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Invariância Conforme e Teoria de Campo de Liouville

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Conformal Invariance and Liouville Field Theory

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Abstract

In this work, we make a brief review of the Conformal Field Theory in two dimensions, in order to understand some basic definitions in the study of the Liouville Field Theory, which has many application in theoretical physics like string theory, general relativity and supersymmetric gauge field theories.

In particular, we focus on the analytic continuation of the Liouville Field Theory, context in which an interesting relation with the Chern-Simons Theory arises as an extension of its well-known relation with the Wess-Zumino-Witten model.

Thus, calculating correlation functions by using the complex solutions of the Liouville Theory will be crucial aim in this work in order to test the consistency of this analytic continuation.

We will consider as an application the time-like version of the Liouville Theory, which has several applications in holographic quantum cosmology and in studying tachyon condensates.

Finally, we calculate the three-point function for the Wess-Zumino-Witten model for the standard Kac-Moody level $k > 2$ and the particular case $0 < k < 2$, the latter has an interpretation in time-dependent scenarios for string theory. Here we will find an analogue relation we find by comparing the correlation function of the time-like and space-like Liouville Field Theory.

Keywords

Conformal Field Theory, Liouville Field Theory, Analytic Continuation, Time-like Liouville Theory, Wess-Zumino-Witten model.

Resumo

Neste trabalho, nós fazemos uma breve revisão da Teoria de Campo Conforme em duas dimensões, a fim de entender algumas definições básicas do estudo da Teoria de Campo de Liouville, que tem muitas aplicações em física teórica como a teoria das cordas, a relatividade geral e teorias de campo de calibre supersimétricas.

Em particular, vamos nos concentrar sobre a continuação analítica da Teoria de Campo de Liouville, contexto no qual uma interessante relação com a Teoria de Chern-Simons surge como uma extensão de sua relação conhecida com o modelo de Wess-Zumino-Witten.

Assim, o cálculo das funções de correlação usando as soluções complexas da Teoria Liouville será o objectivo fundamental neste trabalho, a fim de testar a consistência da continuação analítica.

Vamos considerar como uma aplicação a versão time-like da Teoria de Liouville, que tem várias aplicações em cosmologia quântica holográfica e no estudo de condensados de tachyon.

Finalmente, calculamos a função de três pontos para o modelo de Wess-Zumino-Witten no nível de Kac-Moody $k > 2$ e o caso particular $0 < k < 2$, este último tem uma interpretação em cenários dependentes do tempo para a teoria das cordas. Aqui nós vamos encontrar uma relação análoga ao que temos para a função de correlação do space-like e time-like na Teoria de Campo de Liouville.

Keywords

Teoria de Campo Conforme, Teoria de Campo de Liouville, Continuação Analítica , Teoria de Liouville Time-like, Modelo de Wess-Zumino-Witten.

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Chapter 1

Introduction

Liouville field theory has many applications on several areas of theoretical physics, including string theory [2] [3] [4], general relativity and supersymmetric gauge theories [6] [5] [36] [48] [56].

In this work, we are interested in studying the analytic continuation of Liouville theory, or in other words to make complex its parameters. Thus, a principal ingredient for us will be the correlation function for three points of a product of three primary fields on the Riemann sphere, that has an exact formula known as the DOZZ formula [55] [50].

Liouville theory is suitably parametrized in terms of a quantity b and its central charge $c = 1 + 6Q^2$, where $Q = b + b^{-1}$. In the semiclassical limit, when $b \rightarrow 0$, we can consider two cases for the momentum α_i , heavy operators whose $\alpha_i = \eta_i/b$ with η_i fixed as $b \rightarrow 0$. Insertions of these operators change the saddle points that dominate the functional integral; and light operators $\alpha_i = \sigma_i b$, with σ_i fixed as $b \rightarrow 0$. These operators do not affect the saddle points in the semiclassical limit. A saddle point is simply a real solution of the classical field equations [55]. We define the physical region as that where $\eta_i < 1/2$ and $\sum_i \eta_i > 1$. Here the semiclassical limit at the path integral is determined by a real saddle point [22].

In Chapter 2, we give a brief review of Conformal field theory (CFT), focusing on the case of two-dimensions. In order to introduce some basic concepts that we need to understand the rest of this work. [1] [9]

In Chapter 3, we introduce the Liouville field theory and calculate the three-point

function following the seminal work by Zamolodchikov and Zamolodchikov [55], we also compute the four-point function in a modern approach as was proposed by Teshner to obtain the DOZZ formula [50]. The relation proposed by Teshner for the structure constants will be crucial in the Chapter 6.

In the Chapter 4, after making a convenient parametrization, we show the form of the complex solution of the Liouville field theory (LFT) and analyze the two-point and three-point functions in this context. Also we show that the analytic continuation of the DOZZ formula in a restricted region of η , can be interpreted in terms of complex classical solutions [22].

In Chapter 5, we consider singular solutions coming from our last parametrization, which will be interpreted. It is interesting to see that those solutions are related to the Chern-Simons theory [54].

In Chapter 6, we consider the so-called time-like version of Liouville theory, that has other interesting applications. Time-like LFT has been considered in holographic quantum cosmology [15] [43], studying tachyon condensates [19] [41] [44], and time-dependent scenarios for string theory [24]. Here review the proposal given by Harlow, Maltz and Witten [22] as well as the well-known free field method for the DOZZ formula. Finally, in the Chapter 7, we study briefly the Wess-Zumino-Novikov-Witten (WZNW) model focusing on the computation of the three-point function for $k > 2$ and $0 < k < 2$, by means of the free-field method.

Chapter 2

Conformal Field Theories

Conformal field theories (CFT's) are a particular class of field theories characterized by a type of symmetry transformation whose net effect on the metric is to multiply it by a positive function, thus preserving angles. The approach to study conformal field theories is somewhat different from the usual approach for quantum field theories. Indeed, instead of starting with a classical action for the fields and quantising them via the canonical quantisation or the path integral method, one employs the symmetries of the theory [9].

For example, in statistical mechanics as a system approaches a second order phase transition its correlation length diverges. At the critical point the theory possesses no dimensional parameter and is scale or dilatation invariant; in two dimensions the field theory describing the critical point turns out to be, not just dilatation invariant, but conformally invariant.

The operator product $\phi(x)\phi(0)$ of some quantum field theories becomes independent of mass in the limit of small x . This has led to the suggestion that the elementary constituents of matter, which are the relevant degrees of freedom at very small distance, may be described by theories with conformally invariant small distance limits. The short distance behavior of field theories is intimately related to their renormalisation properties. The renormalisation properties of correlation functions are constrained by the need to obey the Callan-Symanzik renormalisation group equations. A necessary, though insufficient, condition for a theory to possess conformal invariance is that the renormalisation group flow has a fixed point; this means that the Callan-Symanzik function $\beta(g)$ has a zero [37].

The main object of a field theory is the calculation of correlation functions, which are the physically measured quantities. In general, non-trivial theories with conformal invariant correlation functions are very difficult to find. However, in two dimensions, where the conformal group is infinite dimensional, the situation is somewhat better [1].

2.1 Conformal Invariance

Let us consider a d dimensional space-time with flat metric $g_{\mu\nu} = \eta_{\mu\nu}$. The conformal group is defined as a subgroup of the coordinate transformations that leaves the metric tensor invariant up to a scale, i.e.

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x), \quad (2.1)$$

$$g'_{\rho\sigma}(x') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x)g_{\mu\nu}(x).$$

The conformal transformation is locally equivalent to a rotation and a dilatation, that also leave the metric invariant up to a scale. The conformal transformation preserves the angle between two crossing curves. The set of conformal transformations forms a group which has the Poincaré group as a subgroup corresponding to the scale factor $\Lambda(x) = 1$.

Under an infinitesimal coordinate transformation $x'^\rho = x^\rho + \epsilon^\rho(x) + \mathcal{O}(\epsilon^2)$. Such a transformation induces a variation of the metric tensor of the form

$$\begin{aligned} \eta'_{\rho\sigma}(x') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} &= \eta_{\rho\sigma} \left(\delta_\mu^\rho + \frac{\partial \epsilon^\rho}{\partial x^\mu} + \mathcal{O}(\epsilon^2) \right) \left(\delta_\nu^\sigma + \frac{\partial \epsilon^\sigma}{\partial x^\nu} + \mathcal{O}(\epsilon^2) \right), \\ &= \eta_{\mu\nu} + \eta_{\mu\sigma} \frac{\partial \epsilon^\sigma}{\partial x^\nu} + \eta_{\rho\nu} \frac{\partial \epsilon^\rho}{\partial x^\mu} + \mathcal{O}(\epsilon^2), \\ &= \eta_{\mu\nu} + \left(\frac{\partial \epsilon_\mu}{\partial x^\nu} + \frac{\partial \epsilon_\nu}{\partial x^\mu} \right) + \mathcal{O}(\epsilon^2). \end{aligned} \quad (2.2)$$

To get a conformal transformation, the change of variables must satisfy the requirement (2.1), implying

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x)\eta_{\mu\nu}, \quad (2.3)$$

where $f(x)$ is determined by taking the trace on both sides of last equation

$$\begin{aligned}\eta^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) &= f(x)d, \\ f(x) &= \frac{2}{d}\partial \cdot \epsilon.\end{aligned}\tag{2.4}$$

So we find the following restriction on the infinitesimal transformation to be conformal

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu},\tag{2.5}$$

These are the *Killing-Cartan equations* and the functions ϵ_μ satisfying these equations are called the *conformal Killing vectors*.

Other useful relations are

$$(d-1)\square(\partial \cdot \epsilon) = 0,\tag{2.6}$$

$$2\partial_\mu\partial_\nu\epsilon_\rho = \frac{2}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)(\partial \cdot \epsilon).\tag{2.7}$$

After having obtained the condition for the infinitesimal transformations that satisfy (2.1), let us determine the conformal group in the case of dimension $d > 2$.

We see that (2.6) implies that $(\partial \cdot \epsilon)$ is at most linear in x^μ , then it follows that ϵ_μ is at most quadratic in x^ν and we can introduce the ansatz

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho,\tag{2.8}$$

where $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho} \ll 1$ are constants.

- The term a_μ describes infinitesimal translations $x'^\mu = x^\mu + a^\mu$ with generator is the momentum operator $P_\mu = -i\partial_\mu$.
- If we insert the last expression in (2.5), we get

$$\begin{aligned}\partial_\mu(a_\nu + b_{\nu\mu}x^\mu + c_{\nu\mu\rho}x^\mu x^\rho) + \partial_\nu(a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho) \\ = \frac{2}{d}\partial^\mu(a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho)\eta_{\mu\nu}, \\ b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}(\partial^\rho b_{\rho\sigma}x^\sigma)\eta_{\mu\nu}, \\ b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}(\eta^{\rho\sigma}b_{\rho\sigma})\eta_{\mu\nu},\end{aligned}$$

$b_{\mu\nu}$ can be split a symmetric and antisymmetric part

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu},$$

where $m_{\mu\nu} = -m_{\nu\mu}$ and corresponds to infinitesimal rotations $x'^\mu = (\delta_\nu^\mu + m_\nu^\mu)x^\nu$, for which the generator is the angular momentum operator $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$. The symmetric term $\alpha\eta_{\mu\nu}$ describes infinitesimal scale transformations $x'^\mu = (1 + \alpha)x^\mu$ with generator $D = -ix^\mu\partial_\mu$.

- The quadratic term can be studied using (2.7),

$$\begin{aligned} \partial_\mu\partial_\nu(a_\rho + b_{\rho\sigma}x^\sigma + c_{\rho\sigma\gamma}x^\sigma x^\gamma) &= \partial_\mu\{b_{\rho\sigma}\delta_\nu^\sigma + c_{\rho\sigma\gamma}\delta_\nu^\sigma x^\gamma + c_{\rho\sigma\gamma}x^\sigma\delta_\nu^\gamma\}, \\ &= c_{\rho\sigma\gamma}\delta_\nu^\sigma\delta_\mu^\gamma + c_{\rho\sigma\gamma}\delta_\mu^\sigma\delta_\nu^\gamma, \\ &= 2c_{\rho\mu\nu}. \end{aligned} \tag{2.9}$$

$$\begin{aligned} \partial \cdot \epsilon &= \partial^\mu\{a_\mu + b_{\mu\sigma}x^\sigma + c_{\mu\sigma\gamma}x^\sigma x^\gamma\}, \\ &= b_{\mu\sigma}\partial^\mu x^\sigma + c_{\mu\sigma\gamma}(\partial^\mu x^\sigma)x^\gamma + c_{\mu\sigma\gamma}x^\sigma(\partial^\mu x^\gamma), \\ &= b_\mu^\mu + 2c_{\mu\gamma}^\mu x^\gamma. \end{aligned} \tag{2.10}$$

Replacing (2.9) and (2.10) in (2.7)

$$\begin{aligned} c_{\rho\mu\nu} &= \frac{1}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)(c_{\mu\gamma}^\mu x^\gamma), \\ &= \frac{1}{d}c_{\lambda\gamma}^\lambda\{-\eta_{\mu\nu}\delta_\rho^\gamma + \eta_{\rho\mu}\delta_\nu^\gamma + \eta_{\nu\rho}\delta_\mu^\gamma\}, \\ &= \frac{1}{d}\{\eta_{\rho\mu}c_{\lambda\nu}^\lambda + \eta_{\nu\rho}c_{\lambda\mu}^\lambda - \eta_{\mu\nu}c_{\lambda\rho}^\lambda\}, \end{aligned}$$

if $b_\mu = \frac{1}{d}c_{\lambda\mu}^\lambda$

$$c_{\mu\nu\rho} = \eta_{\mu\nu}b_\rho + \eta_{\mu\rho}b_\nu - \eta_{\nu\rho}b_\mu.$$

These transformations are called *Special Conformal Transformations* (SCT) with the following infinitesimal form

$$x'^\mu = x^\mu + 2(x \cdot b)x^\mu - (x \cdot x)b^\mu, \tag{2.11}$$

for which the generator is written as $K_\mu = -i(2x_\mu x^\nu\partial_\nu - (x \cdot x)\partial_\mu)$.

We identified the infinitesimal conformal transformation but, in order to determine the conformal group, we will need the finite version of the last transformation which is

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2}. \quad (2.12)$$

Summarizing,

$$\begin{aligned} x \rightarrow x'^\mu &= x^\mu + a^\mu, \\ x \rightarrow x'^\mu &= \Lambda^\mu_\nu x^\nu \quad (\Lambda^\nu_\mu \in SO(d+1, 1)), \\ x \rightarrow x'^\mu &= \alpha x^\mu, \\ x \rightarrow x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}. \end{aligned} \quad (2.13)$$

The first and second lines correspond to the Poincaré transformations. The third is a dilatation and the fourth are named *special conformal transformations*.

Comutation Rules:

$$\begin{aligned} \bullet [D, P_\mu] &= (-ix^\rho \partial_\rho)(-i\partial_\mu) - (-i\partial_\mu)(-ix^\rho \partial_\rho), \\ &= -x^\rho \partial_\rho \partial_\mu + \delta_\mu^\rho \partial_\rho + x^\rho \partial_\rho \partial_\mu = i(-i\partial_\mu), \\ [D, P_\mu] &= iP_\mu. \end{aligned} \quad (2.14)$$

$$\begin{aligned} \bullet [D, L_{\mu\nu}] &= (-ix^\rho \partial_\rho) \{i(x_\mu \partial_\nu - x_\nu \partial_\mu)\} - \{i(x_\mu \partial_\nu - x_\nu \partial_\mu)\} (-ix^\rho \partial_\rho), \\ &= x_\rho \delta_\mu^\rho \partial_\nu + x_\rho x_\mu \partial^\rho \partial_\nu - x_\rho \delta_\nu^\rho \partial_\mu - x_\rho x_\nu \partial^\rho \partial_\mu, \\ &\quad -x_\mu \delta_\nu^\rho \partial_\rho - x_\mu x^\rho \partial_\nu \partial_\rho + x_\nu \delta_\mu^\rho \partial_\rho + x_\nu x^\rho \partial_\mu \partial_\rho, \\ [D, L_{\mu\nu}] &= 0. \end{aligned} \quad (2.15)$$

$$\begin{aligned} \bullet [D, K_\mu] &= (-ix^\rho \partial_\rho) \{2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\} + \{2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\} (x^\rho \partial_\rho), \\ &= -2x_\rho \delta_\mu^\rho x^\nu \partial_\nu - 2x^\rho x_\mu \delta_\rho^\nu \partial_\nu - 2x^\rho x_\mu x^\nu \partial_\rho \partial_\nu \\ &\quad + x^\rho \partial_\rho (x^2) \partial_\mu + x^\rho x^2 \partial_\rho \partial_\mu + 2x_\mu x^\nu \delta_\nu^\rho \partial_\rho \\ &\quad + 2x_\mu x^\nu x^\rho \partial_\nu \partial_\rho - x^2 \delta_\mu^\rho \partial_\rho - x^2 x^\rho \partial_\mu \partial_\rho, \\ &= -2x_\mu x^\nu \partial_\nu + x^2 \partial_\mu, \\ [D, K_\mu] &= -iK_\mu. \end{aligned} \quad (2.16)$$

$$\begin{aligned}
\bullet [K_\mu, P_\nu] &= -\{2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu\} \{\partial_\nu\} + \partial_\nu \{2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu\}, \\
&= -2x_\mu x^\rho \partial_\rho \partial_\nu + x^2 \partial_\mu \partial_\nu + 2\eta_{\mu\nu} x^\rho \partial_\rho + 2x_\mu \delta_\nu^\rho \partial_\rho \\
&\quad + 2x_\mu x^\rho \partial_\rho \partial_\nu - 2x_\nu \partial_\mu - x^2 \partial_\mu \partial_\nu, \\
&= 2\eta_{\mu\nu} x^\rho \partial_\rho + 2x_\mu \partial_\nu - 2x_\nu \partial_\mu, \\
[K_\mu, P_\nu] &= 2i \{\eta_{\mu\nu} D - L_{\mu\nu}\}. \tag{2.17}
\end{aligned}$$

$$\begin{aligned}
\bullet [K_\rho, L_{\mu\nu}] &= \{2x_\rho x^\lambda \partial_\lambda - x^2 \partial_\rho\} \{x_\mu \partial_\nu - x_\nu \partial_\mu\} - \{x_\mu \partial_\nu - x_\nu \partial_\mu\} \{2x_\rho x^\lambda \partial_\lambda - x^2 \partial_\rho\}, \\
&= 2x_\rho x^\lambda \partial_\lambda (x^\lambda \eta_{\mu\lambda}) \partial_\nu + 2x_\rho x_\mu x^\lambda \partial_\lambda \partial_\mu - 2x_\rho x^\lambda \partial_\lambda (x^\lambda \eta_{\nu\lambda}) \partial_\mu \\
&\quad - 2x_\rho x_\nu x^\lambda \partial_\lambda \partial_\mu - x^2 \partial_\rho (x^\rho \eta_{\mu\rho}) \partial_\nu - x^2 x_\mu \partial_\nu \partial_\rho \\
&\quad + x^2 \partial_\rho (x^\rho \eta_{\nu\rho}) \partial_\mu + x^2 x_\nu \partial_\mu \partial_\rho - 2x_\mu \partial_\nu (x^\nu \eta_{\nu\rho}) x^\lambda \partial_\lambda \\
&\quad - 2x_\mu x_\rho \delta_\nu^\lambda x_\lambda - 2x_\nu x_\rho x^\lambda \partial_\lambda \partial_\nu + 2x_\mu x_\nu \partial_\rho \\
&\quad + x^2 x_\mu \partial_\nu \partial_\rho + 2x_\nu \partial_\mu (x^\mu \eta_{\mu\rho}) x^\lambda \partial_\lambda + 2x_\nu x_\rho \delta_\mu^\lambda x_\lambda \\
&\quad + 2x_\nu x_\rho x^\lambda \partial_\lambda \partial_\mu - 2x_\nu x_\mu \partial_\rho - x^2 x_\nu \partial_\mu \partial_\rho + \dots, \\
&= \eta_{\mu\rho} (2x_\nu x^\lambda \partial_\lambda - x^2 \partial_\nu) - \eta_{\nu\rho} (2x_\mu x^\lambda \partial_\lambda - x^2 \partial_\mu) + 2x_\rho (x_\mu \partial_\nu - x_\nu \partial_\mu), \\
[K_\rho, L_{\mu\nu}] &= i \{\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu\}. \tag{2.18}
\end{aligned}$$

We also get some other ones

$$\bullet [P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu). \tag{2.19}$$

$$\bullet [L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}). \tag{2.20}$$

In order to put the above commutation rules into a simpler form, we define the following generators

$$\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu}, \\
J_{-1,0} &= D, \\
J_{-1,\mu} &= \frac{1}{2} (P_\mu - K_\mu), \\
J_{0,\mu} &= \frac{1}{2} (P_\mu + K_\mu),
\end{aligned}$$

where $J_{ab} = -J_{ba}$, with $\{a, b \in \{-1, 0, 1, \dots, d\}\}$. These new generators obey the $SO(d+1, 1)$ commutation relations:

$$[J_{ab}, J_{cd}] = i \{\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}\}. \tag{2.21}$$

The metric used above is $\eta_{ab} = \text{diag}(-1, 1, \dots, 1)$ for Euclidean d -dimensional space $\mathbb{R}^{d,0}$. For the case of dimension $d = p + q \geq 3$ the conformal group on $\mathbb{R}^{p,q}$ is $SO(p+1, q+1)$. Notice that the Poincare group plus with dilations form a subgroup of the full conformal group. This means that a theory invariant under translations, rotations and dilations is not necessarily invariant under special conformal transformation (SCT).

2.2 Conformal symmetry in 2-dimensions

Conformal invariance in $d = 2$ is more interesting since it implies stronger restrictions on the correlation functions. We begin studying the conformal group of two dimensions. Using the condition (2.5) for invariance, in two dimensions we have

$$\partial_0 \epsilon_0 = +\partial_1 \epsilon_1, \quad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0.$$

These equations are similar to the Cauchy-Riemann equations for holomorphic complex functions.

We introduce complex variables in the following way:

$$\begin{aligned} z &= x^0 + ix^1 & \epsilon &= \epsilon^0 + i\epsilon^1 & \partial_z &= \frac{1}{2}(\partial_0 - i\partial_1), \\ \bar{z} &= x^0 - ix^1 & \bar{\epsilon} &= \epsilon^0 - i\epsilon^1 & \partial_{\bar{z}} &= \frac{1}{2}(\partial_0 + i\partial_1). \end{aligned}$$

We express the infinitesimal conformal transformations as

$$z \rightarrow f(z) \quad \text{and} \quad \bar{z} \rightarrow \bar{f}(\bar{z}), \tag{2.22}$$

which implies that the metric tensor transforms as

$$ds^2 = dzd\bar{z} \mapsto \frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial \bar{z}} dzd\bar{z}.$$

Every analytic map on the complex plane is known to be conformal and to preserve the angles.

The conformal algebra in $d = 2$ is infinite dimensional as can be seen from an infinitesimal transformation $z' \rightarrow z + \epsilon(z)$, then we perform a Laurent expansion of $\epsilon(z)$ around say $z = 0$,

$$\begin{aligned} z' &= z + \epsilon(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n (-z^{n+1}), \\ \bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n (-\bar{z}^{n+1}). \end{aligned}$$

We have also introduced the generators

$$l_n = -z^{n+1}\partial_z, \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \quad (n \in \mathbb{Z}). \quad (2.23)$$

Let us calculate some commutators

$$\begin{aligned} \bullet [l_m, l_n] &= z^{m+1}\partial_z(z^{n+1}\partial_z) - z^{n+1}\partial_z(z^{m+1}\partial_z) \\ &= (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z \\ &= (m-n)l_{m+n}, \\ \bullet [\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n}, \\ \bullet [l_m, \bar{l}_n] &= 0. \end{aligned} \quad (2.24)$$

Since we can show that the holomorphic and antiholomorphic parts commute, the algebra is a direct sum of two isomorphic subalgebras and we can treat z and $\bar{z}$ as independent. The algebra (2.24) is named *de-Witt algebra*.

There are only three generators (2.23) that are globally defined on the Riemann sphere $S^2 = \mathbb{C} \cup \infty$. This set of conformal transformations correspond to the special conformal group with associated subalgebra given l_{-1}, l_0, l_1 . It can be seen that l_{-1} generate the translations, l_1 generate the special conformal transformations and $l_0 + \bar{l}_0$ generate the dilatations while $l_0 - \bar{l}_0$ generate the rotations on the real plane. The eigenvalue of the operator $l_0(\bar{l}_0)$ is called the holomorphic (antiholomorphic) conformal dimension $h(\bar{h})$. The conformal dimension (or weight) Δ and the spin s are respectively the eigenvalues $\Delta = h + \bar{h}$ and $s = h - \bar{h}$.

The finite form of the global conformal transformations is

$$f(z) = \frac{az + b}{cz + d}, \quad (2.25)$$

with $a, b, c, d \in \mathbb{C}$. For these transformations to be invertible, we have to require that $ad - bc$ is non zero. So we can scale that constant such that $ad - bc = 1$. Note that the expression is invariant under $(a, b, c, d) \mapsto (-a, -b, -c, -d)$. These are the three global holomorphic and the three global antiholomorphic conformal transformations and form

the *Mobius group* $SL(2, \mathbb{C})/\mathbb{Z}_2 \approx SO(3, 1)$ in correspondence with the conformal transformation in $d > 2$ explained in the previous section, where there are only global conformal transformations.

Virasoro Algebra: $Virasoro = Witt \oplus \mathbb{C}$ The central extension $\tilde{g} = g \oplus \mathbb{C}$ of a Lie algebra g by $\mathbb{C}$ is characterized by the commutations rules,

$$\begin{aligned} [\tilde{x}, \tilde{y}]_{\tilde{g}} &= [x, y]_g + c p(x, y) \quad \tilde{x}, \tilde{y} \in \tilde{g}, \\ [\tilde{x}, c]_{\tilde{g}} &= 0 \quad x, y \in g, \\ [c, c]_{\tilde{g}} &= 0 \quad c \in \mathbb{C}, \end{aligned}$$

where $p : g \times g \rightarrow \mathbb{C}$ is bilinear (linear to "g and g").

For the deWitt algebra,

$$[\hat{L}_m, \hat{L}_n] = (m - n)\hat{L}_{m+n} + c_{m,n}, \quad m, n \in \mathbb{Z}.$$

Using the Jacobi identity, we find

$$\begin{aligned} [[\hat{L}_m, \hat{L}_n], \hat{L}_p] + [[\hat{L}_n, \hat{L}_p], \hat{L}_m] + [[\hat{L}_p, \hat{L}_m], \hat{L}_n] &= 0, \\ (m - n)c_{m+n,p} + (n - p)c_{n+p,m} + (p - m)c_{p+m,n} &= 0. \end{aligned}$$

The algebra does not change if one adds a constant operator,

$$\hat{L}_m \rightarrow \hat{L}'_m = \hat{L}_m + b(m).$$

Proof.

$$\begin{aligned} [L'_m, L'_n] &= [L_m + b(m), L_n + b(n)] = [L_m, L_n], \\ &= (m - n)L_{m+n} + c_{m,n}, \\ &= (m - n)L'_{m+n} - (m - n)b(m + n) + c_{m,n}, \\ &= (m - n)L'_{m+n} + c'_{m,n}. \end{aligned}$$

Therefore $c'_{m,n} = c_{m,n} - (m - n)b(m + n)$. Choosing

$$\left. \begin{aligned} b(m) &= \frac{c_{m,0}}{m} \rightarrow c'_{m,0} = 0 \\ b(0) &= \frac{c_{-1,1}}{2} \rightarrow c'_{-1,1} = 0 \end{aligned} \right\} \begin{array}{l} \text{without lost} \\ \text{of generality} \end{array}$$

thus, from the beginning one can choose $c_{m,0}$ for all m and $c_{-1,1} = 0$. From the Jacobi identity we find

$$(m-n)c_{m+n,0} + nc_{n,m} + (-m)c_{m,n} = 0.$$

Since the algebra has to be satisfied, we have $c_{m,n} = -c_{n,m}$, then,

$$(m+n)c_{m,n} = 0 \Rightarrow c_{m,n} = c(m)\delta_{m,-n}.$$

In order to find the expression for $c(m)$, we put $p = -(m+1)$ and $n = 1$ in the Jacobi identity,

$$\begin{aligned} (m-1)c_{m+1,-(m+1)} + (m+2)c_{-m,m} + (-(2m+1))c_{1,1} &= 0, \\ (m-1)c(m+1) + (m+2)c(-m) &= 0, \\ (m-1)c(m+1) - (m+2)c(m) &= 0, \\ c(m+1) = \frac{m+2}{m-1}c(m) &\begin{cases} c(1) = 0 \\ c(0) = 0, \end{cases} \end{aligned}$$

for $m > 1$

$$\begin{aligned} \bullet c(3) &= \frac{4}{1}c(2) = \frac{4}{1}c(2)\frac{3 \cdot 2}{3 \cdot 2} = \frac{4!}{3!}c(2), \\ \bullet c(4) &= \frac{5}{2}c(3) = \frac{5}{2} \cdot \frac{4}{1} \cdot c(2)\frac{3 \cdot 2}{3 \cdot 2} = \frac{5!}{3!2!}c(2), \\ \bullet c(5) &= \frac{6}{3}c(4) = \frac{6}{3} \cdot \frac{5}{2} \cdot \frac{4}{1} \cdot c(2)\frac{3 \cdot 2}{3 \cdot 2} = \frac{6!}{3!3!}c(2). \end{aligned}$$

Solution: $c(m) = \frac{(m+1)!}{3!(m-2)!}c(2)$, defining $c(2) = \frac{c}{2}$,

$$c(m) = \frac{m(m^2-1)}{12} \cdot c.$$

Finally

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}.$$

The central extension of the Witt algebra has the following form

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n} + \frac{c}{12}m(m^2-1), \\ [\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n} + \frac{c}{12}m(m^2-1), \\ [\bar{l}_m, l_n] &= 0. \end{aligned}$$

This is the quantum version of the Witt algebra. We can note that $c_{m,n} = 0$ for $m, n = -1, 0, 1$. It is still true that l_{-1} generates translation, l_0 generates dilations and rotations and l_1 generates the special conformal transformations. So $\{l_{-1}, l_0, l_1\}_{VIR}$ are generators of $SL(2, \mathbb{C})/\mathbb{Z}_2$ transformations.

2.3 Conformal Invariance in Quantum Field Theory

There are certain functions that are invariant under the transformations (2.13), which are important in the construction of n -point correlation functions. Invariance under translations and rotations enforce these functions to depend on the relative distance between pairs of different points $|x_i - x_j|$. Scale invariance allows these functions to be only quotients between these distances, $\frac{|x_i - x_j|}{|x_k - x_l|}$. Finally, special conformal transformations invariance imposes that these functions depend only on the expressions

$$\frac{|x_1 - x_2| |x_3 - x_4|}{|x_1 - x_3| |x_2 - x_4|}, \quad \frac{|x_1 - x_2| |x_3 - x_4|}{|x_2 - x_3| |x_1 - x_4|}, \quad (2.26)$$

called anharmonic ratios or cross-ratios.

Conformal invariance at the quantum level not only implies invariance of the action but also invariance of the measure of path integrals. That means that the expectation value of the trace of the energy-momentum tensor must be zero.

We show how conformal invariance is imposed on n -point correlation functions. Under a conformal transformation $x \rightarrow x'$ a field $\phi(x)$ of spin zero transforms as

$$\phi(x) \rightarrow \phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \phi(x), \quad (2.27)$$

where Δ is the conformal dimension of ϕ and $\left| \frac{\partial x'}{\partial x} \right|$ is the Jacobian corresponding to the coordinate transformation. Fields transforming in this way are named quasi-primaries.

Correlation functions of a theory covariant under the transformation (2.27) must satisfy

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\Delta_1/d} \dots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{\Delta_n/d} \langle \phi_1(x'_1) \dots \phi_n(x'_n) \rangle. \quad (2.28)$$

The expectation value $\langle \mathcal{O} \rangle$ is defined as $\langle 0 | \mathcal{O} | 0 \rangle$, where $| 0 \rangle$ is the vacuum. By definition $\langle \mathcal{O} \rangle = \int \mathcal{D}\phi e^{-S} \mathcal{O}$, where S is the Euclidean action.

The vacuum $|0\rangle$ must be invariant under the conformal group. Due to (2.28) a 2-point correlation function invariant under translations, rotations and dilatations satisfies

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\Delta_1/d} \left| \frac{\partial x'}{\partial x} \right|_{x=x_2}^{\Delta_2/d} \langle \phi_1(x'_1)\phi_2(x'_2) \rangle.$$

If we have a dilatation transformation, that is, $x \rightarrow \lambda x$

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\phi_2(\lambda x_2) \rangle.$$

For rotation and translation invariance

$$\begin{aligned} \langle \phi_1(x_1)\phi_2(x_2) \rangle &= f(|x_1 - x_2|), \\ \rightarrow f(\lambda x) &= \frac{f(x)}{\lambda^{\Delta_1+\Delta_2}}. \end{aligned}$$

Therefore

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \quad (2.29)$$

Recall that the Jacobian for dilations and special conformal transformations is given by

$$\begin{aligned} \left| \frac{\partial x'}{\partial x} \right| &= \frac{1}{(1 - 2b \cdot x + b^2 x^2)^d} = \frac{1}{\gamma^d}, \\ \langle \phi_1(x_1)\phi_2(x_2) \rangle &= \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \langle \phi_1(x'_1)\phi_2(x'_2) \rangle, \\ \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} &= \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \end{aligned}$$

if $\Delta_1 = \Delta_2 = \Delta$, or must be 0 if $\Delta_1 \neq \Delta_2$. C_{12} is a constant that depends on the normalization of the fields. In other words, two quasi-primary fields are correlated only if they have the same scaling dimension.

The 3-point functions are as follows,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{C_{123}}{x_{12}^a x_{23}^b x_{13}^c},$$

where a, b, c is restricted such that $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$. As in the previous case, we have now

$$\frac{C_{123}}{x_{12}^a x_{23}^b x_{13}^c} = \frac{C_{123}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_1\gamma_3)^{c/2}}{x_{12}^a x_{23}^b x_{13}^c},$$

the covariance under the special conformal transformations requires that

$$\begin{aligned} a + c &= 2\Delta_1, \\ a + b &= 2\Delta_2, \\ b + c &= 2\Delta_3. \end{aligned}$$

Therefore the 3-point function depends only on C_{123} :

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1 + \Delta_2 - \Delta_3} |x_2 - x_3|^{\Delta_2 + \Delta_3 - \Delta_1} |x_1 - x_3|^{\Delta_1 + \Delta_3 - \Delta_2}}. \quad (2.30)$$

But ($n \geq 4$)-functions are not completely determined by conformal invariance. We know them up to a factor depending on the cross-ratios (2.26). In general, the 4-point functions are

$$\langle \phi_1(x_1)\dots\phi_4(x_4) \rangle = f\left(\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|}, \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|}\right) \prod_{i < j}^4 |x_i - x_j|^{\Delta/3 - \Delta_i - \Delta_j}, \quad (2.31)$$

where $\Delta = \sum_{i=1}^4 \Delta_i$ and f is an arbitrary function of cross ratios.

2.4 Conformal theories in 2-dimensions

The scale invariance in $d = 2$ is equivalent to conformal invariance at the classical level. But for a theory to be conformally invariant at the quantum level the integration measure of path integrals must be also invariant. Furthermore, if the theory is to be used in string theory, conformal invariance must be preserved by integrating over every two dimensional manifold.

Considering conformally invariant theory at the quantum level, we analyze the restrictions to the correlation functions. Under a conformal map $z \rightarrow w(z)$, $\bar{z} \rightarrow \bar{w}(\bar{z})$, the line element $ds^2 = dzd\bar{z}$ transforms as

$$ds^2 \rightarrow \left(\frac{\partial f}{\partial z}\right) \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) ds^2.$$

We can generalize this transformation to the form

$$\Phi(z, \bar{z}) \rightarrow \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})). \quad (2.32)$$

This property under conformal transformations defines the primary fields Φ of conformal weight $(h, \bar{h})$. Fields not transforming in this way are known as secondary fields. A primary field is always a quasi-primary field because it satisfies (2.27) under global transformations. A secondary field may or may not be a quasi-primary field. Conformal invariance forces the n -point functions of n primary fields to transform as

$$\langle \Phi_1(f_1, \bar{f}_1) \dots \Phi_n(f_n, \bar{f}_n) \rangle = \prod_{i=1}^n \left(\frac{df}{dz} \right)_{f=f_i}^{-h_i} \left(\frac{d\bar{f}}{d\bar{z}} \right)_{\bar{f}=\bar{f}_i}^{-\bar{h}_i} \langle \Phi_1(z_1, \bar{z}_1) \dots \Phi_n(z_n, \bar{z}_n) \rangle. \quad (2.33)$$

This relation fixes the form of 2 and 3-point functions. In contrast to the previous section, primary fields can have spin. The spin value is incorporated in the difference $s = h_i - \bar{h}_i$. Therefore, 2-point functions are

$$\langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}}, \quad (2.34)$$

if $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$. In any other case it is zero. The sum of the spins within the correlation function must be zero. The 3-point functions are

$$\langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \Phi_3(z_3, \bar{z}_3) \rangle = C_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}, \quad (2.35)$$

where $z_{ij} = z_i - z_j$. Constants C_{12} and C_{123} can also be determined due to the conformal invariance.

The 4-point functions are not completely fixed. In $d = 2$ there are only three independent cross-ratios invariant under global conformal transformations. They can be written as

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad 1 - x = \frac{z_{14}z_{23}}{z_{13}z_{24}}, \quad \frac{x}{1-x} = \frac{z_{12}z_{34}}{z_{14}z_{23}}. \quad (2.36)$$

Therefore, the functional form of a 4-point function is

$$\langle \Phi_1(x_1) \dots \Phi_4(x_4) \rangle = f(x, \bar{x}) \prod_{i < j}^4 z_{ij}^{\frac{h}{3} - h_i - h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3} - \bar{h}_i - \bar{h}_j}, \quad (2.37)$$

where $h = \sum_{i=1}^4 h_i$ and $\bar{h} = \sum_{i=1}^4 \bar{h}_i$.

Radial quantization

To probe carefully the consequences of conformal invariance in a two dimensional quantum field theory, we explain some of the details of the quantization procedure. We

can consider it either on a plane or on a cylinder with coordinates $\sigma \in [0, 2\pi]$ and $\tau \in [-\infty, +\infty]$. The first one is a Lorentzian variety $\mathbb{R}^2$ and the second one is an Euclidean variety $\mathbb{R} \times U(1)$.

The first step is to perform a Wick rotation $\sigma^\pm = \tau \pm \sigma \rightarrow -i(\tau \pm i\sigma)$ where τ and σ are two space-time coordinates. Next step is to define complex coordinates on the cylinder

$$\begin{aligned} z' &= \tau - i\sigma, \\ \bar{z}' &= \tau + i\sigma. \end{aligned} \tag{2.38}$$

These coordinate transformations applied respectively to the left and the right moving fields in two-dimensional Minkowski space-time transform them into Euclidean fields depending on holomorphic or anti-holomorphic coordinates.

$$\begin{aligned} z &= e^{z'} = e^{\tau - i\sigma}, \\ \bar{z} &= e^{\bar{z}'} = e^{\tau + i\sigma}. \end{aligned} \tag{2.39}$$

The infinite past and future in the cylinder ($\tau = \mp\infty$) are mapped to the points $|z| = 0, \infty$ on the plane. Equal time lines in the cylinder ($\tau = cte$) correspond to circles with center at the origin in the plane, and the time inversion in the cylinder ($\tau \rightarrow -\tau$) correspond to the map $z \rightarrow 1/\bar{z}$ in the complex plane.

It is important to realize that dilatations in the complex plane correspond to temporal translations in the cylinder. Consequently, the generator of dilatations in the plane can be thought as the Hamiltonian of the system and the Hilbert space is built from concentric circles.

This method of defining a quantum theory in the plane is called radial quantization.

