Universidade de São Paulo
Instituto de Física

**Deformações de Yang-Baxter da supercorda em
 $AdS_5 \times S^5$ no formalismo de espinores puros**

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Dissertação de doutorado apresentada ao Instituto de Física para a obtenção do título de Doutor em Ciências.

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Sao Paulo
2018

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**Yang-Baxter deformation of the $AdS_5 \times S^5$ pure
spinor superstring**

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Thesis submitted to the Physics Institute of the University of Sao Paulo in partial fulfilment of the requirements for the degree of Doctor of Science.

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Sao Paulo
2018

Abstract

We construct an integrable deformation of the $AdS_5 \times S^5$ pure spinor action based on homological perturbation theory. We show that the resulting model describes a pure spinor string moving in an η -target superspace. In that sense, we establish a one-to one map between vertex operators in the cohomology of the undeformed BRST charge and the η target space.

Resumo

Construímos uma deformação integrável da ação de espinores puros no espaço $AdS_5 \times S^5$ baseada na teoria de perturbação homológica. Mostramos que o modelo resultante descreve uma corda movendo-se em um superespaço η . Nesse sentido, estabelecemos um mapa entre operadores de vértices na cohomologia da carga BRST em $AdS_5 \times S^5$ e o superespaço η .

I will tell you the deeper significance of this, which otherwise might seem a banal hydraulic joke. Caus knew that if one fills a vessel with water and seals it at the top, the water, even if one then opens a hole in the bottom, will not come out. But if one opens a hole in the top, also, the water spurts out below.

"Isn't that obvious?" I said. "Air enters at the top and presses the water down".

"A typical scientific explanation, in which the cause is mistaken for the effect, or vice versa. The question is not why the water comes out in the second place, but why it refuses to come out in the first case."

"And why does it refuse?" Garamond asked eagerly.

"Because, if it came out, it would leave a vacuum in the vessel, and nature abhors a vacuum."

— Umberto Eco, Foucault's Pendulum

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Chapter 1

Introduction

Since the early days of string theory, it was recognized that a suitable candidate for the graviton, the hypothetical particle that mediates the gravitational interaction, is included in its closed sector spectrum as an oscillation mode around the flat space. Soon, it was realized that in order to describe a consistent realistic theory, including either bosonic and fermionic fields, the target space of the theory should exhibit supersymmetry; a fermionic symmetry which relates the bosonic and fermionic degrees of freedom. The quantization of superstrings in ten-dimensional background geometries can be carried out consistently in only five ways, generating the superstring theories known as type I, IIA, IIB and heterotic $E_8 \times E_8$ and $SO(32)$, so called because their massless spectra contain states of the corresponding supergravity theories. Since then, many exciting developments have recast our understanding of the inherent nature of string theory. Even though string theory describe a consistent formulation of quantum gravity, it was never shown how it unambiguously implies, as a low energy effective description, general relativity and the standard model of particle physics.

In recent years a duality between a string theory in an Anti-de Sitter (AdS) spacetime in $d + 1$ dimension, and a conformal gauge theory in a flat d -dimensional space have received much attention. The AdS/CFT correspondence conjectures that observables and correlation functions of both theories are equivalent. In its original formulation, this duality relates the type IIB superstring in an $AdS_5 \times S^5$ background to the maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM), a conformal field theory in four dimensions living on the boundary of AdS with gauge group $SU(N_c)$. The correspondence relates the coupling constant of the SYM theory g_{YM} and the number of colors N_c with the string coupling constant g_s and the Regge slope α' as follows

$$\lambda = N_c g_{YM}^2 = 4\pi N_c g_s = (R^2/\alpha')^{1/2}, \quad (1.1)$$

where R the radius of AdS_5 and λ the 't Hooft coupling.

A particular interesting limit occurs when N_c is very large and λ is kept fixed. In this limit, known as the planar limit, both gauge and string theories show their integrable structure(for a review [1] [2]). Roughly speaking, a field theory is called integrable if there exist an infinite number of conserved charges. This property allows us to exactly solve the system. In this limit the quantum theory of $\mathcal{N} = 4$ is greatly simplified since only planar Feynman diagrams, contribute at leading order. The problem of computing the spectrum of local operators was solved by realizing that the dilaton operator could be identified with the Hamiltonian of an integrable spin chain. In such a way, the quantum corrected scaling dimensions of these operators can be read by diagonalizing the spin-chain Hamiltonian using the Bethe ansatz. Correspondingly, the string coupling g_s goes to zero as $N_c \rightarrow 0$ and only the world-sheet of lowest genus survives. Therefore, the gravity sector describes a non-interacting string on $AdS_5 \times S^5$. Nevertheless, this is a highly non-linear sigma model which receive α' corrections at quantum level. Again, the fundamental observation on the gravity sector is that superstrings propagating in AdS can be formulated by an integrable model. The $AdS_5 \times S^5$ space arises as a solution of type IIB supergravity when supported by a self-dual Ramond-Ramond five-form flux. A string moving on a curved background including a RR flux can be correctly formulated by using the Green-Schwarz formalism or the pure spinor formalism which present manifest target space supersymmetry. In these formulations, the equations of motion, either in the Green-Schwarz (GS) or in the pure spinor (PS) formulation, can be cast into a zero curvature equation satisfied by a Lax pair which ensures the existence of an infinite number of conserved charges [3, 4]. Although the complete quantization of the superstring in $AdS_5 \times S^5$ has never been fulfilled, significant progress has been made in understanding the excitation spectrum using integrability techniques.

One may wonder about the origin of this exact solvability. To go further in this direction it is quite natural to seek deformations of $AdS_5 \times S^5$ superstring and its dual $\mathcal{N} = 4$ SYM which preserve the integrable structure while relaxing some of these symmetries. The first known example of such an integrable deformation was the so-called β -deformation of $\mathcal{N} = 4$ SYM associated with strings on the Lunin-Maldacena background [5]

More recently, using the GS formulation of the superstring, two significant approaches have been developed to deform the $AdS_5 \times S^5$ structure while preserving integrability and the fermionic κ -symmetry. On one hand, the GS λ -deformed model [6] is based on a G/G gauged WZW model. The deformation is implemented by using the first order formulation of the supercoset sigma model on $G = PSU(2,2|4)$ and replacing the coupling of the Lagrange

multiplier which impose the zero curvature condition with a gauged G/G WZW action. The resulting model yields a deformed target superspace which always corresponds to a supergravity solution. On the other hand, η -deformation of the $AdS_5 \times S^5$ GS superstring [7–9] is a generalization of the Yang-Baxter (YB) deformation of the principal chiral model introduced in the context of Poisson-Lie duality [10]. For this deformations, the main ingredient is a linear operator on $\mathfrak{psu}(2,2|4)$ which solves the (modified) classical Yang-Baxter equation (m)CYBE

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = c[X, Y]. \quad (1.2)$$

Even though κ -symmetry is preserved for the η -deformation, its target superspace does not solve in general the equations of motion of type IIB supergravity [11–13] but rather a generalization of them which depend on two vector fields, K_a and X_a , instead of a scalar field ϕ . It was also argued that, even without a dilaton to preserve Weyl invariance at one-loop level, the generalized supergravity backgrounds still define a two-dimensional scale invariant theory [14]. This apparent conflict was solved by Tseytlin and Wulff [15], who showed that, contrary to the standard assumption that κ -symmetry implies the equations of motion of the type IIB supergravity, the constraints imposed on the target space by the requirement of kappa-symmetry leads, in fact, to the generalized supergravity equations of motion. In particular, they found that the vector fields K_a and X_a are subject to the relations (when the fermionic fields are set to zero)

$$\nabla_{(a} K_{b)} = 0, \quad X^a K_a = 0, \quad 2\nabla_{[a} X_{b]} + K^c H_{abc} = 0. \quad (1.3)$$

In this framework, the supergravity equations can be recovered when then killing vector K_a vanishes and $X_a = \nabla_a \phi$. In the context of the η -model, the condition $K_a = 0$ translates into an algebraic relation on the R -matrices, the so-called unimodular condition [16]

$$R^{AB} f_{AB}^C = 0. \quad (1.4)$$

where f_{AB}^C is the structure constant of $\mathfrak{psu}(2,2|4)$. The generalized backgrounds are also related to the standard supergravity equations by T -dualizing a supergravity target space in a isometric direction which is a symmetry of all the fields except for the dilaton transforming linearly in this direction [17, 18]. The relation between homogeneous Yang-Baxter deformation and T-duality has been extensively studied [19–22]. Remarkably it has been shown that these deformations and the (non-abelian) T-duality transformations of the original model can be formulated in a unified description [24–26]. The emergence of generalized supergravity

backgrounds has been further explored in the context of AdS/CFT correspondence and open-closed string map [27–33].

For pure spinors on the other side, the β -deformation, obtained by TsT transformations on the supergravity background [35], is a only known deformations of $AdS_5 \times S^5$. An attempt to λ -deform the $AdS_5 \times S^5$ space was performed in [34], by considering only the matter sector of the pure spinor model. This is an unsatisfactory scenario from many points of view; in the pure spinor formalism, the world-sheet metric is in the conformal gauge and the problematic κ -symmetry of the GS superstring is replaced by a global BRST symmetry, therefore it avoids the issues with the light-cone gauge. By exploiting these powerful tools many interesting problems have been overcome. For instance, it was proven that the theory remains conformally invariant at one loop level in perturbation theory [36]. Moreover, it was argued that conformal invariance and integrability of the pure spinor string in $AdS_5 \times S^5$ persists at all loops in the quantum theory [37]. In addition, several progress have been made to study vertex operators and correlation functions [38–40].

In this thesis, we consider the approach developed in [41] to deform the pure spinor model in $AdS_5 \times S^5$ in a BRST invariant way based on homological perturbation theory. In this construction, one starts with an infinitesimal deformation corresponding to the massless vertex operator

$$V[R](\varepsilon, \varepsilon') = R^{AB} (g^{-1}(\varepsilon \lambda_1 - \varepsilon \lambda_3)g)_A (g^{-1}(\varepsilon' \lambda_1 - \varepsilon' \lambda_3)g)_B, \quad (1.5)$$

parametrized by a constant antisymmetric tensor B^{AB1} , which belongs to the BRST cohomology of $AdS_5 \times S^5$. At the linearized level, the deformation of the action is given by its integrated vertex operator, namely V_1 , which is obtained from (1.5) by applying the standard descent procedure. Once V_1 is known, the deformed action S_{def} and the deformed BRST operator Q_{def} can be constructed as a series expansion in the deformation parameter η

$$S_{def} = S_0 + \eta \int V_1 + \eta^2 \int V_2 + \dots, \quad (1.6)$$

$$Q_{def} = Q_0 + \eta Q_1 + \eta^2 Q_2 + \dots. \quad (1.7)$$

In reference [41], this procedure was implemented to obtain the first two terms in the expansion (1.7) by considering the antifield formulation of the PS in string in $AdS_5 \times S^5$ [59] which allows to define a BSRT nilpotent operator off-shell. Further, this construction

¹The indices A and B are used for the adjoint representation of $\mathfrak{psu}(2,2|4)$, λ_1 and λ_3 are the usual Lie-algebra-valued pure spinor fields and ε and ε' are Grassmannian parameters.

was able to describe the β -deformation and opened the possibility to formulate pure spinor superstrings in backgrounds which do not correspond to any supergravity solution [43, 45]. Furthermore, the cohomology of the deformed BRST charge in the flat limit does not reproduce a supergravity background but rather a generalization involving a pair of vectors which satisfy the conditions [45]

$$\partial_{(M}A_{N)}^- = 0, \quad \partial_{[M}A_{N]}^+ = 0. \quad (1.8)$$

In the light of η deformations, these states could be interpreted as the pair of vectors arising in generalized supergravity. However, this endeavour is still incomplete and its relation to GS deformations not well understood.

In this thesis we will follow [41] to construct the full deformed pure spinor action [42]. We will show that the BRST charge expansion (1.7) stops at first order. Remarkably, the deformed BRST action on the matter sector possess the same structure as the η -deformed GS superstring, which allows the preservation of kappa symmetry [16]

$$\delta g = g\varepsilon, \quad \varepsilon = (1 - \eta R_g)\lambda_1 + (1 + \eta R_g)\lambda_3 \quad (1.9)$$

thus suggesting that the pure spinor deformed model reproduces the η -deformed background. More, its nilpotency holds only if the R -matrices solve the mCYBE. For the sake of clarity and simplicity we will work with R -matrices which solve the CYBE. The case $c \neq 0$ was also considered in [41] and could be constructed in a straightforward way.

A troublesome issue of this construction is that Q_1 acts through a non-local operator on the anti-field sector turning the full deformed action non-polynomial. We show that it is possible to remove the anti-field sector so that the complete deformed action is a polynomial expansion in the R -matrices. As a consequence the BRST operator has now an infinite series expansion in the ghost sector and its nilpotency holds only on-shell. Even so, this local action allows us to read the background superfields using the general action of Berkovits and Howe [54]. By rewriting the PS deformed action in terms of the GS deformed variables we show that both, the GS and the PS, η -deformed models have the same geometry and target space fields [42]

As it was found for the flat space limit of this deformation [43], BRST invariance is not enough to characterize the linearised equations of motion for type IIB supergravity. The solution for this conflict lies in conformal invariance of the deformed pure spinor action. In

order to preserve the conformal symmetry for the deformed worldsheet theory the vertex (1.5) must be given by a primary operator of $AdS_5 \times S^5$. This means that the double pole of the vertex operator with the energy-momentum tensor which is proportional to the action of the Laplacian on $\mathfrak{psu}(2,2|4)$ must vanishes [44]. In our case this requirement implies the unimodular condition on the R -matrix [41].

The η deformation of the GS superstring can be constructed in a way which manifestly preserves their integrability [8]. On the pure spinor side the main criterion to find the deformation is BRST invariance. Nevertheless it is possible to recast the *deformed* equations of motion into the same algebraic structure of the *undeformed* equations of motion allowing us to find a Lax representation for them [42].

Outline

This thesis is organized as follows. In Chapter 2 we introduce the pure spinor formalism for the superstring in the simplest case of flat ten dimensions. We explain the BRST cohomology equations for massless vertex operators. We recall the GS and pure spinor non-linear sigma models in a curved type II supergravity background and the relation between superspace constraints and consistency of the world-sheet theory. We present the equations of motion imposed on the ten-dimensional target superspace geometry by the requirement of classical kappa-symmetry of the Green-Schwarz superstring derived in [15]. We re-derive these results starting from the constraints imposed on the supergeometry by the nilpotency and holomorphicity of the BRST charge.

Chapter 3 is devoted to the coset sigma model of the $AdS_5 \times S^5$ superstring. We briefly discuss some relevant issues on supergroups and supercosets in order to construct the GS and PS action on $PSU(2,2|4)/SO(1,5) \times SO(6)$. Their equations of motion, Lax connections and symmetries are discussed. The construction of physical vertex operators for the PS string, corresponding to on-shell fluctuations on the $AdS_5 \times S^5$ background, is reviewed.

In Chapter 4 we present the Yang-Baxter deformation of the GS superstring. The deformed field equations, its integrability, and kappa-symmetry transformations are discussed. We follow [16] to derive the target space supergeometry by computing the superspace torsion and comparing with the standard results obtained in Chapter 2. We show how the unimodular condition arises in this context. We conclude this chapter by reviewing recent progress to understand the Yang-Baxter deformations in the context of (non)abelian T-duality and AdS/CFT correspondence.

In Chapter 5 we present the full deformation of the $AdS_5 \times S^5$ PS superstring. We show how to get rid of the awkward non-polynomial terms appearing in the original deformed action. We prove classical integrability of the deformed PS action by constructing a suitable Lax connection. At the end of this chapter, we make contact with the GS η -model and show that the target space superfields of our model correspond to those of η -deformation.

In Chapter 6 we summarize and discuss the possibilities for future work.

Chapter 2

The sigma model description of superstrings

The main purpose of this chapter is to introduce the pure spinor formalism for the superstring in a general ten-dimensional background. In the first part of this chapter, we present the salient features of the GS and pure spinor superstrings in the simple case of flat ten dimensional space. We review the GS non-linear sigma model in a curved background and the emergence of generalized supergravity. At the end of this chapter, we present the Berkovits-Howe action for the pure spinor superstring in a curved background. In addition, we will show that the constraints imposed on the target space supergeometry by the requirement of classical nilpotency of the BRST charge leads to the generalized supergravity equations of motion. In appendix A the reader can find the conventions and some important identities for the ten-dimensional gamma matrices.

2.1 Superstrings in flat space

The classical GS superstring on a flat space-time describes a two-dimensional sigma model mapping into the quotient manifold $SUSY(N=2)/SO(9,1)$ [47]. The two-dimensional space is parametrized by world-sheet coordinates σ^i ($i = 1, 2$), and the target space is parametrized by the ten-dimensional vector $X^a(\sigma)$ with $a = 0, \dots, 9$, and two Majorana-Weyl spinors $\theta^{\alpha I}$, with $I = 1, 2$ ¹ and $\alpha = 1, \dots, 16$. The action is given by

$$S_{GS} = -\frac{1}{4\pi\alpha'} \int d^2\sigma (L_{kin} + L_{WZ}), \quad (2.1)$$

¹Alternatively, we will name $\theta^{\alpha 1} = \theta^\alpha$ and $\theta^{\alpha 2} = \hat{\theta}^{\hat{\alpha}}$.

where

$$\begin{aligned} L_{kin} &= \sqrt{-g} g^{ij} \Pi_i \cdot \Pi_j, \\ L_{WZ} &= -2\varepsilon^{ij} (\bar{\theta}^1 \gamma^a \partial_i \theta^1) (\bar{\theta}^2 \gamma_a \partial_j \theta^2) + 2i\varepsilon^{ij} \partial_i X_a (\bar{\theta}^1 \gamma^a \partial_j \theta^1 - \bar{\theta}^2 \gamma^a \partial_j \theta^2), \end{aligned} \quad (2.2)$$

here $\Pi_i^a = \partial_i X^a - i\delta_{IJ} \bar{\theta}^I \gamma^a \partial_i \theta^J$, g^{ij} is the world-sheet metric and ε^{ij} is the two-dimensional skew-symmetric tensor. This is invariant under bosonic local reparametrizations and the equation of motion of g^{ij} give us the Virasoro constraints

$$T_{ij} = -\frac{1}{2} g_{ij} \Pi^k \cdot \Pi_k + \Pi_i \cdot \Pi_j, \quad (2.3)$$

which allows us to eliminate two bosonic degrees of freedom.

The insertion of the WZ term in (2.1) is needed for the action to be invariant under a fermionic local symmetry, known as κ -symmetry which allows to get rid of half of the fermionic propagating modes. This symmetry is given by the following transformations

$$\delta_\kappa x^a = 2i\theta^I \gamma^a \delta \theta^I, \quad (\delta_\kappa \Pi^a = 0), \quad (2.4)$$

$$\delta_\kappa \theta^I = 2i\gamma_a \kappa^{Ia}, \quad (2.5)$$

$$\delta_\kappa (\sqrt{-g} g^{ij}) = -16\sqrt{-g} (P_-^{il} \kappa^{1j} \partial_l \theta^1 + P_+^{il} \kappa^{2j} \partial_l \theta^2), \quad (2.6)$$

where $P_-^{ij} = \frac{1}{2} [g^{ij} - \frac{\varepsilon^{ij}}{\sqrt{-g}}]$ is a two-dimensional projector and κ_i^I is a space-time spinor which transforms as a vector under local reparametrizations. However, due to the nature of the constraints of the theory, the covariant quantization for this model is not possible; there is a pair of 16 primary constraints ($d_{\alpha I}$) that relates the conjugate momenta of $\theta^{\alpha I}$, namely $p_{\alpha I}$, with other phase space variables and combine 8 first-class and 8 second-class constraints. However, it is not possible to separate the first-class constraints in a covariant way; one is forced to fix kappa symmetry in the light cone gauge, thus losing manifest Lorentz covariance.

