second, we can provide the elements of matter with small rotational velocities (without perturbing the mass density) which will also be accompanied by perturbations of the gravitational field. This is called cosmological rotational perturbations. And third, we can perturb gravitational field itself. These are called cosmological gravitational waves. In contrast to gravitational waves emitted by localized sources, we will be dealing with waves as excitations of the gravitational field in the entire Universe.

General Relativity couples geometry with matter by the Einstein equations. Therefore, matter perturbations are perturbations in the geometry described by the metric $g_{\mu\nu}$. The perturbation theory in General Relativity is not a trivial task, since this theory has coordinate transformation symmetry. The main problem is this freedom of choice of coordinates, or gauge freedom, used to describe the perturbations. In contrast to the homogeneous and isotropic Universe, where the preferable coordinate system is fixed by the symmetry properties of the background, there are no obvious preferable coordinates for analyzing perturbations. The gauge freedom leads to the appearance of fictitious perturbation modes which do not describe any real inhomogeneity, only reflecting the properties of the coordinate system that is used.

To avoid this problem we must clarify the matter and metric perturbations introducing gauge-invariant variables, which do not depend on the particular choice of coordinates. A brief summary of the theory of cosmological perturbations will be shown. For a more extensive and detailed study, see [10, 81]. We begin postulating small perturbations in matter which induce perturbations in the metric.

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad \text{where} \quad |\bar{g}_{\mu\nu}| \gg |\delta g_{\mu\nu}|, \tag{4.1}$$

where $\bar{g}_{\mu\nu}$ is the unperturbed flat (k = 0) FLRW metric. First, notice that since $\delta g_{\mu\nu}$ must be symmetric as $\bar{g}_{\mu\nu}$ is, hence, the degrees of freedom are reduced to ten. Let's see the irreducible parts of the perturbation. The component δg_{00} transforms as a scalar; it can be written in terms of 3-scalar function ϕ in the following way:

$$\delta g_{00} = 2a^2(\eta)\phi(\eta, \mathbf{x}). \tag{4.2}$$

The off-diagonal components of the perturbed metric δg_{0i} are the vector part of it. The components δg_{0i} can be decomposed into the sum of the spatial gradient of some scalar B and a vector S_i with null divergence $S_i^{,i} = 0$, this is because if $S_i^{,i} \neq 0$ it could be decomposed in a vector part with null divergence and a scalar. The component δg_{0i} transforms as a vector:

$$\delta g_{0i} = a^2(\eta)(B_{,i} + S_i) \tag{4.3}$$

In a similar way, the components of δg_{ij} can be written as the sum of their irreducible pieces:

$$\delta g_{ij} = a^2(\eta)(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}). \tag{4.4}$$

where ψ and E are scalars that can determine a tensor in different ways, being multiplied by an unitary tensor or applying a Laplacian. To construct a tensor from a vector F_i with an irreducible description, it must have a null divergence $(F_i^{,i} = 0)$ as stated above. And finally the term h_{ij} is a irreducible tensor; it does not decompose into new scalars or vectors; it must be a transverse-traceless tensor

$$h_i^i = 0, \quad h_{j,i}^i = 0.$$
 (4.5)

Counting the number of independent functions used to $\delta g_{\mu\nu}$, we have four functions for the scalar perturbations, four functions for the vector perturbations (two 3-vectors with one constraint each), and two functions for the tensor perturbations (a symmetric 3-tensor has six independent components and there are four constraints). Thus we have ten functions altogether, which are the number of independent components of $\delta g_{\mu\nu}$. Now we can clearly divide the perturbations depending on its nature. For the scalar perturbations:

$$\delta g_{\mu\nu}^{\text{scalar}} = a^2 \begin{pmatrix} 2\phi & B_{,i} \\ B_{,i} & 2(\psi\delta_{ij} + E_{,ij}) \end{pmatrix}$$
(4.6)

These perturbations are associated with the large scale structure formation because they are coupled with inhomogeneities of the energy density. Then we have for the vector perturbations:

$$\delta g_{\mu\nu}^{\text{vector}} = a^2 \begin{pmatrix} 0 & S_i \\ S_i & F_{i,j} + F_{j,i} \end{pmatrix}$$
(4.7)

Not coupled to the energy density $(\delta g_{00}^{\text{vector}} = 0)$. And for the tensor perturbations

$$\delta g_{\mu\nu}^{\text{tensor}} = a^2 \begin{pmatrix} 0 & 0\\ 0 & h_{ij} \end{pmatrix}$$
(4.8)

These perturbations represent the two polarizations of the gravitational waves. In this formalism we will see that they do not couple to the matter perturbations. From here we will only focus in the tensor perturbations which represent the gravitational waves. Some aspects concerning tensor fluctuations of the geometry will be summarized. The background geometry as stated before will be assumed to be the spatially flat FLRW.

The perturbed tensor line element can be written as

$$ds^{2} = a^{2}(\eta)(-d\eta^{2} + (\delta_{ij} + h_{ij})dx^{i}dx^{j}).$$
(4.9)

We first begin with the definition of the Christoffel connection,

$$\Gamma^{\alpha}_{\gamma\beta} = \frac{1}{2} g^{\alpha\delta} \left(\partial_{\beta} g_{\gamma\delta} + \partial_{\gamma} g_{\delta\beta} - \partial_{\delta} g_{\gamma\beta} \right).$$
(4.10)

Perturbing up to first order, the fluctuations of Christoffel connections can be computed as

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\bar{g}^{\mu\nu}(-\partial_{\nu}\delta g_{\alpha\beta} + \partial_{\beta}\delta g_{\nu\alpha} + \partial_{\alpha}\delta g_{\beta\nu}) + \frac{1}{2}\delta g_{\mu\nu}(-\partial_{\nu}\bar{g}_{\alpha\beta} + \partial_{\beta}\bar{g}_{\nu\alpha} + \partial_{\alpha}\bar{g}_{\beta\nu}).$$
(4.11)

The Riemann tensor is defined as

$$R^{\alpha}_{\beta\mu\nu} \equiv \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta}.$$
(4.12)

Then the Ricci tensor, which is the contraction of the Riemann tensor is

$$R^{\alpha}{}_{\mu\alpha\nu} \equiv R_{\mu\nu} = \frac{1}{2} (\partial^{\alpha}\partial_{\mu}g_{\nu\alpha} + \partial^{\alpha}\partial_{\nu}g_{\mu\alpha} - \partial_{\mu}\partial_{\nu}g - \partial_{\alpha}\partial^{\alpha}g_{\mu\nu}).$$
(4.13)

The first-order fluctuation of the Ricci scalar can also be computed as

$$\delta R_{\mu\nu} = \partial_{\alpha} \delta \Gamma^{\alpha}_{\mu\nu} - \partial_{\nu} \delta \Gamma^{\beta}_{\mu\beta} + \delta \Gamma^{\alpha}_{\mu\nu} \bar{\Gamma}^{\beta}_{\alpha\beta} + \bar{\Gamma}^{\alpha}_{\mu\nu} \delta \Gamma^{\beta}_{\alpha\beta} - \delta \Gamma^{\beta}_{\alpha\mu} \bar{\Gamma}^{\alpha}_{\beta\nu} - \bar{\Gamma}^{\beta}_{\alpha\mu} \delta \Gamma^{\alpha}_{\beta\nu}.$$
(4.14)

The fluctuations of the Ricci tensor with one contravariant index and one covariant index, as well as the fluctuations of the Ricci scalar, are then

$$\delta R^{\nu}_{\mu} = \delta g^{\nu\alpha} \overline{R}_{\alpha\mu} + \overline{g}^{\nu\alpha} \delta R_{\alpha\mu}, \qquad (4.15)$$

$$\delta R = \delta g^{\alpha\beta} \overline{R}_{\alpha\beta} + \overline{g}^{\alpha\beta} \delta R_{\alpha\beta}. \tag{4.16}$$

Finally, the fluctuations of the components of the Einstein tensor can be easily deduced using equations (4.15) and (4.16)

$$\delta G^{\nu}_{\mu} = \delta R^{\nu}_{\mu} - \frac{1}{2} \delta^{\nu}_{\mu} \delta R.$$
(4.17)

Formally, the perturbation of the covariant conservation equation is

$$\partial_{\mu}\delta T^{\mu\nu} + \overline{\Gamma}^{\mu}_{\mu\alpha}\delta T^{\alpha\nu} + \delta\Gamma^{\mu}_{\mu\alpha}\overline{T}^{\alpha\nu} + \overline{\Gamma}^{\nu}_{\alpha\beta}\delta T^{\alpha\beta} + \delta\Gamma^{\nu}_{\alpha\beta}\overline{T}^{\alpha\beta} = 0.$$
(4.18)

To obtain the explicit equations for the tensor perturbations, it is necessary to have the values of the Christoffel connections of the background

$$\overline{\Gamma}_{00}^{0} = \mathcal{H}, \quad \overline{\Gamma}_{ij}^{0} = \mathcal{H}\delta_{ij}, \quad \overline{\Gamma}_{0i}^{j} = \mathcal{H}\delta_{i}^{j}.$$
(4.19)

The term \mathcal{H} denotes the Hubble parameter function in terms of the conformal time, that is, $\mathcal{H} = a'/a = Ha$. The components of the Ricci tensor and the Ricci scalar of the background are then:

$$\overline{R}_{00} = -3\mathcal{H}', \quad \overline{R}_{0}^{0} = -\frac{3}{a^{2}}\mathcal{H}',$$

$$\overline{R}_{ij} = (\mathcal{H}' + 2\mathcal{H}^{2})\delta_{ij}, \quad \overline{R}_{i}^{j} = -\frac{1}{a^{2}}(\mathcal{H}' + 2\mathcal{H}')\delta_{i}^{j},$$

$$\overline{R} = -\frac{6}{a^{2}}(\mathcal{H}' + \mathcal{H}')\delta_{i}^{j}.$$
(4.20)

Consider now the case of the tensor modes of the geometry. First, according to the equations (4.5) we have

$$\delta g_{ij} = -a^2 h_{ij}, \quad \delta g^{ij} = \frac{h_{ij}}{a^2} \tag{4.21}$$

From the equation (4.11) the tensor contribution to the fluctuation of the connections can be expressed as

$$\delta\Gamma_{ij}^{0} = \frac{1}{2}(h_{ij}' + 2\mathcal{H}h_{ij}),$$

$$\delta\Gamma_{i0}^{j} = \frac{1}{2}h_{i}^{j\prime},$$

$$\delta\Gamma_{ij}^{k} = \frac{1}{2}(\partial_{j}h_{i}^{k} + \partial_{i}h_{j}^{k} - \partial^{k}h_{ij}).$$
(4.22)

Inserting these results into the perturbed expressions of the Ricci tensors we obtain:

$$\delta R_{ij} = \frac{1}{2} [h_{ij}'' + 2\mathcal{H}h_{ij} + 2(\mathcal{H}' + 2\mathcal{H}^2)h_{ij} - \nabla^2 h_{ij}], \qquad (4.23)$$

$$\delta R_i^j = -\frac{1}{2a^2} [h_i^{j\prime\prime} + 2\mathcal{H}h_i^{j\prime} - \nabla^2 h_i^j].$$
(4.24)

Now, let's see what happens to the perturbations of the material content. The background energy-momentum tensor is necessarily a perfect fluid:

$$\bar{T}^{\mu\nu} = (\bar{\rho} + \bar{p})\bar{u}^{\mu}\bar{u}^{\nu} + \bar{p}\bar{g}^{\mu\nu}, \qquad (4.25)$$

$$\bar{T}^{\mu}_{\nu} = (\bar{\rho} + \bar{p})\bar{u}^{\mu}\bar{u}_{\nu} + \bar{p}\delta^{\mu}_{\nu}.$$
(4.26)

Because of homogeneity, $\bar{\rho} = \bar{\rho}(\eta)$ and $\bar{p} = \bar{p}(\eta)$. Because of isotropy, we choose a coordinate system where the fluid is at rest, $\bar{u}^{\mu} = (\bar{u}^0, 0, 0, 0)$ in the background universe. Since

$$\bar{u}^{\mu}\bar{u}_{\mu} = \bar{g}_{\mu\nu}\bar{u}^{\mu}\bar{u}^{\nu} = a^{2}\eta_{\mu\nu}\bar{u}^{\mu}\bar{u}^{\nu} = -a^{2}(\bar{u}^{0})^{2} = -1, \qquad (4.27)$$

we have

$$\bar{u}^{\mu} = a^{-1}(1, 0, 0, 0), \text{ and } \bar{u}_{\mu} = -a(1, 0, 0, 0)$$
 (4.28)

The energy tensor of the perturbed universe is

$$T^{\mu}_{\nu} = \bar{T}^{\mu}_{\nu} + \delta T^{\mu}_{\nu}. \tag{4.29}$$

Just like the metric perturbation, the energy-momentum tensor perturbation has 10 degrees of freedom, of which 6 are physical and 4 are gauge. It can likewise be divided into independent scalar, vector and tensor parts, with 4+4+2 degrees of freedom, of which 2+2+2 are physical. The perturbation can also be divided into perfect plus non-perfect fluids, with 5+5 degrees of freedom. The perfect fluid degrees of freedom in δT^{μ}_{ν} are those which keep T^{μ}_{ν} in the perfect fluid form

$$T^{\mu}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + p\delta^{\mu}_{\nu}. \tag{4.30}$$

Thus they can be taken as the density perturbation, pressure perturbation, and velocity perturbation

$$\rho = \bar{\rho} + \delta\rho, \quad p = \bar{p} + \delta p, \quad \text{and} \quad u^i = \bar{u}^i + \delta u^i = \delta u^i \equiv a^{-1} v^i.$$
(4.31)

The δu^0 is not an independent degree of freedom, because of the constraint $u^{\mu}u_{\mu} = -1$. We shall call

$$v^i = \frac{dx^i}{d\eta} = au^i,\tag{4.32}$$

the velocity perturbation. It is equal to the coordinate velocity up to first order. To express u^{μ} and u_{ν} in terms of v^{i} , we write them as

$$u^{\mu} = \bar{u}^{\mu} + \delta u^{\mu} = (a^{-1} + \delta u^{0}, a^{-1}v_{1}, a^{-1}v_{2}, a^{-1}v_{3}), \qquad (4.33)$$

$$u_{\nu} = \bar{u}_{\nu} + \delta u_{\nu} = (-a + \delta u_0, \delta u_1, \delta u_2, \delta u_3).$$
(4.34)

These are related by $u_{\nu} = g_{\mu\nu}u^{\mu}$ and $u_{\mu}u^{\mu} = -1$. Using the complete metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2 \begin{pmatrix} -1 + 2\phi & B_{,i} + S_i \\ B_{,i} + S_i & (1 + 2\psi) + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij} \end{pmatrix}$$
(4.35)

we get

$$u_{0} = g_{0\mu}u^{\mu} = a^{2}(-1+\phi)(a^{-1}+\delta u^{0}) - \delta^{ij}a^{2}(B_{,i}+S_{i})a^{-1}v_{j}$$

= $-a - a^{2}\delta u^{0} + 2a\phi.$ (4.36)

The last equality was achieved considering only the first order terms, which yields $\delta u^0 = a^{-1}\phi$; likewise, $\delta u_0 = a\phi$. Thus we have for the 4-velocity

$$u^{\mu} = a^{-1}(1+\phi, v_i)$$
 and $u_{\mu} = a(-1+\phi, v_i + B_{,i} + S_i).$ (4.37)

Inserting this into the perturbed energy-momentum tensor (4.30) we get

$$T^{\mu}_{\nu} = \bar{T}^{\mu}_{\nu} + \delta T^{\mu}_{\nu}$$
$$= \begin{pmatrix} -\bar{\rho} & 0\\ 0 & \bar{p}\delta^{i}_{j} \end{pmatrix} + \begin{pmatrix} -\delta\rho & (\bar{\rho} + \bar{p})(v_{i} + B_{,i} + S_{i})\\ -(\bar{\rho} + \bar{p})v_{i} & \delta p\delta^{i}_{j} \end{pmatrix}$$
(4.38)

There are 5 remaining degrees of freedom in the space part, δT_j^i , corresponding to perturbations away from the perfect fluid from. We can write it as

$$\delta T^i_j = \delta p \delta^i_j + \Pi^i_j, \tag{4.39}$$

where Π_{ij} is called anisotropic stress. It is symmetric and traceless, and for a perfect fluid $\Pi_{ij} = 0$. The energy tensor perturbation δT^{μ}_{ν} is built out of the scalar perturbations $\delta \rho$, δp , the 3-vector v_i and the traceless 3-tensor $\Pi_{ij} = 0$. Just like for the metric perturbations, we can extract a scalar perturbation out of v_i :

$$v_i = -v_{,i} + w_i$$
 where $w_i^{,i} = 0.$ (4.40)

And the anisotropic stress can be decomposed into

$$\Pi_{ij} = \Pi_{ij}^{\text{scalar}} + \Pi_{ij}^{\text{vector}} + \Pi_{ij}^{\text{tensor}}, \qquad (4.41)$$

where $\Pi_{ij}^{\text{scalar}} = (\partial_i \partial_j - \frac{1}{2} \delta_{ij} \nabla) \Pi$, $\Pi_{ij}^{\text{vector}} = -\frac{1}{2} (\Pi_{i,j}^{\text{vector}} + \Pi_{j,i}^{\text{vector}})$ and $\delta^{ik} \Pi_{ij,k}^{\text{tensor}} = 0$. Gathering all these quantities now we can decompose the energy-momentum tensor perturbation into its scalar, vector and tensor components $\delta T_{\mu\nu} = \delta T_{\mu\nu}^{\text{scalar}} + \delta T_{\mu\nu}^{\text{vector}} + \delta T_{\mu\nu}^{\text{tensor}}$:

$$\delta T_{\mu\nu}^{\text{scalar}} = \begin{pmatrix} -\delta\rho & (\bar{\rho} + \bar{p})(-\nu + B)_{,i} \\ (\bar{\rho} + \bar{p})v_{,i} & \delta p\delta_j^i + \Pi_{ij}^{\text{scalar}} \end{pmatrix}$$
(4.42)

$$\delta T_{\mu\nu}^{\text{vector}} = \begin{pmatrix} 0 & (\bar{\rho} + \bar{p})(w_i + S_i) \\ -(\bar{\rho} + \bar{p})w_i & \Pi_{ij}^{\text{vector}} \end{pmatrix}$$
(4.43)

$$\delta T_{\mu\nu}^{\text{tensor}} = \begin{pmatrix} 0 & 0 \\ 0 & \Pi_{ij}^{\text{tensor}} \end{pmatrix}$$
(4.44)

For a metric with small perturbations, the Einstein tensor can be written as $G_{\mu\nu}$ =

 $\bar{G}_{\mu\nu} + \delta G_{\mu\nu} + \cdots$, where $\delta G_{\mu\nu}$ denotes the linear fluctuations. The energy-momentum tensor can be split in a similar way $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} + \cdots$. Then we have shown that both sides of the perturbed Einstein equations can be separated into their scalar, vector and tensor components, hence the linearized tensor equations for the perturbations are:

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \quad \Rightarrow \quad \delta G_{\mu\nu}^{\text{tensor}} = 8\pi G \delta T_{\mu\nu}^{\text{tensor}}. \tag{4.45}$$

Combining the equations (4.17), (4.24) and (4.44) with the last equation we have

$$h_j^{i''} + 2\mathcal{H}h_j^{i'} - \nabla^2 h_j^i = -16\pi G a^2 \Pi_j^{i \text{ tensor}}.$$
 (4.46)

We can see that perfect fluid perturbations do not have a tensor perturbation component ($\Pi_{ij}^{\text{tensor}} = 0$). Finally the equations of motion of the gravitational waves (tensor perturbations) are:

$$h_j^{i''} + 2\mathcal{H}h_j^{i'} - \nabla^2 h_j^i = 0.$$
(4.47)

As $h_i^i = 0$ and $h_{j,i}^i = 0$, the z axis is chosen along the propagation direction, in this case the two physical polarizations of the gravitational waves will be

$$h_1^1 = -h_2^2 \equiv h_+, \quad h_1^2 = h_2^1 \equiv h_\times,$$
(4.48)

To find the solution of (4.47), it is natural to propose a solution in terms of Fourier spatial harmonics in the following way

$$h_{ij}(\eta, \mathbf{x}) = \frac{\sqrt{16\pi G}}{(2\pi)^{3/2}} \int d^3 \mathbf{n} \sum_{r=+,\times} {}^r_{\epsilon_{ij}}(\mathbf{n}) \left[h_n^r(\eta) e^{i\mathbf{n}\cdot\mathbf{x}} c_{\mathbf{n}}^r + h_n^r(\eta) e^{-i\mathbf{n}\cdot\mathbf{x}} c_{\mathbf{n}}^{r\dagger} \right].$$
(4.49)

For a classical gravitational field, **n** is the comoving wave vector, $\vec{c}_{\mathbf{n}}$ and $\vec{c}_{\mathbf{n}}^{r\dagger}$ are complex coefficients and $\vec{h}_n(\eta)$ are the mode functions. The polarization tensor $\vec{\epsilon}_{ij}(\mathbf{n})$, inherits the tensor properties and symmetries of h_{ij} , i.e. $\vec{\epsilon}_{ij}(\mathbf{n})$, is symmetric $(\vec{\epsilon}_{ij}(\mathbf{n}) = \vec{\epsilon}_{ji}(\mathbf{n}))$, traceless $(\vec{\epsilon}_{ii}(\mathbf{n}) = 0)$, and transverse $(n_i \vec{\epsilon}_{ij}(\mathbf{n}) = 0)$. We also choose a circular-polarization basis in which $\vec{\epsilon}_{ij}(\mathbf{n}) = (\vec{\epsilon}_{ij}(-\mathbf{n}))^*$, that normalize the basis

$$\sum_{i,j} \stackrel{r}{\epsilon}_{ij}(\mathbf{n}) (\stackrel{s}{\epsilon}_{ij}(\mathbf{n}))^* = 2\delta^{rs}, \tag{4.50}$$

The physical wave number k is given by

$$n = |\mathbf{n}| = \frac{2\pi a(\eta)}{\lambda} = k \, a(\eta), \tag{4.51}$$

 λ is the physical wavelength. Substituting (4.49) in (4.47) and making the calculations, we obtain the mode function evolution equation

$$\ddot{h}_{n}'' + 2\mathcal{H}\ddot{h}_{n}' + n^{2}\ddot{h}_{n} = 0.$$
(4.52)

This equation has the form of a damped harmonic oscillator equation. The two polarizations h_+ and h_{\times} satisfy independently the evolution equation (4.52). Defining the auxiliary function:

$${}^{r}_{\mu_{n}}(\eta) \equiv {}^{r}_{h_{n}}(\eta)a(\eta), \qquad (4.53)$$

we can rewrite (4.52) in the following way

$$\mu_n^{r''} + \left(n^2 - \frac{a''}{a}\right)^r \mu_n = 0.$$
(4.54)

The equation (4.54) is the master equation to be studied. It is satisfied independently for each polarization, $r = +, \times$, from here we will omit the polarization index. This is a general result where we have only assumed a flat FLRW geometry with small perturbations. Knowing the functional form of scale factor $a(\eta)$ we can in principle solve the equation for $\mu_n(\eta)$ for each mode n.