2.5 The stress-tensor and the Virasoro algebra

The Noether theorem establishes that local transformations of coordinates are generated by charges built from the energy-momentum tensor $T_{\mu\nu}$, and in $d = 2$ every such transformation is a conformal transformation. In this case the energy-momentum tensor is not only symmetric but also traceless and it results in a two component tensor that can be

written as

$$T(z) \equiv T_{zz}(z), \quad \bar{T}(\bar{z}) \equiv \bar{T}_{\bar{z}\bar{z}}(\bar{z}), \quad (2.40)$$

where the holomorphic and the anti-holomorphic parts are separated. The operator $T(z)$ is related to the trace of the energy-momentum tensor. The following expansion of the energy-momentum tensor

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \quad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n, \quad (2.41)$$

is important since the modes L_n do generate the local conformal transformations at the quantum level in the equivalence to what the generators (2.23) do at the classical level.

They satisfy the famous *Virasoro algebra*,

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\ [L_n, \bar{L}_m] &= 0, \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \end{aligned} \quad (2.42)$$

where the parameter c is the *central charge*.

The correlation functions can have singularities when the positions of two or more fields coincide. The operator product expansion (OPE) is the representation of a product of two operators inserted in points z and w given by a finite sum of terms, each being a single operator, well defined as $z \rightarrow w$, multiplied by a function of $z - w$.

The OPE between the energy-momentum tensor and a primary field ϕ of conformal dimension h is

$$T(z)\phi(w, \bar{w}) = \frac{h}{(z-w)^2}\phi(w, \bar{w}) + \frac{1}{z-w}\partial_w\phi(w, \bar{w}) + \dots \quad (2.43)$$

and similarly for the anti-holomorphic part with h substituted by $\bar{h}$ and w and $\bar{w}$ interchanged. The dots ... express that regular terms have been ignored in the right side of the equation.

One last advantage of radial quantization to be mentioned is the relation that it establishes between commutators and OPEs. Let us consider two operators A and B that are integrals over space at fixed time of the fields $a(z)$ and $b(z)$, respectively,

$$A = \oint a(z)dz, \quad B = \oint b(z)dz, \quad (2.44)$$

where the contours of integration are circles centered around the origin. If we perform an equal time commutator between A and B we can express it in terms of both integrations. This calculation imposes the circles to be, one infinitesimally bigger than the other.

Operating with these contours we end up with the following expression,

$$[A, B] = \oint_0 dw \oint_w dz a(z)b(w). \quad (2.45)$$

In this way, only the term in $1/(z - w)$ of the OPE between $a(z)$ and $b(z)$ contributes to the commutator, by the theorem of residues. Therefore, OPEs establish equal time commutators.

An example of the above are the modes of the energy-momentum tensor,

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z). \quad (2.46)$$

We can deduce the Virasoro algebra if we consider the OPE between the energy-momentum tensor and itself.

The central charge

Not all fields satisfy the transformation law (2.32) under conformal transformations. For example a derivative of a primary field, in general have more complicated transformation properties. They are called secondary fields.

Another example is the energy-momentum tensor. Its OPE with itself has the form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \dots, \quad (2.47)$$

where the factor present in the $(z-w)^{-4}$ term is the central charge c and its value in general will depend on the particular theory under consideration. This parameter cannot be determined by symmetry considerations. It is imposed by the behavior at short distances and it represent somehow an extensive measure of the number of degrees of freedom of the system. It is also related with the *soft* breaking of conformal symmetry under the introduction of a macroscopic scale via a local conformal transformation (it is trivial to see that restricting to just the global conformal group corresponding to $n = -1, 0, 1$ in (2.42) the central charge does not appear).

Chapter 3

Liouville Theory and Correlation Functions

In 1981 A. Polyakov studied a string theory equivalent to Feynman diagram summation [40], instead of summing over line diagrams with an increasing number of loops. His idea consists in summing over closed (two-dimensional) Riemann surfaces with an increasing genus. In this context (uniformization theorem) the Liouville theory was introduced as an attempt to discover the proper measure for the path integral in the closed bosonic string formulation in a non-critical dimension. This leads to a Lagrangian that looks like

$$S_L = \frac{1}{4\pi} \int d^2\xi \sqrt{\tilde{g}} \left[\partial_a \phi \partial_b \phi \tilde{g}^{ab} + Q \mathcal{R} \phi + 4\pi \mu e^{2b\phi} \right]. \quad (3.1)$$

We identify the following terms: kinetic term for the free scalar field, the metric $\tilde{g}$ is referred to as the "reference" metric ($\mathcal{R}$ is its scalar curvature with coupling constant $Q = b + 1/b$). The quantity $g_{ab} = e^{\frac{2}{Q}\phi} \tilde{g}_{ab}$ is referred to as the "physical" metric and an exponential potential term.

We begin by calculating the stress tensor T for the free field component and will complete the other terms of the Liouville Lagrangian, so we notice how it changes and interpret the terms we have added.

We will also derive the form of the central charge and of the conformal dimensions of primary fields. Finally, we present the correlation functions for three- and four-points.

3.1 Free Field Theory

We consider the action of a free bosonic field, invariant under a translation $\phi \mapsto \phi + a$,

$$\mathcal{S} = \frac{1}{4\pi} \int d^2\xi \sqrt{\tilde{g}} \tilde{g}^{ab} \partial_a \phi \partial_b \phi. \quad (3.2)$$

The n primary fields correlator is given by the expression

$$\langle e^{2\alpha_1 \phi} \dots e^{2\alpha_n \phi} \rangle. \quad (3.3)$$

In order to achieve invariance under translation (since this is one of the conformal symmetries), we demand the condition

$$\sum_i \alpha_i = 0, \quad (3.4)$$

displaying, under translation, a phase $\exp\{\sum_i \alpha_i\}$. We use also the *Ward Identity*

$$-\frac{1}{2\pi} \int_{\epsilon} d^2\xi \partial_a \left\langle J^a(\xi) \prod_i \mathcal{O}_i(\xi_i) \right\rangle = \left\langle \delta \prod_i \mathcal{O}_i(\xi_i) \right\rangle, \quad (3.5)$$

where J^a is the current associated with a transformation, $\mathcal{O}_i$ are a set of operators, ϵ is an infinitesimal transformation. This general relation can be applied to our case by making two simplifications. First, using the fact that, for any vector J^a ,

$$\int_{\epsilon} \partial_a J^a = \oint_{\partial\epsilon} J_a \hat{n}^a = \oint_{\partial\epsilon} (J_1 d\xi^2 - J_2 d\xi^1) = -i \oint_{\partial\epsilon} (J_z dz - J_{\bar{z}} d\bar{z}), \quad (3.6)$$

where $J_z = \frac{1}{2}(J_1 - iJ_2)$ and $J_{\bar{z}} = \frac{1}{2}(J_1 + iJ_2)$. We get,

$$\frac{i}{2\pi} \oint_{\partial\epsilon} dz \left\langle J_z(z, \bar{z}) \prod_i \mathcal{O}_i(\xi_i) \right\rangle - \frac{i}{2\pi} \oint_{\partial\epsilon} d\bar{z} \left\langle J_{\bar{z}}(z, \bar{z}) \prod_i \mathcal{O}_i(\xi_i) \right\rangle = \left\langle \delta \prod_i \mathcal{O}_i(\xi_i) \right\rangle. \quad (3.7)$$

Second, since we consider only conformal transformations; one can make J_z holomorphic and $J_{\bar{z}}$ anti-holomorphic, then the contour integrals only pick up the residues of the product of the J 's with the first operator. Thus, for our case, a translation induces a variation in the vertex operator $\delta V = 2\alpha a V$, and we have

$$\frac{i}{2\pi} \oint_{\partial\epsilon} dz \left\langle \partial\phi \prod_i \mathcal{O}_i(\xi_i) \right\rangle - \frac{i}{2\pi} \oint_{\partial\epsilon} d\bar{z} \left\langle \bar{\partial}\phi \prod_i \mathcal{O}_i(\xi_i) \right\rangle = 2 \left\langle \prod_i \mathcal{O}_i(\xi_i) \right\rangle \left(\sum_i \alpha_i \right). \quad (3.8)$$

for infinitesimal a .

Since the contours enclose all of space, and there are no operators inserted at infinity, the LHS in (3.8) is zero and for non-zero correlators, we obtain

$$\sum_i \alpha_i = 0. \quad (3.9)$$

The stress-energy tensor is defined as usual,

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\delta \mathcal{S}}{\delta g^{\alpha\beta}} \Big|_{\eta_{\alpha\beta}}. \quad (3.10)$$

Let us consider the following action for free field in flat space-time

$$\mathcal{S} = \frac{1}{4\pi} \int d^2\xi \partial_a \phi \partial^a \phi = -\frac{1}{4\pi} \int d^2\xi \phi \partial^2 \phi. \quad (3.11)$$

The propagator is easily obtained, and is given by the expression

$$\langle \phi(\xi) \phi(\xi') \rangle = -\frac{1}{2} \ln(\xi - \xi')^2. \quad (3.12)$$

Using complex coordinates, the action is

$$\mathcal{S} = \frac{1}{4\pi} \int dz d\bar{z} \partial \phi \bar{\partial} \phi, \quad (3.13)$$

so that the equation of motion is $\partial \bar{\partial} \phi = 0$. This allows us to split ϕ into left- and right-moving pieces: $\phi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$. The product of ϕ 's can be written as

$$\phi(\xi) \phi(\xi') = (\phi(z) + \bar{\phi}(\bar{z}))(\phi(w) + \bar{\phi}(\bar{w})) = \phi(z) \phi(w) + \bar{\phi}(\bar{z}) \bar{\phi}(\bar{w}), \quad (3.14)$$

where the mixed terms vanish in the correlation function. We also have that

$$\begin{aligned} \log(\xi - \rho)^2 &= \log((\xi^1 - \rho^1)^2 + (\xi^2 - \rho^2)^2) \\ &= \log \left(\left(\frac{1}{2}(z + \bar{z}) - \frac{1}{2}(w + \bar{w}) \right)^2 + \left(\frac{1}{2i}(z - \bar{z}) - \frac{1}{2i}(w - \bar{w}) \right)^2 \right) \\ &= \log(z - w) + \log(\bar{z} - \bar{w}). \end{aligned} \quad (3.15)$$

We see that, in complex coordinates, our left-mover propagator is

$$\langle \phi(z) \phi(w) \rangle = -\frac{1}{2} \log(z - w). \quad (3.16)$$

The *stress-energy tensor* is

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\delta \mathcal{S}}{\delta g^{\alpha\beta}} \Big|_{\eta_{\alpha\beta}} = \frac{1}{2} \eta_{\alpha\beta} (\partial\phi)^2 - \partial_\alpha \phi \partial_\beta \phi. \quad (3.17)$$

Therefore, in flat space with complex coordinates we find (remembering that $ds^2 = (d\xi^1)^2 + (d\xi^2)^2 = dzd\bar{z}$) that $T_{\alpha\beta}$ can be expressed as

$$\begin{aligned} T_{z\bar{z}} &= 0, \\ T_{zz} &= T(z) = -\partial\phi\partial\phi, \\ T_{\bar{z}\bar{z}} &= \bar{T}(\bar{z}) = -\bar{\partial}\phi\bar{\partial}\phi. \end{aligned} \quad (3.18)$$

3.2 Coupling the Curvature

We can modify the above action by adding a term that couples to the curvature,

$$\mathcal{S} = \frac{1}{4\pi} \int d^2\xi \sqrt{\tilde{g}} (\tilde{g}^{ab} \partial_a \phi \partial_b \phi + Q \mathcal{R} \phi). \quad (3.19)$$

where $\mathcal{R}$ is the Ricci scalar and Q is a coupling parameter. We look for a momentum conservation rule similar to (3.9). Now the action is no longer invariant under translation; it instead changes as

$$\delta \mathcal{S} = \frac{Qa}{4\pi} \int d^2\xi \sqrt{g} \mathcal{R}. \quad (3.20)$$

It is known that for two-dimensional compact, boundaryless, orientable manifolds, the Gauss-Bonnet theorem states that

$$\frac{1}{4\pi} \int d^2\xi \sqrt{g} \mathcal{R} = \chi, \quad (3.21)$$

where χ is the Euler characteristic of the surface under consideration, which is 2 in the case of the sphere. So, we have a variation of the action equal to $2aQ$, which for infinitesimal a adds a term $2Q \langle \prod_i \mathcal{O}_i(\xi_i) \rangle$ to the LHS of (3.8). Then the condition (3.9) becomes

$$\sum_i \alpha_i = Q. \quad (3.22)$$

This result can be interpreted as a background charge $-Q$ at "infinity".

The contribution to the stress tensor of this new term, $Q\mathcal{R}\phi$, can be calculate. Since we are going to take the flat-metric limit, at the end, we drop any term that will lead to a derivative or variation of $g_{\mu\nu}$. Now we calculate the contribution to the variation of the action, that is

$$\begin{aligned}
\delta S' &= \frac{Q}{4\pi} \int d^2\xi \sqrt{g} \phi \delta \mathcal{R}, \\
&= \frac{Q}{4\pi} \int d^2\xi \sqrt{g} \phi g^{\mu\nu} \delta R_{\mu\nu}, \\
&= \frac{Q}{4\pi} \int d^2\xi \sqrt{g} \phi \partial_\mu \{ g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\mu - g^{\alpha\mu} \Gamma_{\alpha\tau}^\tau \}, \\
&= \frac{Q}{4\pi} \int d^2\xi \sqrt{g} (\partial_\mu \phi) \delta \left\{ \frac{1}{\sqrt{g}} \partial_\tau (g^{\mu\tau} \sqrt{g}) + g^{\tau\mu} \partial_\tau (\log \sqrt{g}) \right\}, \\
&= -\frac{Q}{4\pi} \int d^2\xi (\partial_\mu \partial_\tau \phi) \delta g^{\mu\tau}.
\end{aligned} \tag{3.23}$$

As a consequence, the contribution to $T_{\alpha\beta}$ will be $T_{\alpha\beta}' = Q\partial_\alpha \partial_\beta \phi$. In complex coordinates, it reads as

$$T'_{zz} = Q\partial^2 \phi. \tag{3.24}$$

Adding (3.18) and (3.24), we find that the total stress-energy tensor is

$$\begin{aligned}
T_{z\bar{z}} &= 0, \\
T_{zz} &= T(z) = -(\partial\phi)^2 + Q\partial^2 \phi, \\
T_{\bar{z}\bar{z}} &= \bar{T}(\bar{z}) = -(\bar{\partial}\phi)^2 + Q\bar{\partial}^2 \phi.
\end{aligned} \tag{3.25}$$

3.3 Computing the Central Charge

With the previous results, we can now calculate the central charge, from (3.12)

$$\begin{aligned}
\langle \partial\phi(z) \phi(w) \rangle &= \frac{1}{z-w} + \dots \\
\langle \partial^2 \phi(z) \phi(w) \rangle &= -\frac{1}{(z-w)^2} + \dots \\
\langle \partial\phi(z) \partial\phi(w) \rangle &= \frac{1}{(z-w)^2} + \dots \\
\langle \partial^2 \phi(z) \partial^2 \phi(w) \rangle &= -\frac{6}{(z-w)^4} + \dots
\end{aligned} \tag{3.26}$$

We see now from (2.47), that the central charge is the coefficient in front of the $(z - w)^{-4}$ term in the OPE of $T(z)T(w)$,

$$\begin{aligned} T(z)T(w) &= \left(-(\partial\phi(z))^2 + Q\partial^2\phi(z)\right) \left(-(\partial\phi(w))^2 + Q\partial^2\phi(w)\right) \\ &= \left(2\langle\partial\phi(z)\partial\phi(w)\rangle^2 + Q^2\langle\partial^2\phi(z)\partial^2\phi(w)\rangle\right) + \dots \\ &= \frac{(1 + 6Q^2)/2}{(z - w)^{-4}} + \dots \end{aligned} \quad (3.27)$$

We can read off that the central charge is

$$c = 1 + 6Q^2. \quad (3.28)$$

3.4 Primary Fields and their Conformal Dimension

Now we derive the primary fields in this theory. Using the complex euclidean coordinates $z = \xi^1 + i\xi^2$. The field $\phi(z, \bar{z})$ varies under holomorphic coordinate transformations $z \rightarrow w(z)$ as

$$\phi(w, \bar{w}) = \phi(z, \bar{z}) - \frac{Q}{2} \log \left| \frac{\partial w}{\partial z} \right|^2. \quad (3.29)$$

We can see that the field ϕ is not scalar.

Given the transformation (3.29) for ϕ , we construct the primary field as the exponential of ϕ . We have

$$\begin{aligned} V_\alpha = e^{2\alpha\phi(w, \bar{w})} &= \exp \left\{ 2\alpha \left(\phi(z, \bar{z}) + \frac{1}{2} \log \left[\left(\frac{\partial w}{\partial z} \right) \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right) \right]^{-Q} \right) \right\} \\ &= \left(\frac{\partial w}{\partial z} \right)^{-\alpha Q} \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)^{-\alpha Q} e^{2\alpha\phi(z, \bar{z})}, \end{aligned} \quad (3.30)$$

and, recalling our transformation law for primary field (2.32), we identify the conformal dimension $(\Delta, \bar{\Delta}) = (\alpha Q, \alpha Q)$ of V_α . To compute the quantum conformal dimension of our primaries V_α , we write our primary as

$$V_\alpha(z) =: e^{2\alpha\phi(z)} : \quad (3.31)$$

The OPE between the stress tensor (3.18) and primary field (3.30) is

$$T(z)V_\alpha(w) = \frac{\Delta_\alpha}{(z - w)^2} V_\alpha(w) + \frac{1}{z - w} (L_{-1}V_\alpha)(w) + \dots \quad (3.32)$$

and we find

$$\begin{aligned}
T(z)V_\alpha(w) &= \left(-(\partial\phi(z))^2 + Q\partial^2\phi(z)\right) \left(\sum_{j=0}^{\infty} \frac{1}{j!} ((2\alpha)\phi(w))^j\right) \\
&= -\left(0 + 0 + \frac{1}{2}(2\alpha)^2 \cdot 2 \langle \partial\phi(z)\phi(w) \rangle^2 + \frac{1}{6}(2\alpha)^3 \cdot 3 \cdot 2 \langle \partial\phi(z)\phi(w) \rangle^2 \phi(w) + \dots\right) \\
&\quad + Q\left(0 + (2\alpha) \langle \partial^2\phi(z)\phi(w) \rangle + \frac{1}{2}(2\alpha)^2 \cdot 2 \langle \partial^2\phi(z)\phi(w) \rangle \phi(w) + \dots\right) + \dots \\
&= \left[-(2\alpha)^2 \left(-\frac{1}{2} \frac{1}{z-w}\right)^2 + Q(2\alpha) \left(\frac{1/2}{(z-w)^2}\right)\right] \left(\sum_{j=0}^{\infty} \frac{1}{j!} (2\alpha\phi(w))^j\right) + \dots \\
&= (-\alpha(\alpha - Q)) \frac{1}{(z-w)^2} V_\alpha(w) + \dots
\end{aligned} \tag{3.33}$$

We thus arrive at the expression

$$\Delta_\alpha = \alpha(Q - \alpha). \tag{3.34}$$

3.5 Adding the Liouville Exponential

Adding an exponential term to the action (3.19), we get

$$\mathcal{S} = \frac{1}{4\pi} \int d^2\xi \sqrt{\tilde{g}} \left[\tilde{g}^{ad} \partial_a \phi \partial_d \phi + Q\mathcal{R}\phi + 4\pi\mu e^{2b\phi} \right], \tag{3.35}$$

with an arbitrary parameter b . In order to preserve conformal invariance, the extra term must be a marginal deformation with conformal dimensions $(\Delta, \bar{\Delta}) = (1, 1)$.

We see that this is true if and only if $b(Q - b) = 1$, or $Q = b + 1/b$, moreover, we find that our primary conformal dimensions, stress-energy tensor, field transformation law, and central charge do not change with the addition of this extra term. The reason for this is, because our theory does not depend on the particular value of the field ϕ , but rather only on the form of the action. By making $\phi \ll 0$, the Liouville interaction term vanishes, and we find that our results are those from before the addition of the potential term.

Let us summarize our results in the Table 3.1 (3.1).

Background Charge	$Q = b + 1/b$
Central Charge	$c = 1 + 6Q^2$
Primary Field	$: \exp 2\alpha\phi(z, \bar{z}) :$
Conformal Dimension	$\alpha(Q - \alpha)$

Table 3.1: Important elements from Liouville CFT

We will study this theory on a two-sphere S^2 . We take the reference metric to be the flat metric $ds^2 = dzd\bar{z}$ with

$$\phi = -2Q \log(r) + \mathcal{O}(1) \quad \text{as } r \rightarrow \infty, \quad r = |z|, \quad (3.36)$$

then the physical metric is smooth on S^2 . This ensures that ϕ is nonsingular at infinity with respect (3.29). The motivation for the condition (3.36) is that there is an operator insertion at infinity representing the curvature of S^2 , which has been suppressed in taking the reference metric to be flat.

Though the use of a flat reference metric is convenient, with this choice there is some subtlety in computing the action; we need to introduce a boundary term.

Let D be a disk of radius R , the action for large R is of the form

$$S_L = \frac{1}{4\pi} \int_D d^2\xi [\partial_a \phi \partial_a \phi + 4\pi\mu e^{2b\phi}] + \frac{Q}{\pi} \oint_{\partial D} \phi d\theta + 2Q^2 \log(R). \quad (3.37)$$

The last two terms ensure finiteness of the action and also invariance under $R \rightarrow \infty$.

3.6 The semi-classical limit

In the semi-classical limit $b \rightarrow 0$, we study the theory (3.35) on flat space (i.e. $\mathcal{R} = 0$) with the rescaled field $\phi_c = 2b\phi$, in terms of which the action becomes

$$\begin{aligned} S_{Liouville}[\phi] &= b^2 S_L = \frac{1}{16\pi} \int d^2\xi [\partial_a \phi_c \partial_a \phi_c + 16\pi\mu b^2 e^{\phi_c}] \\ &\quad + \frac{1}{2\pi} \oint_{\partial D} \phi_c d\theta + 2\log(R) + \mathcal{O}(b^2). \end{aligned} \quad (3.38)$$

The boundary condition will be

$$\phi_c(z, \bar{z}) = -2\log(z\bar{z}) + \mathcal{O}(1), \quad |z| \rightarrow \infty. \quad (3.39)$$

The field $\phi_c(z, \bar{z})$ satisfies the classical Liouville equation,

$$\partial\bar{\partial}\phi_c = 2\pi\mu b^2 e^{\phi_c} \quad (3.40)$$

and locally describes a surface of constant negative curvature $-8\pi\mu b^2$.

3.7 Semiclassical Correlators

Based on the articles of Dorn, Otto [12] and Al.B., A.B.Zamolodchikov [55], in this section we present some details that led to calculate the well-known DOZZ formula.

Now, we consider the correlation functions of primary fields V_{α_i} ,

$$\langle V_{\alpha_1}(z_1, \bar{z}_1) \dots V_{\alpha_n}(z_n, \bar{z}_n) \rangle \equiv \int \mathcal{D}\phi_c e^{-S_L} \prod_{i=1}^n \exp\left(\frac{\alpha_i \phi_c(z_i, \bar{z}_i)}{b}\right). \quad (3.41)$$

We see that the action (3.38) scales like b^{-2} . This is an important detail because we want to use the method of steepest descent to approximate the path integral in (3.41) for small b , so we need to know how the α_i 's scale with b . Thus, the insertion of any V_{α_i} affects the classical field dynamics saddle point, only if the Liouville momentum α scales as b^{-1} , i.e., if $\alpha = \eta/b$ and keeping η fixed for $b \rightarrow 0$. This is called "the heavy" Liouville primary field. Also, we define "light" operators with $\alpha = b\sigma$. Now σ is kept fixed for $b \rightarrow 0$, and the insertion of such an operator has no effect on the saddle point ϕ_c .

When we insert a heavy operator an additional delta function term appears in the equation of motion,

$$\partial\bar{\partial}\phi_c = 2\pi\mu b^2 e^{\phi_c} - 2\pi \sum_i \eta_i \delta^2(\xi - \xi_i). \quad (3.42)$$

If we assume that in the vicinity of one of the operator insertions the exponential term can be ignored, this equation then becomes Poisson's equation,

$$\nabla^2 \phi_c = -8\pi\eta_i \delta^2(\xi - \xi_i). \quad (3.43)$$

The solution will be

$$\phi_c(z, \bar{z}) = C - 4\eta_i \log |z - z_i|, \quad (3.44)$$

so we find that in a neighborhood of a heavy operator we have

$$\phi_c(z, \bar{z}) = -4\eta_i \log |z - z_i| + \mathcal{O}(1) \quad \text{as } z \rightarrow z_i. \quad (3.45)$$

The physical metric in this region has the form

$$ds^2 = \frac{1}{r^{4\eta_i}}(dr^2 + r^2 d\theta^2). \quad (3.46)$$

Substituting the solution (3.44) in the equation of motion, we find the following condition, if we consider that the exponential term is subleading,

$$\text{Re}(\eta_i) < \frac{1}{2}. \quad (3.47)$$

When this inequality is not satisfied, the interactions affect the behavior of the field arbitrarily close to the operator. This condition is referred to as Seiberg bound for "good" Liouville operators [42].

We can notice that α and $Q - \alpha$ correspond to the same quantum operator and has the same dimension $\alpha(Q - \alpha)$,

$$V_{Q-\alpha} = R(\alpha)V_\alpha, \quad (3.48)$$

where $R(\alpha)$ is called the reflection coefficient [42]. Then α or $Q - \alpha$ will always obey the Seiberg bound.

We consider $\eta_i < \frac{1}{2}$ and doing a simple change of variables to find the metric, we have,

$$ds^2 = dr'^2 + r'^2 d\theta'^2, \quad (3.49)$$

where $r' \in (0, \infty)$ and $\theta' \in (0, (1 - 2\eta_i)2\pi)$. Thus we can notice that the operator produces a conical singularity in the physical metric. Now, we have the following conditions: $0 < \eta_i < \frac{1}{2}$, when we have a conical deficit and $\eta_i < 0$ when we have a conical excess. Finding real solutions of the equation of motion in presence of heavy operators with real η 's is equivalent to finding metrics of constant negative curvature on the sphere punctured by conical singularities because of the presence of η_i .

However, if we remember the Gauss-Bonnet theorem, in order to have positive Euler character, we require that the integrated curvature is positive. Then, in the case of our punctured

sphere of constant negative curvature we must introduce enough positive curvature to cancel the negative curvature we have. Thereby we obtain an extra condition for the existence of real solutions ϕ_c that together with the Seiberg bound restrict the Liouville momentum α . By integrating (3.42) and using (3.39), we get

$$\sum_i \eta_i > 1. \quad (3.50)$$

In this way, the inequalities (3.47) and (3.50) define the so-called physical region. We say that the condition (3.50) will imply that a product of light fields on S^2 will not be a real solutions ϕ_c , and in this case $\eta_i = 0$.

At this point we can make some comments, since in chapter four we will be interested in complex solutions ϕ_c .

In general, for complex η 's, the saddle points ϕ_c will be complex and the condition (3.50) will not be valid.

We can see that when we insert the solution (3.45) inside the action (3.38), both the kinetic and the source terms are divergent. To solve this, we follow [55] and perform the action integral only over the part of the disk D that excludes a disk d_i of radius ϵ about each of the heavy operators. Then the "semiclassically renormalized" operators are

$$V_{\frac{\eta_i}{b}}(z_i, \bar{z}_i) \approx \epsilon^{\frac{2\eta_i^2}{b^2}} \exp \left(\frac{\eta_i}{2\pi} \oint_{\partial d_i} \phi_c d\theta \right). \quad (3.51)$$

The prefactor $\epsilon^{\frac{2\eta_i^2}{b^2}}$ in (3.51) contributes a term $-2\eta_i^2/b^2$ to the scaling dimension of the operator $V_{\eta_i/b}$; this is a contribution of $-\eta_i^2/b^2$ to both Δ and $\bar{\Delta}$, consistent with the quantum shift $-\alpha_i^2$ of the operator weights. We can define the regularized Liouville action on the remaining part D of the complex plane incorporating the effects of all the heavy operators

$$\begin{aligned} b^2 \tilde{S}_L &= \frac{1}{16\pi} \int_{D - \cup_i d_i} d^2\xi (\partial_a \phi_c \partial_a \phi_c + 16\lambda e^{\phi_c}) + \frac{1}{2\pi} \oint_{\partial D} \phi_c d\theta + 2 \log R \\ &\quad - \sum_i \left(\frac{\eta_i}{2\pi} \oint_{\partial d_i} \phi_c d\theta_i + 2\eta_i^2 \log \epsilon_i \right). \end{aligned} \quad (3.52)$$

The equations of motion for this action now include both Liouville's equation (3.40) and the boundary condition (3.39) and (3.45). The *semiclassical expression* for the expectation

value of a product of heavy and light primary field is [22]

$$\left\langle V_{\frac{\eta_1}{b}}(z_1, \bar{z}_1) \dots V_{\frac{\eta_n}{b}}(z_n, \bar{z}_n) V_{b\sigma_1}(x_1, \bar{x}_1) \dots V_{b\sigma_m}(x_m, \bar{x}_m) \right\rangle \approx e^{-\tilde{S}_L[\phi_\eta]} \prod_{i=1}^m e^{\sigma_i \phi_\eta(x_i, \bar{x}_i)}. \quad (3.53)$$

Where we have n heavy operators and m light operators, and ϕ_η is the solution (3.42) obeying the correct boundary conditions.

3.8 DOZZ Formula

The formula DOZZ is a proposal for the Liouville three-point correlation function on S^2 , which is the fundamental building block of the Liouville theory as a CFT. We saw that the operators V_α are primaries of weight $\Delta = \alpha(Q - \alpha)$, so their three-point function takes the general form

$$\langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_3}(z_3, \bar{z}_3) \rangle = \frac{C(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{2(\Delta_1 + \Delta_2 - \Delta_3)} |z_{23}|^{2(\Delta_1 + \Delta_3 - \Delta_2)} |z_{31}|^{2(\Delta_2 + \Delta_3 - \Delta_1)}}, \quad (3.54)$$

where $z_{ij} = z_i - z_j$. Then, the DOZZ formula is an analytic expression for C in Liouville theory. In particular, this solution is unique due to the recursion relation that were derived by Teschner in [48] [50], and this will not be the case when we make the analytic continuation later. The DOZZ formula is

$$\begin{aligned} C(\alpha_1, \alpha_2, \alpha_3) &= \left[\pi \mu \gamma(b^2) b^{2-2b^2} \right]^{(Q - \sum_{i=1}^3 \alpha_i)/b} \\ &\times \frac{\Upsilon_0 \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1 + \alpha_2 + \alpha_3 - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b(\alpha_1 - \alpha_2 + \alpha_3) \Upsilon_b(-\alpha_1 + \alpha_2 + \alpha_3)}, \end{aligned} \quad (3.55)$$

where $V_\alpha = e^{2\alpha\phi}$, $\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)$ and $\Upsilon_b(x)$ is a function of x defined (for real and positive b) by

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left(\frac{Q}{2} - x \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right], \quad 0 < \text{Re}(x) < Q. \quad (3.56)$$

Though this integral representation is limited to the strip $0 < \text{Re}(x) < Q$, $\Upsilon_b(x)$ has an analytic continuation to an entire function of x . From the definition of Υ function, we can

see:

$$\Upsilon(x) = \Upsilon(Q - x), \quad (3.57)$$

$$\Upsilon(Q/2) = 1, \quad (3.58)$$

and we also use the notation: $\Upsilon_0 = \frac{d\Upsilon(x)}{dx} \big|_{x=0}$.

We can calculate the DOZZ-formula following the Dorn-Otto [12] and Zamolodchikov-Zamolodchikov [55] procedure or the Trick of Teschner [50].

Let us start by showing the original derivation that uses the free field approach.

Using the Knizhnik-Polyakov-Zamolodchikov (KPZ) scaling law,

$$\langle e^{2\alpha_1\phi} e^{2\alpha_2\phi} \dots e^{2\alpha_n\phi} \rangle_g \propto \mu^{\frac{(1-g)Q - \sum_i \alpha_i}{b}}, \quad (3.59)$$

where we denote $s \equiv (Q - \sum_i \alpha_i)/b$. Using (3.59) we can determine the power of μ for the general correlation function in the Liouville Theory. We need to adjust α or b to the power of μ to become an integer, when we consider the perturbative calculations.

By assuming that the 3-point function can be computed in a perturbative series in the cosmological constant μ , we have

$$G_{\alpha_1, \alpha_2, \alpha_3}(z_1, z_2, z_3) = \sum_{n=0}^{\infty} G_{\alpha_1, \alpha_2, \alpha_3}^n(z_1, z_2, z_3), \quad (3.60)$$

$$\begin{aligned} G_{\alpha_1, \alpha_2, \alpha_3}(z_1, z_2, z_3) &= \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle = \int D\phi \prod_{i=1}^3 e^{2\alpha_i\phi(z_i)} e^{-S_L} \\ &\sim \sum_{n=0}^{\infty} \int D\phi \prod_{i=1}^3 e^{2\alpha_i\phi(z_i)} \frac{(-\mu)^n}{n!} \left(\int d^2z e^{2b\phi} \right)^n e^{-\frac{1}{4\pi} \int (\partial_a \phi)^2 d^2z}. \end{aligned} \quad (3.61)$$

Separating the zero-mode of the path integration over $\phi(z) = \phi_0 + \bar{\phi}(z)$ and integrate over the zero mode first,

$$\begin{aligned} G_{\alpha_1, \alpha_2, \alpha_3}(z_1, z_2, z_3) &\sim - \sum_{n=0}^{\infty} \frac{(-\mu)^n}{n!} \frac{1}{2b(s-n)} \\ &\quad \int D\bar{\phi} \prod_{i=1}^3 e^{2\alpha_i\bar{\phi}(z_i)} \left(\int d^2z e^{2b\bar{\phi}} \right)^n e^{-\frac{1}{4\pi} \int (\partial_a \bar{\phi})^2 d^2z}. \end{aligned} \quad (3.62)$$

Using the formula for the correlation function of the Coulomb gas representation ($\mu = 0$) [36]

$$\langle e^{2\alpha_1\phi_1} \dots e^{2\alpha_N\phi_N} \rangle_{C.G;Q} = \delta \left(Q - \sum_i \alpha_i \right) \prod_{i>j} |z_{ij}|^{-4\alpha_i\alpha_j},$$

this free field functional integral for the n th term in (3.62) matches the spherical boundary condition (3.36) only if $\sum_{i=1}^3 \alpha_i = Q - nb$, this can be interpreted as a kind of "on-mass-shell" condition. We see that (3.62) has a singularity. Hence we need to find the residue in $s = n$. We obtain the residues $G_{\alpha_1, \alpha_2, \alpha_3}^n(z_1, z_2, z_3)$ of the correlator

$$\begin{aligned} \text{res}_{\sum \alpha_i = Q - nb} G_{\alpha_1, \alpha_2, \alpha_3}(z_1, z_2, z_3) &= G_{\alpha_1, \alpha_2, \alpha_3}^n(z_1, z_2, z_3) |_{\sum \alpha_i = Q - nb} \\ &= |z_{12}|^{2\Delta_{12}} |z_{23}|^{2\Delta_{23}} |z_{31}|^{2\Delta_{31}} I_n(\alpha_1, \alpha_2, \alpha_3), \end{aligned} \quad (3.63)$$

while we can obtain $I_n(\alpha_1, \alpha_2, \alpha_3)$ by [14],

$$I_n(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{-\pi\mu}{\gamma(-b^2)} \right)^n \frac{\prod_{j=1}^n \gamma(-jb^2)}{\prod_{k=0}^{n-1} \gamma(2\alpha_1 b + kb^2) \gamma(2\alpha_2 b + kb^2) \gamma(2\alpha_3 b + kb^2)}. \quad (3.64)$$

Now, the on-mass-shell condition reads,

$$\text{res}_{\sum \alpha_i = Q - nb} C(\alpha_1, \alpha_2, \alpha_3) = I_n(\alpha_1, \alpha_2, \alpha_3), \quad (3.65)$$

in order to require the $b \rightarrow b^{-1}$ duality, we have the following relation

$$\frac{C(\alpha_1 + b, \alpha_2, \alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} = \frac{-\gamma(-b^2)}{\pi\mu} \frac{\gamma(b(2\alpha_1 + b))\gamma(2b\alpha_1)\gamma(b(\alpha_2 + \alpha_3 - \alpha_1 - b))}{\gamma(b(\alpha_1 + \alpha_2 + \alpha_3 - Q))\gamma(b(\alpha_1 + \alpha_2 - \alpha_3))\gamma(b(\alpha_1 + \alpha_3 - \alpha_2))}. \quad (3.66)$$

We can see that $C(\alpha_1, \alpha_2, \alpha_3)$ satisfies this functional relation, if we consider the following relations for the Υ function,

$$\Upsilon(x + b) = \gamma(bx)b^{1-2bx}\Upsilon(x), \quad (3.67)$$

$$\Upsilon(x + 1/b) = \gamma(x/b)b^{2x/b-1}\Upsilon(x). \quad (3.68)$$

We have several comments on the DOZZ formula:

1. Using $\Upsilon(x) = \Upsilon(Q-x)$, this expression has the reflection property $C(Q-\alpha_1, \alpha_2, \alpha_3) = S(P)C(\alpha_1, \alpha_2, \alpha_3)$ where,

$$S(i\alpha_1 - iQ/2) = S(P) = - [\pi\mu\gamma(b^2)]^{-2iP/b} \frac{\Gamma(1 + 2iP/b)\Gamma(1 + 2iPb)}{\Gamma(1 - 2iP/b)\Gamma(1 - 2iP/b)};$$

this justifies the reflection formula (3.48) and later we are going to show that this function will be associated to the two-point function of the operators $V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)$ in the Liouville field theory.

2. Consider the invariance of the expression (3.55) under $b \rightarrow \frac{1}{b}$ and also $\mu \rightarrow \tilde{\mu}$, we obtain the relation

$$\pi\tilde{\mu}\gamma(1/b^2) = [\pi\mu\gamma(b^2)]^{\frac{1}{b^2}}, \quad (3.69)$$

thereby the "dual" cosmological constant $\tilde{\mu}$ is related to μ .

3.9 Four-Point Functions and Degenerate Operators

We know that in two-dimensional CFT, position dependence of the four-point function on S^2 is not completely determined by conformal symmetry. However in this section we study the four-point function because it will be useful to calculate the DOZZ formula by means of the so-called trick of Teschner. The recurrence relation derived by Teschner will be crucial for our study of the time like Liouville theory. We define the four point function of exponential operators,

$$\begin{aligned} & \langle V_{\alpha_1}(z_1, \bar{z}_1)V_{\alpha_2}(z_2, \bar{z}_2)V_{\alpha_3}(z_3, \bar{z}_3)V_{\alpha_4}(z_4, \bar{z}_4) \rangle \\ &= |z_{42}|^{2(\Delta_1 - \Delta_4 - \Delta_3 - \Delta_2)} |z_{41}|^{2(\Delta_3 + \Delta_2 - \Delta_4 - \Delta_1)} |z_{31}|^{-4\Delta_3} |z_{21}|^{2(\Delta_4 + \Delta_3 - \Delta_2 - \Delta_1)} G_{1234}(z, \bar{z}). \end{aligned} \quad (3.70)$$

The four-point function depends on the harmonic ratio z due to the projective invariance,

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}. \quad (3.71)$$

We properly choose: $z_1 \rightarrow \infty, z_2 \rightarrow 1, z_3 \rightarrow z$, and $z_4 \rightarrow 0$. We can calculate G_{1234} :

$$G_{1234}(z, \bar{z}) = \lim_{z_1, \bar{z}_1 \rightarrow \infty} z_1^{2\Delta_1} \bar{z}_1^{2\bar{\Delta}_1} \langle V_{\alpha_1}(z_1, \bar{z}_1)V_{\alpha_2}(1, 1)V_{\alpha_3}(z, \bar{z})V_{\alpha_4}(0, 0) \rangle. \quad (3.72)$$

to define the functions as in [8], we can express (3.72) as

$$G_{1234}(z, \bar{z}) = \langle \alpha_1 \mid V_{\alpha_2}(1, 1) V_{\alpha_3}(z, \bar{z}) \mid \alpha_4 \rangle. \quad (3.73)$$

We can also define C as

$$C(\alpha_1, \alpha_2, \alpha_3) = \langle \alpha_3 \mid V_{\alpha_2}(1, 1) \mid \alpha_1 \rangle. \quad (3.74)$$

The crossing symmetry condition in terms of $G_{1234}(z, \bar{z})$ is

$$G_{34}^{21}(z, \bar{z}) = G_{32}^{41}(1 - z, 1 - \bar{z}) = z^{-2\Delta_3} \bar{z}^{-2\bar{\Delta}_3} G_{31}^{24} \left(\frac{1}{z}, \frac{1}{\bar{z}} \right). \quad (3.75)$$

These last equations are fundamental "objects" for the conformal bootstrap program in [8] for a conventional two-dimensional CFT (Rational CFT). Using this program, we can express the four-point function as a product of three-point functions by taking the operator product expansion to replace the product of two fields with a single field, or by inserting a complete set of states between $V_{\alpha_3}(1, 1)$ and $V_{\alpha_2}(z, \bar{z})$.