This failure motivated Siegel to formulate a superstring action with a set of only first-class constraints [69]. The full set of constraints would include the generators of reparametrizations and κ -transformations. In conformal gauge, the chiral generators are given by

$$A = -\frac{1}{2} \Pi^a \cdot \Pi_a - d_\alpha \partial \theta^\alpha, \quad B^\alpha = \Pi^a (\gamma_a)^{\alpha\beta} d_\beta, \quad (2.7)$$

and must be supplemented by other first-class generators to furnish a closed algebra which reproduces the superstring spectrum. The fundamental operators of this first-class constraints

must realize the current algebra

$$\Pi_a(z)\Pi_b(w) \sim -\frac{\eta_{ab}}{(z-w)^2}, \quad (2.8)$$

$$d_\alpha(z)d_\beta(w) \sim 2\frac{\gamma_{\alpha\beta}^a\Pi_a(w)}{z-w}, \quad (2.9)$$

$$d_\alpha(z)\Pi_a(w) \sim 2i\frac{(\gamma_a)_{\alpha\beta}\partial\theta^\beta(w)}{z-w}, \quad (2.10)$$

$$d_\alpha(z)\partial\theta^\beta(w) \sim -\frac{\delta_\alpha^\beta}{z-w}. \quad (2.11)$$

It is important to note that, as a consequence of (2.8)-(2.11), d_α works as a supersymmetric derivative when acting on superfields

$$d_\alpha(z)\Phi(x, \theta)(w) \sim \frac{D_\alpha\Phi(x, \theta)(w)}{z-w}, \quad (2.12)$$

where, $D_\alpha = \partial_\alpha + (\gamma^a\theta)_\alpha\partial_a$,

and can be solved for

$$d_{A\alpha} = p_{A\alpha} - (\gamma_a\theta)_{A\alpha}(i\partial x^a + \frac{1}{2}\theta^1\gamma^a\partial\theta^1 + \frac{1}{2}\theta^2\gamma^a\partial\theta^2). \quad (2.13)$$

The algebra (2.8)-(2.11) can be interpreted as reflecting the chiral symmetry of the gauge-fixed action

$$S_{Siegel} = -\frac{1}{4\pi\alpha'} \int d^2\sigma (\partial x^a \bar{\partial} x_a + p_\alpha \bar{\partial} \theta^\alpha + \hat{p}_{\hat{\alpha}} \partial \hat{\theta}^{\hat{\alpha}}), \quad (2.14)$$

Even though the Siegel's approach was implemented to precisely quantize the superparticle [70], the correct set of first-class constraints for reproducing the superstring spectrum was never found. Nevertheless, there are important issues we can learn from this approach. First of all, the action (2.14) describes a free conformal field theory with central charge (-22,-22). Moreover, taking into account (2.13), the action (2.14) can be written as

$$S_{Siegel} = S_{GS} - \frac{1}{4\pi\alpha'} \int d^2\sigma (d_\alpha \bar{\partial} \theta^\alpha + d_{\hat{\alpha}} \partial \hat{\theta}^{\hat{\alpha}}), \quad (2.15)$$

which coincides with the GS action if the global currents $d_{I\alpha}$ are constrained to be zero. However, the chiral currents $d_{I\alpha}$ do not realize a closed subalgebra, confirming their second-class nature, and usual BRST construction is forbidden. Berkovits overcame this problem

defining the following BRST operators [49]

$$Q_L = \int dz \lambda^\alpha d_\alpha, \quad Q_R = \int d\bar{z} \hat{\lambda}^{\hat{\alpha}} d_{\hat{\alpha}}, \quad (2.16)$$

where the bosonic spinorial ghosts λ^α and $\hat{\lambda}^{\hat{\alpha}}$ are constrained to satisfy the pure spinor condition

$$\lambda \gamma^a \lambda = \hat{\lambda} \gamma^a \hat{\lambda} = 0. \quad (2.17)$$

These ten-dimensional constraints are reducible and can be solved by decomposing λ^α and $\hat{\lambda}^{\hat{\alpha}}$ with regard to the $U(5)$ subgroup of $SO(9,1)$ (see, for example, [50]). As a result, the number of degrees of freedom of a pure spinor in ten dimensions are reduced to 11 complex components.

In addition, $d_{A\alpha}$ possesses conformal weight 1, therefore λ^α is a conformal field of weight 0. Hence, they must compose a β - γ system of conformal weight (1,0) with their conjugate momenta ω_α (of opposite chirality to λ^α). It follows that the central charge of the ghost system is $2 \times 11 = 22$ [49]. Hence, the pure spinor action

$$S_{PS} = -\frac{1}{4\pi\alpha'} \int d^2z (\partial x^a \bar{\partial} x_a + p_\alpha \bar{\partial} \theta^\alpha + \hat{p}_{\hat{\alpha}} \partial \hat{\theta}^{\hat{\alpha}} - \lambda^\alpha \bar{\partial} \omega_\alpha - \hat{\lambda}^{\hat{\alpha}} \partial \hat{\omega}_{\hat{\alpha}}), \quad (2.18)$$

describes a conformal field theory with vanishing central charge. The left and right-moving stress tensors are

$$T = \frac{1}{2} \partial x^a \partial x_a + p_\alpha \partial \theta^\alpha + T_\lambda, \quad \bar{T} = \frac{1}{2} \bar{\partial} x^a \bar{\partial} x_a + \bar{p}_{\hat{\alpha}} \bar{\partial} \hat{\theta}^{\hat{\alpha}} + \bar{T}_{\hat{\lambda}}, \quad (2.19)$$

where T_λ and $\bar{T}_{\hat{\lambda}}$ are the $c = 22$ stress-tensor of the ghost sector. The pure spinor constraints (2.17) are first-class and generate the gauge transformations

$$\delta \omega_\alpha = \Lambda^a (\gamma_a \lambda)_\alpha, \quad \delta \hat{\omega}_{\hat{\alpha}} = \Lambda^a (\gamma_a \hat{\lambda})_{\hat{\alpha}}. \quad (2.20)$$

Therefore, ω_α must appear only in gauge invariant combinations. These are the Lorentz currents $N_{ab} = \frac{1}{2} \omega \gamma_{ab} \lambda$ and the ghost number current $J_{gh} = \omega_\alpha \lambda^\alpha$, as well as the combinations $\lambda^\alpha \partial \omega_\alpha$ and $\lambda^\alpha \bar{\partial} \omega_\alpha$.

In the rest of this section we will show that the quantization of this model describes the correct spectrum of the type II superstrings by constructing the massless vertex operators for the cohomology of the BRST charge (2.16).

The massless states of the PS superstring are characterized by a vertex operator of ghost number two [49]

$$U = \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} A_{\alpha\hat{\alpha}}(x, \theta, \hat{\theta}), \quad (2.21)$$

where $A_{\alpha\hat{\alpha}}(x, \theta, \hat{\theta})$ is a bispinor which only depends on the matter fields. The cohomology condition for a physical vertex operator is imposed as

$$Q_L U = Q_R U = 0. \quad (2.22)$$

Exact vertex operators, which represent states that correspond to pure gauge transformations, take the form

$$\delta U = Q\Lambda + \hat{Q}\hat{\Lambda}, \quad (2.23)$$

for $\Lambda = \hat{\lambda}^{\hat{\alpha}} \hat{\Omega}_{\hat{\alpha}}$ and $\hat{\Lambda} = \lambda^\alpha \Omega_\alpha$ such that $Q_R \Lambda = Q_L \hat{\Lambda} = 0$. Demanding these conditions on (2.21), the bispinor $A_{\alpha\hat{\alpha}}$ must satisfy the equations

$$\gamma_{abcde}^{\alpha\beta} D_\beta A_{\alpha\hat{\alpha}} = \gamma_{abcde}^{\hat{\alpha}\hat{\beta}} D_{\hat{\beta}} A_{\alpha\hat{\alpha}} = 0, \quad (2.24)$$

and, the gauge invariance is re-written as

$$\delta A_{\alpha\hat{\alpha}} = D_\alpha \hat{\Omega}_{\hat{\alpha}} + D_{\hat{\alpha}} \Omega_\alpha, \quad (2.25)$$

such that $\gamma_{abcde}^{\alpha\beta} D_\beta \Omega_\alpha = \gamma_{abcde}^{\hat{\alpha}\hat{\beta}} D_{\hat{\beta}} A_{\hat{\alpha}} = 0$. This freedom can be used to find a gauge choice such that

$$\begin{aligned} A_{\alpha\hat{\alpha}} = & (\gamma^a \theta)_\alpha (\gamma^b \hat{\theta})_{\hat{\alpha}} h_{ab}(x) + (\gamma^a \theta)_\alpha (\hat{\theta} \gamma^{bcd} \hat{\theta}) (\gamma_{bcd} \hat{\Psi}_a(x))_{\hat{\alpha}} + \\ & + (\theta \gamma^{bcd} \theta) (\gamma^a \hat{\theta})_{\hat{\alpha}} (\gamma_{bcd} \Psi_a(x))_\alpha + (\hat{\theta} \gamma_{abc} \hat{\theta}) \gamma_{\beta\alpha}^{abc} (\theta \gamma_{def} \theta) \gamma_{\hat{\beta}\hat{\alpha}}^{def} F^{\beta\hat{\beta}} + \dots, \end{aligned} \quad (2.26)$$

where the dots above contain only auxiliary fields. It follows that the condition (2.22) imposes the linearized type IIB equations of motion on the background fields

$$\begin{aligned} \partial^a (\partial_a h_{bc} - \partial_b h_{ac}) &= \partial^a (\partial_a h_{bc} - \partial_c h_{ab}) = 0, \\ \partial^a (\partial_a \Psi_b^\alpha - \partial_b \Psi_a^\alpha) &= \partial^a (\partial_a \hat{\Psi}_b^{\hat{\alpha}} - \partial_b \hat{\Psi}_a^{\hat{\alpha}}) = 0, \\ \gamma_{\alpha\beta}^a \partial_a \Psi_b^\beta &= \gamma_{\hat{\alpha}\hat{\beta}}^a \partial_a \hat{\Psi}_b^{\hat{\beta}} = 0, \\ \gamma_{\alpha\beta}^a \partial_a F^{\beta\hat{\gamma}} &= \gamma_{\hat{\alpha}\hat{\beta}}^a \partial_a F^{\hat{\beta}\gamma} = 0. \end{aligned} \quad (2.27)$$

The tensor h_{ab} decomposes in irreducible representations of $SO(9, 1)$ as $h_{ab} = h_{(ab)} + h_{[ab]} + \delta_{ab} h$, and (2.27) give us the linearized equations of motion of the metric, the Kalb-Ramond

field and dilaton. Analogously, Ψ_a^α decomposes as $\Psi_a^\alpha = \psi_a^\alpha + \gamma_a^{\alpha\beta} \chi_\beta$, and describes the gravitino and dilatino, as well as the bispinor $F^{\beta\gamma}$ which describes the Ramond-Ramond fluxes:

$$F = \gamma^a \mathcal{F}_a + \frac{1}{3!} \gamma^{abc} \mathcal{F}_{abc} + \frac{1}{2.5!} \gamma^{abcde} \mathcal{F}_{abcde}. \quad (2.28)$$

In our discussion on superstrings in $AdS_5 \times S^5$ we will demand that the vertex operator to be a world-sheet primary field. In a flat background, the condition of no double poles of the vertex operator (2.21) with the energy-momentum tensors (2.19) implies that

$$\partial^a \partial_a A_{\alpha\hat{\alpha}} = 0. \quad (2.29)$$

The remaining gauge freedom which leaves (2.29) invariant is given by

$$\delta A_{\alpha\hat{\alpha}} = D_\alpha \hat{\Omega}_{\hat{\alpha}} + D_{\hat{\alpha}} \Omega_\alpha, \quad (2.30)$$

such that,

$$\partial^m \partial_m \hat{\Omega}_{\hat{\alpha}} = \partial^m \partial_m \Omega_\alpha = 0. \quad (2.31)$$

In many instances, we also need vertex operators in integrated form $\int d^2z V$. In bosonic strings V is usually obtained from the unintegrated vertex operator by anti-commuting with the b ghost, which is required for making the energy-momentum tensor BRST trivial. Since there is no fundamental b field in this formalism, we use an alternative procedure for constructing V where it can be written as left-right product of the open superstring vertex operators [49]

$$\begin{aligned} V = \int d^2z [& \Pi^a \bar{\Pi}^b A_{ab}(x, \theta^{\alpha I}) + \partial \theta^\alpha \bar{\partial} \hat{\theta}^{\hat{\alpha}} A_{\alpha\hat{\alpha}}(x, \theta^{\alpha I}) + \Pi^a \bar{\partial} \hat{\theta}^{\hat{\alpha}} A_{a\hat{\alpha}}(x, \theta^{\alpha I}) + \partial \theta^\alpha \bar{\Pi}^a A_{a\alpha}(x, \theta^{\alpha I}) \\ & + d_\alpha (\bar{\partial} \hat{\theta}^{\hat{\alpha}} A_{\hat{\alpha}}^\alpha(x, \theta^{\alpha I}) + \Pi^a A_a^\alpha(x, \theta^{\alpha I})) + d_{\hat{\alpha}} (\partial \theta^\alpha A_\alpha^{\hat{\alpha}}(x, \theta^{\alpha I}) + \Pi^a A_a^{\hat{\alpha}}(x, \theta^{\alpha I})) + d_\alpha d_{\hat{\alpha}} P^{\alpha\hat{\alpha}}(x, \theta^{\alpha I}) \\ & + N_{ab} (\bar{\partial} \hat{\theta}^{\hat{\alpha}} \Omega_{\hat{\alpha}}^{ab}(x, \theta^{\alpha I}) + \bar{\Pi}^c \Omega_c^{ab}(x, \theta^{\alpha I})) + \bar{N}_{ab} (\partial \theta^\alpha \Omega_\alpha^{ab}(x, \theta^{\alpha I}) + \Pi^c \bar{\Omega}_c^{ab}(x, \theta^{\alpha I})) \\ & + \bar{N}_{ab} \hat{d}_{\hat{\gamma}} \hat{C}^{ab\hat{\gamma}}(x, \theta^{\alpha I}) + \bar{N}_{ab} d_\gamma C^{ab\gamma}(x, \theta^{\alpha I}) + N_{ab} \bar{N}_{cd} S^{abcd}(x, \theta^{\alpha I})]. \end{aligned} \quad (2.32)$$

The sigma model for the Type II superstring can be constructed starting from the flat action and adding the integrated vertex operator.

2.2 Superstrings in curved backgrounds

Now that we have introduced the main features of the GS and PS superstrings in flat space, we are ready to generalize these formulations to a curved target superspace.

2.2.1 Green-Schwarz sigma model

Let us try to discuss the most general way to construct a superstring sigma model as a direct generalization of the flat Green-Schwarz action we described above. Considering a target superspace provided with curved superspace coordinates $Z^M = \{x^m, \theta^\alpha, \hat{\theta}^{\hat{\alpha}}\}$, where just as we considered before, the Grassmann fermionic coordinates are Majorana-Weyl spinors of the same chirality for type IIB superstring, and $M = (m, \alpha, \hat{\alpha})$; $m = 0, \dots, 9$. The Green-Schwarz Type II superstring can be naturally extended to curved backgrounds [48]

$$S_{GS} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{-g} g^{ij} G_{MN}(Z) + \epsilon^{ij} B_{NM}(Z) \right) \partial_i Z^M \partial_j Z^N. \quad (2.33)$$

where, G_{MN} and B_{NM} are background superfields. The action presented above represents a generalization of the action (2.1) we constructed before. The first term is the kinetic term while the second one corresponds to the Wess-Zumino term.

As we mentioned before, the superspace can be regarded as a supermanifold, so at every point Z we can define a tangent superspace with flat metric η_{ab} and a cotangent superspace. The latter admits coordinate dual basis $\{dZ^M\}$ and orthonormal basis $\{E^A\}$, where $A = (a, \alpha, \hat{\alpha})$ with $a = 0, \dots, 9$ and $\alpha, \hat{\alpha} = 1, \dots, 16$ are indices on the tangent superspace. The change of basis define the *supervielbein* E_M^A as follows

$$E^A = E_M^A dZ^M. \quad (2.34)$$

The superfield G_{MN} represents a generalization of the metric to the superspace, then we can write

$$G_{MN}(Z) = E_M^a(Z) E_N^b(Z) \eta_{ab}. \quad (2.35)$$

On the other hand, in a generic supergravity background the Wess-Zumino term is given by the 2-superform

$$B = \frac{1}{2} B_{MN}(Z) dZ^M \wedge dZ^N = \frac{1}{2} B_{AB} E^A \wedge E^B, \quad (2.36)$$

where

$$B_{MN}(Z) = E_M^A(Z) E_N^B(Z) B_{AB}(Z). \quad (2.37)$$

The Wess-Zumino term can be written as an integral of the 2-superform B as follows

$$S_{WZ} = \frac{1}{2\pi\alpha'} \int B = \frac{1}{4\pi\alpha'} \int B_{MN} dZ^M \wedge dZ^N. \quad (2.38)$$

The world-sheet pullback of (2.34) reads as $E_i^A = E_M^A \partial_i Z^M$, so using equations (2.35) and (2.38), the Green-Schwarz sigma model action can be written as follows

$$S_{GS} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{-g} g^{ij} \eta_{ab} E_i^a E_j^b + \epsilon^{ij} B_{AB} E_i^A E_j^B \right). \quad (2.39)$$

Varying the action (2.39) with respect g^{ij} we find the Virasoro constraints

$$T_{ij} = \frac{1}{2} g_{ij} (g^{lm} E_l^a E_m^b \eta_{ab}) - E_i^a E_j^b \eta_{ab} = 0 \quad (2.40)$$

Defining $G_{ij} = E_i^a E_j^b \eta_{ab}$, we can use the Virasoro constraints to show that that

$$\frac{G_{ij}}{\sqrt{-G}} = \frac{1}{2} \frac{g_{ij}}{\sqrt{-g}}. \quad (2.41)$$

This relation allows us to eliminate the world-sheet metric from the superstring action. The classical GS is written in the Nambu-Goto form as

$$S_{GS} = -\frac{1}{4\pi\alpha'} \left(\int d^2\sigma \sqrt{-G} - \int B \right). \quad (2.42)$$

In order to get the correct number of degrees of freedom, the type IIB GS action is required to be invariant under the following κ -transformations

$$\begin{aligned} \delta_\kappa z^M E_M^a &= 0, \\ \delta_\kappa z^M E_M^{\alpha I} &= \frac{1}{2} (1 + \Gamma)^{\alpha I}{}_{\beta J} k^{\beta J}, \quad \Gamma = \frac{1}{2\sqrt{-G}} \epsilon^{ij} E_i^a E_j^b \gamma_{ab} \sigma^3, \end{aligned} \quad (2.43)$$

here the Pauli matrix σ^3 acts on the indices $I = 1, 2$. Varying the action one obtain

$$\delta_\kappa S = \frac{1}{4\pi\alpha'} \int d^2\sigma \delta_\kappa z^M E_M^{\alpha I} \left(\sqrt{-G} G^{ij} \eta_{ab} E_i^a E_j^c T_{C\alpha I}^b + \frac{1}{2} \epsilon^{ij} E_i^c E_j^B H_{BC\alpha I} \right), \quad (2.44)$$

where $T^A = dE^A + E^B \wedge \Omega_B^A$ is the torsion and $H = dB$. The structure group is $SO(9, 1)$ and the non-zero components of the super-connection are Ω_a^b and $\Omega_\alpha^\beta = \frac{1}{4} \delta_{IJ} (\gamma^{ab})_\alpha^\beta \Omega_{ab}$.