4.2 Amplification mechanism

The first thing that catches the attention of the definition (4.53) is that it shows that the modes functions $h_n(\eta) = a^{-1}(\eta)\mu_n(\eta)$ consists of two parts: the solution $\mu_n(\eta)$ and the inverse of the scale factor a^{-1} , as this last is always an increasing function, it means that the cosmic expansion damps or dilutes the evolution of $\mu_n(\eta)$ in some way. At first glance, the solution of the equation (4.54) seems to be oscillatory which would imply that the evolution of the Universe dilutes the perturbations of the young Universe, making them practically insignificant for the today's old Universe. But in some particular cases this effect can be diminished or nullified depending on characteristics of the cosmic expansion, this is what is called the *amplification mechanism* of the primordial GWs [82]. Let's see how it works.

Rewriting the above equation (4.54) as $\mu_n'' + \omega_n^2(\eta)\mu_n = 0$, the equation describes a

harmonic oscillator with variable frequency. This variable frequency $\omega_n^2(\eta) = n^2 - a''/a$, has some interesting properties like that in some cases an external agent can inject energy driving to an amplification of the oscillations. To illustrate this idea let's consider, for example the simple pendulum: for small swings the equation of motion is $\ddot{\theta} + \sqrt{g/l\theta} = 0$. If now we assume that its frequency is variable $(\omega^2(t) = g/l(t))$ then we have a variable pendulum length l = l(t) (assuming that the gravitational acceleration g is constant). The stretching and shortening the pendulum's length in a suitable way can cause the amplitude of oscillation to be greater than at the beginning. Particularly, if it is shortened quickly then we have a magnification of the amplitude. This simple example is schematically illustrated in Figure (4.1) [83].



Figure 4.1: Harmonic oscillator with variable frequency, $\ddot{\theta} + \omega^2(t)\theta = 0$. **a**) The pendulum has an initial frequency and amplitude. **b**) Varying the length of the pendulum, the frequency and amplitude can be modified. **c**) The final state of the pendulum is with the same pendulum length but with greater amplitude. The amplification of the oscillations is due the addition of energy by stretching and shortening of the pendulum's length.

Another illustrative way to interpret equation (4.54) is to compare it with the singleparticle, time-independent, 1-d Schroedinger equation:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x)\right)\psi = E\psi, \quad \Rightarrow \quad \frac{d^2\psi}{dx^2} + \left(\frac{2m}{\hbar}E - \frac{2m}{\hbar}U(x)\right)\psi = 0.$$

In this spirit, the conformal time η plays the role of spatial coordinate x, the square of the wave number n^2 the energy $\frac{2m}{\hbar}E$ and a''/a the potential $\frac{2m}{\hbar}U(x)$. Due to this analogy, is common to see in the specialized literature the term a''/a be named as the "potential"; we will keep following this nomenclature. The perturbation is like a particle moving through the space with a certain energy to then hit and interact with a potential barrier (or well) which changes its nature. The interaction between them will depend on the relation of the energy (frequency) and the potential (Universe acceleration) i.e. the nature of the variable frequency $\omega_n(\eta)$.

Lets see the limit cases. For high-frequency perturbations or, more precisely, for waves such that $n^2 \gg a''/a$, the master equation becomes the common harmonic oscillator equation:

$$\mu'' + n^2 \mu = 0, \qquad \Rightarrow \qquad \mu(\eta, n) = a_1 \cos(n\eta) + a_2 \sin(n\eta),$$

$$\Rightarrow \qquad h_n(\eta) = \frac{a_1 \cos(n\eta) + a_2 \sin(n\eta)}{a(\eta)}. \tag{4.55}$$

Here, when the potential doesn't play any role, the nature of the perturbations is unavoidable: the GWs are diluted by the cosmic expansion. In other words, there is no amplification. Naturally, if a'' = 0, a non accelerating Universe, there is not amplification for any frequency.

In the opposite side, for low frequencies, $n^2 \ll a''/a$, we get two solutions of the form:

$$\mu'' - \frac{a''}{a}\mu = 0, \quad \Rightarrow \quad \mu_1 = a(\eta), \quad \mu_2 = a(\eta) \int^{\eta} a^{-2}(\eta') d\eta'. \tag{4.56}$$

For an expanding Universe, the dominant solution is μ_1 , and hence, as long as the wave stays under the potential a''/a, its amplitude will remain practically constant $h_n = \mu_1/a \propto$ constant. Here is where the amplification occurs.

Consider first two waves with the same amplitude but with different frequencies evolving from an epoch where the potential is negligible. Both are damped by the cosmic expansion with the same rate, independent of its frequency. Then they enter an epoch where the potential is dominant for one of the waves, but not to the other. The first one when it enters the potential maintains its amplitude constant, the other one continues its evolution as if there was no potential. Finally, the waves enter again an epoch without potential. The wave which passed trough the potential now has a greater amplitude than the wave which evolved without entering to the potential. Grishchuk has called this phenomenon the "superadiabatic amplification of GWs" [59, 60, 82]. In figure 4.2 it is schematically shown the GW amplification mechanism.

Up to now we have only considered classical perturbations. These are postulated in the sense that they "appear" to then interact with the gravitational background. There



Figure 4.2: Amplification mechanism of GWs.

is a more clear origin of these perturbations from the quantum vacuum fluctuations based on the Uncertainty Principle of Heisenberg. But when we introduce this in the theory the have to quantize the perturbations in some way. This will be discussed in the next section.

4.3 Quantized tensor perturbations

It is now clear what we are going to quantize: the fluctuating part $h_{\mu\nu}$ of the gravitational field. The theory of small perturbations is essentially linear, but the fundamental nonlinearity of the gravitational dynamics provides us the interaction of the quantized perturbations with the background acting as a pump field. Making an analogy, in contrast to quantum optics we do not need any intervening nonlinear optical medium in order to couple the quantized part of the field with the pump field, gravity is inherently nonlinear [80].

As we want to quantize the system, it is necessary to find the correct canonical quantization variables. They will give us an effective quadratic action in the form of harmonic oscillators to then be quantized. In this way the study of the vacuum (as in the quantum harmonic oscillator) will be in terms of annihilation and creation operators. As we saw in the classical perturbation theory, for tensor perturbations there is only one variable that describe the perturbations (h_{ij}) . Our goal is then to find this variable in this context to then quantize it.

First of all we need the action for the GWs. This action can be derived by expanding the Einstein action up to the second order in transverse, traceless metric perturbations h_{ij} , the result is

$$S = \frac{1}{64\pi} \int a^2 \left((h_j^i)'(h_i^j)' - h_{j,l}^i h_i^{j,l} \right) d\eta d^3 x.$$
(4.57)

Substituting the expansion of equation (4.49) into the action we have

$$S = \frac{1}{64\pi} \int a^2 \epsilon_j^i \epsilon_i^j \left[h'_{\mathbf{n}} h'_{-\mathbf{n}} - n^2 h_{\mathbf{n}} h_{-\mathbf{n}} \right] d\eta d^3 x.$$
(4.58)

Rewriting in terms of the new variable

$$\mu_{\mathbf{n}} = \sqrt{\frac{\epsilon_j^i \epsilon_i^j}{32\pi}} a h_{\mathbf{n}}, \qquad (4.59)$$

the action becomes

$$S = \frac{1}{2} \int \mathcal{L} d\eta d^3 x = \frac{1}{2} \int \left[\mu'_{\mathbf{n}} \mu'_{-\mathbf{n}} - \left(n^2 - \frac{a''}{a} \right) \right] d\eta d^3 x.$$

$$(4.60)$$

This new form of the action is quadratic. The quantization of the perturbations with the action (4.60) is formally equivalent to the quantization of a free scalar field μ with a time dependent "mass" $m^2 = a''/a$ in Minkowski space. The time dependence of this mass is due to the interaction of the perturbations with the homogeneous expanding background. The energy of the perturbations is not conserved and they can be excited by borrowing energy from the Hubble expansion. Here the canonical quantization variable is μ_n .

Making the Fourier transformation we obtain an equation of motion

$$\mu_{\mathbf{n}}'' + \omega_n^2(\eta)\mu_{\mathbf{n}} = 0, \quad \omega_n^2(\eta) = n^2 - \frac{a''}{a}, \tag{4.61}$$

that is the same equation of motion of the classical tensor cosmological perturbations. This is an important result, both classical and quantum perturbations follow the same equation motion. We can reformulate this in the Hamiltonian formalism. This gives a different scope of the system, since, instead of using into second order equations in the Lagrangian formalism, it provides us first order equations, the Hamilton's equations, defined in phase space. For this we have to define the canonical conjugate momentum

$$\Pi_{\mu}(\eta, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \mu'} = \mu' \tag{4.62}$$

And making the Legendre transformation we obtain the Hamiltonian of the system

$$H = \int (\mu' \Pi_{\mu} - \mathcal{L}) d^3 x = \frac{1}{2} \int \left(\Pi_{\mu}^2 + (\mu_{,i})^2 \frac{a''}{a} \mu^2 \right) d^3 x.$$
(4.63)

The dynamics of the variables are given by the Hamilton's equations defined as

$$[\mu, H] = i\mu', \quad [\Pi_{\mu}, H] = i\Pi'_{\mu}. \tag{4.64}$$

With the quadratic action we can proceed to the canonical quantization. Let's start with a general variable $Q(\eta, \mathbf{x})$ to be quantized. The treatment presented below can be used for both scalar and tensor perturbations. Our Lagrangian for this variable is

$$S[Q] = \int \left[(Q')^2 - Q_i Q^i + \frac{a''}{a} Q^2 \right] d\eta d^3 x.$$
(4.65)

The Euler-Lagrange equations provides us the equations of motion

$$\frac{\partial \mathcal{L}}{\partial Q} - \frac{\partial}{\partial \mu} \left[\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} Q)} \right] = 0, \quad \Rightarrow \quad Q'' + \left(\nabla^2 - \frac{a''}{a} \right) Q = 0. \tag{4.66}$$

Making the canonical quantization, the canonical variables turns into quantum operators

$$Q \rightarrow \hat{Q}, \quad \Pi \rightarrow \hat{\Pi},$$
 (4.67)

that must satisfy the commutation relations

$$[\hat{Q}(\eta, \mathbf{x}), \hat{\Pi}(\eta, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \quad [\hat{Q}(\eta, \mathbf{x}), \hat{Q}(\eta, \mathbf{y})] = [\hat{\Pi}(\eta, \mathbf{x}), \hat{\Pi}(\eta, \mathbf{y})] = 0.$$
(4.68)

Consequently for the variables in the Fourier space

$$[\hat{Q}_{\mathbf{n}}(\eta), \hat{\Pi}_{\mathbf{m}}(\eta)] = i\delta(\mathbf{n} - \mathbf{m}).$$
(4.69)

A general solution for the equation (4.61) in terms of $\hat{Q}_{\mathbf{n}}(\eta)$ can be written as a decomposition

$$\hat{Q}(\eta, \mathbf{x}) = \int \frac{d^3n}{(2\pi)^3/2} [Q_{\mathbf{n}}(\eta)\hat{a}_{\mathbf{n}}e^{-i\mathbf{n}\mathbf{x}} + Q_{\mathbf{n}}^*(\eta)\hat{a}_{\mathbf{n}}^{\dagger}e^{i\mathbf{n}\mathbf{x}}], \qquad (4.70)$$

where $\hat{a}_{\mathbf{n}}$ and $\hat{a}_{\mathbf{n}}^{\dagger}$ are the creation and annihilation particle operators. The introduction of these operators turns the field variables into quantum operator,

$$\mu \to \hat{\mu}.\tag{4.71}$$

This is known as *second quantization*. The commutation relations stated before define the commutation relations of these new operators at equal times

$$[\hat{a}_{r',\mathbf{n}}, \hat{a}_{r,\mathbf{m}}^{\dagger}] = \delta_{r'r} \delta^{3}(\mathbf{n} - \mathbf{m}), \quad [\hat{a}_{r',\mathbf{n}}, \hat{a}_{r,\mathbf{m}}] = [\hat{a}_{r',\mathbf{n}}^{\dagger}, \hat{a}_{r,\mathbf{m}}^{\dagger}] = \delta_{r'r} \delta^{3}(\mathbf{n} - \mathbf{m}) = 0.$$
(4.72)

We have reintroduced the polarization index to emphasize that the commutation relations are not independent of the polarization. These commutation rules are valid only if the quantum modes $\hat{\mu}_{r,\mathbf{n}}(\eta)$ obey the normalization

$$\hat{\mu}_{r,\mathbf{n}}'\hat{\mu}_{r,\mathbf{n}}^* - \hat{\mu}_{r,\mathbf{n}}^{*'}\hat{\mu}_{r,\mathbf{n}} = i, \qquad (4.73)$$

derived from the equation (4.68), which is the Wronskian for the classical solutions. This normalization fixes the amplitude of $\hat{\mu}$ to be compatible with the Uncertainty Principle of Heisenberg. The Hamiltonian operator of the system can now be easily written in terms of the creation and annihilation operators in the following way

$$\hat{H}_{\mathbf{n}} = \omega_{r,\mathbf{n}}^2(\eta) \hat{a}_{\mathbf{n}}^{\dagger} \hat{a}_{\mathbf{n}}, \qquad (4.74)$$

here $\omega_{r,\mathbf{n}}^2(\eta)$ acts as a frequency. With the Hamiltonian defined, the next step is to define the Fock space where the operators act. The Hamiltonian has eigenstates with eigenvalues given by the energy of the system. As the Hamiltonian is quadratic and definite-positive it must exist a minimum energy state $|0\rangle$ with energy E_0 , such that $\hat{H}|0\rangle = E_0|0\rangle$. This allows us to define a vacuum state

$$\hat{a}_{r,\mathbf{n}}|0\rangle = 0 \quad \forall \ \mathbf{n},\tag{4.75}$$

This is the state with no particles. If we consider the expectation value of the bilinear number operator $\hat{N}_{\mathbf{n}} \equiv \hat{a}_{\mathbf{n}}^{\dagger} \hat{a}_{\mathbf{n}}$ of the vacuum state and in a many particle state we have

$$\langle 0|\hat{N}_{\mathbf{n}}(\eta)|0\rangle = 0, \quad \langle n|\hat{N}_{\mathbf{n}}(\eta)|n\rangle = n_{\mathbf{n}}, \tag{4.76}$$

respectively. The expectation value of the operator $N_{\mathbf{n}}$ is the number of particles with wave number \mathbf{n} at a time η . The quantization procedure presented here is very similar to the free fields in the presence of a translation-invariant external field. In these cases sometimes is possible to define an unique vacuum state $|0\rangle$ for all the space points and for all times, from which the other states can be built using the creation operators. But in an Universe in expansion, the time symmetry is broken, preventing the definition of an unique orthonormal basis. That is, the vacuum state defined at some time η_1 would not necessarily be the vacuum state at some time η_2 . $\hat{a}_{r,\mathbf{n}}|0,\eta_1\rangle = 0$ does not necessarily imply $\hat{a}_{r,\mathbf{n}}|0,\eta_2\rangle = 0$. In this way the creation and annihilation operators in different times define different vacuum states. The fact that we cannot define uniquely the vacuum state is responsible for the creation of particle from the vacuum.

We can relate the creation and annihilation operators at a time η_2 with their counterparts of a previous time η_1 through the *Bogoliubov Transformations*:

$$\begin{pmatrix} \hat{a}_{r,\mathbf{n}}(\eta_2) \\ \hat{a}_{r,\mathbf{n}}^{\dagger}(\eta_2) \end{pmatrix} = \begin{pmatrix} \alpha_n & \beta_n^* \\ \beta_n & \alpha_n^* \end{pmatrix} \begin{pmatrix} \hat{a}_{r,\mathbf{n}}(\eta_1) \\ \hat{a}_{r,\mathbf{n}}^{\dagger}(\eta_1) \end{pmatrix},$$
(4.77)

where α_n and β_n are the Bogoliubov coefficients. These transformations play a fundamental role in the particle creation from the ambiguous definition of the vacuum state.

In practical terms, when we consider quantized perturbations on a classic background, the complex functions $c_{\mathbf{n}}^{r}$ and $c_{\mathbf{n}}^{r\dagger}$ of the equation (4.49) could be promoted to creation and annihilation operators which satisfy equal-time commutation relations (4.72), [84]. To find the equation of motion of the mode functions we use commutation relations in the damped harmonic oscillator equation (4.47). This give the same equation obtained previously (4.52) in a classical fashion.

To clarify the role of the Bogoliubov transformations and coefficients, let us consider a complete set of mode solutions $u_i(x)$ of the field equation (for details see [16]). The index *i* represents the set of quantities necessary to label the modes. These modes are orthonormal in the product $(u_1, u_2) \equiv -i(u_1\partial_t u_2^* - u_2\partial_t u_1^*)$ satisfying:

$$(u_i, u_j) = \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i, u_j^*) = 0.$$
 (4.78)

The field ϕ may be expanded as:

$$\phi(x) = \sum_{i} [a_{i}u_{i}(x) + a_{i}^{\dagger}u_{i}^{*}(x)], \qquad (4.79)$$

where a_i and a_i^{\dagger} are the annihilation and creation operators, respectively. With this we can define a vacuum state $a_i|0\rangle = 0$, $\forall i$. But we can also consider a second complete set of modes $\bar{u}_i(x)$ that expand the same field as

$$\phi(x) = \sum_{j} [\bar{a}_{j} \bar{u}_{j}(x) + \bar{a}_{j}^{\dagger} \bar{u}_{j}^{*}(x)].$$
(4.80)

This new expansion defines a new vacuum state $\bar{a}_j |\bar{0}\rangle = 0$, $\forall j$. As both sets are assumed complete, the new modes \bar{u}_j can be written in terms of the old ones

$$\bar{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*). \tag{4.81}$$

Conversely, the old ones in terms of the new

$$u_{i} = \sum_{j} (\alpha_{ji}^{*} \bar{u}_{j} - \beta_{ji} \bar{u}_{j}^{*}).$$
(4.82)

These relations are known as the Bogoliubov transformations and the terms α_{ji} and β_{ji} are matrices called Bogoliubov coefficients. Equating the expansion (4.79) and (4.80) and making use of (4.81) and (4.82) we obtain

$$a_{i} = \sum_{j} (\alpha_{ji} \bar{a}_{j} + \beta_{ji}^{*} \bar{a}_{j}^{\dagger}), \quad \bar{a}_{j} = \sum_{i} (\alpha_{ji}^{*} a_{i} + \beta_{ji}^{*} a_{j}^{\dagger}).$$
(4.83)

From the last equations follows that the two Fock spaces based in the two choices of modes u_i and \bar{u}_j are different as long as $\beta_{ji} \neq 0$, And hence the state $|\bar{0}\rangle$ will not be annihilated by a_i

$$a_i |\bar{0}\rangle = \sum_j \beta_{ji}^* |\bar{1}\rangle \neq 0.$$
(4.84)

And for the expectation value of the number operator $N_i = a_i^{\dagger} a_i$ for the number of u_i -mode particles in the state $|\bar{0}\rangle$ is

$$\langle N_i \rangle \equiv \langle \bar{0} | N_i | \bar{0} \rangle = \sum_j |\beta_{ji}|^2,$$
(4.85)

This means that the vacuum state $|\bar{0}\rangle$ contains " u_i particles" and vice versa. This is the basic principle behind the particle (graviton) creation by the cosmic expansion [60]. In our particular case, for a particular cosmic era and for each mode n, the solution is

$$h_p(\eta) = c_p v_{ij} + c_p^{\dagger} v_{ij}^*, \qquad (4.86)$$

where the index p denotes a particular cosmic era and c_p and c_p^{\dagger} are the annihilation and creation operators for this era, and the function v_{ij} is

$$v_{ij}(\mathbf{k}, \mathbf{x}) = \frac{\sqrt{16\pi G}}{(2\pi)^{3/2}} \epsilon_{ij}(\mathbf{n}) \frac{\mu_p(n, \eta)}{a_p(\eta)} e^{i\mathbf{n}\cdot\mathbf{x}}$$
(4.87)

where μ satisfies the equation (4.54). For the previous era p-1, we can write a solution analogue to (4.87) with modes u_{ij} . For simplicity, we will omit the indices ij and we will not write the dependence for the specific mode **n**. So, the functions v are the mode solutions for the era p and u is the mode solution for the era p-1. As every mode solution χ is complete and orthonormal, from the scalar product follows

$$(\chi,\chi) \equiv -i(\chi\chi^{*\prime} - \chi'\chi^{*}) = 1, \quad \Rightarrow \quad \chi\chi^{*\prime} - \chi'\chi^{*} = i.$$
(4.88)

We can write v in terms of u using the Bogoliubov transformations, $v = \alpha u + \beta u^*$, so that

$$(v,v) = -i(vv^{*\prime} - v'v^{*}) = -i(|\alpha|^{2} - |\beta|^{2})(uu^{*\prime} - u'u^{*})$$

$$\Rightarrow |\alpha|^{2} - |\beta|^{2} = 1,$$
(4.89)

which is a normalization condition. Now let's calculate the Bogoliubov coefficients. Using again the scalar product we have

$$\alpha(n, t^*) = -i(vu^{*\prime} - v'u^*) = (v, u), \qquad (4.90)$$

$$\beta(n,t^*) = i(vu' - v'u) = -(v,u^*).$$
(4.91)

The Bogoliubov coefficients $\alpha(n, t^*)$ and $\beta(n, t^*)$ are evaluated in the transition time t^* between the two eras and they depend in the mode n. The calculation of these

coefficients is important for the calculation of the number of gravitons created.