Using the operator product expansion [8],

$$\begin{aligned} & G_{1234}(z, \bar{z}) \\ &= \underbrace{\lim_{z_1, \bar{z}_1 \rightarrow \infty} z_1^{2\Delta_1} \bar{z}_1^{2\bar{\Delta}_1}} \langle 0 \mid V_{\alpha_1}(z, \bar{z}) V_{\alpha_2}(1, 1) \sum_q \sum_{\{l, \bar{l}\}} \frac{C_{34}^q \{l, \bar{l}\} V_q^{\{l, \bar{l}\}}(0, 0)}{(z_3 - z_4)^{\Delta_3 + \Delta_4 - \Delta_q - \Delta_l} (\bar{z}_3 - \bar{z}_4)^{\bar{\Delta}_3 + \bar{\Delta}_4 - \bar{\Delta}_q - \bar{\Delta}_l}} \mid 0 \rangle, \\ &= \langle \alpha_1 \mid V_{\alpha_2}(1, 1) \sum_q \sum_{\{l, \bar{l}\}} \frac{C_{34}^q \beta_{34}^q \{l\} \bar{\beta}^q \{\bar{l}\} V_q^{\{l, \bar{l}\}}(0, 0)}{(z_3 - z_4)^{\Delta_3 + \Delta_4 - \Delta_q - \Delta_l} (\bar{z}_3 - \bar{z}_4)^{\bar{\Delta}_3 + \bar{\Delta}_4 - \bar{\Delta}_q - \bar{\Delta}_l}} \mid 0 \rangle, \\ &= \langle \alpha_1 \mid V_{\alpha_2}(1, 1) \sum_q \frac{C_{34}^q \Psi_q(z, \bar{z} \mid 0, 0)}{z^{\Delta_3 + \Delta_4 - \Delta_q} \bar{z}^{\bar{\Delta}_3 + \bar{\Delta}_4 - \bar{\Delta}_q}} \mid 0 \rangle, \\ &= \sum_q C_{34}^q z^{\Delta_q - \Delta_3 - \Delta_4} \bar{z}^{\bar{\Delta}_q - \bar{\Delta}_3 - \bar{\Delta}_4} \langle \alpha_1 \mid V_{\alpha_2}(1, 1) \Psi_q(z, \bar{z} \mid 0, 0) \mid 0 \rangle, \end{aligned} \quad (3.76)$$

where

$$\Psi_q(z, \bar{z} \mid 0, 0) = \sum_{\{l, \bar{l}\}} \beta_{34}^q \{l\} \bar{\beta}^q \{\bar{l}\} z^{\Delta_l} \bar{z}^{\bar{\Delta}_l} V_q^{\{l, \bar{l}\}}(0, 0).$$

We can also define

$$\begin{aligned} A_{34}^{21}(q \mid z, \bar{z}) &= (C_{12}^q)^{-1} z^{\Delta_q - \Delta_3 - \Delta_4} \bar{z}^{\bar{\Delta}_q - \bar{\Delta}_3 - \bar{\Delta}_4} \langle \alpha_1 \mid V_{\alpha_2}(1, 1) \Psi_q(z, \bar{z} \mid 0, 0) \mid 0 \rangle, \\ &= \mathcal{F}_{34}^{21}(q \mid z) \bar{\mathcal{F}}_{34}^{21}(q \mid \bar{z}), \end{aligned}$$

where, in general, the function $\mathcal{F}$ is given by

$$\mathcal{F}_{nm}^{lk}(q | z) = z^{\Delta_q - \Delta_n - \Delta_m} \sum_{\{k\}} \beta_{nm}^{q\{k\}} z^{\sum k} \frac{\langle k | \phi_l(1, 1) L_{-k_1} \dots L_{-k_N} | q \rangle}{\langle k | \phi_l(1, 1) | q \rangle}. \quad (3.77)$$

These functions are completely determined by the conformal symmetry and we shall call them conformal blocks [57] [58], because any correlation function (3.70) is built up of them.

Finally we have, for (3.76),

$$G_{34}^{21}(z, \bar{z}) = \sum_q C_{34}^q C_{12q} A_{34}^{21}(q | z, \bar{z}). \quad (3.78)$$

However, in the Liouville model α is continuous and complex. Hence the interpretation of a complete set of states is not clear. So when we make OPE, and expand the product of two fields in terms of a complete set of fields, this poses a problem too. This situation was solved by Seiberg, who used the fact that states with $\alpha = \frac{Q}{2} + iP$ are normalizable delta-function for real and positive P . Those states and its descendants form a complete basis of normalizable states. We can demonstrate this using the DOZZ formula

$$\lim_{\epsilon \rightarrow 0} C(Q/2 + iP_1, \epsilon, Q/2 + iP_2) = 2\pi \delta(P_1 - P_2) G(Q/2 + iP_1), \quad (3.79)$$

where the normalization $G(\alpha)$ is given by

$$G(\alpha) = \frac{1}{R(\alpha)} = \frac{1}{b^2} [\pi \mu \gamma(b^2)]^{(Q-2\alpha)/b} \gamma(2\alpha/b - 1 - 1/b^2) \gamma(2b\alpha - b^2). \quad (3.80)$$

Seiberg also argued that semiclassically, the state $V_{\alpha_2}(z, \bar{z}) | \alpha_1 \rangle$ with both α 's real and less than $Q/2$ is normalizable, if and only if, $\alpha_1 + \alpha_2 > \frac{Q}{2}$. If we assume that α_1 and α_2 are in this range, then we can expand this state in terms of the renormalizable states $| Q/2 + iP, k, \bar{k} \rangle$. In the same way when $\alpha_3 + \alpha_4 > \frac{Q}{2}$, the state $\langle \alpha_4 | V_{\alpha_3}(1, 1)$ is also normalizable, so we can evaluate (3.70) by inserting a complete set of normalizable states. Recalling (3.78) and using (3.74) and (3.79), we have

$$\begin{aligned} G_{1234}(z, \bar{z}) &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dP}{2\pi} C(\alpha_1, \alpha_2, Q/2 + iP) C(\alpha_3, \alpha_4, Q/2 + iP) \\ &\quad \times \mathcal{F}_{1234}(\Delta_i, \Delta_P, z) \mathcal{F}_{1234}(\Delta_i, \Delta_P, \bar{z}), \end{aligned} \quad (3.81)$$

where, $\Delta_P = P^2 + Q^2/4$, and the Virasoro conformal block (3.77) is:

$$\mathcal{F}_{1234}(\Delta_i, \Delta_P, \bar{z}) = z^{\Delta_P - \Delta_1 - \Delta_2} \sum_{k=0}^{\infty} \beta_{12}^{P,k} \frac{\langle \alpha_4 | V_{\alpha_3}(1, 1) | Q/2 + iP, k, 0 \rangle}{C(\alpha_3, \alpha_4, Q/2 + iP)} z^k. \quad (3.82)$$

Here the sum over k represents a sum over the full conformal family descended from $V_{Q/2+iP}$. The power of z for each term is given by the level of the descendant operator, so for example $L_{-1}L_{-2} | Q/2+iP \rangle$ contributes at order z^3 . $\beta_{12}^{P,k}$ is defined in [8], and appears in the expansion of $V_{\alpha_2} | \alpha_1 \rangle$ via

$$\begin{aligned} V_{\alpha_2}(z, \bar{z}) | \alpha_1 \rangle &= \int_0^\infty \frac{dP}{2\pi} C(\alpha_1, \alpha_2, Q/2+iP) R(Q/2+iP) | y |^{2(\Delta_P - \Delta_1 - \Delta_2)} \\ &\times \sum_{k, \bar{k}=0}^\infty \beta_{12}^{P,k} \beta_{12}^{P, \bar{k}} z^k \bar{z}^{\bar{k}} | Q/2+iP, k, \bar{k} \rangle. \end{aligned} \quad (3.83)$$

The conformal block and $\beta_{12}^{P,k}$ are universal building blocks for two-dimensional CFT's, and conformal invariance determines how they depend on the conformal weights and central charge.

Now we will make some comments about correlation functions that include degenerate fields. This is the fundamental idea that led Teshner to deriving the DOZZ formula (Teshner's Tricks) [50].

We begin by defining a highest weight state of the primary field with the conformal dimension Δ . It has the properties

$$\begin{aligned} L_n | \Delta \rangle &= 0 \quad \text{for } n > 0, \\ L_0 | \Delta \rangle &= \Delta. \end{aligned} \quad (3.84)$$

The application of L_n on the state $| \Delta \rangle$ for $n > 0$ creates new states. The set of all these states is called a Verma module and the lowest level states in the Verma module are

$$L_{-1} | \Delta \rangle, \quad L_{-2} | \Delta \rangle, \quad L_{-1}L_{-1} | \Delta \rangle, \quad L_{-3} | \Delta \rangle, \dots$$

In fact the Verma module is the set of states corresponding to the conformal family $[\phi(z)]$ of a primary field $\phi(z)$ with conformal dimension Δ .

Depending on the combination of the central charge and the conformal dimension, there can be states of vanishing or even of negative norm in a Verma module. The first case should be absent and the second case should be removed from the Verma module in the unitary theories, however the states with negative norm generate a Verma module orthogonal to the parent one [9]. To determine the existence of zero-norm states, we are

going to compute the Kac-determinant at level N ,

$$\det M_N(c, \Delta) = \alpha_N \prod_{\substack{p, q \leq N \\ p, q > 0}} (\Delta - \Delta_{p,q}(c))^{P(N-pq)}, \quad (3.85)$$

with

$$\Delta_{p,q}(m) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}, \quad m = -\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{25-c}{1-c}}.$$

The appearance of null states in the unitary models with $0 < c < 1$ restricts the form of the OPEs between the primary fields $\phi_{(p,q)}$. Let us denote the degenerate primary field $\phi_{(p,q)}$ with dimension $\Delta_{p,q}$ as $\psi_{(p,q)}$. Let us consider the null states at level $N = 2$. In general such a state is a linear combination given by

$$L_{-2} | \Delta \rangle + a L_{-1} L_{-1} | \Delta \rangle = 0. \quad (3.86)$$

If we apply L_1 to this equation, we can find a , then (3.86) is

$$L_{-2} | \Delta \rangle - \frac{3}{2(2\Delta + 1)} L_{-1}^2 | \Delta \rangle = 0, \quad (3.87)$$

and if we apply L_2 to (3.86), we obtain the central charge $c = \frac{2\Delta}{2\Delta+1}(5 - 8\Delta)$ [9], so the null states at level $N = 2$ satisfy this final equation.

Let us consider the following correlation function:

$$\langle \psi_{(p,q)}(z) \phi_1(z_1) \dots \phi_N(z_N) \rangle. \quad (3.88)$$

An important property of such correlation function is that it satisfies a linear partial differential equations. To make this evident let us recall that the correlator involving a descendent field $\hat{L}_{-n} \psi_{(p,q)}$ can be expressed in terms of the correlator involving the corresponding primary field $\psi_{(p,q)}$ by applying the differential operator $\mathcal{L}_{-n}$ as

$$\left\langle \hat{L}_{-n} \psi_{(p,q)}(z) \phi_1(z_1) \dots \phi_N(z_N) \right\rangle = \mathcal{L}_{-n} \langle \psi_{(p,q)}(z) \phi_1(z_1) \dots \phi_N(z_N) \rangle, \quad (3.89)$$

where the operator $\mathcal{L}_{-n}$ has the form

$$\mathcal{L}_{-n} = \sum_{i=1}^N \left(\frac{(n-1)\Delta_i}{(z_i - z)^n} - \frac{1}{(z_i - z)^{n-1}} \partial_{z_i} \right). \quad (3.90)$$

For example, using (3.87) and (3.90) for the degenerate field $\psi_{(1,2)}(z)$ one gets

$$\left\{ \frac{3}{2(2\Delta + 1)} \frac{\partial^2}{\partial z^2} - \sum_{i=1}^N \frac{\Delta_i}{(z - z_i)^2} - \sum_{i=1}^N \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right\} \times \langle \psi_{(1,2)}(z) \phi_1(z_1) \dots \phi_N(z_N) \rangle = 0. \quad (3.91)$$

The Liouville theory has degenerate fields that can be light or heavy:

$$\alpha = -\frac{n}{2b} - \frac{mb}{2}, \quad (3.92)$$

where n and m are nonnegative integers.

Back to the equation (3.78), we see that it cannot be solved for the Liouville theory because it has an infinity of intermediate states. The trick is to set $\alpha_2 = -\frac{b}{2}$ so that it becomes a degenerate operator. So for the special case that $\alpha_2 = -b/2$ the equation (3.87) in our Liouville theory is

$$\frac{\partial^2}{\partial z^2} V_{\alpha_2}(z, \bar{z}) = b^2 (T_<(z) V_{\alpha_2}(z, \bar{z}) + V_{\alpha_2}(z, \bar{z}) T_>(z)),$$

where

$$T_<(z) = \sum_{n=-\infty}^{-2} z^{-n-2} L_n \quad T_>(z) = \sum_{n=-1}^{\infty} z^{-n-2} L_n, \quad (3.93)$$

and we have also the equation (3.91) for the light degenerate field $V_{-b/2}$

$$\left\{ \frac{3}{2(2\Delta + 1)} \frac{\partial^2}{\partial z^2} - \sum_{i=1}^N \frac{\Delta_i}{(z - z_i)^2} - \sum_{i=1}^N \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right\} \times \langle V_{-b/2}(z, \bar{z}) V_{\alpha_1}(z_1, \bar{z}_1) \dots V_{\alpha_n}(z_n, \bar{z}_n) \rangle = 0, \quad (3.94)$$

where Δ is the conformal weight of the field $V_{-b/2}$. An identical equation holds for correlators involving $V_{-\frac{1}{2b}}$, with Δ now being the weight of $V_{-\frac{1}{2b}}$.

If we apply this equation to the three-point function $\langle V_{-b/2} V_{\alpha_1} V_{\alpha_2} \rangle$ and consider also the fact that it must take the form (3.54), we find that this function vanishes unless $\alpha_2 = \alpha_1 \pm b/2$. We can verify that the DOZZ formula indeed vanishes if $\alpha_3 = -b/2$ and consider generic α_1, α_2 , but if we set $\alpha_2 = \alpha_1 \pm b/2$ and $\alpha_3 = -b/2$ the value of $C(\alpha_1, \alpha_2, \alpha_3)$ is indeterminate. So correlators with degenerate fields cannot always be

obtained by considering generic correlators for particular values.

After this comment, we can write G explicitly using the equation (3.78),

$$G_{\alpha_4\alpha_3\alpha_2\alpha_1}(z, \bar{z}) = \sum_{\alpha} C(\alpha_4, \alpha_3, \alpha) C(Q - \alpha, \alpha_2, \alpha_1) \left| \mathcal{F}_{\alpha} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} (z) \right|^2. \quad (3.95)$$

It satisfies the ordinary differential equation

$$\left(-\frac{1}{b^2} \frac{d^2}{dz^2} + \left(\frac{1}{z-1} + \frac{1}{z} \right) \frac{d}{dz} - \frac{\Delta_3}{(z-1)^2} - \frac{\Delta_1}{z^2} + \frac{\Delta_3 + \Delta_2 + \Delta_1 - \Delta_4}{z(z-1)} \right) G(z, \bar{z}) = 0. \quad (3.96)$$

From this equation, we note that the only α that is contained in (3.78) is $\alpha_1 + sb/2$ with $s = \pm 1$. By applying (3.96) to (3.78), we can determine the conformal blocks in terms of the hypergeometric function as

$$\mathcal{F}_{1234}(\Delta_i, \Delta_{\pm}, z) = z^{\alpha_{\mp}} (1-z)^{\beta} F(A_{\mp}, B_{\mp}, C_{\mp}, z), \quad (3.97)$$

with

$$\begin{aligned} \Delta_{\pm} &= (\alpha_1 \pm b/2)(Q - \alpha_1 \mp b/2), \\ \Delta &= -\frac{1}{2} - \frac{3b^2}{4} \\ \alpha_{\mp} &= \Delta_{\pm} - \Delta - \Delta_1, \\ \beta &= \Delta_{\alpha_3-b/2} - \Delta - \Delta_3, \\ A_{\mp} &= \mp b(\alpha_1 - Q/2) + b(\alpha_3 + \alpha_4 - b) - 1/2, \\ B_{\mp} &= \mp b(\alpha_1 - Q/2) + b(\alpha_3 - \alpha_4) + 1/2, \\ C_{\mp} &= 1 \mp b(2\alpha_1 - Q). \end{aligned}$$

Here, we use the following identity of the hypergeometric function

$$\begin{aligned} F(A, B; C; z) &= \frac{\Gamma(C)\Gamma(B-A)}{\Gamma(B)\Gamma(C-A)} (-z)^{-A} F(A, 1-C+A; 1-B+A, 1/z) \\ &\quad \times \frac{\Gamma(C)\Gamma(A-B)}{\Gamma(A)\Gamma(C-B)} (-z)^{-B} F(B, 1-C+B; 1-A+B, 1/z), \end{aligned} \quad (3.98)$$

which yields the relation

$$\mathcal{F}_s \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} (z) = z^{-2\Delta_2} \sum_{t=+,-} B_{st} \mathcal{F}_s \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_1 & \alpha_4 \end{bmatrix} (1/z). \quad (3.99)$$

By exploiting the crossing symmetry (3.75), one finds

$$\frac{C(\alpha_4, \alpha_3, \alpha_1 + b/2)}{C(\alpha_4, \alpha_3, \alpha_1 - b/2)} = -\frac{C_{12}^-}{C_{12}^+} \frac{B_{-+} \bar{B}_{--}}{B_{++} \bar{B}_{+-}}, \quad (3.100)$$

where we use the notation $C_{12}^s = C(\alpha_1, -b/2, Q - (\alpha_1 + sb/2))$.

The explicit form of the B_{st} is found from the inversion formula (3.98),

$$\begin{aligned} B_{+-} &= \frac{\Gamma(C_+) \Gamma(B_+ - A_+)}{\Gamma(B_+) \Gamma(C_+ - A_+)}, \\ B_{--} &= \frac{\Gamma(C_-) \Gamma(B_- - A_-)}{\Gamma(B_-) \Gamma(C_- - A_-)}, \\ B_{++} &= \frac{\Gamma(C_+) \Gamma(A_+ - B_+)}{\Gamma(A_+) \Gamma(C_+ - B_+)}, \\ B_{-+} &= \frac{\Gamma(C_-) \Gamma(A_- - B_-)}{\Gamma(A_-) \Gamma(C_- - B_-)}. \end{aligned}$$

Remember that $\gamma(z) = \Gamma(z)/\Gamma(1-z)$ and $\frac{\Gamma^2(2-z)}{\Gamma^2(z)} = -\frac{\gamma(2-z)}{\gamma(z)}$, then we have

$$\begin{aligned} \frac{B_{-+} \bar{B}_{--}}{B_{++} \bar{B}_{+-}} &= -\frac{\gamma(b(2\alpha_1 - b))}{\gamma(2 - b(2\alpha_1 - b))} \\ &\times \frac{\gamma(b(-\alpha_1 + \alpha_3 + \alpha_4 - b/2))}{\gamma(b(\alpha_1 - b/2 + \alpha_3 + \alpha_4 - Q)) \gamma(b(\alpha_1 - \alpha_3 + \alpha_4) - b/2) \gamma(b(\alpha_1 + \alpha_3 - \alpha_4 - b/2))}. \end{aligned}$$

The $C_{12}^s(\alpha)$ have been computed in [14] with the Coulomb gas computation in free field theory. Explicitly,

$$\begin{aligned} C_{12}^+ &= -\frac{\pi\mu}{\gamma(-b^2) \gamma(2\alpha_1 b) \gamma(2 + b^2 - 2b\alpha_1)}, \\ C_{12}^- &= 1. \end{aligned} \quad (3.101)$$

From the functional equation for $C(\alpha_4, \alpha_3, \alpha_1)$,

$$\begin{aligned} \frac{C(\alpha_3, \alpha_4, \alpha_1 + b/2)}{C(\alpha_3, \alpha_4, \alpha_1 - b/2)} &= -\frac{\gamma(-b^2)}{\pi\mu} \\ &\times \frac{\gamma(2\alpha_1 b) \gamma(2b\alpha_1 - b^2) \gamma(b(\alpha_3 + \alpha_4 - \alpha_1) - b^2/2)}{\gamma(b(\alpha_1 + \alpha_4 - \alpha_3) - b/2) \gamma(b(\alpha_1 + \alpha_3 - \alpha_4) - b^2/2) \gamma(b(\alpha_1 + \alpha_3 + \alpha_4) - 1 - 3b^2/2)}. \end{aligned} \quad (3.102)$$

Teschner compared it with the corresponding equation satisfied by the equation (3.14) in [55] and used the functional equation for the Υ function,

$$\Upsilon(x + b) = \gamma(bx) b^{1-2bx} \Upsilon(x). \quad (3.103)$$

He found that

$$\begin{aligned} \frac{C(\alpha_3, \alpha_2, \alpha_1 + b)}{C(\alpha_3, \alpha_2, \alpha_1)} &= -\frac{\gamma(-b^2)}{\pi\mu} \\ &\times \frac{\gamma(2\alpha_1 b)\gamma(2b\alpha_1 + b^2)\gamma(b(\alpha_2 + \alpha_3 - \alpha_1) - b^2)}{\gamma(b(\alpha_1 + \alpha_2 + \alpha_3 - Q))\gamma(b(\alpha_1 + \alpha_2 - \alpha_3))\gamma(b(\alpha_1 + \alpha_3 - \alpha_2))}. \end{aligned}$$

We can check that the DOZZ formula indeed obeys this recursion relation. In fact, Teschner showed that the DOZZ formula is the unique structure constant that allows both four point functions to be singlevalued.

Chapter 4

Complex solutions of the Liouville Theory

In this chapter, we start determining the most general complex-valued solutions of the Liouville equation on the Riemann sphere S^2 with two and three heavy operators as extensions of the real solutions (3.45) for real η 's [55].

Next, we study the new features that arise once complex η 's are allowed and find the uniqueness of the solutions. A remarkable issue that will appear for the three-point function is that, for many regions of the parameters η_1, η_2, η_3 , there are singular solutions of the Liouville equation with the desired properties, even complex-valued ones. Finally, we determine the analytic forms of those singularities that appear.

4.1 General Form of Complex Solutions

We begin by determining the form of a solution of the Liouville equation with flat reference metric (or fiducial). The equation of motion that can be derived from the action (3.40) is known as Liouville equation [55]

$$\partial\bar{\partial}\phi_c = 2\lambda e^{\phi_c}, \tag{4.1}$$

where we have promoted ϕ_c to complex values and $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$. We have also defined $\lambda = \pi\mu b^2$, which is fixed for $b \rightarrow 0$ in the so-called semiclassical

limit. Thus, the parameterization of the solution ϕ_c in terms of a new field f is

$$e^{\phi_c(z, \bar{z})} = \frac{1}{\lambda} \frac{1}{f(z, \bar{z})^2}. \quad (4.2)$$

Now the equation of motion (4.1) becomes

$$\partial \bar{\partial} f = \frac{1}{f} (\partial f \bar{\partial} f - 1). \quad (4.3)$$

We want to split this partial differential equation into two ordinary ones. Using the fact, that the stress tensor $T(z) = -(\partial\phi)^2 + Q\partial^2\phi$ obtained from the action (3.40) is holomorphic, and similarly for the antiholomorphic case, these components of the stress tensor are proportional to two functions $W = -\partial^2 f/f$ and $\tilde{W} = -\bar{\partial}^2 f/f$, respectively. Let us see how this result is obtained from the definition (4.2):

$$\partial^2 f = -\frac{1}{2} \left(\frac{1}{\lambda} \right)^{1/2} \left\{ \partial^2 \phi_c - \frac{1}{2} (\partial \phi_c)^2 \right\} e^{-\phi_c/2} = -\frac{1}{2} f \left\{ \partial^2 \phi_c - \frac{1}{2} (\partial \phi_c)^2 \right\}.$$

Therefore,

$$W = -\frac{\partial^2 f}{f} = \frac{1}{2} \left\{ \partial^2 \phi_c - \frac{1}{2} (\partial \phi_c)^2 \right\}.$$

We can see that this is proportional to $T(z) = Q\partial^2\phi - (\partial\phi)^2$. Thus, we get

$$\partial^2 f + W(z)f = 0, \quad (4.4)$$

$$\bar{\partial}^2 f + \tilde{W}(\bar{z})f = 0, \quad (4.5)$$

with W and $\tilde{W}$ holomorphic and antiholomorphic functions, that allow to decouple (4.3).

In (4.4) and (4.5), we treat z and $\bar{z}$ independently. We need to write f locally, therefore, we introduce the Ansatz:

$$f = u(z)\tilde{u}(\bar{z}) - v(z)\tilde{v}(\bar{z}). \quad (4.6)$$

So we express f as a sum of the two linearly independent holomorphic solutions of the equation (4.4) with coefficients depending only on $\bar{z}$ and for the equation of the function antiholomorphic $\tilde{W}$, we can insert this ansatz and see that now $\tilde{u}$ and $\tilde{v}$ are anti-holomorphic solutions of that equation. We can write (4.3) as

$$\partial f \bar{\partial} f - f \partial \bar{\partial} f = 1, \quad (4.7)$$

and plugging the equation (4.6) into (4.3), we get

$$(u\partial v - v\partial u)(\tilde{u}\bar{\partial}\tilde{v} - \tilde{v}\bar{\partial}\tilde{u}) = 1. \quad (4.8)$$

We identify that both factors are constants because they are Wronskians evaluated on (4.6). Given this relation, we obtain a basis of the two linearly independent holomorphic or antiholomorphic solutions of their corresponding equations (4.4) and (4.5), respectively. Notice that this solution is valid as long as the fiducial metric is $ds^2 = dzd\bar{z}$. Replacing (4.6) into (4.2), we obtain

$$e^{\phi_c} = \frac{1}{\lambda} \frac{1}{(u(z)\tilde{u}(\bar{z}) - v(z)\tilde{v}(\bar{z}))^2}, \quad (4.9)$$

with $u, \tilde{u}$ and $v, \tilde{v}$ obeying

$$\partial^2 g + W(z)g = 0, \quad (4.10)$$

$$\bar{\partial}^2 \tilde{g} + \tilde{W}(\bar{z})\tilde{g} = 0, \quad (4.11)$$

where

$$g = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad \tilde{g} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}.$$

The representation in (4.9) is not unique. In order to specify a particular solution, we need to choose W and $\tilde{W}$ and a basis to solve (4.10) and (4.11). These changes are restricted by the boundary conditions

$$\begin{cases} \phi_c(z, \bar{z}) = -2\log(z\bar{z}) + \mathcal{O}(1) & \text{at } |z| \rightarrow \infty, \\ \phi_c(z, \bar{z}) = -4\eta_i \log|z - z_i| + \mathcal{O}(1) & \text{at } z \rightarrow z_i, \end{cases} \quad (4.12)$$

where z_i will label the position of operator insertions, as we will see later.

The presence of the heavy operators requires that the solution ϕ_c be singular at specific points, z_i , then we determine the following boundary condition of f on S^2 ,

$$f(z, \bar{z}) \sim |z - z_i|^{2\eta_i} \quad \text{as } z \rightarrow z_i. \quad (4.13)$$

From (4.9), we can see that there are two possible sources of the singularities. One is due to at least one of $u, v, \tilde{u}, \tilde{v}$ being singular. The other one is that all four functions, in

the denominator of (4.9), are nonzero but $u\tilde{u} - v\tilde{v} = 0$. For the second case, we expand f as

$$f \sim A(z - z_0) + B(\bar{z} - \bar{z}_0) + \mathcal{O}(|z - z_0|^2) \quad \text{as } z \rightarrow z_0. \quad (4.14)$$

Plugging this into (4.7), we find that $AB = 1$, so we cannot produce the desired behavior (4.13).

For now, we focus on singularities that occur in the first case. If we want to obtain the behavior (4.13) from singularities of u and v , we see that they must behave as linear combinations of $(z - z_i)^{n_i}$ and $(z - z_i)^{1-n_i}$ for $z \rightarrow z_i$, similarly for $\tilde{u}, \tilde{v}$. Consequently, W and $\tilde{W}$ must have double poles at $z = z_i$. Thus, we define a double pole of W in a differential equation of the form (4.4) as a regular singular point. This agrees with the behavior of the stress tensor at a point with insertion of a primary field, as usual in CFT (see (2.43)).

Because of the boundary condition at infinity on S^2 , $\phi_c(z, \bar{z}) = -2\log(z\bar{z}) + \mathcal{O}(1)$ as $|z| \rightarrow \infty$, f must behave as

$$f(z, \bar{z}) \sim |z|^2 \quad \text{as } |z| \rightarrow \infty. \quad (4.15)$$

To obtain this, the two solutions of (4.10) should behave as 1 and z , respectively. Requiring this equation to have a solution of the form $a_1 z + a_0 + a_{-1} z^{-1} + \dots$, implies that W vanishes for $z \rightarrow \infty$ at least as fast as $1/z^4$. Let us try to understand this statement a little better first by analyzing the equation (4.10) as an ordinary differential equation and then by connecting the results with those we learned from the expected behavior of the stress tensor. We need to impose a boundary condition at infinity because we want equations (4.10) and (4.11) to hold over the entire Riemann sphere not just over the complex plane which does not include the point at infinite. So, the only singularities of the stress tensor should be at insertions not at infinity which is just another point. Therefore, the stress tensor W must have a specific form for the differential equation (4.10) and must have a regular singular point at infinity. To check this we use the coordinate chart defined by $z = 1/t$, $t = 0$ is $z = \infty$. This is the stereographic projection of the sphere when the north pole of the sphere is centered at the origin, as opposed to the south pole, which is

the normal case. The coordinate z covers the entire sphere, except the north pole, and the t covers the entire sphere except the south pole. Now, let us see how (4.10) looks like in terms of t ,

$$z = 1/t, \quad \text{so} \quad dz = -\frac{1}{t^2}dt, \quad \text{and hence} \quad \frac{d}{dz} = -t^2 \frac{d}{dt}.$$

The equation (4.10) becomes

$$t^4 \frac{d^2 g}{dt^2} + 2t^3 \frac{dg}{dt} + Wg = 0. \quad (4.16)$$

The W has a power series expansion around $z = \infty$ (it is holomorphic there) so what happens if W vanishes slower than $1/z^4$, say $1/z^3$. Then in terms of t , W looks like $W \sim t^3 + \dots$, then (4.10) is

$$t^4 \frac{d^2 g}{dt^2} + 2t^3 \frac{dg}{dt} + (t^3 + \dots)g = 0, \quad (4.17)$$

and divide by t^2 to put (4.10) in the form necessary to check if $t = 0$ is a regular singular point,

$$t^2 \frac{d^2 g}{dt^2} + 2t \frac{dg}{dt} + (t + \dots)g = 0. \quad (4.18)$$

We see that $t = 0$ is not a regular singular point. It means that there is no solution such that g can be written as $g = t^a h$, which solves (4.10) where a is not necessarily an integer and h is a holomorphic function. Since we need $t = 0$ to be a regular singular point, we claim that W must behave as t^m with $m \geq 4$. From the conformal theory side, think of 2D. The stress tensor is symmetric, traceless, and is an energy density. It has spin and dimension 2. Since it is defined over the whole sphere it has to transform covariantly as a conformal tensor under global conformal transformations over the whole sphere.

We perform the conformal transformation $z \rightarrow t = 1/z$. Then the stress tensor $T(z) \rightarrow T'(t) = (dt/dz)^{-2}T(z) = z^4 T(z)$. Then, $T'(0) = \lim_{z \rightarrow \infty} z^4 T(z)$ must be finite (the stress tensor is finite at infinity, there are no insertions there) so $T(z)$ must decay faster than $1/z^4$.

We analyzed the behavior of the stress tensor in the presence of finitely many operator insertions on $\mathbb{R}^2 = \mathbb{C}$ (the projected surface of $\mathcal{S}^2$). For this behavior, we say that the differential equation (4.10), has a regular singular point at $z \rightarrow \infty$ and other singularities

in W or $\tilde{W}$ are not wanted, because they have not physical interpretation. In other words, a pole in W leads to a delta function, and since the Liouville equation implies that $\bar{\partial}W = 0$, the delta function correction to this equation implies the existence of a delta function source term, i.e, an operator insertion of some kind.

In conclusion, for a finite number of operator insertions, W and $\tilde{W}$ have only finitely many second order poles. In particular, W and $\tilde{W}$ are rational functions. The parameters of these rational functions must be adjusted to achieve the desired behavior near operator insertions and at infinity. In the following subsections, we study this problem for the cases with two and three heavy operators.

4.1.1 Two-Point Complex Solutions

Let us focus in the case of solutions with insertions of two heavy operators. For this, we have to determine $u, v, \tilde{u}, \tilde{v}$ to construct f and find ϕ_c . Of the previous section, we know that W should have two double poles and should vanish as $1/z^4$ for $z \rightarrow \infty$; $\tilde{W}$ should behave in the same way. Using (4.14), (4.4) and (4.5), we obtained these functions:

$$\begin{aligned} W(z) &= \frac{w(1-w)z_{12}^2}{(z-z_1)^2(z-z_2)^2}, \\ \tilde{W}(\bar{z}) &= \frac{\tilde{w}(1-\tilde{w})\bar{z}_{12}^2}{(\bar{z}-\bar{z}_1)^2(\bar{z}-\bar{z}_2)^2}. \end{aligned} \tag{4.19}$$

Notice that if we make $z \rightarrow z_1$, $W(z_1)$ can be obtained from (4.4), when $z_i = z_1$ in (4.14).

For solving the ordinary differential equations (ODE's) (4.4) with (4.19) in terms of elementary functions, we make a convenient parameterization of the constant, so a particular basis of solutions is conveniently chosen:

$$\begin{aligned} g_1(z) &= (z-z_1)^w(z-z_2)^{1-w}, \\ g_2(z) &= (z-z_1)^{1-w}(z-z_2)^w, \\ \tilde{g}_1(\bar{z}) &= (\bar{z}-\bar{z}_1)^{\tilde{w}}(\bar{z}-\bar{z}_2)^{1-\tilde{w}}, \\ \tilde{g}_2(\bar{z}) &= (\bar{z}-\bar{z}_1)^{1-\tilde{w}}(\bar{z}-\bar{z}_2)^{\tilde{w}}. \end{aligned} \tag{4.20}$$

We need that these equations satisfy (4.13) and that the product of the Wronskians obeys (4.8). So we obtain $\eta_1 = \eta_2 = w = \tilde{w} \equiv \eta$. This result is expected, because in

conformal field theory, the two-point function for operators of distinct conformal weights always vanishes as we saw before in (2.34).

Then, the u 's and v 's can be written in terms of this basis by

$$\begin{aligned} u(z) &= g_1(z), \\ v(z) &= g_2(z), \\ \tilde{u}(\bar{z}) &= \kappa \tilde{g}_1(\bar{z}), \\ \tilde{v}(\bar{z}) &= \frac{\tilde{g}_2(\bar{z})}{\kappa(1-2\eta)^2 |z_{12}|^2}. \end{aligned} \tag{4.21}$$

Finally, the desired solution is

$$e^{\phi_c} = \frac{1}{\lambda} \frac{1}{\left(\kappa |z - z_1|^{2\eta} |z - z_2|^{2-2\eta} - \frac{1}{\kappa(1-2\eta)^2 |z_{12}|^2} |z - z_1|^{2-2\eta} |z - z_2|^{2\eta} \right)^2}. \tag{4.22}$$

The constant κ is an arbitrary complex number, but if we impose the condition that f be non vanishing away from the operator insertions, κ will be slightly restricted because the denominator in (4.22) vanish only if κ is found on a certain real curve l in the complex plane, where l is the real axis if η is real. If we do not consider the curve l from the complex κ plane, and bearing in mind the fact that the sign of κ is irrelevant, we obtain a moduli space of solutions that has complex dimension one and that as a complex manifold is a copy of the upper half-plane $\mathbf{H}$. This description coincides with the hyperbolic regimes of the Riemman surface, here the uniformization theorem states that every simply connected Riemann surface is conformally equivalent to the open disk $D := z \in \mathbb{C} : |z| < 1$ or equivalently the upper half-plane $H := z \in \mathbb{C} : \text{Im}(z) > 0$.

We are going back to the general solution (4.22) and make two comments about it:

- If we consider that η is real, and to avoid singularities, κ must be purely imaginary. Then, e^{ϕ_c} will be real and negative, and then ϕ_c will be complex. The geometric interpretation for this is the following: we can define a new metric, $-e^{\phi_c} \delta_{ab}$, which is indeed the genuine metric on the sphere, and because of the sign change, it has positive curvature! Following [51], there are two conical singularities, which can be described as the geometry of an American football ball.

Figure 4.1: Standard recipe to make an American football ball out of an European one.

- In eqn. (4.22) we give the most general form of e^{ϕ_c} , but there can be performed another transformation $\phi \rightarrow \phi + ik\phi/b$, where $k \in \mathbb{Z}$. Thus the moduli space of solutions will be actually isomorphic to $\mathbb{H} \times \mathbb{Z}$.

4.1.2 Three-Point Complex Solutions

Now we can determine the solution with three heavy operator insertions. For this we proceed in the same way, as we did in the last section. So, we begin to define the potentials W and $\tilde{W}$, and in this case, these functions must be rational functions with three double poles, as is expected from the behavior of the stress tensor.