So, the requirement of κ -symmetry imposes dynamical constraints on the background. By using (2.43) it is possible to show that (3.69) vanishes if

$$H_{\alpha I \beta J \gamma K} = 0, \quad T_{\alpha I \beta J}^a = \delta_{IJ} \gamma_{\alpha\beta}^a, \quad H_{a\alpha I \beta J} = \sigma_{IJ}^3 (\gamma_a)_{\alpha\beta}. \quad (2.45)$$

As it was shown in [15] these constraints do not imply in general, the type IIB supergravity equations of motion but a weaker set of equations involving, instead of a dilaton, two vector fields K_a and X_a . This is not totally surprising as the condition of classical κ -symmetry does not take into account the Fradkin-Tseytlin term, $\int d^2\sigma R^2\Phi$, required for making the momentum-tensor traceless. The corresponding equations of generalized supergravity can be written as [15]

$$2\nabla_{[a}X_{b]} + K^c H_{abc} + \psi_{ab}\chi = 0, \quad (2.46)$$

$$\nabla_{(a}K_{b)} = 0, \quad (2.47)$$

$$K^a X_a + \frac{1}{4}\chi\gamma^a\sigma^3\nabla_a\chi - \frac{1}{96}\chi\gamma^{abc}\chi H_{abc} = 0, \quad (2.48)$$

$$\nabla^c H_{abc} - 4\nabla_{[a}X_{b]} - 2X^c H_{abc} - 2\psi_{ab}\chi - \frac{1}{64}\text{Tr}(\mathcal{P}\gamma_a\mathcal{P}\gamma_b) = 0, \quad (2.49)$$

$$R_{ab} + 2\nabla_{(a}X_{b)} - \frac{1}{4}H_{acd}H_b^{cd} + \frac{1}{128}\text{Tr}(\mathcal{P}\gamma_a\mathcal{P}\gamma_b) = 0, \quad (2.50)$$

$$\begin{aligned} \nabla^a X_a - 2X_a X^a - 2K_a K^a + \frac{1}{12}H_{abc}H^{abc} + \chi\gamma^a\nabla_a\chi - \frac{1}{3}\chi\gamma^{abc}\chi H_{abc} \\ - \frac{1}{256}\text{Tr}(\mathcal{P}\gamma_a\mathcal{P}\gamma^a) - \chi\gamma^a\nabla_a\chi + \frac{1}{24}\chi\gamma^{abc}\sigma^3\chi H_{abc} = 0, \end{aligned} \quad (2.51)$$

$$\begin{aligned} (\gamma^a\nabla_a\mathcal{P})_{\alpha I}{}^{\beta J} - (\gamma^a(X_a + \sigma^3 K_a)\mathcal{P})_{\alpha I}{}^{\beta J} + \left(\frac{1}{8}(\gamma^a\sigma^3\gamma^{bc}\mathcal{P})_{\alpha I}{}^{\beta J}\right) + \frac{1}{24}(\gamma^{abc}\sigma^3\mathcal{P})_{\alpha I}{}^{\beta J})H_{abc} \\ - \chi_{\alpha I}(\mathcal{P}\chi)^{\beta J} + (\sigma^3\chi)_{\alpha I}(\sigma^3\mathcal{P}\chi)^{\beta J} + 2(\gamma^{cd}\chi)_{\alpha I}\psi_{cd}^{\beta J} - 2(\gamma^{cd}\sigma^3\chi)_{\alpha I}(\sigma^3\psi_{cd})^{\beta J} = 0. \end{aligned} \quad (2.52)$$

Here, $\mathcal{P}^{\alpha I\hat{\alpha}J}$ are the analog of the Ramond-Ramond bispinor which decomposes as

$$\mathcal{P} = \gamma^a \mathcal{F}_a + \frac{1}{3!}\gamma^{abc}\mathcal{F}_{abc} + \frac{1}{2.5!}\gamma^{abcde}\mathcal{F}_{abcde}. \quad (2.53)$$

It is worth noting that when the fermionic components fields are set to zero the whole bosonic background $(G, H, \mathcal{F})$ remains K_a -isometric. For instance, canceling the fermionic fields for (2.46)-(2.48) we have that

$$2\nabla_{[a}X_{b]} + K^c H_{abc} = 0, \quad \nabla_{(a}K_{b)} = 0, \quad K^a X_a = 0, \quad (2.54)$$

which imply that $d(\mathcal{L}_K B) = \mathcal{L}_K dB = \mathcal{L}_K H = 0$. On the other hand $\mathcal{L}_K X = i_K dX = i_K i_K H = 0$.

By choosing, via gauge transformation, an isometric B -field, the equation (2.54) can be solved for X as

$$X_a = \partial_a \phi - B_{ab} K^b, \quad K^m \partial_m \phi = 0, \quad (2.55)$$

It leads us to consider that the generalized supergravity equations of motions involve the standard fields of type IIB supergravity plus an extra Killing vector K^a . They reduce to the standard type IIB supergravity equations if

$$K_a = 0, \quad X_a = \nabla_a \phi. \quad (2.56)$$

On the other hand, the bosonic contribution of the equation (2.52) can be written in terms of (2.53) as

$$\nabla_{[a_1} \mathcal{F}_{a_2 \dots a_{2n+2}]} + \frac{2n+1}{2} T_{[a_1 a_2}{}^b \mathcal{F}_{b|a_3 \dots a_{2n+2}]} - X_{[a_1} \mathcal{F}_{a_2 \dots a_{2n+2}]} \quad (2.57)$$

$$- \frac{1}{2n+2} K^b \mathcal{F}_{ba_1 \dots a_{2n+2}} + \frac{2n(2n+1)}{3!} H_{[a_1 a_2 a_3} \mathcal{F}_{a_4 \dots a_{2n+2}]} = 0. \quad (2.58)$$

or, in curved space coordinates,

$$d\mathcal{F}_{2n+1} + X \wedge \mathcal{F}_{2n+1} - H \wedge \mathcal{F}_{2n-1} - i_K \mathcal{F}_{2n+3} = 0. \quad (2.59)$$

And taking into account that $dX + i_K H = 0$, it is possible to show that $\mathcal{L}_K \mathcal{F}_{2n+1} = (i_K X) \mathcal{F}_{2n+1} = 0$. Moreover, (2.58) reproduce the same equation of motion for the RR fluxes which must be obeyed to preserve scale invariance of the GS superstring on a curved background [14].

These results can be generalized to superspace by lifting the Killing vector K_a and the form fields $\mathcal{F}$ to superspace vector field and superspace forms. This is done by lifting the one-form X_m to a super-one-form $X = X_M dz^M$ in such a way that

$$X_{\alpha I} = \chi_{\alpha I}. \quad (2.60)$$

The superisometries are generated by the Killing super-vector field $K^A = (K^a, \Theta^{\alpha I})$ where

$$\Theta^{\alpha I} = \frac{1}{4} (\gamma^a \nabla_a - 2\gamma^a (X_a - \sigma^3 K_a) - \frac{1}{24} \gamma^{abc} \sigma^3 H_{abc} - \frac{1}{4} \mathcal{P}) \sigma^3 \chi, \quad (2.61)$$

and the Ramond-Ramond form fields can be lifted to superspace and setting

$$\mathcal{F}'_{a_1 \dots a_n} = \mathcal{F}_{a_1 \dots a_n} - \chi^1 \gamma_{a_1 \dots a_n} \chi^2, \quad (2.62)$$

meanwhile, one has to impose the following constraints

$$\mathcal{F}_{\alpha I \beta J c} = \sigma_{ij}^1(\gamma_c)_{\alpha\beta}, \quad \mathcal{F}_{\alpha I \beta J c d e} = -\sigma_{ij}^2(\gamma_{c d e})_{\alpha\beta}, \quad (2.63)$$

$$\mathcal{F}_{\alpha I} = (\sigma^2 \chi)_{\alpha I}, \quad \mathcal{F}_{\alpha I b c} = -(\sigma^1 \gamma_{bc} \chi)_{\alpha I}, \quad \mathcal{F}_{\alpha I b c d e} = (\sigma^2 \gamma_{bc d e} \chi)_{\alpha I}. \quad (2.64)$$

One can show that $\mathcal{F}'$ satisfies the superspace version of the equations (2.58) [15].

2.2.2 Pure spinor superstring in curved backgrounds

The type IIB pure spinor superstring is described by the standard Berkovits-Howe action [54] which can be constructed by adding the massless vertex operator of (2.32) to the flat action and then covariantizing with respect to $N = 2, D = 10$ super-reparameterization invariance

$$\begin{aligned} S_{BH} = & \frac{1}{2\pi\alpha'} \int dz^2 \left(\frac{1}{2} E^a \bar{E}^b \eta_{ab} + \frac{1}{2} E^A \bar{E}^B B_{AB} + d_\alpha \bar{E}^\alpha + d_{\hat{\alpha}} \bar{E}^{\hat{\alpha}} + d_\alpha d_{\hat{\alpha}} P^{\alpha\hat{\alpha}} + \right. \\ & \left. + \Omega_\alpha^\beta \lambda^\alpha \omega_\beta + \hat{\Omega}_{\hat{\alpha}}^{\hat{\beta}} \hat{\lambda}^{\hat{\alpha}} \hat{\omega}_{\hat{\beta}} + \lambda^\alpha \omega_\beta \hat{d}_{\hat{\gamma}} C_\alpha^{\beta\hat{\gamma}} + \hat{\lambda}^{\hat{\alpha}} \hat{\omega}_{\hat{\beta}} d_{\hat{\gamma}} \tilde{C}_{\hat{\alpha}}^{\hat{\beta}\gamma} + \lambda^\alpha \omega_\beta \hat{\lambda}^{\hat{\alpha}} \hat{\omega}_{\hat{\beta}} S_{\alpha\hat{\alpha}}^{\beta\hat{\beta}} + S_{gh} \right). \end{aligned} \quad (2.65)$$

This is the most general action which possesses BRST symmetry, classical world-sheet conformal invariance and zero ghost number. Here, E^A and $(\Omega_\alpha^\beta, \hat{\Omega}_{\hat{\alpha}}^{\hat{\beta}})$ are the supervielbeins, and the left and right-moving spin connection. $A = (a, \alpha, \hat{\alpha})$ is a tangent space index. The action also includes the ghosts $(\lambda^\alpha, \omega_\beta, \hat{\lambda}^{\hat{\alpha}}, \hat{\omega}_{\hat{\beta}})$ and the world-sheet auxiliary fields $(d_\alpha, d_{\hat{\alpha}})$. These world-sheet fields are coupled through target space superfields. The superfield B_{AB} is a superspace two-form. The leading component of $P^{\alpha\hat{\alpha}}$ is the Ramond-Ramond bispinor. The $(C_\alpha^{\beta\hat{\gamma}}, \tilde{C}_{\hat{\alpha}}^{\hat{\beta}\gamma})$ are related to the gravitino and dilatino, and $S_{\alpha\hat{\alpha}}^{\beta\hat{\beta}}$ is related to the Riemann curvature.

Classically, the BRST transformations does not take into account the Fradkin-Tseytlin term $\int d^2z \Phi(Z) R$ required to preserve Weyl invariance quantum mechanically. We will comeback to this point at the end of this chapter.

As stated above, the pair $(d_\alpha, d_{\hat{\alpha}})$ are auxiliary fields and can be integrated out when $P^{\alpha\hat{\alpha}}$ is invertible. Defining its inverse as $P_{\alpha\hat{\alpha}} P^{\beta\hat{\beta}} = \delta_\alpha^\beta$, the equations of motion for d_α and $\hat{d}_{\hat{\alpha}}$ give us

$$d_\alpha = P_{\alpha\hat{\alpha}} (E^{\hat{\alpha}} + \lambda^\rho \omega_\beta C_\rho^{\beta\hat{\alpha}}), \quad (2.66)$$

$$\hat{d}_{\hat{\alpha}} = -P_{\alpha\hat{\alpha}} (\bar{E}^\alpha + \hat{\lambda}^{\hat{\rho}} \hat{\omega}_{\hat{\beta}} \tilde{C}_{\hat{\rho}}^{\hat{\beta}\alpha}). \quad (2.67)$$

The pure spinor constraints $\lambda \gamma^a \lambda = 0$ and $\hat{\lambda} \gamma^a \hat{\lambda} = 0$ enforce the relations

$$\begin{aligned} (\gamma^{abcd})_{\beta}^{\alpha} \Omega_{\alpha}^{\beta} &= (\gamma^{abcd})_{\beta}^{\alpha} C_{\alpha}^{\hat{\rho}\beta} = (\gamma^{abcd})_{\hat{\beta}}^{\hat{\alpha}} \tilde{\Omega}_{\hat{\alpha}}^{\hat{\beta}} = (\gamma^{abcd})_{\hat{\beta}}^{\hat{\alpha}} C_{\hat{\alpha}}^{\rho\hat{\beta}} = 0, \\ (\gamma^{abcd})_{\beta}^{\alpha} S_{\alpha\hat{\alpha}}^{\beta\hat{\beta}} &= (\gamma^{abcd})_{\hat{\beta}}^{\hat{\alpha}} S_{\alpha\hat{\alpha}}^{\beta\hat{\beta}} = 0. \end{aligned} \quad (2.68)$$

As a consequence of these constraints, the spin connections decompose as

$$\Omega_{\alpha}^{\beta} = \Omega \delta_{\alpha}^{\beta} + \frac{1}{4} \Omega^{ab} (\gamma_{ab})_{\alpha}^{\beta}, \quad \tilde{\Omega}_{\hat{\alpha}}^{\hat{\beta}} = \tilde{\Omega} \delta_{\hat{\alpha}}^{\hat{\beta}} + \frac{1}{4} \tilde{\Omega}^{ab} (\gamma_{ab})_{\hat{\alpha}}^{\hat{\beta}}. \quad (2.69)$$

The action (2.65) is also invariant under the local gauge transformations

$$\begin{aligned} \delta E_M^a &= \eta_{bc} \Lambda^{ab} E_M^c, \quad \delta E_M^{\alpha} = \Sigma_{\beta}^{\alpha} E_M^{\beta}, \quad \delta E_M^{\hat{\alpha}} = \hat{\Sigma}_{\hat{\beta}}^{\hat{\alpha}} E_M^{\hat{\beta}}, \\ \delta \Omega_{M\beta}^{\alpha} &= \partial_M \Sigma_{\beta}^{\alpha} + \Sigma_{\beta}^{\gamma} \Omega_{M\gamma}^{\alpha} - \Sigma_{\gamma}^{\alpha} \Omega_{M\beta}^{\gamma}, \quad \delta \tilde{\Omega}_{M\hat{\beta}}^{\hat{\alpha}} = \partial_M \hat{\Sigma}_{\hat{\beta}}^{\hat{\alpha}} + \hat{\Sigma}_{\hat{\beta}}^{\gamma} \tilde{\Omega}_{M\gamma}^{\hat{\alpha}} - \hat{\Sigma}_{\gamma}^{\hat{\alpha}} \tilde{\Omega}_{M\hat{\beta}}^{\gamma}, \\ \delta \lambda^{\alpha} &= \Sigma_{\beta}^{\alpha} \lambda^{\beta}, \quad \delta \omega_{\alpha} = -\Sigma_{\alpha}^{\beta} \omega_{\beta}, \quad \delta \hat{\lambda}^{\hat{\alpha}} = \hat{\Sigma}_{\hat{\beta}}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}}, \quad \delta \hat{\omega}_{\hat{\alpha}} = -\hat{\Sigma}_{\hat{\alpha}}^{\hat{\beta}} \hat{\omega}_{\hat{\beta}}, \end{aligned} \quad (2.70)$$

where $\Sigma_{\alpha}^{\beta} = \Sigma^{(s)} \delta_{\alpha}^{\beta} + \frac{1}{2} \Sigma^{ab} (\gamma_{ab})_{\alpha}^{\beta}$ and $\hat{\Sigma}_{\hat{\alpha}}^{\hat{\beta}} = \hat{\Sigma}^{(s)} \delta_{\hat{\alpha}}^{\hat{\beta}} + \frac{1}{2} \hat{\Sigma}^{ab} (\gamma_{ab})_{\hat{\alpha}}^{\hat{\beta}}$. It is important to remark that Λ^{ab} , $\hat{\Sigma}_{\hat{\alpha}}^{\hat{\beta}}$ and Σ_{α}^{β} parametrize independent transformations on the vector and spinor sectors respectively. Moreover, the action (2.65) is invariant under the shift symmetry

$$\begin{aligned} \delta \Omega_{\alpha}^s &= (\gamma)_{\alpha\beta} h^{a\beta}, \quad \delta \Omega_{\alpha}^{ab} = 2(\gamma^{[a})_{\alpha\beta} h^{b]\beta}, \quad \delta d_{\alpha} = \delta \Omega_{\alpha\beta}^{\gamma} \lambda^{\beta} \omega_{\gamma}, \\ \delta \tilde{\Omega}_{\hat{\alpha}}^s &= (\gamma)_{\hat{\alpha}\hat{\beta}} \tilde{h}^{a\hat{\beta}}, \quad \delta \tilde{\Omega}_{\hat{\alpha}}^{ab} = 2(\gamma^{[a})_{\hat{\alpha}\hat{\beta}} \tilde{h}^{b]\hat{\beta}}, \quad \delta d_{\hat{\alpha}} = \delta \tilde{\Omega}_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} \hat{\lambda}^{\hat{\beta}} \hat{\omega}_{\hat{\gamma}}, \\ \delta C_{\alpha}^{\beta\hat{\gamma}} &= P^{\sigma\hat{\gamma}} \delta \Omega_{\sigma\alpha}^{\beta}, \quad \delta \tilde{C}_{\hat{\alpha}}^{\hat{\beta}\gamma} = -P^{\sigma\hat{\gamma}} \delta \tilde{\Omega}_{\hat{\sigma}\hat{\alpha}}^{\hat{\beta}}, \quad \delta S_{\beta\hat{\rho}}^{\gamma\hat{\sigma}} = \tilde{C}_{\hat{\gamma}}^{\hat{\sigma}\varepsilon} \delta \Omega_{\varepsilon\alpha}^{\beta} + C_{\alpha}^{\beta\varepsilon} \delta \Omega_{\hat{\varepsilon}\hat{\gamma}}^{\hat{\sigma}}. \end{aligned} \quad (2.71)$$

The left and right energy-momentum tensors for the pure spinor action (2.65) read as follows

$$T = -\frac{1}{2} \Pi^a \Pi_a - d_{\alpha} \Pi^{\alpha} - \omega_{\alpha} \nabla \lambda^{\alpha}, \quad (2.72)$$

$$\bar{T} = -\frac{1}{2} \bar{\Pi}^a \bar{\Pi}_a - \hat{d}_{\hat{\alpha}} \hat{\Pi}^{\hat{\alpha}} - \hat{\omega}_{\hat{\alpha}} \bar{\nabla} \hat{\lambda}^{\hat{\alpha}}. \quad (2.73)$$

In order that the action (2.65) presents BRST symmetry, we must impose some constraints on the background fields [54]. These constraints allow us to solve the Bianchi identities in order to obtain the equations of motion satisfied by the background fields.

Let us recall that the torsion T^A , the curvature R_A^B and the 3-form H are defined as

$$T^A = dE^A + E^B \wedge \Omega_B^A, \quad R_B^A = d\Omega_B^A + \Omega_B^C \wedge \Omega_C^A, \quad H = dB, \quad (2.74)$$

satisfy the Bianchi identities

$$\nabla T^A = E^B \wedge R_B^A, \quad \nabla R_B^A = 0, \quad dH = 0. \quad (2.75)$$

In components they read as

$$(\nabla T + TT)_{ABC}{}^D \equiv \nabla_{[A} T_{BC]}{}^D + T_{[AB}{}^E T_{E|C]}{}^D - R_{[ABC]}{}^D = 0, \quad (2.76)$$

$$(\nabla R + TR)_{ABC}{}^D \equiv \nabla_{[A} R_{BC]}{}^D + T_{[AB}{}^E R_{E|C]}{}^D = 0, \quad (2.77)$$

$$(\nabla H + TH)_{ABCD} \equiv \nabla_{[A} H_{BCD]} + \frac{3}{2} T_{[AB}{}^E H_{E|CD]} = 0. \quad (2.78)$$

The BRST charge is defined through the BRST currents j_B and $\hat{j}_B$ as

$$Q_L = \oint j_B, \quad j_B = \lambda^\alpha d_\alpha, \quad (2.79)$$

$$Q_R = \oint \hat{j}_B, \quad \hat{j}_B = \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}. \quad (2.80)$$

Following [54], we have that nilpotency of the BRST charge Q_L implies the following conditions

$$\lambda^\alpha \lambda^\beta T_{\alpha\beta}^A = \lambda^\alpha \lambda^\beta H_{A\alpha\beta} = \lambda^\alpha \lambda^\beta \lambda^\gamma R_{\alpha\beta\gamma}{}^\rho = \lambda^\alpha \lambda^\beta \hat{R}_{\alpha\beta\gamma}{}^\rho = 0. \quad (2.81)$$

The nilpotency of Q_R implies that

$$\hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} T_{\hat{\alpha}\hat{\beta}}^A = \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} H_{A\hat{\alpha}\hat{\beta}} = \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} \hat{\lambda}^{\hat{\gamma}} \hat{R}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^{\hat{\rho}} = \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} \hat{R}_{\hat{\alpha}\hat{\beta}\gamma}{}^{\hat{\rho}} = 0. \quad (2.82)$$

And the nilpotency of Q_L and Q_R implies that

$$T_{\hat{\alpha}\beta}^A = H_{A\hat{\alpha}\beta} = \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} \hat{R}_{\gamma\hat{\alpha}\hat{\beta}}{}^{\hat{\rho}} = \lambda^\alpha \lambda^\beta R_{\gamma\alpha\beta}{}^\rho = 0. \quad (2.83)$$