4.4 The energy and power spectrum

One important quantity is the so-called *power spectrum*. For a given signal, the power spectrum gives a plot of the signal's power (energy per unit time) within given frequency bins. The most common way of calculating a power spectrum is by using a discrete Fourier transform. To calculate it, first the two-point function for the canonical variable must be calculated. This is the so-called Green function. In general we have

$$\langle \hat{Q}(\eta, \mathbf{x}) \hat{Q}(\eta', \mathbf{y}) \rangle \equiv \langle 0 | h_{ij}(\eta, \mathbf{x}) h^{ij}(\eta, \mathbf{x}) | 0 \rangle = g(x, y)$$
(4.92)

Making the calculation for $h_{ij}(\eta, \mathbf{x})$ in the Fourier space using (4.49) with (4.72) we obtain

$$\langle 0|h_{ij}(\eta, \mathbf{x})h^{ij}(\eta, \mathbf{x})|0\rangle = \frac{16\pi G}{(2\pi)^3} \int \sum_{r} 2|\overset{r}{h_n}(\eta)|^2 d\mathbf{n} = \frac{16\pi G}{(2\pi)^3} \int 4|h_n(\eta)|^2 4\pi n^2 dn$$

$$= \frac{32G}{\pi} \int_0^\infty n^3 |h_n(\eta)|^2 \frac{dn}{n}.$$

$$(4.93)$$

We have taken into account that the modulus of each polarization is the same, $|\overset{*}{h}|^2 = |\overset{\times}{h}|^2 = |h|^2$. The last expression can be rewritten in the following way

$$\langle 0|h_{ij}(\eta, \mathbf{x})h^{ij}(\eta, \mathbf{x})|0\rangle = \int d\ln n \,\mathcal{P}(n, \eta),\tag{4.94}$$

where $\mathcal{P}(n,\eta)$ by definition is the (dimensionless) tensor power spectrum

$$\mathcal{P}(n,\eta) \equiv \frac{d\langle 0|h_{ij}(\eta,\mathbf{x})h^{ij}(\eta,\mathbf{x})|0\rangle}{d\ln n} = \frac{32G}{\pi}n^3|h_n(\eta)|^2.$$
(4.95)

The last expression says that the tensor power spectrum is the quadratic mean value of the perturbations. Another important quantity is the energy spectrum defined in the following way

$$\Omega_{\rm gw}(n,\eta) \equiv \frac{1}{\rho_{\rm crit}} \frac{d\langle 0|\rho_{\rm gw}(\eta)|0\rangle}{d\ln n}$$
(4.96)

Which represents the gravitational wave energy density ρ_{gw} , per logarithmic wave number interval in units of the critical density $\rho_{crit}(\eta) = \frac{3H^2(\eta)}{8\pi G}$. The gravitational wave density is [85]

$$\rho_{\rm gw} = T_0^0 = \frac{1}{64\pi G} \frac{(h'_{ij})^2 + (\nabla h_{ij})^2}{a^2},\tag{4.97}$$

which has vacuum expectation value

$$\langle 0|\rho_{\rm gw}|0\rangle = \int_0^\infty \frac{n^3}{2\pi^2} \frac{|h_n'(\eta)|^2 + n^2|h_n(\eta)|^2}{a^2} \frac{dn}{n},\tag{4.98}$$

so that the energy spectrum is given by

$$\Omega_{\rm gw}(n,\eta) = \frac{8\pi G}{3H^2(\eta)} \frac{n^3}{2\pi^2} \frac{|h_n'(\eta)|^2 + n^2 |h_n(\eta)|^2}{a^2}.$$
(4.99)

Both power spectrum (4.95) and energy spectrum (4.99) are of great physical relevance for future observations (if ever detected) of primordial gravitational waves. These quantities have the imprint of the Universe at the time when were they created. Also, they could give some clues about the Quantum Gravity Theory because the graviton production in the early Universe couple quantum mechanics (by the particle creation due the vacuum fluctuation) to the Universe expansion (General Relativity) which stretches it to cosmic scales.

Chapter 5

Decaying vacuum models

5.1 Accelerating-expansion problem

One of the most important discoveries of the 20th century is that the Universe is expanding. And not less surprising is the more recent discovery of the accelerating expansion of the Universe [7, 8] which broke the idea of a slowing down expansion accepted for decades. This important characteristic of the dynamics of the Universe is one of the most challenging open problems of the cosmology because actually there is not a satisfactory model which explains it. We know that at large scales gravity is the dominant force, an attractive force. So, one way to have an accelerating Universe is to assume a new component of the Universe with the feature to be gravitationally repulsive or change gravity.

By this way, the most common explanation to the accelerated expansion of the Universe is to propose a new kind of cosmic component called dark energy. This component must have some exotic characteristics like having negative pressure and permeate every part of the Universe to have a global repulsive effect.

There are several astronomical observations of different types that support the existence of the dark energy: Ia-type supernovae [86], CMB [64], baryon acoustic oscillations [87], late-time integrated Sachs-Wolfe effect [88] and the Hubble parameter [89]. All these evidence makes dark energy to be widely accepted by the scientific community in the area. But despite the good observational arguments there is no clear clue to its theoretical support. Observations have not said anything about its origin neither the fundamental physics.

From the point of view of theoretical physics, there is still no conclusive understanding

of dark energy that allows consensus on its physical basis. This allows various types of candidates like quintessence [90], Chaplygin gas [91]; and proposals in the context of string theory and brane cosmology [92] among others; for a nice review about theoretical models of dark energy see [93] and references therein. But the most natural candidate for dark energy is the cosmological constant (sometimes called a vacuum energy), Λ , in Einsteins field equations. As seen in chapter 3, the cosmological constant Λ has negative pressure equal to its energy density ($p = -\rho$) causing the expansion of the Universe to accelerate. Assuming Λ to be the dark energy with cold dark matter gives the Λ CDM or Standard Cosmological Model which is the most accepted and more successful cosmological model.

But the choice of Λ to be the dark energy has some disadvantages. One of them is the Cosmological Constant Problem that in few words says that comparing the observed (vacuum) energy density of the cosmological constant $\rho_{\Lambda} \leq 10^{-47}$ GeV⁴ with its Quantum Field Theory (QFT) counterpart, $\rho_{\rm vac}^{\rm QFT} \sim 10^{71}$ GeV⁴ leads a discrepancy of more than 100 orders of magnitude. Another is the Cosmic Coincidence Problem, why if the matter energy density decays as $\rho_{\rm mat} \propto a^{-3}$ and $\rho_{\Lambda} = constant$, both are of the same order today?

5.2 Time-dependent cosmological "constant"

The cosmological constant Λ was introduced by Einstein in 1917, soon after formulating the General Relativity, into their field equations to make it possible a static Universe preventing it from collapse due to the gravitational force. A few months later, de Sitter presented a cosmological model with only the contribution of the cosmological constant, here the test particles separate from each other due to the repulsive effect of Λ ; this is maybe the first expanding cosmological model. Then after Hubble discovered the Universe was expanding, Einstein discards Λ from his equations while Lemaître proposes a model mixing the advantages of the Einstein's and the de Sitter's models.

During the decades of 1930's and 1940's the Λ term appeared and disappeared of the Einstein equations for reasons of either simplicity, completeness or because some researchers liked it or not. It was not until the 1960's when measurements of the age of the universe led to adopt the cosmological constant. At the end of that decade Gliner began the study of the effects of vacuum in cosmology [94] and then a couple of years later Zeldovich justified the nonzero value of Λ showing that the sum of the vacuum energy densities of the quantum fields act as a cosmological constant [26, 95]. From these works, it was accepted the contribution of the vacuum energy density as an effective cosmological term. Landau stated that from the Einstein equations $G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu}$ we know that due to the geometric properties $\nabla_{\nu} G^{\mu\nu} = \nabla_{\nu} g^{\mu\nu} = 0$ and the energy conservation $\nabla_{\nu} T^{\mu\nu} = 0$, consequently $\nabla_{\nu} \Lambda = \partial_{\nu} \Lambda = 0$, i.e., Λ is a constant in spacetime, it has no dynamics.

When the Λ term is on the left side of the Einstein equations, it can be interpreted as a pure geometrical term that contributes to the curvature of the spacetime. When it is on the right side of the equations, it can be interpreted as the vacuum component of a more general energy-momentum tensor,

$$G^{\mu\nu} = 8\pi G T^{\mu\nu} - \Lambda g^{\mu\nu} = 8\pi G (T^{\mu\nu} + T^{\mu\nu}_{\Lambda}) = 8\pi G T^{\mu\nu}_{\text{tot}},$$
(5.1)

where $T_{\Lambda}^{\mu\nu} = -\Lambda/(8\pi G)g^{\mu\nu}$. As the energy-momentum tensor is Lorentz invariant $T_{\Lambda}^{\mu\nu} = T_{\Lambda}^{\mu'\nu'} = \Lambda_{\alpha}^{\mu'}\Lambda_{\beta}^{\nu'}T_{\Lambda}^{\alpha\beta}$, then it can be assumed $T_{\Lambda}^{\mu\nu} = T_{\Lambda}^{00}\eta^{\mu\nu}$ and using the principle of covariance, we have $T_{\Lambda}^{\mu\nu} = T_{\Lambda}^{00}g^{\mu\nu}$. Notice that all the non-diagonal components are null. The zero-zero component is interpreted as the energy density and in FLRW models, so that, in the most general form, the Lorentz invariance imposes

$$T^{\mu\nu}_{\Lambda} = -\rho_{\Lambda} g^{\mu\nu}, \quad \Rightarrow \Lambda = 8\pi G \rho_{\Lambda}, \tag{5.2}$$

the last relation links the vacuum energy density to the cosmological constant. Finally, if it is assumed that $T^{\mu\nu}_{\Lambda}$ is a perfect fluid, as all the other material components in cosmological models, it satisfies the relation $T^{\mu\nu}_{\Lambda} = (p_{\Lambda} + \rho_{\Lambda})U^{\mu}U^{\nu} + p_{\Lambda}g^{\mu\nu}$ which immediately indicates that the EoS of the vacuum energy density is $\rho_{\Lambda} = -p_{\Lambda}$.

Assuming $G^{\mu\nu} = 8\pi G T_{\text{tot}}^{\mu\nu}$, the energy conservation $\nabla_{\nu} T_{\text{tot}}^{\mu\nu} = 0$ doesn't necessarily imply that the components conserve independently $\nabla_{\nu} T^{\mu\nu} = \nabla_{\nu} T_{\Lambda}^{\mu\nu} = 0$, but yes together. This is the crucial point of the decaying vacuum models, because to conserve the total energy-momentum tensor, the vacuum must interact with the other material fields. It will be shown later in the next section how this interaction occurs, but we can anticipate that for a young universe the vacuum energy density is very large to decay into other fields as the universe expands. Another important remark is that as ρ_{Λ} is an isotropic and homogeneous fluid (because $T^{\mu}_{(\Lambda)\nu} \propto \delta^{\mu}_{\nu}$), when perturbed at first order it vanishes $\delta T^{i}_{(\Lambda)j} = 0$ and it does not contribute to δG^{i}_{j} , by this way the tensor perturbation equation (4.47) is maintained.

Historically, the idea of a time dependent cosmological "constant" was first put forward by M. Bronstein [96] in 1932 and criticized by Landau. Then, for years this idea was practically forgotten until Ozer and Taha reintroduced it in the 1980's with a $\Lambda \propto a^{-2}$ model [97]. From there, in the subsequent years several phenomenological $\Lambda(t)$ models were proposed and studied, but originally these approaches did not worry too much about the fundamental physics origin of the models, they only tried to reproduce the evolution of the Universe taking account $\Lambda(t)$.

To cite some of them, in 1992 Carvalho et al. [98] studied a model $\Lambda = 3\beta H^2 + 3\alpha/a^2$ where the first term contributes to increase the age of the Universe. In 1994 Lima and Maia [99] studied a nonsingular cosmological model with $\Lambda = 3\beta H^2 + 3(1-\beta)H^3/H_I$ and then Lima and Trodden [100] generalized it adding a curvature term. For examples of several phenomenological $\Lambda(t)$ models see [27] and references therein.

However, there exist some theoretical insights based on fundamental physics that support the time dependence of Λ . Shapiro and Solà justified it from renormalization group arguments [101, 102, 103]. These models were also confronted with observations: in [104], Basilakos showed that a slowly running of Λ is compatible with the observations, with supernovae data [28], and later on with the modern observations of supernovae, baryonic acoustic oscillations, CMB and structure formation in [29, 30, 31].

So, the $\Lambda(t)$ decaying vacuum models models are a good option to overcome the drawbacks of the dark energy. They are based on the idea of relaxation of the vacuum energy density starting from a high energy density state which relaxes with the evolution of the Universe reaching the observed small value today $\Lambda_0 = \Lambda(t_0)$. This means that Λ_0 is small because the Universe is old [105]. The relaxation or decaying of the vacuum energy density avoids the Cosmic Coincidence Problem and provides a possibility to solve the Cosmological Constant Problem dynamically without relying on to fine-tuning [32].

5.3 Phenomenological arguments for $\Lambda(t)$

First, let us model the expanding Universe as a mixture of N = 1, 2, ... perfect fluids, with 4-velocities $U_{\mu}^{(N)}$ and total energy momentum tensor given by

$$T_{\mu\nu} = \sum_{N} T_{\mu\nu}^{(N)} = \sum_{N} [(\rho^{(N)} + p^{(N)})U_{\mu}^{(N)}U_{\nu}^{(N)} + p^{(N)}g_{\mu\nu}], \qquad (5.3)$$

with components

$$T_0^0 = -\sum_N \rho^{(N)} \equiv -\rho_{\text{tot}}, \quad T_j^i = \sum_N p^{(N)} \delta_j^i \equiv p_{\text{tot}} \delta_j^i, \tag{5.4}$$

where ρ_{tot} and p_{tot} are the total energy density and pressure in the comoving reference frame. Consider now the local conservation law

$$\nabla_{\mu}T^{\mu\nu} = 0, \quad \Rightarrow \quad U_{\nu}^{(N)}\nabla_{\mu}T^{\mu\nu} = \sum_{N} [U_{\mu}^{(N)}\nabla^{\mu}\rho^{(N)} + (\rho^{(N)} + p^{(N)})\nabla^{\mu}U_{\mu}^{(N)}] = 0.$$
(5.5)

where we had used $U_{\nu}^{(N)} \nabla_{\mu} U^{(N)\nu} = 0$ and $U_{\nu}^{(N)} U^{(N)\nu} = -1$. This general form can be reduced more in the FLRW geometry taking account that for a comoving frame $U^{(N)\mu} = \delta_0^{\mu}$ one gets $\nabla_{\mu} U^{(N)\mu} = 3H$. Substituting this in the previous equation (5.5):

$$\sum_{N} [\dot{\rho}^{(N)} + 3H(\rho^{(N)} + p^{(N)})] = 0.$$
(5.6)

To continue, we proceed specifying the type of perfect fluids. For simplicity, let's assume that we have a mixture of two fluids: a material fluid (relativistic and non-relativistic matter, i.e. dark and baryonic matter with radiation) and vacuum energy. This material fluid could be a radiation-dominated fluid in the early Universe after inflation, $\rho_{\rm rad}$, or matter-dominated after the equilibrium time $\rho_{\rm mat}$. For simplicity it will be denoted only as ρ and the vacuum energy-density continue being ρ_{Λ} . The corresponding pressures are p and p_{Λ} , and equations of state $p = \omega \rho$ and $p_{\Lambda} = -\rho_{\Lambda}$ respectively. As usual, the values of the parameter ω are $\omega = 1/3$ for radiation and $\omega = 0$ for matter. The corresponding Einstein equations of the system formed by the material component and the vacuum fluid are:

$$8\pi G\rho_{\rm tot} = 8\pi G\rho + \Lambda = 3H^2, \qquad (5.7)$$

$$8\pi G p_{\rm tot} = 8\pi G p - \Lambda = -2\dot{H} - 3H^2.$$
(5.8)

Substituting these equations in the conservation law equation (5.6), we find

$$\dot{\rho}_{\Lambda} + \dot{\rho} + 3(1+\omega)\rho H = 0, \qquad (5.9)$$

notice that clearly when ρ_{Λ} is a constant, this relation recovers the standard matter conservation law $\dot{\rho} + 3(1 + \omega)\rho H = 0$ of the Λ CDM model. But we are interested in the case of a no constant vacuum energy density $\rho_{\Lambda} = \frac{\Lambda}{8\pi G}$. While we assume that the Newton's constant G remains strictly constant, the only chance is to assume Λ to be time dependent. Now we define the ratio between the fluids energy densities in the following way:

$$\beta(t) = \frac{\rho_{\Lambda} - \rho_{\Lambda 0}}{\rho + \rho_{\Lambda}} = \frac{\rho_{\Lambda} - \rho_{\Lambda 0}}{\rho_{\text{tot}}},\tag{5.10}$$

where $\rho_{\Lambda 0}$ is a constant vacuum density, which defines the observed cosmological constant Λ_0 today. The $\beta(t)$ parameter quantifies the time variation of the vacuum energy density. If $\rho_{\Lambda} = \rho_{\Lambda 0} = constant$, then $\beta = 0$ and we recover the Λ CDM model; if $\rho_{\Lambda 0} = 0$, then $\beta(t)$ defines the fraction of the vacuum to the total density. If this fraction is constant in the course of the Universe evolution we have $\rho_{\Lambda} = \beta \rho_{tot}$ and substituting into equation (5.7) we have:

$$\Lambda = 3\beta H^2. \tag{5.11}$$

This form of the decaying vacuum has the characteristic that the Universe is always accelerating or decelerating depending on the sign of β . Despite it does not recover Λ CDM model because $\Lambda(t)$ never reaches Λ_0 , it is useful when modeling the early Universe when $\Lambda_0 \ll H^2$ and Λ_0 could be neglected. This kind of model was discussed before by [98].

On the other hand, if we now consider the ratio (5.10) as a constant, then we have:

$$\Lambda(t) = c_0 + 3\beta H^2(t), \tag{5.12}$$

where c_0 is a constant. In this framework the present value of the cosmological constant is $\Lambda_0 = c_0 + 3\beta H_0^2$. In contrast to the equation (5.11), the presence of the additive term allows the existence of a transition from deceleration to acceleration and a smooth connection with the Λ CDM model is possible in the limit $\beta \rightarrow 0$. In general the ratio (5.10) may not remain constant during the cosmic evolution, hence β should be a time-dependent quantity giving for the vacuum energy the form:

$$\Lambda(t) = c_0 + 3\beta(t)H^2.$$
(5.13)

Now the value of the cosmological constant is $\Lambda_0 = c_0 + 3\beta(t_0)H_0^2$. As β is now variable let's assume that we can expand it as a constant plus a time-dependent term:

$$\beta(t) = \nu + \alpha \left(\frac{H}{H_I}\right)^n,\tag{5.14}$$

where ν , α and H_I are constants; n is a positive integer. Now the time dependence of the vacuum is

$$\Lambda(t) = c_0 + 3\nu H^2 + 3\alpha \frac{H^k}{H_I^{k-2}}, \quad \text{where} \quad k = n+2.$$
(5.15)

The constant term c_0 , in the last equation represents the dominant contribution at very low energies, $H \approx \mathcal{O}(H_0) \ll H_I$. The H^2 term represents a small correction (if $\nu \ll 1$) to the dominant term at the present time, providing the behavior to the vacuum energy density at intermediate times. And finally the H^k for $k \geq 3$ acquires great relevance in the very early universe, near the H_I energy scale (interpreted as the Hubble parameter in the inflationary era).