As we saw in the last section, for two operator insertions, W and $\tilde{W}$ had to have the correct behavior, when we approach to the positions of the insertions and also the required behavior at infinity that, since the potential W is proportional to the stress tensor, must be as $1/z^4$. We also want to construct the most general second order differential equation with three regular singular points. This is the Riemann P equation, as we will see in short, which becomes the hypergeometric equation, when the points are $0, 1$ and ∞ . Remember the solution to an ODE with a regular singular point, say z_0 , will be of the form $(z - z_0)^2 Q(z)$, where $Q(z)$ is the function that is not proportional to some derivative in the ODE. In order to have a regular singular point, it must be analytic. Thus, to make W give the

right monodromies gives

$$\begin{aligned} W(z) &= \left[\frac{\eta_1(1-\eta_1)z_{12}z_{13}}{z-z_1} + \frac{\eta_2(1-\eta_2)z_{21}z_{23}}{z-z_2} + \frac{\eta_3(1-\eta_3)z_{31}z_{32}}{z-z_3} \right] \frac{1}{(z-z_1)(z-z_2)(z-z_3)}, \\ \tilde{W}(\bar{z}) &= \left[\frac{\eta_1(1-\eta_1)\bar{z}_{12}\bar{z}_{13}}{\bar{z}-\bar{z}_1} + \frac{\eta_2(1-\eta_2)\bar{z}_{21}\bar{z}_{23}}{\bar{z}-\bar{z}_2} + \frac{\eta_3(1-\eta_3)\bar{z}_{31}\bar{z}_{32}}{\bar{z}-\bar{z}_3} \right] \frac{1}{(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_2)(\bar{z}-\bar{z}_3)}. \end{aligned} \quad (4.23)$$

Replacing these potentials in (4.4), the differential equation becomes a hypergeometric equation (see Appendix B.2). Its solution can be given in terms of Riemann P functions. Remember that (4.10) and (4.11) have a regular singular point at infinity. Here we have a problem, because the P functions are solutions of a differential equation with three regular singularities at specified points, and with no singularity at infinity. To solve this, we define $g(z) = (z-z_2)h(z)$ and $\tilde{g}(\bar{z}) = (\bar{z}-\bar{z}_2)\tilde{h}(\bar{z})$. We can check that the equations, that h and $\tilde{h}$ obey, are special cases of Riemann's equation (see Appendix B.3),

$$\begin{aligned} f'' + \left\{ \frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3} \right\} f' + \\ \left\{ \frac{\alpha\alpha'z_{12}z_{13}}{z-z_1} + \frac{\beta\beta'z_{21}z_{23}}{z-z_2} + \frac{\gamma\gamma'z_{31}z_{32}}{z-z_3} \right\} \frac{f}{(z-z_1)(z-z_2)(z-z_3)} = 0, \end{aligned}$$

with the parameters given by

$$\begin{aligned} \alpha &= \eta_1, & \alpha' &= 1-\eta_1, \\ \beta &= -\eta_2, & \beta' &= \eta_2-1, \\ \gamma &= \eta_3, & \gamma' &= 1-\eta_3. \end{aligned} \quad (4.24)$$

The boundary conditions (4.13) ensure that we can choose $u, v, \tilde{u}$, and $\tilde{v}$ to diagonalize the monodromy about, say $z = z_1$, as

$$\begin{aligned} u(z) &= (z-z_2)P^{\eta_1}(x), \\ v(z) &= (z-z_2)P^{1-\eta_1}(x), \\ \tilde{u}(\bar{z}) &= a_1(\bar{z}-\bar{z}_2)P^{\eta_1}(\bar{x}), \\ \tilde{v}(\bar{z}) &= a_2(\bar{z}-\bar{z}_2)P^{1-\eta_1}(\bar{x}), \end{aligned} \quad (4.25)$$

where a_1, a_2 are complex numbers and $x = z_{23}(z-z_1)/z_{13}(z-z_2)$. The P functions are

related to hypergeometric functions by (see Appendix B.4)

$$\begin{aligned} P^{\eta_1}(x) &= x^{\eta_1}(1-x)^{\eta_3}F(\eta_1+\eta_3-\eta_2, \eta_1+\eta_2+\eta_3-1, 2\eta_1, x), \\ P^{1-\eta_1}(x) &= x^{1-\eta_1}(1-x)^{1-\eta_3}F(1-\eta_1+\eta_2-\eta_3, 2-\eta_1-\eta_2-\eta_3, 2-2\eta_1, x). \end{aligned} \quad (4.26)$$

By imposing (4.8) around $z = z_1$, we can determine the product a_1a_2 . This task is easy, we use the usual expansion for the hypergeometric function near $x = 0$, which leads to

$$\begin{aligned} &\left((z-z_2)P^{\eta_1}(x)\frac{\partial}{\partial z}[(z-z_2)P^{1-\eta_1}] - (z-z_2)P^{1-\eta_1}\frac{\partial}{\partial z}[(z-z_2)P^{\eta_1}(x)] \right) \\ &\left(a_1a_2(\bar{z}-\bar{z}_2)P^{\eta_1}(\bar{x})\frac{\partial}{\partial \bar{z}}[(\bar{z}-\bar{z}_2)P^{1-\eta_1}(\bar{x})] - a_1a_2(\bar{z}-\bar{z}_2)P^{1-\eta_1}(\bar{x})\frac{\partial}{\partial \bar{z}}[(\bar{z}-\bar{z}_2)P^{\eta_1}(\bar{x})] \right) = 1, \\ &(z-z_2)^2(\bar{z}-\bar{z}_2)^2a_1a_2 \left[P^{\eta_1}(x)\frac{\partial P^{1-\eta_1}(x)}{\partial z} - P^{1-\eta_1}(x)\frac{\partial P^{\eta_1}(x)}{\partial z} \right] \\ &\left[P^{\eta_1}(\bar{x})\frac{\partial P^{1-\eta_1}(\bar{x})}{\partial \bar{z}} - P^{1-\eta_1}(\bar{x})\frac{\partial P^{\eta_1}(\bar{x})}{\partial \bar{z}} \right] = 1. \end{aligned}$$

We know that the harmonic ratio is $x = \frac{z_{23}(z-z_1)}{z_{13}(z-z_2)}$. Then $\frac{\partial x}{\partial z} = \frac{z_{23}z_{12}}{z_{13}(z-z_2)^2}$, and we can write the above equation as

$$\begin{aligned} &(z-z_2)^2(\bar{z}-\bar{z}_2)^2a_1a_2 \frac{z_{23}z_{12}}{z_{13}(z-z_2)^2} \frac{\bar{z}_{23}\bar{z}_{12}}{\bar{z}_{13}(\bar{z}-\bar{z}_2)^2} \\ &\times \left[P^{\eta_1}(x)\frac{\partial P^{1-\eta_1}(x)}{\partial x} - P^{1-\eta_1}(x)\frac{\partial P^{\eta_1}(x)}{\partial x} \right] \left[P^{\eta_1}(\bar{x})\frac{\partial P^{1-\eta_1}(\bar{x})}{\partial \bar{x}} - P^{1-\eta_1}(\bar{x})\frac{\partial P^{\eta_1}(\bar{x})}{\partial \bar{x}} \right] = 1, \\ &a_1a_2 \frac{z_{23}z_{12}}{z_{13}} \frac{\bar{z}_{23}\bar{z}_{12}}{\bar{z}_{13}} \left[P^{\eta_1}(x)\frac{\partial P^{1-\eta_1}(x)}{\partial x} - P^{1-\eta_1}(x)\frac{\partial P^{\eta_1}(x)}{\partial x} \right] \\ &\times \left[P^{\eta_1}(\bar{x})\frac{\partial P^{1-\eta_1}(\bar{x})}{\partial \bar{x}} - P^{1-\eta_1}(\bar{x})\frac{\partial P^{\eta_1}(\bar{x})}{\partial \bar{x}} \right] = 1. \end{aligned}$$

We use (4.26) and make $x = 0$, $z = z_1$,

$$\begin{aligned} P^{\eta_1}(x)\frac{\partial P^{1-\eta_1}(x)}{\partial x} &= F_{\eta_1}F_{1-\eta_1}(1-\eta_1), \\ P^{1-\eta_1}(x)\frac{\partial P^{\eta_1}(x)}{\partial x} &= F_{\eta_1}F_{1-\eta_1}(\eta_1), \end{aligned}$$

where $F_{\eta_1} = F_{1-\eta_1} = 1$ see Appendix B.1. Then, we obtain

$$a_1a_2 = \frac{|z_{13}|^2}{|z_{12}|^2|z_{23}|^2(1-2\eta_1)^2}. \quad (4.27)$$

We can see from the last formulas that $f = u\tilde{u} - v\tilde{v}$ is singlevalued around $z = z_1$. In case we wish this to be true also for z_2 and z_3 , we need to write the connection formulas

$$\begin{aligned} P^\alpha(x) &= a_{\alpha\gamma}P^\gamma(x) + a_{\alpha\gamma'}P^{\gamma'}(x), \\ P^{\alpha'}(x) &= a_{\alpha'\gamma}P^\gamma(x) + a_{\alpha'\gamma'}P^{\gamma'}(x). \end{aligned}$$

For example, from (4.25) we obtain

$$f|z - z_2|^{-2} = a_1 P^{\eta_1}(x) P^{\eta_1}(\bar{x}) - a_2 P^{1-\eta_1}(x) P^{1-\eta_1}(\bar{x}).$$

This function will be singlevalued near $z = z_3$, which corresponds to $x = 1$ only if

$$a_1 a_{\eta_1, \eta_3} a_{\eta_1, 1-\eta_3} = a_2 a_{1-\eta_1, \eta_3} a_{1-\eta_1, 1-\eta_3}. \quad (4.28)$$

The connection coefficients a_{ij} are given by

$$\begin{aligned} a_{\alpha\gamma} &= a_{\eta_1, \eta_3} = \frac{\Gamma(2\eta_1)\Gamma(1-2\eta_3)}{\Gamma(\eta_1 + \eta_2 - \eta_3)\Gamma(\eta_1 - \eta_2 - \eta_3 + 1)}, \\ a_{\alpha\gamma'} &= a_{\eta_1, 1-\eta_3} = \frac{\Gamma(2\eta_1)\Gamma(2\eta_3 - 1)}{\Gamma(\eta_1 - \eta_2 + \eta_3)\Gamma(\eta_1 + \eta_2 + \eta_3 - 1)}, \\ a_{\alpha'\gamma} &= a_{1-\eta_1, \eta_3} = \frac{\Gamma(2-2\eta_1)\Gamma(1-2\eta_3)}{\Gamma(2-\eta_1-\eta_2-\eta_3)\Gamma(1-\eta_1-\eta_3+\eta_2)}, \\ a_{\alpha'\gamma'} &= a_{1-\eta_1, 1-\eta_3} = \frac{\Gamma(2-2\eta_1)\Gamma(2\eta_3 - 1)}{\Gamma(1-\eta_1-\eta_2+\eta_3)\Gamma(\eta_3 + \eta_2 - \eta_1)}. \end{aligned}$$

Thus, combining the equations above with (4.27) and recalling $\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$, we find

$$(a_1)^2 = \frac{|z_{13}|^2}{|z_{12}|^2 |z_{23}|^2} \frac{\gamma(\eta_1 + \eta_2 - \eta_3)\gamma(\eta_1 + \eta_3 - \eta_2)\gamma(\eta_1 + \eta_2 + \eta_3 - 1)}{\gamma(2\eta_1)^2 \gamma(\eta_2 + \eta_3 - \eta_1)}. \quad (4.29)$$

So the solution is completely determined. Using (4.28) the solution is also singlevalued near z_2 . Thus, we have indeed, a well-defined product of a_1, a_2 . The solution is then given by

$$e^{\phi_c} = \frac{1}{\lambda} \frac{|z - z_2|^{-4}}{[a_1 P^{\eta_1}(x) P^{\eta_1}(\bar{x}) - a_2 P^{1-\eta_1}(x) P^{1-\eta_1}(\bar{x})]^2}. \quad (4.30)$$

This result is the analytic continuation in η_i of the real solution given in [55], whose uniqueness has been established by our last arguments.

This is not the end of the story because the coefficients a_1, a_2 were determined in (4.27) and (4.29), without taking into account the cancellations of the denominator in (4.30), so

for generic η 's, we are not convinced that the denominator does not have zeros. So, we assume (4.14) and consider that a singularity is present at, say, $z = z_0$.

When the η 's are real, we can see from (4.29) that

- If a_1 is imaginary, it is easy to see from (4.27) that a_2 is also imaginary with opposite sign for its imaginary part. Furthermore, when a_1 is purely imaginary, there will be no singularities coming from cancellation of the denominator because $P^n(x)P^n(\bar{x})$ and $P^{1-n}(x)P^{1-n}(\bar{x})$ are positive. In this case, the metric $e^{\phi_c}\delta_{ab}$ will be negative definite and lead to a complex saddle point for ϕ_c .
- If a_1 is real, then a_2 is real with the same sign as a_1 . We know that only if (3.47) and (3.50) are satisfied, the real solutions ϕ_c exist and leads to a real metric of constant negative curvature in this case, the denominator in (4.30) is positive definite away from the operator insertions (see [55]). But if these inequalities are not satisfied, the denominator in (4.30) vanishes somewhere.

To finish this section, we can observe that as the denominator of (4.30) is symmetric under exchanging x and $\bar{x}$, the singularities near a zero of this denominator actually come in pairs. We can provide some comments about the stability of these singularities under perturbations.

Near a zero, the expansion (4.14) implies complex coefficients A and B .

- If $|A| \neq |B|$, a zero of f is stable under small perturbations. To explain this, we define the complex function $f = se^{i\psi}$, where ψ is real and s is a positive function. If f has an isolated zero at $z = z_0$ and $e^{i\psi}$ is defined on the circle $z = z_0 + \epsilon e^{i\theta}$ for a real θ and some small positive ϵ , then the winding number is $\frac{1}{2\pi} \oint_0^{2\pi} d\theta d\psi/d\theta$, which is invariant under small changes in f . In the case of (4.14), the winding number is 1, when $|A| > |B|$, and -1, when $|A| < |B|$. This is the case of a zero of the denominator in (4.30), if the η 's are complex. The isolated singularities are stable against small perturbations, but when the η 's are real, the behavior near singularities of this type requires more examination.
- If $|A| = |B|$, the winding number depends on higher terms in the expansion .

4.2 Analytic Continuation and Stokes Phenomena

In this section, we interpreted the analytic continuation of the two- (3.80) and three- (3.55) point functions using the solution we have found in the last section.

We will see that for the two-point function there is a satisfactory picture in terms of complex saddle points, which improves on the old fixed-area results in the semiclassical approximation. For the three-point function we will find that the treatment is more subtle; we will be able to "improve" on the fixed-area result here as well, but to understand the full analytic continuation we will need to take care of the singularities at which the denominator of the solution vanishes. We focus only on the part of the analytic continuation that avoids those singularities, and postpone the discussion of them until the next section.

4.2.1 Analytic Continuation of the Two-Point Function

In the Section 3.9, we saw that our calculation for the DOZZ formula, using the four-point functions, implies that the Liouville two-point function can be written as

$$\langle V_\alpha(z_1, \bar{z}_1) V_\alpha(z_2, \bar{z}_2) \rangle = |z_{12}|^{-4\alpha(Q-\alpha)} \frac{2\pi}{b^2} [\pi\mu\gamma(b^2)]^{(Q-2\alpha)/b} \gamma(2\alpha/b - 1 - 1/b^2) \gamma(2b\alpha - b^2) \delta(0). \quad (4.31)$$

Here we consider the same Liouville momentum and $\delta(0)$ represents the continuum normalization of the operators α . Since the space of physical states of the Liouville theory consists of a continuum set of primary states that correspond to operator V_α with $\alpha = \frac{Q}{2} + iP$, the analytic continuation we make could lack of physical sense, but this not the case. In reference [42], we saw that for the two-point function on S^2 , the path integral over real Liouville fields does not converge.

Seiberg solved this divergence by using the fixed area method, that consists in restricting the path integral to configurations of fields that obey $\int d^2\xi e^{\phi_c} = A$. If we simply keep A fixed, we would not expect to get a local quantum field theory [42].

However, Harlow, Maltz and Witten give an alternative proposal to this problem. They claim that (4.31) can be calculated by a local path integral over a complex integration cycle.

They showed that the semiclassical limit of (4.31), with $\alpha = \eta/b$, is reproduced by a sum over the complex saddle points with two heavy operators (Subsection 4.1.1). Inspired by the famous work of Witten [54], they interpret their previous result as suggesting that the path integral is evaluated over a cycle that is a sum of cycles attached to complex saddle points and also find that the set of contributing saddle points jumps discontinuously as η crosses the real axis. This is known as Stokes phenomenon.

Evaluation of the Action for Two-Point Solutions

For obtain the solution ϕ_c in (4.22), we need to take the logarithm. This is nontrivial, because of the branch-cut in the logarithm. We relabel $\kappa = i\tilde{\kappa}$, ensuring that the denominator has no zeros coming from the complex nature of κ . Consequently, (4.22) has changed sign and that is why we choose $\tilde{\kappa}$ to have positive real part. We obtain:

$$e^{\phi_c} = -\frac{1}{\lambda \tilde{\kappa}^2} \frac{1}{\left(|z - z_1|^{2\eta} |z - z_2|^{2-2\eta} + \frac{1}{\tilde{\kappa}^2 (1-2\eta)^2 |z_{12}|^2} |z - z_1|^{2-2\eta} |z - z_2|^{2\eta} \right)^2}. \quad (4.32)$$

Imposing the Seiberg bound, we have $\text{Re}(1 - 2\eta) > 0$. If η is real in (4.32), then $\tilde{\kappa}$ is real and positive. Then our prescription, we take the logarithm and obtain:

$$\begin{aligned} \phi_{c,N}(z, \bar{z}) = & i\pi + 2\pi i N - \log \lambda - 2\log \tilde{\kappa} \\ & - 2\log \left(|z - z_1|^{2\eta} |z - z_2|^{2-2\eta} + \frac{1}{\tilde{\kappa}^2 (1 - 2\eta)^2 |z_{12}|^2} |z - z_1|^{2-2\eta} |z - z_2|^{2\eta} \right). \end{aligned} \quad (4.33)$$

We choose the branch cut such that this logarithm behaves like $-4\eta \log |z - z_1| + (4\eta - 4)\log |z_{12}|$ near z_1 . In this case, no problem, if we do this because a choice of branch is equivalent only to shifting the integer N in (4.33). This logarithm can be extended throughout the z -plane (punctured at z_1 and z_2) because away from z_1 it is defined by continuity. In the three-point function, as we will see, zeros of the logarithm are relevant !.

Remember that the behavior of $\phi_{c,N}$ near of the singularity is given by (3.44). In general we have:

$$\phi_{c,N}(z, \bar{z}) \rightarrow -4\eta \log |z - z_i| + C_i \quad \text{as } z \rightarrow z_i \quad (4.34)$$

and knowing, that the behavior of (4.39) near z_1 and z_2 is

$z \rightarrow z_1$

$$\phi_{c,N}(z, \bar{z}) \rightarrow 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda - 2\log \tilde{\kappa} - 4\eta \log |z - z_1| + (4\eta - 4)\log |z_{12}|$$

$z \rightarrow z_2$

$$\phi_{c,N}(z, \bar{z}) \rightarrow 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda - 2\log \tilde{\kappa} - 4\eta \log |z - z_2| + 4\eta \log |z_{12}| + 4\log (1 - 2\eta),$$

by comparing this with (4.34), we get,

$$\begin{aligned} C_1 &= 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda - 2\log \tilde{\kappa} + (4\eta - 4) \log |z_{12}|, \\ C_2 &= 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda + 2\log \tilde{\kappa} + 4\eta \log |z_{12}| + 4\log (1 - 2\eta). \end{aligned} \quad (4.35)$$

We can check that the integer N is the same in both C_1 and C_2 , when η and $\tilde{\kappa}$ are real, since then the final logarithm in (4.33) has no imaginary part; then, in general, the statement that N is the same in C_1 and C_2 follows by continuity.

Calculation of the Liouville action (3.52) can be simplified by the trick used by Zamolodchikov in [55]. Here

$$\frac{d\tilde{S}_{Liouville}}{d\eta_i} = -\phi_i - 4\eta_i \log \epsilon_i,$$

where ϕ_i is defined $\phi_i = \frac{1}{2\pi} \oint_{\partial d_i} \phi_c d\theta_i$. Near the singular point z_i the solution behaves as (4.34), therefore in the limit $\epsilon_i \rightarrow 0$, we have:

$$\frac{d\tilde{S}_{Liouville}}{d\eta_i} = -C_i \quad (4.36)$$

This equation implies that $d\tilde{S}_{Liouville} = -\sum_{i=1}^n C_i d\eta_i$ can be integrated up to a constant independent on η_i .

Using (4.36) and (4.35), we arrive at the equation

$$\begin{aligned} b^2 \frac{d\tilde{S}_L}{d\eta} &= -C_1 - C_2 \\ &= -2\pi i (2N + 1) + 2\log \lambda + (4 - 8\eta) \log |z_{12}| - 4\log (1 - 2\eta). \end{aligned} \quad (4.37)$$

Integrating out this form, we find $\tilde{S}_L[\phi_{c,N}]$:

$$\begin{aligned} b^2 \tilde{S}_L &= -2\pi i (2N + 1) \eta + 2\eta \log \lambda + 4(\eta - \eta^2) \log |z_{12}| \\ &\quad + 2[(1 - 2\eta) \log (1 - 2\eta) - (1 - 2\eta)] + \text{cte}. \end{aligned} \quad (4.38)$$

We can determine the constant of integration by comparing with an explicit evaluation of the action (3.52), when $\eta = 0$, i.e., when the saddle point (4.33) is

$$\phi_{c,N}(z, \bar{z}) = i\pi + 2\pi iN - \log \lambda - 2\log \tilde{\kappa} - 2\log \left(|z - z_2|^2 + \frac{1}{\tilde{\kappa}^2 |z_{12}|^2} |z - z_1|^2 \right). \quad (4.39)$$

Because of the $SL(2, \mathbb{C})$ transformation, $ad - bc = 1$. In this case, with $\tilde{\kappa} = 1$, we have

$$\begin{aligned} a &= \frac{1}{\tilde{\kappa}(z_1 - z_2)}, \\ b &= -\frac{z_1}{\tilde{\kappa}(z_1 - z_2)}, \\ c &= 1, \\ d &= -z_2. \end{aligned}$$

The saddle point (4.39) becomes an $SL(2, \mathbb{C})$ transformation of a metric. Now, we can write (4.39) as

$$\phi_c = i\pi + 2i\pi N - \log \lambda - 2\log(1 + z\bar{z}). \quad (4.40)$$

Explicitly, the action (3.52) for this solution is:

$$b^2 \tilde{S}_0 = 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda - 2. \quad (4.41)$$

Evaluating (4.38) with $\eta = 0$ and comparing with (4.41), we obtain the constant of integration $2\pi i(N + 1/2) - \log \lambda$. Our final result for the action (3.52) with nonzero η is

$$\begin{aligned} b^2 \tilde{S}_L &= 2\pi i(N + 1/2)(1 - 2\eta) + (2\eta - 1)\log \lambda + 4(\eta - \eta^2)\log |z_{12}| \\ &\quad + 2[(1 - 2\eta)\log(1 - 2\eta) - (1 - 2\eta)]. \end{aligned} \quad (4.42)$$

If we remember the two-point function (4.31), we see that its z_{12} dependence is $|z_{12}|^{-4\alpha(Q-\alpha)}$, which corresponds to a scalar operator of weight $\Delta = \alpha(Q - \alpha) = (\eta - \eta^2)/b^2$ (with $\alpha = \eta/b$ and $Q = 1/b$). This is consistent with the previous result in (4.42), so when we integrate over it, this will yield a divergent factor, which can be interpreted as the factor $\delta(0)$.

The other observation is that this action (4.42) is multivalued as a function of η . If $\eta = 1/2$, we obtain $b^2 \tilde{S}_L = \log |z_{12}|$, then, when $|z_{12}| \rightarrow 0$, we have a branch point emanating from $\eta = 1/2$, where the original solution (4.32) is not well-defined.

Under monodromy around this point ($\eta = 1/2$), N shifts by 2, thus all even and likewise all odd values of N are linked by this monodromy. Of course, to see the monodromy, we have to consider paths in the η plane that violate the Seiberg bound $\text{Re}(\eta) < \frac{1}{2}$

Comparison with the Limit of the Exact Two-Point Function

We study the semiclassical asymptotics of the expression (4.31), with $\alpha = \eta/b$, $\lambda = \mu\pi b^2$ and $Q = 1/b + b$, which can be written as

$$\begin{aligned} \langle V_\alpha(z_1, \bar{z}_1) V_\alpha(z_2, \bar{z}_2) \rangle &= \delta(0) |z_{12}|^{-4\eta(1-\eta)/b^2 - 4\eta} \left(\frac{2\pi}{b^2} \right) [\pi\mu\gamma(b^2)] \\ &\quad [\pi\mu\gamma(b^2)]^{(1-2\eta)/b^2} \gamma\left(\frac{(2\eta-1)}{b^2} - 1\right) \gamma(2\eta - b^2). \end{aligned}$$

Consider $\mu \sim \frac{1}{b^2}$ and

$$\gamma\left(\frac{(2\eta-1)}{b^2} - 1\right) \gamma(2\eta - b^2) = \gamma\left(\frac{(2\eta-1)}{b^2}\right) \gamma(2\eta - b^2 - 1)b^4.$$

We can see this using the definition of $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$ and the fact that $\Gamma(x+1) = x\Gamma(x)$. The b^4 is canceled by the b^2 under the 2π and the b^2 coming from the factor of μ with exponent 1. The μ in $[\pi\mu\gamma(b^2)]^{(1-2\eta)/b^2}$ is what makes $[\pi\mu\gamma(b^2)]^{(1-2\eta)/b^2} \sim \left[\frac{\gamma(b^2)}{b^2}\right]^{(1-2\eta)/b^2}$ as $b \rightarrow 0$. Then $\gamma(2\eta - b^2 - 1) \rightarrow \gamma(2\eta - 1)$ as $b \rightarrow 0$, which is a number. This is what gives:

$$\langle V_\alpha(z_1, \bar{z}_1) V_\alpha(z_2, \bar{z}_2) \rangle \sim \delta(0) |z_{12}|^{-4\eta(1-\eta)/b^2} \lambda^{(1-2\eta)/b^2} \left[\frac{\gamma(b^2)}{b^2} \right]^{(1-2\eta)/b^2} \gamma\left(\frac{(2\eta-1)}{b^2}\right). \quad (4.43)$$

To compare this with the result (4.42), we need to analyze the last two terms that involve the gamma functions. First, the factor $\gamma(b^2)$ is asymptotic for small positive b to $\exp\left\{-\frac{4(1-2\eta)\log b}{b^2}\right\}$, but for the other factor, we need to understand some about the asymptotic limit of the Γ function when its complex argument becomes large. So we have:

$$\Gamma(x) = \begin{cases} e^{x \log x - x + \mathcal{O}(\log x)} & \text{Re}(x) > 0, \\ \frac{1}{e^{i\pi x} - e^{-i\pi x}} e^{x \log(-x) - x + \mathcal{O}(\log(-x))} & \text{Re}(x) < 0. \end{cases} \quad (4.44)$$

There are different forms to obtain this result [42]. However, we follow [22] in the derivation of this by using the machinery of critical points and Stokes lines.

We begin to analyze the argument of the last term in (4.43). We obtain

$$\begin{aligned} x &= \frac{2\eta}{b^2} - \frac{1}{b^2}, & \operatorname{Re}(x) < 0, \\ 1-x &= \frac{2\eta}{b^2} + \left(1 + \frac{1}{b^2}\right), & \operatorname{Re}(1-x) > 0. \end{aligned}$$

Now, using (4.44),

$$\begin{aligned} \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} &= \frac{1}{e^{i\pi x} - e^{-i\pi x}} e^{x \log(-x) - x - (1-x) \log(1-x) + (1-x)}, \\ &= \frac{1}{e^{i\pi x} - e^{-i\pi x}} e^{x(\log(-x) + \log(1-x)) - \log(1-x) - 2x + 1}. \end{aligned} \quad (4.45)$$

Replacing x by $\frac{2\eta-1}{b^2}$,

$$\begin{aligned} \gamma\left(\frac{2\eta-1}{b^2}\right) &= \frac{1}{e^{i\pi\left(\frac{2\eta-1}{b^2}\right)} - e^{-i\pi\left(\frac{2\eta-1}{b^2}\right)}} e^{\frac{2\eta-1}{b^2}(2\log(1-2\eta) - 4\log b - 2) - \log(1 - \frac{2\eta-1}{b^2}) + 1}, \\ \gamma\left(\frac{(2\eta-1)}{b^2}\right) &\sim \frac{1}{e^{i\pi(2\eta-1)/b^2} - e^{-i\pi(2\eta-1)/b^2}} \exp\left[\frac{(4\eta-2)}{b^2}(\log(1-2\eta) - 2\log b - 1)\right]. \end{aligned} \quad (4.46)$$

Thus, we can write the semiclassical limit of (4.43) as

$$\begin{aligned} \langle V_\alpha(z_1, \bar{z}_1) V_\alpha(z_2, \bar{z}_2) \rangle &\sim \delta(0) |z_{12}|^{-4\eta(1-\eta)/b^2} \lambda^{(1-2\eta)/b^2} \\ &\times e^{-\frac{2}{b^2}[(1-2\eta)\log(1-2\eta) - (1-2\eta)]} \frac{1}{e^{i\pi(2\eta-1)/b^2} - e^{-i\pi(2\eta-1)/b^2}}. \end{aligned} \quad (4.47)$$

Denoting $y = e^{i\pi(2\eta-1)/b^2}$, we see that the function $1/(y - y^{-1})$ can be expanded in two ways,

$$\frac{1}{y - y^{-1}} = \sum_{k=0}^{\infty} y^{-(2k+1)} = - \sum_{k=0}^{\infty} y^{2k+1}. \quad (4.48)$$

This expansion is valid for $|y| > 1$ and for $|y| < 1$. So there is a set T of integers with

$$\frac{1}{e^{i\pi(2\eta-1)/b^2} - e^{-i\pi(2\eta-1)/b^2}} = \pm \sum_{N \in T} e^{2\pi i(N \mp 1/2)(2\eta-1)/b^2}. \quad (4.49)$$

where T consists of nonnegative integers if $\operatorname{Im}((2\eta-1)/b^2) > 0$ and of nonpositive ones if $\operatorname{Im}((2\eta-1)/b^2) < 0$. The line $\operatorname{Im}((2\eta-1)/b^2) = 0$ can be interpreted as a Stokes line along

which the representation of the integration cycle, as a sum of cycles associated to critical points, changes discontinuously. This procedure is similar to [54]. The sign in (4.49) has an analog for the Gamma function and can be interpreted in terms of the orientations of critical points cycles. Finally, we consider the exponential of (4.42) and show explicitly that

$$e^{-\tilde{S}_L} = e^{2\pi i(N+\frac{1}{2})(\frac{2\eta-1}{b^2})} \lambda^{\frac{1-2\eta}{b^2}} |z_{12}|^{-4\eta(1-\eta)/b^2} e^{-\frac{2}{b^2}[(1-2\eta)\log(1-2\eta)-(1-2\eta)]} \quad (4.50)$$

So, the relation between (4.47) and (4.50) is evident.

Relationship to Fixed-Area Results

Restricting to real α 's, we know from [42] that the fixed-area expectation value for a generic correlator in Liouville theory is

$$\langle V_{\alpha_1} \dots V_{\alpha_n} \rangle_A \equiv (\mu A)^{(\sum_i \alpha_i - Q)/b} \frac{1}{\Gamma((\sum_i \alpha_i - Q)/b)} \langle V_{\alpha_1} \dots V_{\alpha_n} \rangle, \quad (4.51)$$

or

$$\langle V_{\alpha_1} \dots V_{\alpha_n} \rangle = \langle V_{\alpha_1} \dots V_{\alpha_n} \rangle_A \Gamma((\sum_i \alpha_i - Q)/b) (\mu A)^{-(\sum_i \alpha_i - Q)/b}.$$

We know that $\Gamma(z) = \int_0^\infty dt t^{z-1} \exp(-t)$ with $\text{Re} z > 0$, [39] then we can write the last expression as

$$\langle V_{\alpha_1} \dots V_{\alpha_n} \rangle = \int_0^\infty dA \mu(\mu A)^{(\sum_i \alpha_i - Q)/b-1} \langle V_{\alpha_1} \dots V_{\alpha_n} \rangle_A e^{-\mu A} (\mu A)^{-(\sum_i \alpha_i - Q)/b}.$$

We assume that $\text{Re}(\sum_i \alpha_i - Q) > 0$. An equivalent formula is given by

$$\langle V_{\alpha_1} \dots V_{\alpha_n} \rangle = \int_0^\infty \frac{dA}{A} e^{-\mu A} \langle V_{\alpha_1} \dots V_{\alpha_n} \rangle_A. \quad (4.52)$$

This is only a definition, but if we compare this to the original Liouville path integral, given by

$$\langle V_{\alpha_1} \dots V_{\alpha_n} \rangle = \int V_{\alpha_1} \dots V_{\alpha_n} \exp^{-S_L[\phi]} \mathcal{D}\phi,$$

we obtain the motivation that led to Harlow-Maltz and Witten to propose an alternative for calculating the fixed-area expectation value: calculate the path integral for the Liouville theory without considering the cosmological constant term and fixing the physical

area explicitly as $\int d^2\xi e^{\phi_A} = A$. We could perform the same process by using Lagrange multipliers, modifying the equation of motion as

$$\partial\bar{\partial}\phi_A = \frac{2\pi}{A} \left(\sum_i \eta_i - 1 \right) e^{\phi_A} - 2\pi \sum \eta_i \delta^2(\xi - \xi_i). \quad (4.53)$$

It is important to note that when $\sum_i \eta_i < 1$, defining $\phi_{c,N} = i\pi + 2\pi iN + \phi_A$ and $\lambda = \pi(\sum_i \eta_i - 1)/A$, the solutions of this equation are mapped exactly into the complex saddle points.

We can prove explicitly that, for the semiclassical two-point function we computed above, combine the factors on the r.h.s of (4.51) join to eliminate the evidence of the complex saddle points and lead to the usual fixed-area result, given in [55]. Replacing (4.47) in (4.51), we get

$$\begin{aligned} \langle V_{\eta/b}(1,1)V_{\eta/b}(0,0) \rangle_A &\equiv (\mu A)^{(\sum_i \alpha_i - Q)/b} \frac{1}{\Gamma((\sum_i \alpha_i - Q)/b)} \langle V_{\eta/b}(1,1)V_{\eta/b}(0,0) \rangle \\ &\equiv (\mu A)^{(2\eta-1)/b^2} \frac{1}{\Gamma(2\eta-1)/b^2} \langle V_{\eta/b}(1,1)V_{\eta/b}(0,0) \rangle, \\ \langle V_{\eta/b}(1,1)V_{\eta/b}(0,0) \rangle_A &\approx (\mu A)^{(2\eta-1)/b^2} \frac{1}{\Gamma((2\eta-1)/b^2)} \delta(0) \left(\frac{(2\eta-1)\pi}{A} \right)^{(1-2\eta)/b^2} \\ &\quad \times e^{-\frac{2}{b^2}[(1-2\eta)\log(1-2\eta)-(1-2\eta)]} \frac{1}{e^{i\pi(2\eta-1)/b^2} - e^{-i\pi(2\eta-1)/b^2}}, \end{aligned}$$

where $\lambda = \pi\mu b^2 = \pi(\sum_i \eta_i - 1)/A$. Then $\mu A = (\sum_i \eta_i - 1)/b^2$,

$$\Gamma\left(\frac{2\eta-1}{b^2}\right) = \frac{e^{\frac{(2\eta-1)}{b^2}\log\left(-\left(\frac{2\eta-1}{b^2}\right)\right) - \frac{(2\eta-1)}{b^2}}}{e^{i\pi\left(\frac{2\eta-1}{b^2}\right)} - e^{-i\pi\left(\frac{2\eta-1}{b^2}\right)}},$$

and we obtain

$$\begin{aligned} \langle V_{\eta/b}(1,1)V_{\eta/b}(0,0) \rangle_A &\approx ((2\eta-1)/b^2)^{(2\eta-1)/b^2} \frac{e^{-\log\frac{A}{\pi}(1-2\eta)/b^2}}{e^{\frac{(2\eta-1)}{b^2}\log\left(-\left(\frac{2\eta-1}{b^2}\right)\right) - \frac{(2\eta-1)}{b^2}}} \\ &\quad \times \delta(0)(2\eta-1)^{(1-2\eta)/b^2} e^{-\frac{2}{b^2}[(1-2\eta)\log(1-2\eta)-(1-2\eta)]}, \end{aligned}$$

$$\langle V_{\eta/b}(1,1)V_{\eta/b}(0,0) \rangle_A \equiv 2\pi\delta(0)G_A(\eta/b) \approx 2\pi\delta(0)e^{-\frac{1}{b^2}(1-2\eta)(\log\frac{A}{\pi} + \log(1-2\eta)-1)}. \quad (4.54)$$

Originally, (4.51) was used as a way to define the correlator for the Liouville field theory when $\sum_i \eta_i < 1$ holds, but it was not clear that this condition would be valid beyond the semiclassical case. We can see now how it emerges naturally from the analytic continuation of the Liouville path integral!

4.2.2 Analytic Continuation of the Three-Point Function

It is time to analyze the three-point function on two particular regions of the parameter space of the variables η_i .

- In the Region I, we assume that the imaginary parts $\text{Im}(\eta_i)$ are small to ensure that (4.30) does not have singularities arising from zeros of the denominator. This is the well-known "physical region" that we defined in the last chapter. Here, we have the following assumptions:

$$\begin{aligned} \text{Re}(\eta_i) &< \frac{1}{2}, \\ \sum_i \text{Re}(\eta_i) &> 1, \end{aligned} \tag{4.55}$$

In this region the Liouville equation has real nonsingular solutions as was studied in [55].

- In the Region II, we assume

$$\begin{aligned} 0 &< \text{Re}(\eta_i) < \frac{1}{2}, \\ \sum_i \text{Re}(\eta_i) &< 1, \\ 0 &< \text{Re}(\eta_i + \eta_j - \eta_k) \quad (i \neq j \neq k). \end{aligned} \tag{4.56}$$

Here, we also assume that the imaginary parts are small to avoid the singularities from the zeros of the denominator in (4.30). The last inequality of (4.56) is a new expression, the so-called triangle inequality, and its meaning will be clear later. It is automatically satisfied, when $\sum_i \text{Re}(\eta_i) > 1$ and $\text{Re}(\eta_i) < \frac{1}{2}$, but when $\sum_i \text{Re}(\eta_i) < 1$ it becomes a nontrivial additional constraint. To understand this constraint, we can note that in this region, when the imaginary parts are all zero, it follows from (4.27) and (4.29) that a_1 and a_2 will be purely imaginary and there will be no singularities arising from the zeros of the denominator (Section 4.1.2). Here, the metric $-e^{\phi_c} \delta_{ab}$ is well defined with constant positive curvature (note that the sign $'-'$ is required, because the metric $e^{\phi_c} \delta_{ab}$ is defined with a constant negative curvature when η 's are real and a_1, a_2 are imaginary). Such metrics has three conical angular deficits and

can be constructed geometrically as follows (3.49). If we consider a triangle on S^2 (see figure 4.2). We have the following expression for the area A of the spherical triangle:

$$A = \theta_1 + \theta_2 + \theta_3 - \pi. \quad (4.57)$$

Because the edges of the triangle are geodesics of the thorny sphere, we can consider an adjacent triangle, which we call "complementary triangle B", that matches without producing any singularities along the edges, except at the vertices ¹. From (3.49), we can construct a geodesic triangle on S^2 with angles $\theta_i = (1 - 2\eta_i)\pi$ (this is the expected behavior to find solutions of the equation of motion with insertions of heavy operators η_i/b). Now we must prove the condition for the existence of these angles in our geometric construction. In order that (4.57) is well-defined, we assume that $\sum_i \theta_i > \pi$, which leads to $\sum_i \eta_i < 1$. From Figure 4.2 we see that the triangle A and its complement, B, form the so-called diangle. Let us label the vertex of the triangle A in the north pole with η_1 , and so the vertex in the south pole in B is the same. Consider now the area of the triangle B:

$$B = x + y + \theta_1 - \pi. \quad (4.58)$$

Knowing that the area of the diangle is $2\theta_1$, we can get B, if we consider

$$B = A_{Diangle} - A = \theta_1 - \theta_2 - \theta_3 + \pi. \quad (4.59)$$

Comparing this with (4.58), we have $x + y = 2\pi - \theta_2 - \theta_3$ and assuming that $x + y + \theta_1 > \pi$, we have $\pi - \theta_2 - \theta_3 + \theta_1 > 0$, finally replacing $\theta_i = (1 - 2\eta_i)\pi$, we obtain:

$$\eta_2 + \eta_3 - \eta_1 > 0. \quad (4.60)$$

As we tend to saturate the inequality, the complementary triangle B becomes smaller and smaller and the original triangle A degenerates into a diangle. Once the inequality is violated, no metric with only the three desired singularities exists.

¹The thorny sphere: that is, 2-dimensional compact surfaces which are everywhere locally isometric to a round sphere S^2 except for a finite number of isolated points where they have conical singularities

Figure 4.2: Diangle that consisting of the two spherical triangular regions A and B .

Now, we will analyze the semiclassical actions of the complex saddle points (4.30) in these two regions and we will compare this with the DOZZ formula.

Evaluation of the Action for Three-Point Solutions

Now, let us evaluate the action for a saddle point contributing to the three-point function. First, we need to find the asymptotic behavior of the solution $\phi_{c,N}$ corresponding to (4.30) near z_i . We have determined this behavior before in (4.34)

$$\phi_{c,N}(z, \bar{z}) \rightarrow -4\eta \log |z - z_i| + C_i \text{ as } z \rightarrow z_i. \quad (4.61)$$

In order to determine C_i , we need to define the logarithm of $\phi_{c,N}$ to (4.30).

