UNIVERSIDADE DE SÃO PAULO
ESCOLA POLITÉCNICA

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**Modeling the flow around a cylinder using sensitivity analysis and
reduced spaces**

São Paulo
July 23, 2018

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reduced spaces**

Doctor's thesis presented to Escola Politécnica
da Universidade de São Paulo in fulfillment of
the requirements for the Doctor of Science de-
gree.

Area: Mechanical Engi-
neering of Energy
and Fluids

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São Paulo

July 23, 2018

Este exemplar foi revisado e corrigido em relação à versão original, sob responsabilidade única do autor e com a anuência de seu orientador.

São Paulo, _____ de _____ de _____

Assinatura do autor: _____

Assinatura do orientador: _____

Catálogo-na-publicação

Patiño Ramirez, Gustavo Alonso

Modeling the flow around a cylinder using sensitivity analysis and reduced spaces / G. A. Patiño Ramirez -- versão corr. -- São Paulo, 2018. 112 p.

Tese (Doutorado) - Escola Politécnica da Universidade de São Paulo. Departamento de Engenharia Mecânica.

1.Sensitivity Analysis 2.Adjoint Method 3.Reduced Order Models
4.Landau Equation I.Universidade de São Paulo. Escola Politécnica.
Departamento de Engenharia Mecânica II.t.

ACKNOWLEDGEMENTS

My most sincere gratitude to my advisor Professor Rafael Gioria for his unconditional support and motivation. It has been an immense pleasure to work and learn at his side. I admire his privileged intelligence and sympathy. I am also very grateful with my other advisor Professor Julio Romano Meneghini for his support and guidance throughout all these years and to the Professor Jose Aranha for sharing with me part of his interesting models and his vast knowledge in fluid mechanics.

I also want to thank my friends of the fluid dynamics lab (NDF): Prof. Adson Agrico, Prof. Bruno Carmo, Prof. Gustavo Assi, Alessandro de Lima, Murilo Cicolin, Ivan korkischko, Lucia Mesa and Victor Silva for their open and supportive attitude and especially I am very grateful with the friends of my room: Sara Malvar, Joao Isler, Daiane Dolci and Gabriel Moreira, for all the fun moments.

I can not leave aside all my dear friends in Brazil: Bruno Aragaki and his family, Rafael Avila, Miguel Paez and Jackson Calhau for making this PhD a nutritious and outstanding adventure. I would like to especially thank Ana Semeano for his patience, affection, for making happy my day-to-day, and especially for helping me to be “a stable system”, far away from the instability region.

My most loving thanks to my family: my mother Nubia Ramirez my father Alonso Patino and my sister Natalia Patino for their love and support, always encouraging me to follow my dreams.

Finally, I would like to express my gratitude to Petrobras and FUSP-Shell for the economic support throughout my doctorate, providing the necessary conditions to do my research.

ABSTRACT

This thesis concerns the wake control and flow dynamic analysis for a flow around a circular cylinder at different Reynolds numbers using reduced models. The wake control and dynamics in the reduced space were addressed using the sensitivity theory and the adjoint method. In the case of wake control, it was possible to predict the physical parameters of the active and passive controllers on the wake of the main cylinder. On the other hand, in the construction of the reduced space, a new shift mode calculated from a perturbation of the mean flow was proposed using the sensitivity to base flow modifications. The mathematical basis of the reduced space was constructed using a Fourier modal decomposition of the flow enriched by the shift mode. The proposed reduced space made possible the recomposition of the flow and the comparison with the physical parameters calculated in the physical space. Additionally, using the reduced space, it was possible to determine the transition dynamics between the equilibrium point of the Navier Stokes equation and the non-linear saturation state using the Landau coefficients obtained in the reduced model, opening the possibility of solving the flow around a 2D and 3D cylinder with low computational cost

Keywords: *Landau equation, stability, sensitivity analysis, reduced space.*

RESUMO

Esta tese trata sobre o controle de esteira assim como a análise dinâmica do escoamento em torno de um cilindro a diferentes números de Reynolds usando modelos reduzidos. O controle de esteira e a dinâmica no espaço reduzido foram abordados usando a teoria da sensibilidade e o método adjunto. No caso de controle de esteira, foi possível prever os parâmetros físicos dos controladores ativos e passivos no escoamento do cilindro principal. Por outro lado, na construção do espaço reduzido, foi proposto um novo modo de deslocamento (shift mode) calculado a partir de uma perturbação do campo médio usando a sensibilidade às modificações do campo base. A base matemática do espaço reduzido foi construída usando uma decomposição modal de Fourier do escoamento enriquecido pelo modo de deslocamento (shift mode). O espaço reduzido proposto possibilitou a recomposição do escoamento e a comparação com os parâmetros físicos calculados no espaço físico. Além disso, usando o espaço reduzido, foi possível determinar a dinâmica de transição entre o ponto de equilíbrio da equação de Navier Stokes e o estado de saturação não linear usando os coeficientes de Landau obtidos no modelo reduzido, abrindo a possibilidade de resolver o escoamento em torno de um cilindro 2D e 3D com baixo custo computacional

Palavras-chave: *Equação de Landau, estabilidade, análise de sensibilidade, espaço reduzido.*

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LIST OF ABBREVIATIONS

LE	<u>L</u> andau <u>E</u> quation
GLE	<u>G</u> inzburg <u>L</u> andau <u>E</u> quation
NSE	<u>N</u> avier <u>S</u> tokes <u>E</u> quation
VIV	<u>V</u> ortex <u>I</u> nduced <u>V</u> ibration
FEM	<u>F</u> inite <u>E</u> lement <u>M</u> ethod
FEM	<u>R</u> oot <u>M</u> ean <u>S</u> quare
BDF	<u>B</u> ackward <u>D</u> ifferentiation <u>F</u> ormulas
BC	<u>B</u> oundary <u>C</u> ondition

LIST OF SYMBOLS

ω	Vorticity $\omega = \nabla \times \mathbf{u} = \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} \hat{i}_k$
St	Strouhal number
Re	Reynolds number
Re_{cr}	Critical Reynolds number
$\mathbf{u}$	Velocity field
p	Pressure
ν	Dynamic viscosity
$\mathbf{u}'$	Velocity perturbation field
λ	Complex eigenvalue
σ	Linear growth rate
ω_{st}	Oscillation frequency of the wake
$\Delta\omega$	Shift in frequency
ω	Linear oscillation frequency of the wake
μ	Landau coefficient
A_{lc}	Amplitude limit cycle
c_3	Ginzburg-Landau coefficient
U_∞	Farfield flow velocity
L	Characteristic length of the body
$\mathbf{U}$	Base flow
σ	Stress tensor

1 INTRODUCTION

The flow around circular cylinders has been one of the most studied topics in fluid mechanics, not only because a lot of engineering applications are related with cylindrical structures immersed in flows, but because this geometry is one of the best archetypes to study interesting physical phenomena inherent to open flows. In spite of the many studies focused on this topic, many phenomena present in the flow around cylinders remain still open. This work has as main objective to address two fundamental problems: the problem of passive and active control of the cylinder wake, and the construction of a reduced order model to study the stability properties and to model the linear and non linear dynamics of the flow around a cylinder at different Reynolds numbers.

Figure 1.1: (a) Flow around a cylinder at $Re = 9.6$. The flow is slightly different from the creeping flow. (b) Flow around a cylinder at $Re = 13.1$. We clearly can see the formation of a recirculation bubble. Figures taken from Van-Dyke (1988)

From a theoretical point of view, changes in the flow around a cylinder are associated with bifurcations which trace the transition path from a stationary flow to a periodic vortex shedding state (see Fig.1.1) and subsequently to a turbulent flow (see Fig.1.2), associated with linear and non-linear stability problems. The alternating vortex shedding is associated with an oscillatory pressure field around the cylinder. In this way, the fluid forces the cylinder in the cross-stream direction in an alternating manner generating a periodic oscillation in the structure, a phenomenon called vortex-induced vibration (VIV) (BLEVINS, 1977; CHEN, 1977; ASSI

et al., 2006). If the structure is forced in one of its natural frequencies, the system could enter into resonance, progressively increasing the oscillation amplitude of the structure and eventually leading to collapse. In this sense, it is necessary to understand the phenomena involved in the flow around cylinders to project structures that resist the continuous action of the fluid in a safe manner. The VIV phenomenon is of special interest in the oil industry, since oil often exists underneath the seabed, requiring the use of long tubes (called risers) to connect the wells with the fluctuating platforms. Several numerical and experimental studies at different Reynolds numbers have been carried out to try to solve the VIV problem using control techniques. The control of vortex shedding can lead to a significant reduction in the forces acting on a body and the frequency of vortex shedding, drastically reducing the oscillation amplitude of the structure. The wake control is classified into two large groups: active and passive control.

Figure 1.2: Flow around a cylinder at $Re = 26$. Recirculation bubble completely developed. (b) Flow around a cylinder at $Re = 10,000$. The boundary layer becomes turbulent. Figures taken from Van-Dyke (1988)

The passive control techniques are associated with geometric changes of the body and do not require the action of external forces. Some types of passive suppressors include: splitter plates (SZEPESSY; BEARMAN, 1992), strakes (KORKISCHKO; MENEGHINI, 2010), hydrofoils (GALVAO et al., 2008), small cylinders located on the wake of the main cylinder (STRYKOWSKI; SREENIVASAN, 1990), among others. As in the case of passive controllers, the active controllers have been well studied, among them we find: acoustic excitations (BLEVINS, 1985), forced oscillations of the main cylinder (TOKUMARU; DIMOTAKIS, 1993), moving surface boundary-layer control (MSBC) (KORKISCHKO; MENEGHINI, 2012) among others. In general, the active control MSBC and the passive control using small control cylinders have been efficient techniques to suppress the cylinder vortex shedding at different Reynolds numbers, however the control degree achieved with these devices depends on many parameters related to the control cylinder: the number of control cylinders, the diameter, the gap between the main cylinder and the control cylinder, and in the active case, the rotation rate of the control cylinder. To optimize these parameters by making continuous direct simulations would require an immense amount of computational time, for this reason in chapter 4 it is shown a way of approaching this problem

using the sensitivity theory, which is mathematically structured and numerically implemented in chapter 3. Sensitivity theory is a practical manner to evaluate the effect of an external force on the transition wake with few computational simulations: base field (Navier Stokes stationary solution), stability and adjoint problem.

From the experimental and numerical point of view, analyzing the flow around a two-dimensional and three-dimensional cylinder is a very complicated task. Experimentally, measurements are limited to the ability to reproduce marine currents. Numerically, the challenge is to solve properly the different scales of the fluid in a relatively large domain, and on the other hand, to model the bifurcations that outline the dynamic path towards turbulence. In this context, one of the most successful approaches has been the reduced models (NOACK et al., 2003; TADMOR et al., 2011). The reduced models are constructed by projecting the Navier Stokes equations (NSE) and the stability differential equations into a reduced dimension space, based on the modes obtained from the temporal solution of the NSE. The base modes are also called as coherent structures and determine the effectiveness of the reduced model. The mathematical definition of the bases for a reduced model is not closed, Noack et al. (2003) and Tadmor et al. (2011) define the base using a proper orthogonal decomposition (POD). Additionally, Noack et al. (2003) and Tadmor et al. (2011) introduced the concept of shift mode with the purpose of modeling the transient behavior in the flow around a cylinder. The base modes used in these works are constructed from a combination of the linear stability modes of the NSE around the base field, plus a perturbation of the base flow. This perturbation enriches the base of the reduced space, because it is defined in a different space from that generated by the POD modes. The dependence of the base modes with the base field could represent a problem in the dynamic study of a cylinder beyond the first bifurcation if the base flow is defined as the stationary solution of the Navier Stokes equations, given that we propose a model in which the flow fields in a cylinder are calculated using an asymptotic model, and the growth rate of a disturbance is restricted to values much smaller than one. This point represents the second great challenge of this work. In chapter 5, one proposes a methodology to build a base that does not depend on the stationary solution of NSE in the physical space, but on an optimal perturbation of the mean flow, constructed using sensitivity theory. Furthermore, the space generated by the modes is built using Fourier modes and not POD modes, which is convenient since the Fourier modes are easily comparable with an asymptotic expansion around the growth rate of the disturbance (chapter 5).

The non linear stability problem in the flow around a cylinder is addressed in chapter 5 using the Landau equation. Around $Re \approx 47$ the flow around a cylinder undergoes a supercritical Hopf bifurcation in which the stationary base field loses its stability and evolves towards

a global oscillation of the wake sustained by the flow, defined as limit cycle (JACKSON, 1987). This fact opens the possibility of modeling the global system with an amplitude equation for the perturbation, where the Landau equation represents one of the most successful models. In the three-dimensional case, Albaredo & Provansal (1995) presents the Ginzburg Landau equation as a suitable phenomenological model to represent the flow around long cylinders, by virtue of the fact that oscillations in the transverse planes occur in a distributed manner along the length of the cylinder. Although the focus of this paper is to study the two-dimensional field in a cylinder using the 2D model of the Landau equation, the coefficients obtained and the construction of the reduced model open up the possibilities for modeling the remaining coefficients of the Ginzburg-Landau equation and thus have a complete model of a long cylinder as a riser.

2 LITERATURE REVIEW

Understanding the behavior of the flow around an arbitrary geometry is an extremely complicated task, independently whether the problem is approached from the experimental or numerical point of view. On account of this fact, just certain particular aspects of fluid systems can be explained in detail. On the one hand, experimental observations have the problem of disturbing the system with external perturbations, being challenging to obtain fluid's parameters with a high degree of precision. On the other hand, the analytical and numerical approaches are limited by the inherent complexity of the Navier Stokes equations (NSE), whose numerical solution began to be viable just at the end of the 20th century. Despite the implicit complexity exhibited by the fluids and the diversity of geometries present in nature, there are certain simple geometries, such as the case of a cylinder, whose behavior when immersed in a fluid flow allows obtaining relevant conclusions for the real fluids world. These conclusions are valuable in various engineering applications, such as the case of fluid control, as explored in chapter 4.

Flows can be classified into two wide groups: open and closed flows. Open flows are characterized by a set of particles that are continuously advected by entering and exiting the fluid domain. Oppositely, in closed flows, the fluid particles are always limited within the fluid domain at all times. A fundamental characteristic of open flows is that they exhibit strong instabilities (DRAZIN, 2002). Then the unstable nature of the open flows leads them to be very sensitive to external disturbances, resulting in unstable non-observable stationary states, and in non-stationary observable dynamics with coherent structures that vary in time (DRAZIN, 2002), which can eventually evolve towards turbulence. Open flows encompass different configurations such as: wakes, jets, boundary layers, Couette flows, Poiseuille flows, among others. Some examples of open flows that can be found in nature are the Von Karman wake that appears in the air flow around an island Fig.2.1(a), the Von Karman wake in the flow around a cylinder Fig.2.1(b), the flow around a sphere Fig.2.2(a), or even in the flow around more complex geometries such as an airfoil illustrated in Fig.2.2(b). Open flows can be categorized as oscillators, which present an intrinsic dynamics with self-sustained oscillations (as the flow around a cylinder), and as amplifiers, which exhibit a strong sensitivity to external disturbances (CHOMAZ, 2005), the jets are a clear example of this type of systems.

Figure 2.1: (a) Von Karman vortex street by Socorro Island, taken from Snashot (2011). (b) Vortices in the flow around a cylinder, taken from Van-Dyke (1988)

A key aspect to be studied in our archetypes is the flow stability. Hydrodynamic stability analysis is the process of examining the response of a fluid to infinitesimal perturbations. These disturbances can potentially transition the flow to a final state completely different from the initial state, which in the case of the cylinder can be the transition from a steady to a periodic, or in jets it can be the transition from a steady state to a turbulent one. Stability analysis has been the focus of studies of eminent scientists for a long time. Reynolds (1883), for example, studied the transition of a stationary flow to a turbulent flow inside a tube and Rayleigh (1878) examined, among other flows, the instability in jets. Computationally, the dynamic analysis of the evolution of a perturbation can be approximated using two points of view: local, in each streamwise place and global in the complete domain.

Figure 2.2: (a) Flow around a sphere, taken from Van-Dyke (1988). (b) Flow around an airfoil, taken from Van-Dyke (1988)

2.1 Conservation of momentum and Navier Stokes equations

In this section Newton's second law will be applied to an infinitesimal fluid element with the purpose of determining the differential equation that governs the dynamics of a fluid. Let us consider the movement of an infinitesimal fluid element as in Fig.2.3. Newton's second law requires that the net force on the element must be equal to the product between the mass of the element and its acceleration. The sum of forces in the direction x_1 in Fig.2.3 is given by

Figure 2.3: Surface stresses on an element moving with the flow. Adapted from Kundu & Cohen (2004)

$$\begin{aligned}
 & \left(\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} \frac{dx_1}{2} - \tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} \frac{dx_1}{2} \right) dx_2 dx_3 \\
 & \left(\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2} \frac{dx_2}{2} - \tau_{21} + \frac{\partial \tau_{21}}{\partial x_2} \frac{dx_2}{2} \right) dx_1 dx_3 \\
 & \left(\tau_{31} + \frac{\partial \tau_{31}}{\partial x_3} \frac{dx_3}{2} - \tau_{31} + \frac{\partial \tau_{31}}{\partial x_3} \frac{dx_3}{2} \right) dx_1 dx_2
 \end{aligned} \tag{2.1}$$

Which can be simplified as

$$\left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} \right) dx_1 dx_2 dx_3 = \frac{\partial \tau_{j1}}{\partial x_j} dV \tag{2.2}$$

With dV a differential volume element. This equation can be generalized for the other two components x_2 and x_3 , so that the net surface force per unit volume due to surface stress is

given by $\frac{\partial \tau_{ij}}{\partial x_j}$. Neglecting body forces, Newton's second law can be expressed as

$$\rho \frac{Du_i}{Dt} = \frac{\partial \tau_{ij}}{\partial x_j} \quad (2.3)$$

D/Dt is the material derivative operator defined as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad (2.4)$$

Eq.(2.3) is the movement equation and is fulfilled for any continuum, solid or fluid independent of the relationship between the stress tensor and the deformation field. Eq.(2.3) is defined as Cauchy's motion equation.

The relationship between the stress and the deformation field in the continuum is called as constitutive relation. In a fluid at rest there are only normal components of stress on the surface. This stress does not depend on the orientation of the surface. In other words, the stress tensor is isotropic or spherically symmetric (KUNDU; COHEN, 2004). An isotropic tensor is defined as one whose components do not change under rotation of the coordinate system. The only second-order isotropic tensor is the Kronecker delta tensor

$$\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.5)$$

Any isotropic tensor of second order must be proportional to delta, therefore as a fluid at rest is isotropic, it is satisfied

$$\tau_{ij} = -p\delta_{ij} \quad (2.6)$$

where p is the thermodynamic pressure. The negative sign in Eq.(2.6) is due to the fact that the normal components of τ are considered positive if they indicate tension instead of compression (KUNDU; COHEN, 2004). A fluid in motion develops additional components in the stress tensor denoted by σ_{ij} , thus for a moving fluid gives

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij} \quad (2.7)$$

The non-isotropic σ tensor is called as deviatoric stress tensor, and it is related to the velocity gradients $\partial u_i / \partial x_j$. The gradient velocity tensor can be decomposed into a symmetric and anti-symmetric part

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \quad (2.8)$$

The antisymmetric part represents a fluid in rotation without deformation which can not generate stress, then the stress must be generated by the strain rate tensor

$$e_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2.9)$$

A linear relationship between the stress and the strain rate will be assumed

$$\sigma_{ij} = K_{ijmn} e_{mn} \quad (2.10)$$

where $\mathbf{K}$ is a tensor with 81 components among where only two components are relevant in the system. Since an isotropic medium has no preferential direction, the stress-strain tensor must be independent of rotations in the coordinate system, this implies that the tensor $\mathbf{K}$ is isotropic. All isotropic tensor of even order can be represented by products of δ tensors (ARIS, 1962), then for the fourth order tensor $\mathbf{K}$ is met

$$K_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu \delta_{im} \delta_{jn} + \gamma \delta_{in} \delta_{jm} \quad (2.11)$$

where λ , γ and μ are scalars. Since $\boldsymbol{\sigma}$ is a symmetric tensor, $\mathbf{K}$ must also be symmetric in the indices ij which is consistent with the condition $\gamma = \mu$. In this way, we conclude that only two of the 81 components survive under the restrictions of symmetry and isotropy. The substitution of Eq.(2.11) in Eq.(2.10) results in

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{mm} \delta_{ij} \quad (2.12)$$

Where $e_{mm} = \nabla \cdot \mathbf{u}$. The complete stress tensor is

$$\tau_{ij} = -p \delta_{ij} + 2\mu e_{ij} + \lambda e_{mm} \delta_{ij} \quad (2.13)$$

The coefficients μ and λ can also be related. Assuming $i = j$, summing over the repeated indexes and noting that $\delta_{ii} = 3$ we obtain

$$\tau_{ii} = -3p + (2\mu + 3\lambda) e_{mm} \quad (2.14)$$

defining an equation for the pressure p

$$p = -\frac{1}{3} \tau_{ii} + \left(\frac{2}{3} \mu + \lambda \right) \nabla \cdot \mathbf{u} \quad (2.15)$$

We can take an average of the diagonal terms of e_{ij} and define a mean pressure $\bar{p}$ (different from the thermodynamic pressure p)

$$\bar{p} \equiv -\frac{1}{3} \tau_{ii} \quad (2.16)$$

Substituting in Eq.(2.15) we get

$$p - \bar{p} = \left(\frac{2}{3}\mu + \lambda \right) \nabla \cdot \mathbf{u} \quad (2.17)$$

For incompressible fluids $e_{mm} = \nabla \cdot \mathbf{u} = 0$ then the constitutive equation becomes

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} \quad (2.18)$$

Where p is a mean pressure since there are no state equations to determine a thermodynamic pressure (KUNDU; COHEN, 2004). For a compressible fluid the thermodynamic pressure can be defined and $p, \bar{p}$ are different according to the Eq.(2.17). The difference between the pressures in Eq.(2.17) is determined by the factor

$$\kappa = \lambda + \frac{2}{3}\mu \quad (2.19)$$

defined as bulk viscosity factor. In principle, the factor κ is measurable, however measures must be taken in extremely large values of $D\rho/Dt$, such as within the shock waves. For practical purposes the Stokes hypothesis Eq.(2.20) is valid (TRUESDELL, 1952)

$$\lambda + \frac{2}{3}\mu = 0 \quad (2.20)$$

With the condition $\kappa = 0$ the constitutive equation becomes

$$\tau_{ij} = - \left(p + \frac{2}{3}\mu \nabla \cdot \mathbf{u} \right) \delta_{ij} + 2\mu e_{ij} \quad (2.21)$$

The linear relationship between stress and the deformation rate in Eq.(2.21) is consistent with the definition of a Newtonian fluid. In non-Newtonian fluids, the shear stress is not a linear function of the local strain rate. It depends not only on the local strain rate but also on its history. Such memory gives the fluid a elastic property in addition to its viscous property.

The equation of motion for a Newtonian fluid is obtained from the substitution of the Eq.(2.21) in the Cauchy's equation Eq.(2.3) obtaining

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[2\mu e_{ij} - \frac{2}{3}\mu (\nabla \cdot \mathbf{u}) \delta_{ij} \right] \quad (2.22)$$

Eq.(2.22) is a general version of the Navier Stokes equations neglecting body forces (such as gravity). If the temperature difference within a fluid is small, the viscosity can leave the derivative in Eq.(2.22) arriving to the differential equation

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \left[\nabla^2 u_i + \frac{1}{3} \frac{\partial}{\partial x_i} (\nabla \cdot \mathbf{u}) \right] \quad (2.23)$$

Where

$$\nabla^2 u_i \equiv \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad (2.24)$$

is the Laplacian of the velocity field. For incompressible fluids the Navier Stokes equation is reduced to

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} \quad (2.25)$$

Using the following normalization

- $\nabla \rightarrow L \nabla$
- $\mathbf{u} \rightarrow \frac{\mathbf{u}}{U}$
- $t \rightarrow \frac{t}{L/U}$
- $p \rightarrow \frac{p}{\rho U^2}$

The Navier-Stokes equation takes the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} \quad (2.26)$$

very common in fluid dynamics. With

$$\text{Re} = \frac{\rho U L}{\mu} \quad (2.27)$$

The Reynolds number, defined from a far field velocity U , an arbitrary reference length L (for example a cylinder's diameter) and the dynamic viscosity μ

2.2 Stability

The hydrodynamic stability theory aims to determine the ability of a system to return to the undisturbed reference state (called as base field) after being perturbed. Stability analysis can be treated using modal and non-modal approaches. The modal approach consists of linearizing the NSE equation around the base flow and evaluating the eigenvectors and eigenvalues of the NSE linear operator. For the linearization of a perturbation $\mathbf{u}'$ around the base flow $\mathbf{U}$ we assume $\mathbf{u} = \mathbf{U} + \mathbf{u}'$ in the NSE Eq.(2.26)

$$\frac{\partial \mathbf{u}'}{\partial t} = -(\mathbf{U} \cdot \nabla) \mathbf{u}' - \mathbf{u}' \cdot \nabla \mathbf{U} - \nabla p' + \frac{1}{\text{Re}} \nabla^2 \mathbf{u}', \quad \nabla \cdot \mathbf{u}' = 0 \quad (2.28)$$

therefore the NSE for the perturbation can be written as

$$\frac{\partial \mathbf{u}'}{\partial t} = L \mathbf{u}' \quad (2.29)$$

with L the linear operator defined by the operators in the right of Eq.(2.28). The flow is defined as asymptotically unstable if there is an eigenvector of L with a positive growth rate, calculated by the sign of the real part of the corresponding eigenvalue.