Since H_I is presumably large, it is clear that $\beta(t_0) \simeq \nu$ for any n and hence the value of the cosmological constant today is essentially $\Lambda_0 = c_0 + 3\nu H_0^2$ for all models of the type (5.15).

5.4 Renormalization group arguments for $\Lambda(H)$

We have motivated the decaying vacuum energy density models $\Lambda(t)$ using only phenomenological arguments, basically assuming only that $\dot{\rho}_{\Lambda} \neq 0$ and assuming a dependence of the vacuum energy on the Hubble parameter. Moreover, the time dependence of the vacuum energy can be substantiated with arguments more fundamental via QFT in curved spacetimes [28].

The arguments in this context focus on running vacuum energy models, this means that not only ρ_{Λ} is a time dependent quantity but there is a dynamical variable $\mu = \mu(t)$ (not confuse with the μ of the tensor perturbations) linked to the vacuum energy in the following way $\Lambda = \Lambda(\mu(t))$ rather than from a direct phenomenological law of the type $\Lambda = \Lambda(t)$. This works in the same way as the renormalization group running of the effective charges in gauge theories but now in the context of cosmology.

Running couplings in flat QFT, such as Quantum Electrodynamics or Quantum Chromodynamics, provide theoretical tools to investigate the running vacuum $\Lambda(\mu(t))$. In these theories, the gauge coupling constants g run with a scale μ associated to a typical energy $g = g(\mu)$. In a similar way the effective action of QFT in curved spacetime, the vacuum energy density ρ_{Λ} should be an effective coupling depending on a mass scale μ . We should expect that the running of ρ_{Λ} from the quantum effects of the matter fields is associated with the change of the spacetime curvature due the Universe expansion and hence with the change of the typical energy of the classical gravitational external field linked to the FLRW metric.

The authors in [102, 105] argue that is natural to associate μ^2 to the scalar curvature R. In the flat FLRW metric is written as

$$|R| = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 12H^2 + 6\dot{H}.$$
(5.16)

It follows that μ^2 is associated with H^2 and \dot{H} (which has the same dimension as H^2). First for simplicity we will only focus on the case $\mu^2 = H^2$. With this assumption, the corresponding renormalization group equation is:

$$(4\pi)^2 \frac{d\rho_{\Lambda}}{d\ln\mu^2} = \sum_{m=1,2,\dots}^{\infty} A_{(2m)}\mu^{2m} = A_{(2)}\mu^2 + A_{(4)}\mu^4 + A_{(6)}\mu^6 + \dots$$
(5.17)

where the indexes in parenthesis are the labels of the coefficients. Only even powers of μ are involved because the expanding variable is μ^2 . The coefficients $A_{(2m)}$ in equation (5.17) can be obtained after summing over the loop contributions of fields of different masses M_i [105]. The general behavior of the coefficients is

$$A_{(2m)} \sim \sum_{i} a_i^{(2m)} M_i^{4-2m}.$$
(5.18)

Notice that $A_{(0)}$ is not considered because this term scales as $A_0 \sim M_i^4$ triggering a toofast running of ρ_{Λ} . The first contributing term is $A_{(2)} \sim \sum_i a_i^{(2)} M_i^2$ where the sum is over the masses of all the fields and its multiplicities with dimension of mass square. In the same way all the coefficients $A_{(2m)}$, except $A_{(4)}$, are dimensionful. Rewriting (5.17) presenting explicitly the mass dimensions and $\mu = H$ we have:

$$(4\pi)^2 \frac{d\rho_{\Lambda}(H)}{d\ln H^2} = \sum_i \left[a_i^{(2)} M_i^2 H^2 + a_i^{(4)} H^4 + a_i^{(6)} \frac{H^6}{M_i^2} + \cdots \right].$$
(5.19)

The series (5.17) became now an expansion in even powers of the Hubble parameter H. Integrating to obtain Λ we have

$$\Lambda = 8\pi G \rho_{\Lambda} = \frac{8\pi c_1}{M_{\rm Pl}^2} + 3\left(\frac{1}{6\pi}\sum_i a_i^{(2)} \frac{M_i^2}{M_{\rm Pl}^2}\right) H^2 + 3\left(\frac{1}{12\pi}\frac{H_I^2}{M_{\rm Pl}^2}\sum_i a_i^{(4)}\right) \frac{H^4}{H_I^2} + \cdots, \quad (5.20)$$

where c_1 is an integration constant and remembering that $G = M_{\rm Pl}^{-2}$. The last equation (5.20) is qualitatively similar to the phenomenological expression (5.15) showing that the RG formulation may provide a fundamental link of QFT in curved spacetime to dynamical cosmological constant.

As the Hubble parameter is time dependent H(t), the series of $\Lambda(H)$ has different characteristics depending on the Universe epoch. For the old (present) Universe, the value of the Hubble parameter is small $H \sim H_0$, and much smaller than any particle mass $H \ll M_i$, making (5.20) converge very fast to the constant term of the series. In the early Universe, because the GUT scale M_X is typically a few orders of magnitude below the Planck scale $M_{\rm Pl} \sim 10^{19}$ GeV, no term of the series beyond H^2 contribute significantly to (5.15) at any stage of the cosmological history after inflation. But if we are interested in studying the physics of the very early Universe, like inflation era is, when H is close but below $M_i \sim M_X$ we should keep the expansion at least the H^4 term.

To find the explicit relation between $\Lambda(H)$ obtained via the RG equation (5.20) with the phenomenological counterpart (5.15) let us consider the particular case of k = 4 of this last equation, for which the highest power of the Hubble parameter is H^4 . We have for the coefficients:

$$c_0 = \frac{8\pi c_1}{M_{\rm Pl}^2}, \quad \nu = \frac{1}{6\pi} \sum_i a_i^{(2)} \frac{M_i^2}{M_{\rm Pl}^2} \quad \text{and} \quad \alpha = \frac{1}{12\pi} \frac{H_I^2}{M_{\rm Pl}^2} \sum_i a_i^{(4)}. \tag{5.21}$$

Immediately we can say something about the physical interpretation of the last quantities. First, the constant c_0 is associated with the vacuum energy observed today, where the Universe has evolved enough so that the terms in $\Lambda(H)$ with powers of the Hubble parameter H can be neglected giving the fiducial cosmological constant $\Lambda_0 \simeq c_0$. Second, the coefficient ν is important to ρ_{Λ} at lower energies, that is, when the Universe has evolved enough to not depend on the terms associated with high energies (terms of $\mathcal{O}(H^4)$ or more) but still has not come close to its state of minimum energy (Λ_0). Finally, the dimensionless coefficient α plays a similar role at high energies, i.e. for the very early Universe where the high order powers of H dominate.

Both coefficients are predicted to be naturally small because $M_i^2 \ll M_{\rm Pl}^2$ for all the particles, even for the typical GUT fields. For the low-energy coefficient ν , an estimate within a generic GUT is found in the range $|\nu| \sim 10^{-6} - 10^{-3}$ [103]. Similarly, α is predicted to be small $|\alpha \ll 1|$ because the inflationary energy scale H_I is smaller than the Planck mass $M_{\rm Pl}$.

5.5 Decaying vacuum cosmologies

In the work [106] the cosmic evolution with a class of decaying vacuum models were described, as presented before, in a detailed way. In this section it will be summarized the ideas presented in that work.

They use a $\Lambda(H)$ that in practice consists of a constant, an additive H^2 -term and a H^k -term with (k > 2) which is responsible for the transition from the inflationary stage to the FLRW radiation epoch. Their model predicts that the Universe starts from a nonsingular state, then the early accelerated regime associated with the inflation has a natural ending by virtue of the faster decrease of the vacuum energy density. The Universe evolves from an initial de Sitter epoch to another late time de Sitter epoch. The mechanism for inflation in that case is different from the usual inflationary models, in this sense it provides an alternative to them. The cosmic evolution is divided into two parts, the early and late Universe which will be described below.

5.5.1 From the early de Sitter stage to the ω -dominated phase

Combining the equations (5.7), (5.9) and (5.15) we obtain the following differential equation for the time evolution of the Hubble parameter:

$$\dot{H} + \frac{3}{2}(1-\omega)H^2 \left[1-\nu - \frac{c_0}{3H^2} - \alpha \left(\frac{H}{H_I}\right)^n\right] = 0.$$
(5.22)

At early stages of the Universe, the term associated with c_0 can be neglected, then equation (5.22) becomes

$$\dot{H} + \frac{3}{2}(1-\omega)H^2 \left[1 - \nu - \alpha \left(\frac{H}{H_I}\right)^n\right] = 0.$$
(5.23)

The integration of the above equation gives

$$H(a) = \frac{\tilde{H}_I}{[1 + Da^{n\xi}]^{1/n}}$$
(5.24)

where $\xi \equiv 3(1+\omega)(1-\nu)/2$ and $\tilde{H}_I \equiv H_I[(1-\nu)/\alpha]^{1/n}$. We stress that in our analysis we consider epochs of the cosmic evolution where matter is dominated by the relativistic or the nonrelativistic components, i.e. epochs where we have $\omega = 1/3$ and $\omega = 0$ respectively, without considering the interpolation regime between the two. Therefore, in practice for all the considerations in this section, we have $\omega = 1/3$ and so $\xi = 2(1 - \nu)$ as our discussion is related to the transition from the initial de Sitter to the radiation dominated universe.

In equation (5.24), D is an integration constant that can be fixed using the condition $H(a_{\star}) \equiv H_{\star}$ (where $a_{\star} = a(t_{\star})$, typically corresponding to the initial time t_{\star} of the ω -fluid dominated era). Thus,

$$D = a_{\star}^{-n\xi} \left[\left(\frac{\tilde{H}_I}{H_{\star}} \right)^n - 1 \right], \qquad (5.25)$$

and it is greater than zero for $\tilde{H}_I > H_{\star}$. Note that if D = 0 the solution remains always de Sitter. Using the auxiliary variable

$$u = -\frac{1}{Da^{n\xi}},\tag{5.26}$$

we write equation (5.24) as

$$\dot{u} = -n\xi \tilde{H}_I u^{1+1/n} (u-1)^{-1/n}, \qquad (5.27)$$

and its inversion results:

$$\frac{dt}{du} = -\frac{1}{n\xi\tilde{H}_I}u^{-(1+1/n)}(u-1)^{1/n}.$$
(5.28)

The second derivative may be put in the form:

$$u(1-u)\frac{d^2t}{du^2} + \left[1 + \frac{1}{n} - u\right]\frac{dt}{du} = 0.$$
 (5.29)

Hence, we have the hypergeometric equation with parameters a = 0, b = 0, and c = 1 + 1/n. Its integration yields

$$t(u) = B - Anu^{-1/n} {}_{2}F_{1}\left[-\frac{1}{n}, -\frac{1}{n}; 1 - \frac{1}{n}; u\right], \qquad (5.30)$$

where B and A are integration constants. We can set B = 0 if the origin of time is placed just after the inflation period and t is then the cosmic time in the FLRW epoch. Using Eulers relation for the hypergeometric function and the boundary condition (when $t = t_{\star}$ at the end of the inflationary period) for the Hubble parameter H the above solutions can be rewritten as:

$$t(a) = B + \frac{(1 + Da^{n\xi})^{\frac{1+n}{n}}}{\xi \tilde{H}_I Da^{n\xi}} {}_2F_1\left[1, 1; 1 - \frac{1}{n}; -\frac{1}{Da^{n\xi}}\right],$$
(5.31)

Using the Einstein equations and the above solutions we can obtain the corresponding energy densities:

$$\rho_{\Lambda}(a) = \tilde{\rho}_{I} \frac{1 + \nu D a^{n\xi}}{[1 + D a^{n\xi}]^{1+2/n}}, \qquad (5.32)$$

$$\rho(a) = \tilde{\rho}_I \frac{(1-\nu)Da^{n\xi}}{[1+Da^{n\xi}]^{1+2/n}},$$
(5.33)

$$\rho_{\rm tot}(a) = \tilde{\rho}_I \frac{1}{[1 + Da^{n\xi}]^{2/n}}, \qquad (5.34)$$

with $\tilde{\rho}_I \equiv 3\tilde{H}_I^2/8\pi G$. These expressions reproduce the energy densities for the primeval de Sitter and radiation dominated epochs, for details see Figure 5.1.

5.5.2 From the ω -dominated era to the residual vacuum stage

The corresponding formulas for the more recent universe when the ω -fluid plus decaying vacuum will be derived under the condition $H \ll H_I$. In this case the evolution equation for the Hubble parameter (5.22) can be approximated as

$$aH\frac{dH}{da} + \xi H^2 - \frac{1+\omega}{2}c_0 = 0, \qquad (5.35)$$

the first integral of this equation gives

$$H^{2}(a) = \frac{c_{0}}{3(1-\nu)} \left[\left(\frac{C_{1}}{a}\right)^{2\xi} + 1 \right], \qquad (5.36)$$

where the constant

$$C_1^{2\xi} = a_0^{2\xi} \left[\frac{3H_0^2(1-\nu)}{c_0} - 1 \right], \qquad (5.37)$$

is obtained from the condition $H(a_0) \equiv H_0$ for the present time. Using the above solutions, the Friedmann equations provide the total and the ω -fluid densities


Figure 5.1: Plots and description taken from [106]. Left panel: The evolution of the vacuum and radiation energy densities during the primordial era, where $H^2 \gg c_0$. We normalize the densities with respect to the primeval critical value $\tilde{\rho}_I$. The plots show that the decay of the vacuum density, as well as the production and subsequently dilution of radiation, occur in a faster way for large values of the parameter n, thereby ensuring the universality of the graceful exit for any $n \geq 2$. Additionally, in this figure we can see that the vacuum density always decays faster than it does the radiation density after the transition period. Right panel: The behavior of the vacuum density with the variation of the parameter ν for n = 2. In this graph, we can see that during the radiation dominated era the vacuum density ceases to decay; it only dilutes with time (in a similar way as the radiation energy density) due to the effect of the expansion. The precise instant when this change occurs is earlier for larger values of the parameter ν . On the other hand, the evolution of the radiation energy density is affected very little by the variation of the parameter ν , for $\nu \leq 10^3$. In this figure we show the radiation energy density for $\nu = 10^4$.

$$8\pi G\rho_{\rm tot}(a) = \frac{c_0}{1-\nu} \left[\left(\frac{C_1}{a}\right)^{2\xi} + 1 \right], \qquad (5.38)$$

$$8\pi G\rho(a) = c_0 \left(\frac{C_1}{a}\right)^{2\xi}.$$
 (5.39)

In a more explicit form, the Hubble function (5.36) reads

$$H^{2}(a) = \frac{H_{0}^{2}}{1 - \nu} \left[\Omega_{X}^{0} a^{-2\xi} + \Omega_{\Lambda}^{0} - \nu \right], \qquad (5.40)$$

where we have the sum rule $\Omega_X^0 + \Omega_{\Lambda}^0 = 1$, and we have set $\omega = 0$ since we are in the matter-dominated epoch. The ω -fluid density (5.39) can be expressed as

$$\rho(a) = \rho^0 a^{-2\xi}, \tag{5.41}$$

where ρ^0 is the current value. We can see that for $\nu = 0$ we retrieve the standard scaling $\rho = \rho^0 a^{3(1+\omega)}$. The departure from this law caused by a nonvanishing ν is related to the exchange of energy between matter and vacuum. A evolves as

$$\Lambda(a) = \frac{c_0}{1-\nu} \left[\nu \left(\frac{C_1}{a}\right)^{2\xi} + 1 \right].$$
(5.42)

The corresponding vacuum energy density is the following:

$$\rho_{\Lambda}(a) = \rho_{\Lambda}^{0} + \frac{\nu \rho^{0}}{1 - \nu} [a^{-2\xi} - 1].$$
(5.43)

We see that only for $\nu = 0$ we recover $\Lambda = c_0 = \text{const.}$ and $\rho_{\Lambda}(a) = \rho_{\Lambda}^0 = \text{const.}$, as in the Λ CDM case. Furthermore, we can easily check that Eqs. (5.41) and (5.43) satisfy the overall local conservation law (5.9), which can be rewritten in terms of the scale factor as follows:

$$\frac{\rho_{\Lambda}(a)}{da} + \frac{\rho(a)}{da} + \frac{3}{a}(1+\omega)\rho(a) = 0.$$
 (5.44)

We can integrate the last equation (5.36) to obtain the time evolution of the scale factor a(t):

$$a(t) = C_1 \sinh^{1/\xi} \left[\sqrt{3c_0(1-\nu)}(1+\omega)(t-C_2)/2 \right].$$
(5.45)

Without losing generality we can set $C_2 = 0$. Substituting (5.45) in the previous equations we immediately get the time-evolving functions $\rho = \rho(t)$ and $\Lambda = \Lambda(t)$.

Let us finally mention for completeness that there are cases where we have to deal with a mixture of cold matter and radiation. Defining Ω_{mat}^0 and Ω_{rad}^0 as the standard nonrelativistic and radiation density parameters at the present time, one can show that the complete Hubble function reads

$$H^{2}(a) = \frac{H_{0}^{2}}{1-\nu} \left[\Omega_{\text{mat}}^{0} a^{-3(1-\nu)} + \Omega_{\Lambda}^{0} + \Omega_{\text{rad}}^{0} a^{-4(1-\nu)} - \nu\right], \qquad (5.46)$$

where the density parameters satisfy the extended sum $\Omega_{\text{mat}}^0 + \Omega_{\text{rad}}^0 + \Omega_{\Lambda}^0 = 1$.

Chapter 6

Gravitational waves with decaying vacuum

6.1 Scale factor evolution for $\Lambda(t)$ models

To study the cosmological tensor perturbations in a $\Lambda(t)$ model, our main goal is to solve the equation $\mu'' + (n^2 - V)\mu = 0$. To calculate the power spectrum and other physical quantities describing the primordial GWs we must solve the differential equation and obtain an explicit solution of $\mu(\eta)$. To do this, the first step is to calculate the scale factor $a(\eta)$ from the Friedmann equations in order to have an explicit form of the potential $V(\eta) = a''/a$. As we are interested in cosmologies with decaying vacuum where the evolution of the scale factor depends on it, we must choose a specific decaying vacuum model to then solve the Friedmann equations for the scale factor.

Let us begin choosing a specific decaying vacuum model. Many phenomenological functional forms have been proposed in the literature for describing a time-varying $\Lambda(t)$ vacuum as discussed in detail in chapter 5. For mathematical simplicity we will use the following expression:

$$\Lambda(t) = \Lambda_0 + 3\beta H^2, \tag{6.1}$$

where Λ_0 is the common cosmological constant, β is a dimensionless constant parameter and the factor 3 was added for mathematical convenience. Despite its simplicity, this particular functional form of $\Lambda(t)$ contains the main features of a more general form as shown in (5.15). It contains the dominant decaying term proportional to H^2 and the residual constant term Λ_0 for advanced stages of the Universe. As will be shown later, this choice of the decaying vacuum term modifies the evolution of the scale factor giving interesting new results for the primordial GWs, different from the no-decaying vacuum models. For early times of the Universe, the term proportional to H^2 rules the decaying vacuum evolution and for later times it is the constant Λ_0 term who dominates. Taking this account we will omit the constant term from the calculations because the origin of the primordial GWs is in the very early Universe during inflation, so we will begin using just $\Lambda(t) = 3\beta H^2$. We must remember that this is only an approximation for our calculations, in other contexts the constant term cannot be omitted because do not recover the Λ CDM model for the present time.

Having this in mind, the Friedmann equations (3.6) and (3.7) including the decaying vacuum term $\Lambda(t) = 3\beta H^2$ are:

$$8\pi G\rho + \Lambda(t) = 8\pi G\rho + 3\beta H^2 = 3H^2,$$
(6.2)

$$8\pi Gp - \Lambda(t) = 8\pi Gp - 3\beta H^2 = -2\frac{a}{a} - H^2.$$
(6.3)

Now we need to specify the perfect fluid equation of state. We will assume

$$p = \omega \rho, \tag{6.4}$$

where ω is a particular constant for each cosmological era. Combining the equations (6.2), (6.3) and (6.4) we obtain the equation of evolution for the scale factor:

$$a\ddot{a} + \Delta \dot{a}^2 = 0$$
, where $\Delta \equiv \frac{3(1+\omega)(1-\beta)-2}{2}$. (6.5)

This parameter Δ carries the physical information about the cosmic era and the decay rate of the vacuum. Rewriting this equation in terms of the conformal time we have:

$$aa'' + (\Delta - 1)a'^2 = 0, (6.6)$$

integrating the last equation we find the general solution for the scale factor

$$a(\eta) = \begin{cases} b_1 e^{b_2 \eta} & \Delta = 0\\ \\ c_1 (\Delta \eta - c_2)^{1/\Delta} & \Delta \neq 0 \end{cases}$$
(6.7)

where b_1 , b_2 , c_1 and c_2 are integration constants. We will only consider solutions for $\Delta \neq 0$ i.e. the power law solution. To find the solution for the different cosmological eras we first have to specify the different values of the constant ω of the equation of state (6.4) of each era. We will use the well-known values for de Sitter inflation, radiation and matter era, that is $\omega_{inf} = -1$, $\omega_{rad} = 1/3$ and $\omega_{mat} = 0$ respectively.