- For the Region I, we define:

$$\begin{aligned} \phi_{c,N}(z, \bar{z}) &= 2\pi i N - \log \lambda - 4 \log |z - z_2| \\ &\quad - 2 \log (a_1 P^{\eta_1}(x) P^{\eta_1}(\bar{x}) - a_2 P^{1-\eta_1}(x) P^{1-\eta_1}(\bar{x})). \end{aligned} \quad (4.62)$$

The branch in the logarithm $z \rightarrow z_1$, ($x = 0$) is chosen so that using (4.29) and (4.26), the series expansion of P^{η_1} are:

$$\begin{aligned} P^{\eta_1}(x) &= \left(\frac{z_{23}}{z_{12}z_{13}} \right)^{\eta_1} (z - z_1)^{\eta_1}, \\ P^{1-\eta_1}(x) &= \left(\frac{z_{23}}{z_{12}z_{13}} \right)^{1-\eta_1} (z - z_1)^{1-\eta_1}. \end{aligned}$$

We get

$$\begin{aligned}
& [a_1(P^{\eta_1}(x)P^{\eta_1}(\bar{x})) - a_2(P^{1-\eta_1}(x)P^{1-\eta_1}(\bar{x}))] = a_1 \left[P^{\eta_1}(x)P^{\eta_1}(\bar{x}) - \frac{a_2}{a_1}P^{1-\eta_1}(x)P^{1-\eta_1}(\bar{x}) \right] \\
& = a_1 \left[\frac{a_1}{a_2}P^{\eta_1}(x)P^{\eta_1}(\bar{x}) - \frac{|z_{13}|^2}{|z_{12}|^2|z_{23}|^2(1-2\eta)^2a_1^2}P^{1-\eta_1}(x)P^{1-\eta_1}(\bar{x}) \right] \\
& = a_1 \left\{ \left(\frac{|z_{23}|^2}{|z_{12}|^2|z_{13}|^2} \right)^{\eta_1} (|z - z_1|^2)^{\eta_1} - \frac{|z_{13}|^2}{|z_{12}|^2|z_{23}|^2} \frac{(|z - z_1|^2)^{1-\eta_1}}{(1-2\eta_1)^2a_1^2} \left(\frac{|z_{23}|^2}{|z_{13}|^2|z_{12}|^2} \right)^{1-\eta_1} \right\},
\end{aligned}$$

and when $z \rightarrow z_1$,

$$[a_1(P^{\eta_1}(x)P^{\eta_1}(\bar{x})) - a_2(P^{1-\eta_1}(x)P^{1-\eta_1}(\bar{x}))] = a_1 \left\{ \left(\frac{|z_{23}|^2}{|z_{12}|^2|z_{13}|^2} \right)^{\eta_1} (|z - z_1|^2)^{\eta_1} \right\}.$$

By replacing in (4.62), we get

$$\begin{aligned}
\phi_{c,N}(z, \bar{z}) &= 2\pi iN - \log \lambda - 4 \log |z_{12}| \\
&\quad - \log a_1^2 - 2\eta_1 \log \left(\frac{|z_{23}|^2}{|z_{12}|^2|z_{13}|^2} \right) - 2\eta_1 \log |z - z_1|^2.
\end{aligned}$$

Using (4.29)

$$\begin{aligned}
\phi_{c,N}(z, \bar{z}) &= 2\pi iN - \log \lambda - \log |z_{12}|^4 - \log \left(\frac{|z_{23}|^2}{|z_{12}|^2|z_{13}|^2} \right) - \log \left(\frac{|z_{23}|^2}{|z_{12}|^2|z_{13}|^2} \right)^{2\eta_1} \\
&\quad - \log \frac{\gamma(\eta_1 + \eta_2 - \eta_3)\gamma(\eta_1 + \eta_3 - \eta_2)\gamma(\eta_1 + \eta_2 + \eta_3 - 1)}{\gamma(2\eta_1)^2\gamma(\eta_2 + \eta_3 - \eta_1)} - 4\eta_1 \log |z - z_1|,
\end{aligned}$$

and comparing with (4.61), we obtain

$$\begin{aligned}
C_1 &= 2\pi iN - \log \lambda - (1 - 2\eta_1) \log \frac{|z_{12}|^2|z_{13}|^2}{|z_{23}|^2} \\
&\quad - \log \frac{\gamma(\eta_1 + \eta_2 - \eta_3)\gamma(\eta_1 + \eta_3 - \eta_2)\gamma(\eta_1 + \eta_2 + \eta_3 - 1)}{\gamma(2\eta_1)^2\gamma(\eta_2 + \eta_3 - \eta_1)}. \tag{4.63}
\end{aligned}$$

In order to find C_2 and C_3 , we can permute the indices finding

$$\begin{aligned}
C_2 &= 2\pi iN - \log \lambda - (1 - 2\eta_2) \log \frac{|z_{12}|^2|z_{23}|^2}{|z_{13}|^2} \\
&\quad - \log \frac{\gamma(\eta_1 + \eta_2 - \eta_3)\gamma(\eta_2 + \eta_3 - \eta_1)\gamma(\eta_1 + \eta_2 + \eta_3 - 1)}{\gamma(2\eta_2)^2\gamma(\eta_1 + \eta_3 - \eta_2)}, \tag{4.64}
\end{aligned}$$

$$\begin{aligned}
C_3 &= 2\pi iN - \log \lambda - (1 - 2\eta_3) \log \frac{|z_{23}|^2|z_{13}|^2}{|z_{12}|^2} \\
&\quad - \log \frac{\gamma(\eta_3 + \eta_2 - \eta_1)\gamma(\eta_1 + \eta_3 - \eta_2)\gamma(\eta_1 + \eta_2 + \eta_3 - 1)}{\gamma(2\eta_3)^2\gamma(\eta_1 + \eta_2 - \eta_3)}. \tag{4.65}
\end{aligned}$$

Here, we consider that N is the same in the vicinity of different points. We observe that the logarithm with the functions γ has its argument real and positive because in this region, $\sum_i \eta_i > 1$. We also see that for any set of η_i 's, which are in Region I, all of the arguments of $\gamma(\cdot)$ have real part between zero and one. $\gamma(\cdot)$ has no zeros or poles in this strip, so any loop in this region can be contracted to a point without changing the monodromy of the logarithm. Thus, there is no monodromy. In order to calculate the action, we integrate

$$b^2 \frac{\partial \tilde{S}_L}{\partial \eta_i} = -C_i. \quad (4.66)$$

So, let us start with $i = 1$:

$$b^2 \frac{\partial \tilde{S}_L}{\partial \eta_1} = -C_1, \quad (4.67)$$

with (4.63), then we integrate the η_1 variable from $1/2$ to η_1 .

Now, to do that, we should use some consequences of the following definition

$$F(\eta) \equiv \int_{\frac{1}{2}}^{\eta} \log \gamma(x) dx, \quad (4.68)$$

with the contour in $0 < \text{Re}(x) < 1$, and also

$$\delta_i \equiv \eta_i(1 - \eta_i).$$

For an arbitrary constant p ,

$$\begin{aligned} \int_{1/2}^{\eta} dx \log \gamma(x+p) &= \int_{1/2+p}^{\eta+p} dy \log \gamma(y) = \int_{1/2}^{\eta+p} dy \log \gamma(y) - \int_{1/2}^{p+1/2} dy \log \gamma(y) \\ &= F(\eta+p) - F(p+1/2), \end{aligned}$$

where $y = x+p$ and p is constrained so that the contours stays in $0 < \text{Re}(y) < 1$. Similarly:

$$\begin{aligned} \int_{1/2}^{\eta} dx \log \gamma(p-x) &= - \int_{p-1/2}^{p-\eta} dy \log \gamma(y) = - \int_{1/2}^{p-\eta} dy \log \gamma(y) + \int_{1/2}^{p-1/2} dy \log \gamma(y) \\ &= -F(p-\eta) + F(p-1/2), \end{aligned}$$

with $y = p-x$. Also,

$$\begin{aligned} \int_{1/2}^{\eta} dx \log \gamma(2x) &= - \int_1^{2\eta} dy \log \gamma(y) = \int_{1/2}^{2\eta} dy \log \gamma(y) - \int_{1/2}^1 dy \log \gamma(y) \\ &= F(2\eta) - F(1), \end{aligned}$$

with $y = 2x$. Lastly,

$$F(0) = \int_{1/2}^0 dx \log \gamma(x) = \int_1^{1/2} dy \log \gamma(y) = - \int_{1/2}^1 dy \log \gamma(y) = -F(1)$$

with $y = x + 1/2$. So

$$\begin{aligned} b^2 \tilde{S}_L &= -2\pi i N (\eta_1 - 1/2) + \log \lambda (\eta_1 - 1/2) + [(\eta_1 - \eta_1^2) - 1/4] \log \frac{|z_{12}|^2 |z_{13}|^2}{|z_{23}|^2} \\ &\quad + F(\eta_1 + \eta_2 - \eta_3) - F(\eta_2 - \eta_3 + 1/2) + F(\eta_1 + \eta_3 - \eta_2) \\ &\quad - F(\eta_3 - \eta_2 + 1/2) + F(\eta_1 + \eta_2 + \eta_3 - 1) - F(\eta_2 + \eta_3 - 1/2) - 2F(2\eta_1) - 2F(0) \\ &\quad + F(\eta_2 + \eta_3 - \eta_1) - F(\eta_2 + \eta_3 - 1/2) 2\pi + I(\eta_2, \eta_3). \end{aligned} \quad (4.69)$$

Here I is a function of η_2 , and η_3 which is an integration constant with respect to η_1 . It is the term that comes from the fact:

$$\int_{1/2}^{\eta_1} d\tilde{\eta}_1 \quad b^2 \frac{\partial \tilde{S}(\tilde{\eta}_1, \eta_2, \eta_3)_L}{\partial \tilde{\eta}_1} = b^2 S(\eta_1, \eta_2, \eta_3) - b^2 S(1/2, \eta_2, \eta_3) \quad (4.70)$$

We called the second term $I(\eta_2, \eta_3)$. Now, differentiating (4.69) with respect to η_2 , we get and compare to $b^2 \frac{\partial \tilde{S}_L}{\partial \eta_2} = -C_2$. This gives an expression for $\frac{\partial I(\eta_2, \eta_3)}{\partial \eta_2}$. Integrate η_2 in this expression from $1/2$ to η_2 . This gives an expression for (η_2, η_3) with an undetermined function of $J(\eta_3)$. We put the expression I into (4.69) and differentiate this with respect to η_3 . Then we compare the result to $b^2 \frac{\partial \tilde{S}_L}{\partial \eta_3} = -C_3$. This gives an expression for $\frac{\partial J(\eta_3)}{\partial \eta_3}$. Integrate, $\frac{\partial I(\eta_3)}{\partial \eta_3}$ with respect to η_3 from $1/2$ to η_3 , and combine all terms in (4.69). This gives the expression

$$\begin{aligned} b^2 \tilde{S}_L &= \left(\sum_i \eta_i - 1 \right) \log \lambda + (\delta_1 + \delta_2 - \delta_3) \log |z_{12}|^2 + (\delta_1 + \delta_3 - \delta_2) \log |z_{13}|^2 \\ &\quad + (\delta_2 + \delta_3 - \delta_1) \log |z_{23}|^2 + F(\eta_1 + \eta_2 - \eta_3) + F(\eta_1 + \eta_3 - \eta_2) + F(\eta_2 + \eta_3 - \eta_1) \\ &\quad + F(\eta_1 + \eta_2 + \eta_3 - 1) - F(2\eta_1) - F(2\eta_2) - F(2\eta_3) - F(0) \\ &\quad + 2\pi i N \left(1 - \sum_i \eta_i \right). \end{aligned} \quad (4.71)$$

A quick way to check formula (4.69) is to use the indefinite integral $F(\eta) = \int d\eta \log \gamma(\eta) + \Lambda$, where λ is an undetermined coefficient and repeat the integration and differentiation procedure on (4.66). This will yield (4.71) up to an undetermined η_i independent constant.

The constant will turn out to be $-F(0)$, which you can get by evaluating the action explicitly with $\sum_i \eta_i = 1$, to yield $b^2 \tilde{S}_L = \sum_{i < j} 2\eta_i \eta_j \log |x_i - x_j|^2$ as was done in [55]. We can see then that the z_i -dependence in (4.71) has the correct form for a conformal three-point function.

- For the Region II, the calculations are similar, but the branch is defined as

$$\begin{aligned} C_1 = & 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda - (1 - 2\eta_1) \log \frac{|z_{12}|^2 |z_{13}|^2}{|z_{23}|^2} + 2 \log \left(1 - \sum_i \eta_i \right) \\ & - \log \frac{\gamma(\eta_1 + \eta_2 - \eta_3) \gamma(\eta_1 + \eta_3 - \eta_2) \gamma(\eta_1 + \eta_2 + \eta_3)}{\gamma(2\eta_1)^2 \gamma(\eta_2 + \eta_3 - \eta_1)}. \end{aligned} \quad (4.72)$$

Where we have used $\gamma(x-1) = -\frac{1}{(x-1)^2} \gamma(x)$ to ensure that if the η 's are real, the only imaginary parts come from the first term. Permuting, we obtain:

$$\begin{aligned} C_2 = & 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda - (1 - 2\eta_2) \log \frac{|z_{12}|^2 |z_{23}|^2}{|z_{13}|^2} + 2 \log \left(1 - \sum_i \eta_i \right) \\ & - \log \frac{\gamma(\eta_1 + \eta_2 - \eta_3) \gamma(\eta_2 + \eta_3 - \eta_1) \gamma(\eta_1 + \eta_2 + \eta_3)}{\gamma(2\eta_2)^2 \gamma(\eta_1 + \eta_3 - \eta_2)}, \end{aligned} \quad (4.73)$$

$$\begin{aligned} C_3 = & 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda - (1 - 2\eta_3) \log \frac{|z_{23}|^2 |z_{13}|^2}{|z_{12}|^2} + 2 \log \left(1 - \sum_i \eta_i \right) \\ & - \log \frac{\gamma(\eta_3 + \eta_2 - \eta_1) \gamma(\eta_1 + \eta_3 - \eta_2) \gamma(\eta_1 + \eta_2 + \eta_3)}{\gamma(2\eta_3)^2 \gamma(\eta_1 + \eta_2 - \eta_3)}. \end{aligned} \quad (4.74)$$

With these results, we can again integrate and arrive to

$$\begin{aligned} b^2 \tilde{S}_L = & \left(\sum_i \eta_i - 1 \right) \log \lambda + (\delta_1 + \delta_2 - \delta_3) \log |z_{12}|^2 + (\delta_1 + \delta_3 - \delta_2) \log |z_{13}|^2 \\ & + (\delta_2 + \delta_3 - \delta_1) \log |z_{23}|^2 + F(\eta_1 + \eta_2 - \eta_3) + F(\eta_1 + \eta_3 - \eta_2) + F(\eta_2 + \eta_3 - \eta_1) \\ & + F(\eta_1 + \eta_2 + \eta_3) - F(2\eta_1) - F(2\eta_2) - F(2\eta_3) - F(0) \\ & + 2 \left[\left(1 - \sum_i \eta_i \right) \log \left(1 - \sum_i \eta_i \right) - \left(1 - \sum_i \eta_i \right) \right] \\ & + 2\pi i (N + 1/2) \left(1 - \sum_i \eta_i \right). \end{aligned} \quad (4.75)$$

In this case, we consider $\eta_i = 0$ to determine the constant, giving an action $b^2 \tilde{S}_0 = 2\pi i (N + 1/2) - \log \lambda - 2$.

We are ready to compare these expressions with the asymptotics of the DOZZ formula in Regions I and II, however, before doing this, it is worth to mention some details about their multivaluedness.

We can see from (4.68) that $F(\eta)$ has branch points at each integer η . The form of the branch points for $\eta = n$ with $n = 0, 1, 2, \dots$ is $-(\eta - n)\log(\eta + n)$, while for $\eta = m$ with $m = 1, 2, \dots$ is $(\eta - m)\log(m - \eta)$. Thus we find that the monodromy of $F(\eta)$ around any loop in the η -plane is given by

$$F(\eta) \rightarrow F(\eta) + \sum_{m=1}^{\infty} (\eta - m)2\pi i N_m - \sum_{n=0}^{\infty} (\eta + n)2\pi i N_n, \quad (4.76)$$

where N_n and N_m count the number of times the loop circles the branch points in a counterclockwise direction. To better understand the expression (4.76), we remember some definitions: a branch of $\log(x)$ has a branch cut that goes from zero to infinity. Picking a branch determines the direction of this cut. For the principle branch of the logarithm, the cut is along the negative real axis x or positive real axis, depending on the convention for the $\arg(z)$ (the argument go from $-\pi$ to π or from 0 to 2π , etc). Checking where $\gamma(x) = 0$ or ∞ , will determine cuts in the integrand and looking for how the function changes around such a branch point η will determine the monodromies of F . The monodromy of $\log x$ is $2\pi i$ and the integral $\int dx \log x$ is $x \log x - x$. So, if we expand $\gamma(x)$ around a zero or pole of $\gamma(x)$ and then integrate the leading term, this is one way of getting the monodromy of F around that point.

Observing (4.71), this can have nonsingular solutions by the multivaluedness of the term $2\pi i N(1 - \sum_i \eta_i)$, and if we apply (4.76) to (4.71) around a loop in the η -plane, this produce shifts of the action from the terms F , such as $2\pi i N(\eta_1 + \eta_2 - \eta_3)$. Then, there seems to be a discrepancy between the branches of the action (4.71) and the saddle points. One could interpret this multivaluedness as indicating the existence of extra solutions, but as we explained in 4.1.2 actually there are no more solutions. A mechanism to clarify this additional multivaluedness will be suggested later.

However, for the Region I or Region II the situation is simpler, because a continuation will only activate the branch cuts in $F(\sum_i \eta_i - 1)$, and this leads to the kind of multivaluedness that can be accounted by the known nonsingular solutions.

Comparison with Asymptotics of the DOZZ Formula

Now we calculate the semiclassical limit of the DOZZ formula (3.55) with three heavy operators in Regions I and II.

First, we analyze the semiclassical behavior of the prefactor of the DOZZ formula. We deduce that,

$$\left[\lambda \gamma(b^2) b^{-2b^2} \right]^{(Q - \sum_i \alpha_i)/b} \rightarrow \exp \left[-\frac{1}{b^2} \left\{ \left(\sum_i \eta_i - 1 \right) \log \lambda - 2 \left(\sum_i \eta_i - 1 \right) \log b \right\} \right]. \quad (4.77)$$

The other terms of (3.55) contain the functions $\Upsilon_b(\eta/b)$. Using the formulas given in Appendix A, we can show that when $b \rightarrow 0$ this functions are:

$$\Upsilon_b(\eta/b) = e^{\frac{1}{b^2} [F(\eta) - (\eta - 1/2)^2 \log b + \mathcal{O}(b \log b)]}, \quad 0 < \text{Re}(\eta) < 1. \quad (4.78)$$

- For the Region I, all the functions Υ_b are in the region where (4.78) is well defined,

$$\begin{aligned} \Upsilon_b \left(\frac{2\eta_1}{b} \right) &= e^{\left[\frac{1}{b^2} (F(2\eta_1) - (2\eta_1 - 1/2)^2 \log b + \mathcal{O}(b \log b)) \right]}, \\ \Upsilon_b \left(\frac{2\eta_2}{b} \right) &= e^{\left[\frac{1}{b^2} (F(2\eta_2) - (2\eta_2 - 1/2)^2 \log b + \mathcal{O}(b \log b)) \right]}, \\ \Upsilon_b \left(\frac{2\eta_3}{b} \right) &= e^{\left[\frac{1}{b^2} (F(2\eta_3) - (2\eta_3 - 1/2)^2 \log b + \mathcal{O}(b \log b)) \right]}, \\ \Upsilon_b \left(\frac{\eta_1 + \eta_2 - \eta_3}{b} \right) &= e^{\left[\frac{1}{b^2} (F(\eta_1 + \eta_2 - \eta_3) - (\eta_1 + \eta_2 - \eta_3 - 1/2)^2 \log b + \mathcal{O}(b \log b)) \right]}, \\ \Upsilon_b \left(\frac{\eta_1 - \eta_2 + \eta_3}{b} \right) &= e^{\left[\frac{1}{b^2} (F(\eta_1 - \eta_2 + \eta_3) - (\eta_1 - \eta_2 + \eta_3 - 1/2)^2 \log b + \mathcal{O}(b \log b)) \right]}, \\ \Upsilon_b \left(\frac{\eta_2 + \eta_3 - \eta_1}{b} \right) &= e^{\left[\frac{1}{b^2} (F(\eta_2 + \eta_3 - \eta_1) - (\eta_2 + \eta_3 - \eta_1 - 1/2)^2 \log b + \mathcal{O}(b \log b)) \right]}, \\ \Upsilon_0 &= e^{\left[\frac{1}{b^2} (F(0) - \frac{1}{4} \log b) \right]}. \end{aligned}$$

So we find that they asymptotically tend to

$$\begin{aligned} & \frac{\Upsilon_0 \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1 + \alpha_2 + \alpha_3 - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b(\alpha_1 - \alpha_2 + \alpha_3) \Upsilon_b(-\alpha_1 + \alpha_2 + \alpha_3)} = \\ & \exp \left[-\frac{1}{b^2} \left\{ F(2\eta_1) + F(2\eta_2) + F(2\eta_3) + F(0) - F \left(\sum_i \eta_i - 1 \right) \right. \right. \\ & \left. \left. - F(\eta_1 + \eta_2 - \eta_3) - F(\eta_1 + \eta_3 - \eta_2) - F(\eta_2 + \eta_3 - \eta_1) - 2 \left(\sum_i \eta_i - 1 \right) \log b \right\} \right]. \end{aligned} \quad (4.79)$$

Then, considering (4.77) and (4.79), we obtain that this is in agreement with (4.71), when $N = 0$. Thus, in this region, the contribution comes only from one saddle point, and the path integral can be interpreted as being evaluated on a single integration cycle passing through it.

- For the Region II, we can see that for the term $\Upsilon_b(\sum_i \alpha_i - Q)$ we can no longer apply (4.78). But we can move the argument of Υ_b to the region where (4.78) is applicable. Then

$$\begin{aligned} \Upsilon_b \left(\left(\sum_i \eta_i - 1 \right) / b \right) &= \gamma \left(\left(\sum_i \eta_i - 1 \right) / b^2 \right)^{-1} b^{1-2(\sum_i \eta_i - 1)/b^2} \Upsilon_b \left(\sum_i \eta_i / b \right) \\ &= \frac{\Gamma \left(1 - \left(\frac{\sum_i \eta_i - 1}{b^2} \right) \right)}{\Gamma \left(\frac{\sum_i \eta_i - 1}{b^2} \right)} b^{1-2(\sum_i \eta_i - 1)/b^2} \\ &\quad \times \exp \left\{ \frac{1}{b^2} \left[F \left(\sum_i \eta_i \right) - \left(\sum_i \eta_i - 1/2 \right)^2 \log b \right] \right\}. \end{aligned} \quad (4.80)$$

Then we have

$$\begin{aligned} &\frac{\Upsilon_0 \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1 + \alpha_2 + \alpha_3 - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b(\alpha_1 - \alpha_2 + \alpha_3) \Upsilon_b(-\alpha_1 + \alpha_2 + \alpha_3)} = \\ &\gamma \left(\left(\sum_i \eta_i - 1 \right) / b^2 \right) b^{2(\sum_i \eta_i - 1)/b^2 - 1} \exp \left[-\frac{1}{b^2} \left\{ -F(2\eta_1) - F(2\eta_2) - F(2\eta_3) \right. \right. \\ &\quad \left. \left. - F(0) + F \left(\sum_i \eta_i \right) + F(\eta_1 + \eta_2 - \eta_3) + F(\eta_1 + \eta_3 - \eta_2) + F(\eta_2 + \eta_3 - \eta_1) \right\} \right]. \end{aligned} \quad (4.81)$$

For the asymptotics of the Γ function (and hence from $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$) we use (4.44) and (4.45),

$$\begin{aligned} \gamma \left(\left(\sum_i \eta_i - 1 \right) / b^2 \right) &= e^{\left(\frac{\sum_i \eta_i - 1}{b^2} \right) \left(2 \log \left(\frac{1 - \sum_i \eta_i}{b^2} \right) \right) - \log \left(1 - \left(\frac{\sum_i \eta_i - 1}{b^2} \right) \right) - 2 \left(\frac{\sum_i \eta_i - 1}{b^2} \right) + 1} \\ &\quad \times \frac{1}{e^{i\pi(\sum_i \eta_i - 1)/b^2} - e^{-i\pi(\sum_i \eta_i - 1)/b^2}} \\ &= e^{\left(\frac{\sum_i \eta_i - 1}{b^2} \right) \left(2 \log \left(1 - \sum_i \eta_i \right) - 4 \log b - 2 \right) - \log \left(1 - \sum_i \eta_i \right) + 2 \log b} \\ &\quad \times \frac{1}{e^{i\pi(\sum_i \eta_i - 1)/b^2} - e^{-i\pi(\sum_i \eta_i - 1)/b^2}}. \end{aligned} \quad (4.82)$$

Finally, using (4.77),(4.81) and (4.82), we obtain:

$$\begin{aligned}
C(\eta_i/b) &\sim \exp \left[-\frac{1}{b^2} \left\{ \left(\sum_i \eta_i - 1 \right) \log \lambda - F(2\eta_1) - F(2\eta_2) - F(2\eta_3) - F(0) \right. \right. \\
&\quad + F \left(\sum_i \eta_i \right) + F(\eta_1 + \eta_2 - \eta_3) + F(\eta_1 + \eta_3 - \eta_2) + F(\eta_2 + \eta_3 - \eta_1) \\
&\quad \left. \left. + 2 \left[\left(1 - \sum_i \eta_i \right) \log \left(1 - \sum_i \eta_i \right) - \left(1 - \sum_i \eta_i \right) \right] \right\} \right] \\
&\quad \times \frac{1}{e^{i\pi(\sum_i \eta_i - 1)/b^2} - e^{-i\pi(\sum_i \eta_i - 1)/b^2}}, \tag{4.83}
\end{aligned}$$

which agrees with (4.75). We can interpret the final term as coming from a sum over infinitely many complex saddle points. The saddle points that contribute are $N = \{-1, -2, \dots\}$ when $\text{Im}((\sum_i \eta_i - 1)/b^2) < 0$ and $N = \{0, 1, 2, \dots\}$ when $\text{Im}((\sum_i \eta_i - 1)/b^2) > 0$, and the condition $\text{Im}((\sum_i \eta_i - 1)/b^2) = 0$ defines a Stokes wall.

4.2.3 Three-Point Function with Light Operators

Finally, we compute the semiclassical limit of the DOZZ formula (3.55) with three light operators and then compare it with a semiclassical computation based on equation (3.53). In this case, we review in some detail the derivation of the DOZZ formula using the fixed-area calculation in [55] and then we use the machinery of complex saddle points.

By computing the asymptotics of the DOZZ formula with three light operators; we need to consider higher order in b because now $\alpha_i = b\sigma_i$ and this led to a trivial analysis.

To calculate the prefactor not involving Υ_b 's, we use the Weierstrass product formula,

$$\Gamma(x) = \frac{1}{x} e^{-\gamma_E x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n} \right)^{-1} e^{x/n}.$$

Where γ_E is the Euler-Mascheroni constant $\gamma_E \equiv \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right)$. By taking the logarithm of both sides of the formula above, we get

$$-\log(\Gamma(x)) = \log(x) + \gamma_E x + \sum_{p=1}^{\infty} \left(\log \left(1 + \frac{x}{p} \right) - \frac{x}{p} \right).$$

Using this to analyze of the function $\gamma(b^2)$, we find

$$\begin{aligned}\gamma(b^2) &\approx \frac{\Gamma(b^2)}{\Gamma(1-b^2)} \approx e^{\log \Gamma(b^2) - \log \Gamma(1-b^2)} \\ &\approx \exp \left\{ -\log b^2 - \gamma_E b^2 + \log(1-b^2) + \gamma_E - b^2 \gamma_E \right. \\ &\quad \left. - \sum_{p=1}^{\infty} \left(\log \left(1 + \frac{b^2}{p} \right) - \frac{b^2}{p} \right) + \sum_{p=1}^{\infty} \left(\log \left(1 + \frac{(1-b^2)}{p} \right) - \frac{(1-b^2)}{p} \right) \right\},\end{aligned}$$

and when $\gamma(b^2) \approx \exp^{\gamma_E(1-2b^2)/b^2}$ as $b \rightarrow 0$, then

$$\begin{aligned}\left[\lambda \gamma(b^2) b^{-2b^2} \right]^{(Q - \sum_i \alpha_i)/b} &= \lambda^{1+1/b^2 - \sum_i \sigma_i} (e^{-2b^2 \gamma_E} b^{-2b^2 - 2})^{1+1/b^2 - \sum_i \sigma_i} \\ &= b^{-2/b^2 + 2 \sum_i \sigma_i - 4} \lambda^{1/b^2 + 1 - \sum_i \sigma_i} e^{-2 \gamma_E + \mathcal{O}(b \log b)}.\end{aligned}\quad (4.84)$$

Now, the asymptotics limit of $\Upsilon_b(\sigma b)$ as $b \rightarrow 0$ is given by equation (A.10)

$$\Upsilon_b(b\sigma) = \frac{C b^{1/2 - \sigma}}{\Gamma(\sigma)} \exp \left[-\frac{1}{4b^2} \log b + \frac{F(0)}{b^2} + \mathcal{O}(b^2 \log b) \right]. \quad (4.85)$$

Here C is a constant that will cancel at the end. However, for the Υ_b involving $\sum_i \sigma_i$, we have to use the following recursion relation

$$\Upsilon_b \left[\left(\sum_i \sigma_i - 1 \right) b - 1/b \right] = \gamma \left(\sum_i \sigma_i - 1 - 1/b^2 \right)^{-1} b^{3+2/b^2 - 2 \sum_i \sigma_i} \Upsilon_b \left[\left(\sum_i \sigma_i - 1 \right) b \right]. \quad (4.86)$$

Now, to determine the semiclassical limit of this, we consider Euler's reflection formula

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

$$\gamma(x - 1/b^2) = \frac{\pi}{\Gamma(1-x + 1/b^2)^2 \sin(\pi(x - 1/b^2))}. \quad (4.87)$$

We note that the Γ function that appears in this equation always has positive real part as $b \rightarrow 0$, so we can simply include the first subleading terms in Stirling's formula to get

$$\Gamma(1-x + 1/b^2) = \sqrt{2\pi} b^{-2/b^2 + 2x-1} e^{-1/b^2} (1 + \mathcal{O}(b^2)). \quad (4.88)$$

This implies

$$\gamma \left(\sum_i \sigma_i - 1 - 1/b^2 \right) = \frac{i}{e^{i\pi(\sum_i \sigma_i - 1 - 1/b^2)} - e^{-i\pi(\sum_i \sigma_i - 1 - 1/b^2)}} b^{4/b^2 - 4 \sum_i \sigma_i + 6} e^{2/b^2} (1 + \mathcal{O}(b^2)). \quad (4.89)$$

By using (4.85) for the Υ_b functions, we have

$$\begin{aligned}
\Upsilon(2b\sigma_1) &= \frac{Cb^{1/2-2\sigma_1}}{\Gamma(2\sigma_1)} \exp \left[-\frac{1}{4b^2} \log b + \frac{F(0)}{b^2} + \mathcal{O}(b^2 \log b) \right], \\
\Upsilon(2b\sigma_2) &= \frac{Cb^{1/2-2\sigma_2}}{\Gamma(2\sigma_2)} \exp \left[-\frac{1}{4b^2} \log b + \frac{F(0)}{b^2} + \mathcal{O}(b^2 \log b) \right], \\
\Upsilon(2b\sigma_3) &= \frac{Cb^{1/2-2\sigma_3}}{\Gamma(2\sigma_3)} \exp \left[-\frac{1}{4b^2} \log b + \frac{F(0)}{b^2} + \mathcal{O}(b^2 \log b) \right], \\
\Upsilon(b\sigma_1 + b\sigma_2 - b\sigma_3) &= \frac{Cb^{1/2-(b\sigma_1+b\sigma_2-b\sigma_3)}}{\Gamma(b\sigma_1 + b\sigma_2 - b\sigma_3)} \exp \left[-\frac{1}{4b^2} \log b + \frac{F(0)}{b^2} + \mathcal{O}(b^2 \log b) \right], \\
\Upsilon(b\sigma_2 + b\sigma_3 - b\sigma_1) &= \frac{Cb^{1/2-(b\sigma_2+b\sigma_3-b\sigma_1)}}{\Gamma(b\sigma_2 + b\sigma_3 - b\sigma_1)} \exp \left[-\frac{1}{4b^2} \log b + \frac{F(0)}{b^2} + \mathcal{O}(b^2 \log b) \right], \\
\Upsilon(b\sigma_3 + b\sigma_1 - b\sigma_2) &= \frac{Cb^{1/2-(b\sigma_3+b\sigma_1-b\sigma_2)}}{\Gamma(b\sigma_3 + b\sigma_1 - b\sigma_2)} \exp \left[-\frac{1}{4b^2} \log b + \frac{F(0)}{b^2} + \mathcal{O}(b^2 \log b) \right], \\
\Upsilon_0 &= \frac{C^{1/2}}{b} \exp \left[-\frac{1}{4b^2} \log b + \frac{F(0)}{b^2} + \mathcal{O}(b^2 \log b) \right].
\end{aligned}$$

Redefining $\Upsilon(b(\sum_i \sigma_i - 1) - \frac{1}{b})$, using (4.86, 4.87, 4.88 and 4.89) with $x = \sum_i \sigma_i - 1$, we have

$$\begin{aligned}
\Upsilon(b(\sum_i \sigma_i - 1) - \frac{1}{b}) &= \left(\frac{i}{e^{i\pi(\sum_i \sigma_i - 1 - \frac{1}{b^2})} - e^{-i\pi(\sum_i \sigma_i - 1 - \frac{1}{b^2})}} b^{\frac{4}{b^2} - 4\sum_i \sigma_i + 6} e^{2/b^2} (1 + \mathcal{O}(b^2)) \right)^{-1} \\
&\times b^{3 + \frac{2}{b^2} - 2\sum_i \sigma_i} \frac{Cb^{-(\sigma_1 + \sigma_2 + \sigma_3) + \frac{3}{2}}}{\Gamma(\sigma_1 + \sigma_2 + \sigma_3 - 1)} \exp \left[-\frac{1}{4b^2} \log b + \frac{F(0)}{b^2} + \mathcal{O}(b^2 \log b) \right].
\end{aligned}$$

Replacing all the above in $C(\sigma_1 b, \sigma_2 b, \sigma_3 b)$ and using (4.84), we obtain

$$\begin{aligned}
C(b\sigma_1, b\sigma_2, b\sigma_3) &= \left[\lambda \gamma(b^2) b^{-2b^2} \right]^{(Q - \sum_i \alpha_i)/b} \\
&\times \frac{\Upsilon_0 \Upsilon(2b\sigma_1) \Upsilon(2b\sigma_2) \Upsilon(2b\sigma_3)}{\Upsilon(b\sigma_1 + b\sigma_2 + b\sigma_3 - (b + \frac{1}{b})) \Upsilon(b\sigma_1 + b\sigma_2 - b\sigma_3) \Upsilon(b\sigma_2 - b\sigma_3 - b\sigma_1) \Upsilon(b\sigma_3 + b\sigma_1 - b\sigma_2)} \\
&= ib^{-3} \lambda^{1/b^2 + 1 - \sum_i \sigma_i} e^{2/b^2 - 2\gamma_E + \mathcal{O}(b \log b)} \frac{1}{e^{i\pi(\sum_i \sigma_i - 1 - 1/b^2)} - e^{-i\pi(\sum_i \sigma_i - 1 - 1/b^2)}} \\
&\times \frac{\Gamma(\sigma_1 + \sigma_2 - \sigma_3) \Gamma(\sigma_1 + \sigma_3 - \sigma_2) \Gamma(\sigma_2 + \sigma_3 - \sigma_1) \Gamma(\sigma_1 + \sigma_2 + \sigma_3 - 1)}{\Gamma(2\sigma_1) \Gamma(2\sigma_2) \Gamma(2\sigma_3)}.
\end{aligned} \tag{4.90}$$

Before comparing this result with (3.53), we have to consider that: if there are no "heavy" operators at all the relevant saddle point is the S^2 sphere (4.40). It is equivalent to

$\phi_0(z, \bar{z}) = \log \frac{A}{\pi(1+z\bar{z})^2}$ with $A = (\sum \eta_i - 1)/(\mu b^2)$ in [55]. The light operator insertions require that this solution has to be integrated over its $SL(2, \mathbb{C})$ orbit parameterized by four complex numbers $\alpha, \beta, \gamma, \delta$ with the constraint $\alpha\delta - \beta\gamma = 1$,

$$\phi_{c,N}(z, \bar{z}) = 2\pi i(N + 1/2) - \log \lambda - 2 \log(|az + \beta|^2 + |\gamma z + \delta|^2). \quad (4.91)$$

This leads to the following expression. For the 3-point function of "light" operators (3.53) that is appropriate for comparison with (4.90),

$$\langle V_{b\sigma_1}(z_1, \bar{z}_1) V_{b\sigma_2}(z_2, \bar{z}_2) V_{b\sigma_3}(z_3, \bar{z}_3) \rangle \approx A(b) \sum_{N \in T} e^{-S_L[\phi_{c,N}]} \int d\mu(\alpha, \beta, \gamma, \delta) \prod_{i=1}^3 e^{\sigma_i \phi_{c,N}(z_i, \bar{z}_i)}, \quad (4.92)$$

where $d\mu(\alpha, \beta, \gamma, \delta)$ stands for the invariant measure on $SL(2, \mathbb{C})$ (see [55]).

$$d\mu(\alpha, \beta, \gamma, \delta) = 4\delta^2(\alpha\delta - \beta\gamma - 1) d^2\alpha d^2\beta d^2\gamma d^2\delta$$

Here T is some set of integers and $A(b)$ is introduced to represent the fluctuation determinant, because we need to include all effects of $\mathcal{O}(b^0)$ order and also the Jacobian to transform the integral over ϕ_c into an integral over the parameters $\alpha, \beta, \gamma, \delta$.

If we observe the subleading terms from (3.37), specifically, the expressions multiplying powers of Q , remember that $Q = b + 1/b$ and $Q^2 = b^2 + 2 + 1/b^2$ so in (3.37) $\frac{Q}{\pi} \oint_{\partial D} \phi d\theta + 2Q^2 \log R = \frac{b+1/b}{\pi} \oint_{\partial D} \phi d\theta + 2(b^2 + 2 + 1/b^2) \log R$, now $\phi = \phi_c/2b$ this is equal to $\frac{1+1/b^2}{2\pi} \oint_{\partial D} \phi_c d\theta + 2(b^2 + 2 + 1/b^2) \log R$, anything proportional to $1/b^2$ is leading term $\frac{1}{2\pi} \oint \phi_c d\theta + 2 \log R$ things of order b^0 are the subleading term $\frac{1}{2\pi} \oint_{\partial D} \phi_c d\theta + 4 \log R$. For the saddle point in (4.91) the leading part was calculated above (4.41), and including now the subleading term we obtain

$$S_L[\phi_{c,N}] = \frac{1}{b^2} [2\pi i(N + 1/2) - \log \lambda - 2] + 2\pi i(N + 1/2) - \log \lambda + \mathcal{O}(b^2). \quad (4.93)$$

Following the steps in [55], we calculate the integral (4.92) for the case $n = 3$,

$$\begin{aligned} & \int d\mu(\alpha, \beta, \gamma, \delta) \prod_{i=1}^3 e^{\sigma_i \phi_{c,N}(z_i, \bar{z}_i)} = \lambda^{-\sum_i \sigma_i} e^{2\pi i(N+1/2) \sum_i \sigma_i} \\ & \times \int \frac{d\mu(\alpha, \beta, \gamma, \delta)}{(|\alpha z_1 + \beta|^2 + |\gamma z_1 + \delta|^2)^{2\sigma_1} (|\alpha z_2 + \beta|^2 + |\gamma z_2 + \delta|^2)^{2\sigma_2} (|\alpha z_3 + \beta|^2 + |\gamma z_3 + \delta|^2)^{2\sigma_3}}. \end{aligned} \quad (4.94)$$

The position dependence of this integral can be extracted by using its $SL(2, \mathbb{C})$ transformation properties, so that this integral takes the form,

$$\int d\mu \prod_{i=1}^3 e^{\sigma_i \phi_{c,N}} = \lambda^{-\sum_i \sigma_i} e^{2\pi i(N+1/2)\sum_i \sigma_i} |z_{12}|^{2\nu_3} |z_{23}|^{2\nu_1} |z_{13}|^{2\nu_2} I(\sigma_1, \sigma_2, \sigma_3),$$

where $\nu_1 = \sigma_1 - \sigma_2 - \sigma_3$, $\nu_2 = \sigma_2 - \sigma_1 - \sigma_3$, $\nu_3 = \sigma_3 - \sigma_1 - \sigma_2$, and

$$I(\sigma_1, \sigma_2, \sigma_3) \equiv \int \frac{d\mu(\alpha, \beta, \gamma, \delta)}{(|\beta|^2 + |\delta|^2)^{2\sigma_1} (|\alpha + \beta|^2 + |\gamma + \delta|^2)^{2\sigma_2} (|\alpha|^2 + |\gamma|^2)^{2\sigma_3}}. \quad (4.95)$$

This integral was calculate in [55], the result is,

$$I(\sigma_1, \sigma_2, \sigma_3) = \pi^3 \frac{\Gamma(\sigma_1 + \sigma_2 - \sigma_3) \Gamma(\sigma_1 + \sigma_3 - \sigma_2) \Gamma(\sigma_2 + \sigma_3 - \sigma_1) \Gamma(\sigma_1 + \sigma_2 + \sigma_3 - 1)}{\Gamma(2\sigma_1) \Gamma(2\sigma_2) \Gamma(2\sigma_3)}. \quad (4.96)$$

Using this result along with (4.93) and (4.95), we find that (4.92) gives

$$\begin{aligned} C(\sigma_i b) &\approx \pi^3 A(b) \lambda^{1/b^2 + 1 - \sum_i \sigma_i} e^{2/b^2} \sum_{N \in T} e^{2\pi i(N+1/2)(\sum_i \sigma_i - 1 - 1/b^2)} \\ &\times \frac{\Gamma(\sigma_1 + \sigma_2 - \sigma_3) \Gamma(\sigma_1 + \sigma_3 - \sigma_2) \Gamma(\sigma_2 + \sigma_3 - \sigma_1) \Gamma(\sigma_1 + \sigma_2 + \sigma_3 - 1)}{\Gamma(2\sigma_1) \Gamma(2\sigma_2) \Gamma(2\sigma_3)}. \end{aligned} \quad (4.97)$$

Comparing this with the DOZZ asymptotics formula we wrote in (4.90), we find complete agreement, with the saddle points included depending on the sign of $\text{Im}(\sum_i \sigma_i - 1/b^2)$. We also see that apparently $A(b) = i\pi^{-3} b^{-3} e^{-2\gamma_E}$, which would be interesting to check by explicitly treating the measure.

