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Dinâmica quase-linear e não-linear de automodos de Alfvén excitados por íons energéticos

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Quasilinear and nonlinear dynamics of energetic-ion-driven Alfvénic eigenmodes

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Abstract

The destabilization of plasma waves upon their interaction with fast ions is studied using a kinetic framework. The work consists of two parts: (I) a study of the applicability of quasilinear theory using a perturbative, early-time nonlinear evolution of a mode and its prediction with respect to chirping oscillations, and (II) the resonance-broadened quasilinear formulation of the evolution of unstable modes.

In part I, we have developed predictive capabilities for the type of fast-ion-induced transport by means of a criterion for the likelihood of a mode to oscillate at a constant frequency or to evolve to a bifurcation consisting of nonlinear chirping oscillations. The proposed criterion is derived and evaluated using the linear codes NOVA and NOVA-K. The criterion was shown to be in agreement with experimentally observed modes in the tokamaks DIII-D and NSTX. The analysis reveals that micro-turbulence is a key mediator for suppressing chirping and therefore allowing quasilinear theory to be applicable.

In part II, a system of resonance-broadened quasilinear equations (RBQ) was derived using action and angle variables, which takes advantage of system symmetries by using the invariants of the unperturbed motion as variables when accounting for the effects of perturbations due to modes. The equations capture information on mode structures and on resonances that are spread over phase space. We then expressed them in terms of NOVA code notation. The RBQ model is presented, along with the finite-difference scheme used for numerical integration. Numerical results and future developments are also described.

Resumo

A desestabilização de ondas em plasmas quando de sua interação com íons rápidos é estudada utilizando a abordagem cinética. Este trabalho consiste em duas partes: (I) um estudo sobre a aplicabilidade da teoria quase-linear, utilizando a evolução não linear perturbada em seu estágio inicial de um modo e sua previsão com respeito às oscilações do tipo gorjeio (*chirping*), e (II) a formulação quase-linear de ressonâncias alargadas da evolução de modos instáveis.

Na parte I, desenvolvemos capacidades preditivas em relação ao tipo de transporte induzido por íons rápidos por meio de um critério a respeito da probabilidade de um modo oscilar com uma frequência constante ou evoluir para uma bifurcação que consista de oscilações de gorjeio não lineares. O critério proposto é derivado e calculado utilizando os códigos lineares NOVA e NOVA-K. Mostramos que o critério obtido concorda com modos observados experimentalmente nos tokamaks DIII-D e NSTX. A análise revela que a microturbulência é um mediador-chave na supressão dos gorjeios e que, portanto, permite que a teoria quase-linear seja aplicável.

Na parte II, um sistema de equações quase-lineares com ressonâncias alargadas (RBQ) foi derivado utilizando variáveis de ângulo e ação, tirando-se proveito das simetrias do sistema ao se tomar como variáveis os invariantes do movimento não perturbado quando consideramos os efeitos das perturbações aos modos. As equações capturam as informações sobre a estrutura dos modos e sobre as ressonâncias que se espalham sobre o espaço de fase. Expressamo-las, então, em termos da notação do código NOVA. O modelo RBQ é apresentado, juntamente com um esquema de diferenças finitas, utilizado para a integração numérica. Resultados numéricos e desenvolvimentos futuros também são descritos.

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Chapter 1

Introduction

1.1 The research on controlled thermonuclear fusion

Since the early 1950s, research on thermonuclear controlled fusion energy for peaceful purposes is being actively pursued [1]. In spite of the overwhelming scientific and technical challenges imposed by Nature to reproduce the power source of the stars in laboratories, significant progress has been achieved in the two main schemes employed in this area, magnetically confinement fusion and inertial confinement fusion. Indeed in the 1990s, experiments in the devices Joint European Torus (JET, at Culham Laboratory, UK) and Tokamak Fusion Test Reactor (TFTR, at Princeton Plasma Physics Laboratory, USA) clearly demonstrated controlled production of fusion energy in high temperature tokamak plasmas of deuterium and tritium [2, 3, 4]. More recently, experiments carried out in the National Ignition Facility (NIF), at the Lawrence Livermore National Laboratory, have demonstrated net energy gain in inertial confinement fusion, in a scheme where 192 power-

ful laser beams are focused on a centimeter-scale capsule filled with deuterium and tritium. The capsules are compressed to extremely high pressures, overcoming the Coulomb repulsion between deuterons and tritons, and triggering fusion reactions [5]. None of these schemes, however, has yet achieved the so-called ignition condition. In this regime, the high energy alpha particles (helium nuclei) produced in the fusion reactions deliver most of their energy to the electrons through Coulomb collisions, heating the plasma and balancing the ubiquitous energy losses due to turbulent transport and radiation. Under this condition, the fusion processes can be maintained without external power input to the reacting medium. This regime is called the breakeven.

As an international effort to pursue fusion power, the European Union, India, Japan, China, Russia, South Korea and the United States are funding the construction of a multi-billion enterprise in southern France, the International Thermonuclear Experimental Reactor (ITER), whose artistic view is shown in figure 1.1. It is designed to be a next-generation burning plasma experiment. ITER is expected to operate in a self-sustained way, with the alpha particles fusion products being the leading source of heating. ITER is planned to have a fusion energy gain factor (Q-value) of 10, i.e., deliver ten times more than the power input of 50 MW for periods up to 500 seconds and eventually work in steady state. ITER's main goal is to demonstrate the feasibility of fusion power and, if successful, may be followed by DEMO, designed to be the first commercial demonstration fusion power plant.

Figure 1.1: Artistic view of ITER tokamak, and the comparison with the size of a person on the bottom left part [6].

1.2 Basic physics of tokamak confinement

Tokamaks are plasma confinement devices in which the plasma acts like the secondary of a transformer, as shown in figure 1.2. Consequently, as a result of a driven electric field, an axial (toroidal) current (usually referred to simply as plasma current) is generated inductively, which in turn gives rise to an azimuthal (poloidal) field. Several coils displaced along the toroidal direction (appearing in green in figure 1.2) produce a strong toroidal magnetic field. The resulting magnetic field is therefore helical. Tokamaks were originally conceived during the early 1950s by the Soviet physicists Igor Tamm and Andrei Sakharov.

Figure 1.2: Tokamak diagrammatic view [7].

Plasma species, which mainly follow magnetic field lines, are also allowed to move in directions perpendicular to the field by means of a number of orbit drifts, in addition to the fast cyclotron motion. As depicted in figure 1.3, there is a gradient of the toroidal field intensity increasing in the direction of the axis of symmetry, which, in combination with the toroidal field itself, generates a drift in the vertical direction, the so-called ∇B drift. Whether it is up or down depends on the particle charge. This drives charge separation and, consequently, an electric field. Tokamak are designed so that the resulting twisted, helical field lines (the combination of the toroidal field due to induction and the poloidal field component generated by the toroidal current) cancel those vertical drifts, as will be explained shortly.

As long as helical field lines are present, it is possible to think of a plasma cross section in an axisymmetric tokamak as a set of nested surfaces on which the magnetic vector field lies, but not necessarily with a constant field intensity on them. Those surfaces are called

flux surfaces since they indicate the amount of field flux enclosed by a given surface. Some of the plasma in a flux surface will be on the outside (or low-field side) of the torus and will drift to other flux surfaces farther from the magnetic axis of the torus. Other portions of the plasma in the flux surface will be on the inside region (or high-field side). Then, the outward drift is compensated by an inward drift on the same flux surface. In other words, the electric field between the top and the bottom of the torus, which tends to cause the outward drift, is shorted out because there are magnetic field lines connecting the top to the bottom, through which the plasma particles are allowed to move. Spherical tokamaks (STs) may not need such an intense toroidal current drive as conventional tokamaks need. This is because in STs have a larger fraction of bootstrap current and the neutral beam injection (NBI) current drive could be sufficient due to large plasma beta.

For most MHD applications, tokamak equilibrium plasmas are nearly axisymmetric (i.e., symmetry with respect to the toroidal angle). This occurs when the effect of the discreteness of the toroidal field coils (see figure 1.2) is not important. This assumption will be used in the remainder of this work. In case the coils discreteness has an appreciable effect, the resulting bumpy magnetic field can lead to ripple diffusion [8, 9].

Throughout this work, Gaussian units will be used.

Figure 1.3: Key particle orbit drifts in a tokamak, as a result of field inhomogeneities [10].

Figure 1.4: Particle orbits in a tokamak [11].

1.3 Energetic particles and their interaction with low-frequency Alfvénic eigenmodes

In the fusion context, energetic ions, in this thesis referred simply as energetic particles (EPs), are ionic plasma components with suprathermal energies (ranging from tens of keV to MeV), i.e., greater than the thermal energy of the bulk ions that will eventually undergo fusion thermonuclear reactions. Sources of EPs are essentially auxiliary heating mechanisms, such as neutral beam injection (NBI) and ion cyclotron resonant heating (ICRH), and fusion-born ions (alpha particles) [8, 12, 13, 14, 15]. In ITER, the alpha particle pressure is expected to significantly contribute to the total plasma pressure. The free energy associated with their pressure gradient can destabilize MHD modes, increasing the transport and jeopardizing the confinement. ITER is expected to operate in a self-sustained mode, generating a 500MW of power from a 50MW input, where the alphas are the main source of heating. For this regime to be achieved, it is essential to avoid EP losses and to control their resonant interaction, which occurs if the EP thermal velocity is comparable with the phase velocity of the wave, with modes propagating in the plasma. In this work, we investigate the interaction of EPs with Alfvénic waves, which are known to lead to substantial energy and particle losses.

One of the fundamental types of hydrodynamic modes in magnetized plasmas is the shear Alfvén wave, which is a result of the coupling between electrodynamic and fluid equations. This wave is sustained by a self-consistent mechanism, in which fluid perturbed motions give rise to an electromotive force that drives a current density, which in turn acts on the fluid through the Lorentz force [16]. In its simplest version for homogeneous

plasmas, the dispersion relation of shear Alfvén waves is given by $\omega_A = k_{\parallel} v_A$, where $k_{\parallel}$ is the component of the wave number vector parallel to the magnetic field $\mathbf{B}$ and $v_A \equiv B/\sqrt{\mu_0 \rho}$ the Alfvén speed. ρ is the fluid mass density and μ_0 is the magnetic permeability in a vacuum. In addition to the shear branch, there are also fundamental compressional branches, such as the slow and fast magnetosonic modes [17]. As previously mentioned, due to equilibrium and stability requirements, the confining magnetic field in magnetically confined plasmas has a helical structure. This means that in a cylindrical plasma column, for example, the magnetic field has a longitudinal component B_z and an azimuthal (poloidal) component B_{θ} . The pitch of the field lines varies with the radial coordinate r and is characterized by the safety factor, defined by $q(r) = 2\pi r B_z(r)/L B_{\theta}(r)$, where L is the (periodic) length of the plasma column. Taking this into account, the dispersion relation for shear Alfvén waves in a cylindrical plasma column becomes $\omega_A^2 = [2\pi v_A/L(n - m/q)]^2$, where m and n are the azimuthal and longitudinal mode numbers of the electromagnetic wave, respectively. The dispersion relation is written for ω^2 to allow for waves propagating in the positive and negative longitudinal directions.

Since in a confined plasma the mass density ρ normally varies from a minimum at the edge to a maximum at the center, the Alfvén speed v_A correspondingly varies from a maximum at the edge to a minimum at the center, defining the so-called Alfvén continuum for ω_A . Inside this continuum, the solutions of the eigenmode equation for the Alfvén waves are singular, so that discrete global Alfvén eigenmodes cannot be excited; these exist only for frequencies somewhat below the minimum of the continuum [18]. In tokamaks, however, the plasma column is bent in a torus, to avoid particle losses along the magnetic field lines. Although the simplified description of Alfvén waves in cylindrical plasmas can,

in principle, be simply extended to toroidal geometry by considering the periodic length given by $L = 2\pi R$, where R is the major radius of the torus, the toroidal curvature of the configuration introduces a new dependence on the equilibrium quantities. In fact, it can be shown that both its poloidal, B_θ , and toroidal, B_z , components must have a periodic dependence on the poloidal angle θ [19]. This means that this coordinate is not anymore ignorable, as in the simple cylindrical model of magnetic confinement, and the poloidal mode number m becomes not anymore a “good quantum number” to decompose the perturbations of the physical quantities.

Therefore, modes with different values of the mode number m in cylindrical geometry get coupled in toroidal geometry, giving rise to different mode structures. In particular, in a process similar to the band structure that appears in the solution of the Schrödinger equation in periodic lattices, the mode coupling gives rise to gaps in the Alfvén continuum of cylindrical geometry [20, 21]. Beta-induced (BAEs), toroidal (or toroidicity-induced) and ellipticity-induced (EAEs) Alfvén Eigenmodes (TAEs) are examples of discrete eigenmodes that can be excited in these gaps with a global radial structure [22]. In its simplest version, valid for plasma columns with circular cross section, the TAE gaps are mainly due to the coupling between the m and $m + 1$ cylindrical harmonics and occur at the radial positions r_0 where $q(r_0) = (m + 1/2)/n$. Its angular frequency is given by $\omega_{TAE} = v_A/2Rq(r_0)$.

It turns out that the phase velocity of the Alfvénic modes can be of the same order of the velocities of EPs in the plasma, either externally injected for heating purposes or fusion-born alpha particles. In this case, these modes can be excited through the mechanism of resonant wave-particle interaction, growing at the expense of the free energy available in the tail of the velocity distribution of the energetic particles [23, 24]. This process

is worrisome for fusion plasmas not only because it can take away energy of the EPs, which would otherwise be available for plasma heating, but also because it may lead to substantial losses of EPs through nonlinear processes [25]. Indeed, in experiments carried out in the TFTR and DIII-D tokamaks with EP injection, at low values of the magnetic field (to decrease somewhat the value of the Alfvén speed), large amplitude Alfvénic modes were excited, triggering substantial fast ion losses [26]. Therefore, the investigation of EP excitation and interaction with Alfvénic waves is a highly relevant subject for present-day tokamaks and also for ITER. In ITER, it is expected that fusion alphas and beam ions will be superalfvénic, therefore being capable of strongly exciting TAEs and RSAEs, the reversed shear Alfvén eigenmodes [25].

1.4 Particle invariants for an axisymmetric tokamak equilibrium

Noether’s theorem states that if the Hamiltonian is invariant with respect to translations of an angle (being then an ignorable generalized coordinate), its conjugate action is conserved. Consequently, there is a conservation law associated to each degree of symmetry of the system. For the case of an axisymmetric plasma equilibrium, the Hamiltonian is independent of the toroidal angle φ , which implies that the toroidal canonical momentum P_φ is conserved. The guiding center approximation is justified whenever the cyclotron timescale is much smaller than other characteristic times of the system and spatial inhomogeneities are negligible. When the equilibrium fields change slowly (both in time and in space) as compared to the gyromotion, the Hamiltonian is independent of the gyroangle θ_B and the

magnetic moment μ is conserved. Periodicity also leads to a poloidal adiabatic invariance. Since the orbit of a particle is closed in the (P_θ, θ) plane, the bounce motion closed integral $J_p = \oint P_\theta d\theta$ is a constant of motion [9]. In addition to the three conserved actions, for the case of a stationary equilibrium, the Hamiltonian does not depend explicitly on time (time symmetry), which implies that the energy $\mathcal{E}$ of each individual particle is conserved.

Note that, under the axisymmetric, guiding-center approximation, P_φ and μ are local invariants, having the same value at every single point of a particle trajectory. However J_p is only a global adiabatic invariant that requires integration over the entire poloidal orbit. It is more convenient to make the transformation $(P_\varphi, J_p, \mu) \rightarrow (P_\varphi, \mathcal{E}, \mu)$ since $\mathcal{E}$ is a particle property regardless of its position in phase space. Each point in $(P_\varphi, \mathcal{E}, \mu)$ space defines a distinct particle orbit. Co-passing and counter-passing particle contributions should then be summed up in order to account for the two possible trajectories. We use the constants of the unperturbed motion $P_\varphi, \mathcal{E}, \mu$ because of the simplicity in treating various particle orbits (banana, passing, stagnant,...) in a unified way. They are defined here the same way as in NOVA:[27]

$$\mathcal{E} = \frac{v^2}{2},$$

$$\mu = \frac{v_\perp^2}{2B},$$

$$P_\varphi = \frac{z\psi(R, Z)}{Mc} - \underbrace{\sigma_\parallel \sqrt{2\mathcal{E}}}_{=v_\parallel/v} \sqrt{1 - \frac{\mu B}{\mathcal{E}}} \frac{RB_\varphi}{B},$$

where z is the EP charge, M is the EP mass, v is the EP speed, $\psi(R, Z) = -RA_\varphi$, and $\sigma_\parallel$ equals $+1$ for co-passing and -1 for counter-passing particles, respectively, with respect to the magnetic field orientation. The relative strength of the two terms of P_φ can be estimated as

$$\frac{Rv_\varphi}{z\psi/Mc} \sim \frac{Mc v_\varphi}{zB_\theta r} \sim \frac{\rho_\theta}{r},$$

where $\rho_\theta = Mv_\varphi/(zB_\theta c)$ is the poloidal gyroradius. Typically this ratio is small unless the particle speed is too large or if it is close to the magnetic axis, where $B_\theta = 0$. Thus, P_φ is a good indication of the particle radial location.

1.5 Particle invariants in the presence of a perturbation

The generalized momenta are defined as $P_i = \frac{\partial L}{\partial \dot{q}_i}$. Therefore, the Euler-Lagrange equations $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$ imply that $\frac{dP_i}{dt} = \frac{\partial L}{\partial q_i}$. After a perturbation is imposed to the plasma, $\mathcal{E}$, P_φ and μ are not necessarily particle invariants anymore. In NOVA code, the Lagrangian is assumed to depend on time, toroidal angle, and gyroangle in the form $e^{i(-n\varphi + l\theta_B - \omega t)}$. Therefore, it follows that

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \Rightarrow \frac{d\mathcal{E}}{dt} = i\omega L,$$

$$\frac{dP_\varphi}{dt} = \frac{\partial L}{\partial \varphi} = -inL,$$

$$\frac{d\mu'}{dt} = \frac{m}{z} \frac{d\mu}{dt} = \frac{\partial L}{\partial \theta_B} = ilL,$$

where μ' has units of action (defined per unit mass). Consequently

$$\frac{d\mathcal{E}}{dt} = \frac{\omega}{n} \frac{dP_\varphi}{dt} = \frac{\omega B}{l\omega_c} \frac{d\mu}{dt}.$$

For low-frequency waves, i.e., $\omega/\omega_c \ll 1$, the modes are not able to have cyclotron resonances, which implies $l = 0$. In this case, $\frac{d\mu}{dt}$ is identically zero:

$$\boxed{\mu = \text{const}},$$

and a new constraint for the perturbed single particle dynamics arises:

$$\boxed{\mathcal{E}' = \mathcal{E} + \omega P_\varphi/n}. \quad (1.1)$$

Resonant particle motion occurs along paths of constant value of $\mathcal{E}'$ as long as the ansatz $e^{i(-n\varphi - \omega t)}$ holds.

1.6 Operator $\partial/\partial I$ in $(\mathcal{E}, P_\varphi, \mu)$ space

In this section the independent variables are changed from (J_p, P_φ, μ) to $(\mathcal{E}, P_\varphi, \mu)$. The former is the set of actions associated to the guiding center Langrangian while the latter is the usual set of invariants used to determine univocally particle orbits. The action of the poloidal motion J_p is an adiabatic invariant for slow evolving systems (with characteristic time scale much greater than the inverse poloidal bounce - or transit - frequency) in which particles perform several poloidal transits before the system evolves appreciably. This is because J_p is a Poincaré invariant, i.e., an integral invariant over the entire particle orbit.

We are interested in working with variables that are invariant in every point of particle trajectory, which motivates the variable change $J_p \rightarrow \mathcal{E}(J_p, P_\varphi, \mu) = \mathcal{E}(J_1, J_2, J_3)$. The following operator is intended to act on a function $f = f(J_p, P_\varphi, \mu)$:

$$\frac{\partial}{\partial I} = l_1 \frac{\partial}{\partial J_1} + l_2 \frac{\partial}{\partial J_2} + l_3 \frac{\partial}{\partial J_3},$$

where I can be chosen to be

$$I = -P_\varphi/n$$

for constant $\mathcal{E}' = \mathcal{E} + \omega P_\varphi/n$. Now the above operator is rewritten to act on a function $g = g(\mathcal{E}, P_\varphi, \mu)$ ¹:

$$\frac{\partial}{\partial I} = \frac{\partial \mathcal{E}}{\partial I} \frac{\partial}{\partial \mathcal{E}} + l_2 \frac{\partial}{\partial J_2} + l_3 \frac{\partial}{\partial J_3} = (l_1 \omega_1 + l_2 \omega_2 + l_3 \omega_3) \frac{\partial}{\partial \mathcal{E}} + l_2 \frac{\partial}{\partial J_2} + l_3 \frac{\partial}{\partial J_3}.$$

$\mathcal{E}$ is the Hamiltonian of the unperturbed motion and satisfies $\omega_1 = \frac{\partial \mathcal{E}}{\partial J_1}$, $\omega_2 = \frac{\partial \mathcal{E}}{\partial J_2}$ and $\omega_3 = \frac{\partial \mathcal{E}}{\partial J_3}$, which are the poloidal, toroidal and gyro frequencies, respectively. For resonant particles, $l_1 \omega_1 + l_2 \omega_2 + l_3 \omega_3$ is equal to the mode frequency ω . l_2 is minus the toroidal mode number and l_3 needs to be taken as zero for low-frequency modes, as compared to the cyclotron resonance. Consequently,

$$\boxed{\frac{\partial}{\partial I} = \omega \frac{\partial}{\partial \mathcal{E}} - n \frac{\partial}{\partial P_\varphi} = \omega \frac{\partial}{\partial \mathcal{E}} \Big|_{P'_\varphi} = -n \frac{\partial}{\partial P_\varphi} \Big|_{\mathcal{E}'},} \quad (1.2)$$

where $P'_\varphi = P_\varphi + n\mathcal{E}/\omega$ and $\mathcal{E}' = \mathcal{E} + \omega P_\varphi/n$. Another way of deriving the above operator

¹Note that this transformation involves the use of a new basis which is not orthogonal, although the variables are linearly independent. After the transformation is made, $\mathcal{E}$ is to be treated as an independent variable not related to P_φ and μ . The transformation can be formally understood as if J_p were the new Hamiltonian.

is to project the gradient $\left(\frac{\partial}{\partial \mathcal{E}}, \frac{\partial}{\partial P_\varphi}, \frac{\partial}{\partial \mu}\right)$ onto the path that preserves condition (1.1), which is given by $(\omega, -n, 0)$. Consequently, $\frac{\partial}{\partial I} = \omega \frac{\partial}{\partial \mathcal{E}} - n \frac{\partial}{\partial P_\varphi}$. Therefore, it may be useful to make a transformation $(P_\varphi, \mathcal{E}) \rightarrow (I(P_\varphi, \mathcal{E}), \mathcal{E}'(P_\varphi, \mathcal{E}))$, where $I = -P_\varphi/n$ and $\mathcal{E}' = \mathcal{E} + \omega P_\varphi/n$.