The non-modal stability analysis consists in the use of adjoint modes, that is, the eigenvectors of $L^\dagger$ ($\dagger$ the conjugate transpose), in order to determine the optimal perturbations and the optimal response of the flow (SCHMID, 2007).

The eigenvectors of the linearized operator L can be evaluated in the framework of global and local stability. The local stability just considers the eigenmodes in the orthogonal direction of the base flow, while in the global approach the modes in the longitudinal direction are also considered. Local stability analysis has historically been used in the case of parallel flows, i.e fluids that are homogeneous in the streamwise direction, using a length scale based in the wavelength of the instability waves. In the case of open flows around complex geometries, the hypothesis of parallel flows is no longer valid. Therefore, in this type of systems it is necessary to apply a global stability analysis as introduced by Jackson (1987).

2.2.1 Local analysis

The local stability theory was originally developed in the context of plasma physics (BRIGGS., 1964) and only in the 1980s appears in the framework of hydrodynamics (HUERRE; MONKEWITZ, 1990), where Cossu & Chomaz (1997) gives a detailed theoretical description of the methodology. In the local stability theory, the susceptibility of a parallel flow to be unstable in a particular streamwise station is analyzed.

In hydrodynamic stability, a fluid can be classified into three categories: locally stable, convectively unstable, and absolutely unstable. At a particular streamwise station, the fluid is considered stable if the fluid section passing through the streamwise section is stable as shown in Fig.2.4(a). So the disturbance in the streamwise station dissipates over time. If this condition is not fulfilled, the flow is called unstable, within this category we can distinguish two types of instability: convective instability and absolute instability, depending of the competition between the advection of the disturbance and its growth rate. In the case of convective instability the advection of the base field (defined as the stationary solution of NSE) is higher than the growth rate of the disturbance, so that the disturbance is advected away from the initial point as in Fig2.4(b). On the other hand, a flow is absolutely unstable when the disturbance grows indefinitely at the initial point because the growth rate is much greater than the advection (Fig2.4c). This dichotomy presented in the hydrodynamic instability is related to the ability of open flows to behave as amplifiers or oscillators. In effect, oscillators such as wakes of 2D and 3D objects are associated, using a local point of view, with the presence of an area of absolute instability

Figure 2.4: Evolution of a perturbation located in the neighborhood of $x = 0$ at time $t = 0$ for a one dimensional base state. The base flow could be (a) stable, (b) convectively unstable or (c) absolutely unstable. Taken from Charu (2011)

(MONKEWITZ; HUERRE; CHOMAZ, 1993; CHOMAZ; HUERRE; REDEKOPP, 1991) and in a global context with the existence of a global instability mode. Oscillators have an intrinsic dynamics, so self-sustained instabilities are developed in response to an initial disturbance. This type of behavior is opposite to that presented in amplifier-type systems, such as jets, where only instabilities develop under the influence of external forcing. The amplifiers develop convective instabilities in a local context and are asymptotically stable in a global point of view.

The analysis of local stability is formally valid only for parallel flows, by contrast inflexional velocity profiles, such as boundary layers and wakes do not necessarily satisfy this condition, consequently it is not possible to make an approximation of local stability in these type of flows. For non-parallel flows (such as the flow around cylinders), a global stability analysis is required, in this way the system is globally stable or unstable according to the sign of the dominant eigenvalue of the linear operator in Eq.(2.29)

2.2.2 Global analysis

In this type of analysis, the stability is defined as the susceptibility of the system to infinitesimal perturbations from a equilibrium point, which in our case is the stationary solution of the NSE, defined as the base field. The seminal works about the use of global stability in cylinder wakes go back to Noack & Eckelmann (1994) and Barkley & Henderson (1996). The global behavior of the flow depends on the interaction of the stability in local zones, then a fluid can be globally stable but locally it can present convective instabilities (as in the case of a jet). In contrast, a globally unstable flow must necessarily present a slowly-changing base flow (CHOMAZ; HUERRE; REDEKOPP, 1991) (see Fig.2.5), although this condition is not sufficient, since a fluid with a small package of absolute local instability can remain globally stable. This type of behavior was shown by Giannetti & Luchini (2007) in the flow around a cylinder, in which a region

Figure 2.5: Convective and global instability regions in the flow around a cylinder. Figure modified from Van-Dyke (1988)

Figure 2.6: Harmonic spectra of velocity time series in the vortex shedding regime at $Re = 100$. (a) v -component at $x = 2.2$, $y = 0.7$. (b) u -component at $x = 0.2$, $y = 2.1$. Taken from Pier (2002)

of absolute instability develops in the domain at $Re \approx 25$, while the fluid remains globally stable up to $Re \approx 47$. In general, the open globally unstable flows show an intrinsic nonlinear dynamics and develop self-sustained oscillations at defined frequencies (HUERRE; MONKEWITZ, 1985), a classic example of this type of systems is the flow around a cylinder. The resonance shown by the flow around a cylinder can be seen clearly in the spectrum of frequencies shown in Fig.2.6, where the predominant peak shows the frequency of vortex shedding of the system. The global stability is based in the modal analysis, which has the advantage of allowing to predict the onset of the instabilities when a dominant unstable eigenmode appears, characterized by the most unstable eigenvalue of the linear operator Eq.(2.29). The most unstable mode describes the structure and frequency of the perturbation when it begins to grow, later the disturbance becomes very large and the non-linear effects (not included in the global stability analysis) begin to take effect. Although the modal stability analysis works well to describe oscillators, it fails in the dynamic description of amplifiers since in these type of systems the eigenvalue spectrum

is always stable.

2.3 Non-normality and adjoint equations

The concept of non-normality directly is related with the notion of adjoint operator (CHOMAZ, 2005) defined by the equation $\langle \mathbf{u}_1, X \mathbf{u}_2 \rangle = \langle X^\dagger \mathbf{u}_1, \mathbf{u}_2 \rangle$. Where X is an arbitrary operator, $\mathbf{u}_1, \mathbf{u}_2$ are arbitrary fields, and $\langle, \rangle$ an internal product defined in the space. In this way, an operator is considered as normal if it satisfies the condition $X X^\dagger = X^\dagger X$. In the case of non-normality, the modal stability analysis, which is valid for asymptotic behavior of the linearized equations, provides an incomplete description of the stability problem, since the perturbation could have a transient growth in short times caused by the non-orthogonality of the direct modes (HENNINGSON, 2001). This transient growth may be accompanied by an energy saturation which could distort the instability thresholds predicted by the modal analysis, caused by the non-linear effects. In normal systems, the adjoint mode represents the optimal disturbance that maximizes the amplitude of the direct mode associated with long times. On the other hand, in non-normal and marginally stable systems, the adjoint mode is related to the notion of receptivity, that is, to the susceptibility of a system respond due to the effect of an external force. Additionally, adjoint modes are used to evaluate the sensitivity of an instability to modification of a variable (HILL, 1995). Giannetti & Luchini (2007) showed that the regions of greater sensitivity are obtained from the superposition of the direct and the associated adjoint mode calculated from the linear stability problem. Apart from this, the adjoint field allows to build the sensitivity fields to changes in the base field and external forces (Giannetti & Luchini (2007); Marquet, Sipp & Jacquin (2008)), defining the critical instability regions, and opening the possibility of applying efficiently active and passive control methods (Marquet, Sipp & Jacquin (2008); Patino, Gioria & Meneghini (2017)). With sensitivity analysis and passive controllers Marquet, Sipp & Jacquin (2008) successfully reproduce the experimental results obtained by Strykowski & Sreenivasan (1990), which determined the regions where it was necessary to act to suppress the Von Karman's instabilities in the flow around a cylinder at low Reynolds numbers.

When the operator is not normal, the disturbance can be constructed as the superposition of damped global modes, or in other words, as the singular modes of the resolvent (SIPP et al., 2010) $(i\omega \mathbf{I} - X)^{-1}$, where ω is the frequency of the harmonic forcing. Therefore, evaluating the non-normality of an operator in a local and global context is relevant, because this property defines the dynamic of a perturbation in finite times.

Figure 2.7: (a) Frequencies and (b) growth rate of a perturbation in function of the Reynolds number. Bold curve represents the Strouhal number. Thin curves are for the base flow. Triangles are from the mean flow. Vertical dashed lines indicate the critical Reynolds. Taken from Barkley (2006)

Figure 2.8: Vorticity of the leading eigenmode for. (a) base flow and (b) mean flow at $Re = 100$. The streamwise spacing of vorticity maxima for the mean-flow eigenvalue is very similar to that of the vortex shedding pattern. This is not the case for the base flow eigenmode. Taken from Barkley (2006)

2.4 Nonlinear dynamics

In many systems the linear stability analysis fails due to the non-linear characteristics of the NSE. Among this type of systems we can categorize the flow around a cylinder. In this case the laminar flow becomes unstable through a supercritical Hopf bifurcation above $Re \approx 47$ (JACKSON, 1987). Applying the linear stability analysis around the base flow (see Fig.2.7), we see that the modal analysis represents properly the spatial structure and the frequency of the perturbation near to the bifurcation point $Re \approx 47$, however it fails in the prediction of the vortex shedding at higher Reynolds numbers. On the other hand, we see that the global stability analysis around the mean flow provides an adequate description of the system in terms of the spatial structure of the disturbance (see Fig.2.8) and the frequency of vortex shedding (see Fig.2.7) (BARKLEY, 2006), although with a null growth rate (see Fig.2.7), which has serious implications in the saturation process and the amplitude of the oscillations. The inability of the linear stability analysis around the base flow to predict the frequency of vortex shedding beyond the bifurcation point is probably related with the non-linearities captured by the Reynolds stress tensor, which modifies the base flow into the mean flow (WALEFFE, 1995). Hence the importance of using nonlinear

Figure 2.9: Fluctuating velocity in the point $(x, y) = (2, 1)$ obtained from the solution of the NSE. $Re = 100$.

models such as Landau equation to represent the amplitude of the perturbations

$$\frac{dA}{dt} - \sigma A + \mu |A|^2 A = 0 \quad (2.30)$$

For a complex amplitude A , and coefficients σ, μ that depend on the Reynolds number. This model, despite its apparent simplicity, is capable of modeling the saturation mechanism in a supercritical Hopf bifurcation as the flow changes from a stable state to an unstable saturated state (Fig.2.9). To determine the time evolution of an unstable perturbation (Eq.2.30), Landau & Lifshitz (1987a) assume the system is in the neighborhoods of the critical Reynolds (guaranting the disturbance is enough small) and start taking the squared amplitude of the disturbance $|A|^2$. Derivating it in relation to the time it gets $d|A|^2/dt = 2\sigma|A|^2$. This expression is just the first term in an expansion in power series of A and A^* (* the complex conjugate). In general we can expand the temporal derivate of the disturbance in power series of A and A^* . Below is the expansion up to fourth order

$$\frac{d|A|^2}{dt} = a_0|A|^2 + a_1|A|^3 + a_2|A|^4 \quad (2.31)$$

We are not interested in the exact value of the time derivation of A but with its temporal average. Making a temporal average of the terms in the series we see that the term in third order disappears because it contains the periodic factor. However the fourth order term proportional to $A^2 A^{*2} = |A|^4$ does not disappear in average, then until fourth order and defining $a_0 = 2\sigma$

and $a_2 = -\mu$ we have

$$\frac{d|A|^2}{dt} = 2\sigma|A|^2 - \mu|A|^4 \quad (2.32)$$

where μ represents the Landau coefficient. This coefficient can be either positive or negative. The solution of the differential equation Eq.(2.32) is given by (LANDAU; LIFSHITZ, 1987a)

$$\frac{1}{|A|^2} = \frac{\mu}{2\sigma} + \text{constant } e^{-2\sigma t} \quad (2.33)$$

The Landau equation can be also written as

$$\frac{dA}{dt} - \sigma A + \mu|A|^2 A = 0 \quad (2.34)$$

This model was formulated with the purpose of describing universal characteristics of an unstable system in a small but finite region around a control parameter, which in our case is represented by the Reynolds number. From this model we can obtain two important results: the amplitude of the limit cycle A_{lc} and the shift in frequency $\Delta\omega = \omega_{st} - \omega$ owing to the difference between the oscillation frequency of the wake in the nonlinear saturation and the frequency of linear oscillation. Generally $A(t)$ is a complex function $A(t) = |A|e^{i\phi(t)}$.

Experimentally the amplitude $A(t)$ represents any fluid variable that usually is the flow velocity. In our case $A(t)$ represents the amplitude of the unstable mode. Then in first order the velocity field can be expanded on the basis of modes $\mathbf{u} = A(t)\mathbf{E}(\mathbf{x}, t)e^{i\omega t}$, being $\mathbf{E}(\mathbf{x}, t)$ the most unstable mode and $A(t)$ is governed by the Landau equation Eq.(2.34).

In general the Landau coefficient μ is complex $\mu = \mu_r + i\mu_i$. Separating the real and imaginary parts of the Landau equation we obtain

$$\frac{1}{|A|} \frac{d|A|}{dt} = \sigma - \mu_r|A|^2 \quad (2.35)$$

$$\frac{d\phi}{dt} = -\mu_i|A|^2 \quad (2.36)$$

From the real part is possible to determine the amplitude of the limit cycle $d|A|/dt = 0$ given by

$$A_{lc} = \sqrt{\frac{\sigma}{\mu_r}} \quad (2.37)$$

According to Eq.2.35 the stable states are given by the condition $\sigma < 0$ and $\mu_r > 0$. The saturation is the state in which the nonlinear effects become relevant until the point of limiting the system. The field $\mathbf{u}(\mathbf{x}, t)$ oscillates with frequency $\omega + d\phi/dt$ with $d\phi/dt$ given by Eq.(2.36). In the saturation $\omega_{lc} = \omega + \Delta\omega$ where $\Delta\omega$ is the variation in frequency due to nonlinear effects given by

$$\Delta\omega = -\mu_i A_{lc}^2 = -\sigma \frac{\mu_i}{\mu_r} \quad (2.38)$$

using this definition the Landau equation can be rearranged and written as

$$\frac{dA}{dt} = \sigma A - \mu_r(1 - ic_3)|A|^2 A \quad (2.39)$$

where

$$c_3 = -\frac{\mu_i}{\mu_r} \quad (2.40)$$

$$\Delta\omega = \sigma c_3 \quad (2.41)$$

This way of describing the Landau equation allows a more natural extension to the Ginzburg-Landau equation (GLE) when thinking in a three-dimensional model for a wake.

In general σ is a function of the Reynolds number. Expanding the coefficient in power series in the neighborhood of Re_{cr} we get until first order

$$\sigma = constant (Re - Re_{cr}) \quad (2.42)$$

Substituting the asymptotic solution Eq.(2.33) we can see that module $|A|$ is proportional to the square root $(Re - Re_{cr})$

$$|A|_{max} \propto \sqrt{Re - Re_{cr}} \quad (2.43)$$

when $\mu < 0$ in Eq.(2.32) both terms in the expansion are not enough to determine the limit amplitude of the saturation. Then the following term proportional to $-\beta|A|^5$ with the $\beta > 0$ should be included. The dependence is shown in Fig.2.10b. Fig.2.10a shows the case $\mu > 0$. When $Re > Re_{cr}$ there is no steady flow; when $Re = Re_{cr}$ the disturbance achieves discontinuously a non-zero amplitude. In the range $Re'_{cr} < Re < Re_{cr}$ the unperturbed flow is metastable, being stable in relation to infinitesimal perturbations and unstable in relation to perturbations of finite amplitude.

Figure 2.10: Amplitude of the Landau equation in function of Re . (a) Case $\mu > 0$. (b) Case $\mu < 0$. Taking from Landau & Lifshitz (1987a)

2.5 Reduced models

One way to understand the complete dynamics of the flow around a cylinder at different Reynolds numbers is to create reduced models. These reduced models are generally constructed in Galerkin's framework (Tadmor et al. (2011); Noack et al. (2003)). The reduced models are based on a set of coherent structures obtained from the base field.

Noack et al. (2003) built a nicely reduced hierarchical model for the incompressible flow around a cylinder at low Reynolds numbers. The model is based on a Karhunen-Loeve decomposition of the numerical solution and incorporates a disturbed mode representing a correction of the mean flow. Noack et al. (2003) found that the inclusion of the mean flow correction improves the resolution of the transient dynamics from the onset of the vortex shedding to the Von Karman periodic flow, and can be even enhanced by enriching the space with the linear stability modes of the system (see Fig.2.11). The full Galerkin model proposed by Noack et al. (2003)

Figure 2.11: Transient dynamics between the base flow and the mean flow accounted by the shift mode, plotted in the vertical axis. The shift mode is not contained in the space generated by the base flow modes (a_1, a_2) . Taken from (TADMOR et al., 2011)

(Eq.2.44) is constructed based on the mean flow, a fluctuation formed by the POD modes calculated using the steady solution of the Navier Stokes equation, and a shift-mode obtained from the correction of the mean flow after a Gram-Schmidt orthonormalization technique in relation to the new Galerkin ansatz.

$$\mathbf{u} = \mathbf{u}_0 + a_1 \mathbf{u}_1^s + a_2 \mathbf{u}_2^s + a_\Delta \mathbf{u}_\Delta^s \quad (2.44)$$

Although the reduced space of Noack et al. (2003) works very well for the studied Reynolds numbers, it has the difficulty of being linked with the steady flow and the solution of the linear stability problem around it, a task that can become complicated for high Reynolds numbers. As the Reynolds number increases the recirculation bubble in the steady state grows considerably

with the Reynolds number (BARBEIRO, 2012) increasing the computational meshes and the computational cost. Additionally, to find the converged steady state at high Reynolds numbers using the Newton method implies to suppose a good initial guess to ensure the convergence of the iterative method. The initial guess must be the steady solution of a slightly smaller Reynolds number. Then, for example, to find the steady solution for $Re = 300$ an internal loop must be performed to solve a chain of steady states for $Re < 300$, which is not difficult but a bit tedious. The original mean flow can not be considered in the construction of the Galerkin model because the transition time between the fixed point of the model and the limit cycle is enormous, given that the system is linearly stable to infinitesimal perturbations (BARKLEY, 2006). Bearing in mind the idea of Noack et al. (2003), in chapter 5 a new Galerkin model is proposed using the mean flow plus an optimal perturbation of it. The optimization criterion will be to determine the optimum direction in which the most unstable eigenvalue tends to have the greatest linear growth, following the sensitivity theory developed by Marquet, Sipp & Jacquin (2008). In this fashion, the ansatz used in the construction of the Galerkin model is considered as a perturbation of the mean flow and not as a disturbance of the steady state. This approach avoids the use of the steady state at high Reynolds numbers, and at the same time allows defining a fixed point in the model that respects the restrictions in the growth rate of the disturbance as shown in chapter 5.

2.6 Outline of this work

The main objective of this work is to use the sensitivity theory to face two important problems in fluid mechanics: the problem of passive and active control wake in the flow around a cylinder, and the construction of an appropriate reduced space to elucidate the linear and non-linear dynamics of a perturbation at different Reynolds numbers.

The sensitivity theory is constructed in the framework of the adjoint method, for this reason, in chapter 3 and appendix A a detailed mathematical deduction is made for the adjoint Navier Stokes equations, additionally it is shown the theoretical construction of sensitivity theory as well as the numerical implementation using the programming language Freefem++. The sensitivity and adjoint results had already been shown in the renamed articles by Giannetti & Luchini (2007) and Marquet, Sipp & Jacquin (2008), so this chapter serves as a reference to show that our computational codes were correctly implemented. After computationally building the adjoint method and the sensitivity theory, the problem of wake control is addressed in chapter 4. In this chapter an extension of the theory developed by Marquet, Sipp & Jacquin (2008) is made with the purpose of including active wake control using rotating cylinders. Using the sensitiv-

ity theory, it was possible to predict the minimum gap as well as the optimal rotation rate of the control cylinders to suppress the wake in the flow around a cylinder at different Reynolds numbers, the results agree well with experimental findings of Mittal (2001) and Korkischko & Meneghini (2012). In chapter 5 we use the sensitivity theory to build a reduced model. The range of validity of the reduced models is restricted since they do not model, in principle, the deformations that happened during the wake transition. These limitations were outlined by Noack et al. (2003) and Tadmor et al. (2011) enriching the space with a shift mode, in view of the shift mode exists outside the space generated by the base field modes. This methodology allowed to predict the behavior of the mean flow as well as the transition from the stationary base flow to the periodic field developed in the flow around a cylinder. However, it presents the problem of setting the fixed point of the model as the stationary solution of the Navier Stokes equations, which is inconvenient for our archetype since the model presented in Chapter 5 represents the flow variables using an asymptotic expansion up to third order in the linear growth rate of a disturbance, forcing this parameter to be a small number, or in other words forcing the fixed point of the model to be close to the limit cycle for an arbitrary Reynolds number. For this reason the fixed point of the model was sought as an optimal perturbation of the mean flow. The optimization criterion was based on determining the optimal direction in which the unstable eigenvalues tend to have the highest linear growth rate, using the sensitivity to base flow modifications. Additionally, the space generated by the base flow modes were constructed using Fourier modes and not POD as in Noack et al. (2003), showing the potential use of this technique because it separates the flow frequencies, being possible to select the main frequencies in the flow spectrum. Finally in chapter 6 the final conclusions and future works are outlined. In Fig.2.12 a flow diagram of the structure of this thesis can be seen.

Figure 2.12: Flow diagram of this thesis

3 THEORETICAL FRAMEWORK AND NUMERICAL IMPLEMENTATION

3.1 Global modes of the stability problem

The motion of a fluid in a certain domain is described by the variables $\mathbf{Q} = (\mathbf{u}, P)$ that satisfy the incompressible Navier Stokes equations (NSE) Eq.(2.26). The velocity field in the computational mesh is defined in a cartesian coordinates system with components $\mathbf{u} = (u, v)$. The NSE Eq.(2.26) fulfills with the boundary conditions

$$\begin{aligned} \mathbf{u} &= (1, 0) \text{ at the inlet and lateral sides} \\ P\hat{n} - \text{Re}^{-1}(\nabla\mathbf{u}) \cdot \hat{n} &= 0 \text{ at the outlet} \end{aligned} \quad (3.1)$$

We are interested in studying the onset of the instability in the NSE applying a linear theory and analysis of normal modes. The field $\mathbf{Q}$ is decomposed into the sum of a stationary part $(\mathbf{U}, p)$ and a small non-stationary disturbance $(\mathbf{u}', p')$ as

$$\begin{aligned} \mathbf{u}(x, y, t) &= \mathbf{U}(x, y) + \mathbf{u}'(x, y, t) \\ P(x, y, t) &= p(x, y) + p'(x, y, t) \end{aligned} \quad (3.2)$$

replacing Eq.3.2 in Eq.2.26 and linearizing around the field $\mathbf{U}$ one gets two problems which describe the spatial structure of the base field and the temporal evolution of the disturbance. In particular, the base field is governed by the steady Navier Stokes equation $\text{SNE} = 0$, with

$$\text{SNE}(\mathbf{U}, p) = \nabla\mathbf{U} \cdot \mathbf{U} + \nabla p - \text{Re}^{-1}\nabla^2\mathbf{U} - \mathbf{F}; \quad \nabla \cdot \mathbf{U} = 0 \quad (3.3)$$

while the perturbation field is described by a set of Navier Stokes equations linearized around the base field

$$\partial_t \mathbf{u}' + \mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U} - \text{Re}^{-1}\nabla^2 \mathbf{u}' + \nabla p' = \mathbf{f}; \quad \nabla \cdot \mathbf{u}' = 0 \quad (3.4)$$

where $\mathbf{f}$ has been introduced to take account of an external forcing.