In addition, the holomorphicity of the BRST left-current ($\bar{\partial} j_B = 0$) implies that

$$\begin{aligned} T_{\alpha(ab)} &= H_{\alpha ab} = T_{\alpha\beta a} - H_{\alpha\beta a} = \lambda^\alpha \lambda^\beta R_{a\alpha\beta\gamma}{}^\rho = T_{a\alpha}{}^{\hat{\beta}} + P^{\gamma\hat{\beta}} T_{\gamma\alpha a} = 0, \\ \tilde{R}_{a\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}} - \tilde{C}_{\hat{\beta}}{}^{\gamma\hat{\rho}} T_{\hat{\rho}\hat{\alpha}a} &= T_{\hat{\gamma}\hat{\alpha}}{}^{\hat{\beta}} + \frac{1}{2} P^{\beta\hat{\rho}} H_{\hat{\rho}\hat{\alpha}\hat{\beta}} = \hat{R}_{\rho\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}} + \frac{1}{2} \hat{C}_{\hat{\beta}}{}^{\hat{\gamma}\epsilon} H_{\epsilon\rho\alpha} = \nabla_\alpha P^{\gamma\hat{\beta}} + C_\alpha{}^{\beta\hat{\gamma}} - P^{\rho\hat{\beta}} T_{\rho\alpha}{}^{\hat{\beta}} = 0, \\ \nabla_\alpha \tilde{C}_{\hat{\beta}}{}^{\hat{\gamma}\rho} + \tilde{C}_{\hat{\beta}}{}^{\hat{\gamma}\sigma} T_{\sigma\alpha}{}^\rho + P^{\rho\hat{\sigma}} \tilde{R}_{\hat{\sigma}\alpha\hat{\beta}}{}^{\hat{\gamma}} - S_{\alpha\hat{\beta}}{}^{\rho\hat{\gamma}} &= \lambda^\alpha \lambda^\beta (\nabla_\alpha C_\beta{}^{\gamma\hat{\rho}} - P^{\sigma\hat{\rho}} R_{\sigma\alpha\beta}{}^\gamma) = 0, \\ \lambda^\alpha \lambda^\beta (\nabla_\alpha S_{\beta\hat{\rho}}{}^{\gamma\hat{\sigma}} + C_{\hat{\beta}}{}^{\gamma\hat{\epsilon}} \tilde{R}_{\hat{\epsilon}\alpha\hat{\rho}}{}^{\hat{\sigma}} + \tilde{C}_{\hat{\rho}}{}^{\hat{\sigma}\epsilon} R_{\epsilon\alpha\hat{\rho}}{}^\gamma) &= 0. \end{aligned} \quad (2.84)$$

And the anti-holomorphicity of the BRST right-current ($\partial \bar{j}_B = 0$) implies that

$$\begin{aligned} T_{\hat{\alpha}(ab)} &= H_{\alpha ab} = T_{\hat{\alpha}\hat{\beta}a} + H_{\hat{\alpha}\hat{\beta}a} = \lambda^{\hat{\alpha}} \lambda^{\hat{\beta}} \tilde{R}_{a\hat{\alpha}\hat{\beta}}^{\hat{\rho}} = T_{a\hat{\alpha}}^{\hat{\beta}} - P^{\hat{\beta}\hat{\gamma}} T_{\hat{\gamma}\hat{\alpha}a} = 0, \\ R_{a\alpha\hat{\beta}}^{\gamma} - \hat{C}_{\hat{\beta}}^{\hat{\gamma}\rho} T_{\rho\alpha a} &= T_{\gamma\alpha}^{\hat{\beta}} + \frac{1}{2} P^{\rho\hat{\beta}} T_{\rho\alpha\beta} = \hat{R}_{\rho\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} + \frac{1}{2} \hat{C}_{\hat{\beta}}^{\hat{\gamma}\epsilon} H_{\epsilon\rho\alpha} = \nabla_{\alpha} P^{\gamma\hat{\beta}} + C_{\alpha}^{\beta\hat{\gamma}} - P^{\rho\hat{\beta}} T_{\rho\alpha}^{\beta} = 0, \\ \nabla_{\hat{\alpha}} C_{\beta}^{\gamma\hat{\rho}} + C_{\beta}^{\gamma\hat{\sigma}} T_{\hat{\sigma}\hat{\alpha}}^{\hat{\rho}} + P^{\hat{\rho}\sigma} R_{\sigma\hat{\alpha}\hat{\beta}}^{\gamma} - S_{\hat{\alpha}\hat{\beta}}^{\hat{\rho}\gamma} &= \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} (\nabla_{\hat{\alpha}} \tilde{C}_{\hat{\beta}}^{\hat{\gamma}\rho} - P^{\hat{\sigma}\rho} \tilde{R}_{\hat{\sigma}\hat{\alpha}\hat{\beta}}^{\hat{\gamma}}) = 0, \\ \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} (\nabla_{\hat{\alpha}} S_{\beta\hat{\rho}}^{\gamma\hat{\sigma}} + \tilde{C}_{\hat{\beta}}^{\hat{\gamma}\epsilon} R_{\epsilon\hat{\alpha}\rho}^{\sigma} + C_{\rho}^{\sigma\hat{\epsilon}} \tilde{R}_{\hat{\epsilon}\hat{\alpha}\rho}^{\hat{\gamma}}) &= 0. \end{aligned} \quad (2.85)$$

By using the above symmetries and constraints we will show, by reducing the structure group of local symmetries to Lorentz, that the set of constraints on the torsion are equivalent to the equations of motions of generalized supergravity obtained in [15], after solving the Bianchi identities (2.76)-(2.78) in order of increasing dimension of the component superfields.

Dimension 1/2

As it was shown in [54] it is possible to fix $T_{\alpha\beta}^a = \gamma_{\alpha\beta}^a$ and $T_{\hat{\alpha}\hat{\beta}}^a = \gamma_{\hat{\alpha}\hat{\beta}}^a$ by using the scale and one of the local Lorentz transformations. However, to derive the correct equations from the constraints we need to get rid of the scalar connections in (2.69), as it was suggested in [54], by redefining $T_{\alpha\beta}^{\gamma}$ as

$$T_{\alpha\beta}^{\gamma} \rightarrow T_{\alpha\beta}^{\gamma} - 2\delta_{(\alpha}^{\gamma} \Omega_{\beta)}, \quad T_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} \rightarrow T_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} - 2\delta_{(\hat{\alpha}}^{\hat{\gamma}} \tilde{\Omega}_{\hat{\beta})}. \quad (2.86)$$

Now we can use shift symmetry (2.71) in a convenient way to fix $T_{\alpha\beta}^{\gamma}$. Considering (2.83), $T_{\alpha\beta}^{\gamma}$ can be decomposed in irreducible components of $SO(9, 1)$ as

$$T_{\alpha\beta}^{\gamma} = \gamma_{\alpha\beta}^{\rho} S_{\rho}^{\gamma} + \gamma_{\alpha\beta}^{\rho} \gamma_{\rho}^{\gamma\sigma} S_{\sigma} - 2\delta_{(\alpha}^{\gamma} \Omega_{\beta)}. \quad (2.87)$$

Taking into account the decomposition $h^{a\alpha} = h^{a\alpha} + (\gamma^a)^{\alpha\beta} h_{\beta}$, it is possible to shift $\Omega_{\alpha\beta}^{\gamma}$ such that

$$T_{\alpha\beta}^{\gamma} = \gamma_{\alpha\beta}^{\rho} \gamma_{\rho}^{\gamma\sigma} \Omega_{\sigma} - 2\delta_{(\alpha}^{\gamma} \Omega_{\beta)}. \quad (2.88)$$

An analogous result can be obtained for $T_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}}$

$$T_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} = \gamma_{\hat{\alpha}\hat{\beta}}^{\rho} \gamma_{\rho}^{\hat{\gamma}\hat{\sigma}} \tilde{\Omega}_{\hat{\sigma}} - 2\delta_{(\hat{\alpha}}^{\hat{\gamma}} \tilde{\Omega}_{\hat{\beta})}. \quad (2.89)$$

Thereafter, we can obtain $T_{\alpha a}{}^b$ by using the condition $T_{\alpha(ab)} = 0$ and the Bianchi identity $(\nabla T + TT)_{\alpha\beta\gamma}{}^b$. They imply that $T_{(\alpha\beta}{}^\varepsilon \gamma_{\sigma)\varepsilon}^b = 0$ and can be written as

$$\gamma_{(\alpha\gamma}^a T_{\sigma)[ab]} = 0. \quad (2.90)$$

which leads to the solution

$$T_{\alpha a}{}^b = 0, \quad T_{\hat{\alpha} a}{}^b = 0. \quad (2.91)$$

Dimension 1

One can set the dimension three torsion $T_{abc} = 0$ to zero by redefining the connection as

$$\Omega_{abc} \rightarrow \Omega_{abc} + \frac{1}{2}H_{abc}, \quad (2.92)$$

$$\tilde{\Omega}_{abc} \rightarrow \tilde{\Omega}_{abc} - \frac{1}{2}H_{abc}. \quad (2.93)$$

As a consequence, $T_{\alpha\alpha}{}^\beta$ and $T_{a\hat{\alpha}}{}^{\hat{\beta}}$ are shifted as

$$T_{\alpha\alpha}{}^\beta = \frac{1}{8}(\gamma^{bc})_{\alpha}{}^\beta H_{abc}, \quad T_{a\hat{\alpha}}{}^{\hat{\beta}} = -\frac{1}{8}(\gamma^{bc})_{\hat{\alpha}}{}^{\hat{\beta}} H_{abc}. \quad (2.94)$$

Using the Bianchi identity $(\nabla T + TT)_{\alpha\beta a}{}^b$ and $(\nabla T + TT)_{\hat{\alpha}\hat{\beta} a}{}^b$, it is easy to obtain

$$R_{\alpha\beta ab} = \frac{1}{2}(\gamma^c)_{\alpha\beta} H_{abc}, \quad R_{\hat{\alpha}\hat{\beta} ab} = -\frac{1}{2}(\gamma^c)_{\hat{\alpha}\hat{\beta}} H_{abc}. \quad (2.95)$$

The Bianchi identities $(\nabla T)_{\alpha\beta\gamma}{}^\rho$ and $(\nabla T)_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^\rho$ imply that the derivative of the superfields Ω_α and $\tilde{\Omega}_{\hat{\alpha}}$ is given by

$$\nabla_\alpha \Omega_\beta = \Omega_\alpha \Omega_\beta - \frac{1}{2}\gamma_{\alpha\beta}^a Z_a + \frac{1}{24}\gamma_{\alpha\beta}^{abc} H_{abc}, \quad (2.96)$$

$$\nabla_{\hat{\alpha}} \tilde{\Omega}_{\hat{\beta}} = \tilde{\Omega}_{\hat{\alpha}} \tilde{\Omega}_{\hat{\beta}} - \frac{1}{2}\gamma_{\hat{\alpha}\hat{\beta}}^a Z'_a - \frac{1}{24}\gamma_{\hat{\alpha}\hat{\beta}}^{abc} H_{abc}, \quad (2.97)$$

where $Z_a = \frac{1}{8}\gamma_a^{\alpha\beta}\nabla_{(\alpha}\Omega_{\beta)}$ and $Z'_a = \frac{1}{8}\gamma_a^{\hat{\alpha}\hat{\beta}}\nabla_{(\hat{\alpha}}\tilde{\Omega}_{\hat{\beta})}$ are two vector superfields. Using $(\nabla T + TT)_{\alpha\beta\hat{\alpha}}{}^{\hat{\beta}}$ and $(\nabla T + TT)_{\hat{\alpha}\hat{\beta}\alpha}{}^\beta$ imply ²

$$T_{a\alpha}{}^{\hat{\alpha}} = \frac{1}{8}(\gamma_a \mathcal{P})_{\alpha}{}^{\hat{\alpha}}, \quad T_{a\hat{\alpha}}{}^{\alpha} = \frac{1}{8}(\gamma_a \mathcal{P})_{\hat{\alpha}}{}^{\alpha}, \quad (2.98)$$

²In order to make contact with [15], we scale the RR bispinor $P^{\alpha\hat{\alpha}}$ defined for the PS formulation as $\mathcal{P} = 8P$.

The identity $(\nabla T + TT)_{\alpha\hat{\beta}a}{}^b$ implies that

$$R_{\alpha\hat{\beta}}{}^{ab} = 2\gamma_{\alpha\beta}^{[a} \mathcal{P}^{\beta\hat{\alpha}} \gamma_{\hat{\alpha}\hat{\beta}}^{b]}. \quad (2.99)$$

The remaining one-dimensional Bianchi identities are $(\nabla T + TT)_{\alpha\hat{\alpha}\beta}{}^{\gamma}$ and $(\nabla T + TT)_{\hat{\alpha}\alpha\hat{\beta}}{}^{\hat{\gamma}}$ and they imply

$$\nabla_{\hat{\alpha}} \Omega_{\alpha} = \frac{1}{16} (\gamma_a \mathcal{P} \gamma^a)_{\alpha\hat{\alpha}}, \quad \nabla_{\alpha} \tilde{\Omega}_{\hat{\alpha}} = \frac{1}{16} (\gamma_a \mathcal{P} \gamma^a)_{\hat{\alpha}\alpha}. \quad (2.100)$$

Dimension 3/2

Identifying $T_{ab}{}^{\gamma I}$ with the gravitino field strength $(T_{ab}{}^{\gamma I} = \psi_{ab}{}^{\gamma I})$ the Bianchi identity $(\nabla T + TT)_{\alpha ab}{}^c$ gives us

$$(\gamma_c \psi_{ab})_{\alpha I} = 2R_{\alpha I [ab]c}. \quad (2.101)$$

Using the identity

$$2R_{\alpha abc} = 2R_{\alpha [ab]c} + 2R_{\alpha [ca]b} - 2R_{\alpha [bc]a}, \quad (2.102)$$

we can show that

$$R_{\alpha I abc} = (\psi_{a[c} \gamma_{b]})_{\alpha I} - \frac{1}{2} (\gamma_a \psi_{bc})_{\alpha I}. \quad (2.103)$$

The other Bianchi identities at dimension 3/2 are $(\nabla T + TT)_{a\alpha\beta}{}^{\gamma}$ and $(\nabla T + TT)_{a\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}}$, and give us the equations

$$\nabla_{\alpha I} H_{abc} + 3(\gamma_{[a} \sigma^3 \psi_{bc]})_{\alpha I} = 0, \quad (2.104)$$

$$\nabla_a \Omega_{\alpha I} + \frac{1}{8} (\gamma^{bc} \sigma^3 \Omega)_{\alpha I} H_{abc} + \frac{1}{2} (\gamma^b \psi_{ab})_{\alpha I} = 0. \quad (2.105)$$

The identities $(\nabla T + TT)_{a\alpha\beta}{}^{\hat{\alpha}}$ and $(\nabla T + TT)_{a\hat{\alpha}\hat{\beta}}{}^{\alpha}$ can be read as

$$\nabla_{\alpha} \mathcal{P}^{\beta\hat{\gamma}} - \delta_{\alpha}^{\beta} \mathcal{P}^{\varepsilon\hat{\gamma}} \Omega_{\varepsilon} - \mathcal{P}^{\beta\hat{\gamma}} \Omega_{\alpha} + \mathcal{P}^{\varepsilon\hat{\gamma}} \gamma_{\alpha\varepsilon}^a \gamma^{\beta\gamma} \Omega_{\gamma} + 2(\gamma^{ab})_{\alpha}{}^{\beta} \psi_{ab}^{\hat{\gamma}} = 0, \quad (2.106)$$

$$\nabla_{\hat{\alpha}} \mathcal{P}^{\beta\hat{\gamma}} - \delta_{\hat{\alpha}}^{\beta} \mathcal{P}^{\varepsilon\hat{\gamma}} \Omega_{\varepsilon} - \mathcal{P}^{\beta\hat{\gamma}} \Omega_{\hat{\alpha}} + \mathcal{P}^{\beta\hat{\varepsilon}} \gamma_{\hat{\alpha}\hat{\varepsilon}}^a \gamma^{\hat{\sigma}\hat{\gamma}} \Omega_{\hat{\sigma}} - 2(\gamma^{ab})_{\hat{\alpha}}{}^{\hat{\beta}} \psi_{ab}^{\hat{\gamma}} = 0, \quad (2.107)$$

Applying ∇_σ and $\nabla_{\hat{\sigma}}$ on (2.96) and (2.97) respectively and symmetrizing the derivatives, it is possible to show

$$\begin{aligned}\nabla_\alpha Z_a - (\gamma_a \gamma^b \Omega)_\alpha Z_b - \frac{1}{2}(\gamma^b \gamma_a \nabla_b \Omega)_\alpha - \frac{1}{8}(\gamma^{bc} \sigma^3 \Omega)_\alpha H_{abc} - \frac{1}{48}(\gamma_a \gamma^{bcd} \sigma^3 \Omega)_\alpha H_{bcd} &= 0, \\ \nabla_{\hat{\alpha}} Z'_a - (\gamma_a \gamma^b \Omega)_{\hat{\alpha}} Z'_b - \frac{1}{2}(\gamma^b \gamma_a \nabla_b \Omega)_{\hat{\alpha}} - \frac{1}{8}(\gamma^{bc} \sigma^3 \Omega)_{\hat{\alpha}} H_{abc} - \frac{1}{48}(\gamma_a \gamma^{bcd} \sigma^3 \Omega)_{\hat{\alpha}} H_{bcd} &= 0.\end{aligned}\quad (2.108)$$

Now, applying ∇_σ and $\nabla_{\hat{\sigma}}$ on (2.97) and (2.96) respectively and symmetrizing the derivatives, we have

$$\nabla_{\hat{\alpha}} Z_a - \nabla_a \Omega_{\hat{\alpha}} - \frac{1}{8}(\gamma^{bc} \sigma^3 \Omega)_{\hat{\alpha}} H_{abc} - \frac{1}{8}(\gamma_a \mathcal{P} \Omega)_{\hat{\alpha}} = 0, \quad (2.109)$$

$$\nabla_\alpha Z'_a - \nabla_a \Omega_\alpha - \frac{1}{8}(\gamma^{bc} \sigma^3 \Omega)_\alpha H_{abc} - \frac{1}{8}(\gamma_a \mathcal{P} \Omega)_\alpha = 0. \quad (2.110)$$

At this point, we make contact with the standard notation and define $X_a = \frac{1}{2}(Z_a + Z'_a)$ and $K_a = \frac{1}{2}(Z_a - Z'_a)$. From the above equations, it follows that

$$\begin{aligned}\nabla_{\alpha I} X_a &= \nabla_a \Omega_{\alpha I} - \frac{1}{4}(\gamma_a \gamma^b \nabla_b \Omega)_{\alpha I} + \frac{1}{2}(\gamma_a \gamma^b (X_b + \sigma^3 K_b) \Omega)_{\alpha I} + \frac{1}{16}(\gamma_a \mathcal{P} \Omega)_{\alpha I} \\ &\quad + \frac{1}{8}(\gamma^{bc} \sigma^3 \Omega)_{\alpha I} H_{abc} + \frac{1}{96}(\gamma_a \gamma^{bcd} \sigma^3 \Omega)_{\alpha I} H_{bcd}, \\ \nabla_{\alpha I} K_a &= -\frac{1}{4}(\gamma_a \gamma^b \nabla_b \Omega)_{\alpha I} + \frac{1}{2}(\gamma_a \gamma^b (X_b + \sigma^3 K_b) \Omega)_{\alpha I} + \frac{1}{16}(\gamma_a \mathcal{P} \Omega)_{\alpha I} \\ &\quad + \frac{1}{96}(\gamma_a \gamma^{bcd} \sigma^3 \Omega)_{\alpha I} H_{bcd}.\end{aligned}\quad (2.111)$$

Dimension 2

It is easy to see that $(\nabla T + TT)_{abc}^d$ give us $R_{[abc]}^d = 0$. The relations $(\nabla T + TT)_{ab\alpha I}^{\delta J}$ imply

$$\nabla_{\alpha I} \psi_{ab}^{\delta J} + \frac{1}{4}(\gamma_{[a} \nabla_{b]} \mathcal{P})_{\alpha I}^{\delta J} - \frac{1}{32}(\gamma_{[a} \mathcal{P} \gamma_{b]} \mathcal{P})_{\alpha I}^{\delta J} - \psi_{ab}^{\delta J} \Omega_{\alpha I} = 0, \quad (2.112)$$

when $I \neq J$. For $I = J$, we have

$$\begin{aligned}
\nabla_{\alpha I} \psi_{ab}^{\delta I} = & -\frac{1}{4} \sigma^3 (\gamma^{cd})_{\alpha}{}^{\delta} \nabla_{[a} H_{b]cd} + \frac{1}{32} \sigma^3 ((\gamma^{cd} \gamma_{[a} \mathcal{P})_{\alpha}{}^{\delta} H_{b]cd} - (\gamma_{[a} \mathcal{P} \gamma^{cd})_{\alpha}{}^{\delta} H_{b]cd}) \\
& + \frac{1}{8} (\gamma^{cd})_{\alpha}{}^{\delta} H_{ace} H_{bd}{}^e - \delta_{\alpha}^{\delta} (\psi_{ab} \Omega) + (\gamma^c \psi_{ab})_{\alpha I} (\gamma_c \Omega)^{\delta J} + \frac{1}{4} R_{ab}{}^{cd} (\gamma_{cd})_{\alpha}{}^{\delta} .
\end{aligned} \tag{2.113}$$

Now, we can use (2.112), (2.113) and (2.105) to obtain the equations(2.49)-(2.52). The remaining generalized equations (2.46)-(2.48) can be obtained by applying a symmetrized spinor derivative on (2.111), which tell us that the classical pure spinor constraints imply the generalized supergravity equations.