Chapter 5

Singular solutions of the Liouville Field Theory

After the presentation of the complex solutions of the Liouville field theory in the previous chapter, we now interpret the singular solutions produced by the analytic continuation in two different contexts, the analysis of the four-point functions and the interesting relation between the complex solutions of the Liouville theory and the Chern-Simons theory.

5.1 Four-Point Functions and the Interpretation of Singular Saddle Points

We return to the Subsection 4.1.2, considering more complex values of the η_i . In this situation, we obtain singular solutions of the Liouville equation, because we have zeros from the denominator in (4.2). The explicit solution is

$$\phi_c = -2\log f - \log \lambda, \tag{5.1}$$

which is singular and generically multivalued. The function f can be expanded as

$$f(z, \bar{z}) = A(z, z_0) + B(\bar{z} - \bar{z}_0) + \dots \tag{5.2}$$

Then the singular solution changes by $-4\pi i k$ around a zero of f with winding number k .

Therefore, it is a problem to interpret the full analytic continuation of the DOZZ formula in terms of the conventional path integral.

However, Harlow, Maltz and Witten argue that the expression (4.30) still makes sense and controls the asymptotic behavior of the DOZZ formula, even when ϕ_c is multivalued. They give the following arguments for this:

- An analytic continuation of (3.52) is possible because this formula agrees, with up to a minor redefinition of the action, when there are no singularities, and it is finite even in the presence of zeros of the denominator.
- The presence of singularities actually allows the full multivaluedness of the action (4.75) to be realized by analytic continuation of the "solutions".
- The saddle points that dominate the three-point function by including a fourth light degenerate operator are in agreement with the (4.30) for all values of the η_i .

5.1.1 Finiteness of the "Action"

In Subsection 4.2.2, when we defined the Region I, we assumed that the imaginary parts of the η_i 's are small in order to ensure that the denominator (4.30) did not vanish away from the operator insertions. On the other hand, independently of the imaginary part of η , if we consider the multivaluedness for C_i and $\tilde{S}_L$, conditions $\text{Re}(\sum_i \eta_i) < 1$ or $\text{Re} \eta_i < \frac{1}{2}$ do not coincide. We know that if we consider an expansion near a zero of the denominator, we have

$$\phi_c(z, \bar{z}) \approx -2\log [A(z - z_0) + B(\bar{z} - \bar{z}_0)], \quad (5.3)$$

which has a logarithmic singularity as well as a branch cut discontinuity, which will lead to a divergence in the kinetic term of the action. This divergence can be avoided, if we write the action in terms of f because this is a continuous function.

Considering solutions with no additional singularities and using (5.1) we can write (3.52) as

$$b^2 \tilde{S}_L = \frac{1}{\pi} \int_{D \cup d_i} d^2 \xi [\partial f \bar{\partial} f / f^2 + 1 / f^2] + \text{boundary terms}. \quad (5.4)$$

Here, Harlow, Maltz and Witten propose that this action is valid in the presence of zeros of the denominator of (4.2) with the integral defined by removing a small disc d_i of radius ϵ centered around each zero and then taking $\epsilon \rightarrow 0$. Using the expansion (5.2) near a zero $z = z_0$, in the disc, we have $d^2\xi = r dr d\theta$. We write $f = r(Ae^{i\theta} + Be^{-i\theta})$ and the action in the vicinity of z_0 is

$$b^2 \tilde{S}_L = \frac{1}{\pi} \int_{\epsilon} \frac{dr}{r} \int_0^{2\pi} d\theta \frac{AB + 1}{(Ae^{i\theta} + Be^{-i\theta})^2} + \text{boundary terms.} \quad (5.5)$$

The radial integral gives $\log r|_{r=\epsilon}$, which is divergent when $\epsilon \rightarrow 0$, and the angular integral is zero as long as $|A| \neq |B|$. But if we consider $|A| = |B|$, the contribution of the angular integral $\int_0^{2\pi} (\csc\theta)^2 d\theta$ is divergent.

Furthermore, the expression (4.75) for the action can easily be continued to values of η_i , where the denominator vanishes, and its value is finite there. So in order to keep using the trick of differentiating with respect to η_i with which we get the action (4.75), we must demonstrate that (5.5) is in agreement with this.

With this aim, we calculate the variation of the action (5.5) under $f \rightarrow f + \delta f$ with δf continuous; thus, a multivalued "solution" is interpreted as a stationary point of this action. However we must be careful, because when we vary the action (5.5) with respect to f ,

$$\int d^2z \frac{\partial \delta f \bar{\partial} f + \partial f \bar{\partial} \delta f}{f^2} + \text{terms that do not have } \delta f \text{ under a derivative.}$$

This is equal to

$$\int d^2z \left(\partial \left(\frac{\delta f \bar{\partial} f}{f^2} \right) - \delta f \frac{\bar{\partial} f}{f^2} \right) + \text{the bar term} + \text{terms that do not have } \delta f \text{ under a derivative,}$$

so the first term and the analogous $\bar{\partial}$ are what lead to a contour term

$$\Delta \tilde{S}_L = -\frac{\epsilon}{2\pi b^2} \int_0^{2\pi} d\theta \frac{\partial_r f}{f^2} \delta f|_{(z-z_0)=\epsilon e^{i\theta}}. \quad (5.6)$$

They are not zero as usually because f has a branch cut emanating from the center of the disk $f(e^{-i\epsilon}) \neq f(e^{i(2\pi-\epsilon)})$, so it does not vanish, it becomes (5.6). This boundary term is also present near each of the operator.

Remember (4.13), near the insertion of the operator at z_i we have $f \sim r^{2\eta_i}$. This f in (5.6)

produces $-\frac{\eta_i}{2\pi} \int_0^{2\pi} d\theta \phi_c$. However, when f is given by (5.2), this boundary term is zero by itself since the angular integral vanishes. Then a multivalued "solution" is a stationary point of the action. Indeed, a singlevalued ϕ with singularities away from the operator insertions can never be such a stationary point.

For the case when $|A| = |B|$, the higher order terms near the singularity are important, and the singularity may be non-isolated. We will see this just as a degenerate limit of the more general situation. In particular, we can continue from Region I to anywhere else in the η_i -plane without passing through a configuration with a singularity with $|A| = |B|$, so this subtlety should not affect our picture of the analytic continuation of (3.55).

Let us summarize in the Table 4.1. We see that there exist two different kinds of multivaluedness, one is with respect to η_i , and the other is with respect to $z, \bar{z}$.

	$z, \bar{z}$ behavior at fixed η_i	η_i behavior at fixed $z, \bar{z}$
e^{ϕ_c}	singlevalued	singlevalued
$b^2 \tilde{S}_L$	trivial	defined up to addition of $2\pi i (\sum_i \eta_i m_i + n)$ with m_i all even or all odd
C_i	trivial	defined up to addition of $2\pi i$
a_1	trivial	defined up to multiplication by a sign
a_1/a_2	trivial	singlevalued
f	singlevalued	defined up to multiplication by a $z, \bar{z}$ -independent sign
ϕ_c	possibly singlevalued, possibly monodromy of addition of $4\pi i$ about points where $f = 0$	defined up to addition of $2\pi i$

Table 5.1: The multivaluedness of various quantities

5.1.2 Multivaluedness of the Action

Remember the action in (4.71), we saw this is highly multivalued as a function of the η_i , with the multivaluedness arising from the function $F(\eta)$ defined in (4.68).

Now this multivaluedness of the action can be interpreted as a consequence of the multivaluedness in $z, \bar{z}$ that ϕ_c acquires in the presence of zeroes of f . One more time, we emphasize that this multivaluedness does not affect the kinetic and potential terms of the action because they depend only on f , which is singlevalued as a function of $z, \bar{z}$.

However, the terms $-\sum_i \frac{\eta_i}{2\pi} \int_0^{2\pi} d\theta \phi_c$ that we saw in the previous subsection, are sensitive to such multivaluedness and gives

$$\Delta \tilde{S}_L = -\frac{1}{b^2} \sum_i \eta_i C_i, \quad (5.7)$$

Remembering (4.63), (4.64) and (4.65) we can see that the constant C_i can be shifted by an integer multiple of $2\pi i$ by a closed path in the parameter space of the η_i . This produce a change in the action by an integer linear combination of the quantities $2\pi i \eta_i$. So the action in (4.71) have a shift in the C_i or an equivalent shift in the function F in this formula, leading to the same multivaluedness.

In the Subsection (4.2.2), the full multivaluedness of the action in η can be realized only because ϕ_c can be multivalued as a function of $z, \bar{z}$. Here, any continuation in η_i that passes only trough continuous ϕ_c 's cannot produce monodromy for the difference of any two C_i 's because of continuity.

Moreover, if the paths in η_i pass through multivalued ϕ_c 's, these differences can have the nontrivial monodromy necessary to produce the full set of branches of the action. "The multivaluedness of the action in η_i has a natural interpretation once we allow solutions of the complex Liouville equations that are multivalued in $z, \bar{z}$ ".

5.1.3 Degenerate Four-Point Function as a Probe

In the Section 3.9, we studied the Teschner's formula for the exact four-point function of a light degenerate field $V_{-b/2}$ with three generic operators V_{α_i} . Choosing $\alpha_i = \eta_i/b$'s, we can evaluate the semiclassical limit for any values of the η_i . We can compare this result with the formula proposed by Harlow, Maltz and Witten in (3.53) for calculating the correlation function in the semiclassical limit. Considering the operator $V_{-b/2}$, we have $\sigma = -1/2$, so in (3.53), we have the function $\exp(-\phi_c/2)$, where ϕ_c is saddle point determined by the three heavy operators, then (3.53) gives

$$\langle V_{\eta_4/b}(z_4, \bar{z}_4) V_{\eta_3/b}(z_3, \bar{z}_3) V_{-b/2}(z_2, \bar{z}_2) V_{\eta_1/b}(z_1, \bar{z}_1) \rangle \approx \sum_N e^{-\phi_{c,N}(z_2, \bar{z}_2)/2} e^{-\tilde{S}_{L,N}}.$$

Using the definitions (3.54) and (3.70) and also (4.2) this implies

$$G_{1234}(x, \bar{x}) \approx \sqrt{\lambda} \frac{|z_{14}| |z_{34}|}{|z_{13}| |z_{24}|^2} \sum_N f_N(z_2, \bar{z}_2) e^{-\tilde{S}_{134,N}}, \quad (5.8)$$

where $\tilde{S}_{134,N}$ is obtained by replacing $\eta_2 \rightarrow \eta_4$ in (4.71)

$$\begin{aligned} b^2 \tilde{S}_{134,N} &= (\eta_1 + \eta_3 + \eta_4 - 1) \log \lambda + F(\eta_1 + \eta_4 - \eta_3) + F(\eta_1 + \eta_3 - \eta_4) + F(\eta_3 + \eta_4 - \eta_1) \\ &\quad + F(\eta_1 + \eta_3 + \eta_4 - 1) - F(2\eta_1) - F(2\eta_3) - F(2\eta_4) - F(0) \\ &\quad + 2\pi i(n + m_1\eta_1 + m_3\eta_3 + m_4\eta_4), \end{aligned} \quad (5.9)$$

where n, m_i are integers determined by the branch N . On the other hand, in (3.95) Teschner's proposal is

$$G_{1234}(x, \bar{x}) = C^-_{12} C_{34-} \left[\mathcal{F}_-(x) \mathcal{F}_-(\bar{x}) + \frac{C^+_{12} C_{34+}}{C^-_{12} C_{34-}} \mathcal{F}_+(x) \mathcal{F}_+(\bar{x}) \right]. \quad (5.10)$$

Using (3.101) and (3.102), Teschner's recursion relation reads

$$\begin{aligned} \frac{C_{34+} C^+_{12}}{C_{34-} C^-_{12}} &= \frac{\gamma(2b\alpha_1 - b^2)}{\gamma(b(\alpha_1 + \alpha_4 - \alpha_3) - b/2) \gamma(b(\alpha_1 + \alpha_3 - \alpha_4 - b/2))} \\ &\quad \times \frac{\gamma(b(\alpha_3 + \alpha_4 - \alpha_1 - b/2))}{\gamma(b(\alpha_1 + \alpha_3 + \alpha_4 - Q - b/2)) \gamma(2 + b^2 - 2b\alpha_1)}, \end{aligned}$$

where $\gamma(b(\alpha_1 + \alpha_3 + \alpha_4) - 1 - 3b^2/2) = \gamma(b(\alpha_1 + \alpha_3 + \alpha_4) - Q - b/2)$ and

$$\frac{\gamma(2b\alpha_1 - b^2)}{\gamma(2 - (2b\alpha_1 - b^2))} = \frac{\Gamma(2b\alpha_1 - b^2) \Gamma(2b\alpha_1 - b^2 - 1)}{\Gamma(1 - 2b\alpha_1 + b^2) \Gamma(2 - 2b\alpha_1 + b^2)} = -\frac{\gamma^2(2b\alpha_1 - b^2)}{(1 - 2b\alpha_1 + b^2)^2}.$$

$$\begin{aligned} \frac{C_{34+} C^+_{12}}{C_{34-} C^-_{12}} &= -\frac{1}{(1 - b(2\alpha_1 - b))^2} \\ &\quad \times \frac{\gamma^2(b(2\alpha_1 - b)) \gamma(b(\alpha_3 + \alpha_4 - \alpha_1 - b/2))}{\gamma(b(\alpha_1 + \alpha_4 - \alpha_3 - b/2)) \gamma(b(\alpha_1 + \alpha_3 - \alpha_4 - b/2)) \gamma(b(\alpha_1 + \alpha_3 + \alpha_4 - Q - b/2))}, \end{aligned} \quad (5.11)$$

and using (4.27) and (4.28) we obtain

$$-\frac{a_1}{a_2} = -\frac{\gamma^2(2\eta_1) \gamma(\eta_4 + \eta_3 - \eta_1)}{\gamma(\eta_1 + \eta_4 - \eta_3) \gamma(\eta_1 + \eta_3 - \eta_4) \gamma(\eta_1 + \eta_4 + \eta_3 - 1)} \frac{1}{(1 - 2\eta_1)^2}.$$

Taking the semiclassical limit ($b \rightarrow 0$), $\frac{C_{34+} C^+_{12}}{C_{34-} C^-_{12}}$ this becomes

$$\frac{C_{34+} C^+_{12}}{C_{34-} C^-_{12}} = -\frac{\gamma^2(2\eta_1) \gamma(\eta_4 + \eta_3 - \eta_1)}{\gamma(\eta_1 + \eta_4 - \eta_3) \gamma(\eta_1 + \eta_3 - \eta_4) \gamma(\eta_1 + \eta_4 + \eta_3 - bQ)} \frac{1}{(1 - 2\eta_1)^2}, \quad (5.12)$$

also $bQ \rightarrow 1$, then

$$\frac{C_{34+} C^+_{12}}{C_{34-} C^-_{12}} \rightarrow -\frac{a_2}{a_1}, \quad (5.13)$$

where a_1 and a_2 are the constants in the semiclassical solution (4.30), considering that now $\eta_2 \rightarrow \eta_4$. We can also see that in this limit (4.26) gives

$$\begin{aligned}\mathcal{F}_+(x) &\rightarrow P^{1-\eta_1}(x), \\ \mathcal{F}_-(x) &\rightarrow P^{\eta_1}(x).\end{aligned}\tag{5.14}$$

Where, by using (3.97), we can see

$$\begin{aligned}\mathcal{F}_+(x) &= x^{\alpha_-}(1-x)^{\beta}F(A_-, B_-, C_-, x), \\ &= x^{\Delta_+ - \Delta - \Delta_1}(1-x)^{\Delta_{\alpha_3 - b/2} - \Delta - \Delta_3}F(-b(\alpha_1 - Q/2) + b(\alpha_3 + \alpha_4 - b) - 1/2 \\ &\quad -b(\alpha_1 - Q/2) + b(\alpha_3 - \alpha_4) + 1/2, 1 - b(2\alpha_1 - Q), x),\end{aligned}$$

with (semiclassical limit $b \rightarrow 0$)

$$\begin{aligned}\Delta_+ - \Delta - \Delta_1 &= \left(\alpha_1 + \frac{b}{2}\right)\left(Q - \alpha_1 - \frac{b}{2}\right) + \frac{1}{2} + \frac{3b^2}{4} - \alpha_1(Q - \alpha_1) = 1 - \eta_1, \\ \Delta_{\alpha_3 - b/2} - \Delta - \Delta_3 &= \left(\alpha_3 - \frac{b}{2}\right)\left(Q - \alpha_3 + \frac{b}{2}\right) + \frac{1}{2} + \frac{3b^2}{4} - \alpha_3(Q - \alpha_3) = 1 - \eta_3, \\ A_- &= -b\left(\alpha_1 - \frac{Q}{2}\right) + b(\alpha_3 + \alpha_4 - b) - 1/2 = 1 - \eta_1 - \eta_3 + \eta_4, \\ B_- &= -b\left(\alpha_1 - \frac{Q}{2}\right) + b(\alpha_3 - \alpha_4) + 1/2 = 2 - \eta_3 - \eta_1 - \eta_4, \\ C_- &= 1 - b(2\alpha_1 - Q) = 2 - 2\eta_1.\end{aligned}$$

Replacing (5.13) and (5.14) in (5.10), we find that in the semiclassical limit we have

$$G_{1234}(x, \bar{x}) = C_{34-} \left[P^{\eta_1}(x)P^{\eta_1}(\bar{x}) - \frac{a_2}{a_1}P^{1-\eta_1}(x)P^{1-\eta_1}(\bar{x}) + \mathcal{O}(b) \right]. \tag{5.15}$$

Using (4.30) in (5.8), we can compare the result above with the one we got before in (5.15), to see that

$$e^{-\tilde{S}_{34-,N}} = a_{1,N} \sqrt{\lambda} \frac{|z_{14}| |z_{34}|}{|z_{13}|} e^{-\tilde{S}_{134,N} + \mathcal{O}(b)}. \tag{5.16}$$

Semiclassically the structure constants C_{134} and C_{34-} are in the same region of the η_i plane, thus we assume they are both a sum over the same set of branches N . Using (5.9), we see that

$$\begin{aligned}\tilde{S}_{134,N} - \tilde{S}_{34-,N} &= \frac{1}{2} \log \lambda + i\pi m_1 + \frac{1}{2} [\log \gamma(\eta_1 + \eta_4 - \eta_3) + \log \gamma(\eta_1 + \eta_3 - \eta_4) \\ &\quad + \log \gamma(\eta_1 + \eta_3 + \eta_4 - 1) - \log \gamma(\eta_3 + \eta_4 - \eta_1) - 2 \log \gamma(2\eta_1)] \\ &\quad + \mathcal{O}(b).\end{aligned}\tag{5.17}$$

Evaluating this in (5.16), then comparing with (4.29), we obtain that $a_{1,N}$ is satisfied up to an overall branch-dependent sign. Now we define $a_{1,N}$ depending of the branch as

$$a_{1,N} = \frac{|z_{13}|}{|z_{14}||z_{34}|} \exp \left[\log \gamma(\eta_1 + \eta_4 - \eta_3) + \log \gamma(\eta_1 + \eta_3 - \eta_4) + \log \gamma(\eta_1 + \eta_3 + \eta_4 - 1) \right. \\ \left. - \log \gamma(\eta_3 + \eta_4 - \eta_1) - 2 \log \gamma(2\eta_1) + i\pi \tilde{m}_1 \right]. \quad (5.18)$$

This logarithms are defined by continuation from real η 's in Region I along a specific path, which gives an unambiguous meaning to $\tilde{m}_1$. So the signs will match in (5.16) if $m_1 = \tilde{m}_1$. In order to understand this, remember that

$$\phi_{c,N} = -4\eta_1 \log |z - z_1| + C_{1,N} \quad \text{when } z \rightarrow z_1, \quad (5.19)$$

with

$$C_{1,N} = -2\pi i m_1 - \log \lambda - (1 - 2\eta_1) \log \frac{|z_{14}|^2 |z_{13}|^2}{|z_{34}|^2} - \log \gamma(\eta_1 + \eta_4 - \eta_3) \\ - \log \gamma(\eta_1 + \eta_3 - \eta_4) - \log \gamma(\eta_1 + \eta_3 + \eta_4 - 1) \\ + \log \gamma(\eta_3 + \eta_4 - \eta_1) + 2 \log \gamma(2\eta_1), \quad (5.20)$$

where the logarithms are defined by analytic continuation along the same path as in defining $a_{1,N}$ because $\frac{\partial \tilde{S}_{L,N}}{\partial \eta_1} = -C_{1,N}$, we are justified using m_1 in this formula. Near $z = z_1$ we have

$$e^{-\phi_{c,N}/2} \equiv \sqrt{\lambda} f_N = |z - z_1|^{2\eta_1} e^{-C_{1,N}} [1 + \mathcal{O}(|z - z_1|)], \quad (5.21)$$

thus, in (5.8) we should choose the branch of f_N , and thus of $a_{1,N}$, with $\tilde{m}_1 = m_1$.

This is the demonstration given by Harlow, Maltz and Witten of the consistency of (5.16). So, for the four-point function to degenerate, the Liouville path integral is controlled by singular "solutions" throughout the full η_i three-plane.

5.2 Interpretation in Chern-Simons Theory

The main argument given by Gaiotto and Witten [16] in order to verify directly that the Jones polynomial of a knot in three dimensions can be computed by counting the

solutions of certain gauge theory equations in four dimensions was the link between the four-dimensional gauge theory equations in the question and conformal blocks for degenerate representations of the Virasoro algebra in two dimensions. In this context arise the better understanding of the relation between Chern-Simons gauge theory and Virasoro conformal blocks [49]. Here, Gaiotto and Witten remark that the relation between Virasoro conformal blocks and Chern-Simons theory has a simple extension to a relation between Liouville theory and Chern-Simons Theory. In this section, we present the interpretation of the singular solution of the Liouville theory in the Chern-Simons theory.

5.2.1 Liouville Solutions and Flat Connections

We know that a solution of the Liouville equation can be parametrized in terms of f . The equation of motion gives a holomorphic and antiholomorphic differential equation for this function f ,

$$\left(\frac{\partial^2}{\partial z^2} + W(z)\right)f = 0, \quad (5.22)$$

$$\left(\frac{\partial^2}{\partial \bar{z}^2} + \tilde{W}(\bar{z})\right)f = 0. \quad (5.23)$$

Both equations locally have a two-dimensional space of solutions. We have a solution of Liouville equation from a basis $\begin{pmatrix} u \\ v \end{pmatrix}$ of holomorphic solutions of (5.22) and a basis $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$ of antiholomorphic solutions of (5.23). This can be applied on any Riemann surfaces Σ .

However, the insertion of the heavy operator produces singularities; so, globally in going around a singularity in Σ , $\begin{pmatrix} u \\ v \end{pmatrix}$ has in general non-trivial monodromy. The monodromy maps this pair to another basis of the same two-dimensional space of solutions,

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix}, \quad (5.24)$$

here M is a constant 2×2 matrix. Since the Wronskian $u\partial v - v\partial u$ is independent of z , this leads to $\det M = 1$, so this takes values in $SL(2, \mathbb{C})$. We can also see that the differential equation (5.22) may be expressed in terms of an $SL(2, \mathbb{C})$ flat connection. So we define the complex gauge field $\mathcal{A}$,

$$\mathcal{A}_z = \begin{pmatrix} 0 & -1 \\ W(z) & 0 \end{pmatrix}, \quad \mathcal{A}_{\bar{z}} = 0. \quad (5.25)$$

For that $\binom{f}{g}$ to be covariantly constant with respect to this connection, we require that f is a holomorphic solution of the equation (5.22) and $g = \partial f / \partial z$. Thus, parallel transport of this doublet around a loop, which we accomplish by multiplying by $U = P e^{-\oint A_z dz}$, is the same as analytic continuation around the same loop. In the same way, the antiholomorphic differential (5.23) have monodromies valued in $SL(2, \mathbb{C})$, in this case the flat connection is,

$$\tilde{\mathcal{A}}_z = 0, \quad \tilde{\mathcal{A}}_{\bar{z}} = \begin{pmatrix} 0 & \tilde{W}(\bar{z}) \\ -1 & 0 \end{pmatrix}. \quad (5.26)$$

Near points with heavy operators insertions, $\mathcal{A}$ and $\tilde{\mathcal{A}}$ have singularities. Remembering that the solutions of (5.22) behave as z^η and $z^{1-\eta}$ near an operator insertion at $z = 0$. We can define the invariant way of the monodromies, $\exp(\pm 2\pi i \eta)$ of these functions under a circuit in the counterclockwise direction around $z = 0$ as

$$\text{Tr } M = 2\cos(2\pi\eta). \quad (5.27)$$

In the same way, the monodromy of the antiholomorphic equation (5.23) around $z = 0$ obeys (5.27).

In general, the flat connections $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are actually gauge-equivalent and have conjugate monodromies around all cycles. Thus, a solution of Liouville's equations, being either real or complex, gives us a flat $SL(2, \mathbb{C})$ connection over Σ that can be put in the gauge (5.25) and in the gauge (5.26).

To be more precise, we have to introduce the concept "oper" defined in [16]. So, locally $\mathcal{A}$ is a flat connection, after picking a local coordinate, z , can be put in the form (5.26), in such a way that in the intersection of coordinate patches, the gauge transformation required to compare the two descriptions is lower triangular

$$g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \quad (5.28)$$

A flat connection with this property is known as an oper. Similarly, the global characterization of $\tilde{\mathcal{A}}$ has the same form with upper triangular matrices replacing lower triangular ones. The proposal of Harlow, Maltz and Witten is that a classical solution of Liou-

ville theory is a flat connection whose holomorphic structure is that of an oper, while its antiholomorphic structure is also that of an oper.

5.2.2 Interpretation in Chern-Simons Theory

For the case of a compact symmetry group, it is already known that Chern-Simons on $C \times I$ produces the WZW model on C . Based on this relationship that exists between Virasoro conformal blocks and Chern-Simons theory, we can do a simple extension to relate the Liouville theory on a Riemann surface Σ to Chern-Simons theory on $\Sigma \times I$ [53] [16]. The Chern-Simons action with an $SL(2, \mathbb{C})$ connection $\mathcal{A}$ is

$$S_{CS} = \frac{1}{4\pi i b^2} \int_M \text{Tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right), \quad (5.29)$$

defined on a three-manifold M . This action is invariant under gauge transformations that are continuously connected to the identity, but not under homotopically non-trivial gauge transformations. For $M = S^3$, the homotopically non-trivial gauge transformations are parametrized by $\pi_3(SL(2, \mathbb{C})) = \mathbb{Z}$. Under a homotopically nontrivial gauge transformation on S^3 , S_{CS} transforms as

$$S_{CS} \rightarrow S_{CS} + \frac{2\pi i n}{b^2}, \quad (5.30)$$

where n is the invariant of a gauge transformations called winding number and b is the Liouville coupling parameter, this matches the multivaluedness of Liouville theory that comes from the trivial symmetry $\phi_c \rightarrow \phi_c + 2\pi i$.

At this point, we must emphasize that the Chern-Simons action (5.29) is normalized, in the context of the Analytic Continuation of the Chern-Simons Theory of Witten in [54]. Here we are not restricted to assume that b^2 is the inverse of an integer the normalization, so in the path integral, we consider integration cycles that are not invariant under homotopically non-trivial gauge transformations, and we do not view homotopically nontrivial gauge transformations as symmetries of the theory. A basis of the possible integration cycles is given by the cycles that arise by steepest descent from a critical point of the action [54]. The Yang-Mills field strength is defined by $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. The classical equations of motion of Chern-Simons theory are simply

$$\mathcal{F} = 0. \quad (5.31)$$

One considers Chern-Simons theory $M = \Sigma \times I$, where I is a unit interval and Σ is the Riemann surface on which we do Liouville theory. Although the fundamental group of M is the same as that of Σ , we introduce the unit interval because we need that an $SL(2, \mathbb{C})$ flat connection Σ obeys: it can be described by a holomorphic differential equation, and it can also be described by an antiholomorphic differential equation. It is possible to consider boundary conditions at the two ends of I , one imposes the Nahm pole boundary condition and at the other end, one imposes a variant of the Nahm pole boundary condition with z and $\bar{z}$ exchanged [16].

In conclusion, Liouville partition functions and correlation functions are built by combining holomorphic and anti-holomorphic Virasoro conformal blocks, which arise natural from Chern-Simons on $\Sigma \times I$ with boundary conditions just stated.

Figure 5.1: Monodromy defects in $\Sigma \times I$

5.2.3 Liouville Primary Fields and Monodromy Defects

The argument principle in this section is that a primary field of the Liouville theory, inserted at a point $p \in \Sigma$, corresponds to monodromy defects on $p \times I \subset \Sigma \times I$ [16].

So we demand that $\mathcal{A}$ has a suitable singularity along Σ . Considering that $z = re^{i\theta}$ is a local coordinate that vanishes at p , the singular behavior of $\mathcal{A}$ is

$$\mathcal{A} = i d\theta \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix} + \dots, \quad (5.32)$$

where the eigenvalues of the monodromy are $\exp(\pm 2\pi i \eta)$.

The Ansatz (5.32) is only invariant under diagonal gauge transformations along p . Now gauge transformation g is constrained: at the ends of $\Sigma \times I$, it equals ± 1 , while along the monodromy defects it is diagonal, and away from the boundary and the monodromy

defects, it takes arbitrary values in $SL(2, \mathbb{C})$.

Let us consider the following diagonal gauge transformation

$$g = \begin{pmatrix} \rho e^{i\vartheta} & 0 \\ 0 & \rho^{-1} e^{-i\vartheta} \end{pmatrix} \quad (5.33)$$

with ρ positive and ϑ real. Parametrizing the interval I by $y = 0$ at the left end of I and $y = 1$ at the right end. Constraining ϑ to vanish at $y = 0$ (where $g = 1$), and to πm at $y = 1$, where m is even if $g = 1$ on the right end of $\Sigma \times I$, and m is odd if $g = -1$ there. The winding number m along the monodromy defect should be

$$m = \frac{1}{\pi} \int_0^1 dy \frac{d\vartheta}{dy}. \quad (5.34)$$

In the presence of a monodromy defect, we need to add one more term to the action because we would like a flat bundle on the complement of the monodromy defect that has the singularity (5.32) along the defect to be a classical solution. So such flat bundle obeys

$$\mathcal{F} = 2\pi i \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix} \delta_K, \quad (5.35)$$

where K labels the monodromy defect, and δ_K is a two-form delta function. Remembering that the equation of motion of the Chern-Simons action (5.29) is simply $\mathcal{F} = 0$ rather than (5.35), we add to the action a term

$$S_K = -\frac{1}{b^2} \int_K \text{Tr} \mathcal{A} \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}. \quad (5.36)$$

For $K = p \times I$, only the component of $\mathcal{A}$ in the y direction, appears in (5.35). So under a diagonal gauge transformation (5.33), the diagonal matrix elements of $\mathcal{A}_y$ are shifted by $\mp d \log(\rho e^{i\vartheta})/dy$. Taking the trace and integrating out over y , we find that S_K is shifted by

$$\frac{2i\eta}{b^2} \int_0^1 dy \frac{d\vartheta}{dy} = \frac{2\pi i \eta m}{b^2}. \quad (5.37)$$

Generally, consider s heavy operator insertions, with the Liouville parameters $\eta_i, i = 1, \dots, s$, inserted at points $p_i \in \Sigma$. In Chern-Simons theory, they correspond to monodromy

defects, supported on $K_i = p_i \times I$. Introducing a winding number m_i associated to each monodromy defect with a bulk winding number n . The shift in the total action $S = S_{CS} + \sum_i S_{K_i}$ under a gauge transformation is

$$S \rightarrow S + \frac{2\pi i}{b^2} \left(n + \sum_{i=1}^s m_i \eta_i \right). \quad (5.38)$$

5.2.4 Interpretation

In the Chern-Simons theory, critical points corresponds to flat connections on $\Sigma \times I$, with prescribed behavior near the ends and near monodromy defects, modulo *topologically trivial* gauge transformations. Topologically nontrivial gauge transformations do not correspond to symmetries because they do not leave the action invariant. Instead, they generate new critical points from old ones.

For the case with three heavy operators on S^2 , all critical points are related to each other only by topologically nontrivial gauge transformations. This leads to a simple way to compare the path integrals over cycles associated to different critical points.

Let $\mathcal{A}$ be a connection that represents a critical point ρ . Suppose a gauge transformation g with winding numbers n and $m_1, \dots, m_s$ acts on $\mathcal{A}$ to produce a new critical point $\mathcal{A}'$. Let Z_ρ and $Z_{\rho'}$ be the path integrals over integration cycles associated to $\mathcal{A}$ and to $\mathcal{A}'$, respectively. $Z_{\rho'}$ and Z_ρ are not equal, since the gauge transformation g does not preserve the action, as we mentioned. But since g transforms the action by a simple additive c -number (5.38), there is a simple exact formula that express the relation between $Z_{\rho'}$ and Z_ρ

$$Z_{\rho'} = Z_\rho \exp \left(- (2\pi i / b^2) \left(n + \sum_i m_i \eta_i \right) \right). \quad (5.39)$$

The interpretation of this formula in Liouville theory will be seen later in the Section 6.1.6.

Chapter 6

Timelike Liouville

In this chapter, we calculate the correlation functions for the timelike version of the Liouville theory. This theory has application in holographic quantum cosmology, in the study of tachyon condensation and in other time-dependent scenarios of string theory [34]. After an extensive analysis about the analytic continuation of the Liouville field theory, the question is whether the case of the timelike Liouville theory can actually be regarded as a conformal field theory [29] [30] [31]. Here arises the importance of calculating the three-point function, because this observable is nontrivial since, contrary to naive expectations, the structure constants of timelike Liouville theory are not the analytic extension of the structure constants of spacelike Liouville theory. We review the procedure of Harlow, Maltz and Witten. They calculate this within the path integral approach. Finally we obtain the correlation function using the free-fields formalism. We obtain interesting relations between the spacelike version and the timelike version for the three-point function in the Liouville theory.

6.1 Proposal of Harlow, Maltz and Witten

In this section, we consider the path integral with complex values of all parameters and then analyze, which integration cycles to use in order to reproduce the analytic continuation from the physical region. We can see some numerical results in [55] for complex b , then we can confirm the crossing symmetry of the four-function based DOZZ formula.

We begin with the following redefinitions for the parameters

$$b = -i\hat{b} \quad \text{then} \quad \alpha = i\hat{\alpha}. \quad (6.1)$$

$$\phi = i\hat{\phi}. \quad (6.2)$$

$$Q = i\left(\frac{1}{\hat{b}} - \hat{b}\right) \equiv i\hat{Q}. \quad (6.3)$$

Now the action (3.1) is

$$S_L = \frac{1}{4\pi} \int d^2\xi \sqrt{\tilde{g}} \left[-\partial_a \hat{\phi} \partial_b \hat{\phi} \tilde{g}^{ab} - \hat{Q} \tilde{R} \hat{\phi} + 4\pi\mu e^{2\hat{b}\hat{\phi}} \right]. \quad (6.4)$$

Since the kinetic term in this action has the wrong sign, the theory is called to as "timelike" Liouville theory. The central charge in terms of the redefined parameters is

$$c_L = 1 - 6\hat{Q}^2, \quad (6.5)$$

and the physical metric now is $g_{ab} = e^{\frac{2}{\hat{Q}}\hat{\phi}} \tilde{g}_{ab}$. The solution $\hat{\phi}$ has the following boundary condition at infinity

$$\hat{\phi}(z, \bar{z}) = -2\hat{Q} \log |z| + \mathcal{O}(1). \quad (6.6)$$

Finally, we define the conformal weights as

$$\Delta \left(e^{-2\hat{\alpha}\hat{\phi}} \right) = \bar{\Delta} \left(e^{-2\hat{\alpha}\hat{\phi}} \right) = \hat{\alpha}(\hat{\alpha} - \hat{Q}). \quad (6.7)$$

The regularized Liouville action (3.52) with the insertion of heavy operators $\hat{\alpha}_i = \eta_i/\hat{b}$, using the rescaled field $\phi_c = 2b\phi = 2\hat{b}\hat{\phi}$ with a flat reference metric is

$$\begin{aligned} \tilde{S}_L = & -\frac{1}{16\pi\hat{b}^2} \int_{D-\cup_i d_i} d^2\xi \left(\partial_a \phi_c \partial_a \phi_c - 16\hat{\lambda} e^{\phi_c} \right) - \frac{1}{\hat{b}^2} \left(\frac{1}{2\pi} \oint_{\partial D} \phi_c d\theta + 2 \log R \right) \\ & + \frac{1}{\hat{b}^2} \sum_i \left(\frac{\eta_i}{2\pi} \oint_{\partial d_i} \phi_c d\theta_i + 2\eta_i^2 \log \epsilon \right), \end{aligned} \quad (6.8)$$

where $\hat{\lambda} = \pi\mu\hat{b}^2 = -\lambda$, then the new equation of motion is

$$\partial\bar{\partial}\phi_c = -2\hat{\lambda}e^{\phi_c} - 2\pi \sum_i \eta_i \delta^2(\xi - \xi_i), \quad (6.9)$$

when μ is positive, we have the equation of motion for constant positive curvature with conical deficits at the heavy operators. If we consider that $\hat{b}$ is real and η_i is in Region II, described by (4.56), and also is real, we obtain a real solution.

6.1.1 The Timelike DOZZ formula

The DOZZ formula when b is real contains the especial functions $\Upsilon_b(x)$, the integral that defines this functions does not converge for any x . For this, when b is imaginary, the argument of this functions changes and we have a problem, if we want to do an analytic continuation of the DOZZ formula from its expression for b real. However, we can solve this by deforming the contour of integration. Consider the function $H_b(x)$ as defined in [59]:

$$H_b(x) = \Upsilon_b(x)\Upsilon_{ib}(-ix + ib). \quad (6.10)$$

In order that both functions Υ 's are defined by (3.56) we take b to have positive real part and negative imaginary part. The functions H_b obey the recursion relations

$$\begin{aligned} H_b(x+b) &= e^{\frac{i\pi}{2}(2bx-1)} H_b(x), \\ H_b(x+\frac{1}{b}) &= e^{\frac{i\pi}{2}(1-2x/b)} H_b(x). \end{aligned} \quad (6.11)$$

Following the analysis in [59], we introduced the θ -function. This is defined as

$$\theta_1(z, \tau) = i \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau(n-1/2)^2 + 2\pi iz(n-1/2)}, \quad \text{Im}\tau > 0, \quad (6.12)$$

this function obeys the following relations:

$$\begin{aligned} \theta_1(z+1, \tau) &= e^{-i\pi} \theta_1(z, \tau), \\ \theta_1(z+\tau, \tau) &= e^{i\pi(1-\tau-2z)} \theta_1(z, \tau). \end{aligned} \quad (6.13)$$

If we evaluate the terms $n = 1, 2, \dots$ with $n = 0, -1, \dots$ in (6.12), the θ -function has a zero at $z = 0$. If we apply the recursion relations, it has zeros for all $z = m + n\tau$ with $m, n \in \mathbb{Z}$. These zeros are simple and they are the only zeros. Then we can obtain the following expression,

$$e^{\frac{i\pi}{2}(x^2+x/b-xb)} \theta_1(x/b, 1/b^2). \quad (6.14)$$

This term has the same recursion relations and the same zeros as $H_b(x)$. If $x = \frac{b}{2} + \frac{1}{2b}$ and $\Upsilon_b(Q/2) = 1$, we can express $H_b(x)$ as

$$H_b(x) = e^{\frac{i\pi}{2}(x^2+x/b-xb+\frac{b^2}{4}-\frac{3}{4b^2}-\frac{1}{4})} \frac{\theta_1(x/b, 1/b^2)}{\theta_1(\frac{1}{2} + \frac{1}{2b^2}, 1/b^2)}. \quad (6.15)$$

Using the definition (6.10) and the last construction of the function $H_b(x)$, we can have $\Upsilon_{ib}(-ix + ib) = \frac{H_b(x)}{\Upsilon_b(x)}$ and use this to analyze the behavior of Υ_b near imaginary b .