We seek by the global modes of the linearized NSE, that is, in non-trivial solutions of Eq.3.4 attempted in the form $(\mathbf{u}'(x, y, t), p') = (\hat{\mathbf{u}}, \hat{p})(x, y) \exp(\sigma t)$. σ is a complex number and the complex field $\hat{\mathbf{q}} = (\hat{\mathbf{u}}, \hat{p})$ satisfies the homogeneous equations

$$\sigma \hat{\mathbf{u}} + \nabla \hat{\mathbf{u}} \cdot \mathbf{U} + \nabla \mathbf{U} \cdot \hat{\mathbf{u}} + \nabla \hat{p} - \text{Re}^{-1} \nabla^2 \hat{\mathbf{u}} = 0; \quad \nabla \cdot \hat{\mathbf{u}} = 0 \quad (3.5)$$

with Dirichlet boundary conditions. The system of equations Eq.3.5 and defined as $\mathbf{P}(\sigma, \hat{\mathbf{q}}, \mathbf{Q}) = 0$ give rise to a generalized eigenvalues equation whose most unstable eigenvalue σ has a real part λ that represents the growth rate of the perturbation and a imaginary part ω that represents the frequency. In Eq.3.5 the perturbation is influenced by the base field by means of the two terms that arise from the linearization of the nonlinear term in Eq.2.26. These terms are $\nabla \hat{\mathbf{u}} \cdot \mathbf{U}$ representing the downstream transport of the disturbance by the base flow $\mathbf{U}$ and $\nabla \mathbf{U} \cdot \hat{\mathbf{u}}$ related with energy transfer of the perturbation from the base flow (MARQUET; SIPP; JACQUIN, 2008)

3.2 Adjoint equations and sensitivity analysis

The adjoint and sensitivity equations will be obtained using a Lagrange functional $\mathbf{L}$ defined introducing two Lagrange multipliers $\hat{\mathbf{q}}^\dagger = (\hat{\mathbf{u}}^\dagger, p^\dagger)$ and $\mathbf{Q}^\dagger = (\mathbf{U}^\dagger, P^\dagger)$. The Lagrange multipliers are required to accomplish with the constraints given by both the modal NSE of the perturbation $\mathbf{P}(\sigma, \hat{\mathbf{q}}, \mathbf{Q}) = 0$ (Eq.3.5) and the linearized NSE ($\mathbf{LNE}(\mathbf{Q})$) forced by an external force $\mathbf{Q}_F$. In this way the Lagrange functional is defined as

$$\mathbf{L}(\mathbf{Q}, \sigma, \hat{\mathbf{q}}, \mathbf{Q}_F, \mathbf{Q}^\dagger, \hat{\mathbf{q}}^\dagger) = \sigma - (\hat{\mathbf{q}}^\dagger, \mathbf{P}(\sigma, \hat{\mathbf{q}}, \mathbf{Q})) - (\hat{\mathbf{Q}}^\dagger, \mathbf{LNE}(\mathbf{Q}) - \mathbf{Q}_F) \quad (3.6)$$

where σ is the objective function, that is, the function whose optimum equilibrium point is sought. The functional Eq.3.6 depends on six independent variables: $\mathbf{Q}$ represents the velocities and pressure fields, σ the eigenvalue associated with the stability problem, $\hat{\mathbf{q}}$ the modes associated with the global stability problem, $\mathbf{Q}_F = (\mathbf{F}, 0)$ an arbitrary external force and $\hat{\mathbf{q}}^\dagger$, $\mathbf{Q}^\dagger$ the Lagrange multipliers needed to satisfy the constraints of the system. According to the definition of the gradient of a function with respect to a variable $\mathbf{a}$, the gradient is defined by (MARQUET; SIPP; JACQUIN, 2008)

$$\frac{\partial L}{\partial \mathbf{a}} \delta \mathbf{a} = \lim_{s \rightarrow 0} \frac{L(\mathbf{a} + s \delta \mathbf{a}) - L(\mathbf{a})}{s} \quad (3.7)$$

According to this definition the gradient of the Lagrange functional is

$$\begin{aligned} & \underbrace{-(\hat{\mathbf{q}}^\dagger, \mathbf{P}(\sigma, \hat{\mathbf{q}}, \delta \mathbf{Q})) - (\hat{\mathbf{Q}}^\dagger, \mathbf{LNE}(\delta \mathbf{Q}))}_{I} + \underbrace{\delta \sigma - (\hat{\mathbf{q}}^\dagger, \mathbf{P}(\delta \sigma, \hat{\mathbf{q}}, \mathbf{Q}))}_{II} \\ & + \underbrace{(\hat{\mathbf{q}}^\dagger, \mathbf{P}(\sigma, \delta \hat{\mathbf{q}}, \mathbf{Q}))}_{III} + \underbrace{(\mathbf{Q}^\dagger, \delta \mathbf{Q}_F)}_{IV} \end{aligned} \quad (3.8)$$

where the linear operator $\mathbf{LNE}(\delta \mathbf{Q})$ is explicitly done by

$$\mathbf{LNE}(\delta \mathbf{Q}) = \nabla \delta \mathbf{U} \cdot \mathbf{U} + \nabla \mathbf{U} \cdot \delta \mathbf{U} + \nabla \delta p - \text{Re}^{-1} \nabla^2 \delta \mathbf{U} = 0 \quad (3.9)$$

Differentiating Eq.3.6 in relation to the Lagrange multipliers $\hat{\mathbf{q}}^\dagger$ and $\mathbf{Q}^\dagger$ we obtain directly the constraints $\mathbf{P}(\sigma, \hat{\mathbf{q}}, \mathbf{Q}) = 0$ and $\mathbf{LNE}(\mathbf{Q}) = \mathbf{Q}_F$.

3.3 Adjoint equations

To determine the adjoint Navier Stokes equations we must consider the expression marked as III in Eq.3.8. The internal product in III is given by

$$III = \int_{\Omega} \hat{\mathbf{u}}^\dagger \cdot (\sigma \delta \hat{\mathbf{u}} + \nabla \delta \hat{\mathbf{u}} \cdot \mathbf{U} + \nabla \mathbf{U} \cdot \delta \hat{\mathbf{u}} - \text{Re}^{-1} \nabla^2 \delta \hat{\mathbf{u}} + \nabla \delta \hat{p}) + p^\dagger (\nabla \cdot \delta \hat{\mathbf{u}}) \quad (3.10)$$

and shows an explicit dependence on variable $\hat{\mathbf{u}}^\dagger$ and implicit in $\delta \hat{\mathbf{q}}$. Taking into account that in the optimum equilibrium point $III = 0$, we can exchange the explicit dependency on $\hat{\mathbf{u}}^\dagger$ by an explicit dependency on $\delta \hat{\mathbf{q}}$ performing several integration by parts, and determine the adjoint operators using the condition that $\delta \hat{\mathbf{q}}$ must not be null to avoid trivial solutions. Integrating by parts each of the terms in Eq.3.10 (see appendix A) and considering that $\delta \hat{\mathbf{q}} \neq 0$ we obtain

$$\sigma^* \hat{\mathbf{u}}^\dagger - \mathbf{U} \cdot \nabla \hat{\mathbf{u}}^\dagger + \nabla \mathbf{U} \cdot \hat{\mathbf{u}}^\dagger - \text{Re}^{-1} \nabla^2 \hat{\mathbf{u}}^\dagger + \nabla p^\dagger = 0; \quad \nabla \cdot \hat{\mathbf{u}}^\dagger = 0 \quad (3.11)$$

with boundary conditions

$$\begin{aligned} p^\dagger \hat{n} - \text{Re}^{-1} (\nabla \hat{\mathbf{u}}^\dagger \cdot \hat{n}) &= (\mathbf{U} \cdot \hat{n}) \hat{\mathbf{u}}^\dagger \text{ at the outlet} \\ \hat{\mathbf{u}}^\dagger &= 0 \text{ otherwise} \end{aligned} \quad (3.12)$$

The set of equations Eq.3.11 and Eq.3.12 define the set of adjoint Navier Stokes equations, where the adjoint field is interpreted as the Lagrange multiplier needed to find an optimal equilibrium point in the variation of the most unstable eigenvalue subject to the constraint that the modal Navier Stokes equation must be fulfilled.

3.4 Normalization

Eliminating the term II in Eq.3.8

$$II = \int_{\Omega} \hat{\mathbf{u}}^\dagger \cdot (\delta\sigma \hat{\mathbf{u}}) - \delta\sigma \quad (3.13)$$

we obtain the normalization condition between the direct and adjoint modes of the stability problem defined by

$$(\hat{\mathbf{u}}^\dagger \cdot \hat{\mathbf{u}}) = 1 \quad (3.14)$$

3.5 Sensitivity to a steady force

Now we consider the term I in Eq.3.8.

$$I = -(\hat{\mathbf{q}}^\dagger, \mathbf{P}(\sigma, \hat{\mathbf{q}}, \delta\mathbf{Q})) - (\hat{\mathbf{Q}}^\dagger, \mathbf{LNE}(\delta\mathbf{Q})) \quad (3.15)$$

From Eq.3.15 we see that I is composed by two terms. Doing successive integration by parts (see appendix A) considering that the variations in $\delta\mathbf{Q}$ are non-zero by definition one leads to the set of base field adjoint equations

$$\begin{aligned} -\nabla \mathbf{U}^\dagger \cdot \mathbf{U} + \nabla \mathbf{U} \cdot \mathbf{U}^\dagger - \nabla p^\dagger - \text{Re}^{-1} \nabla^2 \mathbf{U}^\dagger &= -\nabla \hat{\mathbf{u}}^* \cdot \hat{\mathbf{u}}^\dagger + \nabla \hat{\mathbf{u}}^\dagger \cdot \hat{\mathbf{u}}^* \\ \nabla \cdot \mathbf{U}^\dagger &= 0 \end{aligned} \quad (3.16)$$

with boundary conditions

$$\begin{aligned} \mathbf{U}^\dagger &= 0 \text{ at the inlet and solid walls} \\ (\partial_y U^\dagger, V^\dagger) &= (0, 0) \text{ on the lateral boundaries} \\ p^\dagger \hat{n} + \text{Re}^{-1} (\nabla \mathbf{U}^\dagger \cdot \hat{n}) &= -(\mathbf{U} \cdot \hat{n}) \mathbf{U}^\dagger + (\hat{\mathbf{u}}^* \cdot \hat{n}) \hat{\mathbf{u}}^\dagger \text{ at the outlet} \end{aligned} \quad (3.17)$$

keeping in mind that

$$\frac{\partial L}{\partial \mathbf{Q}_F} \delta \mathbf{Q}_F = (\mathbf{Q}^\dagger, \delta \mathbf{Q}_F) \quad (3.18)$$

and by definition

$$\frac{\partial L}{\partial \mathbf{Q}_F} \delta \mathbf{Q}_F = (\nabla_{\mathbf{Q}_F} \sigma, \delta \mathbf{Q}_F) \quad (3.19)$$

we note that the base field adjoint equations represent the sensitivity of the eigenvalue to an external force.

$$\nabla_{\mathbf{Q}_F} \sigma = \mathbf{U}^\dagger \quad (3.20)$$

3.6 Sensitivity to base flow modifications

Unlike the procedure of the Lagrangian functional Eq.3.6, in this case the state of the system is defined by the variables $\{\sigma, \hat{\mathbf{q}}\}$ (MARQUET; SIPP; JACQUIN, 2008) and the control variable is now the base field $\mathbf{Q}$. The same Lagrangian functional technique is used to determine the sensitivity equations to base flow modifications. The Lagrange multiplier is $\hat{\mathbf{q}}^\dagger$, the objective function is the eigenvalue and the constraint is the Navier Stokes equation for the perturbation Eq.3.5.

$$L(\mathbf{Q}, \sigma, \hat{\mathbf{q}}, \hat{\mathbf{q}}^\dagger) = \sigma - (\hat{\mathbf{q}}^\dagger, \mathbf{P}(\sigma, \hat{\mathbf{q}}, \mathbf{Q})) \quad (3.21)$$

Considering variations in the base field, we obtain

$$\frac{\partial L}{\partial \mathbf{Q}} \delta \mathbf{Q} = \frac{\partial \sigma}{\partial \mathbf{Q}} \delta \mathbf{Q} - (\hat{\mathbf{q}}^\dagger, \mathbf{P}(\sigma, \hat{\mathbf{q}}, \delta \mathbf{Q})) \quad (3.22)$$

The second term in Eq.3.22 is precisely the forcing term in the equation of sensitivity to external forces Eq.3.16. Defining $\frac{\partial \sigma}{\partial \mathbf{Q}} = \nabla_{\mathbf{U}} \sigma$ one obtains

$$\nabla_{\mathbf{U}} \sigma = -\nabla \hat{\mathbf{u}}^* \cdot \hat{\mathbf{u}}^\dagger + \nabla \hat{\mathbf{u}}^\dagger \cdot \hat{\mathbf{u}}^* \quad (3.23)$$

The first term in the right of Eq.3.23 is interpreted as the sensitivity to modifications in the transport of the disturbance by the base flow and the second term as the sensitivity to modifications in the production of the perturbation by the base flow (MARQUET; SIPP; JACQUIN, 2008)

3.7 Reduced models

The reduced model constructed in chapter 5 focuses on the representation of a two-dimensional flow at low to moderate Reynolds numbers. The model is built by the projection of the incompressible flow equations into an appropriate orthonormal basis, conformed by the Fourier modes plus a shift mode calculated from a perturbation of the mean flow using the sensitivity to base flow modifications (MARQUET; SIPP; JACQUIN, 2008). The temporal dynamics, the hydrodynamic forces as well as the stability properties are calculated in the reduced space and compared with the results of the physical space in chapter 5.

The shift mode $\delta \hat{\mathcal{E}}$ proposed is given by a perturbation of the mean flow. The field $\delta \hat{\mathcal{E}}$ is calculated from the sensitivity to changes in the base field defined by

$$\mathcal{S} = -\nabla \hat{\mathbf{u}}^* \cdot \hat{\mathbf{u}}^\dagger + \nabla \hat{\mathbf{u}}^\dagger \cdot \hat{\mathbf{u}}^*$$

Since the sensitivity field $\mathcal{S}$ is a complex entity we define $\mathcal{S} = \mathcal{R} + i\mathcal{C}$. With this fact in mind, the change in the most unstable eigenvalue of the global problem is given by the projection

$$\Delta\lambda = (\mathcal{R}_s + i\mathcal{C}_s) \cdot \hat{\mathcal{E}}$$

where the subscript s corresponds to the solenoidal components of the vector in order to accomplish with the null divergence condition in the NSE. In this manner, $\hat{\mathcal{E}}$ is the field needed to optimize the variations in the eigenvalue, i.e, it represents the direction of greatest sensitivity to changes in the mean flow. In this way, the particular solution of the NSE is given by $\mathbf{U}_p = \mathbf{U}_m + \delta\hat{\mathcal{E}}$ with $\mathbf{U}_m$ representing the mean flow. Solving the stability problem around the field $\mathbf{U}_p$ we obtain the most unstable mode $\hat{\mathbf{U}}_{mod}$ with which we construct the reduced space modes

$$\begin{aligned} \mathbf{J}_f &= [\mathbf{U}_\alpha, \text{Real}(\hat{\mathbf{U}}_{mod}), \text{Imag}(\hat{\mathbf{U}}_{mod})] \\ \alpha &= 1, 2, \dots, N \end{aligned}$$

with $\mathbf{U}_\alpha$ the Fourier modes. Finally, using a Gram-Schmidt orthogonalization process we build the reduced space base $\mathbf{T}$. This base allows to project all the NSE operators in the physical space into the reduced space as $\mathbf{O}_f = \mathbf{T}^t \cdot \mathbf{O} \cdot \mathbf{T}$ where $\mathbf{O}_f$ is an arbitrary operator in the reduced space and $\mathbf{O}$ an arbitrary operator in the physical space. Bearing this in mind, one has a dynamic and stability model in the reduced space. The details of these calculations are shown in chapter 5. With this procedure, it was possible to determine the linear and non-linear dynamics of a disturbance in the flow around a cylinder without the need of using the stationary field as in Noack et al. (2003).

3.8 Numerical implementation of the adjoint and sensitivity equations

3.8.1 Direct and adjoint modes

The solution of the differential equations involved in the global mode analysis, adjoint and sensitivity equations are obtained using the finite element method implemented using the Freefem++ programming language (HECHT, 2012). Fig.3.1 shows the steady base field solution obtained using the finite element method for Eq.3.3 at $\text{Re} = 50$. A boundary layer separation and the consequent formation of a recirculation bubble are observed. To study the stability of the system, we evolved a small perturbation in the base flow and analyzed the linear growth rate near the bifurcation point, which should happen around $\text{Re} = 47$ (BARKLEY; HENDERSON, 1996). From

Figure 3.1: Base field around a cylinder $Re = 50$. Base field streamlines are represented in white. Color is the velocity magnitude.

Figure 3.2: Eigenvalue spectrum isolated cylinder $Re = 50$

the eigenvalue spectrum given by equation Eq.3.5 and shown in Fig.3.2, we choose the most unstable eigenvalue whose corresponding eigenvector is depicted in Fig.3.3. The most unstable eigenvalue has a real part of $\lambda = 0.0136$ and an imaginary part of $\omega = 0.741$ ($f = 0.117$), agreeing with the results presented by Marquet, Sipp & Jacquin (2008). From the mode shown in Fig.3.3, the perturbation is transported downstream in the flow. With a similar procedure, it is possible to obtain the adjoint field from Eq.3.11 and Eq.3.12. Given that the transport terms in equations Eq.3.5 and Eq.3.11 have opposite signs, the perturbations in the adjoint system will be transported upstream, as can be seen in Fig.3.4. This subtle difference has serious implications in the definition of the regions of stability in the flow, as will be shown in the following sections.

Figure 3.3: Most unstable global mode $Re = 50$. Color represents the normalized velocity magnitude

Figure 3.4: Adjoint mode related to the most unstable mode $Re = 50$. Color is the normalized adjoint velocity magnitude

3.8.2 Wavemaker

Giannetti & Luchini (2007) introduced the concept of wavemaker by identifying regions of the fluid where structural modifications of the stability problem produce the greatest change in the most unstable eigenvalue. To determine this wavemaker, they studied the variation of the most unstable eigenvalue by forcing the global stability problem with a structural modification. To introduce this structural modification, we must consider the disturbed eigenvalue problem Eq.3.5 forced by the differential operator $\delta\mathbf{H}$ that expresses a structural perturbation of the differential equation to obtain

$$\sigma \hat{\mathbf{u}} + \nabla \hat{\mathbf{u}} \cdot \mathbf{U} + \nabla \mathbf{U} \cdot \hat{\mathbf{u}} + \nabla \hat{p} - Re^{-1} \nabla^2 \hat{\mathbf{u}} = \delta\mathbf{H}(\hat{\mathbf{u}}, \hat{p}); \quad \nabla \cdot \hat{\mathbf{u}} = 0 \quad (3.24)$$

A shift in the eigenvalue and in general in the eigenfunction $\delta\hat{\mathbf{q}} \equiv \{\delta\hat{\mathbf{u}}, \delta\hat{p}\}$ can be obtained using an expansion in terms of the unperturbed problem (GIANNETTI; LUCHINI, 2007). Assuming $\hat{\mathbf{u}} \rightarrow \hat{\mathbf{u}} + \delta\hat{\mathbf{u}}$, $\hat{p} \rightarrow \hat{p} + \delta\hat{p}$, $\sigma \rightarrow \sigma + \delta\sigma$ inserting in Eq.3.24 and neglecting quadratic terms in

the perturbation, we arrive at

$$\sigma \delta \hat{\mathbf{u}} + \nabla \delta \hat{\mathbf{u}} \cdot \mathbf{U} + \nabla \mathbf{U} \cdot \delta \hat{\mathbf{u}} + \nabla \delta \hat{p} - \text{Re}^{-1} \nabla^2 \delta \hat{\mathbf{u}} = -\delta \sigma \hat{\mathbf{u}} + \delta \mathbf{H}(\hat{\mathbf{u}}, \hat{p}); \quad \nabla \cdot \delta \hat{\mathbf{u}} = 0 \quad (3.25)$$

Multiplying the perturbed NSE Eq.3.25 by the adjoint field we obtain

$$\begin{aligned} \sigma \hat{\mathbf{u}}^\dagger \cdot \delta \hat{\mathbf{u}} + (\hat{\mathbf{u}}^\dagger \cdot \nabla \delta \hat{\mathbf{u}}) \cdot \mathbf{U} + (\hat{\mathbf{u}}^\dagger \cdot \nabla \mathbf{U}) \cdot \delta \hat{\mathbf{u}} + \hat{\mathbf{u}}^\dagger \cdot \nabla \delta \hat{p} - \text{Re}^{-1} \hat{\mathbf{u}}^\dagger \cdot \nabla^2 \delta \hat{\mathbf{u}} + p^\dagger (\nabla \cdot \delta \hat{\mathbf{u}}) = \\ -\delta \sigma \hat{\mathbf{u}}^\dagger \cdot \hat{\mathbf{u}} + \hat{\mathbf{u}}^\dagger \cdot \delta \mathbf{H}(\hat{\mathbf{u}}, \hat{p}) \\ \nabla \cdot \delta \hat{\mathbf{u}} = 0 \end{aligned} \quad (3.26)$$

Integrating by parts to change the explicit dependence to the variable $\delta \hat{\mathbf{u}}$, satisfying the constraint Eq.3.11 and taking into account the boundary conditions Eq.3.12 we arrive at (see appendix A)

$$\delta \sigma = \frac{\int_{\Omega} \hat{\mathbf{u}}^\dagger \cdot \delta \mathbf{H}(\hat{\mathbf{u}}, \hat{p})}{\int_{\Omega} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}}^\dagger} \quad (3.27)$$

This approach can be used to investigate the structural stability of the flow around a cylinder since the Eq.3.27 is valid for a generic structural perturbation. The expression Eq.3.27 can be even more simplified if we consider localized perturbations in the space proportional to the local velocity field. This type of localized structural modifications can proceed from different sources, including changes in the base field, in the geometry of the body or by inducing a localized feedback in the space (GIANNETTI; LUCHINI, 2007). Precisely the last approach will be used in this section to introduce the concept of wavemaker. Such localized feedback can be physically produced by introducing a small device which exerts a force whose direction and magnitude depend on the local velocity field, in this sense the device can be a small control cylinder in the fluid. Mathematically the localized feedback can be expressed as (MARQUET; SIPP; JACQUIN, 2008)

$$\delta \mathbf{H} = \delta(x - x_0, y - y_0) \mathbf{C}_o \cdot \hat{\mathbf{u}} \quad (3.28)$$

Where $\delta(x - x_0, y - y_0)$ is the Kronecker symbol and $\mathbf{C}_o$ is a constant matrix of coupling coefficients (for example drag coefficients in the case of a control cylinder). An upper bound for the eigenvalue drift due to a localized feedback mechanism can be determined using the Lagrange identities. In this way and using the normalization condition Eq.3.14, we obtain

$$|\delta \sigma| = \frac{\int_{\Omega} \hat{\mathbf{u}}^\dagger \cdot \mathbf{C}(x, y) \cdot \hat{\mathbf{u}}}{\int_{\Omega} \hat{\mathbf{u}}^\dagger \cdot \hat{\mathbf{u}}} \leq \|\mathbf{C}_o\| W(x_0, y_0) \quad (3.29)$$

Here we have defined $\mathbf{C} \equiv \delta(x - x_0, y - y_0) \mathbf{C}_o$ and the wavemaker function $W(x, y)$ as

$$W(x, y) = \frac{\|\hat{\mathbf{u}}^\dagger\| \cdot \|\hat{\mathbf{u}}\|}{\int_{\Omega} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}}^\dagger} = \|\hat{\mathbf{u}}^\dagger\| \cdot \|\hat{\mathbf{u}}\| \quad (3.30)$$

Figure 3.5: Wavemaker to spatially localized feedbacks $Re = 50$

The function $W(x, y)$ shows that the product between the direct and the adjoint mode provides the maximum coupling of the velocity components. The function $W(x, y)$ can be used to determine the regions where the wavemaker is strongest, identifying in this way the regions where the stability mechanism acts. The Fig.3.5 shows that the highest value of wavemaker is reached in two lobes located symmetrically along the separation bubble (GIANNETTI; LUCHINI, 2007). Note that in the upstream and downstream regions the overlap between the direct and the adjoint is small, showing that these areas are not important in the stability analysis of the system. Fig.3.5 recovers the results of Giannetti & Luchini (2007) and Marquet, Sipp & Jacquin (2008) and helps to validate our computational codes.

3.8.3 Sensitivity to base flow modifications

The sensitivity to base flow modifications is defined by $\nabla_{\mathbf{U}}\sigma$ in Eq.3.23. The real part of $\nabla_{\mathbf{U}}\sigma$ defined as the growth rate sensitivity $\nabla_{\mathbf{U}}\lambda$ is plotted in Fig.3.6 and the frequency sensitivity defined as $\nabla_{\mathbf{U}}\omega$ is plotted in Fig.3.7. The quantities $\nabla_{\mathbf{U}}\lambda$ and $\nabla_{\mathbf{U}}\omega$ are vector entities, hence the streamlines in Fig.3.6 and Fig.3.7 represent the direction of the vector and the colors the magnitude. In this sense the sensitivity to base flow modifications contains more information than the wavemaker defined by Giannetti & Luchini (2007) since the wavemaker is a scalar entity. The relationship between the sensitivity analysis and the wavemaker remains still unknown (MARQUET; SIPP; JACQUIN, 2008).

Near to the rear stagnation point of the cylinder the magnitude of the sensitivity to base flow modifications exhibits a maximum value, and the direction is upstream, therefore the presence of backflows in the recirculation region have a destabilizing effect. On the other side according to Fig.3.7, backflows have an opposite effect on the frequency sensitivity, that is, the frequency tends to decrease since the streamlines in Fig.3.7 are oriented downstream.