1.7 Jacobian of the transformation $(J_p, P_\varphi, \mu) \rightarrow (\mathcal{E}, P_\varphi, \mu)$

In modeling fast ion interaction with eigenmodes, it is often necessary to express a phase-space volume element in terms of integrations over the variables $(\mathcal{E}, P_\varphi, \mu)$. This involves a Jacobian, which is calculated in this section. Let us start with

$$\int d^3 J = \int dJ_p \int dP_\varphi \int d\mu = \int d\mathcal{E} \int dP_\varphi \int d\mu (\det \underline{\underline{J}})^{-1},$$

where $\det \underline{\underline{J}}$ is the determinant of the Jacobian matrix

$$\det \begin{bmatrix} \frac{\partial \mu}{\partial \mu} & \frac{\partial \mu}{\partial P_\varphi} & \frac{\partial \mu}{\partial J_p} \\ \frac{\partial P_\varphi}{\partial \mu} & \frac{\partial P_\varphi}{\partial P_\varphi} & \frac{\partial P_\varphi}{\partial J_p} \\ \frac{\partial \mathcal{E}}{\partial \mu} & \frac{\partial \mathcal{E}}{\partial P_\varphi} & \frac{\partial \mathcal{E}}{\partial J_p} \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial \mathcal{E}}{\partial \mu} & \frac{\partial \mathcal{E}}{\partial P_\varphi} & \frac{\partial \mathcal{E}}{\partial J_p} \end{bmatrix} = \frac{\partial \mathcal{E}}{\partial J_p} = \bar{\omega}_\theta,$$

where the last equality relation follows from Hamilton's equations. There $\bar{\omega}_\theta$ is the mean poloidal bounce frequency. If the integrand is cyclic²,

$$\int d^3 J \int d^3 \Theta \dots = \int d^3 r \int d^3 v \dots = (2\pi)^2 \sum_{\sigma_\parallel} \frac{B}{\omega_c} \int d\mathcal{E} \int dP_\varphi \int \frac{d\mu}{\bar{\omega}_\theta} \dots,$$

with dt being associated to the fast particle poloidal drift motion.

²Note that the actions are defined without a mass.

1.8 Bounce frequency and nonlinear Landau damping

Let us consider the Hamiltonian of a particle interacting with a mode (with toroidal number n and frequency ω_0) being expressed as a combination of an unperturbed Hamiltonian H_0 that determines unperturbed orbits and a perturbation component H_1 due to the mode. If H_0 is integrable, it can be expressed only in terms of the system actions, and H reads

$$H = H_0(J_1, J_2, J_3) + H_1(J_1, J_2, J_3, \xi_1, \xi_2, \xi_3, t); \quad H_1 \ll H_0$$

with

$$H_1(J_1, J_2, J_3, \xi_1, \xi_2, \xi_3, t) = 2C_n(t) \sum_{l_1, l_2, l_3} V_{l_1, l_2, l_3}(J_1, J_2, J_3) \cos(l_1 \xi_1 + l_2 \xi_2 + l_3 \xi_3 - \omega_0 t),$$

where J_i are the actions, ξ_i are the angles, C_n stands for the mode amplitude, l_i are integers, and V_{l_1, l_2, l_3} represents matrix elements that provide mode structure information through the projection of the particle velocity onto the wave electric field. The coefficients V_{l_1, l_2, l_3} can be found by inverting the previous equation. The particle Hamiltonian for nearly resonant particles is

$$H(\mathbf{p}, \mathbf{r}; t) = \frac{1}{2M} \left[\mathbf{p} - \frac{z}{c} \mathbf{A}(\mathbf{r}, t) \right]^2 + z\phi(\mathbf{r}, t).$$

$\mathbf{A}$ is the magnetic vector potential and ϕ is the scalar electrostatic potential. The perturbed Hamiltonian, for $\mathbf{p} = M\mathbf{v}$, then reads

$$\delta H(\mathbf{p}, \mathbf{r}; t) = -\frac{z}{c} \mathbf{v} \cdot \delta \mathbf{A}(\mathbf{r}, t) + z\delta\phi(\mathbf{r}, t),$$

which, in the gauge in which the electrostatic potential vanishes, can be written simple as $\delta H(\mathbf{p}, \mathbf{r}; t) = -2\frac{z}{c}Re[\mathbf{v} \cdot \delta \mathbf{A}(\mathbf{r}, t)] = 2zRe[C_n(t)\frac{\mathbf{e} \cdot \mathbf{v}}{\omega_0}e^{-i\omega_0 t}]$. The δ symbol represents the perturbed quantities. Therefore, the Fourier coefficients can be calculated³:

$$V_{l_1, l_2, l_3} = \frac{iz}{\omega} \int \frac{d^3\xi}{(2\pi)^3} \mathbf{e} \cdot \mathbf{v} e^{-i(l_1\xi_1 + l_2\xi_2 + l_3\xi_3)},$$

where $\mathbf{e}$ is the eigenmode structure, which only depends on position. Let us now consider the contribution of a single, non-overlapped resonance. This is equivalent to treating separately a given set of integers $\mathbf{l} = (l_1, l_2, l_3)$ (with a single resonance stripped from the total Hamiltonian). For this case we can make a canonical transformation in order to express a new angle $\zeta \equiv l_1\xi_1 + l_2\xi_2 + l_3\xi_3$ with its corresponding action being I . Therefore, the effect of a single isolated resonance can be described as

$$H(I, \zeta, t) = H_0(I) + H_1(I, \zeta, t); \quad H_1(I, \zeta, t) = 2C_n(t)V_{n,l}(I)\cos(\zeta - \omega_0 t).$$

Note that the other two actions are omitted in the above expression since we are only interested in the phase space dynamics in (ζ, I) space. Using the operator $\partial/\partial I \equiv \mathbf{l} \cdot \partial/\partial \mathbf{J}$, Hamilton's equations are

$$\Omega = \dot{\zeta} = \frac{\partial H_0}{\partial I} + \frac{\partial H_1}{\partial I}$$

and

$$\dot{I} = -\frac{\partial H}{\partial \zeta} = -\frac{\partial H_1}{\partial \zeta} = 2C_n(t)V_{n,l}(I)\sin(\zeta - \omega_0 t).$$

³Note that $\mathbf{e}$ has dimensions of electric field, C is dimensionless and V has dimensions of energy.

The equation of oscillatory motion around a stable resonant point can be obtained by computing $\ddot{\zeta}$ via the chain rule:

$$\ddot{\zeta} = \frac{\partial \Omega}{\partial I} \dot{I} + \frac{\partial \Omega}{\partial \zeta} \dot{\zeta} + \frac{\partial \Omega}{\partial t} = 2C_n(t)V_{n,l}(I)\sin(\zeta - \omega_0 t) \frac{\partial \Omega}{\partial I} + \underbrace{\frac{\partial^2 H_1}{\partial \zeta \partial I} \Omega + \frac{\partial^2 H_1}{\partial t \partial I}}_{-(\Omega - \omega_0) \frac{\partial}{\partial I} [2C_n(t)V_{n,l}(I)\sin(\zeta - \omega_0 t)]}.$$

For resonant particles, $\Omega(I_r) = \omega_0$ and consequently, up to a phase, a **pendulum equation** is obtained:

$$\ddot{\zeta} + \omega_{b,l}^2 \sin(\zeta - \omega_0 t + \zeta_0) = 0,$$

where

$$\omega_{b,(n,l)} = \left| 2C_n(t)V_{n,l}(I_r) \frac{\partial \Omega_{n,l}}{\partial I} \right|_{I=I_r}^{1/2}$$

is the **nonlinear bounce frequency of the most deeply trapped particles**.⁴ The bounce frequency is a characteristic of a given point at each resonance, and not of the mode n as a whole. The fact that individual particle motion satisfies a pendulum equation (at different frequencies for different particles) is a sufficient condition for particles to undergo phase mixing and exchange energy with the wave. Whenever one has nonlinear oscillator equations for particles, particles with different initial energies bounce at different frequencies. This means that, in phase space, they rotate around an equilibrium point at different speeds. This produces phase mixing, which is reflected in a change of the phase

⁴In the literature, the term “bounce frequency” can be related to two distinct quantities: (i) the frequency of the particle poloidal motion in a toroidal cross section and (ii) the frequency of the pendular motion of the particles that are trapped by a wave. In this thesis, we follow the definition (ii).

space distribution, δf . Then one could remap this change into the original variables $(\mathbf{r}, \mathbf{v})$ and calculate the change of the kinetic energy, $\int (\delta f)(mv^2/2)d\mathbf{r}d\mathbf{v}$. The only source where this energy can come from is the wave. Hence, phase mixing implies energy exchange with the wave.

For a purely electrostatic case, it is clear which force is enforcing particles to satisfy the pendulum equation $(-e\nabla\phi)$ and it is also clear that particle movement is restricted by the potential walls of the wave. For the general case, the physical picture can be somewhat blurred, even though the forces on particles can be calculated using Hamilton's equations. Therefore, particle dynamics does not always have a transparent explanation in terms of variables $(\mathbf{r}, \mathbf{v})$.

Landau damping needs rearrangement of particles giving rise to a modification (amplification or damping) of the field. In electrostatic waves, the field part of the energy is stored in structure of the charge density (equivalently the electrostatic field), but in electromagnetic (EM) waves the field part of the energy can just as well be stored in currents (i.e., the perturbed magnetic field) rather than charges [28]. There are also analogs of Landau damping that do not involve EM phenomena whatsoever, e.g., see Ref. [29]. In fact, any collection of oscillators with dense enough spectrum would work for that purpose [28].

1.9 Goals and structure of this thesis

For present-day and also for future-generation tokamaks, it is instructive to be able to anticipate the type of nonlinear evolution that a mode will follow. This is because the nonlinear properties of the mode have a direct influence on the nature of the transport it may

induce. In experiments, an Alfvénic mode typically saturates at a nearly constant frequency or evolves to a harder nonlinear phase in which the frequency rapidly changes in repetitive cycles and the mode amplitude bursts.⁵ The ability to predict the nonlinear features of a mode allows a more effective approach when modeling and interpreting experiments. For example, in case an Alfvénic mode does not chirp, it is likely that reduced modeling, such as a quasilinear modeling, should be enough to account for the relevant properties of fast ion transport. However, if a mode is predicted to hit a virulent, hard nonlinear level, fully nonlinear modeling should need to be employed, which is much more computationally expensive than the quasilinear approach. In Chapter 2, a criterion for the applicability of quasilinear theory is derived, evaluated and shown to reproduce experimental scenarios. The validation exercise provides fair degree of confidence to employ resonance-broadened quasilinear (RBQ)⁶ in regions of parameters where it is indeed expected to be valid. In Chapter 2, it is shown that fast ion micro-turbulence can be a crucial element in determining the likely nonlinear evolution of a mode [40, 41].

In case modes are to be modeled by reduced models, it is important to predict whether a mode will saturate alone or overlap with other modes, which may generate large-scale diffusion. When modes grow, neighboring resonances can overlap (see Fig. (1.5)). In that case, particles that are trapped in one resonance can be expelled to another one, therefore implying long-range particle excursions. In case particles are lost beyond the last closed surface, an initial distribution (called classical in Fig. (1.5)), will be lowered below a marginal level where the instability will be benign. Eventually, collisions may restore the

⁵Other types of evolution are also observed, although less frequently, e.g., nonlinear frequency splitting [30] and avalanches [31, 32, 33, 34, 35, 36].

⁶In previous publications, such as Refs. [37, 38, 39], the term line-broadened was used in lieu of resonance-broadened.

Figure 1.5: Scheme showing intermittent relaxation of a bump-on-tail distribution function that relaxes due to the effect of several resonances. The classical distribution refers to the situation without the modes while the metastable case is the threshold for island overlap. Fast ion losses can lower the classical distribution below a marginal level (at the point when the mode damping balances linear mode drive) while collisions can act to restore the initial distribution function. Even below the marginal distribution for linear instabilities, nonlinear excitation may be triggered sub-critically. Figure taken from [42].

distribution function, by replenishing the resonance region with more particles, and a new cycle of mode growth can start. The marginal condition, which is a threshold for the sudden increase of transport, is observed in experiments. The stiffness in fast ion transport has been modeled using critical gradient models [43, 44, 45] that we generalize in this thesis.

While the conventional quasilinear (QL) theory only applies to a situation with multiple modes overlapping, the RBQ model is designed to address the particle interaction with both isolated and overlapping modes. This is done by using the same structure of the QL equations for the fast ions distribution function, but with the diffusive delta function broadened in both P_φ and $\mathcal{E}$ directions. It is a key element of the proposed diffusion model that the parametric dependencies of the broadened window are to reproduce the expected effects of the isolated modes. Such dependencies of the broadened width are adjusted for better prediction of the expected saturation levels.

We propose to revisit the RBQ model for practical cases of Alfvénic instabilities excited by energetic ions. In ITER it is expected that fusion alphas and beam ions will excite toroidicity-induced Alfvén eigenmodes (TAEs) and reversed shear Alfvén Eigenmodes (RSAEs). The RBQ model was proposed by Berk *et al* [37, 46] and was implemented for the case of the bump-on-tail instability by Fitzpatrick in his thesis [47] and Ghantous [39, 38] benchmarked the RBQ model with the Vlasov code BOT. The RBQ model can be a valuable predictive tool needed at the design stages of burning plasma experiments. It gives physical insight into one of the most critical mechanisms that need to be understood for building a viable fusion power plant. Intrinsic to RBQ is that mode structure and resonances are unchanged with time insofar as they are treated linearly. For this reason, it does not capture chirping, which are anharmonic, fully nonlinear oscillations. In Chapter 3,

the conventional quasilinear diffusion theory is presented in action and angle variables. In Chapter 4, essential information on the NOVA code is presented and it is shown how it is possible to use it for the purpose of providing the necessary input for the quasilinear system of equations. In Chapter 5, the RBQ model is presented, along with the numerical scheme used to integrate the equations and numerical results. Discussions and future perspectives are given in Chapter 6.

Chapter 2

Nonlinear chirping and the quasilinear applicability

2.1 Introduction

The presence of variable oscillation frequencies is inherent to nonlinear dynamical systems. In tokamak plasmas, the nonlinear dynamics of phase space structures and their associated frequency chirping are topics of major interest in connection with mechanisms for fast ion losses. Supra-thermal fast ions exist in fusion-grade tokamaks as a result of neutral beam injection (NBI), resonant heating and fusion reactions. This population of energetic particles (EPs) can strongly resonate with Alfvénic modes and excite instabilities that can seriously damage the confinement [14, 8, 12, 13]. The control of this interaction is necessary for the achievement of burning plasmas scenarios, in which the fast ions need to have sufficient time to transfer their energy to the background - mostly from drag (slowing

down) on electrons - in order to ensure high temperature for the continuation of fusion reactions. This energy transfer mechanism is considered essential for the good performance of ITER.

Theoretically, the time evolution of the amplitude of a mode interacting with EPs can exhibit a variety of patterns as the mode departs from an initial linear phase towards its nonlinear response. During this evolution, several bifurcations take place, with typical phases being steady, regular, chaotic and chirping oscillations [48]. Upon the kinetic interaction of particles with an eigenmode, nonlinear phase-space structures may spontaneously emerge in the resonance regions of the particle distribution, depending on the competition between drive, damping and collisionality [49]. These disturbances can self-consistently support anharmonic oscillations, in a generalization of BGK modes [50] which, in the presence of wave damping, are pushed towards lower energy states. These self-trapped structures consist of accumulation and depletion of particles in phase-space and are commonly referred to as clumps and holes, respectively.

Frequency chirping can emerge just above the threshold for marginal stability, where the energy extracted from resonant EPs slightly exceeds the power being absorbed by the background dissipation, as discussed in Ref. [49]. The initial nonlinear response is to relax the EP distribution in its resonance region, which would reduce the drive which can then damp the mode. However, the plasma-EP system can also find an alternate option, of slightly shifting its frequency, thereby still tapping the free energy of previously untapped neighboring non-resonant particles, that then become resonant due to the changed frequency. In the fully developed chirping state, the resonant region of an enhanced number of particles (clump) or of a deficient number of particles (holes) are trapped by the finite amplitude

wave, and these regions of phase space shift the resonant distribution to lower energy regions of phase space, with the released energy being absorbed by the background dissipation mechanisms that are present while keeping the nonlinear amplitude hardly changed. Therefore the frequency variation itself allows for the moving structures to access phase space regions with distribution function gradients otherwise inaccessible which leads to *convective* losses over an extended region.

For the sake of clarity, we distinguish between the terminologies frequency *chirping* and frequency *sweeping*, that often appear in the literature. The distinction often implicitly assumed in the literature is as follows [8]:

a) **frequency sweeping**: slow evolution ($\sim 100ms$) of Alfvénic modes, in the presence of a non-monotonic q profile, that relies on the time variation of q_{min} . Those events, shown in Fig. 2.1, are associated with a modification of the plasma equilibrium (and consequently, of the dispersion relation) and are often referred to as Alfvén Cascades (AC) or Reversed Shear Alfvén Eigenmodes (RSAE). It is an important tool in order to control the q profile and is often used to study internal transport barriers.

Figure 2.1: Sweeping observation of RSAEs on JET [51].

b) **frequency chirping**: rapid events ($\sim ms$), as shown in Fig. 2.2, in which several cycles of frequency evolution are observed, all starting at roughly the same mode frequency. This indicates that the original mode properties are preserved and the equilibrium remains virtually unaffected. Those events are associated with the appearance of BGK-like modes, i.e., modes whose very existence is linked to an accumulation or depletion of particles (holes and clumps). Holes and clumps appear near the original resonance and tap the free energy of the distribution in order to balance background dissipation. Chirping is faster than sweeping and harder to suppress using external control.

Figure 2.2: Chirping observation of an $n = 1$ TAE on MAST [52].

Chirping modes can have frequency shifts greater than the linear growth rate γ_L and are observed to be precursors of even worst scenarios, known as avalanches. A spectrogram showing repetitive chirping cycles followed by avalanches for toroidal Alfvén eigenmodes (TAEs) in NSTX, for several toroidal mode numbers, is presented in Fig. 2.3(a). The inset shows four of the chirping events and indicates that it consists mostly of a down-chirping. The system preference for a direction (up or down in frequency) has been theoretically linked with the competition between different collisional processes [53]. Fig. 2.3(b) shows very significant neutron rate drop correlating with the avalanches.

The long-range chirping evolution was described by the Berk-Breizman prediction for the frequency variation $\delta\omega$ scaling with the bounce frequency ω_b to the power of $3/2$ [49]. It has been successfully used for applications that include the inference of mode amplitude on MAST [52] and the estimation of kinetic parameters such as drive and damping in JT-60U [54] and in NSTX [36].

The present work however focuses on establishing the conditions for chirping onset rather than modeling their long-term evolution in order to predict the likely character of

EP transport. The EP losses are typically diffusive (e.g., due to mode overlap, turbulence, radiofrequency fields, and collisional scattering) or convective (e.g., as a result of chirping oscillations, collisional drag, and the redistribution induced by sawteeth and fishbone oscillations due to convective cells). We describe the methodology for the generalization of previous works and we show how it is possible to include micro-turbulent stochasticity, which is shown to compete, and even greatly exceed, collisional scattering in many tokamak experiments and therefore needs to be added to the stochasticity introduced by pitch-angle scattering.

Chirping and quasilinear (QL) regimes correspond to two opposite limits of kinetic theory. Since they may be competing mechanisms in the modification of the distribution of fast ions in tokamaks (and their consequent transport), their parameter-space regions of applicability need to be carefully addressed. The derivation of the QL diffusion equations [55, 56] relies on averages, over a statistical ensemble, that smooth out the distribution function. In order to justify the resulting smooth, coarse-grained distribution, stochastic processes (which can be intrinsic due to mode overlap or extrinsic due to collisions) need to be invoked. The fast-varying response associated to the ballistic term is disregarded, which implies that entropy is no longer conserved. Consequently, QL theory kills phase correlations and cannot capture chirping events, since chirping needs time coherence from one bounce to the next in order to move nonlinear structures altogether over phase space. QL diffusion theory needs phase decorrelation, i.e. particles need to be expelled from a phase-space resonant island at a time less than the nonlinear bounce time. This means that there are no particles effectively trapped. Due to the reduced dimensionality of phase space, the QL description is less computationally demanding than the full nonlinear description

needed to capture chirping. It also is much less computationally expensive than particle codes. A criterion for chirping likelihood is an important element for identification of parameter space for QL theory applicability for practical cases and consequent validation of reduced models. An example of such models is the Resonance-Broadened Quasilinear (RBQ) code [57]. It uses the usual structure of the QL system written in action-angle variables [58] with a broadened resonance width that scales with bounce frequency, growth rate and collisional frequency [37, 39].

In this work, we build predictive capabilities regarding the likelihood of the nonlinear regime, which can be useful for burning plasma scenarios [40, 41, 59, 60]. If further validated and verified, the developed methodologies could be of practical importance for predictive tools of EP distribution relaxations in the presence of Alfvénic instabilities. This is especially important for the development of the reduced models since their methodologies critically depend on that.

2.2 The early nonlinear evolution of a mode amplitude

The onset of a mode amplitude evolution can be studied using perturbation theory within the kinetic framework. Starting with the kinetic equation,

$$\frac{df}{dt} = C[f],$$

with $C[f]$ is given as shown in Appendix A, and an energy balance equation (i.e., the rate of mode energy change equals the power nonlinearly extrated from the particles to the mode minus the damping rate γ_d times the mode energy), Refs. [48, 61, 62, 63] showed that,

Figure 2.3: (a) Spectrogram showing chirping associated with toroidal Alfvén eigenmodes (TAEs) for several toroidal mode numbers and (b) neutron loss rates in NSTX shot 141711 correlating with the TAE avalanches. The small inset shows a zoomed region with mostly down-chirping [40].

for $|\gamma_L - \gamma_d| \ll \gamma_L$, truncation of mode amplitude at third order is justified for the case when the mode amplitude is small but already considered to be entering a nonlinear stage, i.e., when $\omega_b/(\gamma_L - \gamma_d) \ll 1$.¹ Taking ν_{stoch} (the overall stochasticity felt by EPs, which includes ν_{scatt}) and ν_{drag} independent of time but dependent on phase space localization (as described in Appendix A), the equation for the early-time perturbed mode (a mode that exists without accounting for the kinetic component) amplitude $A(t)$ evolution can be written as a time-delayed, integro-differential cubic equation [48, 61, 62, 63, 40, 41]

¹Note that the nonlinear trapping frequency ω_b is proportional to the square root of the mode amplitude.

$$\begin{aligned}
 \frac{dA(t)}{dt} - A(t) = & -e^{i\alpha} \sum_j \int d\Gamma \mathcal{H} \int_0^{t/2} d\tau \tau^2 A(t-\tau) \times \\
 & \times \int_0^{t-2\tau} d\tau_1 e^{-\dot{\nu}_{stoch}^3 \tau^2 (2\tau/3 + \tau_1) + i\dot{\nu}_{drag}^2 \tau(\tau + \tau_1)} \times \\
 & \times A(t-\tau-\tau_1) A^*(t-2\tau-\tau_1)
 \end{aligned} \tag{2.1}$$

where $\mathcal{H} = 2\pi\omega\delta(\Omega_j) |V_{n,j}|^4 \left(\frac{\partial\Omega_j}{\partial I}\right)^3 \frac{\partial f}{\partial\Omega}$ and

$$V_{l_1, l_2, l_3}(I) = \frac{iq_{EP}}{\omega} \int \frac{d\xi_1 d\xi_2 d\xi_3}{(2\pi)^3} e^{-i(l_1\xi_1 + l_2\xi_2 + l_3\xi_3)} \mathbf{v} \cdot \mathbf{e},$$

or

$$V_{n,j}(I) = \frac{iq_{EP}}{\omega} \int \frac{d\varphi d\theta}{(2\pi)^2} e^{-i(j\theta - n\varphi)} \mathbf{v} \cdot \mathbf{e}, \tag{2.2}$$

accounts for the wave-particle energy exchange, where $\mathbf{e}$ is the electric field eigenstructure and $\mathbf{v}$ is the velocity of a resonant particle. l_1, l_2, l_3 are integers and ξ_1, ξ_2, ξ_3 are angles conjugate to the invariants of motion (actions of the Hamiltonian). In equation (2.1), the circumflex denotes normalization with respect to $\gamma = \gamma_L - \gamma_d$ (growth rate minus damping rate) and t is the time normalized with the same quantity. A is the normalized complex mode amplitude of an eigenmode oscillating with frequency ω . In equation 2.2, the quantum number corresponding to the gyroangle was taken equals to zero since low-frequency Alfvénic waves do not have the cyclotron resonance.