Inspired by the works of Al. B. Zamolodchikov on the three-point function in the minimal models, and Kostov and Petkova with the Non-rational 2D quantum gravity, Harlow, Maltz and Witten propose to continue Teschner's recursion relations to imaginary b .

For a generic complex b , the solution of Teschner's recursion relations is not unique, but when b is imaginary, we have again a unique solution. This result is not given by analytic continuation of the DOZZ formula, because, if one multiplies this solution by an $\hat{\alpha}$ -independent arbitrary function of b , the recursion relations are unaffected. Then we obtain the solutions, which is called the timelike DOZZ formula

$$\begin{aligned} \hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) &= \frac{2\pi}{\hat{b}} \left[-\pi\mu\gamma(-\hat{b}^2)\hat{b}^{2+2\hat{b}^2} \right]^{(\sum_i \hat{\alpha}_i - \hat{Q})/\hat{b}} e^{-i\pi(\sum_i \hat{\alpha}_i - \hat{Q})/\hat{b}} \\ &\frac{\Upsilon_{\hat{b}}(\hat{\alpha}_1 + \hat{\alpha}_2 - \hat{\alpha}_3 + \hat{b})\Upsilon_{\hat{b}}(\hat{\alpha}_1 + \hat{\alpha}_3 - \hat{\alpha}_2 + \hat{b})\Upsilon_{\hat{b}}(\hat{\alpha}_2 + \hat{\alpha}_3 - \hat{\alpha}_1 + \hat{b})\Upsilon_{\hat{b}}(\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 - \hat{Q} + \hat{b})}{\Upsilon_{\hat{b}}(\hat{b})\Upsilon_{\hat{b}}(2\hat{\alpha}_1 + \hat{b})\Upsilon_{\hat{b}}(2\hat{\alpha}_2 + \hat{b})\Upsilon_{\hat{b}}(2\hat{\alpha}_3 + \hat{b})}. \end{aligned} \quad (6.16)$$

Compare this result with the one obtained by Zamolodchikov in [59]. We can see some differences between these two results: first, the power of $\pi\mu\gamma(-\hat{b}^2)$, this difference is needed to interpret of timelike Liouville as being a different integration cycle of the conventional Liouville and second difference is that the result in [59] is divided by a factor of $\frac{\hat{b}^3}{2\pi}\gamma(1 - \hat{b}^2)\gamma(2 - 1/\hat{b}^2)$; these choices do not affect the recursion relations and can be interpreted as an ambiguity in the normalization of the operators.

In contrast with the three-point function, the two-point function in (3.80) have a suitable timelike two-point function with a good analytic continuation to imaginary b , name by

$$\begin{aligned} \hat{G}(\hat{\alpha}) &= -\frac{\hat{b}}{2\pi}\hat{C}(0, \hat{\alpha}, \hat{\alpha}) = -\frac{1}{\hat{b}^2} \left[-\pi\mu\gamma(-\hat{b}^2) \right]^{(2\hat{\alpha} - \hat{Q})/\hat{b}} e^{-i\pi(2\hat{\alpha} - \hat{Q})/\hat{b}} \\ &\times \gamma(2\hat{\alpha}\hat{b} + \hat{b}^2)\gamma\left(\frac{1}{\hat{b}^2} - \frac{2\hat{\alpha}}{\hat{b}} - 1\right). \end{aligned} \quad (6.17)$$

This result has the following particular characteristics:

When we consider real $\hat{\alpha}$, this result is not positive as expected from the wrong-sign kinetic term. Unlike the spacelike Liouville in (3.79), the relation between this two-point function

and the three-point function is somewhat arbitrary. In order to analyze the two-point function, when we set one of the $\hat{\alpha}$'s to zero in the timelike DOZZ formula, does NOT produce a δ -function. So if we reduce the timelike DOZZ formula and consider $\hat{\alpha}_1 \rightarrow 0$, this function has a finite and nonzero limit even if $\hat{\alpha}_2 \neq \hat{\alpha}_3$. In [59], this was observed as part of a larger issue, whereby the degenerate fusion rules mentioned below equation (3.94) are not automatically satisfied by the timelike DOZZ formula. In [35], this was interpreted as the two-point function being genuinely non-diagonal in the operator weights.

6.1.2 Semiclassical Tests of the Timelike DOZZ formula

Now we test, if in the semiclassical limit, (6.16) and (6.17) are consistent with the demand that they are produced by the usual Liouville path integral on a different integration cycle. We will obtain an expression for a solution of (6.9) considering a solution $\phi_{c,N}(\eta_i, \lambda, b, z, \bar{z})$ of the old equation of motion (3.42). It is given by

$$\hat{\phi}_{c,N}(\eta_i, \hat{\lambda}, \hat{b}, z, \bar{z}) \equiv \phi_{c,N}(\eta_i, \hat{\lambda}, \hat{b}, z, \bar{z}) - i\pi. \quad (6.18)$$

In the same way, we defined the action function of the original action (3.52) as $\tilde{S}_L = [\eta_i, \lambda, b, z_i, \bar{z}_i; \phi_{c,N}(\eta_i, \lambda)]$. Thus we have

$$\begin{aligned} \tilde{S}_L &= [\eta_i, -\hat{\lambda}, -i\hat{b}, z_i, \bar{z}_i; \hat{\phi}_{c,N}(\eta_i, \hat{\lambda})] \equiv \hat{\tilde{S}}_L = [\eta_i, \hat{\lambda}, \hat{b}, z_i, \bar{z}_i; \hat{\phi}_{c,N}(\eta_i, \lambda)] \\ &= -\tilde{S}_L = [\eta_i, \hat{\lambda}, \hat{b}, z_i, \bar{z}_i; \hat{\phi}_{c,N}(\eta_i, \hat{\lambda})] + \frac{i\pi}{\hat{b}^2} \left(1 - \sum_i \eta_i \right). \end{aligned} \quad (6.19)$$

We can see that the left hand side of this is just (6.8), so we can compute the action for timelike Liouville theory by an easy modification of the old results in the Chapter 4.

6.1.3 Two-point Function

With the previous considerations (6.19) and the use (4.42) of, is easy to find the timelike action of the saddlepoint (4.39):

$$\begin{aligned} \hat{\tilde{S}}_L &= -\frac{1}{\hat{b}^2} [2\pi i N(1-2\eta) + (2\eta-1)\hat{\lambda} + 2((1-2\eta)\log(1-2\eta) - (1-2\eta)) \\ &\quad + 2(\eta-\eta^2)\log|z_{12}|^2]. \end{aligned} \quad (6.20)$$

When α is heavy, the semiclassical limit of (6.17) is

$$\hat{G}(\eta) \rightarrow \left(e^{2\pi i(1-2\eta)/\hat{b}^2} - 1 \right) \exp \left\{ \frac{1}{\hat{b}^2} \left[-(1-2\eta)\log\hat{\lambda} + 2((1-2\eta)\log(1-2\eta) - (1-2\eta)) \right] \right\}, \quad (6.21)$$

which is matched by a sum over the two saddle points $N = 0$ and $N = 1$ with actions given by (6.20). Note that the integral over the moduli would again produce a divergence, but that, unlike in the DOZZ case, this divergence does not seem to be produced by the limit $\alpha_1 \rightarrow 0$, $\alpha_2 = \alpha_3$.

6.1.4 Three-point Function with Heavy Operators

In the Region I, the timelike version of (4.62) has the action

$$\begin{aligned} \hat{S}_L = & -\frac{1}{\hat{b}^2} \left[-\left(1 - \sum_i \eta_i \right) \log \hat{\lambda} - (\hat{\delta}_1 + \hat{\delta}_2 - \hat{\delta}_3) \log |z_{12}|^2 - (\hat{\delta}_1 + \hat{\delta}_3 - \hat{\delta}_2) \log |z_{13}|^2 \right. \\ & \left. - (\hat{\delta}_2 + \hat{\delta}_3 - \hat{\delta}_1) \log |z_{23}|^2 + F(\eta_1 + \eta_2 - \eta_3) + F(\eta_1 + \eta_3 - \eta_2) + F(\eta_2 + \eta_3 - \eta_1) \right. \\ & \left. + F(\eta_1 + \eta_2 + \eta_3 - 1) - F(2\eta_1) - F(2\eta_2) - F(2\eta_3) - F(0) + 2\pi i(N - 1/2) \left(1 - \sum_i \eta_i \right) \right], \end{aligned} \quad (6.22)$$

here $\delta \equiv \eta_i(\eta_i - 1)$, is consistent with (6.7). The action in other regions is always an analytic continuation of this action along some path.

In the Region II the timelike action is

$$\begin{aligned} \hat{S}_L = & -\frac{1}{\hat{b}^2} \left[-\left(1 - \sum_i \eta_i \right) \log \hat{\lambda} - (\hat{\delta}_1 + \hat{\delta}_2 - \hat{\delta}_3) \log |z_{12}|^2 - (\hat{\delta}_1 + \hat{\delta}_3 - \hat{\delta}_2) \log |z_{13}|^2 \right. \\ & \left. - (\hat{\delta}_2 + \hat{\delta}_3 - \hat{\delta}_1) \log |z_{23}|^2 + F(\eta_1 + \eta_2 - \eta_3) + F(\eta_1 + \eta_3 - \eta_2) + F(\eta_2 + \eta_3 - \eta_1) \right. \\ & \left. + F(\eta_1 + \eta_2 + \eta_3) - F(2\eta_1) - F(2\eta_2) - F(2\eta_3) - F(0) \right. \\ & \left. + 2 \left\{ \left(1 - \sum_i \eta_i \right) \log \left(1 - \sum_i \eta_i \right) - \left(1 - \sum_i \eta_i \right) \right\} + 2\pi i N \left(1 - \sum_i \eta_i \right) \right]. \end{aligned} \quad (6.23)$$

In order to compare them with the timelike version of the DOZZ formula, we use the asymptotic formula (4.78). The other terms of (6.16) that are different to the $\Upsilon_{\hat{b}}$ approach

are

$$e^{\frac{1-\sum_i \eta_i}{\hat{b}^2}(i\pi+2\log\hat{b}-\log\hat{\lambda})+\mathcal{O}(1/\hat{b})}. \quad (6.24)$$

Using (4.78), we find that in Region I the $\Upsilon_{\hat{b}}$'s combine with this to give

$$\begin{aligned} \hat{C}(\eta_i/\hat{b}) \sim & \exp\left\{\frac{1}{\hat{b}^2}\left[-\left(1-\sum_i \eta_i\right)\log\hat{\lambda}+F(\eta_1+\eta_2-\eta_3)+F(\eta_1+\eta_3-\eta_2)\right.\right. \\ & +F(\eta_2+\eta_3-\eta_1)+F(\eta_1+\eta_2+\eta_3-1)-F(2\eta_1)-F(2\eta_2) \\ & \left.\left.-F(2\eta_3)-F(0)+i\pi\left(1-\sum_i \eta_i\right)\right]\right\}. \end{aligned} \quad (6.25)$$

Comparing this with (6.22), we see that only the saddle point with $N = 0$ contributes.

In Region II (4.83) we use (A.4) to shift one of the $\Upsilon_{\hat{b}}$'s before using the asymptotic formula (4.78), giving

$$\Upsilon_{\hat{b}}\left(\frac{\sum_i \eta_i - 1}{\hat{b}} + 2\hat{b}\right) \sim \gamma \left(\left(\sum_i \eta_i - 1\right)/\hat{b}^2\right)^{-1} \hat{b}^{\frac{1}{\hat{b}^2}(2(1-\sum_i \eta_i)-(\sum_i \eta_i - 1/2)^2)} e^{\frac{1}{\hat{b}^2}F(\sum_i \eta_i)}. \quad (6.26)$$

In the Region II we have,

$$\begin{aligned} \hat{C}(\eta_i/\hat{b}) \sim & \exp\left\{\frac{1}{\hat{b}^2}\left[-\left(1-\sum_i \eta_i\right)\log\hat{\lambda}+F(\eta_1+\eta_2-\eta_3)+F(\eta_1+\eta_3-\eta_2)\right.\right. \\ & +F(\eta_2+\eta_3-\eta_1)+F(\eta_1+\eta_2+\eta_3-1)-F(2\eta_1)-F(2\eta_2) \\ & -F(2\eta_3)-F(0)+2\left(1-\sum_i \eta_i\right)\log\left(1-\sum_i \eta_i\right)-2\left(1-\sum_i \eta_i\right) \\ & \left.+i\pi\left(1-\sum_i \eta_i\right)\right]\right\} \left(e^{i\pi(1-\sum_i \eta_i)/\hat{b}^2}-e^{-i\pi(1-\sum_i \eta_i)/\hat{b}^2}\right). \end{aligned} \quad (6.27)$$

Comparing this with (6.23), we have the contributions of two saddle points with $N = 0$ and $N = 1$. Unlike for the spacelike DOZZ formula, there are no Stokes walls in Region II.

6.1.5 Three-point Function with Light Operators

Finally, we calculate the semiclassical limit, when $b \rightarrow 0$ of the timelike DOZZ with three light operators. Now, we define $\sigma_i = \frac{\alpha_i}{b} = -\frac{\hat{\alpha}_i}{\hat{b}}$, which gives $\Delta \rightarrow \sigma$ as $\hat{b} \rightarrow 0$. Following a

procedure similar to Subsection 4.2.3, so the timelike version of (4.90) is

$$\begin{aligned} \hat{C}(-\sigma_1 \hat{b}, -\sigma_2 \hat{b}, -\sigma_3 \hat{b}) &= -2\pi i \hat{b}^{-3} \hat{\lambda}^{1-\sum_i \sigma_i - 1/\hat{b}^2} e^{-2/\hat{b}^2 - 2\gamma_E + \mathcal{O}(\hat{b} \log \hat{b})} \left(e^{2\pi i (\sum_i \sigma_i - 1 + 1/\hat{b}^2)} - 1 \right) \\ &\times \frac{\Gamma(1-2\sigma_1)\Gamma(1-2\sigma_2)\Gamma(1-\sigma_3)}{\Gamma(1+\sigma_1-\sigma_2-\sigma_3)\Gamma(1+\sigma_2-\sigma_1-\sigma_3)\Gamma(1+\sigma_3-\sigma_1-\sigma_2)\Gamma(2-\sum_i \sigma_i)}. \end{aligned} \quad (6.28)$$

Applying the Euler reflection formula $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$ to each Γ -functions, we have

$$\begin{aligned} \hat{C}(-\sigma_1 \hat{b}, -\sigma_2 \hat{b}, -\sigma_3 \hat{b}) &= \hat{b}^{-3} \hat{\lambda}^{1-\sum_i \sigma_i - 1/\hat{b}^2} e^{-2/\hat{b}^2 - 2\gamma_E + \mathcal{O}(\hat{b} \log \hat{b})} \left(e^{2\pi i (\sum_i \sigma_i - 1 + 1/\hat{b}^2)} - 1 \right) \\ &\times \frac{\Gamma(\sigma_1 + \sigma_2 - \sigma_3)\Gamma(\sigma_1 + \sigma_3 - \sigma_2)\Gamma(\sigma_2 + \sigma_3 - \sigma_1)\Gamma(\sum_i \sigma_i - 1)}{\Gamma(2\sigma_1)\Gamma(2\sigma_2)\Gamma(2\sigma_3)} \\ &\times \frac{(e^{2\pi i (\sigma_1 + \sigma_2 - \sigma_3)} - 1)(e^{2\pi i (\sigma_1 + \sigma_3 - \sigma_2)} - 1)(e^{2\pi i (\sigma_2 + \sigma_3 - \sigma_1)} - 1)(e^{2\pi i (\sigma_1 + \sigma_2 + \sigma_3)} - 1)}{(e^{4\pi i \sigma_1} - 1)(e^{4\pi i \sigma_2} - 1)(e^{4\pi i \sigma_3} - 1)}. \end{aligned} \quad (6.29)$$

For this correlation function, we have the semiclassical formula (4.92):

$$\langle e^{\sigma_1 \phi_c(z_1, \bar{z}_1)} e^{\sigma_2 \phi_c(z_2, \bar{z}_2)} e^{\sigma_3 \phi_c(z_3, \bar{z}_3)} \rangle \approx A(-i\hat{b}) \sum_{N \in T} e^{-S_L[\hat{\phi}_{c,N}]} \int d\mu(\alpha, \beta, \gamma, \delta) \prod_{i=1}^3 e^{\sigma_i \hat{\phi}_{c,N}(z_i, \bar{z}_i)}, \quad (6.30)$$

where $A(-i\hat{b})$ is only the analytic continuation of their spacelike version. Explicitly, the timelike "solution" in (6.18) is

$$\hat{\phi}_{c,N}(z, \bar{z}) = 2\pi i N(z, \bar{z}) - \log \hat{\lambda} - 2\log(|\alpha z + \beta|^2 + |\gamma z + \delta|^2). \quad (6.31)$$

The dependence of N with the positions suggest the different branches of the action to be realized and if we remember the Subsection 5.1 this agrees with the situation, where discontinuous "solutions" must be included even though single-valued solutions exist.

Following the same procedure as in subsection (4.2.3), we find the structure constant with three-light operators

$$\hat{C}(-\sigma_1 \hat{b}, -\sigma_2 \hat{b}, -\sigma_3 \hat{b}) \approx A(-i\hat{b}) \hat{\lambda}^{1-\sum_i \sigma_i - 1/\hat{b}^2} e^{-2/\hat{b}^2} I(\sigma_1, \sigma_2, \sigma_3) \sum_{N \in T} e^{-2\pi i (\sum_i m_i \sigma_i + n/\hat{b}^2)}, \quad (6.32)$$

where $-n$ is the value of N at ∞ and $-m$ is its value near the various insertions.

In the Subsection 4.2.3, we obtained $I(\sigma_1, \sigma_2, \sigma_3)$ in (4.96). So we can compare (6.12) with (6.29). We find that $A(-i\hat{b}) = \hat{b}^{-3}\pi^{-3}e^{-2\gamma_E}$ is in agreement with the spacelike case.

Finally, we mention the differences between our results in the spacelike case studied in the Subsection 4.2.3 and the timelike Liouville: First, in the timelike case, the set T of branches is complex, because of the dependence on the position of $N(z, \bar{z})$; a consequence that can be seen in the term of the last line in (6.29). So in the timelike version, many branches that correspond to discontinuous "solutions" are now definitely needed. We have to remember that the modular integral over $SL(2, \mathbb{C})$ converges only, when the σ 's obey certain inequalities (F20) in [22]. In spacelike with the σ 's in Region II the integral is convergent, so we do not need to use a contour deformation for its evaluation. Instead, in timelike Liouville case, when the σ 's are in Region II, many of the inequalities are violated and the integral must be defined by analytic continuation. This continuation results in an additional Stokes phenomena, which changes the contributing saddle points.

6.1.6 An Exact Check

After we reviewed the tests of Harlow, Maltz and Witten for their solution of the timelike DOZZ formula in the semiclassical limit, we present the exact argument given by them, where the timelike DOZZ formula (6.16) is obtained by evaluating the usual Liouville path integral on a new integration cycle. Also, they showed that the ratio of the spacelike and timelike DOZZ formula must have a specific form.

By definition, we know that

$$Z_\rho(\alpha_i, z_i, \bar{z}_i) = \int_{\mathcal{C}_\rho} \mathcal{D}\phi_c V_{\alpha_1}(z_1, \bar{z}_1) \dots V_{\alpha_n}(z_n, \bar{z}_n) e^{-S_L}, \quad (6.33)$$

where ρ is a critical point of the action with heavy operators as sources, and the path integral is evaluated on the steepest descent cycle $\mathcal{C}_\rho$ that passes through ρ . Remembering the procedure of Witten in [54], the analytic continuation of the Chern-Simons theory can be carried out by generalizing the usual integration cycle of the Feynman path integral in the framework of the Picard-Lefschetz theory. Here, we mention that the quantity (6.33) is *not* in general equal to the Liouville correlator because we need to sum over such cycles

with integer coefficients a^ρ .

After giving some details of (6.33), we proceed with our object. Remember the original action (3.38),

$$S_L = \frac{1}{16\pi b^2} \int_D d^2\xi [(\partial\phi_c)^2 + 16\lambda e^{\phi_c}] + \frac{1}{2\pi b^2} (1+b^2) \oint_{\partial D} \phi_c d\theta + \frac{2}{b^2} (1+2b^2+b^4) \log(R). \quad (6.34)$$

If we consider the transformation $\phi_c \rightarrow \phi_c + 2\pi i N$, we have $S_L \rightarrow S_L + \frac{2\pi i N}{b^2} (1+b^2)$ and in the semiclassical limit, the operator V_α defined in (3.51) transforms as $V_\alpha \rightarrow V_\alpha e^{2\pi i \alpha/b}$. A important concept here is that the Seiberg bound ensures that the renormalization needed to define this operator precisely is the same as in free field theory, this is actually the exact transformation of V_α . On the other hand the path-integral measure $\mathcal{D}\phi_c$ is invariant under the shift. With this arguments, if the difference between two ρ 's is only by adding $2\pi i N$, we have the following relation

$$Z_{\rho+2\pi i N} = e^{(\sum_i \alpha_i/b - 1/b^2)} Z_\rho. \quad (6.35)$$

This result is exact, and shows that the result of integrating over a sum of integration cycles of this type can be factored out from the correlator

$$Z = \sum_{N=-\infty}^{\infty} a^{\rho+2\pi i N} Z_{\rho+2\pi i N} = Z_\rho \sum_{N=-\infty}^{\infty} a^{\rho+2\pi i N} e^{2\pi i N (\sum_i \alpha_i/b - 1/b^2)}. \quad (6.36)$$

Thus, in general, the ratio of Z 's which are computed on different cycles, both of the form $\sum_{N=-\infty}^{\infty} a^{\rho+2\pi i N} \mathcal{C}_{\rho+2\pi i N}$, will be expressible as a ratio of Laurent expansions in $e^{2\pi i N (\sum_i \alpha_i/b - 1/b^2)}$ with integer coefficients [22]. This will lead to nontrivial constraint as the ratio invariant under shifting by $\alpha_i \rightarrow \alpha_i + b$ and more subtle invariance as $b \rightarrow \frac{b}{\sqrt{1+b^2}}$ and $\alpha_i \rightarrow \frac{\alpha_i}{\sqrt{1+b^2}}$.

However if we want to consider the DOZZ formula in the full range of α_i 's, this is insufficient because in (5.3) was necessary to consider discontinuous "solutions" that differ by different multiplies of $2\pi i$ at the different operator insertions. Harlow, Maltz and Witten proposed that the action (6.34) in terms of f as in (5.4), which changes only by an overall c -number, if we shift the field configuration by $2\pi i N$ with a position-dependent $N \in \mathbb{Z}$. So, the change in the action depends on the value of N at infinity, and the contributions of operator insertions also shift in a way that depends on the value of N in their vicinity.

On the other hand in the Chern-Simons theory these additional "solutions" were just as valid and conventional as the usual ones. Harlow, Maltz and Witten assumed that the relationship between Z_ρ and $Z_{\rho'}$ is given by a formula analogous to (5.39) in the Chern-Simons version

$$Z_{\rho'} = Z_\rho e^{-\frac{2\pi i}{b^2}(n + \sum_i m_i \alpha_i b)}, \quad (6.37)$$

where n and m_i are the differences in N at infinity and near the various operator insertions, and m_i are either all even or all odd.

Finally, the ratio of the spacelike and timelike DOZZ formulas for a region of b where both make sense, can be obtain using (6.16) as well as (3.55), (6.15), and (6.10). We have

$$\begin{aligned} \frac{\hat{C}(-i\alpha_1, -i\alpha_2, -i\alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} &= -\frac{2\pi i}{b} \lim_{\epsilon \rightarrow 0} \frac{\Upsilon_b(\epsilon)}{\Upsilon_0 H_b(\epsilon)} e^{i\pi(1/b-b)(Q - \sum_i \alpha_i)} \\ &\times \frac{H_b(\sum_i \alpha_i - Q) H_b(\alpha_1 + \alpha_2 - \alpha_3) H_b(\alpha_1 + \alpha_3 - \alpha_2) H_b(\alpha_2 + \alpha_3 - \alpha_1)}{H_b(2\alpha_1) H_b(2\alpha_2) H_b(2\alpha_3)} \\ &= -2\pi i e^{-\frac{2\pi i}{b^2}(\sum_i \alpha_i b - (1+b^2)/2)} \\ &\frac{\theta_1\left(\frac{\sum_i \alpha_i - Q}{b}, \frac{1}{b^2}\right) \theta_1\left(\frac{\alpha_1 + \alpha_2 - \alpha_3}{b}, \frac{1}{b^2}\right) \theta_1\left(\frac{\alpha_1 + \alpha_3 - \alpha_2}{b}, \frac{1}{b^2}\right) \theta_1\left(\frac{\alpha_2 + \alpha_3 - \alpha_1}{b}, \frac{1}{b^2}\right)}{\theta'_1\left(0, \frac{1}{b^2}\right) \theta_1\left(2\alpha_1/b, \frac{1}{b^2}\right) \theta_1\left(2\alpha_2/b, \frac{1}{b^2}\right) \theta_1\left(2\alpha_3/b, \frac{1}{b^2}\right)}, \end{aligned} \quad (6.38)$$

where $\theta'_1(z, \tau) \equiv \frac{\partial \theta_1}{\partial z}(z, \tau)$. Using (6.13), we have the simplified version of the above result,

$$\frac{\hat{C}(-i\alpha_1, -i\alpha_2, -i\alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} = 2\pi i \frac{\theta_1\left(\frac{\sum_i \alpha_i}{b}, \frac{1}{b^2}\right) \theta_1\left(\frac{\alpha_1 + \alpha_2 - \alpha_3}{b}, \frac{1}{b^2}\right) \theta_1\left(\frac{\alpha_1 + \alpha_3 - \alpha_2}{b}, \frac{1}{b^2}\right) \theta_1\left(\frac{\alpha_2 + \alpha_3 - \alpha_1}{b}, \frac{1}{b^2}\right)}{\theta'_1\left(0, \frac{1}{b^2}\right) \theta_1\left(2\alpha_1/b, \frac{1}{b^2}\right) \theta_1\left(2\alpha_2/b, \frac{1}{b^2}\right) \theta_1\left(2\alpha_3/b, \frac{1}{b^2}\right)}. \quad (6.39)$$

With this result, Harlow, Maltz and Witten affirmed that the timelike DOZZ formula is given by the ordinary Liouville path integral evaluated on a different integration cycle. Here we can see the reason, why an analytic continuation directly between purely real and purely imaginary b , fails.

6.2 Free-field method

We calculate the three-point function in the timelike version of the Liouville field following the proposal of Giribet [19], using the free field method. We define the action

$$S_L[\varphi, \mu] = \frac{1}{4\pi} \int_{\mathcal{C}} d^2x \sqrt{g} (\sigma g^{ab} \partial_a \varphi \partial_b \varphi + Q_\sigma R \varphi + 4\pi \mu e^{2b\varphi}), \quad (6.40)$$

where μ and b are two real parameters, and $Q_\sigma = b + \sigma b^{-1}$ with $\sigma = \pm 1$. For convenience, we use $\sigma = +1$, for the spacelike Liouville theory and $\sigma = -1$ for the wrong sign kinetic term. Now the central charge is

$$c = 1 + 6\sigma Q_\sigma^2. \quad (6.41)$$

Also, the primary operators of the theory $V_\alpha(z) = e^{2\alpha\varphi(z)}$, creates states with the following conformal dimension

$$\Delta_\alpha = \sigma \alpha (Q_\sigma - \alpha). \quad (6.42)$$

6.2.1 Three-point correlation function in the free-field representation

To calculate this, we set

$$\begin{aligned} S_L &= S_0 + S_{int}, \\ S_0 &= \frac{1}{4\pi} \int_{\mathcal{C}} d^2x \sqrt{g} (\sigma g^{ab} \partial_a \varphi \partial_b \varphi + Q_\sigma R \varphi), \\ S_{int} &= \int_{\mathcal{C}} d^2x \mu e^{2b\varphi}. \end{aligned}$$

The n -point correlation functions of local operators on a curve $\mathcal{C}$ are defined by

$$\left\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \right\rangle_{\mathcal{C}} = \int_{\varphi(\mathcal{C})} \mathcal{D}\varphi e^{-S_L[\varphi, \mu]} \prod_{i=1}^n e^{2\alpha_i \varphi(z_i)}. \quad (6.43)$$

So, three-point correlation functions on the sphere are

$$\left\langle \prod_{i=1}^3 V_{\alpha_i}(z_i) \right\rangle_{\mathbb{S}^2} = C_b(\alpha_1, \alpha_2, \alpha_3) \prod_{i < j}^3 |z_i - z_j|^{\Delta_{ij}},$$

with $\Delta_{ij} = \Delta_{\alpha_1} + \Delta_{\alpha_2} + \Delta_{\alpha_3} - 2\Delta_{\alpha_i} - 2\Delta_{\alpha_j}$. Using the Goulian-Li paper [20], we define $\tilde{\varphi}(z) \equiv \varphi(z) - \varphi_0$ and integrate over the zero-mode. Then, fixing the insertions points of the three vertex operators $(z_1, z_2, z_3) = (0, 1, \infty)$. The structure constants are

$$\begin{aligned} C_b^{(\sigma)}(\alpha_1, \alpha_2, \alpha_3) &= \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\mathbb{S}^2}, \\ &= \int_{\varphi_{\mathbb{CP}^1}} \mathcal{D}\varphi e^{S_L[\varphi, \mu]} e^{2\alpha_1 \varphi(0)} e^{2\alpha_2 \varphi(1)} e^{2\alpha_3 \varphi(\infty)}, \\ &= \mu^{s_\sigma} \Gamma(-s_\sigma) b^{-1} \int_{\varphi_{\mathbb{CP}^1}} \mathcal{D}\tilde{\varphi} e^{-S_L[\tilde{\varphi}, \mu=0]} e^{2\alpha_1 \tilde{\varphi}(0)} e^{2\alpha_2 \tilde{\varphi}(1)} e^{2\alpha_3 \tilde{\varphi}(\infty)} \prod_{r=1}^{s_\sigma} e^{2b\tilde{\varphi}(w_r)}, \end{aligned} \quad (6.44)$$

where $s_\sigma = b^{-1}(Q_\sigma - \alpha_1 - \alpha_2 - \alpha_3)$. We can calculate this integral using the Dotsenko and Fateev's paper [14], because we can obtain the 3-point function from the 4-point function.

$$C^{(\sigma)}(\alpha_1, \alpha_2, \alpha_3) = \mu^{s_\sigma} \Gamma(-s_\sigma) b^{-1} \prod_{l=1}^{s_\sigma} \int_{\mathbb{C}} d^2 w_l \prod_{r=1}^{s_\sigma} |w_r|^{-4\sigma b \alpha_1} |1 - w_r|^{-4\sigma b \alpha_2} \prod_{t' < t}^{s_\sigma} |w_{t'} - w_t|^{-4\sigma b^2}. \quad (6.45)$$

Using the Fateev-Dotsenko integral (B.9): $(\frac{1}{2}i dz d\bar{z} = d^2 z, \alpha = -2\sigma b \alpha_1, \beta = -2\sigma b \alpha_2, \rho = -\sigma b^2)$

$$\begin{aligned} &\int \prod_1^m d^2 z \prod_1^m |z_i|^{2\alpha} |1 - z_i|^{2\beta} \prod_{i < j}^m |z_i - z_j|^{4\rho} = m! \pi^m \left(\frac{\Gamma(1 - \rho)}{\Gamma(\rho)} \right)^m \prod_1^m \frac{\Gamma(i\rho)}{\Gamma(1 - i\rho)} \\ &\times \prod_0^{m-1} \frac{\Gamma(1 + \alpha + i\rho) \Gamma(1 + \beta + i\rho) \Gamma(-1 - \alpha - \beta - (m - 1 + i)\rho)}{\Gamma(-\alpha - i\rho) \Gamma(-\beta - i\rho) \Gamma(2 + \alpha + \beta + (m - 1 + i)\rho)}. \end{aligned} \quad (6.46)$$

Thus we can write (6.45) as

$$\begin{aligned} C^{(\sigma)}(\alpha_1, \alpha_2, \alpha_3) &= \mu^{s_\sigma} \Gamma(-s_\sigma) b^{-1} \left\{ (s_\sigma)! \pi^{s_\sigma} \left(\frac{\Gamma(1 + \sigma b^2)}{\Gamma(-\sigma b^2)} \right)^{s_\sigma} \prod_{t=1}^{s_\sigma} \frac{\Gamma(-t\sigma b^2)}{\Gamma(1 + t\sigma b^2)} \right. \\ &\quad \times \prod_{t=0}^{s_\sigma-1} \frac{\Gamma(1 - 2\sigma b \alpha_1 - t\sigma b^2) \Gamma(1 - 2\sigma b \alpha_2 - t\sigma b^2)}{\Gamma(2\sigma b \alpha_1 + t\sigma b^2) \Gamma(2\sigma b \alpha_2 + t\sigma b^2)} \\ &\quad \times \frac{\Gamma(-1 + 2\sigma b \alpha_1 + 2\sigma b \alpha_2 + (s_\sigma - 1 + t)\sigma b^2)}{\Gamma(1 + (1 - 2\sigma b \alpha_1 - 2\sigma b \alpha_2 - (s_\sigma - 1 + t)\sigma b^2))} \left. \right\}. \end{aligned} \quad (6.47)$$

Considering $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$, we have

$$C^{(\sigma)}(\alpha_1, \alpha_2, \alpha_3) = \mu^{s_\sigma} \pi^{s_\sigma} \Gamma(1 + s_\sigma) \Gamma(-s_\sigma) b^{-1} \left\{ (\gamma(1 + \sigma b^2))^{s_\sigma} \prod_{t=1}^{s_\sigma} \gamma(-t\sigma b^2) \right\}$$

$$\times \prod_{i=0}^{s_\sigma-1} \gamma(1 - 2\sigma b\alpha_1 - t\sigma b^2)\gamma(1 - 2\sigma b\alpha_2 - t\sigma b^2)\gamma(-1 + 2\sigma b\alpha_1 + 2\sigma b\alpha_2 + (s_\sigma - 1 + t)\sigma b^2) \}. \quad (6.48)$$

Using the property $\gamma(x)\gamma(-x) = -x^{-2}$, we find $\gamma(1 + \sigma b^2) = -b^4\gamma(\sigma b^2)$ and $\gamma(x) = \gamma^{-1}(1 - x)$. Hence

$$\begin{aligned} & C^{(\sigma)}(\alpha_1, \alpha_2, \alpha_3) \\ &= (-1)^{s_\sigma} \Gamma(1 + s_\sigma) \Gamma(-s_\sigma) b^{-1} (\pi \mu b^4 \gamma(\sigma b^2))^{s_\sigma} \frac{\prod_{t=1}^{s_\sigma} \gamma(-t\sigma b^2)}{\prod_{i=1}^3 \prod_{t=1}^{s_\sigma} \gamma(2\sigma b\alpha_i + \sigma(t-1)b^2)}, \\ &= (-1)^{s_\sigma} \Gamma(1 + s_\sigma) \Gamma(-s_\sigma) b^{-1} (\pi \mu b^4 \gamma(\sigma b^2))^{s_\sigma} \prod_{l=1}^{s_\sigma} \gamma(-\sigma l b^2) \prod_{i=1}^3 \left(\frac{\prod_{r=1}^{\beta_i^\sigma} \gamma(-r b^2)}{\prod_{t=1}^{\beta_i^\sigma + s_\sigma} \gamma(t b^2)} \right)^\sigma, \end{aligned}$$

with $\beta_i^\pm = 2\alpha_i b^{-1} - 1 + (1 \mp 1)b^{-2}/2$. Consider

$$\prod_{r=1}^n \gamma(r b^2) = \frac{\Upsilon_b(nb + b)}{\Upsilon_b(b)} b^{n(b^2(n+1)-1)}, \quad (6.49)$$

here $\Upsilon_b(x)$ is the known special function defined in the Appendix A, which has the following inversion relations:

$$\Upsilon_b(x) = \Upsilon_{b^{-1}}(x), \quad \Upsilon_b(x) = \Upsilon_b(b + b^{-1} - x). \quad (6.50)$$

Now we show explicitly the three-point functions of the spacelike and timelike Liouville theory.

6.2.2 The three-point function on the spacelike Liouville

For this case, we denote $\sigma = +1 = +$ and using the relation (6.49) and the properties in (6.50) and (A.3), we have

$$C_b^{(+)}(\alpha_1, \alpha_2, \alpha_3) = (\pi \mu \gamma(b^2) b^{2-2b^2})^{s_+} \frac{\Upsilon'_b(0)}{\Upsilon_b(\sum_{k=1}^3 \alpha_k - Q_+)} \prod_{i=1}^3 \frac{\Upsilon_b(2\alpha_i)}{\Upsilon_b(\sum_{j=1}^3 \alpha_j - 2\alpha_i)}, \quad (6.51)$$

where $\Upsilon'_b(x) = \frac{\partial}{\partial x} \Upsilon_b(x)$, with $Q_+ = b + b^{-1}$, $s_+ = 1 + b^{-2} - b^{-1} \sum_{i=1}^3 \alpha_i$. It is worth mentioning that the function behaves like $\Upsilon_b(Q_+) = \Upsilon_b(0) \sim \Upsilon'_b(0)/\Gamma(0)$. So this $\Gamma(0)$ cancels the one coming from $\Gamma(-s_+) \sim (-1)^{s_+} \Gamma(0)/\Gamma(1 + s_+)$.

6.2.3 The three-point function on the timelike Liouville

Now, we have $\sigma = -1 = -$ following the same procedure than for the spacelike case, we obtain

$$C_b^-(\alpha_1, \alpha_2, \alpha_3) = (-\pi\mu\gamma(-b^2)b^{2+2b^2})^{s_-} \frac{\Upsilon_b(-\sum_{k=1}^3 \alpha_k + Q_- + b)}{b\Upsilon_b(b)} \times \prod_{i=1}^3 \frac{\Upsilon_b(2\alpha_i - \sum_{j=1}^3 \alpha_j + b)}{\Upsilon_b(b - 2\alpha_i)}, \quad (6.52)$$

where $Q_- = b - b^{-1}$, $s_- = 1 - b^{-2} - b^{-1} \sum_{i=1}^3 \alpha_i$. In this case unlike the spacelike case, there is no $\Upsilon_b(0)$ factor, and the contribution of the $\Gamma(-s_-) \sum (-1)^{s_-} \Gamma(0)/\Gamma(1+s_-)$ factor do not cancel. This result is achieved by excluding the divergent overall factor $\Gamma(0)$.

Conclusions: Up to a phase $e^{-i\pi s_-}$, we can see that (6.52) reproduces exactly (6.16), which we obtained in the Section 6.1. In [19] we can see the explanation about this phase. The most interesting result here is that if we compare (6.51) and (6.52), the timelike structure constants, the inverse of spacelike structure constant, in the sense that the product of both timelike and spacelike quantities yields a simple factorized expression. So, this feature is nicely in the free-field calculation as it follows from the property $\gamma(x) = \gamma^{-1}(1-x)$ of the Γ functions.