Figure 3.6: Growth rate sensitivity $\nabla_U \lambda$, $Re = 50$ Figure 3.7: Frequency sensitivity $\nabla_U \omega$, $Re = 50$

3.8.4 Sensitivity to a steady force

Fig.3.8 shows the sensitivity of the most unstable eigenvalue σ as a function of a stationary force $\mathbf{F}$ denoted by $\nabla_F \sigma$, and calculated in the proximity of the critical Reynolds number of the primary transition. The calculations were made according to that presented in section.3.5. Two maxima in the sensitivity to a steady force may be observed in Fig.3.8. The first is localized in the separation zone and the second one in the recirculation region (region formed by the separation bubble shown in Fig.3.1). The variation of an eigenvalue $\delta\sigma$ due to variations in an external force is formally defined by

$$\delta\sigma = (\nabla_F \sigma, \delta\mathbf{F}) \quad (3.31)$$

This operation involves the inner product between two complex vectors and it is defined by an integral of the type (MARQUET; SIPP; JACQUIN, 2008)

$$(\mathbf{A}_1, \mathbf{A}_2) = \int_{\Omega} (\mathbf{A}_1^* \cdot \mathbf{A}_2) d\Omega \quad (3.32)$$

with $\mathbf{A}_1, \mathbf{A}_2$ arbitrary vector fields. If the external force comes from a body whose size is much smaller than the size of the main cylinder, it can be assumed that the external force is localized at a point and being represented by a Dirac delta function. Under these conditions,

Figure 3.8: Sensitivity to a steady force $\nabla_F \sigma$ of the most unstable eigenvalue σ for $\text{Re} = 50$. Color is sensitivity magnitude and lines indicate its direction.

Figure 3.9: Frequency sensitivity to a steady force $\nabla_F \omega$ of the most unstable eigenvalue σ for $\text{Re} = 50$

the integral given in Eq.3.31 becomes an inner product between vectors. The sensitivity to an external force shown in Fig.3.8 allows inferring that a force with a considerable lift and applied locally in the separation region could also stabilize the flow. This idea will be explored in the following chapter. In the same way from Fig.3.8, it can thus be induced that a drag force applied locally in the outer region (including the point $(x, y) = (1.2, 1)$) may stabilize the flow, because the projection of the local drag vector on the sensitivity to a steady force (black line) would be at maximum, agreeing qualitatively with the experimental results of Strykowski & Sreenivasan (1990). The spatial distribution of the frequency sensitivity $\nabla_F \omega$ is shown in Fig.3.9. It is noted that a local force oriented in the opposite direction to the arrows shown in Fig.3.9 induces a negative growth in the eigenvalue frequency. In the recirculation region and in the separation region (red region in Fig.3.8) a downstream force would have a stabilizing effect on the growth rate of the eigenvalue and a destabilizing effect on the frequency since the flow lines in Fig.3.9 are always oriented downwards. On the other hand, in the outer region (green region in Fig.3.8) a stabilizing upstream force in the growth rate also has an effect of frequency decrease. Consequently, a drag body such as a small control cylinder in the outer

region of the cylinder would have a stabilizing effect coinciding with the optimal regions found by Strykowski & Sreenivasan (1990).

4 SENSITIVITY ANALYSIS APPLIED TO WAKE CONTROL

This chapter shows an attempt to apply the sensitivity theory to the problem of wake flow control in a cylinder. At $Re = 47$ the flow around a cylinder presents a Hopf bifurcation and consequently the system exhibits an alternating vortex shedding. An effective way to control the cylinder wake at low Reynolds numbers is to place small control cylinders in the fluid with and without rotation. The numerical and experimental problem with these type of devices is related with the number of control variables. In the particular case of active and passive control with cylinders, the control variables are: the diameter of the cylinders, the optimum cylinder's position and the rotation rate. Varying all these parameters until obtaining the set of optimal variables becomes an expensive and tedious computationally task. This chapter is based on the paper "Evaluating the control of a cylinder wake by the method of sensitivity analysis" published in Physics of fluids (doi.org/10.1063/1.4979482) and shows that the sensitivity theory can be used to choose efficiently the optimal control parameters using few numerical simulations and the solution of few linear stability problems.

Paper: *Evaluating the control of a cylinder wake by the method of sensitivity analysis*

Evaluating the control of a cylinder wake by the method of sensitivity analysis

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(Dated: 30 November 2017)

Numerical simulations and sensitivity analysis are carried out regarding control vortex shedding from a circular cylinder using small rotating and non-rotating control cylinders. Prediction of the changes in unsteady wake, instability growth rate and frequency produced by the control devices is presented in the framework of sensitivity analysis to steady field force. In the case of the rotating cylinders, this prediction relies on the observation that the small control cylinders introduce a steady force whose direction is not aligned to the base flow direction. This external force direction depends on the control cylinders rotation rates having an additional parameter to attain flow control. The evaluation of the control cylinders rotation rate within sensitivity analysis framework is a novel contribution in this work. The results of sensitivity analysis to a steady force are then compared to DNS computations of the flow fields serving as supporting data for discussion.

Keywords: Control cylinders. Sensitivity analysis. Stability analysis.

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4.1 Introduction

Understanding and controlling wakes of bluff bodies represents one of the most active fields in engineering given the practical importance of these objects in different branches of industry. The bluff bodies present several undesirable physical phenomena when immersed in a flow, such as high levels of drag and lift fluctuations. The proximity between the vortex shedding frequency and the natural frequency of the structure may lead to oscillations of large amplitudes known as vortex induced vibrations (VIV), and may result in structural fatigue, thus controlling the vortex shedding is an important way to prevent VIV in engineering problems. Several numerical and experimental studies in a wide range of Reynolds numbers have long focused on this problem (WILLIAMSON, 1996). The vortex shedding control leads to a reduction in the unsteady forces acting on the body or changes in its frequency which may significantly reduce the vibration amplitude. The controlled flow can be achieved by momentum transfer, either in boundary layer or wake, resulting in changes of wake dynamics.

The control mechanisms of vortex wakes can be classified as passive, both active with closed and open loop. The passive techniques are related with changes in the body geometry, affecting the formation of vortex shedding, and they do not require external power to be supplied. Some geometric modifications to suppress the vortex shedding include end-plates (SZEPESSY; BEARMAN, 1992), and helical strakes (KORKISCHKO; MENEHINI, 2010). Another typical example of passive controllers is small control cylinders strategically located in the wake of the main cylinder (STRYKOWSKI; SREENIVASAN, 1990; KUO; CHIOU; CHEN, 2007; DIPANKAR; SENGUPTA; TALLA, 2007). In an effort to study passive controllers, Strykowski & Sreenivasan (1990) reported that the vortex shedding generated by the flow around a circular cylinder can be controlled in a limited range of Reynolds numbers by locating a small control cylinder in the wake of the main cylinder. The region where the control cylinder is located is slightly above the higher vorticity region. The size of this region decreases with the Reynolds number and at the limit, it reaches a critical region defined by a point. Mittal & Raghuvanshi (2001) numerically verified the experimental results of Strykowski & Sreenivasan (1990) and concluded that the control cylinder generates a favorable pressure gradient in the wake region, responsible for locally stabilizing the flow by injecting momentum in the flow.

As in the case of passive control, the active control mechanisms have also been intensively studied. There is a wide variety of active control devices to suppress the vortex shedding in bluff bodies. Among them we find: acoustic excitations (BLEVINS, 1985), changes in the temperature of the body (LECORDIER; HAMMA; PARANTHOEN, 1991), oscillations in the cylinder modulating it with an appropriate frequency (TOKUMARU; DIMOTAKIS, 1993), moving sur-

face boundary layer control (MSBC)(KORKISCHKO; MENEGHINI, 2012).

Moving surface boundary layer control (MSBC)(KORKISCHKO; MENEGHINI, 2012) injects momentum into the flow field near the body by means of small rotating cylinders. In this type of active control, one of the most important control parameters is established by the ratio between the tangential velocity of the rotating cylinders and the free flow velocity U_c/U . Modi (1997) applied the MSBC method to different bluff bodies, finding that for a rotation rate $U_c/U \geq 4$ a narrowing in the wake, a delay in the separation, and a considerable reduction in the drag are observed. The MSBC was applied by Patnaik & Wei (2002) to the flow around a cylinder of D section for $Re = 200$ e $Re = 400$ and $U_c/U = 1.25$. Later, Muddada & Patnaik (2010) used this control strategy applied to the case of a cylinder with two rotating actuators located at 120° in relation to the upstream stagnation point and by Zhu et al. (2015) locating the cylinders at 45° , -45° from downstream point attenuating the VIV response in high Reynolds numbers. Other examples of MSBC to suppress vortex shedding may be found in Beaudoin et al. (2006).

Mittal (2001) studied the effect of the gap between the control cylinder and the main body in the control flow around the principal cylinder. The control cylinders were located at 90° and the Reynolds numbers studied were $Re = 100$ and $Re = 10000$. The rotating cylinders generate circulation and the momentum injection is more effective to affect the main cylinder wake. If the gap is too large, the effect in the main cylinder is expected to be small even at high rotation rates of the control cylinders. On the other hand, if the gap is too small, the interaction between the control cylinder and the main cylinder could be great, yet the control cylinders could not generate enough circulation to control the wake at a high Reynolds number. The gap is thus an important parameter in the control of the system. Mittal (2001) found that for $Re = 100$ and $U_c/U = 5$ the flow reached a steady state, while it was not possible to reach it at $Re = 10000$. However, in this regime the effect of the actuators was to organize and to narrow the wake of the main cylinder when compared with the case without a control.

Korkischko & Meneghini (2012) carried out a parametric study based on the variation of the ratio U_c/U in the flow around a circular cylinder controlled by two small cylinders. The controllers were located 90° from the main cylinder to experimentally verify the effectiveness of the MSBC as a VIV suppressor with low drag. In general, the main contributions of the MSBC are: delaying the boundary layer growth by minimizing the relative motion between the surface and the free flow, transferring momentum into the boundary layer and creating a high suction accelerating region close to the boundary layer (MUDDADA; PATNAIK, 2010).

The grade of control in the MSBC depends on several parameters. Some of these include the number of control cylinders, the diameter of the control cylinders, their rotation rate

and the gap between the main cylinder and the control devices. To vary all these parameters would be extremely expensive from the experimental and numerical point of view. Each parameter variation would require a DNS of the flow to evaluate its effect resulting in a massive amount of computational time. This issue has motivated the use of sensitivity theory introduced by Hill (1992), which is a practical manner to evaluate the effects of an added steady force on the primary transition of the wake with much fewer simulations: base flow, stability and adjoint problem. Sensitivity analysis to steady force has been applied thoroughly by Giannetti & Luchini (2007); Marquet, Sipp & Jacquin (2008); Pralits, Brandt & Giannetti (2010) and Hill (1992) was first in presenting a framework for analyzing the stabilization of wake flows. Giannetti & Luchini (2007) introduced the concept of wavemaker region in the wake of a circular cylinder, already characterizing regions where instabilities were amplified and regions where flow control could be achieved. Marquet, Sipp & Jacquin (2008) have presented the theoretical formulation to assess the effect of a steady force on the base flow and on the unstable mode, quantifying the change on the growth and frequency of the instability. As an example of application, Marquet, Sipp & Jacquin (2008) have compared their prediction of a circular cylinder wake control using a small non-rotating cylinder to the experimental observations of Strykowski & Sreenivasan (1990) satisfactorily. Pralits, Brandt & Giannetti (2010) have extended the sensitivity analysis to the wake of a rotating circular cylinder, also quantifying base flow parameters and different regions of instability growth due to the modified base flow around the rotating main cylinder. The authors focused on the passive control of the wake instabilities in the flow around the rotating cylinder. Herein, we focus, as in Marquet, Sipp & Jacquin (2008) on the wake of a non-rotating circular cylinder in the presence of small rotating and non-rotating control cylinders. Hence, common data of Pralits, Brandt & Giannetti (2010) and the present work are particularly focused on the drag and lift forces of a rotating circular cylinder and the theoretical formulation of sensitivity analysis.

The main contribution of this work is the sensitivity analysis of wake instability growth rate and frequency to the force generated by small rotating circular cylinders. At constant rotation, the small circular cylinder introduces a steady force in the flow whose direction is not aligned to the base flow direction. This steady force depends on the cylinder rotation rates, hence one has an additional resource to attain flow control. As far as we are aware, this analysis with rotating cylinders is a novel contribution although several papers have studied wake control with control cylinders either experimentally (MUDDADA; PATNAIK, 2010; KORKISCHKO; MENEGHINI, 2012) or by CFD (MITTAL, 2001; ZHU et al., 2015). Within the framework of sensitivity analysis to steady forces, the required rotation rate to suppress vortex shedding is predicted and compared to DNS flow computations in this paper. Although these computations

are presented at low Reynolds number range close to the primary instability of a circular cylinder wake, Reynolds numbers from 50 to 120 are brought into discussions. As the main results, sensitivity analysis to steady force from rotating small circular cylinders is applied to address the transition of a circular cylinder wake with low computational cost and to discuss the physics of MSBC flow control found in literature. In fact, this work shows that the best position for a given rotation rate of the control cylinders is not close to the boundary layer, although many authors have used this as argument for positioning their control devices

The paper is organized in the following manner: the next section presents methodology and computational results validation, which is followed by the problem definition and discussions, and finally closed by the conclusions section.

4.2 Theoretical formulation

We consider the viscous flow around a body in a closed domain with boundary conditions (BC) properly defined. The flow is described by the incompressible Navier-Stokes equation

$$\partial_t \mathbf{u} + \nabla \mathbf{u} \cdot \mathbf{u} + \nabla p - Re^{-1} \nabla^2 \mathbf{u} = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad (4.1)$$

Eq.(4.1) is non-dimensional and the Reynolds number (Re) $Re = \rho U_\infty L / \mu$ is the control parameter. L represents a characteristic length of the body, U_∞ the farfield flow velocity and μ the dynamic viscosity. The boundary conditions (BC) applied to Eq.(4.1) are: on the inlet and lateral sides $\mathbf{u} = (1, 0)$, on the body boundary $\mathbf{u} = (0, 0)$ and on the outlet $p \hat{n} - Re^{-1} (\nabla \mathbf{u}) \cdot \mathbf{n} = 0$. Eq.(4.1) with the BC described above allows a steady solution $(\mathbf{U}, P)$ that satisfies

$$\nabla \mathbf{U} \cdot \mathbf{U} + \nabla P - Re^{-1} \nabla^2 \mathbf{U} = \mathbf{F}, \quad \nabla \cdot \mathbf{U} = 0 \quad (4.2)$$

with the same BC as Eq.(4.1). In order to study the stability of this solution, the flow $(\mathbf{u}, p)$ is described as the superposition of $(\mathbf{U}, P)$ considered as a base field and an infinitesimal perturbation $(\mathbf{u}', p')$, which time evolution, neglecting second order terms, is governed by the equation

$$\frac{\partial \mathbf{u}'}{\partial t} = -(\mathbf{U} \cdot \nabla) \mathbf{u}' - (\mathbf{u}' \cdot \nabla) \mathbf{U} - \nabla p' + Re^{-1} \nabla^2 \mathbf{u}', \quad \nabla \cdot \mathbf{u}' = 0 \quad (4.3)$$

In order to find a solution of the stability problem, we can use the finite elements method and penalties getting to

$$\mathbf{M} \dot{\mathbf{u}}' + \mathbf{N}(\mathbf{U}) \mathbf{u}' + \mathbf{U}(\mathbf{N}) \mathbf{u}' + Re^{-1} \mathbf{D} \mathbf{u}' - \mathbf{R} p' = 0, \quad \mathbf{R}^T \mathbf{u}' = \epsilon p' \quad (4.4)$$

with $\mathbf{N}(\mathbf{U}) \equiv \mathbf{U} \cdot \nabla(\cdot)$, $\mathbf{U} \cdot \mathbf{N} \equiv \nabla \mathbf{U} \cdot (\cdot)$, $\mathbf{M}$ the mass matrix, $\mathbf{D}$ the Laplacian matrix (diffusion) and $\mathbf{R}$ the gradient matrix. The parameter ϵ was defined as 1×10^{-8} as shown in section 5.2. Putting the solution in modes, we obtain a generalized eigenvalue equation with the form

$$\mathbf{A}\hat{\mathbf{u}} = \sigma\hat{\mathbf{u}}\mathbf{B} \quad (4.5)$$

in which σ is the eigenvalue associated to the fields $(\hat{\mathbf{u}}, \hat{p})$. The solutions $(\hat{\mathbf{u}}, \hat{p})$ obtained from Eq.(4.5) compose the set of global linear modes of the stability problem. Each mode has a complex eigenvalue $\sigma = \lambda + i\omega$ where the real part λ represents the linear growth rate of the mode and the imaginary part corresponds to the circular frequency.

4.2.1 Sensitivity to a steady force

In this case, the eigenvalue σ is interpreted as a function of the steady force $\mathbf{F}$. In order to build this analysis, we must consider small variations in the base flow $\delta\mathbf{U}$ induced by small variations in the force $\delta\mathbf{F}$ via Eq.(4.1). Variations in the eigenvalue $\delta\sigma$ owing to small variations in the force $\delta\mathbf{F}$ are determined by (MARQUET; SIPP; JACQUIN, 2008)

$$\delta\sigma = (\nabla_F \sigma, \delta\mathbf{F}) \quad (4.6)$$

where $\nabla_F \sigma$ defines the sensitivity to a steady force modifications. The change in the growth rate $\nabla_F \lambda$ is given by the real part of $\nabla_F \sigma$ and the sensitivity in frequency by $\nabla_F \omega = -\text{Im}(\nabla_F \sigma)$. The coefficient $\nabla_F \sigma$ can be calculated as (MARQUET; SIPP; JACQUIN, 2008)

$$\nabla_F \sigma = \mathbf{U}^\dagger \quad (4.7)$$

where $\mathbf{U}^\dagger$ is the adjoint of the base flow satisfying

$$-\nabla \mathbf{U}^\dagger \cdot \mathbf{U} + (\nabla \mathbf{U}^T) \cdot \mathbf{U}^\dagger - \nabla P^\dagger - Re^{-1} \nabla^2 \mathbf{U}^\dagger = \nabla_U \sigma, \quad \nabla \cdot \mathbf{U}^\dagger = 0 \quad (4.8)$$

with BC defined as

$$\mathbf{U}^\dagger = \mathbf{0}, \quad \text{On the inlet} \quad (4.9)$$

$$(\partial_y U^\dagger, V^\dagger) = (0, 0), \quad \text{on the lateral boundaries} \quad (4.10)$$

and on the outlet side

$$\hat{P}^\dagger \hat{\mathbf{n}} - Re^{-1} (\nabla \mathbf{U}^\dagger) \cdot \hat{\mathbf{n}} = -(\mathbf{U} \cdot \hat{\mathbf{n}}) \mathbf{U}^\dagger + (\hat{\mathbf{u}}^* \cdot \hat{\mathbf{n}}) \hat{\mathbf{u}}^\dagger \quad (4.11)$$

Note that this field involves sensitivity to changes in the base field which acts as a forcing to sensitivity equation. $\nabla_U \sigma$ is determined as

$$\nabla_U \sigma = -(\nabla \hat{\mathbf{u}})^H \cdot \hat{\mathbf{u}}^\dagger + \nabla \hat{\mathbf{u}}^\dagger \cdot \hat{\mathbf{u}}^* \quad (4.12)$$

H represents the conjugate transpose operator and $(\hat{\mathbf{u}}^\dagger, \hat{p}^\dagger)$ is the adjoint mode associated with $(\hat{\mathbf{u}}, \hat{p})$ and satisfies

$$\sigma^* \hat{\mathbf{u}}^\dagger - \nabla \hat{\mathbf{u}}^\dagger \cdot \mathbf{U} + (\nabla \mathbf{U})^T \cdot \hat{\mathbf{u}}^\dagger = -\nabla \hat{p}^\dagger + Re^{-1} \nabla^2 \hat{\mathbf{u}}^\dagger, \quad \nabla \cdot \hat{\mathbf{u}}^\dagger = 0 \quad (4.13)$$

The BC are: $\hat{\mathbf{u}}^\dagger = 0$ on the inlet and body surface. On the lateral sides, we impose

$$(\hat{u}^\dagger, \hat{v}^\dagger) = (1, 0) \quad (4.14)$$

And on the outlet, the conditions are established by

$$\hat{p}^\dagger \hat{\mathbf{n}} - Re^{-1} (\nabla \hat{\mathbf{u}}^\dagger) \cdot \hat{\mathbf{n}} = (\mathbf{U} \cdot \hat{\mathbf{n}}) \hat{\mathbf{u}}^\dagger \quad (4.15)$$

Finally, the modes are normalized with the condition

$$(\hat{\mathbf{u}}^\dagger, \hat{\mathbf{u}}) = 1 \quad (4.16)$$

4.3 Methodology

The accuracy of the finite element method, as many other computational methods, depends on the discretization of the computational domain. Therefore, convergence tests must be performed in order to obtain an adequate mesh, defined as the mesh by which we can obtain accurate results within a tolerance range using the lowest degree of refinement. The simulations are DNS and the velocity and pressure fields were computed using FreeFem (HECHT, 2012). The same method was used in He et al. (2017) in DNS and flow stability analysis. We use second order for the velocity and first order polynomial pressure fields in the Taylor-Hood spatial discretization. This space satisfies the div-stability condition and it has 2nd-order overall accuracy (GUNZBURGER, 1989).

The mesh presents five boundaries (Fig.4.1): Γ_i defines the circular inlet boundary, Γ_u defines the upper horizontal boundary, Γ_l defines the lower horizontal boundary, Γ_o the outlet vertical boundary and Γ_w the cylinder boundary. On the circular boundary Γ_i , upper horizontal Γ_u and lower horizontal Γ_l , we imposed a BC $\mathbf{u} = (1, 0)$. The right vertical boundary Γ_o represents the outlet zone; hence, the BC imposed on Eq.(4.1) was used. Finally, on the cylinder

Figure 4.1: Geometrical description of the mesh. Length in diameters. Not in scale. Γ_i : inlet boundary; Γ_u : upper horizontal boundary; Γ_l : lower horizontal boundary; Γ_o : vertical boundary; Γ_w : cylinder boundary; γ_t and γ_b : density of nodes in the wake region; γ_c : density of nodes in the boundary layer region

surface, a boundary condition of a wall without slip is imposed.

Several convergence tests were performed in order to prove the influence of the external boundary conditions on the physical properties of the base flow by modifying the limits of the computational domain. The main features of the stationary flow were validated with data found in the literature.

Table 5.2 compares the drag coefficient calculated for different meshes with the results reported by Henderson (1995) for $Re = 50$. Four different meshes denoted by M1 to M4 were used, they exhibit different spatial extensions and different densities. The extent of the meshes is governed by the parameters Γ_i and Γ_u while the densities by the parameters (see Fig. 4.1) γ_i (defined in Γ_i), γ_u defined in (Γ_u) , γ_o (defined in the output Γ_o), γ_t and γ_b that define the density in the wake region, and γ_c that defines the density of nodes in the boundary layer region. M1 is the mesh of largest size and density tested. M2 has the same density as M1 but is smaller. M3 and M4 have the same spatial extent but M3 is denser. Finally mesh M2 was chosen for the calculation of the base field, given that it is the smallest mesh that meets the convergence criterion defined as a variation smaller than 1% in the calculation of the base flow forces with a reasonable computational cost.

The stability and sensitivity analysis was fulfilled in domains of different sizes to verify its influence in the eigenvalue growth and frequency. Four meshes from M1s to M4s were tested. As in the case of the base flow calculation, the extent of the meshes is governed by the parameters Γ_i and Γ_u . The density of the meshes was fixed at the mesh density M2 and only the sizes of the domains were varied. The eigenvalue growth rate and frequency used as reference are those reported by Giannetti & Luchini (2007). In addition, the sensitivity to external forces

Table 4.1: Drag sensitivity to the size of the computational mesh. The drag coefficient reference is $C_d = 1.4121$ reported by Henderson(HENDERSON, 1995). Γ 's length in diameters and γ 's density in nodes per diameter

Mesh	Γ_i	Γ_u	γ_c	γ_i	γ_u	γ_o	γ_t	C_d	$C_d(\%)$
M1	64	200	270	15	30	90	300	1.400	0.8%
M2	32	100	270	15	30	90	300	1.398	1.0%
M3	32	50	270	15	30	90	300	1.397	1.06%
M4	32	50	180	10	20	60	200	1.396	1.13%

at an arbitrary point $(x_0, y_0) = (1.0, 0.0)$ was calculated observing a good convergence as can be seen in Table 5.3. The results are in agreement with those obtained by Marquet, Sipp & Jacquin (2008). The final mesh adopted for the stability and sensitivity analysis was M2s. The mesh used in the stability and sensitivity calculations has been chosen for flow quantities with variation less than 2% in relation to Giannetti & Luchini (2007) and Marquet, Sipp & Jacquin (2008).

Table 4.2: Eigenvalue σ and sensitivity to external forces $\nabla_F \sigma$ variations as function of the computational mesh. The value of $\mathcal{Re}(\sigma) = 0.01295$ and $\mathcal{Im}(\sigma) = 0.7502$ taken as reference were reported by Giannetti and Luchini(GIANNETTI; LUCHINI, 2007). $\mathcal{Re}(\sigma)$ and $\mathcal{Im}(\sigma)$ represent the real and imaginary part of the eigenvalue. The Γ 's lengths are measured in diameters D .