The phase factor α is a measure of the non-perturbative aspect of the linear problem. Here we limit ourselves to the case $\alpha = 2\pi h$, $h \in \mathbb{Z}$, which is the case where a positive energy mode exists in absence of the kinetic response of EPs. Then the kinetic response leads to a perturbative response of the positive energy wave to the kinetic interaction. Note that the case $\alpha = \pi + 2\pi h$ leads to a perturbative response of a negative energy wave to the

kinetic interaction, while other values of α imply a non-perturbative response, where there is no linear wave present in absence of the EPs. In this last case the EP population is very close to the marginal stability point of the mode. In other words, α represents the ratio between the dissipative and the reactive responses of the perturbing field. A detailed study of the explosive solutions of (2.1) in terms of the parameter α has been recently reported in Ref. [64].

The solutions of eq. (2.1) can exhibit several bifurcations and therefore several phases, as shown for a bump-on-tail configuration in Ref. [48]. Interestingly, eq. (2.1) allows for the excitation of sub-critical instabilities and for nonlinear frequency splitting [30]. If the nonlinearity in equation (2.1) is weak, the system will most likely saturate close to the linear stability state, where the trapping frequency ω_b satisfies $\omega_b \ll \gamma_L \simeq \gamma_d$. However, in case the solution of the cubic equation explodes, the system enters a strong nonlinear phase, which may lead to chirping modes. Indeed, long-range numerical simulations indicate the explosive behavior of A as the precursor to the formation of holes and clumps structures [65]. Furthermore, chirping events are significantly enhanced by the coherence introduced by dynamical friction (i.e, particle drag) [63, 53] and are inhibited by stochasticity from diffusive processes, such as resonant particle heating [66], collisional pitch-angle scattering and from background turbulence, all of which contribute to causing particles to detune from a resonance. Stochastic events lead to loss of phase information that contribute to destroy coherent structures.

Eq. (2.1) was originally derived for a bump-on-tail system with Krook collisions [48] and later generalized to complex tokamak geometries and also to include collisional scattering [62]. Lilley, Breizman and Sharapov [63] included the effects of drag on the bump-on-

tail cubic equation and derived a criterion to determine stable and unstable regions of solutions of the cubic equation in drag *vs* scattering collisional parameter space. Our work aims at improving this prediction by using realistic resonant surfaces and mode structure information, coming from the NOVA and NOVA-K codes. To this end, orbit and phase space averages are employed in order to account for the effective Fokker-Planck collisional coefficients. Experimental data are analyzed in order to verify whether chirping events coincide with the occurrence with the "explosive" phase of the cubic equation, as predicted by the theory.

2.3 A criterion for chirping onset

It has been shown [40] that a simplified bump-on-tail approach that only accounts for a single representative value for the collisional coefficients is insufficient to make predictions for the nonlinear nature of a mode in a tokamak. The missing physics were shown to be the absence of non-uniform mode structures, (multiple) resonance surfaces and poloidal bounce averages that account for particle trajectories on a poloidal cross section.

A necessary but not sufficient condition for the existence of fixed-frequency, steady-state solutions would be that the real component of the right-hand side of Eq. (2.1) be negative at late times when the response is stationary, i.e. when the nonlinear term is allowed to balance the linear growth. The delta function $\delta(\Omega_j(P_\varphi, \mathcal{E}, \mu))$ that restricts the integration to the resonance condition can be exploited and the following criterion for the existence of

fixed-frequency oscillations is obtained [40]:

$$Crt = \frac{1}{N} \sum_{j, \sigma_{\parallel}} \int dP_{\varphi} \int d\mu \frac{|V_{n,j}|^4}{\omega_{\theta} \nu_{drag}^4} \left| \frac{\partial \Omega_j}{\partial I} \right| \frac{\partial f}{\partial I} Int > 0, \quad (2.3)$$

where

$$Int \equiv Re \int_0^{\infty} dz \frac{z}{\frac{\nu_{stoch}^3}{\nu_{drag}^3} z - i} \exp \left[-\frac{2}{3} \frac{\nu_{stoch}^3}{\nu_{drag}^3} z^3 + i z^2 \right] \quad (2.4)$$

and N is a normaliation for Crt , which consists in the same sums and integrations that appear in the numerator of (2.3) but in the absence of Int . In eqs. (2.3) and (2.4), the quantities τ_b , ν_{drag} , ν_{stoch} , $V_{n,j}$ and Ω_j are understood to be evaluated at $\mathcal{E} = \mathcal{E}' - \omega P_{\varphi}/n$. Criterion (2.3) was incorporated into NOVA-K making use of a polynomial interpolation for Int . Crt provides a prediction for the likelihood of a fully nonlinear phenomenon obtained only from pure linear physics elements and therefore can be tested in linear codes. This is a considerable advantage in efficiency for making a prediction of a nonlinear property. The integrand of Int is plotted in figure (2.4) for different values of ν_{stoch}/ν_{drag} .

Figure 2.4: Integrand of Int , given by eq. (2.4) as a function of ν_{stoch}/ν_{drag} . It has a strong oscillating behavior for small values of ν_{stoch}/ν_{drag} , which makes evident the changing sign introduced by drag in the kernel of the cubic equation (2.1) (part a). In this regime the sign of the integral flips recurrently and prevents a steady solution to being established. For moderately higher values of ν_{stoch}/ν_{drag} , the integrand is less oscillatory but the integral is still negative (part b). After ν_{stoch}/ν_{drag} exceeds 1.04, the integral becomes positive. Part c was taken close to the peak positive value of the integral.

Drag introduces an oscillatory behavior of the integrand of Int . This has an effect of flipping the sign of the integral kernel of the cubic equation. This causes dA/dt to vary more abruptly and therefore prevents a steady state solution from being achieved. Instead nonlinear chirping solution is more likely to be achieved. It is interesting to note that formally, drag enters the kinetic equation in a mathematically similar way as a chirping frequency does, with $kv\nu_{drag}$ replaced by $d\omega/dt$.

Int , which is a function of phase space, is plotted in Fig. (2.5) as a function of ν_{stoch}/ν_{drag} . Note that the positive and negative domains of Int are roughly an order of magnitude different in the interval $0 \leq \nu_{stoch}/\nu_{drag} \lesssim 2$, while for $\nu_{stoch}/\nu_{drag} \gg 1$, $Int \approx 1.022 (\nu_{stoch}/\nu_{drag})^{-4}$.

Figure 2.5: Numerical values for Int , the time integral in equation (2.4), as a function of ν_{stoch}/ν_{drag} . An interpolation of this function is provided as an input to NOVA-K code, which performs the phase space integrals *a posteriori*.

For our investigation, we focus on the criterion for the non-existence of a fixed-frequency, steady solution. To determine the physical parameters needed in the calculation described below, the global transport code TRANSP [67, 68] is used. The code determines the

needed particle diffusivity that matches the observed plasma parameters to the particle and heat sources. We interpret this empirically obtained result as due to both classical (which includes neoclassical) transport and due to micro-turbulence. The resulting thermal particle diffusivity is then used together with a model to scale the EP diffusivity given the turbulent background diffusivity, to estimate the diffusivity of EPs, which will then contribute to the value of ν_{stoch} . Here we perform a quantitative study based on the time delayed cubic equation using experimental results to determine whether the theoretically predicted nonlinear character of the response correlates with the observation.

2.4 Kinetic-MHD perturbative computations

NOVA [27] is a nonvariational ideal MHD equilibrium code primarily used to integrate non-Hermitian integro-differential eigenmode equations in the presence of EPs, using a general flux coordinate system. It uses realistic numerically calculated MHD equilibria and hence it is suited to study both conventional and spherical tokamaks. The code uses Fourier expansion in θ and cubic spline finite elements in the radial ψ direction. NOVA provides the eigenstructures used in our analysis.

NOVA-K [69, 70] hybrid kinetic-MHD stability code that takes into account finite Larmor radius and orbit width effects to study the destabilization of MHD modes from the EPs free energy. We use NOVA-K to calculate the resonance surfaces, in $(P_\varphi, \mathcal{E}, \mu)$ space, associated with the modes. These surfaces are needed to calculate the growth and damping rates of eigenmodes in the presence of EPs as well as the integral Int . TRANSP is used to provide the distribution function which contains the necessary information on the most

representative EP population as input for NOVA-K runs.

The quadratic form of MHD is particularly useful when stability analysis is addressed. If the mode frequency is assumed to be $\omega + i\gamma$, with $|\gamma| \ll |\omega|$ where ω is the mode eigenfrequency and γ is its growth rate, it is obtained [27, 69]

$$\gamma \approx \frac{Im\delta W_k}{2\delta K}\omega$$

where $\delta K = \omega^2 \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r}/2$ is the inertial (kinetic) energy, δW_k is the potential energy associated to the nonadiabatic component of the distribution function and $\omega\delta W_k$ is the power released by the resonant particles. The growth rate can be expressed as [27]

$$\begin{aligned} \gamma = & \frac{2M^2\pi^3c}{q_{EP} \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r}} \sum_{\sigma_{\parallel}} \int dP_{\varphi} d\mu d\mathcal{E} \sum_{m,m',j} G_{m'j}^* \mathcal{E}^2 \tau_b \\ & \left(\frac{\partial f}{\partial \mathcal{E}} \right) (1 - \omega_*/\omega) G_{mj} \delta(\omega + n\omega_{\varphi} - j\omega_{\theta}), \end{aligned} \quad (2.5)$$

with the diamagnetic frequency being defined by

$$\omega_* = -i \frac{\frac{\partial f}{\partial P_{\varphi}}}{\frac{\partial f}{\partial \mathcal{E}}} \frac{\partial}{\partial \varphi}.$$

G represents mode structure matrix elements, which are associated to the projection of the resonant particle current onto the wave electric field. For each isolated resonance, the reduced Hamiltonian can be written as [62]

$$\begin{aligned} H(I, \zeta, t) &= H_0(I) + H_1(I, \zeta, t); \\ H_1(I, \zeta, t) &= 2A(t)V_{n,j}(I)\cos(\zeta - \omega t), \end{aligned}$$

where $\zeta \equiv l_1\xi_1 + l_2\xi_2 + l_3\xi_3$. The bounce frequency of the most deeply trapped particles

can be shown to be

$$\omega_b = \left| 2A(t)V_{n,j}(I_r) \frac{\partial \Omega_j}{\partial I} \Big|_{I=I_r} \right|^{1/2},$$

where the subscript r denotes the resonance location and the derivative is defined by eq. (1.2). The G matrices are related to V (given by equation (2.2)) via

$$\left(\frac{M\mathcal{E}}{\sqrt{2}} \right)^2 \sum_{m,m'} G_{m'j}^* G_{mj} = |V_{n,j}|^2.$$

2.5 Mode structure identification

In order to characterize the mode being observed in the experiment, NSTX and TFTR reflectometer measurements are compared to the mode structures computed by NOVA. Reflectometry diagnostic measures the density fluctuation of the plasma at the location where the launched wave has a cutoff. The fluid displacement times the local density gradient is equivalent to the density fluctuation. All poloidal harmonics calculated by NOVA are summed up for this analysis. Modes are categorized according to their mode structure and whether their frequencies fall into a given gap of the continuum. The reflectometer cannot resolve the density in the core for the cases of flat density profile and especially when the peak of the density is displaced from the plasma center, as typically observed during the H-mode regime. In DIII-D, Electron Cyclotron Emission is used for the purpose of measuring the mode structure, following a methodology described in [71]. An interesting aspect of the observation is that, in spite of the intrinsic nonlinear nature of chirping events, the

structure of the chirping mode is not substantially changed during the time evolution of the system [36, 31, 72]. This gives credence to the use of a linear code for the eigenstructure identification at early times.

2.6 Averaging implementations in NOVA-K

In order to calculate the above expressions for the effective collisional coefficients (A.4) and (A.5), NOVA-K is employed to perform bounce averaged calculations since the bounce motion is much faster than the perturbative mode evolution. Then, a phase-space average needs to be performed to account for the contribution of resonance surfaces spread over phase space, as described below.

2.6.1 Bounce averaging

The period of a particle poloidal bounce or transit motion is given by $\tau_b = \oint dt = \oint d\theta r / (\mathbf{v}_{\parallel} + \mathbf{v}_d) \cdot \hat{\theta}$, where the drift velocity $\mathbf{v}_d$ is due to curvature and ∇B terms. Then, the bounce average of the effective collisional frequencies is given by

$$\langle \dots \rangle = \frac{1}{\tau_b} \oint \frac{d\theta r (\dots)}{(\mathbf{v}_{\parallel} + \mathbf{v}_d) \cdot \hat{\theta}}.$$

For each set $(\mathcal{E}, P_{\varphi}, \mu)$, the trajectories are *a priori* known, hence there is no additional need to explicitly follow the particles motion within the code.

2.6.2 Phase space averaging

The average over phase space is taken along the surfaces over which the resonance condition is satisfied, for different poloidal bounce harmonics. The phase space volume elements are weighted in accord to their relative contribution to the overall growth rate. Specifically, we evaluate

$$\overline{(\dots)} = \frac{\int (\dots) Q d\Gamma}{\int Q d\Gamma},$$

where $Q = \frac{1}{\omega} \sum_j \langle q_{EP} \mathbf{e} \cdot \mathbf{v} \rangle^2 \frac{\partial f}{\partial I} \delta(\Omega_j)$ is the contribution to the growth rate, γ_L , from a given phase space location that satisfies $\Omega_j(\mathcal{E}, P_\varphi, \mu) = 0$. In the present study, the phase-space averages are taken over several harmonics of a given mode. This averaging technique was previously used to predict the TAE amplitude saturation in TFTR experiments [70].

2.7 Comparison with simplified bump-on-tail prediction as evaluated by NOVA

In this thesis, we show that a previous approach that attempted to simplify the needed input that the theory requires [63] is insightful but limited for making accurate predictions of experimental scenarios. Here we employ a generalized formulation and show that its predictions are in accordance with observations. This analysis reveals that micro-turbulence, even while producing no observable effect on the beam ion transport, provides the vital mechanism in determining which non-linear regime is more likely for a mode as well as the mode transition from one regime to the other, as parameters of an experiment change in

time.

In Ref. [63], Eq. (2.1) was simplified by using characteristic values for the collisional ν_{scatt} and ν_{drag} and conditions for the existence and stability of solutions of the cubic equation were derived. In Fig. 2.6, we test for the first time this prediction against modes measured in different tokamaks. We see in Fig. 2.6 that about half of the chirping NSTX modes lie in a region where stable steady modes are predicted by Ref. [63]. For the DIII-D experimental cases that produced fixed-frequency modes, the predictions of Ref. [63] are mostly in agreement although one point is borderline and another one may be unstable enough to be in a chirping regime. Hence we see that using the simplified, although elaborate, modeling akin to that used in [63], might be in satisfactory agreement with DIII-D data but is generally not satisfactory for much of the NSTX and TFTR data. This comparison indicates that the use of a single characteristic value, as being representative of the entire phase space, for ν_{scatt} (considered the only contribution to ν_{stoch}) and ν_{drag} , although insightful, appears insufficient to provide quantitative predictions for practical tokamak cases. This conclusion motivated the pursuit of a theoretical method to take into account important missing elements, such as spatial mode structures and local phase-space contributions on multiple resonant surfaces of the wave-particle interaction terms, all of which are needed in toroidal geometry. The appropriate weightings for the various needed quantities can be expressed in the action-angle formulation.

In NOVA-K we use the contributions from each resonance weighted in accord with the appropriate eigenfunction (that fits the measured field structure) and the position in phase space of the resonant interaction. We will see that this procedure produces quite a different

Figure 2.6: Comparison between analytical predictions with experiment when single characteristic values for phase space parameters are chosen. The dotted line delineates the region of existence of steady amplitude solutions of the cubic equation (2.1) while the solid line delineates the region of stability, as predicted by [63]. Modes that chirped are represented in red and the ones that were steady are in black, as experimentally observed for TAEs, RSAEs and BAAEs in DIII-D (circular discs), TAEs in NSTX (diamonds) and TAE in TFTR (square).

conclusion from the less detailed method that uses a single characteristic factor, as is the case in Fig. 2.6.

Non-steady oscillations, with the likelihood of chirping, are predicted to occur if $Crt < 0$ while a steady (fixed-frequency) solution exists if $Crt > 0$. However, we see from Fig. 2.5 that the magnitude of Int could be an order of magnitude larger in phase space regions where $\nu_{stoch}/\nu_{drag} \lesssim 1.04$ compared with regions where $\nu_{stoch}/\nu_{drag} \gtrsim 1.04$. Hence, because of this disparity, it can turn out that a choice of the use of a single characteristic value for ν_{stoch}/ν_{drag} , would lead to a positive value for Crt while the use of the appropriately weighted average leads to a negative value for Crt . Such a change is indeed the case for all the TFTR and DIII-D modes and for most of the NSTX modes shown in Fig. 2.6, where ν_{stoch} was considered simply as ν_{scatt} . The reason for this sensitivity is that there will always be a contribution to Crt from a phase space region where $\nu_{scatt}/\nu_{drag} \ll 1$ because our modeling of the pitch angle scattering coefficient goes to zero as μ vanishes. Hence even when the characteristic value of ν_{scatt}/ν_{drag} is substantially greater than unity, one still can find that $Crt < 0$.

In considering the classical transport processes, we only included the dominant transport process of pitch angle scattering, which is larger than energy scattering by a factor $\sim E_{EP}/T_i$. Also, the collisional effects from beam-beam interactions have been neglected since they are smaller than the beam-background interactions by a factor $\sim n_b/n_i$, with n_b being the density of the EPs injected by the neutral beam and n_i the background ion density. RF fields have been shown to be a determining component for the suppression of chirping in CTX [66]. Our analysis, however, was only focused on shots where the EPs were created by neutral beam injection, hence there was no need to account for diffusion

from RF waves.

2.8 Study of the frequency chirping prediction for eigenmodes

In this section, we show more detailed representations of the proposed criterion evaluation for different modes, for both cases of chirping and fixed-frequency, in three different tokamaks. The analysis shows the importance of fast ion micro-turbulence in decreasing the likelihood of wave chirping.

Tokamak Fusion Test Reactor (TFTR)

In order to show the importance of micro-turbulence as a chirping suppression mechanism, an $n = 4$ TAE mode driven by alpha particles in the Tokamak Fusion Test Reactor (TFTR) shot 103101 was studied. This mode, which was observed to oscillate at a constant frequency, was analyzed in terms of the proposed criterion discussed above. Its frequency and strength relative to the background field are presented in Fig. (2.7). The corresponding mode structure obtained with NOVA code is shown in Fig. (2.8). Shown in Fig. (2.9) are scans over the inferred experimental values of ν_{stoch} and ν_{drag} , denoted by $\nu_{stoch}^{(exp)}$ and $\nu_{drag}^{(exp)}$, for the situations (a) without the inclusion of micro-turbulent stochasticity and (b) with its inclusion.

A detailed visualization of how far the experiment is from the expected boundary that separates the steady and chirping regions, as predicted by the model, can be provided by scanning the criterion integral for several values multiplying the actual ν_{drag} and ν_{stoch}

(as shown in Fig. (2.9)). The vertical axis represents the constants that are multiplying ν_{stoch} while the horizontal one represents constants that are multiplying ν_{drag} . The point $\left(\nu_{stoch}/\nu_{stoch}^{(exp)} = 1, \nu_{drag}/\nu_{drag}^{(exp)} = 1\right)$ corresponds to the inferred experimental condition and the criterion boundary is represented by the 0.0 countour. Contours of negative values represent regions in which chirping is expected while the positive ones represent expected steady-state (fixed frequency). If experimental conditions imply a positive Crt sufficiently far from this transition boundary, steady solutions should arise in experiment.

Each contour plot corresponds to a given value of Crt (eq. (2.3)), as labeled in the two plots in Fig. (2.9). In part (a), where Crt is evaluated only in terms of collisional pitch-angle scattering and drag, Crt is a negative number. Upon the addition of micro-turbulent scattering, the re-evaluation of Crt shows that the point (1, 1) indicating the experimental conditions now operates where $Crt > 0$, which means that the criterion predicts that the mode should not be able to chirp, being in agreement with the observation. For this case, the estimation for D_{EP} based on the scaling of [73] led to $D_{EP} \approx 0.1m^2/s$, which is consistent with direct measurements using beam blips in similar experiments carried out in TFTR [15]. Numerically calculated $\nu_{scatt}^{(exp)}$ and $\nu_{drag}^{(exp)}$ were $8568s^{-1}$ and $3138s^{-1}$, respectively. $\nu_{stoch}^{(exp)}$ was found to be nearly 10 times larger than $\nu_{scatt}^{(exp)}$ for this particular mode. The 90° degree pitch-angle scattering and the inverse slowing down time were found to be $\nu_{\perp} = 0.19s^{-1}$ and $\tau_s^{-1} = 4.55s^{-1}$.

The NOVA-K code does not account for turbulence stochasticity, and consequently it evaluates the criterion via an indirect method, using the following procedure. First, we note that Crt only depends on the ratio of the overall stochasticity and collisional drag. From equations (A.1), (A.2), (A.4) and (A.5), we see that the effective drag coefficient

is proportional to $T_e^{-3/2}$ while the effective collisional scattering is insensitive to T_e . It is then possible to mimic the inclusion of turbulence into ν_{stoch} by artificially decreasing drag, which can be achieved by multiplying T_e by a corresponding numerical factor that leads to the correct value of the averaged values of ν_{stoch}/ν_{drag} .

Figure 2.7: (a) Spectrogram of fixed-frequency, alpha-particle-driven TAEs in TFTR and (b) relative amplitude of the TAEs with respect to the equilibrium field for toroidal mode numbers $n = 2 - 5$. Reproduced from [74].

Figure 2.8: Mode structure of the dominant poloidal harmonics of a $n = 4$ alpha-particle-driven, core localized TAE in TFTR shot 103101 at $t = 2.92s$. ξ_n is the fluid displacement and ψ_θ is the poloidal flux normalized with its value at the plasma edge.

Figure 2.9: Plots showing contours of constant value of the criterion for chirping existence, Crt , for an $n = 4$ TAE in TFTR shot #103101 (a) before and (b) after the inclusion of micro-turbulence stochasticity. The enhancement of stochasticity makes the point $\left(\nu_{stoch}/\nu_{stoch}^{(exp)} = 1, \nu_{drag}/\nu_{drag}^{(exp)} = 1\right)$, that corresponds to the inferred experimental parameters, to cross the boundary towards the positive region (where no chirping should be allowed), which is consistent with the observation. Micro-turbulence has a strong effect on bringing the mode to the steady state region, thus suppressing the chirping. The contour plots are instructive in order to analyze how far from the boundary (thick black line) the mode is and therefore how prone it is to have its nonlinear character changed.

DIII-D

A series of dedicated shots were performed on DIII-D with the objective of triggering chirping and studying what conditions most strongly determine a mode nonlinear evolution into the chirping regime. These shots used high ion temperature at the core ($10 - 12 keV$), $q_{min} \sim 2$ and strong toroidal rotation (up to $50 kHz$ on axis). A good example is the chirping observed in shot 152828, shown in Figs. (2.10) and (2.11).