It is important to remark that this analysis is based on the constraints imposed by the classical symmetries of the theory. It was shown in [54] that the vanishing of the ghost number anomaly determines that the scale connections Ω_{α} and $\Omega_{\hat{\alpha}}$ must be proportional to the spinorial derivatives of the dilaton superfield Φ .

$$\nabla_{\alpha} \Phi = 4\Omega_{\alpha}, \quad \nabla_{\hat{\alpha}} \Phi = 4\Omega_{\hat{\alpha}}, \tag{2.114}$$

which is equivalent to the solution $K_a = 0$ and $X_a = \partial_a \Phi$ in the generalized supergravity context, implying the standard supergravity background. This condition was used in [55, 56] to consistently quantize heterotic and type IIB pure spinor superstring on a curved background.

Chapter 3

$AdS_5 \times S^5$ Superstring

The $AdS_5 \times S^5$ space arises as a solution of the Type IIB supergravity equations supported by a self-dual Ramond-Ramond five-form flux. Superstring in AdS can be formulated by using the Green-Schwarz formalism or the pure spinor formalism which present manifest target space supersymmetry and allows us to correctly describe a string on a Ramond-Ramond background. Just as in the flat space case where the Green Schwarz-formalism can be interpreted as a Wess-Zumino like sigma model on the coset superspace being a quotient of the ten dimensional super-Poincaré group ($SUSY(\mathcal{N} = 2)$) over its Lorentz subgroup $SO(9, 1)$, one can make use of a similar approach in the $AdS_5 \times S^5$ case but this time with a target space given by the supercoset $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$. This supercoset has a $\mathbb{Z}_4$ structure which can be used to write the action as a bilinear form of currents. In order to construct the pure spinor model we determine the $AdS_5 \times S^5$ background superfields and plug them in the Berkovits-Howe action.

3.1 Type IIB supergravity solution for $AdS_5 \times S^5$

The type IIB supergravity equations of motion allows a maximally supersymmetric solution on an $AdS_5 \times S^5$ target space supported by a Ramond-Ramond 5-form flux

$$ds^2 = \exp \phi / R (-dt^2 + (dx^i)^2) + d\phi^2 + R^2 d\Omega_5^2, \quad (3.1)$$

$$\mathcal{F}_5 = (1 + *)Vol(S^5)N_c/R, \quad (3.2)$$

where N_c are the units of the RR flux through the five-sphere and $R^2 = \alpha'(4\pi g_s N_c)^{1/2}$ is the radius of the sphere and of AdS. Let us introduce the coset formulation of $AdS_5 \times S^5$ by

embedding it into $\mathbb{R}^n$ as

$$AdS_5 = \{X \in \mathbb{R}^6 \mid X \cdot X = -1\}, \quad S^5 = \{Y \in \mathbb{R}^6 \mid Y \dot{Y} = 1\}, \quad (3.3)$$

In S^5 , the points Y transforms canonically under $SO(6)$ and isotropic under $SO(5)$. In AdS_5 , the isometry group is $SO(4,2)$ and the isotropic subgroup $SO(4,1)$.

Considering that $SO(4,2)$ and $SO(6)$ are locally isomorphic to $SU(2,2)$ and $SU(4)$ respectively, the super-isometry group combines the groups $SU(2,2)$ and $SU(4)$ plus 32 fermionic generators into the supergroup $PSU(2,2|4)$. Then, the $AdS_5 \times S^5$ superspace is obtained by dividing by the bosonic isotropy group

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}. \quad (3.4)$$

The construction of the coset sigma model of the superstring relies on the properties of this supergroup and will be summarized in the following section. Here we will determine the relevant background superfields to formulate the world-sheet non-linear sigma model action on $AdS_5 \times S^5$, based on the discussion in chapter 2. Considerig the five-form (3.2), the Ramond-Ramond bispinor can be written as

$$\mathcal{P}^{\alpha\hat{\alpha}} = 8 \frac{\eta^{\alpha\hat{\alpha}}}{(g_s N_c)^{1/4}}, \quad \eta^{\alpha\hat{\alpha}} = (\gamma^{01234})^{\alpha\hat{\alpha}}. \quad (3.5)$$

The B field can be read from the definition of $H = dB$. In flat components it reads as $H = \nabla B + TB$. The NS-NS B-field vanishes, hence $H_{abc} = 0$. Considering the constraint $H_{a\alpha\beta} = -(\gamma_a)_{\alpha\beta}$, it follows that $H_{a\alpha\beta} = T_{a\alpha}{}^{\hat{\gamma}} B_{\hat{\gamma}\beta}$, which implies that $H_{a\alpha\beta} = \frac{1}{8}(\gamma_a)_{\alpha\sigma} \mathcal{P}^{\sigma\hat{\gamma}} B_{\hat{\gamma}\beta}$. It follows that

$$B_{\alpha\hat{\alpha}} = \frac{1}{2}(g_s N_c)^{1/4} \eta_{\alpha\hat{\alpha}}. \quad (3.6)$$

The fields $(C_{\alpha}^{\beta\hat{\gamma}}, \tilde{C}_{\hat{\alpha}}^{\hat{\beta}\gamma})$ are related to the gravitino and dilatino, and must therefore vanish. The $S_{\alpha\hat{\alpha}}^{\beta\hat{\beta}}$ is related to the Riemann curvature as $S_{\alpha\hat{\alpha}}^{\beta\hat{\beta}} = R_{abcd}(\gamma^{a[b})_{\alpha}^{\beta}(\gamma^{c]d})_{\hat{\alpha}}^{\hat{\beta}}$, and the Riemann curvature can be written from the Bianchi identity $(\nabla T + TT)_{ab\alpha}{}^{\beta}$ as

$$R_{abcd} = \frac{2}{(g_s N_c)^{1/2}} \eta_{a[b} \eta_{c]d}. \quad (3.7)$$

3.2 Basic properties of the $\mathfrak{psu}(2, 2|4)$ superalgebra

In this section we shall summarize the notation and some relevant facts about the superalgebra $\mathfrak{psu}(2, 2|4)$ in order to better understand the supercoset construction of the superstring sigma model of $AdS_5 \times S^5$ [51].

Let $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ be a $\mathbb{Z}_2$ -graded vector space, where $\dim(\mathcal{V}_0) = m$ and $\dim(\mathcal{V}_1) = n$. We define a supermatrix M as an element of $\text{End}(\mathcal{V})$, that is $M : \mathcal{V} \rightarrow \mathcal{V}$ which may be decomposed as follows

$$M \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_1 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_1 \end{pmatrix}, \quad M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}. \quad (3.8)$$

where m and n are of even grading and of dimension $m \times m$ and $n \times n$ respectively. In contrast, θ and η are of odd grading and of dimension $m \times n$ and $n \times m$ respectively.

We define the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ as the set of all supermatrices (3.8) over the field $\mathbb{C}$. The supertrace and supertranspose are defined as follows

$$\text{str} M \equiv \text{tr} m - \text{tr} n, \quad M^{st} = \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}. \quad (3.9)$$

Then, in analogy to the theory of classical Lie algebras, one can define the special unitary Lie superalgebra by taking a subalgebra of $\mathfrak{gl}(m|n)$. Let us define the matrix Lie superalgebra $\mathfrak{sl}(m|n)$ over $\mathbb{C}$ as follows

$$\mathfrak{sl}(m|n) = \{M \in \mathfrak{gl}(m|n); \quad \text{str} M = 0\}. \quad (3.10)$$

We now specialize to the case of the superalgebra $\mathfrak{sl}(4, 4)$ over the field $\mathbb{C}$. The defining representation of this superalgebra is given by a graded vector space which is spanned by 8×8 supermatrices (3.8), which are constructed in terms of 4×4 blocks, where m and n are regarded as even (bosonic) and θ and η are regarded as odd (fermionic). As we established in (3.10), we define the superalgebra $\mathfrak{sl}(4, 4)$ as spanned by the matrices M with vanishing supertrace $\text{str} M \equiv \text{tr} m - \text{tr} n = 0$.

Furthermore, the superalgebra $\mathfrak{su}(2, 2|4)$ is defined as a non-compact real form of $\mathfrak{sl}(4, 4)$. We consider the so-called *Cartan involution*

$$\phi(M) \equiv M^* = M, \quad (3.11)$$

where $M^\star$ is defined as follows

$$M^\star = -HM^\dagger H^{-1}, \quad (3.12)$$

here $M^\dagger$ stands for the adjoint of the supermatrix M defined by $M^\dagger = (M^t)^\star$. Combining equations (3.11) and (3.12) we find that a supermatrix M from $\mathfrak{su}(2,2|4)$ satisfies the following reality condition

$$MH + HM^\dagger = 0. \quad (3.13)$$

The hermitian matrix H is defined as follows

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & 1_4 \end{pmatrix}, \quad (3.14)$$

Here 1_n stands for the $n \times n$ identity matrix and Σ is a 4×4 matrix defined as

$$\Sigma = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}. \quad (3.15)$$

We can find how the reality condition (3.13) acts on the block entries of M by expanding it as follows

$$M = \begin{pmatrix} m\Sigma & \theta \\ \eta\Sigma & n \end{pmatrix} = \begin{pmatrix} -\Sigma m^\dagger & -\Sigma \eta^\dagger \\ -\theta^\dagger & -n^\dagger \end{pmatrix}, \quad (3.16)$$

then the condition (3.13) implies

$$m^\dagger = -\Sigma m \Sigma, \quad n^\dagger = -n, \quad \eta^\dagger = -\Sigma \theta, \quad (3.17)$$

from these conditions we can easily see that the matrix blocks m and n span the unitary algebras $\mathfrak{u}(2,2)$ and $\mathfrak{u}(4)$ respectively. From this simple analysis we can deduce that the bosonic subalgebra of $\mathfrak{su}(2,2|4)$ is given by

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1), \quad (3.18)$$

where we have added the final factor since the $\mathfrak{u}(1)$ generator $i1_4$ is also part of $\mathfrak{su}(2,2|4)$ because it satisfies (3.13) and possess vanishing supertrace. Finally, We define the $\mathfrak{psu}(2,2|4)$ superalgebra as a quotient algebra of $\mathfrak{su}(2,2|4)$ over this $\mathfrak{u}(1)$ factor.

The most important property of $\mathfrak{psu}(2, 2|4)$ is that it admits a fourth order automorphism $\Omega : M \mapsto \Omega(M) \in \mathfrak{psu}(2, 2|4)$ which can be defined as follows

$$\Omega(M) = \begin{pmatrix} J m^t J & -J \theta^t J \\ J \eta^t J & J n^t J \end{pmatrix}; \quad J = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix}. \quad (3.19)$$

It is easy to check that this definition satisfies $\Omega^4(M) = M$, that is

$$\Omega^4 = I, \quad (3.20)$$

Hence, the linear map Ω has eigenvalues $\pm 1, \pm i$. Thus, if we denote $\mathfrak{g}_k$ as the eigenspace associated to the eigenvalue i^k ($k = 0, 1, 2, 3$), we have

$$\Omega(H_k) = i^k H_k \quad ; \quad H_k \in \mathfrak{g}_k. \quad (3.21)$$

As a consequence, we can use this automorphism to construct a decomposition that exhibits explicitly the $\mathbb{Z}_4$ grading structure of $\mathfrak{psu}(2, 2|4)$.

$$\mathfrak{psu}(2, 2|4) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3. \quad (3.22)$$

Furthermore, one can note that:

$$\Omega([H_k, H_l]) = [\Omega(H_k), \Omega(H_l)] = i^{k+l} [H_k, H_l], \quad (3.23)$$

thus,

$$[H_k, H_l] \in \mathfrak{g}_{k+l} \mod 4, \quad (3.24)$$

which is compatible with the supertrace operation

$$\text{str}(H_k H_l) = 0 \quad , \quad \text{unless, } k + l = 0 \mod 4 \quad (3.25)$$

Since we want to construct some homogeneous space that can be identified with the $AdS_5 \times S^5$ manifold, we need to study the explicit form of $\mathfrak{g}$. By definition $\Omega(H_0) = H_0$, which applied to the block structure of the supermatrices gives

$$m^t J + J m = 0, \quad n^t J + J n = 0, \quad \theta = \eta = 0. \quad (3.26)$$

These means that $m, n \in \mathfrak{sp}(4)$, which is the Lie algebra of the symplectic group $Sp(4)$. From the discussion we had before $m \in \mathfrak{u}(2, 2)$ and $n \in \mathfrak{u}(4)$. Defining the Lie algebra $\mathfrak{usp}(n)$ of

the unitary symplectic group $USp(n) \equiv SU(n) \cap Sp(n)$, we can conclude that:

$$m \in \mathfrak{usp}(2, 2), \quad n \in \mathfrak{usp}(4). \quad (3.27)$$

Since $\theta = \eta = 0$, then $\mathcal{H}_0$ is a bosonic subalgebra that can be identified with

$$\mathfrak{g}_0 = \mathfrak{usp}(2, 2) \oplus \mathfrak{usp}(4), \quad (3.28)$$

which is the Lie algebra of the subgroup $USp(2, 2) \times USp(4)$ of $PSU(2, 2|4)$. From classical group theory $USp(2, 2) \cong SO(4, 1)$ and $USp(4) \cong SO(5)$, thus the fixed point set of Ω is given by $SO(4, 1) \times SO(5)$.

Let us introduce the generators for the fundamental representation of $\mathfrak{psu}(2, 2|4)$ according to their grading. We split the generators $\{t_A\}$ by using the following convention

$$\{t_{[ab]}\}, \quad \text{generators of } \mathfrak{g}_0 \quad \{t_\alpha\}, \quad \text{generators of } \mathfrak{g}_1 \quad (3.29)$$

$$\{t_a\}, \quad \text{generators of } \mathfrak{g}_2 \quad \{t_{\hat{\alpha}}\}, \quad \text{generators of } \mathfrak{g}_3 \quad (3.30)$$

where $a = 0, \dots, 9$ and $\alpha, \hat{\alpha} = 1, \dots, 16$. In this basis the non-vanishing supertraces are

$$\text{Str}(t_a t_b) = \eta_{ab}, \quad \text{Str}(t_{[ab]} t_{[cd]}) = \frac{1}{2} \eta_{a[b} \eta_{c]d}, \quad \text{Str}(t_\alpha t_{\hat{\alpha}}) = \eta_{\alpha\hat{\alpha}}. \quad (3.31)$$

The non-vanishing structure constants of $\mathfrak{psu}(2, 2|4)$ algebra are

$$\begin{aligned} f_{[cd][ef]}^{[ab]} &= \frac{1}{2} (\eta_{ce} \delta_d^{[a} \delta_f^{b]} - \eta_{cf} \delta_d^{[a} \delta_e^{b]} + \eta_{df} \delta_c^{[a} \delta_e^{b]} - \eta_{de} \delta_c^{[a} \delta_f^{b]}), \\ f_{\alpha\hat{\alpha}}^{[ab]} &= \frac{1}{2} (\gamma^{ab})_\alpha^\beta \eta_{\beta\hat{\alpha}}, \quad f_{cd}^{[ab]} = \delta_c^{[a} \delta_d^{b]}, \\ f_{\alpha\beta}^a &= \gamma_{\alpha\beta}^a, \quad f_{[ab]d}^c = \eta_{d[a} \delta_{b]}^c, \\ f_{\hat{\alpha}c}^\alpha &= -\gamma_c^{\alpha\beta} \eta_{\beta\hat{\alpha}}, \quad f_{[ab]\beta}^\alpha = \frac{1}{2} (\gamma_{ab})_\alpha^\beta, \\ f_{\hat{\alpha}c}^{\hat{\alpha}} &= -\gamma_c^{\hat{\alpha}\hat{\beta}} \eta_{\alpha\hat{\beta}}, \quad f_{[ab]\hat{\beta}}^{\hat{\alpha}} = \frac{1}{2} (\gamma_{ab})_{\hat{\alpha}}^{\hat{\beta}}, \end{aligned} \quad (3.32)$$

3.3 Integrability and Lax pair

The requirement for classical integrability is the existence of an infinite number of conserved quantities. Let us define a Lax connection $L(z)$ depending on the dynamical fields and on a complex spectral parameter z and satisfying the flatness condition

$$\partial_i L_j(z) - \partial_j L_i(z) - [L_i(z), L_j(z)] = 0 \quad (3.33)$$

Having obtained a flat current, we can construct a monodromy matrix $T(\tau, z)$ which is the Wilson loop of the Lax connection around the closed string

$$T(\tau, z) = \mathcal{P} \exp \oint_C (d\sigma L_\sigma(\sigma, \tau; z)), \quad (3.34)$$

where C is a non-contractible loop around the world-sheet. The eigenvalues of the monodromy matrix, which depend on the complex spectral parameter, form an infinite set of conserved quantities and $T(\tau, z)$ satisfies

$$\partial_\tau T(\tau, z) = [L_\tau(0, \tau; z), T(\tau, z)], \quad (3.35)$$

thus, the spectral properties of the model are encoded into the monodromy matrix. Thereafter, we say the systems is classically integrable when the equations of motion can be cast a zero curvature representation.

3.4 Green-Schwarz superstring in $AdS_5 \times S^5$

Based on our discussions in the last sections, we expect that the action for the Type IIB superstring propagating in $AdS_5 \times S^5$ will be given by a Wess-Zumino-Witten like sigma-model Lagrangian, where the first term corresponds to a kinetic term and the second one is obtained as the integral of a closed 3-form Ω_3 over a three dimensional manifold which has the string world-sheet as its boundary [51, 52].

The first step is to define the Maurer-Cartan one-form. Let g be an element of the supergroup $SU(2, 2|4)$, we define the Maurer-Cartan form as $A = -g^{-1}dg$, since it takes values in $\mathfrak{su}(2, 2|4)$ we can write it as follows

$$A = -g^{-1}dg = A_0 + A_1 + A_2 + A_3, \quad (3.36)$$

where we have exhibited the $\mathbb{Z}_4$ -decomposition of A . We should observe that, by construction A satisfy the Maurer-Cartan equation $dA - A \wedge A = 0$, which in components can be written as follows

$$\partial_i A_j - \partial_j A_i - [A_i, A_j] = 0. \quad (3.37)$$

where i, j are world-sheet coordinates. Let us consider local right $SO(4, 1) \times SO(5)$ multiplication on the elements of the superalgebra $g \rightarrow gh$, where h belongs to $SO(4, 1) \times SO(5)$. Under this action, the Maurer-Cartan form transforms as

$$\begin{aligned} A &\rightarrow -(gh)^{-1} (dgh + gdh) \\ &\rightarrow h^{-1} A h - h^{-1} dh. \end{aligned} \quad (3.38)$$

which in $\mathbb{Z}_4$ components reads

$$A_{(1,2,3)} \rightarrow h^{-1} A_{(1,2,3)} h, \quad A^0 \rightarrow h^{-1} A^0 h - h^{-1} dh. \quad (3.39)$$

In order to formulate the Lagrangian, we need to make a key observation here. The transformation on A_0 is typical of a gauge field, so we can understand A_0 as the $SO(4, 1) \times SO(5)$ gauge field, while $A_{(1,2,3)}$ transform according to the adjoint representation of $SO(4, 1) \times SO(5)$. Furthermore, since only the components $A_{(1,2,3)}$ undergo a similarity transformation, then any gauge invariant Lagrangian in the supercoset is given by a bilinear in the A 's which can not contain A_0 , thus the Lagrangian should only depend on a coset element and not on the group element g .

We now proceed to construct the Green-Schwarz action on the supercoset $PSU(2, 2|4)/SO(4, 1) \times SO(5)$. We consider the kinetic term for the bosonic components A_2 , but we can not allow a kinetic term for the fermionic components because it would break kappa symmetry. In this way, the fermionic components of the Maurer-Cartan form enter through the Wess-Zumino term. The sigma model Type II superstring action on $AdS_5 \times S^5$ is given by [51]

$$S_{GS} = -\frac{1}{4\pi\alpha'} \int d^2\sigma [\gamma^{ij} \text{str}(A_{2i}A_{2j}) + \kappa \epsilon^{ij} \text{str}(A_{1i}A_{3j})], \quad (3.40)$$

where $\gamma^{ij} = g^{ij} \sqrt{-g}$ is the Weyl invariant combination of the world-sheet metric g_{ij} such that $\det \gamma = 1$. The parameter κ is a constant number which must be real in order to guarantee the reality of the Lagrangian. Furthermore, since this formulation pretends to preserve the local structure of the flat Green-Schwarz superstring, we actually need to require this action to have kappa-symmetry and this leaves only two possibilities $\kappa = \pm 1$.