Mesh	Γ_i	Γ_u	$\mathcal{Re}(\sigma)$	$\mathcal{Re}(\sigma)(\%)$	$\mathcal{Im}(\sigma)$	$\mathcal{Im}(\sigma)(\%)$
M1s	20	50	0.01300	0.38%	0.7435	0.90%
M2s	20	30	0.01321	2.00%	0.7352	2%
M3s	10	20	0.01324	2.24%	0.7326	2.35%
M4s	5	10	0.01502	15.98%	0.7397	1.40%

According to Eq.4.4 the stability problem depends on the choice of the penalty coefficient ϵ . With the aim of establishing how much the most unstable eigenvalue is sensitive with this parameter, we carried out a new convergence test using the mesh M2s and varying the coefficient ϵ as can be noted in Table 5.4. We can observe that the eigenvalue obtained with the coefficient $\epsilon = 1 \times 10^{-8}$ presents a good agreement with the test case and will be used for all the stability calculations.

Finally in order to accomplish a complete convergence analysis, we tested the variation of the most unstable eigenvalue as a function of the downstream boundary condition and Reynolds number, since some results presented here explore the stabilization in a cylinder wake for several regimes. Table 4.4 presents the variation in the real part and frequency of the most unstable eigenvalue. For certain meshes one can note a good convergence with the results

Table 4.3: Eigenvalue sensitivity to variations in the penalty coefficient ϵ . Calculations were made in the mesh M2s

ϵ	$\mathcal{Re}(\sigma)$	$\mathcal{Re}(\sigma)(\%)$	$\mathcal{Im}(\sigma)$	$\mathcal{Im}(\sigma)(\%)$
1×10^{-10}	0.01321	2%	0.7351	2%
1×10^{-8}	0.01321	2%	0.7352	2%
1×10^{-5}	0.01370	5.7%	0.7352	2%

shown in Giannetti & Luchini (2007) for different Reynolds numbers and Marquet, Sipp & Jacquin (2008) for $Re = 50$. As can be seen in Table 4.4 the mesh M_{2s} satisfies the convergence criteria for each Reynolds number.

Table 4.4: Eigenvalue sensitivity to variations in the downstream boundary condition and Reynolds number. Γ 's length in diameters. Reference values are (GIANNETTI; LUCHINI, 2007)

Re = 50					
Mesh	Γ_u	$\mathcal{Re}(\sigma)$	$\mathcal{Re}(\sigma)(\%)$	$\mathcal{Im}(\sigma)$	$\mathcal{Im}(\sigma)(\%)$
M1s	50	0.013	0.38%	0.7435	0.9%
M2s	30	0.01321	2.40%	0.7352	2%
M3s	10	0.01512	16.8%	0.7395	1.42%
Re = 100					
Mesh	Γ_u	$\mathcal{Re}(\sigma)$	$\mathcal{Re}(\sigma)(\%)$	$\mathcal{Im}(\sigma)$	$\mathcal{Im}(\sigma)(\%)$
M1s	50	0.123	0.81%	0.7316	0.38%
M2s	30	0.1264	1.93%	0.7415	1.75%
M3s	10	0.097	21.7%	0.7512	3.01%

4.4 Results

4.4.1 Choosing The Gap

The gap is considered one of the most important parameters for seeking an optimal wake control. In principle, there are several possible gaps that can lead to stabilization of the wake. The main purpose of this section is to use the sensitivity theory to determine an appropriate gap, defined as the gap closest to the main cylinder and located at the cylinder poles i.e. in the region $x = 0$ of Fig.4.2(a). The region $x = 0$ was chosen to keep as much symmetry as possible in relation to the incident flow. Fig.4.2(a) shows the variation of the eigenvalue growth rate $\delta\lambda$

Figure 4.2: (a) Growth rate variation contours $\delta\lambda = -0.01321$. $\delta\lambda$ is given by the real part of Eq.(4.6). Colors represent different rotation rates. Gray $\alpha = 0$, red $\alpha = 2$, black $\alpha = 4$. $Re = 50$, $d/D = 0.06$. The points are the most sensitive regions. Gray point coordinates: $(x_0, y_0) = (0.8, 0.9)$. Red and black: $(x_0, y_0) = (0.0, 0.07)$. The rotation rate is positive so clockwise rotating cylinder is represented. (b) Values of $\delta\lambda$ along $x = 0$

contours given by the real part of Eq.(4.6) as function of the external force modeled by Eq.(4.17)

$$\delta\mathbf{F} = -\frac{1}{2}dC_d \parallel \mathbf{U} \parallel (U_i\hat{e}_i + U_j\hat{e}_j)\delta(x - x_0, y - y_0) + \frac{1}{2}dC_l \parallel \mathbf{U} \parallel (-U_j\hat{e}_i + U_i\hat{e}_j)\delta(x - x_0, y - y_0) \quad (4.17)$$

with (x_0, y_0) as an arbitrary point where the external force can be applied. Eq.(4.17) represents the combination of the drag and lift force and can be interpreted as the external force exerted by a small control cylinder, where the drag and lift coefficients C_d and C_l directly depend on the rotational rate of the controllers $\alpha = \Omega d/2U_\infty$. Here Ω represents the angular velocity of the control cylinder. d refers to the diameter of the small control cylinders. $\hat{e}_i$ and $\hat{e}_j$ are unit vectors parallel and perpendicular to the free stream. U_i and U_j are the components of the local

velocity vector in the direction of the unit vectors.

The contour lines shown in Fig.4.2(a) were calculated for three different rotation rates and show the regions where the eigenvalue decreases in a rate equal to the growth rate of the most unstable eigenvalue λ defined as $\lambda = 0.01321$ for $Re = 50$. The points marked in Fig.4.2(a) highlight the regions where the eigenvalue is most susceptible to external forces. It is remarkable to show that when $\alpha = 0$ (pure drag force) the largest decrease in the eigenvalue occurs at $(x_0, y_0) = (0.8, 0.9)$, which is close to the region $(x_0, y_0) = (1.2, 1.0)$ found experimentally by Strykowski & Sreenivasan (1990). It is also noted that these positions (x_0, y_0) are not in the boundary layers. As the rotation rate increases from zero to $\alpha = 2$ (represented by a red line) and $\alpha = 4$ (represented by a black line) the stability regions and the minimum sensitivity to steady forces (red and black dots) collapse in particular regions. In order to find the appropriate gap, contour values along $x = 0$ were stored and plotted in Fig.4.2(b). The most unstable eigenvalue $\lambda = 0.01321$ was added to the variation of the growth rate $\delta\lambda$ in Fig.4.2(b), so that the zero-crossing of the curves show the appropriate gaps for the different rotation rates. As in the case shown in Fig.4.2(a), the minimum gap between the main cylinder and the controllers for different rotation rates begins to accumulate around a particular region defined by $g/D \approx 0.07$.

4.4.2 Choosing The Rotation Rate

It is very interesting to achieve wake control manipulating the rotation rate of the control cylinders. For this, we calculated the minimum rotation rate α_{min} ($\alpha = \Omega d/2U_\infty$) that stabilizes the flow for the chosen gap. This minimum rotation rate can be obtained using two different methods: First, calculating the growth rate of the most unstable eigenvalue λ as function of the rotation rate α , computed for the main cylinder in interference with two control cylinders located in the minimum gap $g = 0.07D$ for different Reynolds numbers. The results obtained are shown in Fig.4.3(a) for $Re = 50$. From Fig.4.3(a), it can be seen that the minimum rotation rate is approximately $\alpha_{min} \approx 1.89$ for $Re = 50$ and $g = 0.07D$. At this limit, the most unstable eigenvalue is marginally stable as shown in Fig.4.3(b). The second method used to find the minimum rotation rate, involved avoiding the solution of the stability problem for several rotation rates. This was done, by using the sensitivity analysis method only solving a single stability and sensitivity problem for each Reynolds number. According to Eq.(4.6), we have a relation between a particular eigenvalue, the sensitivity field and the external force, where the external force is modeled with Eq.(4.17). Having determined the sensitivity and velocity values, we can estimate the coefficients C_d and C_l needed to stabilize the flow through a linear relation by using Eq.(4.6). This linear relation is the straight line plotted in Fig.4.4. The circular points

Figure 4.3: (a) Growth rate λ of the most unstable eigenvalue as function of rotation rate of two control cylinders $\alpha = \Omega d/2U_\infty$. (b) Eigenvalue spectrum of main cylinder in interference with control cylinders, $Re = 50$. Gap $g/D = 0.07$, $d/D = 0.06$

in Fig.4.4 represent the C_d and C_l coefficients as function of the rotation rate for the control cylinders in interference with the main cylinder. The intersection between the force coefficients curve with the sensitivity line in Fig.4.4 determines the minimum rotation required to stabilize the flow. The minimum rotation rate obtained using sensitivity was $\alpha = 1.9$ for point (1) in Fig. 4.4. This value is very close to the value previously obtained through solving the stability problem $\alpha = 1.89$ depicted in Fig.4.3(a).

Even though the wake control near the first instability ($Re = 50$) was our main study interest, the linear stability theory employed can be used to predict the control of vortex shedding in systems in which the Reynolds number is $Re > 50$. In Fig.4.5 we present the calculated minimum rotation rate required to control wakes in systems with different Reynolds numbers obtained using sensitivity analysis with the same procedure summarised in Fig.4.4. The continuous line in Fig.4.5 corresponds to the spline curve through the points of minimum rotation and the diamond

Figure 4.4: $C_l(\alpha)$ vs $C_d(\alpha)$ curve. Straight line represents the sensitivity line. Circular points are the force coefficients of the bottom control cylinder in interference with the main cylinder. Intersection (1) $\alpha \approx 1.9$. Gap $g/D = 0.07$, $d/D = 0.06$, $Re = 50$

point refers to the numerical result of Mittal (2001). One can see that the proposed interpolation passes relatively close to the Mittal (2001) result for $Re = 100$. The small divergence is due to the fact that the geometrical conditions are not exactly the same. The gap and the control cylinder diameter used in Mittal (2001) are $g/D = 0.075$, $d/D = 0.05$, values slightly different to the ones used in this research. Although the wake control works well for the Reynolds numbers studied, it is not possible to guarantee wake control for arbitrarily high Reynolds numbers. This limitation was observed by Mittal (2001) when he attempted to control the flow around a cylinder in $Re = 10000$. Mittal (2001) found a remarkable effect for $\alpha = 5$. In this regime, the effect of the actuators is to organize and narrow the wake of the main cylinder when compared with the case without controllers, but the system was not stabilized. Mittal (2001) tested different values $\alpha > 5$ obtaining similar behavior when $\alpha = 5$ without obtaining a full control of the wake.

Given that the sensitivity to an external force $\nabla_F \sigma$ is a complex entity, the net effect of an external force on a particular eigenvalue given in Eq.(4.6) is also a complex entity. The real part represents the variation in linear growth of the eigenmode and gives an indication of the system's stability. The imaginary part on the other hand, represents a shift in the frequency which also contains information about stability properties. For example, if the frequency is moved out of the lock-in region it is possible to attenuate a VIV amplitude response even without completely suppressing the wake. This is considered a plausible cause of the VIV attenuation studied by Zhu et al. (2015). They placed two control rotating cylinders at 45° and

Figure 4.5: Minimum rotation rate for several Reynolds numbers. $Re_{cr} = 47.8$, $50 \leq Re \leq 150$, gap $g/D = 0.07D$, $d/D = 0.06$. Diamond correspond to Mittal (2001) results, continuous line present work.

-45° relative to the downstream part of the main cylinder, resulting in a smaller oscillation amplitude even though vortex shedding was still observed.

By applying sensitivity analysis to external forces modeled by Eq.(4.17) it is possible to obtain the variation in the frequency of a particular eigenvalue $\delta\omega$. Both Fig.4.6(a) and Fig.4.6(b) show the variation in frequency to external forces with $\alpha = 0$ and $\alpha = 1.9$ respectively. When $\alpha = 0$, Fig.4.6(a) shows that the force tends to decrease the frequency $\delta\omega$ in a region that includes the optimal position of the control cylinder with respect to growth rate (gray point in Fig.4.2a). The eigenvalue frequency decreases due to the effect of a drag external force for almost all the near regions of the cylinder except the recirculation region. Fig.4.6(b) shows the behavior of the system with the minimum rotation rate obtained in present work. Since the rotation rate is positive, we are focusing on a small control cylinder positioned at the top of the main cylinder. In this case the system acts in a opposite way than the $\alpha = 0$ case, in the region where the control cylinder must be positioned, the frequency in the eigenvalue tends to increase as shown by the strong red region in the upper part of the main cylinder in Fig.4.6(b), That is, the resulting eigenvalue after applying the external force is marginally stable but with a frequency greater than frequency without flow control.

4.4.3 Passive Control with cylinders

The growth rate variation contour to a drag force (gray contour in Fig.4.2a) shows that there are several configurations that can lead to control wake if drag forces are placed near to the main

Figure 4.6: Variation in the frequency $\delta\omega$ as function of the external force modeled by Eq.(4.17). $\delta\omega$ is calculated from the imaginary part of Eq.(4.6). $Re = 50$, $d/D = 0.06$. (a) Case $\alpha = 0$. (b) Case $\alpha = 1.9$, minimum rotation rate obtained with sensitivity and stability analysis

cylinder. The geometric disposition of the contours in Fig.4.2(a) shows an asymmetry between the front and rear part of the cylinder. This asymmetry can be observed noting that the regions where the eigenvalue is more susceptible to decrease are placed approximately a diameter upstream of the main cylinder while the rear regions are practically attached to it. In order to find

Figure 4.7: Growth rate variation contour for $\delta\lambda = -0.012$ given by the real part of Eq.(4.6) as a function of a drag force, case $C_l = 0$ in Eq.(4.17). $Re = 50$, $d/D = 0.06$. Force direction is a function of local flow velocity direction.

a more symmetrical region, we seek a combined effect of control cylinders to achieve the flow control. Fig.4.7 shows the contours for $\delta\lambda = -0.012$. In principle, a single drag-generating body such as a stationary cylinder may not stabilize the wake. However the combination of a few bodies in the regions of highest sensitivity can have a combined effect and stabilize the flow. With this idea in mind, we placed several cylinders around the main cylinder avoiding

regions that are not encompassed by the contours in Fig.4.7. In order to find the optimum distribution of the control cylinders, the stability problem (Eq. 4.5) was calculated for a system composed of the main cylinder and a small control cylinder whose position was uniformly varied in $\pi/8 \text{ rad}$ around the main cylinder. The cylinder's diameter and the gap are $d/D = 0.06$ and $g/D = 0.07$, Reynolds $\text{Re} = 50$. These parameters were chosen according to the sensitivity analysis results shown in section 4.4.1. Fig.4.8 shows the distribution of the growth rate λ of the most unstable eigenvalue as a function of angle θ measured between the centers of the main cylinder and the center of the control cylinder. Zero degrees are defined in the main cylinder downstream part, and the angles were measured anticlockwise. Fig.4.8 shows that, in all the coupled bodies configurations, the system is still unstable (the eigenvalues are always positive). However, it is observed that for some positions of the control cylinders, the eigenvalues of the two cylinders in interference are smaller compared to the most unstable eigenvalue of a single cylinder $\lambda = 0.013$ (continuous straight line). Here it is possible to say that the system was

Figure 4.8: Distribution of the most unstable eigenvalues as a function of the angle θ . The horizontal line is the eigenvalue without the control device. Continuous line is an interpolation of the discrete points shown. The Diameter and the gap are $d/D = 0.06$, $g/D = 0.07$. $\text{Re} = 50$, $\alpha = 0$

slightly stabilized by the cylinders positioned below the continuous line. This fact opens up the possibility of stabilizing the flow using a combination of the “stabilizing” cylinders. Fig.4.9(a) allows to observe the base flow of the main cylinder with the best control cylinders (the ones shown below the horizontal line in Fig.4.8). In Fig.4.9(b), the eigenvalues spectrum of the system is observed. Note that the superposition between the appropriate cylinders changed the most unstable eigenvalue from the linear unstable region to the stable region.

Figure 4.9: Velocity magnitude of the base flow for main cylinder with passive control cylinders, $Re = 50$, gap $g/D = 0.07$, $d/D = 0.06$. White lines are streamlines. (b) Eigenvalue spectrum main cylinder in interference with passive control cylinders, $Re = 50$. $g/D = 0.07$, $d/D = 0.06$

4.5 Summary and Conclusions

The sensitivity analysis was applied to the case of the flow around a cylinder. Sensitivity theory was used to identify regions where the external force produces the largest variation in the most unstable eigenvalue. This variation is computed by Eq.(4.6) where $\nabla_F \sigma$ is the sensitivity to external forces and $\delta \mathbf{F}$ is an arbitrary external force. The external force can be created by introducing small rotating and non-rotating control cylinders modeled by Eq.(4.17). C_l and C_d represent drag and lift coefficients and depend on the rotation rate of the cylinders. In the case of the non-rotating cylinders ($C_l = 0$) the control device exerts a force opposite to the local velocity direction of the base flow. The external force provided by the rotating cylinder has two components, one opposite to the local base flow velocity direction and another perpendicular to it. The direction of the external force depends on the local direction of the velocity, then

the force can be applied at all points of the domain and sensitivity maps of the most unstable eigenvalue to external forces can be built, with the help of these maps we know the best regions to place a control device. The results obtained were compared with the experimental result of Strykowski & Sreenivasan (1990) and numerical results of Marquet, Sipp & Jacquin (2008) for the case of non-rotating cylinders, $Re = 50$ and Mittal (2001) for the case of rotating cylinders $Re=100$. Using the contours of the sensitivity to external forces shown in Fig.4.2 it was possible to determine the appropriate gap to locate the control cylinders as $g/D = 0.07$. The appropriate gap was defined as the minimum distance between the main and the control cylinder along the axis $x = 0$ in Fig.4.2. After defining the appropriate gap using sensitivity analysis, we proceeded to determine the minimum rotation rate using sensitivity and linear stability theory. The values of the minimum rotation rate obtained with the two methods are in good agreement, however the results determined from the sensitivity theory have a lower computational cost especially when different Reynolds numbers are explored. In Fig.4.5 we show the interpolation of the minimum rotation rates to stabilize the wake in different Reynolds numbers. The curve obtained closely matches the Mittal (2001) numerical result for $Re = 100$. The application of sensitivity analysis to steady forces produced by rotating cylinders is a novel contribution and interpretation of flow control.

On the other hand, we explored the wake control using passive control cylinders, with the help of the sensitivity and stability theory, we were able to determine the appropriate positions to locate the controllers as shown in Fig.4.7 and Fig.4.8. The final configuration obtained effectively stabilizes the wake as we can see in Fig.4.9(b). From the results we note that the non-rotating control cylinders should not be positioned near to the boundary layer of the main cylinder, in this way we conclude that the main effect of the actuators is not necessary to transfer momentum from the outer flow to the cylinder boundary layer but the presence of a source of momentum in the Navier Stokes equations. The physical effect of the control devices is to reduce the circulation within the recirculation region of the main cylinder inducing stability.

5 SENSITIVITY ANALYSIS APPLIED TO THE CONSTRUCTION OF A REDUCED ORDER MODEL

This chapter concerns with the construction of a reduced order model by means of projecting the flow field onto a base generated from the sensitivity to base flow modifications. The reduced order models are widely studied in fluid mechanics since they allow to solve complex problems in spaces with fewer degrees of freedom. In the construction of reduced spaces it is essential to define the base flow of the system and the fixed point in the model. The mean is an interesting candidate given its stability properties, however it presents the problem of having a zero growth rate. With the purpose of continuing to use the mean flow as a base field, an auxiliar field is built using sensitivity analysis in order to disturb the mean flow in the direction of the greatest growth rate of the most unstable eigenvalue, allowing to recover the saturation path of the system. This chapter is based on the paper: *Building a low-dimensional model using Fourier modes and sensitivity analysis: An approach to study the flow around a cylinder beyond the first transition*. Submitted to Physics of fluids.

Paper: *Building a low-dimensional model using Fourier modes and sensitivity analysis: An approach to study the flow around a cylinder beyond the first transition*

Building a low-dimensional model using Fourier modes and sensitivity analysis: An approach to study the flow around a cylinder beyond the first transition

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(Dated: 27 February 2018)

A low-dimensional model is introduced to study the flow around a cylinder beyond the first transition ($Re \approx 47$). The flow variables at arbitrary Reynolds numbers are obtained from an asymptotic expansion around the growth rate of a perturbation, being possible to deduce the Landau equation in the third order term. Since the growth rate of the disturbance must be a small number for arbitrarily large Reynolds numbers, the fixed point in the model is defined very close to the limit cycle. With this in mind, the base of the reduced space is enriched with a mode disturbed in an optimal direction defined by the sensitivity to changes in the base flow. The other part of the base set is built with the most energetic Fourier modes obtained through snapshots of the temporal simulation. The reduced space constructed with this base reproduces the limit cycle as well as different hydrodynamic and fluid stability properties of the system in the physical space.

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5.1 Introduction

In the present work, a low dimensional model is constructed to simulate the two-dimensional flow around a cylinder at different Reynolds numbers. The reduced models are relevant to understand the main physical processes exhibited by the system, studied in a space with few degrees of freedom. Numerical results have shown that many complex flows exhibit low dimensionality (DEANE et al., 1991; MA; KARAMANOS; KARNIADAKIS, 2000). Different approximations of reduced models have been proposed for flow around cylinders, many of them focused on studying the primary wake transition. Next to the primary transition (STRYKOWSKI; SREENIVASAN, 1990) or Hopf bifurcation the system presents a vortex shedding dynamics which can be described using Galerkin methods (REMPFER, 2003). These methods are appropriate to obtain relevant physical information such as: the temporal dynamics of the system, the exchange of energy between small and large scales (NOACK; ECKELMANN, 1994) and understand the physical mechanisms involved in the transition. They are also a good alternative to develop 2D global stability analysis with low computational cost. The size and properties of the Galerkin models depend on the choice of the expansion modes in the Galerkin approximation. These modes must satisfy the incompressibility condition and the boundary conditions (NOACK et al., 2003). Deane et al. (1991) and Cao & Aubri (1993) constructed the spaces based on empirical eigenfunctions derived from an orthogonal POD decomposition, achieving the validation of the reduced model in low to moderate Reynolds numbers in complex geometries, and understanding some relevant physical aspects in the system as: mechanisms of frequencies selection, transition and bifurcation processes. The orthogonal POD decomposition (BERKOOZ et al., 1993) is a methodology that first identifies the most energetic modes of a system and second provides a way to obtain a low-order description of the dynamical system. Ma & Karniadakis (2002) constructed a reduced space based on the hierarchy of the most energetic scales of the fluid, resulting in the construction of spaces with few modes. Noack et al. (2003) proposes a hierarchy of reduced models for incompressible flows around a cylinder in order to combine the strengths of Galerkin's empirical and mathematical models in a hybrid approach. The model is based on a Karhunen-Loeve decomposition (POD) (ROWLEY, 2005) of order 8, and incorporates a mean flow correction as a reference field improving the resolution of the transient dynamics that develops until the flow attains a periodic Von Karman wake. Dynamic Mode Decomposition (DMD) is another widely used technique for the decomposition of data obtained from numerical simulation and experiments in the field of fluid mechanics. The method is introduced by means of the spectral analysis of the Koopman operator (ROWLEY et al., 2009; MEZIC, 2005) and was initially introduced by Schmid (2010). The method is based on the eigendecomposition of a best-fit linear

operator that approximates the numerical data (TAIRA et al., 2017). By definition, the eigenvalues analysis of the DMD provides the frequency and growth rate of the corresponding modes. In some way the DMD can be seen as a combination of the favorable aspects of both the POD and discrete Fourier decomposition (TAIRA et al., 2017; ROWLEY et al., 2009; CHEN; TU; ROWLEY, 2012) resulting in coherent structures of the fluid similar to the Fourier analysis, hence it has been used as a good alternative for the construction of reduced models (TISSOT et al., 2014; NOACK et al., 2016). Alternatively to the POD and DMD methods, we will use in this work a Fourier decomposition for building the reduced model, taking advantage of the hierarchical scale of energy in the Fourier modes for the flow around a cylinder, which allows comparing the Fourier series decomposition with an asymptotic expansion around the growth rate of a disturbance. This asymptotic expansion, arranged by order of magnitude of the perturbation, enables modeling the flow around a cylinder using the solution of the Landau equation and the Fourier modes (see section 5.3.3), where the Landau coefficients are easily calculated in the reduced space. The fluid dynamics in the reduced space is obtained by the projection of the incompressible flow equations into an appropriate orthonormal basis, constructed from the Fourier modes plus a perturbation of the mean flow using the sensitivity analysis (MARQUET; SIPP; JACQUIN, 2008; GIANNETTI; LUCHINI, 2007). The temporal dynamics, the hydrodynamic forces as well as the stability properties are calculated in the reduced space and compared with the results of the physical space showing a significant similarity. Additionally, the Landau equation (LANDAU; LIFSHITZ, 1987b) is used to understand the transient dynamics of the system in the reduced space at higher Reynolds numbers.