Chapter 7

WZNW and the Liouville Theory

The relation between the Sine-Liouville model and the Wess-Zumino-Novikov-Witten (WZNW) is given by the Fateev-Zamolodchikov-Zamolodchikov (FZZ) duality, this conjecture was proven by Hikida and Schomerus by combining the well-known self-duality of Liouville theory with an intriguing correspondence between the cigar and Liouville field Theory. The prove consist in to show that correlations of all vertex operators agree for the cigar conformal field theory dual to the Sine-Liouville model. This prove is extended to the case of higher genus and also extended to the case with boundaries by Hikida. In [24], Hikida and Takayanagi investigated the $SL(2, \mathbb{R})/U(1)$ WZW model with level $0 < k < 2$ as a solvable time-dependent background in string theory. Unlike of Hikida and Takayanagi that used the conjecture of FZZ in order to implement the method of extrapolation of Schomerus to obtain the three-point function in the timelike Liouville. In this chapter we calculate the three-point function in the conventional level $k > 2$ and $0 < k < 2$ level for the WZNW model by using the free-field method [18] and following the idea of Giribet for calculating the three-point function in the Timelike Liouville [19].

7.1 The WZNW model and the primary operators

Bosonic string propagation in a AdS_3 background is described by a nonlinear sigma model equivalent to a WZNW model on $SL(2, \mathbb{R})$ (actually, its Euclidean version on

$SL(2, \mathbb{C})/SU(2)$) [47] [46] [45] [26] [25] [27]. The action of the theory is given by

$$S = \frac{k}{8\pi} \int d^2z (\partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma}), \quad (7.1)$$

where $\{\phi, \gamma, \bar{\gamma}\}$ are the Poincare coordinates of the (euclidean) AdS_3 spacetime, where ϕ is a real field and $\{\gamma, \bar{\gamma}\}$ are complex coordinates parametrizing the boundary of AdS_3 , which is located at $\phi \rightarrow \infty$. The constant $k = l^2/l_s^2$, where l being related to the scalar curvature of AdS_3 as $\mathcal{R} = -2/l^2$ and l_s being the fundamental string length. [26] [33] [32]. The equation (7.1) can be obtained by integrating out the one-form auxiliary fields β and $\bar{\beta}$ in the following action,

$$S = \int d^2z \left[\sigma \partial\phi\bar{\partial}\phi - \frac{2}{\alpha_+} \mathcal{R}^{(2)}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - M\beta\bar{\beta}e^{-(2/\alpha_+)\phi} \right], \quad (7.2)$$

. We write the action (7.2) for the WZNW model in terms of σ to differentiate the cases when we use the usual level for $k > 2$ ($\sigma = +1$) or the level when this is $0 < k < 2$ ($\sigma = -1$). Notice that the WZNW model $\sigma = -1$ can be obtained from the standard case $\sigma = +1$ by going to imaginary values of the parameter $\alpha_+ \rightarrow i\alpha_+$ and the field $\phi \rightarrow i\phi$. Here $\mathcal{R}^{(2)}$ is the scalar curvature of the world sheet. And the relation between α_+ and the Kac-Moody level is k

$$\alpha_+^2 = 2\sigma(k - 2), \quad (7.3)$$

The WZNW model is invariant under the action of two copies of the $SL(2, \mathbb{R})$ current algebra at level k [28]. We will denote the currents generating this algebra by $J^a(z)$ and $\bar{J}^a(\bar{z})$, where $a = 1, 2, 3$. Introducing the complex base $(J^\pm, J^3)$ given by

$$J^\pm = J^1 \pm iJ^2, \quad \text{and} \quad (J^\pm)^\dagger = J^\mp. \quad (7.4)$$

The Kac-Moody algebra take the following form

$$\begin{aligned} [J_m^3, J_n^3] &= -\frac{k}{2}m\delta_{m+n,0}, \\ [J_m^3, J_n^\pm] &= \pm J_{m+n}^\pm, \\ [J_m^\pm, J_n^\mp] &= -2J_{m+n}^3 + km\delta_{m+n,0}. \end{aligned} \quad (7.5)$$

The currents algebra $SL(2, \mathbb{R})$ can be described in terms of the free fields (ϕ, β, γ) by using the Wakimoto representation [7]

$$\begin{aligned} J^+(z) &= \beta(z), \\ J^3(z) &= -\beta(z)\gamma(z) - \sigma \frac{\alpha_+}{2} \partial\phi(z), \\ J^-(z) &= \beta(z)\gamma^2(z) + \sigma\alpha_+\gamma(z)\partial\phi(z) + k\partial\gamma(z), \end{aligned} \quad (7.6)$$

where the propagators are given by

$$\begin{aligned} \langle \phi(z)\phi(w) \rangle &= -\sigma \ln(z-w), \\ \langle \gamma(z)\beta(w) \rangle &= -\frac{1}{(z-w)}. \end{aligned} \quad (7.7)$$

The Sugawara construction gives rise to the following energy-tensor

$$T_{SL(2, \mathbb{R})} = -\frac{1}{2}\sigma\partial\phi\partial\phi - \frac{1}{\alpha_+}\partial^2\phi + \beta\partial\gamma, \quad (7.8)$$

The vertex operators are essential elements to calculate correlation functions that describing scattering amplitudes in string theory. These represent incoming and outgoing states of the strings interacting in a particular scattering process. A suitable representation of the vertex operator creating these states is

$$V_{j,m,\bar{m}} = \gamma(z)^{j-m} \bar{\gamma}(z)^{j-\bar{m}} e^{2j\phi(z)/\alpha_+}. \quad (7.9)$$

In this case, we consider the winding number $w = 0$. The conformal dimension of $V_{j,m,\bar{m}}$, is obtained from the OPE with the energy tensor (7.8).

In the Coulomb formalism it is mandatory to insert, in addition, some operators in order to screen the charges of these vertices and the background charge. The so-called screening operators in the $SL(2, \mathbb{R})$ WZNW model are the following,

$$\mathcal{S}_+ = \int d^2w \beta(w) \bar{\beta}(\bar{w}) e^{-\frac{2}{\alpha_+}\phi(w, \bar{w})}, \quad (7.10)$$

$$\mathcal{S}_- = \int d^2w (\beta(w) \bar{\beta}(\bar{w}))^{(k-2)} e^{-\sigma(\alpha_+ \phi(w, \bar{w}))}. \quad (7.11)$$

Because we are only working with operators of type $\mathcal{S}_-$, for theirs is the following relation between the number of screenings, s , with the moments j_i ,

$$s = \left(\sum_{i=1}^3 j_i + 1 \right) / (k-2), \quad (7.12)$$

for $\sigma = +1$,

$$s_{+1} = \frac{2}{\alpha_+^2} \left(\sum_{i=1}^3 j_i + 1 \right), \quad (7.13)$$

for $\sigma = -1$,

$$s_{-1} = -\frac{2}{\alpha_+^2} \left(\sum_i j_i + 1 \right). \quad (7.14)$$

7.2 Three-point correlation function in the free-field representation

In the context of the Coulomb gas formalism one has to compute expectation values of the form

$$\mathcal{A}_n = \langle V_{j_1, m_1, \bar{m}_1}(z_1, \bar{z}_1) \dots V_{j_n, m_n, \bar{m}_n}(z_n, \bar{z}_n) \mathcal{S}_- \rangle. \quad (7.15)$$

Using the vertex operator (7.9) and the screening $\mathcal{S}_-$ in (7.11), we obtain the Dotsenko-Fateev integral. Using Goulian-Li paper [20], we define $\tilde{\phi}(z) \equiv \phi(z) - \phi_0$ and integrate over the zero-mode. Then, fixing the insertions points of the three vertex operators $(z_1, z_2, z_3) = (0, 1, \infty)$, the equation (7.15) is

$$\begin{aligned} \mathcal{A}_{m_1, m_2, m_3}^{j_1, j_2, j_3} &= M^{s_\sigma} \Gamma(-s_\sigma) \langle V_{j_1, m_1, \bar{m}_1}(0) V_{j_2, m_2, \bar{m}_2}(1) V_{j_3, m_3, \bar{m}_3}(\infty) \\ &\times \int \prod_{t=1}^{s_\sigma} d^2 w_t (\beta(w_t) \bar{\beta}(\bar{w}_t))^{(k-2)} e^{-\sigma(\alpha_+ \tilde{\phi}(w_t, \bar{w}_t))} \rangle_{\mathbb{S}^2}, \end{aligned} \quad (7.16)$$

Consider that this correlation function contains a state of highest weight, $j_1 = m_1$. We can calculate the first term by performing all possible Wick contractions and using the free field propagator(7.7). Then, we have

$$\begin{aligned} \mathcal{A}_{j_1, m_2, m_3}^{j_1, j_2, j_3} &= M^{s_\sigma} \Gamma(-s_\sigma) (-1)^{\sigma \frac{\alpha_+^2}{2} s_\sigma} \Delta(j_2 - m_2 + 1) \Delta(j_3 - m_3 + 1) \\ &\times \int \prod_{t=1}^{s_\sigma} d^2 w_t |w_t|^{4j_1} |1 - w_t|^{4j_2 - \sigma \alpha_+^2} \prod_{t < j} |w_t - w_j|^{-2\sigma \alpha_+^2}. \end{aligned} \quad (7.17)$$

Using the Fateev-Dotsenko integral [14] and consider $\Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$,

$$\begin{aligned} \mathcal{A}_{j_1, m_2, m_3}^{j_1, j_2, j_3} &= M^{s_\sigma} \Gamma(-s_\sigma) (-1)^{\sigma \frac{\alpha_+^2}{2} s_\sigma} \Delta(j_2 - m_2 + 1) \Delta(j_3 - m_3 + 1) \{ \Gamma(1 + s_\sigma) \pi^{s_\sigma} \\ &\times \left(\Delta(1 + \sigma \frac{\alpha_+^2}{2}) \right)^{s_\sigma} \prod_{t=1}^{s_\sigma} \Delta(-t\sigma \frac{\alpha_+^2}{2}) \prod_{t=0}^{s_\sigma-1} \Delta(1 + 2j_1 - t\sigma \frac{\alpha_+^2}{2}) \Delta(1 + 2j_2 - \sigma \frac{\alpha_+^2}{2} - t\sigma \frac{\alpha_+^2}{2}) \\ &\times \Delta(-1 - 2j_1 - 2j_2 + (s_\sigma + t)\sigma \frac{\alpha_+^2}{2}) \}. \end{aligned} \quad (7.18)$$

7.3 The three-point function on the nivel $k > 2$

Defining the properties

$$\Delta(1+x) = \frac{G_k(x - (k-2))}{G_k(x)} (k-2)^{2x+1}, \quad (7.19)$$

$$G_k(x) = G_k(-1-x-(k-2)). \quad (7.20)$$

Now we calculate explicitly the amplitude for the case when $\sigma = +1$. Finally $\Gamma(-s_+) \sim \frac{(-1)^{s_+} \Gamma(0)}{\Gamma(1+s_+)}$,

$$\begin{aligned} \mathcal{A}_{j_1, m_2, m_3}^{j_1, j_2, j_3} &= M^{s_+} \pi^{s_+} (-1)^{\frac{\alpha_+^2}{2} s_+} \left(\frac{\alpha_+^2}{2} \right)^2 \left(\Delta\left(\frac{\alpha_+^2}{2}\right) \right)^{s_+} \\ &\times \Delta(j_2 - m_2 + 1) \Delta(j_3 - m_3 + 1) \Delta(-1 - \sum_i j_i) \Delta(1 + 2j_1) \Delta(-j_1 - j_2 + j_3) \Delta(j_2 - j_1 - j_3) \\ &\times \frac{G_k(-2 - \sum_i j_i)}{G_k(-1)} \frac{G_k(j_1 - j_2 - j_3 - 1) G_k(j_2 - j_1 - j_3 - 1) G_k(-j_1 - j_2 + j_3 - 1)}{G_k(-1 - 2j_1) G_k(-1 - 2j_2) G_k(-1 - 2j_3)}. \end{aligned} \quad (7.21)$$

Using the relation between G_k and Υ_b

$$G_k(x) = b^{-b^2 x^2 - (b^2+1)x} \Upsilon_b^{-1}(-bx), \quad (7.22)$$

with $b^{-2} = k - 2$, then in our last expression, we also consider $\alpha_+^2 = 2b^{-2}$. In a simplified notation we see that

$$\mathcal{A}_{j_1, m_2, m_3}^{j_1, j_2, j_3} = (-1)^{b^{-2} s_+} \Delta(1 + j_2 - m_2) \Delta(1 + j_3 - m_3) \tilde{\mathcal{I}}(j_1, j_2, j_3, b^{-2}). \quad (7.23)$$

$$\tilde{\mathcal{I}}(j_1, j_2, j_3, b^{-2}) = M^{s_+} \pi^{s_+} (b^{-2})^2 \left(\Delta(b^{-2}) \right)^{s_+} \mathcal{D}(j_1, j_2, j_3). \quad (7.24)$$

$$\mathcal{D}(j_1, j_2, j_3) = \Delta(-1 - \sum_i j_i) \Delta(1 + 2j_1) \Delta(j_2 - j_1 - j_3) \Delta(j_3 - j_1 - j_2) C(j_1, j_2, j_3). \quad (7.25)$$

$$C(j_1, j_2, j_3) = \frac{b^{-2b^2(\sum_i j_i + 1) + 1}}{\Upsilon_b(b(2 + \sum_i j_i))} \frac{\Upsilon_b(b) \Upsilon_b(b(2j_1 + 1)) \Upsilon_b(b(2j_2 + 1)) \Upsilon_b(b(2j_3 + 1))}{\Upsilon_b(b(j_1 + j_2 - j_3 + 1)) \Upsilon_b(b(j_3 + j_1 - j_2 + 1)) \Upsilon_b(b(j_2 + j_3 - j_1 + 1))}. \quad (7.26)$$

7.4 The three-point function on the level $0 < k < 2$

We find that the three-point function in (7.18) we need an integer, but negative, number of s_- screenings. Dotsenko [13] introduced a trick to give meaning to the product of a negative number of screening charges.

$$\prod_{i=1}^l f(i) = \prod_{i=0}^{|l|-1} \frac{1}{f(-i)}. \quad (7.27)$$

This very general trick can be applied succesfully to our case, and thus we can give meaning to any three-point correlation function.

Let us consider $\hat{b} = \frac{1}{\sqrt{\tilde{\alpha}}}$ where $\tilde{\alpha} = 2 - k > 0$, then $\hat{s}_- = -\hat{b}^2(\sum_i j_i + 1)$. Since $\Gamma(-\hat{s}_-) \sim \frac{(-1)^{\hat{s}_-} \Gamma(0)}{\Gamma(1 + \hat{s}_-)}$, the difference between the expression for levels $k > 2$ and $0 < k < 2$ is given by $s_+ = -\hat{s}_-$, $C(j_1, j_2, j_3)$ and $\hat{C}(j_1, j_2, j_3)$

$$\hat{\mathcal{A}}_{j_1, m_2, m_3}^{j_1, j_2, j_3} = (-1)^{(-\hat{b}^{-2})\hat{s}_-} \Delta(1 + j_2 - m_2) \Delta(1 + j_3 - m_3) \hat{\mathcal{I}}(j_1, j_2, j_3, -\hat{b}^{-2}). \quad (7.28)$$

$$\hat{\mathcal{I}}(j_1, j_2, j_3, -\hat{b}^{-2}) = M^{\hat{s}_-} \pi^{\hat{s}_-} (-\hat{b}^{-2})^2 \left(\Delta(-\hat{b}^{-2}) \right)^{\hat{s}_-} \hat{\mathcal{D}}(j_1, j_2, j_3). \quad (7.29)$$

$$\hat{\mathcal{D}}(j_1, j_2, j_3) = \Delta(-1 - \sum_i j_i) \Delta(1 + 2j_1) \Delta(j_2 - j_1 - j_3) \Delta(j_3 - j_1 - j_2) \hat{C}(j_1, j_2, j_3). \quad (7.30)$$

$$\begin{aligned} \hat{C}(j_1, j_2, j_3) &= \hat{b}^{2+2\hat{b}^2+2\hat{b}^2(\sum_i j_i)} \Gamma(0) \frac{\Upsilon_{\hat{b}}(-\hat{b}(\sum_i j_i + 1))}{\Upsilon_{\hat{b}}(\hat{b})} \\ &\times \frac{\Upsilon_{\hat{b}}(-\hat{b}(-j_1 + j_2 + j_3)) \Upsilon_{\hat{b}}(-\hat{b}(-j_2 + j_1 + j_3)) \Upsilon_{\hat{b}}(-\hat{b}(-j_3 + j_1 + j_2))}{\Upsilon_{\hat{b}}(-\hat{b}(2j_1)) \Upsilon_{\hat{b}}(-\hat{b}(2j_2)) \Upsilon_{\hat{b}}(-\hat{b}(2j_3))}. \end{aligned} \quad (7.31)$$

Here we obtain an interesting behavior if we compare our results for the WZNW model at the Kac-Moody level $k > 2$ and $0 < k < 2$ in a similar form to (6.39) for the Liouville Field Theory.

$$\begin{aligned} \frac{\hat{C}_{b=ib}(j_1, j_2, j_3)}{C_b(j_1, j_2, j_3)} &= e^{\frac{i\pi}{2}(2b^2(\sum_i j_i + 1) + 2)} \frac{\Theta_1(\sum_i j_i + 1, b^{-2})}{\Theta_1(-1, b^{-2})} \\ &\times \frac{\Theta_1(-j_1 + j_2 + j_3, b^{-2}) \Theta_1(-j_2 + j_1 + j_3, b^{-2}) \Theta_1(-j_3 + j_1 + j_2, b^{-2})}{\Theta_1(2j_1, b^{-2}) \Theta_1(2j_2, b^{-2}) \Theta_1(2j_3, b^{-2})}. \end{aligned} \quad (7.32)$$

Conclusions

After obtaining a complex solution of the Liouville equation on the Riemann sphere with two and three heavy operators, we see that the two-point function emerge naturally from the analytic continuation of the Liouville integral, which in turns improves the obtained results for fixed-area, in which we require that $\sum_i \eta_i < 1$.

However, for the three-point function, the analysis is more subtle since, if we want a total analytic continuation, it is necessary to include multivalued solutions (discontinuous). These singularities, on one side, suggest that there are singularities in the four-point function, but if we consider the degenerate four-point function, the Liouville path integral is still controlled by the "singular" solutions, proving then that (5.16) is still consistent.

On the other hand, a natural interpretation emerges in the Chern-Simons theory in the context of its analytic continuation [54], in which the normalization does not restrict k to be an integer. In this case, k will be related with the parameter b of the Liouville field theory. Here, a solution of the Liouville equation, either real or complex, gives a flat connection over Σ . Thus partition and correlation functions that can be constructed by means of Virasoro conformal blocks arise naturally in the Chern-Simons theory on $\Sigma \times I$ with boundary conditions just stated. Moreover, in this interpretation, insertions of the operators in the Liouville field theory will correspond to defects of the monodromy in Chern-Simons theory.

Finally, we reviewed proposals for obtaining the DOZZ formula in the timelike version of the Liouville field theory. One of them was proposed by Harlow, Maltz and Witten; they suggest an analytic continuation from the recurrence relation of Teschner and not directly from the DOZZ formula itself. After making a test for different cases, they conclude that the timelike DOZZ formula is given by an ordinary path integral evaluated in different

"integration cycles". This reminds the Witten procedure for the analytic continuation of the Chern-Simons theory in [54]. On the other hand, we also analyze the Giribet proposal after a simple redefinition and using the free-fields method to obtain the timelike version of the DOZZ, which is the "inverse" of the one obtained in the conventional Liouville theory: this behavior would be obtained also for the WZNW model, when we analyze the three-point function for $k > 2$ and $0 < k < 2$. For us, this feature consists in the well-known relation between the Liouville and the WZNW models.

After all this through analysis of the analytic continuation of the Liouville theory and its relation with other theories, a question arises, to what extent the peculiar case of the timelike Liouville Field Theory can actually be regarded as a CFT.

Appendix A

Properties of the Υ_b Function

The function Υ can be defined by

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[(Q/2 - x)^2 e^{-t} - \frac{\sinh^2 \left((Q/2 - x) \frac{t}{2} \right)}{\sinh \frac{tb}{2} \sinh \frac{t}{2b}} \right] \quad 0 < \operatorname{Re}(x) < \operatorname{Re}(Q) \quad (\text{A.1})$$

Here $Q = b + \frac{1}{b}$ and follow the definition $\Upsilon_b(Q - x) = \Upsilon_b(x)$. When $x = 0$ the second term in the integral diverges logarithmically at large t , and at small but finite x it behaves like $\log x$. Υ_b therefore has a simple zero at $x = 0$ as well as $x = Q$.

To extend the function over the whole x -plane, we can use the identity

$$\log \Gamma(x) = \int_0^\infty \frac{dt}{t} \left[(x - 1) e^{-t} - \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \right] \quad \operatorname{Re}(x) > 0 \quad (\text{A.2})$$

to show that in its range of definition Υ_b obeys

$$\begin{aligned} \Upsilon_b(x + b) &= \gamma(bx) b^{1-2bx} \Upsilon_b(x) \\ \Upsilon_b(x + 1/b) &= \gamma(x/b) b^{\frac{2x}{b} - 1} \Upsilon_b(x) \end{aligned} \quad (\text{A.3})$$

where:

$$\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$$

The recursion relations show that the simple zeros at $x = 0, Q$ induce more simple zeros at $x = -mb - n/b$ and $x = (m' + 1)b + (n' + 1)/b$, with m, m' and n, n' all non-negative

integers. Also we have the inverse recursion relations:

$$\begin{aligned}\Upsilon_b(x+b) &= \gamma(bx-b^2)^{-1}b^{2bx-1-2b^2}\Upsilon_b(x) \\ \Upsilon_b(x+1/b) &= \gamma(x/b-1/b^2)^{-1}b^{1+\frac{2}{b^2}-\frac{2x}{b}}\Upsilon_b(x)\end{aligned}\quad (\text{A.4})$$

We will also need various semiclassical limits of Υ_b . Rescaling t by b and using the identity

$$\log x = \int_0^\infty \frac{dt}{t} [e^{-t} - e^{-xt}] \quad \text{Re}(x) > 0,$$

we see that

$$\begin{aligned}b^2 \log \Upsilon_b(\eta/b + b/2) &= -\left(\eta - \frac{1}{2}\right)^2 \log b \\ &+ \int_0^\infty \frac{dt}{t} \left[\left(\eta - \frac{1}{2}\right)^2 e^{-t} - \frac{2}{t} \left(1 - \frac{t^2 b^4}{24} + \dots\right) \frac{\sinh^2 \left[(\eta - \frac{1}{2}) t/2\right]}{\sinh \left(\frac{t}{2}\right)} \right]\end{aligned}\quad (\text{A.5})$$

When $0 < \text{Re}(\eta) < 1$, the subleading terms in the series $1 + t^2 b^4 + \dots$ can be integrated term by term, with only the 1 contributing to nonvanishing order in b . From the identity A.2, we can find

$$F(\eta) \equiv \int_{1/2}^\eta \log \gamma(x) dx = \int_0^\infty \frac{dt}{t} \left[\left(\eta - \frac{1}{2}\right)^2 e^{-t} - \frac{2}{t} \frac{\sinh^2 \left[(\eta - \frac{1}{2}) t/2\right]}{\sinh \left(\frac{t}{2}\right)} \right] \quad 0 < \text{Re}(\eta) < 1,$$

so using this we find the asymptotic formula:

$$\Upsilon_b(\eta/b + b/2) = e^{\frac{1}{b^2} [-(\eta-1/2)^2 \log b + F(\eta) + \mathcal{O}(b^4)]} \quad 0 < \text{Re}(\eta) < 1 \quad (\text{A.6})$$

For the case of the heavy operator when $b \rightarrow 0$, we have

$$\Upsilon_b(\eta/b) = e^{\frac{1}{b^2} [F(\eta) - (\eta-1/2)^2 \log b + \mathcal{O}(b \log b)]} \quad 0 < \text{Re}(\eta) < 1 \quad (\text{A.7})$$

For light operator in the $b \rightarrow 0$ limit of the first recursion relation A.3 we find

$$\Upsilon_b((\sigma+1)b) \approx \frac{1}{\sigma b} \Upsilon_b(\sigma b) \quad (\text{A.8})$$

One solution to this relation is

$$\Upsilon_b(\sigma b) \approx \frac{b^{-\sigma}}{\Gamma(\sigma)} h(b) \quad (\text{A.9})$$

where $h(b)$ is independent of σ . This solution is not unique since we can multiply it by any periodic function of σ with period one and still obey the recursion relation. We see

however that it already has all of the correct zeros at $\sigma = 0, -1, -2, \dots$ to match the Υ_b function, so we might expect that this periodic function is a constant.

This periodic function in any case is nonvanishing and has no poles, so it must be the exponential of an entire function. If the entire function is nonconstant then it must grow as $\sigma \rightarrow \infty$, which seems to be inconsistent with the nice analytic properties of Υ_b .

In particular (A.6) shows no sign of such singularities in η as $\eta \rightarrow 0$. We can derive $h(b)$ analytically, up to a b -independent constant which we determine numerically. The manipulations are sketched momentarily in a footnote, the result is

$$\Upsilon_b(\sigma b) = \frac{C b^{1/2-\sigma}}{\Gamma(\sigma)} \exp \left[-\frac{1}{4b^2} \log b + F(0)/b^2 + \mathcal{O}(b^2 \log b) \right] \quad (\text{A.10})$$

The numerical agreement of this formula with the asymptotics of the integral (A.1) is excellent; in particular we find $C = 2.50663$. The constant C will cancel out of all of our computations since we are always computing ratios of equal numbers of Υ_b 's.

This precise numerical agreement also confirms our somewhat vague argument for the absence of an additional periodic function in σ . As an application of this formula we can find the asymptotics of Υ_0 from the DOZZ formula:

$$\Upsilon_0 = \frac{C}{b^{1/2}} \exp \left[-\frac{1}{4b^2} \log b + F(0)/b^2 + \mathcal{O}(b^2 \log b) \right] \quad (\text{A.11})$$

Appendix B

Theory of Hypergeometric Functions

B.1 Hypergeometric Series

We begin by studying the series

$$F(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \quad (\text{B.1})$$

Here $(x)_n \equiv x(x+1)\dots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}$. It is easy to see using the ratio test that if c is not a negative integer, then for any complex a and b the series converges absolutely for $|z| < 1$, diverges for $|z| > 1$, and is conditional for $|z| = 1$. It is also symmetric in a and b , and we can observe that if either a or b is a nonpositive integer then the series terminates at some finite n . One special case which is easy to evaluate is

$$F(1, 1, 2, z) = -\frac{\log(1-z)}{z}$$

which shows that the analytic continuation outside of the unit disk is not necessarily singlevalued.

We will also need the value of the series at $z = 1$, see section (B.5) in [22]

$$F(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{Re}(c-a-b) > 0 \quad (\text{B.2})$$

B.2 Hypergeometric Differential Equation

By direct substitution one can check that the function $F(a, b, c, z)$ obeys the following differential equation:

$$z(1-z)f'' + (c - (a+b+1)z)f' - abf = 0 \quad (\text{B.3})$$

This second-order equation has three regular points, at 0, 1, and ∞ . Since $F(a, b, c, z)$ is manifestly nonsingular at $z = 0$, its analytic continuation has potential singularities only at 1 and ∞ . There is the possibility of a branch cut running between 1 and ∞ . We saw this in the special case we evaluated above, and it is standard to choose this branch cut to lie on the real axis. We can determine the monodromy structure of a general solution of this equation by studying its asymptotic behavior in the vicinity of the singular points. By using a power-law ansatz it is easy to see that any solution generically takes the form

$$f(z) \sim \begin{cases} A_0(z) + z^{1-c}B_0(z) & \text{as } z \rightarrow 0 \\ z^{-a}A_\infty(1/z) + z^{-b}B_\infty(1/z) & \text{as } z \rightarrow \infty \\ A_1(1-z) + (1-z)^{c-a-b}B_1(1-z) & \text{as } z \rightarrow 1 \end{cases} \quad (\text{B.4})$$

with $A_i(\cdot), B_i(\cdot)$ being holomorphic functions in a neighborhood of their argument being zero. The solution of (B.3) defined by the series (B.1) is a case of (B.4), with $A_0(z) = F(a, b, c, z)$ and $B_0 = 0$. We will determine the A_i and B_i at the order two singular points later. These expressions confirm that a solution of (B.3) will generically have branch points at 0, 1 and ∞ .

B.3 Riemann's Differential Equation

It will be very convenient for our work on Liouville to make use of Riemann's hypergeometric equation, of which (B.3) is a special case. This more general differential equation is:

$$f'' + \left\{ \frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3} \right\} f' + \left\{ \frac{\alpha\alpha'z_{12}z_{13}}{z-z_1} + \frac{\beta\beta'z_{21}z_{23}}{z-z_2} + \frac{\gamma\gamma'z_{31}z_{32}}{z-z_3} \right\} \frac{f}{(z-z_1)(z-z_2)(z-z_3)} = 0 \quad (\text{B.5})$$

along with a constraint:

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1 \quad (\text{B.6})$$

Here $z_{ij} \equiv z_i - z_j$, the parameters $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are complex numbers, and the constraint is imposed to make the equation nonsingular at infinity. The points z_i are regular singular points. This is in fact the most general second-order linear differential equation with three regular singular points and no singularity at infinity. To see that this reduces to the hypergeometric equation (B.3), one can set $z_1 = 0, z_2 = \infty, z_3 = 1, \alpha = \gamma = 0, \beta = a, \beta' = b$, and $\alpha' = 1 - c$.

Solutions to Riemann's equation can always be written in terms of solutions of the hypergeometric equation; this is accomplished by first doing an $SL(2, \mathbb{C})$ transformation to send the three singular points to 0, 1 and ∞ , followed by a nontrivial rescaling. To see this explicitly, say that $g(a, b, c, z)$ is a solution of the differential equation (B.3), not necessarily the solution given by (B.1). Then a somewhat tedious calculation shows that

$$f = \left(\frac{z - z_1}{z - z_2} \right)^\alpha \left(\frac{z - z_3}{z - z_2} \right)^\gamma g \left(\alpha + \beta + \gamma, \alpha + \beta' + \gamma, 1 + \alpha - \alpha', \frac{z_{23}(z - z_1)}{z_{13}(z - z_2)} \right) \quad (\text{B.7})$$

is a solution of the differential equation (B.5).

Near the singular points any solution behaves as

$$f(z) \sim \begin{cases} A_1(z - z_1)^\alpha + B_1(z - z_1)^{\alpha'} & \text{as } z \rightarrow z_1 \\ A_2(z - z_2)^\beta + B_2(z - z_2)^{\beta'} & \text{as } z \rightarrow z_2 \\ A_3(z - z_3)^\gamma + B_3(z - z_3)^{\gamma'} & \text{as } z \rightarrow z_3 \end{cases} \quad (\text{B.8})$$

so the monodromies are simply expressed in terms of $\alpha, \alpha', \beta, \dots$

B.4 Particular Solutions of Riemann's Equation

We now construct explicit solutions of Riemann's equation that have simple monodromy at the three singular points in terms of the hypergeometric function (B.1).

Given equation (B.7), the most obvious solution we can write down is

$$f^{(\alpha)}(z) \equiv \left(\frac{z - z_1}{z - z_2} \right)^\alpha \left(\frac{z - z_3}{z - z_2} \right)^\gamma F \left(\alpha + \beta + \gamma, \alpha + \beta' + \gamma, 1 + \alpha - \alpha', \frac{z_{23}(z - z_1)}{z_{13}(z - z_2)} \right)$$

We denote it $f^{(\alpha)}$ because the holomorphy of the series (B.1) at 0 ensures that any non-trivial monodromy near z_1 comes only from the explicitly factor $(z - z_1)^\alpha$. The differential equation is invariant under interchanging $\alpha \leftrightarrow \alpha'$, so we can easily write down another solution that is linearly independent with the first (assuming that $\alpha \neq \alpha'$):

$$f^{(\alpha')}(z) \equiv \left(\frac{z - z_1}{z - z_2} \right)^{\alpha'} \left(\frac{z - z_3}{z - z_2} \right)^\gamma F \left(\alpha' + \beta + \gamma, \alpha' + \beta' + \gamma, 1 + \alpha' - \alpha, \frac{z_{23}(z - z_1)}{z_{13}(z - z_2)} \right)$$

This solution has the alternate monodromy around $z = 0$.

The differential equation is also invariant under $\beta \leftrightarrow \beta'$ and $\gamma \leftrightarrow \gamma'$: the former leaves the solutions $f^{(\alpha)}, f^{(\alpha')}$ invariant and can be ignored but the latter apparently generates two additional solutions. We can find even more solutions by simultaneously permuting $\{z_1, \alpha, \alpha'\} \leftrightarrow \{z_2, \beta, \beta'\} \leftrightarrow \{z_3, \gamma, \gamma'\}$, so combining these permutations we find a total of $4 \times 6 = 24$ solutions, known as "Kummer's Solutions". Since these are all solutions of the same 2nd-order linear differential equation, any three of them must be linearly dependent. To pin down this redundancy, it is convenient to define a particular set of six solutions, each of which has simple monodromy about one of the singular points. The definition is somewhat arbitrary as one can change the normalization at will as well as move around the various branch cuts. We will choose expressions that are simple when all z -dependence is folded into the harmonic ratio

$$x \equiv \frac{z_{23}(z - z_1)}{z_{13}(z - z_2)} \tag{B.9}$$

Our explicit definitions are the following:

$$\begin{aligned}
P^\alpha(x) &= x^\alpha(1-x)^\gamma F(\alpha + \beta + \gamma, \alpha + \beta' + \gamma, 1 + \alpha - \alpha', x) \\
&= x^\alpha(1-x)^{-\alpha-\beta} F(\alpha + \beta + \gamma, \alpha + \beta + \gamma', 1 + \alpha - \alpha', \frac{x}{x-1}) \\
P^{\alpha'}(x) &= x^{\alpha'}(1-x)^{\gamma'} F(\alpha' + \beta + \gamma', \alpha' + \beta' + \gamma', 1 + \alpha' - \alpha, x) \\
&= x^{\alpha'}(1-x)^{-\alpha'-\beta} F(\alpha' + \beta + \gamma, \alpha' + \beta + \gamma', 1 + \alpha' - \alpha, \frac{x}{x-1}) \\
P^\gamma(x) &= x^\alpha(1-x)^\gamma F(\alpha + \beta + \gamma, \alpha + \beta' + \gamma, 1 + \gamma - \gamma', 1-x) \\
&= x^{\alpha'}(1-x)^\gamma F(\alpha' + \beta + \gamma, \alpha' + \beta' + \gamma, 1 + \gamma - \gamma', 1-x) \\
P^{\gamma'}(x) &= x^\alpha(1-x)^{\gamma'} F(\alpha + \beta + \gamma', \alpha + \beta' + \gamma', 1 + \gamma' - \gamma, 1-x) \\
&= x^{\alpha'}(1-x)^{\gamma'} F(\alpha' + \beta + \gamma', \alpha' + \beta' + \gamma', 1 + \gamma' - \gamma, 1-x) \\
P^\beta(x) &= x^\alpha(1-x)^{-\alpha-\beta} F(\alpha + \beta + \gamma, \alpha + \beta + \gamma', 1 + \beta - \beta', \frac{1}{x-1}) \\
&= x^{\alpha'}(1-x)^{-\alpha'-\beta} F(\alpha' + \beta + \gamma, \alpha' + \beta + \gamma', 1 + \beta - \beta', \frac{1}{x-1}) \\
P^{\beta'}(x) &= x^\alpha(1-x)^{-\alpha-\beta'} F(\alpha + \beta' + \gamma, \alpha + \beta' + \gamma', 1 + \beta' - \beta, \frac{1}{x-1}) \\
&= x^{\alpha'}(1-x)^{-\alpha'-\beta'} F(\alpha' + \beta' + \gamma, \alpha' + \beta' + \gamma', 1 + \beta' - \beta, \frac{1}{x-1})
\end{aligned} \tag{B.10}$$

These formulas are somewhat intimidating, but they follow from the simple permutations just described. For convenience in the following derivation we give two equivalent forms of each. More symmetric integral expressions for them will be described in section (B.5) in [22].

Since only two of these can be linearly independent, there must exist coefficients a_{ij} such that

$$\begin{aligned}
P^\alpha(x) &= a_{\alpha\gamma} P^\gamma(x) + a_{\alpha\gamma'} P^{\gamma'}(x) \\
P^{\alpha'}(x) &= a_{\alpha'\gamma} P^\gamma(x) + a_{\alpha'\gamma'} P^{\gamma'}(x)
\end{aligned} \tag{B.11}$$

These coefficients are called connection coefficients. To determine them we can evaluate these two equations at $x = 0$ and $x = 1$, which gives

$$\begin{aligned}
a_{\alpha\gamma} &= \frac{\Gamma(1 + \alpha - \alpha') \Gamma(\gamma' - \gamma)}{\Gamma(\alpha + \beta + \gamma') \Gamma(\alpha + \beta' + \gamma')} \\
a_{\alpha\gamma'} &= \frac{\Gamma(1 + \alpha - \alpha') \Gamma(\gamma - \gamma')}{\Gamma(\alpha + \beta + \gamma) \Gamma(\alpha + \beta' + \gamma)} \\
a_{\alpha'\gamma} &= \frac{\Gamma(1 + \alpha' - \alpha) \Gamma(\gamma' - \gamma)}{\Gamma(\alpha' + \beta + \gamma') \Gamma(\alpha' + \beta' + \gamma')} \\
a_{\alpha'\gamma'} &= \frac{\Gamma(1 + \alpha' - \alpha) \Gamma(\gamma - \gamma')}{\Gamma(\alpha' + \beta + \gamma) \Gamma(\alpha' + \beta' + \gamma)}
\end{aligned} \tag{B.12}$$

In solving these equations one uses (B.2). Similarly one can find:

$$\begin{aligned}
a_{\alpha\beta} &= \frac{\Gamma(1 + \alpha - \alpha') \Gamma(\beta' - \beta)}{\Gamma(\alpha + \beta' + \gamma) \Gamma(\alpha + \beta' + \gamma')} \\
a_{\alpha\beta'} &= \frac{\Gamma(1 + \alpha - \alpha') \Gamma(\beta - \beta')}{\Gamma(\alpha + \beta + \gamma) \Gamma(\alpha + \beta + \gamma')} \\
a_{\alpha'\beta} &= \frac{\Gamma(1 + \alpha' - \alpha) \Gamma(\beta' - \beta)}{\Gamma(\alpha' + \beta' + \gamma) \Gamma(\alpha' + \beta' + \gamma')} \\
a_{\alpha'\beta'} &= \frac{\Gamma(1 + \alpha' - \alpha) \Gamma(\beta - \beta')}{\Gamma(\alpha' + \beta + \gamma) \Gamma(\alpha' + \beta + \gamma')}
\end{aligned} \tag{B.13}$$

Finally we note that our expressions for the connection coefficients allow us to derive some beautiful facts about the original hypergeometric function $F(a, b, c, z)$.

First making the replacements mentioned below (B.5), we see that (B.11) gives:

$$\begin{aligned}
F(a, b, c, z) &= \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} F(a, b, 1 + a + b - c, 1 - z) \\
&\quad + \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} (1 - z)^{c - a - b} F(c - a, c - b, 1 + c - a - b, 1 - z)
\end{aligned} \tag{B.14}$$

This gives explicit expressions for $A_1(1 - z)$ and $B_1(1 - z)$ for $F(a, b, c, z)$, as promised above. We can also set $\alpha = 0, \alpha' = 1 - c, \beta = 0, \beta' = c - a - b, \gamma = a, \gamma' = b, z_1 = 0, z_2 = 1$, and $z_3 = \infty$, in which cases (B.11) gives:

$$\begin{aligned}
F(a, b, c, z) &= \frac{\Gamma(c) \Gamma(b - a)}{\Gamma(c - a) \Gamma(b)} (-z)^{-a} F(a, 1 - c + a, 1 - b + a, z^{-1}) \\
&\quad + \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(c - b) \Gamma(a)} (-z)^{-b} F(b, 1 - c + b, 1 + b - a, z^{-1})
\end{aligned} \tag{B.15}$$

This expressions gives $A_\infty(1/z)$ and $B_\infty(1/z)$ for $F(a, b, c, z)$, and in fact it gives the full analytic continuation of the series (B.1) in the region $|z| > 1$, since the hypergeometric series on the right hand side converge in this region. We can thus observe that indeed the only singular behavior of the function $F(a, b, c, z)$ is a branch cut running from one to infinity.

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