Equations of motion

Let us find the equations of motion which follow from the Lagrangian (3.40). In order to do this, we will make use of the following property

$$\text{str} \left(\Omega^k(M_1)M_2 \right) = \text{str} \left(M_1 \Omega^{4-k}(M_2) \right). \quad (3.41)$$

So, taking the variation of the Lagrangian (3.40) we get

$$\delta L = -\frac{1}{4\pi\alpha'} \left[2\gamma^{ij} \text{str}(\delta A_{2i}A_{2j}) + \kappa \varepsilon^{ij} \text{str}(\delta A_{1i}A_{3j} + A_{1i}\delta A_{3j}) \right]. \quad (3.42)$$

With the help of the results (3.41) we can write for the variation of the Lagrangian

$$\delta L = -\text{str}(\delta A_i \Lambda^i), \quad (3.43)$$

where

$$\Lambda^i = \frac{1}{2\pi\alpha'} \left[\gamma^{ij} A_{2j} - \frac{1}{2} \kappa \varepsilon^{ij} (A_{1j} - A_{3j}) \right]. \quad (3.44)$$

Besides that, the variation of A_i is given by

$$\delta A_i = -\delta(g^{-1}\partial_i g) = -g^{-1}\delta g A_\alpha - g^{-1}\partial_\alpha(\delta g), \quad (3.45)$$

and plugging the above result into the variation for the Lagrangian, we get

$$\begin{aligned} \delta L &= +\text{str} \left[g^{-1}\delta g A_i \Lambda^i + g^{-1}\partial_i(\delta g) \Lambda^i \right] \\ &= -\text{str} \left[g^{-1}\delta g (\partial_i \Lambda^i - [A_i, \Lambda^i]) \right], \end{aligned} \quad (3.46)$$

where we made use of integration by parts, and we arrived to the second line by neglecting total derivatives. Since equation (3.46) must vanish for an arbitrary group element g , and if we consider $\partial_i \Lambda^i - [A_i, \Lambda^i]$ as an element of $\mathfrak{su}(2,2|4)$ which contains the central element $i1$, we must require that

$$\partial_i \Lambda^i - [A_i, \Lambda^i] = ci1 \quad (3.47)$$

where the parameter c can be found by taking the trace of both sides in the expression above. Since we defined $\mathfrak{psu}(2,2|4)$ as the quotient algebra of $\mathfrak{su}(2,2|4)$ over its central element, we may identify any elements proportional to the identity as zero, in this way we can write the equations of motion as follows

$$\partial_i \Lambda^i - [A_i, \Lambda^i] = 0. \quad (3.48)$$

We can use the $\mathbb{Z}_4$ decomposition and rearrange all the terms of the equation above according to its $\mathbb{Z}_4$ grading. For instance, the 0-grading term is given by

$$\gamma^{jj} [A_{2i}, A_{2j}] - \frac{1}{2} \kappa \varepsilon^{ij} ([A_{3i}, A_{1j}] - [A_{1i}, A_{3j}]) = 0. \quad (3.49)$$

This equation is trivial, since each of the parts of the left hand side vanishes when using the symmetry and anti-symmetry properties of $\gamma^{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ respectively leading to $0 = 0$. The 2-grading component is given by

$$\partial_i (\gamma^{jj} A_{2j}) - \gamma^{jj} [A_{0i}, A_{2j}] + \frac{1}{2} \kappa \varepsilon^{ij} ([A_{1i}, A_{1j}] - [A_{3i}, A_{3j}]) = 0. \quad (3.50)$$

The 1-grading and 3-grading components of the equations of motion are given respectively by

$$\gamma^{jj} [A_{3i}, A_{2j}] + \kappa \varepsilon^{ij} [A_{2i}, A_{3j}] = 0, \quad (3.51)$$

$$\gamma^{jj} [A_{1i}, A_{2j}] + \kappa \varepsilon^{ij} [A_{2i}, A_{1j}] = 0, \quad (3.52)$$

where we made use of the Maurer-Cartan equation (3.37).

When varying the Lagrangian, we have not considered the variations of the world-sheet metric, so we can easily derive the equations of motion for the world-sheet metric by varying the Lagrangian with respect to γ^{ij} , obtaining the so-called Virasoro constraints

$$\text{Str} (A_{2i} A_{2j}) - \frac{1}{2} \gamma_{ij} \gamma^{kl} \text{str} (A_{2k} A_{2l}) = 0. \quad (3.53)$$

Lax pair

A remarkable feature of this construction is that the existence of a Lax representation for the classical equations of motion, only for the values $\kappa = \pm 1$ which also implies kappa symmetry [3]. Let us introduce the tensors

$$P_{\pm}^{ij} = \frac{1}{2} (\gamma^{ij} \pm \kappa \varepsilon^{ij}). \quad (3.54)$$

Then, we define

$$A_+^i = P_+^{ij} A_j, \quad A_-^i = P_-^{ij} A_j. \quad (3.55)$$

In terms of A_+^i and A_-^i the Maurer-Cartan equation (3.37) reads

$$\mathcal{L} \equiv \partial_i A_+^i - \partial_i A_-^i - [A_+^i, A_{-i}] = 0. \quad (3.56)$$

The equations of motion (3.50) can be re-written as

$$\partial_i A_{2+}^i + [A_{0-i}, A_{2+}^i] + [A_{3-i}, A_{3+}^i] = 0, \quad (3.57)$$

$$\partial_i A_{2-}^i + [A_{0+i}, A_{2-}^i] - [A_{1-i}, A_{1+}^i] = 0, \quad (3.58)$$

while equations (3.51) and (3.52) can be written as

$$P_-^{ij} [A_{2i}, A_{3j}] = 0, \quad (3.59)$$

$$P_+^{ij} [A_{2i}, A_{1j}] = 0. \quad (3.60)$$

Furthermore, for $\kappa = \pm 1$, the tensors $P_{\pm}^{ij}$ are orthogonal projectors, that is, they satisfy the relations

$$P_+^{ij} + P_-^{ij} = \gamma^{ij}, \quad P_{\pm}^{ik} P_{\pm k}^j = P_{\pm}^{ij}, \quad P_{\pm}^{ik} P_{\mp k}^j = 0. \quad (3.61)$$

The next step consists of introducing an ansatz for the Lax pair

$$L_+^i = A_{0+}^i + l_1 A_{1+}^i + l_2 A_{2+}^i + l_3 A_{3+}^i, \quad (3.62)$$

$$L_-^i = A_{0-}^i + l'_1 A_{1-}^i + l'_2 A_{2-}^i + l'_3 A_{3-}^i, \quad (3.63)$$

where l_i and l'_i are undetermined constants, which depends on the spectral parameter z , which will be determined by imposing the zero curvature condition

$$\partial_i L_+^i - \partial_i L_-^i + [L_{i-}, L_+^i] = 0. \quad (3.64)$$

Using the equations of motion (3.49)-(3.52), the zero curvature condition (3.64) gives

$$\begin{aligned} l_1 &= z, & l_2 &= z^{-2}, & l_3 &= z^{-1}, \\ l'_1 &= z, & l'_2 &= z^2, & l'_3 &= z^{-1}. \end{aligned} \quad (3.65)$$

The existence of the Lax connection allows us to construct an infinite number of conserved quantities.

Kappa-symmetry

We saw in the previous chapter that a classical GS model must possess a local fermionic symmetry to ensure the space-time supersymmetry of the physical spectrum. For (3.40) this local symmetry can be realized as right local action of $G = \exp \varepsilon$ acting on the elements of

the coset (see, for example, [53])

$$g.G = g'h \quad (3.66)$$

where h is a compensating element from $SO(4,1) \times SO(5)$ and ε is a local fermionic parameter taking values in $\mathfrak{g}_1 \oplus \mathfrak{g}_3$. Infinitesimally it reads as

$$\delta_\kappa g = g\varepsilon, \quad \varepsilon = \varepsilon_1 + \varepsilon_3. \quad (3.67)$$

Then, the κ transformation of the Maurer-Cartan form is given by

$$\delta_\kappa A = -d\varepsilon + [A, \varepsilon]. \quad (3.68)$$

This transformation on g has to be accompanied with the κ -transformation of the world-sheet metric. The κ -variation of the Lagrangian gives

$$\delta_\kappa L = -\frac{1}{4\pi\alpha'} [(\delta_\kappa \gamma^{ij}) \text{Str}(A_{2i}A_{2j}) - 4\text{Str}([A_{1+}^i, A_{i,2-}] \varepsilon_1 + [A_{3+}^i, A_{i,2+}] \varepsilon_3)]. \quad (3.69)$$

This transformation represents a symmetry of the theory if

$$\begin{aligned} \varepsilon_1 &= i\kappa_{1+i}A_{2-}^i + iA_{2-}^i\kappa_{1+i}, \\ \varepsilon_3 &= i\kappa_{3-i}A_{2+}^i + iA_{2+}^i\kappa_{3-i}, \end{aligned} \quad (3.70)$$

Moreover, for any grade 2 traceless matrix $A_\pm^2$ the following relation holds

$$A_{\pm i}^2 A_{\pm j}^2 = \frac{1}{8} \text{Str}(A_{\pm i}^2 A_{\pm j}^2 W) + c_{ij} 1_8, \quad (3.71)$$

where, W is a diagonal 8×8 matrix, $W = \text{diag}(1_4, -1_4)$. They have to be supplemented by the transformation of the world-sheet metric to lead to a κ -invariant theory

$$\delta_\kappa \gamma^{ij} = \frac{1}{2} (\text{Str}(W[i\kappa_{1+}^i, A_{+1}^j]) + \text{Str}(W[i\kappa_{3-}^i, A_{-3}^j])), \quad (3.72)$$

where κ_{1+} and κ_{3-} are independent parameters of gradings 1 and 3 respectively.

3.5 Pure spinor superstring in $AdS_5 \times S^5$

Being g an element of the supergroup $PSU(2,2|4)$ we define $A = -dg g^{-1}$. Since it takes values in $\mathfrak{psu}(2,2|4)$ we can decompose it as

$$A = -dg g^{-1} = A_0 + A_1 + A_2 + A_3. \quad (3.73)$$

The pure spinor action in $AdS_5 \times S^5$ can be constructed by plugging the supervielbeins, spin connection and the background superfields (3.5), (3.6), (3.7) in the Berkovits-Howe action (2.66). For instance, the matter contribution in first line of (2.66), can be written as

$$S_{matter} = \frac{1}{2\pi} \int d^2z \left(\frac{1}{2} A_-^a A_+^b \eta_{ab} - \frac{1}{4} A_-^\alpha A_+^{\hat{\alpha}} \eta_{\alpha\hat{\alpha}} + \frac{1}{4} A_+^\alpha A_-^{\hat{\alpha}} \eta_{\alpha\hat{\alpha}} \right. \\ \left. + d_\alpha A_+^\alpha + \hat{d}_{\hat{\alpha}} A_+^{\hat{\alpha}} + \frac{1}{4} d_\alpha \hat{d}_{\hat{\alpha}} \eta^{\alpha\hat{\alpha}} \right), \quad (3.74)$$

Because the Ramond-Ramond bispinor is invertible (4.37), we can integrate out the auxiliary fields d_α and $\hat{d}_{\hat{\alpha}}$ by using their equations of motion:

$$d_\alpha = A_-^{\hat{\alpha}} \eta_{\alpha\hat{\alpha}}, \quad d_{\hat{\alpha}} = -A_+^\alpha \eta_{\alpha\hat{\alpha}}. \quad (3.75)$$

The matter contribution remains as

$$S_{matter} = \frac{1}{2\pi} \int d^2z \left(\frac{1}{2} A_-^a A_+^b \eta_{ab} - \frac{1}{4} A_-^\alpha A_+^{\hat{\alpha}} \eta_{\alpha\hat{\alpha}} - \frac{3}{4} A_+^\alpha A_-^{\hat{\alpha}} \eta_{\alpha\hat{\alpha}} \right), \quad (3.76)$$

The remaining contribution contains the pure ghost sector and the coupling ghost- spin connection,

$$S_{ghost} = \frac{1}{2\pi} \int d^2z \left(\omega_\alpha (\nabla_+ \lambda)^\alpha + \hat{\omega}_{\hat{\alpha}} (\nabla_- \hat{\lambda})^{\hat{\alpha}} - \frac{1}{2} N^{ab} \hat{N}_{ab} \right). \quad (3.77)$$

Using the supertraces of the $\mathfrak{psu}(2,2|4)$ generators, the pure spinor action in $AdS_5 \times S^5$ [44] reads

$$S_{AdS} = \int \text{Str} \left(\frac{1}{4} A_+ d_{PS} A_- + \omega_{1+} \partial_- \lambda_3 + \omega_{3-} \partial_+ \lambda_1 + N_{0+} A_{0-} + N_{0-} A_{0+} - N_{0-} N_{0+} + S_{gh} \right), \quad (3.78)$$

where S_{gh} is the free ghost action and $d_{PS} = P_1 + 2P_2 + 3P_3$, where P_i projects an element of the superalgebra $\mathfrak{g}$ on its $\mathfrak{g}_i$ -component. The Lie algebra valued ghost fields are defined as

$$\lambda_1 = \lambda^\alpha t_\alpha, \quad \omega_{3-} = \omega_\alpha \eta^{\alpha\hat{\alpha}} t_{\hat{\alpha}}, \quad \lambda_3 = \hat{\lambda}^{\hat{\alpha}} t_{\hat{\alpha}}, \quad \omega_{3+} = \hat{\omega}_{\hat{\alpha}} \eta^{\alpha\hat{\alpha}} t_\alpha. \quad (3.79)$$

The bosonic ghosts λ^α and $\hat{\lambda}^{\hat{\alpha}}$ are constrained to satisfy the pure spinor condition

$$\lambda \gamma^\alpha \lambda = \hat{\lambda} \gamma^{\hat{\alpha}} \hat{\lambda} = 0, \quad (3.80)$$

and the pure spinor Lorentz generators are given by

$$N_{0-} = -\{\omega_{1+}, \lambda_3\}, \quad N_{0+} = -\{\omega_{3-}, \lambda_1\}. \quad (3.81)$$

The action is invariant under a BRST symmetry whose classical charge $Q = Q_L + Q_R$ is given by

$$Q_L = \oint \text{Str}(\lambda_1 A_{3-}), \quad Q_R = \oint \text{Str}(\lambda_3 A_{1+}), \quad (3.82)$$

and acts on a group element as a derivative

$$\varepsilon Q(g) = (\varepsilon \lambda_1 + \varepsilon \lambda_3)g, \quad (3.83)$$

$$\varepsilon Q(w_{3-}) = -\varepsilon A_{3-}, \quad (3.84)$$

$$\varepsilon Q(w_{1+}) = -\varepsilon A_{1+}, \quad (3.85)$$

implying

$$\varepsilon Q(A_{i-}) = \delta_{i,1} \partial_- (\varepsilon \lambda_1) + [A_{i+3;-}, \varepsilon \lambda_1] + \delta_{i,3} (\partial_- \varepsilon \lambda_3) + [A_{i+1;-}, \varepsilon \lambda_3], \quad (3.86)$$

$$\varepsilon Q(A_{i+}) = \delta_{i,1} \partial_+ (\varepsilon \lambda_1) + [A_{i+3;+}, \varepsilon \lambda_1] + \delta_{i,3} (\partial_+ \varepsilon \lambda_3) + [A_{i+1;+}, \varepsilon \lambda_3], \quad (3.87)$$

$$\varepsilon Q(N_-) = [A_{3-}, \lambda_1], \quad (3.88)$$

$$\varepsilon Q(N_+) = [A_{1+}, \lambda_3]. \quad (3.89)$$

For the matter sector the equations of motion are obtained from small variations $\delta g = g \xi_i$, $i = 1, 2, 3$, where ξ_i is an element of $\mathfrak{g}_i$ is

$$\frac{1}{4} \int \text{Str}(\delta g g^{-1}, \mathcal{E}_0), \text{ where,}$$

$$\mathcal{E}_0 \equiv \partial_+ (d_{PS} A_-) + \partial_- (\hat{d}_{PS} \bar{A}_+) + [\bar{A}_+, dA_-] + [\bar{A}_+, \hat{d}A_-] + [A_-, N_{0+}] + [A_+, N_{0-}] = 0. \quad (3.90)$$

where $d_{PS} = P_1 + 2P + 3P_3$ and $\hat{d}_{PS} = 3P_1 + 2P + P_3$. Defining the covariant derivatives as

$$D_\pm = \partial_\pm + [A_{0\pm},], \quad (3.91)$$

we can express the equations of motion as

$$D_- A_{1+} + [A_{1-}, N_{0+}] - [N_{0-}, A_{1+}] = 0, \quad (3.92)$$

$$D_- A_{2+} + [A_{1-}, A_{1+}] + [A_{2-}, N_{0+}] - [N_{0-}, A_{2+}] = 0, \quad (3.93)$$

$$D_- A_{3+} + [A_{1-}, A_{2+}] + [A_{2-}, A_{1+}] - [A_{3-}, N_{0+}] - [N_{0-}, A_{3+}] = 0, \quad (3.94)$$

$$D_+ A_{1-} + [A_{2+}, A_{3-}] + [A_{3+}, A_{2-}] + [A_{1-}, N_{0+}] - [N_{0-}, A_{1+}] = 0, \quad (3.95)$$

$$D_+ A_{2-} + [A_{3+}, A_{3-}] + [A_{2-}, N_{0+}] - [N_{0-}, A_{2+}] = 0, \quad (3.96)$$

$$D_- A_{3+} + [A_{3-}, N_{0-}] - [N_{0-}, A_{3+}] = 0. \quad (3.97)$$

Similarly, the equations of motion for the ghost sector are obtained by varying the action with respect of λ and ω and expressing the result in terms of the Lorentz currents

$$D_- N_{0+} - [N_{0-}, N_{0+}] = 0, \quad D_+ N_{0-} - [N_{0+}, N_{0-}] = 0. \quad (3.98)$$

Classical integrability can be proven by constructing the Lax pair [4]

$$\begin{aligned} L_+(z) &= A_{0+} + z^{-3}A_{1+} + z^{-2}A_{2+} + z^{-1}A_{3+} + (z^{-4} - 1)N_{0+}, \\ L_-(z) &= A_{0-} + zA_{1-} + z^2A_{2-} + z^3A_{3-} + (z^4 - 1)N_{0-}, \end{aligned} \quad (3.99)$$

where z is the spectral parameter, in such a way that the equations of motion (3.92)-(3.97) and (3.98) are equivalent to the zero curvature condition

$$\partial_- L_+(z) - \partial_+ L_-(z) + [L_-(z), L_+(z)] = 0. \quad (3.100)$$

Defining $z = e^l$ it is possible to express the density of the local conserved charges as

$$j = g^{-1} \left(\frac{dL}{dl} \Big|_{l=0} \right) g, \text{ such that, } \partial_+ j_- - \partial_- j_+ = 0. \quad (3.101)$$

Explicitly, the $j_{\pm}$ currents are

$$\begin{aligned} j_- &= g^{-1}(A_{1-} + 2A_{2-} + 3A_{3-} + 4N_{0-})g, \\ j_+ &= -g^{-1}(3A_{1+} + 2A_{2+} + A_{3+} + 4N_{0+})g. \end{aligned} \quad (3.102)$$

It is also useful to find the BRST transformation of the global currents

$$\begin{aligned} \varepsilon Q(j_+) &= \partial_+ \Lambda(\varepsilon) + 4g^{-1}(D_+ \varepsilon \lambda_1 - [N_{0+}, \varepsilon \lambda_1])g, \\ \varepsilon Q(j_-) &= \partial_- \Lambda(\varepsilon) - 4g^{-1}(D_- \varepsilon \lambda_3 - [N_{0-}, \varepsilon \lambda_3])g, \end{aligned} \quad (3.103)$$

where,

$$\Lambda(\varepsilon) = g^{-1}(\varepsilon\lambda_1 - \varepsilon\lambda_3)g. \quad (3.104)$$

Note that $\Lambda(\varepsilon)$ is BRST invariant because of the pure spinor condition (3.80)

$$\varepsilon' Q\Lambda(\varepsilon) = g([\varepsilon\lambda_1, \varepsilon'\lambda_1] - [\varepsilon\lambda_3, \varepsilon'\lambda_3])g^{-1} = 0. \quad (3.105)$$

It is possible to generalize the construction of vertex operators in flat space to the $AdS_5 \times S^5$ background [44]. An unintegrated vertex operator is defined as

$$U = \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} A_{\alpha\hat{\alpha}}(x, \theta, \hat{\theta}), \quad (3.106)$$

where $A_{\alpha\hat{\alpha}}(x, \theta, \hat{\theta})$ is a bispinor superfield which only depends on the matter fields. The cohomology conditions for a physical vertex operator are imposed by

$$Q_- U = Q_+ U = 0. \quad (3.107)$$

$$\delta U = Q_- \Lambda + Q_+ \hat{\Lambda}. \quad (3.108)$$

such that $Q_+ \Lambda = Q_- \hat{\Lambda} = 0$. Demanding these conditions on (3.106), the bispinor $A_{\alpha\hat{\alpha}}$ must satisfy

$$\gamma_{abcde}^{\alpha\beta} \nabla_\beta A_{\alpha\hat{\alpha}} = \gamma_{abcde}^{\hat{\alpha}\hat{\beta}} \nabla_{\hat{\beta}} A_{\alpha\hat{\alpha}} = 0, \quad (3.109)$$

$$\delta A_{\alpha\hat{\alpha}} = \nabla_\alpha \hat{\Omega}_{\hat{\alpha}} + \nabla_{\hat{\alpha}} \Omega_\alpha, \quad (3.110)$$

such that $\gamma_{abcde}^{\alpha\beta} \nabla_\beta \Omega_\alpha = \gamma_{abcde}^{\hat{\alpha}\hat{\beta}} \nabla_{\hat{\beta}} A_{\hat{\alpha}} = 0$. Here, we have considered the covariant derivatives $\nabla_\alpha = E_\alpha^M (\partial_M + \Omega_M)$ and $\nabla_{\hat{\alpha}} = E_{\hat{\alpha}}^M (\partial_M + \Omega_M)$, where, $\Omega_M^{[cd]} = A_M^{[cd]}$ ($A_0^{[cd]} = A_M^{[cd]} dz^M$).