This article is organized as follows: Initially in chapter 5.3 we expose two relevant concepts in the conception of the reduced model: the linear stability analysis around a base flow (section 5.3.2) and the relationship between the Navier Stokes equations and the Landau equation (section 5.3.3). This last topic defines largely the main characteristics of the reduced space, given that the dynamics in the reduced space is developed using an asymptotic expansion up to third order in the growth rate of a disturbance. Chapter 5.4 shows the formulation of the reduced space. In this part we performed a Fourier decomposition (section 5.4.1) and a sensitivity analysis to changes in the base flow (section 5.4.2) in order to construct the space in section 5.4.3. Furthermore, section 5.4.4 shows the linear stability analysis in the reduced space. Finally chapter 5.5 shows the numerical solution in the reduced space. Section 5.5.1 shows the transient dynamics and section 5.5.2 shows the flow fields and forces in the cylinder.

5.2 Methodology

The computational mesh is defined in a Cartesian coordinate system with origin at the cylinder center. According to the converged mesh shown in Patino, Gioria & Meneghini (2017) and in the results in Table.5.1, Table.5.2 and Table.5.3, we can infer that the computational mesh defined by $-20D \leq x \leq 50D$, $-35D \leq y \leq 35D$ (with cylinder diameter $D = 1$) agrees with values shown in Norberg (2003) and Henderson (1995) for the hydrodynamic properties of the fluid, and with the results shown by Giannetti & Luchini (2007) and Barkley (2006) for the stability properties of the system. The NSE are discretized and solved using the finite element method. Taylor-Hood elements ($P2, P2, P1$) are used to represent velocity and pressure fields (u, v, P) in the spatial discretization. The space defined by these elements satisfies the condition of div-stability and has a 2nd-order overall accuracy (GUNZBURGER, 1989). A Dirichlet boundary condition $(u, v) = (1, 0)$ is used at the input and at the upper and lower sides of the domain. A non-slip boundary condition is used on the cylinder surface and in the outlet the standard outflow boundary condition is adopted. The temporal discretization of the NSE is given using the scheme Eq.(5.4) described in section 5.3.1. FreeFem (HECHT, 2012) is used to solve the systems of linear and nonlinear differential equations, to build the sparse matrices involved in the stability analyzes and in the construction of the operators in both the physical and the reduced spaces.

The convergence results are validated according to that shown in Table.5.1, Table.5.2 and Table.5.3. The convergence parameters used are: the drag coefficient, the lift coefficient and the frequency of vortex shedding for different Reynolds numbers, in addition to the growth rate and the frequency of the most unstable mode resulting from the linear stability analysis in the NSE for different Reynolds numbers. From the Table.5.2 we can verify that the results for the vortex shedding frequency and the hydrodynamic forces agree very well with the results shown by Norberg (2003) and Henderson (1995). From the linear stability analysis for different Reynolds numbers we see that the results agree with those reported by Giannetti & Luchini (2007) in the case where the base flow is the stationary solution of NSE, and with the frequency values reported by Barkley (2006) in the case where the base flow is the mean flow.

Table 5.1: Components of the most unstable eigenvalue $\lambda = \sigma + i\omega$ for several Reynolds numbers, calculated in the mesh $-20D \leq x \leq 50D$, $-35D \leq y \leq 35D$. Errors in the magnitudes are calculated based in the results reported by Giannetti & Luchini (2007)

Re	σ	ω	$\sigma(\%error)$	$\omega(\%error)$
50	0.013	0.7435	0.38	0.90
60	0.049	0.7512	1.40	0.40
80	0.095	0.7576	1.05	0.39
100	0.126	0.7415	1.93	1.75

5.3 Preliminar topics

5.3.1 Spatial and temporal discretization of the Navier Stokes equations

The Navier Stokes equations (NSE) for incompressible 2D fluids are given by Eq.(5.1).

$$\begin{aligned}\partial_t \mathbf{U} + \mathcal{N}(\mathbf{U}) + \mathcal{L}(\mathbf{U}, P) &= 0 \\ \nabla \cdot \mathbf{U} &= 0\end{aligned}\tag{5.1}$$

Here $\mathbf{U} = (u, v)$ is the velocity with components (u, v) in the directions (x, y) and $\mathcal{N}(\mathbf{U})$, $\mathcal{L}(\mathbf{U}, P)$ are the non-linear and linear operators done by

$$\mathcal{N}(\mathbf{U}) = (\mathbf{U} \cdot \nabla) \mathbf{U}, \quad \mathcal{L}(\mathbf{U}, P) = \nabla P - \frac{1}{\text{Re}} \nabla^2 \mathbf{U}\tag{5.2}$$

Re is the Reynolds number $\text{Re} = \frac{\rho U_\infty L}{\mu}$, ρ is the density, L represents a characteristic length of the body, U_∞ is the farfield velocity, and μ is the dynamic viscosity. The spatial discretization was done by the finite element method solved in the FreeFem software (HECHT, 2012) with boundary conditions (B.C): on the inlet and lateral sides $\mathbf{U} = (1, 0)$, on the body boundary $\mathbf{U} = (0, 0)$ and on the outlet $P \hat{n} - \text{Re}^{-1}(\nabla \mathbf{U}) \cdot \mathbf{n} = 0$. Eq.(5.1) can be discretized using the finite element method obtaining

$$\mathbf{M} \cdot \dot{\mathbf{U}} + \mathbf{N}(\mathbf{U}) \cdot \mathbf{U} + \frac{1}{\text{Re}} \mathbf{D} \cdot \mathbf{U} - \mathbf{R}P = 0\tag{5.3}$$

$\mathbf{M}$ represents the mass matrix, $\mathbf{N}(\mathbf{U})$ the non-linear convection matrix, $\mathbf{D}$ the linear divergence matrix and $\mathbf{R}$ the gradient matrix, where $\mathbf{U}$ and P are the discrete velocity and pressure variables.

The temporal discretization of the NSE was calculated using the Adams methods derived from temporal integration of Eq.(5.1). The integral of the linear operator $\mathcal{L}(\mathbf{U}, \mathbf{P})$ is calculated using the implicit method of Adams-Moulton and the nonlinear operator $\mathcal{N}(\mathbf{U})$ with an explicit

Figure 5.1: (a) Temporal solution of the NSE Eq.(5.1). Vorticity, $Re = 200$. (b) Drag and lift coefficients for $Re = 200$

Adams-Bashford method summarized in the formula

$$\mathbf{u}^{n+1} - \mathbf{u}^n = -\Delta t \sum_{i=0}^{k-1} \beta_i^1 \mathcal{L}(\mathbf{u}^{n+1-i}, p^{n+1-i}) - \Delta t \sum_{i=1}^k \beta_i^2 \mathcal{N}(\mathbf{u}^{n+1-i}) \quad (5.4)$$

Whose coefficients of second order ($k = 2$ in Eq.5.4) are

$$\beta_0^1 = 0.5, \beta_1^1 = 0.5, \beta_1^2 = 1.5, \beta_2^2 = -0.5 \quad (5.5)$$

Fig.5.1(a) shows the NSE solution for $Re = 200$. The drag and lift coefficients (C_d) and (C_l) along with the Strouhal number (S_t) are shown in Fig.5.1(b). In Table.5.2 the mean drag coefficient C_{dmean} , the RMS of the lift coefficient C_{lRMS} and the Strouhal number S_t for $Re = 100, 200, 300$ are shown. The results for the Strouhal number and C_{lRMS} are consistent with the results reported in Norberg (2003) and with the results for C_{dmean} reported in Henderson

(1995).

Table 5.2: Mean drag coefficient force C_{dmean} , RMS lift coefficient force C_{lRMS} and Strouhal number S_t for several Reynolds numbers

Re	S_t	C_{dmean}	C_{lRMS}
100	0.166	1.35	0.24
200	0.195	1.33	0.49
300	0.213	1.39	0.66

5.3.2 Stability analysis around a particular base flow $\mathbf{U}_p$

The spatial discretization of the NSE Eq.(5.1) leads to the system Eq.(5.3). An equivalent Navier Stokes equation for a field $\mathbf{U}_0(t)$ ($\mathbf{U}(t) = \mathbf{U}_0(t) + \mathbf{U}_p$, $P(t) = P_0(t) + P_p$) can be written in relation to a base field $\mathbf{U}_p$ as

$$\mathbf{M} \cdot \dot{\mathbf{U}}_0 + \mathbf{N}(\mathbf{U}_0) \cdot \mathbf{U}_0 + \left[\overline{\mathbf{N}}(\mathbf{U}_p) + \frac{1}{\text{Re}} \mathbf{D} \right] \cdot \mathbf{U}_0 - \mathbf{R}P_0 = \mathbf{F}(\mathbf{U}_p) \quad (5.6)$$

with operators

$$\begin{aligned} \overline{\mathbf{N}}(\mathbf{U}_p) \cdot \mathbf{U}_0 &= \mathbf{N}(\mathbf{U}_p) \cdot \mathbf{U}_0 + \mathbf{N}(\mathbf{U}_0) \cdot \mathbf{U}_p \\ \mathbf{F}(\mathbf{U}_p) &= - \left[\mathbf{N}(\mathbf{U}_p) \cdot \mathbf{U}_p + \frac{1}{\text{Re}} \mathbf{D} \cdot \mathbf{U}_p - \mathbf{R}P_p \right] \end{aligned} \quad (5.7)$$

$\overline{\mathbf{N}}(\mathbf{U}_p)$ represents the linear convection operator around the base field $\mathbf{U}_p$ and $\mathbf{F}(\mathbf{U}_p)$ the forcing term in the NSE. Using a modal decomposition $\mathbf{U}_0(x, y, t) = \hat{\mathbf{U}}_b(x, y) * e^{\lambda t}$, neglecting non linear terms in Eq.(5.6) and assuming $\mathbf{F}(\mathbf{U}_p) = 0$, a generalized eigenvalue problem is obtained

$$\left(\lambda \mathbf{M} + \overline{\mathbf{N}}(\mathbf{U}_p) + \frac{1}{\text{Re}} \mathbf{D} - \frac{1}{\epsilon} \mathbf{R}^T \cdot \mathbf{R} \right) \hat{\mathbf{U}}_b = 0 \quad (5.8)$$

with $\lambda = \sigma + i\omega$. For obtaining Eq.(5.8) we couple the velocity and pressure fields by means of a very small relaxation term $\epsilon = 1 \times 10^{-10}$, whose convergence was delineated in Patino, Gioria & Meneghini (2017). The linear growth rate of the perturbation σ is obtained from the real part of the most unstable eigenvalue λ ; while the frequency corresponds to the imaginary part of the eigenvalue. The onset of the vortex shedding instability can be obtained from a linear stability analysis in which the base field corresponds to a stationary solution of the NSE, i.e $\mathbf{U}_p = \mathbf{U}_{st}$, the results are shown in section 5.2. In Table5.3 we can see the stability results for different Reynolds numbers in the case $\mathbf{U}_p = \mathbf{U}_m$ being $\mathbf{U}_m$ the mean flow. The stability results agree with the findings obtained by Zielinska et al. (1997) and Barkley (2006) in the study of stability of a cylinder around the mean flow, where they observed that time-mean wake flows

are marginally stable to linear mechanisms and the associated instability frequencies are very close to those obtained in a DNS. Formally the mean flow is not a stationary solution of NSE Eq.(5.3) because the term $\mathbf{F}(\mathbf{U}_p)$ in Eq.(5.6) is not null by definition. In spite of this fact, linear stability analysis around the mean flow using Eq.(5.8) is allowed as shown in Barkley (2006). The term $\mathbf{F}(\mathbf{U}_m)$ is a solution of the NSE forced by a term due to the Reynolds stress tensor generated by the fluctuating field (BARKLEY, 2006), and considering infinitesimal perturbations in the forced NSE, one obtains the same linear equation Eq.(5.8) because the forcing term does not appear in the linear stability analysis since this is taken as a constant. We note that in all the cases studied the principal eigenvalue is marginally stable and the frequency of vortex shedding is very close to the frequency obtained from a non-linear DNS simulation as shown in Table5.3.

Table 5.3: Frequency component of the most unstable eigenvalue $\lambda = \sigma + i\omega$ for several Reynolds numbers. Strouhal error calculated from the DNS simulations shown in Table.5.2

Re	ω	$S_t(\%error)$
100	1.05	0.67
200	1.21	1.24
300	1.32	1.34

5.3.3 Navier Stokes and Landau equations in a reduced space

Consider a reduced space constructed from a base $\mathbf{T}$. The dynamic equation in the reduced space can be obtained by projecting the NSE Eq.(5.6) in the reduced space using the base $\mathbf{T}$ and the tranformation $\mathbf{U}_o = \mathbf{T} \cdot \mathbf{q}$ arriving to

$$\mathbf{M}_f \cdot \dot{\mathbf{q}} + \mathbf{N}_f(\mathbf{q}) \cdot \mathbf{q} + \left[\overline{\mathbf{N}}_f(\mathbf{U}_{st}) + \frac{1}{\text{Re}} \mathbf{D}_f \right] \cdot \mathbf{q} = 0 \quad (5.9)$$

with the condition $\mathbf{T}^T \cdot \mathbf{F}(\mathbf{U}_{st}) = 0$ and operators

$$\begin{aligned} \mathbf{M}_f &= \mathbf{T}^t \cdot \mathbf{M} \cdot \mathbf{T} \\ \mathbf{D}_f &= \mathbf{T}^t \cdot \mathbf{D} \cdot \mathbf{T} \\ \overline{\mathbf{N}}_f &= \mathbf{T}^t \cdot \mathbf{N}(\mathbf{U}_{st}) \cdot \mathbf{T} \\ \mathbf{N}_f(\mathbf{q}) \cdot \mathbf{q} &= \left(\sum_{i=0}^{N-1} \mathbf{q}_i * \mathbf{N}_i \right) \cdot \mathbf{q} \\ \mathbf{N}_i(\mathbf{q}) &= \mathbf{T}^t \cdot \mathbf{N}(\mathbf{T}_i) \cdot \mathbf{T} \end{aligned} \quad (5.10)$$

The field $\mathbf{U}_{st}$ is mapped from the reduced space using the identity $\mathbf{U}_{st} = \mathbf{U}_p + \mathbf{T} \cdot \mathbf{q}_{st}$, with $\mathbf{q}_{st}$ the fixed point in the reduced space. The solution of Eq.(5.9) naturally develops a limit cycle that according to the properties of the reduced space can be equal to the limit cycle in the physical space. The equilibrium point in the reduced space defined by the stationary solution of Eq.(5.9) has an unstable mode $\{\mathbf{U}_e; \lambda_e = \sigma + i\omega_e\}$ described by the solution of the normal and adjoint eigenvalue problem

$$\begin{aligned} \left[\lambda_e \mathbf{M}_f + \overline{\mathbf{N}}_f(\mathbf{U}_{st}) + \frac{1}{\text{Re}} \mathbf{D}_f \right] \cdot \mathbf{q}_e &= \mathbf{0} \\ \mathbf{U}_e &= \mathbf{T} \cdot \mathbf{q}_e, \quad \mathbf{U}_e^T \cdot \mathbf{M} \cdot \mathbf{U}_e^* = 1 \end{aligned} \quad (5.11)$$

$$\begin{aligned} \left[\lambda_e \mathbf{M}_f + \overline{\mathbf{N}}_f^T(\mathbf{U}_{st}) + \frac{1}{\text{Re}} \mathbf{D}_f \right] \cdot \mathbf{q}_{ae} &= \mathbf{0} \\ \mathbf{U}_{ae} &= \mathbf{T} \cdot \mathbf{q}_{ae}, \quad \mathbf{U}_{ae}^T \cdot \mathbf{M} \cdot \mathbf{U}_{ae} = 1 \end{aligned} \quad (5.12)$$

with $\mathbf{U}_{ae}$ the corresponding most unstable adjoint mode and $\mathbf{q}_e, \mathbf{q}_{ae}$ the direct and adjoint modes in the reduced space. Eq.(5.11) is the reduced space version of Eq.(5.8) for the normal modes $\mathbf{q} = \mathbf{q}_e e^{\lambda_e t}$. Given a small initial perturbation different to the optimal perturbation of the transient growth path, the orbit of the system initially follows the unstable mode $\mathbf{U}_e$ with an amplitude $\tilde{a}$ that grows exponentially with time while oscillating with an ω_e frequency, i.e $\tilde{a} \propto e^{\sigma t} e^{i\omega_e t}$. Therefore $\mathbf{q} \cong \tilde{a} \mathbf{U}_e e^{i\omega_e t}$ with $\frac{d\tilde{a}}{dt} \cong \sigma \tilde{a}$. This exponential growth is limited by nonlinear terms in an asymptotic expansion around σ . The first limiter term is the cubic interactions $|\tilde{a}| \tilde{a} e^{i\omega_e t}$, since it is the first term in oscillating at the same frequency of the unstable mode (see Eq.5.13). In this way, the oscillation amplitude in the limit cycle fulfills the condition $\sigma \tilde{a} \approx |\tilde{a}|^2 \tilde{a} \Rightarrow \tilde{a} \cong \mathcal{O}(\sigma^{1/2})$. Considering σ as a small number $\sigma \ll 1$, the nonlinear hydrodynamic problem can be asymptotically transformed into a sequence of linear problems using the value of $\sigma^{1/2}$ as an order parameter. Until third order the asymptotic expansion takes the form

$$\begin{aligned} \mathbf{U}_o(t) &= [\mathbf{U}_e \tilde{a} e^{i\omega_e t} + (*)] + [\mathbf{U}_{20} |\tilde{a}|^2 + (\mathbf{U}_{22} \tilde{a}^2 e^{2i\omega_e t} + (*))] + \\ &[(\mathbf{U}_{31} |\tilde{a}|^2 \tilde{a} e^{i\omega_e t} + (*))] + [(\mathbf{U}_{33} \tilde{a}^3 e^{3i\omega_e t} + (*))] + \mathcal{O}(\sigma^2) \end{aligned} \quad (5.13)$$

Where $\{\mathbf{U}_e; \mathbf{U}_{\alpha\beta}\}, \alpha, \beta = 1, 2, \dots, N$ represents an arbitrary basis of the asymptotic expansion (in this notation α represents the order of the base and β the frequency, for example, $\mathbf{U}_{22}$ represents the mode of order 2 with terms that oscillate with twice the frequency of the most

unstable mode). Using the identity $i\omega_e = \lambda_e - \sigma$ we obtain

$$\begin{aligned} \dot{\mathbf{U}}_o(t) = & \left[\lambda_e \mathbf{U}_e \cdot \tilde{a} e^{i\omega_e t} + \left(\frac{d\tilde{a}}{dt} - \sigma \tilde{a} \right) \mathbf{U}_e e^{i\omega_e t} + (*) \right] + [2i\omega_e \mathbf{U}_{22} \tilde{a}^2 e^{2i\omega_e t} + (*)] + \\ & [(i\omega_e \mathbf{U}_{31} \cdot \tilde{a} \tilde{a} e^{i\omega_e t} + (*)) + (3i\omega_e \mathbf{U}_{33} \tilde{a}^3 e^{3i\omega_e t} + (*))] \end{aligned} \quad (5.14)$$

The nonlinear term in the discrete NSE can be expressed as

$$\mathbf{N}(\mathbf{U}_o) \mathbf{U}_o = [\mathbf{N}_{20} \cdot \tilde{a} \tilde{a} + (\mathbf{N}_{22} \tilde{a}^2 e^{2i\omega_e t} + (*))] + [\mathbf{N}_{31} \cdot \tilde{a} \tilde{a} e^{i\omega_e t} + \mathbf{N}_{33} \tilde{a}^3 e^{3i\omega_e t} + (*)] \quad (5.15)$$

with the definitions

- $\mathbf{N}_{20} = \mathbf{N}(\mathbf{U}_e) \cdot \mathbf{U}_e^* + \mathbf{N}(\mathbf{U}_e^*) \cdot \mathbf{U}_e$
- $\mathbf{N}_{22} = \mathbf{N}(\mathbf{U}_e) \cdot \mathbf{U}_e$
- $\mathbf{N}_{31} = (\mathbf{N}(\mathbf{U}_e) \cdot \mathbf{U}_{20} + \mathbf{N}(\mathbf{U}_{20}) \cdot \mathbf{U}_e) + (\mathbf{N}(\mathbf{U}_e^*) \cdot \mathbf{U}_{22} + \mathbf{N}(\mathbf{U}_{22}) \cdot \mathbf{U}_e^*)$
- $\mathbf{N}_{33} = \mathbf{N}(\mathbf{U}_e) \cdot \mathbf{U}_{22} + \mathbf{N}(\mathbf{U}_{22}) \cdot \mathbf{U}_e$

Placing the asymptotic expansion Eq.(5.13), the derivative Eq.(5.14) and the non-linear term Eq.(5.15) within the NSE Eq.(5.6) and separating the terms of the same order in σ we obtain the cascade of linear problems (up to third order in $\sigma^{1/2}$)

$\mathcal{O}(\sigma^{1/2}) :$

$$\left(\left[\lambda_e \mathbf{M} + \overline{\mathbf{N}}(\mathbf{U}_{st}) + \frac{1}{\text{Re}} \mathbf{D} \right] \cdot \mathbf{U}_e - \mathbf{R}P_e \right) \tilde{a} e^{i\omega_e t} = 0$$

$\mathcal{O}(\sigma) :$

$$\left(\overline{\mathbf{N}}(\mathbf{U}_{st}) + \frac{1}{\text{Re}} \mathbf{D} \right) \cdot \mathbf{U}_{20} + \mathbf{N}_{20} - \mathbf{R}P_{20} = 0$$

$$\left(\left[2i\omega_e \mathbf{M} + \overline{\mathbf{N}}(\mathbf{U}_{st}) + \frac{1}{\text{Re}} \mathbf{D} \right] \cdot \mathbf{U}_{22} + \mathbf{N}_{22} - \mathbf{R}P_{22} \right) \tilde{a}^2 e^{2i\omega_e t} = 0$$

$\mathcal{O}(\sigma^{3/2}) :$

$$\left(\left[3i\omega_e \mathbf{M} + \overline{\mathbf{N}}(\mathbf{U}_{st}) + \frac{1}{\text{Re}} \mathbf{D} \right] \cdot \mathbf{U}_{33} + \mathbf{N}_{33} - \mathbf{R}P_{33} \right) \tilde{a}^3 e^{3i\omega_e t} = 0$$

$$\left(\left[\lambda_e \mathbf{M} + \overline{\mathbf{N}}(\mathbf{U}_{st}) + \frac{1}{\text{Re}} \mathbf{D} \right] \cdot \mathbf{U}_{31} + \mathbf{N}_{31} - \mathbf{R}P_{31} \right) \cdot \tilde{a} \tilde{a} e^{i\omega_e t} = \left(\frac{d\tilde{a}}{dt} - \sigma \tilde{a} \right) \mathbf{M} \cdot \mathbf{U}_e e^{i\omega_e t} \quad (5.16)$$

The third order term in the expansion ($\sigma^{3/2}$) is of special interest. Defining the operators

$$\begin{aligned}\hat{\mathbf{N}}_{31} &= \mathbf{N}_{31} - (\mathbf{U}_{ae}^T \cdot \mathbf{N}_{31}) \mathbf{M} \cdot \mathbf{U}_e \\ \mathbf{U}_{ae}^T \cdot \mathbf{N}_{31} &= \mu(1 - ic_3) \\ \mathcal{G}(\tilde{a}) &= \frac{d\tilde{a}}{dt} - \sigma\tilde{a} + \mu(1 - ic_3) |\tilde{a}|^2 \tilde{a}\end{aligned}\quad (5.17)$$

the term of third order in Eq.(5.16) (applying the operator $\mathbf{T}^T$ in the left and considering $\mathbf{U}_{31} = \mathbf{T} \cdot \mathbf{q}_{31}$) can be written as

$$\left[\lambda_e \mathbf{M}_f + \overline{\mathbf{N}}_f(\mathbf{U}_{st}) + \frac{1}{\text{Re}} \mathbf{D}_f \right] \cdot \mathbf{q}_{31} |\tilde{a}|^2 \tilde{a} = \mathcal{G}(\tilde{a}) \mathbf{T}^T \cdot \mathbf{M} \cdot \mathbf{U}_e - \mathbf{T}^T \cdot \hat{\mathbf{N}}_{31} |\tilde{a}|^2 \tilde{a} \quad (5.18)$$

which multiplied on the left by the adjoint mode $\mathbf{q}_{ae}^T$ together with the identities $\mathbf{q}_{ae}^T \cdot \mathbf{T}^T = \mathbf{U}_{ae}^T$, $\mathbf{U}_{ae}^T \cdot \mathbf{M} \cdot \mathbf{U}_{ae} = 1$, $\mathbf{U}_{ae}^T \cdot \hat{\mathbf{N}}_{31} = 0$ becomes

$$\mathcal{G}(\tilde{a}) = 0 \Leftrightarrow \frac{d\tilde{a}}{dt} - \sigma\tilde{a} + \mu(1 - ic_3) |\tilde{a}|^2 \tilde{a} = 0 \quad (5.19)$$

So far we have obtained an important conclusion: the equation that modulates the amplitude of the modes in the asymptotic expansion could be defined by the Landau equation.