The integration constants for each era can be found using the continuity junction conditions at the transition times between each era, i.e. $a_n(\eta_i) = a_{n+1}(\eta_i)$ and $a'_n(\eta_i) = a'_{n+1}(\eta_i)$. The continuity of the scale factor and its first derivative prevents the presence of divergences in the pump field, given by $V(\eta) = a''/a$. In first approximation the transition between two eras will be assumed instantaneous. This is justified because the duration of an era dominated by a material component is much larger than the transition period. This assumption is advantageous because it simplifies the calculations but a more careful analysis must take account the physics of the transition times. Taking account all these considerations, we calculate the explicit form of the scale factor:

$$a(\eta) = \begin{cases} -l\eta^{-1}, & \eta \le \eta_1 \text{ and } \eta < 0\\ la_{0r}(\Delta_{\mathrm{rad}}\eta - \eta_{\mathrm{rad}})^{1/\Delta_{\mathrm{rad}}}, & \eta_1 \le \eta \le \eta_{\mathrm{eq}}\\ la_{0m}(\Delta_{\mathrm{mat}}\eta - \eta_{\mathrm{mat}})^{1/\Delta_{\mathrm{mat}}}, & \eta \ge \eta_{\mathrm{eq}} \end{cases}$$
(6.8)

where l is a constant, the parameter Δ_{α} is evaluated with the value of ω_{α} , η_1 is the transition time between inflation and radiation and η_{eq} between radiation and matter. The values of the integration constants are

$$\eta_{\rm rad} = (\Delta_{\rm rad} + 1)\eta_1,$$

$$a_{0r} = (-\eta_1)^{-(1+1/\Delta_{\rm rad})},$$

$$\eta_{\rm mat} = (\Delta_{\rm mat} - \Delta_{\rm rad})\eta_{\rm eq} + \eta_{\rm rad},$$

$$a_{0m} = a_{\rm rad} \frac{(\Delta_{\rm rad}\eta_{\rm eq} - \eta_{\rm rad})^{1/\Delta_{\rm rad}}}{(\Delta_{\rm mat}\eta_{\rm eq} - \eta_{\rm mat})^{1/\Delta_{\rm mat}}},$$
(6.9)

Notice that for the special case $\beta = 0$ we recover the solution of the scale factor calculated by Grishchuk in [107].

Now we have to estimate values of the transition times η_1 and η_{eq} . To do this we compare the scale factors at the present time η_0 with the scale factor at the end of inflation $a(\eta_0)/a(\eta_1) \simeq 10^{21}$ and for the end of the radiation era $a(\eta_0)/a(\eta_{eq}) \simeq 10^4$ [19]. Using the solution for the scale factor (6.8) and solving the equation system we obtain that approximately $\eta_1 \simeq -10^{-17}$ and $\eta_{eq} \simeq 3/1000$. From here to the front we will adopt these values. This is a rude approximation, in fact the value of η_{eq} must slightly depend on the parameter β but for computational purposes we will adopt the constant values.

Finally, we must calculate the value of the constant l. To do this, notice that the Hubble parameter of inflation is $H_{\rm inf} = a'_{\rm inf}/a^2_{\rm inf} = l^{-1}$. This means that the value of l is related with the inverse of the energy scale of the inflation which is approximately $H_{\rm inf} \leq 3.7 \times 10^{-5} M_{\rm Pl} = 4.51733 \times 10^{23}$ eV [108], therefore $l \simeq 2.2137 \times 10^{-24} ({\rm eV})^{-1} \simeq 1.12183 \times 10^{-17}$ m. We will adopt the length units in ${\rm eV}^{-1}$.



Figure 6.1: Evolution of the scale factor $a(\eta)$ for some values of β . For $\eta < \eta_1$ the scale factor is independent of the parameter β . For $\eta > \eta_1$ the scale factor grows faster as β increases.

Performing a joint likelihood analysis of supernovae type Ia data, the CMB shift parameter and the baryonic acoustic oscillations, the numerical value of the parameter β was constrained $|\beta| = \mathcal{O}(10^{-3})$ [104, 30]. With values of this order the differences in the evolution of the scale factor are very small making hard to see their consequences for the GW production. For this reason we use relative big values of β accentuating its effect because we are interested in studying more qualitatively than quantitatively primordial GWs produced in the early universe due to the dynamics of cosmic expansion.

In the figure 6.1 we show the behavior of the scale factor for some selected values of β

and for the case of no-decaying vacuum ($\beta = 0$). For a given value of ω , the scale factor grows faster for higher values of β . This happens because the vacuum component have negative pressure with a repulsive gravitational effect contributing to the expansion of the Universe.

We emphasize that for inflation (times $\eta < \eta_1$), from the definition of Δ the parameter β plays no role in the scale factor and subsequently in the potential a''/a in this approach. Having the complete solution for the scale factor we can easily calculate the potential $V(\eta) = a''/a$:.

$$V(\eta) = \begin{cases} \frac{2}{\eta^2}, & \eta \le \eta_1 \text{ and } \eta < 0\\ \\ \frac{2\beta}{[(\eta - 2\eta_1)(1 - 2\beta)]^2}, & \eta_1 \le \eta \le \eta_{\text{eq}} \\ \\ \frac{2 + 6\beta}{[\eta(1 - 3\beta) - (1 - \beta)(4\eta_1 - \eta_{\text{eq}})]^2}, & \eta \ge \eta_{\text{eq}} \end{cases}$$
(6.10)

The potential is plotted in figure 6.2. One important observation is that in the limit case of no-decaying vacuum, $\beta = 0$, the potential vanish in the radiation era ($\eta_1 < \eta < \eta_{eq}$). This fact means that there is not gravitational wave amplification in this era.



Figure 6.2: Potential $V(\eta) = a''/a$ for some values of β . For $\eta < \eta_1$ the potential is independent of the parameter β . In the radiation era $(\eta_1 < \eta < \eta_{eq})$ for $\beta = 0$ there is no adiabatic amplification since a'' = 0; and for $\beta > 0$ duration of the potential is longer as β increases. For the matter era $(\eta > \eta_{eq})$ the potential is higher depending on β . Notice the discontinuities in each transition time.

With the explicit solutions for the scale factor $a(\eta)$ and the potential $V(\eta)$ we are

in conditions to calculate the solutions for $\mu(\eta)$. We will show in detail each calculation for the different cosmic epochs. Then, having an explicit solution for the mode functions $h = \mu/a$ we will calculate the power spectrum and energy spectrum for each cosmological era.

6.2 Inflation era

The GW production in inflation has been studied by several authors in the past (see [10, 19] and references therein). Inflation plays a fundamental role in the primordial GWs physics because in this era the primordial GWs are produced and settle the initial physical conditions for their subsequent evolution. This is because in this era the physical lengths were stretched from quantum to macroscopic distances in a very short time, allowing the tensor fluctuations of quantum origin be amplified, becoming classical perturbations. Another interpretation is to think that the virtual gravitons produced by the vacuum fluctuations become real due the pump of energy of the gravitational background separates them.

As summarized in section 3.5, the inflation era was proposed to solve some cosmological problems and it is well supported by several cosmological observations, but nowadays there exists a large variety of theoretical models of it. A careful analysis of primordial GWs in different inflationary scenarios combined with the decaying vacuum models seems to be very interesting and fructiferous, but is too extensive and goes beyond the scope and purposes of this work. From the numerous inflationary models, the de Sitter inflation seems to emerge quite naturally in the framework of the slow-roll approximation. For its simplicity and good approximation we will use this particular type of inflation.

The particular case of $\omega = -1$ for an exponential (de Sitter) inflation gives a potential $a''/a = -2/\eta^2$ regardless of the values assumed by the β parameter. This is of especial importance because means that the initial spectra of the radiation-dominated era are the same (epoch where the decaying vacuum effects on GWs are of our special interest). With this the equation (4.54) in particular is:

$$\mu_{\inf}''(n,\eta) + \left(n^2 - \frac{2}{\eta^2}\right)\mu_{\inf}(n,\eta) = 0, \qquad (6.11)$$

where for simplicity, we have suppressed the polarization index r. The general solution of the last equation can be expressed in terms of Bessel's functions

$$\mu_{\inf}(n,\eta) = \sqrt{\eta} \left[A_i J_{3/2}(n,\eta) + B_i J_{-3/2}(n,\eta) \right], \tag{6.12}$$

where A_i and B_i are integration constants to be specified. For this we use the fact that the solution of the inflation era at the limit of high frequencies reaches the so-called *adiabatic* vacuum condition [16, 107]. This limit essentially says that for high frequencies there is no gravitational wave amplification because when $n^2 \gg V$, the equation of $\mu(\eta)$ becomes the common harmonic oscillator. Choosing the positive frequency solution of the adiabatic vacuum we have

$$\lim_{n \to \infty} \mu_{\inf}(n, \eta) \propto e^{-in\eta}.$$
(6.13)

In order to calculate the constants we use the continuity junction conditions, $\lim_{n\to\infty} \mu_{\inf}(n,\eta) = e^{-in\eta}$ and $\lim_{n\to\infty} \mu'_{\inf}(n,\eta) = -ine^{-in\eta}$. The Bessel functions for large arguments asymptotically are

$$\lim_{x \to \infty} J_{\pm \alpha}(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x \mp \frac{\alpha \pi}{2} - \frac{\pi}{4}\right),\tag{6.14}$$

substituting equations (6.13) and (6.14) in the solution (6.12) and making some simple algebra we obtain the following equation system:

$$A_i\left(-\sqrt{\frac{2}{\pi}}\cos(n\eta)\right) + B_i\left(-\sqrt{\frac{2}{\pi}}\sin(n\eta)\right) = e^{-in\eta}, \qquad (6.15)$$

$$A_i\left(n\sqrt{\frac{2}{\pi}}\sin(n\eta)\right) + B_i\left(n\sqrt{\frac{2}{\pi}}\cos(n\eta)\right) = -ine^{-in\eta}.$$
 (6.16)

Solving the equation system we find the values of the constants, $A_i = -\sqrt{\pi/2}$ and $B_i = i\sqrt{\pi/2}$. The Bessel functions of this particular case can be expressed in term of trigonometric functions, $J_{\pm 3/2}(x) = \sqrt{2/\pi x}(-\cos(x) \pm \sin(x)/x)$. Joining all the parts and using the normalization (4.73), the complete normalized inflation solution is:

$$\mu_{\inf}(n,\eta) = \frac{e^{-in\eta}}{\sqrt{2n}} \left(1 - \frac{i}{n\eta}\right).$$
(6.17)

This particular solution is our start point in our analysis. We can see that μ_{inf} is a complex function implying that its modulus is the one with physical meaning.

6.2.1 Modulus of the mode function $|h_{inf}|$

With the solution μ_{inf} and the scale factor a_{inf} , the modulus of the mode function now can be easily calculated:

$$|h_{\rm inf}(n,\eta)| = \frac{\sqrt{\mu_{\rm inf}\,\mu_{\rm inf}^*}}{|a_{\rm inf}|} = H_{\rm inf}\sqrt{\frac{1+n^2\eta^2}{2n^3}}.$$
(6.18)

As expected, the modulus in the inflation era is independent of β because the scale factor doesn't either. This independence will be reflected in the power and energy density spectra. In figure 6.3 is plotted the evolution of $|h_{inf}|$ for different values of the wave number n. The value of $|h_{inf}|$ increases as the frequency diminishes and decreases in time monotonically reaching a final value which depends on n. The decrease naturally comes from the cosmic expansion (the factor a^{-1}).



Figure 6.3: Evolution of the modulus of the mode function $|h_{inf}|$ for different values of n in the inflation era.

The figure 6.4 shows the dependence on $|h_{\text{inf}}|$ with the wave number. The spectrum shows that as n grows the $|h_{\text{inf}}|$ diminishes as a power law $\propto n^{-3/2}$ for low frequencies and $\propto n^{-1/2}$ for high frequencies. Notice that the big values of $|h_{\text{inf}}|$ for small frequencies apparently could be troublesome, but small values of n implies big values of the physical wavelength defined as $\lambda = 1/2\pi k$ (remember k = n/a) and consequently at some frequency it would be bigger than the Hubble radius being outside of the horizon.



Figure 6.4: Spectrum of $|h_{inf}|$ at the end of the inflation era $\eta = \eta_1$.

6.2.2 Power spectrum \mathcal{P}_{inf}

The power spectrum for this era can be calculated using the equation (4.95):

$$\mathcal{P}_{\rm inf}(n,\eta) = \frac{32G}{\pi} n^3 |h_{\rm inf}(n,\eta)|^2 = \frac{16G}{\pi} H_{\rm inf}^2 [1+(n\eta)^2].$$
(6.19)

From the last equation immediately we perceive two limiting situations, when $1 \gg (n\eta)^2$ and $1 \ll (n\eta)^2$. To study these limit cases in a more clear way, let's rewrite the last equation in terms of the physical wavenumber k = n/a:

$$\mathcal{P}_{\rm inf} = \frac{16G}{\pi} H_{\rm inf}^2 \left[1 + \left(\frac{k}{H_{\rm inf}}\right)^2 \right]. \tag{6.20}$$

In the high frequencies condition, $(k/H_{inf})^2 \gg 1$, the power spectrum behaves as $\mathcal{P}_{inf} = \frac{16G}{\pi}n^2a^{-2}$. For a fixed (relative large) conformal wave number n, the power spectrum diminishes as the Universe expands. This condition of high frequencies corresponds to GWs with wavelengths much smaller that the horizon $k/H_{inf} \gg 1 \Rightarrow \lambda \ll H_{inf}^{-1}$.

In the other limit, the condition $\lambda \gg H_{\text{inf}}^{-1} \Rightarrow 1 \gg k/H_{\text{inf}}$ implies that we are outside the horizon. This limit condition for long wavelength GWs gives flat (constant) power spectrum proportional to the square of the Hubble parameter of the inflation:

$$\mathcal{P}_{\inf}^{\text{out}} = \frac{16G}{\pi} H_{\inf}^2 \simeq 7 \times 10^9.$$
 (6.21)

To illustrate the stated before, in figure 6.5 is plotted the power spectrum versus the conformal time η . The power spectrum is bigger as n grows, decreasing monotonically in

time and when it is close to the transition time η_1 , it decays in an abrupt way. The final value of the power spectrum at the transition time to the radiation era is very close to its value outside the horizon, $\mathcal{P}_{inf}(\eta_1) \sim \mathcal{P}_{inf}^{out}$, this because $(n\eta_1)^2 \ll 1$ for most of the values of n due the small numeric value of η_1 .

The power spectrum in function of the wavenumber n at the transition time $\eta = \eta_1$ is shown in figure 6.6. Clearly we can see the two differentiated behaviors of high and low frequencies. First for high frequencies the power spectrum has a square dependence with the wavenumber, $\mathcal{P}_{inf} \propto n^2$ growing without limit as the value of n increase. This suggests that there is a maximum value of the wavenumber, i.e. a minimum GW wavelength. On the opposite side, we can appreciate that for low frequencies it clearly approximates to the flat spectrum \mathcal{P}_{inf}^{out} showing that in this frequency regime the power of the GWs is the same for each wavelength.



Figure 6.5: Evolution of the power spectrum \mathcal{P}_{inf} for different values of n in the inflation era.



Figure 6.6: Power spectrum \mathcal{P}_{inf} at the end of the inflation era $\eta = \eta_1$.

6.2.3 GW energy density $\Omega_{gw}^{(inf)}$

Using the general equation of the GW energy density (4.99) with the solution (6.17) and the scale factor of the inflation era (6.8) we have:

$$\Omega_{\rm gw}^{\rm (inf)}(n,\eta) = \frac{4GH_{\rm inf}^2}{3\pi}(n\eta)^2 [1+(n\eta)^2].$$
(6.22)

First, notice that $\Omega_{gw}^{(inf)} \sim (n\eta)^2 \mathcal{P}_{inf}$. This means that the overall behavior of $\Omega_{gw}^{(inf)}$ is roughly similar to \mathcal{P}_{inf} , it diminishes in time having an abrupt fall when approaching the end of inflation η_1 and have bigger values of the GW energy density for bigger values of n. The figure 6.7 clearly schematize the said before.

As in the power spectrum, we have two limits related to high and low frequencies: $1 \ll (n\eta)^2$ and $1 \gg (n\eta)^2$. For a particular time (fixed η) we can see that for low frequencies $\Omega_{gw}^{(inf)} \propto n^2$ and for high frequencies $\Omega_{gw}^{(inf)} \propto n^4$. This can be seen in the spectrum shown in figure 6.8 at the time of the transition $\eta = \eta_1$. Again, we can see that the value of $\Omega_{gw}^{(inf)}$ increases disproportionately as the wavenumber n increases, reinforcing supposition that there is a maximum frequency of the GWs.

All these are standard results well studied independent of the decaying vacuum model. This is very important because gives the same initial conditions for the radiation era where the potential of the GWs depend on the parameter β . Another important point is that the exit power spectrum is also mostly independent of the frequency being \mathcal{P}_{inf}^{out} . So for practical purposes, the entering power spectrum of the radiation is the same for any type of decaying vacuum and frequency. In the next section we will calculate the modulus, power and energy density spectra of the GWs in the radiation era.



Figure 6.7: Evolution of the GW energy density spectrum $\Omega_{gw}^{(inf)}$ for different values of n in the inflation era.



Figure 6.8: GW energy density spectrum $\Omega_{gw}^{(inf)}$ at the end of the inflation era $\eta = \eta_1$.

6.3 Radiation era

The evolution of the GWs produced by the rapid acceleration of the Universe in the inflation is modified in the radiation era. This is because the evolution of the GWs depends on the nature of the scale factor which is different for each cosmic era. In the radiation era we have for the equation of state $\omega_{\rm rad} = 1/3$ which immediately implies $\Delta_{\rm rad} = 1 - 2\beta$ resulting that the scale factor has a dependence on the parameter β like $a_{\rm rad} \propto \eta^{\frac{1}{1-2\beta}}$. This particular dependence has a very important role. To see this, let's calculate the potential V = a''/a using the equation (6.6) with $\Delta_{\rm rad}$:

$$aa'' + (\Delta_{\rm rad} - 1)a'^2 = aa'' - 2\beta a'^2 = 0$$

$$\Rightarrow \frac{a''}{a} = 2\beta \left(\frac{a'}{a}\right)^2.$$
(6.23)

In the standard cosmology of $\beta = 0$, because the potential is null in all the radiation era, the solution of the mode functions is very simple $h_n(\eta) \sim e^{in\eta}/a$ an oscillatory function damped by the expansion of the universe, independent of n. This independence is very important because it means that the GW amplification will never happen in any case.

At this point, we have to stress one of the most important results of this work: clearly in the decaying vacuum model used here if $\beta \neq 0$ then the potential $V \neq 0$, condition implying that under certain conditions the GW amplification happens in the radiation era. This is very important because it establishes a clear differentiation between the evolution of the primordial GWs in decaying and non-decaying vacuum cosmological models in the radiation era. Without any calculation we can say that in some frequency range the contribution of the amplified GWs in the radiation era will affect the subsequent signal of the matter era because enters as initial conditions. This would distantly be observed providing information about the nature of $\Lambda(t)$.