The frequency of the chirping mode was too low to be a TAE. At first, NOVA was run in a mode that enabled only the Alfvénic branch to be captured and no reasonable mode structure was found. Then, NOVA was run allowing both the Alfvénic and the sonic branches. A mode that matches experimental evidence was obtained and identified to have the characteristics of a beta-induced Alfvén-acoustic eigenmode (BAAE) [75] and is shown in Fig. 2.12. The mode was identified using the data available on mode localization from the electron cyclotron emission (ECE), frequency, local rotation and toroidal mode number ($n = 1$ for the chirping mode). For the comparison with ECE data for electron temperature fluctuation, the fluid displacement ξ calculated by NOVA was post-processed using the relation

$$\frac{\delta T_e}{T_e} = -(\gamma - 1) \nabla \cdot \xi - \frac{\nabla T_e}{T_e} \cdot \xi.$$

Figure 2.10: (a) time variation of the spectrum magnetic field dB/dt for DIII-D shot 152828, (b) root mean square of dB/dt during and after the chirping and (c) spectrogram showing the chirping fundamental and second harmonics that exist before strong neoclassical tearing mode starts at around $t = 992$ ms.

Figure 2.11: (a) Thermal ion conductivity and (b) spectrogram for DIII-D shot #152828 as the mode undergoes transition from fixed-frequency to the chirping regime. The chirping time window is indicated by the bold black dashed lines.

Figure 2.12: Electron temperature fluctuation reconstructed from NOVA $n = 1$ BAAE mode structure for DIII-D shot 152828 at $t = 0.97s$ and its comparison with electron cyclotron emission (ECE) measurements. The peak fluctuation at the high field side, at around $R = 1.4m$ is mostly due to fluid compressibility while the peak at the low field side, close to $R = 2.0m$ is mostly due to the fluid displacement itself, projected onto the temperature gradient.

The time-delayed cubic equation (2.1) is derived assuming that the nonlinear bounce frequency ω_b is much less than the growth rate in absence of dissipation. This implies a limitation on the cubic equation, with its domain of validity being restricted to the early nonlinear phase, when mode amplitude, represented by ω_b , is still small compared to γ_L . From DIII-D observations, we have noted that the chirping behavior typically starts when χ_i drops to values lower than $0.25m^2/s$ [40] but chirping persists even when χ_i raises to

values that would not admit the onset of the chirping process, according to the theory. The chirping cycle appears to involve some degree of hysteresis since once the first chirp happens, there is a tendency of the system to continue chirping. Thus this observation appears to indicate that once the chirping structures are already embedded in the system, continued new chirping can still arise even though the apparent diffusivity has increased to the point that chirping would not occur if phase space structures were not already present. This hysteresis can be noted in Fig. (2.11) as the chirping persists up to the point where $\chi_i \approx 1.5m^2/s$.

Chirping does not restart when χ_i drops for the second time, at $t = 1.05s$. This is probably because of the effect of strong neoclassical tearing modes (NTM) that are present at this time (as can be seen from the parallel horizontal spectral lines that begin at around $t = 992ms$ in Fig. (2.11)). The suppression then probably occurs because the NTMs are detuning the EPs from the resonance and therefore contributes to extinguishing the drive.

Fig (2.13) (a) shows the experimental condition for the mode in DIII-D shot 152828 shown in Fig. (2.12) before chirping starts (at $t = 900ms$, when $D_{th,i} \approx 0.9m^2/s$) and Fig. (2.13) (b) shows it during chirping (at $t = 960ms$, when $D_{th,i} \approx 0.25m^2/s$), when the point has transitioned from the positive to the negative region, in agreement with the generalized criterion prediction. The criterion evaluated close to the onset of chirping, at $t = 960ms$, used the mode structure calculated at $t = 970ms$ (Fig. 2.12) since trustworthy equilibria and profiles could not be made exactly at the time of its onset.

Figure 2.13: Plots showing contours of constant value of the criterion for chirping existence, Crt , for an $n = 1$ BAAE in DIII-D shot #152828 shown on the spectrogram of Fig. (2.11). The point $\left(\nu_{stoch}/\nu_{stoch}^{(exp)} = 1, \nu_{drag}/\nu_{drag}^{(exp)} = 1\right)$ corresponds to the inferred experimental situation. (a) Before chirping starts (at $t = 900ms$, when $D_{th,i} \approx 0.9m^2/s$) the criterion integral Crt is positive and (b) during chirping (at $t = 960ms$, when $D_{th,i} \approx 0.25m^2/s$) the criterion integral is negative. The point crosses the steady/chirping boundary represented by the thick black line and has its nonlinear behavior changed.

National Spherical Tokamak eXperiment (NSTX)

Similar framework of analysis was also applied to NSTX shot 141711, whose spectrogram (Fig. (2.3)) showed chirping for several TAEs with different toroidal mode numbers. A number of chirping modes were studied, with the evaluation of Crt for all of them leading to negative numbers. A representative example is a TAE that was catalogued by comparing the mode structures calculated by NOVA with the reflectometer measurements, as shown in Fig. (2.14). The contour plot for Crt for this mode is presented in Fig. (2.15). During the chirping time window in NSTX shot 141711, the ion transport was dominated by neoclassical processes. It was observed that micro-turbulence was not strong enough to suppress the bucket structures that sustain chirping.

Figure 2.14: Comparison between the mode structure (quantified by the ratio of the perturbed density δn and the background density n) inferred by the reflectometer (in black, with actual measurement locations shown by the cyan diamonds) and the one calculated by NOVA code (in red) for an $n = 3$ TAE in NSTX shot 141711 at $t = 450\text{ms}$.

Figure 2.15: Plots showing contours of constant value of the criterion for chirping existence, Crt , for an $n = 3$ TAE in NSTX shot #141711 at $t = 450ms$ shown in Figs. (2.3) and (2.14). The point $\left(\nu_{stoch}/\nu_{stoch}^{(exp)} = 1, \nu_{drag}/\nu_{drag}^{(exp)} = 1\right)$ corresponds to the inferred experimental situation. .

2.9 The role of micro-turbulence in the transition to chirping in DIII-D

Fig. 2.16 shows values of $|Crt|^{1/4}$ multiplied by the sign of Crt , as a function of the ratio of phase-space averaged stochasticity and drag for modes of Fig. 2.6. This representation provides better visualization than simply plotting Crt , especially close to the steady/chirping boundary and is chosen because of the fourth power dependence $Int \approx 1.022 (\nu_{stoch}/\nu_{drag})^{-4}$ for $\nu_{stoch}/\nu_{drag} \gg 1$. Fig. 2.16(a) shows chirping modes in NSTX and Fig. 2.16(b) shows steady modes in DIII-D and TFTR. The curved arrows represent how the prediction for a mode is affected by micro-turbulence-induced scattering of EPs.

It has a strong effect on DIII-D and TFTR (bringing the modes to the steady region, or at least very close to it) while its effect is almost imperceptible for the chirping modes in NSTX. This is because, unlike in conventional tokamaks, thermal ion transport in spherical tokamaks (STs) is usually very close to neoclassical levels [76, 77] even though the electron transport is anomalous. NSTX modes in Fig. 2.16(a) are only able to transition to the fixed-frequency region when ν_{stoch} is artificially multiplied by a factor from 10 to 50, depending on the specific mode, which indicates the robustness of the chirping prediction. For the analyzed DIII-D discharges, the background turbulence is believed to be mostly in the ITG range. In Fig. 2.16(b), we artificially multiply the predicted turbulent stochasticity for the resonant energetic ions by factors of 2 and 1/2 in order to understand how sensitive the evaluated criterion is with respect to uncertainties in the inferred EP turbulence level. It turns out that Crt can be rather sensitive near the positive/negative transition due to the dependence of Int on the ratio ν_{stoch}/ν_{drag} but becomes quite insensitive to deviations of χ_i as the point moves away from this borderline [40].

Guided by the theory, we have then examined chirping modes that rarely appear in DIII-D tokamak. A series of dedicated shots were performed on DIII-D to study the transition to the chirping regime. These shots had high ion temperature in the core ($10 - 12keV$), low q_{min} (~ 2) and strong toroidal rotation (up to $50kHz$ on axis). We find that the chirping onset correlates very closely with conditions where the thermal ion transport had drastically decreased (more specifically, near the L to H transition), as shown in Fig. 2.17. This is attributed to the decrease in micro-turbulence-induced transport, which also causes decreased EP transport. Alfvénic modes only started chirping when the thermal ion conductivity dropped to values lower than $0.3m^2/s$. An example of the evaluation of the

Figure 2.16: Numerical values for $|Crt|^{1/4}$ multiplied by the sign of Crt as a function of $\langle \nu_{stoch} \rangle / \langle \nu_{drag} \rangle$. Modes that chirped are represented in red and the ones that were steady are in black, as experimentally observed in (a) NSTX (diamonds) and (b) DIII-D (discs) and TFTR (squares). The arrows represent the effect of micro-turbulence. The bars represent by how much the prediction for the modes change if we double the turbulent diffusivity (upper bars) or divide it by 2 (lower bars). In NSTX case, the points hardly move upon the addition of spatial diffusion to collisional scattering.

criterion (2.3) is DIII-D shot 152828 (Fig. 2.17 (c)). Before chirping starts (at $t = 920ms$, when $D_{th,i} \approx 0.55m^2/s$) the calculated criterion is $Crt = +0.001$. During the early phase of chirping (at $t = 955ms$, when $D_{th,i} \approx 0.25m^2/s$) the value is $Crt = -0.013$, i.e. the mode has transitioned from the positive (steady) region to the negative region of Crt , therefore allowing chirping, in agreement with the observation. We note that mode stability can be sensitive to ion and electron kinetic effects, not captured by NOVA, and to the time evolving equilibrium as the modes transition to chirping. However, even though the growth rate changes, the most relevant parameter for the chirping criterion in Fig. 2.17 is the substantial drop microturbulence levels. This is because microturbulence stochasticity enters directly the chirping-relevant ratio ν_{stoch}/ν_{drag} , while the growth rate affects the phase-space averaging, which has a minor effect.

The micro-turbulence interaction with EPs is a key factor that can determine the nature of mode saturation regime (quasi-steady and chirping) and also the transition between them. It also explains the longstanding question of why chirping Alfvénic modes are ubiquitous in STs and rare in conventional tokamaks. Experimentally, the EP transport due to micro-turbulence is too low compared to Alfvénic-induced transport [78, 79]. Yet, its effect on EP transport can be crucial in determining the character of emerging Alfvénic oscillations and the type of transport caused by them. This suggests that micro-turbulence simulations employed to predict the thermal plasma transport of future burning plasma devices must also be factored in to considerations of the drive and saturation of modes driven by EPs.

This work provides a means for choosing which of the two extreme scenarios is most likely to be relevant for predicting the character of the energetic particle transport, based on the sign of Crt . For a negative Crt the physical conditions are established to enable

Figure 2.17: Correlation in DIII-D between the emergence of chirping and the development of low diffusivity, as calculated by TRANSP at the radius where the mode is peaked.

a nonlinear BGK-like mode [50] to form, where the frequency remains locked to a particle resonance frequency as particles trapped by the wave are convected in phase space which, for the Alfvénic instabilities, primarily causes resonant energetic particles to flow across field lines. Alternatively, a positive Crt represents the lack of chirping and indicates that the details of the nonlinear particle transport might be described by a quasilinear diffusion theory [56, 37, 55]. Therefore, the application of this criterion should be important in the planning and modeling of scenarios for future fusion plasma experiments.

2.10 Chirping likelihood in terms of the beam injection speed: the criterion predictions for abrupt large events in JT-60U

In planning and interpreting experiments, it can be meaningful to understand the likelihood of observing wave chirping in terms of the resonant particle speed. This is because the injected NB ions can be either supra- or sub-Alfvénic which, for the case of TAEs, would lead to dominant resonances located around $v_{\parallel} = v_A$ and $v_{\parallel} = v_A/3$, respectively. In particular, a puzzling observation in JT-60U is that the abrupt large events (ALEs) and their associated bursts and chirps usually happen when the beam is injected supra-Alfvénically, using negative-ion-based neutral beams [80, 81, 82, 83, 84]. The ALEs have been intensively studied over the past decade [85, 86, 87, 88, 89, 90]. Here we propose an explanation for the likelihood of these events in terms of the framework adopted in this work and we show that it qualitatively agrees with JT-60U observations.

Let us analyze a TAE for which the resonance condition is $\Omega = \omega_{TAE} + nv_{\parallel}/R - jv_{\parallel}/qR = 0$. For scenarios in which micro-turbulence can be ignored in comparison with collisional scattering, the relevant ratio for the criterion, eq. (2.3), would be simply $(\nu_{scatt}/\nu_{drag})^3$, with the effective scattering given by eq. (A.4),

$$\nu_{scatt}^3 \simeq \nu_{\perp} R^2 v_{\perp}^2 \left(\frac{jv_{\parallel}}{\omega_c q R r} \frac{dq}{dr} \right)^2,$$

where ω_c is the EP cyclotron frequency and $dP_{\varphi} \simeq -(\omega_c/q) r dr$. The effective drag is given by (A.5),

$$\nu_{drag}^2 \simeq \frac{jRv_{\parallel}^2}{\tau_s \omega_c q r} \frac{dq}{dr} \left(1 + \frac{v_c^3}{v^3} \right),$$

with v_c given by (A.3). Therefore

$$\frac{\nu_{scatt}^3}{\nu_{drag}^3} = \frac{v_0^2}{v^2} \frac{\sin^2 \theta / \cos \theta}{\left(1 + \frac{v_c^3}{v^3} \right)^{3/2}}, \quad (2.6)$$

where

$$v_0^2 \equiv \frac{1}{2} \frac{\bar{A}_i}{[Z]} \frac{\langle Z \rangle}{A_{EP}} v_c^3 \left(\frac{j\tau_s}{\omega_c q r} \frac{dq}{dr} \right)^{1/2}$$

and θ is the angle between $\mathbf{v}$ and $\mathbf{B}$. The ratio between the chirping-criterion-relevant parameter $(\nu_{scatt}/\nu_{drag})^3$ prescribed by eq. (2.6), for the cases in which the main resonance of a TAE is at $v_{\parallel} = v_A$ and $v_{\parallel} = v_A/3$ is plotted in Fig. (2.18). The ratio $(\nu_{scatt}/\nu_{drag})_{v_{\parallel}=v_A}^3 / (\nu_{scatt}/\nu_{drag})_{v_{\parallel}=v_A/3}^3$ is 1/9 for $v_c/v_A \approx 0$. If $v_c \cos \theta / v_A \lesssim 0.5$, it means that supra-Alfvénic injection implies more likelihood for chirping. It is a direct implication of the chirping criterion (eq. (2.3)) and the form of Int (Fig. (2.4)). Considering deuterium as the only background ion species, $n_e \approx n_i$, $\ln \Lambda_i \approx \ln \Lambda_e$, and using typical parameters for

JT-60U, such as $B = 3T$, $n_i = 3 \cdot 10^{19} m^{-3}$ and $T_e = 3keV$ one obtains $\frac{v_c}{v_A} \approx 0.3$. The factor $\cos\theta$ contributes to reduce the variable $v_c \cos\theta / v_A$ used in Fig. (2.18). Therefore, because of the resonant velocity dependence, if all other parameters are held fixed, supra-Alfvénic injection implies more chances of observing wave chirping for TAEs in JT-60U than in the sub-Alfvénic situation, provided that the mode is the same in the two situations. Similar conclusion would also hold for other conventional tokamaks, such as DIII-D.

Figure 2.18: Ratio between the chirping-criterion-relevant parameter $(\nu_{scatt} / \nu_{drag})^3$ for the cases in which the main resonance of a TAE is at $v_{||} = v_A$ and $v_{||} = v_A/3$, as a function of $v_c \cos\theta / v_A$.

The previous conclusion was based on the fact that the stochasticity coming from micro-turbulence is low compared to the one coming from pitch-angle scattering. However, the inclusion of micro-turbulence using the scalings found in [73] does not change the conclusion that higher resonant velocities imply more likelihood for chirping, provided that v_c / v_A is sufficiently small. In fact, turbulence even contributes to strengthen this interpretation. This is because [73] found that the micro-turbulence diffusivity goes as v^{-2} for passing par-

ticles and v^{-4} for trapped particles, which is, similar to the effective pitch-angle scattering, an inverse velocity dependence. Therefore micro-turbulence contributes to further reduce the ratio $(\nu_{scatt}/\nu_{drag})_{v_{\parallel}=v_A}^3 / (\nu_{scatt}/\nu_{drag})_{v_{\parallel}=v_A/3}^3$, which makes chirping for supra-Alfvénic injection even more probable than in the purely collisional case.

2.11 Discussions

We have performed a study of the early phase of chirping events in tokamak plasmas by means of realistic calculations of eigenstructures and collisional coefficients. The proposed methodology has been described in fair detail. In order to generalize the theoretical predictions from the previous bump-on-tail approach, we have employed an action-angle formalism for the general problem, with a similar perturbative procedure used in [62]. It leads to the generalized criterion for the likelihood of chirping and fixed-frequency solutions. It should be noted that it only allows a steady solution to exist but does not guarantee that the system will necessarily evolve to such a state. It may happen that the evolution to steady states be only accessible when higher-order nonlinear terms are taken into account, which are not captured by the lowest nonlinear order perturbative approach used by the framework of the time-delayed cubic equation (eq. (2.1)).

An interesting effect unveiled by the methodology used in this work is that the disparity of magnitudes of the positive and negative regions of Int (equation (2.4)), typically leads to the prediction that a given mode should chirp, even for $\langle \nu_{stoch}/\nu_{drag} \rangle > 1$, unless a strong diffusive mechanism implies $\langle \nu_{stoch}/\nu_{drag} \rangle \gg 1$. This additional mechanism is observed to be often related to turbulent processes. The contribution from regions of low μ turns out to

be quite important since the magnitude of Int is larger for $\nu_{stoch}/\nu_{drag} \ll 1$. This shows the importance of using a weighted average of the transport coefficients and not only a single representative point.

The employed approach is applicable for the modes near the stability threshold. We implement it perturbatively using the mode structures computed by the ideal MHD code NOVA. The perturbative approach is justified when the growth rates evaluated for TAEs and RSAEs in NSTX, DIII-D and TFTR plasmas are typically much smaller than the eigenfrequencies. The case of unstable BAAEs requires more careful stability analysis that is beyond the scope of this work. Nevertheless observations of those modes in DIII-D shown in Fig. (2.11)(b) indicate that these modes are slowly growing, suggesting that they are near the instability threshold.

We have set up plausible rules for determining the effective diffusivity and drag. The main uncertainty in the diffusivity chosen for the EPs, is the value of the diffusivity contribution from the background fluctuations. We have chosen to base this value to be in accordance with what TRANSP predicts to be the diffusion needed to produce the observed energy confinement time. However, an additional uncertainty still remains in determining how this thermal diffusion coefficient would be related to the effective diffusion coefficient for the large orbit EP. We chose to use an empirical formula obtained in ref. [73], based on electrostatic turbulence. However, the value chosen is rough, and the actual scaling for a given experiment might differ considerably from our choice. Using the estimates for EP diffusivity, we brought in the model the effect of background turbulence, that adds up to collisional scattering and contribute to suppress chirping.

A number of factors that may influence the chirping formation are not captured by

the current theory. For example, static 3D fields [91] have been shown to affect bursting Alfvén modes and reduce chirping. Besides that, effects such as toroidal field ripples, energy diffusion, radio frequency heating fields [66, 92], electromagnetic turbulence and neoclassical tearing modes can be important in some scenarios to prevent chirping formation due to randomization of phase space, with consequent resonance detuning. Other limitations are that the cubic equation assumes small mode amplitude, which is not necessarily the case in the experiment, and also that no mode overlap has taken place.

Chirping events require self-trapped resonant particles to remain locked with the excited chirping frequency for successive wave-trapping bounce times. The maintenance of this time-dependent resonance condition can induce significant convective transport over an extended region of phase space. The present work can be helpful in addressing the likelihood of convective and diffusive transport of fast ions and therefore, be useful as a predictive tool for present-day and next-generation devices. Besides, the criterion can be a valuable tool to address the applicability of reduced fast ion transport models, such as the quasilinear diffusion framework.

Chapter 3

Quasilinear diffusion theory in action variables

Quasilinear theory can be a very convenient reduced modeling tool when one takes advantage of the symmetries present in the system. Kaufman [58] developed a quasilinear formalism for the case in which the unperturbed system can be described by three adiabatically invariant actions (constants of the unperturbed motion) $\mathbf{J} \equiv (J_1, J_2, J_3)$, with canonically conjugate variables of angle $\boldsymbol{\Theta} \equiv (\Theta_1, \Theta_2, \Theta_3)$ being ignorable generalized coordinates. The angle variables are assumed cyclic for the perturbations, so any function of them can be expanded in Fourier series:

$$g(\boldsymbol{\Theta}) = \sum_l g_l e^{i\mathbf{l} \cdot \boldsymbol{\Theta}},$$

$$g_1 = \int \frac{d^3\Theta}{(2\pi)^3} g(\Theta) e^{-i\mathbf{l} \cdot \Theta},$$

where $\mathbf{l}$ represents a list of integers (l_1, l_2, l_3) and $(\mathbf{p}, \mathbf{r})$ denotes position and momentum of a particle corresponding to $(\mathbf{J}, \Theta)$.

3.1 Linearized Vlasov equation

Throughout this work we will assume that the distribution function integrated over all actions and angles is dimensionless. This is the case when the action are defined without the mass, i.e. in units of length squared by time, as assumed in NOVA. In $(\mathbf{J}, \Theta)$ variables, the linearized Vlasov equation for a distribution function $f(\mathbf{J}, \Theta; t) = f_0(\mathbf{J}) + \delta f(\mathbf{J}, \Theta; t)$ reads

$$\frac{\partial \delta f}{\partial t} + \underbrace{\frac{d\Theta}{dt}}_{\omega(\mathbf{J}) = \frac{1}{M} \frac{\partial H_0}{\partial \mathbf{J}}} \cdot \frac{\partial \delta f}{\partial \Theta} + \underbrace{\frac{d\mathbf{J}}{dt}}_{\delta \dot{\mathbf{J}} = -\frac{1}{M} \frac{\partial \delta H}{\partial \Theta}} \cdot \frac{\partial f_0}{\partial \mathbf{J}} = 0, \quad (3.1)$$

where the Hamiltonian is written as $H(\mathbf{J}, \Theta; t) = H_0(\mathbf{J}) + \delta H(\mathbf{J}, \Theta; t)$ ¹. Using $\delta H(\mathbf{J}, \Theta; t) = \sum_1 \delta H_1(\mathbf{J}; t) e^{i\mathbf{l} \cdot \Theta}$, $\delta f(\mathbf{J}, \Theta; t) = \sum_1 \delta f_1(\mathbf{J}; t) e^{i\mathbf{l} \cdot \Theta}$, $M\delta \dot{\mathbf{J}} = -\frac{\partial \delta H}{\partial \Theta} = -i \sum_1 \mathbf{l} \delta H_1(\mathbf{J}, t) e^{i\mathbf{l} \cdot \Theta}$ and Fourier transforming in time,²

$$\delta f_1(\mathbf{J}, \omega) = \frac{1}{M} \frac{\delta H_1(\mathbf{J}; \omega)}{\mathbf{l} \cdot \omega(\mathbf{J}) - \omega} \mathbf{l} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \quad (3.2)$$

¹Defining Poisson brackets as $\{a, b\} = \frac{\partial a}{\partial \mathbf{J}} \frac{\partial b}{\partial \Theta} - \frac{\partial b}{\partial \mathbf{J}} \frac{\partial a}{\partial \Theta}$, to lowest order in the perturbation it can be put in the form

$$\frac{\partial \delta f}{\partial t} + \frac{1}{M} \{H, \delta f\} = 0 \Rightarrow \frac{\partial \delta f}{\partial t} + \frac{1}{M} \{H_0, \delta f\} + \frac{1}{M} \{\delta H, f_0\} = 0$$

²Fourier transforms in time will be chosen as $\hat{g}(\omega) = \int g(t) e^{i\omega t} dt$ so that $\frac{\partial}{\partial t} \rightarrow -i\omega$.