The solution of the eigenvalue problem Eq.(5.11) is not unique, in this way, if the mode $\mathbf{U}_e$ is solution of the eigenvalue problem, $\delta e^{i\phi} \mathbf{U}_e$ will also be a solution, then

$$\begin{aligned}\tilde{\mathbf{U}}_e &= \delta e^{i\phi} \mathbf{U}_e \\ \tilde{\mathbf{U}}_{ae} &= \frac{1}{\delta} e^{-i\phi} \mathbf{U}_{ae}\end{aligned}\quad (5.20)$$

are solutions of the direct and adjoint problem with $\tilde{\mathbf{U}}_{ae}^T \cdot \mathbf{M} \cdot \tilde{\mathbf{U}}_e = 1$. From the definitions Eq.(5.17) one obtains the identities

$$\tilde{\mathbf{N}}_{31} = \delta^3 e^{i\phi} \mathbf{N}_{31}$$

$$\tilde{\mathbf{U}}_{ae}^T \cdot \tilde{\mathbf{N}}_{31} = \delta^2 \mathbf{U}_{ae}^T \cdot \mathbf{N}_{31} = \delta^2 \mu(1 - ic_3)$$

therefore the Landau equations in Eq.(5.21) are equivalent.

$$\begin{aligned}\frac{d\tilde{a}}{dt} - \sigma\tilde{a} + \mu(1 - ic_3) |\tilde{a}|^2 \tilde{a} &= 0 \\ \Downarrow \\ \frac{da}{dt} - \sigma a + \delta^2 \mu(1 - ic_3) |a|^2 a &= 0\end{aligned}\quad (5.21)$$

In particular, by choosing the delta value as $\delta = \sqrt{\sigma/\mu}$, it is obtained the renormalized Landau equation

$$\begin{aligned} \frac{da}{dt} - \sigma a + \sigma(1 - ic_3) |a|^2 a &= 0 \\ a(t) &= e^{i(\sigma c_3 t)} \end{aligned} \quad (5.22)$$

From the asymptotic expansion Eq.(5.13), we see that until second order (σ^2) the terms $\mathbf{U}_e$ and $\mathbf{U}_{31}$ must be equal to the first harmonic Fourier

$$(\tilde{\mathbf{U}}_e + |a|^2 \tilde{\mathbf{U}}_{31}) a e^{i\omega_e t} + \mathcal{O}(\sigma^2) = \mathbf{U}_1 e^{i\omega_s t}$$

that is, $\sigma c_3 + \omega_e = \omega_s$. Consequently, it follows that

$$c_3 = \frac{\omega_s - \omega_e}{\sigma} \quad (5.23)$$

In summary, using renormalization, the Landau equation becomes Eq.(5.22) and depends on two coefficients: the linear growth rate of the perturbation in the reduced space σ and the coefficient c_3 estimated by Eq.(5.23).

Comparing term by term the modes in the asymptotic expansion Eq.(5.13) and a Fourier modes decomposition, we can establish relations between the bases $\{\mathbf{U}_{st}; \mathbf{U}_{20}; \mathbf{U}_{22}; \mathbf{U}_{31}; \mathbf{U}_{33} \dots\}$ and $\{\mathbf{U}_m; \mathbf{U}_1; \mathbf{U}_2; \mathbf{U}_3 \dots\}$, with $\mathbf{U}_\alpha, \alpha = 1, 2, \dots N$ Fourier modes. In this perspective, the solution of the velocity field can be obtained from the expansion

$$\begin{aligned} \mathbf{U}(t) &= [\mathbf{U}_{st} + |a|^2 (\mathbf{U}_m - \mathbf{U}_{st})] + \{[\tilde{\mathbf{U}}_e + |a|^2 (\mathbf{U}_1 - \tilde{\mathbf{U}}_e)] a e^{i\omega_e t} + (*)\} + \\ &\{[\tilde{\mathbf{U}}_{22} + |a|^2 (\mathbf{U}_2 - \tilde{\mathbf{U}}_{22})] a^2 e^{2i\omega_e t} + (*)\} + \{[\tilde{\mathbf{U}}_{33} + |a|^2 (\mathbf{U}_3 - \tilde{\mathbf{U}}_{33})] a^3 e^{3i\omega_e t} + (*)\} + \\ &\sum_{n=4}^f (\mathbf{U}_n a^n e^{ni\omega_e t} + (*)) \end{aligned} \quad (5.24)$$

Using a Poisson equation (see section 5.5.2), the pressure field can be calculated from the velocity field Eq.(5.24), and consequently the forces in the cylinder can be determined using only the Landau equation with coefficients obtained in a reduced model. The main point in determining the velocity field with Eq.(5.24) is that the hydrodynamic variables of the fluid: velocity fields, pressure, Strouhal number, forces can be determined in a short computational time using a reduced space with few degrees of freedom.

5.4 Theoretical formulation of the reduced model

An appropriate definition for the fixed point is essential when building a reduced model. The most natural approach would be to define the fixed point as the stationary solution of the NSE $\mathbf{U}_{st}$. This fixed point was efficiently used by Noack et al. (2003) to model the transient dynamics in a Galerkin model, however it is not appropriate for our model given that the asymptotic expansion Eq.(5.13) forces the σ growth rate to be a small value. $\sigma \ll 1$ is only satisfied at Reynolds numbers near to the bifurcation point $Re \approx 47$, but it fails for larger Reynolds numbers as can be seen in table.5.2. The mean flow is not a good candidate either, since the system has practically a zero growth rate, making the transient solution to take a long time in the path of leaving the equilibrium point and reaching the limit cycle. Additionally, the oscillation frequency obtained from the linear stability problem around the mean flow coincides very well with the non-linear frequency of the system (see table.5.3 and Barkley (2006)) so the coefficient c_3 in Eq.(5.23) will diverge. According to the results of sensitivity to changes in the base flow shown in Marquet, Sipp & Jacquin (2008), a backflow in the rear region of a cylinder tends to increase the growth rate of the most unstable eigenvalue of the system and at the same time tends to decrease the frequency. Our idea is to slightly disturb the mean flow in an optimal direction defined by the sensitivity to base flow modifications, in this way we preserve the condition of non null σ with $\sigma \ll 1$ for arbitrary Reynolds numbers, allowing to define a Landau path in a reduced space manifold. The optimization criterion will be to determine the optimum direction in which the most unstable eigenvalue tends to have the greatest linear growth. Obviously, the path of the transient solution in the manifold of the physical space and the manifold of the reduced space will be different, however the final state (the limit cycle) will be the same in both spaces. Taking this into account, our model would fail to reproduce the hydrodynamic variables during the transition, but would reproduce adequately these variables in a developed flow. In section 5.4.2, we show how the optimal perturbation is defined using sensitivity to base flow modifications, and in section 5.4.3, we show how to construct the base of the reduced space using the perturbation and Fourier modes. The NSE solution is decomposed in Fourier modes in section 5.4.1 taking advantage of the energy hierarchy shown by the modes, this enables to compare the Fourier expansion with the asymptotic expansion Eq.(5.24).

5.4.1 Fourier modes decomposition

Considering the ansatz

$$\mathbf{U}(t) = \mathbf{U}_p + \mathbf{U}_0(t) \quad (5.25)$$

into the Navier-Stokes equations, we can approximate the flow by the finite Fourier expansion

$$\mathbf{U}_0(t) = (\mathbf{U}_m - \mathbf{U}_p) + \sum_{\alpha=1}^N (\mathbf{U}_\alpha e^{i\alpha\omega_s t} + c.c) \quad (5.26)$$

where $\mathbf{U}_m$ represents the mean flow, $\mathbf{U}_\alpha$ are the N first Fourier modes and $c.c$ the complex conjugate. The Fourier coefficients are expressed as

$$\mathbf{U}_{\alpha cos} = \frac{2}{T} \int_{t_0}^{t_0+T} \mathbf{U}(t) \cos(\omega_\alpha t) dt \quad (5.27)$$

$$\mathbf{U}_{\alpha sin} = \frac{2}{T} \int_{t_0}^{t_0+T} \mathbf{U}(t) \sin(\omega_\alpha t) dt \quad (5.28)$$

In this study the Fourier modes are calculated from 256 snapshots evenly distributed over a simulation period.

Figure 5.2: (a) u component of $\mathbf{U}_{\alpha cos}$, $\alpha = 1$. (b) v component of $\mathbf{U}_{\alpha cos}$, $\alpha = 1$, $\text{Re} = 100$

Figure 5.3: (a) u component of $\mathbf{U}_{\alpha cos}$, $\alpha = 2$. (b) v component of $\mathbf{U}_{\alpha cos}$, $\alpha = 2$, $\text{Re} = 100$

Fig.5.2 and Fig.5.3 show the cosine component of the first two Fourier modes. Fourier

odd modes are antisymmetric in relation to x axis

$$\begin{aligned} u(x, -y) &= -u(x, y) \\ v(x, -y) &= v(x, y) \end{aligned} \tag{5.29}$$

while the even modes are symmetric

$$\begin{aligned} u(x, -y) &= u(x, y) \\ v(x, -y) &= -v(x, y) \end{aligned} \tag{5.30}$$

Given that the basis of the reduced space is constructed using Fourier modes, it is essential

Figure 5.4: Energy of the first non null frequency fourier modes. $Re = 100$

to define the number of Fourier modes required to set the reduced space with enough degrees of freedom, so that the dynamic and stability properties in the original physical space are well defined. Calculating the energy of the Fourier modes E_ω as $E_\omega = \int_{\Omega} U_\alpha^2 d\Omega$, we can deduce from the behavior of the energy in function of the modes for $Re = 100$ (Fig.5.4) that the energy decays abruptly in the higher frequency modes. A similar behavior can be observed from the scale of the velocities in Fig.5.2 and Fig.5.3 where the second Fourier mode has already a much smaller magnitude than the first one, consequently, 10 Fourier modes ($N = 10$) can be defined as an appropriate measure in the construction of the reduced space.

5.4.2 Calculation of the sensitivity field $\hat{\mathcal{E}}$

The sensitivity to base flow modifications is defined as (MARQUET; SIPP; JACQUIN, 2008)

$$\mathcal{S} = -\nabla \mathbf{U}_e^* \cdot \mathbf{U}_{ae} + \nabla \mathbf{U}_{ae} \cdot \mathbf{U}_e^* \quad (5.31)$$

with $\mathbf{U}_{ae}$ the adjoint field associated to the most unstable mode $\mathbf{U}_e$ resulting of the linear stability analysis around the mean flow. The sensitivity field shows the regions where the most unstable eigenvalue is more susceptible to suffer alterations. This field is not solenoidal by construction, therefore the solenoidal component of $\mathcal{S}$ must be obtained in order to satisfy the condition of zero divergence in the NSE and to be able to disturb the mean flow in the construction of the particular solution $\mathbf{U}_p$. Assuming that $\mathcal{S}$ is a complex entity $\mathcal{S} = \hat{\mathcal{R}} + i\hat{\mathcal{C}}$ and has a solenoid and irrotational components, we have

$$\begin{aligned} \hat{\mathcal{R}} &= \hat{\mathcal{R}}_s + \hat{\mathcal{R}}_g \\ \hat{\mathcal{C}} &= \hat{\mathcal{C}}_s + \hat{\mathcal{C}}_g \end{aligned} \quad (5.32)$$

The gradient components can be calculated from the potential functions ϕ_g^r, ϕ_g^c with: $\hat{\mathcal{R}}_g = \mathbf{R}_0 \phi_g^r, \hat{\mathcal{C}}_g = \mathbf{R}_0 \phi_g^c$. Where $\mathbf{R}_0$ corresponds to the gradient matrix without the Dirichlet boundary conditions. Applying the divergence to the Eq.(5.32) on arrives to the Poisson discrete system

$$\begin{aligned} (\mathbf{R}_0^T \cdot \mathbf{R}_0) \phi_g^r &= \mathbf{R}_0^T \cdot \hat{\mathcal{R}} \\ (\mathbf{R}_0^T \cdot \mathbf{R}_0) \phi_g^c &= \mathbf{R}_0^T \cdot \hat{\mathcal{C}} \end{aligned} \quad (5.33)$$

After determining the potential function from Eq.(5.33) and calculating the $\hat{\mathcal{R}}_g$ and $\hat{\mathcal{C}}_g$ components, we perform the operation Eq.(5.32) to obtain the solutions $\hat{\mathcal{R}}_s$ and $\hat{\mathcal{C}}_s$.

Let us assume the arbitrary field $\hat{\mathcal{E}}$ given by the linear combination of the auxiliary fields $\hat{\mathcal{E}}_\sigma, \hat{\mathcal{E}}_\omega$

$$\hat{\mathcal{E}} = \alpha \hat{\mathcal{E}}_\sigma + \beta \hat{\mathcal{E}}_\omega \quad (5.34)$$

whose components are defined in the vector space generated by the vectors $\hat{\mathcal{R}}_s, \hat{\mathcal{C}}_s$.

$$\begin{aligned} \hat{\mathcal{E}}_\sigma &= \cos(\theta_\sigma) \hat{\mathcal{R}}_s + \sin(\theta_\sigma) \hat{\mathcal{C}}_s \\ \hat{\mathcal{E}}_\omega &= \sin(\theta_\omega) \hat{\mathcal{R}}_s + \cos(\theta_\omega) \hat{\mathcal{C}}_s \end{aligned} \quad (5.35)$$

$\cos(\theta_\sigma), \sin(\theta_\sigma), \sin(\theta_\omega), \cos(\theta_\omega)$ are arbitrary coefficients given by the projection of $\hat{\mathcal{E}}_\sigma, \hat{\mathcal{E}}_\omega$ in the base vectors $\hat{\mathcal{R}}_s, \hat{\mathcal{C}}_s$. In order to calculate these coefficients, we consider the variation of the most unstable eigenvalue given by the projection of $\hat{\mathcal{E}}$ onto the sensitivity field $\hat{\mathcal{S}}$

$$\Delta\lambda = \Delta\sigma + i\Delta\omega = (\hat{\mathcal{R}}_s + i\hat{\mathcal{C}}_s) \cdot \hat{\mathcal{E}} \quad (5.36)$$

Defining the field $\hat{\mathcal{E}}_\sigma$ as the auxiliary field which represents only variations in the eigenvalue growth rate and not in the frequency one has

$$\begin{aligned}\Delta\sigma &= \|\hat{\mathcal{R}}_s\|^2 \cos(\theta_\sigma) + \hat{\mathcal{R}}_s^T \cdot \hat{\mathcal{C}}_s \sin(\theta_\sigma) \\ \Delta\omega &= 0\end{aligned}\tag{5.37}$$

which allows to calculate the angle θ_σ as

$$\tan \theta_\sigma = \frac{-\hat{\mathcal{R}}_s \cdot \hat{\mathcal{C}}_s}{\|\hat{\mathcal{C}}_s\|^2}$$

In a similar way the field $\hat{\mathcal{E}}_\omega$ is defined as the auxiliary field which represents variations in the frequency and not in the growth rate

$$\begin{aligned}\Delta\sigma &= 0 \\ \Delta\omega &= \|\hat{\mathcal{C}}_s\|^2 \cos(\theta_\omega) + \hat{\mathcal{R}}_s^T \cdot \hat{\mathcal{C}}_s \sin(\theta_\omega)\end{aligned}\tag{5.38}$$

allowing to calculate the angle θ_ω as

$$\tan \theta_\omega = \frac{-\hat{\mathcal{R}}_s \cdot \hat{\mathcal{C}}_s}{\|\hat{\mathcal{R}}_s\|^2}$$

Taking into account the Eq.(5.34) and Eq.(5.36) the variation in the eigenvalue is done by

$$\begin{pmatrix} \Delta\sigma \\ \Delta\omega \end{pmatrix} = \begin{pmatrix} \|\hat{\mathcal{R}}_s\|^2 & \hat{\mathcal{R}}_s^T \cdot \hat{\mathcal{C}}_s \\ \hat{\mathcal{R}}_s^T \cdot \hat{\mathcal{C}}_s & \|\hat{\mathcal{C}}_s\|^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\tag{5.39}$$

the eigenvalues of the matrix in Eq.(5.39) represent the intensity of the perturbation and the eigenvectors the respective directions. The optimum mean flow perturbation is given by the eigenvalue of greater value in Eq.(5.40) and its eigenvectors define the optimal direction of growth in σ and decrease in ω . The shift in σ is necessary to model the linear growth rate on the path between the state $\mathbf{U}_p$ and the saturated state. The shift in ω is a prerequisite for modeling the amplitude of the saturation using the Landau model as shown in section 5.3.3. Subsequently the coefficients α, β arise from the eigenvalue problem

$$\begin{pmatrix} \hat{\mathcal{R}}_s^T \cdot \hat{\mathcal{R}}_s & \hat{\mathcal{R}}_s^T \cdot \hat{\mathcal{C}}_s \\ \hat{\mathcal{C}}_s^T \cdot \hat{\mathcal{R}}_s & \hat{\mathcal{C}}_s^T \cdot \hat{\mathcal{C}}_s \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mu \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\tag{5.40}$$

It must be guaranteed that the mean flow perturbation is much smaller than the mean flow itself. For this purpose we normalize the field $\hat{\mathcal{E}}$ using the standard norm

$$\|\hat{\mathcal{E}}\|_1 = \|\mathbf{U}_m\|_1\tag{5.41}$$

with $\| \mathbf{U}_m \|_1^2 = \mathbf{U}_m^H \cdot \mathbf{D} \cdot \mathbf{U}_m$ (being $\mathbf{D}$ the diffusion matrix). The norm 1 has been chosen since it does not depend on the domain size used in the discretization of the physical domain. With this in mind the perturbation is defined as

$$\mathbf{U}_p = \mathbf{U}_m + \delta \hat{\mathcal{E}} \quad (5.42)$$

Where δ is an arbitrarily small number. The sensitivity mode $\hat{\mathcal{E}}$ points in the direction of greatest variation of the most unstable eigenvalue and since the mode was normalized according to the Eq.(5.41), the magnitude of the disturbance remains defined by the factor δ . This factor is arbitrary because its function is to disturb the mean flow allowing a route for the saturation, however, it should not be close to 1. Different values of δ influence the time that the system takes to leave the particular solution $\mathbf{U}_p$ and fall into the saturated solution, but it not must affect the limit cycle. As an example in these simulations a value $\delta = 0.05$ was chosen.

5.4.3 Building the reduced space

Solving the linear stability problem Eq.(5.8) around the particular solution $\mathbf{U}_p = \mathbf{U}_m + \delta \hat{\mathcal{E}}$ we obtain the complex mode $\hat{\mathbf{U}}_{mod}$ whose real and imaginary part are used to enrich the space generated by the fourier modes. The vectors used in the construction of the base for the reduced space J_f are then

$$\begin{aligned} \mathbf{J}_f &= [\mathbf{U}_\alpha, \text{Real}(\hat{\mathbf{U}}_{mod}), \text{Imag}(\hat{\mathbf{U}}_{mod})] \\ \alpha &= 1, 2, \dots, N \end{aligned} \quad (5.43)$$

The base of the reduced space is built with the following Gram-Schmidt procedure beginning with the first fourier mode

$$\begin{aligned} P_{\mathbf{u}}(\mathbf{v}) &= \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} \\ \mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} P_{\mathbf{u}_j}(\mathbf{v}_k) \end{aligned} \quad (5.44)$$

In this way the base of the new reduced space will be the orthonormal system $\mathbf{T}$ constructed from the vectors Eq.(5.43) and the Gram-Schmidt procedure

$$\mathbf{T} = [\mathbf{T}_1, \dots, \mathbf{T}_{N+2}] \quad (5.45)$$

Considering that the transformation between the reduced space and the physical space is done by

$$\mathbf{U}_0(t) \cong \mathbf{T} \cdot \mathbf{q} \quad (5.46)$$

with $\mathbf{q}$ the degrees of freedom in the reduced space, the new dynamic equations can be derived from the projection of the NSE into the reduced space J_f . Following Ma & Karniadakis (2002) and Noack et al. (2003) the projection of the pressure term in the NSE can be neglected, therefore the new NSE in the reduced space is given by Eq.(5.9). The reduced model considered has an asymptotic stable limit cycle as can be seen from the dynamic solution $\mathbf{q}_1$ vs $\mathbf{q}_2$ Fig.5.5. The higher order Fourier coefficients solve the major harmonics of the dynamics. The lissajous figures for the higher frequency harmonics can be seen in the same Fig.5.5. The amplitude of the lissajous figures is arbitrary and depends on the normalization of the transformation matrix $\mathbf{T}$.

Figure 5.5: Phase portraits of $\mathbf{q}$ coefficients in the reduced space versus the first coefficient of the system. $\text{Re} = 200$

5.4.4 Stability analysis in the reduced space

The linear stability properties in the reduced space are carried out in two stages: First calculating the steady solution $\mathbf{q}_{st}$ of the NSE in the reduced space J_f using the Newton method, and second performing a linear stability analysis around of this stationary solution.

Neglecting the temporal variable in Eq.(5.9) we can define the functional

$$\mathcal{F}_s(\mathbf{q}) = \left[\mathbf{N}_f(\mathbf{q}) + \overline{\mathbf{N}}_f(\mathbf{U}_p) + \frac{1}{\text{Re}} \mathbf{D}_f \right] \cdot \mathbf{q} \quad (5.47)$$

with which we can solve the steady state in the reduced space using the Newton's iterative method

$$\mathbf{q}_{n+1} = \mathbf{q}_n - \mathbf{J}^{-1} \mathcal{F}(\mathbf{q}_n) \quad (5.48)$$

where the Jacobian matrix is defined by

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathcal{F}}{\partial q_0} & \frac{\partial \mathcal{F}}{\partial q_1} & \frac{\partial \mathcal{F}}{\partial q_2} & \cdots & \frac{\partial \mathcal{F}}{\partial q_{N_v}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

and the derivatives as

$$\frac{\partial \mathcal{F}}{\partial q_n} \approx \frac{\mathcal{F}(\mathbf{q} + \Delta \mathbf{q}_n) - \mathcal{F}(\mathbf{q})}{|\Delta \mathbf{q}_n|} \quad (5.49)$$

To study the stability properties in the reduced space we apply a perturbation to the stationary solution. Putting this perturbation in the NSE of the reduced space Eq.(5.9) neglecting nonlinear terms and considering a perturbation of the type

$$\mathbf{q}' = \mathbf{q}_e * e^{(\lambda t)} \quad (5.50)$$

we obtain the generalized eigenvalue equation Eq.(5.11). $\mathbf{q}_e$ is the most unstable mode in the reduced space, whose corresponding eigenvalues are given by

$$\lambda_e = \sigma_e + i\omega_e \quad (5.51)$$

In Fig.5.6 we can observe the eigenvalues spectrum of the problem Eq.(5.11) when the par-

Figure 5.6: Eigenvalues spectrum in the reduced space. $\text{Re} = 200$ (a) $\mathbf{U}_p = \mathbf{U}_m$. (b) $\mathbf{U}_p = \mathbf{U}_m + \delta \hat{\mathcal{E}}$.

ticular solution $\mathbf{U}_p$ is the mean flow and when the mean flow is disturbed by the sensitivity field. One sees that in the reduced space the system is also marginally stable to infinitesimal perturbations when the base field is the mean flow. The behavior of the disturbed spectrum is

very similar to that of the mean flow, but the pair of the most unstable eigenvalues are in the region of instability, allowing a transient analysis of the perturbation and a natural path to the Landau equation analysis in section 5.3.3.

Table 5.4: Stability analysis around a perturbation of the mean flow $\mathbf{U}_p = \mathbf{U}_m + \delta\hat{\mathcal{E}}$ in the reduced space (RS) for different Reynolds numbers

Re	σ_{rs}	ω_{rs}
100	1.075×10^{-2}	1.045
200	1.339×10^{-2}	1.187
300	1.142×10^{-2}	1.281

Table.5.4 shows the linear stability analysis in the reduced space J_f for different Reynolds numbers. According to the Fig.5.6(b) the flow around the cylinder in the reduced space becomes unstable face an infinitesimal perturbation and remains governed by an oscillatory instability of the particular solution. The stability spectrum consists of a pair of complex conjugated eigenvalues with positive growth while the remaining eigenvalues represent damped modes.

5.5 Numerical results in the reduced model

5.5.1 Landau dynamics

This section aims to show the dynamic transition between the unstable base flow $\mathbf{U}_p$ and the fully saturated system with vortex shedding using the Landau model in a reduced space. During this transition, the base flow is slightly modified with the progressive saturation of the instability associated with the reduction in the growth rate of the most unstable eigenvalue (MANTIC-LUGO; ARRATIA; GALLAIRE, 2014). The amplitude A is defined as a dynamic variable that reflects the time evolution of the most unstable eigenvalue amplitude whose instantaneous growth rate is given by $\sigma = \text{Real}(\lambda)$. In the transition, the relevant range of amplitudes is given by the domain defined between the solution of the base field $\mathbf{U}_p$ where A is close to zero and the saturated vortex shedding solution A^* where it holds $\sigma(A^*) = 0$ (MANTIC-LUGO; ARRATIA; GALLAIRE, 2015). In other words the path leading from A to A^* is the path required to advance from the base field $\mathbf{U}_p$ to the stable limit cycle capturing the transient dynamics. The DNS of the perturbation amplitude in the physical space $A = |\rho_{pert} e^{i\theta_{per}}|$ and the perturbation amplitude in

the reduced space J_f , $A = |\rho_{j_f} e^{i\theta_{j_f}}|$ are depicted in Fig.5.7 and calculated as

$$\begin{aligned} \mathbf{U}_0^T \cdot \mathbf{M} \cdot \hat{\mathbf{U}}_{mod}^* &= \rho_{per} e^{i\theta_{per}} \\ \mathbf{q}^T \cdot \mathbf{M}_f \cdot \hat{\mathbf{q}}_{mod}^* &= \rho_{J_f} e^{i\theta_{j_f}} \end{aligned} \quad (5.52)$$

with $\mathbf{U}_0$ the DNS temporal solution of the NSE, $\mathbf{M}$ the mass matrix and $\hat{\mathbf{U}}_{mod}^*$ the most unsta-

(a)

Figure 5.7: Amplitude of the perturbation in the physical space (red curves) and in the reduced space (black curves) for $\text{Re} = 300$

ble mode resulting of the stability analysis Eq.(5.8) with $\mathbf{U}_p = \mathbf{U}_m + \delta\hat{\mathcal{E}}$ and $\mathbf{q}$, $\mathbf{M}_f$, $\mathbf{q}_{mod}$ the corresponding variables in the reduced space. As a result of the calculations shown in Fig.5.7 we can observe that the perturbation in the physical space and in the space J_f are not equal in the initial transition range, however they converge to the same amplitude value in the final state of the transition. Consequently, the Landau coefficients obtained in the reduced space allow adequate modeling the limit cycle.