Let's calculate the physical quantities of interest. In the radiation era the master equation (4.54) acquires the following form:

$$\mu_{\rm rad}''(n,\eta) + \left(n^2 - \frac{2\beta}{[(\eta - 2\eta_1)(1 - 2\beta)]^2}\right)\mu_{\rm rad}(n,\eta) = 0.$$
(6.24)

Integrating, the general solution is expressed in terms of the Bessel's functions:

$$\mu_{\rm rad}(n,\eta) = \sqrt{\Delta_{\rm rad}\eta - \eta_1(\Delta_{\rm rad}+1)} \left[A_r J_{\alpha_r} \left(\frac{n}{\Delta_{\rm rad}} (\Delta_{\rm rad}\eta - \eta_1(\Delta_{\rm rad}+1)) \right) + B_r J_{-\alpha_r} \left(\frac{n}{\Delta_{\rm rad}} (\Delta_{\rm rad}\eta - \eta_1(\Delta_{\rm rad}+1)) \right) \right],$$
(6.25)

where A_r and B_r are integration constants and we have defined the index as $\alpha_r = \frac{1}{\Delta_{rad}} - \frac{1}{2}$. To calculate the integration constants, A_r and B_r , we must use here the continuity junction conditions with the solution of inflation. In the transition time η_1 the first continuity junction condition is $\mu_{inf}(\eta_1) = \mu_{rad}(\eta_1)$:

$$\frac{e^{-in\eta_1}}{\sqrt{2n}} \left(1 - \frac{i}{n\eta_1}\right) = \sqrt{-\eta_1} \left[A_r J_{\alpha_r} \left(\frac{-n\eta_1}{\Delta_{\rm rad}}\right) + B_r J_{-\alpha_r} \left(\frac{-n\eta_1}{\Delta_{\rm rad}}\right)\right].$$
(6.26)

The second continuity junction condition is $\mu'_{inf}(\eta_1) = \mu'_{rad}(\eta_1)$:

$$\frac{e^{-in\eta_1}}{\sqrt{2n}} \left(\frac{i - n\eta_1 - in^2 \eta_1^2}{n\eta^2} \right) = A_r F + B_r G$$
(6.27)

where we have defined the following auxiliary functions:

$$F = \frac{1}{\sqrt{-\eta_1}} \left[J_{\alpha_r} \left(\frac{-n\eta_1}{\Delta_{\rm rad}} \right) + k\eta_1 J_{\alpha_r+1} \left(\frac{-n\eta_1}{\Delta_{\rm rad}} \right) \right], \tag{6.28}$$

$$G = \frac{1}{\sqrt{-\eta_1}} \left[(\Delta_{\text{rad}} - 1) J_{-\alpha_r} \left(\frac{-n\eta_1}{\Delta_{\text{rad}}} \right) + k\eta_1 J_{-\alpha_r + 1} \left(\frac{-n\eta_1}{\Delta_{\text{rad}}} \right) \right], \quad (6.29)$$

and renaming

$$\gamma_1 = \frac{e^{-in\eta_1}}{\sqrt{2n}} \left(1 - \frac{i}{n\eta_1}\right), \tag{6.30}$$

$$\gamma_2 = \frac{e^{-in\eta_1}}{\sqrt{2n}} \left(\frac{i - n\eta_1 - in^2 \eta_1^2}{n\eta^2} \right).$$
 (6.31)

we have the following equation system for ${\cal A}_r$ and ${\cal B}_r$

$$\sqrt{-\eta_1} J_{\alpha_r} \left(\frac{-n\eta_1}{\Delta_{\text{rad}}}\right) A_r + \sqrt{-\eta_1} J_{-\alpha_r} \left(\frac{-n\eta_1}{\Delta_{\text{rad}}}\right) A_r = \gamma_1, \tag{6.32}$$

$$FA_r + GB_r = \gamma_2, \tag{6.33}$$

which can be immediately solved giving:

$$A_{r} = \frac{1}{\sqrt{-\eta_{1}}} \left(\frac{\gamma_{1}G - \gamma_{2}\sqrt{-\eta_{1}}J_{-\alpha_{r}}\left(\frac{-n\eta_{1}}{\Delta_{\mathrm{rad}}}\right)}{J_{\alpha_{r}}\left(\frac{-n\eta_{1}}{\Delta_{\mathrm{rad}}}\right)G - J_{-\alpha_{r}}\left(\frac{-n\eta_{1}}{\Delta_{\mathrm{rad}}}\right)F} \right), \qquad (6.34)$$
$$B_{r} = \frac{1}{\sqrt{-\eta_{1}}} \left(\frac{\gamma_{2}\sqrt{-\eta_{1}}J_{\alpha_{r}}\left(\frac{-n\eta_{1}}{\Delta_{\mathrm{rad}}}\right) - \gamma_{1}F}}{J_{\alpha_{r}}\left(\frac{-n\eta_{1}}{\Delta_{\mathrm{rad}}}\right)G - J_{-\alpha_{r}}\left(\frac{-n\eta_{1}}{\Delta_{\mathrm{rad}}}\right)F} \right). \qquad (6.35)$$

Simplifying the above expressions we have:

$$A_{r} = \frac{e^{-in\eta_{1}}\pi \sec(\pi/\Delta_{rad})}{\Delta_{rad}(-2n\eta_{1})^{3/2}} \left[(-2i + n\eta_{1}(2 + in\eta_{1}) + \Delta_{rad}(i - k\eta_{1}))J_{-\alpha_{r}}\left(\frac{-n\eta_{1}}{\Delta_{rad}}\right) + n\eta_{1}(i - n\eta_{1})J_{-\alpha_{r}+1}\left(\frac{-n\eta_{1}}{\Delta_{rad}}\right) \right],$$

$$B_{r} = \frac{e^{-in\eta_{1}}\pi \sec(\pi/\Delta_{rad})}{\Delta_{rad}(-2n\eta_{1})^{3/2}} \left[-in^{2}\eta_{1}^{2}J_{\alpha_{r}}\left(\frac{-n\eta_{1}}{\Delta_{rad}}\right) - (in\eta - n^{2}\eta_{1}^{2})J_{\alpha_{r}+1}\left(\frac{-n\eta_{1}}{\Delta_{rad}}\right) \right] 6.37)$$

Replacing these expressions in the solution (6.25), we have the complete solution for $\mu_{\rm rad}$. The entire simplified expression will not be written explicitly because they are very large and cumbersome. This solution is normalized since satisfies $\mu_{\rm rad} \mu'_{\rm rad} - \mu^*_{\rm rad} \mu'_{\rm rad} = i$. We can see that this solution for the radiation era is quite more complicated that the inflation solution. Not only it depends on the conformal time η and wavenumber n, now the transition time η_1 and β are involved. Also the constants A_r and B_r are more complicated being only independent on time, but not on n and β .

6.3.1 Modulus of the mode function $|h_{rad}|$

The calculation of the modulus of the mode functions $|h_{\rm rad}| = \sqrt{\mu_{\rm rad} \mu_{\rm rad}^*} / a_{\rm rad}$ was performed, but for the same reasons of the complete expression of $\mu_{\rm rad}$ will not be presented explicitly. Instead, to study $|h_{\rm rad}|$ we have plotted it in some selected cases where we can appreciate their different behaviors depending both on decaying vacuum and wavenumber (frequency).

Figure 6.9 display the evolution of $|h_{\rm rad}|$ as a function of the conformal time η for some selected values of the parameter β and a relative low value of the wavenumber. In general terms in this case the modulus decreases slightly in time. We noticed an interesting feature for low frequencies if we have two perturbations, say $|h_1(\beta_1)|$ and $|h_2(\beta_2)|$, and if $\beta_1 > \beta_2$; then $|h_1| > |h_2|$. In the low frequency regime, due to the condition $n^2 \ll a''/a$, the solution is almost $\mu \propto a$ so that the perturbations remain nearly constant because $h = \mu/a$ and are greater as the value of β increase due the dependence of the scale factor with it. It is exactly in this frequency regime where the GW amplification occurs even during the radiation phase.

In the other side, for a relative big value of the wavenumber, the plot 6.10 display the evolution of $|h_{\rm rad}|$ as a function of the conformal time η for some selected values of the parameter β . Similarly to the low frequency regime, the modulus decreases in time



Figure 6.9: Evolution of the modulus of the mode function $|h_{\rm rad}|$ for different values of β with fixed value n = 100 in the radiation era.

but in a much more pronounced and in an oscillatory way. Inversely to the low frequency regime, if two decaying vacuum parameters satisfy $\beta_1 > \beta_2$ then their corresponding modules $|h_1| < |h_2|$.

The basic point here is that in the high frequency regime, we have the condition $n^2 \gg a''/a$, therefore the solution of μ is an oscillating function making the amplitudes to be damped by the scale factor even more intensively as the decaying vacuum increase since the scale factor expands faster for higher values of $\beta \neq 0$ (see figure 6.1). We can conclude that for high frequencies, in the radiation era, the decaying vacuum contributes more to the GW damping than to its amplification.

To make this more clear, we plot the spectrum of $|h_{\rm rad}|$ in figure 6.11. The spectrum diminishes uniformly as the wavenumber grows presenting two different behaviors for high and low frequencies. In the small *n* regime for different values of β , the differences are subtle but important, presenting more GWs amplification for bigger values of the parameter β . For big values of *n*, the decay of the spectrum of $|h_{\rm rad}|$ is faster because the expansion of the Universe dominates the amplification, diluting the GWs being more significant for bigger values of β .



Figure 6.10: Evolution of the modulus of the mode function $|h_{\rm rad}|$ for different values of β with fixed value $n = 10^5$ in the radiation era.



Figure 6.11: Spectrum of $|h_{\rm rad}|$ for different values of β at the end of the radiation era $\eta = \eta_{\rm eq}$.

6.3.2 Power spectrum \mathcal{P}_{rad}

The power spectrum of GWs in the radiation era $\mathcal{P}_{rad}(n,\eta) = \frac{32G}{\pi}n^3|h_{rad}(n,\eta)|^2$ was calculated. It has some similar general features present in the modulus of the mode function $|h_{rad}|$.

In the low frequency regime for bigger values of the decaying vacuum parameter β we have a slightly bigger power spectrum, as illustrated in the figure 6.12. Oppositely, in the high frequency regime, for bigger values of β we have a clearly smaller spectrum shown in figure 6.13. The reason of these two behaviors is the same as explained before for the modulus of the mode functions. Also the evolution of the power spectrum in the conformal time is always decreasing, being monotonous and almost constant for low frequencies, and oscillatory with an abrupt fall for high frequencies.



Figure 6.12: Evolution of the power spectrum \mathcal{P}_{rad} for different values of β with fixed value n = 100 in the radiation era.



Figure 6.13: Evolution of the power spectrum \mathcal{P}_{rad} for different values of β with fixed value $n = 10^5$ in the radiation era.

In figure 6.14 we show the power spectrum as a function of n for a fixed time. Notice that for low frequencies there is a plateau in the power spectrum, these are the long wavelength constant GWs of inflation that still outside of the horizon remaining "frozen" until re-enter in a later time. It is said that they are frozen because they don't "feel" the evolution of the scale factor and its spectrum is not damped by it. Here the contribution of β is almost imperceptible but increases the value of \mathcal{P}_{rad} .

Nevertheless, after some transition wavenumber (corresponding to $n \sim 10^3$) the GWs begin to be strongly damped and the associated power spectrum is no longer flat. This means that the decaying vacuum in this regime contributes more to increase the scale factor than to the GW amplification. In this high-frequency regime, \mathcal{P}_{rad} decreases as a power law and the effect is even more pronounced for larger values of β .



Figure 6.14: Power spectrum \mathcal{P}_{rad} for different values of β at the end of the radiation era $\eta = \eta_{eq}$.

6.3.3 GW energy density $\Omega_{gw}^{(rad)}$

The GW energy density in the radiation era was calculated, being approximately:

$$\Omega_{\rm gw}^{\rm (rad)} \simeq \frac{n^2}{12\mathcal{H}^2} \mathcal{P}_{\rm rad}.$$
(6.38)

An interesting characteristic of the energy density spectrum in the radiation era is that unlike the modulus $|h_{\rm rad}|$ and power spectrum $\mathcal{P}_{\rm rad}$ the energy density $\Omega_{\rm gw}^{\rm (rad)}$ always maintains the behavior that if for the decaying vacuum parameter $\beta_1 > \beta_2$, then $\Omega_{\rm gw}^{\rm (rad)}(\beta_1) < \Omega_{\rm gw}^{\rm (rad)}(\beta_2)$ for all values of the wavenumber.

Looking out from the limit of low frequencies, $n^2 \ll a''/a$, we have seen that the power spectrum is almost constant, giving for the energy density, $\Omega_{gw}^{(rad)} \propto n^2 \mathcal{H}^{-2}$, which is an increasing function in time and grows slower as β increase. This behavior is illustrated in figure 6.15.

In the high frequency limit, $n^2 \gg a''/a$, we have that the GW energy density behaves as $\Omega_{gw}^{(rad)} \propto (a')^{-2}$ which is also a decreasing function in time and decreases faster for bigger values of β . The figure 6.16 shows how the GW energy density in this regime is an oscillatory function with a decreasing upper envelope, almost flat for $\beta = 0$ and more pronounced for bigger values of β .



Figure 6.15: Evolution of the GW energy density $\Omega_{gw}^{(rad)}$ for different values of β with a fixed value n = 100 in the radiation era.



Figure 6.16: Evolution of the GW energy density $\Omega_{gw}^{(rad)}$ for different values of β with a fixed value $n = 10^5$ in the radiation era.

In the figure 6.17 it is plotted the spectrum of the energy density of GWs in the radiation era. Going from relative small values of the wavenumber to big ones, the GW energy density increases monotonically as a power law, being smaller for bigger values of β but having the same growth rate. Up to some value $n \sim 500$, where $\Omega_{gw}^{(rad)}$ begin to decay and becomes oscillatory. For values $\beta \sim 0$ it is approximately flat and as the values of β increase, the decay is steeper and strongly dependent on this parameter.



Figure 6.17: GW energy density spectrum $\Omega_{gw}^{(rad)}$ in function of n for different values of β at the end of the radiation era $\eta = \eta_{eq}$.

6.4 Matter era

The evolution of the primordial GWs produced in the inflation, some amplified and another diluted during the radiation era, continues into the matter era; where, due to the specific nature of the scale factor in this era their evolution again is modified. As we saw in the expression of the potential, equation (6.10), it is not null even in the particular case of no-decaying vacuum. This means that there is GW amplification at least some range of n, regardless of the value of β .

In the matter era we assume that the material component of the Universe does not have pressure. This implies that the equation of state is p = 0 i.e. $\omega_{\text{mat}} = 0$. With this, we have $\Delta_{\text{mat}} = (1 - 3\beta)/2$, which reduces the scale factor to:

$$a_{\rm mat}(\eta) = l \, a_{0m} \left(\frac{1 - 3\beta}{2}\eta - \eta_{\rm mat}\right)^{2/(1 - 3\beta)},\tag{6.39}$$

Calculating the potential and substituting into the master equation (4.54) we have:

$$\mu_{\rm mat}''(n,\eta) + \left(n^2 - \frac{2+6\beta}{[\eta(1-3\beta) - (1-\beta)(4\eta_1 - \eta_{\rm eq})]^2}\right)\mu_{\rm mat}(n,\eta) = 0.$$
(6.40)

Solving, we obtain the general solution in terms of the Bessel's functions

$$\mu_{\rm mat}(n,\eta) = \sqrt{\Delta_{\rm mat}\eta - \eta_{\rm mat}} \left[A_m J_{\alpha_m} \left(\frac{n(\Delta_{\rm mat}\eta - \eta_{\rm mat})}{\Delta_{\rm mat}} \right) + B_m J_{-\alpha_m} \left(\frac{n(\Delta_{\rm mat}\eta - \eta_{\rm mat})}{\Delta_{\rm mat}} \right) \right], \qquad (6.41)$$

with the index $\alpha_m = \frac{1}{\Delta_{\text{mat}}} - \frac{1}{2}$. Again the integration constants $A_m(n,\beta)$ and $B_m(n,\beta)$ will be calculated using the continuity condition at the transition time η_{eq} between the radiation and matter era, i.e. $\mu_{\text{rad}}(\eta_{\text{eq}}) = \mu_{\text{mat}}(\eta_{\text{eq}})$ and $\mu'_{\text{rad}}(\eta_{\text{eq}}) = \mu'_{\text{mat}}(\eta_{\text{eq}})$. The full calculation was performed giving two long and awkward expressions that will not be presented here.

Once calculated the normalized solution μ_{mat} , we are in condition to do the subsequent calculation of the modulus of the mode function, power and energy density spectra. The explicit forms of these quantities will not be written because they are cumbersome and their main characteristics can be seen plotting it for particular cases.

6.4.1 Modulus of the mode function $|h_{mat}|$



Figure 6.18: Evolution of the modulus of the mode function $|h_{\text{mat}}|$ for different values of β with fixed value n = 0.1 in the matter era.

The modulus of the mode function in the matter era presents a similar behavior as in the radiation era. For low frequencies if we have the condition for the decaying vacuum parameter $\beta_1 > \beta_2$ then for two mode functions we have $|h_1(\beta_1)| > |h_2(\beta_2)|$. The figure 6.18 shows this behavior for this condition. Again, similarly as in the radiation era, this condition inverts for the high frequency regime. If $\beta_1 > \beta_2$ then the modulus of the mode function $|h_1(\beta_1)| < |h_2(\beta_2)|$ as is shown in the figure 6.19. The reason is the same as stated before, in the high frequency regime the decaying vacuum contributes more to the Universe expansion than to the GW amplification.

The spectrum of $|h_{\text{mat}}|$ is shown in figure 6.20. For any value of β , the spectrum presents the same general characteristics. It diminishes uniformly as the wave number grows, for small values of n the spectrum diminish monotonously until a transition wave number 0.1 < n < 1 above which the spectra become oscillatory decreasing rapidly.



Figure 6.19: Evolution of the modulus of the mode function $|h_{\text{mat}}|$ for different values of β with fixed value $n = 10^3$ in the matter era.



Figure 6.20: Spectrum of $|h_{\text{mat}}|$ for different values of β at the end of the mater era $\eta = 11$.

6.4.2 Power spectrum \mathcal{P}_{mat}

As is expected, the \mathcal{P}_{mat} presents the same high and low frequencies general properties of the radiation era. It starts with an almost flat spectrum and also decreases faster as long as the vacuum contribution is relatively larger (higher values of β).



Figure 6.21: Evolution of the power spectrum \mathcal{P}_{mat} for different values of β with fixed value n = 0.1 in the matter era.

For low frequencies the power spectrum has the same behavior as the modulus of the mode function, if the decaying vacuum parameter $\beta_1 > \beta_2$ then the spectra have the relation $\mathcal{P}_{mat}(\beta_1) > \mathcal{P}_{mat}(\beta_2)$ as can be clearly appreciated in figure 6.21. Despite the differences due β the spectra diminish slowly in time.

For high frequencies, the last condition inverts; if $\beta_1 > \beta_2$ then the spectra $\mathcal{P}_{mat}(\beta_1) < \mathcal{P}_{mat}(\beta_2)$ as is shown in the figure 6.22. In this regime, \mathcal{P}_{mat} is oscillatory with a fast decay in time being more abrupt for higher values of β .

The power spectrum is shown in the figure 6.23. In general, in the low frequency regime, the power spectrum is almost constant as the frequency increases and the spectra are bigger for bigger values of the parameter β . At a certain characteristic frequency, the behavior changes and instead of growing it diminish oscillating. In this regime (high frequencies) the decay is faster for bigger values of β .



Figure 6.22: Evolution of the power spectrum \mathcal{P}_{mat} for different values of β with fixed value $n = 10^3$ in the matter era.



Figure 6.23: Power spectrum \mathcal{P}_{mat} for different values of β at the end of the radiation era $\eta = 11$.

6.4.3 GW energy density $\Omega_{gw}^{(mat)}$

The evolution of $\Omega_{gw}^{(mat)}$ follows similar trends to those of its counterpart of the radiation era. As in the radiation case, in matter era if we have the condition for the decaying vacuum parameter $\beta_1 > \beta_2$, then the GW energy density $\Omega_{gw}^{(mat)}(\beta_1) < \Omega_{gw}^{(mat)}(\beta_2)$ for all values of the wave number.

First, for low frequencies the GW energy density grows monotonously being lower for bigger values of β as can be seen in figure 6.24. Here the behavior of $\Omega_{gw}^{(mat)}$ is qualitatively the same of $\Omega_{gw}^{(rad)}$. However, as is shown in figure 6.25, in the high frequency regime the spectrum always decreases but varies differently as a function of the β parameter. As happens with \mathcal{P}_{mat} , the damping is very abrupt and faster for bigger values of β . A notable difference in this regime from $\Omega_{gw}^{(rad)}$ where in the particular case of no-decaying vacuum $\beta = 0$ have an almost constant upper envelope, here $\Omega_{gw}^{(mat)}(\beta = 0)$ have a clear decay.

The GW energy density spectrum $\Omega_{gw}^{(mat)}$ is presented in figure 6.26. Going from small to big values of the wave number, we can see first that the spectra for different values of β grows with the same linear power law until reach a characteristic wave number of 0.1 < n < 1 where they stop growing to start an oscillating decrease. The upper envelopes of the spectra are also linear power laws, but are more pronounced for bigger values of β . Up to here all the features of $\Omega_{gw}^{(mat)}$ are the same as $\Omega_{gw}^{(rad)}$, but at another characteristic wave number of $n \sim 500$ the rate of damping of the spectra change, continuing to decrease but not as fast as before.



Figure 6.24: Evolution of the GW energy density $\Omega_{gw}^{(mat)}$ for different values of β with a fixed value n = 0.1 in the matter era.



Figure 6.25: Evolution of the GW energy density $\Omega_{gw}^{(mat)}$ for different values of β with a fixed value $n = 10^3$ in the matter era.



Figure 6.26: GW energy density spectrum $\Omega_{gw}^{(mat)}$ in function of n for different values of β at the end of the matter era $\eta = 11$.

We investigated the primordial GWs in a particular $\Lambda \propto H^2$ model in the Friedmann equations. We calculated the scale factor in its general form in three cosmological eras, inflation, radiation and matter. The GW equation of $\mu(\eta)$ was derived, integrated, calculated the integration constants and finally obtained an explicit solution for each era. The modulus of the mode functions, power spectrum and energy density spectrum were calculated, plotted and analyzed for different values of β . The main results obtained were published in [33].

| Chapter

GWs in a nonsingular $\Lambda(H)$ -Cosmology

7.1 Nonsingular $\Lambda(H)$ -Cosmology

In the last chapter we studied the primordial GWs of the simplest non trivial decaying vacuum model $\Lambda = 3\beta H^2$. It was shown that this model has important consequences that makes it different from the standard approach, mainly because there is adiabatic amplification of GWs in the radiation era. Also, we have seen that this is only one particular case of the family of decaying vacuum models shown in chapter 5. Special particular cases of the general form of the decaying vacuum give new interesting features about the cosmic evolution and primordial GWs depending on the order of truncation of the power law and the simplifications assumed. More general models should be studied further, giving solutions that extend and recover the previous particular results, but the special cases are important because they give clues about what to expect about a general decaying vacuum cosmological model.