3.2 Conductivity tensor

The contribution of this particle for the current density at position $\mathbf{x}$ is

$$\mathbf{j}(\mathbf{x}|\mathbf{J}, \boldsymbol{\Theta}) = z\mathbf{v}(\mathbf{J}, \boldsymbol{\Theta}) \delta[\mathbf{x} - \mathbf{r}(\mathbf{J}, \boldsymbol{\Theta})],$$

The overall perturbed current density (already Fourier transformed) is

$$\delta \langle \mathbf{j} \rangle(\mathbf{x}, \omega) = \int d^3J \int d^3\Theta \delta f(\mathbf{J}, \boldsymbol{\Theta}; \omega) \mathbf{j}(\mathbf{J}, \boldsymbol{\Theta}; \omega).$$

Using $\delta f(\mathbf{J}, \boldsymbol{\Theta}; \omega) = \sum_{\mathbf{l}} \delta f_{\mathbf{l}}(\mathbf{J}; \omega) e^{i\mathbf{l} \cdot \boldsymbol{\Theta}}$ and $\mathbf{j}(\mathbf{x}|\mathbf{J}, \boldsymbol{\Theta}) = \sum_{\mathbf{l}'} \mathbf{j}_{\mathbf{l}'}(\mathbf{x}|\mathbf{J}) e^{i\mathbf{l}' \cdot \boldsymbol{\Theta}}$, the integral over the angles is $\int d^3\Theta e^{i(\mathbf{l}+\mathbf{l}') \cdot \boldsymbol{\Theta}} = (2\pi)^3 \delta(\mathbf{l} + \mathbf{l}')$ and consequently only $\mathbf{l}' = -\mathbf{l}$ survives:³

$$\delta \langle \mathbf{j} \rangle(\mathbf{x}; \omega) = (2\pi)^3 \int d^3J \sum_{\mathbf{l}} \mathbf{j}_{\mathbf{l}}^*(\mathbf{x}|\mathbf{J}) \delta f_{\mathbf{l}}(\mathbf{J}; \omega). \quad (3.3)$$

The exact Hamiltonian of a particle is

$$H(\mathbf{p}, \mathbf{r}; t) = \frac{1}{2M} \left[\mathbf{p} - \frac{z}{c} \mathbf{A}(\mathbf{r}, t) \right]^2 + z\phi(\mathbf{r}, t).$$

The perturbed Hamiltonian, for $\mathbf{p} = M\mathbf{v}$, then reads

$$\delta H(\mathbf{p}, \mathbf{r}; t) = -\frac{z}{c} \mathbf{v} \cdot \delta \mathbf{A}(\mathbf{r}, t) + z \delta \phi(\mathbf{r}, t) = -\frac{1}{c} \int d^3x \mathbf{j}(\mathbf{x}|\mathbf{p}, \mathbf{r}) \cdot \delta \mathbf{A}(\mathbf{x}, t) + \int d^3x \rho(\mathbf{x}|\mathbf{p}, \mathbf{r}) \delta \phi(\mathbf{x}, t)$$

Unlike quasistatic perturbations, a time-dependent field allows a gauge in which $\delta\phi = 0$ to

³Physically $\mathbf{j}$ should be assumed real, so that the reality relation $\mathbf{j}_{-\mathbf{l}} = \mathbf{j}_{\mathbf{l}}^*$ follows.

be chosen, which implies $\delta \mathbf{E}(\mathbf{x}, \omega) = \frac{i\omega}{c} \delta \mathbf{A}(\mathbf{x}, \omega)$ and consequently

$$\delta H(\mathbf{J}, \boldsymbol{\Theta}; \omega) = \frac{i}{\omega} \int d^3x \mathbf{j}(\mathbf{x} | \mathbf{J}, \boldsymbol{\Theta}) \cdot \delta \mathbf{E}(\mathbf{x}, \omega),$$

and

$$\delta H_1(\mathbf{J}; \omega) = \frac{i}{\omega} \int d^3x \mathbf{j}_1(\mathbf{x} | \mathbf{J}) \cdot \delta \mathbf{E}(\mathbf{x}, \omega), \quad (3.4)$$

which, substituted together with (3.2) in (3.3), gives

$$\delta \langle \mathbf{j} \rangle(\mathbf{x}, \omega) = \int d^3x' \underline{\underline{\sigma}}'(\mathbf{x}, \mathbf{x}'; \omega) \cdot \delta \mathbf{E}(\mathbf{x}', \omega),$$

where

$$\underline{\underline{\sigma}}(\mathbf{x}, \mathbf{x}'; \omega) = \frac{(2\pi)^3}{M} \frac{i}{\omega} \int d^3J \sum_{\mathbf{l}} \frac{\mathbf{j}_{\mathbf{l}}^*(\mathbf{x} | \mathbf{J}) \mathbf{j}_{\mathbf{l}}(\mathbf{x}' | \mathbf{J})}{\mathbf{l} \cdot \boldsymbol{\omega}(\mathbf{J}) - \omega} \left(\mathbf{l} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \right).$$

The field energy dissipation to particles is given by its Hermitian part in the limit $Im\omega \rightarrow 0+$. Using Plemelj's formula, we get

$$\underline{\underline{\sigma}}'(\mathbf{x}, \mathbf{x}'; \omega) = \frac{(2\pi)^3}{M} \frac{(-1)}{\omega} \int d^3J \sum_{\mathbf{l}} \mathbf{j}_{\mathbf{l}}^*(\mathbf{x} | \mathbf{J}) \mathbf{j}_{\mathbf{l}}(\mathbf{x}' | \mathbf{J}) \left(\mathbf{l} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \right) \pi \delta(\omega - \mathbf{l} \cdot \boldsymbol{\omega}), \quad (3.5)$$

which requires the resonance condition $\mathbf{l} \cdot \boldsymbol{\omega}(\mathbf{J}) - \omega = 0$ to be fulfilled.

3.3 Linear growth rate

The linearized Faraday and Ampère-Maxwell (in Gaussian units) are

$$\begin{aligned}\nabla \times \delta \mathbf{E}(\mathbf{x}, \omega) - \frac{i\omega}{c} \delta \mathbf{B}(\mathbf{x}, \omega) &= 0, \\ \nabla \times \delta \mathbf{B}(\mathbf{x}, \omega) + \frac{i\omega}{c} \delta \mathbf{E}(\mathbf{x}, \omega) &= \frac{4\pi}{c} \int d^3x' \underline{\underline{\sigma}}(\mathbf{x}, \mathbf{x}'; \omega) \cdot \delta \mathbf{E}(\mathbf{x}', \omega).\end{aligned}$$

They can be put together as

$$\underline{\underline{\mathcal{K}}}(\omega) \cdot \delta \mathbf{E}(\mathbf{x}, \omega) = -\delta \mathbf{E}(\mathbf{x}, \omega),$$

with the operator $\underline{\underline{\mathcal{K}}}$ defined through

$$\underline{\underline{\mathcal{K}}}(\mathbf{x}, \omega) \cdot \mathbf{F}(\mathbf{x}, \omega) \equiv -\frac{c^2}{\omega^2} \nabla \times \nabla \times \mathbf{F}(\mathbf{x}, \omega) + \frac{4\pi i}{\omega} \int d^3x' \underline{\underline{\sigma}}(\mathbf{x}, \mathbf{x}'; \omega) \cdot \mathbf{F}(\mathbf{x}', \omega).$$

Now we shall proceed to look at weakly damped or growing waves (as compared to the mode frequency) so that the operator $\underline{\underline{\mathcal{K}}}$ is assumed nearly Hermitian. The frequency is then $\omega_r + i\gamma$ with $\frac{\gamma}{\omega_r} \ll 1$ where ω_r is the real part of the frequency and γ is its imaginary part. The problem can be treated perturbatively by splitting the operator as $\underline{\underline{\mathcal{K}}} = \underline{\underline{\mathcal{K}}} + i\underline{\underline{\mathcal{K}}}''$, with $\underline{\underline{\mathcal{K}}}'$ and $\underline{\underline{\mathcal{K}}}''$ being purely Hermitian and anti-Hermitian, respectively, with $|\underline{\underline{\mathcal{K}}}''| \ll |\underline{\underline{\mathcal{K}}} + i\underline{\underline{\mathcal{K}}}''|$. Expanding $\underline{\underline{\mathcal{K}}}$ around $\omega = \omega_r$, we have

$$\begin{aligned}\underline{\underline{\mathcal{K}}}(\mathbf{x}, \omega) &= \underline{\underline{\mathcal{K}}}(\mathbf{x}, \omega_r + i\gamma) \simeq \underline{\underline{\mathcal{K}}}(\mathbf{x}, \omega_r) - i\gamma \frac{\partial \underline{\underline{\mathcal{K}}}(\mathbf{x}, \omega_r)}{\partial \omega} = \\ &= \underline{\underline{\mathcal{K}}}(\mathbf{x}, \omega_r) - i\gamma \frac{\partial \underline{\underline{\mathcal{K}}}(\mathbf{x}, \omega_r)}{\partial \omega} + i\underline{\underline{\mathcal{K}}}''(\mathbf{x}, \omega_r) + \gamma \frac{\partial \underline{\underline{\mathcal{K}}}''(\mathbf{x}, \omega_r)}{\partial \omega}.\end{aligned}$$

We shall express the electric field as $\delta \mathbf{E} = C(t)\mathbf{e}$, where $C(t)$ is the dimensionless mode amplitude and $\mathbf{e}$ is the eigenfunction. To lowest order of the eigenfunction $\mathbf{e}(\mathbf{x}, \omega_r)$ we then have:

$$\underline{\underline{\mathcal{K}}}'(\mathbf{x}, \omega_r) \cdot \mathbf{e}(\mathbf{x}, \omega_r) = -\mathbf{e}(\mathbf{x}, \omega_r),$$

$$\left[-\gamma \frac{\partial \underline{\underline{\mathcal{K}}}'(\mathbf{x}, \omega_r)}{\partial \omega} + \underline{\underline{\mathcal{K}}}''(\mathbf{x}, \omega_r) \right] \cdot \mathbf{e}(\mathbf{x}, \omega_r) = 0.$$

Multiplying both equations by $\mathbf{e}^*$ from the left side and integrating, we get

$$\int d^3x \mathbf{e}^* (\mathbf{x}, \omega_r) \cdot \underline{\underline{\mathcal{K}}}'(\mathbf{x}, \omega_r) \cdot \mathbf{e}(\mathbf{x}, \omega_r) = - \int d^3x |\mathbf{e}(\mathbf{x}, \omega_r)|^2$$

and

$$\gamma = - \frac{\int d^3x \mathbf{e}^* (\mathbf{x}, \omega_r) \cdot \underline{\underline{\mathcal{K}}}''(\mathbf{x}, \omega_r) \cdot \mathbf{e}(\mathbf{x}, \omega_r)}{\int d^3x \mathbf{e}^* (\mathbf{x}, \omega_r) \cdot \frac{\partial \underline{\underline{\mathcal{K}}}'(\mathbf{x}, \omega_r)}{\partial \omega} \cdot \mathbf{e}(\mathbf{x}, \omega_r)}.$$

$\underline{\underline{\sigma}}'$ and $\underline{\underline{\sigma}}''$ are responsible for the reactive part of the response and for the dissipation, respectively. The mode energy at $C(t) = 1$ is

$$\sigma_a = \frac{\omega_r}{4\pi} \int d^3x \mathbf{e}^* (\mathbf{x}, \omega_r) \cdot \frac{\partial \underline{\underline{\mathcal{K}}}'(\mathbf{x}, \omega_r)}{\partial \omega} \cdot \mathbf{e}(\mathbf{x}, \omega_r)$$

Therefore

$$\gamma = - \frac{\omega_r}{4\pi \sigma_a} \int d^3x \mathbf{e}^* (\mathbf{x}, \omega_r) \cdot \underline{\underline{\mathcal{K}}}''(\mathbf{x}, \omega_r) \cdot \mathbf{e}(\mathbf{x}, \omega_r).$$

Unlike Kaufman, we do not express γ per unit of energy. We have

$$\underline{\underline{\mathcal{K}}}''(\mathbf{x}, \omega) \cdot \mathbf{F}(\mathbf{x}, \omega) = \frac{4\pi}{\omega} \int d^3x' \underline{\underline{\sigma}}'(\mathbf{x}, \mathbf{x}'; \omega) \cdot \mathbf{F}(\mathbf{x}', \omega)$$

and therefore

$$\gamma = -\frac{1}{\sigma_a} \int d^3x' \int d^3x \mathbf{e} * (\mathbf{x}, \omega_r) \cdot \underline{\underline{\sigma}}' (\mathbf{x}, \mathbf{x}'; \omega) \cdot \mathbf{e} (\mathbf{x}', \omega_r)$$

with

$$\underline{\underline{\sigma}}' (\mathbf{x}, \mathbf{x}'; \omega) = \frac{(2\pi)^3}{M} \frac{(-1)}{\omega} \int d^3J \sum_{\mathbf{l}} \mathbf{j}_{\mathbf{l}}^* (\mathbf{x} | \mathbf{J}) \mathbf{j}_{\mathbf{l}} (\mathbf{x}' | \mathbf{J}) \left(\mathbf{l} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \right) \pi \delta (\omega - \mathbf{l} \cdot \boldsymbol{\omega}).$$

Hence

$$\begin{aligned} \gamma &= \frac{(2\pi)^3}{\omega_r M \sigma_a} \int d^3J \sum_{\mathbf{l}} \left(\int d^3x \mathbf{e} * (\mathbf{x}, \omega_r) \cdot \mathbf{j}_{\mathbf{l}}^* (\mathbf{x} | \mathbf{J}) \right) \times \\ &\times \left(\int d^3x' \mathbf{j}_{\mathbf{l}} (\mathbf{x}' | \mathbf{J}) \cdot \mathbf{e} (\mathbf{x}', \omega_r) \right) \left(\mathbf{l} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \right) \pi \delta (\omega - \mathbf{l} \cdot \boldsymbol{\omega}) \end{aligned}$$

or

$$\gamma = \frac{(2\pi)^3}{\omega_r M \sigma_a} \int d^3J \sum_{\mathbf{l}} \alpha_{\mathbf{l}} (\mathbf{J}) \left(\mathbf{l} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \right) \pi \delta (\omega - \mathbf{l} \cdot \boldsymbol{\omega}), \quad (3.6)$$

with the mode structure contribution (matrix elements) given by

$$\alpha_{\mathbf{l}} (\mathbf{J}) \equiv \left| \int d^3x \mathbf{e} (\mathbf{x}, \omega_r) \cdot \mathbf{j}_{\mathbf{l}} (\mathbf{x} | \mathbf{J}) \right|^2 \quad (3.7)$$

3.4 Quasilinear diffusion equation

QL is a mean-field theory and, as such, involves averages over fast oscillations. Therefore its use is only justified when particle motion is randomized within timescales shorter

than timescales for particle trapping to take effect. By averaging over the angles, the QL formulation accounts for a slowly changing $f_0(t)$ while keeping mode structures unchanged, i.e. modes remain linear. Those averages kill the fast correlation described by the ballistic term. Therefore QL theory cannot capture phase-space structures that support chirping modes, which are fully nonlinear phenomena. Using the identity $\frac{\partial}{\partial \Theta} \cdot \dot{\Theta} + \frac{\partial}{\partial \mathbf{J}} \cdot \dot{\mathbf{J}} = 0$, which follows from Hamilton's equations, Vlasov equation for the full f can be put in the form:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \Theta} \cdot (\dot{\Theta} f) + \frac{\partial}{\partial \mathbf{J}} \cdot (\dot{\mathbf{J}} f) = 0,$$

which, averaged over Θ , gives

$$\frac{\partial}{\partial t} \langle f \rangle (\mathbf{J}; t) = - \frac{\partial}{\partial \mathbf{J}} \cdot \langle \delta \dot{\mathbf{J}} \delta f \rangle. \quad (3.8)$$

The lowest (zeroth) order f is assumed to depend only on the actions $\mathbf{J}$ in addition to a slow dependence in time, $\langle f \rangle = \langle f \rangle (\mathbf{J}; t)$ while δf is the perturbed (first order) component. Now we need all quantities expressed with their time dependencies (not Fourier transformed), which makes the algebra more involved. The solution of (3.1) can be formally written as⁴

$$\delta f (\mathbf{J}, \Theta; t) = - \int_0^t d\tau \delta \dot{\mathbf{J}} (\mathbf{J}, \Theta - \omega \tau; t - \tau) \cdot \frac{\partial f_0}{\partial \mathbf{J}} \quad (3.9)$$

⁴This can be seen by noticing that inside the integral in (3.9), $\delta \dot{\mathbf{J}} (\mathbf{J}, \Theta - \omega \tau; t - \tau) = \frac{d}{d(t-\tau)} \delta \mathbf{J} (\mathbf{J}, \Theta - \omega \tau; t - \tau) = - \frac{d}{d\tau} \delta \mathbf{J} (\mathbf{J}, \Theta - \omega \tau; t - \tau)$. Using the fundamental theorem of calculus, setting the initial condition to zero and using Hamilton's equations, it becomes straightforward to check that this form of solution indeed satisfies (3.1).

Therefore, a diffusion equation can be written

$$\frac{\partial}{\partial t} f_0(\mathbf{J}; t) = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\underline{\underline{D}}'(\mathbf{J}; t) \cdot \frac{\partial f_0}{\partial \mathbf{J}} \right] \quad (3.10)$$

with the diffusion coefficient being given by

$$\underline{\underline{D}}'(\mathbf{J}; t) = \frac{1}{M^2} \sum_{\mathbf{l}} \mathbf{l} \int d^3x \int d^3x' \mathbf{j}_{\mathbf{l}} * (\mathbf{x}|\mathbf{J}) \mathbf{j}_{\mathbf{l}}(\mathbf{x}'|\mathbf{J}) : \int_0^\infty d\tau \delta \mathbf{A}(\mathbf{x}, t) \delta \mathbf{A}(\mathbf{x}', t - \tau) e^{-i\mathbf{l} \cdot \boldsymbol{\omega} \tau}$$

or

$$\underline{\underline{D}}'(\mathbf{J}; t) = \frac{2\pi}{M^2} \sum_n \frac{C_n^2(t)}{\omega_n^2} \sum_{\mathbf{l}} \mathbf{l} \delta(\omega_n - \mathbf{l} \cdot \boldsymbol{\omega}(\mathbf{J})) \alpha_{\mathbf{l}}^n(\mathbf{J}) \quad (3.11)$$

The diffusion coefficient is proportional to the mode amplitude squared, so the transport is expected to be more pronounced close to the peaks of the mode structure. EPs are usually concentrated at the center, so they will mostly interact via resonant curves that are present there. Nonlinear effects naturally broaden the resonances and they mostly overlap in regions where the mode amplitude is greater, since the resonant island is proportional to the bounce frequency.

As nonlinear Landau damping takes place, the equations contain a ballistic response term (before the QL approximation is made). This term retains information such as the wave frequency even when the wave gets completely extinguished. It gives rise to plasma echos, which were the first experimental confirmation of Landau damping [93]. QL theory relies on averaging out over short timescales, which implies washing out the ballistic term. This averaging procedure implies that some stochastic randomization mechanism (intrinsic due to mode overlap or background turbulence or extrinsic due to collisions) are

invoked to justify smoothing out the distribution (i.e. averaging over fast oscillations). As a consequence, QL theory misses mode coupling, kills phase space correlations and does not conserve entropy. In other words, in QL theory, particles need to forget their phases from one bounce to another. Therefore formation of coherent phase-space structures cannot be captured.

Chapter 4

NOVA code as input for quasilinear modeling

4.1 Quadratic form and kinetic-MHD perturbative approach

By perturbing MHD equations, one can write an eigenvalue equation like $\underline{\underline{F}} \cdot \boldsymbol{\xi} = \omega^2 \boldsymbol{\xi}$ for the ideal fluid displacement $\boldsymbol{\xi}$, with $\underline{\underline{F}}$ being a force-like operator. For the special case of an ideal equilibrium in the absence of flows, it can be shown that $\underline{\underline{F}}$ has the property of being self-adjoint (Hermitian). In this case, $\omega^2 \in \mathbb{R}$ which implies that ω is either real or purely imaginary. Then, considerations regarding the stability of a system can be reduced by analyzing the sign of ω^2 using a variational energy principle [94]. However, more realistic models, such as resistive MHD, kinetic MHD or MHD with flows, break the Hermitianity

and a variational principle can no longer be established.

NOVA [27, 20, 69] is a nonvariational, ideal MHD code primarily used to integrate non-Hermitian eigenmode equations in the presence of EPs, using a general flux coordinate system. NOVA-K is a stability code used to study the destabilization of TAEs by EPs free energy stored in the gradients of the distribution. The resonance response of energetic particles enter the system through the perturbed pressure associated with them. NOVA makes no use of inverse aspect ratio approximation and hence is well suited to study spherical tokamaks. The code uses Fourier expansion in θ and cubic spline finite elements in the radial ψ direction. We are interested in using the code for calculating the growth rate of TAEs in the presence of EPs. Following the procedure of Cheng [69] we may start with the MHD formalism where the fast particle contribution is treated perturbatively, with all perturbed quantities represented in the form

$$A(\mathbf{r}, t) = \sum_m A_m e^{i(S_m - \omega t)}; \quad S_m \equiv m\theta - n\varphi,$$

We shall start with the momentum conservation equation. This equation can be put in a quadratic form if dot multiplied by $\boldsymbol{\xi}^*$ and integrated over the whole plasma volume. If eqs. (3.15) of [27] are used we get

$$D(\omega) = \delta W_f + \delta W_k - \delta K = 0,$$

where the inertial (kinetic) energy is given by $\delta K = \omega^2 \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r}$. δW_f is the total fluid potential energy and δW_k is the EPs potential energy. ¹The quadratic form is particularly

¹In NOVA-K, both potential and kinetic energies are defined as twice their actual values. This can be seen from comparing equations (3.69) and (3.70) of [69] with equations (4.19) and (4.31) of [9]. This choice

useful when stability analysis is addressed. For example, in the study of EP-TAE resonances, if the mode frequency is assumed to be $\omega = \omega_r + i\gamma$, with $|\gamma| \ll |\omega_r|$, the growth rate is then found to be [69]

$$\gamma \approx \frac{\Im \delta W_k}{2\delta K} \omega_r.$$

In the last equation, it was already used that the inertial energy is close to the mode energy, as shown in the next section. $\Re$ and $\Im$ denote the real and imaginary parts, respectively. $\delta K = \omega^2 \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r}$ is just a number (also provided by NOVA-K), being a global factor for each toroidal mode number n . Since ρ is the total plasma density, it is hardly affected by the fast ions density. For TAEs, the growth rate is not simple, being an integral over the resonant curve in phase space and depending on the mode structure. In order to compute $\Im \delta W_k$, and consequently γ , the non-adiabatic part of the distribution function, g , must be calculated. This function can be redefined as $\hat{g}$ to satisfy

$$\frac{d\hat{g}}{dt} = \frac{z}{M} \frac{\partial f}{\partial \mathcal{E}} (\omega - \omega_*) \hat{X},$$

where

$$\omega_* = -i \frac{\frac{\partial f}{\partial P_\varphi}}{\frac{\partial f}{\partial \mathcal{E}}} \frac{\partial}{\partial \varphi} \rightarrow n \frac{\frac{\partial f}{\partial P_\varphi}}{\frac{\partial f}{\partial \mathcal{E}}}$$

is a plasma diamagnetic frequency-like, being a measure of the relation of the radial gradient in the energetic in EP profiles to the velocity gradient. Since $dP_\varphi = -\frac{qr}{\omega_c} dr$ and the EP density decreases with radius, we have $\frac{\partial f}{\partial P_\varphi} > 0$. On the other hand, the average energy of

does not introduce changes to the growth rate.