A close look at the disturbance growth rate shows an interesting behavior. Initially, the perturbation has a linear dynamic very similar in both the reduced and physical space, growing with a rate defined by the most unstable eigenvalue obtained from the linear stability analysis around the disturbed mean flow. Later, the nonlinear terms dominate the transitory growth of the disturbance making the trajectories in the spaces to be different. This phenomenon probably occurs because the reduced space is much poorer than the physical space, so that the dynamics of the system is limited to the combination of the most energetic Fourier modes. Finally, the perturbations converge again to the same limit cycle since the final state is well represented by the direct and conjugated fourier mode that oscillate with the non-linear frequency ω .

The Landau equation was given in Eq.(5.22) and can be written as

$$\frac{da}{dt} - \sigma a + \mu |a|^2 a = 0 \quad (5.53)$$

with $\mu = \mu_r + i\mu_i$; $\mu_r = \sigma$, $\mu_i = -\sigma c_3$. Assuming $a(t) = |a|e^{i\phi t}$ and separating the real and imaginary part of Eq.(5.53) we have

$$\frac{1}{|a|} \frac{d|a|}{dt} = \sigma - \mu_r |a|^2 \quad (5.54)$$

$$\frac{d\phi}{dt} = -\mu_i |a|^2 \quad (5.55)$$

Considering the condition $d|a|/dt = 0$ in the real part Eq.(5.54) we see that the saturation amplitude in the limit cycle is given by

$$\rho_{lc} = \sqrt{\frac{\sigma}{\mu_r}} \quad (5.56)$$

The velocity field will oscillate with a frequency $\omega + \frac{d\phi}{dt}$ with $\frac{d\phi}{dt}$ defined in Eq.(5.55), so that the shift in the frequency is interpreted as a variation in the frequency caused by the nonlinearities and it can be determined as

$$\Delta\omega = -\mu_i \rho_{lc}^2 = -\sigma \frac{\mu_i}{\mu_r} \quad (5.57)$$

allowing to determine the Landau coefficients. Knowing the coefficients, the Landau equation can be solved and with this we can calculate the solution of the Eq.(5.24) obtaining the velocity field using only parameters of the reduced space.

5.5.2 Hydrodynamic fields and forces

The hydrodynamic fields can be obtained using the Landau equation and the Fourier modes of the system as in Eq.(5.24). Besides to this approach, it is possible to calculate the fluid variables taking advantage of the properties of the reduced space. The Eq.(5.9) is the equivalent NSE in the reduced space, whose solution gives the degrees of freedom $\mathbf{q}$ in the reduced space. Knowing the dynamic solution in the reduced space and the base $\mathbf{T}$, the velocity field in the physical space can be calculated from the transformation $\mathbf{U}(t) \cong \mathbf{T} \cdot \mathbf{q} + \mathbf{U}_p$ with $\mathbf{U}_p = \mathbf{U}_m + \delta\mathcal{E}$. Both approaches (the Landau model and the transformation) are equivalent in the two-dimensional case. On the other side, in the three-dimensional case, this model opens the possibility of calculating the flow variables of a thin 3D cylinder with low computational cost, using only the Ginzburg Landau Eq.(5.58) fed by the 2D Landau coefficients obtained in the reduced space and the coefficient $\gamma(1 + ic_1)$ to be determined.

$$\frac{\partial a}{\partial t} - \sigma a + \sigma(1 - ic_3) |a|^2 a - \gamma(1 + ic_1) \frac{\partial^2 a}{\partial z^2} = 0 \quad (5.58)$$

In Fig.5.8 we see the velocity field reassembled from the reduced space. It is notorious the

Figure 5.8: Velocity fields $Re = 200$. (a) Velocity calculated using the dynamic solution in the reduced space. (b) Velocity calculated from the solution of the NSE in the physical space

similarity between the fields, considering that the velocity field in Fig.5.8(b) was obtained from the direct solution of NSE in a rich physical space with more than 50000 degrees of freedom and the reduced space only has 22. The pressure field can be obtained directly from the NSE. Applying the operator $\mathbf{R}^T$ (divergence) in Eq.(5.1) yields to a discrete Poisson equation

$$\mathbf{R}^T \cdot \mathbf{R}P = -\mathbf{R}^T \cdot \left(\mathbf{M} \cdot \dot{\mathbf{U}} + \mathbf{N}(\mathbf{U}) \cdot \mathbf{U} + \frac{1}{Re} \mathbf{D} \right) \quad (5.59)$$

where it is worth to emphasize that the matrix $\mathbf{R}$ is modified to adjust the Dirichlet boundary conditions for the velocity, then the lines of $\mathbf{R}$ corresponding to those degrees of freedom are eliminated in the discrete space. In Fig.5.9 we see the pressure field obtained from the reduced

Figure 5.9: Pressure fields $Re = 200$. (a) Pressure calculated using the dynamic solution in the reduced space. (b) Pressure calculated from the solution of the NSE in the physical space

space in comparison with the pressure in the physical space. Again the similarity between the

fields is remarkable. These results allow to infer that the hydrodynamic forces acting on the cylinder using the solution in the reduced space must be similar to the forces in the physical space, as can be corroborated in Fig.5.10 when compared with the forces in Fig.5.1(b). The

Figure 5.10: Hydrodynamic forces calculated using the dynamic solution in the reduced space. $Re = 200$

forces on the cylinder are obtained from the integration of the stress tensor over the entire cylinder surface.

5.6 Conclusion

In the present work a guideline for the construction of an appropriate reduced model is given to simulate the dynamics and stability properties of a flow around a cylinder for different Reynolds numbers. The trial bases were constructed based on hierarchical modes extracted from a temporal simulation of the NSE following a Fourier decomposition plus the mean flow disturbed by the sensitivity to base flow modifications mode. In this context, and taking advantage of the energy flow from the lower frequency modes to the higher ones, the reduced model was built with a base of just 10 Fourier modes plus the sensitivity mode. The reduced space constructed allows to obtain accurately several dynamic properties such as: vortex shedding frequency, hydrodynamic forces, pressure and velocity fields. In the same way, it allows to capture the most unstable mode as well as the saturated state of the system at different Reynolds numbers. As the model has few degrees of freedom compared to the physical space, the manifold in the reduced space becomes deformed, so that the transition path between the steady state and the saturation are not exactly the same in both spaces, however the low order model was able to

represent the saturated state for different Reynolds numbers. To obtain the transition path we use a Landau equation whose coefficients were previously obtained in the reduced space. Using the Landau model to study fluid properties at Reynolds numbers beyond the primary transition is quite useful, since the reduced model opens the possibility of modeling the flow around a slender cylinder using the Ginzburg-Landau equation with low computational cost.

6 CONCLUSIONS AND FUTURE WORK

This thesis addressed two very relevant problems in the field of fluid mechanics: the wake control problem in the flow around cylinders and the dynamic and stability solution in a reduced space. Both problems were addressed using the sensitivity theory and the adjoint method. First, for the wake control problem, an extension of the model presented by Marquet, Sipp & Jacquin (2008) was made to stabilize the flow around a cylinder at low and moderate Reynolds numbers. The model presented allows the calculation of drag and lift forces with good accuracy, enabling the study of passive and active wake suppressors. The results for the passive suppressors were in agreement with the numerical findings of Marquet, Sipp & Jacquin (2008) and Giannetti & Luchini (2007) and the experiments of Strykowski & Sreenivasan (1990). In the case of active suppressors, the results approached very well with the experimental cases studied by Mittal (2001) and Korkischko & Meneghini (2012). The presented numerical results are very relevant since they allow to elucidate the optimal position of the control cylinders in the passive case, and the position, gap and rotating rate of the cylinders in the active case. All of this doing sensitivity analysis, avoiding the use of multiple solutions of linear stability problems which is expensive from the computational point of view.

Later, the sensitivity analysis was used to disturb the mean flow to determine the shift mode in the construction of a reduced space. A Reduced model with a shift mode was an idea successfully used by Noack et al. (2003) and Tadmor et al. (2011) however it had the problem of depending on the stationary solution of the NSE and the solution of the linear stability problem around this field. The dependence of the fixed point with the stationary solution of the Navier Stokes equations is not compatible with the model of asymptotic expansion proposed in Chapter 5, since the growth rate of the disturbance is only a small number near to the bifurcation point $Re \approx 47$. The method proposed here allows to determine the shift mode from an optimal perturbation of the mean flow which is easily calculated at any Reynolds number. The optimal disturbance is delineated by the sensitivity to changes in the mean flow, allowing to define the optimal direction of greater growth rate of the eigenvalues. This modification restricts the fixed point of the model to be close to the limit cycle for arbitrary Reynolds numbers. The inclusion of this shift mode enables to make a transition path from the base field to the non-linear satura-

tion state, which can be modeled with the Landau equation. The proposed reduced space and the Landau model allowed to properly calculate different physical properties such as Strouhal number, drag and lift forces as well as velocity and pressure fields.

6.0.1 Future work

The solution Eq.(5.24) assumes that the vortices are thrown in phase along the cylinder's span, which is an unlikely physical condition in slender cylinders. In a complete 3D description, the phase $e^{i\phi}$ of the amplitude must vary in the direction z , describing the disturbance along the span. The pressure varies with z causing a longitudinal flow $w \propto \partial a / \partial z$ ($\mathbf{u} = (u, v, w)$) that at the same time disturbs the incompressibility condition with a term $\partial w / \partial z \propto \partial^2 a / \partial z^2$. Assuming a smooth variation along the span, that is, $\partial a / \partial z \ll 1$, the correction in the incompressibility condition must modify the Landau equation with a term $\partial^2 a / \partial z^2$. In this way, the 3D flow could be described by the Ginzburg Landau equation Eq.(5.58). Where most of the coefficients are inherited from the solution of the bidimensional landau equation in the reduced space. This reduced space also opens the possibility of numerically determining the remaining Ginzburg-Landau coefficient $\gamma(1 + ic_1)$ in Eq.(5.58), allowing to determine the dynamic of a 3D cylinder with low computational cost.

On the other hand, an important step was taken in the determination of the Landau coefficients in a reduced space, this fact opens the possibility of numerically determining the coefficients of a modified Landau equation to model the fluid structure interaction between the flow and a bidimensional cylinder with low computational cost.

From the sensitivity point of view, there are still several types of forces to explore wake control such as harmonic forces and forces generated by finite bodies. This type of forces represents a challenge since the Kronecker delta function approach used in Marquet, Sipp & Jacquin (2008) and Patino, Gioria & Meneghini (2017) can not be used.

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APPENDIX A – IMPLEMENTATION DETAILS OF THE SENSITIVITY ANALYSIS

A.1 Details of the adjoint equations

Integration by parts of each of the terms of the Eq.3.10:

$$\int_{\Omega} \hat{\mathbf{u}}^{\dagger} \cdot (\nabla \delta \hat{\mathbf{u}} \cdot \mathbf{U}) d\Omega = \int_{\Gamma} \delta \hat{\mathbf{u}} \cdot [\hat{\mathbf{u}}^{\dagger} (\mathbf{U} \cdot \hat{\mathbf{n}})] d\Gamma - \int_{\Omega} \delta \hat{\mathbf{u}} \cdot (\nabla \hat{\mathbf{u}}^{\dagger} \cdot \mathbf{U}) d\Omega \quad (\text{A.1})$$

$$\int_{\Omega} \nabla^2 \delta \mathbf{u} \cdot \hat{\mathbf{u}}^{\dagger} d\Omega = \int_{\Gamma} \hat{\mathbf{u}}^{\dagger} \cdot (\nabla \delta \hat{\mathbf{u}} \cdot \hat{\mathbf{n}}) d\Gamma - \int_{\Omega} \nabla \delta \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}}^{\dagger} d\Omega \quad (\text{A.2})$$

$$\int_{\Omega} \nabla \delta \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}}^{\dagger} d\Omega = \int_{\Gamma} \delta \hat{\mathbf{u}} \cdot (\nabla \hat{\mathbf{u}}^{\dagger} \cdot \hat{\mathbf{n}}) d\Gamma - \int_{\Omega} \delta \hat{\mathbf{u}} \cdot \nabla^2 \hat{\mathbf{u}}^{\dagger} d\Omega \quad (\text{A.3})$$

$$\int_{\Omega} \hat{\mathbf{u}}^{\dagger} \cdot \nabla \delta \hat{p} d\Omega = \int_{\Gamma} \delta \hat{p} (\hat{\mathbf{u}}^{\dagger} \cdot \hat{\mathbf{n}}) d\Gamma - \int_{\Omega} \delta \hat{p} \nabla \cdot \hat{\mathbf{u}}^{\dagger} d\Omega \quad (\text{A.4})$$

$$\int_{\Omega} p^{\dagger} (\nabla \cdot \delta \hat{\mathbf{u}}) d\Omega = \int_{\Gamma} p^{\dagger} \hat{\mathbf{n}} \cdot \delta \hat{\mathbf{u}} d\Gamma - \int_{\Omega} \nabla p^{\dagger} \cdot \delta \hat{\mathbf{u}} d\Omega \quad (\text{A.5})$$

To satisfy the boundary conditions of the adjoint equations Eq.3.12 it is necessary to impose a boundary condition on the variation of the mode $\delta \hat{\mathbf{q}}$ done by

$$-\delta p \hat{\mathbf{n}} + \text{Re}^{-1} \frac{\partial \delta \hat{\mathbf{u}}}{\partial n} = 0 \quad (\text{A.6})$$

A.2 Details of the sensitivity to a steady force calculations

The first term in Eq.3.15 is explicitly done by

$$(\hat{\mathbf{q}}^\dagger, \mathbf{P}(\sigma, \hat{\mathbf{q}}, \delta \mathbf{Q})) = \int_{\Omega} \hat{\mathbf{u}}^\dagger \cdot (\nabla \hat{\mathbf{u}} \cdot \delta \mathbf{U} + \nabla \delta \mathbf{U} \cdot \hat{\mathbf{u}}) d\Omega \quad (\text{A.7})$$

$$\int_{\Omega} \hat{\mathbf{u}}^\dagger \cdot (\nabla \delta \mathbf{U} \cdot \hat{\mathbf{u}}) d\Omega = \int_{\Gamma} \delta \mathbf{U} \cdot \hat{\mathbf{u}}^\dagger (\hat{\mathbf{u}}^* \cdot \hat{\mathbf{n}}) d\Gamma - \int_{\Omega} \delta \mathbf{U} \cdot (\nabla \hat{\mathbf{u}}^\dagger \cdot \hat{\mathbf{u}}^*) d\Omega \quad (\text{A.8})$$

$$- (\hat{\mathbf{q}}^\dagger, \mathbf{P}(\sigma, \hat{\mathbf{q}}, \delta \mathbf{Q})) = \int_{\Omega} \delta \mathbf{U} \cdot (-\nabla \hat{\mathbf{u}}^\dagger \cdot \hat{\mathbf{u}}^* + \hat{\mathbf{u}}^\dagger \cdot \nabla \hat{\mathbf{u}}^*) d\Omega - \int_{\Gamma} \delta \mathbf{U} \cdot (\hat{\mathbf{u}} \cdot \hat{\mathbf{n}}) \hat{\mathbf{u}}^\dagger d\Gamma \quad (\text{A.9})$$

While the second part of the Eq.3.15 is

$$(\hat{\mathbf{Q}}^\dagger, \mathbf{LNE}(\delta \mathbf{Q})) = \int_{\Omega} [\hat{\mathbf{U}}^\dagger \cdot (\nabla \delta \mathbf{U} \cdot \mathbf{U} + \nabla \mathbf{U} \cdot \delta \mathbf{U} - \text{Re}^{-1} \nabla^2 \delta \mathbf{U} + \nabla \delta p) + p^\dagger (\nabla \cdot \delta \mathbf{U})] d\Omega \quad (\text{A.10})$$

$$\int_{\Omega} \mathbf{U}^\dagger \cdot (\nabla \delta \mathbf{U} \cdot \mathbf{U}) d\Omega = \int_{\Gamma} \delta \mathbf{U} \cdot \mathbf{U}^\dagger (\mathbf{U} \cdot \hat{\mathbf{n}}) d\Gamma - \int_{\Omega} \delta \mathbf{U} \cdot (\nabla \mathbf{U}^\dagger \cdot \mathbf{U}) d\Omega \quad (\text{A.11})$$

$$\int_{\Omega} \mathbf{U}^\dagger \cdot (\nabla^2 \delta \mathbf{U}) d\Omega = \int_{\Gamma} \mathbf{U}^\dagger (\nabla \delta \mathbf{U} \cdot \hat{\mathbf{n}}) d\Gamma - \int_{\Omega} \nabla \mathbf{U}^\dagger \cdot (\nabla \delta \mathbf{U}) d\Omega \quad (\text{A.12})$$

$$\int_{\Omega} \nabla \mathbf{U}^\dagger \cdot \nabla \delta \mathbf{U} d\Omega = \int_{\Gamma} \delta \mathbf{U} \cdot (\nabla \hat{\mathbf{U}}^\dagger \cdot \hat{\mathbf{n}}) d\Gamma - \int_{\Omega} \delta \mathbf{U} \cdot \nabla^2 \mathbf{U}^\dagger d\Omega \quad (\text{A.13})$$

$$\int_{\Omega} \mathbf{U}^\dagger \cdot (\nabla \delta p) d\Omega = \int_{\Gamma} \delta p \mathbf{U}^\dagger \cdot \hat{\mathbf{n}} d\Gamma - \int_{\Omega} \delta p (\nabla \cdot \mathbf{U}^\dagger) d\Omega \quad (\text{A.14})$$

$$\int_{\Omega} p^\dagger (\nabla \cdot \delta \mathbf{U}) d\Omega = \int_{\Gamma} \delta \mathbf{U} \cdot p^\dagger \hat{\mathbf{n}} d\Gamma - \int_{\Omega} \delta \mathbf{U} \cdot \nabla p^\dagger d\Omega \quad (\text{A.15})$$

$$\begin{aligned}
-(\hat{\mathbf{Q}}^\dagger, \mathbf{LNE}(\delta \mathbf{Q})) &= \int_{\Omega} \delta \mathbf{U} \cdot (-\nabla \mathbf{U}^\dagger \cdot \mathbf{U} + \nabla \mathbf{U} \cdot \mathbf{U}^\dagger - \nabla p^\dagger - \text{Re}^{-1} \nabla^2 \mathbf{U}^\dagger) d\Omega \\
&+ \int_{\Gamma} \delta \mathbf{U} \cdot [p^\dagger \hat{n} + \text{Re}^{-1} (\nabla \mathbf{U}^\dagger \cdot \hat{n}) + (\mathbf{U} \cdot \hat{n}) \mathbf{U}^\dagger] d\Gamma
\end{aligned} \tag{A.16}$$

putting together the expressions Eq.A.9 and Eq.A.16 and bearing in mind that $\delta \mathbf{Q} \neq 0$ we arrive to the definition of the adjoint equations of the base flow Eq.3.16. In order to define the boundary conditions it is necessary to suppose

$$\text{Re}^{-1} \frac{\partial \delta \mathbf{U}}{\partial n} - \delta p \hat{n} = 0 \tag{A.17}$$

A.3 Details wavemaker equations

$$\int_{\Omega} (\hat{\mathbf{u}}^\dagger \cdot \nabla \delta \hat{\mathbf{u}}) \cdot \mathbf{U} d\Omega = \int_{\Gamma} \delta \hat{\mathbf{u}} \cdot (\mathbf{U} \cdot \hat{n}) \hat{\mathbf{u}}^\dagger d\Gamma - \int_{\Omega} \delta \hat{\mathbf{u}} \cdot (\nabla \hat{\mathbf{u}}^\dagger \cdot \mathbf{U}) d\Omega \tag{A.18}$$

$$\int_{\Omega} \hat{\mathbf{u}}^\dagger \cdot \nabla^2 \delta \hat{\mathbf{u}} d\Omega = \int_{\Gamma} \hat{\mathbf{u}}^\dagger \cdot (\nabla \delta \hat{\mathbf{u}} \cdot \hat{n}) d\Gamma - \int_{\Omega} \nabla \hat{\mathbf{u}}^\dagger \cdot \nabla \delta \hat{\mathbf{u}} d\Omega \tag{A.19}$$

$$\int_{\Omega} \nabla \hat{\mathbf{u}}^\dagger \cdot \nabla \delta \hat{\mathbf{u}} d\Omega = \int_{\Gamma} \delta \hat{\mathbf{u}} \cdot (\nabla \hat{\mathbf{u}}^\dagger \cdot \hat{n}) d\Gamma - \int_{\Omega} \delta \hat{\mathbf{u}} \cdot \nabla^2 \hat{\mathbf{u}}^\dagger d\Omega \tag{A.20}$$

$$\int_{\Omega} \hat{\mathbf{u}}^\dagger \cdot \nabla \delta \hat{p} d\Omega = \int_{\Gamma} \hat{\mathbf{u}}^\dagger \cdot (\delta \hat{p} \hat{n}) d\Gamma - \int_{\Omega} \delta \hat{p} \nabla \cdot \hat{\mathbf{u}}^\dagger d\Omega \tag{A.21}$$

$$\int_{\Omega} p^\dagger \cdot (\nabla \cdot \delta \hat{\mathbf{u}}) d\Omega = \int_{\Gamma} \delta \hat{\mathbf{u}} \cdot p^\dagger \hat{n} d\Gamma - \int_{\Omega} \delta \hat{\mathbf{u}} \cdot \nabla p^\dagger d\Omega \tag{A.22}$$

$$\begin{aligned}
&\int_{\Omega} \delta \hat{\mathbf{u}} \cdot [\sigma \hat{\mathbf{u}}^\dagger - \nabla \hat{\mathbf{u}}^\dagger \cdot \mathbf{U} + \nabla \mathbf{U} \cdot \hat{\mathbf{u}}^\dagger + \nabla p^\dagger - \text{Re}^{-1} \nabla^2 \hat{\mathbf{u}}^\dagger] d\Omega + \\
&\int_{\Gamma} \delta \hat{\mathbf{u}} \cdot [p^\dagger \hat{n} - \text{Re}^{-1} (\nabla \hat{\mathbf{u}}^\dagger \cdot \hat{n}) - (\mathbf{U} \cdot \hat{n}) \hat{\mathbf{u}}^\dagger] d\Gamma
\end{aligned} \tag{A.23}$$

APPENDIX B – PUBLICATIONS AND PARTICIPATION IN CONFERENCES

B.1 International publications

- G. A. Patino, R. S. Gioria, J. R. Meneghini. Evaluating the control of a cylinder wake by the method of sensitivity analysis. *Physics of Fluids* 29, 044103 (2017); doi: 10.1063/1.4979482.
- G. A. Patino, R. S. Gioria, J. A. P. Aranha and J. R. Meneghini. Building a low-dimensional model using Fourier modes and sensitivity analysis: An approach to study the flow around a cylinder beyond the first transition. Submitted to *Physics of Fluids*.

B.2 International Conferences

- G. A. Patino, R. S. Gioria, J. A. P. Aranha and J. R. Meneghini. Building a low dimensional Galerkin model using sensitivity analysis. 13th ERCOFTAC SIG 33 Workshop. Paraty 2018. *Oral presentation*
- G. A. Patino, R. S. Gioria, J. A. P. Aranha and J. R. Meneghini. Modeling the flow around a cylinder using the Landau equation and sensitivity analysis. 1st German-Brazilian Workshop on Computational Mechanics. Sao Paulo 2018. *Oral presentation*
- J Meneghini, G Patino, R Gioria. Sensitivity analysis of small circular cylinders as wake control. APS Meeting Portland 2016.
- G. Patino, M. Silva-Ortega, R.S. Gioria, G.S.R. Assi, J.R. Meneghini. Investigation of circular cylinder VIV passive control device using flow sensitivity analysis. 6th International Symposium on Bifurcations and instabilities in Fluid Dynamics - BIFD2015. Paris. *Oral presentation*

- G Patino, R Gioria, J Meneghini. Sensitivity analysis applied to stalled airfoil wake and steady control. Bulletin of the American Physical Society. San Francisco. 2014. *Oral presentation*