In this spirit, in this chapter it will be discussed, in a general way, another particular case of the general decaying vacuum model which is interesting because it presents a smooth transition (ST) from de Sitter inflation to radiation phase, avoiding the *graceful exit problem*¹. This characteristic simplifies the early Universe models having a natural link between inflation and the standard cosmic expansion, and includes the decaying vacuum alleviating the Cosmological Constant Problem.

¹A pure de Sitter inflation model consist in an exponentially expanding Universe with H = constant, with the problem that inflation never ends. Quantum-mechanically, tunnelling from the false vacuum to the true vacuum ends inflation locally, but the post-inflationary Universe looks nothing like our Universe. The Universe is either empty or much too inhomogeneous. This is the graceful exit problem of old inflation. Any successful inflationary mechanism has to include a way of ending inflation and successfully reheating the Universe.

Let's see how this model works. The starting point is the decaying vacuum general equation (5.15)

$$\Lambda(H) = c_0 + 3\nu H^2 + 3\alpha \frac{H^k}{H_I^{k-2}}, \quad \text{where} \quad k = 3, 4, \dots$$
 (7.1)

where c_0 is a constant, α and β are constant parameters of the model and H_I is the typical inflation energy scale of inflation of the model. As we are interested in the GWs of the early Universe, the constant term c_0 (the bare old Universe cosmological constant) can be neglected because the contribution of the *H*-terms dominate over it in that epoch, and then we will take the first possibility of the power-law term k = 3 and assume that this term is dominant over the H^2 -term. With these considerations the general equation of the decaying vacuum reduces to:

$$\Lambda = 3\alpha \frac{H^3}{H_I}.\tag{7.2}$$

Defined the functional form of $\Lambda(H)$, the next step is to insert it into the Friedmann equations for a flat Universe and to derive the dynamical equations of this cosmological model

$$8\pi G\rho + \Lambda(H) = 3H^2 = 8\pi G\rho + 3\alpha \frac{H^3}{H_I},$$
(7.3)

$$8\pi Gp - \Lambda(H) = -2\dot{H} - 3H^2 = 8\pi Gp - 3\alpha \frac{H^3}{H_I}.$$
(7.4)

Assuming an equation of state of the type $p = \omega \rho$ with ω constant, substituting in the two previous equations and combining them with some algebra we obtain the evolution equation of the Hubble parameter:

$$\dot{H} + \frac{3(\omega+1)}{2}H^2 \left[1 - \alpha \frac{H}{H_I}\right] = 0.$$
(7.5)

Let's examine the limits of this equation: the first limit is when the Hubble parameter is of the order of the inflation energy scale $\alpha H \simeq H_I$, the equation boils down to $\dot{H} \simeq 0 \Rightarrow H = constant$, which is precisely the de Sitter expansion. This condition is expected for the very early Universe when H is big, then as the expansion continues this condition doesn't hold anymore finishing the de Sitter epoch. It is important to remark that this occurs for any value of ω , except for $\omega = -1$ (that is the equation of state of a pure de Sitter Universe). The second limit occurs when the inflation energy scale dominates over the Hubble parameter $H_I \gg H$, here is restored the well-known standard equation $\dot{H} + 3(\omega + 1)H^2/2 = 0$, this condition is expected for the old Universe when H is small. Notice that this limit is also reached for the no-decaying vacuum, i.e. for $\alpha = 0$.

The solution of H (and consequently for the scale factor) begins being de Sitter to then pass smoothly to a period where the decaying vacuum is important and finally ends in the solution of the standard cosmology. This is the main characteristic that makes it interesting, the solution passes smoothly from inflation to the standard cosmology. So far we have not said anything about ω , but naturally we expect to be radiation, because it goes after inflation in the standard cosmological model.

To study the GWs in this model we must proceed as it was done previously: first, the scale factor must be calculated explicitly in order to obtain the potential and then integrate the GW equation. Due to the problem is more complicated, unlike the previous calculations for $\Lambda = 3\beta H^3$, now we will only calculate the GW mode functions in the interval of time where the decaying vacuum is significant and assume the other cosmological eras to be as the standard form. The solution calculated have to be joined to the other eras by the continuity junction conditions at the transition times.

7.2 Scale factor, potential and comoving Hubble parameter

In this section it will be derived the scale factor a, the potential V = a''/a and the comoving Hubble parameter $\mathcal{H} = a'/a$. These quantities give us some characteristics of the model studied and allows us to anticipate some results about the GWs before solving the GW equation directly. We have to calculate first the scale factor. To do this, we use the relation $\dot{H} = \ddot{a}/a - \dot{a}^2/a^2 = a''/a^3 - 2a'^2/a^4$ into (7.5), giving evolution equations of the scale factor in functions of the physical and conformal time

$$a\ddot{a} + \left[\frac{3(\omega+1)}{2} - 1\right]\dot{a}^2 - \frac{3\alpha(\omega+1)}{2H_I}\frac{\dot{a}^3}{a} = 0,$$
(7.6)

$$2aa'' + (3\omega - 1)a'^2 - \frac{3\alpha(\omega + 1)}{H_I}\frac{a'^3}{a^2} = 0.$$
 (7.7)

Because the model recovers the inflation for early times for any ω , we can assume initially a Universe filled with radiation ($\omega = 1/3$) without problem. The Hubble parameter and scale factor equations reduce to

$$\dot{H} + 2H^2 \left[1 - \alpha \frac{H}{H_I} \right] = 0, \tag{7.8}$$

$$a'' - \frac{2\alpha}{H_I} \left(\frac{a'}{a}\right)^3 = 0.$$
(7.9)

Integrating the equation (7.9) directly we have

$$a(\eta) = \frac{H_I(\eta + C_2) \pm \sqrt{H_I^2(\eta + C_2)^2 + 4\alpha H_I C_1}}{2H_I C_1},$$
(7.10)

where C_1 and C_2 are integration constants. Notice that when $\alpha = 0$ the "minus" solution is trivial a = 0, and for the "plus" solution we recover the usual radiation era solution $a \propto \eta$. Because that reason we choose the "plus" solution

$$a(\eta) = \frac{1}{2C_1} \left[\eta + C_2 + \sqrt{(\eta + C_2)^2 + \frac{4\alpha C_1}{H_I}} \right].$$
 (7.11)

Calculating the comoving Hubble parameter

$$\mathcal{H}(\eta) = \frac{a'}{a} = \left[(\eta + C_2)^2 + \frac{4\alpha C_1}{H_I} \right]^{-1/2},$$
(7.12)

we notice that $\mathcal{H}(\eta)$ is maximum for $\eta = -C_2$:

$$\mathcal{H}(-C_2) = \mathcal{H}(\eta_{\max}) \equiv \mathcal{H}_{\max} = \sqrt{\frac{H_I}{4\alpha C_1}},\tag{7.13}$$

$$\Rightarrow \quad C_1 = \frac{H_I}{4\alpha \mathcal{H}_{\max}^2}, \quad C_2 = -\eta_{\max}. \tag{7.14}$$

Automatically we can express the scale factor, potential and comoving Hubble parameter in terms of the new constants \mathcal{H}_{max} and η_{max} :

$$a(\eta) = \frac{2\alpha \mathcal{H}_{\max}}{H_I} \left[\mathcal{H}_{\max}(\eta - \eta_{\max}) + \sqrt{\mathcal{H}_{\max}^2(\eta - \eta_{\max})^2 + 1} \right], \quad (7.15)$$

$$V(\eta) = \mathcal{H}_{\max}^2 \frac{\sqrt{\mathcal{H}_{\max}^2(\eta - \eta_{\max})^2 + 1} - 1}{(\mathcal{H}_{\max}^2(\eta - \eta_{\max})^2 + 1)^{3/2}},$$
(7.16)

$$\mathcal{H}(\eta) = \frac{\mathcal{H}_{\max}}{\sqrt{\mathcal{H}_{\max}^2(\eta - \eta_{\max})^2 + 1}}.$$
(7.17)

The next step is to calculate the numerical values of \mathcal{H}_{max} and η_{max} . For this, let us consider that the Universe becomes matter-dominated (without decaying vacuum) after the conformal time of radiation-matter equilibrium η_{eq} . At this late time the vacuum decay is negligible, thus the scale factor acquires the usual matter-dominated era solution:

$$a_{\rm mat}(\eta) = \frac{H_0^2}{4}\eta^2, \quad \text{for} \quad \eta \ge \eta_{\rm eq}.$$
 (7.18)

We have chosen the unit normalization at the present time $a_{\text{mat}}(\eta_0) \equiv 1$, therefore the conformal present time is $\eta_0 = 2/H_0$ (where H_0 is the present Hubble parameter). From equation (7.18) the transition time η_{eq} can be calculated

$$\frac{a_{\rm mat}(\eta_0)}{a_{\rm mat}(\eta_{\rm eq})} = 1 + z_{\rm eq}, \qquad \Rightarrow \qquad \eta_{\rm eq} = \frac{2}{H_0\sqrt{1 + z_{\rm eq}}} = \frac{\eta_0}{\sqrt{1 + z_{\rm eq}}}.$$
 (7.19)

Where z_{eq} is the radiation-matter equilibrium redshift. Now, imposing the continuity junction conditions, $a(\eta_{eq}) = a_{mat}(\eta_{eq})$ and $a'(\eta_{eq}) = a'_{mat}(\eta_{eq})$ and solving the equation system we have for the constants:

$$\eta_{\max} = \frac{\eta_{eq}}{2} + \frac{2\alpha}{H_I} (1 + z_{eq}), \qquad (7.20)$$

$$\mathcal{H}_{\max} = \frac{H_0^2 H_I \eta_{eq}^2}{2\sqrt{2\alpha(H_0^2 H_I \eta_{eq}^3 - 8\alpha)}} \simeq \frac{H_0}{2} \sqrt{\frac{H_I \eta_{eq}}{2\alpha}}.$$
 (7.21)

The approximation of the last equation was made assuming $\alpha \sim \mathcal{O}(1)$ and for the cosmological constants $H_0 \sim 10^{-17}$, $H_I \sim 10^{35}$ and $z_{eq} \sim 3400$.

For mathematical convenience, we define the variable $\tau = \mathcal{H}_{\max}(\eta_{\max} - \eta)$, here the maximum of \mathcal{H} is reached at $\tau_{\max} = 0$. The scale factor, potential and comoving Hubble parameter in terms of the new variable can be read:

$$a(\tau) = \frac{2\alpha \mathcal{H}_{\max}}{H_I} \left[\tau + \sqrt{\tau^2 + 1} \right], \qquad (7.22)$$

$$V(\tau) = \mathcal{H}_{\max}^2 \frac{\sqrt{\tau^2 + 1} - 1}{(\tau^2 + 1)^{3/2}},$$
(7.23)

$$\mathcal{H}(\tau) = \frac{\mathcal{H}_{\max}}{\sqrt{\tau^2 + 1}}.$$
(7.24)

Now we are in conditions to compare these solutions with the standard forms. For simplicity, we assumed $\alpha = 1$ in all the plots.

First, figure 7.1a presents the evolution of the scale factor during the transition from the de Sitter regime to the radiation phase. For the no-decaying vacuum we have an abrupt transition (AT) passing from inflation $\omega = -1$ to radiation $\omega = 1/3$ instantaneously, the inflation energy scale was fixed in $H_{inf} = 10^{35}$. Notice that when the inflation energy of the ST model is the same of the AT model $H_I = H_{inf}$, the two curves are closely similar. For $H_I < H_{inf}$ the ST scale factor grows faster than the AT and this condition inverts for $H_I > H_{inf}$. This effect is due the repulsive gravity of the vacuum gravity and because for smaller values of H_I the term Λ is bigger.

Figure 7.1b shows the details of the different scale factors around the transition time. The time τ'_{max} is the inflation-radiation transition time of the standard model. Around τ_{max} the differences are manifest, the AT scale factor has a faster grow than the ST near the transition time.

Second, for the potential the ST and AT models are clearly different, figure 7.2a shows the behavior of the potential in the same conditions as presented for the scale factor. As mentioned several times in chapters 5 and 6, after the inflation-radiation transition time the potential is null for the AT model contrasting the ST potential which always have a positive value. The maximum value of the ST potential is reached at τ_{max} to then fall down, the magnitude of the potential depends on the values of the inflation energy scale being higher for bigger values of H_I .

The figure 7.2(b) shows that the maxima of the potentials are not reached at the same time, the maximum of the AT potential is delayed at $\tau'_{\text{max}} > \tau_{\text{max}}$ to then vanish. For the same inflation energy scale, the AT potential has a "peak" which is higher than the ST maximum having important consequences. This means that the AT potential acts over


Figure 7.1: Scale factor $a(\tau)$ for smooth and abrupt transition models. (a) Left panel: comparison between the abrupt transition (AT) scale factor (in blue) with the smooth transition (ST) scale factor (in red) for different values of the initial Hubble parameter H_I of the early de Sitter phase. (b) Right panel: detail of the left panel near τ_{max} .

higher frequencies that its ST counterpart reflecting in the power spectrum a bigger GW amplification at this frequency regime. After the peak, the AT potential is zero stopping the GW amplification, in contrast the ST potential always amplifies but at lower frequencies, whose effect in the power spectrum is minor than the high-frequency AT peak.



Figure 7.2: Potential V = a''/a for smooth and abrupt transition models. (a) Left panel: comparison between the AT potential (in blue) with the ST potential (in red) for different values of the initial Hubble parameter H_I of the early de Sitter phase. (b) Right panel: detail of the left panel near τ_{max} .

Finally, in figure 7.3a is displayed the behavior of the comoving Hubble parameter $\mathcal{H} = a'/a$ during the transition from the early de Sitter stage to the radiation phase. Similarly as for the scale factor, the differences between the AT and ST \mathcal{H} 's are subtle far from the transition time with the same inflation energy scale. The ST \mathcal{H} is higher as the value adopted for H_I is bigger.

The interesting point is clear in 7.3b, that the transition (maximum) at τ_{max} in the decaying vacuum model is smooth and can analytically be followed. Similarly as in the potential, the maximum of the AT \mathcal{H} is a sharp peak delayed at τ'_{max} and for the same inflation energy scale is bigger that the ST maximum.



Figure 7.3: Comoving Hubble parameter $\mathcal{H} = a'/a$ for smooth and abrupt transition models. (a) Left panel: comparison between the AT Comoving Hubble parameter (in blue) with the ST comoving Hubble parameter (in red) for different values of the initial Hubble parameter H_I of the early de Sitter phase. (b) Right panel: detail of the left panel near τ_{max} .

7.3 A multi-stage cosmological model

We have the complete solution of the scale factor of a Universe with decaying vacuum of the type $\Lambda = 3\alpha H^3/H_I$, filled with radiation $\omega = 1/3$ and which reduces to a matterdominated with no-decaying vacuum Universe after the transition time η_{eq} . Now let's see the limiting cases of the scale factor. From equation (7.11) and naming for convenience $z \equiv H_I(\eta + C_2)$ and $d \equiv H_I C_1$:

$$a(z) = \frac{z + \sqrt{z^2 + 4\alpha d}}{2d} = \frac{z}{d} \left[\frac{2\alpha d}{z^2 + z\sqrt{z^2 + 4\alpha d}} + 1 \right].$$
 (7.25)

For very late times $\eta \gg 1$, the first term inside the brackets of the equation (7.25) goes to zero, recovering the radiation solution $a_{\rm rad} \propto \eta$:

$$a(z) \simeq \frac{z}{d} = \frac{\eta + C_2}{C_1}.$$
 (7.26)

On the other side, for very early times, $\eta < 0$ and $|\eta| \gg 1$, defining x = -z with x > 0we recover the de Sitter inflation solution $a_{dS} \propto \eta^{-1}$:

$$a(x) = -\frac{x}{d} \left[\frac{2\alpha d}{x^2 - x\sqrt{x^2 + 4\alpha d}} + 1 \right] \simeq -\frac{x}{d} \left[\frac{2\alpha d}{x^2 - x\sqrt{x^2 + 4\alpha d}} \right]$$
$$= \frac{z}{d} \left[\frac{2\alpha d}{z^2 + z\sqrt{z^2 + 4\alpha d}} \right] \simeq \frac{z}{d} \left[\frac{\alpha d}{z^2} \right] = \frac{\alpha}{z} = \frac{\alpha}{\eta + C_2}, \tag{7.27}$$

Hence, we have explicitly shown that the solution $a(\eta)$ approximates the de Sitter inflation in the limit of early times and to the radiation-dominated Universe for late times limit. We have assumed a matter-dominated epoch after the equilibrium time with the radiation era. In that way, it is possible to construct a multi-stage cosmological model with four cosmological eras: de Sitter inflation, radiation-decaying vacuum, radiation and matter with no-decaying vacuum.

An initial inflationary de Sitter era has advantages because it provides a well-known initial GW spectrum [19] to be used as initial conditions when integrating the GW equation. The late times radiation era limit permits to link the present matter-dominated epoch normalization $a(\eta_0) = 1$ to our solution with the purpose to find the integration constants. To establish the transition times between each era in an appropriate form is very important in order to reach the correct Λ CDM approximations. To construct the multi-stage model we have to delimit the beginning and the end of the radiation-decaying vacuum era in some way. For this, we will use as criteria the deceleration parameter defined as $q = -\frac{\ddot{a}a}{\dot{a}^2}$, where for inflation is $q_{inf} = -1$ and for radiation $q_{rad} = 1$.

Using equation (7.6) with $\omega = 1/3$ we can express the deceleration parameter in terms of the Hubble parameter H:

$$q = 1 - 2\alpha \frac{H}{H_I}.$$
(7.28)

Now expressing the Hubble parameter equation (7.8) in terms of the scale factor and integrating we have:

$$aH\frac{dH}{da} + 2H^2 \left[1 - \alpha \frac{H}{H_I}\right] = 0, \quad \Rightarrow \quad H(a) = \frac{H_I}{\alpha + C_1 H_I a^2}.$$
(7.29)

Combining the last equation with (7.28) we obtain an expression of the scale factor in function of the deceleration parameter:

$$a(q) = \sqrt{\frac{\alpha}{C_1 H_I} \left(\frac{1+q}{1-q}\right)} = \frac{2\alpha \mathcal{H}_{\text{max}}}{H_I} \sqrt{\frac{1+q}{1-q}}.$$
(7.30)

To approximate the beginning of the radiation-decaying vacuum epoch, the deceleration parameter will be assumed to be $q_{-} = -1 + 10^{-\delta}$ and equivalently $q_{+} = 1 - 10^{-\delta}$ for the approximation of the end. The parameter δ is a constant positive number to be adjusted for our approximation purposes. By this way we can calculate the initial and final values of the scale factor.

$$a(q_{-}) \equiv a_{-} = \frac{2\alpha \mathcal{H}_{\max}}{H_{I}} 10^{-\delta/2} \sqrt{\frac{1}{2 - 10^{-\delta}}} \simeq \frac{\sqrt{2}\alpha \mathcal{H}_{\max}}{H_{I}} 10^{-\delta/2},$$
 (7.31)

$$a(q_{+}) \equiv a_{+} = \frac{2\alpha \mathcal{H}_{\max}}{H_{I}} 10^{\delta/2} \sqrt{2 - 10^{-\delta}} \simeq \frac{2\sqrt{2\alpha} \mathcal{H}_{\max}}{H_{I}} 10^{-\delta/2},$$
 (7.32)

where for the approximation was assumed that $2 \gg 10^{-\delta}$, i.e. $\delta \ge 4$ to fulfill the criterion properly. Using (7.15) we calculate initial and final conformal times of the radiation decaying vacuum era:

$$\eta_{-} = \eta_{\max} + \frac{H_{I}a_{-}}{4\alpha\mathcal{H}_{\max}^{2}} - \frac{\alpha}{H_{I}a_{-}} = \eta_{\max} - \frac{10^{\delta/2}}{\sqrt{2}\mathcal{H}_{\max}} \left(1 - \frac{1}{2\sqrt{2}*10^{\delta}}\right) \simeq \eta_{\max} - \frac{10^{\delta/2}}{\sqrt{2}\mathcal{H}_{\max}},$$

$$\eta_{+} = \eta_{\max} + \frac{H_{I}a_{+}}{4\alpha\mathcal{H}_{\max}^{2}} - \frac{\alpha}{H_{I}a_{+}} = \eta_{\max} + \frac{10^{\delta/2}}{\sqrt{2}\mathcal{H}_{\max}} \left(1 - \frac{1}{2*10^{\delta}}\right) \simeq \eta_{\max} + \frac{10^{\delta/2}}{\sqrt{2}\mathcal{H}_{\max}},$$

$$\Rightarrow \quad \eta_{\pm} = \eta_{\max} \pm \frac{10^{\delta/2}}{\sqrt{2}\mathcal{H}_{\max}},$$
(7.33)

again to make the approximation have to be assumed at least $\delta \geq 4$.