EPs also decreases with radius, which implies $\frac{\partial f}{\partial \mathcal{E}} < 0$. Therefore, in a tokamak, the free energy stored in the radial gradient drives the mode while the negative gradient in energy tends to stabilize the mode. The mode is unstable when $\omega_* > \omega$. $\hat{X}$ is given by

$$\hat{X} = \left(\frac{d\mathbf{r}_c}{dt} - v_{\parallel} \hat{b} \right) J_0 \nabla \phi_c + \frac{i\mu\omega M}{z} \frac{df}{d\mu} (J_0 + J_2) B_{\parallel c},$$

where the subscript c means that the quantity is evaluated at the particle gyrocenter. The Bessel functions are understood to have the argument $v_{\perp} \nabla_{\perp} / \omega_c$, which operates on the perturbed quantities². If $\mathbf{E}_{\perp} = -\nabla \phi$ ($\mathbf{A}_{\perp}$ contribution neglected) we can write

$$\hat{X} \simeq \frac{i\omega M}{z} [(2\mathcal{E} - 3\mu B) J_0 \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_{\perp} - \mu B J_0 \nabla \cdot \boldsymbol{\xi}_{\perp}],$$

where $\boldsymbol{\kappa}$ stands for the curvature.

4.2 Growth rate as a result of the nonadiabatic component of the distribution function

The solution of the DKE is given by

$$\hat{g} = \frac{z}{M} \int_0^t \frac{\partial f}{\partial \mathcal{E}} (\omega - \omega_*) \hat{X} dt',$$

where the time integration is along the particle trajectory. Assuming $S_m(t=0) = 0$, one

² $v_{\perp} \nabla_{\perp} / \omega_c \sim v_{\perp} k_{\perp} / \omega_c \ll 1$ which justifies the Taylor expansion of the Bessel functions, $J_{\alpha}(v_{\perp} \nabla_{\perp} / \omega_c) \rightarrow \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j+\alpha+1)} (v_{\perp} \nabla_{\perp} / \omega_c)^{2j+\alpha}$.

can write in terms of the following Fourier series

$$\hat{X} = \sum_{m,l} e^{-i\omega t + i\bar{\omega}_{Dm}t} X_{ml} e^{il\omega_b t},$$

with

$$X_{ml} = \frac{1}{\tau_b} \oint dt' \hat{X} e^{i \int^{t'} (\omega_{Dm} - \bar{\omega}_{Dm} - l\omega_b) dt''}.$$

$\Im \delta W_k$ can then be calculated from

$$\delta W_k = - \int \int \hat{g} \mathbf{v} \cdot (\mathbf{v} \cdot \nabla) \xi^* d\mathbf{r}_c d\mathbf{v}.$$

Hence

$$\delta W_k = -\frac{iz}{\omega} \int \int \hat{X}^* \hat{g} d\mathbf{r}_c d\mathbf{v}.$$

The phase-space integration is given by

$$\begin{aligned} \int d\Gamma \dots &= \int d^3J \int d^3\Theta \dots = (2\pi)^3 \int d^3J \sum_l \dots = \\ &= \int d\mathbf{r}_c \int d\mathbf{v} \dots = (2\pi)^2 \sum_{\sigma_{\parallel}} \frac{B}{\omega_c} \int dP_{\varphi} \int d\mathcal{E} \int d\mu \int dt \dots, \end{aligned}$$

with dt being associated to the fast particle orbital motion. The integrals over φ and θ_g will contribute with 2π each. Note that in NOVA, P_{φ} , $\mathcal{E}$ and μ are defined without the mass. If they had usual units, we would have $\int d\mathbf{r}_c \int d\mathbf{v} \dots = (2\pi)^2 \left(\frac{c}{zM^2}\right) \sum_{\sigma_{\parallel}} \int dP_{\varphi} \int d\mathcal{E} \int d\mu \int dt \dots$.

$$\hat{g} = \frac{iz}{M} \sum_{m,l} \frac{e^{-i\omega t + i\bar{\omega}_{Dm}t + il\omega_b t}}{\omega - \bar{\omega}_{Dm} - l\omega_b} \frac{\partial f}{\partial \mathcal{E}} (\omega - \omega_*) X_{ml},$$

So, Eq. 26 of [27] reads

$$\delta W_k = -\frac{(2\pi M)^2 c\omega}{z} \sum_{\sigma_{\parallel}} \int dP_{\varphi} d\mu d\mathcal{E} dt \sum_{m,m',l,l'} G_{m'l}^* \frac{\mathcal{E}^2 \left(\frac{\partial f}{\partial \mathcal{E}}\right) (1 - \omega_*/\omega)}{\omega - \bar{\omega}_{Dm} - l\omega_b} G_{ml} e^{-it(\bar{\omega}_{Dm'} - \bar{\omega}_{Dm} + (l' - l)\omega_b)},$$

where the “drift” frequency is

$$\omega_{Dm} \equiv \frac{dS_m}{dt}.$$

$\bar{\omega}_{Dm}$ means orbit averaged drift frequency, and

$$G_{ml} = -i \frac{z X_{ml}}{\omega M \mathcal{E}}, \quad G_{ml}^* = i \frac{z X_{ml}^*}{\omega M \mathcal{E}}.$$

For trapped particles the drift trajectory does not encircle the magnetic axis, thus

$$\bar{\omega}_{Dm} = \bar{\omega}_{Dm}|_{m=0} \equiv \bar{\omega}_{D0}.$$

So the time integral becomes $\int dt e^{-it(l' - l)\omega_b} = \int_0^{2\pi} \frac{d(\omega_b t)}{\omega_b} e^{-i(\omega_b t)(l' - l)} = \tau_b \delta_{l,l'}$; $\tau_b \equiv \frac{2\pi}{\omega_b}$ (constant in time, i.e., time-averaged). Consequently, the sum over l' could be eliminated. Moreover, as long as the imaginary part of the frequency is regarded as a small number compared to the real frequency, Plemelj formula³ could be used to develop the denominator. So, the imaginary part of δW_k becomes⁴

³Plemelj formula: $\lim_{\epsilon \rightarrow 0} \frac{1}{x \pm i\epsilon} = \mp i\pi \delta(x) + P \frac{1}{x}$, where P stands for the Cauchy principal value of the integral.

⁴Since only the imaginary part of δW_k contributes to the growth rate, we only keep the term involving the delta function.

$$\Im \delta W_k = \frac{(2M)^2 \pi^3 c \omega_r}{z} \int dP_\varphi d\mu d\mathcal{E} \sum_{m,m',l} G_{m'l}^* \mathcal{E}^2 \tau_b \left(\frac{\partial f}{\partial \mathcal{E}} \right) (1 - \omega_*/\omega_r) G_{ml} \delta(\omega_r - \bar{\omega}_{D0} - l\omega_b)$$

Since

$$\gamma \approx \frac{\Im \delta W_k}{2\delta K} \omega_r,$$

then, it can be written as

$$\gamma = \frac{(2M)^2 \pi^3 c \omega_r^2}{2z\omega_r^2 \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r}} \sum_{\sigma_{\parallel}} \int dP_\varphi d\mu d\mathcal{E} \sum_{m,m',l} G_{m'l}^* \mathcal{E}^2 \tau_b \left(\frac{\partial f}{\partial \mathcal{E}} \right) (1 - \omega_*/\omega_r) G_{ml} \delta(\omega_r - \bar{\omega}_{D0} - l\omega_b). \quad (4.1)$$

4.3 Inertial energy and the energy stored in the fields

Alfvén-like waves involve negligible electric field perturbations. Therefore their energy, in S.I. units,

$$W = W_{fields} + W_{particles} = \int \frac{|\delta \mathbf{B}|^2}{2\mu_0} dV + \int \frac{\rho |\dot{\boldsymbol{\xi}}|^2}{2} dV,$$

is nearly equally divided between the perturbed magnetic energy and the kinetic energy of particles oscillating as a result of the perturbation. This can be noticed if one considers ideal Ohm's law and Faraday's law via an Eulerian fluid picture. Using that $\dot{\boldsymbol{\xi}} \perp \mathbf{B}_0$ and $\mathbf{k} \perp (\dot{\boldsymbol{\xi}} \times \mathbf{B}_0)$ for Alfvén waves, we have

$$\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times (\dot{\boldsymbol{\xi}} \times \mathbf{B}_0) \Rightarrow \omega_A |\delta \mathbf{B}| = k |\dot{\boldsymbol{\xi}}| B_0.$$

Since $v_A = \frac{\omega_A}{k} = \frac{B_0}{\sqrt{\mu_0 \rho}}$,

$$\frac{W_{fields}}{W_{particles}} \approx \frac{|\delta \mathbf{B}|^2}{\rho \mu_0 |\dot{\boldsymbol{\xi}}|^2} = 1.$$

The expression for the inertial energy used in NOVA, $\delta K = \omega^2 \int \rho |\boldsymbol{\xi}|^2 dV$,⁵ takes into account the displacement due to all harmonics of a given mode.

4.4 Correspondence between NOVA's and Kaufman's notations for mode structure information: the relation between G_{ml} and $\int d^3x \mathbf{e}(\mathbf{x}, \omega_r) \cdot \mathbf{j}_l(\mathbf{x} | \mathbf{J})$

Now we want to compare the growth rate in NOVA and in Kaufman's normal mode theory, in order to be able to relate their respective mode structure information (G and α_l) and build a quasilinear theory based on that. Equations (3.6) and (3.7) read

$$\gamma = \frac{(2\pi)^3}{\omega_r M \sigma_a} \int d^3J \sum_l \alpha_l(\mathbf{J}) \left(\mathbf{l} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \right) \pi \delta(\omega - \mathbf{l} \cdot \boldsymbol{\omega}),$$

⁵In NOVA the inertial energy is defined as twice the actual kinetic energy of particles oscillating as a result of the field perturbation.

with the mode structure contribution (matrix elements) given by

$$\alpha_1(\mathbf{J}) \equiv \left| \int d^3x \mathbf{e}(\mathbf{x}, \omega_r) \cdot \mathbf{j}_1(\mathbf{x} | \mathbf{J}) \right|^2.$$

Using $\int d^3J \int d^3\Theta \dots = (2\pi)^3 \int d^3J \sum_l \dots = \int d\mathbf{r}_c \int d\mathbf{v} \dots = (2\pi)^2 \sum_{\sigma_{\parallel}} \frac{B}{\omega_c} \int dP_{\varphi} \int d\mathcal{E} \int d\mu \int dt \dots$,

we get

$$\gamma = \frac{2^2 \pi^3}{\omega_r M \sigma_a} \sum_{\sigma_{\parallel}} \frac{B}{\omega_c} \int dP_{\varphi} \int d\mathcal{E} \int d\mu \tau_b \sum_1 \left| \int d^3x \mathbf{e}(\mathbf{x}, \omega_r) \cdot \mathbf{j}_1(\mathbf{x} | \mathbf{J}) \right|^2 \underbrace{\left(\mathbf{l} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \right)}_{\omega \frac{\partial f}{\partial \mathcal{E}} + n \frac{\partial f}{\partial P_{\varphi}}} \delta(\omega - \mathbf{l} \cdot \boldsymbol{\omega})$$

or

$$\gamma = \frac{2^2 \pi^3 c}{z \sigma_a} \sum_{\sigma_{\parallel}} \int dP_{\varphi} \int d\mathcal{E} \int d\mu \tau_b \sum_1 \left| \int d^3x \mathbf{e}(\mathbf{x}, \omega_r) \cdot \mathbf{j}_1(\mathbf{x} | \mathbf{J}) \right|^2 \left(\frac{\partial f}{\partial \mathcal{E}} + \frac{n}{\omega} \frac{\partial f}{\partial P_{\varphi}} \right) \delta(\omega - \mathbf{l} \cdot \boldsymbol{\omega}).$$

The growth rate in NOVA is given by (4.1):

$$\gamma = \frac{(2M)^2 \pi^3 c \omega^2}{2z \omega^2 \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r}} \sum_{\sigma_{\parallel}} \int dP_{\varphi} d\mu d\mathcal{E} \sum_{m, m', l} G_{m'l}^* \mathcal{E}^2 \tau_b \underbrace{\left(\frac{\partial f}{\partial \mathcal{E}} \right)}_{\frac{\partial f}{\partial \mathcal{E}} + \frac{n}{\omega} \frac{\partial f}{\partial P_{\varphi}}} (1 - \omega_*/\omega) G_{ml} \delta(\omega - \bar{\omega}_{D0} - l\omega_b).$$

Using $\sigma_a = \omega^2 \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r}$, the comparison leads to⁶

$$\boxed{\frac{M \mathcal{E} \omega}{\sqrt{2}} \sum_m G_{ml} = \int d^3x \mathbf{e}(\mathbf{x}, \omega) \cdot \mathbf{j}_1(\mathbf{x} | \mathbf{J}) = \frac{\omega}{i} V_1} \quad (4.2)$$

⁶Note that NOVA, being a linear code, does not have an amplitude C embedded in the fluid displacement $\boldsymbol{\xi}$.

or, alternatively

$$\alpha_1(\mathbf{J}) = \left(\frac{M\mathcal{E}\omega}{\sqrt{2}} \right)^2 \sum_{m,m'} G_{m'l}^* G_{ml} = \omega^2 |V_1|^2 \quad (4.3)$$

Since

$$\begin{aligned} g_1 &= \int \frac{d^3\Theta}{(2\pi)^3} g(\Theta) e^{-i\mathbf{l}\cdot\Theta}, \\ \mathbf{j}(\mathbf{x}|\mathbf{J}, \Theta) &= z\mathbf{v}(\mathbf{J}, \Theta) \delta[\mathbf{x} - \mathbf{r}(\mathbf{J}, \Theta)], \\ \mathbf{j}(\mathbf{x}|\mathbf{J}, \Theta) &= \sum_{\mathbf{l}} \mathbf{j}_{\mathbf{l}}(\mathbf{x}|\mathbf{J}) e^{i\mathbf{l}\cdot\Theta}, \end{aligned}$$

it follows that

$$\mathbf{j}_{\mathbf{l}}(\mathbf{x}|\mathbf{J}) = \int \frac{d^3\Theta}{(2\pi)^3} z\mathbf{v}(\mathbf{J}, \Theta) \delta[\mathbf{x} - \mathbf{r}(\mathbf{J}, \Theta)] e^{-i\mathbf{l}\cdot\Theta}$$

and consequently

$$\int d^3x \mathbf{e}(\mathbf{x}, \omega) \cdot \mathbf{j}_{\mathbf{l}}(\mathbf{x} | \mathbf{J}) = z \int \frac{d^3\Theta}{(2\pi)^3} \mathbf{e}(\mathbf{r}, \omega) \cdot \mathbf{v}(\mathbf{J}, \Theta) e^{-i\mathbf{l}\cdot\Theta}.$$

Therefore

$$\int d^3x \mathbf{e}(\mathbf{x}, \omega) \cdot \mathbf{j}_{\mathbf{l}}(\mathbf{x} | \mathbf{J}) = \frac{\omega}{i} V_1$$

implies

$$V_l = \frac{iz}{\omega} \int \frac{d^3\Theta}{(2\pi)^3} \mathbf{e}(\mathbf{r}, \omega) \cdot \mathbf{v}(\mathbf{J}, \Theta) e^{-i\mathbf{l}\cdot\Theta},$$

which is consistent with the expression given in [62].

Mode structure is provided in NOVA via the G matrices. They are calculated via time integration that account for the eigenmode felt by a particle in a given trajectory. A weighted integration is performing according to how much time a particle spends at each

position of its trajectory. Mirror-trapped particles have their parallel velocity reversed at the tips of a banana orbit, where they tend to remain longer and consequently the local mode structure will have an important contribution for the overall integration. By using equation (4.2) we can now express the quasilinear diffusion equation, (3.10) and (3.11), in terms of NOVA code notation.

Chapter 5

1D slanted RBQ model

In this Chapter, we present the model behind the Resonance Broadened Quasilinear code, along with the finite-difference method used to solve the system and with numerical results. Using (1.2), the 1D quasilinear equations written in NOVA notation, are

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial I} \left(\sum_{n,l,m,m'} D(I;t) \right) \frac{\partial}{\partial I} f + C[f]; \quad D(I;t) = \pi n^2 C_n^2(t) \mathcal{E}^2 \frac{\delta(I - I_r)}{|\frac{\partial \Omega_l}{\partial I}|} G_{m'l}^* G_{ml}.$$

$C[f]$ is a collisional operator acting on the distribution function. The equation for the amplitude of each mode can be written formally without the explicit contributions from other modes. The amplitudes satisfy the equation from linear theory and can be written as

$$\frac{dC_n^2(t)}{dt} = 2(\gamma_{L,n} - \gamma_{d,n}) C_n^2(t), \quad (5.1)$$

where $\gamma_{L,n}$ is the linear growth of the mode and $\gamma_{d,n}$ is the wave damping rate in the

absence of an instability source. $\gamma_{L,\alpha}$ is given by (4.1). The broadening of the resonance can be performed by choosing $\mathcal{F}$ as the **window function** with width ΔI that satisfies $\int_{-\infty}^{\infty} \mathcal{F} dI = 1$ (in principle could be taken as a flat-top function),

$$D(I; t) = \pi n^2 C_n^2(t) \mathcal{E}^2 \frac{\mathcal{F}(I - I_r)}{\left| \frac{\partial \Omega_l}{\partial I} \right|} G_{m'l}^* G_{ml}.$$

The derivatives $\frac{\partial \Omega_l}{\partial I}$ are provided by NOVA outputs.

5.1 Resonance broadening

QL theory assumes that mode amplitudes remain small and therefore particles follow unperturbed orbits. In QL theory, particles are in resonance only if they exactly satisfy the resonance condition. This implies that resonant particles can only diffuse over a point, which is clearly an ill-posed problem. Dupree [95] realized that the turbulent spectrum contributes to diffuse particles away from their original unperturbed trajectories¹. In a pioneering work, his theory accounted for the orbit diffusion due to trapped particle deviation from their original path by means of the resonance broadening. Incorporating resonant island width into QL theory introduces an additional nonlinearity in the problem. The broadened width, that replaces a delta function on the resonance, is a more realistic platform that allows for momentum and energy exchange between particles and waves via Landau damping.

¹In principle, any mechanism that leads to phase randomization should contribute to the broadening.

5.1.1 Parametric dependencies of the frequency broadening $\Delta\Omega$ around resonances

For the RBQ, the window width is chosen to be proportional to

- the difference between linear growth and damping rate ($\gamma = |\gamma_L - \gamma_d|$),² as expected for a wave treated by ordinary quasilinear theory. As long as an imaginary part of the frequency is accounted for, the diffusion coefficient naturally contains a Lorentzian (Cauchy) distribution, which has the property of having a characteristic height of $1/\gamma$ and a full width equals to 2γ at half maximum. The broadening based on γ collapses to a delta function when $\gamma \rightarrow 0$, i.e. when a mode reaches saturation;

$$\pi\delta(\omega - \omega_0) \xrightarrow{\gamma \neq 0} \frac{\gamma}{(\omega - \omega_0)^2 + \gamma^2},$$

where ω_0 is the location parameter that specifies the location of the peak of the distribution

- the separatrix width, as expected for a wave treated by single-mode theory. In phase space, particles that exchange energy with the mode are trapped and bounded by a separatrix of width $2\omega_{b,n}$. Each particle satisfy a nonlinear pendulum equation with

²The overall damping rate is the sum of several contributions, such as electron collisional damping, thermal electron Landau damping, radiative damping, continuum damping and ion Landau damping. The components are calculated in NOVA-K code using approximate analytic expressions.

The radiative damping is normally zero or very close to zero. It relies on the coupling between global and kinetic AEs. The kinetic ones have wavelengths of order of the ion Larmor radius. It turns out that the global ones can only resonate with the kinetic ones for high toroidal mode numbers n . For ITER n could be > 20 and the radiative damping will be quite important. If n is small, it is normal that the radiative damping gives zero for most cases.

a given bounce frequency ω_b related to its energy, which leads to phase mixing ;

- the effective collisional frequency ν_{eff} since collisions imply that particles are redistributed, being kicked in and out of the separatrix, which leads to particle decorrelating the resonance. This increases the effective range of the resonance region since more particles are allowed to interact with the mode via the resonant platform. The presence of collisions is required by QL theory in order to guarantee enough phase information erasure, to assure that the ballistic term can be averaged out to zero.

Consequently

$$\Delta\Omega = a\omega_{b,n} + b|\gamma_n| + c\nu_{scatt}. \quad (5.2)$$

5.1.2 Expected saturation levels from single mode perturbation theory

Analytic results far from and close to marginal stability can be used in order to choose the parameters a , b and c . We should note that the assumed expression for $\Delta\Omega$, eq. (5.2), could in principle be written more generally in terms of a weighted sum, as used in [39, 38]. For a single, isolated mode in a simplified bump-on-tail situation,

$$\frac{\partial f}{\partial t} = \frac{\pi}{2} \frac{\partial}{\partial \Omega} \omega_b^4 \mathcal{F} \frac{\partial}{\partial \Omega} f + \nu_{scatt}^3 \frac{\partial^2 (f - f_0)}{\partial \Omega^2}.$$

At saturation ($\gamma_L \simeq \gamma_d$, $\frac{\partial f}{\partial t} = 0$),

$$\frac{\pi}{2} \omega_b^4 \mathcal{F} \frac{\partial f}{\partial \Omega} + \nu_{scatt}^3 \frac{\partial(f - f_0)}{\partial \Omega} = 0.$$

Exploiting the fact that the bump-on-tail problem consists of a single resonance point,

$$\frac{\gamma_L(t)}{\gamma_{L,0}} = \frac{\frac{\partial f}{\partial \Omega}}{\frac{\partial f_0}{\partial \Omega}},$$

therefore

$$\frac{\pi}{2} \omega_b^4 \mathcal{F} = \frac{\gamma_{L,0} - \gamma_d}{\gamma_d} \nu_{scatt}^3.$$

For $\mathcal{F} = 1/\Delta\Omega$,

$$\Delta\Omega = \frac{\pi}{2} \frac{\omega_b^4}{\nu_{scatt}^3} \frac{\gamma_d}{\gamma_{L,0} - \gamma_d}. \quad (5.3)$$

Near marginal stability

For this case $\Delta\Omega \approx c\nu_{scatt}$. The expected saturation level from analytical theory for the case $\omega_b/\nu_{scatt} \ll 1$ is [70, 38]

$$\omega_b \simeq 1.18 \nu_{scatt} \left(\frac{\gamma_L}{\gamma_d} - 1 \right)^{1/4},$$

which, when substituted in (5.3) leads to

$$a = 2.7.$$

Far from marginal stability

For this case $\Delta\Omega \approx a\omega_b$. The expected saturation level from analytical theory for the case $\omega_b/\nu_{scatt} \gg 1$ is [70, 38]

$$\omega_b \simeq 1.2\nu_{scatt} \left(\frac{\gamma_L}{\gamma_d} \right)^{1/3},$$

which, when substituted in (5.3) leads to

$$c = 2.5.$$

There are no known analytical solutions to rely on in order to determine the constant b . It was simply taken equal to c in previous works [39, 38, 37, 46].