In our discussion on deformations it will be crucial to demand that the vertex operators must be world-sheet primary fields. The left and right moving components of the energy momentum tensor are

$$T_- = \text{Str}(\frac{1}{2}A_{2-}A_{2-} + A_{1-}A_{3-} + \omega D_- \lambda), \quad (3.111)$$

$$T_+ = \text{Str}(\frac{1}{2}A_{2+}A_{2+} + A_{1+}A_{3+} + \hat{\omega} D_+ \hat{\lambda}). \quad (3.112)$$

and satisfy the conditions $\partial_+ T_- = \partial_- T_+ = 0$. The condition of no double poles of the vertex operator (3.106) with the energy-momentum tensors implies that

$$\nabla^A \nabla_A A_{\alpha\hat{\alpha}} = 0. \quad (3.113)$$

The remaining gauge freedom which leaves (2.29) invariant is given by

$$\delta A_{\alpha\hat{\alpha}} = \nabla_{\alpha} \hat{\Omega}_{\hat{\alpha}} + \nabla_{\hat{\alpha}} \Omega_{\alpha}, \quad \text{such that,} \quad \nabla^m \nabla_m \hat{\Omega}_{\hat{\alpha}} = \nabla^m \nabla_m \Omega_{\alpha} = 0. \quad (3.114)$$

Chapter 4

η -Deformation of the GS $AdS_5 \times S^5$

Having reviewed the supercoset formulation of the superstring in $AdS_5 \times S^5$, in this chapter we introduce a family of deformations of the GS superstring in $AdS_5 \times S^5$, the so-called Yang-Baxter or η deformation, which preserve integrability and local symmetries of the original model. The resulting target space fields of the model is derived. It is shown how the unimodular condition for the R -matrices arises to retain a non-linear sigma model interpretation. We briefly discuss recent progress to relate the Yang-Baxter deformations to (non)abelian T-duality and AdS/CFT correspondence.

4.1 The η -deformed action

The η -model of the GS superstring is an example of the so-called Yang-Baxter deformations [7–9]. The main ingredient for constructing the η -deformation is a linear operator $R : \mathfrak{g} \rightarrow \mathfrak{g}$ defined as

$$R = \frac{1}{2} R^{AB} t_A \wedge t_B, \quad R(X) = R^{AB} t_A \text{Str}(t_B X), \quad (4.1)$$

where r is a skew symmetric matrix and $X \in \mathfrak{g}$. This R -operator must satisfy the (modified) classical Yang-Baxter equation, (m)CYBE:

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = c[X, Y], \quad (4.2)$$

where the cases $c = \pm 1$ or $c = 0$, is known as the modified or the homogeneous classical Yang-Baxter equation respectively.

At the level of the action, the deformation is implemented through the Lie algebra operators:

$$\mathcal{O}_{GS-} = 1 - \eta R_g \circ d_{GS}, \quad \mathcal{O}_{GS+} = 1 + \eta R_g \circ \hat{d}_{GS} \quad (4.3)$$

here, R_g is the composite operator $R_g = Ad_g^{-1} \circ R \circ Ad_g$. We have also introduced a pair of operators, d_{GS} and $\hat{d}_{GS}$, which are a combination of projectors onto the gradings of the $\mathfrak{psu}(2,2|4)$ algebra

$$d_{GS} = P_1 + 2\hat{\eta}^{-2}P_2 - P_3, \quad \hat{d}_{GS} = -P_1 + 2\hat{\eta}^{-2}P_2 + P_3. \quad (4.4)$$

where $\hat{\eta} := (1 - c\eta^2)^{1/2}$.

The η deformed classical action is given by

$$S_{GS} = -\frac{(1 + c\eta^2)^2}{4(1 - c\eta^2)} (\gamma^{ij} - \epsilon^{ij}) \int Str(g^{-1} \partial_i g, d_{GS} \circ \mathcal{O}_{GS-}^{-1} g^{-1} \partial_j g), \quad (4.5)$$

Note that one recover the undeformed action (3.40) when $\eta = 0$. Let us define the deformed currents for the GS η -model

$$J_{GS-} = \mathcal{O}_{GS-}^{-1}(g^{-1} dg), \quad J_{GS+} = \mathcal{O}_{GS+}^{-1}(g^{-1} dg). \quad (4.6)$$

The equations of motion are obtained by varying the action respect to g as

$$\int Str(g^{-1} \delta g, \mathcal{E}), \quad (4.7)$$

where,

$$\mathcal{E} = \partial_+(d_{GS} J_{GS-}) + \partial_-(\hat{d}_{GS} J_{GS+}) + [J_{GS+}, dJ_{GS-}] + [J_{GS-}, \hat{d}J_{GS+}] = 0. \quad (4.8)$$

It is important to recall that to derive the Lax connection for the undeformed model, besides the equations of motion, one makes use of the Maurer-Cartan equations. The same is true for the deformed model, therefore, it is crucial to write the zero curvature condition for the Maurer-Cartan forms in terms of J_{GS-} and J_{GS+}

$$\mathcal{Z} = \partial_- J_{GS+} - \partial_+ J_{GS-} + [J_{GS-}, J_{GS+}] + \hat{\eta}^2 [d(J_{GS-}), \hat{d}(J_{GS+})] + \eta R_g(\mathcal{E}) = 0 \quad (4.9)$$

As a consequence of these relations, one can define a Lax pair satisfying the zero curvature condition:

$$L_+^i = J_{GS0+}^i + z\sqrt{1 + \hat{c}\eta^2} J_{GS1+}^i + z^{-2} \frac{1 + \hat{c}\eta^2}{1 - \hat{c}\eta^2} J_{GS2+}^i + z^{-1} \sqrt{1 + \hat{c}\eta^2} J_{GS3+}^i, \quad (4.10)$$

$$L_-^i = J_{GS0-}^i + z\sqrt{1 + \hat{c}\eta^2} J_{GS1-}^i + z^2 \frac{1 + \hat{c}\eta^2}{1 - \hat{c}\eta^2} J_{GS2-}^i + z^{-1} \sqrt{1 + \hat{c}\eta^2} J_{GS3-}^i, \quad (4.11)$$

where z is the spectral parameter.

Kappa-symmetry

As we have emphasized, κ -symmetry is needed to maintain a string theory interpretation. To that end, we will show that kappa invariance is preserved for the deformed model [7]. Consider a local fermionic transformation of the form

$$\delta g = g\varepsilon, \quad \varepsilon = (1 - \eta R_g)\rho_1 + (1 + \eta R_g)\rho_3. \quad (4.12)$$

where ρ_1 and ρ_3 are local fermionic parameter taking values in $\mathfrak{g}_1 \oplus \mathfrak{g}_3$ respectively. The variation of the action due to this transformation give us

$$\delta S_{GS} = \frac{(1 + c\eta^2)^2}{4(1 - c\eta^2)} (\gamma^{ij} - \varepsilon^{ij}) \int Str(\rho_1 P_3 \circ (1 + \eta R_g)(\mathcal{E}) + \rho_3 P_1 \circ (1 - \eta R_g)(\mathcal{E})), \quad (4.13)$$

This result can be put in the standard form when taking into account the following identities

$$P_1 \circ (1 - \eta R_g)(\mathcal{E}) = -4[J_{GS2+}^i, J_{GS3-}^i], \quad (4.14)$$

$$P_3 \circ (1 + \eta R_g)(\mathcal{E}) = -4[J_{GS2-}^i, J_{GS1+}^i]. \quad (4.15)$$

Analogously, this fermionic transformation generates the characteristic kappa symmetry of the GS action if

$$\begin{aligned} \rho_1 &= i\kappa_{+i}^1 J_{GS2-}^i + iJ_{GS2-}^i \kappa_{+i}^1, \\ \rho_3 &= i\kappa_{-i}^3 J_{GS2+}^i + iJ_{GS2+}^i \kappa_{-i}^3, \end{aligned} \quad (4.16)$$

and the two dimensional metric γ^{ij} transforms as

$$\delta_\kappa \gamma^{ij} = \frac{\hat{\eta}^2}{2} (Str(W[i\kappa_{1+}^i, J_{GS+1}^j]) + Str(W[i\kappa_{3-}^i, J_{GS-3}^j])). \quad (4.17)$$

where, W is a diagonal 8×8 matrix, $W = \text{diag}(1_4, -1_4)$.

Isometries

A natural question that arises in this context is whether different choices for R modify the global symmetry $PSU(2, 2|4)$ of the undeformed action, realized as left multiplication of the

coset representative g by a constant group element G

$$Gg = g'h, \quad h \in SO(4,1) \times SO(5). \quad (4.18)$$

A quick inspection of the action (4.5) teach us that any transformation of the form (4.18) that leaves R_g invariant represents an unbroken global symmetry fo the deformed action. In other words, we look for the set of G for which $R_{Gg} = Ad_g^{-1} \circ R_G \circ Ad_g = R_g$, which implies

$$R_G(X) = R(X). \quad (4.19)$$

In terms of the generators of the superalgebra $\mathfrak{psu}(2,2|4)$, it can be written as

$$R([t_A, X]) = [t_A, R(X)], \quad (4.20)$$

Analyzing this equation for a particular R-matrix gives us the (super)charges t_A preserved for the corresponding deformation.

4.2 Reading the target space fields

The target superspace of the η -model, as it was determined in [16], can be achieved in two steps: first, one reads the precise supervielbeins by setting the κ transformations of the deformed model in the standard form. Once the supergeometry in completely determined, the background fields can be read by computing the torsion and comparing to the standard expressions derived in chapter 2.

The supervielbeins

It is convenient to rewrite the action (4.5) as

$$S = -\frac{(1+c\eta^2)^2}{4(1-c\eta^2)} \int (\gamma_{ij} Str(J_{GS2-}^i, (J_{GS2-}^j) + \epsilon^{ij} Str(J_{iGS-}, \hat{B} J_{jGS-})) \quad (4.21)$$

where,

$$\hat{B} = \frac{1}{2}(P_1 - P_3 + \eta \hat{d}_{GS} \circ R_g \circ d_{GS}). \quad (4.22)$$

Let us consider the operator

$$M = O_{GS-}^{-1} O_{GS+} = 1 - 2P_2 + 2O_{GS-}^{-1} P_2, \quad (4.23)$$

which relates J_{GS-} to J_{GS+} as $J_{GS-} = MJ_{GS+}$. It follows that $J_{GS0-} = J_{GS0+}$, $J_{GS1-} = J_{GS1+}$, and $J_{GS3-} = J_{GS3+}$. Further, it is easy to see that P_2MP_2 implements a Lorentz transformation on $\mathfrak{g}_2$, which allows us to relate this operator to an element $h \in \mathfrak{g}_0$, such that $P_2MP_2 = P_2Ad_h^{-1} = Ad_h^{-1}P_2$, thus

$$J_{GS2-} = P_2MJ_{GS2+} = Ad_h^{-1}J_{GS2+}. \quad (4.24)$$

As a consequence of (4.12), (4.16), and (4.24), we can obtain the following kappa transformations

$$\delta_\kappa J_{GS2+} = 0, \quad \delta_\kappa J_{GS1+} = \{\kappa_{3i}, Ad_h J_{GS2+}^i\}, \quad \delta_\kappa J_{GS3-} = \{\kappa_{1i}, J_{GS2+}^i\}, \quad (4.25)$$

$$\delta_\kappa \gamma^{ij} = \frac{\hat{\eta}^2}{2} (Str(W[iAd_h \kappa_{1+}^i, Ad_h J_{GS+1}^j]) + Str(W[i\kappa_{3-}^i, J_{GS-3}^j])). \quad (4.26)$$

Now, one can read the supervielbeins E^A of the deformed geometry from (4.16) and (4.17) by comparing to the standard form of the κ transformations. They are given by [15]

$$E_2^a = A_{GS2+}^a, \quad E_1^\alpha = Ad_h A_{GS1+}^\alpha, \quad E_3^{\hat{\alpha}} = A_{GS3-}^{\hat{\alpha}}. \quad (4.27)$$

Background Superfields

Having obtained the supervielbeins of the deformed model, the next task consists in identifying the connection by comparing with the standard form obtained in chapter 2. For instance, we consider the torsion

$$T^a = dE^a + E^b \wedge \Omega_b^a, \quad (4.28)$$

$$T^{\alpha I} = dE^{\alpha I} + \frac{1}{4}(\gamma^{ab})_\beta^{\alpha I} E^{\beta I} \wedge \Omega_{ab}. \quad (4.29)$$

Accordingly with the discussion in Section 2.2, where the non-zero components of (4.28) and (4.29) were computed, we have that

$$T^a = E^\alpha \gamma_{\alpha\beta}^a E^\beta + E^{\hat{\alpha}} \gamma_{\hat{\alpha}\hat{\beta}}^a E^{\hat{\beta}}, \quad (4.30)$$

$$\begin{aligned} T^{\alpha I} &= \frac{1}{8} E^{\alpha I} (\gamma_a \mathcal{P})_\beta^{\alpha I} E^{\beta J} + E^a \psi_{ab}^{\alpha I} E^b + \frac{1}{8} E^a (\gamma^{bc} H_{abc})_{\beta I}^{\alpha I} E^{\alpha I} \\ &+ E^\gamma (\gamma_{\gamma\beta}^\alpha \gamma_a^{\alpha I \rho} \chi_\rho - 2\delta_{(\gamma}^{\alpha I} \chi_{\beta)}) E^\beta. \end{aligned} \quad (4.31)$$

Comparing (4.28) and (4.30), one recognizes the spin connection, so that, we can read the background superfields by comparing (4.29) and (4.31). The first step to enable this

comparison is to compute dE^a and $dE^{\alpha I}$ from (4.27) as

$$\begin{aligned} dE_2 = & \{J_{GS0+}, E_2\} + \frac{1}{2}\{E_1, E_1\} + \frac{1}{2}\{E_3, E_3\} - 2\hat{\eta}\{E_3, P_3 O_-^{-1} E_2\} \\ & + 4\hat{\eta}^{-1} P_2 O_+^{-1} \{E_2, E_3\} - 8P_2 O_+^{-1} \{E_2, P_3 O_-^{-1} E_2\} \\ & + 2\hat{\eta}^2 \{P_3 O_-^{-1} E^2, P_3 O_-^{-1} E^2\} + 2\eta \hat{\eta}^{-2} P_2 O_+^{-1} R_g \{E_2, E_2\}, \end{aligned} \quad (4.32)$$

and,

$$\begin{aligned} dE_3 = & \{J_{GS0+}, E_3\} - 2\{E_3, P_0 O_-^{-1} E_2\} + 2\hat{\eta} P_3 O_-^{-1} \{E_3, E_3\} \\ & + P_3 (4O_-^{-1} - 1 - 2\hat{\eta}^{-2}) Ad_h^{-1} \{E_2, E_1\} - 2\eta \hat{\eta}^{-1} P_3 O_-^{-1} R_g Ad_h^{-1} \{E_2, E_2\} \\ & + 2\hat{\eta} P_3 (4O_-^{-1} - 1 - 2\hat{\eta}^{-2}) \{Ad_h^{-1} E_2, P_1 O_-^{-1} E_2\}. \end{aligned} \quad (4.33)$$

$$\begin{aligned} dE_1 = & \{Ad_h J_{GS0+} - dhh^{-1}, E_3\} + \hat{\eta} P_1 Ad_h O_+^{-1} Ad_h^{-1} \{E_1, E_1\} \\ & + P_1 Ad_h (4O_-^{-1} - 1 - 2\hat{\eta}^{-2}) \{E_2, E_3\} - 2\eta \hat{\eta}^{-1} P_1 Ad_h O_-^{-1} R_g \{E_2, E_2\} \\ & + 2\hat{\eta} P_1 (4O_-^{-1} - 1 - 2\hat{\eta}^{-2}) \{E_2, P_3 O_-^{-1} E_2\}. \end{aligned} \quad (4.34)$$

Based on the above discussion, the spin connection for the η -background can be found as

$$\Omega_{ab} = J_{GS+ab} - 2\hat{\eta}(\gamma_{[a} E_2)_{\hat{\alpha}} M_{b]}^{\hat{\alpha}} - \frac{3}{2}\hat{\eta}^2 E^c M_{[a}^{\hat{\alpha}}(\gamma_b)_{\hat{\alpha}\hat{\beta}} M_{c]}^{\hat{\beta}} - \frac{1}{2}E^c (2M_{c[a,b]} - M_{ab,c}). \quad (4.35)$$

Now, we can make use of this information to compute explicitly (4.29), to identify the superfields of the η -background by comparison with (4.31):

$$H_{abc} = 3M_{[ab,c]} + 3\hat{\eta}^2 M_{[a}^{\hat{\alpha}}(\gamma_b)_{\hat{\alpha}\hat{\beta}} M_{c]}^{\hat{\beta}}, \quad (4.36)$$

$$P^{\alpha\hat{\alpha}} = 8\eta^{\beta\hat{\alpha}} (Ad_h \circ (1 + \hat{\eta}^{-2} - 4\mathcal{O}_{GS+}^{-1}))_{\beta}^{\alpha}, \quad (4.37)$$

$$\psi_{ab}^{\hat{\alpha}} = \frac{1}{4}\hat{\eta} [Ad_h M]_{[a}^{\hat{\beta}}(\gamma_b) \mathcal{P}]_{\hat{\beta}}^{\hat{\alpha}} - 2\eta \hat{\eta}^{-1} [O_-^{-1} R_g Ad_h^{-1}]_{cd}^{\hat{\alpha}} \eta_{ab}^{cd}, \quad (4.38)$$

$$\psi_{ab}^{\alpha} = -\frac{1}{4}\hat{\eta} M_{[a}^{\beta}(\gamma_b) \mathcal{P}]_{\beta}^{\alpha} + 2\eta \hat{\eta}^{-1} [Ad_h O_+^{-1} R_g]_{cd}^{\alpha} \eta_{ab}^{cd}, \quad (4.39)$$

$$\chi_{\hat{\alpha}} = \hat{\eta} \gamma_{\hat{\alpha}\hat{\beta}} M_a^{\hat{\beta}}, \quad \chi_{\alpha} = \hat{\eta} \gamma_{\alpha\beta} M_a^{\beta}, \quad (4.40)$$

Dilaton

In general, the η -background fields solve the generalized supergravity equations rather than standard supergravity ones. As a consequence, there is no a fundamental scalar field which could be read by using the above prescription.

One of the problems solved in [16] was to determine when a dilaton can be defined for the η -model. By comparison with the λ -model it was argued that the dilaton must be defined as

$$e^{-2\phi} = \text{Sdet}(O_+) = \text{Sdet}(1 + \eta R_g \hat{d}). \quad (4.41)$$

To determine when the η -model gives rise to a standard supergravity background, the dilaton must satisfy the consistency conditions (2.56), (2.114). For instance, taking into account (4.40) and (4.41), it can be shown that [16]

$$\nabla_{\hat{\alpha}} \phi = \chi_{\hat{\alpha}} + \frac{\eta \hat{\eta}}{2} [Ad_h^{-1}]_{\hat{\alpha}}^{\hat{\beta}} \hat{\mathcal{K}}^{AB} \text{Str}([t_A, R t_B] g t_{\hat{\alpha}} g^{-1}). \quad (4.42)$$

The vanishing of the second term on the right hand side can be translated to the following condition on the R -matrices

$$\text{Str}(R \circ \text{ad}_x,), \quad \forall x \in \mathfrak{g}, \quad (\text{i.e. } R^B_A f^A_{BC} = 0). \quad (4.43)$$

the so-called *unimodular* condition. Therefore, if we are to interpret the η -model as a string sigma model, we must restrict our attention to unimodular deformations.