We have delimited the minimum appropriate value of the parameter δ . Now we have to look for a maximum value to fix the values of η_{\pm} . As the radiation era ends at η_{eq} , taking this time as the maximum possible value of η_{\pm} , i.e. the case when the radiation-decaying vacuum era covers all the radiation era, we have:

$$\eta_{+} = \eta_{\rm eq} \Rightarrow \delta_{\rm max} = 2\log[2\mathcal{H}_{\rm max}(\eta_{\rm eq} - \eta_{\rm max})] \simeq 46.48, \tag{7.34}$$

calculated with $H_0 \sim 10^{-17} \text{ s}^{-1}$, $H_I \sim 10^{35} \text{ s}^{-1}$, $z_{eq} = 3400$ and $\alpha = 1$. Finally we have an interval of suitable values $4 \le \delta \le 46$.

We have divided the conformal time of our model in four stages: inflation, radiationdecaying vacuum, radiation and matter. The transition times are $-\infty < \eta_{-} < \eta_{\text{max}} < \eta_{+} \leq \eta_{\text{eq}} < \eta_{0}$ and we have the solutions of the scale factor in each era. The next step is to calculate the mode functions

7.4 Gravitational wave solutions

To calculate the physical GW quantities we have to integrate the equation of the mode functions $h''_n + 2\mathcal{H}h'_n + n^2h_n = 0$ for every cosmological era, use the adiabatic vacuum condition to fix the integration constants of the first (inflation) solution and then use the continuity junction conditions to calculate the integration constants of the subsequent era and so on.

The de Sitter inflation mode function solution was calculated using the adiabatic vacuum condition in section 6.2. Evaluating it at the transition time between inflation and the radiation-decaying vacuum era η_{-} we have:

$$h_{-} \equiv h_{n}^{(\mathrm{dS})}(\eta_{-}) = \frac{A_{0}H_{I}}{\alpha} \left[\frac{i}{n} - (\eta_{-} - \eta_{\mathrm{max}})\right] e^{-in(\eta_{-} - \eta_{\mathrm{max}})},$$
(7.35)

$$h'_{-} \equiv h'^{(\mathrm{dS})}(\eta_{-}) = in \frac{A_0 H_I}{\alpha} (\eta_{-} - \eta_{\mathrm{max}}) e^{-in(\eta_{-} - \eta_{\mathrm{max}})},$$
(7.36)

the constant A_0 must be adjusted at the end of the calculation for $h_{\rm rms} \sim 10^{-5}$ in frequencies of the order of $\nu_0 \sim H_0$. Substituting $\eta_- = \eta_{\rm max} - \frac{10^{\delta/2}}{\sqrt{2}\mathcal{H}_{\rm max}}$ the last two expressions simplify:

$$h_{-} = \frac{A_0 H_I}{\alpha} \left[\frac{i}{n} + \frac{10^{\delta/2}}{\sqrt{2} \mathcal{H}_{\text{max}}} \right] e^{\frac{i n 10^{\delta/2}}{\sqrt{2} \mathcal{H}_{\text{max}}}}, \qquad (7.37)$$

$$h'_{-} = -in \frac{A_0 H_I 10^{\delta/2}}{\sqrt{2} \alpha \mathcal{H}_{\max}} e^{\frac{in 10^{\delta/2}}{\sqrt{2} \mathcal{H}_{\max}}}, \qquad (7.38)$$

The last two equations are the initial conditions for the calculations of the mode functions in the radiation-decaying vacuum era.

We don't have an analytical solution for $h_n(\eta)$ the radiation-decaying vacuum era. It should be calculated numerically in the interval $\eta_- \leq \eta \leq \eta_+$ to then be evaluated at the transition time η_+ , giving $h_n(\eta_+) \equiv h_+$ and its first derivative $h'_n(\eta_+) \equiv h'_+$.

The numerical calculations were performed with the computational program Wolfram Mathematica 10.1 [109]. For the calculations the parameters were fixed: $H_0 = 0.7 * 0.32 * 10^{-17} \text{ s}^{-1}$, $z_{eq} = 3400$, $\delta = 6$, $\alpha = 1$, $A_0 = 10^6$ and $l_{Pl} = 10^{-43} \text{ s}$.

After the radiation-decaying vacuum era, the Universe enters to a pure radiation era in the interval $\eta_+ < \eta < \eta_{eq}$. The solution of the GW equation $(\mu'' + n^2 \mu = 0)$ can be easily performed and the mode functions $h_n = \mu/a$ calculated, to then be evaluated at the end of this epoch, η_{eq} . The mode functions $h_n^{(rad)}(\eta_{eq}) \equiv h_{eq}$ and its first derivative $h_n'^{(rad)}(\eta_{eq}) \equiv h'_{eq}$ are:

$$h_{\rm eq} = \frac{1}{a_{\rm eq}} [b_1 e^{-in(\eta_{\rm eq} - \eta_{\rm max})} + b_2 e^{in(\eta_{\rm eq} - \eta_{\rm max})}],$$
(7.39)

$$h'_{\rm eq} = -\frac{1}{a_{\rm eq}} [b_1(\mathcal{H}_{\rm eq} + in)e^{-in(\eta_{\rm eq} - \eta_{\rm max})} + b_2(\mathcal{H}_{\rm eq} - in)e^{in(\eta_{\rm eq} - \eta_{\rm max})}]$$
(7.40)

$$b_1 = -\frac{a_+}{2in} [h'_+ + (\mathcal{H}_+ - in)h_+] e^{in(\eta_+ - \eta_{\max})}, \qquad (7.41)$$

$$b_2 = \frac{a_+}{2in} [h'_+ + (\mathcal{H}_+ + in)h_+] e^{-in(\eta_+ - \eta_{\max})}, \qquad (7.42)$$

with the constants $a_{eq} \equiv a(\eta_{eq}), a_+ \equiv a(\eta_+), \mathcal{H}_{eq} \equiv \mathcal{H}(\eta_{eq})$ and $\mathcal{H}_+ \equiv \mathcal{H}(\eta_+)$.

Finally, at the present time $\eta_0 = 2/H_0$, the matter-dominated era the mode functions are expressed in terms of the Bessel's functions:

$$h_0 \equiv h_n^{(\text{mat})}(\eta_0) = \sqrt{\eta_0} \left[D_1 J_{3/2}(n\eta_0) + D_2 J_{-3/2}(n\eta_0) \right], \tag{7.43}$$

$$D_1 = \frac{\pi}{2} a_{\rm eq} \sqrt{\eta_{\rm eq}} \left[n h_{\rm eq} J_{-5/2}(n \eta_{\rm eq}) - h_{\rm eq}' J_{-3/2}(n \eta_{\rm eq}) \right], \tag{7.44}$$

$$D_2 = \frac{\pi}{2} a_{\rm eq} \sqrt{\eta_{\rm eq}} \left[n h_{\rm eq} J_{5/2}(n \eta_{\rm eq}) + h_{\rm eq}' J_{3/2}(n \eta_{\rm eq}) \right].$$
(7.45)

The present time mode function solution h_0 has embedded in their constants the solutions of all the previous cosmological eras. As part of these calculations was done numerically, we don't have closed expressions related to the mode functions, like the power and energy density spectra.

In our analysis, we will concentrate in the root mean square (rms) of the amplitude of the GWs today, which is related to the power spectrum by $h_{\rm rms}(\nu, \eta_0) = \sqrt{\mathcal{P}(\nu, \eta_0)}$.

Figure 7.4 displays the $h_{\rm rms}(\nu, \eta_0)$ as a function of the physical frequency ν for the ST model and its AT (de Sitter + rad + mat) counterpart with the same inflation energy scale $H_I = 10^{35} \, {\rm s}^{-1}$. Not including the late Λ_0 dominated epoch would result in the same effect for both models, namely, a little smaller value of $h_{\rm rms}$ for all the spectra. Notice the remarkable superposition of both spectra for almost the entire frequency range. They are distinguishable only at very high frequencies.

In the high-frequency regime, the AT model presents a higher GW amplification than the ST. As shown in figures 7.2 and 7.3, such effect can be understood in terms of the behavior of $\mathcal{H}(\eta)$ and $V(\eta)$. Since the decaying vacuum model evolves smoothly from a de Sitter towards a radiation era, the shape of $\mathcal{H}(\eta)$ for the transition and the consequent lower value of \mathcal{H}_{max} , for the same H_I , result in a lower high frequency GW production. Although its effect is short in time, the AT potential peak reaches higher frequencies contributing to the spectrum. Moreover, we saw that the ST model has a greater amplification than the AT model in the low frequency regime, but their effect is minimal, being almost imperceptible in the present spectrum.

At high frequencies, the two models predict distinct spectra for a given value of H_I . To appreciate the main differences, we display in figure 7.5 the energy density spectrum $\Omega_{\rm gw}(\nu,\eta_0)$. In this figure, we have fixed the value of $H_I = 10^{35} \, {\rm s}^{-1}$ for the AT model and considered some possible values of H_I for the decaying vacuum cosmology. As there is no adiabatic amplification for frequencies $\nu > \mathcal{H}_{\rm max}$ we have introduced a cutoff at



Figure 7.4: Present $h_{\rm rms}(\nu, \eta_0)$ for ST and AT models. Both presented with the same inflation energy scale $H_I = 10^{35} \, {\rm s}^{-1}$.

 $\nu_{\max} = \mathcal{H}_{\max}$ in the graph.

7.5 Final Comments

We investigated the primordial GWs in the context of this nonsingular flat cosmology driven by a continuous decaying vacuum energy density. The model can be interpreted as a particular case of the large class of cosmologies which is termed complete in the sense that the cosmic evolution occurs between two extreme de Sitter stages (early and late time de Sitter phases). The first results obtained were published in [34].

The scale factor, potential and comoving Hubble parameter were derived in the decaying vacuum context (ST model) and were compared to the standard case (AT model). It was performed a cosmological model divided for convenience in four eras: three of the AT model (de Sitter inflation, radiation and matter) and a radiation-decaying vacuum era between the usual inflation and radiation eras. The mode functions h_n were calculated in each cosmological era with the particularity that for the radiation-decaying vacuum era was done numerically.

Obtained the present time mode function $\eta_0 = 2/H_0$, the rms amplitude of the GWs of the ST model were compared with the AT. It was found that the stochastic background of GWs is very similar to the one predicted by the Standard Cosmological Model plus



Figure 7.5: Density parameter $\Omega_{\rm gw}(\nu,\eta_0)$ in the domain of high frequencies. For the AT model we have fixed the Hubble parameter to be $H_{\rm inf} = 10^{35} \, {\rm s}^{-1}$ while in the ST model the arbitrary scale H_I assumed three possible indicated values.

inflation except for the higher frequencies. The calculations show that the model predicts a lower energy density for GWs at these frequencies, which depends on the value of the inflation energy scale of the decaying vacuum model H_I . This is a remarkable signature of the studied vacuum decay model making it potentially distinguishable from the usual inflationary scenarios through observations with the proposed high frequency GW detectors [110, 111], if high frequency gravitational wave detectors become operative and reach the appropriate sensitivity.

Finally, we stress that the model discussed here can be thought as a starting point for the investigation of more complex and rich decaying vacuum cosmologies. This class of models deserves a closer scrutiny since they furnish a complete cosmological history.

Chapter 8

Conclusions and perspectives

In this work we have discussed some properties of the primordial GWs in decaying vacuum cosmologies. We worked out two particular models derived from a general decaying vacuum in a flat FLRW geometry. The principal motivation to investigate the primordial GWs in the context of such kind of models is because they provide possible observational clues about the nature of the decaying vacuum models in the GW spectra. This is interesting because as mentioned before, the decaying vacuum models were proposed to be a possible solution to important cosmological problems.

In the first chapter the interest and relevance of this work were justified. As it was written there, the study of GWs in the cosmological context is the fundamental relevance to study the physics of the very early Universe because it decouples from the other cosmic components at very early times, thereby carrying important information of this epoch. Moreover, the decaying vacuum models provide a possible solution (or at least alleviate) the Cosmological Constant and Coincidence Problems. Joining these two pieces we started this work.

In the second chapter it was established the principal features of the GW theory and complementing it, the third chapter describes the basic concepts of cosmology. The fourth chapter was devoted to summarizing the principal mathematical tools and physical properties of the cosmological tensor perturbations; the classical and quantum tensor perturbation theories were described, we explained the GW amplification mechanism. The definitions of the power spectrum and the energy density were derived. The fifth chapter was dedicated to the general phenomenological and renormalization group motivations of the $\Lambda(t)$ models and the studies were illustrated with a very simple cosmological scenario. In chapter 6, it was presented the principal results for the inflation, radiation and matter era for the $\Lambda = 3\beta H^2$ model. Were calculated explicitly the tensor perturbations, h_{ij} , of the FLRW metric in terms of the mode functions $h(n,\eta)$ of the Fourier expansion in the $\Lambda(t)$ context. For our calculations it was assumed a model where the cosmic history is divided into three distinct eras, namely: inflation, radiation and matter. For each one, we have a particular functional form of the scale factor that depends on the value of ω (from the equation of state $p = \omega \rho$) and the numerical parameter of the decaying vacuum β . This implies that we have a particular solution for the mode function $h(n,\eta) = a^{-1}(\eta)\mu(n,\eta)$ for each cosmic era.

The three solutions were calculated, and linked to the other ones via the junction continuity conditions in the transition times: $h_k^{(1)}(\eta_i) = h_k^{(2)}(\eta_i)$, and in its first derivative $h_k^{\prime(1)}(\eta_i) = h_k^{\prime(2)}(\eta_i)$. The integration constants were fixed first using the adiabatic vacuum solution of the early Universe. Having a complete solution of the mode function $h(n, \eta; \beta, \omega)$ for each cosmological era, the next step was to calculate the modulus of the mode function |h|, the power spectrum $\mathcal{P} \sim n^3 |h|^2$ and the energy density $\Omega \sim n^2 \mathcal{H}^{-2} \mathcal{P}$. Once the physical expressions were obtained, we have also plotted all the quantities for the interesting limit cases (low and high frequencies), in order to show its principal characteristics. We have also pointed out the principal features for each graph.

We list the principal results and perspectives of the $\Lambda = 3\beta H^2$ model presented work:

- The most relevant result until now is the GW amplification in the radiation era. This is very different from results obtained previously in the literature. For the no-decaying vacuum case $\beta = 0$, we have that the second derivative of the scale factor vanishes a'' = 0, giving the harmonic oscillator equation $\mu'' + n^2 \mu = 0$. The solution for the μ equation in this case is very simple, $c_1 e^{in\eta} + c_2 e^{-in\eta}$, which has an adiabatic behavior and consequently without particle production. Independently which inflation model is chosen, when $\beta \neq 0$ the scale factor for the radiation era is not null. This very special characteristic provides a possibly observational test for the $\Lambda(t)$ models. For this reason a more detailed and careful calculation must be made for this era.
- For the inflation era we assumed a de Sitter Universe ($\omega = -1$), thereby making the inflation era independent of the parameter β . An inflationary model with a β -dependent scale factor changes substantially the evolution of the Universe which

change the nature of the amplification of the primordial GWs. Consequently the power spectrum and energy density that goes outside the horizon and must be seen for the present time would be changed. Another possibility to achieve a de Sitter inflation is to fix $\beta = 1$ in this era, making $\Delta_i = -1$ for any value of ω but this condition does not seem to be promising. A more general model must consider more general inflationary scenarios.

- We have assumed that the vacuum decays into particles of the dominant component era. In the radiation era, Λ decays into photons and in the matter era into matter particles (baryonic or dark matter). A more general model should have the form $\Lambda(t) = 3(\beta_{\rm rad} + \beta_{\rm mat})H^2$ where $\beta_{\rm rad}$ and $\beta_{\rm mat}$ are the decay parameters for each era. The assumption that the vacuum only decays into photons ($\Lambda(t) = 3\beta_{\rm rad}H^2$) means that only this term is important in the radiation era. This simplifies the model and the calculations permitting make a complete analysis in a relatively easy way. If $\Lambda = \lambda_0 + 3\beta H^2$ and β is always a constant, the observations in the matter-vacuum phase provide very tight constraints on the β parameter [30]. However, the decaying vacuum channels in radiation and matter may happen in different rates.
- The model has several simplifications which must be improved in a more detailed study. One of them is the transition times η_1 and η_{eq} . At the moment, we only estimated the value of these times and assumed it as constants. In fact, these quantities depend on the parameter β and must affect the evolution of the GWs. The dependence of the vacuum decay of the transition times must be considered for a more consistent study.

In chapter 7, were calculated the today's GW spectra for a $\Lambda = 3\alpha \frac{H^3}{H_I}$ model. The main motivation to study this model is because it naturally avoids the graceful exit problem. The scale factor in the model has a smooth inflation-radiation transition where the decaying vacuum effects in the scale factor are clearly manifest in contrast to the standard scale factor which has an abrupt transition. To study the differences of the standard (AT) and decaying vacuum (ST) models, the scale factor a, the potential V = a''/a and the comoving Hubble parameter $\mathcal{H} = a'/a$ were calculated for both models and plotted. The principal result is that despite the ST has GW amplification after the inflation-radiation transition, the AT model has a high frequency amplification peak bigger than the ST model whose contribution to the power spectrum is more significant.

For our calculations, the cosmic history was divided into four distinct eras in a flat FLRW cosmology: inflation, radiation-decaying vacuum, radiation and matter. In the eras where the decaying vacuum can be considered irrelevant (inflation, radiation and matter) the mode functions $h(n, \eta)$ were calculated explicitly and for the radiation-decaying vacuum, numerical calculation were needed and implemented. The integration constants in this case were fixed with the normalization of the scale factor at the present time, and all the solutions were joined by the junction continuity conditions. The today root mean square of the mode functions $h_{\rm rms}(\eta_0)$ and the energy density spectrum $\Omega_{\rm gw}(\nu, \eta_0)$ were calculated and plotted with the AT results.

We list the principal results and perspectives of the $\Lambda = 3\alpha \frac{H^3}{H_I}$ model:

- The model is a particular case of the large class of cosmologies which is termed complete in the sense that the cosmic evolution occurs between two extreme de Sitter stages (early and late time de Sitter phases). The scale factor for any EoS, has the de Sitter solution for the very early time limit, to then smoothly pass to a solution which late-time limit is the standard no-decaying vacuum scale factor. This feature avoids the graceful exit problem when $\omega = 1/3$ is used and it's the principal motivation to study it in the GW context.
- The calculations to obtain the solution of the mode functions of this model are more difficult than the performed in the first model studied. For that reason it was not possible to calculate the mode functions for the entire cosmic history. To tackle this hassle, it was delimited the interval where the decaying vacuum is significant using the deceleration parameter. The interval is close to the inflation-radiation transition time of the AT model. After numerically calculating the solution, it was joined to the analytical solution to have a present time solution.
- The inflation-radiation AT time is when potential vanishes and the comoving Hubble parameter reaches its maximum. For the ST model, the comoving Hubble parameter maximum occurs before, showing that the AT is delayed from the ST. The potential reflects in a more clear way the two main differences between the AT and ST GW spectra. Comparing both potentials, after the transition time, the AT potential goes to zero meanwhile the ST potential decreases as evolves in time but never vanishes. As in the previous model, exist GW amplification in this decaying vacuum model at the same time when there is not in the other model (AT model), the amplified GWs are of low frequency. Their contribution to the spectrum is minimal.
- The high frequency regime is where appear the substantial difference. The AT potential maximum is higher than the ST, with a pronounced peak to then drop

to zero. This means that the AT model amplify higher frequencies to then stop meanwhile the ST model amplify a continuously as the frequency diminish. In the high-frequency regime is where the substantial difference appears. The AT potential maximum is higher than the ST, with a pronounced peak to then fall to zero. This means that the AT model amplify higher frequencies to then stop, meanwhile the ST model amplifies continuously the GWs as the frequency diminishes. When is plotted the $h_{\rm rms}(\eta_0)$ and $\Omega_{\rm gw}(\nu, \eta_0)$, both models show that their stochastic background of GWs is very similar until high frequencies where the AT GW amplification is manifest presenting bigger values in the spectrum after a certain critical frequency.

• The scale factor, the potential, comoving Hubble parameter, the today root mean square of the mode functions and the energy density spectrum, depend on the value of the inflation energy scale of the decaying vacuum model H_I .

Two general assumptions were adopted in both cases: abrupt transitions between each cosmological era and tensor perturbations without anisotropic contributions.

Finally, we stress that the models discussed in this work are the starting point of the investigation of more complex and richer decaying vacuum cosmologies. The general class of decaying vacuum cosmologies should be studied in the future (recover the achieved results, reach the old Universe limit and generalize them). The general model should be contrasted with the standard Λ CDM and cosmological observations.

We have studied two particular cases of a general class of decaying vacuum cosmologies. Both models have proper characteristic features that imprint on the GW spectra a unique way, making the GW spectrum a fingerprint of the early Universe. A direct observation of the stochastic background of GWs looks far away, but indirect observations imprinted on the CMB look promising in the near future. For that reason, the study of the primordial GWs in decaying vacuum cosmologies is very important because they could be a key to know the physics of the very early Universe that are beyond to the today known physics.

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