5.1.3 QL equations with a broadened coefficient

Using (1.2), the 1D QL equations written in NOVA notation, are

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial I} \left(\sum_{n,l,m,m'} D(I;t) \right) \frac{\partial}{\partial I} f + \frac{\nu_{scatt}^3}{\left(\left| \frac{\partial \Omega_l}{\partial I} \right|_{I_r} \right)^2} \frac{\partial^2 (f - f_0)}{\partial I^2},$$

$$D(I;t) = \pi n^2 C_n^2(t) \mathcal{E}^2 \frac{\delta(I - I_r)}{\left| \frac{\partial \Omega_l}{\partial I} \right|} G_{m'l}^* G_{ml}.$$

$C[f]$ is a collisional operator acting on the distribution function. The equation for the amplitude of each mode can be written formally without the explicit contributions from other modes. The amplitudes satisfy the equation from linear theory and can be written as

$$\frac{dC_n^2(t)}{dt} = 2(\gamma_{L,n} - \gamma_{d,n}) C_n^2(t), \quad (5.4)$$

where $\gamma_{L,n}$ is the linear growth of the mode and $\gamma_{d,n}$ is the wave damping rate in the absence of an instability source. $\gamma_{L,\alpha}$ is given by (4.1). The broadening of the resonance can be performed by choosing $\mathcal{F}$ as the **window function** with width ΔI that satisfies $\int_{-\infty}^{\infty} \mathcal{F} dI = 1$ (in principle could be taken as a flat-top function),

$$D(I; t) = \pi n^2 C_n^2(t) \mathcal{E}^2 \frac{\mathcal{F}(I - I_r)}{\left| \frac{\partial \Omega_l}{\partial I} \right|} G_{m'l}^* G_{ml}.$$

The derivatives $\frac{\partial \Omega_l}{\partial I}$ are provided by NOVA outputs. It only makes sense to think of $\frac{\partial \Omega_l}{\partial P_\varphi}$ at the resonant curve, so we will not account for its variation at constant $\mathcal{E}$ for a given resonant curve. This equation needs to be solved for each value of $\mathcal{E}$ present in the coarse grid. When mode structure information is taken into account, $\omega_{b,n} = \omega_{b,n}(P_\varphi, \mathcal{E}, \mu)$.

Before broadening the resonance, the growth rate is

$$\gamma_{L,n} = \frac{(2M)^2 \pi^3 c \omega^2}{2z\omega^2 \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r}} \sum_{\sigma_{\parallel}} \int dP_\varphi d\mu d\mathcal{E} \sum_{m,m',l} G_{m'l}^* \mathcal{E}^2 \tau_b \underbrace{\left(\frac{\partial f}{\partial \mathcal{E}} \right)}_{\frac{\partial f}{\partial \mathcal{E}} + \frac{n}{\omega} \frac{\partial f}{\partial P_\varphi}} (1 - \omega_*/\omega) G_{ml} \delta(\omega - \bar{\omega}_{D0} - l\omega_b)$$

The way the numerical scheme for computing $\gamma_{L,n}$ was designed in NOVA, it is convenient to use the delta function to eliminate the energy integral. We first integrate over $v = \sqrt{2\mathcal{E}}$ before broadening the resonance.

$$\gamma_{L,n} = \frac{\pi n \omega_n}{4\omega_n^2 \int \rho \frac{|\boldsymbol{\xi}|^2}{2} dV} \frac{\tau_b M^3 c}{z} \sum_{l,m,m'} \int dP_\varphi d\mu \left(\frac{\sqrt{2}\mathcal{E}^{5/2} G_{m'l}^* G_{ml}}{(\mathbf{1} \cdot \boldsymbol{\Omega})_v} \frac{\partial}{\partial P_\varphi} f \right) \Big|_{v_{res}}$$

$(\mathbf{1} \cdot \boldsymbol{\Omega})_v$ means that the denominator is being derived with respect to v . All the integrand is now evaluated at v_{res} . The idea of the broadening is to integrate along the resonance, allowing some width in the direction of P_φ .

5.2 Predicted responses

The time evolution of the mode amplitude could be steady or pulsating. If one starts with a small mode amplitude and an inverted distribution function, the mode will grow up to the point when $\gamma_L = \gamma_d$ and $\frac{dC_n^2(t)}{dt} = 0$. When the drive is exhausted, C_n decreases as a result of a constant γ_d . Low mode amplitude implies low diffusion coefficient. Therefore the change in the distribution function during the low-field phase becomes dominated by collisional processes, that tend to build up (restore) the distribution function again while keeping the diffusion coefficient low. When the distribution acquires large enough gradients, the same cycle restarts, which leads to a pulsating response. If collisions are strong enough to build up the distribution while keeping a drive that compensates for the background damping, the mode is able to reach a quasi-steady saturation that scales with some power of ν_{eff} , depending on the relevant dynamics involved in the saturation process.

5.3 Numerical slanted scheme

5.3.1 Discretized equations

Although in principle one can arbitrarily discretize the system of equations to solve numerically, there are some restrictions that are implied by physical considerations. In

order to conserve the system momentum (i.e., particle plus wave momenta) at all times, the discretized equations must guarantee internal self-consistency by adopting the time flow presented in the following. In other words, this flow is chosen in such a way that both particle diffusion and mode amplitude evolution are calculated using the same window function at each time step.

Figura 5.1: Grids for different quantities entering the finite-difference problem.

RBQ equations are

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial I} \left(\sum_{n,l,m,m'} D(I;t) \right) \frac{\partial}{\partial I} f + \frac{\nu_{scatt}^3}{\left(\left| \frac{\partial \Omega_l}{\partial I} \right|_{I_r} \right)^2} \frac{\partial^2 (f - f_0)}{\partial I^2},$$

where ν_{scatt}^3 is given in eq. A.4 of the Appendix and

$$D(I;t) = \pi n^2 C_n^2(t) \mathcal{E}^2 \frac{\mathcal{F}(I - I_r)}{\left| \frac{\partial \Omega_l}{\partial I} \right|} G_{m'l}^* G_{ml},$$

$$\gamma_{L,n} = \frac{\pi n \omega_n}{4 \omega_n^2 \int \rho \frac{|\xi|^2}{2} dV} \frac{\tau_b M^3 c}{z} \sum_{l,m,m'} \int dP_\varphi d\mu \left(\frac{\sqrt{2} \mathcal{E}^{5/2} G_{m'l}^* G_{ml}}{(1 \cdot \Omega)_v} \right) \bigg|_{v_{res}} \int dI \mathcal{F}_{n,l}(I - I_r) \frac{\partial}{\partial I} f,$$

$$\frac{dC_n^2(t)}{dt} = 2(\gamma_{L,n} - \gamma_{d,n}) C_n^2(t),$$

$$\Delta\Omega = 2\eta\omega_{b,n} + 2\lambda(|\gamma_n| + \nu).$$

Their discretized form is

$$\begin{aligned} \frac{f_i^{t+1} - f_i^t}{\Delta t} = & \sum_{n,l,m,m'} \frac{1}{I_{i+1/2} - I_{i-1/2}} \left\{ (D_{n,l,m,m'})_{i+1/2}^t \frac{f_{i+1}^{t+1} - f_i^{t+1}}{I_{i+1} - I_i} - \right. \\ & \left. - (D_{n,l,m,m'})_{i-1/2}^t \frac{f_i^{t+1} - f_{i-1}^{t+1}}{I_i - I_{i-1}} \right\} + \\ & + \frac{\nu_{scatt}^3}{\left(\left| \frac{\partial \Omega_l}{\partial I} \right|_{I_r} \right)^2} \frac{1}{I_{i+1/2} - I_{i-1/2}} \left(\frac{f_{i+1}^{t+1} - f_i^{t+1}}{I_{i+1} - I_i} - \frac{f_i^{t+1} - f_{i-1}^{t+1}}{I_i - I_{i-1}} \right). \end{aligned}$$

The broadening can be introduced using a window function $\mathcal{F}(I)$ around the resonances, with I being the relevant variable to describe resonant particle dynamics. The diffusion coefficient is

$$\begin{aligned} (D_{n,l,m,m'})_{i\pm 1/2}^t &= \pi n^2 (C_n^2)^t \left(\frac{\mathcal{E}^2 G_{m'l}^* G_{ml}}{\left| \frac{\partial \Omega_l}{\partial I} \right|_{I_r}} \right) \Big|_{v_{res}} (\mathcal{F}_{n,l})_{i\pm 1/2}^t, \\ (\gamma_{L,n})^{t+1} &= \frac{\pi n \omega_n}{4\omega_n^2 \int \rho \frac{|\xi|^2}{2} dV} \frac{\tau_b M^3 c}{z} \sum_{l,m,m'} \int dP_\varphi d\mu \left(\frac{\sqrt{2} \mathcal{E}^{5/2} G_{m'l}^* G_{ml}}{(\mathbf{l} \cdot \mathbf{\Omega})_v} \right) \Big|_{v_{res}} \times \\ &\quad \times \sum_i (I_{i+1} - I_i) (\mathcal{F}_{n,l})_{i+1/2}^t \frac{f_{i+1}^{t+1} - f_i^{t+1}}{I_{i+1} - I_i}, \\ (C_n^2)^{t+1} &= (C_n^2)^t + 2\Delta t \gamma_{L,n}^{t+1} (C_n^2)^t. \end{aligned}$$

As will be shown later, the workflow for solving RBQ equations while enforcing momentum conservation is the following:

f^{t+1} is implicitly calculated from f^t , $\omega_{b,n}^t$, and $\mathcal{F}_{n,l}^t$

$$\frac{f_i^{t+1} - f_i^t}{\Delta t} = \frac{\pi}{2} \sum_{n,l} \frac{(\omega_{b,n}^4)^t}{P_{\varphi,i+1/2} - P_{\varphi,i-1/2}} \left[\mathcal{F}_{n,l,i+1/2}^t \frac{f_{i+1}^{t+1} - f_i^{t+1}}{P_{\varphi,i+1} - P_{\varphi,i}} - \mathcal{F}_{n,l,i-1/2}^t \frac{f_i^{t+1} - f_{i-1}^{t+1}}{P_{\varphi,i} - P_{\varphi,i-1}} \right],$$

γ_n^{t+1} is calculated from f^{t+1} and $\mathcal{F}_{n,l}^t$

$$\gamma_n^{t+1} = \frac{\pi}{4} \sum_{l,i} \mathcal{F}_{n,l,i+1/2}^t \frac{f_{i+1}^{t+1} - f_i^{t+1}}{P_{\varphi,i+1} - P_{\varphi,i}} (P_{\varphi,i+1} - P_{\varphi,i}),$$

$\omega_{b,n}^{t+1}$, is calculated from γ_n^{t+1} and $\omega_{b,n}^t$

$$(\omega_{b,n}^4)^{t+1} = (\omega_{b,n}^4)^t + 2\Delta t \gamma_n^{t+1} (\omega_{b,n}^4)^t,$$

$\Delta\Omega^{t+1}$, the updated broadened resonant region, is calculated from γ_n^{t+1} , $\omega_{b,n}^{t+1}$, and ν

$$\Delta\Omega^{t+1} = 4\eta_n \omega_{b,n}^{t+1} + 2\lambda_n (|\gamma_n^{t+1}| + \nu),$$

$$\mathcal{F}_{n,l}^{t+1} \text{ is calculated based on } \Delta\Omega^{t+1}$$

Under the simplest prescription:

$$\mathcal{F}_{n,l}^{t+1} = 1/\Delta\Omega^{t+1}.$$

5.3.2 Mode amplitude initialization

In order for Landau damping/growth to take place, there should be at least a “seed” mode present in the plasma, which is often associated with the system stochastic noise from a variety of sources. For RBQ’s sake, mode amplitude needs to be finite in order for the simulation to be started. The initial guess for the amplitude is taken to be arbitrarily small since it was shown that the saturation level is insensitive to its exact value. Thermal noise, which arises from effects of particle discreteness, should in principle not be the dominant fluctuation driving mechanism. Turbulence, eddy currents on the vessel and asymmetries should play a much more important role in a tokamak plasma.

5.3.3 Boundary conditions

The matrix can only be inverted (and therefore the system can only be solved) when boundary conditions are added to the discretized system of equations. The code needs to account for a loss boundary in $(\mathcal{E}, P_\varphi)$ that is different for each slice $\mu = \text{const.}$ A **Neumann-type** condition specifies the values that the derivative of the distribution func-

tion is to take at the boundaries of the domain. It imposes a constant flux $\Gamma = -D\frac{\partial f}{\partial x}$ at the edge. On the other hand, **Dirichlet-type** condition imposes a condition on the value of the function itself.

In our solver, we have the option to use either Neumann and Dirichlet boundary conditions. They are chosen based on the following physical arguments:

- at the loss boundary f should be zero and Dirichlet is physically appropriate since it allows for particle loss. A Neumann condition would not even make sense there since the a priori flux is not known at that point;
- for the inner regions of the plasma, particles are allowed to accumulate and a Neumann condition describes the relevant physics (is it equivalent to a reflective condition, i.e., with zero net flux). Particle number over the plasma volume is automatically preserved as soon as particles do not escape at the ends. This however is virtually impossible since collisions relax the distribution function everywhere.

5.4 Matrix representation of the discretized system of equations with appropriate boundary conditions

If $\alpha = 0$ ($\alpha = 1$), we have a Dirichlet-type (Neumann-type) boundary condition at the leftmost end of the domain. If $\beta = 0$ ($\beta = -1$), we have a Dirichlet-type (Neumann-type) boundary condition at the rightmost end of the domain. κ and λ give the values of the

distribution function, or its derivative, at the edges of the domain. For most applications we can take $\kappa = 0$ and $\lambda = 0$ which implies a reflection boundary condition that f or its derivative will be zero at the edge.

$$\begin{aligned}
 & \begin{pmatrix} -1 & \alpha & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & D_{-1} & 1+D_0 & D_{+1} & 0 & 0 & \dots & 0 \\ 0 & 0 & D_{-1} & 1+D_0 & D_{+1} & 0 & \dots & 0 \\ 0 & 0 & 0 & D_{-1} & 1+D_0 & D_{+1} & \dots & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & \dots & D_{-1} & 1+D_0 & D_{+1} & 0 \\ 0 & 0 & 0 & \dots & 0 & D_{-1} & 1+D_0 & D_{+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & \beta & 1 \end{pmatrix} \cdot \begin{pmatrix} f_0^{t+1} \\ f_1^{t+1} \\ f_2^{t+1} \\ f_3^{t+1} \\ \vdots \\ f_{N-2}^{t+1} \\ f_{N-1}^{t+1} \end{pmatrix} + \\
 & + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_{-1} & E_0 & E_{+1} & 0 & 0 & \dots & 0 \\ 0 & 0 & E_{-1} & E_0 & E_{+1} & 0 & \dots & 0 \\ 0 & 0 & 0 & E_{-1} & E_0 & E_{+1} & \dots & 0 \\ & & & \dots & \dots & & & \\ 0 & 0 & 0 & \dots & E_{-1} & E_0 & E_{+1} & 0 \\ 0 & 0 & 0 & \dots & 0 & E_{-1} & E_0 & E_{+1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} f_0^{t=0} \\ f_1^{t=0} \\ f_2^{t=0} \\ f_3^{t=0} \\ \vdots \\ \vdots \\ f_{N-2}^{t=0} \\ f_{N-1}^{t=0} \end{pmatrix} = \begin{pmatrix} \kappa \\ f_1^t \\ f_2^t \\ f_3^t \\ \vdots \\ \vdots \\ f_{N-2}^t \\ \lambda \end{pmatrix}
 \end{aligned}$$

where, following a notation similar to that of Ghantous [38], we can write

$$\begin{aligned}
D_{-1} &= -\frac{\Delta t}{\Delta P_\varphi^2} \frac{\nu_{scatt,eff}^3}{\left(\left|\frac{\partial \Omega_l}{\partial P_\varphi}\right|_{\mathcal{E}'}\right)^2} - \sum_n \frac{\Delta t}{\Delta P_\varphi^2} \pi n^2 (C_n^2)^t \left(\frac{\mathcal{E}^2 G_{m'l}^* G_{ml}}{\left|\frac{\partial \Omega_l}{\partial I}\right|_{I_r}} \right) \Big|_{v_{res}} (\mathcal{F}_{n,l})_{i-1/2}^t \\
D_0 &= 2 \frac{\Delta t}{\Delta P_\varphi^2} \frac{\nu_{scatt,eff}^3}{\left(\left|\frac{\partial \Omega_l}{\partial P_\varphi}\right|_{\mathcal{E}'}\right)^2} + \sum_n \frac{\Delta t}{\Delta P_\varphi^2} \pi n^2 (C_n^2)^t \left(\frac{\mathcal{E}^2 G_{m'l}^* G_{ml}}{\left|\frac{\partial \Omega_l}{\partial I}\right|_{I_r}} \right) \Big|_{v_{res}} \left[(\mathcal{F}_{n,l})_{i-1/2}^t + (\mathcal{F}_{n,l})_{i+1/2}^t \right] \\
D_{+1} &= -\frac{\Delta t}{\Delta P_\varphi^2} \frac{\nu_{scatt,eff}^3}{\left(\left|\frac{\partial \Omega_l}{\partial P_\varphi}\right|_{\mathcal{E}'}\right)^2} - \sum_n \frac{\Delta t}{\Delta P_\varphi^2} \pi n^2 (C_n^2)^t \left(\frac{\mathcal{E}^2 G_{m'l}^* G_{ml}}{\left|\frac{\partial \Omega_l}{\partial I}\right|_{I_r}} \right) \Big|_{v_{res}} (\mathcal{F}_{n,l})_{i+1/2}^t
\end{aligned}$$

and

$$\begin{aligned}
E_{-1} &= -\frac{\Delta t}{\Delta P_\varphi^2} \frac{\nu_{scatt,eff}^3}{\left(\left|\frac{\partial \Omega_l}{\partial P_\varphi}\right|_{\mathcal{E}'}\right)^2} \\
E_0 &= 2 \frac{\Delta t}{\Delta P_\varphi^2} \frac{\nu_{scatt,eff}^3}{\left(\left|\frac{\partial \Omega_l}{\partial P_\varphi}\right|_{\mathcal{E}'}\right)^2} \\
E_{+1} &= -\frac{\Delta t}{\Delta P_\varphi^2} \frac{\nu_{scatt,eff}^3}{\left(\left|\frac{\partial \Omega_l}{\partial P_\varphi}\right|_{\mathcal{E}'}\right)^2}
\end{aligned}$$

for the case of a pitch-angle scattering operator.

Mode-particle interaction only takes place inside the resonance islands. However collisions take place everywhere. An interesting aspect of the problem is that disturbances of f due to the wave, initially concentrated at the resonance, start to propagate to other regions

of the plasma mediated by collisions.

5.5 Conservation laws

The number of particles is exactly conserved as long as they do not leave the boundaries of plasma confinement. Total momentum is also conserved to machine precision, as will be shown in the following. Let us multiply the discretized diffusion equation by $P_{\varphi,i} (P_{\varphi,i+1/2} - P_{\varphi,i-1/2})$ and sum over all possible values of i :

$$\begin{aligned} & \sum_i f_i^{t+1} P_{\varphi,i} (P_{\varphi,i+1/2} - P_{\varphi,i-1/2}) - \sum_i f_i^t P_{\varphi,i} (P_{\varphi,i+1/2} - P_{\varphi,i-1/2}) = \\ & \frac{\pi}{2} \Delta t \sum_{n,l,i} (\omega_{b,n}^4)^t \left[\mathcal{F}_{n,l,i+1/2}^t \frac{f_{i+1}^{t+1} - f_i^{t+1}}{P_{\varphi,i+1} - P_{\varphi,i}} - \mathcal{F}_{n,l,i-1/2}^t \frac{f_i^{t+1} - f_{i-1}^{t+1}}{P_{\varphi,i} - P_{\varphi,i-1}} \right] P_{\varphi,i}. \end{aligned}$$

Using $P_{\varphi,i} = P_{\varphi,i-1} + (P_{\varphi,i} - P_{\varphi,i-1})$ and the fact that the sum index is dummy, we see that

$$\sum_i \mathcal{F}_{n,l,i+1/2}^t \frac{f_{i+1}^{t+1} - f_i^{t+1}}{P_{\varphi,i+1} - P_{\varphi,i}} P_{\varphi,i} = \sum_i \mathcal{F}_{n,l,i-1/2}^t \frac{f_i^{t+1} - f_{i-1}^{t+1}}{P_{\varphi,i} - P_{\varphi,i-1}} P_{\varphi,i-1} \text{ and then}$$

$$\begin{aligned} & \sum_i f_i^{t+1} P_{\varphi,i} (P_{\varphi,i+1/2} - P_{\varphi,i-1/2}) - \sum_i f_i^t P_{\varphi,i} (P_{\varphi,i+1/2} - P_{\varphi,i-1/2}) = \\ & - \frac{\pi}{2} \Delta t \sum_{n,l,i} (\omega_{b,n}^4)^t \left[\mathcal{F}_{n,l,i-1/2}^t \frac{f_i^{t+1} - f_{i-1}^{t+1}}{P_{\varphi,i} - P_{\varphi,i-1}} \right] (P_{\varphi,i} - P_{\varphi,i-1}). \end{aligned}$$

Now we notice that it is only possible to use the expression for γ_n^{t+1} for the case of a constant $P_{\varphi,i} - P_{\varphi,i-1}$. This condition can be met for the case of a constant grid spacing inside a given resonant island. It does not matter what happens outside resonant islands since the window (or envelope) function $\mathcal{F}$ is zero there. Consequently we can have exact momentum conservation (up to machine precision) even in the case of different grids (coarse plus fine grids), provided that the grid cells have the same size inside a resonant island³ (i.e., over

³This does not exclude the possibility of a conserving scheme for non-equal grids.

the fine grid):

$$\begin{aligned} \sum_i f_i^{t+1} P_{\varphi,i} (P_{\varphi,i+1/2} - P_{\varphi,i-1/2}) - \sum_i f_i^t P_{\varphi,i} (P_{\varphi,i+1/2} - P_{\varphi,i-1/2}) = \\ -2\Delta t \sum_n \gamma_n^{t+1} (\omega_{b,n}^4)^t. \end{aligned}$$

Consequently

$$\sum_i f_i^{t+1} P_{\varphi,i} (P_{\varphi,i+1/2} - P_{\varphi,i-1/2}) + \sum_n (\omega_{b,n}^4)^{t+1} = \sum_i f_i^t P_{\varphi,i} (P_{\varphi,i+1/2} - P_{\varphi,i-1/2}) + \sum_n (\omega_{b,n}^4)^t,$$

where ΔP_φ is the fine grid cell size. A key point for momentum conservation is that **both** $\omega_{b,n}^{t+1}$ **and** f^{t+1} **should be evaluated using the same $\mathcal{F}^t$.**

Momentum conservation is a valuable internal check for self-consistency in the code. In Fig. (5.2), it is shown that the sum of particle distribution momentum and the wave momentum is a constant in time.