Originally, the η -deformation introduced by Vicedo, Magro, and Delduc [7, 8], was based on R -operators solving the mCYBE ($c = 1$). However, none of the R -matrices considered so far [12, 62] are unimodular, not reproducing a standard supergravity background, instead it solves the generalized supergravity equations of motion[14]. Until now, it is not known whether an unimodular solution exist.

It was soon realized that the Yang-Baxter deformations could be generalized also to the case when R solves the CYBE [9]. These equations accepts many solutions for $\mathfrak{psu}(2, 2|4)$ and generate many different integrable deformations of the $AdS_5 \times S^5$ superstring. These R -matrices can be divided into two classes: abelian and non-abelian, depending on whether the associated generators (4.1) commute or not.

By definition the abelian R -matrices are unimodular since $R^{AB}[t_A, t_B] = 0$, and generate several supergravity solutions. These deformations include the Lunin-Maldacena background [65], Schrödinger and dipole type geometries [66] and the gravity dual of noncommutative Super Yang-Mills [67]. It was proven in [19] that the abelian deformations are equivalent to sequences of TsT transformations. It also includes the fermionic version of TsT transformations when the R matrix includes fermionic generators.

For non-abelian R -matrices the analysis is more involved because, in general, they do not give rise the unimodular condition. There is no a complete classification of unimodular R

matrices which satisfy the CYBE on $\mathfrak{psu}(2,2|4)$. This problem was partially solved in [16] for the bosonic subalgebra $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$ by constructing all the rank-four non-abelian R -matrices. It was observed that the resulting deformed model can be achieved by a combination non abelian T-dualities of the original $AdS_5 \times S^5$ model [68, 18, 21]. The subgroup in which one dualizes is determined by the structure of the R -matrix as we explain below.

4.3 Homogeneous Yang-Baxter deformations and non-abelian T-duality

The relation between homogeneous YB deformations and non-abelian T-duality relies on the fact that there is a one-to-one correspondence between solutions of the CYBE and symplectic submanifolds of $\mathfrak{g}$. Let us recall that a Lie algebra $\tilde{\mathfrak{g}}$ is symplectic if it is endowed with a non-degenerate 2-cocycle ω

$$\omega(X, Y) = -\omega(Y, X), \quad \omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0, \quad X, Y, Z \in \tilde{\mathfrak{g}}. \quad (4.44)$$

For any solution of the CYBE ($R = R^{AB}t_A \wedge t_B$) on the algebra $\mathfrak{g}$, there always exist a subalgebra $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ on which R is non-degenerate, and, as a consequence, R defines a symplectic structure on $\tilde{\mathfrak{g}}$, via the 2-cocycle defined as

$$\omega(t_A, t_B) = R_{AB}^{-1}. \quad (4.45)$$

Besides that, the 2 co-cycle ω defines a closed two-form on $\tilde{\mathfrak{g}}$ given by $\tilde{B} = \omega(\tilde{g}^{-1}d\tilde{g}, \tilde{g}^{-1}d\tilde{g})$. As it could be anticipated, the homogeneous YB deformation can be obtained from the undeformed $PSU(2,2|4)$ model by adding a topological term $\zeta \tilde{B}$ and then performing a non-abelian T-duality on $\tilde{G}$ ($Lie \tilde{G} = \tilde{\mathfrak{g}}$) [24, 25]. After a field redefinition, both models can be identified by relating the deformation parameters as $\eta = \zeta^{-1}$. Further, in this formulation, the deformations based on abelian R matrices are also incorporated after one realize that the TsT transformations can be interpreted as a special case of non-abelian duality for the central extension of the abelian subgroup: the deformation parameter γ works as the background contribution of the central generator.

In this framework, the unimodular condition can be translated into a simple condition on the structure constants; considering the definition (4.45), the co-cycle condition (4.44) can be written as $(R^{-1})_{A[B}f^A_{CD]} = 0$. After contracting with R^{EA} and assuming that R satisfies

the unimodular condition, it remains the condition

$$f^A_{AB} = 0, \text{ i.e., } \text{Str}(ad_X) = 0, \forall X \in \tilde{\mathfrak{g}}, \quad (4.46)$$

which is the condition found in [63, 64] in order to obtain a non-abelian dual with vanishing trace anomaly.

4.4 Beyond Yang-Baxter deformations

The emergence of Yang-Baxter deformations and generalized supergravity has been further interpreted in the context of AdS/CFT correspondence and open-closed string map [27–33].

On one hand, inhomogeneous YB deformations [7] replace the original $\mathfrak{psu}(2,2|4)$ symmetry by the classical analog of the quantum group. In the absence of unimodular R -matrices for the mCBYE, inhomogeneous YB deformations [7] do not yield a supergravity solution and a possible interpretation in terms of superstring theory and AdS/CFT correspondence remains elusive. The homogeneous YB deformations are more treatable. It has been argued that they correspond to Drinfeld twists of the superconformal algebra [23]. Using *AdS/CFT* correspondence, the corresponding dual of the YB deformations has been identified as space-time deformations of $\mathcal{N} = 4$ SYM which can be realized as an appropriate Moyal product for non-commutative (NC) theories. The NC theories are characterized by the antisymmetric matrix $\Theta^{\mu\nu}$ through the following commutator

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}, \quad (\mu, \nu = 0, \dots, 3) \quad (4.47)$$

This possibility was exploited in [27] by using the Seiberg-Witten map to show the existence of corresponding non-commutative deformations of $\mathcal{N} = 4$ SYM with structure constant

$$\Theta^{MN} = -2\eta R^{MN}, \quad (M, N = 0, \dots, 3, z), \quad (4.48)$$

where z is the radial direction of AdS. In this scenario, all the information about the YB deformation, as viewed for the open string, is encoded in Θ , while the AdS geometry remains undeformed. From this perspective, the unimodular condition and the Killing vector K are unified in a single equation

$$\nabla_M \Theta^{MN} = K^N. \quad (4.49)$$

Having understood the connection between η -deformations of the superstring and non-abelian T-duality of the undeformed model, it would be interesting to enlarge this understanding

to include the λ -deformations [6]. For the bosonic σ case, it was argued in [71] that the λ deformed model as well as the η deformed one are related via Poisson-Lie T-duality. One consequence of this duality is that the target space geometry of the λ -model and of the Poisson-Lie T-dual of the η -model are equivalent up to an analytic continuation, however the connection to the superstrings remains unsolved.

Chapter 5

Discussion

The aim of this thesis was to construct the Yang-Baxter deformation of the $AdS_5 \times S^5$ pure spinor string. Unlike the GS construction, we deform the action based on the homological perturbation theory which allows us to deform the model in a BRST-invariant manner. We show that the resulting model is classically integrable. reading the target space superfields we showed that it corresponds to the η -target background. Based on our discussion of the GS η -model, the resulting model only describes a type IIB supergravity space if the R matrices satisfy the unimodular condition. It is important to note that there is a one-to-one map between vertex operators in the cohomology of the undeformed BRST charge and the deformed target space. Further, the restriction of this map to the subspace of primary vertex operators maps to the *physical* deformed target space, i.e., those which solve the type IIB supergravity equations.

Having defined the BRST operator for the η -deformation of $AdS_5 \times S^5$ the next step consists in studying the cohomology for massless vertex operators. In principle, it would be possible to construct the corresponding generalization for the radius changing operator of AdS [59, 61], which allowed to define a composite b ghost and the zero mode measure factor for AdS without introducing non-minimal variables. It should emerge from the surface term in $Q_{def}S_{def} = \int d^2z Q_{def}L_{def} = \int d^2z(\partial_- f_+ - \partial_+ f_-)$ as follows. The nilpotency of the BRST charge

$$\partial_- Q_{def}(f_+) = \partial_+ Q_{def}(f_-), \quad (5.1)$$

implies that

$$Q_{def}(f_+) = \partial_+ \mathcal{U}, \quad Q_{def}(f_-) = \partial_- \mathcal{U}. \quad (5.2)$$

which gives us the unintegrated vertex operator $\mathcal{U}$. In our case,

$$f_+ = Str(\lambda_1 \mathcal{J}_{3+} - \lambda_3 \mathcal{J}_{1+}), \quad f_- = Str(\lambda_1 \mathcal{J}_{3-} - \lambda_3 \mathcal{J}_{1-}). \quad (5.3)$$

Since, the η -deformation provides a considerable number of target superspaces with non-degenerate RR fluxes, it would be interesting to study them in the context of [73].

On the other hand, it would be interesting to look for the corresponding generalization of the vertex operator (??) derived in [38], which is responsible for the YB deformation of the pure spinor string in AdS. This vertex should be perturbatively constructed in series of η as

$$V_{def}[B](\varepsilon, \varepsilon') = V_0 + \eta V_1 + \eta^2 V_2 + \dots, \quad (5.4)$$

where V_0 is the original vertex operator (??). In this construction, the terms V_i are to be solved order by order in η after imposing the condition of BRST-closeness of (5.4). The existence of such vertices could help us to understand whether a double deformation of the $AdS_5 \times S^5$ superstring is possible.

In this thesis, we have shown that the mCYBE condition on the R -matrices, needed for integrability of the deformed action, arises by imposing nilpotency of the deformed BRST charge. This fact may reflect a deeper connection between integrable deformations and BRST symmetry. For instance, this possibility can be explored in the context of the λ -deformation of the PS strings in AdS [74]. It is well known that the λ -deformations modifies the original local symmetry $\delta g = h_0 g$ to $\delta_\lambda g = [h_0, g]$ [6]. Let us show how it arises in the PS formalism. The BRST transformation of the λ -model is given by ¹

$$\varepsilon Q(g) = (\varepsilon \lambda_3 + \tilde{\lambda} \varepsilon \lambda_1) g - g (\tilde{\lambda}^{-1} \varepsilon \lambda_3 + \tilde{\lambda}^2 \varepsilon \lambda_1) \quad (5.5)$$

One finds that

$$\varepsilon' Q(\varepsilon Q(F)) = [\Lambda_0(\tilde{\lambda}, \varepsilon, \varepsilon'), g], \quad (5.6)$$

where $\Lambda_0(\tilde{\lambda}, \varepsilon, \varepsilon') = \tilde{\lambda}[\varepsilon \lambda_1, \varepsilon' \lambda_3]$ is an element of the Lorentz algebra. Therefore, we conclude that consistency of the BRST operator for the λ -deformation requires the deformation of the local Lorentz symmetry as

$$\delta_\lambda(g) = [h_0, g], \quad h_0 \in \mathfrak{g}_0. \quad (5.7)$$

In this case, the equations of motion of the deformed theory possesses the same structure the original model, therefore the deformation remains codified in the symplectic structure. These examples suggest that the natural framework to understand integrable deformations in the context of PS formulation should be the Batalin-Vilkovisky formalism [60]

¹Here, the deformation parameter is written as $\tilde{\lambda}$ to not confuse with the graded pure spinor ghosts.

It is important to remark that our analysis is completely classical and it is plausible to expect that radiative corrections may enforce further restrictions; an explicit calculation of the beta function for the YB-deformation is not known. In the PS formulation we expect that the Weyl symmetry can only be recovered when the deformed target space allows a type IIB supergravity solution suggesting that the central charge of the deformed model must be proportional to the unimodular condition.

Appendix A

γ -matrices in ten dimensions

In this appendix we introduce the relevant identities of the ten-dimensional γ -matrices in order to solve the Bianchi identities in Chapter 2.

In ten dimensions, a Dirac spinor Ψ have 32 complex degrees of freedom. This kind of spinors are not the smallest irreducible representation of $SO(9,1)$. Being in an even dimensional space, we can always impose the *Weyl condition* which splits the spinor into left and right handed components which transform independently under Lorentz transformations.

$$\Psi = \begin{pmatrix} \psi_L^\alpha \\ \psi_{\dot{\alpha}R} \end{pmatrix}, \quad (\text{A.1})$$

In ten dimensions, it is also possible to impose the so-called *Majorana condition* which relates the charge-conjugate Dirac spinor to the original one. This condition reduces by half the number of degrees of freedom and the resulting spinor is necessarily real. After imposing these condition in ten dimensions we remains with a 16-dimensional real spinor. Consequently, instead of using the 32×32 ten-dimensional Dirac matrices Γ , we use the γ matrices, of dimension 16×16 defined as

$$\Gamma^a = \begin{pmatrix} 0 & (\gamma^a)^{\dot{\alpha}\beta} \\ (\gamma^a)_{\alpha\dot{\beta}} & 0 \end{pmatrix}, \quad (\text{A.2})$$

Since Weyl spinors of different chirality belongs to inequivalent representations of $SO(9,1)$ it is not possible rise and lower indices. For instance, we will omit the dotted spinors and use only dotted indices.

The Pauli matrices γ satisfy

$$(\gamma_a)_{\alpha\beta}(\gamma_b)^{\beta\gamma} + (\gamma_b)_{\alpha\beta}(\gamma_a)^{\beta\gamma} = -2\eta_{ab}\delta_{\alpha}^{\gamma}. \quad (\text{A.3})$$

Let us define the antisymmetric product of gamma matrices as

$$(\gamma^{a_1 \dots a_n})_{\alpha\beta} = (\gamma^{[a_1} \dots \gamma^{a_n]})_{\alpha\beta}. \quad (\text{A.4})$$

Notice that the γ^a and γ^{abcde} are symmetric in their spinor indices, while γ^{abc} is antisymmetric. Therefore, any bispinor can be decomposed as

$$\psi_{\alpha}\phi^{\beta} = \frac{1}{16}(\delta_{\alpha}^{\beta}(\psi\phi) + \frac{1}{2!}(\gamma_{ab})_{\alpha}^{\beta}(\psi\gamma^{ab}\phi) + \frac{1}{4!}(\gamma_{abcd})_{\alpha}^{\beta}(\psi\gamma^{abcd}\phi)), \quad (\text{A.5})$$

$$\psi_{\alpha}\phi_{\beta} = -\frac{1}{16}(\gamma_{\alpha\beta}^a(\psi\gamma_a\phi) + \frac{1}{3!}\gamma_{\alpha\beta}^{abc}(\psi\gamma_{abc}\phi) + \frac{1}{5!}\gamma_{\alpha\beta}^{abcde}(\psi\gamma_{abcde}\phi)). \quad (\text{A.6})$$

We present some useful identities that were used to solve the Bianchi identities

$$\begin{aligned} &(\gamma^a)_{(\alpha\beta}(\gamma_a)_{\sigma)\gamma} = 0, \\ &(\gamma_{ab})_{\alpha}^{\gamma}(\gamma^{ab})_{\beta}^{\sigma} = -2(\delta_{\alpha}^{\gamma}\delta_{\beta}^{\sigma} + 4\delta_{\beta}^{\gamma}\delta_{\alpha}^{\sigma} + (\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\sigma}), \\ &(\gamma^a)_{(\alpha\beta}(\gamma_{ab})_{\gamma)}^{\sigma} = -(\gamma_b)_{(\alpha}\delta_{\gamma)}^{\sigma}, \\ &(\gamma^{abc})_{\alpha\beta}(\gamma_c)^{\beta\gamma} = -8(\gamma^{ab})_{\alpha}^{\gamma}, \\ &(\gamma^{ab})_{\alpha}^{\beta}(\gamma_b)^{\beta\gamma} = -9(\gamma^a)_{\alpha\gamma}, \\ &(\gamma^{abcd})_{\alpha}^{\beta}(\gamma_b)^{\beta\gamma} = -7(\gamma^{bcd})_{\alpha\gamma}, \\ &(\gamma^{abcde})_{\alpha\beta}(\gamma_{ab})_{\beta}^{\gamma} = -42(\gamma^{cde})_{\alpha}^{\gamma}, \\ &(\gamma^{abc})_{\alpha\beta}(\gamma_{ab})_{\beta}^{\gamma} = -72(\gamma^c)_{\alpha}^{\gamma}, \\ &(\gamma^{abcde})_{\alpha\beta}(\gamma_a)_{\beta\gamma} = -6(\gamma^{bcde})_{\alpha}^{\gamma}, \\ &(\gamma_{ac})_{\beta}^{\alpha}(\gamma_{ab})_{\gamma}^{\beta} = -8(\gamma^b{}_c)_{\gamma}^{\alpha} - 9\delta_c^b\delta_{\gamma}^{\alpha}, \\ &(\gamma_{ab})_{\beta}^{\alpha}(\gamma_{ab})_{\gamma}^{\beta} = -90\delta_{\gamma}^{\alpha}, \\ &(\gamma_{abc})_{\alpha\beta}(\gamma_{abc})^{\beta\gamma} = 720\delta_{\gamma}^{\alpha}. \end{aligned} \quad (\text{A.7})$$

Appendix B

Some properties of $\mathcal{P}$

First of all we note that

$$Str(\mathcal{P}_{13}A_1, A_3) = Str(A_1, \mathcal{P}_{31}A_3). \quad (\text{B.1})$$

$$\begin{aligned} Str(\mathcal{P}_{13}A_1, \mathcal{P}_{31}A_3) &= Str(\mathcal{P}_{13}A_1, A_3 + [\lambda_1, S_2]), \\ &= Str(\mathcal{P}_{13}A_1, A_3) + Str([\mathcal{P}_{13}A_1, \lambda_1], S_2), \\ &= Str(\mathcal{P}_{13}A_1, A_3) \end{aligned} \quad (\text{B.2})$$

on the other hand

$$\begin{aligned} Str(\mathcal{P}_{13}A_1, \mathcal{P}_{31}A_3) &= Str(A_1 + [\lambda_3, S_2], \mathcal{P}_{31}A_3), \\ &= Str(A_1, \mathcal{P}_{31}A_3) + Str([\mathcal{P}_{31}A_3, \lambda_3], S_2), \\ &= Str(A_1, \mathcal{P}_{31}A_3). \end{aligned} \quad (\text{B.3})$$

In order to prove the some important properties of $\mathcal{P}$ we use the following theorem (see section 7 in [41]) If $[\lambda_1, [\lambda_3, S_2]] = 0$, for any S_2 , then it implies that $[\lambda_3, S_2] = 0$. Analogously, if $[\lambda_3, [\lambda_1, S_2]] = 0$, then $[\lambda_1, S_2] = 0$.

$$Q_{0L}\mathcal{P}_{13}(gt_ag^{-1})_1 = 0$$

Proof.

$$\begin{aligned} 0 = Q_{0L}[\lambda_1, \mathcal{P}_{13}(gt_ag^{-1})] &= [\lambda_1, Q_{0L}\mathcal{P}_{13}(gt_ag^{-1})], \\ &= [\lambda_1, [\lambda_3, (gt_ag^{-1})_1 + Q_{0L}S_2]] \end{aligned} \quad (\text{B.4})$$

Hence, from the above theorem

$$0 = [\lambda_3, (gt_ag^{-1})_1 + Q_{0L}S_2] = Q_{0L}((gt_ag^{-1})_1 + [\lambda_3, S_2]), \quad (\text{B.5})$$

Hence, $Q_{0L}\mathcal{P}_{13}(gt_ag^{-1})_1 = 0$. $Q_{0R}\mathcal{P}_{13}(gt_ag^{-1})_1 = [\lambda_1, (gt_ag^{-1})_0]$

Proof.

$$0 = Q_{0R}[\lambda_1, \mathcal{P}_{13}(gt_ag^{-1})_1] = [\lambda_1, [\lambda_1, (gt_ag^{-1})_1] + [\lambda_3, Q_{0R}S_2]], \quad (\text{B.6})$$

As shown above, this equality implies that

$$0 = [\lambda_1, Q_{0R}[\lambda_3, S_2]] \implies Q_{0R}[\lambda_3, S_2] = 0, \quad (\text{B.7})$$

and it follows that $Q_{0R}\mathcal{P}_{13}(gt_ag^{-1})_1 = [\lambda_1, (gt_ag^{-1})_0]$. Analogously, it can be proved that

$$Q_{0R}\mathcal{P}_{31}(gt_ag^{-1})_3 = 0, \quad (\text{B.8})$$

and

$$Q_{0L}\mathcal{P}_{31}(gt_ag^{-1})_3 = [\lambda_3, (gt_ag^{-1})_0]. \quad (\text{B.9})$$

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