$\mathcal{F}$ is stretched to have at least the size of one cell. This may provide an unphysical platform for energy and momentum exchange. This error is made small by considering a reasonably small cell size. Landau damping is an initial-value problem. In the simulations we consider $t = 0$ as the instant of time in which the mode is introduced. Consequently, at $t = 0$ the resonant island has zero size. In order to start the numerical resolution we set the initial $\mathcal{F}$ to be the size of one grid cell. For all subsequent times the system evolution is self-consistent.

Figura 5.2: (a) Particle momentum, (b) wave momentum and (c) overall momentum as a function of time in RBQ.

5.6 Further development of the RBQ code, its interface with NOVA, NOVA-K and TRANSP codes, and the goal of whole device modeling

The Resonance Broadened Quasi-linear (RBQ) code is being developed in two versions, 1D and 2D, which would deal with the EP dynamics in the presence of the Toroidal Alfvén Eigenmodes (TAEs). The modes are being computed by the state-of-the-art ideal MHD numerical code NOVA and its post-processor kinetic extension NOVA-K. RBQ is based on the model that was proposed by Berk and co-authors [37, 46] in 1995 and was initially implemented for the case of the particle distribution functions with a simple bump-on-tail velocity dependence⁴. RBQ captures the most essential physics of the wave-particle interaction resolving the particle diffusion at each of the resonance.

Conventional Quasilinear Theory (CQL) is a powerful and fairly used theory to address energetic particle (EP) relaxation in the presence of unstable waves. Instead of solving for all the particle dynamics interacting with the mode, the kinetic problem is reduced to a diffusion equation for the distribution function. CQL has several limitations, mainly related to the required degree of stochasticity in order to be applicable (Chirikov criterion). To circumvent this limitation, the Resonance Broadened Quasilinear (RBQ) model was developed to address resonant particle interaction with both regimes of isolated and overlapping modes. This was done by using the same structure of the CQL equations for the ion distribution function but with the diffusive delta function broadened in primarily radial

⁴It should also be noted that in Fitzpatrick's thesis [47], he studied the response of a TAE wave in a model tokamak model, with a test eigenmode structure.

direction. RBQ was conceived in such a way as to reproduce the expected saturation levels of the isolated modes.

The dynamics of fast ion quasilinear redistribution, as calculated by RBQ, is illustrated in Figs. 5.3 and 5.4. The broadened resonance provides a platform for momentum and energy exchange between the particle distribution and the wave. The wave grows at the expense of the free energy stored in the distribution, i.e., by means of a gradual flattening of the distribution function. In the absence of damping mechanisms, the wave amplitude saturates when the distribution is locally completely flattened. In RBQ, the saturation level of a single mode is set to reproduce the expected level from bump-on-tail analytical predictions.

Figure 5.3: Illustrations of fast ion redistribution obtained from the 1D RBQ code for a single resonance and no background damping. A plateau in the inverted distribution function is formed around the resonance. The upper plots show the distribution function as a function of the canonical toroidal momentum, P_ϕ , for different times. The lower plots show the time evolution of the nonlinear bounce frequency, which scales with the square root of mode amplitude. The mode saturates at the expected level of $\omega_{b,sat} \approx 3.2\gamma_{L,0}$.

Figure 5.4: Illustrations of fast ion redistribution obtained from the 1D RBQ code with zero background damping for two separate resonances that eventually overlap. A plateau in the inverted distribution function is formed around each resonance. The upper plots show the distribution function as a function of the canonical toroidal momentum, P_ϕ , for different times. The lower plots show the time evolution of the nonlinear bounce frequency for each individual mode, which scales with the square root of mode amplitude. At the time when modes overlap, there is a sudden release of stored fast ion energy, which leads to substantial mode growth. Each mode saturates well above the expected level of $\omega_{b,sat} \approx 3.2\gamma_{L,0}$.

Figure 5.5: Example of mode structure for reversed-shear Alfvén eigenmodes (RSAEs) in DIII-D shot 153072, as calculated by NOVA (left) and its corresponding resonance lines in (v^2, P_φ) space for a fixed μ obtained by NOVA-K code for realistic tokamak discharge parameters. The resonances are broadened in RBQ code using the prescription used in the model.

Figure 5.6: Example of probability distribution function (PDF) on the left, as a function of the magnitude of particle kick in P_φ , corresponding to a given bin, as shown in the resonance plot in the right. The color code indicates how strong each resonant region is. Note that the large central peak in the PDF profile corresponds to the non-resonant particle in that bin.

The RBQ code is being prepared as a set of modules each tested and verified separately. The modules include the interface with the hybrid kinetic code NOVA-K routines to write the probability density function prepared for TRANSP's NUBEAM package. The core part of the RBQ diffusion scattering module is ready. The interface being used is the same as the one used by the kick model [96] uses in its interpretive runs for DIII-D or NSTX discharges. The RBQ code is being applied on the DIII-D plasma #159243 case prepared within the recent critical gradient experiments and published in [97].

Chapter 6

Summary, conclusions and future developments

Energetic-particle-driven instabilities limit the performance of present-day and next-generation fusion devices. Therefore its deleterious effects on plasma confinement need to be carefully addressed. In this thesis, we propose a criterion for the likely nature of losses of fast ions in tokamaks to be in the convective or diffusive nonlinear regimes. The criterion, which is shown to be rather sensitive to collisional and micro-turbulent processes, ultimately translates into a condition for the applicability of quasilinear modelling for fast ion modeling in realistic tokamak eigenmodes scenarios. The criterion is tested and validated against NSTX and DIII-D where the chirping mode behaviour is shown to satisfy this criterion. In the upcoming DIII-D campaign, dedicated experiments are programmed to be performed in order to further test the consequences of the theory and interpretation proposed in this thesis. The experiments will operate the plasma at negative triangularity, which is known

to reduce the overall turbulence level from drift-like instabilities.

We then propose a resonance-broadened quasilinear framework in which the relevant variables are chosen to be the invariants of the unperturbed resonant particle Hamiltonian. The formulation captures both regimes of isolated and overlapping modes. We show how linear perturbative MHD codes can be used to provide information such as eigenstructures, resonance surfaces and mode-particle interaction matrix elements, all of which are necessary for realistic modelling of experimental discharges. Preliminary numerical results are discussed, as well as the prospects of using the resonance-broadened quasilinear model for whole-device simulation through proper interfacing with global transport codes.

In summary, the main results and conclusions that the present work has led to are:

- Derivation of a general criterion for the likelihood of chirping modes in plasmas and its evaluation using NOVA and NOVA-K codes. The numerically-evaluated criterion was shown to reproduce experimental observations;
- Identification of the mechanism for plasma wave transition from steady-frequency to chirping regimes in DIII-D to be related to a substantial micro-turbulent activity reduction. This provided understanding for the first time of longstanding experimental observations;
- A puzzling result observed in JT-60U is that chirping and ALEs usually take place in JT-60U when supra-Alfvénic beams are on while fixed-frequency, quasi-steady excitation is observed when the beams are sub-Alfvénic. The weak nonlinear theory used in this thesis was shown to be in agreement with observations because of the resonant speed dependence of the relevant ratio ν_{stoch}/ν_{drag} for the chirping criterion;

- Development of a Resonance-Broadened Quasilinear (RBQ) Diffusion code with interface with existing magnetohydrodynamic and kinetic codes [57]. This code is of interest to the fusion community in connection with predictive capabilities in burning plasmas and is currently being postprocessed to be integrated in whole-device modeling transport codes, like TRANSP.

The papers published during the graduate studies are listed in Refs. [40, 41, 98, 99, 100, 101, 102]

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Appendix A

Collisional operator for fast ions

The nature and the strength of collisional processes is a very important element that determines the non-linear character of wave oscillations [48, 62, 63]. Stochastic processes, such as pitch-angle scattering, act to destroy the coherent structures that support wave chirping while drag, or slowing down, is formally equivalent to chirping and enhance the convective transport of these non-linear structures.

A.1 Pitch-angle scattering and drag operators

For EPs, the Fokker-Planck collisional operator that enters the kinetic equation can be approximately written as a combination of a pitch-angle scattering and a drag term for the case $v_{th,i} \ll v_{EP} \ll v_{th,e}$ [68]. Up to a source term, it can be written as

$$C[f] = \frac{\partial}{\partial \mathbf{v}} \cdot [\nu_{\perp} (v^2 \mathbb{I} - \mathbf{v}\mathbf{v})] \cdot \frac{\partial f}{\partial \mathbf{v}} + \frac{1}{\tau_s} \frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{v} \left(1 + \frac{v_c^3}{v^3} \right) f \right]$$

where f is the distribution function; $\nu_\perp$ and τ_s^{-1} are the 90 pitch angle scattering rate and the inverse slowing down time, which are given by

$$\nu_\perp = \frac{1}{2} \langle Z \rangle \frac{\bar{A}_i}{[Z]} \frac{1}{A_{EP}} \left(\frac{v_c}{v} \right)^3 \frac{1}{\tau_s} \quad (\text{A.1})$$

$$\tau_s^{-1} = \frac{Z_{EP}^2 e^4 m_e^{1/2} n_e \ln \Lambda_e}{3 (2\pi)^{3/2} \epsilon_0^2 m_{EP} T_e^{3/2}} \quad (\text{A.2})$$

The critical speed, above which electron drag dominates over ion drag, is defined as

$$v_c = \left(\frac{3\sqrt{\pi}}{4} \frac{m_e}{m_p} \frac{[Z]}{\bar{A}_i} \right)^{1/3} \sqrt{\frac{2T_e}{m_e}} \quad (\text{A.3})$$

and

$$\frac{[Z]}{\bar{A}_i} = \sum_\beta \frac{Z_\beta^2 n_\beta \ln \Lambda_\beta}{A_\beta n_e \ln \Lambda_e}$$

$$\langle Z \rangle = \sum_\beta \frac{Z_\beta^2 n_\beta \ln \Lambda_\beta}{n_e \ln \Lambda_e}$$

For a given ion species β , $A_\beta = m_\beta/m_p$, with m_p being the proton mass, $\ln \Lambda$ is the Coulomb logarithm, Z_i and Z_{EP} are the background and energetic ion atomic number, n is the density, T is the temperature, m is the mass and the thermal speed for a given species a is defined as $v_{Ta} = \sqrt{2T_a/m_a}$. The three-dimensional velocity derivatives are projected onto the aforementioned path of the steepest gradient of the distribution function diffusion, where $\mathcal{E}' = \mathcal{E} + (\omega/n) P_\varphi$ and μ is constant, i.e.

$$\frac{\partial}{\partial \mathbf{v}} \rightarrow \frac{\partial \Omega}{\partial P_\varphi} \bigg|_{\mathcal{E}}, \frac{\partial P_\varphi}{\partial \mathbf{v}} \frac{\partial}{\partial \Omega}$$

Since $\frac{\partial P_\varphi}{\partial \mathbf{v}} = R\hat{\varphi}$, and $v_\varphi \approx v_\parallel \frac{B_\varphi}{B}$, it follows that the effective collisional operator for fast ion resonating with a given mode can be approximately cast in the form

$$C[f] = \nu_{scatt}^3 \frac{\partial^2 f}{\partial \Omega^2} + \nu_{drag}^2 \frac{\partial f}{\partial \Omega}$$

where the effective scattering ν_{scatt} and drag ν_{drag} coefficients are

$$\nu_{scatt}^3 \simeq 2\nu_\perp R^2 \left[\mathcal{E} - \frac{B_\varphi^2}{B^2} (\mathcal{E} - \mu B) \right] \left(\frac{\partial \Omega}{\partial P_\varphi} \bigg|_{\mathcal{E}} \right)^2 \quad (\text{A.4})$$

and

$$\nu_{drag}^2 \simeq \frac{\sqrt{2}R}{\tau_s} \sqrt{\mathcal{E} - \mu B} \frac{B_\varphi}{B} \left(1 + \frac{v_c^3}{v^3} \right) \frac{\partial \Omega}{\partial P_\varphi} \bigg|_{\mathcal{E}}, \quad (\text{A.5})$$

In order to be properly evaluated, the above expressions need to be averaged over the orbit bounce motion of a particle.

A.2 Expression for the Coulomb logarithm

In order to increase the precision of NOVA-K computations of the collisional coefficients, we have implemented in it the same prescription for the Coulomb logarithm used in TRANSP. According to NRL Formulary,

$$\ln \Lambda = \ln \left(\frac{r_{max}}{r_{min}} \right)$$

where

$$r_{max} = \frac{1}{\sqrt{\sum_\gamma \lambda_{D,\gamma}^2}}; \quad \lambda_{D,\gamma}^2 = \frac{kT_\gamma}{n_\gamma e_\gamma}$$

In order to include magnetic field effects (the plasma frequency is of the same order of the cyclotron one), this expression is generalized to¹

$$r_{max} = \frac{1}{\sqrt{\sum_{\gamma} \frac{\omega_{p,\gamma}^2 + \omega_{c,\gamma}^2}{T_{\gamma}/m_{\gamma} + 2E_{EP}/m_{EP}}}}$$

We also have

$$r_{min} = \max \left[\frac{e_{\alpha} e_{\beta}}{m_{\alpha,\beta} \bar{u}^2}, \frac{\hbar}{2m_{\alpha,\beta} \bar{u}} \right]$$

where

$$m_{\alpha,\beta} = \frac{m_{\alpha} m_{\beta}}{m_{\alpha} + m_{\beta}}$$

is the reduced mass and²

$$\bar{u}^2 = 3T_{\gamma}/m_{\gamma} + 2E_{EP}/m_{EP}$$

The NRL Formulary can be used in the evaluation of the quantities entering the problem:

$$\omega_{p,\gamma} [\text{rad/s}] = 1.32 \cdot 10^3 Z_{\gamma} \sqrt{n_{\gamma} [\text{cm}^{-3}]} / \sqrt{A_{\gamma}}$$

$$\omega_{p,\gamma}^2 [\text{rad}^2/\text{s}^2] = 1.74 \cdot 10^6 Z_{\gamma}^2 n_{\gamma} [\text{cm}^{-3}] / A_{\gamma}$$

$$\omega_{c,\gamma} [\text{rad/s}] = 9.58 \cdot 10^3 Z_{\gamma} B [G] / A_{\gamma} = 9.58 \cdot 10^7 Z_{\gamma} B [T] / A_{\gamma}$$

¹This is a robust expression in the sense that the criterion $\bar{u}^2 < T_{\gamma}/m_{\gamma}$ does not need to be checked. It has to do with the Balescu-Lenard operator in the limit $v_{EP} > v_{th,e}$. Normally we have $v_{th,i} \ll v_{EP} \ll v_{th,e}$ so the ion Debye length is automatically recovered.

²The factor 3 appears as a result of the averaging over velocities. $\bar{u}^2 = \overline{|\mathbf{v}_{EP} - \mathbf{v}_{\gamma}|^2} = \overline{v_{EP}^2 - 2v_{EP}v_{\gamma} + v_{\gamma}^2}$. In the absence of a preferential direction, $\bar{u}^2 = v_{EP}^2 + \overline{v_{\gamma}^2} = 2E_{EP}/m_{EP} + 3T_{\gamma}/m_{\gamma}$.

$$\omega_{c,\gamma}^2 [\text{rad}^2/\text{s}^2] = 9.18 \cdot 10^{15} Z_\gamma^2 B^2 [T] / A_\gamma^2$$

For r_{max} ,³

$$v_{rel}^2 [\text{cm}^2/\text{s}^2] = 9.58 \cdot 10^{11} (T_\gamma [\text{eV}] / A_\gamma + 2E_{EP} [\text{eV}] / A_{EP})$$

$$r_{min,cl} [\text{cm}] = \frac{e_\alpha e_\beta}{m_{\alpha,\beta} \bar{u}^2} = \frac{(4.8 \cdot 10^{-10})^2}{1.67 \cdot 10^{-24}} \frac{Z_\alpha Z_\beta}{\frac{A_\alpha A_\beta}{A_\alpha + A_\beta}} \frac{1}{v_{rel}^2} = 137964 \frac{Z_\alpha Z_\beta}{\frac{A_\alpha A_\beta}{A_\alpha + A_\beta}} \frac{1}{v_{rel}^2}$$

$$r_{min,qu} [\text{cm}] = \frac{\hbar}{2m_{\alpha,\beta} e^{1/2} v_{rel}} = \frac{1}{2\pi} \frac{6.62 \cdot 10^{-27}}{2e^{1/2} 1.67 \cdot 10^{-24}} \frac{1}{\frac{A_\alpha A_\beta}{A_\alpha + A_\beta}} \frac{1}{v_{rel}} = 0.000192 \frac{1}{\frac{A_\alpha A_\beta}{A_\alpha + A_\beta}} \frac{1}{v_{rel}}$$

For r_{min} ,

$$v_{rel}^2 [\text{cm}^2/\text{s}^2] = 9.58 \cdot 10^{11} (3T_\gamma [\text{eV}] / A_\gamma + 2E_{EP} [\text{eV}] / A_{EP})$$

A.3 Collisional frequency estimates for NSTX

Let us estimate the slowing down frequency and the scattering frequency using NSTX parameters.

$$\nu_\perp = \frac{1}{2m_{EP}} \left(\frac{v_c}{v} \right)^3 \frac{1}{\tau_s} \frac{\sum_i n_i Z_i^2 \ln \Lambda_i}{\sum_{i'} \frac{n_{i'} Z_{i'}^2 \ln \Lambda_{i'}}{m_{i'}}} = \frac{Z_{EP}^2 e^4}{8\pi \epsilon_0^2 m_{EP}^2} \frac{1}{v^3} \sum_i n_i Z_i^2 \ln \Lambda_i$$

$$\tau_s^{-1} = \frac{Z_{EP}^2 e^4 m_e^{1/2} n_e \ln \Lambda_e}{3 (2\pi)^{3/2} \epsilon_0^2 m_{EP} T_e^{3/2}}$$

The critical speed is

³In TRANSP, $T [\text{keV}]$ and $n [\text{m}^{-3}]$. Here $T [\text{eV}]$ and $n [\text{cm}^{-3}]$.

$$v_c \equiv \left(\frac{3\pi^{1/2}}{4} \frac{m_e}{n_e \ln \Lambda_e} \sum_i \frac{n_i Z_i^2 \ln \Lambda_i}{m_i} \right)^{\frac{1}{3}} v_{Te}, \quad v_{Te} = \sqrt{2T_e/m_e}.$$

Now substituting values: $T_e = 1.4keV = 1400 \text{ } 1.6 \text{ } 10^{-19}J = 2.24 \text{ } 10^{-16}J$ in the core for NSTX shot #141117 at t=470ms, and $m_e = 9.1 \text{ } 10^{-31}kg$, we have

$$v_{Te} = \sqrt{2T_e/m_e} = 2.218 \text{ } 10^7 m/s$$

In this shot, $n_{deut} = 3.78 \text{ } 10^{19}m^{-3}$, $n_H = 3.76 \text{ } 10^{17}m^{-3}$, $n_{impur} = 3.11 \text{ } 10^{17}m^{-3}$, $n_e = 4.3 \text{ } 10^{19}m^{-3}$. So,

$$v_c = 1.53 \text{ } 10^6 m/s$$

Note that

$$\frac{m_{EP} v_c^2}{2} = 17.3 T_e$$

Using $\ln \Lambda = 17$, $\epsilon_0 = 8.85 \text{ } 10^{-12} F/m$,

$$\tau_s = 0.09s$$

Consequently

$$\nu_{drag} = \tau_s^{-1} = 11s^{-1}$$

At the injection energy $E = 80keV = 80000 \text{ } 1.6 \text{ } 10^{-19}J = 1.28 \text{ } 10^{-14}J$. So $v = \sqrt{2E/m_{EP}} = 2.76 \text{ } 10^6 m/s$. Hence

$$\frac{v_c}{v} = 0.55$$

$$\nu_{\perp} = 1.18 s^{-1}$$

Scattering time:

$$1/\nu_{\perp} = 0.85 s$$

Therefore, the slowing down occurs on a timescale ten times faster than the scattering.

Appendix B

Inclusion of fast ion micro-turbulent stochasticity

Stochastic diffusion is determined by collisional scattering processes, such as pitch angle scattering, as well as additional processes, such as the effect of the background turbulence that is often dominant in the determination of the global heat outflow. Micro-turbulence is introduced using a procedure that follows the one introduced by Lang and Fu [103], where it is considered that the main contribution comes from electrostatic ion temperature gradient (ITG) turbulence via radial diffusion rather than the velocity diffusion. As with collisional scattering, the turbulent diffusion operator is projected onto the relevant one-dimensional path for particle dynamics, represented by the variable Ω . Since $\frac{\partial}{\partial r} = \frac{\partial \Omega}{\partial P_\varphi} \bigg|_{\mathcal{E}}, \frac{\partial P_\varphi}{\partial r} \frac{\partial}{\partial \Omega}$ with $\frac{\partial P_\varphi}{\partial r} \approx -\frac{q_{EP}}{m_{EP}} \frac{\partial \psi}{\partial r}$, the spatial micro-turbulence diffusion operator along the radial coordinate r can be written as

$$\frac{1}{r} \frac{\partial}{\partial r} r D_{EP} \frac{\partial f}{\partial r} \approx D_{EP} \left(\frac{q_{EP}}{m_{EP}} \frac{\partial \psi}{\partial r} \right)^2 \left(\frac{\partial \Omega}{\partial P_\varphi} \bigg|_{\mathcal{E}'} \right)^2 \frac{\partial^2 f}{\partial \Omega^2} \quad (\text{B.1})$$

where D_{EP} is the EP diffusivity. The diffusion coefficient $r D_{EP}$ is pulled out on the basis that the radial variation in f gives the dominant contribution. Therefore, the ratio R between the effective stochasticities coming from micro-turbulent (Eq. B.1) and collisional processes (Eq. A.4) is

$$R \approx \frac{D_{EP} \left(\frac{q_{EP}}{m_{EP}} \frac{\partial \psi}{\partial r} \right)^2}{2\nu_\perp R^2 \left[\mathcal{E} - \frac{B_\varphi^2}{B^2} (\mathcal{E} - \mu B) \right]} \quad (\text{B.2})$$

A subtlety in applying the model relies on the determination of D_{EP} . It is expected to be lower than thermal ion diffusivity $D_{th,i}$ since the micro-turbulence wavelength is typically smaller than the beam cyclotron orbit. Historically the effect of micro-turbulence on EP transport has been neglected based on the fact that, since EPs have large orbits, they should experience several phases of the turbulent fields in such a way as to cancel out its overall effect. Although fast ions turbulent transport is negligible compared to Alfvénic-induced transport (see, for example, studies on ASDEX-U [79] and DIII-D [78]), turbulence can be an important transport mechanism, as compared to collisions, at the onset of the evolution of modes, when their amplitude are still small. Over the past decade, the modeling of D_{EP} has been studied by several groups [104, 105, 106, 107, 108, 73]. We have chosen to determine D_{EP} from the scalings that follow from gyrokinetic simulations of GTC [73], in which D_{EP} is proportional to the thermal ion diffusivity, D_i , and a function of T_i/E_{EP} , where E_{EP} is the energy of the EPs. The GTC simulations assumed a specific plasma background, that can be considerably different from a given discharge being analyzed.

Therefore, a significant error can be expected to be associated to the inferred value of D_{EP} . Alternative ways of obtaining D_{EP} could be achieved by using Refs. [105, 104]. In our analysis, we infer the value of D_i from the outputs of the global transport code TRANSP [67, 68] at the position where the mode structure is peaked at the time being analyzed. The particle diffusivity is known to have a huge error associated to it because TRANSP cannot resolve well particle sources, especially close to wall. On the other hand, the thermal conductivity χ is reasonably well known and therefore is frequently used [109] as an indication of the actual value of D (the exact relation for a Maxwellian distribution would be $D = 2\chi/3$).